# Study of rigid solids movement in a viscous fluid

Lamis Marlyn Kenedy Sabbagh

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# Étude du mouvement de solides rigides dans un fluide visqueux

**Présentée par**


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**Le 22 Novembre 2018**

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**Dedicated to the soul of my father.**

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# Abstract

This thesis is devoted to the mathematical analysis of the problem of motion of a finite number of homogeneous rigid bodies within a homogeneous incompressible viscous fluid. Viscous fluids are classified into two categories: Newtonian fluids, and non-Newtonian fluids. First, we consider the system formed by the incompressible Navier-Stokes equations coupled with Newton's laws to describe the movement of several rigid disks within a homogeneous viscous Newtonian fluid in the whole space $\mathbb{R}^2$. We show the well-posedness of this system up to the occurrence of the first collision. Then we eliminate all type of contacts that may occur if the fluid domain remains connected at any time. With this assumption, the considered system is well-posed globally in time. In the second part of this thesis, we prove the non-uniqueness of weak solutions to the fluid-rigid body interaction problem in 3D in Newtonian fluid after collision. We show that there exist some initial conditions such that we can extend weak solutions after the time for which contact has taken place by two different ways. Finally, in the last part, we study the two-dimensional motion of a finite number of disks immersed in a cavity filled with a viscoelastic fluid such as polymeric solutions. The incompressible Navier–Stokes equations are used to model the flow of the solvent, in which the elastic extra stress tensor appears as a source term. In this part, we suppose that the extra stress tensor satisfies either the Oldroyd or the regularized Oldroyd constitutive differential law. In both cases, we prove the existence and uniqueness of local-in-time strong solutions of the considered moving-boundary problem.

**Key words:** Fluid-solid interaction, Navier-Stokes equations, strong solutions, contact problem, viscous fluids, Oldroyd model.

# Résumé

Cette thèse est consacrée à l'analyse mathématique du problème du mouvement d'un nombre fini de corps rigides homogènes au sein d'un fluide visqueux incompressible homogène. Les fluides visqueux sont classés en deux catégories: les fluides newtoniens et les fluides non newtoniens. En premier lieu, nous considérons le système formé par les équations de Navier Stokes incompressible couplées aux lois de Newton pour décrire le mouvement de plusieurs disques rigides dans un fluide newtonien visqueux homogène dans l'ensemble de l'espace $\mathbb{R}^2$. Nous montrons que ce probleme est bien posé jusqu'à l'apparition de la première collision. Ensuite, nous éliminons tous les types de contacts pouvant survenir si le domaine fluide reste connexe à tout moment. Avec cette hypothèse, le système considéré est globalement bien posé. Dans la deuxième partie de cette thèse, nous montrons la non-unicité des solutions faibles au problème d'interaction fluide-solide 3D, dans le cas d'un fluide newtonien, après collision. Nous montrons qu'il existe des conditions initiales telles que nous pouvons étendre les solutions faibles après le temps pour lequel le contact a eu lieu de deux manières différentes. Enfin, dans la dernière partie, nous étudions le mouvement bidimensionnel d'un nombre fini de disques immergés dans une cavité remplie d'un fluide viscoélastique tel que des solutions polymériques. Les équations de Navier Stokes incompressible sont utilisées pour modéliser le solvant, dans lesquelles un tenseur de contrainte élastique supplémentaire apparaît comme un terme source. Dans cette partie, nous supposons que le tenseur de contrainte supplémentaire satisfait la loi differentielle d'Oldroyd ou sa version regularisée. Dans les deux cas, nous prouvons l'existence et l'unicité des solutions fortes locales en temps du problème considéré.

**Mots clés:** Interaction fluide-solide, équations de Navier-Stokes, solutions fortes, problème de contact, fluides visqueux, modèle Oldroyd.

# Aperçu de la thèse

Cette thèse est consacrée à l'étude du mouvement d'un nombre fini de corps rigides homogènes dans un fluide visqueux incompressible. Ce problème figure parmi les problèmes les plus populaires en sciences appliquées, car il comprend des applications biologiques telles que la circulation sanguine dans les artères et les veines, la coagulation sanguine et la modélisation de la parole. En outre, il est couramment utilisé pour décrire le comportement des pulvérisations et pour concevoir et développer des implants prothétiques.

Il existe plusieurs approches pour modéliser les interactions des particules avec les fluides. Ces approches dépendent de la taille des structures (petite ou grande), des propriétés des particules (déformables ou rigides), du type de fluide: compressible versus incompressible, newtonien versus non newtonien, etc. On suppose que les particules sont volumineuses, indéformable et avoir des limites lisses. En particulier, nous supposons que les particules sont des disques de dimension 2 et sont des sphères de dimension 3. De plus, nous supposons que le fluide est un fluide visqueux et incompressible homogène et se déplaçant sous l'action d'une force externe du corps.

Le problème d'interaction fluide-solide peut être divisé en trois parties: problème du fluide, problème solide et condition de couplage. Nous utilisons l'approche eulérienne qui est le point de vue habituel en mécanique des fluides pour décrire le mouvement du fluide. Plus précisément, nous utilisons les équations incompressibles de Navier-Stokes pour décrire le mouvement du flux. Nous utilisons également les lois de Newton pour le moment linéaire et angulaire pour décrire le mouvement des particules. Le couplage entre le sous-système fluide et le sous-système solide se fait via la ou les interfaces fluides-solides en supposant que la vitesse du fluide à chaque frontière fluide-solide est égale à la vitesse du solide qui s'y trouve. Le déplacement du corps solide modifie le domaine fluide. Par conséquent, le domaine des fluides est inconnu a priori et nous avons affaire à un problème à frontière libre.

les modèles fluide/particule décrits ci-dessus ont été largement étudiés ces 18 dernières années. Dans un premier temps, des solutions ont été construites localement en temps. «Localement» pour n'avoir considéré le système qu'avant d'éventuelles collisions entre particules. En 2 dimensions, il est même apparu que le contact est le seul phénomène empêchant de construire des solutions globales [41]. Etude de la collision en temps fini s'est faite pour ces 13 dernières années. Des outils pour contrôler la distance entre les particules ont été développées [12, 24]. Ils permettent de mettre en évidence l'absence de contact et ainsi de fournir des solutions globales aux systèmes Navier-Stokes + Newton dans les configurations simplifiées (une sphere ou un disque se déplace dans une cavité de forme simple). Dans [24], l'auteur montre que toute solution forte est globale sous l'absence de forces externes dans le cas d'un disque en mouvement dans le demi-espace $\mathbb{R}^2_+$. Notre premier objectif dans cette thèse est de généraliser les résultats d'existence locale de solutions fortes pour le cas d'un seul solide [12, 40] au cas de plusieurs corps rigides se déplaçant dans un fluide visqueux dans toute l'espace $\mathbb{R}^2$. Plus précisément, nous montrons le caractère bien posé de ce système jusqu'à l'apparition de la première collision. Ensuite, nous éliminons tous les types de contacts pouvant survenir si le domaine fluide reste connexe à tout moment. Avec cette hypothèse, le système considéré est globalement bien posé. Nous soulignons que l'hypothèse sur le domaine des fluides doit être connexe à tout moment est toujours valable dans le cas de deux corps en mouvement et que des contacts multiples sont vraiment improbables si nous partons d'une suspension suffisamment diluée. Cette non-collision est paradoxale. Elle repose sur le fait que les limites des structures solides sont suffisamment régulières. Rien n'est aussi lisse qu'un disque ou une sphère. Par exemple, les auteurs de [20] ont étudié l'effet induit par la rugosité du corps rigide et la limite du domaine sur le processus de collision. Ils montrent que la collision se produit pour le modèle d'un corps singulier qui tombe sur une rampe.

La question de l'existence de solutions faibles au problème d'interaction fluide solide indépendamment de la collision a été posée par San Martin et ses coauteurs dans [34] en dimension 2 et par Feireisl [16] en dimension 3. Par la suite, [39] à étudié la question de l'unicité des solutions faibles dans le cas à deux dimensions. Il a prouvé que l'unicité des solutions faibles ne tient pas après un contact en 2D. Ce résultat de non-unicité peut-être raisonnablement expliqué par le fait qu'il n'y a pas de loi de rebond pour décrire la dynamique après une collision. La question de l'unicité des solutions faibles en 3D est le deuxieme resultat de cette thèse. Nous prouvons la non-unicité des solutions faibles au problème de l'interaction fluide-rigide en 3D pour un le fluide newtonien après la collision. Plus précisément, nous montrons qu'il existe des conditions initiales telles que nous pouvons étendre les solutions faibles après le temps pour lequel le contact a eu lieu de deux manières différentes. Pour la première solution, le corps s'éloigne du bord de la cavité, tandis que la seconde solution est construite de telle sorte que le corps reste en contact avec le bord de la cavité après une collision. La nouveauté de ce travail est que nous prouvons la non-unicité des solutions faibles pour le problème du mouvement d'un corps rigide dans un fluide visqueux en 3D après un contact avec un terme de source raisonnable.

Enfin, nous étudions le mouvement bidimensionnel d'un nombre fini de disques immergés dans une cavité remplie d'un fluide viscoélastique tel que des solutions polymériques. Les équations incompressibles de Navier – Stokes sont utilisées pour modéliser le flux du solvant, dans lesquelles un tenseur de contrainte supplémentaire élastique apparaît comme un terme source. Ici, nous supposons que le tenseur de contrainte supplémentaire satisfait à la loi différentielle d'Oldroyd ou à sa version régularisée. Nous montrons l'existence et l'unicité des solutions fortes locales. En l'absence de particules l'existence globale de solutions faibles au modèle Oldroyd standard sans diffusion dans le cas de corotation uniquement ($a = 0$) est démontré [31] pour toutes les données. Dans le cas général, l'existence et l'unicité des solutions fortes locales ont été présentées dans [22]; de plus, si le fluide n'est pas trop élastique et si les données sont suffisamment petites, alors les solutions sont globales. L'hypothèse de petitesse a été supprimée plus tard dans [33]. Finalement, à notre connaissance, seuls quelques résultats concernent l'existence et l'unicité de solutions solides pour le modèle diffusif d'Oldroyd: l'existence de solutions globales fortes en 2D pour le modèle diffusif Oldroyd-B (soit $a = +1$) et unicité de la solution parmi une classe de solutions fortes [11].

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# Introduction

## Sommaire



Fluid-structure interaction ((FSI) for short) is the interaction of some movable or deformable structure(s) with an internal or surrounding fluid flow. Studying the interaction between fluid and solid structure(s) is a crucial concern in many engineering systems. The success or failure of a product depends on how well it handles interactions between fluids and structures. When fluid solid interaction occurs, the flow of the fluid may cause the structure(s) to move, spin or even change shape due to flow-induced pressure and shear loads, which in turn changes the fluid flow. This two-way interaction loop continues through multiple cycles, possibly resulting in structural damage or less-than-optimal flow. Tacoma Narrows Bridge is one of the famous examples of large-scale failure. The bridge collapsed in 1940 because normal speed winds produced aeroelastic [1] flutter that matched the bridge's natural frequency [6] (see Figure 1).

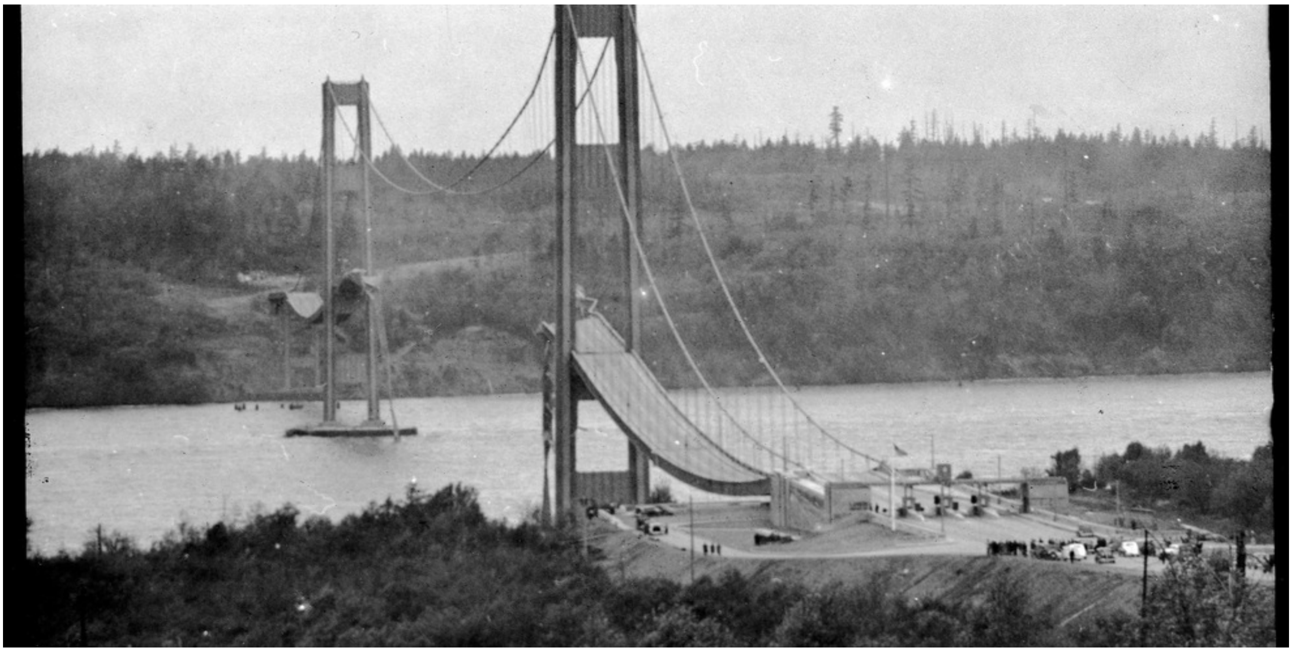

Figure 1.1 – Tacoma Narrows Bridge roadway twisted and vibrated violently under 64 $km/h$ winds on the day of the collapse

Aircraft wings and turbine blades may also break due to resulting oscillations from FSI. In an aircraft, as the speed of the wind increases, there may be a point at which the structural damping is insufficient to damp out the motions which are increasing due to aerodynamic energy being added to the structure. This vibration can cause structural failure and therefore considering flutter characteristics is an essential part of designing aircraft. In wind turbines, the trend nowadays is to design larger turbines to increase the power output. However, manufacturing larger turbines presents new challenges for structural engineers and might require blade materials that are both lighter and stiffer than the ones currently used. Wind turbin designs with larger turbines and relatively softer blades and flexible blades introduces

1. Aeroelasticity is the branch of physics and engineering that studies the interactions between the inertial, elastic, and aerodynamic forces (force exerted on a body by the air or other gases) that occur when an elastic body is exposed to a fluid flow.

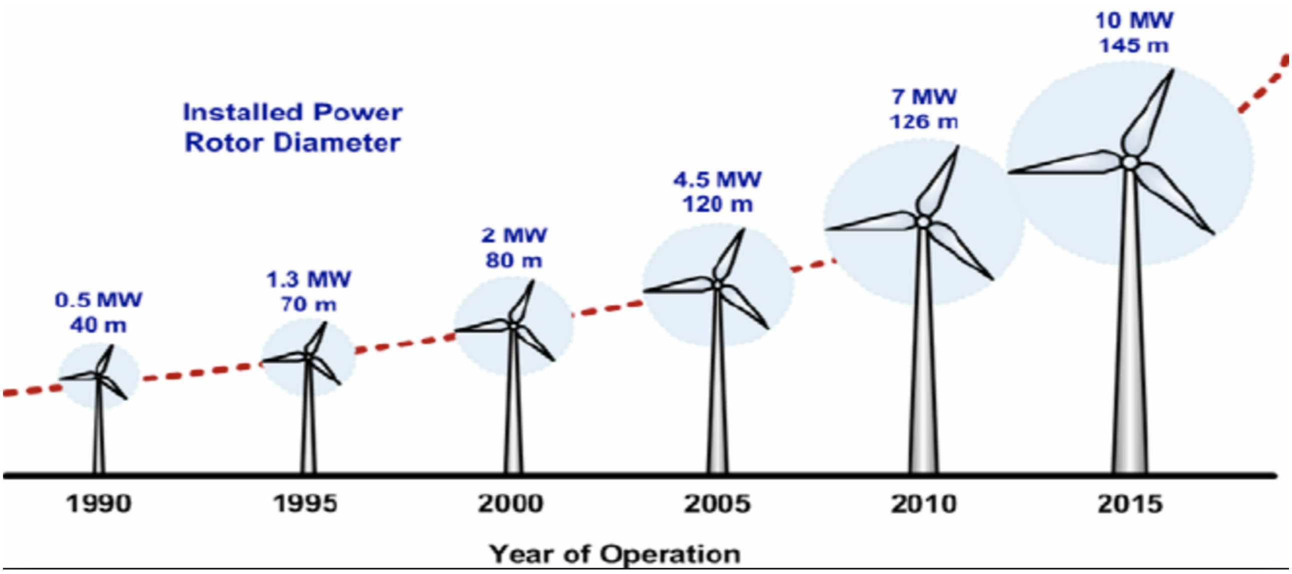


Figure 1.2 – Size evolution of wind turbines over time [32]

considerable aeroelastic effects due to (FSI). These effects might cause aeroelastic instability problems, such as edgewise instability and flutter, which result in devastating the blades and the wind turbine. Therefore, designing larger turbins with flexible blades need a aprecise (FSI) modelling [46]. We remark that such kind of FSI models will be not the subject of this thesis.

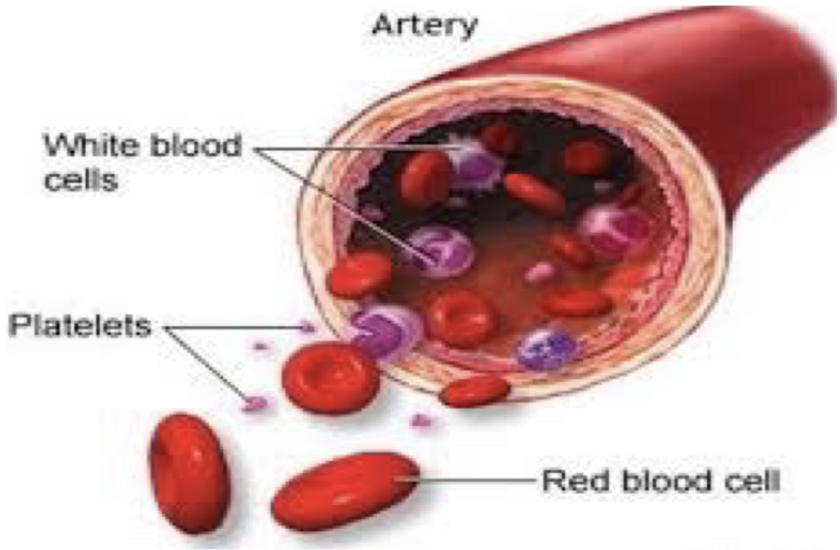


Figure 1.3 – Blood flow in arteries

Moreover, (FSI) is among the most popular problem in applied sciences and includes biological applications such as blood flow in arteries and veins, blood coagulation and speech modelling. More precisely, interactions of biological cells and tissues with flows are important to the circulatory, respiratory and digestive systems. For example, understanding of the complex interaction between the arterial wall and blood which consists of red blood cells, platelets, and white blood cells in a circulatory system is crucial to understand the physiology of the human circulation. In addition, fluid solid interactions are significant in medical field. For example, FSI is commonly used to describe the behaviour of sprays and to design and developing prosthetic implants. We emphasize that the density of air is much smaller than that of the structure, and density of blood is comparable with density of the vessel. Therefore, the coupling nonlinearity is much stronger in the biomedical applications (in a sense that added-mass effect is much stronger) and numerical schemes are therefore different.

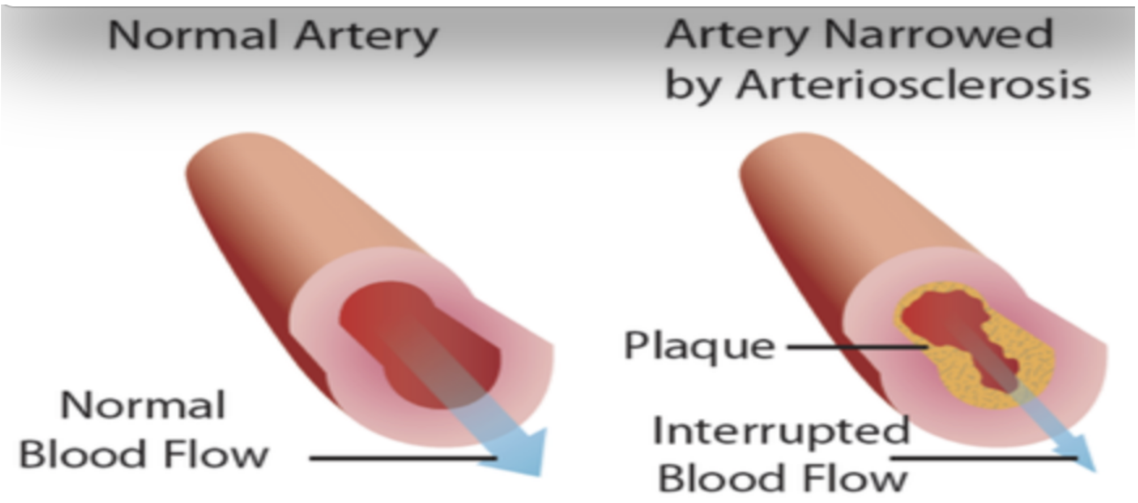


Figure 1.4 – Blood flow in normal and narrowed artery by arteriosclerosis

There are several approaches to model fluid-structure interaction depending on the the size of structures (small versus large) and other properties such as deformable versus rigid, etc... In this thesis, we focus on the macroscopic level to model a finite number of homogeneous rigid bodies immersed in a viscous fluid which is either a Newtonian or viscoelastic fluid in dimension 2 or 3. In such models, the fluid solid interaction problem can be divided into three parts: fluid problem, solid problem, and coupling condition. The equations used to describe the fluid solid interaction problem are complex and challenging due to the high nonlinearity of the problem. Not only the fluid equation exhibits nonlinearity, the displacement of the solid body modifies the fluid domain which generates geometrical nonlinearity as well.

In this chapter, we introduce the general framework of the thesis. In Section 1.1, we write the governing equations of the fluid solids interaction problem. In Section 1.2, we give an overview of recent works on fluid-solid interaction problem. The scope of this work is addressed in Section 1.3. This introductory chapter ends with the outline of the thesis and its main contributions, Section 1.4.

## 1.1 The governing equations

In this section, we write the equations of the motion describing the fluid solid interaction problem. To fix the geometry, we consider $k$ rigid bodies $B_1(t), \dots, B_k(t)$ in $\mathbb{R}^d$, where $k$ is a positive integer and $d = 2$ or 3, immersed in a homogeneous incompressible viscous fluid. We assume that the boundaries $\partial B_i(t)$ for $i = 1, \dots, k$ of the solids are circular in dimension 2 and spheres in dimension 3. The spatial domain occupied by the fluid changes in time and depends on the position of the $k$ rigid bodies. Therefore, the domain occupied by the fluid at time $t$, denoted by $\Omega_F(t)$, is defined as the complement of the set of solids in $\mathbb{R}^d$ or in a bounded domain of $\mathbb{R}^d$ if the solids are moving in a cavity.

We denote by $h_i(t)$ and $\omega_i(t)$ the center of mass and the angular velocity of the $i-$th body at time $t$. We suppose that the rigid bodies are homogeneous and each has a constant density $\overline{\rho}_i$. Hence the mass $m_i$ and the moment of inertia $J_i$ of the $i-$th body related to the center of mass $h_i(t)$ are given by

$$\begin{aligned}
m_i &= \int_{B_i(t)} \overline{\rho}_i dx, \\
J_i &= \int_{B_i(t)} \overline{\rho}_i |x - h_i(t)|^2 dx, \quad \text{if } d = 2 \\
J_i &= \int_{B_i(t)} \overline{\rho}_i \Big( |x - h_i(t)|^2 I_3 - (x - h_i(t)) \otimes (x - h_i(t)) \Big) dx, \quad \text{if } d = 3.
\end{aligned}$$

Since the rigid bodies are homogeneous spheres, then then the above quantities are time independent. Moreover, we have

$$J_i = \Big( \int_{B_i(t)} \overline{\rho}_i |x - h_i(t)|^2 dx \Big) I_3, \quad \text{if } d = 3.$$

For the sake of simplicity, $\Omega_F(0)$ and $B_i(0)$ will be denoted later on by $\Omega_F$ and $B_i$ respectively. Furthermore, we assume that there is no contact initially between the rigid bodies and the boundary of the flow. Precisely, we suppose that $\gamma = \gamma(0) > 0$, where

$$\gamma(t) = \min_{1 \le i \le k} \{ d(B_i(t), \partial\Omega_F(t))) : i \neq j \} > 0. \tag{1.1.1}$$

There are two approaches to describe the motion of a fluid and its associated properties. The first approach is based on following an individual fluid parcel as it moves through space and time. This approach is called Lagrangian approach. More precisely, the fluid parcels are labelled by some vector field $x_0$ which is often chosen to be the center of mass of the parcels at some initial time $t_0$. In the Lagrangian description, the flow is described by a function $\mathbf{X}\left(\mathbf{x}_0, t\right)$ giving the position of the parcel labelled $x_0$ at time t. The second approach is the Eulerian approach which is based on identifying or labelling a certain fixed location in the flow field and follow the change in its property, as different materials pass through that location. For example, the flow velocity is represented by a function $\mathbf{u}\left(\mathbf{x}, t\right)$ at position $x$ and at certain time $t$. The two approaches are related as they describe the velocity of the parcel labelled $x_0$ at time $t$ as follows

$$\mathbf{u}\left(\mathbf{X}(\mathbf{x}_0, t), t\right) = \frac{\partial \mathbf{X}}{\partial t}\left(\mathbf{x}_0, t\right),$$

In this thesis, we use the Eulerian approach which is the usual stand point in fluid mechanics. More precisely, we use the incompressible Navier–Stokes equations to describe the motion of the flow. In terms of fluid velocity $u$, external body force $f$, the incompressible Navier –Stokes equation are given by

$$\rho(\partial_t u + (u \cdot \nabla)u) = \nabla \cdot \sigma + f, \quad x \in \Omega_F(t), \ \ t \in (0, T), \tag{1.1.2}$$

$$\nabla \cdot u = 0, \quad x \in \Omega_F(t), \ \ t \in (0, T), \tag{1.1.3}$$

where $\rho$ denotes the density of the fluid.
Of course, one should add initial data and suitable conditions later at the boundary of the fluid domain. Due to the isotropy, the Cauchy stress tensor writes

$$\sigma = -pI + \tau, \tag{1.1.4}$$

where the scalar $p$ is the hydrostatic pressure (determined by the flow) and $\tau$ is the extra-stress tensor which satisfies a constitutive law depending on the type of the fluid. Mainly, viscous fluids are classified into two categories:

**a) Newtonian fluids:** are fluids with no memory and the stress depends on the instantaneous value of the velocity gradient, not on the prior history of the deformation. Thus the extra-stress is expressed as

$$\tau = 2\eta D[u], \tag{1.1.5}$$

where $\eta$ is the fluid viscosity and $D[u]$ is the rate of deformation tensor defined as follows

$$D[u] = \frac{1}{2}(\nabla u + \nabla u^T).$$

Water, air, mercury... are some of the examples of Newtonian fluids.

**b) Non-Newtonian Fluids:** are those fluids which do not follow the linear law of Newton's law of viscosity (1.1.5) and the relation between the shear stress and the shear rate is non-linear. The viscosity of non-Newtonian fluid is not constant and it depends on other factors such as the rate of shear in the fluid, the container of the fluid and on the previous history of the fluid. The non-Newtonian fluids are further classified into several classes.

For example, *quasi-newtonian fluids* are those for which the viscosity $\eta$ satisfies the power-law model

$$\eta(D) = \eta_0 + \frac{1}{2}\Big(\mathrm{trace}(|D[u]|^2)\Big)^{n-1}.$$

For $n = 1$ we recover the Newtonian fluid, whereas for $n < 1$ this equation describes a shear thinning fluid. Latex paints, blood plasma, and syrup are examples of shear thinning fluid. For $n > 1$, the viscosity is directly proportional with the shear rate and the fluid is called a shear thickening or *dilatant* fluid [35]. Such fluids are rarely encountered, but one common example of this fluid is oobleck which is a mixture of cornstarch and water that hardens upon application of high enough forces, allowing people to run across large pools filled with such mixture. A more natural example is that of wet sand. Walk across it slowly and you will start to sink down but apply enough force by running and the beach will harden beneath you.

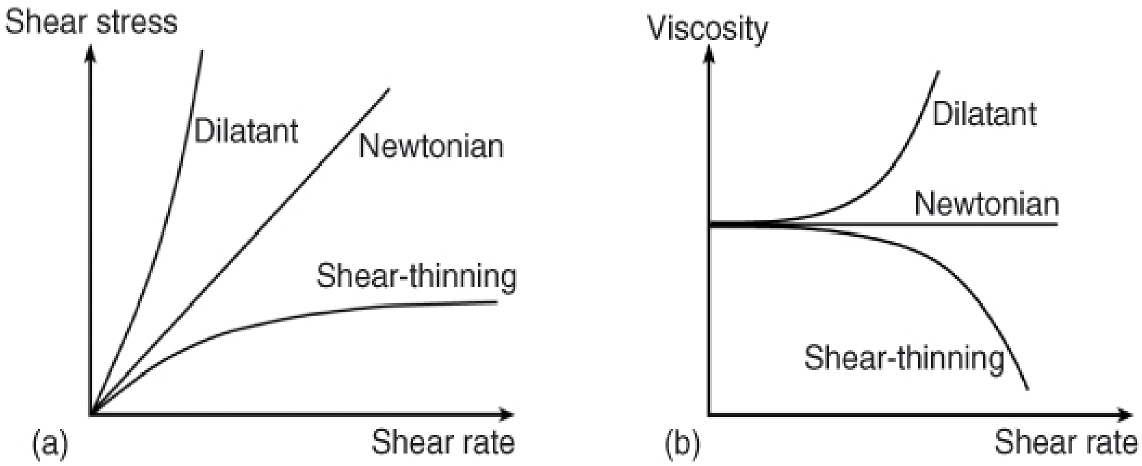


Figure 1.5 – Flow curves for Newtonian, shear thinning and shear thickening (dilatant) fluids: (a) shear stress as a function of shear rate; (b) viscosity as a function of shear rate.

Another category of non- Newtonian fluids contains characteristics of both solids and fluids and exhibit partial elastic recovery after deformation. These are known as *viscoelastic* fluids. Such fluids have memory. In other words, the value of the extra-stress tensor at the present time $t$ of this type of fluids depends on the history of the past deformations and not only on the present deformation. Many important industrial fluids (polymer melts or solutions, metals at very high temperature...) and biological fluids are examples of viscoelastic fluids as they exhibit both viscous and elastic characteristics when undergoing deformation.

In this thesis, we focus mainly on a class of constitutive laws which are widely used in the polymer community due to their relative simplicity, in particular for numerical simulations, the differential models. These differential constitutive laws are basically derived from molecular or continuum mechanics considerations. An important class of them have the following differential form

$$\tau + \lambda_1 \frac{\mathcal{D}_a \tau}{\mathcal{D}t} + \beta(\tau, D[u]) = 2\eta\Big(D[u] + \lambda_2 \frac{\mathcal{D}_a}{\mathcal{D}t}(D[u])\Big), \qquad x \in \Omega_F(t),\ t \in (0,T). \tag{1.1.6}$$

Here, $\lambda_1$ and $\lambda_2$ denote respectively the relaxation and retardation time, such that $0 \leq \lambda_2 < \lambda_1$. As the considered fluid is viscoelastic, we restrict our study to the case when $\lambda_2 > 0$. The operator $\dfrac{\mathcal{D}_a}{\mathcal{D}t}$ is a kind of time derivative which is frame indifferent

$$\frac{\mathcal{D}_a \tau}{\mathcal{D}t} = \Big(\partial_t + (u \cdot \nabla)u\Big)\tau + g_a(\nabla u, \tau),$$

where $g_a$ $(-1 \leq a \leq 1)$ is a bilinear mapping defined as follows

$$g_a(\nabla u, \tau) = \tau W[u] - W[u]\tau - a(D[u]\tau + \tau D[u]),$$

where $W[u] = \dfrac{1}{2}(\nabla u - \nabla u^T)$ is the vorticity tensor.
The particular cases $a = -1, 0, 1$ correspond respectively to the lower convected derivative, Jaumann derivative and upper convected derivative. We focus here in the Oldroyd constitutive law which corresponds to $\beta = 0$ and its transient version, known as the regularized or diffusive Oldroyd model for which an additional dissipative term $\varepsilon\Delta\tau$ appears in the stress equation (1.1.6), see [4] .

The motion of the $i$-th body is governed by the balance equations for linear and angular momentum (Newton's Laws):

$$m_i h_i''(t) = -\int_{\partial B_i(t)} \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} f(t) dx, \qquad t \in (0,T), \tag{1.1.7}$$

$$J_i \omega_i'(t) = -\int_{\partial B_i(t)} (x - h_i(t)) \times \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} (x - h_i(t)) \times f(t) dx, \qquad t \in (0,T). \tag{1.1.8}$$

For $d = 3$, we have denoted by $x \times y$ the classical cross product for $x, y \in \mathbb{R}^3$ whereas for $d = 2$, for $x, y \in \mathbb{R}^2$ and $a \in R$, we have denotes $x \times y = -x_1 y_2 + x_2 y_1$ and $a \times x = a(-x_2, x_1)$. The symbol $\nu_i$ stands for the unit normal vector directed toward the interior of the $i$-th body.

We impose the no-slip boundary conditions at the fluid/rigid body interfaces:

$$u(x,t) = h_i^{'}(t) + \omega_i(t) \times (x - h_i(t)) \;\; x \in \partial B_i(t), \;\; t \in [0,T], \quad i \in \{1,\dots,k\}. \tag{1.1.9}$$

Moreover, if the rigid bodies are suspended in a cavity $\mathcal{O}$ filled with viscous fluid, we assume that

$$u(x,t) = 0, \quad x \in \partial\mathcal{O} \times [0,T]. \tag{1.1.10}$$

To complete the system, we impose initial conditions at $t = 0$ :

$$u(x,0) = u_0(x), \;\; x \in \Omega_F \tag{1.1.11}$$

$$h_i(0) = h_i^0, \;\; h_i'(0) = h_i^1, \;\; \omega_i(0) = \omega_i^0, \;\; i = 1,\dots,k, \tag{1.1.12}$$

$$\tau(x,0) = \tau_0(x), \;\; x \in \Omega_F. \tag{1.1.13}$$

## 1.2 A brief historical overview

In this section we present a short historical overview of the Cauchy theory for fluid solid interaction problem. We recall that we focus on the macroscopic level of modelling as the moving particles contained in the fluid are assumed to be indeformable and sufficiently large and we use the the Eulerian approach to describe the motion of the fluid. The coupling between the fluid and solid subsystems is via the fluid-solid interface(s) by supposing that the fluid velocity at each fluid-solid boundary is equal to the velocity of the solid at it.

The problem of motion of one or several bodies in a viscous Newtoninan incompressible fluid was the interest of many studies (see [10, 13, 14, 27, 29, 36, 37], and references therein). These studies investigated the existence of weak solutions (in a sense which will be defined below) up to "collision" of the fluid rigid body interaction problem. The question of existence of global weak solutions regardless collision, was answered by San Martin and his coauthors in [34] in dimension 2 and by Feireisl [16] in dimension 3. Thereafter, the author in [39] investigated the question of uniqueness of such solutions in the two dimensional case. He proved that uniqueness of weak solutions does not hold after contact in 2D. This non-uniqueness result can be reasonably explained by the fact that there is no bouncing law to describe the dynamics after collision. The method used in [39] to prove the non-uniqueness of weak solutions was to construct a weak solution, colliding in finite time, when the fluid domain and the solid are two disks. Then one may extend this solution after contact by two different ways so that one obtains two weak solutions with different behaviours: for the first solution, the body moves away from the boundary of the flow, whereas the second solution is constructed such that the body stays in contact with the flow boundary after collision. Hillairet and his co-author give an example for which the solid collides with the boundary of the flow in finite time in dimension 3 [25]. The question of uniqueness of weak solutions in 3D is one of our interests in this thesis.

As far as we know, the problem of existence of strong solutions for (FSI) in the case of single moving rigid body of arbitrary shape in a cavity is investigated in several studies. A local-in-time existence result of strong solutions in this case was proved in [21], provided that the inertia of the rigid body is large enough with respect to the inertia of the fluid. Further development in this domain is the work of Takahashi in [40]. The author proves the existence and uniqueness of global strong solution without taking in consideration the assumption in [21]. The first *no collision* result for strong solutions was provided by T. Hesla [23] and M. Hillairet [24]. In [24], the author shows that any strong solution is global under the absence of external forces in the case of a moving disk in the half space $\mathbb{R}^2_+$. Thereafter, it has been studied the roughness-induced effect of the rigid body and the boundary of the domain on the collision process [20].

One of the available results in the case when the fluid domain is the exterior of a single ringid body is due Takahashi and Tucsnak [41]. The authors in [41] prove the existence and uniqueness of strong solutions for an infinite cylinder in dimension 2. A similar result has been proved in Silvestre and Galdi [18] for a rigid body having an arbitrary form. Lately, Cumsile and Takahashi improved the result in [41]. They established the existence and uniqueness of strong solution globally in dimension 2 and also in dimension 3 if the data are small enough [12]. A one-dimensional version of the problem of several rigid bodies is studied in Vázquez and Zuazua [45] where the asymptotic behavior of solutions is also investigated.

Recently, the problem of motion of rigid bodies in non-Newtonian fluids has attracted great attention of many authors due to the wide applications of the fluid solid interaction problem in many biological fields as well as in industrial fields. In the case of non Newtonian fluid of a power-law type, Feireisl, Hillairet, and Nečasová established in [17] the existence of global-in-time weak solutions for the problem describing the motion of several rigid bodies with such type of fluids. Lately, Geissert, Gotze, and Hieber investigated in [19] the fluid solid interaction problem in the case of generalized Newtonian fluid of power-law type of exponent $d \geq 1$. The authors prove the existence of a unique, local, strong solution in the $L^p$ setting of the considered problem when $p > 5$. In the context that a fluid of viscoelastic type occupies the whole cavity, that is in the absence of interaction with obstacles or particles, global existence of weak solutions to the standard Oldroyd model without diffusion in the corotational case only ($a = 0$) was proved in [31] for any data. In the general case, the existence and uniqueness of local strong solutions was shown in [22]; besides, if the fluid is not too elastic and if the data are sufficiently small, then solutions are global. The smallness assumption was removed later on in [33]. However, as far as we know, only few results concerning the existence and uniqueness of strong solutions for the diffusive Oldroyd model: the existence of global strong solutions in 2D for the diffusive Oldroyd-B model (i.e. $a = +1$) and uniqueness of the solution among a class of strong solutions was proved in [11].

## 1.3 Scope of this work

In this thesis, we focus on a common model used to describe the behaviour of sprays which is based on modelling the carrier fluid by the incompressible Navier-Stokes equations and assuming the particles it contains to be bulky and indeformable so that their displacements are described by Euler equations for rigid body dynamics. A strongly coupled system is thus obtained because the displacement of the particles fixes the area occupied by the fluid as well as the value of the fluid velocity field at its edge. On the other hand, the stresses exerted by the fluid influence the particle dynamics.

The first objective of the thesis is to show the well-posed character of the Navier-Stokes

+ Newton system when we consider the displacement of an arbitrary number of spherical particles in an infinite domain. We extend the known result of existence of solution in a context where the cavity is infinite and generalize the tools to control the distance between solids to complex configurations. We emphasize that this result is a major contribution because examples of contacts which has been built previously are for particular configurations. This result illustrates how special they are.

The second objective of this thesis is to investigate the question of uniqueness of weak solutions to (FSI) in dimension 3. We show that there exists some initial conditions for which a weak solution for the (FSI) problem can be extended into two different ways after contact.

Finally, we show the well-posed of the system composed of Navier-Stokes equations for which the extra stress tensor satisfies a differential constitutive law of Oldroyd or regularized Oldroyd type + Newton's laws modelling the displacement of an arbitrary number of spherical particles in a cavity filled with viscoelastic fluid such as polymeric solutions. Here, we extend the known result on the existence of classical solution in a context that the fluid occupies the whole cavity [22].

## 1.4 Thesis outline and main contributions

This PhD thesis is organized as follows. The main contributions of this thesis are reported in Chapters 2, 3 and 4 and are summarized here below. In **Chapter 2**, we show the well-posedness of the problem of motion of several bodies of circular form in a Newtonian viscous fluid globally in time if the the fluid domain is connected domain in $\mathbb{R}^2$ at any time. In **Chapter 3**, we show that uniqueness of weak solutions to the fluid solid interaction problem in three dimensional case does not hold after contact. The non-uniqueness result is due to the lack of collision law in the model under consideration. In **Chapter 4**, we investigate the well-posedness of the problem of interaction of rigid solids with a viscoelastic fluid which obeys either differential constitutive laws of Oldroyd or diffusive Oldroyd type.

### 1.4.1 Chapter 2. On the Motion of Several Disks in an Unbounded Viscous Incompressible Fluid

In this chapter, we study the time evolution of a finite number of homogeneous rigid disks within a viscous homogeneous incompressible Newtonian fluid in the whole domain $\mathbb{R}^2$. The motion of the fluid is governed by Navier-Stokes equations (1.1.2)-(1.1.3) with the Cauchy stress $\sigma$ given as in (1.1.4)-(1.1.5), whereas the movement of each rigid body is described by the standard conservation laws of linear and angular momentum (1.1.7)-(1.1.8).

The regularity of classical solutions to the fluid solid interaction problem is computed through a change of variable $X$ which maps the $\Omega_F$ into $\Omega_F(t)$. The existence of such transform will be discussed in greater details later. More precisely, for a function $u(.,t)$ : $\Omega_F(t) \to \mathbb{R}^2$, we set $U(y,t) = u(X(y,t),t)$ and we use the following notations:

$$\begin{aligned}
L^2(0,T;\mathbf{H}^k(\Omega_F(t))) &= \{u : U \in L^2(0,T;\mathbf{H}^k(\Omega_F))\},\\
H^1(0,T;\mathbf{L}^2(\Omega_F(t))) &= \{u : U \in H^1(0,T;\mathbf{L}^2(\Omega_F))\},\\
\mathcal{C}([0,T],\mathbf{H}^k(\Omega_F(t))) &= \{u : U \in C([0,T],\mathbf{H}^k(\Omega_F))\},\\
L^2(0,T;\dot{H}^1(\Omega_F(t))) &= \{u : U \in L^2(0,T;\dot{H}^1(\Omega_F))\}.
\end{aligned}$$

Moreover, we define $\mathcal{U}(0,T;\Omega_F(t))$ as follows

$$\mathcal{U}(0,T;\Omega_F(t)) = L^2(0,T;\mathbf{H}^2(\Omega_F(t))) \cap \mathcal{C}([0,T],\mathbf{H}^1_\Gamma(\Omega_F(t))) \cap H^1(0,T;\mathbf{L}^2(\Omega_F(t))).$$

We remark here that the definition of the above spaces is independent of the choice of the mapping $X$, see for instance, [40]. In the above spaces, we have denoted the Lebesgue and Sobolev spaces by $L^p(\Omega)$, $1 \le p \le \infty$, with norms $\|\cdot\|_{L^p(\Omega)}$ and $H^k(\Omega)$, with norm $\|\cdot\|_{H^k(\Omega)}$. $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^k(\Omega)$ are spaces of vector valued or tensor valued functions with components in $L^p(\Omega)$ and $H^k(\Omega)$ respectively.

Before stating our main results , it is convenient to introduce the following definition.

**Definition 1.4.1** *Suppose that $T > 0$. We say that $(u, p, (h_i, \omega_i)_{i\in\{1,\ldots,k\}})$ is a strong solution* *of problem (1.1.2)-(1.1.5), (1.1.7)-(1.1.12) if*

$$(u, p, (h_i, \omega_i)_{i\in\{1,\dots,k\}}) \in \mathcal{U}(0,T;\Omega_F(t)) \times L^2(0,T;\dot{H}^1(\Omega_F(t))) \times \Big(H^2(0,T;\mathbb{R}^2)\times H^1(0,T;\mathbb{R})\Big)^k,$$

*and if (1.1.2)-(1.1.5), (1.1.7)-(1.1.12) are satisfied almost everywhere in $(0,T)$ and in $\Omega_F(t)$ or in the trace sense and*

$$\gamma(t) := \min_{1\leq i,j\leq k} \{d(B_i(t), B_j(t)) : i \neq j\} > 0.$$

Our first result is the following existence and uniqueness of strong solutions up to the first collision.

**Theorem 1.4.1** *Suppose that $f \in L^2(0,\infty;\mathbf{L}^2(\mathbb{R}^2))$, $\gamma > 0$, $h_i^0 \in \mathbb{R}^2$, $h_i^1 \in \mathbb{R}^2$, $\omega_i^0 \in \mathbb{R}$, $u_0 \in \mathbf{H}^1(\mathbb{R}^2)$, and that*

$$\begin{aligned} \nabla\cdot u_0 &= 0, && \text{in } \Omega_{\mathrm{F}},\\ u_0(x) &= h_i^1 + \omega_i^0 (x - h_i^0)^\perp, && x \in \partial B_i,\ \forall i \in \{1,\dots,k\}. \end{aligned}$$

*Then there exists $T_0 > 0$ depending on $\gamma$, $h_i^0$, $h_i^1$, $\omega_i^0$, $\|u_0\|_{\mathbf{H}^1(\Omega_F)}$ and $\|f\|_{L^2(0,\infty;\mathbf{L}^2(\mathbb{R}^2))}$ such that problem (1.1.2)-(1.1.5), (1.1.7)-(1.1.12) admits a unique strong solution $(u, p, (h_i, \omega_i)_{i\in\{1,\dots,k\}})$ on $[0,T]$ such that $T < T_0$.*

*Moreover, one of the following alternatives holds true:*

1. *$T_0 = +\infty$,*
2. *$T_0$ is finite and $\limsup_{t\to T_0} \dfrac{1}{\gamma(t)} = +\infty$.*

The proof of Theorem 1.4.1 follows a standard scheme: existence and uniqueness of local solution and analysis of blow up alternative. We emphasize that one key ingredient in the blow up alternative is to show that under the only assumption that $\gamma(t)$ is bounded by below then $\|u(t)\|_{\mathbf{H}^1(\Omega_F(t))}$ is also bounded.

Our second result concerns the global existence and uniqueness of strong solution of the tackled problem. More precisely, we prove:

**Theorem 1.4.2** *Assume that the hypotheses of Theorem 1.4.1 hold true and that*

$$\textit{the fluid domain is connected at any time.} \tag{H1}$$

*Then problem (1.1.2)-(1.1.4),* (1.1.7)-(1.1.12) *admits a unique global strong solution.*

The idea of proof of Theorem 1.4.2 is to act by contradiction. We assume that collision can occur in finite time. We multiply (1.1.2) with a divergence-free vector-field $v$ before collision. We construct a multiplier $v$ locally on the neighbourhood of contact point between the colliding disks and then we extend it by a regular vector field. When two disks approach each other, the viscous term dominates the acceleration term leading to a differential inequality which can be integrated to obtain the *no collision* result.

We emphasize that the assumption (H1) is always valid in the case of two moving bodies and that many body contacts are really unlikely if we start from a sufficiently dilute suspension of bodies. The main difficulty to handle the case of more than two rigid bodies is that collision could possibly divide the fluid domain into several connected components. On such situation, each neighbourhood of the contact point between the colliding particles inside the fluid domain has two connected components. Unfortunately, the flux of the multiplier $v$ which we construct in these neighbourhoods does not vanish on each of the connected components even if their sum does. This prevents us from extending the multiplier $v$ to the whole fluid domain by a divergence free vector field.

### 1.4.2 Chapter 3. Nonuniqueness of Weak Solutions to Fluid Solid Interaction Problem in 3D

This chapter answers the question of uniqueness of weak solutions to the fluid-rigid body problem in dimension three. We consider a single rigid body moving in an incompressible homogeneous Newtonian viscous fluid. The rigid body is supposed to be a ball and the fluid domain has exactly two holes, so that the moving ball can fill exactly the gap between the holes if collision occurs. We prove that uniqueness of weak solutions to the fluid solid interaction problem in three dimensional case does not hold after contact. More precisely, the full system of equations modeling the motion of the fluid and the rigid body reads as

$$\partial_t u + (u\cdot\nabla)u = \nabla\cdot\sigma + f, \quad \text{in } \Omega_F(t),\ t\in(0,T), \tag{1.4.1}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega_F(t),\ t\in(0,T), \tag{1.4.2}$$

$$u(x,t) = \dot G(t) + \omega\times(x-G(t)), \quad x\in\partial B(t)\ t\in(0,T), \tag{1.4.3}$$

$$u(x,t) = 0, \quad x\in\partial\mathcal{O},\ t\in(0,T), \tag{1.4.4}$$

$$m\ddot G(t) = -\int_{\partial B(t)} \sigma\mathbf{n}d\Gamma + \rho_B\int_{B(t)} f(t)dx, \quad t\in(0,T), \tag{1.4.5}$$

$$J\dot\omega(t) = -\int_{\partial B(t)} (x-G(t))\times\sigma\mathbf{n}d\Gamma + \rho_B\int_{B(t)}(x-G(t))\times f(t)dx, \quad t\in(0,T) \tag{1.4.6}$$

In the above system, we denote by $B(t)$ the domain occupied by the moving body with center of mass $G(t)$ at time $t$ and radius 1. The set $\Omega_F(t) = \mathcal{O}\backslash B(t)$ denotes the fluid domain occupied at time $t$, where $\mathcal{O}\subset\mathbb{R}^3$ is a bounded open smooth set. For simplicity, we suppose that the fluid has a constant density 1. To complete the system, we impose initial conditions at $t_0$:

$$u|_{\Omega_F} = u_0,\ G(0) = G_0,\ \dot G(0) = G_1,\ \omega(0) = \omega_0. \tag{1.4.7}$$

We suppose that there is no contact initially between the moving ball and the boundary of the flow; that is $\gamma = \gamma(0) > 0$, where

$$\gamma(t) = d(B(t),\partial\mathcal{O}).$$

Before stating our result, we introduce the notion of weak solutions. To this end, we recall that the global density $\rho$ and the global velocity $\tilde u$ of the system are given respectively by

$$\begin{aligned} \rho(t,x) &= 1_{\Omega_F(t)}(x) + \rho_B 1_{B(t)}(x), \\ \tilde u(t,x) &= u(t,x)1_{\Omega_F(t)}(x) + \Big(\dot G(t) + \omega(t)\times(x-G(t))\Big)1_{B(t)}(x). \end{aligned}$$

For simplicity, we shall denote the global velocity by $u$ instead of $\tilde u$.

Consider domains $B$ and $\mathcal{O}$ in $\mathbb{R}^3$ such that $B \subset \mathcal{O}$. Let

$$\mathcal{V}(O) = \{u \in \mathcal{D}(\mathcal{O}) : \nabla \cdot u = 0 \;\; in \;\; \mathcal{O}\}, \tag{1.4.8}$$

and denote by $H(\mathcal{O})$ and $V(\mathcal{O})$the closure of $\mathcal{V}(\mathcal{O})$ respectively in $\mathbf{L}^2(\mathcal{O})$ and $\mathbf{H}^1(\mathcal{O})$. According to classical results (see [43]) we have

$$\begin{aligned} H(\mathcal{O}) &= \{u \in \mathbf{L}^2(\mathcal{O}) : \;\; \nabla \cdot u = 0 \;\; \text{in } \mathcal{O}, \;\; \mathrm{u} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}, \\ V(\mathcal{O}) &= \{u \in \mathbf{H}_0^1(\mathcal{O}) : \;\; \nabla \cdot u = 0 \;\; \text{in } \mathcal{O}\}. \end{aligned}$$

We introduce the following spaces which will be used in the sequel:

$$\begin{aligned} H(B, \mathcal{O}) &= \{u \in H(\mathcal{O}) : \;\; D[u] = 0 \;\; in \;\; B\}, \\ K(B, \mathcal{O}) &= \{u \in V(\mathcal{O}) : \;\; D[u] = 0 \;\; in \;\; B\}. \end{aligned}$$

By Lemma 1.1 in [44], we have $D[u] = 0$ in $B$ if and only if there exists a vector $a$ and a skew-symmetric tensor $Q \in \mathbb{R}^6$ such that

$$u(x) = a + Qx, \;\; for \;\; x \in B.$$

In particular, there exists a vector $\omega$ such that $Qx = \omega \times x$.

**Definition 1.4.2** *Assume that $G_0 \in \mathcal{O}$ such that $\gamma > 0$ and $u_0 \in H(\mathcal{O})$. We say that $(u, G)$ is a weak solution to problem (3.1.1)-(3.1.8) on $[0, T]$ if the velocity field $u$ and the center of mass of $G$ satisfy*

$$\begin{aligned} G \in W^{1,\infty}(0, T), \;\; with \;\; G(0) = G_0, \\ \gamma(t) \geq 0, \\ u \in L^\infty(0, T; H(\mathcal{O})) \cap L^2(0, T; V(\mathcal{O})), \;\; with \;\; u(0) = u_0, \\ u(x, t) = \dot{G}(t) + \omega(t) \times (x - G(t)), \;\; \forall x \in \partial B(t), \end{aligned}$$

*and*

$$\int_{\mathcal{O}\times[0,T)} \Big(\rho u\cdot\partial_t v+\rho u\otimes u : D[v]-2\nu D[u] : D[v]+\rho f\cdot v\Big)dxds = -\int_{\mathcal{O}} \rho(0)u_0\cdot v(0)dx,\ \forall v\in\mathcal{S}, \tag{1.4.9}$$

*where*

$$\mathcal{S}=\{\varphi\in\mathcal{D}([0,T)\times\mathcal{O}) : \nabla\cdot\varphi=0 \ \textit{on}\ I\times\mathcal{O},\ \ D[\varphi]=0\ \textit{on a neighbourhood of}\ B(t)\}.$$

*We remark that the test function $\varphi$ used in the above weak formulation must be zero when $B$ touches the two holes whereas the velocity $u$ need not.*

Our result is the following:

**Theorem 1.4.3** *There exists initial conditions such that problem (1.4.1)-(1.4.7) admits at least two weak solutions.*

The geometry of the problem is crucial to prove the above theorem. We suppose that the cavity $\mathcal{O}$ is symmetric with respect to some line $(D)$ and has exactly two spherical holes $B^l$ and $B^r$ each of radius 1. We assume that the holes are symmetric with respect to the line $(D)$ and separated by a distance equal to the diameter of the moving ball $B$ so that the ball $B$ can fill exactly the gap between the two holes at collision. Moreover, we assume that $\partial\mathcal{O}$ is flat near $\partial D\cap\partial\mathcal{O}$. An example of such geometry is represented in the following figure:

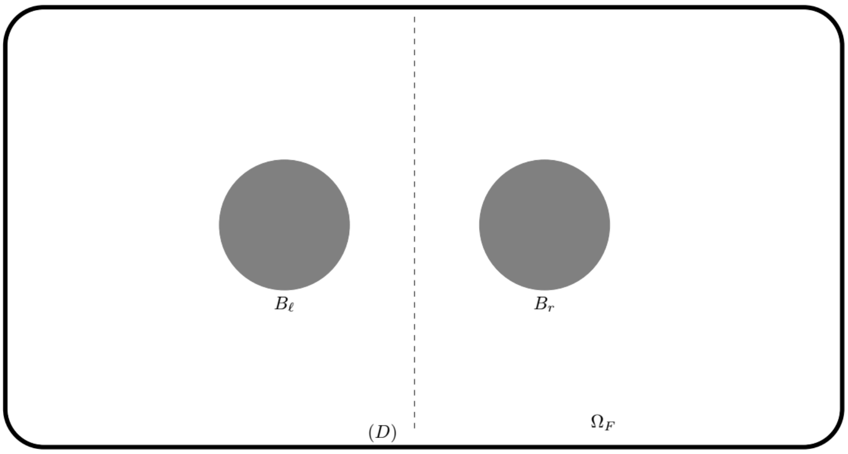


Figure 1.6 – Example of the geometry

The proof of Theorem 1.4.3 follows the same idea as in [39]. The idea of the proof is based on the construction of a weak solution colliding in finite time to problem (1.4.1)-(1.4.7) with $\mathcal{O}$ as described above. Then we extend this solution after contact by two different ways so that one obtains two weak solutions with different behaviours: for the first solution, the body moves away from the boundary of the flow, whereas the second solution is constructed such that the body stays in contact with the flow boundary after collision. The novelty of this result is that we prove the non-uniqueness of weak solutions for the problem of the motion of a rigid body in viscous fluid in 3D after contact with external source term $f \in L^2(0,T;\mathbf{L}^p(\mathcal{O}))$ with $p < 2$.

### 1.4.3 Chapter 4. Existence Results for the Motion of Rigid Bodies in Viscoelastic Fluids

In this chapter, we study the two dimensional motion of a finite number of homogeneous rigid disks in a cavity $\mathcal{O}$ filled with incompressible viscoelastic fluids such as polymeric solutions. We shall consider the system composed of (1.1.2)-(1.1.4) to model the flow of the solvent for which the extra-stress $\tau$ satisfies the Oldroyd or regularized Oldroyd differential constitutive law together with Newton's laws to describe the motion of the disks and the no-slip boundary conditions, and write it in a more mathematical tractable fashion. We decompose the extra-stress tensor $\tau$ into two parts: one corresponding to the Newtonian part $\tau^s$ (the solvent) whereas the other one corresponds to the purely elastic part $\tau^p$ (the polymer). In other words, we write

$$\tau = \tau^s + \tau^p, \tag{1.4.10}$$

with

$$\tau^s = 2\eta_s D[u], \tag{1.4.11}$$

where $\eta_s = \dfrac{\lambda_2}{\lambda_1}\eta$ represents the solvent viscosity and $\eta$ is the total fluid viscosity ($\eta = \eta_s + \eta_p$, $\eta_p$ denotes the polymer viscosity).

Substituting (1.4.10) and (1.4.11) in (1.1.2) and (1.1.6), we obtain that

$$\rho\Big(\partial_t u + (u\cdot\nabla)u\Big) + \nabla p = \eta_s \Delta u + \nabla\cdot\tau_p + f, \qquad \text{in } \Omega_F(t),\ t\in(0,T), \tag{1.4.12}$$

$$\tau_p + \lambda_1 \frac{\mathcal{D}_a \tau_p}{\mathcal{D}t} = 2\eta_p D[u], \qquad \text{in } \Omega_F(t),\ t\in(0,T). \tag{1.4.13}$$

For simplicity, we shall denote from now on $\tau_p$ by $\tau$.

It is more convenient to write the considered problem using dimensionless variables so that the physical parameters appear. We define the Weissenberg number $We = \dfrac{\lambda_1}{\overline{U}L}$, Reyonolds number $Re = \rho\dfrac{\overline{U}L}{\eta}$, $\overline{m}_i = \dfrac{m_i\overline{U}}{\eta L}$ and $\overline{J}_i = \dfrac{J_i\overline{U}}{\eta L^3}$, where $\overline{U}$ and $L$ represent a typical velocity and a typical length of the flow. We set

$$x = \frac{x^*}{L},\ u = \frac{u^*}{\overline{U}},\ t = \frac{\overline{U}}{L}t^*,\ p = \frac{L}{\eta\overline{U}}p^*,\ \tau = \frac{L}{\eta\overline{U}}\tau^*,\ f = \frac{L^2}{\eta\overline{U}}f^*,\ \omega_i = \frac{L}{\overline{U}}\omega_i^*,$$

where stars are attached to dimensional variables.

Hence, in nondimensional variables system (1.1.2)-(1.1.4), (1.1.7)-(1.1.10) reads as (see, for instance, [22]):

$$Re(\partial_t + u\cdot\nabla)u - (1-r)\Delta u + \nabla p = \nabla\cdot\tau + f, \quad x\in\Omega_F(t),\ t\in(0,T), \tag{1.4.14}$$

$$\nabla\cdot u = 0, \quad x\in\Omega_F(t),\ t\in(0,T), \tag{1.4.15}$$

$$u(x,t) = h_i'(t) + \omega_i(t)(x-h_i(t))^{\perp}, \quad x\in\partial B_i(t),\ t\in(0,T), \tag{1.4.16}$$

$$u(x,t) = 0, \quad x\in\partial\mathcal{O},\ t\in(0,T), \tag{1.4.17}$$

$$\overline{m}_i h_i''(t) = -\int_{\partial B_i(t)} \sigma\nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} f(t)dx, \quad t\in(0,T), \tag{1.4.18}$$

$$\overline{J}_i\omega_i'(t) = -\int_{\partial B_i(t)} (x-h_i(t))^{\perp}\cdot\sigma\nu_i d\Gamma_i + \overline{\rho}_i\int_{B_i(t)} (x-h_i(t))^{\perp}\cdot f(t)dx, \quad t\in(0,T), \tag{1.4.19}$$

where the retardation parameter $r$ is defined by $r = 1 - \dfrac{\lambda_2}{\lambda_1}$. We remark the total stress tensor $\sigma$ is given by

$$\sigma = -pI + 2(1-r)D[u] + \tau.$$

Moreover, the Oldroyd model without dimension reads as

$$We\Big(\partial_t\tau + (u\cdot\nabla)\tau + g_a(\nabla u, \tau)\Big) + \tau = 2rD[u], \qquad x\in\Omega_F(t),\ t\in(0,T), \tag{1.4.20}$$

The Oldroyd model has a transient version known as the regularized or diffusive Oldroyd model. In the latter case the elastic extra-stress tensor $\tau$ is expressed as a solution of a second order parabolic partial differential equation:

$$We\Big(\partial_t\tau + (u\cdot\nabla)\tau + g_a(\nabla u, \tau)\Big) + \tau - \varepsilon\Delta\tau = 2rD[u], \qquad x\in\Omega_F(t),\ t\in(0,T), \tag{1.4.21}$$

$$\varepsilon\frac{\partial\tau}{\partial n}(x,t) = 0, \qquad x\in\partial\Omega_F(t),\ t\in(0,T), \tag{1.4.22}$$

The additional dissipative term $\varepsilon\Delta\tau$ in the above stress equation corresponds to a center of mass of diffusion term in the dumbell models. We refer the reader to [4] and the references therein for the derivation of (1.4.21)-(1.4.22). In standard derivation of Oldroyd model from kinetic models for dilute polymers, the diffusive term is routinely omitted on the grounds that it is several orders of magnitude smaller than the other terms in the equation. To complete the system, one should initial data at $t = 0$ :

$$u(x,0) = u_0(x),\ \ x\in\Omega_F \tag{1.4.23}$$

$$h_i(0) = h_i^0,\ \ h_i'(0) = h_i^1,\ \ \omega_i(0) = \omega_i^0,\ \ i = 1,\dots,k, \tag{1.4.24}$$

$$\tau(x,0) = \tau_0(x),\ \ x\in\Omega_F. \tag{1.4.25}$$

To study the system of equations modelling the motion of the fluid coupled with either the Oldroyd model or the regularized Oldroyd model, we introduce the following function spaces:

$$\begin{aligned}
\mathcal{U}(0,T;\Omega_F(t)) &= L^2(0,T;\mathbf{H}^2(\Omega_F(t)))\cap\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t)))\cap H^1(0,T;\mathbf{L}^2(\Omega_F(t))),\\
\mathfrak{T}(0,T;\Omega_F(t)) &= \{\tau\in L^2(0,T;\mathbf{H}^2(\Omega_F(t)))\cap\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t)))\cap H^1(0,T;\mathbf{L}^2(\Omega_F(t))) : \tau = \tau^T\}.
\end{aligned}$$

Before stating our existence result concerning the resolution of the regularized Oldroyd model, we need to introduce the following definition.

**Definition 1.4.3** *Suppose that* $T > 0$. *We say that* $(u, p, \tau, (h_i, \omega_i)_{i\in\{1,\dots,k\}})$ *is a strong solution of problem (1.4.14)-(1.4.19),* (1.4.21)-(1.4.25) *if*

$$u \in \mathcal{U}(0,T;\Omega_F(t)),\ p \in L^2(0,T;\dot{H}^1(\Omega_F(t))),\ \tau \in \mathfrak{T}(0,T;\Omega_F(t)),$$
$$(h_i,\omega_i) \in H^2(0,T;\mathbb{R}^2)\times H^1(0,T;\mathbb{R}),$$

*and if (1.4.14)-(1.4.19),* (1.4.21)-(1.4.25) *are satisfied almost everywhere in* $(0,T)$ *and in* $\Omega_F(t)$ *or in the trace sense and*

$$\gamma(t) = \min_{i\neq j}(d(B_i(t), B_j(t)), d(B_i(t), \partial\mathcal{O})) > 0.$$

We give the first existence result of local-in-time solutions to problem (1.4.14)-(1.4.19) coupled with the regularized Oldroyd constitutive law.

**Theorem 1.4.4 (Regularized Oldroyd model)** *Suppose that* $\partial\mathcal{O} \in \mathcal{C}^2$, $f \in L^2(0,T;\mathbf{L}^2(\Omega_F))$, $u_0 \in \mathbf{H}^1(\Omega_F)$, $\tau_0 \in \mathbf{H}^1(\Omega_F)$, $\gamma > 0$, *and that*

$$\begin{aligned}
\nabla\cdot u_0 &= 0, && \text{in } \Omega_F,\\
u_0(x) &= h_i^1 + \omega_i^0(x-h_i^0)^{\perp}, && x \in \partial B_i,\ \forall i \in \{1,\dots,k\},\\
u_0(x) &= 0, && x \in \partial\mathcal{O}.
\end{aligned}$$

*Then there exists* $T_0 > 0$ *such that problem (1.4.14)-(1.4.19),* (1.4.21)-(1.4.25) *admits unique strong solution on* $[0, T_1]$ *for all* $T_1 \in [0, T_0)$.
*Moreover, one of the following alternatives holds true:*

1. $T_0 = +\infty$,
2. $\limsup_{t\to T_0} \|u(t)\|_{\mathbf{H}^1(\Omega_F(t))} + \|\tau(t)\|_{\mathbf{H}^1(\Omega_F(t))} + \dfrac{1}{\gamma(t)} = +\infty.$

However, when coupling the Navier-Stokes and Newton's laws with the Oldroyd model, classical solutions belong to more regular spaces:

$$
\begin{aligned}
\tilde{\mathcal{U}}(0,T;\Omega_F(t)) &= \Big\{u \in L^2(0,T;\mathbf{H}^3(\Omega_F(t))) \cap \mathcal{C}([0,T],\mathbf{H}^2(\Omega_F(t))) : \\
&\qquad u' \in L^2(0,T;\mathbf{H}^1(\Omega_F(t))) \cap \mathcal{C}([0,T];\mathbf{L}^2(\Omega_F(t)))\Big\}, \\
\tilde{\mathfrak{T}}(0,T;\Omega_F(t)) &= \Big\{\tau \in \mathcal{C}([0,T],\mathbf{H}^2(\Omega_F(t))) : \tau' \in \mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t))) \text{ and } \tau = \tau^T\Big\}.
\end{aligned}
$$

**Definition 1.4.4** *Suppose that $T > 0$. We say that $(u, p, \tau, (h_i, \omega_i)_{i\in\{1,\dots,k\}})$ is a strong solution of problem (1.4.14)-(1.4.20),* (1.4.23)-(1.4.25) *if*

$$
\begin{aligned}
&u \in \tilde{\mathcal{U}}(0,T_1,\Omega_F(t)),\ p \in L^2(0,T_1;H^2(\Omega_F(t))) \cap \mathcal{C}([0,T_1],\dot{H}^1(\Omega_F(t))), \\
&\tau \in \tilde{\mathfrak{T}}(0,T_1;\Omega_F(t)),\ (h_i,\omega_i) \in W^{2,\infty}([0,T_1]\times\mathbb{R}^2) \times W^{1,\infty}([0,T_1]\times\mathbb{R}),
\end{aligned}
$$

*and if (1.4.14)-(1.4.20),* (1.4.23)-(1.4.25) *are satisfied almost everywhere in $(0,T)$ and in $\Omega_F(t)$ or in the trace sense and*

$$
\gamma(t) = \min_{i\neq j}(d(B_i(t),B_j(t)), d(B_i(t),\partial\mathcal{O})) > 0.
$$

Our second result is the following local existence theorem.

**Theorem 1.4.5 (Oldroyd model)** *Suppose that $\partial\mathcal{O} \in \mathcal{C}^3$,$\gamma > 0$, $f \in L^2(0,T;\mathbf{H}^1(\mathcal{O}))$, $f' \in L^2(0,T;H^{-1}(\mathcal{O}))$, $u_0 \in \mathbf{H}^2(\Omega_F)$, $\tau_0 \in \mathbf{H}^2(\Omega_F)$, and that*

$$
\begin{aligned}
\nabla\cdot u_0 &= 0, && \text{in } \Omega_F, \\
u_0(x) &= h_i^1 + \omega_i^0(x-h_i^0)^\perp, && x \in \partial B_i,\ \forall i \in \{1,\dots,k\}, \\
u_0(x) &= 0, && x \in \partial\mathcal{O}.
\end{aligned}
$$

*Then there exists $T_0 > 0$ such that problem (1.4.14)-(1.4.20),* (1.4.23)-(1.4.25) *admits a unique strong solution for all $T_1 \in [0,T_0)$.*

*Moreover, one of the following alternatives holds true:*

1. $T_0 = +\infty$,

2. $\limsup_{t \to T_0} \|u(t)\|_{\mathbf{H}^2(\Omega_F(t))} + \|\tau(t)\|_{\mathbf{H}^2(\Omega_F(t))} + \dfrac{1}{\gamma(t)} = +\infty.$

The movement of rigid bodies modifies the fluid domain and hence the first step to study the models introduced above is to write the equations in a cylindrical domain. To do this, we use a non-linear, local change of coordinates $X$ which only acts on a neighbourhood of the rigid bodies. The method used to prove the above local existence results is similar to the one used in [22, 40]. First, we rewrite the full non-linear transformed problem corresponding to each model as a fixed point of a mapping defined by solving linearzed problems associated to the transformed models. After this reformulation, our approach is based on maximal regularity estimates for the linearized transformed problem. Mainly, we study two linearized problems, one for the velocity and the other for the elastic extra-stress tensor. Existence and uniqueness of classical solutions to the regularized Oldroyd model follows then by implementing classical fixed point theorem. When considering the regularized model, we apply a standard Picard iteration procedure. However, for the Oldroyd model we must turn to a finer version: the Schauder fixed point theorem. We emphasize that one of the critical difficulties in studying the Oldroyd model is that we are dealing with a hyperbolic equation whose transport coefficient does not vanish on the boundaries of the disks. However, the change of variable $X$ which is used to write the model in cylindrical domain has many noteworthy features. Mainly, the transport coefficient in the transformed Oldroyd model is orthogonal to the normal vector on the boundary of the flow.

# 2

# On the Motion of Several Disks in a Viscous Incompressible Fluid

### Sommaire



In this chapter, we consider the system formed by the incompressible Navier-Stokes equations coupled with Newton's laws to describe the motion of a finite number of homogeneous rigid disks within a viscous homogeneous incompressible fluid in the whole space $\mathbb{R}^2$. In Section 2.1, we introduce the model describing the movement of solids in the fluid. Then, we write our model in cylindrical domain in Section 2.2 as we are dealing with a free boundary problem. In Section 2.3, we generalize the existence result of strong solutions of Takahashi in [40] and that of Cumsille and Takahashi in [12] to the case of several rigid bodies. Section 2.4 is devoted to extend solutions up to collision. In the last section, we prove contact between rigid bodies cannot occur for almost arbitrary configurations by studying the distance between solids by a multiplier approach [20].

## 2.1 Statement of the problem

We consider a finite number of homogeneous rigid bodies – each being represented by a closed disk $B_i(t) \subset \mathbb{R}^2$– moving in a viscous homogeneous incompressible fluid which occupies a domain $\Omega_F(t)$ at time $t$, where $\Omega_F(t) = \mathbb{R}^2 \backslash \bigcup_{i=1}^{k} B_i(t)$, with $k \in \mathbb{N}^*$ denoting the number of rigid bodies.

We suppose that the fluid is of viscosity $\nu > 0$, pressure $p$, velocity field $u$ and for simplicity, of density one. The motion of the fluid is governed by the Navier-Stokes equations for incompressible fluids:

$$\partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad x \in \Omega_F(t), \quad t \in (0,T), \tag{2.1.1}$$

$$\nabla \cdot u = 0, \quad x \in \Omega_F(t), \quad t \in (0,T), \tag{2.1.2}$$

where $f \in L^2((0,T) \times \mathbb{R}^2)$ denotes an external body force.

For each rigid body, we define the density $\overline{\rho}_i$, the center of mass $h_i(t)$, the angular velocity $\omega_i(t)$ and the inertia matrix $J_i$ related to the center of mass of the $i$-th body by

$$\overline{\rho}_i = \frac{m_i}{|B_i(0)|}, \quad h_i(t) = \frac{1}{|B_i(0)|} \int_{B_i(t)} x \, dx, \quad J_i(t) = \int_{B_i(t)} \overline{\rho}_i |x - h_i(t)|^2 dx = \int_{B_i(0)} \overline{\rho}_i |y|^2 dy,$$

where $m_i$ denotes the mass of the $i$-th body. Hence $B_i(t)$ is the closed disk of center $h_i(t)$ and radius $r_i$.

The motion of the $i$-th body is governed by the balance equations for linear and angular momentum (Newton's Laws):

$$m_i h_i''(t) = -\int_{\partial B_i(t)} \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} f(t) dx, \qquad t \in (0,T), \tag{2.1.3}$$

$$J_i \omega_i'(t) = -\int_{\partial B_i(t)} (x - h_i(t))^\perp \cdot \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} (x - h_i(t))^\perp \cdot f(t) dx, \quad t \in (0,T). \tag{2.1.4}$$

In the above equations, the matrix $\sigma$ denotes the Cauchy stress tensor in the fluid and is given by

$$\sigma(u,p) = -pI + 2\nu D[u],$$

where $I$ is the identity matrix and $D[u]$ denotes the rate of deformation tensor defined as follows

$$D[u] = \frac{1}{2}(\nabla u + \nabla u^T).$$

We denote by $(x_1, x_2)^\perp = (-x_2, x_1)$ the orthogonal vector of $(x_1, x_2)$ and we use the notation $\partial B_i(t)$ to denote the boundary of the $i$-th body at time $t$. The symbol $\nu_i(x,t)$ stands for the unit normal vector directed toward the interior of the $i$-th body. For simplicity, $\Omega_F(0)$ and $B_i(0)$ will be denoted later on by $\Omega_F$ and $B_i$ respectively.
We impose the no-slip boundary conditions at the fluid/rigid body interfaces

$$u(x,t) = h_i'(t) + \omega_i(t)(x - h_i(t))^\perp, \;\; x \in \partial B_i(t), \;\; t \in [0,T], \quad i \in \{1,\dots,k\}. \tag{2.1.5}$$

To complete the system, we impose initial conditions at $t = 0$ :

$$u|_{\Omega_F} = u_0, \;\; h_i(0) = h_i^0, \;\; h_i'(0) = h_i^1, \;\; \omega_i(0) = \omega_i^0, \;\; \forall i \in \{1,\dots,k\}. \tag{2.1.6}$$

Throughout this chapter, we assume that there is no contact initially between the rigid bodies; that is

$$\gamma = \gamma(0) = \min_{1\le i,j\le k} \{d(B_i(0), B_j(0)) : i \neq j\} > 0. \tag{2.1.7}$$

Since we are dealing with a free boundary problem, the regularity of classical solutions has to be made precise. We recall that the regularity of classical solutions is computed through a change of variable $X$ which maps the fluid domain to its initial shape. More precisely, for a function $u(.,t) : \Omega_F(t) \to \mathbb{R}^2$, we set $U(y,t) = u(X(y,t),t)$ and we use the following notations:

$$\begin{aligned}
L^2(0,T;\mathbf{H}^2(\Omega_F(t))) &= \{u : \; U \in L^2(0,T;\mathbf{H}^2(\Omega_F))\},\\
H^1(0,T;\mathbf{L}^2(\Omega_F(t))) &= \{u : \; U \in H^1(0,T;\mathbf{L}^2(\Omega_F))\},\\
\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t))) &= \{u : \; U \in C([0,T],\mathbf{H}^1(\Omega_F))\},\\
L^2(0,T;\dot{H}^1(\Omega_F(t))) &= \{u : \; U \in L^2(0,T;\dot{H}^1(\Omega_F))\}.
\end{aligned}$$

In the above spaces, we have denoted the Lebesgue and Sobolev spaces by $L^p(\Omega)$, $1 \le p \le \infty$, with norms $\|\cdot\|_{L^p(\Omega)}$ and $H^k(\Omega)$, with norm $\|\cdot\|_{H^k(\Omega)}$. $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^k(\Omega)$ are spaces of vector valued or tensor valued functions with components in $L^p(\Omega)$ and $H^k(\Omega)$ respectively.

Moreover, we define $\mathcal{U}(0,T;\Omega_F(t))$ as follows

$$\mathcal{U}(0,T;\Omega_F(t)) = L^2(0,T;\mathbf{H}^2(\Omega_F(t))) \cap \mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t))) \cap H^1(0,T;\mathbf{L}^2(\Omega_F(t))).$$

This chapter is devoted to prove the following two main results:

**Theorem 2.1.1** *Suppose that $f \in L^2(0,\infty;\mathbf{L}^2(\mathbb{R}^2))$, $\gamma > 0$, $h_i^0 \in \mathbb{R}^2$, $h_i^1 \in \mathbb{R}^2$, $\omega_i^0 \in \mathbb{R}$, $u_0 \in \mathbf{H}^1(\mathbb{R}^2)$, and that*

$$\begin{aligned} \nabla \cdot u_0 &= 0, && \text{in } \Omega_{\mathrm{F}}, \\ u_0(x) &= h_i^1 + \omega_i^0 (x - h_i^0)^\perp, && x \in \partial B_i,\ \forall i \in \{1,\dots,k\}. \end{aligned}$$

*Then there exists $T_0 > 0$ depending on $\gamma$, $h_i^0$, $h_i^1$, $\omega_i^0$, $\|u_0\|_{\mathbf{H}^1(\Omega_F)}$ and $\|f\|_{L^2(0,\infty;\mathbf{L}^2(\mathbb{R}^2))}$ such that problem (2.1.1)-(2.1.6) admits a unique strong solution*

$$(u,p,(h_i,\omega_i)_{i\in\{1,\dots,k\}}) \in \mathcal{U}(0,T;\Omega_F(t)) \times L^2(0,T;\mathbf{H}^1(\Omega_F(t))) \times \Big(H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})\Big)^k,$$

*on $[0,T]$ such that $T < T_0$.*

*Moreover, one of the following alternatives holds true:*

1. *$T_0 = +\infty$,*
2. *$T_0$ is finite and $\limsup\limits_{t\to T_0} \dfrac{1}{\min\limits_{i\neq j} d(B_i(t),B_j(t))} = +\infty$.*

The proof of Theorem 2.1.1 follows a standard scheme: existence and uniqueness of local solution and analysis of blow up alternative. We emphasize that one key ingredient in the blow up alternative is to show that under the only assumption that $\min\limits_{i\neq j} d(B_i(t),B_j(t))$ is bounded by below then $\|u(t)\|_{\mathbf{H}^1(\Omega_F(t))}$ is also bounded.

Then we prove the global existence and uniqueness of strong solution of the tackled problem by adapting the method of Gérard-Varet and Hillairet in [20] to our case and we arrive to the following result:

**Theorem 2.1.2** *Assume that the hypotheses of Theorem 2.1.1 hold true and that*

$$\textit{the fluid domain is connected at any time.} \tag{H1}$$

*Then problem (2.1.1)-(2.1.6) admits a unique global strong solution.*

We act by contradiction and we assume that collision can occur in finite time. We multiply (2.1.1) with a divergence-free vector-field $v$ before collision. When two disks approach each other, the viscous term dominates the acceleration term leading to a differential inequality which can be integrated to obtain the *no collision* result. The main restriction of the global-in-time existence result in Theorem 1.4.1 is that we need the fluid domain to be connected at any time. However, this assumption is always valid in the case when we have just two moving bodies and that many body contacts are really unlikely if we start from a sufficiently dilute suspension of bodies.

The main difficulty to handle the case of more than two rigid bodies is that collision could possibly divide the fluid domain into several connected components. On such situation, each neighbourhood of the contact point between the colliding particles inside the fluid domain has two connected components. Unfortunately, the flux of the multiplier $v$ which we construct in these neighbourhoods does not vanish on each of the connected components even if their sum does. This prevents us from extending the multiplier $v$ to the whole fluid domain by a divergence free vector field.

## 2.2 Equations in cylindrical domain

This section is devoted to write the free boundary value problem (2.1.1)-(2.1.6) in cylindrical domain. First, we introduce a mapping $X$ which maps the fluid domain into its initial shape. Then, we reduce our problem to a problem in cylindrical domain using the transform $X$. We fix $k$ functions $h_i : t \mapsto h_i(t)$ such that for $i \in \{1, \ldots, k\}$, we have $h_i \in H^2(0, T; \mathbb{R}^2)$. Moreover, from now on we fix $\varepsilon$ such that $0 < \varepsilon < \gamma$. With this choice, we have

$$B_i \subset U_i \subset \overline{U}_i \subset B\Big(h_i(0), r_i + \frac{\gamma}{2}\Big),$$

where

$$U_i = B(h_i(0), r_i + \frac{\gamma - \varepsilon}{2}).$$

Consider a family of smooth functions $\psi_1, \ldots, \psi_k$, such that

- supp $\psi_i \subset B(h_i(0), r_i + \frac{\gamma}{2})$
- $0 \leq \psi_i(x) \leq 1$

- $\psi_i \equiv 1$ in $\overline{U}_i$.

Finally, we define the mapping $\Lambda : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$ by

$$\Lambda(x,t) = \sum_{i=1}^{k} \nabla^{\perp}\Big(h_i(t) \cdot x^{\perp}\psi_i(x)\Big). \tag{2.2.1}$$

Since the cut-off functions $\psi_1, \ldots, \psi_k$ are smooth and the center of masses of the rigid bodies are $H^2$ in time, it follows that for all $t \in [0,T]$ the function $\Lambda(t,.) \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and that for all $x \in \mathbb{R}^2$, the function $\Lambda(.,x)$ is of class $H^1(0,T;\mathbb{R}^2)$. Moreover, we have:

**Lemma 2.2.1** *The mapping $\Lambda$ defined in (2.2.1) satisfies the following properties:*

*i.* $\exists r > 0$, *such that* $\Lambda \equiv 0$ *outside* $B(0,r)$, *for all* $t \in [0,T]$,

*ii.* $\nabla \cdot \Lambda = 0$ *in* $\mathbb{R}^2 \times [0,T]$,

*iii.* $\Lambda(x,t) = h_i'(t)$ *in* $B(h_i(0), r_i + \frac{\gamma-\varepsilon}{2})$.

The mapping $X$ is defined as the solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial X}{\partial t}(y,t) & = \Lambda(X(y,t),t), \;\; t \in ]0,T], \\ X(y,0) & = y \in \mathbb{R}^2. \end{cases} \tag{2.2.2}$$

By Cauchy-Lipschitz-Picard theorem, we have:

**Lemma 2.2.2** *For all $y \in \mathbb{R}^2$, the initial-value problem (2.2.2) admits a unique solution $X(y,.) : [0,T] \to \mathbb{R}^2$, which is $\mathcal{C}^1$ on $[0,T]$. Moreover, we have the following properties:*

*i. For all $t \in [0,T]$, the mapping $X(.,t)$ is a $\mathcal{C}^\infty$-diffeomorphism from $\mathbb{R}^2$ onto itself and from $B_i$ onto $B_i(t)$ whenever $B_i(t) \subset B(h_i(0), r_i + \frac{\gamma-\varepsilon}{2})$.*

*ii. Let $Y(.,t)$ be the inverse mapping of $X(.,t)$. Then, for all $x \in \mathbb{R}^2$, the mapping $t \to Y(x,t)$ is a $\mathcal{C}^1$ function in $[0,T]$.*

*iii. For all $(y,t) \in \mathbb{R}^2 \times [0,T]$; the determinant of the jacobian matrix $J_X$ of the mapping $X(.,t)$ is one due to the classical result of Liouville (see, for instance [2]).*

Next, we define the functions $U$, $P$, and $F$ using the transform $X$ defined previously as follows:

$$\begin{aligned} U(y,t) &= J_Y(X(y,t),t)u(X(y,t),t), \\ P(y,t) &= p(X(y,t),t), \\ F(y,t) &= J_Y(X(y,t),t)f(X(y,t),t). \end{aligned}$$

Formal computations implies that $(U, P, (h_i, \omega_i)_{i=1,\dots,k})$ satisfies the following set of equations:

$$\frac{\partial U}{\partial t} - \nu[LU] + [MU] + [NU] + [GP] = F, \quad \text{in } \Omega_F \times ]0,T[, \tag{2.2.3}$$

$$\nabla \cdot U = 0, \quad \text{in } \Omega_F \times ]0,T[, \tag{2.2.4}$$

$$U(y,t) = h_i'(t) + \omega_i(t)(y - h_i(0))^\perp, \quad \text{in } \partial B_i \times [0,T[, \tag{2.2.5}$$

$$U(y,0) = u_0(y), \qquad y \in \Omega_F, \tag{2.2.6}$$

and for all $i \in \{1, \dots, k\}$, we have:

$$m_i h_i''(t) = -\int_{\partial B_i} \Sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{\partial B_i} F(t) dy, \;\; t \in ]0,T[, \tag{2.2.7}$$

$$J_i \omega_i'(t) = -\int_{\partial B_i} (y - h_i(0))^\perp \cdot \Sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{\partial B_i} (y - h_i(0))^\perp \cdot F(t) dy, \;\; t \in ]0,T[, \tag{2.2.8}$$

$$h_i(0) = h_i^0, \;\; h_i'(0) = h_i^1, \;\; \omega_i(0) = \omega_i^0, \;\; \forall i \in \{1, \dots, k\}, \tag{2.2.9}$$

where $\Sigma(U,P)$ is the Cauchy stress tensor field associated to $U$ and $P$. The operators $[LU]$, $[MU]$, $[NU]$ and $[GP]$ that appear in the left hand side of (2.2.3) are defined as follows:

$$[LU]_i = \sum_{j,k=1}^{2} \frac{\partial}{\partial y_j}(g^{jk} \frac{\partial U_i}{\partial y_k}) + 2 \sum_{j,k,\ell=1}^{2} g^{k\ell} \Gamma^i_{j,k} \frac{\partial U_j}{\partial y_\ell} + \sum_{j,k,\ell=1}^{2} \left\{ \frac{\partial}{\partial y_k}(g^{k\ell} \Gamma^i_{j,\ell}) + \sum_{m=1}^{2} g^{k\ell} \Gamma^m_{j,\ell} \Gamma^i_{k,m} \right\} U_j, \tag{2.2.10}$$

$$[MU]_i = \sum_{j=1}^{2} \frac{\partial Y_j}{\partial t} \frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^{2} \left\{ \Gamma^i_{j,k} \frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k} \frac{\partial^2 X_k}{\partial t \partial y_j} \right\} U_j, \tag{2.2.11}$$

$$[NU]_i = \sum_{j=1}^{2} U_j \frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^{2} \Gamma^i_{j,k} U_j U_k, \tag{2.2.12}$$

$$[GP]_i = \sum_{j=1}^{2} g^{ij} \frac{\partial P}{\partial y_j}, \tag{2.2.13}$$

with for all $i, j, k \in \{1, 2\}$, we have denoted

$$g^{ij} = \sum_{k=1}^{2} \frac{\partial Y_i}{\partial x_k} \frac{\partial Y_j}{\partial x_k}, \;\; g_{ij} = \sum_{k=1}^{2} \frac{\partial X_k}{\partial y_i} \frac{\partial X_k}{\partial y_j}, \;\; \Gamma^k_{i,j} = \frac{1}{2} \sum_{\ell=1}^{2} g^{k\ell} \left\{ \frac{\partial g_{i\ell}}{\partial y_j} + \frac{\partial g_{j\ell}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_\ell} \right\}. \tag{2.2.14}$$

**Proposition 2.2.1** *Suppose that for all $i \in \{1, \ldots, k\}$, we have $h_i \in H^2(0, T; \mathbb{R}^2)$ is such that*

$$B_i(t) \subset B(h_i(0), r_i + \frac{\gamma - \varepsilon}{2}), \quad \forall t \in [0, T].$$

*Then,*

$$(u, p, (h_i, \omega_i)_{i=1,\ldots,k}) \in \mathcal{U}(0, T, \Omega_F(t)) \times L^2(0, T, \dot{H}^1(\Omega_F(t))) \times \Big(H^2(0, T; \mathbb{R}^2) \times H^1(0, T; \mathbb{R})\Big)^k$$

*satisfies problem (2.1.1)-(2.1.6) if and only if*

$$(U, P, (h_i, \omega_i)_{i=1,\ldots,k}) \in \mathcal{U}(0, T, \Omega_F) \times L^2(0, T, \dot{H}^1(\Omega_F)) \times \Big(H^2(0, T; \mathbb{R}^2) \times H^1(0, T; \mathbb{R})\Big)^k$$

*satisfies* (2.2.3)-(2.2.9).

Proof. We recall that the equivalence between (2.1.1) and (2.2.3) has been done in [28]. The incompressibility condition follows by noting the following equality:

$$\nabla \cdot U(y, t) = \nabla \cdot u\,(X(y, t), t), \;\; \forall (y, t) \in \Omega_F \times [0, T]. \tag{2.2.15}$$

Noting that $J_X J_Y = I$, we have for all $(y, t) \in \Omega_F \times [0, T]$

$$\begin{aligned} \nabla \cdot U(y, t) &= \sum_{i,j=1}^{2} \frac{\partial}{\partial y_i} \Big\{ \frac{\partial Y_i}{\partial x_j}(X(y, t), t) u_j(X(y, t), t) \Big\} \\ &= \sum_{j=1}^{2} \operatorname{trace}\Big(J_Y{}^{-1} \partial_{x_j} J_Y\Big) u_j(X(y, t), t) + \nabla \cdot u\,(X(y, t), t). \end{aligned}$$

We recall Jacobi's formula which expresses the derivative of the determinant of a matrix $A$ in terms of the adjugate of $A$ and the derivative of $A$. If $A$ is a differentiable map from the real numbers to $n \times n$ matrices, then

$$\frac{d}{dt}\det A(t) = \text{trace}\Big((\text{adj } A(t))\frac{dA(t)}{dt}\Big).$$

Hence Jacobi's formula implies that

$$\nabla \cdot U(y,t) = \sum_{j=1}^{2} \frac{\partial \det(J_Y)}{\partial x_j} u_j(X(y,t),t) + \nabla \cdot u\,(X(y,t),t).$$

Equality (2.2.15) the follows by noting that $\det(J_Y) = 1$.

From the definition of the mapping $X$ in (2.2.2), we get for all $y \in \Omega_F$

$$\begin{aligned} U(y,0) &= J_Y(X(y,0),0)u(X(y,0),0) \\ &= J_Y(y,0)u(y,0) \\ &= u(y,0). \end{aligned}$$

Consequently, the initial condition (2.1.6) is equivalent to (2.2.6).

Using the fact that

$$X(y,t) = y + h_i(t) - h_i(0), \;\; \forall (y,t) \in \partial B_i \times [0,T],$$

it follows that for all $y \in \partial B_i$

$$\begin{aligned} U(y,t) &= u(y + h_i(t) - h_i(0), t) \\ &= h_i'(t) + \omega_i(t)(y - h_i(0))^{\perp}. \end{aligned}$$

Hence, (2.1.5) is equivalent to (2.2.5).

By using the change of variable $X$, we get

$$\int_{\partial B_i(t)} \sigma \nu_i d\Gamma_i = \int_{\partial B_i} \Big( -p(X(y,t),t)I + 2\nu D[u(X(y,t),t)]\Big)\nu_i d\Gamma_i.$$

At this point the equivalence between (2.1.3) and (2.2.7) and that between (2.1.4) and (2.2.8) hold by noticing again that

$$X(y,t) = y + h_i(t) - h_i(0), \ \ \forall (y,t) \in \partial B_i \times [0,T].$$

□

## 2.3 Local existence of solutions

The present section is devoted to prove the existence and uniqueness of strong solutions to problem (2.1.1)-(2.1.6) up to collision or blow up of the $\mathbf{H}^1$ norm of the velocity of the fluid. For sake of simplicity, we suppose that the external body force $f = 0$ throughout this section. First we consider the following linear problem in the fixed domain $\Omega_F$ obtained from (2.2.3)-(2.2.9) by neglecting the non-linear terms.

$$\frac{\partial U}{\partial t} - \nu \Delta U + \nabla P = F, \ \ \text{in } \Omega_F \times ]0,T], \tag{2.3.1}$$

$$\nabla \cdot U = 0, \ \ \text{in } \Omega_F \times ]0,T], \tag{2.3.2}$$

$$U(y,t) = h_i'(t) + \omega_i(t)(y - h_i(0))^\perp, \ \ y \in \partial B_i, \ t \in [0,T], i \in \{1,\ldots,k\}, \tag{2.3.3}$$

$$m_i h_i''(t) = -\int_{\partial B_i} \Sigma(U,P)\nu_i d\Gamma_i, \ \ t \in ]0,T], \ i \in \{1,\ldots,k\}, \tag{2.3.4}$$

$$J_i \omega_i'(t) = -\int_{\partial B_i} (y - h_i(0))^\perp \cdot \Sigma(U,P)\nu_i d\Gamma_i, \ \ t \in ]0,T], \ i \in \{1,\ldots,k\}, \tag{2.3.5}$$

with initial conditions:

$$U(y,0) = u_0(y), \ \ y \in \Omega_F, \tag{2.3.6}$$

$$h_i(0) = h_i^0, \ \ h_i'(0) = h_i^1, \omega_i(0) = \omega_i^0, \ \ i \in \{1,\ldots,k\}, \tag{2.3.7}$$

where we keep the notation $\Sigma = -PI + 2\nu D[U]$ as before and we suppose that there is no contact between the rigid bodies. The unknowns in the above system are $(U, P, (h_i, \omega_i)_{i=1,\ldots,k})$ and whereas $u_0$, $(h_i^0, h_i^1, \omega_i^0)_{i=1,\ldots,k}$ and $F$ are given. Following the same approach used in [41] where the authors studied a similar system, one can show the following theorem:

**Theorem 2.3.1** *Let $F \in L^2(0,T;\mathbf{L}^2(\Omega_F))$ and $u_0 \in \mathbf{H}^1(\Omega_F)$ such that*

$$\begin{aligned} \nabla \cdot u_0 &= 0, && \text{in } \Omega_{\mathrm{F}}, \\ u_0(y) &= h_i^1 + \omega_i^0 (y - h_i^0)^\perp, && y \in B_i,\ \forall i \in \{1,\dots,k\}. \end{aligned}$$

*Then the system (2.3.1)-(2.3.7) admits a unique solution $(U, P, (h_i,\omega_i)_{i=1,\dots,k})$ with*

$$U \in \mathcal{U}(0,T;\Omega_F),\ \ P \in L^2(0,T;\dot{H}^1(\Omega_F)),\ \ h_i \in H^2(0,T;\mathbb{R}^2),\ \ \omega_i \in H^1(0,T;\mathbb{R}).$$

*Moreover, there exists a positive constant $K$ depends only on $\Omega_F$ and $T$; non-decreasing with respect to $T$, such that*

$$\begin{aligned} \|U\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|U\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|U\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} + \|\nabla P\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \\ + \sum_{i=1}^k \|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})} \leq K\Big(\|u_0\|_{\mathbf{H}^1(\mathbb{R}^2)} + \|F\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))}\Big). \end{aligned}$$

Next, we write the solution $(U, P, (h_i,\omega_i)_{i=1,\dots,k})$ of problem (2.2.3)-(2.2.9) as a fixed point of a mapping $\mathcal{N}$ which is defined from the set

$$\begin{aligned} \mathcal{K} = \Big\{(W, Q, (h_i,\omega_i)_{i=1,\dots,k}) \in \mathcal{U}(0,T;\Omega_F) \times L^2(0,T;\mathbf{H}^1(\Omega_F)) \times \Big(H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})\Big)^k : \\ \|W\|_{\mathcal{U}} + \|Q\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^k \|h_i''\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega_i'\|_{L^2(0,T;\mathbb{R})} \leq R\Big\}. \end{aligned}$$

into

$$\mathcal{U}(0,T;\Omega_F) \times L^2(0,T;\mathbf{H}^1(\Omega_F)) \times [H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})]^k$$

as follows

$$\mathcal{N}(W, Q, (h_i,\omega_i)_{i=1,\dots,k}) = (U, P, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}),$$

where $(U, P, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k})$ satisfies:

$$\frac{\partial U}{\partial t} - \nu \Delta U + \nabla P = F, \ \text{ in } \Omega_F \times ]0,T], \tag{2.3.8}$$

$$\nabla \cdot U = 0, \ \text{ in } \Omega_F \times ]0,T], \tag{2.3.9}$$

$$U(y,t) = \tilde{h}_i'(t) + \tilde{\omega}_i(t)(y - h_i(0))^\perp, \ \ y \in \partial B_i,\ t \in [0,T], i \in \{1,\dots,k\}, \tag{2.3.10}$$

$$m_i\tilde{h}_i''(t) = -\int_{\partial B_i} \Sigma(U,P)\nu_i d\Gamma_i, \quad t \in ]0,T], \; i \in \{1,\dots,k\}, \tag{2.3.11}$$

$$J_i\tilde{\omega}_i'(t) = -\int_{\partial B_i} (y - h_i(0))^\perp \cdot \Sigma(U,P)\nu_i d\Gamma_i, \quad t \in ]0,T], \; i \in \{1,\dots,k\}, \tag{2.3.12}$$

with initial conditions at $t = 0$:

$$U(y,0) = u_0(y), \quad y \in \Omega_F, \tag{2.3.13}$$

$$\tilde{h}_i(0) = h_i^0, \;\; \tilde{h}_i'(0) = h_i^1, \;\; \tilde{\omega}_i(0) = \omega_i^0, \quad i \in \{1,\dots,k\}, \tag{2.3.14}$$

and

$$F = \nu[(L-\Delta)W] - [MW] + [(\nabla - G)Q] - [NW]. \tag{2.3.15}$$

We will see later that the source term $F$ is in the good space to apply Theorem 2.3.1 so that the mapping $\mathcal{N}$ is well defined.
It follows from the following proposition that for $T$ small enough and $R$ large enough, the mapping $\mathcal{N}$ has a fixed point in $\mathcal{K}$.

**Proposition 2.3.1** *For $T$ small enough and $R$ large enough, we have:*

*i. $\mathcal{N}(\mathcal{K}) \subset \mathcal{K}$*

*ii. the mapping $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction.*

The rest of this section is devoted then to prove Proposition 2.3.1. In the sequel, we denote by $K_0$ and $C_0$ positive quantities which satisfies the following conditions:

(i) $K_0$ is a positive function of $(\omega_i^0, h_i^1)_{i=1,\dots,k}$, $\|u_0\|_{\mathbf{H}^1(\Omega_F)}, T, r$ and $R$ which is non-decreasing with respect to $T, R$, $\|u_0\|_{\mathbf{H}^1(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

(ii) $C_0$ is a positive function of $(\omega_i^0, h_i^1)_{i=1,\dots,k}$, $\|u_0\|_{\mathbf{H}^1(\Omega_F)}, r$, and $T$ which is non-decreasing with respect to $T$, $\|u_0\|_{\mathbf{H}^1(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

**Proof of Proposition 2.3.1** We start by proving the first assertion in Proposition 2.3.1. Let $(W, Q, (h_i, \omega_i)_{i=1,\dots,k}) \in \mathcal{K}$, and set

$$(U, P, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}) = \mathcal{N}(W, Q, (h_i, \omega_i)_{i=1,\dots,k}).$$

According to Theorem 2.3.1, there exists a positive constant $K$ depends only on $\Omega_F$ and $T$; non-decreasing with respect to $T$ such that

$$\begin{aligned}\|U\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|U\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|U\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} + \|\nabla P\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \\ + \sum_{i=1}^{k} \|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})} \leq K(\|u_0\|_{\mathbf{H}^1(\mathbb{R}^2)} + \|F\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))}), \end{aligned} \tag{2.3.16}$$

where the source term $F$ is given by (2.3.15).

To bound the source term $F$ in $L^2(0,T;\mathbf{L}^2(\Omega_F))$, we need some necessary estimates for the transforms $X$ and $Y$. Using similar arguments as in [40], we get that there exists a constant $K_0$ satisfying condition $(i)$ such that

$$\left\|\frac{\partial X_i}{\partial y_j}\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq K_0, \quad \left\|\frac{\partial Y_i}{\partial x_j}\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq K_0, \tag{2.3.17}$$

$$\left\|\frac{\partial^2 X_i}{\partial y_j \partial y_k}\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0, \quad \left\|\frac{\partial^2 Y_i}{\partial x_j \partial x_k}\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0, \tag{2.3.18}$$

$$\left\|\frac{\partial^3 X_i}{\partial y_j \partial y_\ell \partial y_k}\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0, \quad \left\|\frac{\partial^3 Y_i}{\partial x_j \partial x_\ell \partial x_k}\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0. \tag{2.3.19}$$

Moreover, one has

$$\left\|\frac{\partial X_m}{\partial y_\ell} - \delta_m^\ell\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0, \quad \left\|\frac{\partial Y_m}{\partial x_\ell} - \delta_m^\ell\right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0, \tag{2.3.20}$$

$$\|g^{m\ell} - \delta_m^\ell\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0, \quad \|g_{m\ell} - \delta_m^\ell\|_{L^\infty(\mathbb{R}^2\times[0,T])} \leq TK_0, \tag{2.3.21}$$

where $\delta_m^\ell$ denotes the Kronecker delta (Leopold Kronecker) function. It is easy to check that

$$\begin{aligned}[(L-\Delta)W]_i = \sum_{j,k=1}^{2} \left(g^{jk} - \delta_k^j\right) \frac{\partial^2 W_i}{\partial y_j \partial y_k} + \sum_{j,k=1}^{2} \frac{\partial(g^{jk})}{\partial y_j} \frac{\partial W_j}{\partial y_k} + 2 \sum_{j,k,\ell=1}^{2} g^{k\ell} \Gamma_{j,k}^i \frac{\partial W_j}{\partial y_\ell} \\ + \sum_{j,k,\ell=1}^{2} \left\{ \frac{\partial}{\partial y_k}(g^{k\ell}\Gamma_{j,\ell}^i) + \sum_{m=1}^{2} g^{k\ell} \Gamma_{j,\ell}^m \Gamma_{k,m}^i \right\} W_j.\end{aligned}$$

Combining the above relation with the above estimates, we get that the coefficients of the $W$ and its first derivatives are bounded in $L^\infty([0,T]\times\mathbb{R}^2)$ by $K_0$ and that of its second derivative by $TK_0$.

On the other hand, we have

$$\|W\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} \leq T^{1/2}\|W\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))},$$

and thus for $T$ small enough, we obtain

$$\|[(L-\Delta)W]_i\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq K_0 T^{1/2}. \tag{2.3.22}$$

Using similar arguments, one has

$$\|[MW]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq K_0 T^{1/2}, \tag{2.3.23}$$

$$\|[(\nabla - G)P]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq K_0 T. \tag{2.3.24}$$

It remains to bound the non-linear term $[NW]$ in the expression of the source term $F$. By Holdder's inequality we have

$$\|(W\cdot\nabla)W\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq T^{1/10}\|(W\cdot\nabla)W\|_{L^{5/2}(0,T;\mathbf{L}^2(\Omega_F))}.$$

Lemma 5.2 in [41] implies that there exists $C > 0$ such that

$$\|(W\cdot\nabla)W\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq CT^{1/10}\|W\|^{6/5}_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))}\|W\|^{4/5}_{L^2(0,T;\mathbf{H}^2(\Omega_F))}.$$

Consequently,

$$\|(W\cdot\nabla)W\|_{L^2(0,T;\mathbf{L}^2(\Omega_F)))} \leq K_0 T^{1/10}.$$

Combining the above inequality with the estimates (2.3.17) and (2.3.18), we get for $T$ small enough

$$\|[NW]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq K_0 T^{1/10}.$$

It follows from the above inequality and (2.3.22)-(2.3.24) that

$$\|F\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq K_0 T^{1/10}.$$

Therefore for $T$ small enough, there exists a constant $K_0$ satisfying condition $(i)$ and a constant $C_0$ satisfying condition $(ii)$ such that

$$\|U\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|U\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|U\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} + \|\nabla P\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} + \sum_{i=1}^{k} \|\tilde{h}_i'\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})} \leq C_0 + K_0 T^{1/10}.$$

Thus, for $R > C_0$ and $T$ is small enough, the above estimate implies that

$$\|U\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|U\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|U\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} + \|\nabla P\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} + \sum_{i=1}^{k} \|\tilde{h}_i'\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})} \leq R.$$

Therefore, for $R > C_0$ and $T$ is small enough we have

$$(U, P, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\ldots,k}) \in \mathcal{K}.$$

Thus for $R > C_0$ and $T$ is small enough, the mapping $\mathcal{N}$ maps the set $\mathcal{K}$ into itself.

We turn now to prove that for $R > C_0$ and $T$ is small enough, the mapping $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction. To this end, we consider $(W^1, Q^1, (h_i^1, \omega_i^1)_{i=1,\ldots,k})$ and $(W^2, Q^2, (h_i^2, \omega_i^2)_{i=1,\ldots,k})$ in $\mathcal{K}$ and we denote by $Y^i, X^i, \Gamma^{ik}_{j,\ell}, U^i, P^i$, etc. the terms corresponding to $(W^i, Q^i, (h_j^1, \omega_j^1)_{j=1,\ldots,k})$. Also, we denote by $(U^i, P^i, (\tilde{h}_j^i, \tilde{\omega}_j^i)_{j=1,\ldots,k})$ the image of $(W^i, Q^i, (h_j^1, \omega_j^1)_{j=1,\ldots,k})$ by the mapping $\mathcal{N}$. Moreover, we denote by $Y = Y^1 - Y^2, h_i = h_i^1 - h_i^2$, etc. We get that the difference $(U, P, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\ldots,k})$ satisfies the following system:

$$\frac{\partial U}{\partial t} - \nu \Delta U + \nabla P = F, \ \text{ in } \Omega_F \times ]0,T], \tag{2.3.25}$$

$$\nabla \cdot U = 0, \ \text{ in } \Omega_F \times ]0,T], \tag{2.3.26}$$

$$U(y,t) = \tilde{h}_i'(t) + \tilde{\omega}_i(t)(y - h_{i,0})^\perp, \ \ y \in \partial B_i, \ t \in [0,T], i \in \{1,\ldots,k\}, \tag{2.3.27}$$

$$m_i \tilde{h}_i''(t) = -\int_{\partial B_i} \Sigma \nu_i d\Gamma_i, \ \ t \in ]0,T], \ i \in \{1,\ldots,k\}, \tag{2.3.28}$$

$$J_i \tilde{\omega}_i'(t) = -\int_{\partial B_i} (y - h_{i,0})^\perp \cdot \Sigma \nu_i d\Gamma_i, \ \ t \in ]0,T], \ i \in \{1,\ldots,k\}, \tag{2.3.29}$$

with initial conditions at $t = 0$:

$$U(y,0) = 0, \;\; y \in \Omega_F, \tag{2.3.30}$$

$$\tilde{h}_i(0) = 0, \;\; \tilde{h}_i'(0) = 0, \tilde{\omega}_i(0) = 0, \;\; i \in \{1,\dots,k\}, \tag{2.3.31}$$

and

$$F = \nu[(L^1-\Delta)W] + \nu[LW^2] - [M^1W] - [MW^2] + [(\nabla - G^1)Q] + [GQ^2] + [N^1W^1] - [N^2W^2].$$

Following similar arguments as in [40], there exists a constant $K_0$ such that

$$\left\| \frac{\partial^{i+j} X}{\partial y_1^i \partial y_2^j} \right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \le C_0 T^{1/2} \sum_{\ell=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}, \tag{2.3.32}$$

$$\left\| \frac{\partial^{i+j} Y}{\partial x_1^i \partial x_2^j} \right\|_{L^\infty(\mathbb{R}^2\times[0,T])} \le C_0 T^{1/2} \sum_{\ell=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}. \tag{2.3.33}$$

Consequently,

$$\begin{aligned}
\|[LW^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\le K_0 T^{1/2} \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)},\\
\|[MW^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\le K_0 T^{1/2} \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)},\\
\|[GQ^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\le K_0 T^{1/2} \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)},\\
\|[N^1W^1] - [N^2W^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\le K_0 T^{1/10} \Big( \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)} + \|W\|_{\mathcal{U}(0,T,\Omega_F)} \Big).
\end{aligned}$$

Noticing that the transforms $X^1$ and its inverse $Y^1$ satisfy (2.3.17)-(2.3.21), then one can treat the other terms in the source term $F$ in a similar way as before. Hence, it follows that

$$\|F\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \le K_0 T^{1/10} \Big( \|W\|_{\mathcal{U}(0,T,\Omega_F)} + \|Q\|_{L^2(0,T,\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^{k} \|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})} \Big).$$

According to Theorem 2.3.1, one has

$$\begin{aligned}
&\|U\|_{\mathcal{U}(0,T;\Omega_F)} + \|\nabla P\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} + \sum_{i=1}^{k} \|\tilde{h}_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})}\\
&\le K_0 T^{1/10} \Big( \|W\|_{\mathcal{U}(0,T,\Omega_F)} + \|Q\|_{L^2(0,T,\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^{k} \|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})} \Big).
\end{aligned} \tag{2.3.34}$$

Thus for $T$ is small enough, we get that $\mathcal{N}$ is contraction on $\mathcal{K}$. This ends up the proof. $\square$

**Remark 2.3.1** *By fixed point theorem and Proposition 2.3.1, the mapping $\mathcal{N}$ admits a unique fixed point in the space $\mathcal{K}$. Hence, problem* (2.2.3)-(2.2.9) *admits a unique strong solution. Therefore, according to Proposition 2.2.1 existence of unique strong solution to problem* (2.1.1)-(2.1.6) *follows using the inverse transform $Y$. We remark that according to Theorem 2.3.1, we can extend our solution on $[0, T_1]$ as long as there is no contact between the rigid bodies and $\|u(t)\|_{\mathbf{H}^1(\Omega_F(t))}$ is bounded for all $t \in [0, T_1]$.*

**Remark 2.3.2** *The assumption that all the rigid bodies have to be disks is not essential to obtain the existence and uniqueness of local strong solutions of problem* (2.1.1)-(2.1.6)*. Strong solutions still exist if the rigid bodies are of arbitrary shape but still connected and closed subset of $\mathbb{R}^2$ with boundaries of class $\mathcal{C}^3$. However, to prove the that the solution of problem* (2.1.1)-(2.1.6) *is global, we need the boundaries to be too smooth (see [20]). Therefore, we have to assume that the rigid bodies are closed disks. Moreover, this assumption has simplified the change of variable $X$ as there is no need to introduce the rotation in the definition of the change of variable to transform the fluid domain into its initial shape. Otherwise, on should also transform (otherwise you should also transform $h_i$ and $\omega_i$.*

## 2.4 Existence of solutions up to collision

In the previous section, we have shown that there exists a time $T > 0$ such that problem (2.1.1)-(2.1.6) admits a unique strong solution $(u, p, (h_i, \omega_i)_{i=1,\dots k})$ in $[0, T]$. We define the global velocity as follows:

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{in } \Omega_F(t), \\ h_i'(t) + \omega_i(t)(x - h_i(t))^{\perp} & \text{in } B_i(t), 1 \le i \le k. \end{cases}$$

Moreover, if we define $T_0$ such that:

$$T_0 := \sup \left\{ T \in \mathbb{R}_+^* : \text{ problem } (2.1.1) - (2.1.6) \text{ admits a unique strong solution in } [0, T] \right\},$$

then the following alternative holds true:

1. $T_0 = +\infty$,
2. $\limsup_{t \to T_0} \|\tilde{u}(t)\|_{\mathbf{H}^1(\mathbb{R}^2)} + \dfrac{1}{\min_{i \neq j} d(B_i(t), B_j(t))} = +\infty$.

For simplicity, we denote from now on the global velocity by $u$ instead of $\tilde{u}$ and we fix $\varepsilon$ such that $0 < \varepsilon < \gamma$. We focus here on the blow up alternatives. We prove that the $\mathbf{H}^1$ norm of the solution does not blow up in finite time as long as no the rigid bodies are not in contact.

**Proposition 2.4.1** *If $T_0 < +\infty$ and $\min_{i \neq j} d(B_i(t), B_j(t)) > \varepsilon > 0$ on $[0, T_0]$, then the mapping*

$$t \to \|u(t)\|_{\mathbf{H}^1(\mathbb{R}^2)}$$

*is bounded on $[0, T_0)$ by a constant that depends on $\varepsilon, \gamma, \|f\|_{L^2(0,T_0,\mathbf{L}^2(\mathbb{R}^2))}$ and the initial data.*

**Remark 2.4.1** *Theorem 2.1.1 is an immediate consequence of the above proposition and Remark 2.3.1.*

We split the proof of Proposition 2.4.1 into two lemmas. First, we control the $\mathbf{L}^2$ norm of the solution.

**Lemma 2.4.1** *Let $(u, p, (h_i, \omega_i)_{i=1,\dots k})$ be the strong solution associated to problem (2.1.1)-(2.1.6) on $[0, T]$. If $T_0 < \infty$, then there exists a positive constant $M = M(T_0, (\overline{\rho}_i, B_i)_{i=1,\dots,k})$, such that*

$$\sup_{[0,T_0)} \left( ||u(t)||^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k (|h_i'(t)|^2 + |\omega_i(t)|^2) \right) + 2\nu \int_0^{T_0} ||\nabla u(t)||^2_{\mathbf{L}^2(\mathbb{R}^2)} dt$$
$$\leq M \left( ||f||^2_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))} + ||u_0||^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k (|h_i^1|^2 + |\omega_i^0|^2) \right).$$

Proof. By taking the inner product of equation (2.1.1) with $u$, and integrating over $\Omega_F(t)$, we get that

$$\int_{\Omega_F(t)} \frac{\partial u}{\partial t}(t) \cdot u(t) dx - \nu \int_{\Omega_F(t)} \Delta u(t) \cdot u(t) dx + \int_{\Omega_F(t)} (u(t) \cdot \nabla) u(t) \cdot u(t) dx$$
$$+ \int_{\Omega_F(t)} \nabla p(t) \cdot u(t) dx = \int_{\Omega_F(t)} f(t) \cdot u(t) dx. \quad (2.4.1)$$

By Reynold's theorem, we have

$$\int_{\Omega_F(t)} \frac{\partial u}{\partial t}(t) \cdot u(t) dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega_F(t)} |u(t)|^2 dx - \frac{1}{2} \sum_{i=1}^K \int_{B_i(t)} (u(t) \cdot \nu_i) |u(t)|^2 d\Gamma_i. \quad (2.4.2)$$

By integrating by parts and noting (2.1.2), we have

$$\int_{\Omega_F(t)} (u(t)\cdot\nabla)u(t)\cdot u(t)dx = \frac{1}{2}\sum_{i=1}^{k}\int_{\partial B_i(t)} |u(t)|^2 u(t)\cdot \nu_i d\Gamma_i.$$

The incompressibility condition (2.1.2) implies that

$$\Delta u\cdot u = 2\nabla\cdot(D[u])\cdot u = 2\nabla\cdot(D[u]u) - 2D[u]:D[u]. \tag{2.4.3}$$

The above equality implies that

$$\begin{aligned}\int_{\Omega_F(t)} \Delta u(t)\cdot u(t)dx &= 2\int_{\Omega_F(t)} \nabla\cdot(D[u(t)]u(t))dx - 2\int_{\Omega_F(t)} |D[u(t)]|^2 dx\\ &= 2\sum_{i=1}^{k}\int_{\partial B_i(t)} (D[u(t)]u(t))\cdot u(t)d\Gamma_i - 2\int_{\Omega_F(t)} |D[u(t)]|^2 dx.\end{aligned} \tag{2.4.4}$$

By performing integration by parts; noting (2.1.2), and combining (2.4.2) with (2.4.4), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega_F(t)} |u(t)|^2dx + 2\nu\int_{\Omega_F(t)} |D[u(t)]|^2dx - 2\nu\sum_{i=1}^{k}\int_{\partial B_i(t)} (D[u(t)]u(t))\cdot\nu_i d\Gamma_i&\\ +\sum_{i=1}^{k}\int_{B_i(t)} p(t)u(t)\cdot\nu_i d\Gamma_i = \int_{\Omega_F(t)} f(t)\cdot u(t)dx.&\end{aligned} \tag{2.4.5}$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_F(t)} |u(t)|^2dx + 2\nu\int_{\Omega_F(t)} |D[u(t)]|^2dx = \sum_{i=1}^{k}\int_{\partial B_i(t)} (\sigma u(t))\cdot\nu_i d\Gamma_i + \int_{\Omega_F(t)} f(t)\cdot u(t)dx. \tag{2.4.6}$$

Taking now the inner product of (2.1.3) with $h_i'(t)$ and that of (2.1.4) by $\omega_i(t)$, we get that

$$\begin{aligned}\frac{m_i}{2}\frac{d}{dt}|h_i'(t)|^2 + \frac{J_i}{2}\frac{d}{dt}|\omega_i(t)|^2 = -\int_{\partial B_i(t)} (h_i'(t)+\omega_i(t)(x-h_i(t))^\perp)\cdot\sigma\nu_i d\Gamma_i&\\ +\overline{\rho}_i\int_{B_i(t)} (h_i'(t)+\omega_i(t)(x-h_i(t))^\perp)\cdot f(t)dx,\ \forall i\in\{1,\dots,k\}.&\end{aligned} \tag{2.4.7}$$

The no-slip condition (2.1.5) implies that

$$\frac{m_i}{2}\frac{d}{dt}|h_i'(t)|^2 + \frac{J_i}{2}\frac{d}{dt}|\omega_i(t)|^2 = -\int_{B_i(t)} u(t)\cdot\sigma\nu_i d\Gamma_i \\ + \overline{\rho}_i\int_{B_i(t)}(h_i'(t)+\omega_i(t)(x-h_i(t))^{\perp})\cdot f(t)dx,\ \forall i\in\{1,\dots,k\}. \quad (2.4.8)$$

Combining (2.4.6) with the $k$ equations in (2.4.8) and noticing that the Cauchy stress tensor field $\sigma$ is symmetric, we get that

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_F(t)}|u(t)|^2dx + \sum_{i=1}^{k}(m_i|h_i'(t)|^2 + J_i|\omega_i(t)|^2)\right) + 2\nu\int_{\Omega_F(t)}|D[u(t)]|^2dx \\ = \int_{\Omega_F(t)} f(t)\cdot u(t)dx + \overline{\rho}_i\int_{B_i(t)}(h_i'(t)+\omega_i(t)(x-h_i(t))^{\perp})\cdot f(t)dx. \quad (2.4.9)$$

Integrating the above inequality from 0 to $t$, we get for almost $t\in[0,T_0)$

$$||u(t)||^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k}(m_i|h_i'(t)|^2 + J_i|\omega_i(t)|^2) + 2\nu\int_0^t\int_{\mathbb{R}^2}|\nabla u(t)|^2dx \\ \leq \int_0^t\left(||u(s)||^2_{\mathbf{L}^2(\Omega_F(s))} + \sum_{i=1}^{k}(m_i|h_i'(s)|^2 + J_i|\omega_i(s)|^2)\right)ds + C\Big(\|f(s)\|^2_{L^2(0,T_0,\mathbf{L}^2(\mathbb{R}^2))}ds + ||u(0)||^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big).$$

Therefore the energy estimate is then follows from applying Gronwall lemma. □

In the rest of the section, we keep the constant $M$ as it is defined in the above lemma and we define $K_1$ by

$$K_1 = \left(||f||^2_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))} + ||u_0||^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k}(|h_i^1|^2 + |\omega_i^0|^2)\right)^{\frac{1}{2}}.$$

Proposition 2.4.1 will be then deduced from the following lemma:

**Lemma 2.4.2** *Let $(u,p,(h_i,\omega_i)_{i=1,\dots,k})$ be the strong solution in $[0,T_1]$, where $T_1 < T_0$ is small enough and depends on $\nu$, $M$ and the initial data. Then there exists $K>1$ such that*

$$\sup_{t\in[0,T_1]}\|\nabla u(t)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} \leq K\Big(\|\nabla u(0)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + 1\Big), \quad (2.4.10)$$

*and*

$$\int_0^{T_1} \left\| \frac{\partial u}{\partial t} \right\|^2_{\mathbf{L}^2(\Omega_F(s))} ds + \sum_{i=1}^k \left( \int_0^{T_1} |h_i''(s)|^2 ds + \int_0^{T_1} |\omega_i'(s)|^2 ds \right) \leq K\Big( \|\nabla u(0)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + 1 \Big)^2. \tag{2.4.11}$$

*where the constant $K$ depends on $\Omega_F, B_i, \nu, \overline{\rho}_i, T_0, \|u_0\|_{\mathbf{L}^2(\Omega_F)}, |h_i^1|, |\omega_i^0|$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$.*

**Remark 2.4.2** *As the system is autonomous, then for all $t \geq 0$ the above proposition is still valid on any interval $[t, t+T_1] \subset [0, T_0[$.*

Before giving the proof of Lemma 2.4.2, let us see how it implies Proposition 2.4.1. Lemma 2.4.2 implies that the mapping $t \mapsto \|\nabla u(t)\|_{\mathbf{L}^2(\mathbb{R}^2)}$ is bounded on $[0, T_1]$ for $T_1$ is small enough. We can choose $T_1$ such that $T_0 = NT_1$, for some $N \in \mathbb{N}^*$. This implies that

$$\|\nabla u(t)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} \leq K \|\nabla u((n-1)T_1)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + K, \text{ a.e on } [(n-1)T_1, nT_1[, \ n = 1, \dots, N.$$

By induction, we get that

$$\|\nabla u(t)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} \leq K^n \|\nabla u(0)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \frac{K^{n+1} - K}{K-1}, \text{ a.e on } [(n-1)T_1, nT_1[, \ n = 1, \dots, N,$$

and thus

$$\sup_{t \in [0, T_0[} \|\nabla u(t)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} \leq K^N \|\nabla u(0)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \frac{K^{N+1} - K}{K-1}.$$

Combining this result with Lemma 2.4.1, we get that for $T_0 < +\infty$, the mapping

$$t \to \|u(t)\|_{\mathbf{H}^1(\mathbb{R}^2)}$$

is bounded on $[0, T_0)$ whenever there is no contact between the rigid bodies.

We focus now on the proof of Lemma 2.4.2. To this end, we follow the method of Cumsille and Takahashi in [12] and we start with defining some auxiliary functions.

We consider a family of smooth functions $\{\zeta_i\}_{i=1,\dots,k}$ ; each of compact support contained in $B(h_i(0), r_i + \frac{\delta}{2})$ and equals 1 in a neighbourhood of $B_i$. For a fixed $i$ in $\{1, \dots, k\}$, we set $\hat{\psi}_i(x,t) = \zeta_i(x - h_i(t) + h_i(0))$ and we define the mapping $\hat{\Lambda} : \mathbb{R}^2 \times [0, T] \to \mathbb{R}^2$ by

$$\hat{\Lambda}(x,t) = \sum_{i=1}^k \nabla^\perp \Big( h_i'(t) \cdot x^\perp \hat{\psi}_i(x,t) \Big).$$

Let $\hat{X}$ be the solution of the initial value problem

$$\begin{cases} \dfrac{\partial \hat{X}}{\partial t}(y,t) & = \hat{\Lambda}(\hat{X}(y,t),t), \;\; t \in ]0,T], \\ \hat{X}(y,0) & = y \in \mathbb{R}^2. \end{cases} \tag{2.4.12}$$

Then for $y \in B_i$, we have

$$\hat{X}(y,t) = y + h_i(t) - h_i(0).$$

It is easy to see that

$$\exists C > 0 \text{ such that } ||\hat{\Lambda}||_{\mathbf{W}^{2,\infty}(\Omega_F(t))} \leq C \sum_{i=1}^{k} |h_i'(t)|.$$

By Lemma 2.4.1, we get

$$||\hat{\Lambda}||_{\mathbf{W}^{2,\infty}(\Omega_F(t))} \leq C M^{\frac{1}{2}} K_1.$$

Taking the inner product of equation (2.1.1) with $\partial_t u + (\hat{\Lambda} \cdot \nabla)u - (u \cdot \nabla)\hat{\Lambda}$ yields

$$\begin{aligned} &\int_{\Omega_F(t)} \left|\frac{\partial u}{\partial t}\right|^2 dx + \int_{\Omega_F(t)} \frac{\partial u}{\partial t} \cdot \Big((\hat{\Lambda}\cdot\nabla)u - (u\cdot\nabla)\hat{\Lambda}\Big) dx - \int_{\Omega_F(t)} \nabla\cdot\sigma(u,p)\cdot\Big(\frac{\partial u}{\partial t} + (\hat{\Lambda}\cdot\nabla)u - (u\cdot\nabla)\hat{\Lambda}\Big) dx \\ &= -\int_{\Omega_F(t)} [(u\cdot\nabla)u]\cdot\Big(\frac{\partial u}{\partial t} + (\hat{\Lambda}\cdot\nabla)u - (u\cdot\nabla)\hat{\Lambda}\Big) dx + \int_{\Omega_F(t)} f\cdot\Big(\frac{\partial u}{\partial t} + (\hat{\Lambda}\cdot\nabla)u - (u\cdot\nabla)\hat{\Lambda}\Big) dx. \end{aligned}$$

With similar arguments as in Lemma 4.3 in [12], we have

$$\begin{aligned} &-\int_{\Omega_F(t)} [\nabla\cdot\sigma(u,p)]\cdot(\frac{\partial u}{\partial t} + (\hat{\Lambda}\cdot\nabla)u - (u\cdot\nabla)\hat{\Lambda}) dx \\ &= \nu \frac{d}{dt}\int_{\Omega_F(t)} |D[u]|^2 dx + \sum_{i=1}^{k}\Big(m_i|h_i''(t)|^2 + J_i|\omega_i'(t)|^2 - \int_{B_i(t)} \overline{\rho}_i f(x,t)\cdot(h_i''(t) + \omega_i'(t)(x-h_i(t))^{\perp}\Big) \\ &\qquad + 2\nu\int_{\Omega_F(t)} D[u] : (\nabla u \nabla\hat{\Lambda}) dx - 2\nu\int_{\Omega_F(t)} D[u] : D[(u\cdot\nabla)\hat{\Lambda}]) dx. \end{aligned}$$

It follows that

$$
\begin{aligned}
\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))} & +\nu\frac{d}{dt}\int_{\Omega_F(t)}|D[u]|^2dx+\sum_{i=1}^{k}\Big(m_i|h_i^{''}(t)|^2+J_i|\omega_i^{'}(t)|^2\Big)\\
&=2\nu\int_{\Omega_F(t)}\Big(D[u]:D[(u\cdot\nabla)\hat{\Lambda}]-D[u]:(\nabla u\nabla\hat{\Lambda})\Big)dx\\
&+\sum_{i=1}^{k}\int_{B_i(t)}\overline{\rho}_i f(x,t)\cdot\Big(h_i^{''}(t)+\omega_i^{'}(t)(x-h_i(t))^{\perp}\Big)\\
&-\int_{\Omega_F(t)}\frac{\partial u}{\partial t}\cdot\Big((\hat{\Lambda}\cdot\nabla)u-(u\cdot\nabla)\hat{\Lambda}\Big)dx\\
&-\int_{\Omega_F(t)}[(u\cdot\nabla)u]\cdot\Big(\frac{\partial u}{\partial t}+(\hat{\Lambda}\cdot\nabla)u-(u\cdot\nabla)\hat{\Lambda}\Big)dx\\
&+\int_{\Omega_F(t)}f\cdot\Big(\frac{\partial u}{\partial t}+(\hat{\Lambda}\cdot\nabla)u-(u\cdot\nabla)\hat{\Lambda}\Big)dx,\quad \text{a.e in } (0,T_1).
\end{aligned}
\tag{2.4.13}
$$

Lemma 2.4.1 implies that there exists a positive constant $C_1=C_1(T_0,\nu,(\overline{\rho}_i,B_i)_{i=1,\ldots,k})$, such that the following holds true for almost $t\in(0,T)$

$$
\begin{aligned}
\Big|2\nu\int_{\Omega_F(t)}D[u]:D[(u\cdot\nabla)\hat{\Lambda}]-D[u]:(\nabla u\nabla\hat{\Lambda})dx\Big| &\leq C_1\Big((1+K_1^2)\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}+K_1^4\Big),\\
\Big|\int_{B_i(t)}\overline{\rho}_i f(x,t)\cdot\Big(h_i^{''}(t)+\omega_i^{'}(t)(x-h_i(t))^{\perp}\Big| &\leq \overline{\rho}_i\|f\|^2_{\mathbf{L}^2(B_i(t))}+\frac{J_i}{2}|\omega_i^{'}(t)|^2+\frac{m_i}{2}|h_i^{''}(t)|^2,
\end{aligned}
$$

$$
\begin{aligned}
\Big|\int_{\Omega_F(t)}\frac{\partial u}{\partial t}\cdot\Big((\hat{\Lambda}\cdot\nabla)u-(u\cdot\nabla)\hat{\Lambda}\Big)dx\Big| &\leq \frac{1}{4}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+C_1K_1^2(\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}+K_1^2),\\
\Big|\int_{\Omega_F(t)}[(u\cdot\nabla)u]\cdot\Big(\frac{\partial u}{\partial t}+(\hat{\Lambda}\cdot\nabla)u-(u\cdot\nabla)\hat{\Lambda}\Big)dx\Big| &\leq \frac{1}{8}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+3\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))}\\
&\quad +C_1K_1^2\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}+C_1K_1^4,\\
\Big|\int_{\Omega_F(t)}f\cdot\Big(\frac{\partial u}{\partial t}+(\hat{\Lambda}\cdot\nabla)u-(u\cdot\nabla)\hat{\Lambda}\Big)dx\Big| &\leq \frac{1}{8}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+C_1K_1^2\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}\\
&\quad +\frac{5}{2}\|f\|^2_{\mathbf{L}^2(\Omega_F(t))}+C_1K_1^4.
\end{aligned}
$$

Combining the above estimates with (2.4.13), we obtain for almost $t\in(0,T)$

$$
\begin{aligned}
&\frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+\nu\frac{d}{dt}\int_{\Omega_F(t)}|D[u]|^2dx+\frac{1}{2}\sum_{i=1}^{k}\Big(m_i|h_i^{''}(t)|^2+J_i|\omega_i^{'}(t)|^2\Big)\\
&\qquad\leq C_1\Big(K_1^4+(K_1^2+1)\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}+\|f\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big)+3\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))}.
\end{aligned}
\tag{2.4.14}
$$

We are now in position to estimate the term $(u\cdot\nabla)u$ in terms of the left hand side of inequality (2.4.14). Here, we adapt the method followed by Cumsille and Takahashi in [12] to the case of several rigid bodies. See [13] for an alternative approach. We state the following two technical lemmas and we postpone their proof to Appendix A at the end of the thesis.

**Lemma 2.4.3** *Let $\gamma > \varepsilon > 0$. Then there exists a strong 2-extension operator $E$ for $\Omega_F(t)$. Moreover, there exists a positive constant $k = k(\varepsilon)$ such that for $u \in \mathbf{H}^2(\Omega_F(t))$, we have:*

$$\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))}, \tag{2.4.15}$$

$$\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^1(\Omega_F(t))}, \tag{2.4.16}$$

$$\|Eu\|_{\mathbf{H}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^2(\Omega_F(t))}. \tag{2.4.17}$$

**Lemma 2.4.4** *Let $u$ be the unique strong solution of problem (2.1.1)-(2.1.6). Then there exists $T_1$ small enough, such that for almost every $t \in [0, T_1]$, we have*

$$\|u(t)\|_{\mathbf{H}^2(\Omega_F(t))} \leq K\Big(\Big\|\frac{\partial u}{\partial t}(t)\Big\|_{\mathbf{L}^2(\Omega_F(t))} + \|u(t)\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|\nabla u(t)\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|f(t)\|_{\mathbf{L}^2(\Omega_F(t))} + \|\overline{\Lambda}\|_{\mathbf{H}^2(\mathbb{R}^2)} + 1\Big),$$

*where $K$ is a positive constant that depends on $\Omega_F, B_i, \overline{\rho}_i, \nu, T_0, \|u_0\|_{\mathbf{L}^2(\mathbb{R}^2)}$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$.*

We return now to complete the proof of Lemma 2.4.2. Lemma 2.4.3 implies that there exists a strong 2-extension operator $E$ for $\Omega_F(t)$

$$\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} \leq \|(Eu\cdot\nabla)Eu\|^2_{\mathbf{L}^2(\mathbb{R}^2)} \leq \|Eu\|^2_{\mathbf{L}^4(\mathbb{R}^2)}\|\nabla Eu\|^2_{\mathbf{L}^4(\mathbb{R}^2)}.$$

Moreover, using the continuous embedding of $H^{1/2}(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$ and the interpolation inequality in Lions–Magenes [30], we have that

$$\|z\|_{L^4(\mathbb{R}^2)} \leq C_2\|z\|_{H^{1/2}(\mathbb{R}^2)} \leq C_2\|z\|^{1/2}_{L^2(\mathbb{R}^2)}\|z\|^{1/2}_{H^1(\mathbb{R}^2)}, \;\; \forall z \in H^1(\mathbb{R}^2),$$

where $C_2 = C_2(\mathbb{R}^2)$ is a positive real constant. Hence, we get

$$\begin{aligned}
\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} &\le C_2\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)}\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)}\|\nabla Eu\|_{\mathbf{L}^2(\mathbb{R}^2)}\|\nabla Eu\|_{\mathbf{H}^1(\mathbb{R}^2)}\\
&\le C_2\|u\|_{\mathbf{L}^2(\Omega_F(t))}\|u\|_{\mathbf{H}^1(\Omega_F(t))}\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)}\|Eu\|_{\mathbf{H}^2(\mathbb{R}^2)}\\
&\le C_2\|u\|_{\mathbf{L}^2(\Omega_F(t))}\|u\|^2_{\mathbf{H}^1(\Omega_F(t))}\|u\|_{\mathbf{H}^2(\Omega_F(t))}\\
&\le C_2\|u\|_{\mathbf{L}^2(\Omega_F(t))}\Big(\|u\|^2_{\mathbf{L}^2(\Omega_F(t))}+\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}\Big)\|u\|_{\mathbf{H}^2(\Omega_F(t))}.
\end{aligned} \tag{2.4.18}$$

Let $K > 1$ be a constant that depends on $\Omega_F, B_i, \overline{\rho}_i, \nu, T_0, \|u_0\|_{\mathbf{L}^2(\mathbb{R}^2)}$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$ that may changes between lines.
Combining (2.4.18) with Lemma 2.4.4, we get for $T_1$ small enough

$$\begin{aligned}
\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} \le K\|u\|_{\mathbf{L}^2(\Omega_F(t))}\Big(\|u\|^2_{\mathbf{L}^2(\Omega_F(t))}+\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}\Big)\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F(t))}+\|u\|^2_{\mathbf{L}^2(\Omega_F(t))}\\
+\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}+\|f\|_{\mathbf{L}^2(\Omega_F(t))}+1\Big), \quad a.e\ t\in[0,T_1].
\end{aligned}$$

By Young's inequality, we get for all $\epsilon > 0$

$$\begin{aligned}
\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\Omega_F(t))} \le \frac{K}{\epsilon}\|u\|^2_{\mathbf{L}^2(\Omega_F(t))}\Big(\|u\|^2_{\mathbf{L}^2(\Omega_F(t))}+\|\nabla u\|^2_{\mathbf{L}^2(\Omega_F(t))}\Big)^2+\epsilon\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(t))}+\|u\|^4_{\mathbf{L}^2(\Omega_F(t))}\\
+\|\nabla u\|^4_{\mathbf{L}^2(\Omega_F(t))}+\|f\|^2_{\mathbf{L}^2(\Omega_F(t))}+1\Big), \quad a.e\ t\in[0,T_1].
\end{aligned}$$

By combining (2.4.14) with the above inequality taking $\epsilon = \frac{1}{12}$, then integrating the resulting inequality with respect to $t$, and using Lemma 2.4.1 we get for almost $t$ in $[0, T_1]$

$$\begin{aligned}
\frac{1}{4}\int_0^t \Big\|\frac{\partial u}{\partial t}\Big\|^2_{\mathbf{L}^2(\Omega_F(s))} ds + \frac{\nu}{2}\int_{\mathbb{R}^2}|\nabla u|^2 dx + \frac{1}{2}\sum_{i=1}^k\Big(m_i\int_0^t |h_i''(s)|^2 ds + J_i\int_0^t |\omega_i'(s)|^2 ds\Big)\\
\le \frac{\nu}{2}\|\nabla u(0)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + K\Big(1+\int_0^t \|\nabla u\|^2_{\mathbf{L}^2(\mathbb{R}^2)} ds + \int_0^t \|\nabla u\|^4_{\mathbf{L}^2(\mathbb{R}^2)} ds\Big).
\end{aligned} \tag{2.4.19}$$

By Gronwall lemma and using again Lemma 2.4.1, we get that

$$\|\nabla u(t)\|^2_{\mathbf{L}^2(\mathbb{R}^2)} \le K\Big(\|\nabla u(0)\|^2_{\mathbf{L}^2(\mathbb{R}^2)}+1\Big).\ \text{a.e on } [0,T_1]. \tag{2.4.20}$$

Moreover, by combining (2.4.19) with (2.4.20), estimate (2.4.11) holds. This ends the proof.

□

## 2.5 Mechanism preventing from collision

This section is devoted to accomplish the proof of Theorem 1.2. We follow the approach used in [20] and [24]. We act by contradiction and we assume that collision can take place in finite time $T_0$ under the assumption (H1). For $T < T_0$, we recall that for any divergence free $w \in H^1((0,T)\times\mathbb{R}^2)$, such that $D[w]$ vanishes on the solid domain, we have

$$\int_{\mathbb{R}^2}\Big(\rho u\cdot\partial_t w + \rho u\otimes u : D[w] - 2\nu D[u] : D[w] + \rho f\cdot w\Big)dx = \frac{d}{dt}\int_{\mathbb{R}^2}\rho u\cdot w dx, \qquad (2.5.1)$$

where

$$\rho(x,t) = \rho_F(x,t) + \sum_{i=1}^{k}\rho_i(x,t) = \mathbf{1}_{\Omega_F(t)}(x) + \sum_{i=1}^{k}\overline{\rho}_i\mathbf{1}_{B_i(t)}(x),$$

denotes the global density. The key-idea of the proof is to construct a proper candidate $v$ and use it as a test function in the weak formulation (2.5.1) leading to a differential equation which can be integrated so that we get the *no-collision* result.

### 2.5.1 Construction and estimates for the test function

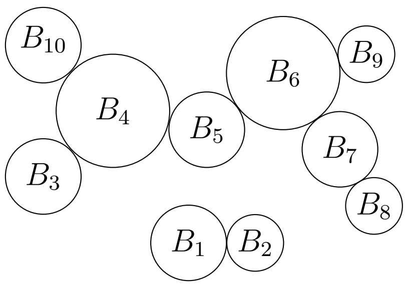


Figure 2.1 – Example of collision at time $T_0$

We suppose that $T_0 < +\infty$ and we start to prove that collision in pair - as that between the disks $B_1$ and $B_2$ or between $B_8$ and $B_7$ in Figure 2.1 - could not take place. Both cases can be summarized by the assumption that one disk has a collision with only one other disk. Up to renumbering, this assumption can be stated as follows:

$$d(B_1(T_0), B_2(T_0)) = 0, \text{ and } d(B_1(T_0), B_i(T_0)) > 0, \forall\, i = 3,\dots,k. \qquad \text{(H2)}$$

Since the disks $B_1$ and $B_2$ collide at time $T_0$, then we can choose an initial time $t_0 < T_0$ such that for all $t \geq t_0$ and all $j \notin J$, we have $d(B_1(t), B_2(t)) < 2r_j$, where $J = \{1,2\}$.

In other words, we can choose initial time $t_0$ such that there is no possibility to find a disk separating the rigid bodies $B_1(t)$ and $B_2(t)$ for all $t \in [t_0, T_0]$. For all $i \in \{2, \ldots, k\}$, we define $d_{1,i}(t) := d(B_1(t), B_i(t))$. Since $d_{1,i}(T_0)$ is positive as long as $i \notin J$, then $\beta := \inf_{t_0 \le t \le T_0} \min_{i \notin J} d_{1,i}(t) > 0$. Also contact at time $T_0$ can only occur at a single point between any pair of disks as the domains of the rigid bodies are convex.

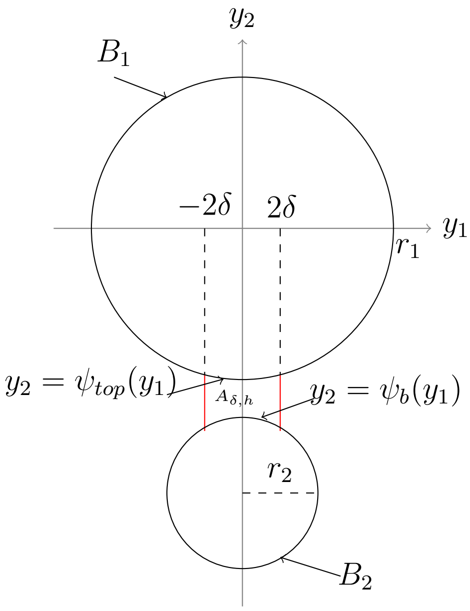


Figure 2.2 – Geometry in the local coordinates

We will see later that the expression of the vector field $v$ involves the boundary functions of the disks in the neighbourhood of the contact point. Hence, it is more convenient to work in a local orthonormal frame of origin attached to the center of mass $h_1$ of the disk $B_1$, and whose associated orthonormal basis $(e_1, e_2)$ are given by: $e_1 = -e_2^{\perp}$ and $e_2 = \dfrac{h_1 - h_2}{|h_1 - h_2|}$. In the new coordinates, $B_1$ is the disk of center $(0, 0)$ and radius $r_1$ whereas $B_2$ is the disk of center $(0, -r_1 - r_2 - h)$ and radius $r_2$, where $h$ denotes the distance between $B_1$ and $B_2$ (see Figure 2.2). Moreover, we can always represent the boundaries of the disks close to the contact point at collision by a suitable boundary functions of simple expressions in the new frame: one is the lower boundary of the disk $B_1$ and the other is the upper boundary of the disk $B_2$.

For any $x \in \mathbb{R}^2$, we denote by $(y_1, y_2)$ the coordinates in the new frame. More precisely, we define $Y = (y_1, y_2)$ as follows:

$$Y(t, x) = \left( -\frac{(x - h_1(t)) \cdot (h_1(t) - h_2(t))^{\perp}}{|h_1(t) - h_2(t)|}, \frac{(x - h_1(t)) \cdot (h_1(t) - h_2(t))}{|h_1(t) - h_2(t)|} \right). \tag{2.5.2}$$

In what follows, we fix $h \in (0, d_{\max})$ where $d_{\max} := \sup\limits_{t_0 \leq t \leq T_0} d_{1,2}(t)$. Also, we fix $\delta > 0$ such that $2\delta < \min(r_1, r_2)$, and we define the bridge $A^i_{\delta,h}$ in the local coordinates by

$$A^i_{\delta,h} := \{y \in \mathbb{R}^2 : \ |y_1| < 2\delta, \ \psi_b(y_1) < y_2 < \psi_{top}(y_1)\},$$

where the boundary functions $\psi_{top}$ and $\psi_b$ of the disks $B_1$ and $B_2$ respectively are given by:

$$\begin{aligned} \psi_{top}(y_1) &:= -\sqrt{r_1^2 - y_1^2}, && \forall y_1 \in [-r_1, r_1], \\ \psi_b(y_1) &:= \sqrt{r_2^2 - y_1^2} - r_1 - r_2 - h, && \forall y_1 \in [-r_2, r_2]. \end{aligned}$$

Moreover, we choose $\delta$ such that

$$A_{\delta, d_{1,2}(t)} \cap B_j(t) = \emptyset, \ \forall t \in [t_0, T_0], \ \forall j \notin \{1,2\}.$$

Before we proceed, we mention some properties of the boundary functions $\psi_{top}$ and $\psi_b$ that will be useful later on. It is easy to see that for all $y \in A^i_{\delta,h}$, we have:

$$y_2 - \psi_b(y_1) \leq \psi_{top}(y_1) - \psi_b(y_1) \quad \text{and} \quad h \leq \psi_{top}(y_1) - \psi_b(y_1). \tag{2.5.3}$$

Moreover, there exists a constant $K = K(\delta, r_1, r_2)$ such that

$$|\psi'_{top}(y_1)| \leq K|y_1|, \ \ |\psi'_b(y_1)| \leq K|y_1|, \ \forall y_1 \in [-2\delta, 2\delta], \tag{2.5.4}$$

$$|\psi''_{top}(y_1)| \leq K, \ \ |\psi''_{top}(y_1)| \leq K, \quad \forall y_1 \in [-2\delta, 2\delta]. \tag{2.5.5}$$

Furthermore, the following inequality

$$\frac{t^2}{2} \leq 1 - \sqrt{1 - t^2} \leq t^2, \ \forall t \in [-1, 1],$$

implies that

$$h + a y_1^2 \leq \psi_{top}(y_1) - \psi_b(y_1) \leq h + 2a y_1^2, \ \forall y_1 \in [-2\delta, 2\delta] \tag{2.5.6}$$

with $a = \frac{1}{2r_1} + \frac{1}{2r_2}$.

We turn now to define the test function $v$. To describe $v$ in the neighbourhood of $B_1$, we define a smooth function $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ with compact support included in $B(0, \alpha)$ such that

$\phi \equiv 1$ in a neighbourhood of $B_1$, where

$$\alpha \leq \min(r_1 + \beta, \sqrt{r_1^2 + \delta^2}).$$

By noting that the distance between the disks $B_1$ and $B(0, \alpha)$ equals to $\alpha - r_1$, we get

$$r_1 + \frac{\alpha - r_1}{2} < \alpha.$$

It follows that
$$\frac{r_1 + \alpha}{2} < \alpha,$$

and hence
$$B_1 \subset B\Big(0, \frac{r_1 + \alpha}{2}\Big) \subset B(0, \alpha).$$

Thus, we can choose $\phi \equiv 1$ on $B(0, \frac{r_1+\alpha}{2})$. Then we introduce a smooth function $\chi : \mathbb{R} \mapsto [0, 1]$ such that
$$\chi(r) = \begin{cases} 1 & \text{if } |r| \leq \delta, \\ 0 & \text{if } |r| \geq 2\delta. \end{cases}$$

We set
$$\overline{v}_h := \nabla^{\perp} \tilde{g}_h, \tag{2.5.7}$$

where $\tilde{g}_h(y) = y_1 \varphi_h$ with $\varphi_h : \mathbb{R}^2 \to \mathbb{R}$ is defined as follows:

$$\begin{aligned}
&\varphi_h = \phi \quad \text{in } \mathbb{R}^2 \backslash \Big(A^i_{\delta,h} \cup \Big(B_2 \cap B(0, \alpha)\Big)\Big), \\
&\varphi_h = (1 - \chi(y_1))\phi(y) + \chi(y_1)\Big(\frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)}\Big)^2 \Big(3 - 2\frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)}\Big) \text{ in } A^i_{\delta,h}, \\
&\varphi_h = 0 \quad \text{in } B_2 \cap B(0, \alpha).
\end{aligned}$$

Finally, we define
$$v(t, x) = J_X(Y(x, t), t)\overline{v}(Y(x, t), t), \tag{2.5.8}$$

where the mapping $\overline{v}$ is defined from $\mathbb{R}^2 \times [0, T_0)$ into $\mathbb{R}^2$ by

$$\overline{v}(y, t) = \overline{v}_{d_{1,2}(t)}(y). \tag{2.5.9}$$

**Remark 2.5.1** *We note that $\varphi_h$ and hence $\overline{v}_h$ are regular up to $h = 0$ outside $A^i_{\delta,h}$, and singularities at $h = 0$ correspond to*

$$g_h(y) = y_1 \left( \frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)} \right)^2 \left( 3 - 2 \frac{y_2 - \psi_b(y_1)}{\psi_{top}(y_1) - \psi_b(y_1)} \right), \quad y \in A^i_{\delta,h}, \tag{2.5.10}$$

*as it involves the difference term between the boundary functions $\psi_{top}$ and $\psi_b$. Hence, all the Sobolev norms of $\overline{v}_h$ are dominated by a constant in $\Omega_{F,\mathbf{h}} \backslash A^i_{\delta,h}$, where $\Omega_{F,\mathbf{h}}$ denotes the fluid domain in the new geometry.*

We state some properties of $\overline{v}_h$ in the following lemma and in this respect, we refer the reader to [24].

**Lemma 2.5.1** *Let $h > 0$, then $\overline{v}_h \in \mathbf{H}^1(\mathbb{R}^2)$ and has a compact support. Moreover, we have:*

*i.* $\nabla \cdot \overline{v}_h = 0$ *in* $\mathbb{R}^2$,

*ii.* $\overline{v}_h = e_2$ *on* $B_1$.

*iii.* $\overline{v}_h = 0$ *on the other disks.*

To prove that collision can not occur between disks in finite time, we need some estimates on the test function $v$. The following lemma shows that we can perform such estimates on the vector field $\overline{v}$ instead of $v$.

**Lemma 2.5.2** *Let $u(t) \in \mathbf{H}^1(\mathbb{R}^2)$ and $v(t) \in \mathbf{L}^p(\mathbb{R}^2)$ be two vector fields with $p \geq 1$. We define $\overline{u}(y,t) = J_Y(X(y,t),t)u(X(y,t),t)$, where $X$ denotes the inverse of the diffeomorphism $Y$ defined in (2.5.2). Then we have:*

$$\|v(t)\|_{\mathbf{L}^p(\mathbb{R}^2)} = \|\overline{v}(t)\|_{\mathbf{L}^p(\mathbb{R}^2)},\ t \in [0, T_0),$$
$$D[u] : D[v] = D[\overline{u}] : D[\overline{v}], \quad \forall v \in \mathbf{H}^1(\mathbb{R}^2).$$

The above lemma is straightforward from the fact that the diffeomorphism $Y$ is an isometry. Next, we state the following lemma which enables to estimate some terms in the weak formulation, such as the non-linear term and the source term.

**Lemma 2.5.3** *Let $h \in (0, d_{\max})$ and consider the vector field $\overline{v}_h$ defined in (2.5.7). Then there exists a constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that the vector field $\overline{v}_h \in \mathbf{L}^p(\mathbb{R}^2)$ for all*

$1 \leq p < 3$ *and we have*

$$\|\overline{v}_h\|_{\mathbf{L}^p(\mathbb{R}^2)} \leq K_m. \tag{2.5.11}$$

Proof. By Remark 2.5.1, all the Sobolev norms of $\overline{v}_h$ in $\Omega_{F,\mathbf{h}} \backslash A^i_{\delta,h}$ are dominated by a constant. From the definition of $g_h$ in (2.5.10), we have

$$\overline{v}_h(y) = \nabla^{\perp}\Big(y_1\,(1-\chi(y))\phi(y)\Big) + g_h(y)\nabla^{\perp}\chi(y) + \chi(y)\nabla^{\perp}g_h(y), \;\; \forall y \in A^i_{\delta,h}.$$

Using the properties of the boundary functions $\psi_{top}$ and $\psi_b$ stated in the previous section, we get that there exists $K = K(\delta, r_1, r_2) > 0$ and $C > 0$ such that

$$|g_h(y)| \leq K, \tag{2.5.12}$$

$$\left|\frac{\partial g_h}{\partial y_1}(y)\right| \leq C + K\frac{y_1^2}{\psi_{top}(y_1) - \psi_b(y_1)}, \tag{2.5.13}$$

$$\left|\frac{\partial g_h}{\partial y_2}(y)\right| \leq C\frac{|y_1|}{\psi_{top}(y_1) - \psi_b(y_1)}. \tag{2.5.14}$$

This implies that for all $y \in A^i_{\delta,h}$, we have:

$$|\overline{v}_{h,1}(y)| \leq C\Big(1 + K + \frac{|y_1|}{\psi_{top}(y_1) - \psi_b(y_1)}\Big) \text{ and } |\overline{v}_{h,2}(y)| \leq C\Big(1 + K + \frac{Ky_1^2}{\psi_{top}(y_1) - \psi_b(y_1)}\Big).$$

For $1 < p < 3$, there exists a positive constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that

$$\|\overline{v}_h(y)\|^p_{\mathbf{L}^p(A^i_{\delta,h})} \leq K_m\Big(1 + \int_0^{2\delta} \frac{y_1^p}{\Big(\psi_{top}(y_1) - \psi_b(y_1)\Big)^{p-1}} dy_1\Big).$$

Using the inequality (2.5.6), we obtain

$$\int_0^{2\delta} \frac{y_1^p}{\Big(\psi_{top}(y_1) - \psi_b(y_1)\Big)^{p-1}} dy_1 \leq \int_0^{2\delta} \frac{y_1^p}{\Big(h + ay_1^2\Big)^{p-1}} dy_1,$$

and thus

$$\|\overline{v}_h(y)\|^p_{\mathbf{L}^p(A^i_{\delta,h})} \leq K_m\Big(1 + \int_0^{2\delta} \frac{dy_1}{y_1^{p-2}}\Big).$$

The integral in the right hand side of the above inequality is finite as $|p-2| < 1$, Therefore (2.5.11) holds. □

To estimate the term that contains $\partial_t v$ in the weak formulation, we need the following lemma:

**Lemma 2.5.4** *Let $h \in (0, d_{\max})$. Then there exists a positive constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that*

$$\left\|\partial_h \tilde{g}_h\right\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq K_m. \tag{2.5.15}$$

Proof. From the definition of $\tilde{g}_h$ in (2.5.7) and by standard calculations, we have for all $y \in A^i_{\delta,h}$:

$$\partial_h \tilde{g}_h(y)(y) = 6y_1\chi(y_1)\Big(\frac{(y_2-\psi_b(y_1))^3}{(\psi_{top}(y_1)-\psi_b(y_1))^4} - 2\frac{(y_2-\psi_b(y_1))^2}{(\psi_{top}(y_1)-\psi_b(y_1))^3} + \frac{y_2-\psi_b(y_1)}{(\psi_{top}(y_1)-\psi_b(y_1))^2}\Big).$$

Hence, there exists some $C > 0$ such that

$$|\partial_h \tilde{g}_h(y)| \leq C\frac{|y_1|}{\psi_{top}(y_1)-\psi_b(y_1)}, \quad \forall y \in A^i_{\delta,h}.$$

Combining the above inequality with the fact that $\tilde{g}_h$ is smooth outside $A^i_{\delta,h}$ and is with compact support, we get that there exists $K_m > 0$ that depends on $\delta, r_1, r_2$ and $d_{\max}$ such that

$$\int_{\mathbb{R}^2} |\partial_h \tilde{g}_h(y)|^2 dy \leq K_m + C\int_0^{2\delta} \frac{y_1^2}{\psi_{top}(y_1)-\psi_b(y_1)} dy_1.$$

Hence,

$$\int_{\mathbb{R}^2} |\partial_h \tilde{g}_h(y)|^2 dy \leq K_m + \frac{C}{a}\int_0^{2\delta} \frac{ay_1^2}{h+ay_1^2} dy_1,$$

and as $ay_1^2 \leq h + ay_1^2$, we get the estimate (2.5.15). $\square$

From now on, we denote by $V_{\overline{u},i} = (V^1_{\overline{u},i}, V^2_{\overline{u},i})$ the translational velocity of the $i$-th rigid body.

The following proposition shows why the vector field $v$ is a good candidate to our problem.

**Proposition 2.5.1** *Let $h \in (0, d_{\max})$ and $\overline{u} \in \mathbf{H}^1(\mathbb{R}^2)$ such that for all $i \in \{1, \ldots, k\}$, we have*

$$\overline{u}(y) = V_{\overline{u},i} + \omega_i(y - y_{G_i})^\perp \ \text{ on } B(G_i, r_i),$$

*where $G_i$ denotes the center of mass of the $i$-th disk in the local coordinates. Then there exists a pressure $q_h : \Omega_{F,\mathbf{h}} \mapsto \mathbb{R}$, such that $q_h \in L^2(\Omega_{F,\mathbf{h}})$ and a positive constant $K_m =$*

$K_m(\delta, r_1, r_2, d_{\max})$ *such that*

$$\left|2\nu \int_{A^i_{\delta,h}} D[\overline{v}_h] : D[\overline{u}]dy - \tilde{n}_1(h)\Big(V_{\overline{u},1} - V_{\overline{u},2}\Big) \cdot e_2\right|$$
$$\leq K_m\Big(\|\overline{u}\|_{\mathbf{L}^2(\Omega_{F,\tilde{H}^n_i})} + \|\overline{u}\|_{\mathbf{L}^\infty(B_1\cup B_2)} + \|\nabla\overline{u}\|_{\mathbf{L}^2(\mathbb{R}^2)}\Big), \quad (2.5.16)$$

*where*

$$\tilde{n}_1(h) = \int_{\partial A^i_{\delta,h}\cap\partial B_1} \Big(2\nu D[\overline{v}_h] - q_h I\Big) n d\Gamma_1 \cdot e_2.$$

*Moreover, there exists an absolute constant* $K = K(\delta, r_1, r_2)$ *such that*

$$\tilde{n}_1(h) \geq \frac{K}{h^{\frac{3}{2}}}.$$

Proof. Without loss, we may assume that $\nu = 1$. By noting that

$$\Delta\overline{v}_h \cdot \overline{u} = 2\nabla\cdot(D[\overline{v}_h]u) - 2D[\overline{v}_h] : D[\overline{u}]$$

and performing integration by parts, we get

$$\int_{A^i_{\delta,h}} \Big(\Delta\overline{v}_h - \nabla q_h\Big) \cdot \overline{u}dy = -\int_{A^i_{\delta,h}} D[\overline{v}_h] : D[\overline{u}]dy + \int_{\partial A^i_{\delta,h}} \Big(D[\overline{v}_h]n - q_h n\Big) \cdot \overline{u}d\Gamma, \quad (2.5.17)$$

for some pressure $q_h$. The idea now is to find a good pressure field $q_h$ on $A^i_{\delta,h}$ such that (2.5.17) holds. We start with computing laplacian of $\overline{v}_h$ and we find that

$$\Delta\overline{v}_h = \begin{pmatrix} -\partial_{112}\tilde{g}_h - \partial_{222}\tilde{g}_h \\ \partial_{111}\tilde{g}_h + \partial_{122}\tilde{g}_h \end{pmatrix}.$$

We construct the pressure field $q_h$ such that

$$\Delta\overline{v}_h - \nabla q_h = \begin{pmatrix} -2\partial_{112}\tilde{g}_h - y_1(1-\chi)\partial_{222}\phi \\ \partial_{111}\tilde{g}_h \end{pmatrix}.$$

To match this property, we define

$$q_h(y,t) = \partial_{12}\tilde{g}_h(y) + \int_{-2\delta}^{y_1} \frac{12\, s\, \chi(s)}{(\psi_{top}(s) - \psi_b(s))^3} ds, \ \ \forall y \in A^i_{\delta,h}. \quad (2.5.18)$$

On the other hand, we have

$$\int_{A^i_{\delta,h}} \Big(-\Delta \overline{v}_h + \nabla q_h\Big)\cdot \overline{u} dy = \int_{A^i_{\delta,h}} \Big(2\partial_{112}\tilde{g}_h \overline{u}_1 - \partial_{111}\tilde{g}_h\overline{u}_2\Big)dy + \int_{A^i_{\delta,h}} y_1(1-\chi(y_1))\partial_{222}\phi(y)\overline{u}_1 dy. \tag{2.5.19}$$

By performing integration by parts, we obtain

$$\int_{A^i_{\delta,h}} \Big(2\partial_{112}\tilde{g}_h\overline{u}_1 - \partial_{111}\tilde{g}_h\overline{u}_2\Big)dy = -\int_{A^i_{\delta,h}} \partial_{11}\tilde{g}_h\Big(2\frac{\partial \overline{u}_1}{\partial y_2} - \frac{\partial \overline{u}_2}{\partial y_1}\Big)dy + \int_{\partial A^i_{\delta,h}} \partial_{11}\tilde{g}_h\Big(2\overline{u}_1 n_2 - \overline{u}_2 n_1\Big)d\Gamma. \tag{2.5.20}$$

For $y \in A^i_{\delta,h}$, we have

$$\partial_{11}\tilde{g}_h(y) = \partial_{11}\Big(y_1(1-\chi(y_1))\phi(y)\Big) + \partial_{11}\chi(y_1)g_h(y) + 2\partial_1\chi\,(y_1)\partial_1 g_h(y) + \chi(y_1)\partial_{11}g_h(y).$$

Hence there exists $C > 0$ and $K = K(\delta, r_1, r_2) > 0$ such that

$$|\partial_{11}\tilde{g}_h(y)| \leq K\Big(1 + \frac{|y_1|}{\psi_{top}(y_1) - \psi_b(y_1)}\Big) + C.$$

This implies that there exists a positive constant $K_m = K_m(\delta, r_1, r_2, d_{\max})$ such that

$$\|\partial_{11}\tilde{g}_h(s)\|^2_{L^2(A^i_{\delta,h})} \leq K_m\Big(1 + \int_0^{2\delta} \frac{y_1^2}{h + ay_1^2}dy_1\Big),$$

and thus

$$\|\partial_{11}\tilde{g}_h(y)\|^2_{L^2(A^i_{\delta,h})} \leq K_m.$$

The above inequality implies that

$$\Big|\int_{A^i_{\delta,h}} \partial_{11}\tilde{g}_h\Big(2\frac{\partial \overline{u}_1}{\partial y_2} - \frac{\partial \overline{u}_2}{\partial y_1}\Big)dy\Big| \leq K_m\Big(\Big\|\frac{\partial \overline{u}_1}{\partial y_2}\Big\|_{L^2(A^i_{\delta,h})} + \Big\|\frac{\partial \overline{u}_2}{\partial y_1}\Big\|_{L^2(A^i_{\delta,h})}\Big). \tag{2.5.21}$$

We turn now to estimate the boundary term in (2.5.20) and in this respect we have

$$\begin{aligned}\Big|\int_{\partial A^i_{\delta,h}} \partial_{11}\tilde{g}_h\overline{u}_2 n_1 d\Gamma\Big| \leq K_m\Big(&||\overline{u}||_{\mathbf{L}^\infty(B_1)}\int_{-2\delta}^{2\delta} \frac{|y_1||\psi'_{top}(y_1)|^3}{(\psi_{top}(y_1) - \psi_b(y_1))^2}dy_1 \\ &+ ||\overline{u}||_{\mathbf{L}^\infty(B_2)}\int_{-2\delta}^{2\delta} \frac{|y_1||\psi'_b(y_1)|^3}{(\psi_{top}(y_1) - \psi_b(y_1))^2}dy_1 + \Big|\int_{\partial A^i_{\delta,h}\cap\{|y_1|=2\delta\}} \partial_{11}\tilde{g}_h\overline{u}_2 n_1 d\Gamma\Big|\Big).\end{aligned} \tag{2.5.22}$$

Noting that $\partial_{11}\tilde{g}_h$ is odd, we get that

$$\begin{aligned}\int_{\partial A^i_{\delta,h}\cap\{|y_1|=2\delta\}}\partial_{11}\tilde{g}_h\overline{u}_2 n_1 d\Gamma &= \int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)}\partial_{11}\tilde{g}_h(2\delta,y_2)(\overline{u}_2(2\delta,y_2)-\overline{u}_2(-2\delta,y_2))dy_2\\ &= \int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)}\partial_{11}\tilde{g}_h(2\delta,y_2)\int_{-2\delta}^{2\delta}\partial_1\overline{u}_2(s,y_2)dsdy_2.\end{aligned}$$

This implies that there exists a positive constant $C$ such that

$$\left|\int_{\partial A^i_{\delta,h}\cap\{|y_1|=2\delta\}}\partial_{11}\tilde{g}_h\overline{u}_2 n_1 d\Gamma\right| \le C\|\partial_1\overline{u}_2\|_{L^2([-2\delta,2\delta]\times[\psi_b(2\delta),\psi_{top}(2\delta)])}.$$

Combining (2.5.22) with the above inequality noting (2.5.4) and (2.5.6), we obtain that

$$\left|\int_{\partial A^i_{\delta,h}}\partial_{11}\tilde{g}_h\overline{u}_2 n_1 d\Gamma\right| \le K_m\Big(||\overline{u}||_{\mathbf{L}^\infty(B_1\cup B_2)}+\|\partial_1\overline{u}_2\|_{L^2([-2\delta,2\delta]\times[\psi_b(2\delta),\psi_{top}(2\delta)])}\Big). \tag{2.5.23}$$

Moreover, we have

$$\begin{aligned}\int_{\partial A^i_{\delta,h}}\partial_{11}\tilde{g}_h\overline{u}_1 n_2 d\Gamma &= \int_{-2\delta}^{2\delta}\partial_{11}\tilde{g}_h(y_1,\psi_{top}(y_1))\Big(V^1_{\overline{u},1}-\omega_1\,\psi_{top}(y_1)\Big)dy_1\\ &\quad+\int_{-2\delta}^{2\delta}\partial_{11}\tilde{g}_h(y_1,\psi_b(y_1))\Big(V^1_{\overline{u},2}-\omega_2\,(\psi_b(y_1)+r_1+r_2+h)\Big)dy_1.\end{aligned}$$

As $\partial_{11}\tilde{g}_h$ is odd with respect to $y_1$ in the time $\psi_{top}$ and $\psi_b$ are even with respect to $y_1$, we get that

$$\int_{\partial A^i_{\delta,h}}\partial_{11}\tilde{g}_h\overline{u}_1 n_2 d\Gamma = 0. \tag{2.5.24}$$

Combining (2.5.21),(2.5.23) and (2.5.24) with (2.5.19) yields to

$$\begin{aligned}\left|\int_{A^i_{\delta,h}}\Big(\Delta\overline{v}_h-\nabla q_h\Big)\cdot\overline{u}dy\right| &\le K_m\Big(\|\overline{u}\|_{\mathbf{L}^2(A^i_{\delta,h})}+\|\overline{u}\|_{\mathbf{L}^\infty(B_1\cup B_2)}+\left\|\frac{\partial\overline{u}_1}{\partial y_2}\right\|_{L^2(A^i_{\delta,h})}\\ &\quad+\left\|\frac{\partial\overline{u}_2}{\partial y_1}\right\|_{L^2(A^i_{\delta,h})}+\left\|\frac{\partial\overline{u}_2}{\partial y_1}\right\|_{L^2([-2\delta,2\delta]\times[\psi_b(2\delta),\psi_{top}(2\delta)])}\Big).\end{aligned} \tag{2.5.25}$$

We turn now to compute the line integral on (2.5.17). It is not difficult to check that

$$q_h,\ \frac{\partial\overline{v}_{h,1}}{\partial y_1}\ \text{ and }\ \frac{\partial\overline{v}_{h,2}}{\partial y_2}$$

are even with respect to $y_1$ whereas

$$\frac{\partial \overline{v}_{h,1}}{\partial y_2} \text{ and } \frac{\partial \overline{v}_{h,2}}{\partial y_1}$$

are odd with respect to $y_1$. This implies that

$$\int_{\partial A^i_{\delta,h}} \Big(2D[\overline{v}_h]n - q_h n\Big) \cdot \overline{u}(t) d\Gamma = \tilde{n}_1(h)\, V_{\overline{u},1} \cdot e_2 + \tilde{n}_2(h)\, V_{\overline{u},2} \cdot e_2 + \int_{\partial A^i_{\delta,h} \cap |y_1|=2\delta} \Big(2D[\overline{v}_h]n - q_h n\Big) \cdot \overline{u} d\Gamma$$

with

$$\tilde{n}_i(h) = \int_{\partial A^i_{\delta,h} \cap \partial B_i} \Big(2D[\overline{v}_h]n - q_h n\Big) d\Gamma_i \cdot e_2, \;\; i = 1, 2.$$

As $\nabla \overline{v}_h$ and $q_h$ are regular on $\partial A^i_{\delta,h} \cap \{|y_1| = 2\delta\}$, then there exists a constant $K$ independent of $h$ such that

$$\int_{\partial A^i_{\delta,h} \cap |y_1|=2\delta} \Big(D[\overline{v}_h]n - q_h n\Big) \cdot \overline{u} d\Gamma \leq K \|u\|_{\mathbf{H}^1(A^i_{\delta,h})}.$$

This implies that

$$\begin{aligned} &\left| 2 \int_{A^i_{\delta,h}} D[\overline{v}_h] : D[\overline{u}] dy - \tilde{n}_1(h) V_{\overline{u},1} \cdot e_2 - \tilde{n}_2(h) V_{\overline{u},2} \cdot e_2 \right| \\ &\qquad \leq K_m \Big( \|\overline{u}\|_{\mathbf{L}^2(\Omega_{F,h})} + \|u\|_{\mathbf{L}^\infty(B_1 \cup B_2)} + \|\nabla \overline{u}\|_{\mathbf{L}^2(\mathbb{R}^2)} \Big). \quad (2.5.26) \end{aligned}$$

By integration by parts, we have

$$\int_{A^i_{\delta,h}} (\Delta \overline{v}_h - \nabla q_h) dy \cdot e_2 = - \int_{\partial A^i_{\delta,h}} \Big(2D[\overline{v}_h]n - q_h n\Big) d\Gamma \cdot e_2.$$

Since

$$\begin{aligned} \int_{\partial A^i_{\delta,h} \cap \{|y_1|=2\delta\}} \Big(2D[\overline{v}_h]n - q_h n\Big) d\Gamma \cdot e_2 &= 2 \int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)} \Big( \frac{\partial \overline{v}_{h,1}}{\partial y_2}(-2\delta, y_2) + \frac{\partial \overline{v}_{h,1}}{\partial y_2}(2\delta, y_2) \Big) dy_2 \\ &+ 2 \int_{\psi_b(2\delta)}^{\psi_{top}(2\delta)} \Big( \frac{\partial \overline{v}_{h,2}}{\partial y_1}(-2\delta, y_2) + \frac{\partial \overline{v}_{h,2}}{\partial y_1}(2\delta, y_2) \Big) dy_2, \end{aligned}$$

and as $\dfrac{\partial \overline{v}_{h,2}}{\partial y_1}$ and $\dfrac{\partial \overline{v}_{h,2}}{\partial y_1}$ are odd with respect to $y_1$, then the above integral vanishes and hence we get

$$\int_{A^i_{\delta,h}} (-\Delta \overline{v}_h + \nabla q_h) dy \cdot e_2 = \tilde{n}_1(h) + \tilde{n}_2(h).$$

Setting $\overline{u} = e_2$ in (2.5.25), we get that

$$\left| \int_{A^i_{\delta,h}} (\Delta \overline{v}_h - \nabla q_h) dy \cdot e_2 \right| \leq K_m.$$

This implies that

$$\tilde{n}_2(h) = -\tilde{n}_1(h) + O(K_m).$$

Combining the above result with (2.5.26), we obtain that

$$\left| 2 \int_{A^i_{\delta,h}} D[\overline{v}_h] : D[\overline{u}] dy - \tilde{n}_1(h) \Big( V_{\overline{u},1} - V_{\overline{u},2} \Big) \cdot e_2 \right| \leq K_m \Big( \|\overline{u}\|_{\mathbf{L}^2(\Omega_{F,\mathbf{h}})} + \|u\|_{\mathbf{L}^\infty(B_1 \cup B_2)} + \|\nabla \overline{u}\|_{\mathbf{L}^2(\mathbb{R}^2)} \Big). \tag{2.5.27}$$

Thus, (2.5.16) holds.

By similar way, one has:

$$\left| 2 \int_{A^i_{\delta,h}} |D[\overline{v}_h]|^2 dy - \tilde{n}_1(h) V_{\overline{v}_h,1} \cdot e_2 - \tilde{n}_2(h) V_{\overline{v}_h,2} \cdot e_2 \right|$$
$$\leq K_m \Big( \|\overline{v}_h\|_{\mathbf{L}^2(A^i_{\delta,h}))} + \|\overline{v}_h\|_{\mathbf{L}^\infty(B_1 \cup B_2)} + \left\| \frac{\partial \overline{v}_{h,1}}{\partial y_2} \right\|_{L^2(A^i_{\delta,h})} + \left\| \frac{\partial \overline{v}_{h,2}}{\partial y_1} \right\|_{L^2(A^i_{\delta,h})} \Big).$$

By Lemma 2.5.1, we have $\overline{v}_h = e_2$ on $B_1$ and vanishes on $B_2$. This implies that

$$\tilde{n}_1(h) \geq 2 \int_{A^i_{\delta,h}} |D[\overline{v}^1_h]|^2 dy - K_m \Big( \|\overline{v}_h\|_{\mathbf{L}^2(A^i_{\delta,h}))} + \|\overline{v}_h\|_{\mathbf{L}^\infty(B_1 \cup B_2)} + \left\| \frac{\partial \overline{v}_{h,1}}{\partial y_2} \right\|_{L^2(A^i_{\delta,h})} + \left\| \frac{\partial \overline{v}_{h,2}}{\partial y_1} \right\|_{L^2(A^i_{\delta,h})} \Big). \tag{2.5.28}$$

Standard calculations show that

$$\frac{\partial \overline{v}_{h,1}}{\partial y_2}(y) = -y_1(1 - \chi(y_1) \partial_{22} \phi - 6 y_1 \chi(y_1) \Big( \frac{1}{(\psi_{top}(y_1) - \psi_b(y_1))^2} - 2 \frac{y_2 - \psi_b(y_1)}{(\psi_{top}(y_1) - \psi_b(y_1))^3} \Big).$$

TIt follows that

$$\left| \frac{\partial \overline{v}_{h,1}}{\partial y_2}(y) \right| \leq C \Big( 1 + \frac{|y_1|}{(\psi_{top}(y_1) - \psi_b(y_1))^2} \Big).$$

Combining the above estimates with the fact that $\dfrac{\partial \overline{v}_{h,2}}{\partial y_1} = -\partial_{11}\tilde{g}_h$, we get that

$$\left\|\frac{\partial \overline{v}_{h,1}}{\partial y_2}\right\|_{L^2(A^i_{\delta,h})} \leq \frac{K_m}{h^{\frac{3}{4}}}, \tag{2.5.29}$$

$$\left\|\frac{\partial \overline{v}_{h,2}}{\partial y_1}\right\|_{L^2(A^i_{\delta,h})} \leq K_m. \tag{2.5.30}$$

To bound from below $D[\overline{v}_h]$ in $\mathbf{L}^2(A^i_{\delta,h})$, it suffices to bound from below $\dfrac{\partial \overline{v}_{h,1}}{\partial y_2}$ in $L^2(A^i_{\delta,h})$. In this respect, there exists $K = K(\delta, r_1, r_2)$ such that

$$\|\frac{\partial \overline{v}_{h,1}}{\partial y_2}\|_{L^2(A^i_{\delta,h})} \geq \frac{K}{h^{\frac{3}{4}}}.$$

Combining (2.5.28) with (2.5.29), (2.5.30), and the above result, we obtain

$$\tilde{n}_1(h) \geq \frac{K}{h^{\frac{3}{2}}}.$$

□

### 2.5.2 No collision result

This subsection is dedicated to prove the following theorem from which we can deduce the proof of Theorem 1.2.

**Theorem 2.5.1** *Assume* (H2) *holds true, then we have* $d(B_1, B_2)(T_0) > 0$.

Proof. Since $(h_1, h_2) \in H^2(0,T)$, the test function $v$ defined in (2.5.8) satisfies $v \in H^1((0,T)\times \mathbb{R}^2))$. Because of Lemma 2.5.1, $v$ is a good candidate to apply (2.5.1). We obtain:

$$\int_{\mathbb{R}^2} \Big(\rho u \cdot \partial_t v + \rho u \otimes u : D[v] - 2\nu D[u] : D[v] + \rho f \cdot v\Big) dx = \frac{d}{dt}\int_{\mathbb{R}^2} \rho u \cdot v dx, \tag{2.5.31}$$

on $(0, T_0)$. We start to estimate each term separately. Lemma 2.5.2 and Lemma 2.5.3 imply that there exists a positive constant $K_m = K_m(\delta, d_{\max})$ such that

$$\left|\int_{\mathbb{R}^2} \rho(s) f(s) \cdot v(s) dx\right| \leq K_m \|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)} \|f\|_{\mathbf{L}^2(\mathbb{R}^2)}. \tag{2.5.32}$$

We turn now to bound the non-linear term and we have

$$\Big|\int_{\mathbb{R}^2}\rho(s)u(s)\otimes u(s):D[v(s)]dx\Big| \leq K_m\|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)}\Big(\|u\|^2_{\mathbf{H}^1(\mathbb{R}^2)}+\Big|\int_{A_{\delta,d_{1,2}(s)}}u(s)\otimes u(s):D[v(s)]dx\Big|\Big).$$

By performing integration by parts, applying Holder inequality and noting that the vector field $v$ is uniformly bounded outside the $A^i_{\delta,h}$, we get

$$\begin{aligned}\Big|\int_{A_{\delta,d_{1,2}(s)}}u(s)\otimes u(s):D[v(s)]dx\Big| &\leq \Big|\int_{A_{\delta,d_{1,2}(s)}}(u(s)\cdot\nabla)u(s)\cdot v(s)dx\Big|+\Big|\int_{\partial A_{\delta,d_{1,2}(s)}}(u(s)\cdot v(s))(u(s)\cdot n)d\Gamma\Big|\\ &\leq C\|u(s)\|^2_{\mathbf{H}^1(\mathbb{R}^2)}\|\overline{v}_{d_{1,2}(s)}\|_{\mathbf{L}^{5/2}(A_{\delta,d_{1,2}(s)})}+K_m(\|u(s)\|^2_{\mathbf{L}^2(\mathbb{R}^2)}+\|\nabla u(s)\|^2_{\mathbf{L}^2(\mathbb{R}^2)}).\end{aligned}$$

Combining the above result with Lemma 2.5.3, we obtain

$$\Big|\int_{\mathbb{R}^2}\rho(s)u(s)\otimes u(s):D[v(s)]dx\Big| \leq K_m\|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)}\Big(\|u\|^2_{L^\infty([0,T_0),\mathbf{L}^2(\mathbb{R}^2))}+\|\nabla u(s)\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big). \tag{2.5.33}$$

For simplicity, we denote $d_{1,2}(t)$ by $h(t)$. With this notation and from the definition of the vector field $v$ in (2.5.8), we have

$$\begin{aligned}\int_{\mathbb{R}^2}\rho(s)u(s)\cdot\partial_t v(s)dx &= \int_{\mathbb{R}^2}\rho(s)u(x,s)\cdot\partial_t\Big(J_X(Y(x,s),s)\overline{v}(Y(x,s),s)\Big)dx\\ &= \int_{\mathbb{R}^2}\rho(s)u(x,s)\cdot\partial_t\Big(J_X(Y(x,s),s)\Big(\nabla_y^\perp\tilde{g}_{h(s)}\Big)(Y(x,s))\Big)dx.\end{aligned}$$

By noting that

$$\frac{\partial\tilde{g}_{h(s)}}{\partial y_i}((Y(x,s),s)=\sum_{j=1}^{2}\frac{\partial X_j}{\partial y_i}(Y(x,s),s)\partial_{x_j}\Big(\tilde{g}_{h(s)}(Y(x,s))\Big),$$

we obtain

$$J_X(Y(x,s),s)\Big(\nabla_y^\perp\tilde{g}_{h(s)}\Big)(Y(x,s),s)=\nabla_x^\perp\Big(\tilde{g}_{h(s)}(Y(x,s))\Big),$$

and thus

$$\int_{\mathbb{R}^2}\rho(s)u(s)\cdot\partial_t v(s)dx=\int_{\mathbb{R}^2}\rho(s)u(x,s)\cdot\partial_t\nabla_x^\perp\Big(\tilde{g}_{h(s)}(Y(x,s))\Big)dx.$$

By performing integration by parts on the space variable, we get that

$$\int_{\mathbb{R}^2} u(x,s)\cdot\partial_t\nabla_x^{\perp}\Big(\tilde{g}_{h(s)}(Y(x,s))\Big)dx = \int_{\mathbb{R}^2}\Big(\frac{\partial u_1}{\partial x_2}(x,s)-\frac{\partial u_2}{\partial x_1}(x,s)\Big)\partial_t\Big(\tilde{g}_{h(s)}(Y(x,s))\Big)dx.$$

By noting that

$$\partial_t\Big(\tilde{g}_{h(t)}(Y(x,s)\Big) = h'(t)\partial_h\tilde{g}_{h(t)}((Y(x,t)) + \sum_{i=1}^{2} Y_i'(x,t)\partial_{y_i}\tilde{g}_{h(t)}(Y(x,t)),$$

and

$$\Big\|\frac{\partial X_i}{\partial y_j}\Big\|_{L^\infty(\mathbb{R}^2)}\le 1,\ \ \Big\|\frac{\partial Y_i}{\partial x_j}\Big\|_{L^\infty(\mathbb{R}^2)}\le 1,\ \ \|Y_i'\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}^2)}\le c\sum_{i=1}^{2}|h_i'(t)|,$$

we get that there exists a positive constant $C$ such that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} u(x,s)\cdot\partial_t\nabla_x^{\perp}\Big(\tilde{g}_{h(s)}(Y(x,s),s)\Big)dx\Big| \le C\|\nabla u(s)\|_{\mathbf{L}^2(\mathbb{R}^2)}\Big\{|h'(s)|\Big(\|\partial_h\tilde{g}_{h(s)}\|_{\mathbf{L}^2(\mathbb{R}^2\setminus A^i_{\delta,h(s)})}\\ +\Big[\int_{A^i_{\delta,h(s)}}|\partial_h\tilde{g}_{h(t)}(y)|^2dy\Big]^{\frac{1}{2}}\Big)+K_m\sum_{i=1}^{2}|h_i'(t)|\Big\}.\end{aligned}$$

By Lemma 2.5.4, we get that

$$\Big|\int_{\mathbb{R}^2}\rho(s)u(s)\cdot\partial_t v(s)dx\Big|\le K_m\|\rho\|_{L^\infty(\mathbb{R}^2\times[0,T_0))}\left(\sup_{s\in[0,T_0)}|h'(s)|+\sum_{i=1}^{2}|h_i'(t)|\right)\|\nabla u(s)\|_{\mathbf{L}^2(\mathbb{R}^2)}. \tag{2.5.34}$$

Adding the term $\tilde{n}_1(d_{1,2}(s))(V_{\overline{u},1}-V_{\overline{u},2})\cdot e_2(h)$ to both sides of the weak formulation (2.5.31) and combining the resulting equation with Proposition 2.5.1, Lemma 2.5.2 and the estimates in (2.5.32), (4.2.13) and (2.5.34), we get that

$$\Big|\frac{d}{dt}\int_{\mathbb{R}^2}\rho(s)u(s)\cdot v(s)dx+\tilde{n}_1(d_{1,2}(s))(V_{\overline{u},1}-V_{\overline{u},2})\cdot e_2(h)\Big|\le K_m'\Big(1+\|u\|^2_{\mathbf{L}^2(\mathbb{R}^2)}+\|\nabla u\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big),$$

where $K_m' = (\delta, d_M, \|\rho\|_{L^\infty([0,T_0)\times\mathbb{R}^2)}, \|f\|_{\mathbf{L}^2(\mathbb{R}^2)})$ is a positive real constant.
By noting that

$$\overline{u}(y,s) = J_Y h_1'(s)+\omega_1(s)y^{\perp},\ \ y\in\partial B(G_1,r_1),$$
$$\overline{u}(y,s) = J_Y h_2'(s)+\omega_2(s)(y-y_{G_2})^{\perp},\ \ y\in\partial B(G_2,r_2),$$

we obtain

$$\left(V_{\overline{u},1} - V_{\overline{u},2}\right) \cdot e_2 = d_{1,2}^{'}(s).$$

This implies that

$$\left|\frac{d}{dt}\int_{\mathbb{R}^2} \rho(s)u(s)\cdot v(s)dx + d_{1,2}^{'}(s)\tilde{n}_1(d_{1,2}(s))\right| \leq K_m^{\prime}\Big(1 + \|u\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \|\nabla u\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big).$$

Integrating the above inequality from $t_0$ to $t < T_0$, we get that

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} \rho(t)u(t)\cdot v(t)dx - \int_{\mathbb{R}^2} \rho(t)u(t_0)\cdot v(t_0)dx + \int_{t_0}^{t} d_{1,2}^{'}(s)\tilde{n}_1(d_{1,2}(s))ds\Big| \\ \leq K_m^{\prime}\Big(T_0 + T_0 \sup_{t\in[0,T_0)} \|u\|^2_{\mathbf{L}^2(\mathbb{R}^2)} + \int_{t_0}^{t} \|\nabla u\|^2_{\mathbf{L}^2(\mathbb{R}^2)}\Big),\end{aligned}$$

Combining together Lemma 2.4.1 with Lemma 2.5.3, we get that there exists $M > 0$ that depends on $T_0$ and the initial data such that

$$\Big|\int_{t_0}^{t} d_{1,2}^{'}(s)\tilde{n}_1(d_{1,2}(s))ds\Big| \leq K_m^{\prime}M.$$

With the change of variable $h(s) = d_{1,2}(s)$, we get that

$$\Big|\int_{d_{1,2}(t_0)}^{d_{1,2}(t)} \tilde{n}_1(h)dh\Big| \leq K_m^{\prime}M.$$

Again by Proposition 2.5.1, we get that

$$\Big|\int_{d_{1,2}(t_0)}^{d_{1,2}(t)} \frac{dh}{h^{\frac{3}{2}}}\Big| \leq K_m^{\prime}M,$$

and thus

$$\frac{1}{[d_{1,2}(t)]^{\frac{1}{2}}} \leq \frac{1}{[d_{1,2}(t_0)]^{\frac{1}{2}}} + K_m^{\prime}M.$$

The last inequality implies that

$$\sup_{t\leq T_0} \frac{1}{[d_{1,2}(t)]^{\frac{1}{2}}} \leq \frac{1}{[d_{1,2}(t_0)]^{\frac{1}{2}}} + K_m^{\prime}M.$$

**Proof of Theorem 2.1.2** It follows from Theorem 2.1.1 that our proof reduces to obtaining that no collision occurs in finite time under the hypothesis (H1). We act by contradiction

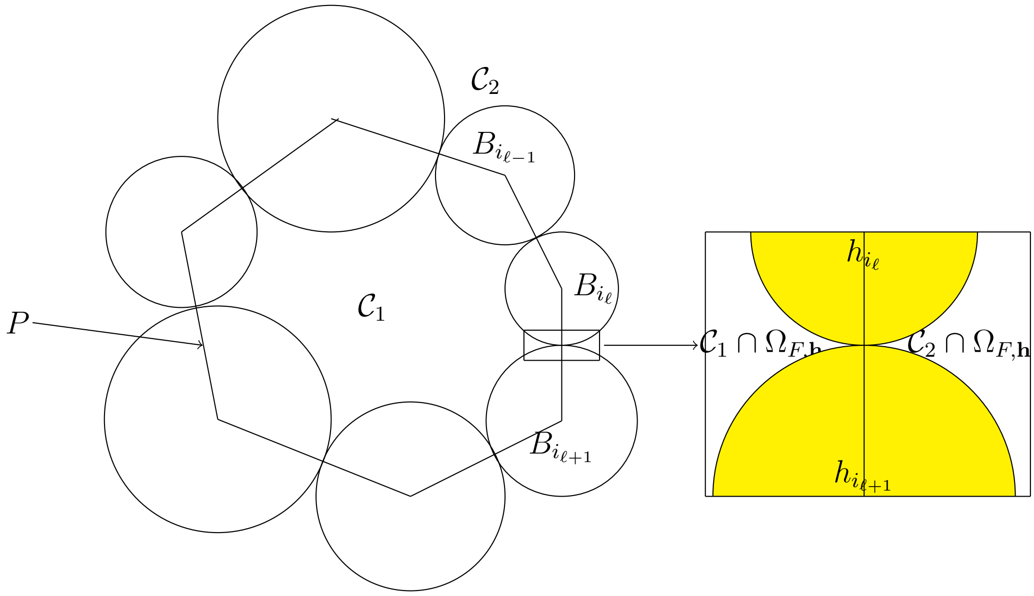


Figure 2.3 – Collision between particles dividing the fluid domain into two connected components.

and we assume that collision could take place in finite time. We define the the non-empty set $J$ of cardinal $2 \leq m \leq k$ as follows

$$J = \Big\{ j \in \{1, \dots, k\} : \exists i \neq j, 1 \leq i \leq k,\ \ d(B_i, B_j)(T_0) = 0 \Big\}.$$

For $i \in J$, we define the non-empty set of indices $J_i$ by

$$J_i = \Big\{ j \in J : \ j \neq i,\ \ d(B_i, B_j)(T_0) = 0 \Big\}.$$

We claim that there exists $i \in J$ such that $\mathrm{card}(J_i) = 1$. Otherwise, we have $\mathrm{card}(J_i) \geq 2$ for all $i \in J$. Hence for a fixed $i_0 \in J$, there exists $i_1 \in J_{i_0}$ and as $\mathrm{card}(J_{i_1}) \geq 2$, then there exists $i_2 \in J_{i_1} \backslash \{i_0\}$. By recurrence, we construct a sequence $\{i_\ell\}_{\ell \in \mathbb{N}}$ such that for all $\ell \in \mathbb{N}$, we have $i_{\ell+1} \in J_{i_\ell} \backslash \{i_{\ell-1}\}$. Since $\mathrm{card}\, J$ is finite, then there exists two positive integers $\ell$ and $p$ such that $i_{\ell+p} = i_\ell$. Moreover, the center of masses $h_{i_\ell}, \dots, h_{i_{\ell+p}}$ of the disks $B_{i_\ell}, \dots, B_{i_{\ell+p}}$ form a set of vertices of a simple polygon $P$, whose complement is the union of at least two connected components $\mathcal{C}_1$ and $\mathcal{C}_2$. Furthermore, the fluid domain $\Omega_{F,\mathbf{h}} \subset P^c$ and we have $\Omega_{F,\mathbf{h}} \cap \mathcal{C}_i \neq \emptyset$, for $i = 1, 2$ (see Figure 2.3). This contradicts the assumption (H1) in Theorem 2.1.1.

Let $j$ denote the index of the disk that the disk $B_i$ only collide with at time $T_0$. Up to a renumbering, we assume that $i = 1$ and $j = 2$, so that (H2) holds true. We apply then Theorem 2.5.1 and we obtain a contradiction. $\square$

**Remark 2.5.2** *To illustrate the difficulties which prevent from ruling out the further connectedness assumption, we study the asymptotic behaviour of a moving body in a rectangular domain when it is approaching the boundary of $\Omega$. In the orthonormal system $(O, \vec{i}, \vec{j})$, we set $\Omega = [0,5] \times [0,5]$ and we suppose that the moving body is a disk of radius 1 and its center of mass $\mathbf{G}$ moves along the line $(D) : y = x$. We assume that there is no contact initially between the rigid disk $B$ and $\partial\Omega$. We denote by $h(t)$ the distance between the ball $B$ and the boundary of the flow at time $t$.*

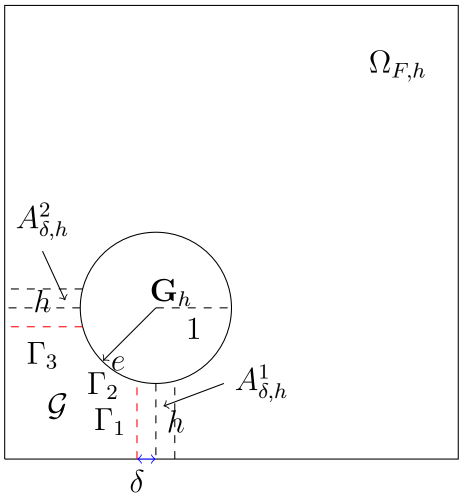


Figure 2.4 – Description of the model

*It follows that*

$$\mathbf{G}(t) = \mathbf{G}_{h(t)} = (1 + h(t), 1 + h(t)).$$

*The domain occupied by the fluid at time $t$ is given by $\Omega_{F,h(t)} = \Omega \backslash B(t)$. If there exists some time $T_*$ such that When $h(T_*) = 0$, then fluid domain has two connected components at time $T_*$. We show that, for solutions to fluid-solid interaction problems, such configurations are excluded. But, we obtain that the mechanism preventing from collision is different to the one exhibited above. Let $t \in [0, T_*)$ such that $h(t) > 0$ on $[0, T_*]$ and let consider $v_h \in \mathbf{H}^1(\Omega)$ such that*

*i.* $\nabla \cdot v = 0$

*ii.* $v(t) = h'(t)e$ *on* $B(t)$, *with* $e = -\frac{1}{\sqrt{2}}(\vec{i} + \vec{j})$.

*iii. $v$ vanishes on $\partial\Omega$.*

*as should be the fluid-velocity-field in our fluid-solid interaction problem. Let For $0 < \delta < 1$ and given $h > 0$, we set*

$$A^1_{\delta,h} = \{x \in \Omega_{F,h} : \ 1 - \delta < x < 1 + \delta, \ 0 < y < \psi_h(x)\},$$

*and*

$$A^2_{\delta,h} = \{x \in \Omega_{F,h} \ 1 < y < 1 + \delta, \ 0 < x < \psi_h(y)\},$$

*where $\psi_h(s) = 1 + h - \sqrt{1 - (s - 1 - h)^2}$.*
*Let us consider the domain*

$$\mathcal{G} = \{x \in \Omega_{F,h} : \ 0 < x < 1 + h - \delta, \ 0 < y < 1 + h - \delta\}$$

*The no-slip condition implies that*

$$\int_{\partial\mathcal{G}} v \cdot n d\Gamma = 0. \tag{2.5.35}$$

*The boundary $\partial\mathcal{G}$ of the domain $\mathcal{G}$ consists of 4 parts:*

$$\partial\mathcal{G} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

*where $\Gamma_1 = \partial\mathcal{G} \cap \{x = 1 + h - \delta\}$, $\Gamma_2 = \partial\mathcal{G} \cap \partial B$, $\Gamma_3 = \partial\mathcal{G} \cap \{y = 1 + h - \delta\}$, and $\Gamma_4 = \partial\mathcal{G} \cap \{x = 0 \ \text{or} \ y = 0\}$. Since $v$ vanishes on $\Gamma_4$, we get that*

$$\int_{\Gamma_2} v \cdot n d\Gamma = -\int_{\Gamma_1} v \cdot n d\Gamma - \int_{\Gamma_3} v \cdot n d\Gamma. \tag{2.5.36}$$

*Since $v = h'(t)e$ on $\partial B$, then*

$$\Big| \int_{\Gamma_2} v \cdot n d\Gamma \Big| = |h'(t)||\mathbf{x}_1 - \mathbf{x}_2|,$$

*where $\mathbf{x}_1 = (1 + h - \delta, 1 + h - \sqrt{1 - \delta^2})$ and $\mathbf{x}_2 = (1 + h - \sqrt{1 - \delta^2}, 1 + h - \delta)$.*

*It follows that*

$$\sqrt{2}(\sqrt{1-\delta^2}-\delta)h'(t) = -\int_0^{1+h-\sqrt{1-\delta^2}} v_1(1+h-\delta,y)dy - \int_0^{1+h-\sqrt{1-\delta^2}} v_2(x,1+h-\delta)dx.$$

*Moreover, since* $0<\delta*<\delta$, *then we get*

$$\sqrt{2}(\sqrt{1-\delta_*^2}-\delta_*)|h'(t)| \leq \int_0^{1+h-\sqrt{1-\delta^2}} |v_1(1+h-\delta,y)|dy + \int_0^{1+h-\sqrt{1-\delta^2}} |v_2(x,1+h-\delta)|dx.$$

*By integrating the above inequality over* $\delta \in (0,r)$ *for* $r \in (0,1)$ *and using Cauchy-Schwartz inequality, we obtain*

$$\begin{aligned}|h'(t)|r \leq \frac{\sqrt{r}}{c(\delta_*)} \sup_{\delta\in(0,r)} \Big(1+h-\sqrt{1-\delta^2}\Big)^{1/2} \Big\{ \Big( \int_0^r \int_0^{1+h-\sqrt{1-\delta^2}} |v_1(1+h-\delta,y)|^2 dy d\delta \Big)^{1/2} \\ + \Big( \int_0^r \int_0^{1+h-\sqrt{1-\delta^2}} |v_2(x,1+h-\delta)|^2 dx d\delta \Big)^{1/2} \Big\},\end{aligned}$$

*where* $c(\delta_*)=\sqrt{2}|\delta-\sqrt{1-\delta^2}|$.
*Since* $v_1(1+h-\delta,0)=v_2(0,1+h-\delta)=0$, *then by Poincaré we get that*

$$\begin{aligned}|h'(t)|r \leq \frac{\sqrt{r}}{c(\delta_*)} \sup_{\delta\in(0,r)} \Big(1+h-\sqrt{1-\delta^2}\Big)^{3/2} \Big\{ \Big( \int_0^r \int_0^{1+h-\sqrt{1-\delta^2}} |\partial_y v_1(1+h-\delta,y)|^2 dy d\delta \Big)^{1/2} \\ + \Big( \int_0^r \int_0^{1+h-\sqrt{1-\delta^2}} |\partial_x v_2(x,1+h-\delta)|^2 dx d\delta \Big)^{1/2} \Big\},\end{aligned}$$

*Therefore, we obtain*

$$|h'(t)|r \leq \frac{\sqrt{r}}{c(\delta_*)} \sup_{\delta\in(0,r)} \Big(1+h-\sqrt{1-\delta^2}\Big)^{3/2} \Big( \int_0^r \int_0^{1+h-\sqrt{1-\delta^2}} |\nabla v|^2 dx dy \Big)^{1/2}$$

*One may check that*

$$1+h-\sqrt{1-\delta^2} \leq \delta^2+h, \text{ for all } \delta \in (0,r).$$

*When* $h<1$, *setting* $r=\sqrt{h}$, *we get that*

$$|h'(t)| \leq \frac{2h^{5/4}}{c(\delta_*)} \|\nabla v\|_{\mathbf{L}^2(\Omega_F)}.$$

*Therefore, we obtain*

$$\left|\frac{h'(t)}{h^{\frac{5}{4}}}\right| \leq \frac{2}{c(\delta_*)}\|\nabla v\|_{\mathbf{L}^2(\Omega_F)}.$$

*To conclude, in fluid interaction systems, we expect the fluid velocity-field to be bounded in* $L^2(0,T;H^1(\Omega_F(t)))$. *The above computations entail that we should have a bound on* $1/h^{1/4}$ *preventing then from collision in finite time. However, we observe that we do not need to involve the Newton's law to prove the no-collision result. Moreover, the exponent (1/4) that is involved is different to the one (1/2) we had in our previous proof. For these reasons, the proof we give above does not adapt easily to this case.*

# 3

# Nonuniqueness of Weak Solutions to Fluid Solid Interaction Problem in 3D

## Sommaire



In this chapter, we consider the three-dimensional motion of a rigid body immersed in an incompressible homogeneous viscous fluid. The rigid body is supposed to be a ball and the fluid domain has exactly two holes so that it can fill exactly the gap between the holes if collision occurs. With this geometrical configuration, we show that there exists at least two weak solutions of different behaviours to this problem: the body either moves away from the boundary of the flow or it remains in contact with it. The novelty of this work is that we prove the non-uniqueness of weak solutions for the problem of the motion of a rigid body in viscous fluid in 3D after contact with external physical source term $f \in L^2(0,T;\mathbf{L}^p(\mathcal{O}))$

with $p < 2$.

The plan of this chapter is as follows: in Section 3.2 we construct a velocity field for which the body touches the boundary of the cavity $\mathcal{O}$ at time $t_* \in (0, T)$. Then, we prove that there exists an external body force $f$ such that the constructed velocity field is a weak solution of problem (3.1.1)-(3.1.8). After collision, we extend this solution so that the body goes away from the boundary. In Section 3.3, we construct another solution with the same function $f$. However, we extend the solution this time such that the body stays attached to the boundary of the flow after contact.

## 3.1 Introduction

In this chapter we investigate the question of uniqueness of weak solutions to the problem of motion of a rigid body immersed in an incompressible homogeneous viscous fluid. We consider a homogeneous rigid ball $B$ moving in a cavity in $\mathcal{O} \subset \mathbb{R}^3$ filled with a homogeneous viscous Newtonian fluid. We assume that the motion of the fluid is described by the classical incompressible Navier–Stokes equations, whereas the motion of the rigid body will be governed by Newton's laws. More precisely, the full system of equations modelling the motion of the fluid and the rigid body reads as:

$$\partial_t u + (u \cdot \nabla)u = \nabla \cdot \sigma + f, \quad \text{in } \Omega_F(t),\ t \in (0, T), \tag{3.1.1}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega_F(t),\ t \in (0, T), \tag{3.1.2}$$

$$u(x, t) = \dot{G}(t) + \omega(t) \times (x - G(t)), \quad x \in \partial B(t)\, t \in (0, T), \tag{3.1.3}$$

$$u(x, t) = 0, \quad x \in \partial\mathcal{O},\ t \in (0, T), \tag{3.1.4}$$

$$m\ddot{G}(t) = -\int_{\partial B(t)} \sigma \mathbf{n} d\Gamma + \rho_B \int_{B(t)} f(t) dx, t \in (0, T), \tag{3.1.5}$$

$$J\dot{\omega}(t) = -\int_{\partial B(t)} (x - G(t)) \times \sigma \mathbf{n} d\Gamma + \rho_B \int_{B(t)} (x - G(t)) \times f(t) dx, \quad t \in (0, T) \tag{3.1.6}$$

In the above system, we denote by $B(t)$ the domain occupied by the moving body with center of mass $G(t)$ at time $t$ and radius 1. The displacement of the solid body $B$ modifies the fluid domain and makes it time dependent. The set $\Omega_F(t) = \mathcal{O} \backslash B(t)$ denotes the fluid domain occupied at time $t$. We suppose that the fluid has a constant density 1 and that it moves under the influence of a given body force $f$. We denote respectively by $u$ and $p$ the velocity and the pressure of the fluid. The total stress tensor $\sigma$ (also called the Cauchy stress) of the fluid is given by

$$\sigma = -pI_3 + 2\nu D[u],$$

where $p$ denotes the pressure, $\nu > 0$ is the fluid viscosity, and $D[u]$ is the rate of deformation tensor defined as follows:

$$D[u] = \frac{1}{2}(\nabla u + \nabla u^T).$$

Moreover, we suppose that the moving body is of constant density $\rho_B$ and has a mass $m = \rho_B|B(0)|$. The moment of inertia matrix $J$ of $B(t)$ related to the center of mass $G(t)$ at any time $t > 0$ is given by

$$J = \Big[\int_{B(t)} \rho_B |x - G(t)|^2 dx\Big] I_3.$$

The vector $\omega$ denotes the angular velocity of the ball $B$ and the sign $\times$ stands for the vector product. To complete the system, we impose initial conditions at $t_0$:

$$u(x,0) = u_0(x), \;\; x \in \Omega_F(0), \tag{3.1.7}$$

$$G(0) = G_0, \;\; \dot{G}(0) = G_1, \;\; \omega(0) = \omega_0. \tag{3.1.8}$$

We suppose that there is no contact initially between the moving ball and the boundary of the flow; that is $\gamma = \gamma(0) > 0$, where

$$\gamma(t) = d(B(t), \partial\mathcal{O}).$$

In this chapter, we show that uniqueness of weak solutions to the fluid solid interaction problem in three dimensional case does not hold after contact. Before stating our result, we introduce the notion of weak solutions. To this end, we recall that the global density $\rho$ and the global velocity $\tilde{u}$ are given respectively by

$$\begin{aligned} \rho(t,x) &= 1_{\Omega_F(t)}(x) + \rho_B 1_{B(t)}(x), \\ \tilde{u}(t,x) &= u(t,x)1_{\Omega_F(t)}(x) + \Big(\dot{G}(t) + \omega(t) \times (x - G(t))\Big) 1_{B(t)}(x). \end{aligned}$$

For simplicity, we denote below the global velocity by $u$ instead of $\tilde{u}$.

Consider domains $B$ and $\mathcal{O}$ in $\mathbb{R}^3$ such that $B \subset \mathcal{O}$. Let

$$\mathcal{V}(O) = \{u \in \mathcal{D}(\mathcal{O}) : \nabla \cdot u = 0 \ \ in \ \ \mathcal{O}\}, \tag{3.1.9}$$

and denote by $H(\mathcal{O})$ and $V(\mathcal{O})$the closure of $\mathcal{V}(\mathcal{O})$ respectively in $\mathbf{L}^2(\mathcal{O})$ and $\mathbf{H}^1(\mathcal{O})$. According to classical results (see [40]) we have

$$\begin{aligned} H(\mathcal{O}) &= \{u \in \mathbf{L}^2(\mathcal{O}) : \ \nabla \cdot u = 0 \ \text{ in } \mathcal{O}, \ \ \mathrm{u} \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}, \\ V(\mathcal{O}) &= \{u \in \mathbf{H}^1_0(\mathcal{O}) : \ \nabla \cdot u = 0 \ \text{ in } \ \mathcal{O}\}. \end{aligned}$$

We introduce the following spaces which will be used in the sequel:

$$\begin{aligned} H(B, \mathcal{O}) &= \{u \in H(\mathcal{O}) : \ D[u] = 0 \ \ in \ \ B\}, \\ K(B, \mathcal{O}) &= \{u \in V(\mathcal{O}) : \ D[u] = 0 \ \ in \ \ B\}. \end{aligned}$$

By Lemma 1.1 in [44], we have $D[u] = 0$ in $B$ if and only if there exists a vector $a$ and a skew-symmetric tensor $Q \in \mathbb{R}^6$ such that

$$u(x) = a + Qx, \ \ for \ \ x \in B.$$

In particular, there exists a vector $\omega$ such that $Qx = \omega \times x$.

**Definition 3.1.1** *Assume that $G_0 \in \mathcal{O}$ such that $\gamma > 0$ and $u_0 \in H(\mathcal{O})$. We say that $(u, G)$ is a weak solution to problem (3.1.1)-(3.1.8) on $[0, T]$ if the velocity field $u$ and the center of mass of $G$ satisfy*

$$\begin{aligned} G \in W^{1,\infty}(0, T), \ \ with \ \ G(0) = G_0, \\ \gamma(t) \geq 0, \\ u \in L^\infty(0, T; H(\mathcal{O})) \cap L^2(0, T; V(\mathcal{O})), \ \ with \ \ u(0) = u_0, \\ u(x, t) = \dot{G}(t) + \omega(t) \times (x - G(t)), \ \ \forall x \in \partial B(t), \end{aligned}$$

*and*

$$\int_{\mathcal{O}\times[0,T)}\Big(\rho u\cdot\partial_t v+\rho u\otimes u : D[v]-2\nu D[u] : D[v]+\rho f\cdot v\Big)dxds=-\int_{\mathcal{O}}\rho(0)u(0)\cdot v(0)dx,\ \forall v\in\mathcal{S}, \tag{3.1.10}$$

*where*

$$\mathcal{S}=\{\varphi\in\mathcal{D}([0,T)\times\mathcal{O}) : \nabla\cdot\varphi=0\ \text{ on }\ I\times\mathcal{O},\ \ D[\varphi]=0\ \text{ on a neighbourhood of } B(t)\}.$$

We remark that the test function $\varphi$ used in the above weak formulation must be zero when $B$ touches the two holes whereas the velocity $u$ need not. Our result is the following:

**Theorem 3.1.1** *There exists initial conditions and a source term $f$ such that problem (3.1.1)-(3.1.8) admits at least two weak solutions.*

The geometry of the problem is crucial to prove the above theorem. We suppose that the cavity $\mathcal{O}$ is symmetric with respect to some line $(D)$ and has exactly two spherical holes $B^l$ and $B^r$ each of radius 1. We assume that the holes are symmetric with respect to the line $(D)$ and separated by a distance equal to the diameter of the moving ball $B$ so that the ball $B$ can fill exactly the gap between the two holes at collision. Moreover, we assume that $\partial\mathcal{O}$ is flat near $\partial D\cap\partial\mathcal{O}$. An example of such geometry is represented in the following figure:

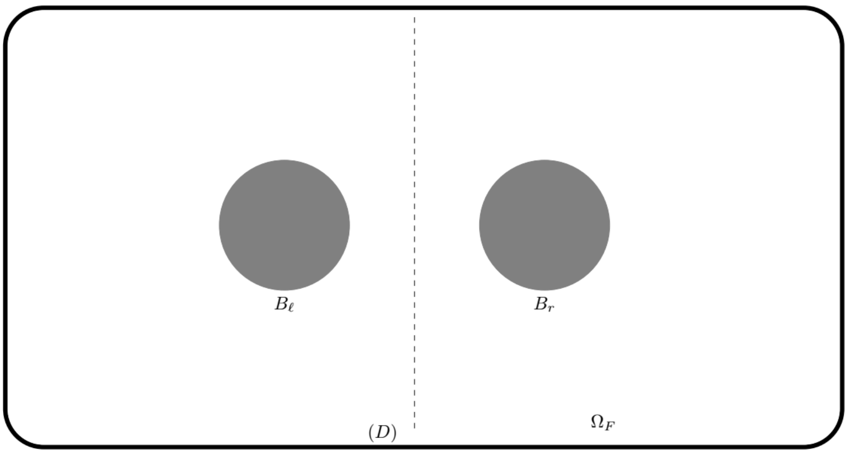


Figure 3.1 – Example of the cavity $\mathcal{O}$

The proof of Theorem 3.1.1 follows the same idea as in [39]. The idea of the proof is based on the construction of a weak solution colliding in finite time to problem (3.1.1)-(3.1.8) with $\mathcal{O}$ as described above. Then we extend this solution after contact by two different ways so that one obtains two weak solutions with different behaviours: for the first solution, the body moves away from the boundary of the flow, whereas the second solution is constructed such that the body stays in contact with the flow boundary after collision.

## 3.2 Construction of the First solution

In this section, we construct a weak solution $u$ for which the moving ball $B$ touches the boundary of the cavity in a finite time then it moves away from the boundary. We assume that the ball touches the boundary of the cavity at time $t_* \in (0, T)$. We start by describing the geometry for which we build the weak solution. Let $(e_1, e_2, e_3)$ denote the orthonormal basis such that $e_1$ is the direction of the line joining the centers of the two holes $B_\ell$ and $B_r$ whereas the unit vector $e_3$ is the direction holding the straight line $(D)$. To be more precise, we shall assume that the line $(D)$ is confounded with the $z - axis$ and the center of the holes, denoted by $G_\ell$ and $G_r$, are placed at $(-2, 0, 0)$ and $(2, 0, 0)$ respectively. Moreover, we suppose that the ball $B$ moves along $(D)$. Hence, its center of mass $G(t)$ at time $t$ is given by the altitude $d(t)$. In other words,

$$G(t) = (0, 0, d(t)).$$

It is important to point out that with this geometry, the only possible contact which may occur is between the ball $B$ and the two holes as other kinds of contact are ruled out due to [26]. If we denote by $h(t)$ the common distance between the ball $B$ and the holes $B_\ell$ and $B_r$ at time $t$ (i.e $h(t) = d(B(t), B^l) = d(B(t), B^r)$), then standard arguments yield

$$d(t) = \sqrt{h(t)^2 + 4h(t)}. \tag{3.2.1}$$

It follows that

$$G(t) = G_{h(t)}(t) = (0, 0, \sqrt{h(t)^2 + 4h(t)}). \tag{3.2.2}$$

From now on we replace $G$ by $G_h$ and we denote by $B_h$ the moving ball of center $G_h$ and radius 1. The fluid domain is then given by $\Omega_{F,h} = \mathcal{O} \backslash B_h$. We emphasize that with the

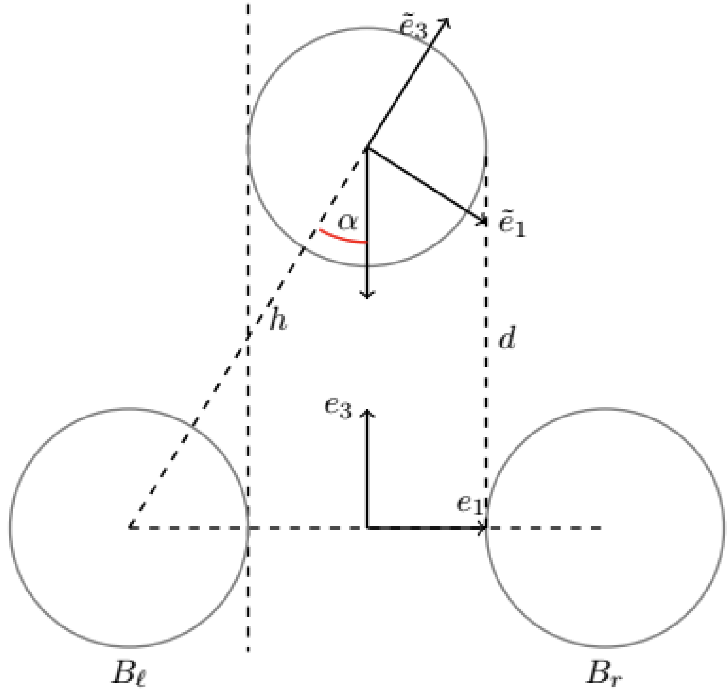


Figure 3.2 – Description of the geometry

above notations, collision occurs when the altitude $d$ or the distance $h$ between the ball $B$ and the holes vanish. Without loss of generality, we may assume that the ball is above the holes before contact, i.e $d(t) > 0$ before collision. Moreover, since collisions between $B$ and $\partial\mathcal{O}\backslash(\partial B_\ell \cup \partial B_r)$ are impossible, then there exists $h_{max} > 0$ such that for all $h \in (0, h_{max}]$ we have

$$d\Big(B_h, \partial\mathcal{O}\backslash(\partial B_\ell \cup \partial B_r)\Big) \geq \delta_0 > 0.$$

We turn now to construct a velocity field $u$ satisfying the weak formulation (3.1.10) in the geometry described above and colliding in finite time. This field should satisfy the following:

$$u = 0 \quad \text{on } \partial\mathcal{O}, \quad \nabla \cdot u = 0 \quad \text{in } \mathcal{O} \quad \text{and} \quad u = \dot{d}(t)e_3 \quad \text{in } B(t). \tag{3.2.3}$$

The authors in [25] show that there is a family of vector fields $(\mathrm{w}[h])_{h>0}$ such that for all $h \in (0, h_{max}]$, $\mathrm{w}[h]$ satisfies the following properties:

i. $\mathrm{w}[h] \in \mathcal{C}(\overline{\mathcal{O}})$
ii. $\nabla \cdot \mathrm{w}[h] = 0$
iii. $\mathrm{w}[h] = e_3$ on $B_h$ and $\mathrm{w}[h] = 0$ on $\partial\mathcal{O}$

Setting

$$u(x,t) = \dot{d}(t)\mathrm{w}[h(t)](x), \tag{3.2.4}$$

we get that $u$ satisfies (3.2.3).

For later purpose, it is more convenient to recall the definition of the vector field $\mathrm{w}[h]$ constructed in [25]. The vector field $\mathrm{w}[h]$ was built first in the half space $\mathcal{P}_\ell := \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1 \leq 0\}$ and it was then extended by symmetry in the other half space $\mathcal{P}_r := \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1 \geq 0\}$. In each half space, it is more convenient to work in a local orthonormal frame attached to the moving ball $B$. The origin of this local frame is $G$ and the associated direct orthonormal basis is $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ are such that $\tilde{e}_2 = e_2$ and $\tilde{e}_1 = \dfrac{G - G_\ell}{2+h}$. We denote the coordinates of any $x \in \mathbb{R}^3$ in the new frame by $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. More precisely, we have

$$\tilde{x} = Q_\alpha(x - G) \text{ or } x = G + Q_\alpha^{-1}\tilde{x},$$

with $Q_\alpha$ is the rotation with axis $\mathbb{R}e_2$ and angle $\alpha = (e_3, G - G_\ell)$.
In the following, for any set $S \subset \mathbb{R}^3$ the following holds:

$$\tilde{S} = Q_\alpha(S - G) \quad \text{or} \quad S = G + Q_\alpha^{-1}\tilde{S}.$$

Actually, in the new frame the ball $B$ is fixed and centered at 0 whereas the center $G_\ell$ of $B_\ell$ has moving coordinates $(0, 0, -2-h)$. For this reason, we prefer to use $\tilde{B}_*$ for the image of $B$ and $\tilde{B}_h$ for the image of $B_\ell$. Relate the cylindrical coordinates $(r, \theta, z)$ to $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ by setting:

$$\tilde{x}_1 = r\cos(\theta),\ \ \tilde{x}_2 = r\sin(\theta),\ \ \tilde{x}_3 = z.$$

When $h = 0$, the fluid domain becomes singular. Hence, in order to surround the singularities we introduce a family of neighbourhoods $\tilde{\Omega}_{h,\delta}$ of the points realizing the distance between $\tilde{B}_*$ and $\tilde{B}_h$. Precisely, for $h \in (0, h_{max})$ and $\delta \in (0,1)$, we define:

$$\tilde{\Omega}_{h,\delta} := \{(r, \theta, z) \in \tilde{\Omega}_{F,h} : r \in [0, \delta), z \in (-(2+h), 0)\}.$$

We remark that there exists $\delta > 0$ such that $\tilde{\Omega}_{h,\delta} \subset \tilde{\mathcal{P}}_\ell$. Moreover, the upper and lower boundary of $\tilde{\Omega}_{h,\delta}$ are parametrized respectively by:

$$(r,\theta,z) \in \partial\tilde{\Omega}_{h,\delta} \cap \tilde{B}_* \iff \{r \in [0,\delta) \text{ and } z = \delta_*(s)\},$$

where

$$\delta_*(s) = -\sqrt{1-s^2}, \ \forall s \in [0,1),$$

and

$$(r,\theta,z) \in \partial\tilde{\Omega}_{h,\delta} \cap \tilde{B}_h \iff \{r \in [0,\delta) \text{ and } z = \delta_h(s)\},$$

where

$$\delta_h(s) = -(2+h) + \sqrt{1-s^2}, \ \forall s \in [0,1).$$

Without loss of generality one may assume $\delta = 1/2$. Since the geometry outside $\tilde{\Omega}_{h,1/4}$ is regular, then for any $h \in [0, h_{max}]$ there exists a width $h_0$ surrounding the boundaries of $\tilde{B}_*$ and the hole $\tilde{B}_h$ such that if $d(\tilde{x}, \tilde{B}_h) \leq h_0$ then $\tilde{x} \in \tilde{\mathcal{P}}_\ell$.
In the local frame of local orthonormal basis $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$, consider the vector field $\tilde{\mathrm{w}}_{\|}[h]$ defined by:

$$\tilde{\mathrm{w}}_{\|}[h] = \begin{cases} \mathrm{curl}\tilde{a}_{\|}, & \text{in } \mathbb{R}^3 \backslash \Big(\tilde{B}_* \cup \tilde{B}_h\Big), \\ \tilde{e}_1, & \text{in } \tilde{B}_*, \\ 0, & \text{in } \tilde{B}_h, \end{cases}$$

where $\tilde{a}_{\|}$ is defined in $\mathbb{R}^3 \backslash \Big(\tilde{B}_* \cup \tilde{B}_h\Big)$ as follows:

$$\tilde{a}_{\|} = \begin{cases} \eta_{1/2}(r)\tilde{a}^d_{\|} + (1-\eta_{1/2}(r))\tilde{a}^s_{\|}, & \text{in } \tilde{\Omega}_{h,1/2}, \\ \tilde{a}^s_{\|}, & \text{in } \mathbb{R}^3 \backslash \Big(\tilde{\Omega}_{h,1/2} \cup \tilde{B}_* \cup \tilde{B}_h\Big), \end{cases}$$

with $\tilde{a}^s_\parallel$ and $\tilde{a}^s_\perp$ are given as follows:

$$\tilde{a}^s_\parallel = \frac{\eta_{h_0}(|\tilde{x}+(0,0,2+h)|-1)}{2}\Big(0,\frac{z+2+h}{2},\frac{r\sin\theta}{2}\Big)+\frac{\eta_{h_0}(|\tilde{x}|-1)}{2}(\tilde{e}_1\times\tilde{x}),\ \forall\tilde{x}\in\mathbb{R}^3, \quad (3.2.5)$$

$$\tilde{a}^d_\parallel(r,\theta,z) = (0,\phi_\parallel(r,z),\frac{1}{2}r\sin\theta),\ \ \forall(r,\theta,z)\in\tilde{\Omega}_{h,1/2}. \quad (3.2.6)$$

For all $\beta>0$ we have denoted by $\eta_\beta=\eta(./\beta)$ where $\eta:[0,\infty)\to[0,1]$ a smooth function such that

$$\eta(s)=\begin{cases}1, & \text{if } s<\frac{1}{2},\\ 0, & \text{if } s>1,\end{cases}$$

We stress that we have chosen $h_0$ such that if $\tilde{x}\notin\tilde{\Omega}_{h,1/4}$, then at most one of the functions $\eta_{h_0}(|\tilde{x}+(0,0,2+h)|-1)$ and $\eta_{h_0}(|\tilde{x}|-1)$ is different from zero. Moreover, $\phi_\parallel$ is a truncation function enabling $\tilde{\mathrm{w}}_\parallel[h]$ to be $\tilde{e}_1$ on $\partial\tilde{B}_*$ and zero on $\partial\tilde{B}_h$. To match this property, we set

$$\phi_\parallel(r,z) = -\frac{P_\parallel(\lambda(r,z))}{4}(\delta_*(r)-\delta_h(r))+\frac{2+h}{4}, \quad (3.2.7)$$

with

$$P_\parallel(s)=2s^2-2s+1,\ \ \forall s\in[0,1], \quad (3.2.8)$$

where $\lambda$ is defined as follows

$$\lambda(r,z)=\frac{z-\delta_h(r)}{\delta_*(r)-\delta_h(r)}. \quad (3.2.9)$$

From [25, Proposition 3], we have for any $h>0$ the vector field $\tilde{\mathrm{w}}_\parallel[h]\in\mathcal{C}(\mathbb{R}^3)$ and satisfies

$$\tilde{\mathrm{w}}_\parallel[h]=\tilde{e}_1 \quad \text{on } \tilde{B}_* \quad \text{and} \quad \tilde{\mathrm{w}}_\parallel[h]=0 \quad \text{on } \tilde{B}_h. \quad (3.2.10)$$

Moreover, in the neighbourhood of $\partial\tilde{\mathcal{P}}_\ell$, we have

$$\tilde{\mathrm{w}}_\parallel[h][\tilde{x}]=\mathrm{curl}_{\tilde{x}}\Big(\frac{\eta_{h_0}(|\tilde{x}|-1)}{2}(\tilde{e}_1\times\tilde{x})\Big).$$

We remark that $\tilde{\mathrm{w}}_\parallel[h]$ is regular up to $h=0$ outside $\tilde{\Omega}_{h,1/2}$ and singularities at $h=0$

corresponds to the "diverging part"

$$\tilde{\mathrm{w}}_{\|}^{d}(r,\theta,z) = \operatorname{curl} \tilde{a}_{\|}^{d}[h] = \Big(\frac{1}{2} - \partial_z \phi_{\|}(r,z), 0, \cos\theta \partial_r \phi_{\|}(r,z)\Big), \quad \forall (r,\theta,z) \in \tilde{\Omega}_{h,1/2}. \quad (3.2.11)$$

We turn now to introduce the normal component $\tilde{\mathrm{w}}_{\perp}[h]$ of the vector field $\tilde{\mathrm{w}}[h]$. We set

$$\tilde{\mathrm{w}}_{\perp}[h] = \begin{cases} \operatorname{curl}\tilde{a}_{\perp}, & \text{in } \mathbb{R}^3 \backslash \Big(\tilde{B}_* \cup \tilde{B}_h\Big), \\ \tilde{e}_3, & \text{in } \tilde{B}_*, \\ 0, & \text{in } \tilde{B}_h, \end{cases}$$

where

$$\tilde{a}_{\perp} = \begin{cases} \eta_{1/2}(r)\tilde{a}_{\perp}^{d} + (1-\eta_{1/2}(r))\tilde{a}_{\perp}^{s}, & \text{in } \tilde{\Omega}_{h,1/2}, \\ \tilde{a}_{\perp}^{s}, & \text{in } \mathbb{R}^3 \backslash \Big(\tilde{\Omega}_{h,1/2} \cup \tilde{B}_* \cup \tilde{B}_h\Big), \end{cases}$$

with

$$\tilde{a}_{\perp}^{s} = \frac{\eta_{h_0}(|\tilde{x}|-1)}{2}(\tilde{e}_3 \times \tilde{x}), \ \ \forall \tilde{x} \in \mathbb{R}^3.$$

In cylindrical coordinates, $\tilde{a}_{\perp}^{d}$ is defined in $\tilde{\Omega}_{h,1/2}$ as follows:

$$\tilde{a}_{\perp}^{d}(r,\theta,z) = \Big(-\phi_{\perp}(r,z)\sin\theta, \phi_{\perp}(r,z)\cos\theta, 0\Big), \quad \forall (r,\theta,z) \in \tilde{\Omega}_{h,1/2},$$

where

$$\phi_{\perp}(r,z) = rP_{\perp}(\lambda)$$

with $\lambda$ as in (3.2.9) and

$$P_{\perp}(s) = \frac{1}{2}s^2(3-2s), \ \ \forall s \in [0,1]$$

Again from [25, Proposition 4] we have for any $h>0$, $\tilde{\mathrm{w}}_{\perp}[h] \in \mathcal{C}(\mathbb{R}^3)$ and satisfies

$$\tilde{\mathrm{w}}_{\perp}[h] = \tilde{e}_3 \quad \text{on } \tilde{B}_* \quad \text{and} \quad \tilde{\mathrm{w}}_{\perp}[h] = 0 \quad \text{on } \tilde{B}_h. \quad (3.2.12)$$

Moreover, in the neighbourhood of $\partial\tilde{\mathcal{P}}_\ell$, we have

$$\tilde{\mathrm{w}}_\parallel[h][\tilde{x}] = \mathrm{curl}_{\tilde{x}}\Big(\frac{\eta_{h_0}(|\tilde{x}|-1)}{2}(\tilde{e}_3 \times \tilde{x})\Big).$$

Furthermore, $\tilde{\mathrm{w}}_\perp[h]$ is regular up to $h=0$ outside $\tilde{\Omega}_{h,1/2}$ and singularities at $h=0$ corresponds to the "diverging part":

$$\tilde{\mathrm{w}}^d_\perp(r,\theta,z) = \mathrm{curl}\,\tilde{a}^d_\parallel[h] = \Big(-\partial_z\phi_\perp\cos\theta, -\partial_z\phi_\perp\sin\theta, \partial_r\phi_\perp + \frac{\phi_\perp}{r}\Big), \quad \forall (r,\theta,z)\in\tilde{\Omega}_{h,1/2}. \tag{3.2.13}$$

Since the ball $B$ is moving along the straight line $(D)$, then the velocity vector field $\tilde{\mathrm{w}}[h]$ should be collinear with the unit vector $e_3$. By noting that

$$e_3 = \cos(\alpha)\tilde{e}_3 - \sin(\alpha)\tilde{e}_1,$$

with $\alpha \in (0,\pi/2)$ given by

$$\sin(\alpha) = \frac{2}{2+h} \ \text{ and } \ \cos(\alpha) = \frac{\sqrt{h^2+4h}}{2+h}, \tag{3.2.14}$$

we set

$$\tilde{\mathrm{w}}[h](\tilde{x}) = \cos(\alpha)\tilde{\mathrm{w}}_\perp[h](\tilde{x}) - \sin(\alpha)\tilde{\mathrm{w}}_\parallel[h](\tilde{x}). \tag{3.2.15}$$

In the global frame of basis $(e_1,e_2,e_3)$, $\tilde{\mathrm{w}}[h]$ reads:

$$\mathrm{w}[h](x) = Q_{-\alpha}\tilde{\mathrm{w}}[h]\Big(Q_\alpha(x-G_h)\Big). \tag{3.2.16}$$

More precisely, for all $x \in \mathcal{P}_\ell$ we have:

$$\mathrm{w}[h](x) = \cos(\alpha)Q_{-\alpha}\tilde{\mathrm{w}}_\perp[h]\Big(Q_\alpha(x-G_h)\Big) - \sin(\alpha)Q_{-\alpha}\tilde{\mathrm{w}}_\parallel[h]\Big(Q_\alpha(x-G_h)\Big).$$

In the remainder of the geometry, we define $\mathrm{w}[h]$ by symmetry. Mainly, we set

$$\mathrm{w}[h](x) = S_D[\mathrm{w}[h]](x), \quad \forall x \in \mathcal{P}^r,$$

where the symmetric mapping $S_D$ around axis $(D)$ of a vector field $v$ is given by

$$S_D[v](x) = (-v_1, -v_2, v_3)(-x_1, -x_2, x_3).$$

Let us study some properties of the vector field $u$.

**Lemma 3.2.1** *For all $t \in [0, t_*)$ the vector field $u$ defined in (3.2.4) satisfies:*

$$\begin{aligned}
\|u\|_{\mathbf{L}^2(\mathcal{O})} &\leq C\frac{\dot{h}(t)}{\sqrt{h(t)}}, && (3.2.17)\\
\|\nabla u\|_{\mathbf{L}^2(\mathcal{O})} &\leq C\frac{\dot{h}(t)}{\sqrt{h(t)}}\Big(1 + \sqrt{\ln(1/h(t))}\Big), && (3.2.18)\\
\|\partial_t u\|^2_{\mathbf{L}^2(\mathcal{O})} &\leq C\left(\frac{(\ddot{h}(t))^2}{h(t)} + \frac{(\dot{h}(t))^4}{h(t)^3} + \frac{(\dot{h}(t))^4}{h(t)}\Big(1 + \frac{1}{h(t)} + \frac{1}{h(t)}\ln(1/h(t))\Big)\right), && (3.2.19)\\
\|(u\cdot\nabla)u\|^2_{\mathbf{L}^2(\mathcal{O})} &\leq C\frac{(\dot{h}(t))^2}{h(t)}\Big(1 + \ln(1/h(t))\Big), && (3.2.20)
\end{aligned}$$

Proof. According to Proposition 5 in [25], for all $h > 0$ the vector field $\tilde{\mathrm{w}}[h]$ satisfies:

$$\begin{aligned}
\|\tilde{\mathrm{w}}[h]\|_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} &\leq C, && (3.2.21)\\
\|\nabla\tilde{\mathrm{w}}[h]\|_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} &\leq C\sqrt{\ln(1/h)}, && (3.2.22)
\end{aligned}$$

where $C$ is a positive positive constant independent of $h$.

Moreover, it was shown in the above reference that

$$|\tilde{\mathrm{w}}^d_{\|}| \leq C, \qquad |\nabla\tilde{\mathrm{w}}^d_{\|}| \leq C\frac{1}{\delta_* - \delta_h},$$
$$|\tilde{\mathrm{w}}^d_{\perp}| \leq C\Big(1 + \frac{r}{\delta_* - \delta_h}\Big), \quad |\nabla\tilde{\mathrm{w}}^d_{\perp}| \leq C\Big(\frac{r}{(\delta_* - \delta_h)^2} + \frac{1}{\delta_* - \delta_h}\Big).$$

Noting that

$$|\cos\alpha| \leq C\sqrt{h} \quad \text{and} \quad |\sin\alpha| \leq 1,$$

we get

$$\Big|(\tilde{\mathrm{w}}[h]\cdot\nabla)\tilde{\mathrm{w}}[h]\Big| \leq C\Big(\frac{1}{(\delta_* - \delta_h)} + \sqrt{h}\frac{r}{(\delta_* - \delta_h)^2} + h\frac{r^2}{(\delta_* - \delta_h)^3}\Big), \quad \text{in } \tilde{\Omega}_{h,1/4}.$$

Lemma 1 in [25] implies that

$$\Big\|(\tilde{\mathrm{w}}[h]\cdot\nabla)\tilde{\mathrm{w}}[h]\Big\|^2_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} \;\le\; C\Big(1+\ln(1/h)\Big).$$

At this point, estimates (3.2.17)-(3.2.18) follow by noting that $Q_\alpha$ is a unit transformation and

$$|\dot{d}(t)| \le C\frac{\dot{h}(t)}{\sqrt{h(t)}}.$$

By chain rule, we compute

$$\partial_t u(t,x) = \ddot{d}(t)\mathrm{w}[h(t)](x) + \dot{d}(t)\dot{h}(t)\partial_h \mathrm{w}[h(t)](x). \tag{3.2.23}$$

By Proposition 6 in [25], we have

$$\|\partial_h \mathrm{w}[h]\|_{\mathbf{L}^2(\mathcal{P}^l\setminus\tilde{\Omega}_{h,1/4})} \le \frac{C}{\sqrt{h}},$$
$$|\partial_h \mathrm{w}[h]| \le C\Big(1+\frac{1}{\sqrt{h}(\delta_*-\delta_h)}+\frac{r}{(\delta_*-\delta_h)^2}\Big), \quad \text{in } \tilde{\Omega}_{h,1/4}.$$

By Young's inequality and noting that

$$h+r^2 \le \delta_*(r)-\delta_h(r) \le h+2r^2, \quad \forall r\in(0,1),$$

we get that

$$\frac{r}{(\delta_*-\delta_h)^2} \le \frac{h+r^2}{2\sqrt{h}(\delta_*-\delta_h)^2} \le \frac{1}{2\sqrt{h}(\delta_*-\delta_h)}.$$

Consequently,

$$|\partial_h \mathrm{w}[h]| \le C\Big(1+\frac{1}{\sqrt{h}(\delta_*-\delta_h)}\Big), \quad \text{in } \tilde{\Omega}_{h,1/4}.$$

Using again Lemma 1 in [25], we get that

$$\|\partial_h \mathrm{w}[h]\|^2_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} \le C\Big(1+\frac{1}{h}\ln(1/h)\Big).$$

Therefore,

$$\|\partial_h \mathrm{w}[h]\|^2_{\mathbf{L}^2(\mathcal{P}^l)} \le C\Big(1+\frac{1}{h}+\frac{1}{h}\ln(1/h)\Big). \tag{3.2.24}$$

Hence, estimate (3.2.19) follows by noting that

$$|\ddot{d}(t)|^2 \leq C\Big(\frac{(\ddot{h}(t))^2}{h(t)} + \frac{(\dot{h}(t))^4}{h(t)^3}\Big).$$

$\square$

In the following lemma, we construct a suitable pressure field so that the external body force $f$ belongs to $L^2(0,T;L^p(\mathcal{O}))$ for all $1<p<2$.

**Lemma 3.2.2** *Given $h>0$, there exists a smooth function $\tilde{q}_h$ such that*

$$-\Delta\tilde{w}[h] + \nabla\tilde{q}_h = \tilde{f}^0 + \tilde{f}^1 + \tilde{f}^2, \tag{3.2.25}$$

*with*

$$\begin{aligned} \|\tilde{f}^0\|^2_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} &\leq C\ln(1/h), \\ \|\tilde{f}^1\|^2_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} &\leq C, \\ \|\tilde{f}^2\|_{\mathbf{L}^p(\tilde{\Omega}_{h,1/4})} &\leq C, \ \forall\ 1<p<2. \end{aligned}$$

Proof. Following similar arguments as those in [26, Lemma 3.8], we construct a pressure $\tilde{q}_\perp$ such that

$$-\Delta\tilde{\mathrm{w}}^d_\perp + \nabla\tilde{q}_\perp = \tilde{f}_\perp, \quad \text{in } \mathcal{P}^l,$$

with

$$|\tilde{f}_\perp| \leq C\Big(\frac{r^2}{(\delta_* - \delta_h)^4} + \frac{1}{(\delta_* - \delta_h)^2}\Big).$$

It follows then from Lemma 1 in [25] that

$$\|\tilde{f}_\perp\|^2_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} \leq C\Big(\frac{1}{h} + \ln(1/h)\Big). \tag{3.2.26}$$

From the definition $\tilde{\mathrm{w}}_\parallel$ we have

$$-\Delta\tilde{\mathrm{w}}_\parallel = \begin{pmatrix} 0 \\ 0 \\ -\cos(\theta)\partial_{rzz}\phi_{//} \end{pmatrix} + \begin{pmatrix} \Delta(\partial_z\phi_{//}) \\ 0 \\ -\cos(\theta)\partial_{rrr}\phi_{//} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \cos(\theta)\Big(\frac{1}{r}\partial_{rr}\phi_{//} - \frac{1}{r^2}\partial_r\phi_{//}\Big) \end{pmatrix}.$$

We remark here that

$$\partial_{rzz}\phi_{//}(r,z) = -\frac{2\delta_h'(r)}{\delta_*(r)-\delta_h(r)}\partial_z\lambda(r,z).$$

Consequently, setting

$$\tilde{q}_{\|}(r,\theta,z) = \cos(\theta)\lambda(r,z)\Big(\frac{2\delta_h'(r)}{\delta_*(r)-\delta_h(r)}\Big),$$

we get

$$\partial_z\tilde{q}_{\|} = -\cos(\theta)\partial_{rzz}\phi_{//}.$$

Noting that

$$\nabla_{\tilde{x}_1,\tilde{x}_2}\tilde{q}_{\|} = \Big(\cos(\theta),\sin(\theta)\Big)\partial_r\tilde{q}_{\|} + \frac{1}{r}\Big(-\sin(\theta),\cos(\theta)\Big)\partial_\theta\tilde{q}_{\|},$$

with

$$\begin{aligned}
\partial_r\tilde{q}_{\|} &= 2\cos(\theta)\Big[\Big(\frac{\delta_h''(r)}{\delta_*(r)-\delta_h(r)} + 2\frac{(\delta_h'(r))^2}{(\delta_*(r)-\delta_h(r))^2}\Big)\lambda(r,z) + \frac{\delta_h'(r)}{\delta_*(r)-\delta_h(r)}\partial_r\lambda(r,z)\Big],\\
\partial_\theta\tilde{q}_{\|} &= -\sin(\theta)\frac{2\delta_h'(r)}{\delta_*(r)-\delta_h(r)}\lambda(r,z),
\end{aligned}$$

Using Lemmas 2 and 4 in [25] we get that

$$|\nabla_{\tilde{x}_1,\tilde{x}_2}\tilde{q}_{\|}| \le C\Big(\frac{1}{\delta_*(r)-\delta_h(r)} + \frac{r^2}{\delta_*(r)-\delta_h(r)}\Big).$$

Consequently,

$$|\cos(\theta)\partial_{rzz}\phi_{\|}\tilde{e}_3 + \nabla_{\tilde{x}}\tilde{q}_{\|}| \le C\Big(\frac{1}{\delta_*(r)-\delta_h(r)} + \frac{r^2}{\delta_*(r)-\delta_h(r)}\Big).$$

Therefore, we get

$$\|\cos(\theta)\partial_{rzz}\phi_{\|}\tilde{e}_3 + \nabla_{\tilde{x}}\tilde{q}_{\|}\|^2_{L^2(\tilde{\Omega}_{h,1/4})} \le C\ln(1/h). \tag{3.2.27}$$

Finally, setting

$$\begin{aligned}
\tilde{q}_h &= \cos\alpha\tilde{q}_{\perp} + \sin\alpha\tilde{q}_{\|},\\
\tilde{f}^0 &= \sin\alpha\Big(-\Delta(\partial_z\phi_{\|}),0,\cos\theta\partial_{rrr}\phi_{\|}\Big),\\
\tilde{f}^1 &= \sin\alpha\Big(\cos(\theta)\partial_{rzz}\phi_{\|}\tilde{e}_3 + \nabla_{\tilde{x}}\tilde{q}_{\|}\Big) + \cos\alpha\tilde{f}_{\perp},\\
\tilde{f}^2 &= \sin\alpha\cos\theta\Big(\frac{1}{r}\partial_{rr}\phi_{\|} - \frac{1}{r^2}\partial_r\phi_{\|}\Big)\tilde{e}_3,
\end{aligned}$$

and noting that

$$\begin{aligned}
|\Delta(\partial_z \phi_{\|})| &\leq \frac{C}{\delta_* - \delta_h}, \\
|\cos\theta \partial_{rrr}\phi_{\|}| &\leq \frac{C}{\delta_* - \delta_h}, \\
\left|\cos\theta\Big(\frac{1}{r}\partial_{rr}\phi_{\|} - \frac{1}{r^2}\partial_r\phi_{\|}\Big)\right| &\leq \frac{C}{r}, \\
|\cos\alpha| &\leq C\sqrt{h},
\end{aligned}$$

and (3.2.26)-(3.2.27), we get that

$$\begin{aligned}
\|\tilde{f}^0\|^2_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} &\leq C\ln(1/h), \\
\|\tilde{f}^1\|^2_{\mathbf{L}^2(\tilde{\Omega}_{h,1/4})} &\leq C\Big(1 + h\ln(1/h)\Big) \leq C, \\
\|\tilde{f}^2\|_{\mathbf{L}^p(\tilde{\Omega}_{h,1/4})} &\leq C, \ \ \forall p < 2.
\end{aligned}$$

$\square$

Consider the following function $h$ given by;

$$h(t) = \frac{(t - t_*)^4}{T^4}. \tag{3.2.28}$$

It is not difficult to check that $|h(t)| \leq 1$ for all $t \in [0, T]$ and vanishes at time $t_*$. With this choice, Lemma 3.2.1 implies that the constructed vector field $u \in L^\infty(0, t_*; H) \cap L^2(0, t_*; V)$ and its time derivative $u_t \in L^2(0, t_*; \mathbf{L}^2(\mathcal{O})$. By Aubin Simon theorem [8], we get $u \in \mathcal{C}([0, t_*], H)$. In other words, we can extend $u$ by continuity to $t = t_*$. For $t > t_*$, we define $u$ by

$$u(x, t) = -\dot{d}(t)\mathrm{w}[h(t)](x), \tag{3.2.29}$$

where $h$ is as in (3.2.28). Therefore, the vector field $u$ has the following regularity:

$$u \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \text{and} \quad u_t \in L^2(0, T; \mathbf{L}^2(\mathcal{O}). \tag{3.2.30}$$

Moreover, we have

$$u \in L^2(0, T; K(B(t), \mathcal{O})), \quad \text{and} \quad (u \cdot \nabla)u \in L^2(0, T; \mathbf{L}^2(\mathcal{O})). \tag{3.2.31}$$

Furthermore, according to Lemma 3.2.2 there exists a pressure $p$ such that

$$-\Delta u + \nabla p \in L^2(0,T;\mathbf{L}^p(\Omega_F(t))), \ \ \forall\, p \text{ such that } \frac{6}{5} < p < 2.$$

Consequently, with $h, u$, and $G$ are given by (3.2.28), (3.2.4),(3.2.29) and (3.2.2) respectively the pair $(u, G)$ is a weak solution to problem (3.1.1)-(3.1.8) with external body force $f \in L^2(0,T;\mathbf{L}^p(\Omega_F(t)))$ ($\frac{6}{5} < p < 2$) defined by

$$f = \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p. \tag{3.2.32}$$

It remains to construct $f$ on the disks. If $f$ is constant, then from the definition of center of mass, we have

$$\int_{B(t)} \rho_B(x - G(t) \times f dx = 0.$$

Also, Since the solution $u$ is symmetric with respect to the $z$-axis, then one has

$$\int_{\partial B(t)} (x - G(t)) \times \sigma \mathbf{n} d\Gamma = 0.$$

Thus, if $f$ is constant then (3.1.6) is satisfied as $\omega = 0$.

Now, we move to construct $F$ such that

$$m\ddot{G}(t) = -\int_{\partial B(t)} \sigma(u,p)\mathbf{n} d\Gamma + F,$$

with $F = \rho_B \int_{B(t)} f(t) dx$.

Again, since the solution is symmetric with respect to the $z$-axis and $G(t) = d(t)e_3$ we get that $F_1 = F_2 = 0$. Therefore, $F = F_3 e_3$ where $F_3$ satisfies

$$F_3 = m\ddot{d}(t) + \dot{d}(t)\int_{\partial B(t)\cap\partial\Omega_{h,1/2}} \sigma(w,q)\mathbf{n}\cdot e_3 d\Gamma.$$

But

$$\int_{\partial\Omega_{h,1/2}} \sigma(w,q)\mathbf{n}\cdot e_3 d\Gamma = -\int_{\Omega_{h,1/2}} (-\Delta w + \nabla q)\cdot w dx + \int_{\Omega_{h,1/2}} |D[w]|^2 dx.$$

By using Lemma 3.2.1 and Lemma 3.2.2, we get that

$$\Big|\int_{\partial B(t)\cap\partial\Omega_{h,1/2}} \sigma(w,q)\mathbf{n}\cdot e_3 d\Gamma\Big| \leq C\Big(1+\ln(\frac{1}{h})\Big).$$

Therefore,

$$|F_3| \leq m|\ddot{d}(t)| + C|\dot{d}(t)|\Big(1+\ln(\frac{1}{h})\Big).$$

This implies that $|F_3| \leq C$. Setting $f = \frac{1}{m}F$ on $B(t)$ with $F = F_3 e_3$, we get that $\rho_B \int_{B(t)} f(t)dx = F$ and hence our solution satisfies Newton's law for linear and angular momentum as $f$ is only time dependent.

## 3.3 Construction of the second solution

In the previous section, we have constructed a weak solution for problem (3.1.1)-(3.1.8) which moves away from the boundary of the flow after $t_*$ with external body force $f \in L^2(0,T;\mathbf{L}^2(\mathcal{O}))$. In this section, we construct a second weak solution of problem (3.1.1)-(3.1.8) with same external body force $f$. The second weak solution, denoted by $v$, is another extension of the vector field $u$ defined in (3.2.4) such that the ball $B$ stays near the boundary after contact. In other words, we set

$$v(t,x) = u(t,x), \quad \text{for all } t \in [0,t_*],$$

and we assume that

$$B(t) = \mathbf{B}_* = B(t_*), \quad \text{for all } t \in (t_*,T].$$

More precisely, the center of mass $G$ of the ball $B$ is defined on $[0,T]$ by:

$$G(t) = \begin{cases} d(t)e_3, & \text{if } 0 < t \leq t_*, \\ 0, & \text{if } t_* < t \leq T, \end{cases}$$

with $d$ is given by (3.2.1)-(3.2.28).

Consequently,

$$v(t,x) = 0, \text{ if } x \in B(t), t \in (t_*,T]. \tag{3.3.1}$$

In the domain $\mathcal{O}\backslash\mathbf{B}_*$, we define the function $v$ to be the weak solution of the Navier-Stokes equation with zero boundary and initial data ($v(x,t_*) = u(x,t_*) = 0$ for any $x \in \mathcal{O}$) and external body force $f \in L^2(0,T;\mathbf{L}^p(\mathcal{O}))$ defined as in (3.2.32). Although the domain $\mathcal{O}\backslash B_*$ is not smooth, we still able to prove that there exists a unique vector field $v \in L^\infty(t_*,T;H(\mathcal{O}\backslash\mathbf{B}_*)) \cap L^2(t_*,T;V(\mathcal{O}\backslash\mathbf{B}_*))$satisfying

$$\int_{t_*}^{T}\int_{\mathcal{O}\backslash\mathbf{B}_*}\Big(\rho v_t\cdot\varphi - (\rho v\otimes v - 2\nu D[v]) : D[\varphi] - \rho f\cdot\varphi\Big)dxdt = 0, \tag{3.3.2}$$

for all $\varphi \in \mathcal{D}((t_*,T)\times\mathcal{O}\backslash\mathbf{B}_*)$ such that div $\varphi = 0$, by using similar arguments as in Proposition 3.1 in [16].

**Proposition 3.3.1** *The pair $(v,G)$ is a weak solution to problem (3.1.1)-(3.1.8) with given source term $f$ defined in* (3.2.32)*.*

Proof. From the definition of the function $v$ and the external body force $f$, it follows that $v$ satisfies

$$\int_0^{t_*}\int_{\mathcal{O}}\Big(\rho v\cdot\partial_t\varphi + \rho v\otimes v : D[\varphi] - 2\nu D[v] : D[\varphi] + \rho f\cdot\varphi\Big)dxdt = -\int_{\mathcal{O}}\rho(0)v(0)\cdot\varphi(0)dx, \tag{3.3.3}$$

for all $\varphi \in \{\varphi \in \mathcal{D}([0,t_*)\times\mathcal{O}) : \nabla\cdot\varphi = 0 \text{ on } [0,t_*)\times\mathcal{O},\ D[\varphi] = 0 \text{ on a neighbourhood of } B_i(t)\}$. Since $v(t_*) = u(t_*) = 0$, then it remains to check that $v$ satisfies

$$\int_{t_*}^{T}\int_{\mathcal{O}}\Big(\rho v_t\cdot\varphi - (\rho v\otimes v - 2\nu D[v]) : D[\varphi] - \rho f\cdot\varphi\Big)dxdt = 0, \tag{3.3.4}$$

for all $\varphi \in \{\varphi \in \mathcal{D}((t_*,T)\times\mathcal{O}) : \nabla\cdot\varphi = 0 \text{ on } (t_*,T)\times\mathcal{O},\ D[\varphi] = 0 \text{ on a neighbourhood of } B_i(t)\}$. Since $v$ satisfies (3.3.2), then one has only to realize that any test function

$$\varphi \in \{\varphi \in \mathcal{D}((t_*,T)\times\mathcal{O}) : \nabla\cdot\varphi = 0\ on\ (t_*,T)\times\mathcal{O},\ D[\varphi] = 0 \text{ on a neighbourhood of } B_i(t)\}.$$

belongs to $\mathcal{D}((t_*,T)\times\mathcal{O}\backslash B_*)$. This end the proof. □

## 3.4 Conclusion

It is clear that the first solution $u$ constructed in Section 3.2 is different from the second solution $v$ which is given in the previous section. It follows that the solution of problem (3.1.1)-(3.1.8) is not unique after contact. Thus, Theorem 3.1.1 is proved.

4

# Existence Results for the Motion of Rigid Bodies in Viscoelastic Fluids

## Sommaire



In this chapter, we study the two dimensional motion of a finite number of homogeneous rigid disks in a cavity filled with incompressible viscoelastic fluids such as polymeric solutions. The incompressible Navier–Stokes equations are used to model the flow of the solvent, in which the elastic extra stress tensor appears as a source term. The extra stress tensor, which stems from the random movement of polymer chains in the solvent, satisfies a differential constitutive law. We focus here in two types of differential constitutive laws: The first one corresponds to the regularized Oldroyd model whereas the other one corresponds to Oldroyd model. The movement of the rigid disks are described by the standard conservation laws of linear and angular momentum. We prove the existence and uniqueness of local-in-time strong solutions of the considered moving-boundary problem in the case of regularized Oldroyd constitutive law as well as in the Oldroyd Model. This chapter is structured as follows: In Section 4.1, we introduce the Oldroyd and the regularized Oldroyd models. Then since we are dealing with a free boundary problem, we rewrite the models in cylindrical domain. Section 4.2 is devoted to study two linear problems associated to the transformed regularized Oldroyd model in cylindrical domain. We end this section by proving the local-in-time existence of strong solutions in the case of the diffusive Oldroyd model. In Section 4.3 we investigate the local-in-time existence of strong solutions in the case of Oldroyd model.

## 4.1 Introduction

In this chapter we study the movement of several homogeneous rigid disks inside a cavity filled with an incompressible viscoelastic fluid such as polymeric solutions which obeys constitutive laws of differential type. Due to the elasticity, viscoelastic fluids have memory and hence in contrast with Newtonian fluids, the dynamic of the flow at a given time depends on the past deformations and not only on the present deformations. We suppose that the considered solvent of the viscoelastic fluid is a homogeneous incompressible, viscous, and Newtonian fluid. Hence, the governing equations for the fluid are the classical incompressible Navier-Stokes equations in which the elastic extra stress tensor appears as a source term, whereas the motion of the rigid bodies is governed by the balance equations for linear and angular momentum (Newton's laws).

Let $\mathcal{O} \subset \mathbb{R}^2$ be an open bounded set representing the domain occupied by the fluid and the $k$ rigid bodies. We recall that $\Omega_F(t)$ denotes the domain occupied by the fluid and by $B_i(t)$, $i = 1, \dots, k$ the domain occupied by the rigid bodies at time $t$. In the sequel, we concentrate on two models: namely the Oldroyd model and its transient version which is known by the regularized Oldroyd model or the diffusive Oldroyd model. In dimensionless variables, the Oldroyd model can be written as (see, for instance, [22]):

$$Re(\partial_t + u \cdot \nabla)u - (1-r)\Delta u + \nabla p = \nabla \cdot \tau + f, \quad x \in \Omega_F(t),\ t \in (0,T), \tag{4.1.1}$$

$$\nabla \cdot u = 0, \quad x \in \Omega_F(t),\ t \in (0,T), \tag{4.1.2}$$

$$u(x,t) = h_i'(t) + \omega_i(t)(x - h_i(t))^{\perp}, \quad x \in \partial B_i(t),\ t \in (0,T), \tag{4.1.3}$$

$$u(x,t) = 0, \quad x \in \partial \mathcal{O},\ t \in (0,T), \tag{4.1.4}$$

$$\overline{m}_i h_i^{''}(t) = -\int_{\partial B_i(t)} \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} f(t) dx, \quad t \in (0,T), \tag{4.1.5}$$

$$\overline{J}_i \omega_i^{'}(t) = -\int_{\partial B_i(t)} (x - h_i(t))^{\perp} \cdot \sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i(t)} (x - h_i(t))^{\perp} \cdot f(t) dx, \quad t \in (0,T), \tag{4.1.6}$$

$$u(x,0) = u_0(x), \quad x \in \Omega_F(0), \tag{4.1.7}$$

$$h_i(0) = h_i^0, \quad h_i'(0) = h_i^1, \quad \omega_i(0) = \omega_i^0, \tag{4.1.8}$$

$$We\Big(\partial_t \tau + (u \cdot \nabla)\tau + g_a(\nabla u, \tau)\Big) + \tau = 2rD[u],\ x \in \Omega_F(t),\ t \in (0,T), \tag{4.1.9}$$

$$\tau(x,0) = \tau_0(x), \quad x \in \Omega_F(0). \tag{4.1.10}$$

However, the elastic extra-stress tensor $\tau$ in the regularized Oldroyd model is expressed as a solution of a second order parabolic partial differential equation. More precisely, in the diffusive model we replace the stress equation (4.1.9) in the above system by

$$We\Big(\partial_t \tau + (u \cdot \nabla)\tau + g_a(\nabla u, \tau)\Big) + \tau - \varepsilon \Delta \tau = 2rD[u], \quad x \in \Omega_F(t),\ t \in (0,T), \tag{4.1.11}$$

$$\varepsilon \frac{\partial \tau}{\partial n}(x,t) = 0, \quad x \in \partial \Omega_F(t),\ t \in (0,T). \tag{4.1.12}$$

The additional dissipative term $\varepsilon \Delta \tau$ with $\varepsilon > 0$ in the stress equation (4.1.11) corresponds to a center of mass of diffusion term in the dumbell models. We refer the reader to [4] and the references therein for the derivation of (4.1.11)-(4.1.12). In standard derivation of Oldroyd model from kinetic models for dilute polymers, the diffusive term is routinely omitted on the grounds that it is several orders of magnitude smaller than the other terms in the equation. However, the omission of the diffusive term in (4.1.11) changes the type of equation from parabolic into hyperbolic (first order transport) equation.

In both models, the unknowns are $u(t,x)$ (the velocity vector field of the fluid), $\tau(t,x)$ (the symmetric elastic extra stress tensor), $p(t,x)$ (the hydrostatic pressure of the fluid),

$h_i(t)$ (the position of mass center of the $i$-th rigid body) and $\omega_i(t)$ (the angular velocity of the $i$-th rigid body). For all $x = (x_1, x_2)$, we denote by $x^\perp$ the vector $x^\perp = (-x_2, x_1)$.

Moreover, we have denoted by $\partial\mathcal{O}$ the boundary of the cavity $\mathcal{O}$, by $\partial B_i(t)$ the boundary of the $i$-th body at time $t$, by $\nu_i$ the unit normal vector directed toward the interior of the $i$-th disk, and by $f(t,x)$ the force acting on the fluid. $Re = \rho\dfrac{\overline{U}L}{\eta}$ and $We = \dfrac{\lambda_1}{\overline{U}L}$ are respectively the well-known Reynolds number and Weissenberg number. Here, $\overline{U}$ and $L$ represent a typical velocity and a typical length of the flow, $\rho$ and $\eta$ are respectively the fluid density and viscosity, and $\lambda_1 > 0$ is a relaxation time. Further, the dimensionless numbers $\overline{m}_i$ and $\overline{J}_i$ are given by $\overline{m}_i = \dfrac{m_i\overline{U}}{\eta L}$, $\overline{J}_i = \dfrac{J_i\overline{U}}{\eta L^3}$, where $\overline{\rho}_i, m_i$ and $J_i$ denote respectively the density, the mass and the moment of inertia of the $i$-th rigid body.
We recall that the total stress tensor $\sigma$ (also called the Cauchy stress) is given by

$$\sigma = -pI + 2(1-r)D[u] + \tau,$$

where $r$ is a retardation parameter and $D[u] = \frac{1}{2}(\nabla u + \nabla u^T)$ is the deformation tensor ($A^T$ denotes the transpose of the matrix $A$). Also, $g_a$ is a bilinear tensor-valued mapping defined by

$$g_a(\nabla u, \tau) = \tau W[u] - W[u]\tau - a(D[u]\tau + \tau D[u]),$$

where $W[u] = \dfrac{1}{2}(\nabla u - \nabla u^T)$ is the vorticity tensor and $a$ is a real number satisfying $-1 \leq a \leq 1$.

Finally, we suppose that there is no contact initially between the rigid bodies and between them and the boundary of the cavity; that is $\gamma = \gamma(0) > 0$, where

$$\gamma(t) = \min_{i=1,\dots,k} d(B_i(t), \partial\Omega_F(t)).$$

The movement of rigid bodies modifies the fluid domain and hence the first step to study the models introduced above is to write the modeling equations in a cylindrical domain. To do this, we use a non-linear, local change of coordinates $X$ introduced in Section 2.2. Next for $(y,t) \in \Omega_F(0) \times [0,T]$, we set

$$\begin{cases} U(y,t) = J_Y(X(y,t),t)u(X(y,t),t), \quad P(y,t) = p(X(y,t),t), \\ F(y,t) = J_Y(X(y,t),t)f(X(y,t),t), \quad \mathcal{T}(y,t) = \tau(X(y,t),t), \end{cases} \tag{4.1.13}$$

where $J_X$ and $J_Y$ are respectively the Jacobian matrix of the diffeomorphism $X(.,t)$ and the Jacobian matrix of inverse $Y(.,t)$, the inverse mapping of $X(.,t)$. We recall that the mapping $Y$ maps the fluid domain into its initial shape $\Omega_F(0)$ [40]. For simplicity, $\Omega_F(0)$ will be denoted throughout this chapter by $\Omega_F$ and $B_i(0)$ by $B_i$.

Formal computations show that $(U, P, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k})$ satisfies the following set of equations:

$$Re\Big(\frac{\partial U}{\partial t} + [MU] + [NU]\Big) - (1-r)[LU] + [GP] = [\operatorname{div} \mathcal{T}] + \mathrm{F}, \quad \text{in } \Omega_F \times ]0,T[, \tag{4.1.14}$$

$$\nabla \cdot U = 0, \quad \text{in } \Omega_F \times ]0,T[, \tag{4.1.15}$$

$$U(y,t) = h_i'(t) + \omega_i(t)(y - h_i(0))^\perp, \quad \text{in } \partial B_i \times [0,T[, \tag{4.1.16}$$

$$U(y,t) = 0, \quad \text{in } \partial\mathcal{O} \times ]0,T[, \tag{4.1.17}$$

$$U(y,0) = u_0(y), \quad y \in \Omega_F, \tag{4.1.18}$$

and for all $i \in \{1, \dots, k\}$, we have:

$$\overline{m}_i h_i''(t) = -\int_{\partial B_i} \Sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i} F(t) dy, \quad t \in ]0,T[, \tag{4.1.19}$$

$$\overline{J}_i \omega_i'(t) = -\int_{\partial B_i} (y - h_i(0))^\perp \cdot \Sigma \nu_i d\Gamma_i + \overline{\rho}_i \int_{B_i} (y - h_i(0))^\perp \cdot F(t) dy, \quad t \in ]0,T[, \tag{4.1.20}$$

$$h_i(0) = h_i^0, \;\; h_i'(0) = h_i^1, \;\; \omega_i(0) = \omega_i^0. \tag{4.1.21}$$

In the case of regularized Oldroyd model, $\mathcal{T}$ solves

$$We\Big(\frac{\partial \mathcal{T}}{\partial t} + \Big(\Big(U + \frac{\partial Y}{\partial t}\Big) \cdot \nabla\Big)\mathcal{T} + [G_a(U, \mathcal{T})]\Big) + \mathcal{T} - \varepsilon[\text{Ł}\mathcal{T}] = 2r[\mathcal{D}U], \text{ in } \Omega_F \times ]0,T[, \tag{4.1.22}$$

$$\varepsilon \frac{\partial \mathcal{T}}{\partial n}(y,t) = 0, \text{ on } \partial\Omega_F \times ]0,T[, \tag{4.1.23}$$

$$\mathcal{T}(y,0) = \tau_0(y), \quad y \in \Omega_F, \tag{4.1.24}$$

whereas in the case of standard Oldroyd model, we have

$$We\Big(\frac{\partial \mathcal{T}}{\partial t} + \Big(\Big(U + \frac{\partial Y}{\partial t}\Big)\cdot\nabla\Big)\mathcal{T} + [G_a(U,\mathcal{T})]\Big) + \mathcal{T} = 2r[\mathcal{D}U], \text{ in } \Omega_F\times]0,T[, \tag{4.1.25}$$

$$\mathcal{T}(y,0) = \tau_0(y), \qquad y \in \Omega_F. \tag{4.1.26}$$

We remark that the total stress tensor field associated to $U, P$ and $\mathcal{T}$ is given by

$$\Sigma(U,P,\mathcal{T}) = -PI + 2(1-r)D[U] + \mathcal{T}.$$

The operator $[LU]$ is the transform of $\Delta u$ , $[MU]$ is the remainder term in the expansion of $\partial_t u$, $[GP]$ is a term related to the pressure $p$, $[\operatorname{div}\mathcal{T}]$ denotes the transform of $\nabla\cdot\tau$, whereas $[NU]$ is a non-linear term corresponding to $(u\cdot\nabla)u$ in equation (4.1.1). Moreover, the operators $[Ł\mathcal{T}], [\mathcal{D}U], [\mathcal{W}U]$ and $[G_a(U,\mathcal{T}))$ in the transformed stress equation are related respectively to rewriting of $\Delta\tau, D[u], W[u]$ and the bilinear function $g_a$. We precise the expression of these operator in due course. We construct below a functional framework in which the equations (4.1.1)-(4.1.6) are rigorously equivalent to (4.1.14)-(4.1.21) and the regularized Oldroyd differential law is equivalent to (4.1.14)-(4.1.24) whereas the Oldroyd differential law is equivalent to (4.1.25)-(4.1.26). Namely for a function $u(.,t) : \Omega_F(t) \to \mathbb{R}^2$, we set $U(y,t) = u(X(y,t),t)$ and we define the following functional spaces as follows:

$$\begin{aligned}
L^2(0,T;\mathbf{H}^k(\Omega_F(t))) &= \{u : U \in L^2(0,T;\mathbf{H}^k(\Omega_F))\},\\
H^1(0,T;\mathbf{L}^2(\Omega_F(t))) &= \{u : U \in H^1(0,T;\mathbf{L}^2(\Omega_F))\},\\
\mathcal{C}([0,T],\mathbf{H}^k(\Omega_F(t))) &= \{u : U \in C([0,T],\mathbf{H}^k(\Omega_F))\},\\
L^2(0,T;\dot{H}^1(\Omega_F(t))) &= \{u : U \in L^2(0,T;\dot{H}^1(\Omega_F))\}.
\end{aligned}$$

In the above spaces, we have denoted the Lebesgue and Sobolev spaces by $L^p(\Omega), 1 \le p \le \infty$, with norms $\|\cdot\|_{L^p(\Omega)}$ and $H^k(\Omega)$, with norm $\|\cdot\|_{H^k(\Omega)}$. $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^k(\Omega)$ are spaces of vector valued or tensor valued functions with components in $L^p(\Omega)$ and $H^k(\Omega)$ respectively. We remark here that the definition of the above spaces is independent of the choice of the mapping $X$.

Moreover, we introduce the following function spaces:

$$\begin{aligned}
\mathcal{U}(0,T;\Omega_F(t)) &= L^2(0,T;\mathbf{H}^2(\Omega_F(t)))\cap\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t)))\cap H^1(0,T;\mathbf{L}^2(\Omega_F(t))),\\
\mathfrak{T}(0,T;\Omega_F(t)) &= \{\tau\in L^2(0,T;\mathbf{H}^2(\Omega_F(t)))\cap\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t)))\cap H^1(0,T;\mathbf{L}^2(\Omega_F(t))):\tau=\tau^T\},\\
\tilde{\mathcal{U}}(0,T;\Omega_F(t)) &= \{u\in L^2(0,T;\mathbf{H}^3(\Omega_F(t)))\cap\mathcal{C}([0,T],\mathbf{H}^2(\Omega_F(t))):\\
&\qquad u'\in L^2(0,T;\mathbf{H}^1(\Omega_F(t)))\cap\mathcal{C}([0,T];\mathbf{L}^2(\Omega_F(t)))\},\\
\tilde{\mathfrak{T}}(0,T;\Omega_F(t)) &= \{\tau\in\mathcal{C}([0,T],\mathbf{H}^2(\Omega_F(t))):\ \tau'\in\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F(t)))\ \text{and}\ \tau=\tau^T\}.
\end{aligned}$$

In the rest of the chapter, we focus mainly in studying the transformed regularized Oldroyd constitutive law and the transformed Oldroyd model. Classical solutions of the original models are then computed through the change of variable $X$.

We focus first in studying the local existence and uniqueness of strong solutions of the regularized Oldroyd model. In this respect, our first result reads:

**Theorem 4.1.1 (Regularized Oldroyd model)** *Suppose that $\partial\mathcal{O}\in\mathcal{C}^2$, $f\in L^2_{loc}(0,\infty;\mathbf{L}^2(\mathcal{O}))$, $u_0\in\mathbf{H}^1(\Omega_F)$, $\tau_0\in\mathbf{H}^1(\Omega_F)$, $\varepsilon>0$, $\gamma>0$, and that*

$$\begin{aligned}
\nabla\cdot u_0 &= 0, && \text{in } \Omega_F,\\
u_0(x) &= h_i^1+\omega_i^0(x-h_i^0)^\perp, && x\in\partial B_i,\ \forall i\in\{1,\dots,k\},\\
u_0(x) &= 0, && x\in\partial\mathcal{O}.
\end{aligned}$$

*Then there exists $T_0>0$ such that problem (4.1.1)-(4.1.8), (4.1.10)-(4.1.12) admits unique strong solution*

$$u\in\mathcal{U}(0,T;\Omega_F(t)),p\in L^2(0,T;\dot{H}^1(\Omega_F(t))),\tau\in\mathfrak{T}(0,T;\Omega_F(t)),(h_i,\omega_i)\in H^2(0,T;\mathbb{R}^2)\times H^1(0,T;\mathbb{R}),$$

*on $[0,T]$ for all $T\in[0,T_0)$.*
*Moreover, one of the following alternatives holds true:*

1. *$T_0=+\infty$,*
2. *$\limsup_{t\to T_0}\|u(t)\|_{\mathbf{H}^1(\Omega_F(t))}+\|\tau(t)\|_{\mathbf{H}^1(\Omega_F(t))}+\dfrac{1}{\gamma(t)}=+\infty.$*

Our second result concerns the resolution of the Oldroyd model.

**Theorem 4.1.2 (Oldroyd model)** *Suppose that $\partial\mathcal{O}\in\mathcal{C}^3$, $f\in L^2_{loc}(0,\infty;\mathbf{H}^1(\mathcal{O}))$,*

*$f' \in L^2_{loc}(0,\infty;\mathbf{H}^{-1}(\mathcal{O}))$, $u_0 \in \mathbf{H}^2(\Omega_F)$, $\tau_0 \in \mathbf{H}^2(\Omega_F)$, $\gamma > 0$, and that*

$$\begin{array}{rcll} \nabla \cdot u_0 & = & 0, & \text{in } \Omega_F, \\ u_0(x) & = & h_i^1 + \omega_i^0 (x - h_i^0)^{\perp}, & x \in \partial B_i,\ \forall i \in \{1,\dots,k\}, \\ u_0(x) & = & 0, & x \in \partial\mathcal{O}. \end{array}$$

*Then there exists $T_0 > 0$ such that problem (4.1.1)-(4.1.10) admits a unique strong solution*

$$u \in \tilde{\mathcal{U}}(0,T,\Omega_F(t)),\ p \in L^2(0,T;H^2(\Omega_F(t))) \cap \mathcal{C}([0,T],\dot{H}^1(\Omega_F(t))),$$
$$\tau \in \tilde{\mathfrak{T}}(0,T;\Omega_F(t)),\ (h_i,\omega_i) \in W^{2,\infty}([0,T]\times\mathbb{R}^2) \times W^{1,\infty}([0,T]\times\mathbb{R}),$$

*for all $T \in [0,T_0)$.*

*Moreover, one of the following alternatives holds true:*

1. *$T_0 = +\infty$,*
2. *$\limsup_{t\to T_0} \|u(t)\|_{\mathbf{H}^2(\Omega_F(t))} + \|\tau(t)\|_{\mathbf{H}^2(\Omega_F(t))} + \dfrac{1}{\gamma(t)} = +\infty$.*

## 4.2 Existence of local solutions to the regularized Oldroyd model

The present section is devoted to prove Theorem 4.1.1. As already mentioned, the method used to prove existence of local unique strong solution to the regularized Oldroyd model is based on the application of the classical fixed point theorem to the mapping $\mathcal{N}$ defined by solving linearzed problems associated to the transformed problem in cylindrical domain. Let us first deal with two linear problems which will be the key ingredient to prove Theorem 4.1.1.

### 4.2.1 The linearized problem

We study here two linearized problems, one for the velocity $U$ and the other for the elastic extra-stress tensor $\mathcal{T}$. In the following, we endow the Banach space $\mathcal{U}(0,T;\Omega_F)$ with the norm

$$\|U\|_{\mathcal{U}(0,T;\Omega_F)} = \|U\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|U\|_{\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F))} + \|U\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))}$$

and the Banach space $\mathfrak{T}(0,T;\Omega_F)$ with the norm

$$\|\mathcal{T}\|_{\mathfrak{T}(0,T;\Omega_F)} = \|\mathcal{T}\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|\mathcal{T}\|_{\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F))} + \|\mathcal{T}\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))}.$$

Let $T$ be a positive real number, we consider the following linear problem associated to the velocity: Given initial data $u_0$, $(h_i^0, h_i^1, \omega_i^0)_{i=1,\dots,k}$ and source terms $F_0$, $F_{1,i}$ and $F_{2,i}$. Find $(U, P, (h_i, \omega_i)_{i=1,\dots,k})$ such that

$$\begin{cases} Re\,\partial_t U - (1-r)\Delta U + \nabla P = F_0, & \text{in } \Omega_F \times ]0,T], \\ \nabla \cdot U = 0, & \text{in } \Omega_F \times ]0,T], \\ U(y,t) = h_i'(t) + \omega_i(t)(y - h_i(0))^\perp, & (y,t) \in \partial B_i \times ]0,T],\ i = 1,\dots,k, \\ U(y,t) = 0, & (y,t) \in \partial\mathcal{O} \times ]0,T], \\ \overline{m}_i h_i''(t) = -\displaystyle\int_{\partial B_i} \overline{\Sigma}\nu_i d\Gamma_i + F_{1,i}, & t \in ]0,T],\ i \in \{1,\dots,k\}, \\ \overline{J}\omega_i'(t) = -\displaystyle\int_{\partial B_i} (y - h_i(0))^\perp \cdot \overline{\Sigma}\nu_i d\Gamma_i + F_{2,i}, & t \in ]0,T],\ i \in \{1,\dots,k\}, \\ U(y,0) = u_0(y), & y \in \Omega_F, \\ h_i(0) = h_i^0,\ \ h_i'(0) = h_i^1,\ \ \omega_i(0) = \omega_i^0, & i = 1,\dots,k, \end{cases} \tag{4.2.1}$$

where $\overline{\Sigma} = -PI + 2(1-r)D[U]$.

The proof of existence of solutions of system (4.2.1) is based on semi-group theory. For more details, we refer the reader to [40] where the authors studied a similar linear problem. In order to state the existence and uniqueness result of system (4.2.1), it is convenient to extend $u_0$ over the rigid disks by setting $u_0(x) = h_i^1 + \omega_i^0(y - h_i^0)^\perp$ over $B_i$. More specifically we have the following.

**Theorem 4.2.1** *Assume that $\partial\mathcal{O} \in \mathcal{C}^2$, $F_0 \in L^2(0,T;\mathbf{L}^2(\Omega_F))$, $F_{1,i} \in L^2(0,T;\mathbb{R}^2)$, $F_{2,i} \in L^2(0,T;\mathbb{R})$ for all $i = 1,\dots,k$, and $u_0 \in \mathbf{H}^1(\Omega_F)$ such that*

$$\begin{aligned} \nabla \cdot u_0 &= 0, && \text{in } \Omega_F, \\ u_0(y) &= h_i^1 + \omega_i^0(y - h_i^0)^\perp, && y \in B_i,\ \forall i = 1,\dots,k, \\ u_0(y) &= 0, && y \in \partial\mathcal{O}. \end{aligned}$$

*Then problem (4.2.1) admits a unique solution $(U, P, (h_i, \omega_i)_{i=1,\dots,k})$ with*

$$U \in \mathcal{U}(0,T;\Omega_F),\ P \in L^2(0,T;\dot{H}^1(\Omega_F)),\ \ h_i \in H^2(0,T;\mathbb{R}^2),\ \ \omega_i \in H^1(0,T;\mathbb{R}).$$

*Moreover, there exists a positive constant $K$ depending only on $\Omega_F$ and $T$; non-decreasing with respect to $T$, such that*

$$\begin{aligned} &\|U\|_{\mathcal{U}(0,T;\Omega_F)} + \|P\|_{L^2(0,T;\dot{H}^1(\Omega_F))} + \sum_{i=1}^k \|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})} \\ &\qquad\qquad \le K\big(\|u_0\|_{\mathbf{H}^1(\mathcal{O})} + \|F_0\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} + \sum_{i=1}^k \|F_{1,i}\|_{L^2(0,T;\mathbb{R}^2)} + \|F_{2,i}\|_{L^2(0,T;\mathbb{R})}\big). \end{aligned}$$

We turn now to study the linearized problem associated to the differential constitutive law of regularized Oldroyd model:

$$\begin{cases} We\dfrac{\partial \mathcal{T}}{\partial t} + \mathcal{T} - \varepsilon\Delta\mathcal{T} = \mathcal{G}, & \text{in } \Omega_F\times]0,T], \\ \varepsilon\dfrac{\partial \mathcal{T}}{\partial n} = 0, & \text{on } \partial\Omega_F\times]0,T], \\ \mathcal{T}(0) = \tau_0, & \text{in } \Omega_F, \end{cases} \tag{4.2.2}$$

where $\mathcal{G}$ and $\tau_0$ are given functions.

Concerning the resolution of the above system, we have the following proposition.

**Proposition 4.2.1** *Let $\mathcal{G} \in L^2(0,T;\mathbf{L}^2(\Omega_F))$ and $\tau_0 \in \mathbf{H}^1(\Omega_F)$ such that $\mathcal{G} = \mathcal{G}^T$ and $\tau_0^T = \tau_0$. Then problem (4.2.2) admits a unique solution $\mathcal{T} \in \mathfrak{T}(0,T;\Omega_F)$. Moreover, there exists a positive constant $K$ depending only on $\Omega_F$ and $T$; non-decreasing with respect to $T$, such that*

$$\|\mathcal{T}\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|\mathcal{T}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|\mathcal{T}\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} \le K\big(\|\tau_0\|_{\mathbf{H}^1(\Omega_F)} + \|\mathcal{G}\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))}\big).$$

The proof of the above proposition follows from semi-group theory and Theorem 3.4.3 in [3].

### 4.2.2 Proof of Theorem 4.1.1

We define the closed set $\mathcal{K}$ for $T$ and $R > 0$ as follows:

$$\mathcal{K} = \Big\{(W,Q,\mathcal{T},(h_i,\omega_i)_{i=1,\dots,k}) \in \mathcal{U}(0,T;\Omega_F)\times L^2(0,T;\dot{H}^1(\Omega_F))\times\mathfrak{T}(0,T;\Omega_F)\times\big(H^2(0,T;\mathbb{R}^2)\times H^1(0,T;\mathbb{R})\big)^k : \\ \|W\|_{\mathcal{U}(0,T;\Omega_F)} + \|Q\|_{L^2(0,T;\dot{H}^1(\Omega_F))} + \sum_{i=1}^{k}\|h_i^{''}\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega_i'\|_{L^2(0,T;\mathbb{R})} + \|\mathcal{T}\|_{\mathfrak{T}(0,T;\Omega_F)} \le R\Big\}.$$

For all $T > 0$, $\mathcal{K} \neq \emptyset$ if $R$ is sufficiently large say $R > \|u_0\|_{\mathbf{H}^1(\Omega_F)} + \|\tau_0\|_{\mathbf{H}^1(\Omega_F)}$. We choose $T$ such that the regularized Oldroyd model in $[0,T]$ is equivalent to the transformed problem (4.1.14)-(4.1.24) in cylindrical domain. We emphasize that this is equivalence always holds as long as $\gamma(t) > 0$, that is in absence of collision so that the fluid domain is non-singular. For simplicity, we assume that the external body force $f = 0$ throughout this chapter. Consider the mapping $\mathcal{N}$ defined from the set $\mathcal{K}$ into

$$\mathcal{U}(0,T;\Omega_F) \times L^2(0,T;\dot{H}^1(\Omega_F)) \times \mathfrak{T}(0,T;\Omega_F) \times \big(H^2(0,T;\mathbb{R}^2) \times H^1(0,T;\mathbb{R})\big)^k$$

as follows:

$$\mathcal{N}(W,Q,\mathcal{T},(h_i,\omega_i)_{i=1,\dots,k}) = (U,P,\tilde{\mathcal{T}},(\tilde{h}_i,\tilde{\omega}_i)_{i=1,\dots,k}),$$

where $(U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k})$ is the unique solution of system (2.3.1)-(4.2.2) with source terms:

$$F_0 = (1-r)[(L-\Delta)W] + [(\nabla - G)Q] - Re[MW] - Re[NW] + [\mathrm{div}\mathcal{T}], \tag{4.2.3}$$

$$F_{1,i} = -\int_{\partial B_i} \mathcal{T}\nu_i d\Gamma_i, \tag{4.2.4}$$

$$F_{2,i} = -\int_{\partial B_i} \mathcal{T}\nu_i \cdot (y - h_i^0)^{\perp} d\Gamma_i, \tag{4.2.5}$$

$$\mathcal{G} = 2r[\mathcal{D}W] - We([G_a(W,\mathcal{T})] + ((W + \partial_t Y)\cdot\nabla)\mathcal{T}) + \varepsilon[(\text{Ł} - \Delta)\mathcal{T}]. \tag{4.2.6}$$

The operators $[LU]$, $[MU]$, $[NU]$, $[GP]$, $[\mathrm{div}\,\mathcal{T}]$, $[\text{Ł}\mathcal{T}]$ and $[G_a(U,\mathcal{T})$ in (4.2.3) and (4.2.6) are given by:

$$[LU]_i = \sum_{j,k=1}^{2} \frac{\partial}{\partial y_j}(g^{jk}\frac{\partial U_i}{\partial y_k}) + 2\sum_{j,k,\ell=1}^{2} g^{k\ell}\Gamma^i_{j,k}\frac{\partial U_j}{\partial y_\ell} + \sum_{j,k,\ell=1}^{2}\left\{\frac{\partial}{\partial y_k}(g^{k\ell}\Gamma^i_{j,\ell}) + \sum_{m=1}^{2} g^{k\ell}\Gamma^m_{j,\ell}\Gamma^i_{k,m}\right\}U_j, \tag{4.2.7}$$

$$[MU]_i = \sum_{j=1}^{2}\frac{\partial Y_j}{\partial t}\frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^{2}\left\{\Gamma^i_{j,k}\frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y_j}\right\}U_j, \tag{4.2.8}$$

$$[NU]_i = \sum_{j=1}^{2} U_j\frac{\partial U_i}{\partial y_j} + \sum_{j,k=1}^{2}\Gamma^i_{j,k}U_jU_k, \tag{4.2.9}$$

$$[GP]_i = \sum_{j=1}^{2} g^{ij}\frac{\partial P}{\partial y_j}, \tag{4.2.10}$$

$$[\mathrm{div}\,\mathcal{T}]_{\mathrm{i}} = \sum_{k,\ell,m=1}^{2}\frac{\partial Y_i}{\partial x_k}\frac{\partial Y_m}{\partial x_\ell}\frac{\partial \mathcal{T}_{k\ell}}{\partial y_m}, \tag{4.2.11}$$

$$[\text{Ł}\mathcal{T}]_{ij} = \sum_{\ell,m=1}^{2} g^{\ell,m}\frac{\partial^2\mathcal{T}_{ij}}{\partial y_\ell\partial y_m} + \sum_{\ell=1}^{2}\Delta Y_\ell\frac{\partial \mathcal{T}_{ij}}{\partial y_\ell}, \tag{4.2.12}$$

$$[G_a(U,\mathcal{T})]_{ij} = \sum_{k=1}^{2}[\mathcal{W}U]_{ik}\mathcal{T}_{kj} - \mathcal{T}_{ik}[\mathcal{W}U]_{kj} - a\Big([\mathcal{D}U]_{ik}\mathcal{T}_{kj} + \mathcal{T}_{ik}[\mathcal{D}U]_{kj}\Big). \tag{4.2.13}$$

where

$$g^{ij} = \sum_{k=1}^{2}\frac{\partial Y_i}{\partial x_k}\frac{\partial Y_j}{\partial x_k},\ g_{ij} = \sum_{k=1}^{2}\frac{\partial X_k}{\partial y_i}\frac{\partial X_k}{\partial y_j},\ \Gamma^k_{i,j} = \frac{1}{2}\sum_{\ell=1}^{2} g^{k\ell}\left\{\frac{\partial g_{i\ell}}{\partial y_j} + \frac{\partial g_{j\ell}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_\ell}\right\}, \tag{4.2.14}$$

where $X(.,t)$ is a $\mathcal{C}^\infty$-diffeomorphism from $\Omega_F$ into $\Omega_F(t)$ for any $t \in [0,T]$ and $Y(.,t)$ denotes its inverse. Moreover, the operators $[\mathcal{D}U]$ and $[\mathcal{W}U]$ represent the symmetric and skew-symmetric part of $\nabla u$ with

$$(\nabla u)_{ij} = \sum_{\ell,m=1}^{2}\frac{\partial^2 X_i}{\partial y_\ell\partial y_m}\frac{\partial Y_\ell}{\partial x_j}U_m + \frac{\partial X_i}{\partial y_m}\frac{\partial Y_\ell}{\partial x_j}\frac{\partial U_m}{\partial y_\ell}. \tag{4.2.15}$$

We will see later that the source terms $F_0, F_1, F_2$ and $G$ are in the good spaces for applying Theorem 4.2.1 and Proposition 4.2.1, and hence the mapping $\tilde{\mathcal{N}}$ is well defined. A fixed point of $\mathcal{N}$ in $\mathcal{K}$ is clearly a solution of problem (4.1.14)-(4.1.24). First, we prove that for $R \gg N_C$ and $T$ small enough, we have so that $\mathcal{N}(\mathcal{K}) \subset \mathcal{K}$. Then, we show that for $T$ small enough, the mapping $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction. Hence, existence and uniqueness of strong solutions of problem (4.1.14)-(4.1.24) follows from the classical Picard's fixed pont theorem to the mapping $\mathcal{N}$ on the convex set $\mathcal{K}$. The conclusion of Theorem 4.1.1 is then obtained using the transform $X$.
In the sequel, we denote by $N_K$ and $N_C$ positive quantities which satisfy the following conditions:

i. $N_K$ is a positive function of $(h_i^1, \omega_i^0)_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^1(\Omega_F)}, \|\tau_0\|_{\mathbf{H}^1(\Omega_F)}, T$ and $R$ which is non-decreasing with respect to $T, R, \|u_0\|_{\mathbf{H}^1(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

ii. $N_C$ is a positive function of $(h_i^1, \omega_i^0, )_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^1(\Omega_F)}, r,$ and $T$ which is non-decreasing with respect to $T, \|u_0\|_{\mathbf{H}^1(\Omega_F)}, \|\tau_0\|_{\mathbf{H}^1(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

We are now in position to prove that for $T$ small enough and $R \gg N_C$, the mapping $\mathcal{N}$ maps $\mathcal{K}$ into itself. Let $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \mathcal{K}$, $F_0, F_{1,i}, F_{2,i}$ and $\mathcal{G}$ be the source terms defined in (4.2.3)-(4.2.6). We focus on the terms involved by the new elastic extra-stress tensor and we move fast on the other terms which are classical due to [40, p. 19-20]. Using Lemma A.1.1 and following similar arguments as in the former reference, we get that there exists a constant $N_K$ satisfying (i) such that

$$\|[(L-\Delta)W]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2}, \tag{4.2.16}$$
$$\|[MW]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2}, \tag{4.2.17}$$
$$\|[(\nabla - G)P]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T, \tag{4.2.18}$$
$$\|[NW]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/10}, \tag{4.2.19}$$

Moreover using Lemma A.1.1 and Holder's inequality, it follows that there exists a constant $N_K > 0$ satisfying (i), such that

$$\begin{aligned}\|[\mathrm{div}\mathcal{T}]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\leq N_K \|\mathcal{T}\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} \\ &\leq N_K T^{1/2} \|\mathcal{T}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))}.\end{aligned}$$

Since $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \mathcal{K}$, we get that

$$\|[\mathrm{div}\mathcal{T}]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2}. \tag{4.2.20}$$

It follows from (4.2.16)-(4.2.20) and the definition of $F_0$ in (4.2.3) that for small $T$, one has

$$\|F_0\|_{L^2(0,T,\mathbb{R}^2)} \leq N_K T^{1/10}.$$

Using the continuity of trace mapping and Holder's inequality, one has

$$\|F_1\|_{L^2(0,T,\mathbb{R}^2)} \leq N_K T^{1/2},$$
$$\|F_2\|_{L^2(0,T,\mathbb{R})} \leq N_K T^{1/2}.$$

By applying Theorem 4.2.1, there exists a constant $N_K$ satisfying (i) and a constant $N_C$ satisfying (ii) such that

$$\|U\|_{\mathcal{U}(0,T,\Omega_F)} + \|\nabla P\|_{L^2(0,T,\mathbf{L}^2(\Omega_F))} + \sum_{i=1}^{k} \|\tilde{h}_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})} \leq N_K T^{1/10} + N_C.$$

Using again Lemma A.1.1, Holder's inequality and classical Sobolev embeddings, we obtain

$$\|[(\text{Ł} - \Delta)\mathcal{T}]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T, \tag{4.2.21}$$
$$\|[(\partial_t Y \cdot \nabla)\mathcal{T}]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2}, \tag{4.2.22}$$
$$\|[(W \cdot \nabla)\mathcal{T}]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/10}, \tag{4.2.23}$$
$$\|[\mathcal{D}\mathcal{T}]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F(t)))} \leq N_K T^{1/2}, \tag{4.2.24}$$

We move now to bound $[G_a(W,\mathcal{T})]$ in $L^2(0,T;\mathbf{L}^2(\Omega_F))$. Lemma A.1.1 implies that

$$\|[\mathcal{W}W]_{ik}\mathcal{T}_{kj}\|^2_{L^2(\Omega_F)} \leq N_K T^2 \int_{\Omega_F} |W(y,t)|^2 |\mathcal{T}(y,t)|^2 dy + N_K \int_{\Omega_F} |\nabla W(y,t)|^2 |\mathcal{T}(y,t)|^2 dy.$$

By Holder's inequality and classical Sobolev embeddings, it follows that

$$\begin{aligned}\|[\mathcal{W}W]_{ik}\mathcal{T}_{kj}\|^2_{L^2(0,T;L^2(\Omega_F))} \leq N_K \Big\{ & T^2 \|W\|^2_{L^\infty(0,T,\mathbf{H}^1(\Omega_F))} \|\mathcal{T}\|^2_{L^2(0,T;\mathbf{H}^1(\Omega_F))} \\ & + T^{1/5} \Big( \|\mathcal{T}\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} \|W\|^{2/5}_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} \|W\|^{8/5}_{L^2(0,T;\mathbf{H}^2(\Omega_F))} \Big) \Big\}. \end{aligned} \tag{4.2.25}$$

Hence,

$$\|[\mathcal{W}W]_{ik}\mathcal{T}_{kj}\|_{L^2(0,T;L^2(\Omega_F))} \leq N_K T^{1/10}.$$

We estimate the other terms of $[G_a(W,\mathcal{T})]$ in similar way and we obtain

$$\|[G_a(W,\mathcal{T})]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F(t)))} \leq N_K T^{1/10}. \tag{4.2.26}$$

It follows from (4.2.21)-(4.2.24), (4.2.26) and the definition of the source term $\mathcal{G}$ in (4.2.6) that for small $T$, we have

$$\|\mathcal{G}\|_{L^2(0,T;\mathbf{L}^2(\Omega_F(t)))} \leq N_K T^{1/10}.$$

Therefore according to Proposition 4.2.1 there exists a constant $N_K$ satisfying (i) and a constant $N_C$ satisfying (ii) such that

$$\|\tilde{\mathcal{T}}\|_{\mathfrak{T}(0,T,\Omega_F)} \leq N_K T^{1/10} + N_C.$$

Choosing $R = 3N_C$, then one has

$$\|U\|_{\mathcal{U}(0,T,\Omega_F)} + \|\nabla P\|_{L^2(0,T,\mathbf{L}^2(\Omega_F))} + \sum_{i=1}^{k} \|\tilde{h}_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})} + \|\tilde{\mathcal{T}}\|_{\mathfrak{T}(0,T,\Omega_F)} \leq N_K T^{1/10} + \frac{2}{3}R.$$

It follows that for $T$ small enough

$$\|U\|_{\mathcal{U}(0,T,\Omega_F)} + \|\nabla P\|_{L^2(0,T,\mathbf{L}^2(\Omega_F))} + \sum_{i=1}^{k} \|\tilde{h}_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})} + \|\tilde{\mathcal{T}}\|_{\mathfrak{T}(0,T,\Omega_F)} \leq R,$$

and thus $(U, P, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\ldots,k}, \tilde{\mathcal{T}}) \in \mathcal{K}$. Therefore, $\mathcal{N}(\mathcal{K}) \subset \mathcal{K}$ holds by choosing $R \gg N_C$ and $T$ small enough.

We turn now to show that for $T$ small enough, the mapping $\mathcal{N} : \mathcal{K} \to \mathcal{K}$ is a contraction. Let $(W^1, Q^1, \mathcal{T}^1, (h_i^1, \omega_i^1)_{i=1,\ldots,k})$ and $(W^2, Q^2, \mathcal{T}^2, (h_i^2, \omega_i^2)_{i=1,\ldots,k})$ in $\mathcal{K}$. Denote by $Y^i, X^i, \Gamma_{j,\ell}^{ik}$, etc. the terms corresponding to $(W^i, Q^i, \mathcal{T}^i, (h_j^i, \omega_j^i)_{j=1,\ldots,k})$, with $i = 1, 2$ and by

$$(U^i, P^i, \tilde{\mathcal{T}}^i, (\tilde{h}_j^i, \tilde{\omega}_j^i)_{j=1,\ldots,k}) = \mathcal{N}(W^i, Q^i, \mathcal{T}^i, (h_j^i, \omega_j^i)_{j=1,\ldots,k}).$$

Also, we denote by $Y = Y^1 - Y^2$, $h_i = h_i^1 - h_i^2$, etc. We get that the difference $(U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\ldots,k})$ satisfies system (4.2.1)-(4.2.2) with zero initial data and source terms given as follows:

$$\begin{aligned}
F_0 &= (1-r)([(L^1 - \Delta)W] + [(L^1 - L^2)W^2]) + [(\nabla - G^1)Q] - [(G^1 - G^2)Q^2] - Re([M^1 W] \\
&\quad + [(M^1 - M^2)W^2] + [N^1 W^1] - [N^2 W^2]) + [\mathrm{div}^1 \mathcal{T}] + [(\mathrm{div}^1 - \mathrm{div}^2)\mathcal{T}^2], \\
\mathcal{G} &= 2r([\mathcal{D}^1 W] + [(\mathcal{D}^1 - \mathcal{D}^2)W^2]) - We\Big([G_a^1(W^1, \mathcal{T}^1)] - [G_a^2(W^2, \mathcal{T}^2)] - (W^1 \cdot \nabla)\mathcal{T}^1 \\
&\quad + (W^2 \cdot \nabla)\mathcal{T}^2 - ([(\partial_t Y^1 \cdot \nabla)\mathcal{T}]) + [(\partial_t Y \cdot \nabla)\mathcal{T}^2]\Big) + \varepsilon([(\text{Ł}^1 - \Delta)\mathcal{T}] + [(\text{Ł}^1 - \text{Ł}^2)\mathcal{T}^2]),
\end{aligned} \tag{4.2.27}$$

$$F_{1,i} = -\int_{\partial B_i} \mathcal{T}\nu_i d\Gamma_i, \tag{4.2.28}$$

$$F_{2,i} = -\int_{\partial B_i} \mathcal{T}\nu_i \cdot (y - h_i^0)^{\perp} d\Gamma_i. \tag{4.2.29}$$

Lemma A.1.2 implies that there exists a constant $N_K$ satisfying (i) such that

$$\begin{aligned}
\|[(L^1 - L^2)W^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\leq N_K T^{1/2} \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}, \\
\|[(M^1 - M^2)W^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\leq N_K T^{1/2} \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}, \\
\|[(G^1 - G^2)Q^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\leq N_K T^{1/2} \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}, \\
\|[N^1W^1] - [N^2W^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} &\leq N_K T^{1/10} \Big( \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)} + \|W\|_{\mathcal{U}(0,T,\Omega_F)} \Big).
\end{aligned} \tag{4.2.30}$$

We refer the reader to [40] for a detailed proof of similar estimates. It is not difficult to check that

$$\begin{aligned}
[(\mathrm{div}^1 - \mathrm{div}^2)\mathcal{T}^2]_i = \sum_{k,\ell,m=1}^{2} \Bigg\{ &\frac{\partial Y_i^1}{\partial x_k}(X^1)\Big(\frac{\partial Y_m}{\partial x_\ell}(X^1) + \frac{\partial Y_m^2}{\partial x_\ell}(X^1) - \frac{\partial Y_m^2}{\partial x_\ell}(X^2)\Big) \\
&+ \left(\frac{\partial Y_i}{\partial x_k}(X^1) + \frac{\partial Y_i^2}{\partial x_k}(X^1) - \frac{\partial Y_i^2}{\partial x_k}(X^2)\right) \frac{\partial Y_m^2}{\partial x_\ell}(X^2) \Bigg\} \frac{\partial \mathcal{T}_{k\ell}^2}{\partial y_m}.
\end{aligned}$$

By applying mean value theorem on the mapping $\dfrac{\partial Y_m^2}{\partial x_\ell}(x,.)$ and using Lemma A.1.1, we get

$$\Big\| \frac{\partial Y_m^2}{\partial x_\ell}(X^1) - \frac{\partial Y_m^2}{\partial x_\ell}(X^2) \Big\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq N_K \|X^1 - X^2\|_{L^\infty([0,T]\times\mathbb{R}^2)}.$$

Noting that the difference $X = X^1 - X^2$ satisfies $\|X\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq N_K T^{1/2} \sum_{i=1}^{k} \|h_i''(t)\|_{L^2([0,T]\times\mathbb{R}^2)}$, we obtain

$$\Big\| \frac{\partial Y_m^2}{\partial x_\ell}(X^1) - \frac{\partial Y_m^2}{\partial x_\ell}(X^2) \Big\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq N_K T^{1/2} \sum_{i=1}^{k} \|h_i''(t)\|_{L^2([0,T]\times\mathbb{R}^2)}$$

It follows from Lemma A.1.2 and the above estimate that there exists a constant $N_K$ satisfying (i) such that

$$\|[(\mathrm{div}^1 - \mathrm{div}^2)\,\mathcal{T}^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2} \sum_{i=1}^{k} \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}. \tag{4.2.31}$$

Combining the above estimate with (4.2.16)-(4.2.20) and (4.2.30), we get that for $T$ small

$$\|F_0\|_{L^2(0,T;\mathbf{L}^2(\Omega_F)} \leq N_K T^{1/10} \Big( \|W\|_{\mathcal{U}(0,T,\Omega_F)} + \|Q\|_{L^2(0,T,\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^{k} \|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})} \Big).$$

Moreover, by the continuity of trace theorem there exists a constant $C > 0$ depending on $\Omega_F$ such that

$$\|F_{1,i}\|_{L^2(0,T,\mathbb{R}^2)} \leq CT^{1/2}\|\mathcal{T}\|_{L^\infty(0,T,\mathbf{H}^1(\Omega_F))},$$
$$\|F_{2,i}\|_{L^2(0,T,\mathbb{R})} \leq CT^{1/2}\|\mathcal{T}\|_{L^\infty(0,T,\mathbf{H}^1(\Omega_F))}.$$

Theorem 4.2.1 implies that

$$\begin{aligned}\|U\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|U\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|U\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} + \|\nabla P\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))}\\ + \sum_{i=1}^k \|\tilde{h}_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})} \leq N_K T^{1/10}\Big(\|W\|_{\mathcal{U}(0,T,\Omega_F)} + \|Q\|_{L^2(0,T,\mathbf{H}^1(\Omega_F))}\\ \|\mathcal{T}\|_{\mathfrak{T}(0,T,\Omega_F)} + \sum_{i=1}^k \|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})}\Big).\end{aligned} \tag{4.2.32}$$

Moreover, it follows from Lemma A.1.2 that there exists a constant $N_K$ satisfying (i) such that

$$\|[(\text{Ł}^1 - \text{Ł}^2)\mathcal{T}^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2}\sum_{i=1}^k \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}, \tag{4.2.33}$$

$$\|[(\partial_t Y \cdot \nabla)\mathcal{T}^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2}\sum_{i=1}^k \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}, \tag{4.2.34}$$

$$\|[(\mathcal{D}^1 - \mathcal{D}^2)W^2]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/2}\sum_{i=1}^k \|h_i''\|_{L^2(0,T,\mathbb{R}^2)}, \tag{4.2.35}$$

$$\|(W^1 \cdot \nabla)\mathcal{T}^1 - (W^2 \cdot \nabla)\mathcal{T}^2\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/10}(\|W\|_{\mathcal{U}(0,T,\Omega_F)} + \|\mathcal{T}\|_{\mathfrak{T}(0,T,\Omega_F)}). \tag{4.2.36}$$

We collect the terms of the difference $[G_a(W^1, \mathcal{T}^1)] - [G_a(W^2, \mathcal{T}^2)]$ as follows:

$$\begin{aligned}[G_a^1(W^1, \mathcal{T}^1)]_{ij} - [G_a^2(W^2, \mathcal{T}^2)]_{ij} = ([\mathcal{W}^1 W^1]_{ik}\mathcal{T}^1_{kj} - [\mathcal{W}^2 W^2]_{ik}\mathcal{T}^2_{kj}) + (\mathcal{T}^1_{ik}[\mathcal{W}^1 W^1]_{kj} - \mathcal{T}^2_{ik}[\mathcal{W}^2 W^2]_{kj})\\ - a\big([\mathcal{D}^1 W^1]_{ik}\mathcal{T}^1_{kj} - [\mathcal{D}^2 W^2]_{ik}\mathcal{T}^2_{kj}\big) - a\big(\mathcal{T}^1_{ik}[\mathcal{D}^1 W^1]_{kj} - \mathcal{T}^2_{ik}[\mathcal{D}^2 W^2]_{kj}\big).\end{aligned} \tag{4.2.37}$$

We rewrite

$$[\mathcal{W}^1 W^1]_{ik}\mathcal{T}^1_{kj} - [\mathcal{W}^2 W^2]_{ik}\mathcal{T}^2_{kj} = [\mathcal{W}^1 W]_{ik}\mathcal{T}^1_{kj} + [\mathcal{W}^1 W^2]_{ik}\mathcal{T}_{kj} + [(\mathcal{W}^1 - \mathcal{W}^2)W^2]_{ik}\mathcal{T}^2_{kj}. \tag{4.2.38}$$

Returning to inequality (4.2.25), we obtain:

$$\|[\mathcal{W}^1 W]_{ik}\mathcal{T}^1_{kj}\|_{L^2(0,T;L^2(\Omega_F))} \leq N_K T^{1/10}\|W\|_{\mathcal{U}(0,T,\Omega_F)}, \tag{4.2.39}$$
$$\|[\mathcal{W}^1 W^2]_{ik}\mathcal{T}_{kj}\|_{L^2(0,T;L^2(\Omega_F))} \leq N_K T^{1/10}\|\mathcal{T}\|_{\mathfrak{T}(0,T,\Omega_F)}. \tag{4.2.40}$$

By Lemma A.1.2 and using classical Sobolev embedding, we have

$$\|[(\mathcal{W}^1 - \mathcal{W}^2)W^2]_{ik}\mathcal{T}^2_{kj}\|_{L^2(0,T;L^2(\Omega_F))} \leq N_K T^{1/2}\sum_{i=1}^{k}\|h_i^{''}(t)\|_{L^2(0,T;\mathbb{R}^2)}.$$

Combining the above estimate with the estimates in (4.2.39) and (4.2.40), we get

$$\|[\mathcal{W}^1W^1]_{ik}\mathcal{T}^1_{kj} - [\mathcal{W}^2W^2]_{ik}\mathcal{T}^2_{kj}\|_{L^2(0,T;L^2(\Omega_F))} \leq N_K T^{1/10}\Big(\|W\|_{\mathcal{U}(0,T;\Omega_F)} + \|\mathcal{T}\|_{\mathcal{T}(0,T;\Omega_F)} + \sum_{i=1}^{k}\|h_i^{''}(t)\|_{L^2(0,T;\mathbb{R}^2)}\Big).$$

By estimating the other terms in (4.2.37) in a similar manner, we obtain

$$\|[G_a^1(W^1,\mathcal{T}^1)] - [G_a^2(W^2,\mathcal{T}^2)]\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} \leq N_K T^{1/10}\Big(\|W\|_{\mathcal{U}(0,T;\Omega_F)} + \|\mathcal{T}\|_{\mathfrak{T}(0,T;\Omega_F)} + \sum_{i=1}^{k}\|h_i^{''}(t)\|_{L^2(0,T;\mathbb{R}^2)}\Big).$$

Combining the above inequality with (4.2.21)-(4.2.24), (4.2.26), (4.2.33)-(4.2.33), it follows that

$$\|\mathcal{G}\|_{\mathfrak{T}(0,T,\Omega_F)} \leq N_K T^{1/10}\Big(\|W\|_{\mathcal{U}(0,T;\Omega_F)} + \|\mathcal{T}\|_{\mathcal{T}(0,T;\Omega_F)} + \sum_{i=1}^{k}\|h_i^{''}(t)\|_{L^2(0,T;\mathbb{R}^2)}\Big).$$

According to Theorem 4.2.1, there exists a constant $N_K$ satisfying (i) such that

$$\|\tilde{\mathcal{T}}\|_{\mathfrak{T}(0,T,\Omega_F)} \leq N_K T^{1/10}\Big(\|W\|_{\mathcal{U}(0,T;\Omega_F)} + \|\mathcal{T}\|_{\mathcal{T}(0,T;\Omega_F)} + \sum_{i=1}^{k}\|h_i^{''}(t)\|_{L^2(0,T;\mathbb{R}^2)}\Big).$$

Gathering the above inequality with inequality (4.2.32), we get that

$$\begin{aligned}
&\|U\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|U\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|U\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} + \|\nabla P\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))} + \|\tilde{\mathcal{T}}\|_{\mathfrak{T}(0,T,\Omega_F)} \\
&\quad + \sum_{i=1}^{k}\|\tilde{h}_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{H^1(0,T;\mathbb{R})} \leq N_K T^{1/10}\Big(\|W\|_{\mathcal{U}(0,T,\Omega_F)} + \|Q\|_{L^2(0,T,\mathbf{H}^1(\Omega_F))} \\
&\qquad \|\mathcal{T}\|_{\mathfrak{T}(0,T,\Omega_F)} + \sum_{i=1}^{k}\|h_i\|_{H^2(0,T;\mathbb{R}^2)} + \|\omega_i\|_{H^1(0,T;\mathbb{R})}\Big).
\end{aligned}$$

It follows that for $T$ small enough, the mapping $\mathcal{N}$ is a contraction mapping. This completes the proof. □

## 4.3 Local existence to standard Oldroyd model

In this section, we show the existence and uniqueness of strong solutions locally in time of the Oldroyd model. The main difficulty here is that the Oldroyd differential law exhibits hyperbolic behaviour and hence the study of strong solutions to the coupled problem requires one to work with velocity fields $u$ that have Sobolev regularity at least in $L^1(0,T,\mathbf{H}^3(\Omega_F(t)))$. We are then enforced to start with more regular data to improve the regularity of the transport coefficient $u$.

First, we start by improving the regularity result in Theorem 4.2.1. To achieve this, we extend the velocity field $U$ over the rigid disks by setting

$$U(y,t) = h_i'(t) + \omega_i(t)(y-h_i^0)^\perp, \text{if } y \in B_i,\ i \in \{1,\dots,k\}.$$

It is thus natural to define the following spaces:

$$\begin{aligned}
H &= \{\phi \in \mathbf{L}^2(\mathcal{O}) : \nabla\cdot\phi = 0 \text{ in } \mathcal{O},\ D[\phi] = 0 \text{ in } B_i,\ i = 1,\dots,k,\ \ \phi\cdot n = 0 \text{ on } \partial\mathcal{O}\},\\
V &= \{\phi \in \mathbf{H}^1_0(\mathcal{O}) : \nabla\cdot\phi = 0 \text{ in } \mathcal{O},\ D[\phi] = 0 \text{ in } B_i,\ i = 1,\dots,k\},\\
D(A) &= \{\phi \in \mathbf{H}^1_0(\mathcal{O}) : \phi|_{\Omega_F} \in \mathbf{H}^2(\Omega_F),\ \ \nabla\cdot\phi = 0 \text{ in } \mathcal{O},\ D[\phi] = 0 \text{ in } B_i,\ \forall i = 1,\dots,k\}.
\end{aligned}$$

According to Lemma 1.1 in [44], for any $\phi \in H$, there exists $(V_{\phi,i},\omega_{\phi,i}) \in \mathbb{R}^2\times\mathbb{R}$ such that

$$\phi(y) = V_{\phi,i} + \omega_{\phi,i}(y-h_i^0)^\perp, \text{ in } B_i,\ \ i \in \{1,\dots,k\}.$$

The space $H$ is equipped with the scalar product

$$(\phi,\psi)_H = Re\int_{\Omega_F}\phi\cdot\psi\,dy + \sum_{i=1}^k \overline{m}_i V_{\phi,i}\cdot V_{\psi,i} + \overline{J}_i\omega_{\phi,i}\,\omega_{\psi,i},$$

and the space $V$ with

$$(\phi,\psi)_V = Re\int_{\Omega_F}\phi\cdot\psi\,dy + \int_{\Omega_F}\nabla\phi\cdot\nabla\psi\,dy + \sum_{i=1}^k \overline{m}_i V_{\phi,i}\cdot V_{\psi,i} + \overline{J}_i\omega_{\phi,i}\,\omega_{\psi,i}.$$

We denote by $V^*$ the dual space of $V$ and we equip the Banach space $V^*$ with the norm

$$\|F\|_{V^*} = \sup_{\|\phi\|_V=1} |\langle F,\psi\rangle_{V^*,V}|.$$

We recall that if $(U,P)$ is a solution of problem (2.3.1), then $U$ is a solution of the following

Cauchy problem [41]:

$$\begin{cases} U' + AU &= \mathbb{P}F, \\ U(0) &= U_0, \end{cases} \tag{4.3.1}$$

where operator $A : D(A) \to H$, such that

$$AU = \mathbb{P}\mathcal{A}U, \ \ \forall U \in D(A), \tag{4.3.2}$$

with $\mathbb{P}$ denotes the orthogonal projection from $\mathbf{L}^2(\mathcal{O})$ onto $H$ in $\|.\|_{\mathbf{L}^2(\mathcal{O})}$, and

$$\mathcal{A}U = \begin{cases} -\dfrac{1-r}{Re}\Delta U, & \text{in } \Omega_F, \\ \dfrac{2(1-r)}{\overline{m}_i}\displaystyle\int_{\partial B_i} D[U]\nu_i d\Gamma_i + \left(\dfrac{2(1-r)}{\overline{J}_i}\displaystyle\int_{\partial B_i}(y-h_i^0)^\perp \cdot D[U]\nu_i d\Gamma_i\right)(y-h_i^0)^\perp, & \text{on } B_i, \end{cases} \tag{4.3.3}$$

and

$$F = \begin{cases} \dfrac{1}{Re}F_0, & \text{in } \Omega_F, \\ \dfrac{1}{\overline{m}_i}F_{1,i} + \dfrac{1}{\overline{J}_i}F_{2,i}(y-h_i^0)^\perp, & \text{on } B_i, \forall i \in \{1,\dots,k\}. \end{cases} \tag{4.3.4}$$

Moreover, the initial velocity $U_0$ is defined as follows:

$$U_0(x) = u_0(x)1_{\Omega_F}(x) + \sum_{i=1}^{k}(h_i^1 + \omega_i^0(y-h_i^0)^\perp)1_{B_i}(x). \tag{4.3.5}$$

In the following we show that a solution of (2.3.1) is more regular, if the data are more regular.

**Proposition 4.3.1** *Suppose that $\partial\mathcal{O} \in \mathcal{C}^3$, $F \in \mathcal{C}([0,T];\mathbf{L}^2(\mathcal{O}))$, $F_{1,i} \in L^2(0,T;\mathbb{R}^2))$, $F_{2,i} \in L^2(0,T;\mathbb{R}))$, for all $i = 1,\dots,k$, $F_0 \in L^2(0,T;\mathbf{H}^1(\Omega_F)), F' \in L^2(0,T;V^*)$ and $u_0 \in \mathbf{H}^2(\Omega_F)$ such that*

$$\begin{aligned} \nabla\cdot u_0 &= 0, && \text{in } \Omega_F, \\ u_0(y) &= h_i^1 + \omega_i^0(y-h_i^0)^\perp, && y \in B_i, \ \forall i = 1,\dots,k, \\ u_0(y) &= 0, && y \in \partial\mathcal{O}. \end{aligned}$$

*Then problem (2.3.1) admits a unique solution $(U, P, (h_i,\omega_i)_{i\in\{1,\dots,k\}})$ with*

$$U \in \tilde{\mathcal{U}}(0,T,\Omega_F), \ P \in L^2(0,T;H^2(\Omega_F)\cap \dot{H}^1(\Omega_F)), \ \nabla P \in \mathcal{C}([0,T],\mathbf{L}^2(\Omega_F)),$$
$$\partial_t\nabla P \in L^2(0,T;\mathbf{H}^{-1}(\Omega_F)), \ h_i \in W^{2,\infty}(0,T;\mathbb{R}^2), \ \omega_i \in W^{1,\infty}(0,T;\mathbb{R}^2).$$

*Moreover, there exists a positive constant $K$ depending only on $\Omega_F$ and $T$; non-decreasing with respect to $T$, such that*

$$
\begin{aligned}
&\|U\|_{\tilde{\mathcal{U}}(0,T;\Omega_F)} + \|P\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|\nabla P\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_F))} + \|\partial_t \nabla P\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega_F))}\\
&+\sum_{i=1}^{k} \|h_i\|_{W^{2,\infty}(0,T;\mathbb{R}^2)} + \|\omega_i\|_{W^{1,\infty}(0,T;\mathbb{R})} \leq K\Big(\|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^{k} \|F_{1,i}\|_{L^2(0,T;\mathbb{R}^2)} + \|F_{2,i}\|_{L^2(0,T;\mathbb{R})}\\
&\qquad + \|F'\|_{L^2(0,T;V^*)} + \|AU(0)\|_H + \|\mathbb{P}F(0)\|_H\Big).
\end{aligned}
$$

Proof. Consider the following Cauchy problem:

$$
\begin{cases}
Z' + AZ &= \mathbb{P}F',\\
Z(0) &= Z_0,
\end{cases}
\tag{4.3.6}
$$

where $Z_0 = \mathbb{P}F(0) - AU_0$ with $U_0$ is defined as in (4.3.5).

By noting that $U_0 \in D(A)$ and $F \in \mathcal{C}([0,T];\mathbf{L}^2(\mathcal{O}))$, we get that $AU_0$ and $\mathbb{P}F$ are well defined and belongs to $H$ and hence $Z_0 \in H$.

Suppose that $Z$ is a classical solution of problem (4.3.6), then we get by continuity and density that

$$
\frac{d}{dt}(Z(t),Y)_H + 2(1-r)\int_{\mathcal{O}} D[Z(t)] : D[Y]dy = \langle F'(t), Y\rangle_{V^*,V}, \ \ \forall\, Y \in V. \tag{4.3.7}
$$

This leads to the following weak formulation:

For $F' \in L^2(0,T;V^*)$ and $Z_0 \in H$, find $Z \in L^2(0,T;V)$ such that $Z(0) = Z_0$ and

$$
\frac{d}{dt}(Z(t),Y)_H + 2(1-r)\int_{\mathcal{O}} D[Z(t)] : D[Y]dy = \frac{d}{dt}(F(t),Y)_H, \ \ \forall\, Y \in V. \tag{4.3.8}
$$

It is important to point out that it is not clear whether $Z(0)$ has sense if $Z$ has the above regularity.

Consider the mapping $A$ defined as follows

$$
\begin{aligned}
A: \ V &\to V^*\\
Z &\to AZ: \ V \to \mathbb{R}\\
&\qquad\qquad Y \to \langle AZ, Y\rangle_{V^*,V} = 2(1-r)\int_{\mathcal{O}} D[Z] : D[Y]dy.
\end{aligned}
$$

The mapping $A$ is a linear and continuous mapping from $V$ into $V^*$. Indeed for $Z \in V$ one has

$$
\|AZ\|_{V^*} = \sup_{\|Y\|_V \leq 1} |\langle AZ, Y\rangle_{V^*,V}| \leq C\|Z\|_V
$$

Moreover, if $Z \in L^2(0,T,V)$, then $AZ \in L^2(0,T,V^*)$ and hence $\mathbb{P}F' - AZ \in L^2(0,T,V^*)$. Furthermore, the weak formulation (4.3.8) implies that

$$\Big\langle \frac{dZ}{dt}, Y\Big\rangle_{V^*,V} = \langle F' - AZ, Y\rangle_{V^*,V}. \tag{4.3.9}$$

Consequently, we get $Z' \in L^2(0,T,V^*)$ with

$$\|Z'\|_{L^2(0,T;V^*)} \leq C\Big(\|Z\|_{L^2(0,T;V)} + \|F'\|_{L^2(0,T;V^*)}\big). \tag{4.3.10}$$

It follows that $Z \in E_{2,2} = \{u \in L^2(0,T;V) \ : \ u' \in L^2(0,T;V^*)\}$. Using the fact that $E_{2,2} \hookrightarrow \mathcal{C}([0,T],H)$ (see, for instance, [8]), we get that $Z \in \mathcal{C}([0,T],H)$ and thus $Z(0)$ has sense in (4.3.8). Therefore, problem (4.3.6) is equivalent to:

For $F' \in L^2(0,T;V^*)$ and $Z_0 \in H$, find $Z \in L^\infty(0,T;H) \cap L^2(0,T;V)$ such that $Z(0) = Z_0$ and

$$\frac{d}{dt}(Z(t),Y)_H + 2(1-r)\int_{\mathcal{O}} D[Z(t)] : D[Y]dy = \langle F'(t), Y\rangle_{V^*,V}, \ \ \forall\, Y \in V. \tag{4.3.11}$$

Since the space $V$ is separable, we use Galerkin method to prove that problem (4.3.11) admits a unique solution $Z \in \mathcal{C}([0,T],H) \cap L^2(0,T);V)$ such that

$$\|Z\|_{L^\infty([0,T],H)} + \|Z\|_{L^2(0,T;V)} \leq C\|F'\|_{L^2(0,T;V^*)} + \|Z_0\|_H).$$

Combining the above inequality with (4.3.10), it follows that

$$\|Z'\|_{L^2(0,T,V^*)} + \|Z\|_{L^\infty([0,T],H)} + \|Z\|_{L^2(0,T;V)} \leq C\big(\|F'\|_{L^2(0,T;V^*)} + \|Z_0\|_H\big). \tag{4.3.12}$$

We set $U(t) = \displaystyle\int_0^t Z(s)ds + U_0$. It follows that

$$\begin{aligned} U'(t) + AU(t) - \mathbb{P}F(t) \quad &= Z(t) + A\int_0^t Z(s)ds + AU_0 - \mathbb{P}\int_0^t \partial_t F(s)ds - \mathbb{P}F(0) \\ &= \int_0^t \Big(Z'(s) + AZ(s) - \mathbb{P}F'(s)\Big)ds + Z_0 + AU_0 - \mathbb{P}F(0). \end{aligned}$$

It follows from (4.3.6) that

$$U'(t) + AU(t) - \mathbb{P}F(t) = 0.$$

Hence, $U$ is the primitive of $Z$ and solves (4.3.1). Therefore, (4.3.12) implies that

$$U' \in \mathcal{C}([0,T],H) \cap L^2(0,T);V), \;\; U'' \in L^2(0,T,V^*),$$

and satisfies

$$\|U''\|_{L^2(0,T,V^*)} + \|U'\|_{L^\infty([0,T],H)} + \|U'\|_{L^2(0,T;V)} \leq C(\|AU_0\|_H + \|F'\|_{L^2(0,T;V^*)} + \|\mathbb{P}F(0)\|_H). \tag{4.3.13}$$

Consequently, we get

$$U' \in L^2(0,T;\mathbf{H}^1(\Omega_F)) \cap \mathcal{C}([0,T],\mathbf{L}^2(\Omega_F)), \; h_i \in W^{2,\infty}(0,T;\mathbb{R}^2), \; \omega_i \in W^{1,\infty}(0,T;\mathbb{R}),$$

and

$$\|U'\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_F))} + \|U'\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^{k} \|h_i''\|_{W^{2,\infty}(0,T;\mathbb{R}^2)} + \|\omega_i'\|_{W^{1,\infty}(0,T;\mathbb{R})}$$
$$\leq K\Big(\|F'\|_{L^2(0,T;V^*)} + \|AU_0\|_H + \|\mathbb{P}F(0)\|_H\Big). \tag{4.3.14}$$

Moreover, we find

$$\begin{cases} -(1-r)\Delta U + \nabla P = F_0 - Re\dfrac{\partial U}{\partial t} \in L^2(0,T;\mathbf{H}^1(\Omega_F)), & \text{in } \Omega_F \times ]0,T], \\ \nabla \cdot U = 0, & \text{in } \Omega_F \times ]0,T], \\ U(y,t) = h_i'(t) + \omega_i(t)(y - h_i(0))^\perp, & (y,t) \in \partial B_i \times \in ]0,T], \; i \in \{1,\dots,k\}, \\ U(y,t) = 0, & (y,t) \in \partial\mathcal{O} \times ]0,T]. \end{cases}$$

Using the regularity results of the steady Stokes equations [43], we deduce that $U \in L^2(0,T;\mathbf{H}^3(\Omega_F))$, $P \in L^2(0,T,H^2(\Omega_F))$ such that $\displaystyle\int_{\Omega_F} P(y)dy = 0$ and

$$\|U\|_{L^2(0,T;\mathbf{H}^3(\Omega_F))} + \|P\|_{L^2(0,T,H^2(\Omega_F))} \leq C\Big(\|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \|U'\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))}$$
$$+ \sum_{i=1}^{k} \|h_i\|_{H^1(0,T;\mathbb{R}^2)} + \|\omega_i\|_{L^2(0,T;\mathbb{R}^2)}\Big),$$

where $C$ is a positive constant that depends on $r$ and $\Omega_F$.

Thanks to (4.3.14) and Theorem 4.2.1,

$$\|U\|_{L^2(0,T;\mathbf{H}^3(\Omega_F))} + \|P\|_{L^2(0,T,H^2(\Omega_F))} \leq K\Big(\|\mathbb{P}F\|_{L^2(0,T;V)} + \|F'\|_{L^2(0,T;V^*)} + \|\mathbb{P}F(0)\|_H + \|AU_0\|_H\Big), \tag{4.3.15}$$

where $K$ is a positive constant that depends on $r, \Omega_F$ and $T$.
Noting that $U \in L^2(0,T;\mathbf{H}^3(\Omega_F))$ with $U' \in L^2(0,T;\mathbf{H}^1(\Omega_F))$, it follows that $U \in \mathcal{C}([0,T],\mathbf{H}^2(\Omega_F))$ and there exists a positive constant $K$ depending on $r, \Omega_F$ and $T$, such that

$$\|U\|_{L^\infty([0,T];\mathbf{H}^2(\Omega_F))} \leq K\big(\|U\|_{L^2(0,T;\mathbf{H}^3(\Omega_F))} + \|U'\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \|U_0\|_H\big).$$

By Proposition 5.3 in [40], we have $\|U\|_{\mathbf{H}^2(\Omega_F)} \leq C\|AU\|_{\mathbf{L}^2(\Omega_F)}$, where $C$ is a positive constant. This implies that

$$\|U\|_{L^\infty([0,T];\mathbf{H}^2(\Omega_F))} \leq K\big(\|U\|_{L^2(0,T;\mathbf{H}^3(\Omega_F))} + \|U'\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \|AU_0\|_H\big).$$

Thanks again to (4.3.14) and (4.3.15), we get

$$\|U\|_{L^\infty([0,T];\mathbf{H}^2(\Omega_F))} \leq K\big(\|\mathbb{P}F\|_{L^2(0,T;V)} + \|F'\|_{L^2(0,T;V^*)} + \|\mathbb{P}F(0)\|_H + \|AU_0\|_H\big).$$

By noting that $\nabla P = F_0 - Re\, U' + (1-r)\Delta U$ and (4.3.14), we get that

$$\nabla P \in \mathcal{C}([0,T],\mathbf{L}^2(\Omega_F)),\ \ \partial_t \nabla P \in L^2(0,T;\mathbf{H}^{-1}(\Omega_F)).$$

Consequently we obtain

$$\begin{aligned}
\|\nabla P\|_{L^\infty([0,T];\mathbf{L}^2(\Omega_F))} &\leq K\big(\|F\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \|F'\|_{L^2(0,T;V^*)} + \|\mathbb{P}F(0)\|_H + \|AU_0\|_H\big),\\
\|\partial_t \nabla P\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega_F))} &\leq K\big(\|F'\|_{L^2(0,T;V^*)} + \|\mathbb{P}F(0)\|_H + \|AU_0\|_H\big).
\end{aligned}$$

Since $\displaystyle\int_{\Omega_F} P(y)dy = 0$, then by Proposition 1.2 in [43] we get that

$$P \in \mathcal{C}([0,T],\dot{H}^1(\Omega_F)),\ \ \partial_t P \in L^2(0,T;L^2(\Omega_F)),$$

and

$$\begin{aligned}
\|P\|_{L^\infty([0,T];H^1(\Omega_F)/\mathbb{R})} &\leq K\big(\|F\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \|F'\|_{L^2(0,T;V^*)} + \|\mathbb{P}F(0)\|_H + \|AU_0\|_H\big),\\
\|\partial_t P\|_{L^2(0,T;L^2(\Omega_F)/\mathbb{R})} &\leq K\big(\|F'\|_{L^2(0,T;V^*)} + \|\mathbb{P}F(0)\|_H + \|AU_0\|_H\big).
\end{aligned}$$

This ends the proof. $\square$

We proceed by defining the mapping $\tilde{\mathcal{N}}$ for which a solution to problem (4.1.14)-(4.1.21),(4.1.25)-(4.1.26) is a fixed point of $\tilde{\mathcal{N}}$. For $T > 0, R > 0$ and $R' > 0$, we define the set

$$\tilde{\mathcal{K}} = \Big\{(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{S}}(0, T, \Omega_F) : \|W\|_{\tilde{\mathcal{U}}(0,T;\Omega_F)} + \|\mathcal{T}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega_F))} + \|Q\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))}$$
$$+ \|\nabla Q\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_F))} + \|\partial_t \nabla Q\|_{L^2([0,T];\mathbf{H}^{-1}(\Omega_F))} + \sum_{i=1}^{k} \|h_i''\|_{L^\infty([0,T]\times\mathbb{R}^2)} + \|\omega_i'\|_{L^\infty([0,T]\times\mathbb{R})} \leq R,$$
$$\|\mathcal{T}'\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} \leq R'\Big\},$$

where

$$\tilde{\mathcal{S}}(0, T; \Omega_F) := \tilde{\mathcal{U}}(0, T; \Omega_F) \times L^2(0, T; H^2(\Omega_F)) \times \tilde{\mathfrak{T}}(0, T; \Omega_F) \times \left(W^{2,\infty}(0, T; \mathbb{R}^2) \times W^{1,\infty}(0, T; \mathbb{R})\right)^k.$$

We note that if $R$ is large enough, then $\tilde{\mathcal{K}} \neq \emptyset$ for all $T > 0$ (see for instance, [22]). Moreover, by Aubin-Simon's theorem [8], the set $\tilde{\mathcal{K}}$ is convex, closed and compact in $X_T$, where

$$X_T = \mathcal{C}([0, T], \mathbf{H}^1(\Omega_F)) \times \mathcal{C}([0, T], L^2(\Omega_F)) \times \mathcal{C}([0, T], \mathbf{H}^1(\Omega_F)) \times \left(\mathcal{C}^1([0, T], \mathbb{R}^2) \times \mathcal{C}([0, T], \mathbb{R})\right)^k.$$

Consider the mapping

$$\begin{aligned} \tilde{\mathcal{N}} : \quad & \tilde{\mathcal{K}} && \to X_T \\ & (W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) && \to (U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}), \end{aligned}$$

where $(U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k})$ satisfies equations (2.3.1), (4.2.3)-(4.2.5) and the following transport equation:

$$\frac{\partial \tilde{\mathcal{T}}}{\partial t} + \Big(\Big(W + \frac{\partial Y}{\partial t}\Big) \cdot \nabla\Big)\tilde{\mathcal{T}} = -[G_a(W, \tilde{\mathcal{T}})] - \frac{1}{We}\tilde{\mathcal{T}} + \frac{2r}{We}[\mathcal{D}W], \quad \text{in } \Omega_F \times ]0, T[, \tag{4.3.16}$$
$$\tilde{\mathcal{T}}(y, 0) = \tau_0(y), \quad y \in \Omega_F. \tag{4.3.17}$$

For the rest of the paper, we use Einstein convention for summation. Moreover, we denote by $K_0$ and $C_0$ which satisfy the following assertions:

i. $K_0$ is a positive function of $(h_i^1, \omega_i^0)_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^2(\Omega_F)}, \|\tau_0\|_{\mathbf{H}^2(\Omega_F)}, T$ and $R$ which is non-decreasing with respect to $T, R, \|u_0\|_{\mathbf{H}^2(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

ii. $C_0$ is a positive function of $(h_i^1, \omega_i^0, )_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^2(\Omega_F)}$, and $T$ which is non-decreasing with respect to $T$, $\|u_0\|_{\mathbf{H}^2(\Omega_F)}$, $\|\tau_0\|_{\mathbf{H}^2(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\ldots,k}$.

Before proceeding, we need some results which will be used in the course of proof Theorem 4.1.2. It can be shown using Lemma A.1.3 the following proposition:

**Proposition 4.3.2** *Suppose that $(W, Q, \mathcal{T}, (\xi_i, \omega_i)_{i=1,\ldots,k}) \in \tilde{\mathcal{K}}$, then there exists a constant $K_0$ satisfying (i) such that*

$$\begin{aligned}
\|[(L-\Delta)W]\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} &\leq K_0 T^{1/2},\\
\|[MW]\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} &\leq K_0 T^{1/2},\\
\|[(\nabla - G)P]\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} &\leq K_0 T,\\
\|[NW]\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} &\leq K_0 T^{1/2},\\
\|[\mathrm{div}\, \mathcal{T}]\|_{\mathrm{L}^2(0,\mathrm{T};\mathbf{H}^1(\Omega_\mathrm{F}))} &\leq \mathrm{K}_0 \mathrm{T}^{1/2},\\
\|F_{1,i}\|_{L^2(0,T,\mathbb{R}^2)} &\leq K_0 T^{1/2},\\
\|F_{2,i}\|_{L^2(0,T,\mathbb{R})} &\leq K_0 T^{1/2}.
\end{aligned}$$

Proof. By Lemma A.1.3, one can prove easily that the terms $[(L-\Delta)W], [MW], [\mathrm{div}\, \mathcal{T}], \mathrm{F}_{1,\mathrm{i}}$ and $F_{2,i}$ are bounded by $K_0 T^{1/2}$ in $L^2(0,T;\mathbf{L}^2(\Omega_F))$. Using similar arguments as in [40], we get that $[(\nabla - G)P]$ is bounded by $K_0 T$ in $L^2(0,T;\mathbf{L}^2(\Omega_F))$, and $[NW]$ is bounded by $K_0 T^{1/10}$ in $L^2(0,T;\mathbf{L}^2(\Omega_F))$. Hence, to complete proof of the above estimates, it remains to bound the first order derivatives in space of these operators.

Let $p \in \{1,2\}$. We start by computing $\partial y_p[(L-\Delta)W]_i$:

$$\begin{aligned}
\partial y_p[(L-\Delta)W]_i = {}& \frac{\partial g^{jk}}{\partial y_p}\frac{\partial^2 W_i}{\partial y_j \partial y_k} + (g^{jk} - \delta_k^j)\frac{\partial^3 W_i}{\partial y_j \partial y_k \partial y_p} + \frac{\partial^2 g^{jk}}{\partial y_p \partial y_j}\frac{\partial W_j}{\partial y_k} + \frac{\partial (g^{jk})}{\partial y_j}\frac{\partial^2 W_j}{\partial y_p \partial y_k}\\
&+ 2\frac{\partial}{\partial y_p}(g^{k\ell}\Gamma_{j,k}^i)\frac{\partial W_j}{\partial y_\ell} + 2 g^{k\ell}\Gamma_{j,k}^i \frac{\partial^2 W_j}{\partial y_\ell \partial y_p}\\
&+ \frac{\partial}{\partial y_p}\left\{\frac{\partial}{\partial y_k}(g^{k\ell}\Gamma_{j,\ell}^i) + g^{k\ell}\Gamma_{j,\ell}^m \Gamma_{k,m}^i\right\} W_j + \left\{\frac{\partial}{\partial y_k}(g^{k\ell}\Gamma_{j,\ell}^i) + g^{k\ell}\Gamma_{j,\ell}^m \Gamma_{k,m}^i\right\}\frac{\partial W_j}{\partial y_p},
\end{aligned}$$

where

$$g^{ij}(y,t) = \frac{\partial Y_i}{\partial x_k}(X(y,t),t)\frac{\partial Y_j}{\partial x_k}(X(y,t),t) \text{ and } \Gamma_{j,k}^i(y,t) = \frac{\partial Y_i}{\partial x_\ell}(X(y,t),t)\frac{\partial^2 X_\ell}{\partial y_k \partial y_j}(y,t).$$

By Lemma A.1.3, Corollary A.1.1, and using the fact that $\|W\|_{\tilde{\mathcal{U}}(0,T;\Omega_F} \leq R$ we get that

$$\|\partial y_p[(L-\Delta)W]_i\|_{L^2(0,T,L^2(\Omega_F))} \leq K_0 T^{1/2}.$$

Hence, estimate (4.3.18) holds true.

By chain rule, we have

$$\partial y_p[MW]_i = \frac{\partial^2 Y_j}{\partial t \partial x_\ell}\frac{\partial X_\ell}{\partial y_p}\frac{\partial W_i}{\partial y_j} + \frac{\partial Y_j}{\partial t}\frac{\partial^2 W_i}{\partial y_p \partial y_j} + \left\{\Gamma^i_{j,k}\frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^2 X_k}{\partial t \partial y_j}\right\}\frac{\partial W_j}{\partial y_p}$$
$$+ \left\{\frac{\partial \Gamma^i_{j,k}}{\partial y_p}\frac{\partial Y_k}{\partial t} + \Gamma^i_{j,k}\frac{\partial^2 Y_k}{\partial t \partial x_\ell}\frac{\partial X_\ell}{\partial y_p} + \frac{\partial^2 Y_i}{\partial x_\ell \partial x_k}\frac{\partial X_\ell}{\partial y_p}\frac{\partial^2 X_k}{\partial t \partial y_j} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^3 X_k}{\partial t \partial y_p \partial y_j}\right\} W_j.$$

Lemma A.1.3 implies that there exists a constant $K_0$ such that

$$\|\partial y_p[MW]_i\|_{L^2(0,T;L^2(\Omega_F))} \leq K_0 T^{1/2}\|W\|_{L^\infty(0,T;\mathbf{H}^2(\Omega_F))}.$$

This implies that $\|\partial y_p[MW]_i\|_{L^2(0,T;L^2(\Omega_F))} \leq K_0 T^{1/2}$.

From the definition of $[GQ]$, we have:

$$\partial y_p[(\nabla - G)Q]_i = \frac{\partial g^{ij}}{\partial y_p}\frac{\partial Q}{\partial y_j} + (g^{ij} - \delta^i_j)\frac{\partial^2 Q}{\partial y_p \partial y_j}.$$

Hence,

$$\|\partial y_p[(\nabla - G)Q]_i\|_{L^2(0,T;L^2(\Omega_F))} \leq T K_0 \|Q\|_{L^2(0,T;H^2(\Omega_F)},$$

and using the fact that $\|Q\|_{L^2(0,T;H^2(\Omega_F)} \leq R$, we get

$$\|\partial y_p[(\nabla - G)Q]_i\|_{L^2(0,T;L^2(\Omega_F))} \leq K_0 T.$$

To show (4.3.18), we calculate:

$$\partial y_p[NW]_i = \frac{\partial W_j}{\partial y_p}\frac{\partial W_i}{\partial y_j} + W_j\frac{\partial^2 W_i}{\partial y_p \partial y_j} + \frac{\partial \Gamma^i_{j,k}}{\partial y_p}W_j W_k + \Gamma^i_{j,k}\Big(\frac{\partial W_j}{\partial y_p}W_k + W_j\frac{\partial W_k}{\partial y_p}\Big).$$

By Holder inequality and the Sobolev injection of $H^1(\Omega_F)$ into $L^4(\Omega_F)$, we get

$$\begin{aligned}
\|W_j W_k\|_{L^2(0,T;L^2(\Omega_F))} &\leq C T^{1/2}\|W\|^2_{L^\infty(0,T;\mathbf{H}^2(\Omega_F))},\\
\Big\|W_j\frac{\partial W_k}{\partial y_p}\Big\|_{L^2(0,T;L^2(\Omega_F))} &\leq C T^{1/2}\|W\|^2_{L^\infty(0,T;\mathbf{H}^2(\Omega_F))},\\
\Big\|\frac{\partial W_i}{\partial y_j}\frac{\partial W_j}{\partial y_p}\Big\|_{L^2(0,T;L^2(\Omega_F))} &\leq C T^{1/2}\|W\|^2_{L^\infty(0,T;\mathbf{H}^2(\Omega_F))},
\end{aligned}$$

where $C$ is a positive constant that depends on $\Omega_F$.

Again by Holder inequality and the Sobolev injection of $H^2(\Omega_F)$ into $L^\infty(\Omega_F)$, there exists $C > 0$

depending on $\Omega_F$ such that:

$$\begin{aligned}\Big\|W_j\frac{\partial^2 W_i}{\partial y_p\partial y_j}\Big\|_{L^2(\Omega_F)} &\leq C\|W_j\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial^2 W_i}{\partial y_p\partial y_j}\Big\|_{L^2(\Omega_F)}\\ &\leq C\|W_j\|_{H^2(\Omega_F)}\|W_i\|_{H^2(\Omega_F)}\\ &\leq C\|W\|^2_{\mathbf{H}^2(\Omega_F)}.\end{aligned}$$

This implies that

$$\Big\|W_j\frac{\partial^2 W_i}{\partial y_p\partial y_j}\Big\|_{L^2(0,T;L^2(\Omega_F))} \leq CT\|W\|^2_{L^\infty(0,T;\mathbf{H}^2(\Omega_F))},$$

and thus

$$\Big\|W_j\frac{\partial^2 W_i}{\partial y_p\partial y_j}\Big\|_{L^2(0,T;L^2(\Omega_F))} \leq K_0 T.$$

We deduce that

$$\|\partial y_p[NW]_i\|_{L^2(0,T;L^2(\Omega_F))} \leq CT^{1/2}\|W\|^2_{\tilde{\mathcal{U}}(0,T;\Omega_F)},$$

and thus estimate (4.3.18) holds.

Estimate (4.3.18) is straightforward by calculating the first order derivatives of $[\operatorname{div}\mathcal{T}]$ and Lemma A.1.3. More precisely, we have:

$$\partial y_p[\operatorname{div}\mathcal{T}]_i = \frac{\partial}{\partial y_p}\Big(\frac{\partial Y_i}{\partial x_k}\frac{\partial Y_m}{\partial x_\ell}\Big)\frac{\partial T_{k\ell}}{\partial y_m} + \frac{\partial Y_i}{\partial x_k}\frac{\partial Y_m}{\partial x_\ell}\frac{\partial^2 T_{k\ell}}{\partial y_p\partial y_m}.$$

The last two estimates are easy to derive and we leave the verification to the reader. □

It can be deduced from the following that the time derivative of the function $F$ defined throughout (4.2.3)-(4.2.5) and (4.3.4) is in the good space to apply Proposition 4.3.1.

**Proposition 4.3.3** *Suppose that* $(W, Q, \mathcal{T}, (\xi_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$, *then there exists a constant* $K_0$ *satisfying (i) such that*

$$\|F'\|_{L^2(0,T;V^*)} \leq K_0(1+R')T^{1/2}. \tag{4.3.18}$$

Proof. Let $F$ be the source term defined throughout (4.2.3)-(4.2.5) and (4.3.4). and let $\phi \in V$. Then we have

$$\begin{aligned}\langle F', \phi\rangle_{V^*,V} &= \frac{d}{dt}\langle F,\phi\rangle_{V^*,V}\\ &= \frac{d}{dt}\langle F,\phi\rangle_{V^*,V}\\ &= \frac{d}{dt}(F,\phi)_H.\end{aligned}$$

This implies that

$$\langle F', \phi\rangle_{V^*,V} \quad = \frac{d}{dt}\int_{\Omega_F} F_0 \cdot \phi dy + \sum_{i=1}^{k} F'_{1,i} \cdot V_{\phi,i} + F'_{2,i}\ \omega_{\phi,i}.$$

It follows that there exists a positive constant $C$ depending on $\Omega_F$ such that

$$|F_{1,i}| \leq C\|\mathcal{T}'\|_{\mathbf{H}^1(\Omega_F)} \quad \text{and} \quad |F_{2,i}| \leq C\|\mathcal{T}'\|_{\mathbf{H}^1(\Omega_F)}.$$

Hence, we obtain

$$\|F'\|_{V^*} \leq \sup_{\|\phi\|_V=1} \Big|\frac{d}{dt}\int_{\Omega_F} F_0 \cdot \phi\, dy\Big| + CR'. \tag{4.3.19}$$

Classical computations implies that $\partial_t[(L-\Delta)W]_i$:

$$\begin{aligned}\partial_t[(L-\Delta)W]_i = \frac{\partial}{\partial y_j}\Big((g^{jk}-\delta^j_k)\frac{\partial^2 W_i}{\partial t\partial y_k}\Big) + \frac{\partial g^{jk}}{\partial t}\frac{\partial^2 W_i}{\partial y_j\partial y_k} + \frac{\partial^2 g^{jk}}{\partial t\partial y_j}\frac{\partial W_j}{\partial y_k} + \frac{\partial g^{jk}}{\partial y_j}\frac{\partial^2 W_j}{\partial t\partial y_k} \\ + 2\partial_t(g^{k\ell}\Gamma^i_{j,k})\frac{\partial W_j}{\partial y_\ell} + 2g^{k\ell}\Gamma^i_{j,k}\frac{\partial^2 W_j}{\partial t\partial y_\ell} + \partial_t\left\{\frac{\partial}{\partial y_k}(g^{k\ell}\Gamma^i_{j,\ell}) + \sum_{m=1}^{2} g^{k\ell}\Gamma^m_{j,\ell}\Gamma^i_{k,m}\right\}W_j \\ + \left\{\frac{\partial}{\partial y_k}(g^{k\ell}\Gamma^i_{j,\ell}) + \sum_{m=1}^{2} g^{k\ell}\Gamma^m_{j,\ell}\Gamma^i_{k,m}\right\}\frac{\partial W_j}{\partial t}.\end{aligned}$$

It follows from Lemma A.1.3 and Corollary A.1.1 in Appendix A and performing integration by parts that

$$\begin{aligned}\Big|\frac{d}{dt}\int_{\Omega_F}[(L-\Delta)W]_i\phi_i dy\Big| \leq \Big| -\int_{\Omega_F}(g^{jk}-\delta^j_k)\frac{\partial^2 W_i}{\partial t\partial y_k}\frac{\partial \phi_i}{\partial y_j}dy + \int_{\partial\Omega_F}(g^{jk}-\delta^j_k)\frac{\partial^2 W_i}{\partial t\partial y_k}\phi_i n_j d\Gamma\Big| \\ + K_0\Big(\|W\|_{\mathbf{H}^2(\Omega_F)} + \|\partial_t W\|_{\mathbf{L}^2(\Omega_F)} + T\Big\|\frac{\partial^2 W_j}{\partial t\partial y_k}\Big\|_{\mathbf{L}^2(\Omega_F)}\Big)\|\phi\|_{\mathbf{L}^2(\Omega_F)}.\end{aligned}$$

Noting that the term $g^{jk}-\delta^j_k$ vanishes on the boundary of the fluid and using again Lemma A.1.3, we get

$$\begin{aligned}\Big|\frac{d}{dt}\int_{\Omega_F}[(L-\Delta)W]_i\phi_i dy\Big| \leq K_0 T\|\partial_t W\|_{\mathbf{H}^1(\Omega_F)}\|\phi_i\|_{H^1(\Omega_F)} \\ + K_0(\|W\|_{\mathbf{H}^2(\Omega_F)} + \|\partial_t W\|_{\mathbf{L}^2(\Omega_F)} + T\|\partial_t W\|_{\mathbf{H}^1(\Omega_F)}),\end{aligned}$$

and thus

$$\sup_{\|\phi\|_V=1}\Big|\frac{d}{dt}\int_{\Omega_F}[(L-\Delta)W]_i\phi_i dy\Big| \leq K_0(\|W\|_{\mathbf{H}^2(\Omega_F)} + \|\partial_t W\|_{\mathbf{L}^2(\Omega_F)} + T\|\partial_t W\|_{\mathbf{H}^1(\Omega_F)}). \tag{4.3.20}$$

Moreover, from the definition of the operator $M, N$ and $[\mathrm{div}\mathcal{T}]$ in (4.2.8),(4.2.9) and (4.2.11), we have

$$
\begin{aligned}
\partial_t[MW]_i &= \partial_t\Big[\Big(\frac{\partial Y}{\partial t}\cdot\nabla\Big)W_i\Big] + \left\{\Gamma^i_{j,k}\frac{\partial Y_k}{\partial t} + \frac{\partial Y_i}{\partial x_k}\frac{\partial^2 X_k}{\partial t\partial y_j}\right\}\frac{\partial W_j}{\partial t} \\
&\quad + \partial_t\left\{\Gamma^i_{j,k}\frac{\partial Y_k}{\partial t}(X(y,t),t) + \frac{\partial Y_i}{\partial x_k}(X(y,t),t)\frac{\partial^2 X_k}{\partial t\partial y_j}(y,t)\right\}W_j \\
\partial_t[NW]_i &= \partial_t[(W\cdot\nabla)W_i] + \frac{\partial\Gamma^i_{j,k}}{\partial t}W_jW_k + \Gamma^i_{j,k}\Big(\frac{\partial W_j}{\partial t}W_k + W_j\frac{\partial W_k}{\partial t}\Big), \\
\partial_t[\mathrm{div}\mathcal{T}]_i &= \Big(\frac{\partial^2 Y_i}{\partial x_k\partial t} + \frac{\partial^2 Y_i}{\partial x_k\partial x_j}\frac{\partial X_j}{\partial t}\Big)\frac{\partial Y_m}{\partial x_\ell}\frac{\partial\mathcal{T}_{k\ell}}{\partial y_\ell} + \frac{\partial Y_i}{\partial x_k}\Big(\frac{\partial^2 Y_m}{\partial x_\ell\partial t} + \frac{\partial^2 Y_m}{\partial x_j\partial x_\ell}\frac{\partial X_j}{\partial t}\Big)\frac{\partial\mathcal{T}_{k\ell}}{\partial y_\ell} + \frac{\partial Y_i}{\partial x_k}\frac{\partial Y_m}{\partial x_\ell}\frac{\partial^2\mathcal{T}_{k\ell}}{\partial y_\ell\partial t}.
\end{aligned}
$$

From Lemma A.1.3 and (4.3.20) it follows that for all $\phi\in V$ such that $\|\phi\|_V = 1$, we have

$$
\begin{aligned}
&\Big|\frac{d}{dt}\int_{\Omega_F}F_0\cdot\phi dy\Big| \le Re\Big|\frac{d}{dt}\int_{\Omega_F}\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)W\cdot\phi dy\Big| + \Big|\frac{d}{dt}\int_{\Omega_F}[(G-\nabla)Q]\cdot\phi dy\Big| + K_0\Big(\|W_jW_k\|_{L^2(\Omega_F)} \\
&+K_0T\Big\|\frac{\partial W_j}{\partial t}W_k\Big\|_{L^2(\Omega_F)} + \|\mathcal{T}\|_{\mathbf{H}^1(\Omega_F)} + \|\mathcal{T}'\|_{\mathbf{H}^1(\Omega_F)} + \|W\|_{\mathbf{H}^2(\Omega_F)} + \|\partial_t W\|_{\mathbf{L}^2(\Omega_F)} + T\|\partial_t W\|_{\mathbf{H}^1(\Omega_F)}\Big).
\end{aligned}
\tag{4.3.21}
$$

We remark that from the definition of $Y$ in Appendix A, we have

$$
\nabla\cdot\Big(\frac{\partial Y}{\partial t}(X(y,t),t) + W(y,t)\Big) = 0.
$$

By performing integration by parts, we get

$$
\begin{aligned}
\int_{\Omega_F}\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)W\cdot\phi\,dy &= -\int_{\Omega_F}W\cdot\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)\phi\,dy + \int_{\partial\Omega_F}W\cdot\phi\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\mathbf{n}\,d\Gamma, \\
\int_{\Omega_F}[(G-\nabla)Q]\cdot\phi\,dy &= -\int_{\Omega_F}\frac{\partial}{\partial y_j}(g^{ij}-\delta^i_j)Q\frac{\partial\phi_i}{\partial y_j}dy + \int_{\partial\Omega_F}(g^{ij}-\delta^i_j)Q\phi_i d\Gamma.
\end{aligned}
$$

We remark that $W = h_i'(t) + \omega_i(t)(y-h_i^0)^\perp$ on $B_i$ and vanishes over $\partial\mathcal{O}$. Hence, (A.1.3) implies that

$$
\Big(\frac{\partial Y}{\partial t}(X(y,t)) + W(y,t)\Big)\cdot\mathbf{n} = 0,\ \text{ for all } (y,t)\in\partial\Omega_F\times[0,T].
$$

Moreover, since $g^{ij}-\delta^i_j$ vanishes on $\partial\Omega_F$ we get

$$
\begin{aligned}
\int_{\Omega_F}\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)W\cdot\phi\,dy &= -\int_{\Omega_F}\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)\phi\cdot W\,dy, \\
\int_{\Omega_F}[(G-\nabla)Q]\cdot\phi\,dy &= -\int_{\Omega_F}\frac{\partial}{\partial y_j}(g^{ij}-\delta^i_j)Q\frac{\partial\phi_i}{\partial y_j}dy.
\end{aligned}
$$

It follows that

$$\frac{d}{dt}\int_{\Omega_F}\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)W\cdot\phi\,dy=-\int_{\Omega_F}\Big(\frac{\partial^2 Y_j}{\partial t^2}+\frac{\partial^2 Y_j}{\partial t\partial x_k}\frac{\partial X_k}{\partial t}\Big)\frac{\partial\phi_i}{\partial y_j}W_i\,dy-\int_{\Omega_F}(\partial_t W\cdot\nabla)\phi\cdot W dy$$
$$-\int_{\Omega_F}\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)\phi\cdot\partial_t W\,dy,$$

and

$$\frac{d}{dt}\int_{\Omega_F}[(G-\nabla)Q]\cdot\phi\,dy=-\int_{\Omega_F}\frac{\partial^2}{\partial t\partial y_j}(g^{ij}-\delta^i_j)Q\frac{\partial\phi_i}{\partial y_j}dy-\int_{\Omega_F}\frac{\partial}{\partial y_j}(g^{ij}-\delta^i_j)\frac{\partial Q}{\partial t}\frac{\partial\phi_i}{\partial y_j}dy.$$

By Holder Inequality and using the embedding of $\mathbf{H}^2(\Omega_F)$ into $\mathbf{L}^\infty(\Omega_F)$ we get that

$$\Big|\frac{d}{dt}\int_{\Omega_F}\Big(\Big(\frac{\partial Y}{\partial t}+W\Big)\cdot\nabla\Big)W\cdot\phi\,dy\Big|\le K_0\Big(\|W\|_{\mathbf{H}^1(\Omega_F)}+\|\partial_t W\|_{\mathbf{L}^2(\Omega_F)}(\|W\|_{\mathbf{H}^2(\Omega_F)}+1)\Big)\|\phi\|_V \quad (4.3.22)$$

$$\Big|\frac{d}{dt}\int_{\Omega_F}[(G-\nabla)Q]\cdot\phi\,dy\Big|\le K_0T\Big(\|Q\|_{\mathbf{L}^2(\Omega_F)}+\|\partial_t Q\|_{\mathbf{L}^2(\Omega_F)}\Big)\|\phi\|_V, \quad (4.3.23)$$

Moreover, by Holder's inequality and the Sobolev embedding of $\mathbf{H}^1(\Omega_F)$ into $\mathbf{L}^4(\Omega_F)$ one can show that

$$\|W_jW_k\|_{L^2(\Omega_F)}\le C\|W\|^2_{\mathbf{H}^1(\Omega_F)}, \quad (4.3.24)$$

$$\Big\|\frac{\partial W_j}{\partial t}W_k\Big\|_{L^2(\Omega_F)}\le C\Big\|\frac{\partial W}{\partial t}\Big\|_{\mathbf{H}^1(\Omega_F)}\|W\|_{\mathbf{H}^1(\Omega_F)}. \quad (4.3.25)$$

Combining (4.3.22)-(4.3.25) with (4.3.21), we obtain

$$\sup_{\|\phi\|_V=1}\Big|\frac{d}{dt}\int_{\Omega_F}F_0\cdot\phi\,dy\Big|\le K_0\Big(\|W\|^2_{\mathbf{H}^1(\Omega_F)}+\|W\|_{\mathbf{H}^2(\Omega_F)}+(1+\|W\|_{\mathbf{H}^2(\Omega_F)})\|\partial_t W\|_{\mathbf{L}^2(\Omega_F)}$$
$$+\|Q\|_{\mathbf{L}^2(\Omega_F)}+\|\mathcal{T}\|_{\mathbf{H}^1(\Omega_F)}+\|\mathcal{T}'\|_{\mathbf{H}^1(\Omega_F)}\Big)$$
$$+K_0T\Big((1+\|W\|_{\mathbf{H}^1(\Omega_F)})\|\partial_t W\|_{\mathbf{H}^1(\Omega_F)}+\|\partial_t Q\|_{\mathbf{L}^2(\Omega_F)}\Big). \quad (4.3.26)$$

Returning to (4.3.19), we get

$$\|F'\|_{V^*}\le K_0\Big(\|W\|^2_{\mathbf{H}^1(\Omega_F)}+\|W\|_{\mathbf{H}^2(\Omega_F)}+(1+\|W\|_{\mathbf{H}^2(\Omega_F)})\|\partial_t W\|_{\mathbf{L}^2(\Omega_F)}+\|Q\|_{\mathbf{L}^2(\Omega_F)}$$
$$+\|\mathcal{T}\|_{\mathbf{H}^1(\Omega_F)}+\|\mathcal{T}'\|_{\mathbf{H}^1(\Omega_F)}\Big)+K_0T\Big((1+\|W\|_{\mathbf{H}^1(\Omega_F)})\|\partial_t W\|_{\mathbf{H}^1(\Omega_F)}+T\|\partial_t Q\|_{\mathbf{L}^2(\Omega_F)}\Big). \quad (4.3.27)$$

Therefore, estimate (4.3.18) follows by noting that $(W,Q,\mathcal{T},(\xi_i,\omega_i)_{i=1,\ldots,k})\in\tilde{\mathcal{K}}$. □

Next, we prove the following lemma.

**Lemma 4.3.1** *Assume that $\partial\mathcal{O}\in\mathcal{C}^1$, $\tau_0\in\mathbf{H}^2(\Omega_F)$, and $(W,Q,(h_i,\omega_i)_{i=1,\ldots,k})\in\tilde{\mathcal{K}}$. Then problem*

*(4.3.16)-(4.3.17) admits a unique solution* $\tilde{\mathcal{T}}$ *in* $\mathcal{C}([0,T], \mathbf{H}^2(\Omega_F))$. *Moreover, there exists a positive constant* $c$ *such that*

$$\|\tilde{\mathcal{T}}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega_F)} \leq \Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)} + \frac{2r}{We}\Big) \exp\Big(K_0 T^{1/2}\Big). \tag{4.3.28}$$

*Furthermore, we get* $\tilde{\mathcal{T}}'$ *belongs to* $\mathcal{C}([0,T], \mathbf{H}^1(\Omega_F))$ *and satisfies*

$$\|\tilde{\mathcal{T}}'\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} \leq K_0\Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)} + \frac{2r}{We}\Big) \exp\Big(K_0 T^{1/2}\Big). \tag{4.3.29}$$

Proof. First we recall that transport coefficient $W + \dfrac{\partial Y}{\partial t} \in L^1(0,T;\mathbf{H}^3(\Omega_F))$ and globally Lipschitz. Moreover, the transport coefficient satisfies (see (A.1.3) in Appendix A)

$$\Big(W(y,t) + \frac{\partial Y}{\partial t}(X(y,t),t)\Big) \cdot \mathbf{n} = 0, \ \ \forall (y,t) \in \partial\Omega_F \times [0,T]. \tag{4.3.30}$$

Therefore the existence of a unique solution of the hyperbolic partial differential equation (4.3.16) follows by applying the method of characteristics. We turn now to prove estimate (4.3.28). To this end, we multiply scalar equation (4.3.16) by $\tilde{\mathcal{T}}$ in $H^2$. By performing integration by parts and using (4.3.30), we get

$$\begin{aligned}
\frac{We}{2}\frac{d}{dt}\|\tilde{\mathcal{T}}(t)\|^2_{\mathbf{H}^2(\Omega_F)} + \|\tilde{\mathcal{T}}(t)\|^2_{\mathbf{H}^2(\Omega_F)} &= 2r([\mathcal{D}W], \tilde{\mathcal{T}})_{\mathbf{H}^2(\Omega_F)} - We([G_a(W,\tilde{\mathcal{T}})], \tilde{\mathcal{T}})_{\mathbf{H}^2(\Omega_F)} \\
&- We\Bigg\{\int_{\Omega_F} \frac{\partial}{\partial y_k}\Big(W_\ell(y,t) + \frac{\partial Y_\ell}{\partial t}(X(y,t),t)\Big)\frac{\partial \tilde{\mathcal{T}}_{ij}}{\partial y_\ell}(y,t)\frac{\partial \tilde{\mathcal{T}}_{ij}}{\partial y_k}(y,t)dy \\
&+ \int_{\Omega_F} \frac{\partial^2}{\partial y_k \partial y_m}\Big(W_\ell(y,t) + \frac{\partial Y_\ell}{\partial t}(X(y,t),t)\Big)\frac{\partial \tilde{\mathcal{T}}_{ij}}{\partial y_\ell}(y,t)\frac{\partial^2 \tilde{\mathcal{T}}_{ij}}{\partial y_k \partial y_m}(y,t)dy \\
&+2\int_{\Omega_F} \frac{\partial}{\partial y_k}\Big(W_\ell(y,t) + \frac{\partial Y_\ell}{\partial t}(X(y,t),t)\Big)\frac{\partial^2 \tilde{\mathcal{T}}_{ij}}{\partial y_\ell \partial y_m}(y,t)\frac{\partial^2 \tilde{\mathcal{T}}_{ij}}{\partial y_k \partial y_m}(y,t)dy\Bigg\}.
\end{aligned} \tag{4.3.31}$$

Using the Sobolev embeddings $H^2(\Omega_F) \hookrightarrow L^\infty(\Omega_F)$ and $H^1(\Omega_F) \hookrightarrow L^4(\Omega_F)$, it follows from Lemma A.1.3 that there exists $K_0$ such that

$$\begin{aligned}
\frac{We}{2}\frac{d}{dt}\|\tilde{\mathcal{T}}(t)\|^2_{\mathbf{H}^2(\Omega_F)} + \|\tilde{\mathcal{T}}(t)\|^2_{\mathbf{H}^2(\Omega_F)} &\leq K_0 We(1 + \|W(t)\|_{\mathbf{H}^3(\Omega_F)})\|\tilde{\mathcal{T}}(t)\|^2_{\mathbf{H}^2(\Omega_F)} \\
&+ 2rK_0\|W(t)\|_{\mathbf{H}^3(\Omega_F)}\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}.
\end{aligned}$$

This implies that

$$\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}\frac{d}{dt}\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)} + \frac{1}{We}\|\tilde{\mathcal{T}}(t)\|^2_{\mathbf{H}^2(\Omega_F)} \leq K_0(1 + \|W(t)\|_{\mathbf{H}^3(\Omega_F)})\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}\Big(\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)} + \frac{2r}{We}\Big).$$

Consequently

$$\frac{1}{2}\frac{d}{dt}\Big(\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)^2 \leq K_0(1+\|W(t)\|_{\mathbf{H}^3(\Omega_F)})\Big(\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)^2. \tag{4.3.32}$$

Integrating (4.3.32) over $(0,t)\subset(0,T)$, we obtain

$$\frac{1}{2}\Big(\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)^2 \leq \frac{1}{2}\Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)^2+K_0\int_0^t(1+\|W(s)\|_{\mathbf{H}^3(\Omega_F)})\Big(\|\tilde{\mathcal{T}}(s)\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)^2 ds.$$

Gronwall lemma implies that

$$\Big(\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)^2 \leq \Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)^2\exp\Big(\int_0^T(1+\|W(s)\|_{\mathbf{H}^3(\Omega_F)})ds\Big),$$

and as $\|W\|_{L^2(0,T;\mathbf{H}^3(\Omega_F))}\leq R$, we get

$$\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We} \leq \Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big)\exp(K_0T^{1/2}). \tag{4.3.33}$$

To prove $\tilde{\mathcal{T}}\in\mathcal{C}([0,T],\mathbf{H}^2(\Omega_F))$, we define the characteristics $y$ that pass through $y_0$ at time $t$ as follows:

$$\begin{aligned}
\frac{d}{ds}y(s,t,y_0) &= W(s,y(s,t,y_0))+\frac{dY}{ds}(s,X(s,y(s,t,y_0))),\ \ y(t,t,y_0)=y_0,\\
\frac{d}{ds}\tilde{\mathcal{T}}(s,y(s,t,y_0)) &= \frac{1}{We}\Big(2r[\mathcal{D}W]-\tilde{\mathcal{T}}-We[G_a(W,\tilde{\mathcal{T}})]\Big)(s,y(s,t,y_0)).
\end{aligned}$$

Integrating the last inequality over $(0,t)\subset(0,T)$, we get

$$\tilde{\mathcal{T}}(t,y(t,t,y_0))-\tilde{\mathcal{T}}(0,y(0,t,y_0))=\frac{1}{We}\int_0^t\Big(2r[\mathcal{D}W]-\tilde{\mathcal{T}}-We[G_a(W,\tilde{\mathcal{T}})]\Big)(s,y(s,t,y_0))ds.$$

Consequently, we obtain

$$\tilde{\mathcal{T}}(t,y_0)=\tau_0\big(y(0,t,y_0)\big)+\frac{1}{We}\int_0^t\Big(2r[\mathcal{D}W]-\tilde{\mathcal{T}}-We[G_a(W,\tilde{\mathcal{T}})]\Big)(s,y(s,t,y_0))ds.$$

The above formula implies that $\tilde{\mathcal{T}}\in\mathcal{C}([0,T],\mathbf{H}^2(\Omega_F))$ (see [7]).
Moreover, since $W\in\mathcal{C}([0,T],\mathbf{H}^2(\Omega_F))$ and $Y'\in\mathcal{C}([0,T],\mathbb{R}^2)$, we get that $\tilde{\mathcal{T}}'$ given by (4.3.16) belongs to $\mathcal{C}([0,T],\mathbf{H}^1(\Omega_F))$ and

$$\|\tilde{\mathcal{T}}'(t)\|_{\mathbf{H}^1(\Omega_F)}\leq K_0\Big(\|W(t)\|_{\mathbf{H}^2(\Omega_F)}+\Big(1+\frac{1}{K_0We}\Big)\Big)\Big(\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)}+\frac{2r}{We}\Big).$$

Inequality (4.3.33) implies that

$$\|\tilde{\mathcal{T}}'(t)\|_{\mathbf{H}^1(\Omega_F)} \leq \Big(K_0 + \frac{1}{We}\Big)\Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)} + \frac{2r}{We}\Big)\exp(K_0T^{1/2}).$$

With these preliminaries in mind, we turn to prove that $\tilde{\mathcal{N}}$ maps $\tilde{\mathcal{K}}$ into itself.

**Corollary 4.3.1** *For $T$ small enough, $R$ and $R'$ sufficiently large, we have $\tilde{\mathcal{N}}(\tilde{\mathcal{K}}) \subset \tilde{\mathcal{K}}$.*

Proof. Let $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$ and denote by $(U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$ its image by the mapping $\tilde{\mathcal{N}}$. It follows from Proposition 4.3.2 that

$$\|F_0\|_{L^2(0,T;\mathbf{H}^1(\Omega_F))} + \|F_{1,i}\|_{L^2(0,T;\mathbb{R}^2)} + \|F_{2,i}\|_{L^2(0,T;\mathbb{R})} \leq K_0T^{1/2}$$

Moreover, since

$$\|\mathbb{P}F(0)\|_H \leq C(\|u_0\|_{\mathbf{H}^2(\Omega_F)} + \|\tau_0\|_{\mathbf{H}^2(\Omega_F)}),$$

it follows from Proposition 4.3.1 and Proposition 4.3.3 that for $T$ small enough, we have

$$\begin{aligned}&\|U\|_{\tilde{\mathcal{U}}(0,T;\Omega_F)} + \|P\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|\nabla P\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_F))} + \|\partial_t \nabla P\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega_F))}\\ &+\sum_{i=1}^{k}\|\tilde{h}_i\|_{W^{2,\infty}(0,T;\mathbb{R}^2)}+\|\tilde{\omega}_i\|_{W^{1,\infty}(0,T;\mathbb{R})} \leq K\Big(\|Au_0\|_H+K_0(1+R')T^{1/2}+\|u_0\|_{\mathbf{H}^2(\Omega_F)}+\|\tau_0\|_{\mathbf{H}^2(\Omega_F)}\Big).\end{aligned}$$

Therefore, by choosing $T$ small enough and $R \gg 2K(\|Au_0\|_H + \|u_0\|_{\mathbf{H}^2(\Omega_F)} + \|\tau_0\|_{\mathbf{H}^2(\Omega_F)})$ we get

$$\begin{aligned}&\|U\|_{\tilde{\mathcal{U}}(0,T;\Omega_F)} + \|P\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|\nabla P\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_F))} + \|\partial_t \nabla P\|_{L^2(0,T;\mathbf{H}^{-1}(\Omega_F))}\\ &\qquad+\sum_{i=1}^{k}\|\tilde{h}_i\|_{W^{2,\infty}(0,T;\mathbb{R}^2)} + \|\tilde{\omega}_i\|_{W^{1,\infty}(0,T;\mathbb{R})} \leq \frac{R}{2}. \qquad (4.3.34)\end{aligned}$$

Moreover, for $T < (\ln 2)^2/\mathrm{K}_0$ in Lemma 4.3.1, we have

$$\begin{aligned}\|\tilde{\mathcal{T}}(t)\|_{\mathbf{H}^2(\Omega_F)} &\leq 2\Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)} + \frac{2r}{We}\Big), \ \forall t \in [0,T],\\ \|\tilde{\mathcal{T}}'(t)\|_{\mathbf{H}^1(\Omega_F)} &\leq 2\Big(K_0 + \frac{1}{We}\Big)\Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)} + \frac{2r}{We}\Big), \ \forall t \in [0,T].\end{aligned}$$

Choosing $R > 4\Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)} + \dfrac{2r}{We}\Big)$ and $R' > 2\Big(K_0 + \dfrac{1}{We}\Big)\Big(\|\tau_0\|_{\mathbf{H}^2(\Omega_F)} + \dfrac{2r}{We}\Big)$ imply that

$$\|\tilde{\mathcal{T}}\|_{L^\infty([0,T],\mathbf{H}^2(\Omega_F))} \leq \frac{R}{2}, \ \text{and} \ \|\tilde{\mathcal{T}}'\|_{L^\infty([0,T],\mathbf{H}^1(\Omega_F))} \leq R'.$$

Combining the last inequality with that in (4.3.34), we get $(U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$. $\square$

Next, we prove the following:

**Proposition 4.3.4** *The mapping $\tilde{\mathcal{N}} : \tilde{\mathcal{K}} \to X_T$ is continuous for the topology of $X_T$.*

Proof. To prove the continuity of the mapping $\tilde{\mathcal{N}}$ for the topology of $X_T$, it suffices to prove the continuity of $\tilde{\mathcal{N}}$ from $(\tilde{\mathcal{K}}, \|.\|_{X_T})$ into $(X_T, \|.\|_{Y_T})$, where

$$Y_T = \mathcal{C}([0,T]; \mathbf{L}^2(\Omega_F)) \times \mathcal{C}([0,T]; L^2(\Omega_F)) \times \mathcal{C}([0,T]; \mathbf{L}^2(\Omega_F)) \times \left(\mathcal{C}^1([0,T]; \mathbb{R}^2) \times \mathcal{C}([0,T]; \mathbb{R})\right)^k.$$

Assume for instance that the mapping $\tilde{\mathcal{N}}$ is continuous from $(\tilde{\mathcal{K}}, \|.\|_{X_T})$ into $(X_T, \|.\|_{Y_T})$. Let $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$ and consider the sequence $(W^n, Q^n, \mathcal{T}^n, (h_i^n, \omega_i^n)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$ such that

$$(W^n, Q^n, \mathcal{T}^n, (h_i^n, \omega_i^n)_{i=1,\dots,k}) \longrightarrow (W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}), \ \text{as } n \to +\infty, \text{ in the topology of } X_T.$$

We denote by $X^n, Y^n, g^{ij,n}, \Gamma_{i,j}^{k,n}, \dots$ the terms corresponding to $(W^n, Q^n, \mathcal{T}^n, (h_i^n, \omega_i^n)_{i=1,\dots,k})$ and by $X, Y, g^{ij}, \Gamma_{i,j}^k, \dots$ the terms corresponding to $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k})$.

Consider the sequence:

$$(U^n, P^n, \tilde{\mathcal{T}}^n, (\tilde{h}_i^n, \tilde{\omega}_i^n)_{i=1,\dots,k}) = \tilde{\mathcal{N}}(W^n, Q^n, \mathcal{T}^n, (h_i^n, \omega_i^n)_{i=1,\dots,k}),$$

and let

$$(U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}) = \tilde{\mathcal{N}}(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}).$$

By Corollary 4.3.1, we obtain

$$(U^n, P^n, \tilde{\mathcal{T}}^n, (\tilde{h}_i^n, \tilde{\omega}_i^n)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}.$$

Since $\tilde{\mathcal{K}}$ is closed, convex and bounded set in $X_T$, then by Ascoli's theorem $\tilde{\mathcal{K}}$ is compact in $X_T$. This implies that there exists $(U^\star, P^\star, \tilde{\mathcal{T}}^\star, (\tilde{h}_i^\star, \tilde{\omega}_i^\star)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$, such that

$$(U^n, P^n, \tilde{\mathcal{T}}^n, (\tilde{h}_i^n, \tilde{\omega}_i^n)_{i=1,\dots,k}) \longrightarrow (U^\star, P^\star, \tilde{\mathcal{T}}^\star, (\tilde{h}_i^\star, \tilde{\omega}_i^\star)_{i=1,\dots,k}), \ \text{as } n \to +\infty,$$

for the topology of $X_T$, and thus for the topology of $Y_T$.

The continuity of the mapping $\tilde{\mathcal{N}}$ in the topology of $Y_T$ implies that

$$(U^n, P^n, \tilde{\mathcal{T}}^n, (\tilde{h}_i^n, \tilde{\omega}_i^n)_{i=1,\dots,k}) \longrightarrow (U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}), \text{ as } n \to +\infty, \text{ in the topology of } Y_T.$$

By uniqueness of the limit, it follows that

$$(U^n, P^n, \tilde{\mathcal{T}}^n, (\tilde{h}_i^n, \tilde{\omega}_i^n)_{i=1,\dots,k}) \longrightarrow (U, P, \tilde{\mathcal{T}}, (\tilde{h}_i, \tilde{\omega}_i)_{i=1,\dots,k}), \text{ as } n \to +\infty, \text{ in the topology of } X_T.$$

Thus the mapping $\tilde{\mathcal{N}}$ is continuous from $(\tilde{\mathcal{K}}, \|.\|_{X_T})$ into $(X_T, \|.\|_{X_T})$.
Now we are in position to prove the continuity of the mapping $\tilde{\mathcal{N}}$ from $(\tilde{\mathcal{K}}, X_T)$ into $(X_T, Y_T)$. To this end, we define:

$$\tilde{U}^n = U - U^n, \ \tilde{P}^n = P - P^n, \ Z^n = \tilde{\mathcal{T}} - \tilde{\mathcal{T}}^n, \ \tilde{H}_i^n = \tilde{h}_i - \tilde{h}_i^n, \text{ and } \tilde{W}_i^n = \tilde{\omega}_i - \tilde{\omega}_i^n.$$

Since the Cauchy stress tensor field $\overline{\Sigma}(\tilde{U}^n, \tilde{P}^n)$ is symmetric and using the fact that

$$\nabla \cdot \Big( W^n(y,t) + \frac{dY^n}{dt}(X^n(y,t),t) \Big) = 0 \text{ and } \Big( W^n(y,t) + \frac{dY^n}{dt}(X^n(y,t),t) \Big) \cdot \mathbf{n}|_{\partial\Omega_F} = 0,$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\Big( Re\|\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k} \overline{m}_i |(\tilde{H}_i^n)'(t)|^2 + \overline{J}_i |\tilde{W}_i^n(t)|^2 \Big) + 2(1-r)\int_{\mathcal{O}} |D[\tilde{U}^n(t)]|^2 dy$$
$$= \int_{\Omega_F} F_0^n(t) \cdot \tilde{U}^n(t) dy + F_{1,i}^n(t) \cdot (\tilde{H}_i^n)'(t) + F_{2,i}^n(t) \tilde{W}_i^n(t), \quad (4.3.35)$$

and

$$\frac{We}{2}\frac{d}{dt}\Big( \|Z^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} \Big) + \|Z^n(t)\|^2_{[L^2(\Omega_F)]^4} \leq \|\mathcal{G}^n(t)\|_{\mathbf{L}^2(\Omega_F)} \|Z^n(t)\|_{\mathbf{L}^2(\Omega_F)}, \quad (4.3.36)$$

where

$$\begin{aligned}
F_0^n &= (1-r)\Big([(L-\Delta)(W-W^n)] + [(L-L^n)W^n]\Big) + [(\nabla - G)(Q-Q^n)] - [(G-G^n)Q^n] \\
&\quad - Re\Big([M(W-W^n)] + [(M-M^n)W^n] + [NW] - [N^n W^n]\Big) + [\mathrm{div}(\mathcal{T}-\mathcal{T}^n)] + [(\mathrm{div}-\mathrm{div^n})\mathcal{T}^\mathrm{n}], \\
F_{1,i}^n &= -\int_{\partial B_i} (\mathcal{T}-\mathcal{T}^n)\nu_i d\Gamma_i, \\
F_{2,i}^n &= -\int_{\partial B_i} (\mathcal{T}-\mathcal{T}^n)\nu_i \cdot (y - h_i^0)^\perp d\Gamma_i, \\
\mathcal{G}^n &= We\Big\{ [G_a^n(W^n, \tilde{\mathcal{T}}^n)] - [G_a(W, \tilde{\mathcal{T}})] - \Big(\Big(W - W^n + \frac{\partial Y}{\partial t}(X(y,t),t) - \frac{\partial Y^n}{\partial t}(X^n(y,t),t)\Big) \cdot \nabla\Big)\tilde{\mathcal{T}} \Big\} \\
&\quad + 2r\Big([\mathcal{D}(W-W^n)] + [(\mathcal{D}-\mathcal{D}^n)W^n]\Big).
\end{aligned}$$

To proceed we need to bound the terms in the right hand side of (4.3.35) and (4.3.36) in terms of the terms in the left hand side of each one of them. However, before deriving these estimates, we shall introduce some new notations which we are going to use fro now on. We define

- $g^{ij}(X^\ell) = \sum_{k=1}^{2} \frac{\partial Y_i}{\partial x_k}(X^\ell(y,t),t)\frac{\partial Y_j}{\partial x_k}(X^\ell(y,t),t), g^{ij,n}(X^\ell) = \sum_{k=1}^{2} \frac{\partial Y_i^n}{\partial x_k}(X^\ell(y,t),t)\frac{\partial Y_j^n}{\partial x_k}(X^\ell(y,t),t),$
- $\bar{g}^{ij,n}(X^\ell) = g^{ij}(X^\ell) - g^{ij,n}(X^\ell),$
- $\Gamma^i_{j,k}(X^\ell) = \sum_{m=1}^{2} \frac{\partial Y_i}{\partial x_m}(X^\ell(y,t),t)\frac{\partial^2 X_m}{\partial y_k \partial y_j}(y,t),\ \ \Gamma^{i,n}_{j,k}(X^\ell) = \sum_{m=1}^{2} \frac{\partial Y_i^n}{\partial x_m}(X^\ell(y,t),t)\frac{\partial^2 X_m^n}{\partial y_k \partial y_j}(y,t).$

**Proposition 4.3.5** *Suppose that* $(W, Q, \mathcal{T}, (h_i,\omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{U}}(0,T;\Omega_F)\times L^2(0,T;\dot{H}^1(\Omega_F))\times\tilde{\mathfrak{T}}(0,T;\Omega_F)\times \left(W^{2,\infty}(0,T;\mathbb{R}^2)\times W^{1,\infty}(0,T;\mathbb{R})\right)^k$. *Then there exists a positive constant* $K_0$ *satisfying (i) such that*

$$\|[(L-L^n)W]\|_{\mathbf{L}^2(\Omega_F)} \le K_0 T \|W\|_{\mathbf{H}^2(\Omega_F)} \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}, \tag{4.3.37}$$

$$\|[(M-M^n)W]\|_{\mathbf{L}^2(\Omega_F)} \le K_0 \|W\|_{\mathbf{H}^1(\Omega_F)} \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}, \tag{4.3.38}$$

$$\|[(G-G^n)Q]\|_{\mathbf{L}^2(\Omega_F)} \le K_0 T \|Q\|_{H^1(\Omega_F)} \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}, \tag{4.3.39}$$

$$\|[(div-div^n)\,\mathcal{T}]\|_{\mathbf{L}^2(\Omega_F)} \le K_0 T \|\mathcal{T}\|_{\mathbf{H}^1(\Omega_F)} \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}, \tag{4.3.40}$$

$$\|[(\mathcal{D}-\mathcal{D}^n)\,W]\|_{\mathbf{L}^2(\Omega_F)} \le K_0 T \|W\|_{\mathbf{H}^1(\Omega_F)} \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}, \tag{4.3.41}$$

$$\|[(\mathcal{W}-\mathcal{W}^n)\,W]\|_{\mathbf{L}^2(\Omega_F)} \le K_0 T \|W\|_{\mathbf{H}^1(\Omega_F)} \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}. \tag{4.3.42}$$

Proof. We start by computing $[(L-L^n)W]$ and we find that

$$[(L-L^n)W]_i = \sum_{j,k=1}^{2} \Big(g^{jk}(X) - g^{jk,n}(X^n)\Big)\frac{\partial^2 W_i}{\partial y_j \partial y_k} + \frac{\partial}{\partial y_j}\Big(g^{jk}(X) - g^{jk,n}(X^n)\Big)\frac{\partial W_i}{\partial y_k}$$
$$+2\sum_{j,k,\ell=1}^{2} \Big(g^{k\ell}(X)\Gamma^i_{j,k}(X) - g^{k\ell,n}(X^n)\Gamma^{i,n}_{j,k}(X^n)\Big)\frac{\partial W_j}{\partial y_\ell} + \sum_{j,k,\ell=1}^{2}\Big\{\frac{\partial}{\partial y_k}\Big(g^{k\ell}(X)\Gamma^i_{j,\ell}(X) - g^{k\ell,n}(X^n)\Gamma^{i,n}_{j,\ell}(X^n)\Big)$$
$$+\sum_{m=1}^{2}\Big(g^{k\ell}(X)\Gamma^m_{j,\ell}(X)\Gamma^i_{k,m}(X) - g^{k\ell,n}(X^n)\Gamma^{m,n}_{j,\ell}(X^n)\Gamma^{i,n}_{k,m}(X^n)\Big)\Big\}W_j. \tag{4.3.43}$$

It is not difficult to check that

$$g^{k\ell}(X)\Gamma^i_{j,k}(X) - g^{k\ell,n}(X^n)\Gamma^{i,n}_{j,k}(X^n) = g^{k\ell}(X)\Big(\Gamma^i_{j,k}(X) - \Gamma^{i,n}_{j,k}(X^n)\Big) + \Big(g^{k\ell}(X) - g^{k\ell,n}(X^n)\Big)\Gamma^{i,n}_{j,k}(X^n).$$

Lemmas A.1.3 and A.1.4 imply that

$$\|g^{k\ell}(X)\Gamma^i_{j,k}(X) - g^{k\ell,n}(X^n)\Gamma^{i,n}_{j,k}(X^n)\|_{L^\infty([0,T]\times\mathcal{O})} \le K_0 T \sum_{m=1}^{k} \|h_m' - (h_m^n)'\|_{L^\infty([0,T]\times\mathcal{O})}.$$

Next, we move to bound the coefficients of $W$ in the expression (4.3.43). To this end, we write

$$\begin{aligned}&\frac{\partial}{\partial y_k}\Big(g^{k\ell}(X)\Gamma^i_{j,\ell}(X)-g^{k\ell,n}(X^n)\Gamma^{i,n}_{j,\ell}(X^n)\Big)\\&=\frac{\partial}{\partial y_k}(g^{k\ell}(X))\Big(\Gamma^i_{j,\ell}(X)-\Gamma^{i,n}_{j,\ell}(X^n)\Big)+g^{k\ell}(X)\frac{\partial}{\partial y_k}\Big(\Gamma^i_{j,\ell}(X)-\Gamma^{i,n}_{j,\ell}(X^n)\Big)\\&+\frac{\partial}{\partial y_k}\Big(g^{k\ell}(X)-g^{k\ell,n}(X^n)\Big)\Gamma^{i,n}_{j,\ell}(X^n)+\Big(g^{k\ell}(X)-g^{k\ell,n}(X^n)\Big)\frac{\partial}{\partial y_k}\Big(\Gamma^{i,n}_{j,\ell}(X^n)\Big).\end{aligned}\tag{4.3.44}$$

By using the estimates in lemmas A.1.3 and A.1.4, we obtain

$$\Big\|\frac{\partial}{\partial y_k}\Big(g^{k\ell}(X)\Gamma^i_{j,\ell}(X)-g^{k\ell,n}(X^n)\Gamma^{i,n}_{j,\ell}(X^n)\Big)\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le K_0T\sum_{m=1}^k\|h'_m-(h^n_m)'\|_{L^\infty([0,T]\times\mathcal{O})}.$$

The second coefficients of $W$ in (4.3.43) can be rewritten as follows

$$\begin{aligned}g^{k\ell}(X)\Gamma^m_{j,\ell}(X)\Gamma^i_{k,m}(X)-g^{k\ell,n}(X^n)\Gamma^{m,n}_{j,\ell}(X^n)\Gamma^{i,n}_{k,m}(X^n)=\Big(g^{k\ell}(X)-g^{k\ell,n}(X^n)\Big)\Gamma^m_{j,\ell}(X)\Gamma^i_{k,m}(X)\\+\Big\{\Gamma^m_{j,\ell}(X)\Big(\Gamma^i_{k,m}(X)-\Gamma^{i,n}_{k,m}(X^n)\Big)+\Big(\Gamma^m_{j,\ell}(X)-\Gamma^{m,n}_{j,\ell}(X^n)\Big)\Gamma^{i,n}_{k,m}(X^n)\Big\}\,g^{k\ell,n}(X^n).\end{aligned}\tag{4.3.45}$$

By using again the estimates in lemmas A.1.3 and A.1.4, we get

$$\|g^{k\ell}(X)\Gamma^m_{j,\ell}(X)\Gamma^i_{k,m}(X)-g^{k\ell,n}(X^n)\Gamma^{m,n}_{j,\ell}(X^n)\Gamma^{i,n}_{k,m}(X^n)\|_{L^\infty([0,T]\times\mathcal{O})}\le K_0T\sum_{m=1}^k\|h'_m-(h^n_m)'\|_{L^\infty([0,T]\times\mathcal{O})}.$$

By combining Lemma A.1.5 and all the proceeding estimates with the expression in (4.3.43), we deduce that

$$\|[(L-L^n)W]\|_{\mathbf{L}^2(\Omega_F)}\le K_0T\|W\|_{\mathbf{H}^2(\Omega_F)}\sum_{i=1}^k\|h'_i-(h^n_i)'\|_{L^\infty([0,T]\times\mathcal{O})}.\tag{4.3.46}$$

One can write

$$\begin{aligned}[(M-M^n)W]_i=\sum_{j=1}^2\Big(\frac{\partial Y_j}{\partial t}(X,.)-\frac{\partial Y^n_j}{\partial t}(X^n,.)\Big)\frac{\partial W_i}{\partial y_j}+\sum_{j,k=1}^2\Big\{\Gamma^i_{j,k}(X)\frac{\partial Y_k}{\partial t}(X,.)-\Gamma^{i,n}_{j,k}(X^n)\frac{\partial Y^n_k}{\partial t}(X^n,.)\\+\frac{\partial Y_i}{\partial x_k}(X,.)\frac{\partial^2X_k}{\partial t\partial y_j}-\frac{\partial Y^n_i}{\partial x_k}(X^n,.)\frac{\partial^2X^n_k}{\partial t\partial y_j}\Big\}W_j.\end{aligned}\tag{4.3.47}$$

We start by estimating the coefficient of $\dfrac{\partial W_i}{\partial y_j}$ in (4.3.47). One can easily prove that

$$\frac{\partial Y_j}{\partial t}(X,.)-\frac{\partial Y^n_j}{\partial t}(X^n,.)=\frac{\partial \bar{Y}^n_j}{\partial t}(X,.)+\frac{\partial Y^n_j}{\partial t}(X,.)-\frac{\partial Y^n_j}{\partial t}(X^n,.).$$

The above identity implies that

$$\Big|\frac{\partial Y_j}{\partial t}(X,.) - \frac{\partial Y_j^n}{\partial t}(X^n,.)\Big| \leq \Big\|\frac{\partial \bar{Y}^n}{\partial t}\Big\|_{L^\infty([0,T]\times\mathcal{O})} + \|\bar{X}^n\|_{L^\infty([0,T]\times\mathcal{O})}\Big\|\frac{\partial^2 Y^n}{\partial t\partial x}\Big\|_{L^\infty([0,T]\times\mathcal{O})}.$$

By writing down the Cauchy problem satisfied by $\bar{Y}^n$, we get

$$\Big\|\frac{\partial \bar{Y}^n}{\partial t}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0 \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})},$$

and thus

$$\Big|\frac{\partial Y_j}{\partial t}(X,.) - \frac{\partial Y_j^n}{\partial t}(X^n,.)\Big| \leq K_0 \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}. \tag{4.3.48}$$

It is easy to verify that

$$\begin{aligned}\Gamma^i_{j,k}(X)\frac{\partial Y_k}{\partial t}(X,.) - \Gamma^{i,n}_{j,k}(X^n)\frac{\partial Y_k^n}{\partial t}(X^n,.) &= \Gamma^i_{j,k}(X)\frac{\partial \bar{Y}_k^n}{\partial t}(X,.)\\ &+ \Gamma^i_{j,k}(X)\Big(\frac{\partial Y_k^n}{\partial t}(X,.) - \frac{\partial Y_k^n}{\partial t}(X^n,.)\Big) + \Big(\Gamma^i_{j,k}(X) - \Gamma^{i,n}_{j,k}(X^n)\Big)\frac{\partial Y_k^n}{\partial t}(X^n,.).\end{aligned}$$

By mean value theorem, we have

$$\begin{aligned}\Big|\Gamma^i_{j,k}(X)\frac{\partial Y_k}{\partial t}(X,.) - \Gamma^{i,n}_{j,k}(X^n)\frac{\partial Y_k^n}{\partial t}(X^n,.)\Big| &\leq K_0 T \sum_{m=1}^{k} \|h_m' - (h_m^n)'\|_{L^\infty([0,T]\times\mathcal{O})}\\ &+ K_0\|\bar{X}^n\|_{L^\infty([0,T]\times\mathcal{O})}\|\partial_t \nabla Y^2\|_{L^\infty([0,T]\times\mathcal{O})} + K_0|\Gamma^i_{j,k}(X) - \Gamma^{i,n}_{j,k}(X^n)|.\end{aligned}$$

Lemmas A.1.3 and A.1.4 imply that

$$\Big|\Gamma^i_{j,k}(X)\frac{\partial Y_k}{\partial t}(X,.) - \Gamma^{i,n}_{j,k}(X^n)\frac{\partial Y_k^n}{\partial t}(X^n,.)\Big| \leq K_0 T \sum_{i=1}^{k} \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}. \tag{4.3.49}$$

By writing

$$\begin{aligned}\frac{\partial^2 X_k}{\partial t\partial y_j}(y,t)\frac{\partial Y_i}{\partial x_k}(X(y,t),t) - \frac{\partial^2 X_k^n}{\partial t\partial y_j}(y,t)\frac{\partial Y_i^n}{\partial x_k}(X^n(y,t),t) &= \frac{\partial^2 X_k}{\partial t\partial y_j}(y,t)\frac{\partial \bar{Y}_i^n}{\partial x_k}(X(y,t),t) +\\ \frac{\partial^2 \bar{X}_k^n}{\partial t\partial y_j}(y,t)\frac{\partial Y_i^n}{\partial x_k}(X(y,t),t) &+ \frac{\partial^2 X_k^n}{\partial t\partial y_j}(y,t)\Big(\frac{\partial Y_i^n}{\partial x_k}(X(y,t),t) - \frac{\partial Y_i^n}{\partial x_k}(X^n(y,t),t)\Big),\end{aligned}$$

and using mean value theorem, we get

$$\Big|\frac{\partial^2 X_k}{\partial t\partial y_j}(y,t)\frac{\partial Y_i}{\partial x_k}(X(y,t),t) - \frac{\partial^2 X_k^n}{\partial t\partial y_j}(y,t)\frac{\partial Y_i^n}{\partial x_k}(X^n(y,t),t)\Big| \leq K_0 \sum_{m=1}^{k} \|h_m' - (h_m^n)'\|_{L^\infty([0,T]\times\mathcal{O})}. \tag{4.3.50}$$

Gathering together the estimates (4.3.48), (4.3.49), and (4.3.50) with the identity in (4.3.47), we obtain the desired estimate (4.3.38).

From the definition of the operator $G$ in (4.2.10), we have:

$$[(G-G^n)Q]_i = \sum_{j=1}^{2}\Big(g^{ij}(X)-g^{ij,n}(X^n)\Big)\frac{\partial Q}{\partial y_j}.$$

By Lemma A.1.5, we get

$$\|[(G^1-G^2)Q]_i\|_{L^2(\Omega_F)} \le K_0T\|\nabla Q\|_{\mathbf{L}^2(\Omega_F)}\sum_{j=1}^{k}\|h_j'-(h_j^n)'\|_{L^\infty([0,T]\times\mathcal{O})}.$$

Moreover, the definition of the operator $[\mathrm{div}\mathcal{T}]$ in (4.2.11) implies that

$$\begin{aligned}[(\mathrm{div}{-}\mathrm{div}^n)\mathcal{T}]_i(y,t) = \Bigg\{&\left(\frac{\partial\bar{Y}_i^n}{\partial x_k}(X(y,t),t)+\frac{\partial Y_i^n}{\partial x_k}(X(y,t),t)-\frac{\partial Y_i^n}{\partial x_k}(X^n(y,t),t)\right)\frac{\partial Y_m}{\partial x_\ell}(X(y,t),t)\\
&+\frac{\partial Y_i^n}{\partial x_k}(X^n(y,t),t)\Big(\frac{\partial Y_m}{\partial x_\ell}(X(y,t),t)-\frac{\partial Y_m}{\partial x_\ell}(X^n(y,t),t)+\frac{\partial\bar{Y}_m^n}{\partial x_\ell}(X^n(y,t),t)\Big)\Bigg\}\frac{\partial\mathcal{T}_{k\ell}}{\partial y_m}(y,t).\end{aligned}$$

Consequently, we get

$$\|[(\mathrm{div}-\mathrm{div}^n)\,\mathcal{T}]\|_{\mathbf{L}^2(\Omega_F)} \le K_0T\|\mathcal{T}\|_{[H^1(\Omega_F)]^4}\sum_{i=1}^{k}\|h_i'-(h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}.$$

The last two estimate follows analogously. □

In the following proposition, we estimate on the difference $[NW]-[N^nW^n]$ in $\mathbf{L}^2(\Omega_F)$.

**Proposition 4.3.6** *Let $(W,Q,\mathcal{T},(h_i,\omega_i)_{i=1,\dots,k})$ and $(W^n,Q^n,\mathcal{T}^n,(h_i^n,\omega_i^n)_{i=1,\dots,k})\in\tilde{\mathcal{K}}$. Then there exists a positive constant $K_0$ satisfying condition (i) such that*

$$\|[NW]-[N^nW^n]\|_{\mathbf{L}^2(\Omega_F)} \le K_0\|W-W^n\|_{\mathbf{H}^1(\Omega_F)}+K_0T\sum_{i=1}^{k}\|h_i'-(h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}. \qquad (4.3.51)$$

Proof. It is easy to check that

$$\begin{aligned}[NW]_i-[N^nW^n]_i &= (W\cdot\nabla)(W_i-W_i^n)+[(W-W^n)\cdot\nabla]W_i^n\\
&+\Big(\Gamma^i_{j,k}(X)-\Gamma^{i,n}_{j,k}(X^n)\Big)W_jW_k+\Gamma^{i,n}_{j,k}(X^n)\Big((W_j-W_j^n)W_k+W_j^n(W_k-W_k^n)\Big).\end{aligned}$$

By Holder inequality and the Sobolev injection of $H^2(\Omega_F)$ into $L^\infty(\Omega_F)$, we get

$$\begin{aligned}\|(W\cdot\nabla)(W_i - W_i^n)\|_{L^2(\Omega_F)} &\leq \|W\|_{\mathbf{L}^\infty(\Omega_F)}\|\nabla(W_i - W_i^n)\|_{\mathbf{L}^2(\Omega_F)}\\ &\leq C\|W\|_{\mathbf{H}^2(\Omega_F)}\|W - W^n\|_{\mathbf{H}^1(\Omega_F)}.\end{aligned}$$

By similar way, one can show that

$$\|[(W - W^n)\cdot\nabla]W_i^n\|_{L^2(\Omega_F)} \leq \|W - W^n\|_{\mathbf{H}^1(\Omega_F)}\|W^n\|_{\mathbf{H}^2(\Omega_F)}.$$

Combining the last two inequalities and using the estimates in Lemma A.1.5 and Lemma A.1.3, we get that

$$\|[NW] - [N^nW^n]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0\|W - W^n\|_{\mathbf{H}^1(\Omega_F)} + K_0T\sum_{i=1}^{k}\|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}.$$

□

We return now to complete the proof of Proposition 4.3.4. Lemma A.1.3 implies that

$$\begin{aligned}\Big|\int_{\Omega_F}[(L-\Delta)(W-W^n)]\cdot\tilde{U}^n dy\Big| \leq \sum_{i,j,k=1}^{2}\Big|\int_{\Omega_F}\frac{\partial}{\partial y_j}\Big((g^{jk}(X)-\delta_k^j)\frac{\partial(W_i - W_i^n)}{\partial y_k}\Big)\tilde{U}_i^n dy\Big|\\ + K_0\|W - W^n\|_{\mathbf{H}^1(\Omega_F)}\|\tilde{U}^n\|_{\mathbf{L}^2(\Omega_F)}.\end{aligned}$$

By performing integration by parts noting that $g_k^j = \delta_k^j$ on $\partial B_i$ for all $i = 1, \ldots, k$ and vanishes on $\partial\mathcal{O}$, we obtain

$$\begin{aligned}\Big|\int_{\Omega_F}[(L-\Delta)(W-W^n)]\cdot\tilde{U}^n dy\Big| \leq \sum_{i,j,k=1}^{2}\Big|\int_{\Omega_F}(g^{jk}(X)-\delta_k^j)\frac{\partial(W_i - W_i^n)}{\partial y_k}\frac{\partial\tilde{U}_i^n}{\partial y_j}dy\Big|\\ + K_0\|W - W^n\|_{\mathbf{H}^1(\Omega_F)}\|\tilde{U}^n\|_{\mathbf{L}^2(\Omega_F)}.\end{aligned}$$

Using again Lemma A.1.3 and Young's inequality, we get that there exists $\varepsilon' > 0$ such that

$$\Big|\int_{\Omega_F}[(L-\Delta)(W-W^n)]\cdot\tilde{U}^n dy\Big| \leq K_0\|W - W^n\|^2_{\mathbf{H}^1(\Omega_F)} + \|\tilde{U}^n\|^2_{\mathbf{L}^2(\Omega_F)} + \frac{\varepsilon'}{2}\|\nabla\tilde{U}^n\|^2_{\mathbf{L}^2(\Omega_F)}.$$

By performing again integration by parts noting that $g_k^j = \delta_k^j$ on $\partial B_i$ for all $i = 1, \ldots, k$ and vanishes on $\partial\mathcal{O}$, we obtain

$$\int_{\Omega_F}[(\nabla - G)(Q-Q^n)]\cdot\tilde{U}^n dy = \int_{\Omega_F}\frac{\partial g^{ij}}{\partial y_j}(Q_i-Q_i^n)\tilde{U}_i^n dy - \int_{\Omega_F}(\delta_j^i - g^{ij})(Q_i-Q_i^n)\frac{\partial \tilde{U}_i^n}{\partial y_j}dy.$$

By Lemma A.1.3 and using Young's inequality, we get that there exist $\varepsilon'>0$ such that

$$\Big|\int_{\Omega_F}[(\nabla - G)(Q-Q^n)]\cdot\tilde{U}^n dy\Big| \leq K_0T\|Q-Q^n\|^2_{L^2(\Omega_F)} + \frac{\varepsilon'}{2}\|\nabla\tilde{U}^n\|^2_{\mathbf{L}^2(\Omega_F)} + \|\tilde{U}^n\|^2_{\mathbf{L}^2(\Omega_F)}.$$

Furthermore, it follows from lemmas A.1.3, and Corollary A.1.1 in Appendix A at the end of this thesis that

$$\begin{aligned} \|[\mathrm{div}(\mathcal{T}-\mathcal{T}^n)]\|_{\mathbf{L}^2(\Omega_F)} &\leq K_0\|\mathcal{T}-\mathcal{T}^n\|_{\mathbf{H}^1(\Omega_F)}, && (4.3.52)\\ \|[M(W-W^n)]\|_{\mathbf{L}^2(\Omega_F)} &\leq K_0\Big(\sum_{i=1}^k\|h_i'-(h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})} + \|W-W^n\|_{\mathbf{L}^2(\Omega_F)}\Big). && (4.3.53)\end{aligned}$$

Combining the above inequalities with Proposition 4.3.5 and Proposition 4.3.6, we get that

$$\begin{aligned}\Big|\int_{\Omega_F}F_0^n(t)\cdot\tilde{U}^n(t)dy\Big| \leq K_0\Big(&\|W-W^n\|^2_{\mathbf{H}^1(\Omega_F)} + \|\nabla(Q-Q^n)\|^2_{\mathbf{H}^{-1}(\Omega_F)} + \|\mathcal{T}-\mathcal{T}^n\|^2_{\mathbf{H}^1(\Omega_F)}\\ &+\sum_{i=1}^k\|h_i'-(h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big) + 3\|\tilde{U}^n\|^2_{\mathbf{L}^2(\Omega_F)} + \varepsilon'\|\nabla\tilde{U}^n\|^2_{\mathbf{L}^2(\Omega_F)},\end{aligned}$$

and thus by Young's inequality, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(Re\|\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k\overline{m}_i|(\tilde{H}_i^n)'(t)|^2 + \overline{J}_i|\tilde{W}_i^n(t)|^2\Big) + 2(1-r)\|\nabla\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\mathcal{O})}\\ &\leq K_0\Big(\|W-W^n\|^2_{\mathbf{H}^1(\Omega_F)} + \|\nabla(Q-Q^n)\|^2_{\mathbf{H}^{-1}(\Omega_F)} + \|\mathcal{T}-\mathcal{T}^n\|^2_{\mathbf{H}^1(\Omega_F)} + \sum_{i=1}^k\|h_i'-(h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big)\\ &\quad+\frac{4}{Re}\Big(\frac{Re}{2}\|\tilde{U}^n\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k\overline{m}_i|(\tilde{H}_i^n)'(t)|^2 + \overline{J}_i|\tilde{W}_i^n(t)|^2\Big) + \varepsilon'\|\nabla\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\Omega_F)}.\end{aligned}$$

By choosing $\varepsilon'$ small enough, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(Re\|\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k\overline{m}_i|(\tilde{H}_i^n)'(t)|^2 + \overline{J}_i|\tilde{W}_i^n(t)|^2\Big) + (2(1-r)-\varepsilon')\|\nabla\tilde{U}^n\|^2_{\mathbf{L}^2(\mathcal{O})}\\ &\leq \frac{4}{Re}\Big(\frac{Re}{2}\|\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k\overline{m}_i|(\tilde{H}_i^n)'(t)|^2 + \overline{J}_i|\tilde{W}_i^n(t)|^2\Big) + K_0\Big(\|W-W^n\|^2_{\mathbf{H}^1(\Omega_F)} + \|\mathcal{T}-\mathcal{T}^n\|^2_{\mathbf{H}^1(\Omega_F)}\\ &\quad+\|\nabla(Q-Q^n)\|^2_{\mathbf{H}^{-1}(\Omega_F)} + \sum_{i=1}^k\|h_i'-(h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big). \qquad (4.3.54)\end{aligned}$$

By integrating the above inequality over $(0,t) \subset [0,T]$, we get that for almost $t \in (0,T)$:

$$
\begin{aligned}
&\frac{1}{2}\Big(Re\|\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k \overline{m}_i|(\tilde{H}_i^n)'(t)|^2 + \overline{J}_i|\tilde{W}_i^n(t)|^2\Big)\\
&\leq \frac{4}{Re}\int_0^t \Big(\frac{Re}{2}\|\tilde{U}^n(s)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k \frac{\overline{m}_i}{2}|(\tilde{H}_i^n)'(s)|^2 + \frac{\overline{J}_i}{2}|\tilde{W}_i^n(s)|^2\Big)ds + K_0T\Big(\|W-W^n\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))}\\
&\qquad + \|\nabla(Q-Q^n)\|^2_{L^\infty(0,T;\mathbf{H}^{-1}(\Omega_F))} + \|\mathcal{T}-\mathcal{T}^n\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^k \|h_i' - (h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big).
\end{aligned}
$$

By Gronwall's lemma, we obtain for almost $t \in (0,T)$

$$
\begin{aligned}
&\frac{1}{2}\Big(Re\|\tilde{U}^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k \overline{m}_i|(\tilde{H}_i^n)'(t)|^2 + \overline{J}_i|\tilde{W}_i^n(t)|^2\Big)\\
&\quad\leq K_0T\Big(\|W-W^n\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|\nabla(Q-Q^n)\|^2_{L^\infty(0,T;\mathbf{H}^{-1}(\Omega_F))} + \|\mathcal{T}-\mathcal{T}^n\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))}\\
&\qquad + \sum_{i=1}^k \|h_i' - (h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big). \qquad (4.3.55)
\end{aligned}
$$

Thus $(\tilde{U}^n, (\tilde{H}_i^n, \tilde{W}_i^n)_{i=1,\ldots,k}) \to 0$, as $n \to +\infty$ in $L^\infty(0,T;\mathbf{L}^2(\Omega_F)) \times \big(L^\infty(0,T;\mathbb{R}^2) \times \infty(0,T;\mathbb{R}^2)\big)^k$. We set

$$
\tilde{\mathcal{U}}^n(t) = \int_0^t \tilde{U}^n(s)ds \;\text{ and }\; \mathcal{F}_0^n(t) = \int_0^t F_0^n(s)ds.
$$

We note that $\tilde{\mathcal{U}}^n$ and $\tilde{\mathcal{F}}_0^n$ belong to $\mathcal{C}([0,T];\mathbf{H}^{-1}(\Omega_F))$. By integrating (4.3.1) from 0 to $t$ and since $\tilde{U}^n(0) = 0$, we get that

$$
(1-r)\int_{\Omega_F} \nabla\tilde{\mathcal{U}}^n : \nabla\phi = \langle Re\,\tilde{U}^n(t) + \mathcal{F}_0^n, \phi\rangle_{\mathbf{H}^{-1}(\Omega_F),\mathbf{H}^1(\Omega_F)},
$$

for all $\phi \in \mathbf{H}^1(\Omega_F)$ such that $\nabla\cdot\phi = 0$ and $t \in [0,T]$. By applying De Rahm theorem, for each $t \in [0,T]$, there exists $\mathcal{P}^n(t)$ in $\mathcal{D}(\Omega_F)$ such that

$$
Re\,\tilde{U}^n(t) - (1-r)\Delta\tilde{\mathcal{U}}^n(t) + \nabla\mathcal{P}^n(t) = \mathcal{F}_0^n(t).
$$

We emphasize that $\nabla\tilde{\mathcal{P}}^n \in \mathcal{C}([0,T];\mathbf{H}^{-1}(\Omega_F))$ and hence $\tilde{\mathcal{P}}^n \in \mathcal{C}([0,T];\mathbf{L}^2(\Omega_F))$. Moreover, it follows (4.3.54) from that $\nabla\tilde{U}^n \to 0$ as $n \to \infty$ in $L^2(0,T;\mathbf{L}^2(\mathcal{O}))$ and hence

$$
\nabla\tilde{\mathcal{P}}^n \to 0,\;\; as\, n \to \infty \;\; in\;\; L^2(0,T;\mathbf{H}^{-1}(\Omega_F)).
$$

Setting

$$\tilde{P}^n = \frac{d\tilde{\mathcal{P}}^n}{dt},$$

we obtain

$$Re\,\partial_t \tilde{U}^n - (1-r)\Delta \tilde{U}^n + \nabla \tilde{P}^n = F_0^n, \text{ in } \Omega_F \times (0,T),$$

and

$$\nabla \tilde{P}^n \to 0, \text{ as } n \to +\infty, \text{ in } H^{-1}(0,T;\mathbf{H}^{-1}(\Omega_F)).$$

The pressure $\tilde{P}^n$, defined as above, appears in general as a distribution on $\Omega_F \times (0,T)$. By Proposition 1.2 in [43], we get that

$$\tilde{P}^n \to 0, \text{ as } n \to +\infty, \text{ in } H^{-1}(0,T;L^2(\Omega_F)/\mathbb{R}).$$

Since $\tilde{\mathcal{N}}$ maps $\tilde{\mathcal{K}}$ into itself, then we have $\tilde{P}^n$ is bounded in $L^\infty(0,T;H^1(\Omega_F))$ and $\partial_t \tilde{P}^n$ is bounded in $L^2(0,T;Ł^2(\Omega_F))$. By Aubin-Simon theorem in [8], there exists a subsequence denoted also by $\{\tilde{P}^n\}_n$ such that

$$\tilde{P}^n \to P^*, \text{ as } n \to +\infty, \text{ in } L^\infty(0,T;L^2(\Omega_F)).$$

By the concept of uniqueness of limit, we get $P^* = 0$.

To conclude the continuity of the mapping $\tilde{\mathcal{N}}$, it remains to show that $Z^n \to 0$, as $n \to +\infty$ in $L^\infty([0,T],\mathbf{H}^1(\Omega_F))$. To this end, we move now to bound $\|\mathcal{G}^n\|_{\mathbf{L}^2(\Omega_F)}$ in the right hand side of (4.3.36).

We collect the terms of the difference $[G_a^n(W^n,\tilde{\mathcal{T}}^n)] - [G_a(W,\tilde{\mathcal{T}})]$ as follows:

$$\begin{aligned}[G_a^n(W^n,\tilde{\mathcal{T}}^n)]_{ij} - [G_a(W,\tilde{\mathcal{T}})]_{ij} &= ([\mathcal{W}^n W^n]_{ik}\tilde{\mathcal{T}}_{kj}^n - [\mathcal{W}W]_{ik}\tilde{\mathcal{T}}_{kj}) + (\tilde{\mathcal{T}}_{ik}^n[\mathcal{W}^n W^n]_{kj} - \tilde{\mathcal{T}}_{ik}[\mathcal{W}W]_{kj}) \\ &\quad - a([\mathcal{D}^n W^n]_{ik}\tilde{\mathcal{T}}_{kj}^n - [\mathcal{D}W]_{ik}\tilde{\mathcal{T}}_{kj}) - a(\tilde{\mathcal{T}}_{ik}^n[\mathcal{D}^n W^n]_{kj} - \tilde{\mathcal{T}}_{ik}[\mathcal{D}W]_{kj}). \quad (4.3.56)\end{aligned}$$

It is not difficult to verify that

$$[\mathcal{W}^n W^n]_{ik}\tilde{\mathcal{T}}_{kj}^n - [\mathcal{W}W]_{ik}\tilde{\mathcal{T}}_{kj} = [\mathcal{W}^n(W^n - W)]_{ik}\tilde{\mathcal{T}}_{kj}^n + [\mathcal{W}^n W]_{ik} Z_{kj}^n + [(\mathcal{W}^n - \mathcal{W})W]_{ik}\tilde{\mathcal{T}}_{kj}.$$

This implies that

$$\begin{aligned}\|[\mathcal{W}^n W^n]_{ik}\tilde{\mathcal{T}}_{kj}^n - [\mathcal{W}W]_{ik}\tilde{\mathcal{T}}_{kj}\|_{L^2(\Omega_F)} &\le \|[\mathcal{W}^n(W^n - W)]_{ik}\|_{L^2(\Omega_F)}\|\tilde{\mathcal{T}}_{kj}^n\|_{L^\infty(\Omega_F)} \\ &\quad + \|[\mathcal{W}^n W]_{ik}\|_{L^\infty(\Omega_F)}\|Z_{kj}^n\|_{L^2(\Omega_F)} + \|[(\mathcal{W}^n - \mathcal{W})W]_{ik}\|_{L^2(\Omega_F)}\|\tilde{\mathcal{T}}_{kj}\|_{L^\infty(\Omega_F)}.\end{aligned}$$

By the Sobolev injection of $H^2(\Omega_F)$ into $L^\infty(\Omega_F)$, we get that

$$\|[\mathcal{W}^n W^n]_{ik}\tilde{\mathcal{T}}^n_{kj} - [\mathcal{W}W]_{ik}\tilde{\mathcal{T}}_{kj}\|_{L^2(\Omega_F)} \leq \|[\mathcal{W}^n(W^n - W)]\|_{\mathbf{L}^2(\Omega_F)}\|\tilde{\mathcal{T}}^n\|_{\mathbf{H}^2(\Omega_F)} + \|[\mathcal{W}^n W]_{ik}\|_{L^\infty(\Omega_F)}\|Z^n\|_{\mathbf{L}^2(\Omega_F)} + \|[(\mathcal{W}^n - \mathcal{W})W]\|_{\mathbf{L}^2(\Omega_F)}\|\tilde{\mathcal{T}}\|_{\mathbf{H}^2(\Omega_F)}.$$

Using again Lemma A.1.3 and Proposition 4.3.5 noting that $\|\tilde{\mathcal{T}}\|_{\mathbf{H}^2(\Omega_F)} \leq R$ as the mapping $\tilde{\mathcal{N}}$ maps $\tilde{\mathcal{K}}$ into itself, we obtain

$$\|[\mathcal{W}^n W^n]_{ik}\tilde{\mathcal{T}}^n_{kj} - [\mathcal{W}W]_{ik}\tilde{\mathcal{T}}_{kj}\|_{L^2(\Omega_F)} \leq K_0\Big(\|W^n - W\|_{\mathbf{H}^1(\Omega_F)} + \|W\|_{\mathbf{H}^3(\Omega_F)}\|Z^n\|_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times‰)}\Big).$$

The other terms in (4.3.56) can be treated in similar way, and thus we obtain

$$\|[G_a^n(W^n, \tilde{\mathcal{T}}^n)] - [G_a(W, \tilde{\mathcal{T}})]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0\Big(\|W^n - W\|_{\mathbf{H}^1(\Omega_F)} + \|W\|_{\mathbf{H}^3(\Omega_F)}\|Z^n\|_{[L^2(\Omega_F)]^4} + \sum_{i=1}^k \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}\Big). \tag{4.3.57}$$

Moreover, by Holder's inequality and using the Sobolev injections noting that $\|\tilde{\mathcal{T}}\|_{\mathbf{H}^2(\Omega_F)} \leq R$, we obtain

$$\Big\|\Big(\Big(W - W^n + \frac{\partial Y}{\partial t}(X(.,t),t) - \frac{\partial Y^n}{\partial t}(X^n(.,t),t)\Big)\cdot\nabla\Big)\tilde{\mathcal{T}}\Big\|_{\mathbf{L}^2(\Omega_F)} \leq K_0\Big(\|W - W^n\|_{\mathbf{H}^1(\Omega_F)} + \sum_{i=1}^k \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}\Big). \tag{4.3.58}$$

Using similar arguments as above, we get

$$\|[(\mathcal{D} - \mathcal{D}^n)\, W^n]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0 T \sum_{i=1}^k \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}, \tag{4.3.59}$$

$$\|[\mathcal{D}(W - W^n)]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0\|W - W^n\|_{\mathbf{H}^1(\Omega_F)}. \tag{4.3.60}$$

Putting together inequalities (4.3.57)-(4.3.60), we deduce that

$$\|\mathcal{G}^n(t)\|_{\mathbf{L}^2(\Omega_F)} \leq K_0\Big\{(We + 2r)\Big(\|W(t) - W^n(t)\|_{\mathbf{H}^1(\Omega_F)} + \sum_{i=1}^k \|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}\Big) + We\|W(t)\|_{\mathbf{H}^3(\Omega_F)}\|Z^n(t)\|_{\mathbf{L}^2(\Omega_F)}\Big\}.$$

By combining the above estimate with inequality (4.3.36), we obtain that

$$\frac{We}{2}\frac{d}{dt}\Big(\|Z^n(t)\|^2_{\mathbf{L}^2(\Omega_F)}\Big) + \|Z^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} \leq K_0 We\|W\|_{\mathbf{H}^3(\Omega_F)}\|Z^n(t)\|^2_{[L^2(\Omega_F)]^4}$$
$$+ K_0\Big(\|W - W^n\|_{\mathbf{H}^1(\Omega_F)} + \sum_{i=1}^{k}\|h_i' - (h_i^n)'\|_{L^\infty([0,T]\times\mathcal{O})}\Big)\|Z^n(t)\|_{\mathbf{L}^2(\Omega_F)}.$$

By Young's inequality, we get that

$$\frac{We}{2}\frac{d}{dt}\|Z^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} \leq K_0\Big(We\|W(t)\|_{\mathbf{H}^3(\Omega_F)}\|Z^n(t)\|^2_{\mathbf{L}^2(\Omega_F)}$$
$$+ \|W(t) - W^n(t)\|^2_{\mathbf{H}^1(\Omega_F)} + \sum_{i=1}^{k}\|h_i' - (h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big).$$

Integrating both sides of the above inequality with respect to time on $(0,t) \subset (0,T)$, we get that

$$\|Z^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} \leq K_0 We\int_0^t \|W(s)\|_{\mathbf{H}^3(\Omega_F)}\|Z^n(s)\|^2_{\mathbf{L}^2(\Omega_F)}ds$$
$$+ K_0 T\Big(\|W - W^n\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^{k}\|h_i' - (h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big).$$

Gronwall's lemma implies that

$$\|Z^n(t)\|^2_{\mathbf{L}^2(\Omega_F)} \leq K_0 T\Big(\|W - W^n\|^2_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \sum_{i=1}^{k}\|h_i' - (h_i^n)'\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big).$$

The above relation implies that

$$\tilde{\mathcal{T}}^n \to \tilde{\mathcal{T}}, \quad \text{as } n \to +\infty \text{ in } L^\infty(0,T;\mathbf{L}^2(\Omega_F)).$$

We deduce from Corollary 4.3.1 and Proposition 4.3.4 that the mapping $\tilde{\mathcal{N}}$ admits at least one fixed point by applying Schauder fixed-point theorem . Consequently, we obtain the local existence of a strong solution of problem (4.1.14)-(4.1.21), (4.1.25)-(4.1.26). Therefore, to prove Theorem 4.1.2 we need to show the uniqueness of the solution in its class of existence.

Suppose that the mapping $\tilde{\mathcal{N}}$ admits two fixed points $(W^1, Q^1, \mathcal{T}^1, (h_i^1, \omega_i^1)_{i=1,\dots,k})$ and $(W^2, Q^2, \mathcal{T}^2, (h_i^2, \omega_i^2)_{i=1,\dots,k})$ in $\tilde{\mathcal{K}}$. This implies that

$$\tilde{\mathcal{N}}(W^1, Q^1, \mathcal{T}^1, (h_i^1, \omega_i^1)_{i=1,\dots,k}) = (W^1, Q^1, \mathcal{T}^1, (h_i^1, \omega_i^1)_{i=1,\dots,k}),$$
$$\tilde{\mathcal{N}}(W^2, Q^2, \mathcal{T}^2, (h_i^2, \omega_i^2)_{i=1,\dots,k}) = (W^2, Q^2, \mathcal{T}^2, (h_i^2, \omega_i^2)_{i=1,\dots,k}).$$

Denote by $Y^i, X^i, g^{k\ell,i}, \Gamma^{k,i}_{j,\ell}, U^i, P^i$, etc the terms corresponding to $(W^i, Q^i, \mathcal{T}^i, (h^1_j, \omega^1_j)_{j=1,\dots,k})$. Moreover, we denote by $Y = Y^1 - Y^2, h_i = h^1_i - h^2_i$, etc. We get that the difference $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k})$ satisfies system (2.3.1) and

$$\frac{\partial \mathcal{T}}{\partial t} + \Big(\Big(W^1 + \frac{\partial Y^1}{\partial t}(X^1(y,t),t)\Big) \cdot \nabla\Big)\mathcal{T} = G, \text{ in } \Omega_F \times ]0,T], \tag{4.3.61}$$

with zero initial conditions and source terms:

$$\begin{aligned}
F_0 &= (1-r)([(L^1-\Delta)W] + [(L^1-L^2)W^2]) + [(\nabla - G^1)Q] - [(G^1-G^2)Q^2] - Re([M^1W] \\
&\quad + [(M^1-M^2)W^2] + [N^1W^1] - [N^2W^2]) + [\mathrm{div}^1\mathcal{T}] + [(\mathrm{div}^1 - \mathrm{div}^2)\mathcal{T}^2], \\
F_{1,i} &= -\int_{\partial B_i} \mathcal{T}\nu_i d\Gamma_i, \\
F_{2,i} &= -\int_{\partial B_i} \mathcal{T}\nu_i \cdot (y - h^0_i)^\perp d\Gamma_i, \\
\mathcal{G} &= -\Big(\Big(W + \frac{\partial Y^1}{\partial t}(X^1(y,t),t) - \frac{\partial Y^2}{\partial t}(X^2(y,t),t)\Big) \cdot \nabla\Big)\tilde{\mathcal{T}}^2 - [G_a(W^1, \tilde{\mathcal{T}}^1)] + [G_a(W^2, \tilde{\mathcal{T}}^2)] \\
&\quad - \frac{1}{We}\tilde{\mathcal{T}} + \frac{2r}{We}[\mathcal{D}^1 W] + \frac{2r}{We}[(\mathcal{D}^1 - \mathcal{D}^2)W^2].
\end{aligned}$$

Using similar arguments as previously, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(Re\|W(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^k \overline{m}_i|h_i'(t)|^2 + \overline{J}_i|\omega(t)|^2\Big) + (1-r)\|\nabla W(t)\|^2_{\mathbf{L}^2(\mathcal{O})} \\
= \int_{\Omega_F} F_0(t) \cdot W(t)dy - \int_{\partial\Omega_F} \mathcal{T}(t)\mathbf{n} \cdot W(t) d\Gamma.
\end{aligned} \tag{4.3.62}$$

By performing integration by parts, one has:

$$\int_{\Omega_F} [\mathrm{div}^1\mathcal{T}] \cdot W\, dy = -\int_{\Omega_F} \frac{\partial}{\partial y_m}\Big(\frac{\partial Y^1_i}{\partial x_k}(X^1)\frac{\partial Y^1_m}{\partial x_\ell}(X^1)W_i\Big)\mathcal{T}_{k\ell}dy + \int_{\partial\Omega_F} \mathcal{T}(t)\mathbf{n} \cdot W\, d\Gamma. \tag{4.3.63}$$

However, we have by Lemma A.1.3

$$\Big|\int_{\Omega_F} \frac{\partial}{\partial y_m}\Big(\frac{\partial Y^1_i}{\partial x_k}(X^1)\frac{\partial Y^1_m}{\partial x_\ell}(X^1)W_i\Big)\mathcal{T}_{k\ell}dy\Big| \le K_0\|W\|^2_{\mathbf{L}^2(\Omega_F)} + \|\mathcal{T}\|^2_{\mathbf{L}^2(\Omega_F)} + \varepsilon'\|\nabla W\|_{\mathbf{L}^2(\Omega_F)}. \tag{4.3.64}$$

Moreover, using again Lemma A.1.3 we get

$$\begin{aligned}
\Big|\int_{\Omega_F} [(L^1-\Delta)W] \cdot W dy\Big| &\le K_0 T\|W\|^2_{\mathbf{H}^1(\Omega_F)}, && (4.3.65) \\
\Big|\int_{\Omega_F} [(\nabla - G^1)Q] \cdot W dy\Big| &\le K_0 T\|\nabla Q\|_{\mathbf{H}^{-1}(\Omega_F)}\|W\|_{\mathbf{L}^2(\Omega_F)}. && (4.3.66) \\
\Big|\int_{\Omega_F} [M^1 W] \cdot W dy\Big| &\le \varepsilon'\|\nabla W\|^2_{\mathbf{L}^2(\Omega_F)} + \|W\|^2_{\mathbf{L}^2(\Omega_F)}. && (4.3.67)
\end{aligned}$$

Replacing $X$ by $X^1$ and $X^n$ by $X^2$ in propositions 4.3.5 and 4.3.6, we get that

$$\|[(L^1 - L^2)W^1]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0 T \sum_{i=1}^{k} \|h_i'\|_{L^\infty([0,T]\times\mathcal{O})},$$

$$\|[(M^1 - M^2)W^1]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0 \sum_{i=1}^{k} \|h_i'\|_{L^\infty([0,T]\times\mathcal{O})},$$

$$\|[(G^1 - G^2)Q^1]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0 T \sum_{i=1}^{k} \|h_i'\|_{L^\infty([0,T]\times\mathcal{O})},$$

$$\|[(\mathrm{div}^1 - \mathrm{div}^2)\,\mathcal{T}^1]\|_{\mathbf{L}^2(\Omega_F)} \leq K_0 T \sum_{i=1}^{k} \|h_i'\|_{L^\infty([0,T]\times\mathcal{O})}.$$

Moreover, we have

$$\Big|\int_{\Omega_F} ([N^1W^1]_i - [N^2W^2]_i)\cdot W dy\Big| \leq K_0\|W\|^2_{\mathbf{L}^2(\Omega_F)} + \varepsilon'\|\nabla W\|^2_{\mathbf{L}^2(\Omega_F)} + K_0 T \sum_{i=1}^{k} \|h_i'\|^2_{L^\infty([0,T]\times\mathcal{O})}. \tag{4.3.68}$$

De Rham theorem implies that

$$\|\nabla Q\|_{\mathbf{H}^{-1}(\Omega_F)} \leq C\big(\|W\|_{\mathbf{H}^1(\Omega_F)} + \|F_0\|_{H^{-1}(\Omega_F)}\big). \tag{4.3.69}$$

Hence, by using the above estimates and choosing $\varepsilon$ and $T$ small enough we get

$$\frac{1}{2}\frac{d}{dt}\Big(Re\|W(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k} \overline{m}_i|h_i'(t)|^2 + \overline{J}_i|\omega(t)|^2\Big)$$
$$\leq K_0\Big(Re\|W(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k} \overline{m}_i\|h_i'(t)\|^2_{L^\infty([0,T]\times\mathcal{O})} + \|\mathcal{T}(t)\|^2_{\mathbf{L}^2(\Omega_F)}\Big) \tag{4.3.70}$$

By multiplying equation (4.3.61) scalar by $\mathcal{T}$ in $\mathbf{L}^2(\Omega_F)$ and using similar arguments as above, we get

$$\frac{We}{2}\frac{d}{dt}\|\mathcal{T}(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \|\mathcal{T}(t)\|^2_{\mathbf{L}^2(\Omega_F)}$$
$$\leq K_0\Big((1 + \|W^1(t)\|_{\mathbf{H}^3(\Omega_F)})\|\mathcal{T}(t)\|^2_{[L^2(\Omega_F)]^4} + \varepsilon'\|W(t)\|^2_{\mathbf{H}^1(\Omega_F)} + \sum_{i=1}^{k} \|h_i'(t)\|^2_{L^\infty([0,T]\times\mathcal{O})}\Big).$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\Big(Re\|W(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k}\overline{m}_i|h_i'(t)|^2 + \overline{J}_i|\omega(t)|^2 + We\|\mathcal{T}(t)\|^2_{\mathbf{L}^2(\Omega_F)}\Big)$$
$$\leq K_0\big(1+\|W^1(t)\|_{\mathbf{H}^3(\Omega_F)}\big)\Big(Re\|W(t)\|^2_{\mathbf{L}^2(\Omega_F)} + \sum_{i=1}^{k}\overline{m}_i\|h_i'(t)\|^2_{L^\infty([0,T]\times\mathcal{O})} + We\|\mathcal{T}(t)\|^2_{\mathbf{L}^2(\Omega_F)}\Big).$$

Integrating the above inequality over $(0,t)\subset[0,T]$, and applying Gronwall lemma we get that for $T$ small enough

$$(W,\nabla Q,\tau,(h_i,\omega_i)_{i=1,\ldots,k}) = 0, \text{ in } \Omega_F\times[0,T].$$

# A

# Appendix

In the first part of this appendix, we recall some properties of the transforms $X$. The second part is devoted to show that there exists a strong 2-extension operator $E$ for $\Omega_F(t)$. We end this part by bounding the $\mathbf{H}^2$ norm of the velocity field $u$. Finally, we prove Proposition 4.2.1 in the last part.

## A.1 Technical details on the change of variables $X$

In this section, we recall the the transform $X$ and some easily verified properties of $X$ and its inverse mapping $Y$. To this end, we fix $k$ functions $h_i : t \mapsto h_i(t)$ such that for $i \in \{1, \dots, k\}$, we assume that $h_i \in H^2(0, T; \mathbb{R}^2)$. Moreover, we define a family of regular cut-off function $\{\psi_i\}_{i=1}^k$ such that each has a compact support contained in $B(h_i(0), r_i + \frac{\gamma}{2})$ and equal 1 in a neighbourhood $V_{B_i}$ of $i$-th disk contained in $B(h_i(0), r_i + \frac{\gamma}{2})$, where $r_i$ denotes the radius of the $i$-th disk. Furthermore, we define the mapping $\Lambda : \mathbb{R}^2 \times [0, T] \to \mathbb{R}^2$ by

$$\Lambda(x, t) = \sum_{i=1}^{k} \nabla^{\perp}(h_i'(t) \cdot x^{\perp} \psi_i(x)).$$

The mapping $X$ is defined as a solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial X}{\partial t}(y,t) & = \Lambda(X(y,t),t), \;\; t \in ]0,T], \\ X(y,0) & = y \in \mathbb{R}^2. \end{cases} \tag{A.1.1}$$

For all $y \in \mathbb{R}^2$, the initial-value problem (A.1.1) admits a unique solution $X(y,.) : [0,T] \to \mathbb{R}^2$, which is $\mathcal{C}^1$ on $[0,T]$. Moreover, the mapping $X(.,t)$ is a $\mathcal{C}^\infty$-diffeomorphism from $\mathcal{O}$ into itself and from $B_i$ onto $B_i(t)$ whenever $B_i(t) \subset V_{B_i}$. Furthermore, the inverse mapping $Y$ of $X$ satisfies

$$\begin{cases} \dfrac{\partial Y}{\partial t}(x,s) & = -\Lambda(Y(x,s),t-s), \;\; t \in ]0,T], \\ Y(x,0) & = x \in \mathbb{R}^2, \end{cases} \tag{A.1.2}$$

Hence, one can easily verify that for all $t$ such that $B_i(t) \subset V_{B_i}$ we have

$$Y(x,t) = x - h_i(t) + h_i(0), \quad \text{if } x \in \partial B_i(t),$$
$$Y(x,t) = 0, \quad \text{if } x \in \partial \mathcal{O}.$$

Moreover, we have

$$\frac{dY}{dt}(x,t) = -h_i'(t), \quad \text{if } x \in \partial B_i(t), \tag{A.1.3}$$
$$\frac{dY}{dt}(x,t) = 0, \quad \text{if } x \in \partial \mathcal{O}. \tag{A.1.4}$$

First, we recall that for $T > 0$,

$$\mathcal{K} = \Big\{(W,Q,\mathcal{T},(h_i,\omega_i)_{i=1,\dots,k}) \in \mathcal{U}(0,T;\Omega_F)\times L^2(0,T;\dot{H}^1(\Omega_F))\times\mathfrak{T}(0,T;\Omega_F)\times\big(H^2(0,T;\mathbb{R}^2)\times H^1(0,T;\mathbb{R})\big)^k : \\ \|W\|_{\mathcal{U}(0,T;\Omega_F)} + \|Q\|_{L^2(0,T;\dot{H}^1(\Omega_F))} + \sum_{i=1}^{k} \|h_i^{''}\|_{L^2(0,T;\mathbb{R}^2)} + \|\omega_i'\|_{L^2(0,T;\mathbb{R})} + \|\mathcal{T}\|_{\mathfrak{T}(0,T;\Omega_F)} \leq R\Big\}.$$

We recall also that $N_K$ and $N_C$ be two positive quantities which satisfy the following conditions (see Chapter 4):

i. $N_K$ is a positive function of $(h_i^1, \omega_i^0)_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^1(\Omega_F)}, \|\tau_0\|_{\mathbf{H}^1(\Omega_F)}, T$ and $R$ which is non-decreasing with respect to $T, R, \|u_0\|_{\mathbf{H}^1(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

ii. $N_C$ is a positive function of $(h_i^1, \omega_i^0,)_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^1(\Omega_F)}, r,$ and $T$ which is non-decreasing with respect to $T, \|u_0\|_{\mathbf{H}^1(\Omega_F)}, \|\tau_0\|_{\mathbf{H}^1(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

The following lemma allows us to bound the coefficients of the operators in the source terms in the linearized problem corresponding to problem (4.1.14)-(4.1.24). We refer the reader to [40] for a similar proof.

**Lemma A.1.1** *Suppose that $(W,Q,\mathcal{T},(h_i,\omega_i)_{i=1,\dots,k})\in\mathcal{K}$, then there exists two constants $N_K$ and $N_C$ satisfying conditions (i) and (ii) respectively, such that*

$$
\begin{aligned}
&\Big\|\frac{\partial X}{\partial y}\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_K, &&\Big\|\frac{\partial^{i+j}X}{\partial y_1^i\partial y_2^j}\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_KT, &&1<i+j\le 3,\\
&\Big\|\frac{\partial Y}{\partial x}\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_K, &&\Big\|\frac{\partial^{i+j}Y}{\partial x_1^i\partial x_2^j}\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_KT, &&1<i+j\le 3,\\
&\Big\|\frac{\partial^2 X_m}{\partial t\partial y_\ell}\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_K, &&\Big\|\frac{\partial^2 Y_m}{\partial t\partial x_\ell}\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_K, &&\ell,m,n\in\{1,2\},\\
&\Big\|\frac{\partial X_m}{\partial y_\ell}-\delta_m^\ell\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_KT, &&\Big\|\frac{\partial Y_m}{\partial x_\ell}-\delta_m^\ell\Big\|_{L^\infty([0,T]\times\mathcal{O})}\le N_KT, &&\ell,m\in\{1,2\},\\
&\|g^{m\ell}-\delta_m^\ell\|_{L^\infty([0,T]\times\mathcal{O})}\le N_KT, &&\|g_{m\ell}-\delta_m^\ell\|_{L^\infty([0,T]\times\mathcal{O})}\le N_KT, &&\ell,m\in\{1,2\}.
\end{aligned}
$$

By using Cauchy-Schwartz inequality and mean value theorem, one can easily check the following.

**Lemma A.1.2** *Suppose that $(W^1,Q^1,\mathcal{T}^1,(h_i^1,\omega_i^1)_{i=1,\dots,k})$ and $(W^2,Q^2,\mathcal{T}^2,(h_i^2,\omega_i^2)_{i=1,\dots,k})$ in $\mathcal{K}$, and let $Y^i,X^i,\Gamma_{j,\ell}^{ik}$, etc. the terms corresponding to $(W^i,Q^i,\mathcal{T}^i,(h_j^i,\omega_j^i)_{j=1,\dots,k})$. Then there exists a constant $N_K$ satisfying condition (i), such that the functions $h_i=h_i^1-h_i^2$, $X=X^1-X^2$, and $Y=Y^1-Y^2$ satisfy the following inequalities:*

$$
\begin{aligned}
\|h_\ell'\|_{L^\infty([0,T]\times\mathcal{O})}&\le N_KT^{1/2}\sum_{i=1}^k\|h_i\|_{L^2(0,T;\mathbb{R}^2)}, &&1\le\ell\le k,\\
\Big\|\frac{\partial^{m+n}X}{\partial y_1^m\partial y_2^n}\Big\|_{L^\infty([0,T]\times\mathcal{O})}&\le N_KT^{1/2}\sum_{i=1}^k\|h_i\|_{L^2(0,T;\mathbb{R}^2)}, &&0\le m+n\le 3,\\
\Big\|\frac{\partial^{m+n}Y}{\partial x_1^m\partial x_2^n}\Big\|_{L^\infty([0,T]\times\mathcal{O})}&\le N_KT^{1/2}\sum_{i=1}^k\|h_i\|_{L^2(0,T;\mathbb{R}^2)}, &&0\le m+n\le 3,\\
\Big\|\frac{\partial^2X_m}{\partial t\partial y_n}\Big\|_{L^\infty([0,T]\times\mathcal{O})}&\le N_KT^{1/2}\sum_{i=1}^k\|h_i\|_{L^2(0,T;\mathbb{R}^2)}, &&m,n\in\{1,2\}\\
\Big\|\frac{\partial^2Y_m}{\partial t\partial x_n}\Big\|_{L^\infty([0,T]\times\mathcal{O})}&\le N_KT^{1/2}\sum_{i=1}^k\|h_i\|_{L^2(0,T;\mathbb{R}^2)}, &&m,n\in\{1,2\}.
\end{aligned}
$$

Next, we recall that

$$\tilde{\mathcal{K}} = \Big\{(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{S}}(0, T, \Omega_F) : \|W\|_{\tilde{\mathcal{U}}(0,T;\Omega_F)} + \|\mathcal{T}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega_F))} + \|Q\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))}$$
$$+ \|\nabla Q\|_{L^\infty(0,T;\mathbf{L}^2(\Omega_F))} + \|\partial_t \nabla Q\|_{L^2(0,T;H^{-1}(\Omega_F))} + \sum_{i=1}^{k} \|h_i^{''}\|_{L^\infty([0,T]\times\mathbb{R}^2)} + \|\omega_i'\|_{L^\infty([0,T]\times\mathbb{R})} \leq R,$$
$$\|\mathcal{T}'\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} \leq R'\Big\},$$

Also, we recall that $K_0$ and $C_0$ are two positive constants which satisfy the following assertions (see Chapter 4):

i. $K_0$ is a positive function of $(h_i^1, \omega_i^0)_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^2(\Omega_F)}, \|\tau_0\|_{[H^2(\Omega_F)]^4}, T$ and $R$ which is non-decreasing with respect to $T, R, \|u_0\|_{\mathbf{H}^2(\Omega_F)}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

ii. $C_0$ is a positive function of $(h_i^1, \omega_i^0, )_{i=1,\dots,k}, \|u_0\|_{\mathbf{H}^2(\Omega_F)}$, and $T$ which is non-decreasing with respect to $T, \|u_0\|_{\mathbf{H}^2(\Omega_F)}, \|\tau_0\|_{[H^2(\Omega_F)]^4}$ and $(|h_i^0|, |h_i^1|, |\omega_i^0|)_{i=1,\dots,k}$.

The following lemma is essential to prove that the source term $F_0$ defined in (4.2.3) is in the good space to apply Proposition 4.3.1. We refer the reader again to [40] for a similar proof.

**Lemma A.1.3** *Suppose that $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$ and let $\Lambda, X$, and $Y$ be the terms corresponding to $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$. Then then there exists a constant $K_0$ satisfying (i) and a constant $C_0$ satisfying (ii) such that*

$$\begin{array}{lll}
\|h_\ell'\|_{L^\infty([0,T]\times\mathcal{O})} \leq C_0 + K_0 T, & \Big\|\dfrac{\partial^{i+j}\Lambda}{\partial x_1^i \partial x_2^j}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq C_0 + K_0 T, & 0 \leq i+j \leq 4, 1 \leq \ell \leq k, \\
\Big\|\dfrac{\partial X}{\partial y}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \Big\|\dfrac{\partial^{i+j}X}{\partial y_1^i \partial y_2^j}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0 T, & 1 < i+j \leq 4, \\
\Big\|\dfrac{\partial Y}{\partial x}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \Big\|\dfrac{\partial^{i+j}Y}{\partial x_1^i \partial x_2^j}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0 T, & 1 < i+j \leq 4, \\
\Big\|\dfrac{\partial^2 X_m}{\partial t \partial y_\ell}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \Big\|\dfrac{\partial^3 X_m}{\partial t \partial y_\ell \partial y_n}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \ell, m, n \in \{1,2\}, \\
\Big\|\dfrac{\partial^2 Y_m}{\partial t \partial x_\ell}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \Big\|\dfrac{\partial^3 Y_m}{\partial t \partial x_\ell \partial x_n}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \ell, m, n \in \{1,2\}, \\
\Big\|\dfrac{\partial^2 X}{\partial t^2}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \Big\|\dfrac{\partial^2 Y}{\partial t^2}\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0, & \\
\Big\|\dfrac{\partial X_m}{\partial y_\ell} - \delta_m^\ell\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0 T, & \Big\|\dfrac{\partial Y_m}{\partial x_\ell} - \delta_m^\ell\Big\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0 T, & \ell, m \in \{1,2\}.
\end{array}$$

We recall that the functions $g^{ij}$, $g_{i,j}$ and $\Gamma_{i,j}^k$ are defined as follows:

$$g^{ij} = \sum_{k=1}^{2} \frac{\partial Y_i}{\partial x_k}\frac{\partial Y_j}{\partial x_k}, \;\; g_{ij} = \sum_{k=1}^{2} \frac{\partial X_k}{\partial y_i}\frac{\partial X_k}{\partial y_j}, \;\; \Gamma_{i,j}^k = \frac{1}{2}\sum_{\ell=1}^{2} g^{k\ell}\left\{\frac{\partial g_{i\ell}}{\partial y_j} + \frac{\partial g_{j\ell}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_\ell}\right\}.$$

By noting that $g^{ij}(0) = g_{ij}(0) = \delta^i_j$ and using mean-value theorem, we get

$$\|g^{ij} - \delta^i_j\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0 T, \ \|g_{ij} - \delta^i_j\|_{L^\infty([0,T]\times\mathcal{O})} \leq K_0 T, \ \forall i, j \in \{1, 2\} \tag{A.1.5}$$

Moreover, we get the following as a direct consequence of Lemma A.1.3 .

**Corollary A.1.1** *There exists a constant $K_0$ satisfying (i) such that*

$$\begin{aligned}
\|g^{ij}\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0, & \|\Gamma^i_{j,k}\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T,\\
\Big\|\frac{\partial g^{ij}}{\partial y_\ell}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T, & \Big\|\frac{\partial \Gamma^i_{j,k}}{\partial y_\ell}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T,\\
\Big\|\frac{\partial^2 g^{ij}}{\partial y_\ell \partial y_m}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T, & \Big\|\frac{\partial^2 \Gamma^i_{j,k}}{\partial y_\ell \partial y_m}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T.\\
\Big\|\frac{\partial g^{jk}}{\partial t}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0, & \Big\|\frac{\partial \Gamma^i_{j,k}}{\partial t}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0,\\
\Big\|\frac{\partial^2 g^{jk}}{\partial t \partial y_k}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0, & \Big\|\frac{\partial^2 \Gamma^i_{j,k}}{\partial t \partial y_\ell}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0.
\end{aligned}$$

We move now to derive some estimates which will be helpful in bounding the terms in the right hand side of (4.3.35) and (4.3.36) in terms of the terms in the left hand side of each one of them. For $(W^n, Q^n, \mathcal{T}^n, (h^n_i, \omega^n_i)_{i=1,\dots,k})$ and $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$, we denote by $X^n, Y^n, g^{ij,n}$, $\Gamma^{k,n}_{i,j}$,... the terms corresponding to $(W^n, Q^n, \mathcal{T}^n, (h^n_i, \omega^n_i)_{i=1,\dots,k})$ and by $X, Y, g^{ij}, \Gamma^k_{i,j}, \dots$ the terms corresponding to $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k})$.
It is important to note that the transforms $X$ and $X^n$ satisfy the estimates in Lemma A.1.3 independent of $n$. We denote by $\bar{X}^n = X - X^n$ and $\bar{Y}^n = Y - Y^n$. Then using arguments identical to that given in [40] shows that $\bar{X}^n$ and $\bar{Y}^n$ satisfy the following.

**Lemma A.1.4** *Assume that $(W, Q, \mathcal{T}, (h_i, \omega_i)_{i=1,\dots,k})$ and $(W^n, Q^n, \mathcal{T}^n, (h^n_i, \omega^n_i)_{i=1,\dots,k}) \in \tilde{\mathcal{K}}$, for all $n \geq 1$.*
*Then there exists a constant $K_0$ satisfying (i) and a positive constant $C$ such that*

$$\begin{aligned}
\Big\|\frac{\partial^{\ell+m}\bar{X}^n}{\partial y_1^\ell \partial y_2^m}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T \sum_{i=1}^k \|h'_i - (h^n_i)'\|_{L^\infty([0,T]\times\mathcal{O})}, \quad 0 \leq \ell+m \leq 3,\\
\Big\|\frac{\partial^{\ell+m}\bar{Y}^n}{\partial x_1^\ell \partial x_2^m}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T \sum_{i=1}^k \|h'_i - (h^n_i)'\|_{L^\infty([0,T]\times\mathcal{O})}, \quad 0 \leq \ell+m \leq 3,\\
\Big\|\frac{\partial^{1+\ell+m}\bar{X}^n}{\partial t \partial y_1^\ell \partial y_2^m}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 \sum_{i=1}^k \|h'_i - (h^n_i)'\|_{L^\infty([0,T]\times\mathcal{O})}, \quad 1 \leq \ell+m \leq 3,\\
\Big\|\frac{\partial^{1+\ell+m}\bar{Y}^n}{\partial t \partial x_1^\ell \partial x_2^m}\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 \sum_{i=1}^k \|h'_i - (h^n_i)'\|_{L^\infty([0,T]\times\mathcal{O})}, \quad 1 \leq \ell+m \leq 3.
\end{aligned}$$

Finally, the following is a direct consequence of Lemma A.1.4.

**Lemma A.1.5** *There exists a positive constant* $K_0$ *satisfying* $(i)$ *such that*

$$\begin{aligned}
\|g^{ij}(X) - g^{ij,n}(X^n)\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T \sum_{\ell=1}^{k} \|h'_\ell - (h^n_\ell)'\|_{L^\infty([0,T]\times\mathcal{O})}, \\
\Big\|\frac{\partial}{\partial y_k}\Big(g^{ij}(X) - g^{ij,n}(X^n)\Big)\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T \sum_{\ell=1}^{k} \|h'_\ell - (h^n_\ell)'\|_{L^\infty([0,T]\times\mathcal{O})}, \\
\|\Gamma^i_{j,k}(X) - \Gamma^{i,n}_{j,k}(X^n)\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T \sum_{\ell=1}^{k} \|h'_\ell - (h^n_\ell)'\|_{L^\infty([0,T]\times\mathcal{O})}, \\
\Big\|\frac{\partial}{\partial y_k}\Big(\Gamma^i_{j,\ell}(X) - \Gamma^{i,n}_{j,\ell}(X^n)\Big)\Big\|_{L^\infty([0,T]\times\mathcal{O})} &\leq K_0 T \sum_{m=1}^{k} \|h'_m - (h^n_m)'\|_{L^\infty([0,T]\times\mathcal{O})}.
\end{aligned}$$

## A.2 Proof of Lemma 2.4.3 and Lemma 2.4.4

Let $X$ be the transform defined in (2.2.2) and consider the operators $[LU], [MU], [NU]$, and $[GP]$ are defined as in (4.2.7)-(4.2.10) (see Chapter 2). First, we show the existence of a strong 2-extension operator $E$ for $\Omega_F(t)$.

**Lemma A.2.1** *There exists a strong 2-extension operator* $E$ *for* $\Omega_F(t)$*. Moreover, there exists a positive constant* $k = k(\varepsilon)$ *such that for* $u \in \mathbf{H}^2(\Omega_F(t))$*, we have:*

$$\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))}, \tag{A.2.1}$$

$$\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^1(\Omega_F(t))}, \tag{A.2.2}$$

$$\|Eu\|_{\mathbf{H}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^2(\Omega_F(t))}. \tag{A.2.3}$$

proof. Let $t < T_0$, $0 < \varepsilon < \gamma$ and $u \in \mathbf{H}^2(\Omega_F(t))$. We consider a family of smooth functions $\{\chi_i\}_{i=1,\dots,k}$ each of compact support included in $[-r_i - \frac{\varepsilon}{2}, r_i + \frac{\varepsilon}{2}]$ and equals to one on $[-r_i, r_i]$. For each $i \in \{0,\dots,k\}$, we define the function $u^{(i)} : \Omega_F(t) \to \mathbb{R}^2$, such that

$$u^{(i)}(x) = \chi_i(|x - h_i(t)|)u(x), \;\; 1 \leq i \leq k$$

and

$$u^{(0)} = u - \sum_{i=1}^{k} u^{(i)}.$$

Moreover, for $i \in \{1, \dots, k\}$, we define the function $v^{(i)} : B(h_i(0), r_i + \frac{\varepsilon}{2}) \backslash B_i(0) \to \mathbb{R}^2$ by

$$v^{(i)} = u^{(i)}(x + h_i(t) - h_i(0)).$$

We note that $v^{(i)} \in \mathbf{H}^2(B(h_i(0), r_i + \frac{\varepsilon}{2}) \backslash B_i(0))$ for all $i \in \{1, \dots, k\}$. We set $\overline{v}^{(i)} = Ev^{(i)}$, where $E$ is a strong 2-extension operator for $\Omega_F$. By Theorem 5.22 in [1], there exists a constant $k = k(\varepsilon)$ such that

$$\begin{aligned} \|\overline{v}^{(i)}\|_{\mathbf{L}^2(\mathbb{R}^2)} &\leq k\|u^{(i)}\|_{\mathbf{L}^2(\Omega_F(t))}, && \text{(A.2.4)} \\ \|\overline{v}^{(i)}\|_{\mathbf{H}^1(\mathbb{R}^2)} &\leq k\|u^{(i)}\|_{\mathbf{H}^1(\Omega_F(t))}, && \text{(A.2.5)} \\ \|\overline{v}^{(i)}\|_{\mathbf{H}^2(\mathbb{R}^2)} &\leq k\|u^{(i)}\|_{\mathbf{H}^2(\Omega_F(t))}. && \text{(A.2.6)} \end{aligned}$$

We note that $\overline{v}^{(i)}$ vanishes outside $B(h_j(t), r_j + \frac{\varepsilon}{2})$ for all $j \neq i$. Finally, we set

$$Eu = \tilde{u}^{(0)} + \sum_{i=1}^{k} \overline{u}^{(i)},$$

where $\overline{u}^{(i)}(x) = \overline{v}^{(i)}(x - h_i(t) + h_i(0)), \forall i \in \{1, \dots, k\}$ and $\tilde{u}^{(0)}$ is the extension of $u^{(0)}$ by zero over the disks. We remark here that $\tilde{u}^{(0)} \in \mathbf{H}^2(\mathbb{R}^2)$ and for simplicity we remove the tilde.
Hence, for $x \in \Omega_F(t)$ we have

$$Eu(x) = u^{(0)}(x) + \sum_{i=1}^{k} \overline{v}^{(i)}(x - h_i(t) + h_i(0)).$$

If $x \in B(h_j(t), r_j + \frac{\varepsilon}{2}) \backslash B_j(t)$, then $x - h_j(t) + h_j(0) \in B(h_j(0), r_j + \frac{\varepsilon}{2}) \backslash B_j(0)$ and $x \notin B(h_i(t), r_i + \frac{\varepsilon}{2}) \backslash B_i(t)$ for all $i \neq j$. Hence, for all $i \neq j$, we have $x - h_i(t) + h_i(0) \notin B(h_i(0), r_i + \frac{\varepsilon}{2}) \backslash B_i(0)$ and thus

$$\begin{aligned} Eu(x) &= u^{(0)}(x) + \overline{v}^{(j)}(x - h_j(t) + h_j(0)) \\ &= u^{(0)}(x) + v^{(j)}(x - h_j(t) + h_j(0)) \\ &= u^{(0)}(x) + u^{(j)}(x) \\ &= u(x). \end{aligned}$$

Now, if $x \in \Omega_F(t) \backslash \bigcup_{i=1}^{k} B(h_i(0), r_i + \dfrac{\varepsilon}{2})$, then $x - h_i(t) + h_i(0) \notin B(h_i(0), r_i + \frac{\varepsilon}{2})$ and thus

$$Eu(x) = u^{(0)}(x) = u(x).$$

Moreover, there exists a positive real constant $k = k(\varepsilon)$ such that

$$\begin{aligned}\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} &\leq \|u^{(0)}\|_{\mathbf{L}^2(\mathbb{R}^2)} + \sum_{i=1}^{k} \|\overline{u}^{(i)}\|_{\mathbf{L}^2(\mathbb{R}^2)} \\ &\leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))} + \sum_{i=1}^{k} \|\overline{v}^{(i)}\|_{\mathbf{L}^2(\mathbb{R}^2)}.\end{aligned}$$

This implies that

$$\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))} + k\sum_{i=1}^{k} \|u^{(i)}\|_{\mathbf{L}^2(\Omega_F(t))}.$$

Hence, we get

$$\|Eu\|_{\mathbf{L}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{L}^2(\Omega_F(t))}. \tag{A.2.7}$$

In a similar way, we can prove

$$\|Eu\|_{\mathbf{H}^1(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^1(\Omega_F(t))}, \tag{A.2.8}$$

$$\|Eu\|_{\mathbf{H}^2(\mathbb{R}^2)} \leq k\|u\|_{\mathbf{H}^2(\Omega_F(t))}. \tag{A.2.9}$$

Next we prove the following Lemma.

**Lemma A.2.2** *Let $u$ be the unique strong solution of problem (2.1.1)-(2.1.6). Then there exists $T_1$ small enough, such that for almost every $t \in [0, T_1]$, we have*

$$\|u(t)\|_{\mathbf{H}^2(\Omega_F(t))} \leq K\Big(\Big\|\frac{\partial u}{\partial t}(t)\Big\|_{\mathbf{L}^2(\Omega_F(t))} + \|u(t)\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|\nabla u(t)\|^2_{\mathbf{L}^2(\Omega_F(t))} + \|f(t)\|_{\mathbf{L}^2(\Omega_F(t))} + 1\Big),$$

*where $K$ is a positive constant that depends on $\Omega_F, B_i, \overline{\rho}_i, \nu, T_0, \|u_0\|_{\mathbf{L}^2(\mathbb{R}^2)}$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$.*

proof. We can consider that the solution $(u, p)$ is a solution of the following problem at a fixed time $t > 0$:

$$\begin{cases} u - \nu\Delta u + \nabla p = \tilde{f}, & \text{in } \Omega_F(t), \\ \nabla \cdot u = 0, & \text{in } \Omega_F(t), \\ u(x,t) = h_i'(t) + \omega_i(t)(x - h_i(t))^{\perp}, & x \in B_i(t),\ \forall i \in \{1, \dots, k\}, \end{cases} \tag{A.2.10}$$

where

$$\tilde{f} = -\frac{\partial u}{\partial t} - (u \cdot \nabla)u + f + u. \tag{A.2.11}$$

## A.2 Proof of Lemma 2.4.3 and Lemma 2.4.4

We define for $(y,t) \in \mathbb{R}^2 \times [0,T]$ and $i \in \{1,\ldots,k\}$, the mappings:

$$\overline{w}_i(y,t) = h_i'(t) \cdot y^{\perp} + \frac{\omega_i(t)}{2}|y - h_i(0)|^2.$$

Finally, we define the mapping $\overline{\Lambda} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}^2$ by

$$\overline{\Lambda}(x_1,x_2,t) = \sum_{i=1}^{k} \nabla^{\perp}(\overline{w}_i \zeta_i). \tag{A.2.12}$$

We note here that

$$\overline{\Lambda}(y,t) = h_i'(t) + \omega_i(t)(y - h_i(0))^{\perp}, \ \ \forall y \in B_i,$$

and

$$||\overline{\Lambda}(t)||_{\mathbf{H}^2(\Omega_F(t))} \leq C \sum_{i=1}^{k}(|h_i'(t)| + |\omega_i(t)|), \ \ \forall t \in [0,T_0).$$

Lemma 2.4.1 implies that

$$||\overline{\Lambda}||_{\mathbf{H}^2(\Omega_F(t))} \leq C M^{\frac{1}{2}} K_1.$$

By using the change of variables $X$ defined in (2.2.2), we see that $(U,P)$ as defined in Chapter 2 (see Section 2.2) satisfies the following problem:

$$\begin{cases} U - \nu \Delta U + \nabla P = \tilde{g}, & \text{in } \Omega_F, \\ \nabla \cdot U = 0, & \text{in } \Omega_F, \\ U|_{\partial B_i} = \overline{\Lambda}|_{\partial B_i}, & \forall i \in \{1,\ldots,k\}, \end{cases} \tag{A.2.13}$$

with

$$\tilde{g} = \nu[(L - \Delta)U] - [(G - \nabla)P] - [MU] - [NU] - \frac{\partial U}{\partial t} + F + U, \tag{A.2.14}$$

where $[LU], [MU], [NU]$, and $[GP]$ are defined as in (4.2.7)-(4.2.10).

By Theorem 2.1 in [15], there exists a unique $(U,P) \in \mathbf{H}^2(\Omega_F) \times \dot{H}^1(\Omega_F)$ solution of problem (A.2.13). Moreover, there exists a constant $C_3 = C_3(\nu,\Omega_F) > 0$ such that

$$\|U\|_{[H^2(\Omega_F)]^2} + \|\nabla P\|_{[L^2(\Omega_F)]^2} \leq C_3\big(\|\tilde{g}\|_{[L^2(\Omega_F)]^2} + \|\overline{\Lambda}\|_{[H^2(\mathbb{R}^2)]^2}\big). \tag{A.2.15}$$

We start with estimating the first term in the expression of $\tilde{g}$. We have:

$$
\begin{aligned}
\big\|[(L-\Delta)U]_i\big\|_{L^2(\Omega_F)} \leq & \sum_{j,k=1}^{2}\|g^{jk}-\delta_k^j\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial^2 U_i}{\partial y_j \partial y_k}\Big\|_{L^2(\Omega_F)} \\
& + \sum_{j,k=1}^{2}\Big\|\frac{\partial g^{jk}}{\partial y_j}\Big\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial U_i}{\partial y_k}\Big\|_{L^2(\Omega_F)} + 2\sum_{j,k,\ell=1}^{2}\|g^{k\ell}\|_{L^\infty(\Omega_F)}\|\Gamma^i_{j,k}\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial U_j}{\partial y_\ell}\Big\|_{L^2(\Omega_F)} \\
& + \sum_{j,k,\ell=1}^{2}\Big\{\Big\|\frac{\partial g^{k\ell}}{\partial y_k}\Big\|_{L^\infty(\Omega_F)}\|\Gamma^i_{j,\ell}\|_{L^\infty(\Omega_F} + \|g^{k\ell}\|_{L^\infty(\Omega_F)}\Big\|\frac{\partial \Gamma^i_{j,\ell}}{\partial y_\ell}\Big\|_{L^\infty(\Omega_F)} \\
& + \sum_{m=1}^{2}\|g^{k\ell}\|_{L^\infty(\Omega_F)}\|\Gamma^m_{j,\ell}\|_{L^\infty(\Omega_F)}\|\Gamma^i_{k,m}\|_{L^\infty(\Omega_F)}\Big\}\|U_j\|_{L^2(\Omega_F)},
\end{aligned}
\tag{A.2.16}
$$

In what follows, we denote by $K$ a positive constant that depends on $\Omega_F, B_i, \overline{\rho}_i, \nu, T_0, \|u_0\|_{\mathbf{L}^2(\mathbb{R}^2)}$ and $\|f\|_{L^2(0,T_0;\mathbf{L}^2(\mathbb{R}^2))}$ that may changes between lines.

From the definition of $g^{ij}$, $g_{i,j}$ and $\Gamma^k_{i,j}$ respectively in (2.2.14), and by applying the same technique of proof of Lemma 6.4 and Corollary 6.5 in [40], we get for all $1 \leq i,j,k \leq 2$:

$$
\begin{aligned}
\|g^{ij}-\delta^i_j\|_{L^\infty(\mathbb{R}^2)} &\leq KT_1, \quad & \|g_{ij}-\delta^i_j\|_{L^\infty(\mathbb{R}^2)} &\leq KT_1, \\
\Big\|\frac{\partial g^{ij}}{\partial y_k}\Big\|_{L^\infty(\mathbb{R}^2)} &\leq KT_1, \quad & \Big\|\frac{\partial g_{ij}}{\partial y_k}\Big\|_{L^\infty(\mathbb{R}^2)} &\leq KT_1, \\
\|\Gamma^k_{i,j}\|_{L^\infty(\mathbb{R}^2)} &\leq KT_1, \quad & \Big\|\frac{\partial \Gamma^k_{i,j}}{\partial y_\ell}\Big\|_{L^\infty(\mathbb{R}^2)} &\leq KT_1.
\end{aligned}
$$

Combining the above estimates with (A.2.16), we obtain that

$$
\big\|[(L-\Delta)U]_i\big\|_{L^2(\Omega_F)} \leq KT_1\|U\|_{\mathbf{H}^2(\Omega_F)}. \tag{A.2.17}
$$

By the same way, we get that there exists some positive constant $C$ such that

$$
\begin{aligned}
\big\|[(\nabla-G)P]_i\big\|_{L^2(\Omega_F)} &\leq KT_1\|\nabla P\|_{\mathbf{L}^2(\Omega_F)}, \\
\big\|[MU]_i\big\|_{L^2(\Omega_F)} &\leq C\|U\|_{\mathbf{H}^1(\Omega_F)}, \\
\big\|[NU]_i\big\|_{L^2(\Omega_F)} &\leq \|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)} + KT_1\|U\|^2_{\mathbf{L}^2(\Omega_F)},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\|\tilde{g}\|_{[L^2(\Omega_F)]^2} \leq \Big\|\frac{\partial U}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F)} + \|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)} + KT_1(\|U\|_{\mathbf{H}^2(\Omega_F)} + \|\nabla P\|_{\mathbf{L}^2(\Omega_F)}) \\
+ C(\|F\|_{\mathbf{L}^2(\Omega_F)} + \|U\|^2_{\mathbf{L}^2(\Omega_F)} + \|\nabla U\|^2_{\mathbf{L}^2(\Omega_F)} + 1).
\end{aligned}
\tag{A.2.18}
$$

Combining the above inequality with the estimate in (A.2.15), we obtain for $T_1$ is small enough:

$$\|U\|_{\mathbf{H}^2(\Omega_F)}+\|\nabla P\|_{\mathbf{L}^2(\Omega_F)} \leq \frac{C_3}{1-KT_1}\Big\{\Big\|\frac{\partial U}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F)}+\|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)}+\|U\|^2_{\mathbf{L}^2(\Omega_F)}+\|\nabla U\|^2_{\mathbf{L}^2(\Omega_F)} + \|F\|_{\mathbf{L}^2(\Omega_F)} + \|\overline{\Lambda}\|_{\mathbf{H}^2(\mathbb{R}^2)} + 1\Big\}. \quad (A.2.19)$$

Bounding the transform $X$ and its derivatives up to order 3 from above as in Lemma 6.4 in [40], we get that

$$\|u\|_{\mathbf{L}^2(\Omega_F(t))} \leq K\|U\|_{\mathbf{L}^2(\Omega_F)}, \quad (A.2.20)$$

$$\|U\|_{\mathbf{L}^2(\Omega_F)} \leq K\|u\|_{\mathbf{L}^2(\Omega_F(t))}, \quad (A.2.21)$$

$$\Big\|\frac{\partial U}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F)} \leq K\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{\mathbf{L}^2(\Omega_F(t))} + \|u\|_{\mathbf{H}^1(\Omega_F(t))}\Big), \quad (A.2.22)$$

$$\|(U\cdot\nabla)U\|_{\mathbf{L}^2(\Omega_F)} \leq K\|u\|^2_{\mathbf{H}^1(\Omega_F(t))}, \quad (A.2.23)$$

$$\|\nabla u\|_{[L^2(\Omega_F)]^4} \leq K\|U\|_{\mathbf{H}^1(\Omega_F)}, \quad (A.2.24)$$

$$\|\nabla^2 u\|_{[L^2(\Omega_F)]^8} \leq K\|U\|_{\mathbf{H}^2(\Omega_F)}. \quad (A.2.25)$$

By combining these estimates with that in (A.2.19), the proof of Lemma 2.4.4 is complete. □

## A.3 Proof of Proposition 4.2.1

For completeness, we recall the statement of Proposition 4.2.1.

**Proposition A.3.1** *Let $\mathcal{G} \in L^2(0,T;\mathbf{L}^2(\Omega_F))$ and $\tau_0 \in \mathbf{H}^1(\Omega_F)$ such that $\tau_0^T = \tau_0$. Then problem*

$$\begin{cases} We\dfrac{\partial \mathcal{T}}{\partial t} + \mathcal{T} - \varepsilon\Delta\mathcal{T} = \mathcal{G}, & \text{in } \Omega_F\times]0,T], \\ \varepsilon\dfrac{\partial \mathcal{T}}{\partial n} = 0, & \text{on } \partial\Omega_F\times]0,T], \\ \mathcal{T}(0) = \tau_0, & \text{in}\, \Omega_F. \end{cases} \quad (A.3.1)$$

*admits a unique solution $\mathcal{T} \in \mathfrak{T}(0,T;\Omega_F)$. Moreover, there exists a positive constant $K$ depending only on $\Omega_F$ and $T$; non-decreasing with respect to $T$, such that*

$$\|\mathcal{T}\|_{L^2(0,T;\mathbf{H}^2(\Omega_F))} + \|\mathcal{T}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega_F))} + \|\mathcal{T}\|_{H^1(0,T;\mathbf{L}^2(\Omega_F))} \leq K\big(\|\tau_0\|_{\mathbf{H}^1(\Omega_F)} + \|\mathcal{G}\|_{L^2(0,T;\mathbf{L}^2(\Omega_F))}\big).$$

Before giving the proof of the above proposition , let us define the following spaces:

$$\begin{array}{rcl} L^2_{\text{sym}} & = & \{\mathcal{T} \in \mathbf{L}^2(\Omega_F) : \mathcal{T}^T = \mathcal{T}\}, \\ H^1_{\text{sym}} & = & \{\mathcal{T} \in \mathbf{H}^1(\Omega_F) : \mathcal{T}^T = \mathcal{T}\}, \\ D(A_N) & = & \{\mathcal{T} \in \mathbf{H}^2(\Omega_F) : \mathcal{T}^T = \mathcal{T},\ \dfrac{\partial \mathcal{T}}{\partial n} = 0 \text{ on } \partial\Omega_F\}. \end{array}$$

For $\mathcal{T}$ and $\sigma$ in $L^2_{\text{sym}}$, we define the scalar product $(.,.)_{L^2_{\text{sym}}}$ as follows:

$$(\mathcal{T}, \sigma)_{L^2_{\text{sym}}} = We \int_{\Omega_F} \mathcal{T} : \sigma dy. \tag{A.3.2}$$

Hence, Problem 2 is equivalent to the following Cauchy problem:

$$\left\{ \begin{array}{rcl} \mathcal{T}' + A_N \mathcal{T} & = & \tilde{\mathcal{G}}, \\ \mathcal{T}(0) & = & \tau_0, \end{array} \right. \tag{A.3.3}$$

where $A_N : D(A_N) \to L^2_{\text{sym}}$ is defined as follows

$$A_N \mathcal{T} = \frac{1}{We}(-\varepsilon \Delta \mathcal{T} + \mathcal{T}),\ \forall \mathcal{T} \in D(A_N),$$

and $\tilde{\mathcal{G}} = \dfrac{1}{We}\mathcal{G}$.

We use semi group theory to solve problem (A.3.3). For $(\mathcal{T}, \sigma) \in [D(A_N)]^2$, we have

$$(A_N \mathcal{T}, \sigma)_{L^2_{\text{sym}}} = \varepsilon \int_{\Omega_F} \nabla \mathcal{T} : \nabla \sigma dy + \int_{\Omega_F} \mathcal{T} : \sigma dy. \tag{A.3.4}$$

Thus $A_N$ is symmetric and we are almost ready to prove that $A_N$ is self -adjoint. To do so, it suffices to show only that it is maximal monotone [9].

Setting $\sigma = \mathcal{T}$ in (A.3.4), we get that $A_N$ is monotone. Now, we are in position to show that $A_N$ is maximal monotone. This is equivalent to show that for all $g \in L^2_{\text{sym}}$, there exists $\mathcal{T} \in D(A_N)$ such that

$$(I + A_N)\mathcal{T} = g. \tag{A.3.5}$$

To this end, we multiply scalar (*A*.3.5) by $\sigma \in L^2_{\text{sym}}$ and we get that

$$(\mathcal{T}, \sigma)_{L^2_{\text{sym}}} + (A_N \mathcal{T}, \sigma)_{L^2_{\text{sym}}} = (g, \sigma)_{L^2_{\text{sym}}}. \tag{A.3.6}$$

In particular for $\sigma \in H^1_{\text{sym}}$, we have

$$(We+1)\int_{\Omega_F} \mathcal{T} : \sigma dy + \varepsilon \int_{\Omega_F} \nabla \mathcal{T} : \nabla \sigma dy = We \int_{\Omega_F} g : \sigma dy. \tag{A.3.7}$$

By Lax-Milligram theorem, there exists unique $\mathcal{T} \in H^1_{\text{sym}}$ satisfying (A.3.7).
Equation (A.3.7) holds for all $\sigma \in \mathcal{D}(\Omega_F)$ such that $\sigma^T = \sigma$. This means that

$$(We+1)\mathcal{T} - \varepsilon \Delta \mathcal{T} = We\, g, \text{ in } \Omega_F,$$

holds in the sense of distribution.
By continuity and density, the above equation implies that for all $\sigma \in \mathbf{H}^1(\Omega_F)$, we have

$$(We+1)\int_{\Omega_F} \mathcal{T} : \sigma dy + \varepsilon \int_{\Omega_F} \nabla \mathcal{T} : \nabla \sigma dy - \varepsilon \int_{\partial\Omega_F} \frac{\partial \mathcal{T}}{\partial n} \cdot \sigma d\Gamma = We \int_{\Omega_F} g : \sigma dy.$$

By comparing the above equation with that in (A.3.7), we get that $\varepsilon \dfrac{\partial \mathcal{T}}{\partial n} = 0$.
Thus $\mathcal{T}$ satisfies the following problem:

$$\begin{aligned} (We+1)\mathcal{T} - \varepsilon \Delta \mathcal{T} &= Weg, && \text{in } \Omega_F, \\ \varepsilon \frac{\partial \mathcal{T}}{\partial n} &= 0, && \text{on } \partial\Omega_F. \end{aligned}$$

By Theorem 3.4.3 in [3], we get that $\mathcal{T} \in \mathbf{H}^2(\Omega_F)$ and thus $A_N$ is a maximal monotone symmetric operator. Consequently, $A_N$ is also self-adjoint. The proof is accomplished by recalling the known identification

$$D(A_N^{1/2}) \equiv \mathbf{H}^1(\Omega_F)$$

from [5] and the application of Propostion 3.3 in [41]. □

# Bibliography


[1] R. A. Adams and J. F. John Fournier, *Sobolev spaces*, Pure Appl. Math., Academic Press, 140 (2003).

[2] V. I. Arnold, *Ordinary Differential Equations*, Springer, Berlin, translated from the third Russian edition, (1992), pp. 249.

[3] V. Barbu, *Partial differential equations and boundary value problems* (Vol. 441). Springer Science & Business Media, (2013).

[4] J. Barrett and E. Suli, *Existence of global weak solutions to some regularized kinetic models for dilute polymers*, Multiscale Model. Simul., (2007), pp. 506-546.

[5] A. Bensoussan, G. Da. Prato, M. C. Delfour, and S. K. Mitter, *Representation and control of infinite dimensional systems*, (Vol. 1). Boston: Birkhäuser, (1993), pp. 432-438.

[6] K. Billah and R. Scanlan, *Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks*, American Journal of Physics, (1991).

[7] J. P. Bourguignon and H. Brezis, *Remarks on the Euler equations*, J. Funct. Analysis (1976) pp. 341-363.

[8] F. Boyer and P. Fabrie, *Eléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles*, Vol. 52. Springer Science & Business Media, (2005).

[9] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer Science and Business Media, 2010.

[10] C. Conca, J.A. San Martin, and M. Tucsnak, *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid*, Comm. Partial Differential Equations, 25 (2000), pp. 99–110.

[11] P. Constantin and M. Kliegl, *Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress*, Arch. Ration. Mech. Anal., (2012), 206(3):725–740.

[12] P. Cumsille and T. Takahashi, *Well posedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid*, Czechoslovak Math. J., 58 (2008), pp. 961–992.

[13] B. Desjardins and M. Esteban, *Existence of weak solutions for the motion of rigid bodies in a viscous fluid*, Arch. Ration. Mech. Anal., 146 (1999), pp. 59–71.

[14] B. Desjardins and M. Esteban, *On weak solutions for fluidrigid structure interaction: Compressible and incompressible models*, Comm. Partial Differential Equations, 25 (2000), pp. 263–285.

[15] R. Farwig and H. Sohr, *The stationary and non-stationary stokes system in exterior domains with non-zero divergence and non-zero boundary values*, Math. methods Appl. Sci., 17 (1994), pp. 269–291.

[16] E. Feireisl, *On the motion of rigid bodies in a viscous fluid*, Appl. Math., 47 (2002), pp. 463–484.

[17] E. Feireisl, M. Hillairet, and Š. Nečasová, *On the motion of several rigid bodies in an incompressible non-Newtonian fluid*, Nonlinearity, (2008), 21(6), 1349.

[18] G. P. Galdi and A. L. Silvestre, *Strong solutions to the problem of motion of a rigid body in a Navier-Stokes liquid under the action of prescribed forces and torques*, Nonlinear Problems in Mathematical Physics and Related Topics I, Springer, (2002), pp. 121–144.

[19] M. Geissert and K. Götze, M. Hieber, *$L^p$ theory for strong solutions to fluid-rigid body interaction in newtonian and generalized newtonian fluids*, Trans. Amer. Math. Soc., 365 (2013), pp. 1393–1439.

[20] D. Gérard-Varet and M. Hillairet, *Regularity issues in the problem of fluid structure interaction*, Arch. for ration. mech. Anal., 195 (2010), pp. 375–407.

[21] C. Grandmont and Y. Maday, *Existence for an unsteady fluid-structure interaction problem*, M2AN Math. Model. Numer. Analy., 34 (2000), pp. 609–636.

[22] C. Guillopé and J.C. Saut, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonlinear Anal., (1990), pp. 849-869.

[23] T. I. Hesla, *Collisions of smooth bodies in viscous fluids: A mathematical investigation*, PhD thesis, University of Minnesota, revised version, (2005).

[24] M. Hillairet, *Lack of collision between solid bodies in a 2d incompressible viscous flow*, Comm. Partial Differential Equations, 32 (2007), pp. 1345–1371.

[25] M. Hilliaret and T. Takahashi, *Blow up and grazing collision in viscous fluid solid interaction systems*, Analyse non linéaire, Elsevier (2010) pp. 291-313.

[26] M. Hilliaret and T. Takahashi, *Collision in 3D fluid structure interaction problems*, SIAM J. Math. Anal., (2009), pp. 2451-2477.

[27] K.H. Hoffmann and V. Starovoitov, *On a motion of a solid body in a viscous fluid. two-dimentional case*, Adv. Math. Sci. Appl., 9 (1999), pp. 633–648.

[28] A. Inoue and M. Wakimoto, *On existence of solutions of the navier-stokes equation in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math, 24 (1977), pp. 303–319.

[29] N. Judakov, *The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid*, Dinamika Splosn. Sredy, 255 (1974), pp. 249–253.

[30] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, vol. 1, Springer-Verlag, (1972).

[31] P. L. Lions and N. Masmoudi, *Global solutions for some Oldroyd models of non-Newtonian flows*, Chinese Ann. Math. Ser. B, (2000), 21(2):131–146.

[32] M. Molina and P. Enrique Mercado, *Modelling and Control Design of Pitch-Controlled Variable Speed Wind Turbines*, (2011).

[33] L. Molinet and R. Talhouk, *On the global and periodic regular flows of viscoelastic fluids with a differential constitutive law*, Nonlinear Differential Equations Appl., 1(2004), 1(3):349–359.

[34] J. A. San Martín, V. Starovoitov, and M. Tucsnak, *Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid*, Arch. Rot. Mech. Anal., 161 (2002), pp. 113–147.

[35] J.C. Saut, *Lectures on the mathematical theory of viscoelastic fluids*, (2012).

[36] D. Serre, *Chute libre d'un solide dans un fluide visqueux incompressible. existence*, Japan J. Appl. Math., 4 (1987), pp. 99–110.

[37] A. Silvestre, *On the self-propelled motion of a rigid body in a viscous liquid and on the attainability of steady symmetric self-propelled motions*, J. Math. Fluid Mech., 4 (2002), pp. 285–326.

[38] V. N. Starovoitov, *Behavior of a rigid body in an incompressible viscous fluid near a boundary*, Birkhäuser, Basel, (2003), pp. 313-327.

[39] V. N. Starovoitov, *Nonuniqueness of a solution to the problem on motion of rigid body in a viscous incompressible fluid*, J. Math. Sci., (2005).

[40] T. Takahashi, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain*, Adv. Differential Equations, 8 (2003), pp. 1499–1532.

[41] T. Takahashi and M. Tucsnak, *Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid*, J. Math. Fluid Mech., 6 (2004), pp. 53–77.

[42] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, SIAM, (1995).

[43] R. Temam, *Navier-Stokes equations: theory and numerical analysis*, volume 343. American Mathematical Soc., (2001).

[44] R. Temam, *Problèmes mathématiques en plasticité* , gauthier-villars, paris, (1983).

[45] J.L. Vàzquez and E. Zuazua, *Large time behavior for a simplified 1D model of fluid-solid interaction*, Comm. Partial Differential Equations, 28(9-10):1705–1738, (2003).

[46] L. Wang, *Nonlinear aeroelastic modelling of large wind turbine composite blades*, University of Central Lancashire, (2015).