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JURY

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Contents

Chapter 0

Introduction

0.1 Quasisymmetric rigidity of $F_{n,p}$

The quasisymmetric geometry of Sierpiński carpets is related to the study of Julia sets in complex dynamics and boundaries of Gromov hyperbolic groups.

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Let $S = \mathbb{S}^2 \setminus \bigcup_{i\geq 1} D_i$, where $D_i, i = 1, 2, \cdots$, are open Jordan domains with $\overline{D}_i \bigcap \overline{D}_j = \emptyset$ for distinct i, j. S is called a (*Sierpiński*) *carpet* if S has empty interior and the spherical diameter $\mathrm{diam}(D_i) \to 0$ as $i \to \infty$. For each i, ∂D_i is called a *peripheral circle* of S. S is called a *round carpet* if each D_i is a round disk.

Topologically all carpets are the same [W58]. Much richer structure arises if we consider quasisymmetric geometry of metric carpets. The famous counjecture of Kapovich-Kleiner predicts that if G is a hyperbolic group with boundary $\partial_\infty G$ homeomorphic to a carpet, then G acts geometrically on a convex subset of $\mathbb{H}^3$ with non-empty totally geodesic boundary. The Kapovich-Kleiner conjecture is equivalent to the conjecture that the carpet $\partial_\infty G$, endowed with a visual metric, is quasisymetriclly equivalent to a round carpet on $\mathbb{S}^2$. The conjecture is true for carpets that can be quasisymmetrically embedding into $\mathbb{S}^2$.

Let us recall the concept of quasisymetric map between metric spaces defined by Tukia and Väisälä [TV80]. Let $f : X \to Y$ be a homeomorphism between two metric spaces (X, d_X) and (Y, d_Y). f is *quasisymmetric* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right), \quad \forall\, x, y, z \in X,\ x \neq z.$$

Let $QS(X)$ be the group of quasisymmetric self-homeomorphisms of X.

Let S_p, $p \geq 3$ odd, be the standard $1/p$-Sierpiński carpets on the Euclidean plane. In [BM13], Bonk and Merenkov proved that for each S_p, $p \geq 3$ odd, $QS(S_p)$ is finite dihedral group. They further conjectured that, for any $p \geq 3$ odd, $QS(S_p)$ is a Euclidean isometry group. When $p = 3$, they has showed that the conjecture is true. They also established that a rigidity theorem that S_p and S_q are quasisymmetrically equivalent if and only if $p = q$.

The aim of Chapter 1 is to extend Bonk-Merenkov's results to a new class of Sierpiński carpets $F_{n,p}$ (see the definition below). In particular, we are able to prove that the group of quasisymmetric self-homeomorphisms of $F_{n,p}$ is a Euclidean isometry group.

Let $5 \leq n, 1 \leq p < \frac{n}{2} - 1$ be integers. Let $Q_{n,p}^{(0)} = [0,1] \times [0,1]$ be the closed unit square in $\mathbb{R}^2$. We first subdivide $Q_{n,p}^{(0)}$ into n^2 subsquares with equal side-length $1/n$ and remove the interior of four subsquares, each has side-length $1/n$ and is of distance $\sqrt{2}p/n$ to one of the four corner points of $Q_{n,p}^{(0)}$.

The resulting set $Q_{n,p}^{(1)}$ consists of $(n^2 - 4)$ squares of side-length $1/n$. Inductively, $Q_{n,p}^{(k+1)}$, $k \geq 1$, is obtained from $Q_{n,p}^{(k)}$ by subdividing each of the remaining squares in the subdivision of $Q_{n,p}^{(k)}$ into n^2 subsquares of equal side-length $1/n^{k+1}$ and removing the interior of four subsquares as we have done above.

The Spierpiński carpet $F_{n,p}$ is the intersection of all sets $Q_{n,p}^{(k)}$, i.e.,

$$F_{n,p} = \bigcap_{k=0}^{+\infty} Q_{n,p}^{(k)}.$$

See Figure 1.

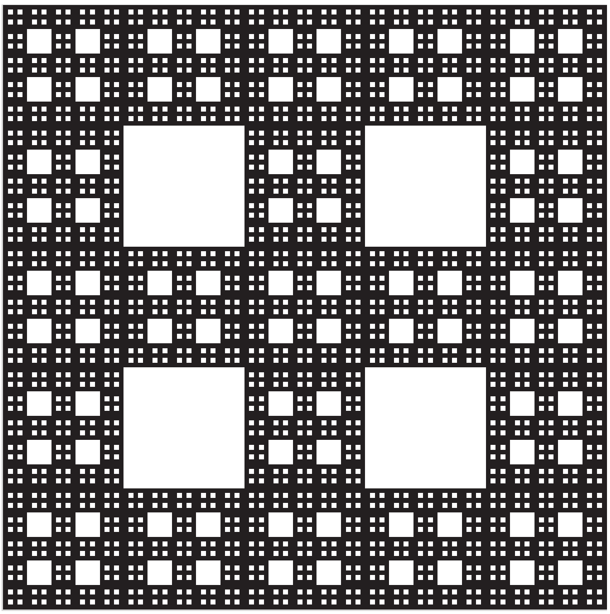

Figure 1: The carpet $F_{5,1}$.

In Chapter 1, we improve the method in Bonk and Merenkov and prove the following theorem,

Theorem 0.1. *Let $n \geq 5, 1 \leq p < \frac{n}{2} - 1$ be integers. Then*

• *the carpet* $F_{n,p}$ *is not quasisymetrically equivalent to any standard Sierpiński carpet* S_m*,* $m \geq 3$ *odd;*

• $\mathrm{QS}(F_{n,p}) = \mathrm{Isom}(F_{n,p})$*, that is, every quasisymmetric self-homeomorphism of* $F_{n,p}$ *is a Euclidean isometry;*

• $F_{n,p}$ *and* $F_{n',p'}$ *are quasisymmetrically equivalent if and only if* $(n, p) = (n', p')$*.*

0.2 Sierpiński carpets arising as Julia sets of rational maps

The first example of Sierpiński carpet as Julia set of a rational map, called *carpet Julia set*, was discovered by Tan Lei [Mi93]. Later, carpet Julia sets appeared in many literatures including examples of McMullen maps, generated McMullen maps and quadratic rational maps, etc. [DFGJ14] [DLU05] [QXY12].

There are two natural questions:

(Q_1) whether any two carpet Julia sets are quasisymmetrically equivalent?

(Q_2) whether any carpet Julia set J_f is quasisymmetrically equivalent to a round carpet?

Let X be a metric space. The *conformal dimension* of X is the infimum of the Hausdorff dimensions of all metric spaces which are quasisymmetrically equivalent to X. For the question (Q_1), Haïssinsky and Pilgrim [HP12] constructed a sequence of hyperbolic rational maps with carpet Julia sets such that their conformal dimensions tend to two. This means there are infinitely many quasisymmetrically inequivalent carpet Julia sets.

For the question (Q_2), Bonk gave a sufficient condition on which the carpet in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is quasisymmetrically equivalent to a round carpet.

The *relative distance* $\Delta(A, B)$ of two subsets A and B in $\overline{\mathbb{C}}$ is defined as

$$\Delta(A, B) := \frac{\mathrm{dist}(A, B)}{\min\{\mathrm{diam}(A), \mathrm{diam}(B)\}}$$

in the spherical metric. A set of Jordan curves $\mathcal{C} = \{\gamma_i\}_{i\in\mathbb{N}}$ in $\overline{\mathbb{C}}$ is called *uniformly relatively separated* if their pairwise relative distances are uniformly bounded away from zero, that is, there exists $\delta > 0$ such that $\Delta(\gamma_i, \gamma_j) \geq \delta$ for every distinct i, j. The set $\mathcal{C}$ are *uniform quasicircles* if there exists $K \geq 1$ such that each γ_i in $\mathcal{C}$ is a K-quasicircle.

Bonk proved that, if the peripheral circles of a carpet S are uniformly relatively separated and are uniform quasicircles, then S is quasisymmetrically equivalent to a round carpet. Recently, Bonk, Lyubich and Merenkov studied the carpet Julia set J_f generated by a postcritically-finite rational map f [BLM14]. They showed that J_f is quasisymmetrically equivalent to a round carpet. They also proved that the group $QS(J_f)$ is a finite group, whose elements are restrictions of Möbius transformations on J_f.

In Chapter 2, we study carpet Julia sets in the case of postcritically-infinite rational maps.

The ω-limit set $\omega(x)$ of a point $x \in \overline{\mathbb{C}}$ under a rational map f is defined as the set of accumulation points of the orbit of x. Obviously $f(\omega(x)) \subseteq \omega(x)$. A critical point c of f is called *recurrent* if $c \in \omega(c)$. A rational map f is called *semi-hyperbolic* if J_f contains neither parabolic periodic points nor recurrent critical points.

We prove the following theorem in Chapter 2.

Theorem 0.2. *Let f be a rational map whose Julia set J_f is a Sierpiński carpet. Let $\mathcal{C}_f$ be the set of all peripheral circles of this carpet. Then*

(1) *If elements in $\mathcal{C}_f$ avoid the ω-limit sets of all critical points, then*

• *J_f is quasisymmetrically equivalent to a round carpet;*

• *$QS(J_f)$ is discrete.*

(2) *If f is semi-hyperbolic, then*

• *elements in $\mathcal{C}_f$ are uniform quasicircles;*

• *$\mathcal{C}_f$ are uniformly relatively separated if and only if elements in $\mathcal{C}_f$ are disjoint with the w-limit sets of all critical points.*

0.3 Criterion for rays landing together

Let f be a polynomials with degree $d \geq 2$. If J_f is connected, then the *basin of infinity* Ω_f, which consists of points with the orbit attracted by ∞, is simply connected. Moreover, there exists a unique holomorphic parameterization $\Psi_f : \Omega_f \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that $\Psi_f(\infty) = \infty$, $\Psi'_f(\infty) = 1$ and

$$\Psi_f \circ f(z) = (\Psi_f(z))^d.$$

Define $R(\theta) := \Psi_f^{-1}\{re^{2\pi i\theta} : r > 1\}$ to be *external ray* with the angle θ. We say that $R(\theta)$ lands at $z \in J_f$ if $\lim_{r\to 1}\Psi_f^{-1}(re^{2\pi i\theta}) = z$. By a theorem of Carathéodory, Ψ_f^{-1} extends continuously to $\partial\mathbb{D}$ with $f(\partial\mathbb{D}) = J_f$ if and only if J_f is locally connected. Throughout this thesis we only consider the case that J_f is connected and locally connected.

Define $\alpha : \mathbb{R}/\mathbb{Z} \to J_f, \theta \mapsto \alpha(\theta)$ where $\alpha(\theta)$ is the landing point of ray $R(\theta)$. We have the following semi-conjugation,

$$f(\alpha(\theta)) = \alpha(\sigma_d(\theta)),$$

where $\sigma_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is given by $\theta \mapsto d\theta \bmod \mathbb{Z}$. Thus, in order to understand the topology of the Julia set and the dynamics of f on J_f, it is necessary to figure out the semi-conjugation α.

Chapter 3 is devoted to give an answer to the following questions.

(Q_1) For any z in J_f, is the fiber $\alpha^{-1}(z)$ finite?

(Q_2) Give a condition under which θ, θ' are in the same fiber?

For the first question, if the orbit of z is finite, the fiber $\alpha^{-1}(z)$ is finite [DH84]. If z is wandering, Kiwi gave an upper bound $\#\alpha^{-1}(z) \leq 2^d$ [Ki02]. We estimate the cardinal number in a more general case: a finite collection of wandering points with disjoint forward orbits. We give a sharp upper bound in Theorem 3.1, which is the same as that in Blokh and Levin's work [BL02], by using a totally different argument.

For the second question, following [BFH92] [Po93] [Ki05], we need the concept of *critical portrait of f*.

• For a critical point c in J_f, let $\Theta(c)$ be the set of angles of external rays landing at c such that σ_d maps $\Theta(c)$ onto exactly one angle and the external ray with this angle landing at $f(c)$.

• For a strictly pre-periodic critical Fatou component U, let $\Theta(U)$ be the collection of $\deg(f|_U)$ angles of external rays *supporting* U such that all of them are pre-image of some external ray supporting $f(U)$.

• For a cycle of critical Fatou component $U_0, \cdots, U_{p-1}$ with $f^i(U_0) = U_i, i = 1, \cdots, p$ and $U_p := U_0$, let $U_{k_0}, \cdots, U_{k_l}, 0 \leq k_0 < \cdots < k_l \leq p-1$ be critical with degree $n_0, \cdots, n_l$. For $0 \leq i \leq p$, choose (z_i, θ_i), $z_i \in \partial U_i$ and $R(\theta_i)$ supporting U_i at z_i such that $f^i(z_0) = z_i, f^i(R(\theta_0)) = R(\theta_i)$ and $f^p(z_p) = z_p$. Let $\Theta(U_{k_j})$ be the set of angles of external rays landing on ∂U_{k_j} which are inverse images of $R(\theta_{k_j+1})$, for $1 \leq j \leq l$.

Let $\mathcal{A} := \{\Theta(c_1), \cdots, \Theta(c_m), \Theta(U_1), \cdots, \Theta(U_n)\}$, where $\{c_1, \cdots, c_m\}$ and $\{U_1, \cdots, U_n\}$ are the set of critical points in J_f and critical Fatou components, respectively. For any $\Theta \in \mathcal{A}$, let

$$\widehat{\Theta} := \bigcup \{\Theta' : \exists \text{ a chain } \Theta_0, \cdots, \Theta_k = \Theta' \text{ in } \mathcal{A} \text{ such that } \Theta_i \bigcap \Theta_{i+1} \neq \emptyset\}.$$

The collection $\widehat{\mathcal{A}} := \{\widehat{\Theta}_1, \cdots, \widehat{\Theta}_N\}$ is called *critical portrait of f*.

A simple case is that f is a polynomial with J_f locally connected and all cycles repelling. In this case the external rays with angles in Θ_i are landing at a same critical point, and f maps these external rays to exact one external ray.

Let $\mathcal{P} := \{I_1, \cdots, I_d\}$ be a partition of the unit circle, where each I_i is a finite union of open intervals in $\mathbb{R}/\mathbb{Z} \setminus \bigcup_{1 \leq i \leq N} \widehat{\Theta}_i$ with total length $1/d$ (See Section 3.5 for details). We say that $\theta, \theta' \in \mathbb{R}/\mathbb{Z}$ *have the same sequence* (*itinerary*) *with respect to* $\mathcal{P}$ if $\sigma_d^k(\theta), \sigma_d^k(\theta') \in I_{i_k}$ for all $k \geq 0$.

Biefield, Fisher and Hubbard showed that, for polynomials with all critical points strictly preperiodic, if θ, θ' have the same sequence with respect to $\mathcal{P}$, then $\alpha(\theta) = \alpha(\theta')$ [BFH92]. Poirier extended the above result to critical finite polynomials that admit periodic Fatou components [Po93]. In [Ki05], Kiwi considered polynomials with all cycles repelling and Julia set connected. He proved that if θ, θ' have the same sequence with respect to $\mathcal{P}$, then the *impressions* of $R(\theta)$ and $R(\theta')$ intersect.

The following result is our main result in Chapter 3.

Theorem 0.3. *Let f be a polynomial with J_f connected and locally connected. Let $\mathcal{P}$ be the partition induced by a critical portrait $\widehat{\mathcal{A}}$ of f. If θ, θ' have the same sequence with respect to $\mathcal{P}$, then either $R(\theta)$, $R(\theta')$ land at the same point or $R(\theta), R(\theta')$ land at the boundary of a Fatou component U, which is eventually iterated to a Siegel disk.*

One of our motivation of the above study is to understand the *core-entropy* of polynomials, which was first introduced and explored by Thurston. The *core-entropy* of polynomial f is the topological entropy of f on its f-invariant set, the *Hubbard tree.* Let $Acc(f)$ be the set of all *biaccessible* angles θ, i.e., there exist at least two rays landing at $\alpha(\theta)$. Then the core-entropy $h(f)$ is related to the Hausdorff dimension of $Acc(f)$ by the following formula:

$$h(f) = \log\, d \cdot \text{H.dim} Acc(f).$$

For more results on the core-entropy, we refer to [Do95] [Ti13] [Ti14] [Ga13] [Li07] [Ju13].

As an application of Theorem 0.3, we prove the monotonicity of core-entropy for a family of quadratic polynomials $\{f_c : z \mapsto z^2 + c, f_c$ has no Siegel disks and J_{f_c} is locally connected $\}$ (see Theorem 3.3). This generalizes Tao Li's result on critical finite quadratic polynomials [Li07].

0.4 A landing theorem on non-recurrent polynomials

Let f be a polynomial with degree $d \geq 2$. Let Ω_f be *basin of infinity* consisting of points in $\overline{\mathbb{C}}$ escaping to ∞. Denote the filled Julia set by $K_f := \overline{\mathbb{C}} \setminus \Omega_f$. There exists a Green function G_f that measures the escape rate of points to ∞, defined by

$$G_f : \overline{\mathbb{C}} \to [0, \infty),\ z \mapsto \lim_{n\to\infty} \frac{\log |f^n(z)|}{d^n}.$$

Note that G_f is positive and harmonic in Ω_f. The derivative of G_f vanishes at z if and only if z is a (pre-)critical point. Each locus $G_f^{-1}(r) = \{z \in \overline{\mathbb{C}}, G_f(z) = r\}$ with $r > 0$ is called an *equipotential curve* around the filled Julia set K_f.

There exists an unique normalized Böttcher map Ψ_f which conjugates f with $z \mapsto z^d$ in a neighborhood of ∞. Ψ_f^{-1} has an unique maximal radial extension to a subset of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. This radial extension terminates at a point w with $|w| > 1$ if and only if Ψ_f^{-1} extends continuously to w and $\Psi_f^{-1}(w)$ is a (pre-)critical point of f. Then *external radius* $R(t)$ with angle t is given by

$$R(t) := \Psi_f^{-1}((r_t, \infty)e^{2\pi i t}),$$

where $\Psi_f^{-1}(r_t e^{2\pi i t})$ is a (pre-)critical point of f if $t_t > 1$. If $r_t = 1$, then $R(t)$ is exactly the external ray defined in Section 0.3.

Let $\mathcal{P}_d$ be the set of monic centered polynomials of degree d. The *shift locus* $\mathcal{S}_d$ is the subset of $\mathcal{P}_d$ formed by polynomials with all critical points escaping to infinity. Let $\mathcal{S}_d(r)$, $r > 0$, consist of polynomials $f \in \mathcal{S}_d$ such that all critical points of f are in the same equipotential curve $G_f^{-1}(r)$ and let $\mathcal{S}'_d := \bigcup_{r>0} \mathcal{S}_d(r)$.

A collection $\Theta = \{\Theta_1, \cdots, \Theta_n\}$ of finite subsets of $\mathbb{R}/\mathbb{Z}$ is called an (*abstract*) *critical portrait* of degree d if the following conditions hold.

(1) For every j, $|\Theta_j| \geq 2$ and $|\sigma_d(\Theta_j)| = 1$, where $\sigma_d : \theta \mapsto d\theta \bmod \mathbb{Z}$ and $|A|$ denotes the cardinal number of the set A.

(2) $\Theta_1, \cdots, \Theta_n$ are pairwise unlinked.

(3) $\sum(|\Theta_j| - 1) = d - 1$.

For another critical portrait $\Theta' = \{\Theta'_1, \cdots, \Theta'_n\}$, we say $\Theta = \Theta'$ if there exists a permutation τ such that $\Theta_i = \Theta'_{\tau(i)}$ for $1 \leq i \leq d$. Let $\mathcal{A}_d$ be the collection of all critical portraits of degree d. In [Ki05], Kiwi gave $\mathcal{A}_d$ a compact-unlinked topology and proved that $\mathcal{A}_d$ is compact and connected. Critical portraits of polynomials with connected Julia sets defined in Section 0.3 are obviously critical portraits defined here.

Now we consider the map

$$\Pi : \mathcal{S}'_d \to \mathcal{A}_d$$

defined as following. Given any $f \in \mathcal{S}'_d$, let $\{c_j\}_{i=1,\cdots,n}$ be set of critical points of f. For each $c_j, j = 1, \cdots, n$, there are exact $\deg_f(c_j)$ external radius terminating at c_j. Let Θ_j be the set of angles of these external radius. Then $\Theta = \{\Theta_1, \cdots, \Theta_n\}$ is a critical portrait in $\mathcal{A}_d$. We set $\Pi(f) := \Theta$.

In [Go94], Goldberg proved that Π is surjective. Kiwi showed that Π is continuous and, for any $\Theta \in \mathcal{A}_d$, the preimage $S_\Theta = \Pi^{-1}(\Theta)$ is a one-dimensional real analytic manifold. Precisely, the map $G : S_\Theta \to (0, \infty)$ which sends f to $G_f(c_i)$ is bijective and analytic. Moreover, given $r > 0$, the restriction $\Pi|_{\mathcal{S}_d(r)} : \mathcal{S}_d(r) \to \mathcal{A}_d$ is a homeomorphism [Ki05].

The *connected locus* $\mathcal{C}_d$ is the set of monic centered polynomials with degree d such that all the critical orbits are bounded. We know that $\mathcal{C}_d$ is a compact and connected subset of $\mathcal{P}_d$ [BH88]. For instance, $\mathcal{C}_2$ is the Mandelbrot set. To describe $\mathcal{C}_d$ we look at it from outside $\mathcal{S}_d$.

The *impression* $I_{\mathcal{C}_d}(\Theta)$ of a critical portrait Θ is a subset of $\mathcal{C}_d$, characterized by the property that $f \in I_{\mathcal{C}_d}(\Theta)$ if and only if there exists a sequence $\{f_n\}$ in $\mathcal{S}'_d$ such that $\Pi(f_n) = \Theta$ and f_n converges to f.

Note that the impression here is slightly different from the definition in [Ki05], which is bigger and containing $I_{\mathcal{C}_d}(\Theta)$ there. Kiwi proved that if all angles in Θ is strictly pre-periodic, then the impression $I_{\mathcal{C}_d}(\Theta)$ is a singleton [Ki05]. He conjectured that there exist critical portraits with aperiodic kneadings and non-trivial impressions.

In Chapter 4, we give an elementary proof of the following theorem based on the tools developed in [CT15].

Theorem 0.4. (1) *The map*

$$P : \mathcal{A}_d \times (0, \infty) \to \mathcal{S}'_d,\ (\Theta, r) \mapsto f_{\Theta,r},$$

such that $f_{\Theta,r} \in \mathcal{S}_d(r)$ *induces the critical portrait* Θ*, is well-defined, one-to-one and continuous.*

(2) *Let* $R_\Theta : (0, \infty) \to \mathcal{S}'_d$*,* $R_\Theta(t) = P(\Theta, t)$*. Then* $R_\Theta(t)$ *forms a simple curve in* $\mathcal{S}'_d$*. (*R_Θ *is called a parameter ray with the angle* Θ*). Let* f *be a polynomial in* $\mathcal{C}_d$ *with no recurrent critical points and all cycles repelling. Then* $R_\Theta(t)$ *lands at* f *if and only if* Θ *is a critical portrait of* f*.*

0.5 On the dynamics of a family of generated renormalization transformations

The statistical mechanical models on hierarchical lattices have attracted many interests recently since they exhibit a deep connection between their limiting sets of the zeros of the partition functions and the Julia sets of rational maps in complex dynamics [BL91, DSI83, Qi11, QL01, QYG10]. The well-known Yang-Lee theorem in statistical mechanics shows that the zeros of the partition function is dense in a line for many magnetic materials in a complex magnetic field plane. This means that the complex singularities of the free energy lie on this line, where the free energy is the logarithm of the partition function [YL52]. By the works of Fisher and others [Fi65], it was generally believed that the zeros of the partition function condense to some simple curve.

Until 1983, Derrida et al. showed that the zeros of the partition function condense to the Julia set of the renormalization transformation of so-called *standard hierarchical lattices* [DSI83]. They proved that the singularities of the free energy lie on the Julia set of the rational map

$$z \mapsto \left(\frac{z^2 + \lambda - 1}{2z + \lambda - 2}\right)^2.$$

This means that the distribution of the singularities of the free energy is not as simple as one desired. For the ideas formulated in renormalization transformation in statistical mechanics, see [Wi71].

Recently, Qiao considered the generalized diamond hierarchical Potts model and proved that the family of rational maps

$$U_{mn\lambda}(z) = \left(\frac{(z + \lambda - 1)^m + (\lambda - 1)(z - 1)^m}{(z + \lambda - 1)^m - (z - 1)^m}\right)^n$$

are actually the renormalization transformation of the *generalized diamond hierarchical*

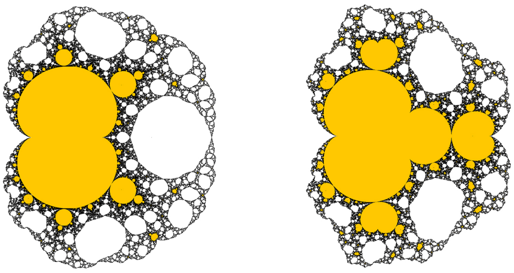

Figure 2: The non-escaping loci $\mathcal{M}_2$ and $\mathcal{M}_3$

Potts model [Qi11, Theorem 1.1], where $m, n \geq 2$ are both integers and $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is a complex parameter. The standard diamond lattice ($m = n = 2$) and the *diamond-like lattice* ($m = 2$ and $n \in \mathbb{N}$) are the special cases.

In this thesis, we will consider the case for $d := m = n \geq 2$, $\lambda \neq 0$. For simplicity, we use $U_{d\lambda}$ to denote $U_{dd\lambda}$ in (5.2). The postcritical set of $U_{dd\lambda}$ is

$$\bigcup_{k \geq 0} U_{d\lambda}^k(0) \bigcup \{1, \infty\}.$$

Since both of $1, \infty$ are superattracting fixed points of $U_{d\lambda}$ with local degree d. The *non-escaping locu* $\mathcal{M}_d$ associated to this family is defined by

$$\mathcal{M}_d = \{\lambda \in \mathbb{C}^* : U_{d\lambda}^{\circ n}(0) \not\to 1 \text{ and } U_{d\lambda}^{\circ n}(0) \not\to \infty \text{ as } n \to \infty\} \bigcup \{0\}.$$

We prove the following theorem in Chapter 5.

Theorem 0.5. *Let $d \geq 2$ be integer and $\lambda \in \mathbb{C}^*$. Denote by $J_{d\lambda}$ the Julia set of $U_{d\lambda}$. Then*

(1) *$J_{d\lambda}$ is connected.*

(2) *If $\lambda \in \mathbb{R}$, then $J_{d\lambda}$ is not a Sierpiński carpet.*

(3) *For sufficiently large $|\lambda|$, $J_{d\lambda}$ is a quasicircle and the Hausdorff dimension of $J_{d\lambda}$ is given by*

$$\dim_H(J_{d\lambda}) = 1 + \frac{1}{4 \log d} |\lambda|^{-\frac{2}{d+1}} + \mathcal{O}(\lambda^{-\frac{3}{d+1}}).$$

(4) *The non-escaping locus $\mathcal{M}_d$ is connected.*

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Chapter 1

Quasisymmetric rigidity of Sierpiński carpets $F_{n,p}$

1.1 Introduction

The quasisymmetric geometry of Sierpiński carpets is related to the study of Julia sets in complex dynamics and boundaries of Gromov hyperbolic groups. For background and research progress, we recommend the survey of M. Bonk [Bo06].

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Let $S = \mathbb{S}^2 \setminus \bigcup_{i\in\mathbb{N}} D_i$ be the complement in $\mathbb{S}^2$ of countably many pair-wise disjoint open Jordan regions $D_i \subset \mathbb{S}^2$. S is called a *(Sierpiński) carpet* if S has empty interior, diam $(D_i) \to 0$ as $i \to \infty$, and $\partial D_i \cap \partial D_j = \emptyset$ for all $i \neq j$. The boundary of D_i, denoted by C_i, is called a *peripheral circle* of S. A *round carpet* is a carpet on $\mathbb{S}^2$ such that all of its peripheral circles are geometric circles. Typical Examples of round carpets are limit sets of convex co-compact Kleinian groups.

Topologically all carpets are the same [W58]. Much richer structure arises if we consider quasisymmetric geometry of metric carpets. The famous conjecture of Kapovich-Kleiner [KK00] predicts that if G is a hyperbolic group with boundary $\partial_\infty G$ homeomorphic to a Sierpiński carpet, then G acts geometrically (the action is isometrical, properly discontinuous and co-compact) on a convex subset of $\mathbb{H}^3$ with non-empty totally geodesic boundary. The Kapovich-Kleiner conjecture is equivalent to the conjecture that the carpet $\partial_\infty G$ (endowed with the "visual" metric) is quasisymetriclly equivalent to a round carpet on $\mathbb{S}^2$. The conjecture is true for carpets that can be quasisymmetrically embedding in $\mathbb{S}^2$ [Bo11].

The concept of quasisymmetric map between metric spaces was defined by Tukia and Väisälä [TV80]. Let $f : X \to Y$ be a homeomorphism between two metric spaces (X, d_X) and (Y, d_Y). f is *quasisymmetric* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta(\frac{d_X(x, y)}{d_X(x, z)}), \ \forall\ x, y, z \in X, x \neq z.$$

It follows from the definition that the quasisymmetric self-maps of X form a group $\mathrm{QS}(X)$.

A homeomorphism $f : X \to Y$ is called *quasi-Möbius* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all 4-tuple (x_1, x_2, x_3, x_4) of distinct points in X, we have

$$[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4]),$$

where

$$[x_1, x_2, x_3, x_4] = \frac{d_X(x_1, x_3)d_X(x_2, x_4)}{d_X(x_1, x_4)d_X(x_2, x_3)}$$

is the *metric cross-ratio*.

It is not hard to check that a quasisymmetric map between metric spaces is quasi-Möbius. Conversely, any quasi-Möbius map between bounded metric spaces is quasisymmetric [TV80].

An important tool in the study of quasisymmetric maps is the conformal modulus of a given family of paths. The notion of conformal modulus (or extremal length) was first introduced by Ahlfors and Beurling [Ah73]. It has many applications in complex analysis and metric geometry [LV, He09].

1.1.1 Motivation

In the work of Bonk and Merenkov [BM11], it was proved that every quasisymmetric self-homeomorphism of the standard 1/3-Sierpiński carpet S_3 is a Euclidean isometry. For the standard $1/p$-Sierpiński carpets S_p, $p \geq 3$ odd, they showed that the groups $\mathrm{QS}(S_p)$ of quasisymmetric self-maps are finite dihedral. They also established that S_p and S_q are quasisymmetrically equivalent if only if $p = q$. The main tool in their proof is the *carpet modulus*, which is a certain discrete modulus of a path family and is preserved under quasisymmetric maps of carpets.

The following question is inspired by the above results of Bonk and Merenkov [BM11]:

Question 1.1. *Determining sufficient condition on a carpet S on $\mathbb{S}^2$ such that* $\mathrm{QS}(S)$ *is* $\mathrm{Isom}(S)$*, the isometry group of S.*

Note that $\mathrm{QS}(S_3) = \mathrm{Isom}(S_3)$ and $\mathrm{QS}(S_p)$ contains $\mathrm{Isom}(S_p)$ as a finite-index subgroup. Bonk and Merenkov [BM11] conjectured that $\mathrm{QS}(S_p) = \mathrm{Isom}(S_p)$ for any p odd. The aim of this chapter is to extend Bonk-Merenkov's results to a new class of Sierpiński carpets $F_{n,p}$ ($5 \leq n, 1 \leq p \leq \frac{n}{2} - 1$). We will show that $\mathrm{QS}(F_{n,p}) = \mathrm{Isom}(F_{n,p})$. This is a further generalization of the work of Bonk and Merenkov [BM11].

1.1.2 Main results

Unless otherwise indicated, we will equip a carpet $S = \mathbb{S}^2 \setminus \bigcup_{i \in \mathbb{N}} D_i$ with the spherical metric. Note that when a carpet is contained in a compact set K of $\mathbb{C} \subset \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$, the Euclidean and the spherical metrics are bi-Lipschitz equivalent on K.

Let $5 \leq n, 1 \leq p < \frac{n}{2} - 1$ be integers. Let $Q_{n,p}^{(0)} = [0,1] \times [0,1]$ be the closed unit square in $\mathbb{R}^2$. We first subdivide $Q_{n,p}^{(0)}$ into n^2 subsquares with equal side-length $1/n$ and remove the interior of four subsquares, each has side-length $1/n$ and is of distance $\sqrt{2}p/n$ to one of the four corner points of $Q_{n,p}^{(0)}$.

The resulting set $Q_{n,p}^{(1)}$ consists of $(n^2 - 4)$ squares of side-length $1/n$. Inductively, $Q_{n,p}^{(k+1)}$, $k \geq 1$, is obtained from $Q_{n,p}^{(k)}$ by subdividing each of the remaining squares in the subdivision of $Q_{n,p}^{(k)}$ into n^2 subsquares of equal side-length $1/n^{k+1}$ and removing the interior of four subsquares as we have done above.

The Spierpiński carpet $F_{n,p}$ is the intersection of all the sets $Q_{n,p}^{(k)}$, i.e.,

$$F_{n,p} = \bigcap_{k=0}^{+\infty} Q_{n,p}^{(k)}.$$

See Figure 1.1.

The following theorem will be proved in Section 1.4. It shows that, from the point of view of quasiconformal geometry, the carpets $F_{n,p}$ are different with the standard Sierpiński carpets $S_m, m \geq 3$ odd (note that the standard Sierpiński carpets S_m is constructed from a similar process, by removing the interior of the middle square in each steps).

Theorem 1.1. *Let $5 \leq n, 1 \leq p < \frac{n}{2} - 1$ be integers. The carpet $F_{n,p}$ is not quasisymmetrically equivalent to the Standard Sierpiński carpet $S_m, m \geq 3$ odd.*

It was proved by Bonk and Merenkov [BM11] that for $m \geq 3$ odd the quasisymmetric group $\mathrm{QS}(S_m)$ is a finite dihedral group. Moreover, when $m = 3$, $\mathrm{QS}(S_3)$ is the Euclidean isometry group of S_3. In Section 1.6, we will show that

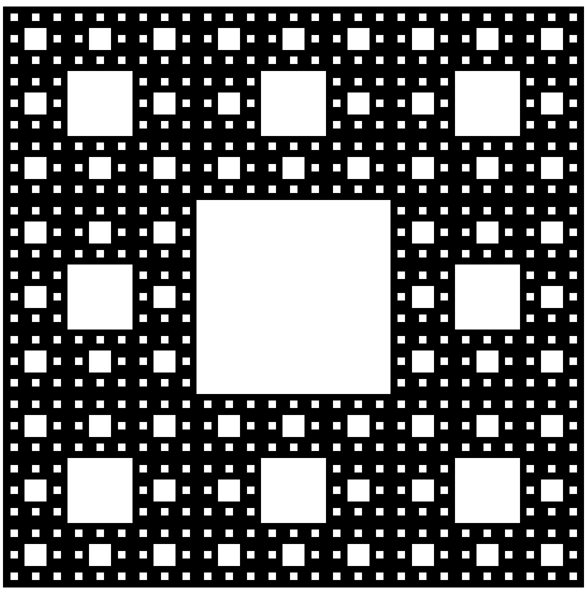

Figure 1.1: The standard Sieipiński carpet S_3.

Theorem 1.2. *Let f be a quasisymmetric self-map of $F_{n,p}$. Then f is a Euclidean isometry.*

Note that the Euclidean isometric group of $F_{n,p}$ (and S_m), consists of eight elements, is the group generated by the reflections in the diagonal $\{(x,y) \in \mathbb{R}^2 : x = y\}$ and the vertical line $\{(x,y) \in \mathbb{R}^2 : x = \frac{1}{2}\}$.

We will also prove that

Theorem 1.3. *Two Sierpiński carpets $F_{n,p}$ and $F_{n',p'}$ are quasisymmetrically equivalent if and only if $(n,p) = (n',p')$.*

1.1.3 Idea of the proofs

The main tools to prove the above theorems are the *carpet modulus* and the *weak tangent*, both of which were investigated in [BM11]. Our arguments follow the same outline as [BM11]. One of the most important observations in [BM11] is that a quasisymmetric self-map f of S_3 should preserve the pair $\{M, O\}$, where M and O are the inner and outer peripheral circle of S_3, respectively. By counting the orbits of points under the action of $\mathrm{QS}(S_3)$, Bonk and Merenkov [BM11] then showed that f maps distinguished points (points of S_3 on the corner or on the middle of peripheral circles) to distinguished points. One the other hand, f induces a "tangent map" Df between weak tangents of distinguished points, which is also quasisymmetry. The study of carpet modulus with respect to the normalized quasisymmetry group of weak tangent shows that f should map M to M and O to O.

We will first concentrate on carpet modulus of the families of curves connecting the boundary of the annulus domains bounded by pairs of distinct peripheral circles of $F_{n,p}$. The extremal mass distribution of such a carpet modulus exists and is unique (Proposition 1.3). This, together with the auxiliary results in Section 1.3, allows us to show that (see Section 4) any quasisymmetric self-map f of $F_{n,p}$ should preserves the set $\{O, M_1, M_2, M_3, M_4\}$, where O is the boundary of the unit square and M_1, M_2, M_3, M_4 are the boundary of the first four squares removed from the unit square.

It is more difficult to see that f should maps O to O. To show this, we first study the weak tangents of the carpets (this is our main work on Section 1.5). In Section 1.6, we prove that $f(O) = O$ by counting the orbit of a corner of O or M_i under the group $\mathrm{QS}(F_{n,p})$. The proofs of Theorem 1.2 and Theorem 1.3 are given at the end of this chapter. Theorem 1.2 is much stronger than the result of Bonk and Merenkov [BM11] for $S_m, m \geq 5$ odd.

1.2 Carpet modulus

In this section, we shall recall the definitions of conformal modulus and carpet modulus. The carpet modulus was introduced by Bonk-Merenkov [BM11] as a quasisymmetric invariant. There are several important properties of the carpet modulus that will be used in the rest of this chapter. In many cases, we will neglect the proof and refer to [BM11] instead.

1.2.1 Conformal modulus

A *path* γ in a metric space X is a continuous map $\gamma : I \to X$ of a finite interval I. Without cause of confusion, we shall identified the map with its image $\gamma(I)$ and denote a path by γ. We say that γ is *open* if $I = (a, b)$. The limits $\lim_{t\to a} \gamma(t)$ and $\lim_{t\to b} \gamma(t)$, if they exist, are called the *end points of* γ. If $A, B \subseteq X$, then we say that γ *connects* A *and* B if γ has endpoints such that one of them lies in A and the other lies in B. If $I = [a, b]$ is a closed interval, then the length of $\gamma : I \to X$ is defined by

$$\text{length}(\gamma) := \sup \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})|$$

where the supremum is taken over all finite sequences $a = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = b$. If I is not closed, then we set

$$\text{length}(\gamma) := \sup_J \text{length}(\gamma|J),$$

where J is taken over all closed subintervals of I and $\gamma|_J$ denotes the restriction of γ on J. We call γ *rectifiable* if its length is finite. Similarly, a path $\gamma : I \to X$ is *locally rectifiable* if its restriction to each closed subinterval is rectifiable. Any rectifiable path $\gamma : I \to X$ has a unique extension $\overline{\gamma}$ to the closure $\overline{I}$ of I.

Let Γ be a family of paths in $\mathbb{S}^2$. Let σ be the spherical measure and ds be the spherical line element on $\mathbb{S}^2$ induced by the spherical metric (the Riemannian metric on $\mathbb{S}^2$ of constant curvature 1). The *conformal modulus* of Γ is defined as

$$\text{mod}(\Gamma) := \inf \int_{\mathbb{S}^2} \rho^2 d\sigma \ ,$$

where the infimum is taken over all nonnegative Borel functions $\rho : \mathbb{S}^2 \to [0, \infty]$ satisfying

$$\int_\gamma \rho ds \geq 1$$

for all locally rectifiable path $\gamma \in \Gamma$. Functions ρ satisfying (1.2.1) for all locally rectifiable path $\gamma \in \Gamma$ are called *admissible*.

It is easy to show that (see [Ah66])

$$\mathrm{mod}(\Gamma_1) \leq \mathrm{mod}(\Gamma_2), \tag{1.1}$$

if $\Gamma_1 \subseteq \Gamma_2$ and

$$\mathrm{mod}(\bigcup_{i=1}^{\infty} \Gamma_i) \leq \sum_{i=1}^{\infty} \mathrm{mod}(\Gamma_i). \tag{1.2}$$

Moreover, if Γ_1 and Γ_2 are two families of paths such that each path γ in Γ_1 contains a subpath $\gamma' \in \Gamma_2$, then

$$\mathrm{mod}(\Gamma_1) \leq \mathrm{mod}(\Gamma_2) \tag{1.3}$$

If $f : \Omega \to \Omega'$ is a continuous map between domains Ω and Ω' in $\mathbb{S}^2$ and Γ is a family of paths contained in Ω, then we denote by $f(\Gamma) = \{f \circ \gamma \mid \gamma \in \Omega\}$.

If $f : \Omega \to \Omega'$ is a conformal map between regions Ω, $\Omega' \subseteq \mathbb{S}^2$ and Γ is a family of paths in Ω, then mod(Γ)=mod($f(\Gamma)$). This is the fundamental property of modulus: conformal maps do not change the conformal modulus of a family of paths.

In this chapter, we shall adopt the metric definition of quasiconformal maps ([HK98], Definition 1.2) and allow them to be orientation-reversing. Suppose that $f : X \to Y$ is a homeomorphism between two metric spaces X and Y. f is *quasiconformal* if there is a constant $H \geq 1$, s.t. $\forall x \in X$,

$$\limsup_{r \to 0^+} \frac{\max\{d(f(x), f(y)) : d(x,y) \leq r\}}{\min\{d(f(x), f(y)) : d(x,y) \geq r\}} \leq H.$$

Quasiconformal maps distort the conformal modulus of path families in a controlled way. Let Ω and Ω' be regions in $\mathbb{S}^2$ and let Γ be a family of paths in Ω. Suppose that $f : \Omega \to \Omega'$ is quasiconformal map. Then

$$\frac{1}{K}\mathrm{mod}(\Gamma) \leq \mathrm{mod}(f(\Gamma)) \leq K\mathrm{mod}(\Gamma), \tag{1.4}$$

where $K \geq 1$ depends on the dilatation of f .

From (1.4), a quasiconformal map preserves the modulus of a path family up to a fixed multiplicative constant. So if $\Gamma_0 \subseteq \Gamma$ and $\mathrm{mod}(\Gamma_0) = 0$, then $\mathrm{mod}(f(\Gamma_0)) = 0$.

1.2.2 Carpet modulus

If a certain property for paths in Γ holds for all paths outside an exceptional family $\Gamma_0 \subseteq \Gamma$ with $\mathrm{mod}(\Gamma_0) = 0$, then we say that it holds for *almost every path in* Γ.

Let $S = \mathbb{S}^2 \backslash \bigcup_{i=1}^{\infty} D_i$ be a carpet with $C_i = \partial D_i$, and let Γ be a family of paths in $\mathbb{S}^2$. A *mass distribution* ρ is a function that assigns to each C_i a non-negative number $\rho(C_i)$.

The *carpet modulus* of Γ with respect to S is defined as

$$\operatorname{mod}_S(\Gamma) = \inf_{\rho} \sum_i \rho(C_i)^2,$$

where the infimum is taken over all *admissible* mass distribution ρ, that is, mass distribution ρ satisfies

$$\sum_{\gamma \bigcap C_i \neq \emptyset} \rho(C_i) \geq 1$$

for all most every path in Γ.

It is straightforward to check that the carpet modulus is momotone and countably subadditive, the same properties as conformal modulus in (1.1), (1.2) and (1.3). An crucial property of carpet modulus is its invariance under quasiconformal maps.

Lemma 1.1 ([BM11]). *Let $D, \widetilde{D} \subset \mathbb{S}^2$ be regions and $f : D \to D$ be a quasiconformal map. Let $S \subseteq D$ be a carpet and Γ be a family of paths such that $\gamma \subset D$ for each $\gamma \in \Gamma$. Then*

$$\operatorname{mod}_{f(S)}(f(\Gamma)) = \operatorname{mod}_S(\Gamma).$$

1.2.3 Carpet modulus with respect to a group

We also need the notion of *carpet modulus with respect to a group.*

Let $S = \mathbb{S}^2 \setminus \bigcup_{i\in\mathbb{N}} D_i$ be a carpet and $C_i = \partial D_i$. Let G be a group of homeomorphisms of S. If $g \in G$ and $C \subseteq S$ is a peripheral circle of S, then $g(C)$ is also a peripheral circle of S. Let $\mathcal{O} = \{g(C) : g \in G\}$ be the orbit of C under the action of G.

Let Γ be a familly of paths in $\mathbb{S}^2$. A admissible G-invariant mass distribution $\rho : \{C_i\} \to [0, +\infty]$ is a mass distribution such that

1. $\rho(g(C)) = \rho(C)$ for all $g \in G$ and all peripheral circles C of S;

2. almost every path γ in Γ satisfies

$$\sum_{\gamma \bigcap C_i \neq \emptyset} \rho(C_i) \geq 1.$$

The *carpet modulus* $\operatorname{mod}_{S/G}(\Gamma)$ with respect to the action of G on S is defined as

$$\operatorname{mod}_{S/G}(\Gamma) := \inf_{\rho} \sum_{\mathcal{O}} \rho(\mathcal{O})^2,$$

where the infimum is taken over all admissible G-invariant mass distributions. In the above definition, $\rho(\mathcal{O})$ is defined by $\rho(C)$ for any $C \in \mathcal{O}$. Since ρ is G-invariant, $\rho(\mathcal{O})$ is well-defined. Note that each orbit contributions with exactly one term to the sum $\sum_{\mathcal{O}} \rho(\mathcal{O})^2$.

Lemma 1.2 ([BM11]). *Let D be a region in $\mathbb{S}^2$ and S be a carpet contained in D. Let G be a group of homeomorphisms on S. Suppose that Γ is a family of paths with $\gamma \subseteq D$ for each $\gamma \in \Gamma$ and $f : D \to \widetilde{D}$ a quasiconformal map onto another region $\widetilde{D} \subseteq \mathbb{S}^2$. We denote $\widetilde{S} = f(S)$, $\tilde{\Gamma} = f(\Gamma)$ and $\widetilde{G} = (f|_S) \circ G \circ (f|_S)^{-1}$, then*

$$\mathrm{mod}_{\widetilde{S}/\widetilde{G}}(\tilde{\Gamma}) = \mathrm{mod}_{S/G}(\Gamma).$$

Lemma 1.3. *Let S be a carpet in $\mathbb{S}^2$ and $\Psi : \mathbb{S}^2 \to \mathbb{S}^2$ be a quasiconformal map with $\Psi(S) = S$, $\psi := \Psi|_S$. Assume that Γ is a Ψ-invariant path family in $\mathbb{S}^2$ such that for every peripheral circle C of S that meets some path in Γ we have $\psi^n(C) \neq C$ for all $n \in \mathbb{Z}$. Then $\mathrm{mod}_{S/\langle\psi^k\rangle}(\Gamma) = k\mathrm{mod}_{S/\langle\psi\rangle}(\Gamma)$ for every $k \in \mathbb{N}$.*

This is ([BM11], Lemma 3.3). In this Lemma, $\langle\psi\rangle$ denotes the cyclic group of homeomorphisms on S generated by ψ, and Γ is called Ψ-*invariant* if $\Psi(\Gamma) = \Gamma$. This lemma gives a precise relationship between the carpet modulus with respect to a cyclic group and its subgroups.

1.2.4 Existence of extremal mass distribution

Let $S = \mathbb{S}^2 \setminus \{D_i\}, C_i = \partial D_i$ be a carpet and Γ be a family of paths on $\mathbb{S}^2$. An admissible mass distribution ρ for a carpet modulus $\mathrm{mod}_S(\Gamma)$ is called *extremal* if $\mathrm{mod}_S(\Gamma)$ is obtained by ρ:

$$\mathrm{mass}(\rho) = \sum_i \rho(C_i)^2 = \mathrm{mod}_S(\Gamma).$$

Similarly, an G-invariant mass distribution that obtains $\mathrm{mod}_{S/G}(\Gamma)$ is also called *extremal*.

A criterion for the existence of an extremal mass distribution for carpet modulus (with respect to the group) is given by [BM11]. Recall that the peripheral circles $\{C_i\}$ are *uniform quasicircles* if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that every C_i is the image of an η-quasisymmetric map of the unit circle.

Proposition 1.1. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles, and let Γ be an arbitrary path family in $\mathbb{S}^2$ with $\mathrm{mod}_S(\Gamma) < +\infty$. Then the extremal mass distribution for $\mathrm{mod}_S(\Gamma)$ exists and is unique.*

This is ([BM11], Proposition 2.4). The uniqueness follows from elementary convexity argument.

Proposition 1.1. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles. Let G be a group of homeomorphisms of S and Γ be a path family in $\mathbb{S}^2$ with $\mathrm{mod}_{S/G}(\Gamma) < +\infty$. Suppose that for each $k \in \mathbb{N}$ there exists a family of peripheral circles $\mathcal{C}_k$ of S and a constant $N_k \in \mathbb{N}$ with the following properties:*

1. *If $\mathcal{O}$ is any orbit of peripheral circles of S under the action of G, then $\#(\mathcal{O} \bigcap \mathcal{C}_k) \leq N_k$ for all $k \in \mathbb{N}$.*

2. *If Γ_k is the family of all paths in Γ that only meet peripheral circles in $\mathcal{C}_k$, then $\Gamma = \bigcup_k \Gamma_k$.*

Then extremal mass distribution for $\operatorname{mod}_{S/G}(\Gamma)$ *exists and is unique.*

This is ([BM11], Proposition 3.2).

1.3 Auxiliary results

In this section, we collect a series of results obtained by M. Bonk and his coauthors [BKM, Bo11]. The theorems and propositions cited here are the cornerstone of our later proof (as well as they were for the proof in [BM11]).

1.3.1 Quasiconformal extention of quasisymmetric map

Proposition 1.2. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniform quasicircles and let f a quasisymmetric map of S onto another carpet $\widetilde{S} \subseteq \mathbb{S}^2$. Then there exists a self-quasiconformal map F on $\mathbb{S}^2$ whose restriction to S is f.*

This is ([Bo11], Proposition 5.1).

1.3.2 Quasisymmetric uniformization and rigidity

The peripheral circles $\{C_i\}$ of S are called *uniformly relatively separated* if the pairwise distances are uniformly bounded away from zero. i.e., there exists $\delta > 0$ such that

$$\Delta(C_i, C_j) = \frac{\operatorname{dist}(C_i, C_j)}{\min\{\operatorname{diam}(C_i), \operatorname{diam}(C_j)\}} \geq \delta$$

for any two distinct i and j. This property is preserved under quasisymmetric maps. See ([Bo11], Corollary 4.6).

Theorem 1.1. *Let S be a carpet in $\mathbb{S}^2$ whose peripheral circles are uniformly relatively separated uniformly quasicircles, then there exists a quasisymmetric map of S onto a round carpet.*

This is ([Bo11], Corollary 1.2). Recall that a carpet $S = \mathbb{S}^2 \setminus \bigcup D_i$ is called *round* if each D_i is an open spherical disk.

Theorem 1.2. *Let S be a round carpet in $\mathbb{S}^2$ of measure zero. Then every quasisymmetric map of S onto any other round carpet is the restriction of a Möbius transformation.*

This is ([BKM], Theorem 1.2). Here by definition a Möbius transformation is a fractional linear transformation on $\mathbb{S}^2 \cong \hat{\mathbb{C}}$ or the complex-conjugate of such a map. So we allow a Möbius transformation to be orientation-reversing.

1.3.3 Three-Circle Theorem

Let $S \subseteq \mathbb{S}^2$ be a carpet. A homeomorphism embedding $f : S \to \mathbb{S}^2$ is called *orientation-preserving* if some homeomorphic extension $F : \mathbb{S}^2 \to \mathbb{S}^2$ of f is orientation-preserving on $\mathbb{S}^2$ (such an extension exists and the definition is independent of the choice of extension, see the proof of Lemma 5.3 in [Bo11]).

Corollary 1.1. *Let S be a carpet in $\mathbb{S}^2$ of measure zero whose peripheral circles are uniformly relatively separated uniform quasicircles and $C_i, i = 1,2,3$ be three distinct peripheral circles of S. Let f and g be two orientation-preserving quasisymmetric self-maps of S. Then we have the following rigidity results:*

1. *Assume that $f(C_i) = g(C_i)$ for $i = 1,2,3$. Then $f = g$.*

2. *Assume that $f(C_i) = g(C_i)$ for $i = 1,2$ and $f(p) = g(p)$ for a given point $p \in S$. Then $f = g$.*

3. *Assume that G is the group of all orientation-preserving quasisymmetric self-maps of S that fix C_1, C_2. Then G is a finite cyclic group.*

4. *Assume that G is the group of all orientation-preserving quasisymmetric self-maps of S that fix C_1 and fix a given point $q \in C_1$, then G is an infinite cyclic group.*

Proof. The proof we given here is contained in [BM11]. Since its conclusion is important for the rest of our chapter, we include it here for completeness.

By Theorem 1.1, there exists a quasisymmetric map h of S onto a round carpet $\widetilde{S}$. Using Proposition 1.2, we can extend h to a quasiconformal map on $\mathbb{S}^2$. Since quasiconformal maps preserve the class of sets of measure zero, $\widetilde{S}$ has measure zero as well. We denote by G_0 and $\widetilde{G_0}$ the group of all orientation-preserving quasisymmetric self-maps of S and $\widetilde{S}$, respectively. By the quasisymmetric rigidity of round carpets (Theorem 1.2), $\widetilde{G_0}$ consists of the restriction of orientation-preserving Möbius transformations that fix $\widetilde{S}$.

Now we look at the homomorphism h_* induced by h:

$$\begin{aligned} h_* : G_0 &\to \widetilde{G_0}, \\ \psi &\mapsto h \circ \psi \circ h^{-1}. \end{aligned}$$

We can check that h_* is well-defined and is an isomorphic. Since $h_*(f)$ and $h_*(g)$ are orientation-preserving Möbius transformation and $h_*(f) \circ (h_*(g))^{-1}$ fixes distinct spherical round circles $h(C_i)$, $i = 1,2,3$, we know that $h_*(f) \circ (h_*(g))^{-1} = \mathrm{id}$ and (1) follows.

We can prove (2) from the fact that any orientation-preserving Möbius transformation fixing distinct spherical round circles and a given non-common center point $p \in \mathbb{S}^2$ is the identity.

To prove (3), it suffices to show that $\widetilde{G} = h_*(G)$ is a finite cyclic. By post-composing fractional linear transformation to h, we can assume that $h(C_1)$ and $h(C_2)$ are distinct spherical round circles with the same center. Note that $\widetilde{G}$ consists of orientation-preserving Möbius transformation, fixing $h(C_1)$, $h(C_2)$ and $\widetilde{S}$. Moreover, $\widetilde{G}$ must be a discrete group as it maps peripheral round circles of S to peripheral round circles. Hence $\widetilde{G}$ is a finite cyclic group, then (3) follows.

For (4), similarly, by post-composing fractional linear transformation to h, we can assume that $h(C_1) = \mathbb{R} \bigcup \{\infty\}$, $h(q) = 0$ and $\widetilde{S}$ is contained in the upper half-plane. Then the maps in $\widetilde{G}$ are of the form: $z \mapsto \lambda z$ with $\lambda > 0$, fixing $\widetilde{S}$. By the same reason as (3), $\widetilde{G}$ is a discrete group. So there exists a $\lambda_0 \geq 1$ such that $\widetilde{G} = \{z \mapsto \lambda_0^n z | n \in \mathbb{N}\}$. It follows that $\widetilde{G}$, and hence also G, is the trivial group consisting only of the identity or an infinite cyclic group. Therefore, (4) follows. □

1.3.4 Square carpets

A $\mathbb{C}^*$-*Cylinder* A is a set of the form

$$A = \{z \in \mathbb{C}; r \leq |z| \leq R\}$$

with $0 < r < R < +\infty$. The metric on A induced by the length element $|dz|/|z|$ which is the flat metric. Equipped with this metric, A is isometric to a finite cylinder of circumference 2π and length $\log(R/r)$. The boundary components $\{z \in \mathbb{C}; |z| = r\}$ and $\{z \in \mathbb{C}; |z| = R\}$ are called the inner and outer boundary components of A, respectively.

A $\mathbb{C}^*$-*square* Q is a Jordan region of the form

$$Q = \{\rho e^{i\theta} : a < \rho < b, \alpha < \theta < \beta\}$$

with $0 < \log(b/a) = \beta - \alpha < 2\pi$. We call the quantity

$$l_{\mathbb{C}^*}(Q) = \log(b/a) = \beta - \alpha$$

its side length. Clearly, two opposite sides of Q parallel to the boundaries of A, while the other two perpendicular to the boundaries of A.

A square carpet T in a $\mathbb{C}^*$-cylinder A is a carpet that can be written as

$$T = A \setminus \bigcup_i Q_i,$$

where the sets Q_i, $i \in I$, are $\mathbb{C}^*$-squares whose closures are pairwise disjoint and contained in the interior of A.

Theorem 1.3. *Let S be a carpet of measure zero in $\mathbb{S}^2$ whose peripheral circles are uniformly relatively separated uniform quasicircles, and C_1 and C_2 two distinct peripheral circles of S. Then there exists a quasisymmetric map f from S onto a square carpet T in a $\mathbb{C}^*$-cylinder A such that $f(C_1)$ is the inner boundary component of A and $f(C_2)$ is the outer one.*

This is ([Bo11], Theorem 1.6).

Let S be a carpet in $\mathbb{S}^2$ and C_1, C_2 be two distinct peripheral circles of S. Soppose that the Jordan regions D_1 and D_2 are the complementary components of S bounded by C_1 and C_2 respectively. We let $\Gamma(C_1, C_2)$ be the family of all open paths in $S^2 \setminus \overline{D}_1 \cup \overline{D}_2$ that connects $\overline{D}_1$ and $\overline{D}_2$.

Proposition 1.3. *Let S be a carpet of measure zero in $\mathbb{S}^2$ whose peropheral circles are uniformly relatively separated uniform quasicircles, and C_1 and C_2 two distinct peripheral circles of S. Then*

(1) $\mathrm{mod}_S(\Gamma(C_1, C_2))$ *has finite and positive total mass.*

(2) *Let f be a quasisymmetric map of S onto a square carpet T in a $\mathbb{C}^*$-cylinder $A = \{z \in \mathbb{C}; r \leq |z| \leq R\}$ such that C_1 corresponds to the inner and C_2 to the outer boundary components of A. Then the extremal mass distribution is given as follows:*

$$\rho(C_1) = \rho(C_2) = 0, \ \ \rho(C) = \frac{l_{\mathbb{C}^*}(f(C))}{\log(R/r)}$$

with the peripheral circles $C \neq C_1, C_2$ of S.

This is ([Bo11], Corollary 12.2).

Let S be a carpet in a closed Jordan region $D \subset \hat{\mathbb{C}}$. S is called *square carpet* if ∂D is a peripheral circle of S and all other peripheral circles are squares with sides parallel to the coordinate axes.

Theorem 1.4. *Let S and $\widetilde{S}$ be square carpets of measure zero in rectangles $K = [0, a] \times [0, 1] \subseteq \mathbb{R}^2$ and $\widetilde{K} = [0, \tilde{a}] \times [0, 1] \subseteq \mathbb{R}^2$, respectively, where a, $\tilde{a} > 0$. If f is an orientation-preserving quasisymmetric homeomorphism form S onto $\widetilde{S}$ that takes the corners of K to the corners of $\widetilde{K}$ with $f(0) = 0$. Then $a = \tilde{a}$, $S = \widetilde{S}$, and f is the identity on S.*

This is ([BM11], Theorem 1.4). Here the expression square carpet S in a rectangle K means that a carpet $S \subset K$ so that ∂K is a peripheral circle of S and all other peripheral circles are squares with four sides parallel to the sides of K, respectively.

1.4 Distinguished peripheral circles

Let $n \geq 5, 1 \leq p < \frac{n}{2} - 1$ be integers. Let $F_{n,p}$ be the Sierpiński carpet as we defined in the introduction. We endow $F_{n,p}$ with the Euclidean metric in $\mathbb{R}^2$. Since $F_{n,p}$ is a subset of $[0,1] \times [0,1]$, the Euclidean metric (measure) is comparable with the spherical metric (measure).

If Q is a peripheral circle of $F_{n,p}$, we denote by ℓ_Q the Euclidean side length of Q. Denote by Q_0 the unit square $[0,1] \times [0,1]$.

Lemma 1.4. *The carpet $F_{n,p}$ is of measure zero. The peripheral circles of $F_{n,p}$ are uniform quasicircles and uniformly relatively separated.*

Proof. It follows from the construction that $F_{n,p}$ is a carpet of Hausdorff dimension

$$\log(n^2 - 4)/\log n < 2.$$

So the measure of $F_{n,p}$ is equal to zero.

Since each peripheral circle of $F_{n,p}$ can be mapped to the boundary of Q_0 by a Euclidean similarity, the peripheral circles of $F_{n,p}$ are uniform quasicircles.

At last, the peripheral circles of $F_{n,p}$ are uniformly relatively separated in the Euclidean metric. Indeed, consider any two distinct peripheral circles C_1, C_2 of $F_{n,p}$. The Euclidean distance between C_1 and C_2 satisfies

$$\begin{aligned} \mathrm{dist}(C_1, C_2) &\geq \min\{\ell(C_1), \ell(C_2)\} \\ &= \frac{1}{\sqrt{2}} \min\{\mathrm{diam}(C_1), \mathrm{diam}(C_2)\}. \end{aligned}$$

□

1.4.1 Distinguished pairs of non-adjacent peripheral circles

We denote by O the boundary of the unit square Q_0. In the first step of the inductive construction of $F_{n,p}$, there are four squares Q_1, Q_2, Q_3, Q_4 of side-length $\frac{1}{n}$, i.e., the lower left, lower right, upper right and upper left squares respectively, removed from Q_0. We denote by $M_i, i = 1, \cdots, 4$ the boundary of $Q_i, i = 1, \cdots, 4$, respectively.

In the following discussions, we call O the *outer circle* of $F_{n,p}$ and $M_i, i = 1, \cdots, 4$ the *inner circles* of $F_{n,p}$. We say that two disjoint peripheral circles C, C' are *adjacent* if there exists a copy F of $F_{n,p}$ (here $F \subset F_{n,p}$ can be considered as a carpet scaled from $F_{n,p}$ by some factor $1/n^k$) such that C, C' are inner circles of F. For example, two distinct inner circles M_i and M_j are adjacent. Two disjoint peripheral circles C, C' which are not adjacent are called *non-adjacent*.

Lemma 1.5. *Let $\{C, C'\}$ be any pair of non-adjacent distinct peripheral circles of $F_{n,p}$. Then*

$$\operatorname{mod}_{F_{n,p}}(\Gamma(C, C')) \leq \operatorname{mod}_{F_{n,p}}(\Gamma(O, M)).$$

Moreover, the equality holds if and only if $\{C, C'\} = \{O, M\}$ for some inner circle $M = M_i$.

Proof. Assume that $\{C, C'\} \neq \{O, M\}$ for any inner circle M. By Lemma 1.4 and Proposition (1.3), $\operatorname{mod}_{F_{p,q}}(\Gamma(C, C'))$ is a finite and positive number. Without loss of generality we may assume that $\ell(C) = 1/n^m \leq \ell(C')$. Note that there exists a copy $F \subset F_{n,p}$, rescaled from $F_{n,p}$ by a factor $1/n^{m-1}$, so that C corresponds to some inner circle, say, M_1 of $F_{n,p}$.

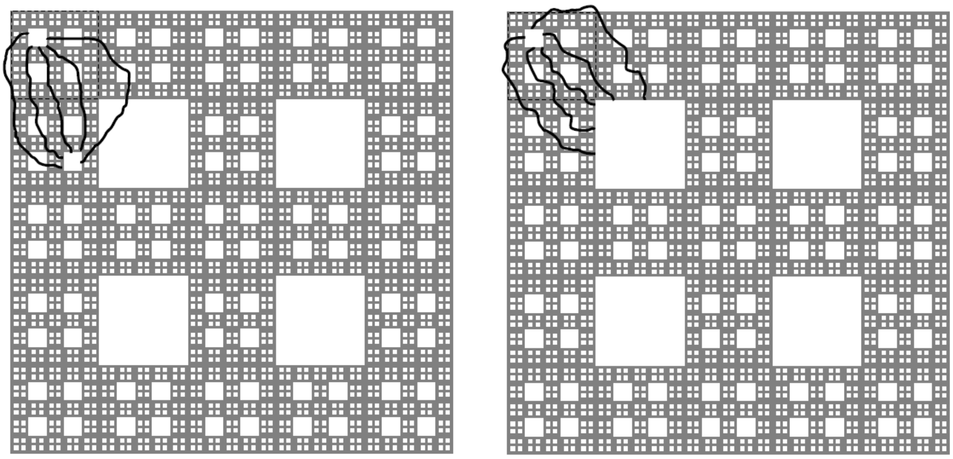

Figure 1.2: Every path in $\Gamma(C, C')$ must intersect with C_0.

Denote the outer circle of F by C_0. Since C and C' are disjoint and $\ell(C) \leq \ell(C')$, C' is disjoint with the interior region of C_0. Hence every path in $\Gamma(C, C')$ must intersect with C_0 and then contains a sub-path in $\Gamma(C, C_0)$. See Figure 1.3. Therefore,

$$\operatorname{mod}_{F_{n,p}}(\Gamma(C, C')) \leq \operatorname{mod}_{F_{n,p}}(\Gamma(C, C_0)). \tag{1.5}$$

On the other hand, since every path in $\Gamma(C, C_0)$ meets exactly the same peripheral circles of F and $F_{n,p}$, we have

$$\operatorname{mod}_{F_{n,p}}(\Gamma(C, C_0)) = \operatorname{mod}_F(\Gamma(C, C_0)).$$

Moreover, by the similarity of $F_{n,p}$ and F,

$$\operatorname{mod}_F(\Gamma(C, C_0)) = \operatorname{mod}_{F_{n,p}}(\Gamma(M, O)).$$

It follows that

$$\operatorname{mod}_{F_{n,p}}(\Gamma(C, C')) \leq \operatorname{mod}_{F_{n,p}}(\Gamma(M_1, O)).$$

We next show that the equality case in (1.7) cannot happen. We argue by contradiction. Assume that

$$\mathrm{mod}_{F_{n,p}}(\Gamma(C, C')) = \mathrm{mod}_{F_{n,p}}(\Gamma(C, C_0)).$$

Note that all carpet modulus considered above are finite by Proposition 1.3 and so there exist unique extremal mass distributions, say ρ and ρ', for $\mathrm{mod}_{F_{n,p}}(\Gamma(C, C'))$ and $\mathrm{mod}_{F_{n,p}}(\Gamma(O, M_1))$, respectively, by Proposition 4.9.

Let $\mathcal{C}$ be the set of all peripheral circles of $F_{n,p}$. According to the description in Proposition 1.3, ρ and ρ' are supported on $\mathcal{C} \setminus \{C, C'\}$ and $\mathcal{C} \setminus \{O, M_1\}$, respectively.

By transplanting ρ' to the carpet F using a suitable Euclidean similarity between F and $F_{n,p}$, we get an admissible mass distribution $\widetilde{\rho}$ for F supported only on the set of peripheral circles of F except C and C_0. Note that the total mass of $\widetilde{\rho}$ is the same as $\mathrm{mass}(\rho')$.

We extend $C \to \widetilde{\rho}(C)$ by zero if C belonging to $\mathcal{C}$ does not intersect the interior region of C_0. Then $\widetilde{\rho}$ is an admissible mass distribution for $\mathrm{mod}_{F_{n,p}}(\Gamma(C, C_0))$, thus for $\mathrm{mod}_{F_{n,p}}(\Gamma(C, C'))$ as well. However, $\widetilde{\rho} \neq \rho$ and $\mathrm{mass}(\widetilde{\rho}) = \mathrm{mod}_{F_{n,p}}(\Gamma(C, C'))$, we arrive at a contradiction by Proposition 4.9.

In summary, we get the following crucial inequality:

$$\mathrm{mod}_{F_{n,p}}(\Gamma(C, C')) < \mathrm{mod}_{F_{n,p}}(\Gamma(O, M_1)) \tag{1.6}$$

where $\{C, C'\} \neq \{O, M_i\}$ $i = 1, 2, 3, 4$ and non-adjacent. So the lemma follows. □

Proof. Assume that $\{C, C'\} \neq \{O, M\}$ for any inner circle M. By Lemma 1.4 and Proposition (1.3), $\mathrm{mod}_{F_{p,q}}(\Gamma(C, C'))$ is a finite and positive number. Without loss of generality we may assume that $\ell(C) = 1/n^m \leq \ell(C')$. Note that there exists a copy $F \subset F_{n,p}$, rescaled from $F_{n,p}$ by a factor $1/n^{m-1}$, so that C corresponds to some inner circle, say, M_1 of $F_{n,p}$.

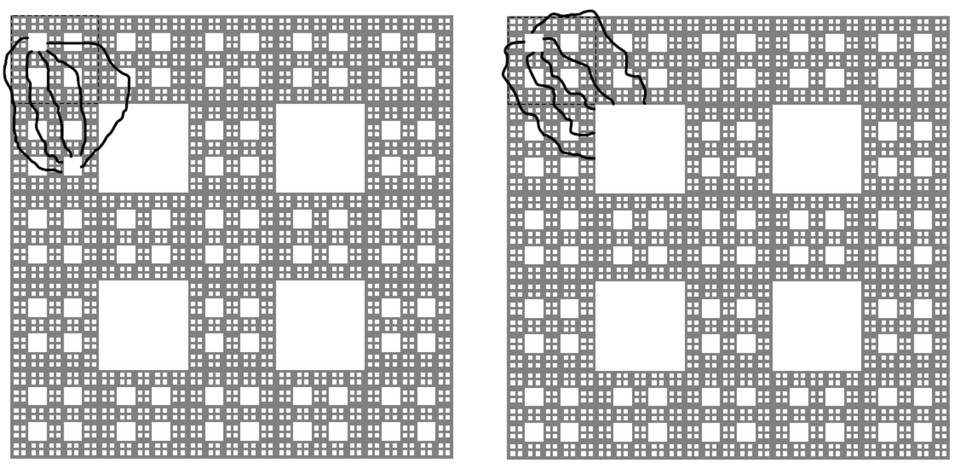

Figure 1.3: Every path in $\Gamma(C, C')$ must intersect with C_0.

Denote the outer circle of F by C_0. Since C and C' are disjoint and $\ell(C) \leq \ell(C')$, C' is disjoint with the interior region of C_0. Hence every path in $\Gamma(C, C')$ must intersect with C_0 and then contains a sub-path in $\Gamma(C, C_0)$. See Figure 1.3. Therefore,

$$\mathrm{mod}_{F_{n,p}}(\Gamma(C, C')) \leq \mathrm{mod}_{F_{n,p}}(\Gamma(C, C_0)). \tag{1.7}$$

On the other hand, since every path in $\Gamma(C, C_0)$ meets exactly the same peripheral circles of F and $F_{n,p}$, we have

$$\mathrm{mod}_{F_{n,p}}(\Gamma(C, C_0)) = \mathrm{mod}_F(\Gamma(C, C_0)).$$

Moreover, by the similarity of $F_{n,p}$ and F,

$$\mathrm{mod}_F(\Gamma(C, C_0)) = \mathrm{mod}_{F_{n,p}}(\Gamma(M, O)).$$

It follows that

$$\mathrm{mod}\ _{F_{n,p}}(\Gamma(C, C')) \leq \quad \mathrm{mod}\ _{F_{n,p}}(\Gamma(M_1, O)).$$

We next show that the equality case in (1.7) cannot happen. We argue by contradiction. Assume that

$$\mathrm{mod}_{F_{n,p}}(\Gamma(C, C')) = \mathrm{mod}_{F_{n,p}}(\Gamma(C, C_0)).$$

Note that all carpet modulus considered above are finite by Proposition 1.3 and so there exist unique extremal mass distributions, say ρ and ρ', for $\mathrm{mod}_{F_{n,p}}(\Gamma(C, C'))$ and $\mathrm{mod}_{F_{n,p}}(\Gamma(O, M_1))$, respectively, by Proposition 4.9.

Let $\mathcal{C}$ be the set of all peripheral circles of $F_{n,p}$. According to the description in Proposition 1.3, ρ and ρ' are supported on $\mathcal{C} \setminus \{C, C'\}$ and $\mathcal{C} \setminus \{O, M_1\}$, respectively.

By transplanting ρ' to the carpet F using a suitable Euclidean similarity between F and $F_{n,p}$, we get an admissible mass distribution $\widetilde{\rho}$ for F supported only on the set of peripheral circles of F except C and C_0. Note that the total mass of $\widetilde{\rho}$ is the same as $\mathrm{mass}(\rho')$.

We extend $C \to \widetilde{\rho}(C)$ by zero if C belonging to $\mathcal{C}$ does not intersect the interior region of C_0. Then $\widetilde{\rho}$ is an admissible mass distribution for $\mathrm{mod}_{F_{n,p}}(\Gamma(C, C_0))$, thus for $\mathrm{mod}_{F_{n,p}}(\Gamma(C, C'))$ as well. However, $\widetilde{\rho} \neq \rho$ and $\mathrm{mass}(\widetilde{\rho}) = \mathrm{mod}_{F_{n,p}}(\Gamma(C, C'))$, we arrive at a contradiction by Proposition 4.9.

In summary, we get the following crucial inequality:

$$\mathrm{mod}_{F_{n,p}}(\Gamma(C, C')) < \mathrm{mod}_{F_{n,p}}(\Gamma(O, M_1)) \tag{1.8}$$

where $\{C, C'\} \neq \{O, M_i\}$ $i = 1, 2, 3, 4$ and non-adjacent. So the lemma follows. $\square$

Corollary 1.2. *Let f be a quasisymmetric self-map of $F_{n,p}$. Then*

$$f(\{O, M_1, M_2, M_3, M_4\}) = \{O, M_1, M_2, M_3, M_4\}.$$

Proof. We argue by contradiction. Assume that f maps $\{O, M_1\}$ to some pair of peripheral circles $\{C, C'\} \nsubseteq \{O, M_1, M_2, M_3, M_4\}$ and $f(O) = C$. By Proposition 1.2, f extends to a quasiconformal homeomorphism on $\mathcal{S}^2$. In particular, $\Gamma(C, C'){=}f(\Gamma(O, M_1))$. Then Lemma 1.1 implies

$$\operatorname{mod}_{F_{n,p}}(\Gamma(O, M_1)) = \operatorname{mod}_{F_{n,p}}(\Gamma(C, C')).$$

We distinguish the argument into two cases depending on the type of the squares C and C', i.e., whether they are adjacent or not.

Case (1): C, C' are non-adjacent. This is only possible if $\{C, C'\} \subseteq \{O, M_1, M_2, M_3, M_4\}$ by Lemma 1.5. Then we get a contradiction.

Case (2): C, C' are adjacent. Suppose C, C' are inner circles of some copy $F \subset F_{n,p}$. Consider $f(M_i)$, $i = 2, 3, 4$. They must be inner circles of F as well. Otherwise, for example, suppose that $f(M_2)$ is not an inner circle of F. Since C and $f(M_2)$ are non-adjacent, we can apply Lemma 1.5 to show that

$$\operatorname{mod}_{F_{n,p}}(\Gamma(C, f(M_2))) < \operatorname{mod}_{F_{n,p}}(\Gamma(O, M_1)),$$

which is contradicted with the fact that

$$\operatorname{mod}_{F_{n,p}}(\Gamma(C, f(M_2))) = \operatorname{mod}_{F_{n,p}}(\Gamma(O, M_2)) = \operatorname{mod}_{F_{n,p}}(\Gamma(O, M_1)).$$

As a result, $\{f(O), f(M_1), f(M_2), f(M_3), f(M_4))\}$ are pairwise adjacent and all of them are inner circles of F. However, F contains exactly four inner circles. So Case (2) can not happen.

By the same argument to pairs O and M_i, $i = 2, 3, 4$, the corollary follows. □

1.4.2 Quasisymmetric group $\mathrm{QS}(F_{n,p})$ is finite

Let H denote the Euclidean isometry group which consists of eight elements: four of them rotate around the center by $\pi/2, \pi, 3\pi/2$, and 2π, respectively; the others are orientation-reserving and reflecting by lines $x = 0$, $x = y$, $y = 0$ and $x{+}y = 0$, respectively. It is obvious that H is contained in $\mathrm{QS}(F_{n,p})$.

Corollary 1.3. *Let $5 \leq n, 1 \leq p < \frac{n}{2} - 1$ be integers. Then the group $\mathrm{QS}(F_{n,p})$ of quasisymmetric self-maps of $F_{n,p}$ is finite.*

Proof. According to Corollary 1.2, $\{O, M_1, M_2, M_3, M_4\}$ are preserved under every quasisymmetric self-map of $F_{n,p}$. The group G of all orientation-preserving quasisymmetric self-maps of $F_{n,p}$ is finite by the proof of Case (1) in Corollary (1.1). Since G is a subgroup of $\mathrm{QS}(F_{p,q})$ with index two, $\mathrm{QS}(F_{p,q})$ is finite. $\square$

1.4.3 Proof of Theorem 1.1

Recall that the standard carpet $S_m, m \geq 3$ odd, is obtained by subdivide $[0,1] \times [0,1]$ into m^2 subsquares of equal size, removing the interior of the middle square, and repeating these operations to every subsquares, inductively.

Proof of Theorem 1.1. Let $\mathcal{M}, \mathcal{O}$ be the inner circle and outer circle of S_m respectively. Lemma 5.1 of [BM11] states that $\mathrm{mod}_{S_m}(\Gamma(\mathcal{O}, \mathcal{M}))$ is strictly larger than the carpet modulus of any other path family $\Gamma(C, C')$ with respect to S_m, where C and C' are peripheral circles of S_m. While for carpet $F_{n,p}$, according to the symmetry, at least two pairs of peripheral circles the maximum of $\{\mathrm{mod}_{F_{n,p}}\Gamma(C_1, C_2) : C_1, C_2 \in \mathcal{C}\}$. Since any quasisymmetric maps from $F_{n,p}$ to S_m must preserve such a maximum property, there is no such quasisymmetric map. $\square$

1.5 Weak tangent spaces

The results in this section generalize the discussion in ([BM11], Section 7).

At first, we explain the definition of weak tangent of a carpet. Then we show that a quasisymmetric map between two carpets $F_{n,p}$ induces a quasisymmetric map between weak tangents.

1.5.1 Weak tangents

In general, the *weak tangents* of a metric space M at a point $p \in M$ can be defined as the Gromov-Hausdorff limits of the pointed metric spaces

$$\lim_{\lambda \to \infty} (\lambda M, p)$$

where λM is the same set of points with M equipped with the original metric multiplied by λ. If the limit is unique up to multiplied by positive constants, then the weak tangents is usually called the *tangent cone* of M at p.

In the following, as in [BM11], we will use a suitable definition of weak tangents for subsets of $\mathbb{S}^2$ equipped with the spherical metric.

Suppose that $a, b \in \mathbb{C}, a \neq 0$ and $M \subseteq \widehat{\mathbb{C}}$. We denote by

$$aM + b := \{az + b : z \in M\}.$$

Let A be a subset of $\widehat{\mathbb{C}}$ with a distinguished point $z_0 \in A$, $z_0 \neq \infty$. We say that a closed set $W_A(z_0) \subseteq \widehat{\mathbb{C}}$ is a *weak tangent* of A if there exists a sequence (λ_n) with $\lambda_n \to \infty$ such that the sets $A_n := \lambda_n(A - z_0)$ converge to $W_A(z_0)$ as $n \to \infty$ in the sense of Hausdorff convergence on $\widehat{\mathbb{C}}$ equipped with the spherical metric. In this case, we use the notation

$$W_A(z_0) = \lim_{n\to\infty}(A, z_0, \lambda_n).$$

Since for every sequence (λ_n) with $\lambda_n \to \infty$, there is a subsequence (λ_{n_k}) such that the sequence of the sets $A_{n_k} = \lambda_{n_k}(A - z_0)$ converges as $k \to \infty$, A has weak tangents at each point $z_0 \in A \setminus \{\infty\}$. In general, weak tangents at a point are not unique. In particular, $\lambda W_A(z_0)$ is also a weak tangent.

Now we apply the notion to our carpets $F_{n,p}$. In fact, the following arguments work for a general class of carpets, such as the standard Sierpiński carpet S_m and carpets which satisfy some self-similarity property.

A weak tangent of a point $z_0 \in F_{n,p}$ is a closed set $W_{F_{n,p}}(z_0) \subseteq \widehat{\mathbb{C}}$ such that

$$W_{F_{n,p}}(z_0) = \lim_{j\to\infty}(F_{n,p}, z_0, n^{k_j}),$$

where $k_j \geq 1$ and $k_j \to \infty$ as $j \to \infty$.

At the point 0 the carpet $F_{n,p}$ has the unique weak tangent

$$W_{F_{n,p}}(0) = \lim_{j\to\infty}(F_{n,p}, 0, n^j) = \{\infty\} \cup \bigcup_{j\in\mathbb{N}_0} n^j F_{n,p}. \tag{1.9}$$

This follows from the inclusions $n^j F_{n,p} \subseteq n^{j+1} F_{n,p}$.

Similarly, at each corner of O there exists a unique weak tangent of $F_{n,p}$ obtained by a suitable rotation of the set $W_{F_{n,p}}(0)$ around 0.

Let $c = p/n + \mathbf{i}p/n$ be the lower-left corner of M_1. Then at c the carpet $F_{n,p}$ has unique weak tangent

$$W_{F_{n,p}}(c) = \lim_{j\to\infty}(F_{n,p}, c, n^j) = \{\infty\} \cup \bigcup_{j\in\mathbb{N}_0} n^j(\mathbf{i}F_{n,p} \cup (-\mathbf{i})F_{n,p} \cup (-1)F_{n,p}).$$

Note that $W_{F_{n,p}}(c)$ can be obtained by pasting together three copies of $W_{F_{n,p}}$. If z_0 is a corner of a peripheral circle $C \neq O$ of $F_{n,p}$, then $F_{n,p}$ has a unique weak tangent at z_0 obtained by a suitable rotation of the set $W_{F_{n,p}}(c)$ around 0.

Lemma 1.6. *Let z_0 be a corner of a peripheral circle of $F_{n,p}$. Then the weak tangent $W_{Fn,p}(z_0)$ is a carpet of measure zero. If $W_{F_{n,p}}(z_0)$ is equipped with the spherical metric, then the family of peripheral circles of $W_{F_{n,p}}(z_0)$ are uniform quasicircles and uniformly relatively separated.*

Proof. We can assume that z_0 equals 0. The proof works for other cases.

First note that (1.9) implies that $W_{F_{n,p}}(0)$ is a carpet of measure zero, since $W_{F_{n,p}}(0)$ is the union of countably many sets of measure zero.

Let $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$. Then $\partial\Omega$ ia a peripheral circle of $W_{F_{n,p}}(0)$. It is easy to construct a bi-Lipschitz map between $\partial\Omega$ and the unit circle (both equipped with the spherical metric). Hence $\partial\Omega$ is a quasicircle. Note that all other peripheral circles of $W_{F_{n,p}}(0)$ are squares. As a result, the peripheral circles of $W_{F_{n,p}}(0)$ are uniformly quasicircles.

To show that the peripheral circles are uniformly relatively separated, we only need to check the following inequality:

$$\mathrm{dist}(C_1, C_2) \geq \min\{\ell(C_1), \ell(C_2)\} \tag{1.10}$$

for any peripheral circles $C_1, C_2 \neq \partial\Omega$. Here $\mathrm{dist}(\cdot,\cdot)$ and $\ell(\cdot)$ denote the Euclidean distance and Euclidean side length.

The inequality implies that the peripheral circles are uniformly relatively separated with respect to the Euclidean metric. To see that they are uniformly relatively separated property with respect to the spherical metric, we can apply an argument of ([BM11], Lemma 7.1).

□

1.5.2 Quasisymmetric maps between weak tangents

We are interested in quasisymmetric maps $g : W \to W'$ between weak tangents W of $F_{n,p}$ and weak tangents W' of $F_{n,p}$. Note that $0, \infty \in W, W'$. We call g *normalized* if

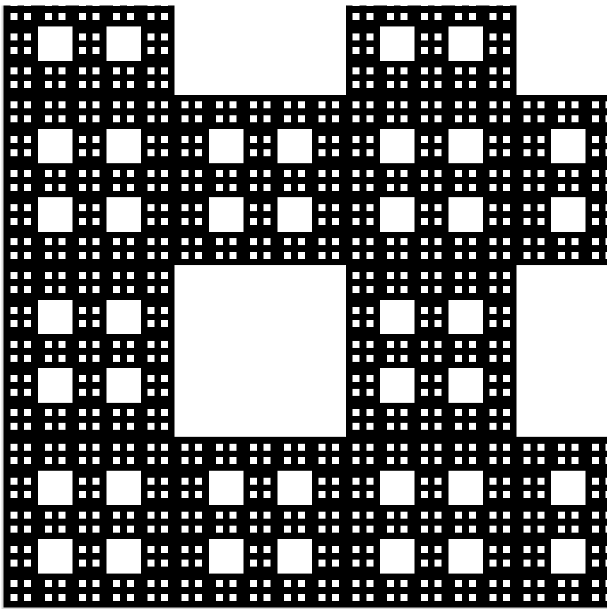

Figure 1.4: The weak tangent $W_{F_{n,p}}(0)$.

$g(0) = 0$ and $g(\infty) = \infty$.

Lemma 1.7. *Let z_0 be a corner of a peripheral circle of F_{n_1,p_1} and let w_0 be a corner of a peripheral circle of F_{n_2,p_2}. Suppose that $f : F_{n_1,p_1} \to F_{n_2,p_2}$ be a quasisymmetric map with $f(z_0) = w_0$. Then f induces a normalized quasisymmetric map g between the weak tangent $W_{F_{n_1,p_1}}(z_0)$ and $W_{F_{n_2,p_2}}(w_0)$.*

Proof. By Proposition 1.2 we can extend f to a quasiconformal self-homeomorphism F of $\widehat{\mathbb{C}}$. There exists a relative neighborhood N_1 of z_0 in F_{n_1,p_1} and a relative neighborhood N_2 of w_0 in F_{n_2,p_2} with $F(N_1) = N_2$ such that

$$W_{F_{n_1,p_1}}(0) \setminus \{\infty\} = \bigcup_{j \in \mathbb{N}_0} n_1^j (N_1 - z_0)$$

and

$$W_{F_{n_2,p_2}}(0) \setminus \{\infty\} = \bigcup_{j \in \mathbb{N}_0} n_2^j (N_2 - w_0)$$

Pick a point $u_0 \in N - z_0$, $u_0 \neq 0$. Then for each $j \in \mathbb{N}_0$ we have $F(z_0 + n_1^{-j} u_0) (\neq w_0, \infty)$ in F_{n_2,p_2}.

We consider the following quasiconformal self-map F_j of $\widehat{\mathbb{C}}$ with $F_j(n_1^j(N_1 - z_0)) = n_2^{k(j)}(N_2 - w_0)$:

$$F_j : u \mapsto n_2^{k(j)}(F(z_0 + n_1^{-j} u) - w_0)$$

for $u \in \widehat{\mathbb{C}}$, where $k(j)$ is the unique integer such that $1 \leq |F_j(u_0)| < n_2$.

Note that $k(j) \to \infty$ as $j \to \infty$ and $F(\infty) \neq w_0$. This implies that $F_j(\infty) \to \infty$ as $j \to \infty$. We also have $F_j(0) = 0$. So the images of $0, \infty$ and u_0 under F_j have mutual spherical distance uniformly bounded from below independent of j. Moreover, F_j is obtained from F by post-composing and pre-composing Möbius transformations. Hence the sequence (F_j) is uniformly quasiconformal, and it follows that we can find a subsequence of (F_j) that converges uniformly on $\widehat{\mathbb{C}}$ to a quasiconformal map F_∞. Without loss of generality, we assume that (F_j) converges uniformly to F_∞.

Note that $F_\infty(0) = 0$ and $F_\infty(\infty) = \infty$. To prove the statement of the lemma, it suffices to show that $F_\infty(W_{F_{n_1,p_1}(z_0)}) = W_{F_{n_2,p_2}}(w_0)$, because then $g := F_\infty | W_{F_{n_1,p_1}}(z_0)$ is an induced normalized quasisymmetric map between $W_{F_{n_1,p_1}}(z_0)$ and $W_{F_{n_2,p_2}}(w_0)$, as desired.

Let u be an arbitrary point in $W_{F_{n_1,p_1}}(z_0)$. There exists a sequence (u_j) with $u_j \in n_1^j(N_1 - z_0)$ converging to u. We have $F_j(u_j) \in n_2^j(N_2 - w_0)$ and a subsequence of $(F_j(u_j))$ converging to some point v in $W_{F_{n_2,p_2}}(w_0)$. By the definition of F_∞, we have $F_\infty(u) = v$. Hence $F_\infty(W_{F_{n_1,p_1}}(z_0)) \subseteq W_{F_{n_2,p_2}}(w_0)$.

For every point v in $W_{F_{n_2,p_2}}(w_0)$, there exists a sequence (u_j) with $u_j \in n_1^j(N_1 - z_0)$ such that $(F_j(u_j))$ converges to v. Then we can choose a subsequence of (u_j) converging to some point u in $W_{F_{n_1,p_1}}(z_0)$ and so $F_\infty(u) = v$.

It follows that $F_\infty(W_{F_{n_1,p_1}}(z_0)) = W_{F_{n_2,p_2}}(w_0)$ and we are done. $\square$

By Corollary 1.2, a quasisymmetric self-map f of $F_{n,p}$ maps $\{O, M_1, M_2, M_3, M_4\}$ to $\{O, M_1, M_2, M_3, M_4\}$. In the remaining part of this section, we will show that there is no quasisymmetric self-map f of $F_{n,p}$ with $f(0) = c$, where c is a corner of an inner circle. By Lemma 1.7, if such an f exists, then it would induce a normalized quasisymmetric map from $W_{F_{n,p}}(0)$ to $W_{F_{n,p}}(c)$. However, the following proposition shows that:

Proposition 1.4. *There is no normalized quasisymmetric map from $W_{F_{n,p}}(0)$ to $W_{F_{n,p}}(c)$.*

To prove the proposition, we need two lemmas.

Let G and $\widetilde{G}$ be the group of normalized orientation-preserving quasisymmetric self-maps of $W_{F_{n,p}}(0)$ and $W_{F_{n,p}}(c)$, respectively. By Corollary 1.1, G and $\widetilde{G}$ are infinite cyclic groups. Note that the map $\mu(z) := nz$ is contained in $G \cap \widetilde{G}$. We assume that $G = <\phi>$ and $\mu = \phi^s$ for some $s \in \mathbb{Z}_+$. Since the peripheral circles of $W_{F_{n,p}}(0)$ are uniformly quasicircles and uniformly relatively separated, there exists a quasiconformal extension $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of ϕ. Let H be the group generated by the reflection in the real and in the imaginary axes. We may assume that Φ is equivalent under the action of H (see Page 42, [BM11] for the discussion).

Let $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) > 0\}$. Then $C_0 := \partial\Omega$ is a peripheral circle of $W_{F_{n,p}}(0)$. Since $\Phi(C_0) = C_0$ and Φ is orientation-preserving, $\Phi(\Omega) = \Omega$.

Let Γ be the family of all open paths in Ω that connects the positive real and positive imaginary axes. Since the paths in Ω are open, they don't intersect with C_0. For any peripheral circle C of $W_{F_{n,p}}(0)$ that meets some path in Γ, note that $\phi^k(C) \neq C$ for all $k \in \mathbb{Z} \setminus \{0\}$ (otherwise, ϕ would be of finite order, contradicted with the fact that ϕ is the generator of the infinite cyclic group G). So we can apply Lemma 1.3 to conclude that

$$\mathrm{mod}_{W_{F_{n,p}}(0)/<\mu>}(\Gamma) = \mathrm{mod}_{W_{F_{n,p}}(0)/<\phi^s>}(\Gamma) = s\mathrm{mod}_{W_{F_{n,p}}(0)/G}(\Gamma).$$

Note that without the action of the group G, the carpet modulus $\mathrm{mod}_{W_{F_{n,p}}(0)}(\Gamma)$ is equal to infinity.

Lemma 1.8. *We have $0 < \mathrm{mod}_{W_{F_{n,p}}(0)/G}(\Gamma) < \infty$.*

Proof. Let us first show that $\mathrm{mod}_{W_{F_{n,p}}(0)/<\mu>}(\Gamma) < \infty$ by constructing an admissible mass distribution of finite mass.

Let $pr : \mathbb{C} \setminus \{0\} \to \mathbb{S}^1$ be the projection $z \mapsto \frac{z}{|z|}$. If $C \neq C_0$ is a peripheral circle of $W_{F_{n,p}}(0)$, we let $\theta(C)$ be the arc length of $pr(C)$. We set

$$\rho(C) := \begin{cases} 0, \text{if } C = C_0; \\ \frac{2}{\pi}\theta(C), \text{if } C \neq C_0. \end{cases}$$

Note that ρ is $< \mu >$-invariant.

Let Γ_0 be the family of paths $\gamma \in \Gamma$ that are not locally rectifiable or for which $\gamma \cap W_{F_{n,p}}(0)$ has positive length. Since $W_{F_{n,p}}(0)$ is a set of measure zero, we have $\text{mod}(\Gamma_0) = 0$, i.e., Γ_0 is an exceptional subfamily of Γ.

For any $\gamma \in \Gamma \setminus \Gamma_0$, note that

$$\sum_{\gamma \cap \mathbb{C} \neq \emptyset} \rho(C) = \frac{2}{\pi} \sum_{\gamma \cap \mathbb{C} \neq \emptyset} \theta(C) \geq 1.$$

As a result, ρ is admissible.

Let $Q_0 = [0,1] \times [0,1]$. Note that every $< \mu >$-orbit of a peripheral circle $C \neq C_0$ has a unique element contained in the set $F = \overline{\mu(Q_0) \setminus Q_0}$. There is a constant $K > 0$ such that

$$\theta(C) \leq K\ell(C)$$

for all peripheral circles $C \subset F$. It follows that

$$\frac{4}{\pi^2} \sum_{C \subset F} \theta(C)^2 \lesssim \sum_{C \subset F} \ell(C)^2 = Area(F) = n^2 - 1.$$

Hence ρ is a finite admissible mass distribution for $\text{mod}_{W_{F_{n,p}}(0)/<\mu>}(\Gamma)$.

To show that $\text{mod}_{W_{F_{n,p}}(0)/<\mu>}(\Gamma) > 0$, we only need to show that the carpet satisfies the assumptions in Proposition 1.1. Then the extremal mass distribution for $\text{mod}_{W_{F_{n,p}}(0)/<\mu>}(\Gamma)$ exists and this is only possible if Γ itself is an exceptional family, that is, $\text{mod}(\Gamma) = 0$.

In fact, for $k \in \mathbb{N}$ we let $\mathcal{C}_k$ be the set of all peripheral circles C of $W_{F_{n,p}}(0)$ with $C \subset F_k = \overline{\mu^k(Q_0) \setminus \mu^{-k}(Q_0)}$. Then

1. Every $< \mu >$-orbit of a peripheral circle $C \neq C_0$ has exactly $2k$ elements in $\mathcal{C}_k$.
2. Let Γ_k be the family of paths in Γ that only meet peripheral circles in $\mathcal{C}_k$. Then $\Gamma = \bigcup_k \Gamma_k$.

As a result, the assumptions in Proposition 1.1 are satisfied. □

Let $\widetilde{\Omega} = \mathbb{C} \setminus \overline{\Omega}$. The closure of $\widetilde{\Omega}$ contains $W_{F_{n,p}}(c)$ and $C_0 = \partial\Omega = \partial\widetilde{\Omega}$ is a peripheral circle of $W_{F_{n,p}}(c)$. Denote $\psi = \Phi|_{W_{F_{n,p}}(c)}$. Then we have $\psi \in \widetilde{G}$. Let $\widetilde{\Gamma}$ be the family of all open paths in $\widetilde{\Omega}$ that join the positive real and the positive imaginary axes.

Lemma 1.9. *We have* $\text{mod}_{W_{F_{n,p}}(c)/<\psi>}(\widetilde{\Gamma}) \leq \frac{1}{3}\text{mod}_{W_{F_{n,p}}(0)/G}(\Gamma)$.

Proof. Let ρ be an arbitrary admissible invariant mass distribution for $\text{mod}_{W_{F_{n,p}}(0)/G}(\Gamma)$, with exceptional family $\Gamma_0 \subset \Gamma$. We set

$$\widetilde{\rho}(\widetilde{C}) := \begin{cases} 0, \text{if } \widetilde{C} = C_0; \\ \frac{1}{3}\rho(\alpha(\widetilde{C})) \end{cases}$$

if there is an $\alpha \in H$ such that $\alpha(\widetilde{C})$ is a peripheral circle of $W_{F_{n,p}}(0)$ (such an α exits and is unique).

Since Φ is H-equivalent and ρ is G-invariant, $\widetilde{\rho}$ is $<\psi>$-invariant.

Let $\widetilde{\Gamma}_0$ be the family of paths in $\widetilde{\Gamma}$ that have a subpath that can be mapped to a path in Γ_0 by an element of $\alpha \in H$. Then $\mathrm{mod}(\widetilde{\Gamma}_0) = 0$.

Let $\gamma \in \widetilde{\Gamma}$. Note that γ has three disjoint open subpaths: one for each quarter-plane of $\widetilde{\Omega}$ and by suitable elements in H, the three subpaths are mapped to paths in Γ. Denote the images by $\gamma_1, \gamma_2, \gamma_3$. If $\gamma \in \widetilde{\Gamma} \setminus \widetilde{\Gamma}_0$, then $\gamma_i \in \Gamma \setminus \Gamma_0, i = 1, 2, 3$ and

$$\sum_{\gamma \cap \widetilde{C} \neq \emptyset} \widetilde{\rho}(\widetilde{C}) \geq \frac{1}{3} \sum_{i=1}^{3} \sum_{\gamma_i \cap C \neq \emptyset} \rho(C) \geq 1.$$

Hence $\widetilde{\rho}$ is admissible for $\mathrm{mod}_{W_{F_{n,p}}(c)/<\psi>}(\widetilde{\Gamma})$ and

$$\mathrm{mod}_{W_{F_{n,p}}(c)/<\psi>}(\widetilde{\Gamma}) \leq \mathrm{mass}_{W_{F_{n,p}}(c)/<\psi>}(\widetilde{\rho}) \leq \frac{1}{3} \mathrm{mass}_{W_{F_{n,p}}(0)/G}(\rho).$$

Since ρ is an arbitrary mass distribution for $\frac{1}{3}\mathrm{mod}_{W_{F_{n,p}}(0)/G}(\Gamma)$, the statement follows.

$\square$

Proof of Proposition 1.4. Suppose not, there exists a normalized quasisymmetric map $f : W_{F_{n,p}}(0) \to W_{F_{n,p}}(c)$. Precomposing f by the reflection in the diagonal line $\{x = y\}$ if necessary, we may assume that f is orientation-preserving. Then $\widetilde{G} = f \circ G \circ f^{-1}$ and $\widetilde{\phi} = f \circ \phi \circ f^{-1}$ is a generator for $\widetilde{G}$.

Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconfomral extension of f. Then $\widetilde{\Gamma} = F(\Gamma)$. By quasisymmetric invariance of carpets modulus,

$$\mathrm{mod}_{W_{F_{n,p}}(c)/\widetilde{G}}(\widetilde{\Gamma}) = \mathrm{mod}_{W_{F_{n,p}}(0)/G}(\Gamma).$$

Assume that $\psi = \widetilde{\phi}^m$. Then similar to our discussion before Lemma 1.8, we have

$$\mathrm{mod}_{W_{F_{n,p}}(c)/<\psi>}(\widetilde{\Gamma}) = |m| \mathrm{mod}_{W_{F_{n,p}}(c)/\widetilde{G}}(\widetilde{\Gamma}).$$

Hence by Lemma 1.9 we have

$$\begin{aligned} \mathrm{mod}_{W_{F_{n,p}}(0)/G}(\Gamma) &= \mathrm{mod}_{W_{F_{n,p}}(c)/\widetilde{G}}(\widetilde{\Gamma}) \\ &= \frac{1}{|m|} \mathrm{mod}_{W_{F_{n,p}}(c)/<\psi>}(\widetilde{\Gamma}) \\ &\leq \frac{1}{3|m|} \mathrm{mod}_{W_{F_{n,p}}(0)/G}(\Gamma). \end{aligned}$$

This is possible only if $\mathrm{mod}_{W_{F_{n,p}}(0)/G}(\Gamma)$ is equal to 0 or ∞. But this is contradicted with Lemma 1.8.

$\square$

1.6 Quasisymmetric rigidity

Let D be the diagonal $\{(x,y) \in \mathbb{R}^2 : x = y\}$ and V be the vertical line $\{(x,y) \in \mathbb{R}^2 : x = \frac{1}{2}\}$. We denote the reflections in D and V by R_D and R_V, respectively. The Euclidean isometry group of $F_{n,p}$ is generated by R_D and R_V.

Let $\mathrm{QS}(F_{n,p})$ be the group of quasisymmetric self-maps of $F_{n,p}$. By Corollary 1.3, $\mathrm{QS}(F_{n,p})$ is a finite group.

Proposition 1.5. *Let f be a quasisymmetric self-map of $F_{n,p}$. Then $f(\{O\}) = \{O\}$ and $f(\{M_1, M_2, M_3, M_4\}) = \{M_1, M_2, M_3, M_4\}$.*

Proof. From Corollary 1.2, we argue by contradiction and assume that there exists a quasisymmetric self-map f of $F_{n,p}$ and some $i \in \{1,2,3,4\}$ such that $f(\{O\}) = \{M_i\}$. By pre-composing and post-composing suitable elements in the Euclidean isometry group, we can suppose that f is orientation-preserving and $f(\{O\}) = \{M_1\}$.

Let G be the subgroup of $\mathrm{QS}(F_{n,p})$,

$$G = \{g \in \mathrm{QS}(F_{n,p}) \mid g(O) = O, g(M_1) = M_1\}.$$

G has a subgroup G' with index two consisting of orientation-preserving elements. Then

$$G = G' \bigsqcup G' \circ R_D.$$

We denote by

$$\mathcal{O}_G(z) = \{g(z) : g \in G\}$$

the orbit of z under the action of G for arbitrary $z \in F_{n,p}$. Let $c = (p/n, p/n)$ and $c' = ((p+1)/n, (p+1)/n)$ be the lower-left and upper-right corners of M_1, respectively.

Now we consider the map

$$\begin{aligned} \Phi_0 \ : \ G' &\longrightarrow \mathcal{O}_G(0) \\ g &\longmapsto g(0). \end{aligned}$$

Note that Φ_0 is an isomorphism. In fact, for any $g(0) \in \mathcal{O}_G(0)$, if g is orientation-preserving, then $\Phi_0(g) = g(0)$; otherwise, $\Phi_0(g \circ R_D) = g(0)$. So Φ_0 is a surjection. On the other hand, if $\Phi_0(g_1) = \Phi_0(g_2)$for any $g_1, g_2 \in G'$, then Case (2) of Corollary 1.1 gives $g_1 = g_2$. So Φ_0 is a injection.

Similarly, we can also define the isomorphism

$$\begin{aligned} \Phi_c \quad : \quad & G' \longrightarrow \mathcal{O}_G(c) \\ & g \longmapsto g(c). \end{aligned}$$

These isomorphisms Φ_0 and Φ_c imply that

$$\#\mathcal{O}_G(0) = \#G' = \#\mathcal{O}_G(c) \tag{1.11}$$

On the other hand, f induces the following isomorphism

$$\begin{aligned} f_* \quad : \quad & G \longrightarrow G \\ & g \longmapsto f \circ g \circ f^{-1}. \end{aligned}$$

We denote by $m = f(0)$. Then

$$\begin{aligned} \mathcal{O}_G(m) &= \{g(m) : g \in G\} = \{f \circ g \circ f^{-1}(m) : g \in G\} \\ &= \{f \circ g(0) : g \in G\} = f(\mathcal{O}_G(0)). \end{aligned}$$

Hence

$$\#\mathcal{O}_G(m) = \#G' = \#\mathcal{O}_G(0)$$

and so the orbits $\mathcal{O}_G(m)$ and $\mathcal{O}_G(c)$ have the same number of elements.

If $G' \neq \{\mathrm{id}\}$, we claim that G' is a cyclic group of order 3. Indeed, for any $g \neq \mathrm{id}$ in G', $g(M_3) \neq M_3$, otherwise Case (1) of Corollary 1.1 implies $g = \mathrm{id}$. By Corollary 1.2, either $g(M_3) = M_4, g(M_4) = (M_2)$ or $g(M_3) = M_2, g(M_2) = (M_4)$. In both cases, g is of order 3, a.e., $g^3 = \mathrm{id}$. Use Corollary 1.2 again we know that G' is generated by g. So the claim follows.

Hence, we have $\#\mathcal{O}_G(m) = \#G' = 1$ or 3. There must be some $h \in G$ with $h(m) = c$ or c'. Otherwise, $\mathcal{O}_G(m)$ does not contain c, c'. For any point $p \in \mathcal{O}_G(m)$, the point $R_D(p) \in \mathcal{O}_G(m)$ and $R_D(p) \neq p$. Then $\#\mathcal{O}_G(m)$ is even, which is impossible.

By Lemma 1.7, $h \circ f$ induces a normalizaed quasisymmetric map between the weak tangent $W_{F_{n,p}}(0)$ and $W_{F_{n,p}}(c)$ or $W_{F_{n,p}}(c')$. This contradicts Proposition 1.4. So we have proved the proposition. $\square$

1.6.1 Proof of main theorems

Proof of Theorem 1.2. We adopt the notations as in previous. The proof of Proposition 1.5 implies that G' is a cyclic group of order 3 or a trivial group. To prove the theorem, it suffices to show that the former case cannot happen. We argue by contradiction and assume that G' is a cyclic group of order 3.

By Theorem 1.1, there exists a quasisymmetric map f from $F_{n,p}$ onto some round carpet S. After post-composing suitable fraction linear transformation, we can assume that the $f(O)$ is the unit disc $\mathbb{D}$ and $f(M_1)$ lies in $\mathbb{D}$ with center $(0,0)$. Then f induces the isomorphism

$$\begin{aligned} f_* \quad : \quad & QS(F_{n,p}) \longrightarrow QS(S) \\ & g \longmapsto f \circ g \circ f^{-1}. \end{aligned}$$

Combined with Theorem 1.2, $f_*(G')$ is a cyclic group of order 3 consisting of Möbius transformations. Moreover, elements in $f_*(G')$ preserves $\partial\mathbb{D}$ and the circle $O_1 = f(M_1)$. Hence we have

$$f_*(G') = \{\text{id}, z \mapsto e^{2\pi i/3}z, z \mapsto \ e^{4\pi i/3}z\}.$$

Claim: $O_2 = f(M_2), O_3 = f(M_3), O_4 = f(M_4)$ are round circles with the same diameter and equidistributed clockwise in the annuli bounded by $\partial\mathbb{D}$ and O_1.

Proof of the claim: In fact, by the proof of Proposition 1.5, we may assume that $G' =< g >$, where $g(M_3) = M_4, g(M_4) = M_2$ and $g(M_2) = M_3$. Note that

$$\begin{aligned} O_3 \quad &= \quad f(M_3) = f \circ g(M_2)) \\ &= \quad f \circ g \circ f^{-1}(O_2) \end{aligned}$$

where $f \circ g \circ f^{-1}$ is equal to the rotation $z \mapsto e^{2\pi i/3}z$. Similarity, one can show that $O_4 = f \circ g \circ f^{-1}(O_3)$. As a result, O_3 is obtained from O_2 by a rotation of angle $2\pi/3$ and O_4 is obtained from O_2 by a rotation of angle $4\pi/3$. The claim follows.

Let R be the rotation in the isometry group of $F_{n,p}$ with $R(M_1) = M_2, R(M_2) = M_3, R(M_3) = M_4$, and $R(M_4) = M_1$. By Theorem 1.2, the composition

$$h = f \circ R \circ f^{-1} : S \to S$$

is also a Möbius transformation which maps $\partial\mathbb{D} \to \partial\mathbb{D}, O_2 \to O_3, O_3 \mapsto O_4$. Such a Möbius transformation must be $\varphi = z \to e^{2\pi i/3}z$. If not, let φ' be other Möbius transformation satisfy the conditions. Then $\varphi' \circ \varphi^{-1}$ fixes three non-concentric circles $\partial\mathbb{D}, O_2$ and O_3 and so $\varphi' \circ \varphi^{-1} = id$. Hence $\varphi' = \varphi$. But $h(O_1) = O_2$, which is impossible. So the theorem follows. $\square$

Proof of Theorem 1.3. Suppose there exists a quasisymmetric map $f : F_{n,p} \to F_{n',p'}$.

Firstly, we claim that $f(O) = O'$, $f(\{M_1, M_2, M_3, M_4\}) = \{M_1', M_2', M_3', M_4'\}$. Indeed, from Theorem 2, we know that every quasisymmetric self-map of $F_{n,p}$ and $F_{n',p'}$ is isometry and so preserves the peripheral circle O and O'. For any g in $QS(F_{n,p})$, $f \circ g \circ f^{-1}$ is a quasisymmetric self-map of $F_{n',p'}$ and $f \circ g \circ f^{-1}(f(O)) = f(O)$. So $f(O)$ is fixed by any element in $QS(F_{n',p'}$. Hence we have $f(O) = O'$. If for some inner circles M_i, say M_1, of $F_{n,p}$, $f(M_1)$ is not an inner circle of $F_{n',p'}$, then by Proposition 1.2, f extension to a quasiconformal self-map of $\mathbb{S}^2$. We have

$$\operatorname{mod}_{F_{n,p}}(\Gamma(M_1, O)) = \operatorname{mod}_{F_{n',p'}}(\Gamma(f(M_1), O'))$$

and

$$\operatorname{mod}_{F_{n',p'}}(\Gamma(M_1', O')) = \operatorname{mod}_{F_{n,p}}\Gamma(f^{-1}(M_1'), O).$$

While Lemma 1.5 implies

$$\operatorname{mod}_{F_{n',p'}}(\Gamma(f(M_1), O)) < \operatorname{mod}_{F_{n',p'}}(\Gamma(M_1', O))$$

and

$$\operatorname{mod}_{F_{n,p}}\Gamma(f^{-1}(M_1'), O) \leq \operatorname{mod}_{F_{n,p}}(\Gamma(M_1, O)).$$

Hence $\operatorname{mod}_{F_{n,p}}(\Gamma(M_1, O)) < \operatorname{mod}_{F_{n,p}}(\Gamma(M_1, O))$ and we get a contradiction.

Secondly, by pre-composing and post-composing with Euclidean isometries, we can assume that f is orientation-preserving and $f(M_1) = M_1'$. We claim that $f((0,0)) = (0,0)$ and $f((1,1)) = (1,1)$ or interchanges them and $f(M_3) = M_3'$. In fact, the orientation-preserving quasisymmetric map

$$f^{-1} \circ R_D \circ f \circ R_D : F_{n,p} \to F_{n,p}$$

fixes peripheral circles O and M_1. Then, by Theorem 1.2, $f^{-1} \circ R_D \circ f \circ R_D$ is a Euclidean isometry and so it is the identity on $F_{n,p}$. This implies $f \circ R_D = R_D \circ f$. Hence the claim follows.

We now distinguish two cases to analyze.

Case (1) $f((0,0)) = (0,0)$ and $f((1,1)) = (1,1)$.

We denote the reflection in the line $\{(x,y) \in \mathbb{R}^2 : x + y = 1\}$ by R_D'. Then the map $f^{-1} \circ R_D' \circ f \circ R_D'$ is an orientation-preserving quasisymmetric map in $QS(F_{n,p})$, fixes peripheral circles O, M_1, and the point $(0,0)$. Hence this map is the identity on $F_{n,p}$ and so $f \circ R_D' = R_D' \circ f$. It follows that f fixes $(1,0)$ and $(0,1)$ or interchanges them. Since f is orientation-preserving, the latter cannot happen. By Theorem 1.4 the map f must be the identity. Hence $(n,p) = (n',p')$.

Case (2) $f((0,0)) = (1,1)$ and $f((1,1)) = (0,0)$.

The map $g = R_D \circ f \circ R_D' : F_{n,p} \to F_{n',p'}$ is an orientation-preserving quasisymmetry which fixes points $(0,0)$ and $(1,1)$ and peripheral circle O and maps M_1 to M_3'. Similar to Case (1), $g^{-1} \circ R_D' \circ g \circ R_D'$ is an orientation-preserving isometry map fixing $(0,0), (1,1)$ and O and so is the identity. Then g fixes $(1,0)$ and $(0,1)$ or interchanges them. The orientation-preserving of g implies the latter case is impossible. By Theorem 1.4 the map g is the identity, which contradicts with $g(M_1) = M_3'$. So case (2) can not happen.

$\square$

1.6.2 Remark

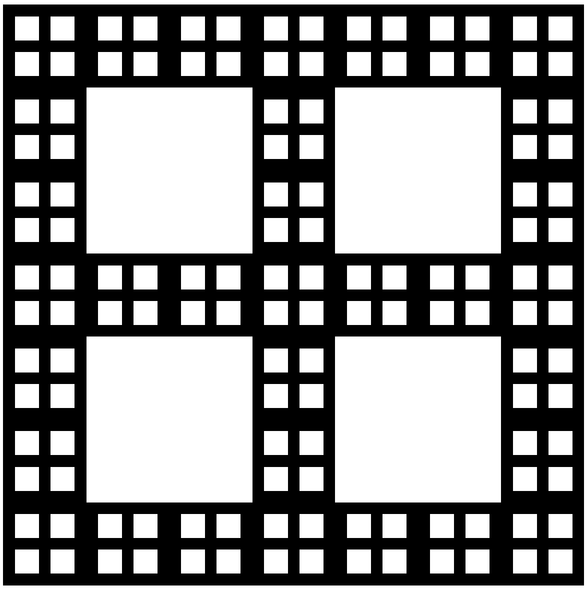

Figure 1.5: The carpet $F_{7,1,2}$.

Our arguments in this chapter apply to a more general class of Sierpiński Carpets $F_{n,p,r}, r \geq 1, p \geq 1, n \geq 5, 1 \leq p + r < \frac{n}{2}$. Let $Q^{(0)}_{n,p,r} = [0, 1] \times [0, 1]$. Subdivide $Q^{(0)}_{n,p,r}$ into n^2 subsquares and remove the interior of four bigger subsquares with side-length r/n and is of distance $\sqrt{2}p/n$ to one of the four corner points of $Q^{(0)}_{n,p,r}$. So the resulting set $Q^{(1)}_{n,p,r}$ has $(n^2 - 4r^2)$ subsquares with side-length $1/n$. Repeating the operation to the subsquares, we obtain $Q^{(2)}_{n,p,r}$. Inductively, we have $Q^{(k)}_{n,p,r}$. Then the carpet $F_{n,p,r} = \bigcap_{k \geq 0} Q^{(k)}_{n,p,r}$. See Figure 1.5. Note that $F_{n,p} = F_{n,p,1}$.

Similarly, $F_{n,p,r}$ is not quasisymmetrically equivalent to $S_m, m \geq 3$ odd and $QS(F_{n,p,r})$ is the isometric group. Moreover, $F_{n,p,r}$ and $F_{n',r',p'}$ are quasisymmetrically equivalent if and only if $(n, p, r) = (n', p', r')$. Since the proof of the above conclusions are of no essential difference from that of $F_{n,p}$, we shall omit it.

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Chapter 2

Quasisymmetric geometry of the carpet Julia sets

2.1 Introduction

Let (X, d_X) and (Y, d_Y) be two metric spaces. If there exist a homeomorphism $f : X \to Y$ and a distortion control function $\eta : [0, \infty) \to [0, \infty)$ which is also a homeomorphism such that

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for every distinct points $x, y, z \in X$, then f is called a *quasisymmetric map* and (X, d_X), (Y, d_Y) are called *quasisymetrically equivalent* to each other. A basic question in quasiconformal geometry is to determine whether two given homeomorphic spaces are quasisymmetrically equivalent to each other.

It is known that the question arises also in the classification of hyperbolic spaces and word hyperbolic groups in the sense of Gromov [BP, Kl]. See also [Bou] for examples of inequivalent spaces modelled on the universal Menger curve. In this chapter, we focus our attention on the Sierpiński carpets that arise as the Julia sets of rational maps.

According to [Wh], a set $S \in \overline{\mathbb{C}}$ is called a *Sierpiński carpet* (*carpet* in short) if S has empty interior and can be expressed as $S = \overline{\mathbb{C}} \setminus \bigcup_{i \in \mathbb{N}} D_i$, where $\{D_i\}$ are pairwise disjoint Jordan disks with $\mathrm{diam}(D_i) \to 0$ as $i \to \infty$. The collection of the boundaries of the Jordan disk $\{\partial D_i\}_{i \in \mathbb{N}}$ are called the *peripheral circles* of S. If each peripheral circle ∂D_i is a round circle, then S is called a *round carpet*. All Sierpiński carpets are homeomorphic to each other, so the question about the quasisymmetric classification of the Sierpiński carpets arises naturally.

Actually, the study of the quasisymmetric equivalences between the Sierpiński carpets and round carpets was partially motivated by the Kapovich-Kleiner conjecture in the geometry group theory. This conjecture is equivalent to the following statement: if the boundary of infinity $\partial_\infty G$ of a Gromov hyperbolic group G is a Sierpiński carpet, then

$\partial_\infty G$ is quasisymmetrically equivalent to a round carpet in $\overline{\mathbb{C}}$.

As the Julia set of a rational map, the first example of Sierpiński carpet was found by Tan [Mi1, Appendix F]. Later, the rational maps whose Julia sets are Sierpiński carpets appeared in many literatures. Such as the McMullen maps [DLU], the generated McMullen maps [XQY] and the quadratic rational maps [DFGJ] etc.

Let f be a rational map whose Julia set J_f is a Sierpiński carpet. Two questions arise naturally: (Q1) Can one give another rational map g whose Julia set J_g is a Sierpiński carpet, but J_g is not quasisymmetrically equivalent to J_f? This question is equivalent to ask whether there exist quasisymmetrically inequivalent carpet Julia sets. (Q2) Can J_f be quasisymmetrically equivalent to a round carpet?

Let X be a metric space. The *conformal dimension* of X is the infimum of the Hausdorff dimensions of all metric spaces which are quasisymmetrically equivalent to X. By definition, it is easy to see the conformal dimension is invariant under the quasisymmetric maps. For the first question stated above, Haïssinsky and Pilgrim constructed a sequence of hyperbolic rational maps with carpet Julia sets and showed that their conformal dimensions tend to two [HP, Theorem 3]. This means that there are infinitely many quasisymmetrically inequivalent Sierpiński carpets as the Julia sets of rational maps.

The *relative distance* $\Delta(A, B)$ of two sets A and B in $\overline{\mathbb{C}}$ is defined as

$$\Delta(A, B) := \frac{\operatorname{dist}(A, B)}{\min\{\operatorname{diam}(A), \operatorname{diam}(B)\}}, \tag{2.1}$$

where $\operatorname{dist}(A, B) := \sup_{a\in A,\, b\in B} |a - b|$ is the *distance* between A and B, and $\operatorname{diam}(A) := \sup_{a_1,a_2\in A} |a_1 - a_2|$ is the *diameter* of A. A set of Jordan curves $\mathcal{C} = \{C_i\}_{i\in\mathbb{N}}$ is called *uniformly relatively separated* if their pairwise relative distances are uniformly bounded away from zero. Specifically, there exists $\delta > 0$ such that $\Delta(C_i, C_j) \geq \delta$ for every two different i and j. The set $\mathcal{C}$ are *uniform quasicircles* if there exists $K \geq 1$ such that each C_i in $\mathcal{C}$ is a K-quasicircle.

For the question (Q2), Bonk gave a sufficient condition on the carpets in $\overline{\mathbb{C}}$ such that they can quasisymmetrically equivalent to some round carpets. He proved that a carpet S in $\overline{\mathbb{C}}$ is quasisymmetrically equivalent to a round carpet if its peripheral circles are uniform quasicircles and is uniformly relatively separated [Bon, Corollary 1.2]. It is worth to mention that quasisymmetric maps preserve the uniform quasicircles and uniformly relatively separated properties. It is not hard to see that the peripheral circles of such S must be uniform quasicircles but are not necessarily uniformly relatively separated.

Recently, Bonk, Lyubich and Merenkov studied the postcritically-finite rational maps whose Julia sets are Sierpiński carpets. They proved that if the Julia set of a subhyperbolic rational map is a Sierpiński carpet, then it is quasisymmetrically equivalent to a round carpet [BLM, Theorem 1.10]. They also consider the quasisymmetric group between the carpet Julia sets of postcritically-finite rational maps and proved that any quasisymmetric map ξ defined from a carpet J_f onto a carpet J_g must be the restriction of a Möbius transformation, where f and g are postcritically-finite rational maps [BLM, Theorem 1.1]. As a corollary, they proved that the group $QS(J_f)$, consisting of quasisymmetric self-map of J_f, is finite [BLM, Corollary 1.2].

In this chapter, we study carpet Julia sets in postcritically-infinite case.

2.1.1 Statement of the main results.

The *ω-limit set* $w(x)$ of a point $x \in \overline{\mathbb{C}}$ under a rational map f is defined as the set of accumulation points in the orbit of x. More precisely, $\omega(x) := \{y \in \overline{\mathbb{C}} :$ there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f^{\circ k_n}(x) = y\}$. Obviously, $\omega(x)$ is f-forward invariant. We establish a sufficient condition on the carpet Julia sets such that they are quasisymmetrically equivalent to some round carpets.

Theorem 2.1. *Let f be a rational map whose Julia set J_f is a Sierpiński carpet. If the boundaries of the periodic Fatou components are disjoint with the ω-limit sets of the critical points, then the peripheral circles of J_f are uniform quasicircles and uniformly relatively separated. In particular, J_f is quasisymmetrically equivalent to a round carpet.*

Recall that a rational map is *sub-hyperbolic* if every critical orbit is either finite or converges to an attracting periodic orbit. Note that the boundary of each Fatou component cannot contain any critical point if the Julia set is a Sierpiński carpet. By Theorem 2.1, we have following immediate corollary.

Corollary 2.1. *Let f be a sub-hyperbolic rational map whose Julia set J_f is a Sierpiński carpet. Then the peripheral circles of J_f are uniform quasicircles and uniformly relatively separated. In particular, J_f is quasisymmetrically equivalent to a round carpet.*

A critical point c of f is called *recurrent* if $c \in \omega(c)$. A rational map f is called *semi-hyperbolic* if and only if the Julia set J_f contains neither parabolic periodic points nor recurrent critical points (see [Ma] and [Yin]). It was known that the Julia set of a semi-hyperbolic rational map is locally connected and has measure zero or equal to $\overline{\mathbb{C}}$.

Theorem 2.2. *Let f be a semi-hyperbolic rational map whose Julia set J_f is a Sierpiński carpet. Then the peripheral circles of J_f are uniform quasicircles. Moreover, they are uniformly relatively separated if and only if the ω-limit sets of the critical points are disjoint with the boundaries of periodic Fatou components.*

If a rational map is not semi-hyperbolic, then the boundary of some Fatou component may not be a quasicircle although it is a Jordan curve. For example, one can construct a rational map f whose Julia set is a Sierpiński carpet but the Julia set J_f contains a parabolic periodic point. The corresponding parabolic Fatou component contains exactly one petal and has infinitely many cusps on its boundary. Thus the boundary of this Fatou component cannot be a quasicircle. In this case, J_f cannot quasisymmetrically equivalent to a round carpet. See Figure 2.1.

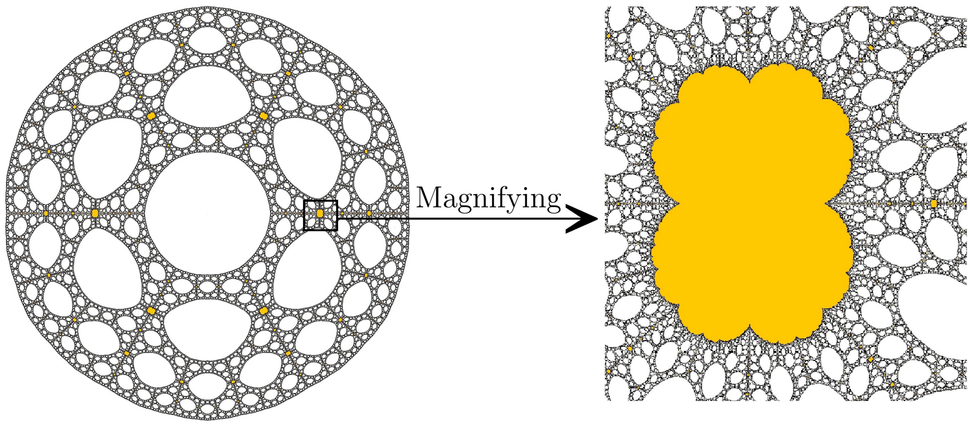

Figure 2.1: The Julia set of $f(z) = z^3 + \lambda/z^3$ and an enlargement of a parabolic Fatou component, where $\lambda \approx 0.02772313$ such that J_f is a Sierpiński carpet containing a parabolic periodic point. The peripheral circles of J_f are not uniform quasicircles but they are uniformly relatively separated.

As a corollary, we have the following theorem.

Theorem 2.3. *Let f be a semi-hyperbolic rational map whose Julia set J_f is a Sierpiński carpet. Then the quasisymmetric group $QS(J_f)$ is discrete.*

2.1.2 Outline of the proof and the organization of this chapter.

We are mainly interested on the condition when a carpet Julia set is quasisymmetrically equivalent to a round carpet. By Bonk's criterion, this motivates us to find the condition when the peripheral circles of a carpet Julia set are uniform quasicircles and when they are uniformly relatively separated.

In order to prove the peripheral circles of some carpet Julia sets are uniform quasicircles, we first discuss the periodic Fatou components and prove that they are quasicircles if their boundaries avoid the parabolic periodic points and the points in the ω-limit sets of the recurrent critical points (Lemma 2.9). Therefore, all peripheral circles are quasicircles by using Sullivan's eventually periodic theorem. In order to prove the uniformity, we discuss two cases. The first case, suppose that all the periodic Fatou components are disjoint with the ω-limit sets of the critical points. Then for each periodic Fatou component U, one can find a large Jordan disk V such that $V \setminus \overline{U}$ is an annulus and all components of the preimages of $V \setminus \overline{U}$ are annuli whose moduli have uniform lower bound. By using a distortion argument, one can prove that all peripheral circles are uniform quasicircles (Proposition 2.1). The second case, suppose that the rational map is semi-hyperbolic. Then the corresponding Julia set (and hence all the periodic Fatou components) contains neither parabolic periodic points nor recurrent critical points. One can also prove that all peripheral circles are uniform quasicircles by using Mañé's theorem and its variation (Theorem 2.5, Lemma 2.6 and Proposition 2.2).

In order to prove the peripheral circles of some carpet Julia sets are uniformly relatively separated, we first establish a lemma which asserts that the modulus can control the relative distance (Lemma 2.3). Then we prove the peripheral circles are uniformly relatively separated by showing that all moduli of the annuli between two different peripheral circles have a lower positive bound (Proposition 2.3).

This chapter is organized as follows: In §2.2, we prepare some distortion lemmas for the proofs of Theorems 2.1 and 2.2. Moreover, we prove that the modulus can control the relative distance. In §2.3, we first prove some propositions about the properties of uniform quasicircles and uniformly relatively separated. Then we prove Theorem 2.1 by using Bonk's criterion and prove Theorem 2.2 by combining Bonk's criterion and Mañé-Yin's characterization on semi-hyperbolic rational maps. In the last section, using the combinatorial method and renormalization theory, we construct a critically-infinite semi-hyperbolic rational map whose Julia set is quasisymmetrically equivalent to a round carpet.

2.2 Some distortion estimations

In this section, we give some distortion estimations and useful lemmas, which will be used in the next section. We use $\mathbb{D} := \{z : |z| < 1\}$ to denote the unit disk on the complex plane $\mathbb{C}$.

Theorem 2.4 (Koebe's distortion theorem, [Pom, p. 9])**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent function. Then for every $z \in \mathbb{D}$, one has*

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z) - f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}; \quad \textit{and} \tag{2.2}$$

$$|f'(0)|\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq |f'(0)|\frac{1+|z|}{(1-|z|)^3}. \tag{2.3}$$

Let A be an annulus with non-degenerated boundary components. Then there exists a conformal map sending A to a standard annulus $\{z \in \mathbb{C} : 0 < r < |z| < 1\}$, where $r > 0$ is uniquely determined by A. As an invariant under conformal maps, the *modulus* of A is defined as $\mathrm{mod}(A) = \frac{1}{2\pi}\log(1/r)$. A set in $\overline{\mathbb{C}}$ is called a *Jordan disk* if it is homeomorphic to the unit disk $\mathbb{D}$ and its boundary is a Jordan curve. Let A and B be two open sets in $\overline{\mathbb{C}}$. We use the notation '$A \Subset B$' if the closure $\overline{A}$ is contained in B.

Lemma 2.1. *Let $U_i \Subset V_i \neq \mathbb{C}$ be a pair of Jordan disks, where $i = 1, 2$. Suppose that $\mathrm{mod}(V_2 \setminus \overline{U}_2) \geq m > 0$ and $f : V_1 \to V_2$ a conformal map with $f(U_1) = U_2$. Then there* *exists a constant $C(m) \geq 1$ depending only on m such that for any $x, y, z, w \in \overline{U}_1$, one has*

$$\frac{1}{C(m)}\frac{|x-y|}{|z-w|} \leq \frac{|f(x)-f(y)|}{|f(z)-f(w)|} \leq C(m)\frac{|x-y|}{|z-w|}.$$

Proof. The proof is based on applying Koebe's distortion theorem. Without loss of generality, suppose that $x \neq y$ and $z \neq w$ are contained in the interior of U_1. If not, we can enlarge U_1 appropriately. By Riemann's mapping theorem, there exists a conformal mapping $g : (\Omega, \mathbb{D}) \to (U_1, V_1)$ which maps the unit disk $\mathbb{D}$ onto V_1 and a simply connected domain Ω onto U_1. In particular, we require that $g(0) = x$.

We claim that there exists a positive constant $r := r(m) < 1$ depending only on m such that $\Omega \subset \mathbb{D}_r := \{z : |z| < r\}$. Let $\zeta \in \partial\Omega$ be the farthest point such that $\mathrm{dist}(0, \partial\Omega) = |\zeta|$. Then $\mathbb{D} \setminus \overline{\Omega}$ is an annulus separating 0 and ζ from the unit circle. By Grötzsch's module theorem [LV, p. 54], we have

$$m \leq \mathrm{mod}(V_1 \setminus \overline{U}_1) = \mathrm{mod}(\mathbb{D} \setminus \overline{\Omega}) \leq \mu(|\zeta|),$$

where $r \mapsto \mu(r)$ is a continuous and strictly decreasing function defined on the interval $(0, 1)$. This means that $|\zeta| \leq \mu^{-1}(m)$ and the claim follows if we set $r = \mu^{-1}(m)$.

Now we consider $f \circ g : \mathbb{D} \to V_2$ and $g : \mathbb{D} \to V_1$. For every $\eta \in \Omega$, by using (2.3) in Theorem 2.4, we have

$$|f'(x)|\,|g'(0)|\frac{1-r}{(1+r)^3} \leq |(f \circ g)'(\eta)| = |f'(g(\eta))|\,|g'(\eta)| \leq |f'(x)|\,|g'(0)|\frac{1+r}{(1-r)^3}. \quad (2.4)$$

Also, we have

$$|g'(0)|\frac{1-r}{(1+r)^3} \leq |g'(\eta)| \leq |g'(0)|\frac{1+r}{(1-r)^3}. \quad (2.5)$$

Combine (2.4) and (2.5), it follows that for every $\xi \in U_1$, we have

$$|f'(x)|\frac{(1-r)^4}{(1+r)^4} \leq |f'(\xi)| \leq |f'(x)|\frac{(1+r)^4}{(1-r)^4}. \quad (2.6)$$

Therefore, for $x, y, z, w \in U_1$, by (2.6), we have

$$|f(x)-f(y)| \leq \frac{(1+r)^4}{(1-r)^4}|f'(x)| \cdot |x-y| \text{ and } |f(z)-f(w)| \geq \frac{(1-r)^4}{(1+r)^4}|f'(x)| \cdot |z-w|.$$

Set $C(m) = (1+r(m))^8/(1-r(m))^8$. The proof is complete. $\square$

Let U be a hyperbolic disk in $\mathbb{C}$ and E a connected and compact subset of U containing at least two points. For any $z_1, z_2 \in E$, the *turning* of E about z_1 and z_2 is defined by

$$\Lambda(E; z_1, z_2) = \frac{\mathrm{diam}(E)}{|z_1 - z_2|}.$$

It is easy to see that $1 \le \Lambda(E; z_1, z_2) \le \infty$ and $\Lambda(E; z_1, z_2) = \infty$ if and only if $z_1 = z_2$.

By definition (see for example, [LV, p. 100]), a Jordan curve C is called a *quasicircle* if there exists a positive constant $K \ge 1$ such that for any different points $x, y \in C$, the turning of γ about x and y satisfies

$$\Lambda(\gamma; x, y) \le K,$$

where γ is one of the two components of $C \setminus \{x, y\}$ with smaller diameter.

Lemma 2.2. *Let $U_i \Subset V_i \neq \mathbb{C}$ be a pair of Jordan disks, where $i = 1, 2$. Suppose that* $\operatorname{mod}(V_2 \setminus \overline{U}_2) \ge m > 0$ *and $f : V_1 \to V_2$ a conformal map with $f(U_1) = U_2$. If ∂U_2 is a K-quasicircle, then there is a constant $C(K, m) \ge 1$ such that ∂U_1 is a $C(K, m)$-quasicircle.*

Proof. By definition, if ∂U_2 is a K-quasicircle, then there exists a constant $C(K) > 0$ such that for any different points $z_1, z_2 \in \partial U_2$, the turning of γ about z_1 and z_2 satisfies

$$\Lambda(\gamma; z_1, z_2) = \frac{\operatorname{diam}(\gamma)}{|z_1 - z_2|} \le C(K), \tag{2.7}$$

where γ is one of the component of $\partial U_2 \setminus \{z_1, z_2\}$ with smaller diameter.

Let $x, y \in \partial U_1$ be two different points which divide the quasicircle ∂U_1 into two closed subcurves α and β. Without loss of generality, let $\alpha \subset \partial U_1$ be the subcurve with smaller diameter. Moreover, let $z, w \in \alpha$ such that $\operatorname{diam}(\alpha) = |z - w|$. By Lemma 2.1, we have

$$\Lambda(\alpha; x, y) = \frac{|z - w|}{|x - y|} \le C(m) \frac{|f(z) - f(w)|}{|f(x) - f(y)|}, \tag{2.8}$$

where $C(m)$ is the constant appeared in Lemma 2.1. Note that $f(x), f(y)$ divide the quasicircle ∂U_2 into two parts $f(\alpha)$ and $f(\beta)$.

If $\operatorname{diam}(f(\alpha)) \le \operatorname{diam}(f(\beta))$, then by (2.7) and (2.8), we have

$$\Lambda(\alpha; x, y) \le C(m) \frac{\operatorname{diam}(f(\alpha))}{|f(x) - f(y)|} \le C(m) C(K). \tag{2.9}$$

If $\operatorname{diam}(f(\alpha)) > \operatorname{diam}(f(\beta))$, let $z', w' \in \beta$ such that $\operatorname{diam}(\beta) = |z' - w'|$. By (2.7) and Lemma 2.1, we have

$$\begin{aligned} \Lambda(\alpha; x, y) \le \Lambda(\beta; x, y) \le \frac{|z' - w'|}{|x - y|} &\le C(m) \frac{|f(z') - f(w')|}{|f(x) - f(y)|} \\ \le C(m) \frac{\operatorname{diam}(f(\beta))}{|f(x) - f(y)|} &\le C(m) C(K). \end{aligned} \tag{2.10}$$

Combine (2.9) and (2.10), the Lemma follows. $\square$

Recall that the *relative distance* $\Delta(A, B)$ of two subsets A and B in $\overline{\mathbb{C}}$ is defined in

(2.1). Now we prove that relative distance of two disjoint Jordan curves can be controlled by the modulus of the annulus between them.

Lemma 2.3 (Modulus controls the relative distance). *Let $A \subset \overline{\mathbb{C}}$ be an annulus with two boundary components C_1 and C_2. If the modulus of A satisfies $\mathrm{mod}(A) \geq m > 0$, then there exists a constant $C(m) > 0$ depending only on m such that the relative distance of C_1 and C_2 satisfies $\Delta(C_1, C_2) \geq C(m) > 0$.*

Proof. Without loss of generality, we assume that $A \subset \mathbb{C}$, C_1, C_2 are not singletons and $0 < \mathrm{diam}(C_1) \leq \mathrm{diam}(C_2)$ and

$$\mathrm{dist}(C_1, C_2) = |x - y| \tag{2.11}$$

for $x \in C_1$ and $y \in C_2$. There exists a point $z \neq x$ in C_1 such that $|x-z| = \sup_{a \in C_1} |a-x|$. Therefore, we have

$$\mathrm{diam}(C_1) \leq 2|x - z|. \tag{2.12}$$

Consider the linear function $h(t) = (t-x)/(x-z)$, which maps x, y, z to $0, (y-x)/(x-z)$ and -1. Then $h(A)$ is an annulus separating the points 0 and -1 from $h(y)$ and ∞, respectively. Let

$$R = |h(y)| = |(y - x)/(x - z)|.$$

By Teichmüller's Module Theorem (see for example, [LV, p. 56]), we have

$$m \leq \mathrm{mod}(A) = \mathrm{mod}(h(A)) \leq 2\,\mu\left(\sqrt{\frac{1}{1+R}}\right),$$

where $r \mapsto \mu(r)$ is a continuous and strictly decreasing map defined on the interval $(0, 1)$. By (2.11) and (2.12), this means that the relative distance of C_1 and C_2 is

$$\Delta(C_1, C_2) = \frac{\mathrm{dist}(C_1, C_2)}{\mathrm{diam}(C_1)} \geq \frac{|x-y|}{2|x-z|} = \frac{R}{2} \geq \frac{1}{2}\left(\frac{1}{(\mu^{-1}(m/2))^2} - 1\right) := C(m).$$

The proof is complete. □

Lemma 2.4 ([KL, Lemma 4.5]). *Let $U_i \Subset V_i \neq \mathbb{C}$ be a pair of Jordan disks, where $i = 1, 2$. Suppose that $g : V_1 \to V_2$ is a proper holomorphic map of degree $d \geq 1$ and U_1 is a component of $g^{-1}(U_2)$. Then*

$$\mathrm{mod}(V_1 \setminus \overline{U}_1) \leq \mathrm{mod}(V_2 \setminus \overline{U}_2) \leq d\ \mathrm{mod}(V_1 \setminus \overline{U}_1).$$

Let U be a hyperbolic disk in $\mathbb{C}$ and $z \in U$. The *shape* of U about z, denoted by

Shape(U, z), is defined as

$$\mathrm{Shape}(U, z) = \frac{\max_{w \in \partial U} |w - z|}{\min_{w \in \partial U} |w - z|} = \frac{\max_{w \in \partial U} |w - z|}{\mathrm{dist}(z, \partial U)}.$$

It is obvious that Shape$(U, z) = \infty$ if and only if U is unbounded and Shape$(U, z) = 1$ if and only if U is a round disk centered at z. In other cases, $1 < \mathrm{Shape}(U, z) < \infty$.

Lemma 2.5 ([QWY, Lemma 6.1]). *Let $U_i \Subset V_i \neq \mathbb{C}$ be a pair of Jordan disks with $\mathrm{mod}(V_2 \setminus \overline{U}_2) \geq m > 0$, where $i = 1, 2$. Suppose that $g : V_1 \to V_2$ is a proper holomorphic map of degree $d \geq 1$ and U_1 is a component of $g^{-1}(U_2)$. Then there are two positive constants $C_1(d, m)$ and $C_2(d, m)$ depending only on d and m, such that*

(1) *For all $z \in U_1$, the shape satisfies*

$$\mathrm{Shape}(U_1, z) \leq C_1(d, m)\ \mathrm{Shape}(U_2, g(z)).$$

(2) *For any connected and compact subset E of U_1 with the cardinal number $\sharp E \geq 2$ and any $z_1, z_2 \in E$, the turning satisfies*

$$\Lambda(E; z_1, z_2) \leq C_2(d, m)\, \Lambda(g(E); g(z_1), g(z_2)).$$

Lemma 2.5 means that the shape and the turning of the interior boundary of an annulus can be controlled under a proper holomorphic map if the modulus of this annulus has a lower bound.

2.3 Proofs of the Main Theorems

If a rational map f whose Julia set J_f is a Sierpiński carpet, then f cannot be a polynomial. In fact, the intersection of the closure of the bounded Fatou components (if any) and the basin of infinity of f is non-empty provided f is a polynomial since the Julia set J_f is the boundary of the basin of infinity. If we want to prove Theorem 2.1, we need to prove that the peripheral circles of the carpets are uniform quasicircles and uniformly relatively separated by Bonk's criterion.

2.3.1 Mañé's Theorem and a lemma.

We first give a theorem due to Mañé, which will be used frequently later.

Theorem 2.5 ([Ma, Theorem II]). *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with degree at least two. If a point $x \in J_f$ is not a parabolic periodic point and is not contained in the ω-limit set of a recurrent critical point, then for any $\epsilon > 0$ there exists an open neighborhood U_x of x such that:*

(C1) *For all* $n \geq 0$, *every component of* $f^{-n}(U_x)$ *has diameter* $\leq \epsilon$;

(C2) *There exists* $d > 0$ *such that for all* $n \geq 0$ *and every connected component* V *of* $f^{-n}(U_x)$, *the degree of* $f^{\circ n} : V \to U_x$ *is* $\leq d$.

When we pull back a Jordan disk U by a rational map f, there maybe exist a component W of $f^{-1}(U)$ which is not simply connected. If the boundary ∂U avoids the critical values, then ∂W is the union of finitely many disjoint Jordan curves $\{C_i\}$. Moreover, we have $f(C_i) = \partial U$ for each i. Note that W is a connected set whose boundary consists of finitely many Jordan curves. We have $\overline{\mathbb{C}} \setminus \overline{W} = \bigcup_i V_i$, where each V_i is a Jordan disk bounded by the Jordan curve C_i. Since the restriction of f on V_i is a holomorphic branched covering and $f(\partial V_i) = \partial U$, we have $f(V_i) = \overline{\mathbb{C}}$ or $f(V_i) = \overline{\mathbb{C}} \setminus \overline{U}$. In other words, the image of each component of the complement of W under f is either $\overline{\mathbb{C}}$ or $\overline{\mathbb{C}} \setminus \overline{U}$. See Figure 2.2 for an example.

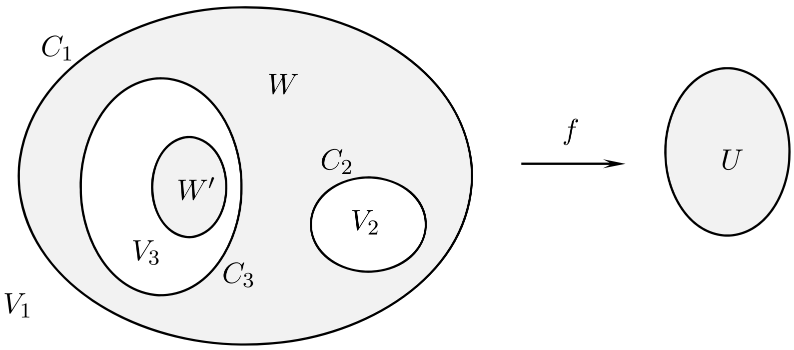

Figure 2.2: The pull back of a simply connected domain U under the rational map f with degree 4, where $f(W) = U$ and $\partial W = C_1 \cup C_2 \cup C_3$. The complement of $\overline{W}$ consists of 3 simply connected components V_1, V_2 and V_3. In particular, $f(V_1) = f(V_2) = \overline{\mathbb{C}} \setminus \overline{U}$ and $f(V_3) = \overline{\mathbb{C}}$. Moreover, W contains 4 critical points of f and $V_3 \setminus W'$ (the white annulus) contains two.

In the rest of this chapter, we only consider the rational maps whose Julia sets are not the whole complex sphere. Therefore, after conjugating f by a suitable Möbius transformation, we always assume that ∞ lies in the Fatou set. This means that J_f is a compact set in $\mathbb{C}$. In the following, we equip J_f the Euclidean metric if not special specified. We use $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$ to denote the round disk in $\mathbb{C}$ with the center $a \in \mathbb{C}$ and radius $r > 0$.

Lemma 2.6. *Let* f *be a rational map with degree at least two and* $J_f \subset \mathbb{C}$. *Suppose that* $x \in J_f$ *is not a parabolic periodic point and is not contained in the* ω*-limit set of a recurrent critical point. Then there exists an open neighborhood* U_x *of* x *such that*

(C3) *For all* $n \geq 0$, *every connected component of* $f^{-n}(U_x)$ *is simply connected.*

Proof. By the assumption that $\infty \notin J_f$, the grand orbit of ∞ lies in the Fatou set of f. Let $\delta_0 > 0$ be a small positive number such that

$$0 < \delta_0 \leq \mathrm{dist}(f^{-1}(\infty), J_f)/2. \tag{2.13}$$

By Theorem 2.5, there exists an open neighborhood U_x' of x such that every component of $f^{-n}(U_x')$ has diameter $\leq \delta_0$ for all $n \geq 0$.

Let $U_x := \mathbb{D}(x, \delta_x)$ be the largest round disk which is contained in U_x'. We claim that every component W_n of $f^{-n}(U_x)$ is simply connected. If not, let V_n be a bounded component of $\mathbb{C} \setminus \overline{W}_n$, where $n \geq 1$. Then $\partial V_n \subset \partial W_n$ and so $\operatorname{diam}(V_n) \leq \delta_0$. This means that V_n cannot intersect $f^{-1}(\infty)$. Inductively, one can easily check that $f^{\circ k}(V_n) \cap f^{-1}(\infty) = \emptyset$ for $0 \leq k \leq n-1$. It follows that $\infty \notin f^{\circ n}(V_n)$, which is a contradiction since $f^{\circ n}(V_n) = \overline{\mathbb{C}}$ or $f^{\circ n}(V_n) = \overline{\mathbb{C}} \setminus \overline{U}_x$. Therefore, such V_n does not exist. This means that W_n is simply connected. The proof is complete. □

Lemma 2.6 is useful in the following since we need to obtain the simply connected preimages of a simply connected domain.

2.3.2 Sufficiency for the property of uniform quasicircles.

In this subsection, we prepare some lemmas and give two sufficient conditions such that the boundaries of the Fatou components are uniform quasicircles. We first discuss the regularity of the boundaries of the periodic Fatou components and then spread the results to their all preimages.

Lemma 2.7. *Let Γ be a Jordan curve in the plane $\mathbb{C}$. Then there exists a constant $\delta_\Gamma > 0$ depending only on Γ such that, for any Jordan subarc $\gamma \subset \Gamma$ with $\operatorname{diam}(\gamma) \leq \delta_\Gamma$, one has $\operatorname{diam}(\gamma) < \operatorname{diam}(\Gamma \setminus \gamma)$.*

Proof. Consider the function $h : \Gamma \times \Gamma \to \mathbb{R}$ which is defined by $h(x, y) = \operatorname{diam}(L'(x, y))$, where $L'(x, y)$ is one of the two components of $\Gamma \setminus \{x, y\}$ with larger diameter. Obviously, the map h is continuous. Since $\Gamma \times \Gamma$ is compact, the function h has a minimum $\delta' > 0$. Then the lemma holds if we set $\delta_\Gamma = \delta'/2$. □

Lemma 2.8. *Let f be a rational map with degree at least two and U a Fatou component which is a Jordan disk. Then $f|_{\partial U}$ is a local homeomorphism.*

Proof. The image $V = f(U)$ is a Fatou component and hence a domain. Since f maps the boundary of U to that of V, it follows that V is a Jordan disk as well and $f(\partial U) = \partial V$. For an annulus A with the outer boundary ∂V and the inner boundary surrounding all the critical values in V, then $A' = (f|_U)^{-1}(A)$ is also an annulus in U with the outer boundary coinciding with ∂U by Riemann-Hurwitz's formula. Then $f : A' \to A$ is an unbranched covering. Thus the restriction of f on ∂U is a local homeomorphism. □

Lemma 2.9 (The boundaries of periodic Fatou components are quasicircles)**.** *Let f be a rational map with degree at least two. Suppose that U is a periodic Fatou component of f whose boundary ∂U is a Jordan curve and ∂U contains neither parabolic periodic points nor the points in $\omega(c)$ for any recurrent critical point c. Then ∂U is a quasicircle.*

Proof. After iterating f by several times, we can assume that the periodic Fatou component U is fixed by f. Without loss of generality, we suppose that $\infty \notin J_f$. Let $\epsilon = \delta_0 > 0$ be the number defined as in (2.13). For any $x \in \partial U$, by Theorem 2.5 and Lemma 2.6, there exists an open neighborhood $U_x := \mathbb{D}(x, \delta_x)$ of x satisfying (C1), (C2) and (C3). Since ∂U is compact and $\partial U \subseteq \bigcup_{x \in \partial U} \mathbb{D}(x, \delta_x/2)$, one can select a collection of finite number of elements $\mathcal{U} = \{\mathbb{D}(x_1, \delta_{x_1}/2), \cdots, \mathbb{D}(x_N, \delta_{x_N}/2)\}$ such that ∂U is covered by $\mathcal{U}$. Let $\delta_1 > 0$ be the Lebesgue number of $\mathcal{U}$. Then every subset of ∂U with diameter $\leq \delta_1$ must be contained in at least one open disk $\mathbb{D}(x_i, \delta_{x_i}/2)$ for some $1 \leq i \leq N$.

By Lemma 2.8, the restriction of f on ∂U is a local homeomorphism. This means that there exists a number $\delta_2 > 0$ such that for any subset $E \subset \partial U$ with $\mathrm{diam}(E) \leq \delta_2$, the restriction of f on E is a homeomorphism. Recall that $\delta_{\partial U} > 0$ is the number depending only on ∂U which is defined in Lemma 2.7. We define

$$\delta := \min \left\{ \frac{\delta_1}{M}, \delta_2, \frac{\delta_{\partial U}}{M} \right\}, \tag{2.14}$$

where $M := 1 + \sup\{|f'(z)| : \mathrm{dist}(z, J_f) \leq \delta_0\} < +\infty$.

Let x, y be two different points in ∂U. We use $\gamma := L(x, y)$ to denote one of the two components of $\partial U \setminus \{x, y\}$ with the smaller diameter. Now we divide the argument into two cases.

Case 1: Suppose that $\mathrm{diam}(\gamma) \geq \delta$. Define $E := \{(\xi, \eta) \in \partial U \times \partial U : \mathrm{diam}(L(\xi, \eta)) \geq \delta\}$. Then E is compact and $(\xi, \xi) \notin E$. The function

$$h : \partial U \times \partial U \to \mathbb{R}^+ \text{ defined by } (\xi, \eta) \mapsto \frac{\mathrm{diam}(L(\xi, \eta))}{|\xi - \eta|}$$

is continuous on E. Then h has a maximum K_1 on E since E is compact. In particular, the turning of γ about x and y satisfies

$$\Lambda(\gamma; x, y) = \frac{\mathrm{diam}(\gamma)}{|x - y|} \leq K_1. \tag{2.15}$$

Case 2: Suppose that $\mathrm{diam}(\gamma) < \delta$. Denote $\gamma_n := f^{\circ n}(\gamma)$ for $n \geq 0$. Note that the forward orbit of γ will eventually cover ∂U. There is a smallest integer $n \geq 0$ such that

$$\mathrm{diam}(\gamma_n) < \delta \text{ and } \mathrm{diam}(\gamma_{n+1}) = \mathrm{diam}(f(\gamma_n)) \geq \delta. \tag{2.16}$$

By the choice of δ in (2.14), we know that $f^{\circ(n+1)}|_\gamma$ is a homeomorphism and so γ_{n+1} is a Jordan arc connecting $f^{\circ(n+1)}(x)$ and $f^{\circ(n+1)}(y)$. Note that there exist two points

$z_1, z_2 \in \gamma_n$, such that

$$\begin{aligned}\operatorname{diam}(\gamma_{n+1}) = |f(z_1) - f(z_2)| &\leq \int_{[z_1,z_2]} |f'(z)|\,|dz| \\ &\leq M|z_1 - z_2| \leq M \operatorname{diam}(\gamma_n) \leq M\delta \leq \min\{\delta_1, \delta_{\partial U}\},\end{aligned} \tag{2.17}$$

where $[z_1, z_2]$ is the straight segment connecting z_1 and z_2.

By the definition of $\delta_{\partial U}$ and Lemma 2.7, the Jordan arc γ_{n+1} is one of the two components of $\partial U \setminus \{f^{\circ(n+1)}(x), f^{\circ(n+1)}(y)\}$ with smaller diameter. Since $\operatorname{diam}(\gamma_{n+1}) \geq \delta$ by (2.16), as discussed in Case 1 above, we have

$$\Lambda(\gamma_{n+1}; f^{\circ(n+1)}(x), f^{\circ(n+1)}(y)) \leq K_1. \tag{2.18}$$

By the definition of δ_1, there exists a disk $\mathbb{D}(x_i, \delta_{x_i}/2)$ such that $\gamma_{n+1} \subset \mathbb{D}(x_i, \delta_{x_i}/2)$ for some $1 \leq i \leq N$ since $\operatorname{diam}(\gamma_{n+1}) \leq \delta_1$ by (2.17). Let $B_{n+1}(x_i, \delta_{x_i}/2)$ and $B_{n+1}(x_i, \delta_{x_i})$, respectively, be the components of $f^{-n-1}(\mathbb{D}(x_i, \delta_{x_i}/2))$ and $f^{-n-1}(\mathbb{D}(x_i, \delta_{x_i}))$ both containing γ. Note that both of them are simply connected by the choice of δ_{x_i}. Applying Lemma 2.5 to the case $(U_1, V_1) = (B_{n+1}(x_i, \delta_{x_i}/2),\, B_{n+1}(x_i, \delta_{x_i}))$, $(U_2,\, V_2) = (\mathbb{D}(x_i, \delta_{x_i}/2),\, \mathbb{D}(x_i, \delta_{x_i}))$ and $m = \frac{1}{2\pi}\log 2$, together with (2.18), we have

$$\Lambda(\gamma; x, y) \leq C_2(d_i)\Lambda(\gamma_{n+1}; f^{\circ(n+1)}(x), f^{\circ(n+1)}(y)) \leq C_2(d_i)K_1, \tag{2.19}$$

where $C_2(d_i)$ is a constant depending only on d_i and $d_i > 0$ is the number appeared in Theorem 2.5 which depends on x_i. Let

$$K = K_1(1 + \max_{1\leq i\leq N} C_2(d_i)).$$

Then $\Lambda(\gamma; x, y) \leq K$ holds for any different $x, y \in \partial U$ by (2.15) and (2.19). By the arbitrariness of x and y, this means that ∂U is a quasicircle. The proof is completed. $\square$

Now we need to consider when the boundaries of all the Fatou components are uniform quasicircles. According to Sullivan [Sul], each Fatou component of a rational map is eventually periodic. It is natural to consider the pull back of the periodic Fatou components and then using some distortion lemmas to control the shape of pre-periodic Fatou components. To do this, it is necessary to construct a larger simply connected domain surrounding the periodic Fatou component such that all components of its preimages under the n-th iteration are still simply connected.

Proposition 2.1 (Uniform quasicircles I)**.** *Let f be a rational map such that the boundary of each Fatou component is a Jordan curve. Suppose that all the boundaries of periodic Fatou components are disjoint with the ω-limit sets of the critical points. Then the boundaries of all the Fatou components of f are uniform quasicircles.*

Proof. If all periodic Fatou components of f are disjoint with the ω-limit sets of the critical points, then f has no parabolic periodic points. By Lemma 2.9 and Sullivan's eventually periodic theorem, all the boundaries of the Fatou components of f are quasicircles. We only need to prove that they are uniform quasicircles.

Let $\mathcal{U}'$ be the collection of all the Fatou components such that each of them is either a critical Fatou component (contains at least one critical point) or a periodic Fatou component. We use $\mathcal{U} := O^+(\mathcal{U}') = \{U_1, \cdots, U_n\}$ to denote the union of the forward orbits of all the Fatou components in $\mathcal{U}'$. Note that the number of Fatou components in $\mathcal{U}$ is finite since $\mathcal{U}'$ is. Therefore, there exists a constant $K_1 > 1$ such that each ∂U_i is a K_1-quasicircle. For each $1 \leq i \leq n$, let V_i be a Jordan disk such that $V_i \setminus \overline{U}_i$ is an annulus which is disjoint with the ω-limit sets of the critical points.

Let $\mathrm{mod}(V_i \setminus \overline{U}_i) = m_i > 0$ for $1 \leq i \leq n$. For each Fatou component $U \notin \mathcal{U}$, there exists a minimal number $k \geq 1$ such that $f^{\circ k}(U) = U_i \in \mathcal{U}$ for some i. Let V be the component of $f^{-k}(V_i)$ containing U. Then $f^{\circ k} : V \to V_i$ is conformal and V is a Jordan disk since V_i contains no points in the critical orbits. By Lemma 2.2, the boundary ∂U is a $C(K_1, m_i)$-quasicircle, where $C(K_1, m_i)$ is a constant depending only on K_1 and m_i.

Let $K = \max_{1 \leq i \leq n} C(K_1, m_i)$. Then the boundary of each Fatou component of f is a K-quasicircle. By the arbitrariness of U, this means that the Fatou components of f are uniform quasicircles. The proof is completed. □

Recall that a rational map f is called *semi-hyperbolic* if and only if the Julia set J_f contains neither parabolic periodic points nor recurrent critical points.

Proposition 2.2 (Uniform quasicircles II). *Let f be a* semi-hyperbolic *rational map such that the boundary of each Fatou component is a Jordan curve. Then the boundaries of all the Fatou components of f are uniform quasicircles.*

Proof. By Lemma 2.9 and Sullivan's eventually periodic theorem, it follows that all the boundaries of the Fatou components of f are quasicircles since f is semi-hyperbolic. We only need to prove that they are uniform quasicircles. According to [Yin, Theorem 1.2], the Julia set J_f is locally connected. Then for any $\epsilon > 0$, there are only finitely many Fatou components with diameter $\geq \epsilon$ [Mi3, Lemma 19.5].

Without loss of generality, we suppose that $\infty \notin J_f$. Let $\epsilon = \delta_0 > 0$ be the number defined as in (2.13). For any $x \in J_f$, by Theorem 2.5 and Lemma 2.6, there exists an open neighborhood $U_x := \mathbb{D}(x, \delta_x)$ of x satisfying (C1), (C2) and (C3). Since J_f is compact, there exists a collection of finite number of elements $\mathcal{U} = \{\mathbb{D}(x_1, \delta_{x_1}/2), \cdots, \mathbb{D}(x_N, \delta_{x_N}/2)\}$ such that J_f is covered by $\mathcal{U}$. We use $\delta > 0$ to denote the Lebesgue number of $\mathcal{U}$. Then every subset of J_f with diameter $\leq \delta$ must be contained in at least one open disk $\mathbb{D}(x_i, \delta_{x_i}/2)$ for some $1 \leq i \leq N$.

We divide the collection of all the Fatou components $\mathcal{F}$ of f into two classes as following. Let $\mathcal{F}_0$ be the collection of all the Fatou components such that each $U \in \mathcal{F}_0$ is one of the following cases: (1) U contains at least one critical point; (2) U is periodic; (3) $\mathrm{diam}(U) \geq \delta$. Let $\mathcal{F}_1' := O^+(\mathcal{F}_0)$ be the set of the union of the forward orbits of all the Fatou components in $\mathcal{F}_0$. Define $\mathcal{F}_1 := \mathcal{F}_1' \cup f^{-1}(\mathcal{F}_1')$. By Sullivan's eventually periodic theorem, the number of Fatou components in $\mathcal{F}_1$ is finite since $\mathcal{F}_0$ is also. Therefore, there exists a constant $K_1 > 1$ such that each Fatou component in $\mathcal{F}_1$ is a K_1-quasicircle.

For any Fatou component $U \in \mathcal{F} \setminus \mathcal{F}_1$, we have $\mathrm{diam}(U) < \delta$. There exists a minimal $n_U \geq 1$ such that $f^{\circ n_U}(U) \in f^{-1}(\mathcal{F}_1') \setminus \mathcal{F}_1' \subset \mathcal{F}_1$ and $\mathrm{diam}(f^{\circ n_U}(U)) < \delta$. Moreover, the map $f^{\circ n_U} : U \to f^{\circ n_U}(U)$ is conformal. By the definition of δ, there exists some disk $\mathbb{D}(x_i, \delta_{x_i}/2)$ in $\mathcal{U}$ such that $f^{\circ n_U}(U) \subset \mathbb{D}(x_i, \delta_{x_i}/2)$. We use B_U and B_U', respectively, to denote the components of $f^{-n_U}(\mathbb{D}(x_i, \delta_{x_i}/2))$ and $f^{-n_U}(\mathbb{D}(x_i, \delta_{x_i}))$ both containing U.

Let $x, y \in \partial U$ be two different points such that $\partial U \setminus \{x, y\} = \gamma_1 \cup \gamma_2$. Then $f^{\circ n_U}(\gamma_1)$ and $f^{\circ n_U}(\gamma_2)$ are both Jordan arcs connecting $f^{\circ n_U}(x)$ with $f^{\circ n_U}(y)$. Applying Lemma 2.5 (2) to the case $(U_1, V_1) = (B_U, B_U')$, $(U_2, V_2) = (\mathbb{D}(x_i, \delta_{x_i}/2), \mathbb{D}(x_i, \delta_{x_i}))$, $m = \frac{1}{2\pi}\log 2$, $g = f^{\circ n_U}$ and $E = \gamma_j$, where $j = 1, 2$, we have

$$\Lambda(\gamma_j; x, y) \leq C_2(d_i)\, \Lambda(f^{\circ n_U}(\gamma_j); f^{\circ n_U}(x), f^{\circ n_U}(y)),$$

where $C_2(d_i)$ is a constant depending only on d_i and $d_i > 0$ is the number appeared in Theorem 2.5 which depends on x_i. Then

$$\min_{j \in \{1,2\}} \{\Lambda(\gamma_j; x, y)\} \leq C_2(d_i) \min_{j \in \{1,2\}} \{\Lambda(f^{\circ n_U}(\gamma_j); f^{\circ n_U}(x), f^{\circ n_U}(y))\} \leq C_2(d_i) K_1.$$

Let $K = \max_{1 \leq i \leq N} C_2(d_i) K_1$. Then ∂U is a K-quasicircle by the arbitrariness of x and y. By the arbitrariness of U, we know that each Fatou component of f is a K-quasicircle and K is a constant depending only on f. The proof is completed. □

Figure 2.1 shows a rational map having a parabolic periodic point whose Julia set is a Sierpiński carpet but the peripheral circles of J_f are not uniform quasicircles. Note that in Propositions 2.1 and 2.2, we do not require the closure of Fatou components are disjoint to each other. They can touch each other at the points on their boundaries. It seems that the conditions in Proposition 2.1 is much stronger than in Proposition 2.2. However, it is not true. One can construct a rational map with recurrent critical points, whose w-limit sets are disjoint with boundaries of Fatou components, using similar method as stated in Section 2.4.

2.3.3 Sufficiency for the property of uniformly relatively separated.

By Lemma 2.3, if the lower bound of the annuli between the boundaries of the Fatou components can be controlled, then one can prove that the peripheral circles of the carpet

Julia set are uniformly relatively separated.

Proposition 2.3 (Uniformly relatively separated)**.** *Let f be a rational map whose Julia set J_f is a Sierpiński carpet. If the boundaries of all periodic Fatou components contain no points in $\omega(c)$ for any critical point $c \in J_f$, then the boundaries of Fatou components are uniformly relatively separated.*

Proof. After iterating f by some times, we can assume that all the periodic Fatou components $X_1, \cdots, X_n$ have period precise one. For $1 \leq i \leq n$, let Y_i be a simply connected domain containing X_i such that $Y_1, \cdots, Y_n$ are mutually disjoint and each annulus $H_i := Y_i \setminus \overline{X}_i$ contains no points in the critical orbits. Define $m = \min_{1 \leq i \leq n} \mathrm{mod}(H_i) > 0$.

For any two different Fatou components U_1 and U_2, there exist two minimal numbers $n_1, n_2 \geq 0$ such that $f^{\circ n_1}(U_1) = X_{k_1}$ and $f^{\circ n_2}(U_2) = X_{k_2}$ for $1 \leq k_1, k_2 \leq n$, where X_{k_1}, X_{k_2} are Fatou components with period one. Since H_{k_i} contains no critical values of $f^{\circ n_i}$ for $i \in \{1, 2\}$ and so the restriction of $f^{\circ n_i}$ on each components of $f^{-n_i}(H_{k_i})$ is an unbranched covering. By Riemann-Hurwitz's formula, it follows that each component of their preimages is an annulus. Therefore, there exist two simply connected domains V_1, V_2 surrounding U_1, U_2 such that $V_i \setminus \overline{U}_i$ is a component of $f^{-n_i}(H_{n_i})$ and $\deg(f^{\circ n_i} : V_i \to Y_{n_i}) = \deg(f^{\circ n_i} : U_i \to X_{n_i})$. Note that $f^{\circ j_1}(U_i) \bigcap f^{\circ j_2}(U_i) = \emptyset$ for $0 \leq j_1 < j_2 \leq n_i$ and f has only finitely many critical points. So the degree of $f^{\circ n_i}|_{U_i}$ is bounded by some number $N \geq 1$ depending only on f. Denote by A the annulus bounded by ∂U_1 and ∂U_2 in $\overline{\mathbb{C}}$. We now divide the arguments into two cases.

Case 1: Suppose that $n_1 = n_2$. Then V_1 and V_2 are two disjoint components of $f^{-n_1}(Y_{n_1} \cup Y_{n_2})$. By Lemma 2.4, we have

$$\mathrm{mod}(A) \geq \mathrm{mod}(V_1 \setminus \overline{U}_1) + \mathrm{mod}(V_2 \setminus \overline{U}_2) \geq \mathrm{mod}(H_{k_1})/N + \mathrm{mod}(H_{k_2})/N \geq 2m/N.$$

Case 2: Suppose that $n_1 > n_2$. We claim that V_1 and U_2 are disjoint. Otherwise, the annulus $V_1 \setminus \overline{U}_1$ intersects U_2 and so $f^{\circ n_2}(V_1 \setminus \overline{U}_1)$ intersects the fixed Fatou component X_{k_2}. Then $H_{k_1} = f^{\circ(n_1 - n_2)}(f^{\circ n_2}(V_1 \setminus \overline{U}_1))$ joints with X_{k_2}, which contradicts with the choice of H_{k_1}. Then we have

$$\mathrm{mod}(A) \geq \mathrm{mod}(V_1 \setminus \overline{U}_1) \geq m/N.$$

Above all, the annulus A has modulus not less than m/N. By Lemma 2.3, U_1 and U_2 are relatively separated with the relative distance $\Delta(\partial U_1, \partial U_2)$ depending only on m and N. By the arbitrariness of U_1 and U_2, the peripheral circles of the carpet Julia set are uniformly relatively separated. The proof is completed. □

Note that the condition in Proposition 2.3 does not exclude the existence of parabolic points on the Julia set. Actually, the peripheral circles of the parabolic rational map appeared in Figure 2.1 are uniformly relatively separated.

2.3.4 The property of non-uniformly relatively separated.

If the peripheral circles of a carpet Julia set are uniformly relatively separated, a natural question is whether it implies that all the boundaries of pre-periodic Fatou components avoid the accumulation points of the critical orbits in the Julia set. We give the answer in the following proposition.

Proposition 2.4 (Non-uniformly relatively separated). *Let f be a semi-hyperbolic rational map whose Julia set is a Sierpiński carpet. Suppose that there exists a Fatou component U of f such that $\partial U \cap \omega(c) \neq \emptyset$ for some critical point $c \in J_f$. Then the boundaries of Fatou components of f are not uniformly relatively separated.*

Proof. Without loss of generality, we suppose that $\infty \notin J_f$. Let $\epsilon = \delta_0 > 0$ be the number defined as in (2.13). Let $x \in \partial U \cap \omega(c)$. By Theorem 2.5 and Lemma 2.6, there exists a number $\delta_x > 0$ such that the open neighborhood $U_x := \mathbb{D}(x, \delta_x)$ satisfies (C1), (C2) and (C3). Since $\partial U \cap \omega(c) \neq \emptyset$ and J_f is a Sierpiński carpet, it follows that the forward orbit of c is infinite. Let $c_{k_n} := f^{\circ k_n}(c)$ be the point in the forward orbit of c converging to x. Set $\epsilon_{k_n} := |x - c_{k_n}|$. We have $\epsilon_{k_n} \to 0$ as $n \to \infty$. Given $0 < \delta < \delta_x$, there exists sufficiently large N such that $\mathbb{D}(c_{k_n}, \delta/2) \subseteq \mathbb{D}(c_{k_n}, \delta) \subseteq \mathbb{D}(x, \delta_x)$ for any $n \geq N$.

Evidently, the round disks $\mathbb{D}(c_{k_n}, \epsilon_{k_n})$, $\mathbb{D}(c_{k_n}, \delta/2)$ and $\mathbb{D}(c_{k_n}, \delta)$ satisfy (C1), (C2) and (C3) in Theorem 2.5 and Lemma 2.6. Pulling these three disks back by $f^{\circ(k_n-1)}$ and $f^{\circ k_n}$ respectively, we denote by $X_{k_n-1}, Y_{k_n-1}, Z_{k_n-1}$, respectively, X_{k_n}, Y_{k_n}, Z_{k_n} the simply connected components of their preimages containing the critical value c_1 and the critical point c respectively. Let U_{k_n-1} be a component of $f^{-(k_n-1)}(U)$ such that $\partial X_{k_n-1} \cap \partial U_{k_n-1} \neq \emptyset$. Then we can choose a point $x_{k_n-1} \in \partial X_{k_n-1} \cap \partial U_{k_n-1}$. Note that such x_{k_n-1} may be not unique. See Figure 2.3.

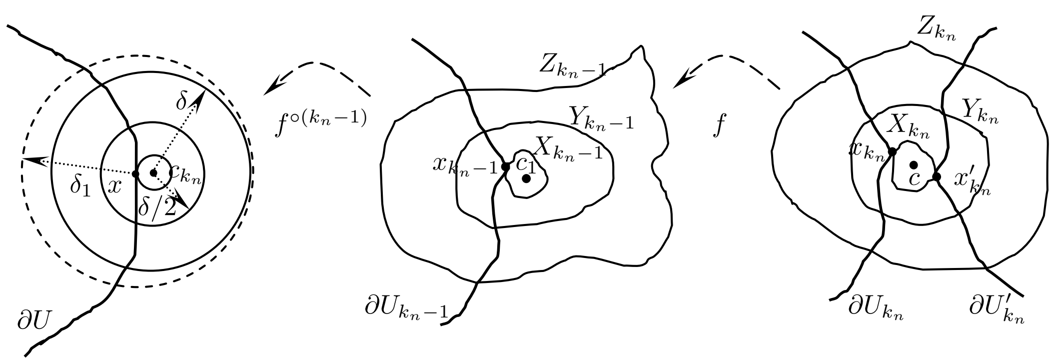

Figure 2.3: Sketch illustration of the mapping relation.

Since c_1 is a critical value, there exist at least two different Fatou components U_{k_n} and U'_{k_n}, which are both the preimages of U_{k_n-1} such that $\overline{X}_{k_n} \cap \overline{U}_{k_n} \neq \emptyset$ and $\overline{X}_{k_n} \cap \overline{U'}_{k_n} \neq \emptyset$. Let $x_{k_n} \in \partial X_{k_n} \cap \partial U_{k_n}$ and $x'_{k_n} \in \partial X_{k_n} \cap \partial U'_{k_n}$ be the preimages of x_{k_n-1}. We will show the relative distance between U_{k_n} and U'_{k_n} converges to zero as ϵ_{k_n} converges to zero.

Applying the Lemma 2.5 (1) to the case $(U_1, V_1) = (Y_{k_n}, Z_{k_n})$, $(U_2, V_2) = (\mathbb{D}(c_{k_n}, \delta/2), \mathbb{D}(c_{k_n}, \delta))$ and $g = f^{\circ k_n}$, we know that there exists a constant $C_1(d) > 0$ such that

$$\mathrm{Shape}(Y_{k_n}, c) = \frac{\max_{w \in \partial Y_{k_n}} |w - c|}{\mathrm{dist}(c, \partial Y_{k_n})} \leq C_1(d), \tag{2.20}$$

where d is the constant appeared in Theorem 2.5.

Similarly, there exists a constant $C_2(d) > 0$ such that

$$\mathrm{Shape}(X_{k_n}, c) = \frac{\max_{w \in \partial X_{k_n}} |w - c|}{\mathrm{dist}(c, \partial X_{k_n})} \leq C_2(d). \tag{2.21}$$

Now we estimate the relative distance of ∂U_{k_n} and $\partial U'_{k_n}$ by (2.20) and (2.21).

$$\begin{aligned} \Delta(\partial U_{k_n}, \partial U'_{k_n}) &= \frac{\mathrm{dist}(\partial U_{k_n}, \partial U'_{k_n})}{\min\{\mathrm{diam}(\partial U_{k_n}), \mathrm{diam}(\partial U'_{k_n})\}} \\ \leq\ & \frac{|x_{k_n} - x'_{k_n}|}{\mathrm{dist}(c, \partial Y_{k_n}) - \max_{w \in \partial X_{k_n}} |w - c|} \leq \frac{2 \max_{w \in \partial X_{k_n}} |w - c|}{\mathrm{dist}(c, \partial Y_{k_n}) - \max_{w \in \partial X_{k_n}} |w - c|} \\ \leq\ & \frac{2C_2(d)\mathrm{dist}(c, \partial X_{k_n})}{C_1^{-1}(d) \max_{w \in \partial Y_{k_n}} |w - c| - C_2(d)\mathrm{dist}(c, \partial X_{k_n})} = \frac{2C_2(d)}{\frac{\max_{w \in \partial Y_{k_n}} |w-c|}{C_1(d)\,\mathrm{dist}(c, \partial X_{k_n})} - C_2(d)}. \end{aligned} \tag{2.22}$$

On the other hand, by Lemma 2.4, the modulus of $Y_{k_n} \setminus \overline{X}_{k_n}$ satisfies

$$\begin{aligned} & \frac{1}{2\pi} \log \frac{\max_{w \in \partial Y_{k_n}} |w - c|}{\mathrm{dist}(c, \partial X_{k_n})} \geq \mathrm{mod}(Y_{k_n} \setminus \overline{X}_{k_n}) \\ \geq\ & \frac{1}{d} \mathrm{mod}(\mathbb{D}(c_{k_n}, \delta/2) \setminus \overline{\mathbb{D}(c_{k_n}, \epsilon_{k_n})}) = \frac{1}{2\pi d} \log \frac{\delta}{2\epsilon_{k_n}}. \end{aligned} \tag{2.23}$$

Note that $\epsilon_{k_n} \to 0$ as $n \to 0$, it follows that the relative distance $\Delta(\partial U_{k_n}, \partial U'_{k_n})$ of ∂U_{k_n} and $\partial U'_{k_n}$ tends to zero as n tends to ∞ by (2.22) and (2.23). This means that the peripheral circles of J_f are not uniformly relatively separated. The proof is completed. □

2.3.5 Proofs of the main results.

We now give the proofs of the main results in the introduction by combining some propositions.

Proof of Theorem 2.1. By Propositions 2.1 and 2.3, the peripheral circles of carpet J_f are uniform quasicircles and uniformly relatively separated. According to Bonk [Bon, Corollary 1.2], J_f is quasisymmetrically equivalent to a round carpet. □

Proof of Theorem 2.2. The theorem follows immediately by Propositions 2.2, 2.3 and 2.4. □

Proof of Theorem 2.3. By Theorem 2.1, let $g : J_f \to S$ be a quasisymmetric map sending

J_f to a round carpet S. According to [Bon, Theorem 1.1], one can extend $g : J_f \to S$ to a quasiconformal map from $\overline{\mathbb{C}}$ to itself. Since f is semi-hyperbolic, the corresponding Julia set J_f has measure zero by [Yin, Theorem 1.3]. It is well known that quasiconformal maps on the plane preserve the measure zero. So the round carpet S has measure zero as well. By the rigidity of Schottky sets (see [BKM, Theorem 1.1]), the quasisymmetric group $QS(S)$ consists of the restriction of Möbius transformations.

Note that g induces a group isomorphism

$$g_* : QS(J_f) \to QS(S) \quad \text{with} \quad g_*(h) = g \circ h \circ g^{-1}.$$

We are left to show that $QS(J_f)$ is *discrete*, i.e., there exists $\delta > 0$ such that

$$\inf_{h \in QS(J_f) \setminus \{\mathrm{id}_{J_f}\}} \big(\max_{z \in J_f} |h(x) - x|\big) \geq \delta.$$

If not, there exists a pairwise distinct sequence $\{h_k\}_{k \geq 1} \subseteq QS(J_f)$ converging to id_{J_f}. Let C_1, C_2 and C_3 be three different peripheral circles of J_f. Then the Hausdorff distance between C_i and $h_k(C_i)$ tends to zero as k tends to ∞. Since all $h_k(C_i)$ are either disjoint or coincides for $k \geq 1$ and $i \in \{1, 2, 3\}$, it follows that $h_k(C_i) = C_i$ for sufficiently large k. This means, for sufficiently large k, the Möbius transformation $g_*(h_k)$ fixes three disjoint round disks bounded by $g(C_i)$, $1 \leq i \leq 3$.

By the rigidity of Möbius transformation, these $g_*(h_k)$ must be the identity id_S. It follows that $h_k = \mathrm{id}_{J_f}$, for sufficiently large k. This contradicts the choice of $\{h_k\}_{k \geq 1}$. The discreteness of $QS(J_f)$ is proved. □

2.4 An example of postcritically-infinite carpet Julia set

In this section, we will construct a carpet Julia set of a rational map such that it is quasisymmetrically equivalent to a round carpet. However, the rational map f is semi-hyperbolic and has an infinite critical orbit in J_f.

Let $q : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the *doubling map* defined by $q(t) = 2t \bmod \mathbb{Z}$ and

$$l(t) := \begin{cases} 2t & \text{if } 0 \leq t < 1/2, \\ 2 - 2t & \text{if } 1/2 \leq t < 1. \end{cases} \tag{2.24}$$

be the length of the component $(\mathbb{R}/\mathbb{Z}) \setminus \{t, 1-t\}$ containing 0. Let $T(t) = \min\{2t, 2-2t\}$ be the *tent map* on the interval $[0, 1]$. One can easily check that

$$T \circ l(t) = l \circ q(t) \tag{2.25}$$

for all $t \in [0,1]$. Actually, the map $l(t)$ is equal to $T(t)$. We use these notations here by following Tiozzo's paper [Ti, p. 24].

Lemma 2.10. *Let $0 \leq \alpha \leq 1$ be a real number. Then α is rational if and only if α is (pre-)periodic under the iteration of the doubling map q.*

Proof. Obviously, this lemma holds for $\alpha = 0$ or 1. Hence we assume that $0 < \alpha < 1$. If α is (pre-)periodic, then there exist two different integers $k_1, k_2 \geq 0$ such that $q^{\circ k_1}(\alpha) \equiv q^{\circ k_2}(\alpha) \bmod \mathbb{Z}$. This means that there exists an integer k_3 such that $2^{k_1}\alpha = 2^{k_2}\alpha + k_3$. Then $\alpha = k_3/(2^{k_1} - 2^{k_2})$ is a rational number.

Conversely, we only need to prove that, if $\alpha = m/n$ is a rational number with the simplest expression, where n is odd, then α is periodic under q. Consider the restriction of q on the set $S := \{0, 1/n, \cdots, (n-1)/n\}$:

$$h := q|_S \,:\, \frac{t}{n} \mapsto \frac{2\,t \bmod n}{n}.$$

We claim that h is injective. Indeed, if $h(t_1/n) = h(t_2/n)$, then $2(t_1 - t_2) = k\,n$ holds for some integer k. Since n is odd, it follows that k is even and $|t_1 - t_2| = |\frac{k}{2}| \cdot n \leq n-1$. This means that $k = 0$ and $t_1 = t_2$. The finiteness of the cardinal number of S implies every element in S is pre-periodic under h. Then each element in S is periodic. Otherwise, there will be at least two elements which are mapped to a same element. This contradicts with that h is a injection. The proof is complete. □

In the following, based on the combinatorial theory of quadratic polynomials and renormalization theory, we shall construct a semi-hyperbolic McMullen map whose Julia set is quasisymmetrically equivalent to a round carpet.

Theorem 2.6. *There exists a suitable parameter $\lambda > 0$ such that the McMullen map*

$$f_\lambda(z) = z^d + \lambda/z^d \tag{2.26}$$

is semi-hyperbolic and the corresponding Julia set is quasisymmetrically equivalent to a round Sierpiński carpet, where $d \geq 3$.

Proof. We divide the construction into three main steps as following.

Step 1. For a given irrational number $\alpha \in (0,1)$, one can write it as an infinite binary sequence $\alpha = 0.a_1a_2a_3\cdots$ by Lemma 2.10, where $a_i \in \{0,1\}$. Define a binary number

$$\theta := 0.0\underbrace{1\cdots1}_{100}\underbrace{0\cdots0}_{b_1}\underbrace{1\cdots1}_{b_2}\underbrace{0\cdots0}_{b_3}\cdots$$

with $b_i = a_i + 1$ for $i \geq 1$. Then $1 \leq b_i \leq 2$ and we have:

• The number $\theta \in (0,1/2)$ is irrational. If not, by Lemma 2.10, the number θ will be eventually periodic under the iteration of the doubling map q. Then there exist $m \geq 2$ and $p \geq 2$ such that

$$q^{\circ n}(\theta) = 0.\overline{\underbrace{1\cdots 1}_{b_m}\underbrace{0\cdots 0}_{b_{m+1}}\cdots\underbrace{0\cdots 0}_{b_{m+p-1}}},$$

where $n = 101+b_1+\cdots+b_{m-1}$. This means that the sequence $b_m, b_{m+1}, \cdots, b_{m+p-1}, b_{m+p}, \cdots$ is periodic with period p. Therefore, the sequence $a_m, a_{m+1}, \cdots, a_{m+p-1}, a_{m+p}, \cdots$ is also periodic with period p since $a_i = b_i - 1$ for each i. Then $\alpha = 0.a_1 \cdots a_{m-1}\overline{a_m a_{m+1} \cdots a_{m+p-1}}$ is a rational number by Lemma 2.10. This is a contradiction since α is irrational.

• Define a rational number with the binary form

$$\theta' = 0.\overline{0\underbrace{1\cdots 1}_{99}}.$$

Then $0 < \theta' < \theta < 1/2$ and θ', θ are very close to $1/2$. We have

$$l(\theta') = 0.\underbrace{1\cdots 1}_{99}\overline{0\underbrace{1\cdots 1}_{99}} \text{ and } l(\theta) = 0.\underbrace{1\cdots 1}_{100}\underbrace{0\cdots 0}_{b_1}\underbrace{1\cdots 1}_{b_2}\underbrace{0\cdots 0}_{b_3}\cdots.$$

For any $n \geq 2$, one can easily check that

$$0 < l(q(\theta')) < l(q^{\circ n}(\theta)), l(q^{\circ n}(\theta')) < l(\theta') < l(\theta). \tag{2.27}$$

• Define a set

$$\mathcal{R} := \{t \in \mathbb{R}/\mathbb{Z} : T^{\circ n}(l(t)) \leq l(t) \text{ for all } n \geq 0\}. \tag{2.28}$$

By (2.25) and (2.27), we have θ', $\theta \in \mathcal{R}$.

Step 2. Construct a quadratic polynomial $P_c(z) = z^2+c$ with the following properties:

(1) The critical orbit $O^+_{P_c}(0) = \{P_c^{\circ n}(0) : n \geq 0\}$ is contained in the Julia set of P_c and the cardinal number of $O^+_{P_c}(0)$ is infinite.

(2) The critical point 0 is non-recurrent and the ω-limit set of 0 does not contain the β *fixed point*. Recall that a β fixed point of a polynomial is the landing point of *dynamical external ray* with angle zero.

The set $\mathcal{R}$ defined in (2.28) is exactly the set of all angles of parameter rays whose *prime-end impression* intersects the subset $\mathbb{R} \cap M = [-2, 1/4]$ of the Mandelbrot set M (see [Ti, Proposition 8.4]). By [Za, Theorem 3.3], there exists a real number $c := c(\theta) \in [-2, 1/4]$ in the boundary of the Mandelbrot set such that c is contained in the prime-end impression of the parameter rays $R_M(\pm\theta)$ since $\theta \in \mathcal{R}$. Moreover, on the dynamical plane, the dynamical rays $R_c(\pm\theta)$ land at the critical value c of $P_c(z) = z^2 + c$.

In fact, such c is unique. Otherwise, suppose that there exists another $c' \neq c$, such that c' is contained in the prime-end impression of the parameter rays $R_M(\pm\theta)$. By the density of hyperbolic parameters in $\mathbb{R}\cap M$ (see [GS] and [Ly]), there is a real hyperbolic parameter $\widetilde{c}$ between c and c' with a pair of rational parameter rays landing at it. This means that c' and c cannot lie in the same prime-end impression of $R_M(\pm\theta)$ at the same time, which is a contradiction.

Now we prove that the quadratic polynomial P_c is the map what we want to find. Again by [Ti, Proposition 8.4], the parameter rays $R_M(\pm\theta')$ land at a parabolic parameter $c_0 \in \mathbb{R}$ since $\theta' \in \mathcal{R}$ is a rational number. These two rays together with their landing point $R_M(\theta') \cup R_M(-\theta') \cup \{c_0\}$ bounds a *wake* $W \ni \{-2\}$ with the following property: The quadratic map $P_\xi(z) = z^2 + \xi$ has a repelling periodic point with exactly two dynamical rays $R_\xi(\pm\theta')$ landing at if and only if $\xi \in W$ (see [Mi2, Theorem 1.2]). By the construction in Step 1, we have $0 < \theta' < \theta < 1/2$. Then $R_M(\pm\theta) \cup \{c\} \subset W$ and hence $R_c(\pm\theta')$ land at a repelling periodic point of P_c on the real line. Also, the image $R_c(\pm q(\theta'))$ of $R_c(\pm\theta')$ land at some point on the real line.

Denote by H the simply connected domain bounding by the four dynamical rays $R_c(\pm\theta')$ and $R_c(\pm q(\theta'))$. The two dynamical rays $R_c(\theta)$ and $R_c(0)$ are contained in different components of $\mathbb{C} \setminus \overline{H}$. Moreover, all the dynamical rays $R_c(\pm q^n(\theta))$, where $n \geq 2$, are contained in H by the definition of $\mathcal{R}$ and θ'. This means that the collection of their landing points $\bigcup_{n\geq 2} P_c^{\circ n}(c)$ are contained in H. Therefore, the critical value c (which is the landing point of $R_c(\pm\theta)$) and the β fixed point of P_c are not contained in the ω-limit set of the origin.

Step 3. Construct the semi-hyperbolic rational map f_λ whose Julia set is quasisymmetrically equivalent to a round carpet. Consider the McMullen map $f_\lambda(z) = z^d + \lambda/z^d$, where $\lambda \in \mathbb{C}\setminus\{0\}$ and $d \geq 3$. The *free* critical points of f_λ are $2d$-th unit roots of λ. They are either escaping to ∞ or have bounded orbits at the same time. The *non-escaping locus* of f_λ is defined as

$$\Lambda_d := \{\lambda \in \mathbb{C} \setminus \{0\} : \text{The free critical orbits of } f_\lambda \text{ are not attracted by } \infty\}.$$

See left picture in Figure 2.4 for the case when $d = 3$.

According to [Ste, Theorem 9], there exists exactly one copy $\mathcal{M}$ of the Mandelbrot set of *order* one in $\Lambda_d \cap \{\lambda \in \mathbb{C}^* : |\arg(\lambda)| < \pi/(d-1)\}$ (Note that there exists a semiconjugacy between f_λ and the rational map discussed in [Ste, Theorem 9]). The copy $\mathcal{M}$ is symmetric with respect to the positive real axis. Moreover, there exists a homeomorphism $\Phi : \mathcal{M} \to M$ such that, for every $\lambda \in \mathbb{R}^+ \cap \mathcal{M} = \mathbb{R}^+ \cap \Lambda_d$, there is a corresponding parameter $\Phi(\lambda) \in [-2, 1/4]$ and the Julia set J_{f_λ} contains an embedded set $\tilde{J}_{P_{\Phi(\lambda)}}$, which is homeomorphic to the Julia set of the quadratic polynomial $P_{\Phi(\lambda)}(z) = z^2 + \Phi(\lambda)$. Moreover, the restriction of f_λ in a neighborhood of $\tilde{J}_{P_{\Phi(\lambda)}}$ is quasiconformally conjugated to the restriction of $P_{\Phi(\lambda)}$ in a neighborhood of $J_{P_{\Phi(\lambda)}}$.

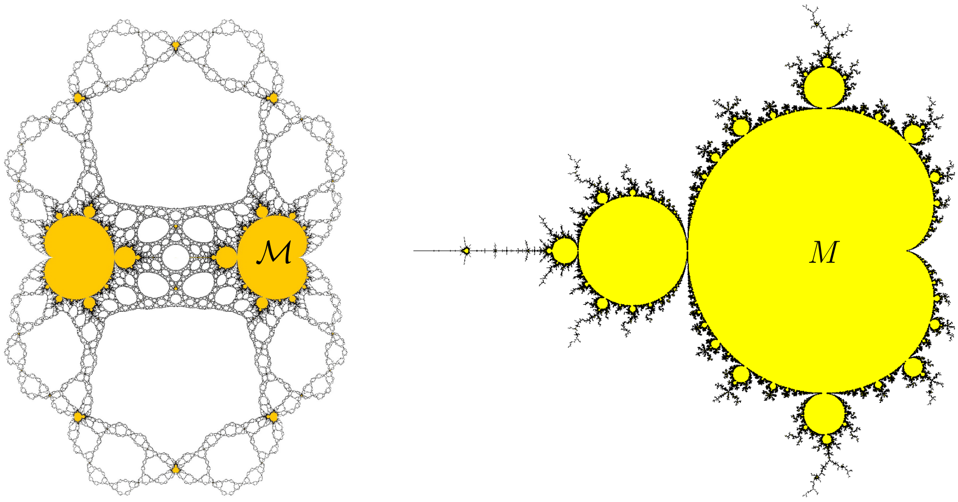

Figure 2.4: The non-escaping locus Λ_3 of f_λ (the left picture) contains infinitely many homeomorphic copies of the Mandelbrot set (the right picture).

Let $\lambda_0 = \Phi^{-1}(c) \in \mathcal{M} \cap \mathbb{R}^+$, where $c = c(\theta) \in (-2, 1/4)$ is the real parameter on the boundary of the Mandelbrot set determined in Step 2. By the symmetry of McMullen maps, all $2d$ free critical points of f_{λ_0} are non-recurrent and they have infinite forward orbits. This means that f_{λ_0} is semi-hyperbolic (and not sub-hyperbolic). Let B_∞ be the immediate attracting basin of ∞ of f_{λ_0}. Then $\tilde{J}_{P_{\Phi(\lambda_0)}} \cap B_\infty = \{z_{\lambda_0}\}$, where z_{λ_0} is the image of the β fixed point of J_{P_c} under the quasiconformal conjugacy stated above [QXY, Lemma 4.1]. Note that B_∞ is the unique periodic Fatou component of f_{λ_0}, it follows that the ω-limit sets of the critical points of f_{λ_0} are disjoint with the periodic Fatou component of f_{λ_0} by the construction of P_c.

By [QXY, Lemma 4.4], the Julia set of f_{λ_0} is a Sierpiński carpet. By Theorem 2.2, the peripheral circles of $J_{f_{\lambda_0}}$ are uniform quasicircles and uniformly relatively separated. By Bonk's criterion ([Bon, Corollary 1.2]), the Julia set of f_{λ_0} is quasisymmetrically equivalent to a round carpet, as required. $\square$

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Chapter 3

Criterion for rays landing together

3.1 Introduction

Let f be a polynomial with degree $d \geq 2$ in the complex plane $\mathbb{C}$. The *filled Julia set* is

$$K_f := \{z \in \mathbb{C} : \text{ The orbit } \{f^n(z)\}_{n\geq 0} \text{ is bounded }\}$$

and the *Julia set* is the topological boundary of the filled Julia set

$$J_f = \partial K_f.$$

Both of them are nonempty and compact, and the filled Julia set is *full*, i.e., the complement $\overline{\mathbb{C}} \setminus K_f$ is connected. We call $\Omega_f := \overline{\mathbb{C}} \setminus K_f$ the *basin of infinity* which consists of points with the orbit attracted by ∞. If J_f is connected. Then Ω_f is a simply connected and there exists an unique holomorphic parameterization $\Psi_f : \Omega_f \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that $\Psi_f(\infty) = \infty$, $\Psi'_f(\infty) = 1$ and

$$\Psi_f \circ f(z) = (\Psi_f(z))^d. \tag{3.1}$$

By the *external ray* $R(\theta)$ we mean the preimage of the radial line $\Psi_f^{-1}\{re^{2\pi i\theta} : r > 1\}$, where $\theta \in \mathbb{R}/\mathbb{Z}$ is the argument of the ray. We say that $R(\theta)$ *lands* at $z \in J_f$ if $\lim_{r\to 1}\Psi_f(re^{2\pi i\theta}) = z$. By the theorem of Carathéodory Ψ_f^{-1} extends continuous to $\partial\mathbb{D}$ with $\Psi_f^{-1}(\partial\mathbb{D}) = J_f$ if and only if J_f is locally connected.

Throughout this chapter we consider the case, J_f is locally connected. Define $\alpha : \mathbb{R}/\mathbb{Z} \to J_f$, $\theta \mapsto \alpha(\theta)$ where $\alpha(\theta)$ is the landing point of ray $R(\theta)$. By (3.1), we have the following semi-conjugation,

$$f(\alpha(\theta)) = \alpha(\sigma_d(\theta)), \tag{3.2}$$

where $\sigma_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with $\theta \mapsto d\,\theta \bmod \mathbb{Z}$. Thus, to study the topology of the Julia set and the dynamics of f on J_f is necessarily to figure out the semi-conjugation α.

There are two questions arising naturally,

(1) For any z in J_f, is the fiber $\alpha^{-1}(z)$ finite ? In other words, are there only finite rays landing at z ?

(2) Give a condition under which θ, θ' are in the same fiber. That is, when two external rays $R(\theta)$, $R(\theta')$ land at a same point ?

For the first question, if the orbit of z is finite, then the fiber $\alpha^{-1}(z)$ is finite [DH84]. If z is *wandering*, i.e., the orbit is infinite, J.Kiwi gave an upper bound $\#\alpha^{-1}(z) \leq 2^d$ [Ki02]. A.Blokh and G.Levin consider the more general problem: counting the number of external rays landing at distinct wandering points with disjoint forward orbits. Blokh and Levin worked the abstract modeling *invariant laminations* and introduced a new tool called *growing tree* [BL02]. In this chapter, inspired by [Ki02], we reprove the inequality in a totally different way.

Theorem 3.1. *Let $z_1, \cdots, z_m$ be wandering branched points such that their forward orbits avoid the critical points and are pairwise disjoint. Then*

$$\sum_{1 \leq i \leq m} (v(z_i) - 2) \leq d - 2.$$

In the above theorem, a point z is called to be a *branched* point if the fiber $\alpha^{-1}(z)$ contains at least three angles and the *valence* $v(z)$ is cardinal number of $\alpha^{-1}(z)$.

For the existence, W. Thurston proved that for quadratic polynomials there is no wandering branched points. He asked a deep question concerning their existence for higher degree in the preprint [Th85]. A.Blokh and L.Oversteegen answered the question by constructing an uncountable family of cubic polynomials, the Julia set of each one is a *dendrite* and containing wandering branched points [BO08].

For the second question, following [BFH92], [Po93] and [Ki05] etc, we need a concept: *critical portrait* associated to a polynomial f.

- For critical point c in J_f, $\Theta(c)$ is the set of arguments of external rays which land at c and are inverse images of one ray landing at critical value $f(c)$. Obviously, $\#\Theta(c)$ is $\deg_f(c)$, the local degree of f at c.
- For strictly pre-periodic critical Fatou component U, $\Theta(U)$ is a collection of $\deg(f|_U)$ arguments whose rays *support* U and are inverse images of one ray *supporting* $f(U)$.
- For Fatou component cycle $U_0, \cdots, U_{p-1}$ with $f^i(U_0) = U_i, U_p := U_0$, let $U_{k_0}, \cdots, U_{k_l}$ with $0 \leq k_0 < \cdots < k_l \leq p-1$ be critical with degree $n_0, \cdots, n_l$. For $0 \leq i \leq p$, choose (z_i, θ_i), $z_i \in \partial U_i$ and $R(\theta_i)$ *supporting* U_i at z_i, such that $f^i(z_0) = z_i$, $f^p(z_p) = z_p$ and $f^i(R(\theta_0)) = R(\theta_i)$. Then $\Theta(U_{k_j})$ is the set of arguments whose external rays land on ∂U_{k_j} and are preimages of $R(\theta_{k_j+1})$, for $0 \leq j \leq l$.

Let $\mathcal{A} := \{\Theta(c_1), \cdots, \Theta(c_m), \Theta(U_1), \cdots, \Theta(U_n)\}$. For any $\Theta \in \mathcal{A}$, set

$$\widehat{\Theta} := \bigcup \{\Theta' : \exists \text{ a chain } \Theta_0 = \Theta, \cdots, \Theta_k = \Theta' \text{ in } \mathcal{A} \text{ such that } \Theta_i \bigcap \Theta_{i+1} \neq \emptyset\}.$$

The collection $\widehat{A} := \{\widehat{\Theta}_1, \cdots, \widehat{\Theta}_N\}$ is called *critical portrait* associated to f. In the unit circle, there is a partition $\mathcal{P} := \{I_1, \cdots, I_d\}$ of $\mathbb{R}/\mathbb{Z} \setminus \bigcup_{1 \leq i \leq N} \widehat{\Theta}_i$. Each I_i is a finite union of open intervals with total length $1/d$.

Given a partition, we say x, x' *have the same itinerary* respect to the partition under a map g if and only if both $g^n(x)$ and $g^n(x')$ lie in the same piece of the partition, for any $n \geq 0$.

For polynomials with all critical points strictly preperiodic, B.Biefield, Y.Fisher and J.H.Hubbard showed that, if θ, θ' have the same sequence respect to the partition $\mathcal{P}$ then $\alpha(\theta) = \alpha(\theta')$ [BFH92]. A.Porier extends this result to crtitical finite polynomials, admitting periodic Fatou component [Po93]. Both of their proofs rely on the *orbifold metric* in Julia set, on which f is expanding. In [Ki05], Kiwi considered the polynomials with all cycle repelling and Julia set connected. Based on the properties of *maximal lamination*, he proved that if θ, θ' have the same sequence respect to $\mathcal{P}$, then the *impressions* of $R(\theta)$ and $R(\theta')$ intersect.

We prove the following theorem, which is the main result of this chapter.

Theorem 3.2 (Main Theorem). *Let f be a polynomial with J_f locally connected. Let $\mathcal{P}$ be the partition induced by critical portrait $\widehat{\mathcal{A}}$. If θ, θ' have the same itinerary respect to $\mathcal{P}$, then either $R(\theta)$,$R(\theta')$ land at the same point or $R(\theta)$, $R(\theta')$ land at the boundary of a Fatou component U, which is eventually iterated to a siegel disk.*

Note that S.Zakeri in [Za00] proved that for *Siegel quadratic polynomial* f, i.e., $f : z \to z^2 + c$ has a fixed Siegel disk, no points has more that two rays landing at and if two rays landing at z then z must eventually hit the critical point 0.

The following Corollary holds immediately.

Corollary 3.1 (No wandering continua in J_f). *Let f be a polynomial with J_f locally connected. Then there is no wandering continua in J_f.*

We have to point out that A.Blokh and G.Levin also proved the above corollary [BL02]. And J.Kiwi proved that, for polynomials without irrational neutral periodic orbits f, J_f is locally connected if and only if f has no wandering continua in J_f. Kiwi's proof relies on constructing a puzzle piece around each pre-periodic or periodic point of a polynomial f with all cycles repelling [Ki04].

3.1.1 Motivation

One of our motivation is to study the *core-entropy* of polynomials. Suppose X is a compact metric space and $g : X \to X$ is continuous. The topological entropy of g is measuring the complexity of iteration from the growth rate of the number of distinguishable orbits. The *core-entropy* of polynomial f is the topological entropy of f on its f-invaritant set *Hubbard tree*, i.e., the convex hull of the critical orbits within the (filled) Julia set. Let $Acc(f)$ be the set of all *biaccessible* angles θ, i.e., there exist at least two rays landing at $\alpha(\theta)$. Then the core-entropy $h(f)$ is related to the Hausdorff dimension of $Acc(f)$ in the following way,

$$h(f) = \log d \cdot \text{H.dim}\, Acc(f). \tag{3.3}$$

These quantities are according to W.Thurston who firstly introduced and explored the core-entropy of polynomials.

For quadratic polynomials, G.Tiozzo showed the continuity of core-entropy along *principal veins* of the Mandelbrot set $\mathcal{M}$ in [Ti13]. This result is generalized by W. Jung to all veins [Ju13]. Recently, G.Tiozzo proves that the function $\theta \mapsto h(f_\theta)$ with $f_\theta(z) = z^2 + c_\theta$ is continuous.

A.Douady proved the monotonicity of core-entropy along real vein $\mathcal{M} \bigcap \mathbb{R}$ [Do95]. The monotonicity for all postcritically finite quadratic polynomials is proved in Tao Li's thesis [Li07]. As an application of theorem 3.2, we extend Tao Li's result to a quadratic family $\mathcal{F} := \{f_c = z^2 + c : f_c$ has no Siegel disks and J_{f_c} is locally connected $\}$.

Theorem 3.3 (Monotonicity of core-entropy)**.** *For any $f_c, f_{c'} \in \mathcal{F}$, if $f_c \prec f_{c'}$, then $Acc(f_c) \subseteq Acc(f_{c'})$ and so $h(f_c) \leq h(f_{c'})$.*

For any f_c, $f_{c'}$ in $\mathcal{F}$, we say $f_c \prec f_{c'}$ if and only if $I_c \supseteq I_{c'}$, where I_c is the *characteristic arc* of f_c. See section 3.7 for details.

3.1.2 Sketch of the proof and outline of the chapter

The proof of main theorem 3.2 is based on the analysis in the dynamical plane. There is a partition $\{\Pi_i\}_{1\leq i\leq d}$ of $\mathbb{C}$, induced by critical portrait. It has nice properties: for any points $x, y \in \Pi_i \bigcap J_f$, the *regulated arc* $[x, y] \subseteq \overline{\Pi}_i$ and $F|_{[x,y]}$ is one-to-one, where F is a topological polynomial which takes the same value as f in $\overline{\Omega}_f$. Thus if $x \neq y$ have the same itinerary respect to $\{\Pi_i\}$, we obtain a sequence $\{F^n[x, y]\}$ of regulated arc. The sequence will eventually meet $\bigcup_{1\leq i\leq d} \partial\Pi_i \bigcap J_f$. However it is difficult to prove that the partition $\{\Pi_i\}_{1\leq i\leq d}$ separates $f^n(x), f^n(y)$ for some n. To overcome this difficult, we use this sequence to construct a wandering arc in J_f, which is a contradiction.

In section 3.2, we prove theorem 3.1. This key result is useful to show the fact of no wandering regulated arcs in Lemma 3.3.

In section 3.3, we give the construction of regulated arcs and describe its properties.

In section 3.4, we explain how to get a desired topological polynomial F by modifying f in Fatou set.

Section 3.5 analysis the properties of partition induced by critical portrait in the dynamic plane.

The main Theorem 3.2 is proved in section 3.6.

In the last section, we discuss characteristic arcs in details and give an application of the main theorem to the monotonicity of core entropy for a quadratic polynomial family.

3.2 Wandering Orbit Portrait

If not otherwise stated, we assume f to be a polynomial with degree $d \geq 2$ and J_f locally connected. Our objective is to prove the Proposition 3.1 in this section.

3.2.1 Portraits

Now we give some definitions by following [Mi00] [GM93] [BFH92] [Ki02] etc.

For a point z in J_f, the *valence* of z, written $v(z)$, is the number of external rays landing at z. Then $1 \leq v(z) \leq \infty$. If $v(z) \geq 3$, z is called to be a *branched* point. z is called to be *wandering* if $f^m(z) \neq f^n(z)$ for $m \neq n \geq 0$.

Let $T := \{\theta_1, \ldots, \theta_n\}$, $\theta_i \in \mathbb{R}/\mathbb{Z}$, $3 \leq n < \infty$. T is called to be a *portrait* of z if all $R(\theta_i)$ land at z. Denote by $\alpha(T) := z$ the base point and $v(T) := n$ the *valence* of T. Obviously, we have $3 \leq v(T) \leq v(z)$.

Let T be a portrait of z. Each connected components of $\mathbb{C} \setminus \bigcup_{\theta \in T} \overline{R}(\theta)$ is called a *sector* of T based at z. Evidently, any sector S of T is bounded by two rays $R(\theta_a), R(\theta_b)$ with $\theta_a, \theta_b \in T$. Let $I(S)$ be the segment of $\mathbb{R}/\mathbb{Z} \setminus \{\theta_a, \theta_b\}$ disjoint with T. Then there is a one-to-one correspondence betweens sectors based at z and the segments of $\mathbb{R}/\mathbb{Z} \setminus T$, characterized by the property that $R(t)$ is contained in S if and only if t is contained in $I(S)$. Denote the correspondence by $I : S \mapsto I(S)$.

We define the *annular size* of a sector S, written $l(S)$, by the length of the corresponding arc $I(S)$ in $\mathbb{R}/\mathbb{Z}$. Number the n sectors of T by $S_1(T), \cdots, S_n(T)$ according to their length:

$$l(S_1(T)) \leq l(S_2(T) \cdots \leq l(S_n(T)).$$

By means of *critical sector* or *critical value sector* if a sector S contains critical points or critical values.

Lemma 3.1 (For portraits with distinct base points). *Let T, T' be two portraits with $\alpha(T) \neq \alpha(T')$. Let S resp. S' be the sector of T resp. T' such that $\alpha(T')$ resp. $\alpha(T)$ is contained in S resp. S'. Then all but S' resp. S of the sectors of T' resp. S' are contained*

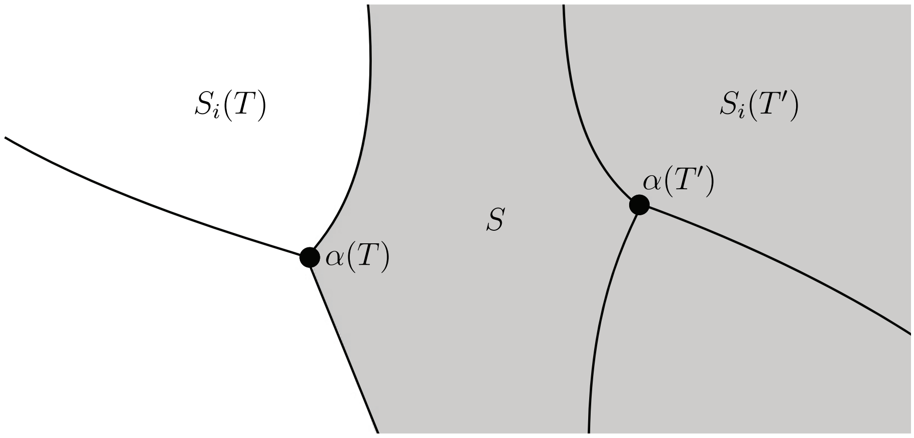

Figure 3.1: Portraits T, T' with distinct base points

in S resp. S' and so we have

$$l(S_i(T')) < l(S) \text{ for } S_i(T') \neq S' \quad \text{ and } \quad l(S_i(T))) < l(S') \text{ for } S_i(T) \neq S.$$

Proof. Set $G := \bigcup_{1 \leq i \leq v(T)} R(\theta_i) \cup \{\alpha(T)\}$ and $G' := \bigcup_{1 \leq i \leq v(T')} R(\theta_i') \cup \{\alpha(T')\}$. They are disjoint close connected subset of $\mathbb{C}$. So G' is contained in exactly one connected component of $\mathbb{C} \setminus G$, that is, some sector of T. Since $\alpha(T') \in S$ and $\alpha(T) \in S'$, we have $G' \subseteq S$ and $G \subseteq S'$. See figure 3.1. Thus all sectors of T' resp. T except S' resp. S are contained in S resp. S'. The lemma follows. $\square$

3.2.2 Sector maps

Lemma 3.2 (Properties of sector maps)**.** *Let $T = \{\theta_1, \cdots, \theta_{v(T)}\}$ be a portrait such that the base point $\alpha(T)$ is not a critical point of f, here θ_i are enumerated in cyclic order around the circle. Then*

(1) *The map $\sigma_d : t \mapsto dt \bmod \mathbb{Z}$ carries T bijectively onto the portrait $T' := \{\sigma_d(\theta_1), \cdots, \sigma_d(\theta_{v(T)})\}$ of $f(\alpha(T))$ preserving cyclic order. Define the* ***portrait map*** *to be*

$$\sigma_d : T \mapsto T'.$$

(2) *Let S be a sector of T bounded by $R(\theta_a)$ and $R(\theta_b)$. Then the* ***sector map***

$$\sigma_d : S \mapsto S',$$

where S' is the sector of T' bounded by $R(\sigma_d(\theta_a))$ and $R(\sigma_d(\theta_b))$, is well defined and one-to-one.

(3) *$l(\sigma_d(S)) = d\,l(S) \bmod \mathbb{Z}$. Moreover, the integer $n_0 := d\,l(S) - l(\sigma_d(S))$ is the number of critical points, counting multiplicity, of f contained in S.*

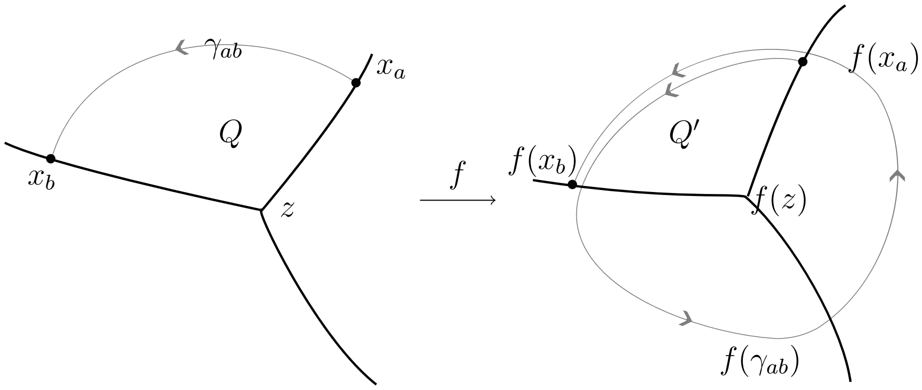

Figure 3.2: Sector maps

(4) *If* $n_0 \geq 1$*, then* $\sigma_d(S)$ *contains at least one critical values.*

(5) *If* $l(S) < 1/d$*, then* $l(\sigma_d(S)) = d\,l(S)$ *and the restriction of* f *on* S *is homeomorphic.*

Note that we distinguish the distinct definitions of σ_d by acting on different categories.

Proof. Let $z := \alpha(T)$. Since z is not critical. f is a locally orientation-preserving homeomorphism at z. Note that the $v(T)$ angles θ_i in $\mathbb{R}/\mathbb{Z}$ and rays $R(\theta_i)$ around z are identical in order. Moreover, the order of $R(\theta_i)$ can be measured within an arbitrarily small neighborhood of z. It follows that all rays with angels in T' land together at $f(\alpha(T)$ and σ_d sends angles in T onto T' bijectively and keeping the order. Thus (1) and (2) follows.

For (3), suppose S is bounded by $R(\theta_a)$, $R(\theta_b)$. Let $\gamma_{ab}(t)$ be a segment of equipotential curve $\{z \in \mathbb{C} : G_f(z) = 1\}$ which lies in $\overline{S}$ with $\gamma(0) = x_a$ and $\gamma(1) = x_b$, where $\{x_a\} := \gamma_{ab} \bigcap R(\theta_a)$ and $\{x_b\} := \gamma_{ab} \bigcap R(\theta_b)$. Let Q be the close domain bounded by $R(\theta_a), \gamma_{ab}$ and $R(\theta_b)$. See figure 3.2.

Consider the image $f(\partial Q)$. It starts at $f(z)$ and goes along the rays $R(\sigma(\theta_a))$ until it arrives at $f(x_a)$, then it rotates $d\ l(S)$ angles, parameterized by angles of external rays, along the equipotential curve $\{z \in \mathbb{C} : G_f(z) = d\}$ to $f(x_b)$, finally it turns to $f(z)$ along $R(\sigma(\theta_b))$ and stops.

Let $G_d := \{z \in \mathbb{C} : G_f(z) < d\}$. Let Q' be the domain $\sigma_d(S) \bigcap G_d$. By the arguments above, it is easy to see that $f(\gamma_{ab})$ surround ∂G_d in n_0 times and overlap $\partial G_d \bigcap \partial Q'$ one time more. Thus,

$$l(\sigma_d(S)) + n_0 = d\,l(S).$$

Moreover, $z \in \partial G_d \setminus \partial Q'$ has n_0 preimages in γ_{ab} and $z \in \partial G_d \bigcap \partial Q'$ has $n_0 + 1$ preimages in γ_{ab}. The winding number of points in $G_d \setminus f(\partial Q')$ are

$$w(z) = \begin{cases} n_0 + 1 & z \in Q' \\ n_0 & z \in G_d \setminus \overline{Q'}. \end{cases} \tag{3.4}$$

By the Arguments Principle, every point $z \in G_d \setminus f(\partial Q)$ has $w(z)$ preimages, counting multiplicity, in Q.

Now claim that every points z in $\partial Q' \setminus \partial G_d$, consisting of two segments of external rays, has $n_0 + 1$ preimages, counting multiplicity, in Q. Since such z can not be a critical value, choose sufficiently small enough neighborhood U_z such that the restriction of f on every component $f^{-1}U_z$ is homeomorphic. Since $U_z \bigcap Q'$ has n_0+1 components in Q and Q is closed, z must have $n_0 + 1$ preimages in Q as well.

Let $v_1, \cdots, v_n \in f(Q)$ be the critical value of $f|_Q$. Let μ_i be the total multiplicity of critical points in Q mapped to v_i. Choose a cell subdivision Δ of $f(Q)$ such that the set of its 0-cells contains $\{f(z), v_1, \cdots, v_n\}$ and the set of 1-cells contains $\partial Q'$. Let $\Delta_1 := \{\text{complexes of } \Delta \text{ contained in } \overline{Q'}\}$ and $\Delta_2 := \Delta \setminus \Delta_1$. It follows that Δ_1 is a cell subdivision of Q'. Set x_i, y_i, z_i to be the number of 0-cell, 1-cell and 2-cell of Δ_i. Computing the Euler characteristic, we have

$$\mathcal{X}(\overline{Q'}) = x_1 - y_1 + z_1 = +1 \tag{3.5}$$

and

$$\mathcal{X}(f(Q)) = (x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = +1. \tag{3.6}$$

After lifting every complexes in Δ by $f|_Q$, we obtain a cell subdivision Δ_0 of Q. Then

$$\begin{aligned}\mathcal{X}(Q) = & [(n_0+1)x_1 + n_0 x_2 - \sum_{1 \le i \le n} \mu_i] - [(n_0+1)y_1 + n_0 y_2] \\ & + [(n_0+1)z_1 + n_0 z_2] = +1.\end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we have

$$\sum_{1 \le i \le n} \mu_i = n_0.$$

Thus (3) is completed.

For (4), we use the notations as above. If not, assume Q' contains no critical values. Then every component of $f|_Q^{-1}(Q')$ is simply connected and f on the closure of which is homeomorphic. Consider the component C with $\partial Q \setminus \gamma_{ab} \subseteq \overline{C}$. $f|_{\overline{C}}$ cannot be one-to-one, a contradiction.

For (5), it follows directly by (3). □

3.2.3 Dynamics of wandering portraits

Portrait T is called to be *wandering* if and only if the point $\alpha(T)$ is wandering and not iterated to critical points of f. We denote by $T_n := \sigma_d^{\circ n}(T)$.

Recall that $S_1(T), \cdots, S_{v(T)}(T)$ are the $v(T)$ sectors of T enumerated by the order of their annular size. We have the following lemma. See also in [Ki02].

Lemma 3.3. *Let T be a wandering portrait. Then*

$$\lim_{n\to\infty} l(S_{v(T)-2}(T_n)) = 0.$$

Proof. If not, there exist a number $a > 0$ and an infinite sequence T_{n_k} such that

$$5a/6 < l(S_{v(T)-2}(T_{n_k}) < 7a/6.$$

The sectors $S_{v(T)-2}(T_{n_k})$ can not be pairwise disjoint. Because otherwise the total length of the infinite many intervals $I(S_{v(T)-2}(T_{n_k})$ would be greater than 1.

Then there exist $n_i \neq n_j$ such that $S_{v(T)-2}(T_{n_i}) \bigcap S_{v(T)-2}(T_{n_j}) \neq \emptyset$. By Lemma 3.1, we can assume $\alpha(T_{n_i}) \in S_{v(T)-2}(T_{n_j})$ and both sectors $S_{v(T)-2}(T_{n_i})$ and $S_{v(T)-1}(T_{n_i})$ are contained in $S_{v(T)-2}(T_{n_j})$. Thus,

$$l(S_{v(T)-2}(T_{n_j})) > l(S_{v(T)-2}(T_{n_i})) + l(S_{v(T)-1}(T_{n_j})) > 5a/3,$$

a contradiction. □

By lemma 3.3, for any wandering portrait T, the annular size of sectors T_n, except the two large ones, will converges to zero. Furthermore, a similar argument can show that $\liminf l(S_{v(T)-1}(T_n)) = 0$. We will not use this fact. We are more interested in the moment when a "wide" critical sector is mapped to a "narrow" critical value sector.

For any sufficiently small $\epsilon > 0$ and $1 \leq k \leq v(T) - 2$, Set

$$n_{\epsilon,k}(T) := \min\{n : l(S_k(T_n)) < \epsilon\}.$$

By lemma 3.3, $l(S_k(T_n))$ will eventually be smaller than ϵ as $n \to \infty$. Thus $n_{\epsilon,k}(T)$ is well defined. We have the following,

Lemma 3.4. *Let T be a wandering portrait. Then There exists $\delta > 0$ such that for any $\epsilon < \delta$, denote by $n_{\epsilon,k} := n_{\epsilon,k}(T)$, $1 \leq k \leq v(T) - 2$, we have $l(S_{k+1}(T_{n_{\epsilon,k}})) > \epsilon$ and there exists at least one critical value sector $S_{k_0}(T_{n_{\epsilon,k}})$ with $1 \leq k_0 \leq k$.*

Proof. By lemma 3.3, there exists an integer $N \geq 1$ such that, for any $n \geq N$,

$$(S_{v(T)-2}(T_n)) < \frac{1}{2v(T)d}.$$

Set

$$\delta := \min_{1\leq i\leq N}\{\, l(S_1(T_i))\,\}.$$

For any $\epsilon < \delta$, since $n_{\epsilon,k}$ is the first time that the k^{th} sector has length strictly less than ϵ. We have

$$\epsilon \leq l(S_k(T_{n_{\epsilon,k}-1})) \leq l(S_{v(T)-2}(T_{n_{\epsilon,k}-1})) < \frac{1}{2v(T)d}.$$

By Lemma 3.2 (5), f maps the $v(T)-2$ sectors $S_1(T_{n_{\epsilon,k}-1}), \cdots, S_{v(T)-2}(T_{n_{\epsilon,k}-1})$ onto sectors of $T_{n_{\epsilon,k}}$ homeomorphic with their length multiplied by d. Then

$$l(\sigma_d(S_k(T_{n_{\epsilon,k}-1}))) \geq d\epsilon > \epsilon \quad \text{and} \quad l(S_k(T_{n_{\epsilon,k}})) < \epsilon.$$

This means σ_d must map at least one of the two sectors $S_{v(T)-1}(T_{n_{\epsilon,k}-1})$ and $S_{v(T)}(T_{n_{\epsilon,k}-1})$ onto a "narrow" sector $S_{k_0}(T_{n_{\epsilon,k}})$ with $l(S_{k_0}(T_{n_{\epsilon,k}})) < \epsilon$. By lemma 3.2 (4), $S_{k_0}(T_{n_{\epsilon,k}})$ is a critical value sector. Actually, there are only one of the above two sectors mapped to such "narrow" sector. Because the total length of the $v(T)-1$ images,

$$l(S_{k_0}(T_{n_{\epsilon,k}})) + \sum_{1 \leq i \leq v(T)-2} l(\sigma_d(S_i(T_{n_{\epsilon,k}-1}))) < \frac{1}{2}.$$

It follows that the other sector is mapped to the widest sector $S_{v(T)}(T_{\epsilon,k})$ with length $> \frac{1}{2}$. Thus, we have

$$S_{k+1}(T_{n_{\epsilon,k}}) = \sigma_d(S_k(T_{n_{\epsilon,k}-1})) \geq d\epsilon > \epsilon \quad \text{and} \quad 1 \leq k_0 \leq k.$$

The proof is completed. $\square$

3.2.4 Proof of theorem 3.1

Proposition 3.1. *Let $T^{(1)}, \cdots, T^{(m)}$ be wandering portraits such that $\alpha(T^{(i)})$ have disjoint forward orbits. Then*

$$\sum_{1 \leq i \leq m} (v(T^{(i)}) - 2) \leq d - 2. \tag{3.8}$$

Proof. Let $\epsilon_0 > 0$ be smaller than any $\delta_{T^{(i)}}$, for $1 \leq i \leq m$, as stated in the Lemma 3.4. Firstly, applying Lemma 3.4 to the case $T = T^{(1)}$, $k = 1$ and $\epsilon = \epsilon_0$, we obtain a critical value sector $S_1(T^{(1)}_{n_{\epsilon_0,1}})$ and

$$\epsilon := l(S_1(T^{(1)}_{n_{\epsilon_0,1}})) < \epsilon_0 < l(S_2(T^{(1)}_{n_{\epsilon_0,1}})). \tag{3.9}$$

Let $n_{k,i} := n_{\epsilon,k}(T^{(i)})$, for $1 \leq i \leq m$, $1 \leq k \leq v(T^{(i)})$. By the definition of $n_{k,i}$ and orbits of $\alpha(T^{(i)})$ disjoint in the condition, it is easy to see that

$$n_{k_1,i} \neq n_{k_2,j} \neq n_{\epsilon_0,1} \quad \text{and} \quad \alpha(T^{(i)}_{n_{i,k_1}}) \neq \alpha(T^{(j)}_{n_{j,k_2}}) \neq \alpha(T^{(1)}_{n_{\epsilon_0,1}}), \tag{3.10}$$

for $1 \leq i, j \leq m$ and $(i, k_1) \neq (j, k_2)$, $1 \leq k_1 \leq v(T^{(i)}), 1 \leq k_2 \leq v(T^{(j)})$. By Lemma 3.4

again, we obtain $N := \sum_{1\le i\le m}(v(T^{(i)})-2)$ critical value sectors, denoted by $S_{\tau(k,i)}(T^{(i)}_{n_{k,i}})$, and we have

$$l(S_{\tau(k,i)}(T^{(i)}_{n_{k,i}})) < \epsilon < l(S_{k+1}(T^{(i)}_{n_{k,i}})), \qquad 1 \le \tau(k,i) \le k. \tag{3.11}$$

By (3.10) and Lemma 3.1, for any distinct two of the $N+1$ critical value sectors $S_1(T^{(1)}_{n_{\epsilon_0},1})$ and $S_{\tau(k,i)}(T^{(i)}_{n_{k,i}})$, they are neither disjoint or one contains the other.

We claim that the latter case can not happen. If not, suppose $S_{\tau(k_1,i_1)}(T^{(i_1)}_{n_{k_1,i_1}})$ are contained in $S_{\tau(k_2,i_2)}(T^{(i_2)}_{n_{k_2,i_2}})$. By Lemma 3.1, we have

$$S_{k_1+1}(T^{(i_1)}_{n_{k_1,i_1}}) \subset S_{\tau(k_2,i_2)}(T^{(i_2)}_{n_{k_2,i_2}}) \quad \text{and} \quad l(S_{k_1+1}(T^{(i_1)}_{n_{k_1,i_1}})) < l(S_{\tau(k_2,i_2)}(T^{(i_2)}_{n_{k_2,i_2}})).$$

This contradicts (3.11). If one of them is $S_1(T^{(1)}_{n_{\epsilon_0},1})$, similarly by (3.9), it is impossible.

Thus the $N+1$ critical values sectors are pairwise disjoint and each of them contains at least one critical value. Since it is known that, for degree d polynomials, there exist at most $d-1$ critical values. So $N+1 \le d-1$. The proof is completed. $\square$

Proof of Theorem 3.1. The theorem follows immediately by Propositions 3.1. $\square$

Actually the result in this section can extended to polynomials with Julia set connected or not connected. We omit the details. See Appendix A in [Ki02].

Corollary 3.2. *Let f be a polynomial with the Julia set J_f locally connected. Then the number of grand orbits of wandering branched points is finite.*

3.3 Regulated arcs

According to Fatou and Sullivan, every bounded Fatou components of polynomials must eventually be mapped to the immediate basin of attraction of an attracting periodic point, or to an attracting petal of a parabolic periodic point, or to a periodic Siegel disk [Mi06] [Su83]. We refer to these cases simply as *hyperbolic*, *parabolic* and *Siegel* cases.

For any two points $x, y \in K_f$ there usually exist more than one arc γ in K_f connecting x and y. In the following, we will give the definition of *internal ray* and *regulated arc* in K_f and show how to choose a canonical embedded arc between any two points in the filled Julia set. Under certain condition, such arc is unique (See Lemma 3.7).

3.3.1 Extended rays

Now consider the polynomial f with J_f locally connected. We have,

Lemma 3.5 (Bounded Fatou components are Jordan domains)**.** *For any bounded Fatou component U, ∂U is a Jordan curve.*

Proof. Since J_f is locally connected, then ∂U is locally connected. Consider the Riemann map: $\Phi_U : \mathbb{D} \to U$, it extends continuously to $\mathbb{D}$ by Carathéodory Theorem. Therefore, ∂U is the curve $\Phi_U(S^1)$. If $\Phi_U|S^1$ is not injective. Then there exists $t < t'$ in S^1 with $\Phi_U(t) = \Phi_U(t')$. The two rays $\Phi_U([0,1]e^{2\pi it})$ and $\Phi_U([0,1]e^{2\pi it'})$ will bound a domain U', which contains subset of the Julia set $\Phi_U(\{e^{2\pi i\eta} : t < \eta < t'\}$. Since J_f is the boundary of infinity attracting domain Ω_f, some points in U' will escape to infinity. This contradicts the Maximum Value Principle. □

Given any bounded Fatou component U, pick a point $c(U)$ in U as *center point* and a Riemann map $\varphi_U : U \to \mathbb{D}$ with $\varphi_U(c(U)) = 0$. Then extend it to a homeomorphism $\varphi_U : \overline{U} \to \overline{\mathbb{D}}$ by Carathédory Theorem.

An arc in $\overline{U}$ of the form $\varphi_U^{-1}\{re^{i\theta} : 0 \leq r \leq 1\}$ is called a *internal ray* of U with angle θ. All these internal rays meet at the center point $c(U)$. Each ray has a well defined landing point in the boundary of U. Conversely, for any point z in the boundary of U, there exists an unique internal ray of U landing at z. We denote this internal ray by $R_U(z)$. For any $\theta \in \mathbb{R}/\mathbb{Z}$, if $\alpha(\theta) = z \in \partial U$, define the *extended ray*

$$\widehat{R}_U(\theta) := R(\theta) \bigcup R_U(z).$$

3.3.2 Components of $J_f \setminus \{x\}$ are arcwise connected

Recall that a topological space X is said to be *arcwise connected* provided that there is a topological embedding of $[0,1]$ into X (called *arc*) joining any two given distinct points. If $p \in X$, then X is said to be *locally arcwise connected* resp. *locally connected* at p, provided that every neighborhood of p contains an arcwise connected neighborhood resp. connected neighborhood of p. The space X is said to be *locally arcwise connected* resp. *locally connected*, provided that X is locally arcwise connected resp. locally connected at every point. We have the following well-know result.

Lemma 3.6. *If a compact metric space X is locally connected, then it is locally arcwise connected.*

It follows directly by the Lemma 17.17 and Lemma 17.18 in [Mi06].

Corollary 3.3. *If a compact metric space X is connected and locally connected, then it is arcwise connected. Moreover, every connected component of $X \setminus \{x\}$ is arcwise connected for any x in X.*

Proof. Fix $p \in X$, define Y as follows

$$Y = \{p\} \bigcup \{x \in X : \text{there is an arc in } X \text{ joining } p \text{ and } x\}$$

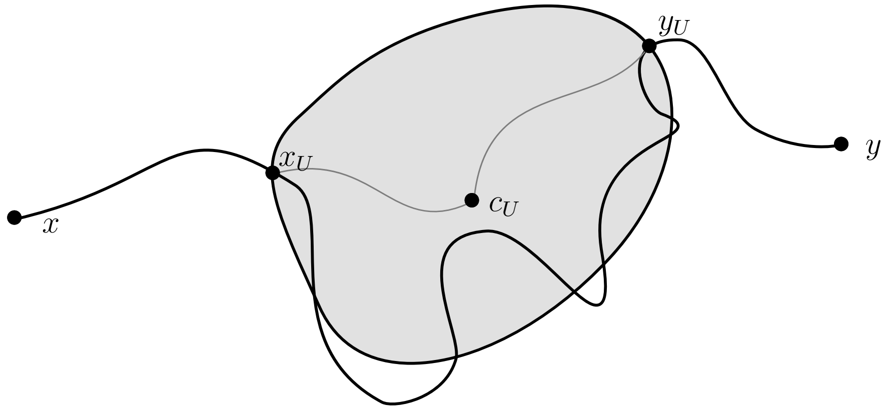

Figure 3.3: Constructing regulated arc

Obviously, $Y \neq \emptyset$. Since X is locally arcwise connected by Lemma 3.6. A simple argument show that both Y and $X \setminus Y$ are open in X. Thus, since $Y \neq \emptyset$ and X is connected, we must have $Y = X$. So X is arcwise connected.

Let C be a connected component of $X \setminus \{x\}$, then C is open in X. Indeed, since X is locally connected, every z in C has a sufficiently small connected neighborhood W_z avoiding x, thus $W_z \subseteq C$.

Since X is locally arcwise connected by Lemma 3.6, C is locally arcwise connected as well. Then one can show that C is arcwise connected in exactly the same way as above.

□

Hence all Julia sets and filled Julia sets discussed in this chapter are locally arcwise connected and arcwise connected.

3.3.3 Uniqueness of regulated arc

An arc γ in K_f is called to be *regulated* if it joins two distinct points in J_f and for any bounded Fatou component U, the intersection $\gamma \bigcap \overline{U}$ is an empty set or a point or exactly two internal rays.

Lemma 3.7 (Uniqueness of regulated arc)**.** *For any two distinct points x, y in J_f, there exists only one regulated arc in K_f joining x and y.*

Proof. Let $\eta(t) : [0,1] \to K_f$ be the arc joining x and y with $\eta(0) = x$ and $\eta(1) = y$. For any Fatou component U whose closure intersects the arc η, set $x_U = \inf_{0 \leq t \leq 1}\{t : \eta(t) \in \overline{U}\}$, i.e., the first time η meets $\overline{U}$, and $y_U = \sup_{0 \leq t \leq 1}\{t : \eta(t) \in \overline{U}\}$, i.e., the last time η meets $\overline{U}$. If $x_U \neq y_U$. Then we replace the segment $\eta((x_U, y_U))$ starting at $\eta(x_U)$ ending at $\eta(y_U)$ by the internal rays $R_U(\eta(x_U)$ and $R_U(\eta(y_U))$, updating $\eta = \eta[0, x_U] \cup R_U(\eta(x_U)) \cup R_U(\eta(y_U)) \cup \eta[y_U, 1]$. After doing these processes for countable many Fatou components, we obtain a regulated arc η connecting x and y as required.

For the uniqueness, if η' is the other one. Then $\mathbb{C} \setminus \eta \cup \eta'$ consists of several disjoint connected components. Let W be one of the bounded component in K_f. Then W is a Jordan domain and $\partial W \subseteq \eta \cup \eta'$. Applying the Maximum value Principle, W belongs to the Fatou set. Let U be the Fatou component containing W. Thus $\overline{W} \subseteq \overline{U}$. Since $(\eta \cup \eta') \bigcap \overline{U}$ consists at most four internal rays and all of the internal rays hit only at the center point $c(U)$. It is impossible for them to bounded a domain W, a contradiction. $\square$

The regulated arc is denoted by $[x, y]$. The open arc (x, y) is defined by $[x, y] \setminus \{x, y\}$, and similarly the semi-open arc $[x, y)$ and $(x, y]$.

3.3.4 Quasi-buried regulated arc

A regulated arc γ is called *quasi-buried* if the intersection between γ and the closure of any bounded Fatou component is either empty or exactly one point. Obviously if $K_f = J_f$, every regulated arc is quasi-buried. But if $K_f \neq J_f$, does there exist quasi-buried arc? We conjecture that for some special locally connected J_f such regulated arc exists.

Similarly as the quadratic case, for high degree polynomials, we still define β *fixed point* as the landing point of external ray $R(0)$. It can be a branched point with at most $d-1$ external rays landing at.

Let $E' := \bigcup_{i \geq 0} \{f^{-i}(\beta)\}$, i.e., the preimages of β fixed points. Set E be the union of E' and branched points in J_f. If J_f is a segment, then $E' = E$. We know that E' is dense in J_f [Mi06] and thus E is dense in J_f. Moreover, we have the following,

Lemma 3.8 (Denseness of E in quasi-buried arcs)**.** *Let $I := [x, y]$ be a quasi-buried regulated arc in K_f. Then E is dense in I.*

Proof. Let p be any point in $I \setminus \{x, y\}$. Since J_f is locally arcwise connected by Lemma 3.6, we can choose sufficiently small arcwise connected neighborhood W_p in J_f such that

$$W_p \bigcap \{x, y\} = \emptyset \quad \text{and} \quad p \in W_p \bigcap I \Subset I. \tag{3.12}$$

See figure 3.4. By the denseness of E in J_f, $W_p \bigcap E$ is not empty. Choose a point z in $W_p \bigcap E$. If z is in I, then we are done. If not, there exists an arc γ_{zp} in W_p joining z and p, because W_p is arcwise connected.

Let ξ be the point at which γ_{zp} meets I at the first time. Then ξ belongs to $I \setminus \{x, y\}$ by (3.12). Let $\gamma_{z\xi}$ be the subarc of γ_{zp} joining z and ξ. It follows that the three arcs $\gamma_{z\xi}$, $[x, \xi]$ and $[y, \xi]$, meeting at ξ, form a "Y" shape.

We are left to show that ξ is a branched point. Due to the Theorem 6.6 in [Mc95], we only have to proof that $K_f \setminus \{\xi\}$ has at least three connected components. Actually we have the following.

Claim that x, y and z lie in distinct connected components of $K_f \setminus \{\xi\}$.

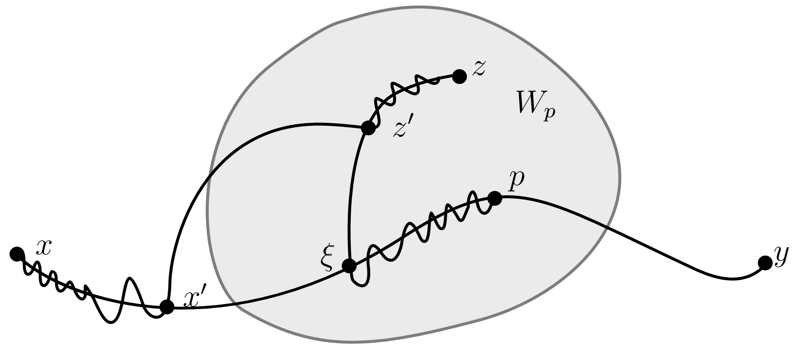

Figure 3.4: illustrating the proof of lemma 3.8

Proof. If not, suppose x, z in the same component C. By Corollary 3.3, C is arcwise connected, thus there exists an arc $\gamma_{xz}(t)$ in C joining x and z with $\gamma_{xz}(0) = x$ and $\gamma_{xz}(1) = y$. Set

$$t_x := \sup_{0 \le t \le 1}\{t : \gamma_{xz}(t) \in [x, \xi]\} \quad \text{and} \quad t_z := \inf_{0 \le t \le 1}\{t : \gamma_{xz}(t) \in \gamma_{z\xi}\}.$$

Denote by $x' = \gamma_{xz}(t_x)$ and $z' = \gamma_{xz}(t_z)$. Note that x', z' are contained in $[x, \xi)$ and $\gamma_{z\xi} \setminus \{\xi\}$ respectively. Let $\gamma_{x'z'}$ be the subarc in γ_{xz} joining x' and z'. It follows that $\eta := \gamma_{x'z'} \cup [z', \xi] \cup [x', \xi]$ bounds a Jordan domain V. By the Maximum Value Principle, V must be contained in some Fatou component U. Then $[x', \xi] \subseteq \partial U$. This contradicts the definition that $I \bigcap \overline{U}$ is either empty or only one point. A same argument show that y, z and x, y cannot lie in the same component of $K_f \setminus \{\xi\}$. The claim is completed. $\square$

Thus ξ is a branched point. The proof is completed. $\square$

3.4 The topological polynomial F

The regulated arcs in K_f may not be preserved by the dynamic of f. In this section, we will construct a nice topological polynomial F by modifying f in each bounded Fatou set. F will coincide with f on the basin of infinity and the Julia set J_f. The above difficulty can be most conveniently overcome by investigating F instead of f. Since we only interest in the Julia set and the combination of external angles. These changes make no essentially differences.

3.4.1 Branched covering map

Let X and Y be domains in $\overline{\mathbb{C}}$, $g : X \to Y$ be a continuous map. Then g is called a *branched covering map* if we can write it locally as the map $z \mapsto z^n$ for some $n \in \mathbb{N}$ after orientation-preserving homeomorphic changes of coordinates in domain and range. More precisely, we require that for each point $q \in Y$ and any preimage p in $g^{-1}(q)$ there exists $n \in \mathbb{N}$, open neighborhoods U of p and V of q, open neighborhoods U' and V' of $0 \in \mathbb{C}$ and orientation-preserving homeomorphisms $\phi : U \to U'$ and $\psi : V \to V'$ with $\phi(p) = 0$ and $\psi(q) = 0$ such that

$$(\psi \circ g \circ \phi^{-1})(z) = z^n \tag{3.13}$$

for all $z \in U'$.

The integer $\deg_g(p) := n \geq 1$ is uniquely determined by g and p and called the *local degree* of g at p. A point $c \in \overline{\mathbb{C}}$ with $\deg_g(c) \geq 2$ is called a *critical point* of g and its image $g(c)$ *critical value*. Moreover, g is an open and surjective mapping. If the set of all critical points only consists of finite isolated points, then g is finite-to-one, i.e., every point has finitely many preimages under g. More precisely, if $\deg(g)$ is the topological degree of g, then

$$\sum_{p \in g^{-1}(q)} \deg_g(p) = \deg(g)$$

for every $q \in Y$. A branched covering with no critical point is called *unbranched covering*. A branched covering map $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called *topological polynomial* if $g^{-1}(\infty) = \infty$, that is , ∞ is a fixed point with local degree $\deg(g)$.

3.4.2 From polynomial f to topological polynomial F

For polynomial f, a bounded Fatou component is called *critical Fatou component* if it contains critical point of f. Its image is *critical value Fatou component*. Given a bounded Fatou component U, f maps U to Fatou component U' holomorphic. $f|_{\partial U} : \partial U \to \partial U'$ is an unbranched covering map with degree $\deg(f|_U)$.

Recall that $\varphi_U : U \to \mathbb{D}$ $c(U) \mapsto 0$ is a conformal parameterization . Set

$$\varphi_{UU'} := \varphi_{U'} \circ f \circ \varphi_U^{-1}|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \partial\mathbb{D}.$$

Now we extend $\varphi_{UU'}$ to be

$$\varphi_{UU'} : \overline{\mathbb{D}} \to \overline{\mathbb{D}} \qquad re^{2\pi i\theta} \mapsto r\varphi_{UU'}(e^{2\pi i\theta}).$$

One can check that $\varphi_{UU'}$ is a branched covering. Define $F_U := \varphi_{U'}^{-1} \circ \varphi_{UU'} \circ \varphi_U : \overline{U} \to \overline{U'}$

by the following communicate diagram,

$$\begin{array}{ccc} (U, c(U)) & \xrightarrow{F_U} & (U', c(U')) \\ {\scriptstyle \psi_U}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{U'}} \\ (\mathbb{D}, 0) & \xrightarrow{\psi_{UU'}} & (\mathbb{D}, 0). \end{array}$$

By the construction, F_U satisfies

• $F_U|_{\partial U} = f|_{\partial U}$.

• F_U sends $c(U)$ to $c(U')$.

• F_U is a branched covering with degree $\deg(f|_U)$ and the critical point can only be $c(U)$.

• F_U sends internal rays to internal rays, more precisely, $F_U(R_U(z)) = R_{U'}(f(z))$.

Now we define the topological polynomial $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$,

$$F(z) := \begin{cases} F_U(z) & \text{If } z \text{ in some bounded Fatou component } U, \\ f(z) & \text{Otherwise.} \end{cases} \tag{3.14}$$

Evidently, F takes the same value as f in the Julia set and the basin of infinity. Furthermore, we have the following.

3.4.3 Properties of the topological polynomial F

Lemma 3.9. (1) *F is continuous.*

(2) *F is a branched covering map.*

(3) *For any $x \neq y \in J_f$, $[F(x), F(y)] \subseteq F([x, y])$.*

(4) *$F(\widehat{R}_U(\theta)) = \widehat{R}_{U'}(\sigma_d(\theta))$, where $U' = F(U)$, for any extended ray $\widehat{R}_U(\theta)$.*

Proof. (1) We only have to show that, for any $z \in J_f$, F is continuous at z. Let $\{z_k\}$ be an arbitrary sequence such that $z_k \to z$ as $k \to \infty$. We continue the discussion into three cases,

• If $\{z_k\} \subseteq \overline{\Omega}_f$. Since $F|_{\overline{\Omega}_f} = f$ and f is continuous, then $F(z_k) \to F(z)$ as $k \to \infty$.

• If $\{z_k\}$ are contained in $\mathbb{C} \backslash \overline{\Omega}_f$. Let $\{U_k\}$ be a sequence of bounded Fatou components such that $z_k \in U_k$ and $\mathcal{U} := \{U_k : k \geq 1\}$. If $\#\mathcal{U} < \infty$, since F is continuous in any Fatou component, we $f(z_k) \to f(z)$ as $k \to \infty$. If $\#\mathcal{U} = \infty$, since J_f is locally connected, the diameter of Fatou component $F(U_k)$ converges to zero as $k \to \infty$ (See for example Lemma 19.5 in [Mi06]). Thus,

$$\begin{aligned} |F(z_k) - F(z)| &\leq |F(z_k) - f(z_k)| + |f(z_k) - F(z)| \\ &\leq \text{diam } F(U_k) + |f(z_k) - f(z)| \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

• In other cases, decompose $\{z_k\}$ into two subsequence $\{z_{k_i}\}$, contained in $\overline{\Omega}_f$, and $\{z_{k'_i}\}$ in Fatou set. By the former arguments, both of the image of the two subsequence converge to $F(z)$ as $k \to \infty$. So $F(z_k) \to 0$ as $k \to \infty$.

Thus F is continuous.

(2) Let $\mathrm{Crit}(F)$ to be the union of critical points of f in J_f and the center of critical Fatou components.

Firstly, claim that $F : \overline{\mathbb{C}} \setminus F^{-1}(F(\mathrm{Crit}(F))) \to \overline{\mathbb{C}} \setminus F(\mathrm{Crit}(F))$ is an unbranched covering. We only have to show that F is locally homeomorphic on $\overline{\mathbb{C}} \setminus \mathrm{Crit}(F)$. For any z in some Fatou component U, It follows by the construction of F_U. For any $z \in J_f \setminus \mathrm{Crit}(F)$, choose a sufficiently small neighborhood W_z such that

• f on W_z is injective,

• $F|_{W_z \bigcap U}$ is injective for any critical Fatou components,

• $f(U) \neq f(U')$ for any distinct Fatou component U and U' which intersect W_z.

By the definition of F, We know that $F|_{W_z}$ is injective. Therefore, $F|_{W_z}$ is a homeomorphism by the domain invariance theorem. The claim follows.

Secondly, consider point z in the finite set $\mathrm{Crit}(F)$. Let W be sufficiently small topological disk around $F(z)$ and

$$\phi : W \to \mathbb{D} \quad F(z) \mapsto 0$$

the topological parameterization. Let W' be one of the component $F^{-1}W$ containing z. Since $F : W' \setminus \{z\} \to W \setminus \{F(z)\}$ is an unbranched covering by the claim. The Riemann Hurwitz formula implies W' is a topological disk around z. Denote by $\delta := \deg(F|_{W' \setminus \{z\}})$. Consider the following communicate diagram,

$$\begin{array}{ccc} W' - \{z\} & \xrightarrow{\psi} & \mathbb{D} - \{0\} \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle z \mapsto z^\delta} \\ W - \{F(z)\} & \xrightarrow{\phi} & \mathbb{D} - \{0\} \end{array}$$

where ψ is a homeomorphism obtained by Lifting ϕ through F and $z \mapsto z^\delta$. Set $\psi(z) = 0$. Thus F satisfies (3.13) at z.

Therefore, F is a branched covering. The critical points set is $\mathrm{Crit}(F)$.

(3) $F([x, y])$, consisting of internal rays, is a curve connecting $F(x)$ and $F(y)$. There exists a regulated arc $\gamma \subseteq F([x, y])$ joining $F(x)$ and $F(y)$. By Lemma 3.7, $\gamma = [F(x), F(y)]$.

(4) Let $z \in \partial U$ to be the landing point of $R(\theta)$. Then $F(z) = \alpha(\sigma_d(\theta)) \in \partial U'$. Since

F_U maps internal ray $R_U(z)$ to internal ray $R_{U'}(F(z))$ and $F(R(\theta)) = R(\sigma_d(\theta))$. Thus $F(\widehat{R}_U(\theta)) = \widehat{R}_{U'}(\sigma_d(\theta))$. The proof is completed. □

3.5 Partitions induced by critical portraits

In this section our objective is to divide the plane into several simple connected domains by external rays and extended rays. These rays land at $\mathrm{Crit}(F)$ and collide together after F. The restriction of F on each pieces is homeomorphic.

3.5.1 Supporting arguments resp. rays

Following [Po93], we give the definition of supporting arguments resp. supporting rays. Let U be a Fatou component and $p \in \partial U$ with total k rays $R(\theta_1), \cdots, R(\theta_k)$ landing at. These rays, numbered in counterclockwise cyclic order, divide the plane into k sectors. Suppose U belong to the sector bounded by $R(\theta_1)$ and $R(\theta_2)$. The argument θ_1 resp. the ray R_{θ_1} is called the *left supporting argument* resp. *left supporting ray* of the Fatou component U. We can also define the *right supporting arguments* resp. *right supporting rays* in analogous way. If only one ray lands at p, then the two supporting rays coincide.

Lemma 3.1. *For any U and $p \in \partial U$, the left resp. right supporting ray of U at p exists and is unique. Let $R(\theta)$ be a ray land at p, then $R(\theta)$ is the left resp. right supporting ray of U at p if and only if $F(R(\theta))$ is the left resp. right supporting ray of $F(U)$ at $F(p)$*

Proof. Firstly, there are at least one and at most finite many rays landing at p by [DH84] and Theorem 3.1. Thus it exists and is unique by definition.

Let $R(\theta')$ be the right (left) supporting ray of U at p. $L_{\theta\theta'} := R(\theta) \bigcup \{p\} \bigcup R(\theta')$ bounds a domain V containing U. The map $F|_V$ is locally homeomorphic at p. So $F(R(\theta))$ and $F(R(\theta'))$ are rays supporting $F(U)$. Since F preserves the orientation. $F(R(\theta)), F(U)$ and $F(R(\theta'))$ are in the same cyclic order around $F(p)$ as $R(\theta), U$ and $R(\theta')$ around p. Thus the lemma follows. □

3.5.2 Definition of critical portraits

Firstly we define $\Theta(c)$, $\Theta(U)$ resp. $\mathcal{R}(c)$, $\mathcal{R}(U)$, for critical point c in J_f and critical Fatou component U by the following way.

• For any critical point $c \in J_f$, we set

$$\Theta(c) := \{\theta_1, \cdots, \theta_{\deg_F(c)}\} \text{ and } \mathcal{R}(c) := \{R(\theta_1), \cdots, R(\theta_{\deg_F(c)})\}$$

such that the total $\deg_F(c)$ external rays meet at c and F maps them onto exactly one external ray.

• For any strictly pre-periodic Fatou component U, we denote by

$$\Theta(U) := \{\theta_1, \cdots, \theta_{\deg_{F|_U}}\} \text{ and } \mathcal{R}(U) := \{\widehat{R}_U(\theta_1), \cdots, \widehat{R}_U(\theta_{\deg(F|_U)})\}$$

such that the $\deg(F|_U)$ external rays $R(\theta_i)$ support U and collide onto one after F. Clearly, by Lemma 3.1, they are supporting U in the same direction.

• For any critical Fatou component cycle $U_0, \cdots, U_{p-1}$ with $F^i(U_0) = U_i$, $U_p := U_0$, it can only be attracting or parabolic [Mi06]. Let $U_{k_0}, \cdots, U_{k_l}$, $0 \le k_0 < \cdots < k_l \le p-1$, be critical with degree $n_0, \cdots, n_l$ respectively.

Firstly, For $1 \le i \le p$, choose (z_i, θ_i), $z_i \in \partial U_i$ and $R(\theta_i)$ landing at z_i, such that $F^i(z_0) = z_i$, $F^p(z_p) = z_p$, $F^i(R(\theta_0)) = R(\theta_i)$ and $R(\theta_p)$ supporting U_p at z_p. Since $F^p : \partial U_0 \to \partial U_0$ is $\delta := n_0 \cdots n_l$ to 1 branched covering, there exist $\delta - 1$ distinct choices of z_p. By Lemma 3.1, all the p external rays supports the Fatou cycle in the same direction.

Secondly, for critical Fatou component U_{k_i}, $0 \le i \le l$, $\Theta(U_{k_i})$ is the set of n_i angles of external rays, which are supporting U_{k_i} and lie in the preimages of $R(\theta_{k_i+1})$, and $\mathcal{R}(U_{k_i})$ is the collection of n_i extended rays of U_{k_i} with angles in $\Theta(U_{k_i})$.

After finishing the choice of $\Theta(U_{k_i})$ and $\mathcal{R}(U_{k_i})$ in critical Fatou cycle, we now state the following lemma by adopting the same notations as above,

Lemma 3.2. *If $z, z' \in \partial U_0$ have the same itinerary respect to $\mathcal{R}(U_{k_0}), \cdots, \mathcal{R}(U_{k_l})$, then $z = z'$.*

Proof. Consider the covering $F^p : \partial U_0 \to \partial U_0$. There are δ preimages of z_p in ∂U_0. These points cut ∂U_0 into open segments $\gamma_0, \cdots, \gamma_{\delta-1}$, numbered in positive cyclic order which starts at z_0. Denote by

$$[s_0, \cdots, s_l] := s_0 n_1 \cdots n_l + s_1 n_2 \cdots n_l + \cdots + s_{l-1} n_l + s_l,$$

where $0 \le s_0 \le n_0 - 1, \cdots, 0 \le s_l \le n_l - 1$.

Let $\gamma_{k_i,0}, \cdots, \gamma_{k_i,n_i-1}$ be the segments of $\partial U_{k_i} \setminus \bigcup_{\theta \in \Theta(U_{k_i})} \alpha(\theta)$, numbered in positive cyclic order which starts at z_{k_i}. Then F maps $\gamma_{k_i,j}$ onto $\partial U_{k_i+1} \setminus \{z_{k_i+1}\}$ one to one.

By the construction above, we can see that $\xi \in \gamma_{[s_0,\cdots,s_l]}$ if and only if $F^{k_i}(\xi) \in \gamma_{k_i s_i}$ for $0 \le i \le l$. Hence by the condition, $\{F^{jp}(z), F^{jp}(z')\}$, for arbitrary $j \ge 0$, are always contained in one segment of $\gamma_0, \cdots, \gamma_{\delta-1}$. Now we show that it is impossible.

Let $\gamma_{zz'}$ be the component of $\partial U_0 \setminus \{z, z'\}$ contained in some segment γ_j. Since F^p is expanding on ∂U_0. There must exist a minimal positive s such that $F^{sp}(\gamma_{zz'})$ can not lie in one of $\gamma_0, \cdots, \gamma_{\delta-1}$. Let $F^{(s-1)p}(\gamma_{zz'}) \subseteq \gamma_{i_0}$. Since $F^p|_{\gamma_{i_0}}$ covers $\partial U_0 \setminus \{z_p\}$ by sticking the two endpoints into z_p, which is the common boundary of γ_j and $\gamma_{(j+1) \bmod \delta}$ for some

$0 \leq j \leq \delta - 1$. Thus $F^{sp}(z)$ and $F^{sp}(z')$ must be in distinct segments. The proof is completed. $\square$

It is easy to see that all the $\mathcal{R}(c)$ and $\mathcal{R}(U)$ defined above are in star shape with a critical point in the center.

Lemma 3.3 (Properties of $\mathcal{R}(c)$ and $\mathcal{R}(U)$)**.** (1) $\mathcal{R}(c) \bigcap \mathcal{R}(c') = \emptyset$, *for distinct critical points* c, c' *in* J_f.

(2) *If* $\mathcal{R}(c) \bigcap \mathcal{R}(U) \neq \emptyset$, *then* $c \in \partial U$ *and the intersection is exactly either a point* $\{c\}$ *or one ray together with the landing point* c. *The latter happens if and only if* $\Theta(c) \bigcap \Theta(U) \neq \emptyset$.

(3) *If* $\mathcal{R}(U) \bigcap \mathcal{R}(U') \neq \emptyset$, *for distinct critical Fatou component* U, U', *then the intersection is exactly either a point* $\{p\} := \partial U \bigcap \partial U'$ *or one ray together with the landing point* p. *The latter happens if and only if* $\Theta(U) \bigcap \Theta(U') \neq \emptyset$.

Proof. By definition, (1) and (2) follow immediately.

(3) Since for any two distinct Fatou component U, U', the intersection $\overline{U} \bigcap \overline{U'}$ is at most one point. $\mathcal{R}(U) \bigcap \mathcal{R}(U') \neq \emptyset$ implies $\overline{U} \bigcap \overline{U'} := \{p\}$. If $\Theta(U) \bigcap \Theta(U') \neq \emptyset$, then the latter case happens. Otherwise, we have $\mathcal{R}(U) \bigcap \mathcal{R}(U') = \{p\}$. $\square$

In $\mathbb{R}/\mathbb{Z}$, let $\mathcal{A} := \{\Theta(c_1), \cdots, \Theta(c_m), \Theta(U_1), \cdots, \Theta(U_n)\}$. For any $\Theta \in \mathcal{A}$, let

$$\widehat{\Theta} := \bigcup \{\Theta' : \exists \text{ a chain } \Theta_0 := \Theta, \cdots, \Theta_k := \Theta' \text{ in } \mathcal{A} \text{ such that } \Theta_i \bigcap \Theta_{i+1} \neq \emptyset\}.$$

The collections $\widehat{\mathcal{A}} := \{\widehat{\Theta}_1, \cdots, \widehat{\Theta}_N\}$ are called *critical portrait* of F. One can check that the following conditions are satisfied.

(1) $\sum_{1 \leq i \leq N} (\# \widehat{\Theta}_i - 1) = d - 1$.

(2) $\widehat{\Theta}_1, \cdots, \widehat{\Theta}_N$ are *pairwise unlinked*, that is, for each $i \neq j$ the sets $\widehat{\Theta}_i$ and $\widehat{\Theta}_j$ are contained in disjoint sub-intervals of $\mathbb{R}/\mathbb{Z}$.

(3) σ_d sends $\widehat{\Theta}_i$ onto exactly one argument.

3.5.3 Critical diagram associated to $\widehat{\mathcal{A}}$

Given critical portrait $\widehat{\mathcal{A}}$, one can construct a *critical diagram* $\mathcal{D} \subseteq \overline{\mathbb{D}}$ as follows. See figure 3.5.

Start with the unit circle $\mathbb{R}/\mathbb{Z}$, for each $\widehat{\Theta}_i$, mark all of the points $e^{2\pi i\theta}$ with $\theta \in \widehat{\Theta}_i$. Let $\widehat{z}_i$ be the center of gravity of the marked points, and join each of these points $e^{2\pi i\theta}$ to $\widehat{z}_i$ by a straight line segment l_θ. Then we obtain a closed set $D_i := \bigcup l_\theta$ in the unit disk. It follows easily by Conditions (2) that distinct D_i and D_j will not cross each other. Let $\mathcal{D} := \bigcup_{1 \leq i \leq d} D_i$ be critical diagram associated to $\widehat{\mathcal{A}}$.

The Condition (1) implies that $\mathbb{D} \setminus \mathcal{D}$ are d simply connected domains $W_1, \cdots, W_d$. Denote by I_i the interior of $\overline{W}_i \bigcap \partial\mathbb{D}$. Then $\{I_i\}_{1 \le i \le d}$ is a partition of $\mathbb{R}/\mathbb{Z}$, each elements of which consists of finite open intervals with total length $1/d$ by Condition (3).

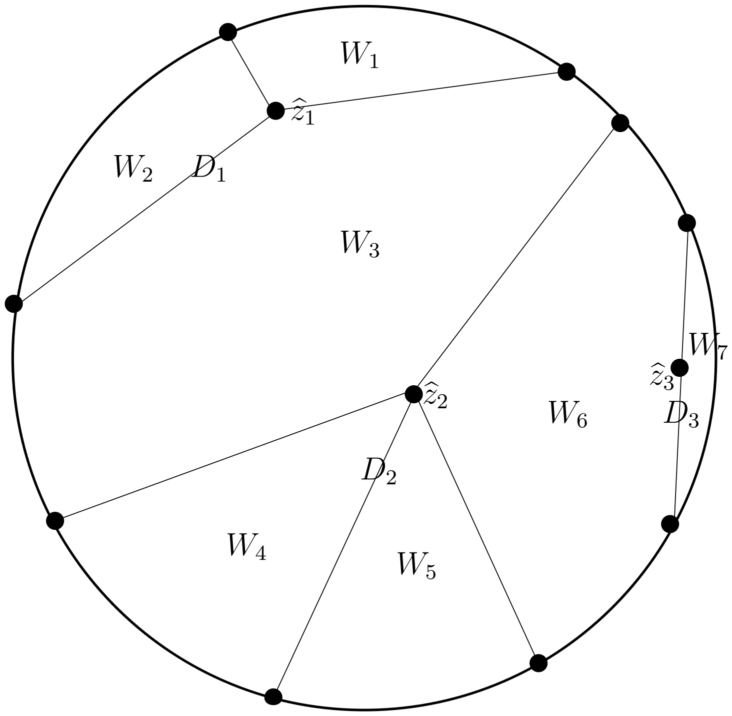

Figure 3.5: An example of critical diagram $\mathcal{D}$

3.5.4 Partition in the dynamic plane

Let $\mathcal{L} := \{\mathcal{R}(c_1), \cdots, \mathcal{R}(c_m), \mathcal{R}(U_1), \cdots, \mathcal{R}(U_n)\}$. For any $\mathcal{R} \in \mathcal{L}$, set $\widehat{\mathcal{R}} := \bigcup\{\mathcal{R}' :$ there exists a chain $\mathcal{R}_0 := \mathcal{R}, \cdots, \mathcal{R}_k := \mathcal{R}'$ in $\mathcal{L}$ such that, for $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$, the latter case in Lemma 3.3(3) happens$\}$.

By Lemma 3.3, each $\widehat{\mathcal{R}}$ corresponds to a $\widehat{\Theta}$, characterized by the property that $R(\theta)$ is in $\widehat{\mathcal{R}}$ if and only if $\theta \in \widehat{\Theta}$.

Lemma 3.10 (Properties of $\widehat{R}$)**.** (1) $T := \widehat{\mathcal{R}} \bigcap K_f$ *is a tree. Namely, any* $z, z' \in T \bigcap J_f$ *can be joined by a regulated arc in* T. *Moreover, the branching points in the tree must be critical points in* J_f *or* $c(U)$ *in critical Fatou component* U.

(2) *Suppose* $R(\theta_1), \cdots, R(\theta_l)$ *be all the external rays in* $\widehat{\mathcal{R}}$, *numbered in counter-clockwise order. Let* $L_{\theta_i\theta_{i+1}} := R(\theta_i) \bigcup R(\theta_{i+1}) \bigcup [\alpha(\theta_i), \alpha(\theta_{i+1})]$, $1 \le i \le l$, $\theta_{l+1} := \theta_1$. *Then* $L_{\theta_i\theta_{i+1}}$ *cuts the plane into two domains* Y, Y'. *Let* Y *be the one disjoint with* $R(\theta_j), 1 \le j \le l$. *Then for any* $x, y \in Y \bigcap J_f$, $[x, y] \subseteq \overline{Y}$ *and* $F|_{[x,y] \bigcap \partial Y}$ *is one-to-one.*

(3) *The image* $F(L_{\theta_i\theta_{i+1}})$ *has only three types:*

- *Type I: one ray union the landing point,*

- *Type II: one extended ray union the landing point,*
- *Type III: two internal rays and one external ray, which looks like "Y".*

(4) *For another $\widehat{\mathcal{R}}'$, if $\widehat{\mathcal{R}} \cap \widehat{\mathcal{R}}' \neq \emptyset$, then the intersection is a point.*

Proof. (1) By the construction of $\widehat{\mathcal{R}}$, it is clear that $[z, z'] \subseteq T$ if $z, z' \in T \bigcap J_f$. The lemma 3.7 implies that regulated arcs cannot form a loop in K_f. Thus T is a tree. Branched point z in Fatou component U is obviously a critical point $c(U)$. If z is in J_f, there are at least three critical Fatou component U_i such that $z \in \mathcal{R}(U_i) \subseteq \widehat{\mathcal{R}}$, $i \in \{1, 2, 3\}$. If z is not critical, $\mathcal{R}(U_i)$ share a common external ray which landing at z. Since one ray supports at most two Fatou components. It is impossible.

(2) Consider $[\alpha(\theta_i), \alpha(\theta_{i+1})]$. It has only three possibilities

(2.1) $\alpha(\theta_i) = \alpha(\theta_{i+1})$, then $[\alpha(\theta_i), \alpha(\theta_{i+1})]$ is degenerated.

(2.2) $[\alpha(\theta_i), \alpha(\theta_{i+1})] \subseteq \overline{U}$ passes through one critical Fatou component, consisting of two internal rays.

(2.3) $[\alpha(\theta_i), \alpha(\theta_{i+1})]$ passes through two critical Fatou component U and U', consisting of four internal rays.

In fact, if $[\alpha(\theta_1), \alpha(\theta_2)]$ passes through more than two critical Fatou component. Let U be one of them with $U \bigcap \{\alpha(\theta_i), \alpha(\theta_{i+1})\} = \emptyset$. Then the supporting properties imply that there exists a external ray in $\mathcal{R}(U)$ contained in Y, impossible.

Assume $[x, y] \setminus Y \neq \emptyset$, otherwise, (2) follows. Let $\gamma(t) := [x, y]$ with $\gamma(0) = x$ and $\gamma(1) = y$. Set $t_1 := \inf_{0 \leq t \leq 1}\{t : \gamma(t) \in \mathbb{C} \setminus Y\}$ and $t_2 := \sup_{0 \leq t \leq 1}\{t : \gamma(t) \in \mathbb{C} \setminus Y\}$. Then $\gamma(t_i) \in \partial Y$, $i \in \{0, 1\}$ and so $[\gamma(t_0), \gamma(t_1)] \subseteq L_{\theta_i \theta_{i+1}}$. We have $[x, y] \bigcap \partial Y = [\gamma(t_0), \gamma(t_1)]$ and $[x, y] = [x, \gamma(t_0)] \bigcup [\gamma(t_0), \gamma(t_1)] \bigcup [\gamma(t_1), y]$. Thus $[x, y] \subseteq \overline{Y}$.

Now we have to show that $F|_{[\gamma(t_0), \gamma(t_1)]}$ is one-to-one. Note that $[\gamma(t_0), \gamma(t_1)]$ consists exactly several internal rays.

In case (2.2), at least one of $R(\theta_i)$, $R(\theta_{i+1})$ is supporting U, because $\mathcal{R}(U) \subseteq \widehat{R}$. So $[\gamma(t_0), \gamma(t_1)] \bigcap \overline{U}$ is either a point or one internal ray. Thus $F|_{[\gamma(t_0), \gamma(t_1)]}$ is one-to-one immediately.

In case (2.3), let $\{p\} = \overline{U} \bigcap \overline{U'}$. We have $R(\theta_i)$, $R(\theta_{i+1})$ supporting U, U' respectively. Otherwise, there exists a ray in $\mathcal{R}(U)$ or $\mathcal{R}(U')$ landing at p contained in Y, impossible. Thus the intersection between $[\gamma(t_0), \gamma(t_1)]$ and $\overline{U}$ resp. $\overline{U}'$ is at most one internal ray.

We are only left to consider the case $[\gamma(t_0), \gamma(t_1)] = [c(U), c(U')]$. Suppose $F|_{[c(U), c(U')]}$ is not one-to-one. Then $F([p, c(U)]) = F([p, c(U')])$, thus p is a critical point. There exists at least a external ray in $\mathcal{R}(p)$ contained in Y. Otherwise, consider the section S of $\mathbb{C} \setminus \mathcal{R}(p)$ containing U, U', $F|_S$ is locally homeomorphic at p, thus it can not paster $[p, c(U)]$ and $[p, c(U)]$ together. This contradicts the choice of $R(\theta_i)$ and $R(\theta_{i+1})$. Therefore $F|_{[c(U), c(U')]}$ is one-to-one.

(3) By the discussion in (2), it follows easily that $F(L_{\theta_i\theta_{i+1}})$ is in Type I, Type II or Type III if and only if $[\alpha(\theta_i), \alpha(\theta_{i+1}]$ is in case (2.1), (2.2) and (2.3), respectively.

(4) It holds directly by the definition and Lemma 3.3. $\square$

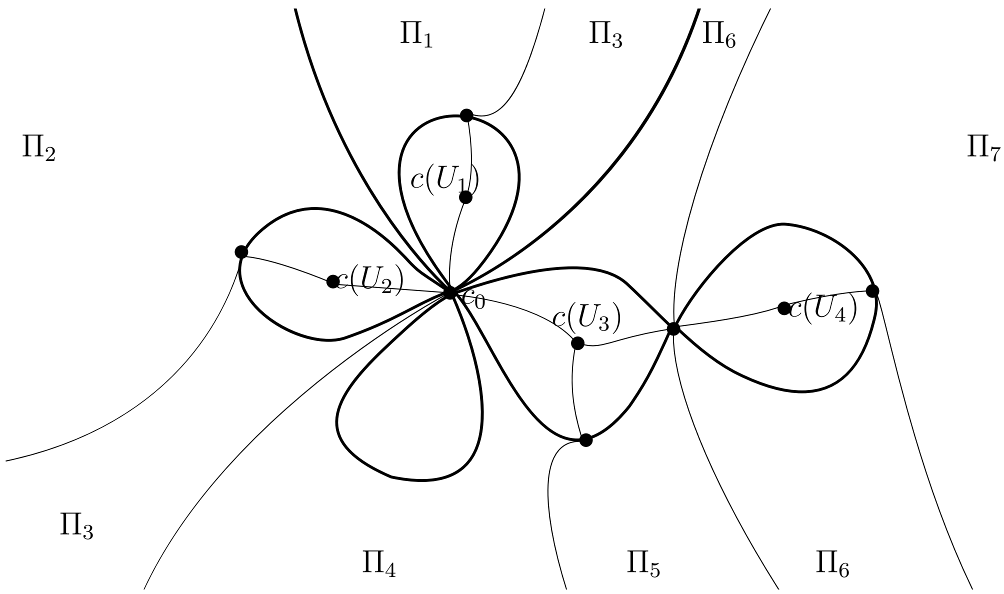

Figure 3.6: An example of partition corresponding to critical diagram in figure 3.5. Here the critical Fatou components are U_1, U_2, U_3 and U_4. There exists a critical points c_0 with $\deg_F(c_0) = 2$.

Let $\widehat{\mathcal{L}} := \{\widehat{\mathcal{R}}_1, \cdots, \widehat{\mathcal{R}}_N\}$. For simplification, the elements are numbered in such fine order that $\widehat{R}_i$ consists of (extended) rays with their arguments in $\widehat{\Theta}_i$. Let $P := \mathbb{C} \setminus \bigcup_{1\le i\le N} \widehat{\mathcal{R}}_i$ consists of finite unbounded pieces $P_1, \cdots, P_s$.

Consider the critical diagram $\mathcal{D}$. Given W_i, suppose it is bounded by $\bigcup_{1\le j\le k_i}(l_{\theta_j} \cup l_{\theta'_j})$ with $\theta_j, \theta'_j \in \widehat{\Theta}_j$ and $l_{\theta_j} \cup l_{\theta'_j} \subseteq D_j$. Then, in the dynamic plane, $L_{\theta_j\theta'_j}$, $1 \le j \le k_i$, in Lemma 3.10 (2) are well defined. As in Lemma 3.10 (2), let $Y_j \bigcup Y'_j := \mathbb{C} \setminus L_{\theta_j\theta'_j}$ where Y'_j be the component disjoint with $\bigcup_{\theta\in I_i} R(\theta)$.

Now we define the *partition* $\{\, \Pi_i \,\}_{1\le i\le d}$ of the dynamical plane by setting

$$\Pi_i := \mathbb{C} \setminus \bigcup_{1\le j\le k_i} \overline{Y'_j}.$$

We have

• $P = \bigcup_{1\le i\le d} \Pi_i$ and $\Pi_i \bigcap \Pi_j = \emptyset$ if $i \ne j$.

• each Π_i, maybe not a domain, consists of finite pieces P_j and $\partial\Pi_i$ are the union of several (extended) rays.

• there is an one-to-one correspondence between $\{\, I_i \,\}_{1\le i\le d}$ and $\{\, \Pi_i \,\}_{1\le i\le d}$ by the property that $\theta \in I_i$ if and only if $R(\theta) \subseteq \Pi_i$. See figure 3.5 and figure 3.6.

Based on the topological argument principle, we shall prove the following,

Proposition 3.2. *The restriction of F on each Π_i is homeomorphic.*

Proof. Recall $G_f : \mathbb{C} \to [0, \infty]$ is the Green's function which vanishes precisely on K_f and $G_t := \{z \in \mathbb{C} : G(z) < t\}$ a simply connected domain. Set $Q_t = G_t \bigcap \Pi_i$, which is bounded by edges in two types,

• The segments of the equipotential cure $G_f(z) = t$ which lies in $\overline{\Pi}_i$. Each one corresponds to an arc in I_i. We denote by Γ_i the union of all these segments.

• The segments of $L_{\theta_j \theta'_j}, 1 \leq j \leq k_i$ satisfying the potential inequality $G_f(z) \leq t$.

Each segments in Γ_i is mapped to equipotential curve $\gamma_{dt} := \{z \in \mathbb{C} : G(z) = dt\}$ locally homeomorphic. Since F pastes the segments of the latter two types together as in Lemma 3.10 (3). It follows that γ_{dt} is covered by Γ_i at least once. We know that $F|_{\gamma_t} : \gamma_t \to \gamma_{dt}$ is d to 1 and γ_t is the union of Γ_i, $1 \leq i \leq d$, with their interiors disjoint. Thus $F(\Gamma_i)$ covers γ_d exactly once.

Let z_0 be any point of $\mathbb{C}$ which does not belong to the image $F(\partial Q_t)$. By the Topological Argument Principle, the number of solutions to the equation $F(z) = z_0$ with $z \in Q_t$, counted with multiplicity, is equal to the winding number of $F(\partial Q_t)$ around z_0. By the arguments above, it is not hard to check that this winding number is $+1$ for z_0 in $G_{dt} \setminus \bigcup_{1 \leq j \leq k_i} F(L_{\theta_j \theta'_j})$ and zeros for z_0 in $\mathbb{C} \setminus \overline{G}_{dt}$. So $F|_{Q_t}$ is one-to-one. By the arbitrariness of t, F on Π_i is homeomorphic.

□

3.5.5 Regulated arcs in the partition

Lemma 3.4. *For any distinct $x, y \in \Pi_i \bigcap J_f$, the regulated arc $[x, y]$ is contained in $\overline{\Pi}_i$. Moreover,*

$$F : [x, y] \to [F(x), F(y)] \text{ is homeomorphic.} \tag{3.15}$$

Proof. We adopt the notations as before. For $1 \leq j \leq k_i$, $x, y \in Y_j$. Then the Lemma 3.10 (2) gives $[x, y] \subseteq \overline{Y}_j$. Thus $[x, y] \subseteq \bigcap_{1 \leq j \leq k_i} \overline{Y}_j = \overline{\Pi}_i$.

Consider the set

$$X := \{z \in F([x, y]) : \text{ there exist } z_1 \neq z_2 \in [x, y] \text{ such that } F(z_1) = F(z_2) = z\}.$$

Since $F|_{\Pi_i}$ is one-to-one by Proposition 3.2, $X \subseteq F([x, y] \bigcap \partial \Pi_i)$.

We claim that $X \subseteq F([x, y] \bigcap \partial \Pi_i \bigcap J_f)$. If not, let $z \in X \bigcap U$ for some bounded Fatou component U. Then there exists two distinct $z_j \in U_j$ such that $F(z_j) = z$. Firstly, If $U_1 = U_2$, then U_1 must be critical. z_1 and z_2 are contained in two internal rays of $\mathcal{R}(U_1)$. It is impossible by Lemma 3.10 (2.2). If $U_1 \neq U_2$, consider the branched covering $F : U_j \to U$. The image $F(U_j \bigcap \Pi_i)$ is either U or $U \setminus R$ for some internal ray. In both of the cases we have

$$F(U_1 \cap \Pi_i) \bigcap F(U_2 \cap \Pi_i) \neq \emptyset.$$

This contradicts the fact that F is one-to-one on Π_i in Proposition 3.2. The claim follows.

Since $[x, y] \bigcap \partial\Pi_i \bigcap J_f$ is finite, then X is finite as well. This means $F([x, y])$ has only finite many self-intersection points. If $X \neq \emptyset$, then we can easily obtain a loop in $F([x, y])$, consisting of regulated arcs by Lemma 3.9 (3). Lemma 3.7 gives a contradiction. Thus we have $X = \emptyset$. Therefore, $F : [x, y] \to [F(x), F(y)]$ is homeomorphic.

□

3.6 Proof of the main theorem

In this section we aim to prove the main theorem, applying the tools prepared in the previous sections.

3.6.1 No wandering regulated arcs

Proposition 3.3. *For any regulated arc $[x, y]$ in K_f, there exist two integer $m \neq n \geq 0$ such that $F^m[x, y] \bigcap F^n[x, y] \neq \emptyset$.*

Proof. For any critical point, if $[x, y]$ is mapped onto it twice, then of course we are done. So, by iterated $[x, y]$ suitable times, we can assume $f^k|_{[x,y]}$ is homeomorphic. We continue the analysis by distinguishing the regulated arc into two case.

- $[x, y]$ is quasi-buried, i.e., $\#[x, y] \bigcap \overline{U} \leq 1$, for any bounded Fatou component U.
- there exists a bounded Fatou component U such that $\#[x, y] \bigcap \overline{U} \geq 2$.

In the first case, $[x, y] \subseteq J_f$. Recall that E is the union of branched points and preimages of β fixed points in J_f. By Lemma 3.8, E is dense in $[x, y]$. If some (pre-)periodic point lies in $[x, y]$, we are done. Then $E \bigcap [x, y]$ contains infinitely many wandering branched points. Since the number of grand orbits of wandering branched point is finite by Corollary 3.2. So there is at least a branched point z such that its grand orbit intersects $[x, y]$ infinitely many times. Choose any two distinct $z_1, z_2 \in [x, y]$ in the grand orbit. Then we have $f^m(z_1) = f^n(z_2)$ for some $m, n \geq 0$. Therefore $f^m[x, y] \bigcap f^n[x, y] \neq \emptyset$. Since $f^m|_{[x,y]}$ and $f^n|_{[x,y]}$ is injective. We must have $m \neq n$.

In the second case, let $[x', y'] := [x, y] \bigcap \overline{U}$, consisting of two internal rays, particularly containing $c(U)$. By Sullivan's no wandering Fatou components, U will eventually be periodic. Then $c(U) \in [x', y']$ is pre-periodic. So there exists $m \neq n$ such that $f^m[x', y'] \bigcap f^n[x', y'] \neq \emptyset$. The proof is completed. □

3.6.2 Quasi-buried case

Proposition 3.4. *Let $\{\Pi_i\}_{1\leq i\leq d}$ be the partition of $\mathbb{C}$ induced by the critical portrait of f. Let $[x, y]$ be quasi-regulated arc in K_f. If x, y have the same itinerary respect to $\{\Pi_i\}_{1\leq i\leq d}$, then $x = y$.*

Proof. We argue by contradiction and suppose $x \neq y$. Denote by $z_n := F^n(z)$ for any $z \in \mathbb{C}$. By Lemma 3.4, for any $m \geq 0, n \geq 1$,

$$F^n : [x_m, y_m] \to [x_{m+n}, y_{m+n}] \text{ is homeomorphic.} \tag{3.16}$$

Firstly, we claim that there exist $M \neq N \geq 0$ and ξ such that
• $\xi \in [x_M, y_M] \bigcap [x_N, y_N]$,
• The orbit of ξ is disjoint with the finite set $X := \bigcup_{1\leq i\leq d}(\partial\Pi_i \cap J_f)$.

Proof of Claim. Consider the set

$$Y := \{z \in [x, y] : \text{ there exist } m, n \geq 0 \text{ and } z' \neq z \in [x, y] \text{ such that } F^m(z) = F^n(z')\}.$$

Since there is no wandering regulated arc by Proposition 3.3, Y is dense in $[x, y]$.

For any $z \in Y$, there exist $m \neq n \geq 0$ such that $z_m \in [x_m, y_m] \bigcap [x_n, y_n]$. If the orbit $\{z_i\}_{i\geq 0}$ never hit X, we are done. If $z_{n_0} \in X$ and the orbit $\{z_{n_0+i}\}_{i\geq 0}$ is infinite, then there exists a large number N_0 such that the orbit $\{z_{N_0+i}\}_{i\geq 0}$ avoids the finite points X. Let $M = N_0 + m, N = N_0 + n$ and $\xi = z_{m+N_0}$, we are done.

Otherwise, we can suppose that all Y are eventually iterated to $X_0 \subseteq X$ and points in X_0 are (pre-)periodic. Then there exist a periodic point w with period p and infinite many points in Y iterated to w. Thus we have (z', n') and (z'', n''), $z' \neq z'' \in Y$, such that $F^{n'}(z') = F^{n''}(z'') = w$ and $n' = n'' \bmod p$. Let $n'' = n' + kp$, $k > 0$. Then $F^{n''}(z') = F^{n''}(z'') = w$, which contradicts (3.16). The claim follows. $\square$

For simplicity we write $[x, y] = [x_M, y_M]$. Let $\xi \in [x, y] \bigcap [x_N, y_N]$, $N \geq 1$, such that its orbit never hits the boundary of the partition $\{\Pi_i\}_{1\leq i\leq d}$. Let

$$H := [x, y] \bigcup [x_N, y_N] \bigcup [x_{2N}, y_{2N}] \bigcup \cdots . \tag{3.17}$$

Then,

• For any $\zeta, \eta \in H$, $[\zeta, \eta] \subseteq H$. Indeed, suppose $\zeta \in [x_{n_1 N}, y_{n_1 N}]$ and $\eta \in [x_{n_2 N}, y_{n_2 N}]$ with integers $n_1 \leq n_2$. Then the path

$$\gamma_{\zeta\eta} := [\zeta, \xi_{n_1 N}] \bigcup [\xi_{n_1 N}, \xi_{(n_1+1)N}] \bigcup \cdots \bigcup [\xi_{n_2 N}, \eta]$$

joins ζ and η. By the uniqueness of regulated arc in Lemma 3.7, It follows that $[\zeta, \eta] \subseteq \gamma_{\zeta\eta}$.

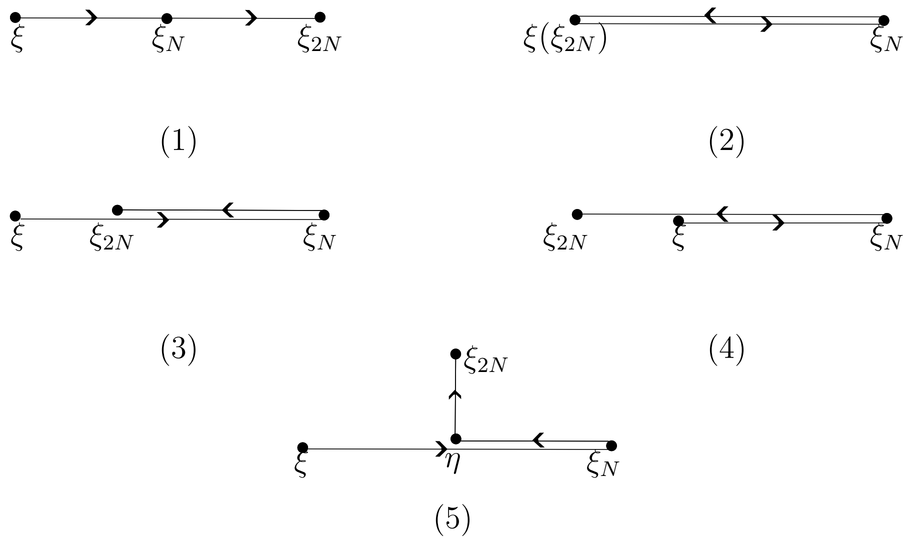

Figure 3.7: Relations of $[\xi, \xi_N]$ and $[\xi_N, \xi_{2N}]$

Since $[\zeta, \xi_{n_1 N}] \subseteq [x_{n_1 N}, y_{n_2 N}]$, $[\xi_{kN}, \xi_{(k+1)N}] \subseteq [x_{kN}, y_{(k+1)N}]$ and $[\zeta_{n_2 N}, \eta] \subseteq [x_{n_2 N}, y_{n_2 N}]$, then $\gamma_{\zeta\eta} \subseteq H$. Thus $[\zeta, \eta] \subseteq H$.

• For any $n \geq 0$, if $F^n(\xi) \in \Pi_{i(n)}$, then $F^n(H) \subseteq \overline{\Pi}_{i(n)}$. Indeed, since ξ is never mapped into $\bigcup_{1 \leq i \leq d} \partial \Pi_i$, such $\Pi_{i(n)}$ exists. We claim that $x_{kN}, \xi_{kN}, \xi_{(k+1)N}, y_{kN}$ have the same itinerary respect to $\{\Pi_i\}$. Since $\xi, \xi_N \in [x, y]$, then $\xi_{kN}, \xi_{(k+1)N} \in [x_{kN}, y_{kN}]$. By Lemma 3.4 and (3.15), $[x_{kN+j}, y_{kN+j}]$ must be contained in some $\overline{\Pi}_{j(n)}$ for any j. In particularly, we have $F^j(x_{kN}), F^j(\xi_{kN}), F^j(\xi_{(k+1)N}), F^j(y_{kN}) \in \Pi_{j(n)}$. By the arbitrariness of j, the claim follows. Therefore we obtain a sequence $\xi, \xi_N, \xi_{2N}, \cdots, \xi_{kN}, x_{kN}, y_{kN}$, which have the same itinerary. Thus if $F^n(\xi) \in \Pi_i$, $F^n[x_{kN}, y_{kN}] \subseteq \overline{\Pi}_i$. By the arbitrariness of k, it follows that $F^n(H) \subseteq \overline{\Pi}_i$.

• For any $n \geq 0$, $F^n|_H$ is homeomorphism and $F^N(H) \subseteq H$. The latter follows immediately by definition. For the former, if not, there exists a minimal number $n_0 \geq 0$ such that we have $\zeta \neq \eta \in F^{n_0}(H)$ with $F(\zeta) = F(\eta)$. By the above conclusions, we see that $[\zeta, \eta] \subseteq F^{n_0}(H)$ and is contained in some $\overline{\Pi}_{i(n_0)}$. Since $[\zeta, \eta]$ is quasi-buried, there exists $[\zeta^{(i)}, \eta^{(i)}] \subseteq [x, y]$ with $\zeta^{(i)}, \eta^{(i)} \in \Pi_{i(n_0)}$ such that $\zeta^{(i)} \to \zeta$, $\eta^{(i)} \to \eta$ as $i \to \infty$. Then $F|_{[\zeta^{(i)}, \eta^{(i)}]}$ is one-to-one by Lemma 3.4. Thus $F[\zeta, \eta]$ is a loop. It is impossible by Lemma 3.7.

Now we pay attention to the two regulated arc $[\xi, \xi_N]$ and $[\xi_N, \xi_{2N}]$. Both of them are contained in H. Their relations are in one of the following five possibilities. See figure 3.7.

(1) $[\xi, \xi_N] \bigcap [\xi_N, \xi_{2N}] = \{\xi_N\}$.

(2) $[\xi, \xi_N]$=$[\xi_N, \xi_{2N}]$.

(3) $[\xi_N, \xi_{2N}] \subset [\xi, \xi_N]$.

(4) $[\xi, \xi_N] \subset [\xi_N, \xi_{2N}]$.

(5) $[\xi, \xi_N] \bigcap [\xi_N, \xi_{2N}] = [\eta, \xi_N]$ for some $\eta \in (\xi, \xi_N)$.

We will show that all of them are impossible and so the proof is completed.

For case (1), we have $[\xi, \xi_{2N}] = [\xi, \xi_N] \bigcup [\xi_N, \xi_{2N}] \subseteq H$. Then $F|_{[\xi,\xi_{2N}]}$ is homeomorphic. Note that $F^N[\xi, \xi_N] = [\xi_N, \xi_{2N}]$. It follows that $[\xi_{2N}, \xi_{3N}] \bigcap [\xi_N, \xi_{2N}] = \{\xi_{2N}\}$. We also have $[\xi_{2N}, \xi_{3N}] \bigcap [\xi, \xi_N] = \emptyset$. Otherwise, the three arcs $[\xi, \xi_N] \bigcup [\xi_N, \xi_{2N}] \bigcup [\xi_{2N}, \xi_{3N}]$ would form a loop. By induction, it follows that $[\xi_{nN}, \xi_{(n+1)N}] \bigcap [\xi, \xi_{nN}] = \{\xi_{nN}\}$ for $n \geq 0$. Then (ξ, ξ_N) is a wandering regulated arc of F^N. By Proposition 3.3, it is impossible.

Case (2) can not happen. Indeed, otherwise $F^N : [\xi, \xi_N] \to [\xi, \xi_N]$ is homeomorphic. Choose any subarc I in $[\xi, \xi_N]$ such that $F^N(I) \bigcap I = \emptyset$. Then I is a wandering regulated arc of F^N.

For case (3), choose an arbitrary subarc I in (ξ, ξ_{2N}). Then $F^N(I) \subseteq (\xi_N, \xi_{2N})$. Since $F^N : [\xi, \xi_N] \to [\xi_N, \xi_{2N}]$ is homeomorphic and $[\xi_N, \xi_{2N}] \subset [\xi, \xi_N]$, I is a wandering regulated arc of F^N, a contradiction.

For case (4), by the intermediate value theorem, there is a fixed point $\nu \in (\xi, \xi_N)$ of F^N. Then $[\nu, \xi] \subset [\nu, \xi_{2N}]$ and the map $F^{2N} : [\nu, \xi] \to [\nu, \xi_{2N}]$ is homeomorphic. Let $\xi_{-2N} \in [\nu, \xi]$ such that $F^{2N}(\xi_{-2N}) = \xi$. Then $[\xi_{-2N}, \xi] \bigcap [\xi, \xi_{2N}] = \{\xi\}$. Similar to case (1), it is impossible.

For case (5), let $\eta_{-N} \in [\xi, \xi_N]$ with $F^N(\eta_{-N}) = \eta$. We distinguish three possibilities to analyze.

(5.1) $\eta_{-N} \in (\xi, \eta)$. Then $\eta_N \in (\eta, \xi_{2N})$. Therefore $[\eta_{-N}, \eta] \bigcap [\eta, \eta_N] = \{\eta\}$. By case (1) again, it is impossible.

(5.2) $\eta_{-N} = \eta$. Then η is a fixed point of F^N. We claim that there exist $\nu \in (\eta, \xi)$ and $n_0 \geq 3$ such that $F^{n_0 N}[\eta, \nu] \subseteq [\eta, \xi]$. Indeed, since $F^{3N}[\eta, \xi] = [\eta, \xi_{3N}]$ and $F^N|_H$ is injective, hence $[\eta, \xi_{3N}] \bigcap ([\eta, \xi_N] \cup [\eta, \xi_{2N}]) = \{\eta\}$. If $[\eta, \xi_{3N}] \bigcap [\eta, \xi] \neq \{\eta\}$, the claim follows. Otherwise, continue the process to $[\eta, \xi_{3N}] \cdots$, until $[\eta, \xi_{kN} \bigcap [\eta, \xi] \neq \{\eta\}$. Otherwise, we obtain an infinity sequence $\{(\eta, \xi_{kN}]\}_{k \geq 0}$ which are pairwise disjoint. This contradict Proposition 3.3. Hence the claim follows.

Choose $\nu' \in (\eta, \nu)$ such that $\nu'_{n_0 N} \neq \nu'$. If $\nu'_{n_0 N} \in (\eta, \nu')$, similarly in case (3), it is impossible. If $\nu'_{n_0 N} \in (\nu, \xi)$, let $\nu'_{-n_0 N} \in (\eta, \nu')$ be the preimage of $F^{n_0 N}|_{[\eta, \nu']}$, then similar in case (1), $(\nu'_{-n_0 N}, \nu')$ is a wandering regulated arc of $F^{n_0 N}$. It is impossible.

(5.3) $\eta_{-N} \in (\xi_N, \eta)$. Applying intermediate value theorem to $F : [\eta_{-N}, \eta] \to [\eta, \eta_N]$, we obtain a fixed point $\nu \in (\eta_{-N}, \eta)$. Since $[\nu, \xi_N] \bigcap [\nu, \xi_{2N}] = \{\nu\}$. So this is the case (5.2), impossible. The proof is completed. $\square$

3.6.3 General cases

Proposition 3.5. *Let $\{\Pi_i\}_{1\leq i\leq d}$ be the partition of $\mathbb{C}$ induced by the critical portrait of f. If $x, y \in J_f$ have the same itinerary respect to $\{\Pi_i\}_{1\leq i\leq d}$, then either $x = y$ or x, y are in the boundary of a Fatou component, which is mapped to a siegel disk.*

Proof. Suppose $x \neq y$. Consider the regulated arc $[x, y]$. Let

$$\mathcal{U} := \{U : U \text{ is a Fatou component such that } U \bigcap [x, y] \neq \emptyset\}.$$

Then $[x, y] \backslash \bigcup_{U\in\mathcal{U}} U$ consists of several disjoint quasi-buried regulated arcs. By Proposition 3.4, each such arcs is a single point.

Firstly, $\mathcal{U}$ is finite. If not, since there is no wandering Fatou components and the number of periodic Fatou components is finite. Infinite many elements in $\mathcal{U}$ will eventually be mapped onto a periodic one. This contradicts (3.16).

Secondly, any $U \in \mathcal{U}$ is mapped to a siegel disk. If not, let $(x', y') = U \bigcap [x, y]$. If there exists $N \geq 0$ such that orbits of x'_N and y'_N avoid the finite set $X := \bigcup_{1\leq i\leq d}(J_f \bigcap \partial\Pi_i)$, then x'_N and y'_N have the same itinerary. Lemma 3.2 gives $x'_N = y'_N$. This contradicts (3.16). Thus there exist N and a periodic point $\xi \in X$ such that $x'_N = \xi$ or $y'_N = \xi$. Suppose $x'_N = \xi$. Let $\xi \in \Theta(U_0)$ and p the period of ξ. Then F^p fixes x'_N and iterates y'_N to at least two distinct segments of $\partial U_0 \setminus \Theta(U_0)$. By properties of supporting rays, x_n, y_n must be separated by $\Theta(U_0)$ for some n, a contradiction.

Finally, $\mathcal{U}$ consists of only one Fatou component. If not, let $U \neq U' \in \mathcal{U}$. Let M, N be integers such that $F^M(U) = F^{M+N}(U), F^M(U') = F^{M+N}(U')$. Then

$$F^{M+N}[c(U), c(U')] = F^M[c(U), c(U')].$$

By Lemma 3.7, $\xi := \partial F^M(U) \bigcap F^M[c(U), c(U')]$ is periodic. Since $F^N|_{\partial F^M(U)}$ conjugates a irrational rotation. Thus ξ can not be periodic, a contradiction. The proof is completed. □

Proof of Theorem 3.2. The theorem follows immediately by Propositions 3.5. □

3.7 Application to core entropy

Consider a quadratic polynomial family $\mathcal{F} := \{f_c = z^2 + c : f_c$ has no Siegel disks and J_{f_c} is locally connected $\}$. As an application of Theorem 3.2, we shall prove the monotonicity of core entropy.

3.7.1 Characteristic arc I_c

In order to introduce a partial order on $\mathcal{F}$, we need the following definition of *characteristic arc* I_c.

$(C1)$ If f_c has a parabolic or attracting Fatou cycle of period $p \geq 1$. Then there exists a unique point z in the boundary of critical value Fatou component U such that $f_c^p(z) = z$. Let S be the sector containing U and bounded by supporting rays of U at z. Then set $I_c := \overline{\{\theta \in \mathbb{R}/\mathbb{Z} : R(\theta) \subseteq S\}}$. Obviously, $I_c = \mathbb{R}/\mathbb{Z}$ if and only if exactly one ray lands at z.

$(C2)$ In other cases, we have $c \in J_{f_c}$. Then there is a unique sector S based at c containing critical point 0. Set $I_c := \overline{\mathbb{R}/\mathbb{Z} \setminus \{\theta \in \mathbb{R}/\mathbb{Z} : R(\theta) \subseteq S\}}$. Evidently, I_c is a single angle if and only if only one ray lands at critical value c.

For any $f_c, f_{c'}$ in $\mathcal{F}$, we say $f_c \prec f_{c'}$ if and only if $I_c \supseteq I_{c'}$.

If $I_c \neq \mathbb{R}/Z$, denote by $[\eta_c, \xi_c] := I_c$. Let $I'_c \bigcup I''_c := \sigma_2^{-1}(I_c)$ with $I'_c := [\eta'_c, \xi'_c]$ and $I''_c := [\eta''_c, \xi''_c]$, where $\{\eta'_c, \eta''_c\} := \sigma_2^{-1}(\eta_c)$ and $\{\xi'_c, \xi''_c\} := \sigma_2^{-1}(\xi_c)$. The above $[\bullet, \bullet]$ are measured in positive cyclic order and we distinguish it from the notation of regulated arc by acting on distinct categories. Evidently, I'_c and I''_c are symmetric respect to origin with length $|I'_c| = |I''_c| = \frac{1}{2}|I_c|$.

Lemma 3.11 (Properties of characteristic arc)**.** *For any $f_c \in \mathcal{F}$, then*

(1) *If f_c is in case $(C2)$, then*

(1.1) *The rays $R(\eta'_c), R(\eta''_c), R(\xi'_c), R(\xi''_c)$ land at critical point 0.*

(1.2) *If I_c is not a single point, let S'_c resp. S''_c be the sectors bounded by $R(\eta'_c)$ and $R(\xi'_c)$ resp. $R(\eta''_c)$ and $R(\xi''_c)$ and S_c the sectors bounded by $R(\eta_c)$ and $R(\xi_c)$ avoiding the critical point. Then $(S'_c \bigcup S''_c) \bigcap S_c = \emptyset$ and f maps S'_c resp. S''_c conformally onto S_c. Denote by $H_c := S'_c \bigcup S''_c$.*

(2) *If f_c is in case $(C1)$ and $I_c \neq \mathbb{R}/\mathbb{Z}$, then*

(2.1) *$L_{\eta_c\xi_c}$ separates critical point 0 and critical value c. Recall $L_{\eta_c\xi_c} := R(\eta_c) \bigcup R(\xi_c) \bigcup \{z\}$. Therefore, $|I_c| < \frac{1}{2}$.*

(2.2) *$R(\eta'_c)$ and $R(\xi''_c)$ resp. $R(\eta''_c)$ and $R(\xi'_c)$ land together at z' resp. z'' with $\{z', z''\} := f^{-1}(z)$.*

(2.3) *Let S_c be the sectors bounded by $R(\eta_c)$ and $R(\xi_c)$ avoiding the critical point and H_c the domain bounded by $L_{\eta'_c\xi''_c}$ and $L_{\eta''_c\xi'_c}$, then $H_c \bigcap S_c = \emptyset$ and $f : H_c \to S_c$ is a branched covering of degree two.*

(3) *For any $f_c, f_{c'} \in \mathcal{F}$, if $f_c \prec f_{c'}$, then $I'_{c'} \bigcup I''_{c'} \subseteq I'_c \bigcup I''_c$.*

Proof. (1) Since both $R(\eta_c)$, $R(\xi_c)$ land at critical value c. Then, at the critical point 0, there exist preimages, rays $R(\eta'_c), R(\eta''_c), R(\xi'_c), R(\xi''_c)$. If I_c is a single point, we have

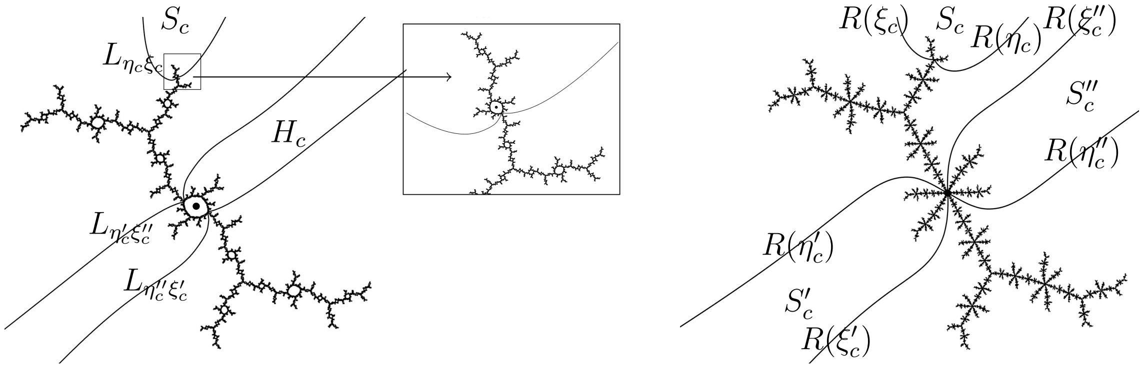

Figure 3.8: illustrating the proof of Lemma 3.11, Left: case (C1). Right: case (C2)

$\eta'_c = \xi'_c, \eta''_c = \xi''_c$. If I_c is not a single point, consider the sectors S'_c and S''_c based at 0. By lemma 3.1 $(S'_c \bigcup S''_c) \bigcap S_c = \emptyset$. Since $f|_{S'_c}$ resp. $f|_{S''_c}$ is conformal. Thus $l(f(S'_c)) = 2l(S'_c) = 2l(S''_c)$. Note that $l(\mathbb{C} \setminus \overline{S_c}) > l(S'_c) + l(S''_c)$. It follows that $f(S'_c) = f(S''_c) = S_c$.

(2) Let p be the period of the critical value Fatou component U and $z_0 := z, z_1 := f(z), \cdots, z_p := f^p(z)$ with $z_p = z_0$. Since this orbit is disjoint with critical point, we can set L_{z_i} the preimage of $f^{-(p-i)}L_{\eta_c\xi_c}$ at each z_i. Obviously, $L_{z_0} = L_{z_p}$, because both of them support Fatou component U. Let S_{z_i} be one of the components $\mathbb{C} \setminus L_{z_i}$ containing 0 and S'_{z_i} the other.

For (2.1), suppose L_{z_0} does not separate 0 and c, then S_{z_0} contains both of them. For $i = p-1$, By Lemma 3.2 (4), the sector map σ_2 must send the critical sector $S_{z_{p-1}}$ to critical value sector S_{z_p}, and thus $\sigma_2(S'_{z_{p-1}}) = S'_{z_p}$. We have $l(S'_{z_{p-1}}) = \frac{1}{2}l(S'_{z_0})$. Claim $L_{z_{p-1}}$ cannot separate 0 and c. Otherwise, using Lemma 3.1 and properties of supporting rays, we have $S'_{z_{p-1}} \supset S'_{z_0}$, thus $l(S'_{z_{p-1}}) > l(S'_{z_0})$, impossible. For $i = p-2, \cdots, 0$, the same argument as above implies $l(S'_{z_i}) = \frac{1}{2}l(S'_{z_{i+1}})$ and S_{z_i} contains both 0 and c. Thus $l(S'_{z_0}) = \frac{1}{2^p}l(S'_{z_0})$, a contradiction.

For (2.2), since z is not a critical value. We have two $z' \neq z''$ preimages of z. We discuss by contradiction and assume $R(\eta'_c), R(\xi'_c)$ resp. $R(\eta''_c), R(\xi''_c)$ land at z' resp. z''. Then consider the sector $S'_c := \bigcup_{\theta\in I'_c} R(\theta)$ resp. $S''_c := \bigcup_{\theta\in I''_c} R(\theta)$. We have $\sigma_2(S'_c) = \sigma_2(S''_c) = S'_{z_0}$. Since $l(S'_c) = l(S''_c) = l(I'_c) = l(I''_c) < \frac{1}{2}$, by Lemma 3.2, $f|_{S''_c}, f|_{S''_c}$ are conformal. Therefore, the image S'_{z_0} cannot contain critical value c. This contradicts (2.1).

For (2.3), note that both of $L_{\eta'_c\xi''_c}$ and $L_{\eta''_c\xi'_c}$ support the critical Fatou component and are symmetry respect the original. Then the fact $|I_c| > |I'_c| = |I''_c|$ implies $H_c \bigcap S_c = \emptyset$.

For (3), one can easily check it by definition. □

3.7.2 Dynamic of biaccessible angles

Given $f_c \in \mathcal{F}$, an angle θ in $\mathbb{R}/\mathbb{Z}$ is called to be *biaccessible*, if there exists $\theta' \neq \theta$ such that $R(\theta)$ and $R(\theta')$ landing together. Evidently, if θ is biaccessible, then the preimages $\sigma_2^{-1}(\theta)$ are biaccessible. Inversely, if θ is biaccessible and $\alpha(\theta)$ is not the critical point, then $\sigma_2(\theta)$ is also biaccessible. Denote by $Acc(f_c)$ the set of all biaccessible angles of f_c. Then if $I_c = \mathbb{R}/\mathbb{Z}$, $Acc(f_c) = \emptyset$ by lemma 3.2.

Lemma 3.12. *Let $I_c \neq \mathbb{R}/\mathbb{Z}$ and not a single angle. Let θ be a biaccessible angle of f_c and the orbit of the landing point $\zeta_0 := \alpha(\theta)$ avoid critical point 0. Then there exists a $N \geq 0$ such that the orbit of $\zeta_N := f^N(\zeta_0)$ is disjoint with H_c, where H_c is defined in Lemma* 3.11 (1.2)(2.3). *Therefore, there exists $\vartheta \neq \theta_N := \sigma_2^N(\theta)$ such that, for any $\nu \in (\eta_c, \xi_c)$, ϑ and θ_N have the same itinerary respect to $\mathbb{R}/\mathbb{Z} \setminus \sigma_2^{-1}(\nu)$.*

Proof. Let $\theta' \neq \theta$ with $\alpha(\theta') = \alpha(\theta) = \zeta_0$. Since ζ_0 will never meet the critical point. For $n \geq 0$, $L_{\theta_n\theta_n'} = f^n(L_{\theta\theta'})$ bounds two sectors S_{ζ_n} and S'_{ζ_n}, where we assume S_{ζ_n} is the one containing 0.

Firstly, there exists a $N \geq 0$ such that $L_{\theta_N\theta'_N}$ separates 0 and c. If not, for each $n \geq 0$, σ_2 must send S_{ζ_n} to $S_{\zeta_{n+1}}$ and S'_{ζ_n} to $S'_{\zeta_{n+1}}$, therefore, $l(S'_{\zeta_{n+1}}) = 2l(S'_{\zeta_n})$ by Lemma 3.2 (2), (3) and (4). It follows that $l(S'_{\zeta_n}) \to \infty$ as $n \to \infty$, impossible.

By Lemma 3.11 (1.2)(2.3), points in H_c will be mapped to S_c. Thus we only have to show that $\zeta_n \notin S_c$, $n \geq N$. Claim $l(S'_{\zeta_n}) \geq l(S_c)$. If not, let $n_0 > N$ be first integer such that $l(S'_{\zeta_{n_0}}) < l(S_c)$. Thus $S'_{\zeta_{n_0}}$ must be a critical value sector. This means $S'_{\zeta_{n_0}} \supseteq S_c$ or $S'_{\zeta_{n_0}} \supseteq \mathbb{C} \setminus \overline{S_c}$, both of which imply $l(S'_{\zeta_{n_0}}) \geq l(S_c)$, a contradiction. Therefore, $\zeta_n \notin S_c, n \geq N$. □

3.7.3 Monotonicity of core-entropy

Proof of Theorem 3.3. If $I_c = \mathbb{R}/\mathbb{Z}$, $Acc(f_c) = \emptyset$.

If $\#I_c = 1$, then $I_c = I_{c'}$, hence $I'_c = I'_{c'}$ and $I''_c = I''_{c'}$. By Theorem 3.2, $Acc(f_c) = Acc(f_{c'})$.

In other cases, we have either $I_c = I_{c'}$ or $I_{c'} \subsetneq I_c$.

If $I_{c'} \subsetneq I_c$. We can assume $\eta_{c'} \in (\eta_c, \xi_c)$. For any $\theta \in Acc(f_c)$, if the orbit of $\alpha(\theta)$ is disjoint with critical point 0, by Lemma 3.12, there exist N and $\theta' \neq \theta_N$ such that θ_N and ϑ have the same itinerary respect to partition $\mathbb{R}/\mathbb{Z} \setminus \sigma_2^{-1}(\eta_{c'})$. By theorem 3.2, in the dynamic plane of $f_{c'}$, external rays with arguments θ_N, ϑ land together. Thus $\theta \in Acc(f_{c'})$. If $\alpha(\theta)$ is iterated to 0, then critical point is not periodic. Evidently, the above N and ϑ exist as well.

If $I_c = I_{c'}$. For any $\theta \in Acc(f_c)$, if $\alpha(\eta_c)$ is not periodic, by the same argument as above, such ϑ and N exist. If $\alpha(\eta_c)$ is periodic. If the orbit of $\alpha(\theta)$ avoids $\alpha(\eta_c)$, then such

ϑ and N exist. If $\alpha(\theta)$ is mapped to $\alpha(\eta_c)$. Then θ is iterated into $\{\eta_c, \xi_c\} \subseteq Acc(f_{c'})$. Thus $\theta \in Acc(f_{c'})$. The proof is completed. $\square$

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Chapter 4

A landing theorem on non-recurrent polynomials

4.1 Introduction

In this section we give some notations and recall some results of polynomial dynamics. We refer to [Mi06], [Go94] and [Ki05] for details.

Let f be a monic polynomial with degree $d \geq 2$. Let Ω_f be *basin of infinity* consisting the set of all points in $\overline{\mathbb{C}}$ escaping to ∞ and the filled Julia set $K_f := \overline{\mathbb{C}} \setminus \Omega_f$. There exists a green function G_f measures the escape rate of points to ∞, defined by

$$G_f : \overline{\mathbb{C}} \to [0, \infty) \quad z \mapsto \lim_{n \to \infty} \frac{\log |f^n(z)|}{d^n}.$$

It is a continuous function which vanishes on the filled Julia set and satisfies

$$G_f(f(z)) = dG_f(z).$$

Moreover, G_f is positive and harmonic in Ω_f. In Ω_f, the derivative of G_f vanishes at z if and only if z is a pre-critical point. Each locus $G_f^{-1}(r) = \{z \in \overline{\mathbb{C}}, G_f(z) = r\}$ with $r > 0$ is called an *equipotential curve* around the filled Julia set K_f.

Near ∞, there exists an unique normalized Böttcher map Ψ_f which conjugates f with $z \to z^d$ in a neighborhood of ∞. Ψ_f^{-1} has an unique maximal radial extension to a subset of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. This radial extension terminates at a point w with $|w| > 1$ if and only if Ψ_f^{-1} extends continuously to w and $\Psi_f^{-1}(w)$ is a (pre)critical point of f. Then *external radius* $R_f(t)$ with argument t is

$$R_f(t) := \Psi_f^{-1}((r_t, \infty)e^{2\pi i t}),$$

where $\Psi_f^{-1}(r_t e^{2\pi i t})$ is (pre)critical point of f. If all critical points has bounded orbits, then

$r_t = 1$ and so Ω_f is simply connected.

We are working in the parameter space $\mathcal{P}_d \cong \mathbb{C}^{d-1}$ of *monic centered* polynomials, that is polynomials $z \mapsto z^d + a_{d-2}z^{d-2} + \cdots + a_0$. The *shift locus* $\mathcal{S}_d$ is the subset of $\mathcal{P}_d$ formed by polynomials with all critical points escaping to infinity. Then K_f is cantor set for each f in $\mathcal{S}_d$. Let $\mathcal{S}_d(r)$, $r > 0$, consist of polynomials in $\mathcal{S}_d$ such that all the critical points are in the same equipotential curve $G^{-1}(r)$ and let $\mathcal{S}'_d := \bigcup_{r>0} \mathcal{S}_d(r)$.

A collection $\Theta = \{\Theta_1, \cdots, \Theta_n\}$ of finite subsets of $\mathbb{R}/\mathbb{Z}$ is called a *critical portrait* of degree d if the following conditions hold,

(1) for every j, $|\Theta_i| \geq 2$ and $|\sigma_d(\Theta_j)| = 1$, where $\sigma_d : \theta \mapsto d \cdot \theta \bmod 1$.

(2) $\Theta_1, \cdots, \Theta_n$ are pairwise unlinked.

(3) $\sum(|\Theta_j| - 1) = d - 1$.

For another critical portrait $\Theta' = \{\Theta'_1, \cdots, \Theta'_n\}$, we say $\Theta = \Theta'$ iff there exist a permutation τ such that $\Theta_i = \Theta'_{\tau(i)}$ for $1 \leq i \leq d$. Let $\mathcal{A}_d$ be the collection of all critical portrait of degree d. In [Ki05], Kiwi gave $\mathcal{A}_d$ a compact-unlinked topology and proved that $\mathcal{A}_d$ is compact and connected.

Now we consider the map

$$\Pi : \mathcal{S}'_d \to \mathcal{A}_d \qquad f \mapsto \Theta.$$

Indeed, since there are $\deg_f(c_j)$ external radius with argument θ_k terminating at c_j. Denote these arguments θ_k by Θ_j. Then $\Pi(f) := \{\Theta_1, \cdots, \Theta_n\}$ is the critical portrait induced by f.

In [Go94], L.R.Goldberg proved that Π is surjective. Indeed, for each $\Theta \in \mathcal{A}$, she constructed a degree d topological polynomial g which maps $X_g(r)$ onto $X_g(dr)$ conformally. All the critical points of g are in $\partial X_g(r)$. Moreover, g induces the prescribed critical portrait Θ. Then g pullbacks the standard complex structure on $X_g(r)$ to the space

$$R = \bigcup_{n\geq 0} g^{-n}(X_r).$$

Thus $g : R \to R$ is a complex analytic map. Since R is a planar Riemann surface, it can be conformally embedded in $\mathbb{C}$ by Koebe's general uniformization Theorem 9.1 in [Sp81]. The complement $\mathbb{C} \setminus R$ is a holomorphically removable Cantor set [SN70]. Thus g extends to a holomorphic map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which is a degree d polynomial with the required critical portrait Θ.

In [Ki05], Kiwi showed that Π is continuous and, for any Θ, the preimage $S_\Theta = \Pi^{-1}(\Theta)$ is a 1-real dimensional analytic manifold. Precisely, the map $G : S_\Theta \to (0, \infty)$ which sends f to $G_f(c_i)$ is bijective and analytic. Moreover, given $r > 0$, the restriction

$$\Pi|_{\mathcal{S}_d(r)} : \mathcal{S}_d(r) \to \mathcal{A}_d$$

is a homeomorphism.

The *connected locus* $\mathcal{C}_d$ is the set of monic centered polynomials with degree d such that all the critical orbits are bounded. We know that $\mathcal{C}_d$ is a compact and connected subset of $\mathcal{P}_d$ [BH88]. For instance, $\mathcal{C}_2$ is the Mandelbrot set. To describe $\mathcal{C}_d$ we look at it from outside $\mathcal{S}_d$.

The *impression* $I_{\mathcal{C}_d}(\Theta)$ of critical portrait Θ is a subset of $\mathcal{C}_d$, characterized by the property that $f \in I_{\mathcal{C}_d}(\Theta)$ if and only if there exists a sequence $\{f_n\}$ in S'_d such that $\Pi(f_n) = \Theta$ and f_n converges to f.

Note that the impression here is slightly different from the definition in [Ki05], which is bigger and containing $I_{\mathcal{C}_d}(\Theta)$. J.Kiwi proved that if all arguments in Θ is strictly pre-periodic, then the impression $I_{\mathcal{C}_d}(\Theta)$ is a singleton [Ki05]. He conjectured that there exist critical portraits with aperiodic kneading and non-trivial impressions.

Main results. In this chapter, we shall give an elementary proof of the following two theorems based on the tools in [CT15].

Theorem 4.1. *The map*

$$P : \mathcal{A}_d \times (0, \infty) \to \mathcal{S}'_d \qquad (\Theta, r) \to f_{\Theta,r}$$

where $f_{\Theta,r} \in \mathcal{S}_d(r)$ induces critical portrait Θ, is well-defined, one-to-one and continuous.

The well-defined and one-to-one properties are proved by quasiconformal surgery. In parameter space we will call the simple curve

$$R_\Theta(t) := P(\Theta, \cdot) : (0, \infty) \to \mathcal{S}'_d$$

parameter ray in $\mathcal{S}_d$ with argument Θ. For quadratic polynomials, R_Θ is exactly parameter ray outside of Mandelbrot set. We will say that R_Θ lands if and only if the impression $I_{\mathcal{C}_d}(\Theta)$ is a singleton.

In dynamical plane, let f be a polynomial in $\mathcal{C}_d$ with J_f locally connected and all cycles repelling. A critical portrait $\Theta = \{\Theta_1, \cdots, \Theta_n\}$ is called a critical portrait of f if each external rays with arguments in Θ_i lands at critical point c_i, for $1 \le i \le n$.

We have the following landing theorem.

Theorem 4.2. *Let f be a polynomial in $\mathcal{C}_d$ with non recurrent critical points and all cycles repelling. Then the parameter ray $R_\Theta(t)$ lands at f if and only if Θ is a critical portrait of f.*

The only if part comes from combinatorial continuity [Ki05] Theorem 1.

4.2 Preliminaries

In this section, we collect some known result on spherical metric, the distortions of modulus, as well as shapes and turning, by holomorphic maps. The convergence of rational maps on $\overline{\mathbb{C}}$ is also discussed. These preliminaries will be used in the rest of this chapter.

4.2.1 Spherical metric

We will denote by $B_e(z,r)$, $\text{dist}_e(x,y)$, $\text{diam}_e W$ $\text{Area}_e S$ the Euclidean balls, distances, diameters and Euclidean area. While $B(z,r)$, $\text{dist}(x,y)$ and $\text{diam}\, W$ are measured in spherical metric.

Recall that the spherical line element and spherical area element on $\overline{\mathbb{C}}$ are

$$ds = \frac{2|dz|}{1+|z|^2} \quad \text{and} \quad d\sigma = \frac{4dxdy}{(1+|z|^2)^2}.$$

So we have

• $\frac{2}{5}|dz| \le ds \le 2|dz|$ on $\overline{B_e(0,2)}$,

• the holomorphic map $\alpha : z \mapsto 1/z$ preserves the spherical distance and spherical area,

• let $\epsilon_0 = \text{dist}\,(B_e(0,1), \partial B_e(0,2))$, then any subset S of $\overline{\mathbb{C}}$ with diam $S < \epsilon_0$ is either contained in $B_e(0,2)$ or $\overline{\mathbb{C}} \setminus B_e(0,1)$.

• $\inf_{x\in\overline{\mathbb{C}}, 0<r<\epsilon_0/2} \left\{\text{mod } B(x,2r) \setminus \overline{B(x,r)}\right\} \ge m_0 > 0.$

4.2.2 Mañé Lemma

Lemma 4.1 ([Ma93],Theorem II)**.** *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map with degree at least two. If a point $x \in J_f$ is not a parabolic periodic point and is not contained in the ω-limit set of a recurrent critical point, then for any $\epsilon > 0$ there exist $\delta = \delta(x,\epsilon) < \epsilon$ and integer $\eta = \eta(x,\epsilon)$ such that for any $r \le \delta$ and any $n \ge 0$,*

(1) *every component of $f^{-n}B(x,r)$ has spherical diameter less than ϵ,*

(2) *for every component W of $f^{-n}B(x,r)$, degree of $f^n : W \to B(x,r)$ is less than η,*

(3) *every component of $f^{-n}B(x,r)$ is a topological disk.*

4.2.3 Distortions of modulus, shape and turning

Lemma 4.2 ([KL09],Lemma 4.5)**.** *Let $U_i \subseteq V_i$ in $\overline{\mathbb{C}}$ be a pair of Jordan disks, where $i = 1, 2$. Suppose that $g : V_1 \to V_2$ is a proper holomorphic map of degree d and U_1 is a component of $g^{-1}(U_2)$. Then*

$$\mathrm{mod}\ V_1 \setminus \overline{U}_1 \leq \mathrm{mod}\ V_2 \setminus \overline{U}_2 \leq d\ \mathrm{mod}\ V_1 \setminus \overline{U}_1.$$

Let U be a domain in $\overline{\mathbb{C}}$ and $z \in U$. The *Shape$_e$* resp. *Shape* of U about z is defined as

$$\mathrm{Shape}_e(U, z) = \frac{\max_{w \in \partial U} \mathrm{dist}_e(w, z)}{\min_{w \in \partial U} \mathrm{dist}_e(w, z)} \quad \text{resp.} \quad \mathrm{Shape}(U, z) = \frac{\max_{w \in \partial U} \mathrm{dist}(w, z)}{\min_{w \in \partial U} \mathrm{dist}(w, z)}$$

Obviously, $B(z, r) \subseteq U \subseteq B(z, kr)$ for some r, where $k := \mathrm{Shape}(U, z)$. Thus U is a round disk centered at z if and only if $\mathrm{Shape}(U, z) = 1$.

Let E be a compact set in $\overline{\mathbb{C}}$ and $z_1, z_2 \in E$, the *turning* is defined as

$$\Lambda_e(E, z_1, z_2) = \frac{\mathrm{diam}_e E}{\mathrm{dist}_e(z_1, z_2)} \quad \text{resp.} \quad \Lambda(E, z_1, z_2) = \frac{\mathrm{diam} E}{\mathrm{dist}(z_1, z_2)}.$$

We have the following lemma,

Lemma 4.3 ([QWY12], Lemma 6.1)**.** *Let $U_i \subseteq V_i$ in $\overline{\mathbb{C}}$ be a pair of Jordan disks with $\mathrm{mod}(V_2 \setminus \overline{U_2}) \geq m > 0$, where $i = 1, 2$. Suppose that $g : V_1 \to V_2$ is a proper holomorphic map of degree $\leq d$ and U_1 is a component of $g^{-1}(U_2)$. Then there are two positive constants $C_1'(d, m)$ and $C_2'(d, m)$ depending only on d and m, such that*

(1) *for all $z \in U_1$, the shape satisfies*

$$\mathrm{Shape}_e(U_1, z) \leq C_1'(d, m) \mathrm{Shape}_e(U_2, g(z)), \tag{4.1}$$

(2) *for any connected and compact subset E of U_1 with the cardinal number $\#E \geq 2$ and any $z_1, z_2 \in E$, the turning satisfies*

$$\Lambda_e(E; z_1, z_2) \leq C_2'(d, m) \Lambda_e(g(E); g(z_1), g(z_2)). \tag{4.2}$$

Remark. If we give additional condition that diam U_i, diam V_i and diam E are less than ϵ_0. Then there exist positive constants $C_1(d, m)$ and $C_2(d, m)$ such that (4.1) and (4.2) hold by replacing Shape_e, Λ_e and $C_i'(d, m)$ with Shape, Λ and $C_i(d, m)$ respectively.

From now on when we apply Lemma 4.3, we always assume U_i, V_i and E satisfying this additional condition.

4.2.4 Convergence of rational map sequences

Throughout this chapter, if not otherwise stated, the convergence of maps on $\overline{\mathbb{C}}$ is measured in spherical metric.

If a sequence of rational maps $\{f_n\}$ uniformly converges on $\overline{\mathbb{C}}$, then it converges to a rational map g and deg $(f_n) =$ deg (g) as n is large enough. Moreover, the coefficients of f_n converges to that of g as well. For more results, we have the following lemma. See also in [CT15].

Lemma 4.4. *Let $\{f_n\}$ be a sequence of rational maps with constant degree $d \geq 1$. Let $U \subseteq \overline{\mathbb{C}}$ be a non-empty open set and $\{f_n\}$ converge uniformly to a map g on U as $n \to \infty$, then g is a rational map and* deg $g \leq d$*. Moreover,* deg $g = d$ *implies that $\{f_n\}$ converges uniformly to g on $\overline{\mathbb{C}}$ as $n \to \infty$.*

Proof. By composing Möbius transformations, we may assume that $\infty \in U$ and $f_n(\infty) \to 1$. Thus as n is large enough, the function f_n has the form

$$f_n(z) = k_n \frac{(z-a_{1,n})\cdots(z-a_{d,n})}{(z-b_{1,n})\cdots(z-b_{d,n})},$$

and

$$k_n \to 1 \text{ as } n \to \infty. \tag{4.3}$$

Since $f_n^{-1}(0) = \{a_{1,n}, \cdots, a_{d,n}\}$, $f_n^{-1}(\infty) = \{b_{1,n}, \cdots, b_{d,n}\}$ and $f_n(\infty) \to 1$, both $\{a_{i,n}\}$ and $\{b_{j,n}\}$ are bounded in $\mathbb{C}$. Passing to a subsequence $\{f_{n_k}\}$, we have

$$(a_{1,n_k}, \cdots, a_{d,n_k}; b_{1,n_k}, \cdots, b_{d,n_k}) \to (a_1, \cdots, a_d; b_1, \cdots, b_d) \text{ as } n_k \to \infty. \tag{4.4}$$

Without loss of generality, we assume that $a_i \neq a_j$, $b_i \neq b_j$ for $i \neq j$ and

$$a_{d_0+1} = b_{d_0+1}, \cdots, a_d = b_d, \quad a_i \neq b_j \text{ for } 0 \leq i, j \leq d_0 \tag{4.5}$$

for some $0 \leq d_0 \leq d$. Let

$$g_1(z) = \frac{(z-a_1)\cdots(z-a_{d_0})}{(z-b_1)\cdots(z-b_{d_0})} \quad \text{or} \quad g_1(z) = 1 \text{ if } d_0 = 0.$$

We claim that f_{n_k} converges locally uniformly to g_1 on $\overline{\mathbb{C}} \setminus \{a_{d_0+1}, \cdots, a_d\}$.

Proof. Consider the metric $d(\cdot,\cdot)$ on $\overline{\mathbb{C}}$

$$d(z,z') := \frac{2|z-z'|}{\sqrt{1+|z|^2}\cdot\sqrt{1+|z'|^2}} \text{ and } d(\infty,z) := \lim_{\xi\to\infty} d(\xi,z)$$

for $z,z'\in\mathbb{C}$. We know that it is equivalent to the spherical metric on $\overline{\mathbb{C}}$.

For any $z\in\mathbb{C}$, the distance $d(f_{n_k}(z),g_1(z))$ equals

$$\frac{2\left|k_n\prod_{1\le i\le d}(z-a_{i,n_k})\prod_{1\le i\le d_0}(z-b_i)-\prod_{1\le i\le d_0}(z-a_i)\prod_{1\le i\le d}(z-b_{i,n_k})\right|}{\sqrt{\left|k_n\prod_{1\le i\le d}(z-a_{i,n_k})\right|^2+\left|\prod_{1\le i\le d}(z-b_{i,n_k})\right|^2}\cdot\sqrt{\left|\prod_{1\le i\le d_0}(z-a_i)\right|^2+\left|\prod_{1\le i\le d_0}(z-b_i)\right|^2}}$$
$$=: \frac{\Delta'_{z,n_k}}{\Delta''_{z,n_k}}$$

By assumption the claim follows at the point ∞.

Let

$$\delta := \frac{1}{3}\cdot\min_{x\neq x'\in\{a_1,\cdots,a_d,b_1,\cdots,b_{d_0},\infty\}} \text{dist }(x,x').$$

For any $x\in\mathbb{C}\setminus\{a_{d_0+1},\cdots,a_d\}$, $\overline{B(x,\delta)}\subseteq\mathbb{C}$. Moreover, by (4.3) (4.4) (4.5),

$$\sup_{z\in B(x,\delta)} \Delta'_{z,n_k}\to 0 \text{ as } k\to\infty$$

and there exists $\delta_x>0$ such that

$$\inf_{z\in B(x,\delta),k\ge 0} \Delta''_{z,n_k}\ge\delta_x.$$

This is because, as k large enough, at most one sequences of $\{a_{i,n_k}\}$ and $\{b_{i,n_k}\}$ stays in the disk $B(x,2\delta)$. Therefore, $\{f_{n_k}\}$ is uniformly convergent on $B(x,\delta)$. The claim is proved. □

Since $\{f_n\}$ converges uniformly to g on U, we have $\{f_n\}$ converges uniformly to $g(=g_1)$ on any compact subset in $\overline{\mathbb{C}}\setminus\{a_{d_0+1},\cdots,a_d\}$ by the claim and hence g is rational map with $\deg(g)=d_0\le d$. Moreover, $\deg(g)=d$ implies $d_0=d$ and $\{f_n\}$ converges uniformly to g on $\overline{\mathbb{C}}$. The lemma is finished. □

4.3 Modulus distortion dominates the spherical distortion

In this section, for our purpose we introduce a new quantity, namely *the maximal distortion of modulus* in [CT15], to control the distance between univalent map and Möbius transformation. Although there are many existent measurements, such as the norm Schwarzian derivative, or the maximal distortion of cross-ratios or extremal lengths etc.

4.3.1 Maximal distortion of modulus

Let V be an open set in $\overline{\mathbb{C}}$ and $\phi : V \to \overline{\mathbb{C}}$ a univalent map. For any two disjoint full continua E_1, E_2 in V, we denote by $A(E_1, E_2) := \overline{\mathbb{C}} \setminus (E_1 \bigcup E_2)$ which is annulus induced by E_1, E_2.

Define *the maximal distortion of modulus* $\mathfrak{D}(\phi, V)$, as follows

$$\mathfrak{D}(\phi, V) := \sup_{E_1, E_2 \subseteq V} |\text{mod } A(E_1, E_2) - \text{mod } A(\phi(E_1), \phi(E_2))|,$$

where E_1, E_2 are disjoint full continua in V with mod $A(E_1, E_2) < \infty$.

This quantity may not be finite, the first example being z^2 acting on the right half plane (see the theorem below for a proof). Also, although it is clear that $\mathfrak{D}(\phi, V) = 0$ if ϕ is the restriction of a Möbius transformation on V, but it is not at all obvious that the converse is also true.

In order to understand the following theorem, let us look at a guiding example: consider V a neighborhood of $\{0, \infty\}$ and define a univalent map ϕ on V as

$$\phi(z) = \begin{cases} \lambda_0 z, & \text{near } 0 \\ z/\lambda_\infty, & \text{near } \infty \end{cases} \quad \text{with } \lambda_0, \lambda_\infty \neq 0.$$

For ϵ sufficiently small, let $E_1 = \{z \in \overline{\mathbb{C}} : |z| \leq \epsilon\}$ and $E_2 = \{z \in \overline{\mathbb{C}} : |z| \geq \frac{1}{\epsilon}\}$. We have $A(\phi(E_1), \phi(E_2)) = \{|\lambda_0|\epsilon < |z| < \frac{1}{|\lambda_\infty|\epsilon}\}$. So

$$|\text{mod } A(E_1, E_2) - \text{mod } A(\phi(E_1), \phi(E_2))| = \frac{1}{2\pi} \log |\lambda_0 \lambda_\infty|| \leq \mathfrak{D}(\phi, V).$$

4.3.2 Properties of maximal distortion of modulus

Theorem 4.3. *Let $\phi : V \to \overline{\mathbb{C}}$ be an univalent map on open set V.*

(1) *For any Möbius transformations α, β, we have*

$$\mathfrak{D}(\alpha \circ \phi \circ \beta, \beta^{-1}(V)) = \mathfrak{D}(\phi, V).$$

(2) *If V contains $0, \infty$ and ϕ fixes $0, \infty$, then, setting $\phi'(\infty) := \lim_{z \to \infty} \frac{z}{\phi(z)}$,*

$$\frac{1}{2\pi} |\log |\phi'(0)\phi'(\infty)|| \leq \mathfrak{D}(\phi, V).$$

(3) *If* V *contains* $0,1,\infty$ *and* ϕ *fixes* $0,1$ *and* ∞*, then, for any* $z \in V \setminus \{\infty\}$*,*

$$\frac{1}{5\pi}\left|\log|\phi'(z)|\right| \le \mathfrak{D}(\phi, V).$$

(4) $\mathfrak{D}(\phi, V) = 0$ *if and only if* ϕ *is the restriction of a Möbius transformation on* V*.*

(5) *If an extension of* ϕ *has a critical point on the boundary of* V*, then* $\mathfrak{D}(\phi, V) = +\infty$*.*

(6) *If* V *contains* $0, \mathbb{D}(1, r_0)$ *and* ∞*. Then there exists constant* $0 < C(r_0) < \infty$ *such that, for any univalent map* ϕ *fixing* $0,1$ *and* ∞*, we have*

$$\sup_{z \in V}\{\text{dist } (\phi(z), z)\} \le C(r_0) \cdot \mathfrak{D}(\phi, V).$$

Proof. (1) It follows evidently. Because Möbius transformations preserve the modulus of annulus.

(2) Let $M_0(\epsilon), m_0(\epsilon)$ be the supremum and infimum of $|\phi(z)|$ on the circle $\{|z| = \epsilon\}$, and $M_\infty(\epsilon)$, $m_\infty(\epsilon)$ be the supremum and infimum of $|\psi(z)|$ on the circle $\{|z| = 1/\epsilon\}$.

Set $E_1 := \{|z| \le \epsilon\}$, $E_2 := \{z \in \overline{\mathbb{C}} : |z| \ge \frac{1}{\epsilon}\}$ and $A_\phi := A(\phi(E_1), \phi(E_2))$. Then

$$\{M_0(\epsilon) < |z| < m_\infty(\epsilon)\} \subseteq A_\phi \subseteq \{m_0(\epsilon) < |z| < M_\infty(\epsilon)\}$$

and

$$\frac{1}{2\pi}\log\frac{\epsilon^2 m_\infty(\epsilon)}{M_0(\epsilon)} \le \text{mod } A_\phi - \text{mod } A \le \frac{1}{2\pi}\log\frac{\epsilon^2 M_\infty(\epsilon)}{m_0(\epsilon)}.$$

Since

$$|\phi'(0)| = \lim_{\epsilon\to 0}\frac{M_0(\epsilon)}{\epsilon} = \lim_{\epsilon\to 0}\frac{m_0(\epsilon)}{\epsilon} \quad \text{and} \quad |\phi'(\infty)| = \lim_{\epsilon\to 0}\frac{1}{\epsilon \cdot M_\infty(\epsilon)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon \cdot m_\infty(\epsilon)},$$

we get

$$\frac{1}{2\pi}\left|\log|\phi'(0)\phi'(\infty)|\right| = \lim_{\epsilon\to 0}|\text{mod } A_\phi - \text{mod } A| \le \mathfrak{D}(\phi, V).$$

(3) For arbitrary distinct $z, w \in V \setminus \{\infty\}$, there exist Möbius transformations

$$\beta_a(\xi) = \frac{w \cdot \xi + z \cdot a}{\xi + a} \quad \text{and} \quad \alpha_b(\xi) = b \cdot \frac{\xi - \phi(z)}{\xi - \phi(w)}$$

with $a, b \in \mathbb{C}^*$ such that $\psi_{a,b} := \alpha_b \circ \phi \circ \beta_a$ fixes 0 and ∞. By (1) and (2), we have

$$\begin{aligned}\mathfrak{D}(\phi, V) = \mathfrak{D}(\psi_{a,b}, \beta_a^{-1}(V)) &\ge \frac{1}{2\pi}\left|\log|\psi'_{a,b}(0)\psi'_{a,b}(\infty)|\right| \\ &= \frac{1}{2\pi}\left|\log\frac{|\phi'(z)\phi'(w)||z-w|^2}{|\phi(z)-\phi(w)|^2}\right| =: \frac{1}{2\pi}\mathfrak{D}_1(\phi)(z,w). \end{aligned} \tag{4.6}$$

Set $\delta := \mathfrak{D}(\phi, V)$, $\lambda_0 = |\phi'(0)|$, $\lambda_1 = |\phi'(1)|$ and $\lambda_\infty = |\phi'(\infty)|$. Apply (4.6) for the pairs

$(0,1)$ and $(1,\xi)$ as $\xi \to \infty$, we get $|\log \lambda_i \lambda_j| \le 2\pi\delta$ for $i \ne j$. It follows that

$$|\log \lambda_i| \le 3\pi\delta \quad \text{for} \quad i \in \{0,1,\infty\}$$

For any $z \in V \setminus \{\infty\}$, choose $w_n \to \infty$ as $n \to \infty$. By (4.6), we have $|\log |\phi'(z)\lambda_\infty|| \le 2\pi\delta$. Thus $|\log |\phi'(z)|| \le 5\pi\delta$.

(4) Without loss of generality, we can assume ϕ fix $0, 1, \infty$ on V by (1). Then (3) implies $\phi'(z) = 1$ on V. Therefore $\phi = id$.

(5) It follows directly by (3).

(6) See Theorem 8.1 (b) in [CT15].

□

4.4 Controlling modulus distortions by areas

In this section, firstly we recall some classic results on modulus of annulus without proof. Then we give a lemma, which provides a way to control the maximal distortion of modulus.

4.4.1 Modulus of annulus

Notations $\mathbb{C}^* := \overline{\mathbb{C}} \setminus \{0,\infty\}$, $\mathbb{D}_t := \{z \in \overline{\mathbb{C}} : |z| < t\}$, $A(z_0; r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ and $A(r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ for $z_0 \in \mathbb{C}$ and $0 < r_1 < r_2$.

Let $A \subseteq \overline{\mathbb{C}}$ be annulus. Let Γ_{Height} be the family of all locally rectifiable curves in A joining the two components of $\overline{\mathbb{C}} \setminus A$ and Γ_{Width} the family of all locally rectifiable closed curves in A separating the two components of $\overline{\mathbb{C}} \setminus A$.

Let $\rho : A \to [0,\infty]$ be a non-negative Borel measure function on A. The *ρ-area* of A is

$$\text{Area } (\rho, A) := \iint_A \rho^2(z)dxdy.$$

The *ρ-length* of a locally rectifiable curve γ in A is

$$\text{Length } (\rho, \gamma) := \int_\gamma \rho(z)|dz|.$$

Height(ρ, A) is the infimum of Length (ρ,γ) over all $\gamma \in \Gamma_{\text{Height}}$. Width (ρ, A) is the infimum of Length (ρ,γ) over all $\gamma \in \Gamma_{\text{Width}}$.

The *modulus of annulus* A is defined to be the modulus of the family of curves Γ_{Width}, that is,

$$\text{mod } A := \inf \text{Area } (\rho, A), \tag{4.7}$$

where the infimum is taken over all nonnegative Borel functions ρ with Width $(\rho, A) \geq 1$.

There are some well-known facts about modulus of annulus.

• It is preserved by conformal maps.

• It equals the *extremal length* of the family of curves Γ_{Height}, that is,

$$\text{mod } A = \sup \frac{\text{Height } (\rho, A)^2}{\text{Area } (\rho, A)} \tag{4.8}$$

where ρ is taken over all nonnegative Borel function ρ with $0 < \text{Area}(\rho, A) < \infty$.

• If mod $A < \infty$, there exists an unique conformal map $\chi_A : A \to A(1, e^{2\pi \text{mod } A})$ up to a post composition of a rotation.

• ρ_0 reaches the infimum in (4.7) if and only if

$$\rho_0(z) = |(\frac{1}{2\pi} \cdot \log \circ \chi_A)'(z)|,$$

which is the pullback of Euclidean metric in $\mathbb{C}$. ρ_0 is called *extremal metric* on A.

• Extremal metric ρ_0 realizes the supremum in (4.8), thus

$$\text{mod } A = \text{Height } (\rho_0, A) = \text{Area } (\rho_0, A) \quad \text{and} \quad \text{Width } (\rho_0, A) = 1. \tag{4.9}$$

• If A in $\mathbb{C}$ is annulus with mod $A > \frac{5log2}{2\pi}$. Then there exists an annulus $A(z_0; r_1, r_2)$ contained essentially in A such that

$$\text{mod } A(z_0; r_1, r_2) \geq \text{mod } A - \frac{5log2}{2\pi}.$$

A *conformal metric* ρ, i.e., nonnegative Borel measurable function, on A is called to be *length increasing* if

$$\text{Width } (\rho, A) \geq 1 \quad \text{and} \quad \text{Height } (\rho, A) \geq \text{Height } (\rho_0, A). \tag{4.10}$$

4.4.2 Area difference controls modulus distortion

Lemma 4.5. *Let Z be a compact set in $\overline{\mathbb{C}}$ and ϕ univalent on $\overline{\mathbb{C}} \setminus Z$. Let E_1, E_2 be disjoint full continua in $\overline{\mathbb{C}} \setminus Z$. Then*

$$|\text{mod } A(E_1, E_2) - \text{mod } A(\phi(E_1), \phi(E_2))| \leq \text{Area}(\rho, A) - \text{Area}(\rho_0, A)$$

for arbitrary length increasing conformal metric ρ on $A(E_1, E_2)$ with $\rho|_Z = 0$.

Proof. We write $A := A(E_1, E_2)$ and $A_\phi := A(\phi(E_1), \phi(E_2))$. Let ρ_0 be an extremal metric on A.

For arbitrary length increasing conformal metric ρ on A with $\rho|_Z = 0$, set

$$\widetilde{\rho}(z) = \begin{cases} 0, & \text{on } A_\phi \setminus \phi(A \setminus Z) \\ \frac{\rho(\phi^{-1}(z))}{|\phi'(z)|}, & \text{on } \phi(A \setminus Z). \end{cases}$$

Then it is not difficult to check that

- Area $(\rho, A) =$ Area $(\widetilde{\rho}, A_\phi)$,
- Width $(\widetilde{\rho}, A_\phi) \geq$ Width (ρ, A),
- Height $(\widetilde{\rho}, A_\phi) \geq$ Height (ρ, A).

Combining with (4.7), (4.8), (4.9) and (4.10), we have

$$\text{mod } A_\phi - \text{mod } A \leq \text{Area } (\widetilde{\rho}, A_\phi) - \text{mod } A = \text{Area } (\rho, A) - \text{Area } (\rho_0, A)$$

and

$$\begin{aligned} \text{mod } A - \text{mod } A_\phi &= \text{Area } (\rho_0, A) - \text{mod } A_\phi \\ &\leq \text{Area } (\rho_0, A) - \frac{\text{Height } (\widetilde{\rho}, A_\phi)^2}{\text{Area } (\widetilde{\rho}, A_\phi)} \\ &\leq \text{Area } (\rho_0, A) - \frac{\text{Height } (\rho, A)^2}{\text{Area } (\rho, A)} \\ &\leq \text{Area } (\rho_0, A) - \frac{\text{Height } (\rho_0, A)^2}{\text{Area } (\rho, A)} \\ &\leq \text{Area } (\rho, A) - \text{Area } (\rho_0, A). \end{aligned}$$

The last inequality holds by the fact Height $(\rho_0, A) =$ Area (ρ_0, A) and the Cauchy Inequality. Thus the lemma is proved. □

4.5 Univalent maps off a finite nested disc system

The result in this section generalizes the discussion in ([CT15], Section 8.2). At first, we give the definition of nested disk systems, the m-nested and λ-scattered properties. Then we pay great effort to prove Theorem 4.4.

4.5.1 Nested disk system

Let Y be a finite set in $\overline{\mathbb{C}}$. A collection of open topological disks $\{D_x\}_{x \in Y}$ is called *nested disk system* if

- each D_x contains x,
- if $D_x \bigcap D_y \neq \emptyset$ then either $D_x \subsetneq D_y$ or $D_x \supsetneq D_y$.

Let $\{D_x\}_{x\in Y}$ be a nested disk system. Let $\{D''_x\}_{x\in Y}$ and $\{D''_x\}_{x\in Y}$ be topological disks. Then $\{D''_x, D'_x, D_x\}_{x\in Y}$ is said to be *m-nested* if

- $x \in D''_x \subseteq D'_x \subseteq D_x$,
- for any $D_y \neq D_x$ with $D_y \bigcap \overline{D''_x} \neq \emptyset$, we have $D_y \subseteq \overline{D'_x}$,
- $\inf_{x\in Y}\{\text{mod } D_x \setminus \overline{D'_x}\} \geq m > 0$.

Let W be an open set in $\overline{\mathbb{C}}$ such that $\overline{\bigcup_{x\in Y} D_x} \subseteq W$. Let V_x be the union of all $D_y(\neq D_x)$ contained in D_x and W_x the component of W containing x. Then the nested disk system $\{D_x\}_{x\in Y}$ is said to be λ-*scattered* in W with $\lambda \in (0,1)$ if and only if for each $x \in Y$ and any univalent map $h : W_x \to \mathbb{C}^*$,

$$\text{Area}(\rho_*, h(V_x)) \leq \lambda \cdot \text{Area}(\rho_*, h(D_x)),$$

where $\rho_* := \frac{1}{2\pi|z|}$ is a planar metric on $\mathbb{C}^*$.

4.5.2 The boundedness of Area$_p(E, W)$

The following lemma will become natural by the end of the proof of Theorem 4.4.

Lemma 4.6. *Let $W \subsetneq \overline{\mathbb{C}}$ be an open set with $\#\overline{\mathbb{C}} \setminus W \geq 2$. Suppose E is a measurable set with $\overline{E} \subsetneq W$. Then*

$$\text{Area}_p(E, W) := \sup_{\phi} \text{Area}\ (\rho_*, \phi(E)) < \infty,$$

where the supermum is taken over all univalent maps $\phi : W \to \mathbb{C}^*$.

Proof. Without loss of generality, assume $W \subseteq \mathbb{C}^*$. Let $d = \text{dist}_e(E, \partial W) > 0$. Let $\phi : W \to \mathbb{C}^*$ be any univalent map. For any $z \in E$, consider the univalent map $\phi : B_e(z, d) \to \mathbb{C}^*$. By the classic Koebe's 1/4-Theorem, we have

$$B_e(\phi(z), \frac{|\phi'(z)|d}{4}) \subseteq \phi(B_e(z, d)) \subseteq \phi(W) \subseteq \mathbb{C}^*.$$

Then

$$|\phi(z)| = \text{dist}_e(\phi(z), 0) \geq \frac{|\phi'(z)d}{4}.$$

We estimate,

$$\text{Area }(\rho_*, \phi(E)) = \iint_{\phi(E)} (\frac{1}{2\pi|\xi|})^2 dxdy = \iint_E \frac{|\phi'(z)|^2}{4\pi^2|\phi(z)|^2} dxdy$$
$$\leq \frac{4}{\pi^2 d^2}\text{Area}_e(E) < \infty \tag{4.11}$$

By the arbitrariness of ϕ, the proof is completed. $\square$

4.5.3 Univalent maps off m-nested and λ-scattered nested disc system

Theorem 4.4. *Let Y be a finite set in $\overline{\mathbb{C}}$. Let $\{(D''_x, D'_x, D_x)\}_{x\in Y}$ be m-nested and λ-scattered disk system in open set $W \subseteq \overline{\mathbb{C}}$. Let $D := \bigcup_{x\in Y} D_x$ and $D'' := \bigcup_{x\in Y} D''_x$. Then For any sufficient large m, there exists $C(m,\lambda) > 0$ with $C(m,\lambda) \to 0$ as $m \to \infty$, such that*

$$\mathfrak{D}(\phi, \overline{\mathbb{C}} \setminus \overline{W}) \leq C(m,\lambda) \cdot \text{Area}_p(D, W)$$

for any univalent map $\phi : \overline{\mathbb{C}} \setminus \overline{D''} \to \overline{\mathbb{C}}$.

Proof. Firstly, we define a partial order on the finite set Y. For any $x, y \in Y$,

$$x \prec y \iff \text{ either } x = y \text{ or } D_x \subsetneq D_y.$$

Let I_1 be the set of all maximal elements in Y. Inductively, I_k, $k \geq 2$, is set of all maximal elements in $Y \setminus \bigcup_{1\leq i\leq k-1} I_i$. Then

$$Y = I_1 \bigcup I_2 \bigcup \cdots \bigcup I_n.$$

We denote by $D_k := \bigcup_{x\in I_k} D_x$ and $D''_k := \bigcup_{x\in I_k} D''_x$. Obviously, it follows that

$$D = D_1 \supsetneq D_2 \supsetneq \cdots \supsetneq D_n. \tag{4.12}$$

Secondly, for $2 \leq k \leq n$, set

$$I'_k := \{x \in I_k : D_x \bigcap \overline{D''_y} = \emptyset \text{ for any } y \in I_1 \bigcup \cdots \bigcup I_{k-1}\} \subseteq I_k$$

and

$$I''_k := \{x \in I_k : D_x \bigcap \overline{D''_y} \neq \emptyset \text{ for some } y \in I_1 \bigcup \cdots \bigcup I_{k-1}\} \subseteq I_k.$$

For $k = 1$, set $I'_1 := I_1$ and $I''_1 := \emptyset$. Obviously, I_k is the union of the two disjoint sets I'_k and I''_k.

Thirdly, for any disjoint full continua E_1, E_2 in $\overline{\mathbb{C}} \setminus \overline{W}$, we write $A := A(E_1, E_2)$. Suppose mod $A < \infty$. Let $\chi_A : A \to A(1, e^{2\pi \text{mod } A})$ be conformal. Then

$$\rho_0(z) = |(\frac{1}{2\pi}\text{log} \circ \chi_A)'(z)|$$

is an extremal metric on A and satisfies (4.9).

Define $\rho_k(z)$ on A for $1 \le k \le n$ inductively by

$$\rho_k(z) = \begin{cases} 0, & z \in \bigcup_{x \in I'_k} D'_x \\ \rho_0(z)(1 - e^{-\pi m})^{-k}, & z \in \bigcup_{x \in I'_k} D_x \setminus D'_x \\ \rho_{k-1}(z), & z \in \bigcup_{x \in I''_k} D_x \\ \rho_{k-1}(z), & z \in A \setminus D_k. \end{cases}$$

When reaching the deepest nest, the metric stabilizes to ρ_n. This is the metric we were looking for.

Claim 1. For $1 \le k \le n$, $\rho_k(z) \le \rho_0(z)(1 - e^{-\pi m})^{-k}$ on A.

Proof. It is obviously true for $k = 1$.

By induction, for $k \ge 2$, we have

$$\begin{aligned} \rho_k(z) &\le \max \{\rho_0(z)(1 - e^{-\pi m})^{-k}, \rho_{k-1}(z)\} \\ &\le \max \{\rho_0(z)(1 - e^{-\pi m})^{-k}, \rho_0(z)(1 - e^{-\pi m})^{-k+1}\} \\ &\le \rho_0(z)(1 - e^{\pi m})^{-k} \end{aligned}$$

for any z in A. Thus the Claim follows. □

Claim 2. For $2 \le k \le n$, $\rho_k(z) = \rho_{k-1}(z) = 0$ on $\bigcup_{x \in I''_k} D_x$.

Proof. For any $z \in \bigcup_{x \in I''_k} D_x$, by (4.12) there exists an unique sequence $\{x_i\}_{1 \le i \le k}$ in Y with $x_i \in I_i$ such that

$$z \in D_{x_k} \subsetneq \cdots \subsetneq D_{x_2} \subsetneq D_{x_1}. \tag{4.13}$$

Since $x_k \in I''_k$, then there exists x_{l_1} with $1 \le l_1 \le k - 1$ such that $D_{x_k} \bigcap \overline{D''_{x_{l_1}}} \ne \emptyset$. By (4.13), for $l_1 - 1 \le i \le k$, we have $D_{x_i} \bigcap \overline{D''_{x_{l_1}}} \ne \emptyset$ and thus $x_i \in I''_i$. Therefore,

$$\rho_k(z) = \rho_{k-1}(z) = \cdots = \rho_{l_1}(z). \tag{4.14}$$

The m-nested property gives

$$z \in D_{x_k} \subsetneq D'_{x_{l_1}} \subsetneq D_{x_{l_1}}. \tag{4.15}$$

If x_{l_1} is contained I'_{l_1}. We have $\rho_{l_1}(z) = 0$ by (4.15) and the definition of ρ_{l_1}. Otherwise we have $x_{l_1} \in I''_{l_1}$. Then we continue the discussion and obtain $l_1 > l_2 > l_3 > \cdots$ satisfying (4.14) and (4.15). After at most $k-1$ steps, we must have some x_{l_j} in I'_{l_j} such that $x \in D'_{x_{l_j}} \subsetneq D_{x_{l_j}}$. This is because $I''_1 = \emptyset$. So $\rho_k(z) = \rho_{l_j}(z) = 0$. The Claim is proved. $\square$

Claim 3. $\rho_n(z) = 0$ on D''.

Proof. Recall that $D'' = \bigcup_{x \in Y} D''_x$. For any $x \in Y$ and any $z \in D''_x$. Suppose $x \in I_k$. By (4.12), there exist a maximal $k \leq l \leq n$ and sequence $\{x_i\}_{k \leq i \leq l}$ with $x_i \in I_i$ and $x_k := x$ such that

$$z \in D_{x_l} \subsetneq D_{x_{l-1}} \subsetneq \cdots \subsetneq D_{x_k}.$$

It follows that $D_{x_i} \bigcap \overline{D''_{x_k}} \neq \emptyset$ and so $x_i \in I''_i$ for $k \leq i \leq l$. By Claim 2, we have

$$0 = \rho_k(z) = \cdots = \rho_l(z) = \cdots = \rho_n(z).$$

The Claim is proved. $\square$

Claim 4. Let $x \in I_k$ with $1 \leq k \leq n$. Then for any curve $\gamma \subseteq \overline{D}_x$ with endpoints $\gamma(0), \gamma(1) \in \partial D_x$, there exists a curve $\tilde{\gamma}$ in $\overline{D}_x$ with $\tilde{\gamma}(0) = \gamma(0)$ and $\tilde{\gamma}(1) = \gamma(1)$ such that

$$\text{Length } (\rho_k, \gamma) \geq \text{Length } (\rho_{k-1}, \tilde{\gamma}). \tag{4.16}$$

Proof. If $x \in I''_k$, let $\tilde{\gamma} := \gamma$, by definition of ρ_k, it follows. Thus we assume $x \in I'_k$. Since $\rho_k(z) = \rho_0(z)(1 - e^{-\pi m})^{-k} > \rho_{k-1}(z)$ by Claim 1. We also assume $\gamma \bigcap D'_x \neq \emptyset$.

Let $h_x : D_x \to \mathbb{C}$ be a univalent branch of the map $\frac{1}{2\pi}\log \circ \chi_A$ (See figure 4.1). Let $H_x = h_x(D_x \setminus \overline{D'_x})$. Then $\rho_0(z) = |h'_x(z)|$ and

$$\text{mod } H_x = \text{mod } D_x \setminus \overline{D'_x} \geq m > \frac{5\log 2}{2\pi}.$$

So there exists an annulus $A(z_x; r_x, R_x)$ essentially contained in H_x such that

$$\text{mod } A(z_x; r_x, R_x) = \frac{1}{2\pi}\log\frac{R_x}{r_x} \geq m - \frac{5\log 2}{2\pi} \geq \frac{m}{2}. \tag{4.17}$$

Let $A_x := h_x^{-1}(A(z_x; r_x, R_x))$. Let $t_0 := \inf\{t : \gamma(t) \in A_x\}$ and $t_1 := \sup\{t : \gamma(t) \in A_x\}$. Set $\xi := \gamma(t_0), \eta := \gamma(t_1), \gamma_1 := \gamma|_{[0,t_0]}, \gamma_2 := \gamma|_{[t_0,t_1]}$ and $\gamma_3 := \gamma|_{[t_1,1]}$. Then $\gamma_1, \gamma_3 \subseteq D_x \setminus D'_x$ and γ_2 crosses A_x at least two times. For $i \in \{1, 3\}$,

$$\text{Length } (\rho_k, \gamma_i) \geq \text{Length } (\rho_{k-1}, \gamma_i). \tag{4.18}$$

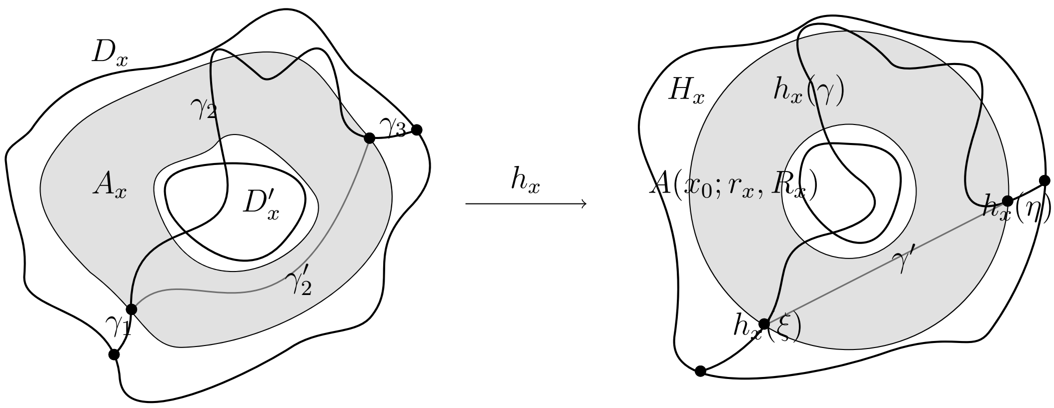

Figure 4.1: Illustrating the proof of Claim 4

Together with (4.17), we estimate

$$\begin{aligned}\text{Length } (\rho_k, \gamma_2) &= \int_{\gamma_2} \rho_0(z)(1-e^{-\pi m})^{-k}|dz| = (1-e^{-\pi m})^{-k}\int_{h_x(\gamma_2)}|dz| \\ &\geq (1-e^{-\pi m})^{-k}(2R_x - 2r_x) \\ &\geq (1-e^{-\pi m})^{-k+1}2R_x \\ &\geq (1-e^{-\pi m})^{-k+1}\int_{\gamma'}|dz| \\ &= \int_{\gamma'_2}\rho_{k-1}(z)|dz| = \text{Length } (\rho_{k-1}, \gamma'_2), \end{aligned} \tag{4.19}$$

where $\gamma' \subseteq h_x(D_x)$ is a straight line segment connecting $h_x(\xi), h_x(\eta)$ and $\gamma'_2 := h_x^{-1}(\gamma')$. Set $\widetilde{\gamma} := \gamma_1 \bigcup \gamma'_2 \bigcup \gamma_3$. By (4.18) and (4.19), the Claim follows. $\square$

Claim 5. ρ_n is a length increasing conformal metric on A with $\rho_n|_{D''} = 0$.

Proof. By Claim 3 $\rho_n|_{D''} = 0$. For arbitrary $\gamma \in \Gamma_{\text{Height}}$, applying Claim 4, we have

$$\begin{aligned}\text{Length } (\rho_n, \gamma) &= \text{Length } (\rho_n, \gamma \bigcap A \setminus D_n) + \sum_{x\in I_n} \text{Length } (\rho_n, \gamma \bigcap \overline{D_x}) \\ &\geq \text{Length } (\rho_{n-1}, \gamma \bigcap A \setminus D_n) + \sum_{x\in I_n} \text{Length } (\rho_{n-1}, \widetilde{\gamma}_x) \\ &= \text{Length } (\rho_{n-1}, \widetilde{\gamma}) \geq \cdots \geq \text{Length } (\rho_0, \gamma').\end{aligned}$$

where $\gamma' \in \Gamma_{\text{Height}}$. Hence it follows that Height $(\rho_n, A) \geq$ Height $(\rho_0, A) \geq 1$. By the same arguments, we have Width $(\rho_n, A) \geq$ Width (ρ_0, A). The Claim is proved. $\square$

Since $\{D_x\}_{x\in Y}$ is λ-scattered. By (4.12), for any $1 \leq k \leq n-1$,

$$\text{Area } (\rho_0, D_{k+1}) \leq \lambda \text{Area } (\rho_0, D_k) \leq \cdots \leq \lambda^k \text{Area } (\rho_0, D_1) \tag{4.20}$$

Now applying Claim 5, Lemma 4.5, Claim 1 and (4.20), we have

$$\begin{aligned}
0 &\leq \text{Area }(\rho_n, A) - \text{Area }(\rho_0, A) \\
&= \iint_{D_n} (\rho_n^2 - \rho_0^2) dxdy + \sum_{1 \leq k \leq n-1} \iint_{D_k \setminus D_{k+1}} (\rho_n^2 - \rho_0^2) dxdy \\
&= \iint_{D_n} (\rho_n^2 - \rho_0^2) dxdy + \sum_{1 \leq k \leq n-1} \iint_{D_k \setminus D_{k+1}} (\rho_k^2 - \rho_0^2) dxdy \\
&\leq \iint_{D_n} \frac{\rho_0^2}{(1-e^{-\pi m})^{2n}} - \rho_0^2 dxdy + \sum_{1 \leq k \leq n-1} \iint_{D_k \setminus D_{k+1}} \frac{\rho_0^2}{(1-e^{-\pi m})^{2k}} - \rho_0^2 dxdy \\
&\leq \sum_{1 \leq k \leq n} \left(\frac{1}{(1-e^{-\pi m})^{2k}} - 1 \right) \text{Area }(\rho_0, D_k) \\
&\leq \sum_{k \geq 1} \left(\frac{1}{(1-e^{-\pi m})^{2k}} - 1 \right) \lambda^{k-1} \text{Area }(\rho_0, D_1).
\end{aligned}$$

If m is sufficient large such that $(1 - e^{-\pi m})^2 > \lambda$. Then the last term of the above inequalities is

$$\frac{e^{-\pi m}(2 - e^{-\pi m})}{(1-\lambda)\left[(1-e^{-\pi m})^2 - \lambda\right]} \text{Area }(\rho_0, D_1) := C(m.\lambda) \cdot \text{Area }(\rho_0, D_1).$$

By the arbitrariness of E_1, E_2 in $\overline{\mathbb{C}} \setminus \overline{W}$, Lemma 4.6 and Lemma 4.5, we have proved the theorem. $\square$

4.6 Application to rational maps

The aim of this section is to construct a sequence of m_k-nested λ-scattered nested disk systems from pullback disk systems with $m_k \to \infty$ as $k \to \infty$. We require the nested disk systems are uniformly λ-scattered. Moreover, each one contains a pullback disk system. See Proposition 4.1 for details.

One of our challenge is to deal with the situation that one critical point c is contained in the w-limit set of another critical point c'. Thus for arbitrary small disk $B(c, r)$, after pulling it back, the preimages will eventually meet c'. This makes it impossible to obtain a desired nested disc system from pullback disk system directly. To overcome this problem, we select nice disks $B(c_i, r_i)$ and discard all of the "bad" components in preimages, namely the component containing points of lower level. The remaining components still contains all the preimages. See (4.32).

Throughout this section, we shall adopt the assumption in the following subsection.

4.6.1 Assumption

Let f be a rational map with degree $d \geq 2$ and no empty Fatou set. Denote by $\mathrm{Orb}(S) := \bigcup_{k\geq 0} f^k(S)$ for set $S \subseteq \overline{\mathbb{C}}$. Let $X_0 := X_0' \bigcup X_0''$ be a finite set in $\overline{\mathbb{C}}$ such that

- X_0' is contained in J_f and X_0'' in Fatou set,
- for distinct $x, x' \in X_0$, $x \notin \mathrm{Orb}(x')$,
- X_0 has no recurrent points, that is, $x \notin \omega(x)$ for any $x \in X_0$,
- $\overline{\mathrm{Orb}(X_0')}$ is disjoint with the ω-limit set of all recurrent critical points,
- $\overline{\mathrm{Orb}(X_0')}$ is disjoint with the parabolic cycles,
- for any critical point $c \notin X_0'$, $\omega(c) \bigcap X_0' = \emptyset$,
- X_0'' contains no periodic points.

Thus X_0'' are disjoint with the immediate rotation domains, such as periodic Siegel disks and Herman rings, and there exists $\delta_0 > 0$ such that

(1) $B(x, \delta_0) \bigcap B(y, \delta_0) = \emptyset$ for any $x \neq y \in X_0$.

(2) For any $x \in X_0'$, $B(x, \delta_0) \bigcap \mathrm{Orb}(c) = \emptyset$ for any critical point $c \notin X_0$.

(3) For any $x \in X_0''$, $B(x, \delta_0) \setminus \{x\}$ is disjoint with critical orbits and Julia set , thus every component of $f^{-n}B(x, \delta_0)$ is open disk and in Fatou set.

(4) For any $x \in X_0''$, $B(x, \delta_0) \bigcap \mathrm{Orb}(fB(x, \delta_0)) = \emptyset$. Because each points in X_0'' is iterated into either periodic rotation domains or converging to periodic points. Hence $f^{-n_1}B(x, \delta_0) \bigcap f^{-n_2}B(x, \delta_0) = \emptyset$ for any $n_1 \neq n_2 \geq 0$.

4.6.2 Shrinking Lemma

Lemma 4.7 (Shrinking lemma). *Let $K := \overline{\mathrm{Orb}(X_0')} \bigcup X_0''$. Then for any $\epsilon > 0$, there exist $0 < \delta \leq \epsilon$ and integer $\eta \geq 1$, such that, for all $x \in K$ and $n \geq 1$,*

(1) *any component of $f^{-n}B(x, \delta)$, written $B^{-n}(x, \delta)$, is an open disk,*

(2) diam $B^{-n}(x, \delta) \leq \epsilon$,

(3) $\sup_{x\in K}\{\text{diam } B^{-n}(x, \delta)\} \to 0$ *as* $n \to \infty$,

(4) *degree of the covering $g_n := f^n : B^{-n}(x, \delta) \to B(x, \delta)$ is less than η,*

(5) *for any ball $B(z, r)$ in $B(x, \delta)$,* $\mathrm{Shape}(B^{-n}(z, r), \xi) \leq C_1 := C_1(\eta, m_0)$, *where $B^{-n}(z, r)$ is a component of $g_n^{-1}B(z, r)$ and $g_n(\xi) = z$,*

(6) *for any two balls $B(z', r') \subseteq B(z, r)$ in $B(x, \delta)$, let $B^{-n}(z', r') \subseteq B^{-n}(z, r)$ be components in $B^{-n}(x, \delta)$, $C_2 := C_2(\eta, m_0)$, then*

$$\frac{\text{diam } B^{-n}(z, r)}{\text{diam } B^{-n}(z', r')} \leq C_2 \cdot \frac{r}{r'}.$$

Recall that m_0 is a constant in subsection 4.2.1 and $C_1(\eta, m_0), C_2(\eta, m_0)$ in Lemma 4.3.

Proof. Firstly, we claim that for any $x \in X_0''$, diam $B^{-n}(x, \delta_0/2) \to 0$ as $n \to \infty$.

Proof. Since $\bigcup_{k\geq 0} f^{-k}B(x,\delta_0)$ contains at most finite critical points. Thus the degree of covering map $f^n : B^{-n}(x,\delta_0) \to B(x,\delta_0)$ is uniformly upper bounded by a integer η_x. By Lemma 4.3,

$$\text{Shape}(B^{-n}(x,\delta_0/2)) \leq C_1(\eta_x, m_0).$$

We know that all disks $\{B^{-n}(x,\delta_0/2)\}_{n\geq 0}$ are pairwise disjoint (see Subsection 4.6.1 (4)). Then the claim follows. $\square$

For any $x \in X_0''$, choose η_x and $\delta_x \leq \delta_0/2$ satisfying (1)-(4) by the claim. For any $x \in \overline{\text{Orb}(X_0')}$, let δ_x, η_x be in Lemma 4.1. Then $\{B(x,\delta_x)\}_{x\in K}$ covers the compact set K. Thus there exists finite set Σ such that $K \subseteq \bigcup_{x\in\Sigma} B(x,\delta_x)$.

Let 3δ be the Lebesgue number of the finite open covering and

$$\eta := \max\ \{\eta_x, x \in \Sigma\}.$$

Then any $B(x, 2\delta)$ with $x \in K$ is contained in a ball $B(\xi, \delta_\xi)$ with $\xi \in \Sigma$. Therefore, (1)(2)(4)(5)(6) hold by Lemma 4.1 and Lemma 4.3.

For (3), if not, there exists a sequence of disks $\{B^{-n_k}(x_k,\delta)\}_{k\geq 0}$, $x_k \in K, n_k \to \infty$ as $k \to \infty$, such that their diameters are greater than some positive number. Since shapes of these disks are bounded by $C_1(\eta, m_0)$. There exist a ball $B(x_\infty, r_0)$ with $x_\infty \in J_f$ contained in infinitely many disks of $\{B^{-n_k}(x_k,\delta)\}$. This contradicts the fact that $J_f \subseteq f^N(B(x_\infty, r_0))$ for sufficiently larger N. $\square$

4.6.3 A sequence of arbitrary small nice disks $\mathcal{N}_x$ around x

Lemma 4.8 (Key lemma)**.** *Let x be a point in X_0 such that $\omega(x)$ is disjoint with the subset Y_x of X_0. Then there exist $\mathcal{N}_x = \{(E_{n,x}, O_{n,x}, U_{n,x})\}_{n\geq 1}, \delta_x > 0$, and $C_x \geq 1$, such that*

(1) *$O_{n,x}, U_{n,x}$ are open disks with $x \in O_{n,x} \subseteq U_{n,x}$,*

(2) *$x \notin \overline{E}_{n,x} \subseteq O_{n,x}$ and $\overline{E}_{n,x}$ is contained in the Fatou set,*

(3) Shape *$(E_{n,x}, \xi) \leq C_x$, for some $\xi \in E_{n,x}$, and* diam $U_{n,x} \leq C_x \cdot$ diam $E_{n,x}$*,*

(4) diam $U_{n,x} \to 0$ *as $n \to \infty$,*

(5) *If $B^{-k}(y,\delta_x) \bigcap \partial U_{n,x} \neq \emptyset$, then $B^{-k}(y,\delta_x) \bigcap O_{n,x} = \emptyset$ for any $y \in Y_x$, $k \geq 1$ and components $B^{-k}(y,\delta_x)$ of $f^{-k}B(y,\delta_x)$,*

(6) *$E_{n,x} \bigcap f^{-k}B(y,\delta_x) = \emptyset$ for any $y \in Y_x$ and $k \geq 1$.*

Proof. If $x \in X_0''$. Then $\{f^n(x)\}$ either locally uniformly converges to a periodic cycle or eventually conjugates to a irrational rotation. Thus there exists a neighborhood $\overline{B(x,\delta)}$

in Fatou set with $\delta \le \delta_0$ such that $\overline{\mathrm{Orb}(fB(x,\delta))} \bigcap Y_x = \emptyset$. Let

$$\delta_x := \frac{1}{2} \cdot \mathrm{dist}\ (\overline{\mathrm{Orb}(fB(x,\delta))}, Y_x).$$

Then $B(x,\delta) \bigcap f^{-k}B(y,\delta_x) = \emptyset$ for any $y \in Y_x$ and $k \ge 1$. Indeed, if not, suppose $\zeta \in B(x,\delta) \bigcap f^{-k}B(y,\delta_x)$. We have $f^k(\zeta) \in \overline{\mathrm{Orb}(fB(x,\delta))} \bigcap B(y,\delta_x)$, a contradiction.

Set

$$O_{n,x} := U_{n,x} := B(x, \delta/2^n), \quad E_{n,x} := B(\xi_n, \delta/(3 \cdot 2^n)) \subseteq O_{n,x} \text{ with } x \notin \overline{E}_{n,x}$$

and $C_x := 3$. Then $\mathcal{N}_x$, δ_x, C_x satisfy the conditions (1)-(6).

If $x \in X_0'$. Denote by $x_n := f^n(x)$ and

$$\epsilon_0 := \frac{1}{2} \cdot \mathrm{dist}\ (\overline{\mathrm{Orb}(f(x))}, Y_x).$$

Applying Lemma 4.7 by setting $\epsilon = \epsilon_0$, we obtain δ and η_1. Let $B^{-n}(x_n, \delta)$ be the component of $f^{-n}B(x_n, \delta)$ containing x. Then we have

$$\epsilon_1 := \inf_{n \ge 1, 1 \le k \le n} \{\mathrm{dist}\ (f^k B^{-n}(x_n, \delta), Y_x)\} > 0. \tag{4.21}$$

Every ball $B(z, \delta/5)$ with $z \in \overline{\mathrm{Orb}(x)}$ intersects the Fatou set. Thus there exists a sufficiently small ball $E_z \subseteq B(z, \delta/5)$ such that $\overline{E_z}$ is contained in the Fatou set and $\overline{\mathrm{Orb}(E_z)} \bigcap Y_x = \emptyset$. Let $\{B(z,\delta/5)\}_{z \in \Sigma}$ be a finite open covering of $\overline{\mathrm{Orb}(x)}$. Using Lemma 4.7 again by setting

$$\epsilon = \min\ \{\epsilon_1, \min_{x \in \Sigma} \mathrm{dist}\ (\overline{\mathrm{Orb}(E_z)}, Y_x), \delta/2\}, \tag{4.22}$$

we obtain δ_x and η_2. Let $\eta := \max\ \{\eta_1, \eta_2\}$.

Now we set $U_{n,x} := B^{-n}(x_n, \delta)$ and $O_{n,x} := B^{-n}(x_n, \delta/2)$, both of which contain x. Since there must exist $\xi \in \Sigma$ such that

$$E_\xi \subseteq B(\xi, \delta/5) \subseteq B(x_n, \delta/2),$$

we can set $E_{n,x}$ to be a component of $f^{-n}E_\xi$ in $O_{n,x}$.

We are left to check that $\mathcal{N}_x := \{(E_{n,x}, O_{n,x}, U_{n,x})\}_{n \ge 1}$ is as required. For (3), by Lemma 4.7 (6),

$$\frac{\mathrm{diam}\ U_{n,x}}{\mathrm{diam}\ E_{n,x}} \le C_2(\eta, m_0) \frac{2\delta}{\mathrm{diam}\ E_\xi} \le C_2(\eta, m_0) \frac{2\delta}{\min_{z \in \Sigma}\{\mathrm{diam}\ E_z\}} =: C_x$$

We claim that for any $y \in Y_x$ and $k \geq 1$, if $U_{n,x} \bigcap f^{-k}B(y, \delta_x) \neq \emptyset$, then $k \geq n+1$. Indeed, if not, we have dist $(f^k U_{n,x}, \{y\}) < \delta_x \leq \epsilon_1$, this contradicts (4.21).

Thus, for (5), we have $f^n B^{-k}(y, \delta_x) = B^{-k'}(y, \delta_x)$ with $k' := k - n \geq 1$ and $B^{-k'}(y, \delta_x) \bigcap \partial B(x_n, \delta) = \emptyset$. Since diam $B^{-k'}(y, \delta_x) < \delta/2$ by (4.22). We have $B^{-k'}(y, \delta_x) \bigcap B(x_n, \delta/2) = \emptyset$. It follows that $B^{-k}(y, \delta_x) \bigcap O_{n,x} = \emptyset$.

For (6), since $\overline{\mathrm{Orb}(E_{n,x})} \bigcap B(y, \delta_x) = \emptyset$ for any $y \in Y_x$ by (4.22) and the claim. It follows evidently. $\square$

4.6.4 Pullback disk systems

For $n \geq 1$, let $X_n := f^{-n}(X_0) \setminus \bigcup_{0 \leq k \leq n-1} X_k$ and $X := \bigcup_{n \geq 0} X_n$. Let $n(x) \geq 0$ be the integer such that $x \in X_{n(x)}$.

Let $\{U_x\}_{x \in X_0}$ be a collection of pairwise disjoint open disks with $x \in U_x$ such that, for any $y \in X$ with $f^{n(y)}(y) = x$, the component U_y of $f^{-n(y)}(U_x)$, which contains y, is an open disk. Then we say that $\{U_x\}_{x \in X}$ is a *pullback disk system* of f induced by $\{U_x\}_{x \in X_0}$.

Let $\{U_x\}_{x \in X}$ be a pullback disk systems. For each $x \in X$ with $y = f^{n(x)}(x)$, define

$$\chi_x := \chi_y \circ f^{n(x)} : U_x \to \mathbb{D} \quad x \mapsto 0,$$

and $U_x(r)$ the component of $\chi_x^{-1}(\mathbb{D}_r)$ containing x, where $\chi_y : U_y \to \mathbb{D}$ with $y \mapsto 0$ is a conformal map. It is easy to check that

(1) $f(U_x(r)) = U_{f(x)}(r)$ for $n(x) \geq 1$, $r \in (0, 1)$,

(2) either $U_x \bigcap U_y = \emptyset$ or $U_x = U_y$ for distinct x, y with $n(x) = n(y) \geq 0$,

(3) if $U_x(r) \bigcap S \neq \emptyset$ resp. $\partial U_x(r) \bigcap S \neq \emptyset$, then $U_y(r) \bigcap f^n(S) \neq \emptyset$ resp. $\partial U_y(r) \bigcap f^{n(x)}(S) \neq \emptyset$, where $y = f^{n(x)}(x)$, for any $x \in X$ and set $S \subseteq \overline{\mathbb{C}}$.

4.6.5 The construction of $\{(U_x, O_x, E_x, U_x(r_x), U_x(r'_x))\}_{x \in X_0}$ in W.

Let W be an open set on $\overline{\mathbb{C}}$ such that $\overline{X} \subseteq W$. Let m be a given positive number. Firstly we define a partial order on X_0. For arbitrary $x, x' \in X_0$

$$x \prec x' \iff \text{ either } x = x' \text{ or } x \in \omega(x').$$

One can check that this order satisfies transitivity property, because of the fact that if $x \in \omega(x')$ then $\omega(x) \subseteq \omega(x')$.

Let L_1 be the set of all maximal elements in X_0. Inductively, $L_k, k \geq 2$, is the set of all maximal elements in $X_0 \setminus \bigcup_{1 \leq i \leq k-1} L_i$. Then

$$X_0 = L_1 \bigcup L_2 \bigcup \cdots \bigcup L_N.$$

It is not difficult to check that $x \notin \omega(x')$ for any $x \in L_k, x' \in L_{k'}$ with $1 \leq k \leq k' \leq N$.

Applying Lemma 4.7 by setting

$$\epsilon = \frac{1}{2} \cdot \min \{\delta_0, \text{dist } (\partial W, \overline{X})\},$$

we obtain $\delta_0' \leq \epsilon$, η. For each x in X_0, let Y_x be the collection of points y in X_0 such that $y \notin \omega(x)$. Let $\mathcal{N}_x = \{(E_{k,x}, O_{k,x}, U_{k,x})\}_{k\geq 0}$, $\delta_x > 0$ and C_x satisfying the conditions (1)-(6) in Lemma 4.8. Set

$$\delta_0'' := \min_{x\in X_0} \{\delta_x\} > 0, \quad C_0 := \max_{x\in X_0} \{C_x\} \geq 1. \tag{4.23}$$

Let A_x be the forward orbits of points $y \in X_0$ with $x \notin \omega(y)$. Set

$$\delta_0''' := \min_{x\in X_0} \text{dist}(x, A_x \setminus \{x\}). \tag{4.24}$$

Inductively, for each $x \in L_k, k = 1, 2, \cdots, N$, by Lemma 4.7 (2) and Lemma 4.8 (4), we

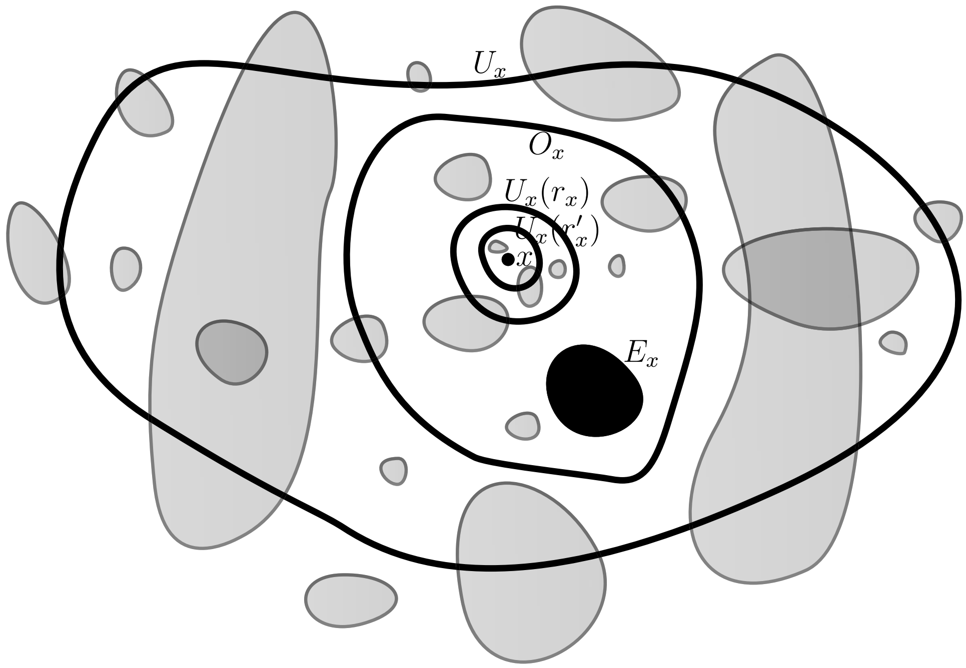

Figure 4.2: A sketch of the open sets $U_x, O_x, E_x, U_x(r_x)$ and $U_x(r_x')$. The shaded domains are pullback disks.

can choose $(E_x, O_x, U_x) \in \mathcal{N}_x$ such that

$$\text{diam } U_x^{-n} \leq \min \{\delta_{k-1}', \delta_{k-1}'', \delta_{k-1}'''\}. \tag{4.25}$$

where U_x^{-n} takes over all components of $f^{-n}U_x$, $n \geq 0$.

Let $\chi_x : U_x \to \mathbb{D}$ be a conformal map with $\chi_x(x) = 0$. We denote by $U_x(r) := \chi_x^{-1}\mathbb{D}_r$. Since $x \notin \overline{E_x}$, we can choose r_x sufficiently small such that $U_x(r_x) \subseteq O_x$ and $U_x(r_x) \bigcap E_x = \emptyset$, moreover,

$$\mathrm{mod}\ O_x \setminus \overline{U_x(r_x)} \geq \eta \cdot m \tag{4.26}$$

For any components U_y^{-n} of $f^{-n}U_y$, $y \in L_1 \bigcup \cdots \bigcup L_k, n \geq 1$, since it can not contain x by (4.24), there exists $r'_x < r_x$ independent of y, n such that if $U_y^{-n} \bigcap \partial U_x(r'_x) \neq \emptyset$ then $U_y^{-n} \subseteq U_x(r_x)$ by the Shrinking Lemma 4.7 (3).

Set

$$\delta'_k := \min_{x \in L_1 \cup \cdots \cup L_k} \{\mathrm{dist}\ (\overline{E}_x, J_f), \mathrm{dist}\ (\overline{E}_x, x)\}, \quad \delta''_k := \min_{x \in L_1 \cup \cdots \cup L_k} \{\mathrm{dist}\ (O_x, \partial U_x)\}, \tag{4.27}$$

and

$$\delta'''_k := \min_{x \in L_1 \cup \cdots \cup L_k} \{\mathrm{dist}\ (x, \partial U_x(r'_x)), \mathrm{dist}\ (U_x(r'_x), \partial U_x(r_x)\}. \tag{4.28}$$

4.6.6 Properties of the desired pullback disk system.

Let $\mathcal{U} := \{U_x\}_{x \in X}$ be the pullback disk system in W induced by $\{U_x\}_{x \in X_0}$ as constructed above. Let X'' be the collection of elements x in X such that there exists some $y \in X$ with $y \in U_x$ and $n(y) \leq n(x)$ and $X' := X \setminus X''$.

Consider the sets $\mathcal{U}' := \{U_x\}_{x \in X'}$ and $\mathcal{U}'' := \{U_x\}_{x \in X''}$.

(1) For any U_x in $\mathcal{U}'$, it does not contain points x' in X of lower level, that is, $x' \notin U_x$ if $n(x') \leq n(x)$.

(2) $\mathcal{U}'$ is a forward invariant set, that is, $f(U_x) = U_{f(x)} \in \mathcal{U}'$ if $U_x \in \mathcal{U}'$ and $n(x) \geq 1$. While $\mathcal{U}''$ is a backward invariant set, that is, $U_x \in \mathcal{U}''$ if $U_{f(x)} \in \mathcal{U}''$.

(3) For $U_x \in \mathcal{U}'$, critical points of the covering mapping

$$f_x := f^{n(x)} : U_x \to U_y$$

can only be the point x, where $y := f_x(x)$. Thus we can define

$$O_x := f_x^{-1}(O_y), r_x := r_y, r'_x := r'_y \tag{4.29}$$

and E_x any one of the component $f_x^{-1}(E_y)$.

(4) For any $x \in X'$, set

$$Z_x(r) := \{y \in X', U_y \bigcap \partial U_x(r) \neq \emptyset \text{ and } \ n(y) \geq n(x)\}, r \in (0, 1].$$

Then the followings hold by (4.25), (4.27), (4.28), subsection 4.6.4(1)(3), and the choice of $U_x(r'_x)$,

$$\bigcup_{y\in Z_x(r'_x)} U_y \subseteq U_x(r_x) \subseteq O_x \subseteq U_x \setminus \bigcup_{y\in Z_x(1)} U_y. \tag{4.30}$$

and

$$U_y \bigcap E_x = \emptyset \quad \text{if} \quad y \in X \setminus \{x\} \text{ and } n(y) \geq n(x). \tag{4.31}$$

(5) For any $x \in X''$, there exists $\xi \in X'$ with $n(\xi) \leq n(x)$ such that $U_x \subseteq U_\xi(r'_\xi)$. Indeed, there exists k such that $f^k U_x \in \mathcal{U}''$ and $f^{k+1} U_x \in \mathcal{U}'$. Then $f^k U_x$ contains a point y in X_0. By (4.24) and (4.28), we have $f^{n(x)}(x) \in \omega(y)$ and $f^k U_x \subseteq U_y(r'_y)$. So $U_x \subseteq U_{y_1}(r'_{y_1})$ with $f^{n(y_1)}(y_1) = y$ and $n(y_1) < n(x)$. If $y_1 \notin X'$, continue the same process to U_{y_1}, U_{y_2}, $\cdots$. Since $n(x) > n(y_1) > n(y_2) > \cdots \geq 0$. After finite steps, we have $\xi := y_k \in X'$. Thus it follows that, for any $n \geq 0$,

$$\bigcup_{x\in X'', n(x)\leq n} U_x \subseteq \bigcup_{x\in X', n(x)\leq n} U_x(r'_x). \tag{4.32}$$

4.6.7 From pullback disk system to nested disk systems.

Let $\mathcal{U}'$ be the pullback disk system of X' as mentioned above. Given $n \geq 0$, for $n(x) = n$ with $x \in X'$, set $D_x := U_x$. Obviously $O_x \subseteq D_x \subseteq U_x$.

Inductively, for $k = n-1, n-2, \cdots, 0$ and $n(x) = k$ with $x \in X'$, let

$$b_k(x) := \{y \in X', k+1 \leq n(y) \leq n \text{ and } D_y \bigcap \partial U_x \neq \emptyset\}.$$

By (4.26) (4.30) and (4.31), there exists a Jordan domain D_x, which is the component of $U_x \setminus \overline{\bigcup_{y\in b_k(x)} U_y}$ containing x, such that the following holds

$$x \in U_x(r'_x) \subseteq U_x(r_x) \subseteq O_x \subseteq D_x \subseteq U_x. \tag{4.33}$$

Moreover,

$$\operatorname{mod} D_x \setminus \overline{U_x(r_x)} \geq m \quad \text{and} \quad E_x \subseteq D_x \setminus \overline{U_x(r_x)}. \tag{4.34}$$

By the construction, it is not difficult to check that $\{D_x\}_{n(x)\leq n, x\in X'}$ is a nested disk system.

In summary, we have the following proposition.

Proposition 4.1. *Let W be an open topological disk in $\overline{\mathbb{C}}$ such that $\overline{X} \subseteq W$. Then there exists uniform $\lambda \in (0,1)$ such that, for any $m > 0$ and any integer $n \geq 0$, we have*

- *m-nested system $\{(D''_x, D'_x, D_x)\}_{x\in Y_n}$ and λ-scattered $\{D_x\}_{x\in Y_n}$ in W, where Y_n is a subset of $\{x \in X; n(x) \leq n\}$,*
- *pullback disk system $\{B_x\}_{x\in X}$ induced by disks $\{B(x,\delta)\}_{x\in X_0}$, where $\delta := \delta(m)$ is independent on n,*

such that

$$\bigcup_{n(x)\le n, x\in X} B_x \subseteq \bigcup_{x\in Y_n} D''_x. \tag{4.35}$$

Proof. Given $m>0$, construct $\{(D_x, U_x, O_x, E_x, U_x(r_x), U_x(r'_x)\}_{x\in X}$ in W as above.

Let $Y_n := \{x\in X'; n(x)\le n\}$, $D'_x := U_x(r_x)$ and $D''_x := U_x(r'_x)$. The equations (4.30), (4.33) and (4.34) imply that $\{(D''_x, D'_x, D_x)\}_{x\in Y_n}$ is m-nested.

For the λ-scattered property, by pre-composing a Möbius transformation, we may assume D is bounded in $\mathbb{C}$ and $\overline{W}\subseteq\mathbb{C}$. Recall that $D := \bigcup_{x\in X} D_x$. Let V_x be the union of all $D_y(\neq D_x)$ contained in D_x and $h: W\to\mathbb{C}^*$ arbitrary univalent map.

By (4.31), (4.33) and (4.34), we have

$$E_x\subseteq D_x \quad\text{and}\quad E_x\bigcap V_x=\emptyset.$$

Consider the map $f^{n(x)}: U_x\to U_y$, where $y := f^{n(x)}(x)$. The Lemma 4.7(5)(6), Lemma 4.8(3) and equations (4.23) (4.33) give

$$\begin{aligned}\text{Shape } (E_x,\xi) &\le C_1\cdot\text{Shape } (E_y, f^{n(x)}(\xi))\\ &\le C_1C_0\end{aligned}$$

and

$$\text{diam } D_x\le \text{diam } U_x\le C_2C_y\cdot\text{diam } E_x\le C_2C_0\cdot\text{diam } E_x.$$

Thus we have

$$\text{Area}_e D_x\le C\cdot\text{Area}_e E_x. \tag{4.36}$$

for some constant $C>1$ independent of x, m and n.

Let W' be another open topological disk in $\overline{\mathbb{C}}$ such that $\overline{D}\subseteq\overline{W'}\subseteq W$. Applying the Koebe distortion therorem to h, we get a constant $C_3\ge 1$ such that

$$\max_{\xi\in\overline{W'}}|h'(\xi)|\le C_3\cdot\min_{\xi\in\overline{W'}}|h'(\xi)|. \tag{4.37}$$

Since h preserves modulus of annulus, we have mod $h(W)\setminus\overline{h(W')}=$ mod $W\setminus\overline{W'}$. There exists constant C_4 independent of h such that

$$\text{diam}_e h(W')\le C_4\cdot\text{dist}_e(\partial h(W),\partial h(W')).$$

Let ξ_h be in $\overline{W'}$ such that $|h(\xi_h)|=\max_{\xi\in\overline{W'}}\text{dist}_e(h(\xi),0)$. Then for any $\xi\in W'$, we have

$$\frac{|h(\xi_h)|}{|h(\xi)|}\le\frac{|h(\xi)|+\text{diam}_e h(W')}{|h(\xi)|}\le 1+\frac{\text{diam}_e h(W')}{\text{dist}_e(\partial h(W),\partial h(W'))}\le 1+C_4 \tag{4.38}$$

Now we estimate, using equations (4.36), (4.37) and (4.38),

$$\begin{aligned}
\text{Area } (\rho_*, h(E_x)) &= \iint_{h(E_x)} \frac{1}{4\pi^2|\xi|^2} dxdy = \iint_{E_x} \frac{|h'(\xi)|^2}{4\pi^2|h(\xi)|^2} dxdy \\
&\geq \frac{1}{C_3^2} \iint_{E_x} \frac{|h'(\xi_h)|^2}{4\pi^2|h(\xi_h)|^2} dxdy \\
&\geq \frac{1}{CC_3^2} \iint_{D_x} \frac{|h'(\xi_h)|^2}{4\pi^2|h(\xi_h)|^2} dxdy \\
&\geq \frac{1}{CC_3^4(1+C_4)} \iint_{D_x} \frac{|h'(\xi)|^2}{4\pi^2|h(\xi)|^2} dxdy \\
&=: (1-\lambda) \text{ Area } (\rho_*, h(D_x))
\end{aligned}$$

Therefore,

$$\text{Area } (\rho_*, h(V_x)) \leq \text{Area } (\rho_*, h(D_x)) - \text{Area } (\rho_*, h(E_x)) \leq \lambda \cdot \text{Area } (\rho_*, h(D_x)).$$

To prove (4.35), let $\{B_x\}_{x\in X}$ be the pullback disk system by choosing

$$\delta := \min_{x\in X_0}\{r'_x\}.$$

Then it follows by (4.29) and (4.32). $\square$

4.6.8 Univalent maps off a pullback disk system

Theorem 4.5 (Controlling distortion of univalent map)**.** *Let V be a Jordan domain on $\overline{\mathbb{C}}$ such that $\overline{X} \subseteq \overline{\mathbb{C}} \setminus \overline{V}$. Let z_1, z_2, z_3 be three distinct points in V. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{z\in V} \{\text{dist } (\phi(z), z)\} \leq \epsilon,$$

for any univalent map $\phi : \overline{\mathbb{C}} \setminus \overline{\bigcup_{n(x)\leq n} B_x} \to \overline{\mathbb{C}}$ fixing z_1, z_2, z_3, where $\{B_x\}_{x\in X}$ is the pullback disk system in $\overline{\mathbb{C}} \setminus \overline{V}$ induced by $\{B(x,\delta)\}_{x\in X_0}$.

Proof. Let W' be Jordan domain on $\overline{\mathbb{C}}$ such that $\overline{W'} \subseteq W := \overline{\mathbb{C}} \setminus \overline{V}$ and $\overline{X} \subseteq W'$. By Proposition (4.1), there exist m-nested system $\{(D''_x, D'_x, D_x)\}_{x\in Y_n}$, λ-scattered $\{D_x\}_{x\in Y_n}$ in W' and pullback disk system $\{B_x\}_{x\in X}$ induced by disks $\{B(x,\delta)\}_{x\in X_0}$.

Let α be the Möbius transformation sending $0, 1, \infty$ to z_1, z_2, z_3 respectively. Suppose $\mathbb{D}(1, r_0) \subseteq \alpha^{-1}(V)$. Denote by $\psi := \alpha^{-1} \circ \phi \circ \alpha$.

Applying Theorem 4.3(1)(6), equation (4.35), Theorem 4.4 and the monotonicity of $\text{Area}_p(D, W)$ on D, we have, for any $z \in \alpha^{-1}(V)$,

$$\begin{aligned}
\text{dist } (\psi(z), z) &\leq C(r_0) \cdot \mathfrak{D}(\psi, \alpha^{-1}(V)) = C(r_0) \cdot \mathfrak{D}(\phi, V) \\
&\leq C(r_0)C(m,\lambda) \cdot \text{Area}_p(D, W)
\end{aligned}$$

$$\leq C(r_0)C(m,\lambda)\cdot \text{Area}_p(W',W).$$

By Proposition 4.1 and Theorem 4.4, we may assume $m \to \infty$ and thus

$$\sup_{z\in\alpha^{-1}(V)} \text{dist }(\psi(z),z) \to 0.$$

Since α is uniformly continuous on $\overline{\mathbb{C}}$ equipped with spherical metric. Therefore, the theorem follows. $\square$

4.7 The existence and uniqueness for shift locus

In this section, we discuss the existence and uniqueness of shift locus in $\mathcal{S}'_d$ for a given critical portrait and escaping rate. The main tool is quasiconformal surgery.

Theorem 4.6. *Let $\Theta = \{\Theta_1, \cdots, \Theta_n\}$ be arbitrary critical portrait and $r > 0$, then there exists an unique f in $\mathcal{S}_d(r)$ such that $\Pi(f) = \Theta$.*

Proof. Based on quasiconformal surgery, we shall prove the theorem by the following four steps.

Step I. Construct a topological polynomial F realizing Θ.

Start with the closed disk $\overline{\mathbb{D}}$, mark all of the points $e^{2\pi i\theta}$ with $\theta \in \Theta_i$. Let z_i be the center of gravity of the marked points, and join each of these points to z_i by a straight line L_θ. $L_{\Theta_i} := \bigcup_{\theta\in\Theta_i} L_\theta$ is a closed subset in $\overline{\mathbb{D}}$. Set $r_0 := e^{dr}$ and $\epsilon > 0$ sufficiently small. Define the quotient map $\pi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ pinching L_{Θ_i} into a point z_i. See figure 4.3.

$$\pi(z) := \begin{cases} z & \text{if } z \in \overline{\mathbb{C}} \setminus \mathbb{D}_{r_0-\epsilon}, \\ z_i & \text{if } z \in L_{\Theta_i} \text{ for } 1 \leq i \leq n, \\ \text{homeomorphism} & \text{otherwise.} \end{cases}$$

Then the interior of $\pi(\overline{\mathbb{D}})$ is a disjoint union of d topological disk $D_1, \cdots, D_d$ where $\overline{D}_i \bigcap \overline{D}_j$ contains at most one point. For any $\theta \in \Theta_j$, $R'(\theta) := \pi(\{t = re^{2\pi i\theta}, t > 1\})$ is a ray landing at z_j. Then $\bigcup_{1\leq j\leq n}\{R'(\theta), \theta \in \Theta_j\}$ cuts $\mathbb{D}_{r_0-\epsilon}$ into $D'_1, \cdots, D'_d$ pieces. Set D''_i Jordan domains such that $\overline{D''_i} \subseteq D_i \subseteq D'_i$. For convenience, we may assume the above $R'(\theta)$, ∂D_i, $\partial D'_i$ and $\partial D''_i$ are C^∞ smooth except at the finite points z_i.

Since D''_j is topological disk, there exists a conformal map $F_j : D''_j \to \mathbb{D}_{r_0-\epsilon}$. Let $v_j = r_0 e^{2\pi i\sigma_d(\Theta_j)}$, $Z = \{z_1, \cdots, z_n\}$ and $Z_j = \overline{D}_j \bigcap Z$. We set F_0 be a differentiable branched

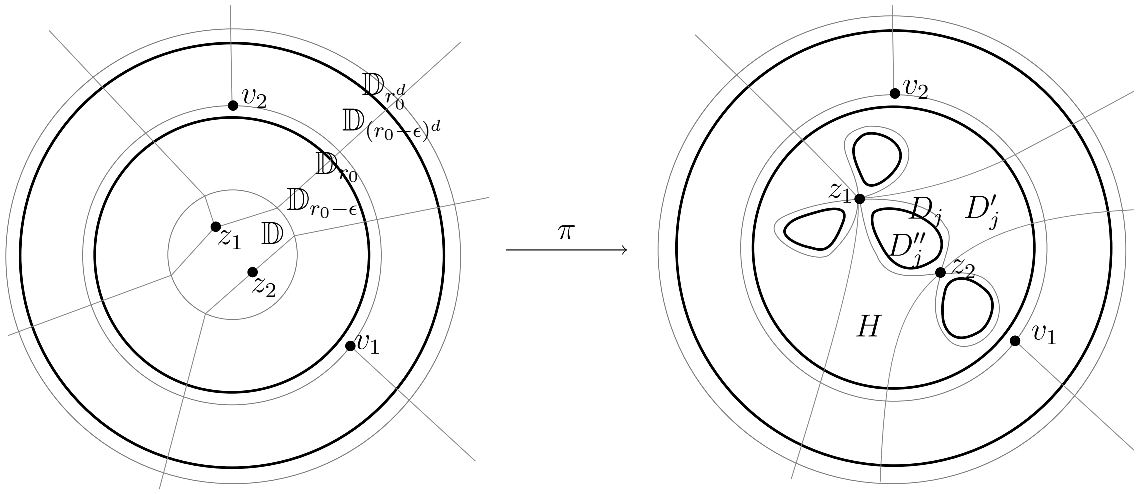

Figure 4.3: Graph of the map π, domains and rays explaining the construction of topological polynomial F.

covering from n-connected domain $H := \overline{\mathbb{D}}_{r_0-\epsilon} \setminus \bigcup_{1\le j\le d} \overline{D}''_j$ to the annulus $\overline{\mathbb{D}}_{(r_0-\epsilon)^d} \setminus \overline{\mathbb{D}}_{r_0-\epsilon}$ satisfying the following condition,

- $F_0|_{\partial\overline{\mathbb{D}}_{r_0-\epsilon}} : z \mapsto z^d$.
- $F_0 = F_j$ on $\partial D''_j$, for $1 \le j \le d$.
- $F_0(R'(\theta) \bigcap \overline{\mathbb{D}}_{r_0-\epsilon}) = R'_{v_j} := \{te^{2\pi i \sigma_d(\theta)}, r_0 < t < (r_0-\epsilon)^d\}$ for $\theta \in \Theta_j$.
- $F_0|_{D'_j \setminus \overline{D}''_j}$: is a diffeomorphism and its image is the set $\mathbb{D}_{(r_0-\epsilon)^d} \setminus \overline{\mathbb{D}}_{r_0-\epsilon}$ minus several R'_{v_i} with $z_i \in \partial D_i$.
- F_0 sends the critical points z_j to v_j locally holomorphic.

Now we can define the topological polynomial F as following

$$F(z) := \begin{cases} z^d & \text{if } z \in \overline{\mathbb{C}} \setminus \mathbb{D}_{r_0-\epsilon}, \\ F_i(z) & \text{if } z \in D''_i \text{ for } 1 \le i \le d, \\ F_0(z) & \text{otherwise.} \end{cases}$$

Step II. Pulling back complex structure to construct polynomial f.

We define a new complex structure μ on $\overline{\mathbb{C}}$ which is preserved by F as follows. Let μ_0 denote the standard complex structure on $\overline{\mathbb{C}}$. Set

$$\mu(z) := \begin{cases} \mu_0(z) & \text{if } z \in \overline{\mathbb{C}} \setminus \mathbb{D}_{r_0-\epsilon}, \\ (F^{n+1})^*(\mu_0)(z) & \text{if } z \in \bigcup_{n\ge 0} F^{-n}(H), \\ \mu_0(z) & \text{otherwise.} \end{cases}$$

Since $F^{-n}(H) \bigcap F^{-m}(H) = \emptyset$, $n \ne m$, μ is well-defined. μ is F-invariant. Actually, for any $z \in F^{-n}(H)$, we have $F(z) \in F^{-n+1}(H)$ and $\mu(z) = (F^{n+1})^*(\mu_0)(z) = (F^n \circ F)^*(\mu_0)(z) = F^* \circ (F^n)^*(\mu_0)(z) = F^*\mu(z)$. Since F is analytic except on H, it follows that μ has bounded distortion on $\overline{\mathbb{C}}$. By the Measurable Riemann Mapping Theorem, there exists an unique normalized quasiconformal map $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $\mu_\psi = \mu$, $\psi(\infty) = \infty$ and $\psi'(\infty) = 1$. Let $f := \psi \circ F \circ \psi^{-1}$. Then $f^*(\mu_0) = \mu_0$. So f is a rational map. As $f^{-1}(\infty) = \infty$, f is a polynomial.

Step III. f realizes the given critical portrait Θ and r.

Since $\mu_\psi|_{\overline{\mathbb{C}} \setminus \mathbb{D}_{r_0}} = \mu_0$, ψ is holomorphic on $\overline{\mathbb{C}} \setminus \mathbb{D}_{r_0}$. By post-composing suitable Möbius transformation, we can assume

$$\psi(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

at ∞. By simple computation, f is a monic centered degree d polynomial. Moreover, ψ^{-1} is the unique Böttcher coordinate of f at the neighborhood $\psi(\mathbb{C} \setminus \mathbb{D}_{r_0})$ of ∞ normalized by tangent at infinity. Since $f \circ \psi = \psi \circ F$, the critical points of f are $\psi(Z)$. Then the critical value are contained in $\psi(\partial \mathbb{D}_{r_0})$ with the same equipotential $\log r_0 = dr$. Thus $f \in \mathcal{S}_d(r)$.

Now we show that $\Pi(f) = \Theta$. Let ϕ be the Böttcher coordinate of f. Then $\phi = \psi^{-1}$ on $\psi(\mathbb{C} \setminus \mathbb{D}_{r_0})$, $\phi \circ \psi(\mathbb{D}_{r_0} \setminus \pi(\mathbb{D})) = \mathbb{D}_{r_0} \setminus \overline{\mathbb{D}}_{e^r}$. Moreover, F and $z \mapsto z^d$ are conjugated by $\phi \circ \psi$ on $\overline{\mathbb{C}} \setminus \pi(\mathbb{D})$. By the definition of F and external ray, it follows that $\psi(R'(\theta))$ is the external ray $R(\theta)$ of f landing at the critical point $\psi(z_i)$, for any $\theta \in \Theta_i$. Thus $\Pi(f) = \Theta$.

Step IV. The uniqueness of f. Assume that f_1 and f_2 are polynomials in $\mathcal{S}_d(r)$ with the same critical portrait Θ. The idea is to construction quasiconformal conjugacy h between f_1 and f_2 which is conformal in the basin of f. Since J_f has Lebesgue measure 0, we can argue that $h : \mathbb{C} \to \mathbb{C}$ is conformal and so h is an affine map.

Denote by $X_{f_i}(t) := \{z \in \overline{\mathbb{C}}, G_{f_i}(z) > t\}$. Recall that G_{f_i} are the Green functions measuring the escape rate of points to ∞ and Ψ_{f_i} are the Böttcher coordinates of f_i. Then the following diagram commutes,

$$\begin{array}{ccccccc}
\mathbb{C} \setminus \overline{\mathbb{D}}_{e^r} & \xleftarrow{\Psi_{f_1}} & X_{f_1}(r) & \xrightarrow{h_0 := \Psi_{f_2}^{-1} \circ \Psi_{f_1}} & X_{f_2}(r) & \xrightarrow{\Psi_{f_2}} & \mathbb{C} \setminus \overline{\mathbb{D}}_{e^r} \\
\Big\downarrow{\scriptstyle z^d} & & \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_2} & & \Big\downarrow{\scriptstyle z^d} \\
\mathbb{C} \setminus \overline{\mathbb{D}}_{e^{dr}} & \xleftarrow{\Psi_{f_1}} & X_{f_1}(dr) & \xrightarrow{h_0} & X_{f_2}(dr) & \xrightarrow{\Psi_{f_2}} & \mathbb{C} \setminus \overline{\mathbb{D}}_{e^{dr}}.
\end{array}$$

Extend h_0 to $\overline{\mathbb{C}} \setminus X_{f_1}(r)$ so that h_0 is a K-quasiconformal map on $\overline{\mathbb{C}}$. Inductively, for $n = 0, 1, \cdots$, use the following diagram to lift h_0,

$$\begin{array}{ccc}
X_{f_1}(d^{-n}r) & \xrightarrow{h_{n+1}} & X_{f_2}(d^{-n}r) \\
\Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_2} \\
X_{f_1}(d^{-n+1}r) & \xrightarrow{h_n} & X_{f_2}(d^{-n+1}r).
\end{array}$$

Actually, we know that $f_i : \overline{\mathbb{C}} \setminus f_i^{-(n+1)} P_{f_i} \to \overline{\mathbb{C}} \setminus f_i^{-n} P_{f_i}$ is an unbranched covering. Since h_n conjugates between f_1 and f_2 on $f_i^{-n} P_{f_i}$, we may choose h_{n+1} to be the unique lift such that, after extended to $f_1^{-(n+1)} P_{f_1}$, it satisfies $h_{n+1} = h_n$ on $f^{-n} P_{f_1}$ [Ha02, Proposition 1.30]. Since f_i is conformal on $\overline{\mathbb{C}} \setminus X_{f_i}(d^{-n} r)$. We can extend h_{n+1} to $\overline{\mathbb{C}} \setminus X_{f_1}(d^{-n} r)$, such that $h_n \circ f_1 = f_2 \circ h_{n+1}$ on $\overline{\mathbb{C}}$. Then h_{n+1} is holomorphic on $X_{f_1}(d^{-(n+1)} r)$ and is a K-quasiconformal map.

Therefore there is a subsequence of $\{h_n\}$ which converges uniformly to a limit quasiconformal map h of $\overline{\mathbb{C}}$. Then $f_2 \circ h = h \circ f_1$ and h is holomorphic in $\bigcup_{n \geq 0} X_{f_1}(d^{-n} r)$ which is the Fatou set. Since the measure of Julia set of f is zero, h is holomorphic on $\overline{\mathbb{C}}$. Since h fixes ∞, we have $h = az + b$. Note that $a = h'(\infty) = \Psi'_{f_1}(\infty) \cdot (\Psi_{f_2}^{-1})'(\infty) = 1$ and $b = 0$ because f_i are centered. Thus $h = id$ and so $f_1 = f_2$. $\square$

4.8 Proof of the results

In this section, we focus on a simple case, that is, all critical points escaping at the same rate (See theorem 4.7). During the proof we apply theorem 4.5 at the situation $X'_0 = \emptyset$ in Assumption 4.6.1. For general cases, the technique is almost the same.

Let us start with a simple lemma.

Lemma 4.9. *Let T be a topology space and S be a subset of T. Let $h_i : T \to T$, $i \in \{1, 2\}$, be maps such that the restrictions of h_i on S have the same image, i.e., $h_1(S) = h_2(S)$, and $h_1|_{T \setminus S} = h_2|_{T \setminus S}$, then for any $n \geq 0$,*

$$\bigcup_{0 \leq k \leq n} h_1^{-k}(S) = \bigcup_{0 \leq k \leq n} h_2^{-k}(S).$$

Proof. If $n = 0$, it follows obviously. We suppose $\bigcup_{0 \leq k \leq n} h_1^{-k}(S) = \bigcup_{0 \leq k \leq n} h_2^{-k}(S)$ by induction.

Firstly, $\bigcup_{0 \leq k \leq n+1} h_1^{-k}(S) \subseteq \bigcup_{0 \leq k \leq n+1} h_2^{-k}(S)$. For any $x \in \bigcup_{0 \leq k \leq n+1} h_1^{-k}(S)$, $h_1(x) \in \bigcup_{0 \leq k \leq n} h_1^{-k}(S) = \bigcup_{0 \leq k \leq n} h_2^{-k}(S)$. If $x \in T \setminus S$ then $x \in h_2^{-1} \circ h_1(x)$ and so $x \in \bigcup_{0 \leq k \leq n+1} h_2^{-k}(S)$. If $x \in S$, then it is obviously true.

Secondly, $\bigcup_{0 \leq k \leq n+1} h_2^{-k}(S) \subseteq \bigcup_{0 \leq k \leq n+1} h_1^{-k}(S)$. It follows by the same arguments as above. The lemma is proved.

$\square$

Let $\mathcal{S}'_d := \bigcup_{0 < r < \infty} \mathcal{S}_d(r)$ and $f_{\Theta,r}$ be the unique polynomial in theorem 4.6. We have the following theorem.

$$
\begin{tikzcd}
f_r \arrow[r, "\text{Step I}", "\text{Surgery}"'] & F_{r+\delta} \arrow[r, "\text{Step II \& Step III}", "c\text{-equi.via } \{\phi_{\delta,n}\}_{n\geq 0}"'] \arrow[d, "\text{Step IV}", "\text{Th.Alg.}\{\eta_{\delta,n}\}_{n\geq 0}"'] & f_{r+\delta} \arrow[dl, "\psi_{\delta,n} := \eta_{\delta,n}\circ\phi_{\delta,n}^{-1}"'] \arrow[d, "\text{Step V}", "\psi_\delta := \alpha_\delta z+\beta_\delta"'] \\
 & f_{\delta,n} \arrow[r, "\text{Step V}", "\text{converging}"'] & \widehat{f}_\delta
\end{tikzcd}
$$

Figure 4.4: Relation of maps in the proof of Theorem 4.7

Theorem 4.7. *For any critical portrait* Θ*,*

$$R_\Theta : (0,\infty) \to \mathcal{S}'_d \qquad r \mapsto f_{\Theta,r}$$

is a simple arc in $\mathcal{S}'_d$*.*

Proof. Given $0 < r < \infty$ and critical portrait Θ, by Theorem 4.6, we only need to show that R_Θ is continuous at r. Given $\delta_0 > 0$ sufficiently small, for any $|\delta| \leq \delta_0$, let

$$f_r(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_1 z + a_0$$

and

$$f_{r+\delta}(z) = z^d + b_{\delta,d-2}z^{d-2} + \cdots + b_{\delta,1}z + b_{\delta,0}$$

such that $f_r \in \mathcal{S}'_d(r)$ and $f_{r+\delta} \in \mathcal{S}'_d(r+\delta)$ with critical portrait Θ.

Step I. Surgery to construct topological polynomial $F_{r+\delta}$.

Let W'_{δ,v_i} be a Jordan domain containing critical value $v_i := \Phi^{-1}(e^{dr}e^{2\pi i\theta_{v_i}})$ such that
- $v_{\delta,i} := \Phi^{-1}(e^{d(r+\delta)}e^{2\pi i\theta_{v_i}}) \in W'_{\delta,v_i}$.
- $W'_{\delta,v_i} \bigcap W'_{\delta,v_{i'}} = \emptyset$, for distinct critical values $v_i, v_{i'}$.
- $\operatorname{diam}(W'_{\delta,v_i}) \to 0$ as $\delta \to 0$.

Then every components of $f_r^{-1}(W'_{\delta,v_i})$ is Jordan domain. Let W_{δ,c_i} be one of them containing critical point c_i. Let $\zeta_\delta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a quasi-conformal map such that ζ_δ is identity outside all $\bigcup_j W'_{\delta,v_j}$ and sends v_i to $v_{\delta,i}$ on each W'_{δ,v_i}. See figure 4.5.

Set $W_\delta := \bigcup_{c\in\mathrm{Crit}(f)} W_{\delta,c}$ and define a quasi-regular map

$$F_{r+\delta}(z) = \begin{cases} f_r(z), & \text{if } z \in \overline{\mathbb{C}} \setminus W_\delta \\ \zeta_\delta \circ f_r(z), & \text{if } z \in W_\delta. \end{cases}$$

Then it satisfies the following,

(1) $F_{r+\delta} = f_r$ on $\overline{\mathbb{C}} \setminus W_\delta$.

(2) $\mathrm{Crit}(F_{r+\delta}) = \mathrm{Crit}(f_r)$ and all the them escape to infinity at the same speed.

(3) $\bigcup_{i\geq 0} F_{r+\delta}^{-i}(W_\delta) = \bigcup_{i\geq 0} f_r^{-i}(W_\delta)$ by Lemma 4.9.

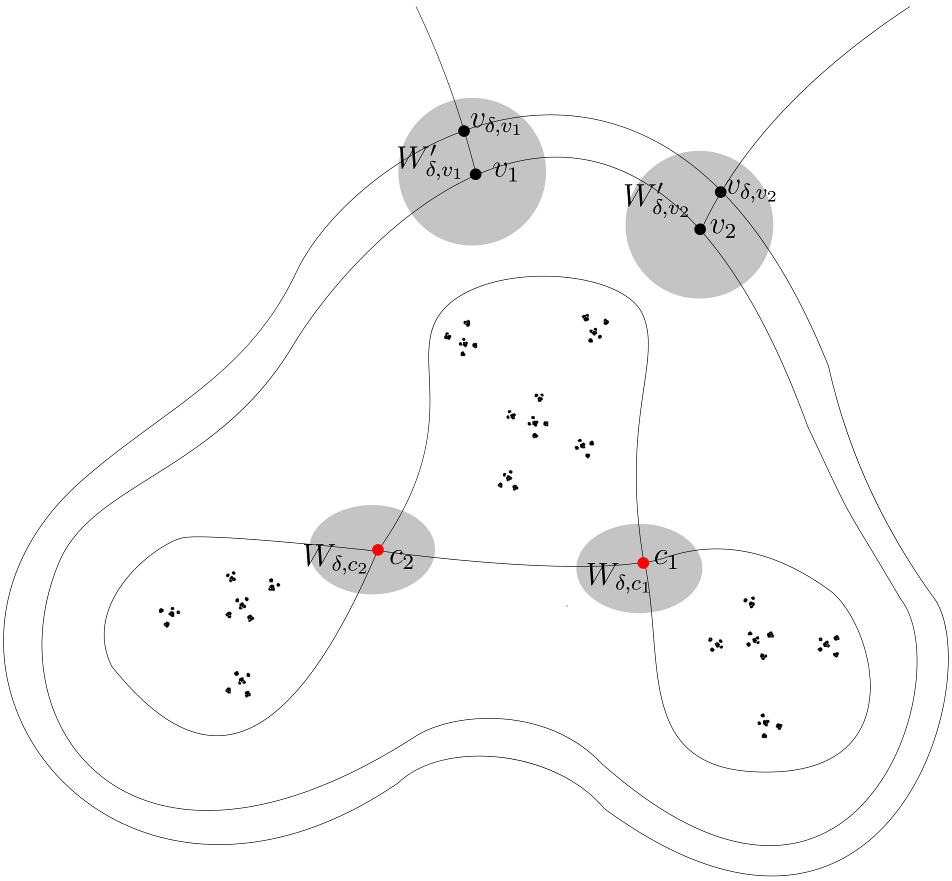

Figure 4.5: Perturbation $F_{r+\delta}$ of $f_r : z \to z^3 - (1.081921 - 0.087513i)\ z + 0.130061 + 1.446914i$

Step II. $F_{r+\delta}$ is c-equivalent to $f_{r+\delta}$ via $(\phi_{\delta,0}, \phi_{\delta,1})$.

Let $\mathcal{U}_\delta := X_{f_r}(d(r+\delta))$ resp. $\mathcal{U}'_\delta := X_{f_{r+\delta}}(d(r+\delta))$. The critical values $F(\mathrm{Crit}(F))$ resp. $f_{r+\delta}$ are in the boundary of $\mathcal{U}_\delta$ resp. $\mathcal{U}'_\delta$. Set

$$\phi_{\delta,0} := \begin{cases} \Psi_{f_{r+\delta}}^{-1} \circ \Psi_{f_r}(z) & \text{if } z \in \mathcal{U}_\delta, \\ g_\delta(z) & \text{if } z \in \overline{\mathbb{C}} \setminus \mathcal{U}_\delta, \end{cases}$$

where $g_\delta : \overline{\mathbb{C}} \setminus \mathcal{U}_\delta \to \overline{\mathbb{C}} \setminus \mathcal{U}'_\delta$ is a quasi-conformal map which coincides with $\Psi_{f_{r+\delta}}^{-1} \circ \Psi_{f_r}$ on the boundary of $\mathcal{U}_\delta$. We can define $\phi_{\delta,1}$ via the following lift

$$\begin{array}{ccc} F_{r+\delta}^{-1}(\mathcal{U}_\delta) \setminus F_{r+\delta}^{-1}P_{F_{r+\delta}} & \xrightarrow{\phi_{\delta,1}} & f_{r+\delta}^{-1}(\mathcal{U}'_\delta) \setminus f_{r+\delta}^{-1}P_{f_{r+\delta}} \\ {\scriptstyle F_{r+\delta}}\Big\downarrow & & \Big\downarrow{\scriptstyle f_{r+\delta}} \\ \mathcal{U}_\delta \setminus P_{F_{r+\delta}} & \xrightarrow{\phi_{\delta,0}} & \mathcal{U}'_\delta \setminus P_{f_{r+\delta}} \end{array}$$

We choose the lift that agree with $\phi_{\delta,0}$ on $\mathcal{U}_\delta$. Since $\phi_{\delta,0} \circ F_{r+\delta} = f_{r+\delta} \circ \phi_{\delta,0}$ on $\mathcal{U}_\delta$, such $\phi_{\delta,1}$ exists. Extending $\phi_{\delta,1}$ to $F_{r+\delta}^{-1}P_{F_{r+\delta}}$ and $\overline{\mathbb{C}} \setminus F_{r+\delta}^{-1}(\mathcal{U}_\delta)$ homeomorphic such that $\phi_{\delta,0} \circ F_{r+\delta} = f_{r+\delta} \circ \phi_{\delta,1}$ on $\overline{\mathbb{C}}$. we know that $\phi_{\delta,1}$ on $\overline{\mathbb{C}} \setminus \mathcal{U}_\delta$ is homeomorphic and so is isotopic to $\phi_{\delta,0}$ rel $\partial\mathcal{U}_\delta$ by the Alexander's trick. Therefore, globally we have

(1) $\phi_{\delta,0} \circ F_{r+\delta} = f_{r+\delta} \circ \phi_{\delta,1}$ on $\overline{\mathbb{C}}$.

(2) $\phi_{\delta,0}$ and $\phi_{\delta,1}$ are isotopic rel $\overline{\mathcal{U}}_\delta$ which contains $\mathcal{P}_{F_{r+\delta}} := \overline{\bigcup_{i\geq 1} F_{r+\delta}^i(\mathrm{Crit}(F_{r+\delta}))}$.

(3) $\phi_{\delta,0} = \phi_{\delta,1}$ on $\overline{\mathcal{U}}_\delta$ and maps it onto $\overline{\mathcal{U}'}_\delta$ holomorphic.

Step III. From c-equivalence to isotopies $\{H_n\}_{n\geq 0}$ and $\{\phi_{\delta,n}\}_{n\geq 0}$.

Let $H_0 : \overline{\mathbb{C}} \times I \to \overline{\mathbb{C}}$ to be an isotopy rel $\overline{\mathcal{U}}_\delta$ such that $H_0(\cdot, 0) = \phi_{\delta,0}$, $H_0(\cdot, 1) = \phi_{\delta,1}$, and $H_0(z,t) = \phi_{\delta,0}(z)$ if $z \in \overline{\mathcal{U}}_\delta$ and $0 \leq t \leq 1$.

It follows from Step II (1) that

$$\phi_{\delta,1}(F_{r+\delta}^{-1}\ (\overline{\mathcal{U}}_\delta)) = f_{r+\delta}^{-1}(\overline{\mathcal{U}'}_\delta)$$

So the map $\phi_{\delta,1}|_{\overline{\mathbb{C}} \setminus F_{r+\delta}^{-1}\ (\overline{\mathcal{U}}_\delta)}$ can be considered as a lift of $H_0(\cdot, 0)|_{\overline{\mathbb{C}} \setminus \overline{\mathcal{U}}_\delta}$ by the non-branched covering maps

$$F_{r+\delta} : \overline{\mathbb{C}} \setminus F_{r+\delta}^{-1}(\overline{\mathcal{U}}_\delta) \to \overline{\mathbb{C}} \setminus \overline{\mathcal{U}}_\delta \quad \text{and} \quad f_{r+\delta} : \overline{\mathbb{C}} \setminus f_{r+\delta}^{-1}(\overline{\mathcal{U}'}_\delta) \to \overline{\mathbb{C}} \setminus \overline{\mathcal{U}'}_\delta.$$

By the homotopy lifting theorem for covering maps, the isotopy $H_0(\cdot, t)|_{\overline{\mathbb{C}} \setminus \overline{\mathcal{U}}_\delta}$ lifts to a unique isotopy $H_1(\cdot, t)$ between $\overline{\mathbb{C}} \setminus F_{r+\delta}^{-1}(\overline{\mathcal{U}}_\delta)$ and $\overline{\mathbb{C}} \setminus f_{r+\delta}^{-1}(\overline{\mathcal{U}'}_\delta)$ such that

$$H_1(\cdot, 0) = \phi_1|_{\overline{\mathbb{C}} \setminus F_{r+\delta}^{-1}(\overline{\mathcal{U}}_\delta)}.$$

Since $H_0(\cdot, t)$ is constant in t on $\overline{\mathcal{U}}_\delta$, each map $H_1(\cdot, t)$ has a continuous extension to $\overline{\mathbb{C}}$, also denoted by $H_1(\cdot, t)$. Then $H_1(\cdot, t)|_{\overline{\mathcal{U}}_\delta}$ does not depend on t. Moreover, each map $H_1(\cdot, t)$ is a homeomorphism, because an inverse of $H_1(\cdot, t)$ can be obtained by lifting the isotopy $H_0(\cdot, t)^{-1}$. Thus we obtain an isotopy $H_1 : \overline{\mathbb{C}} \times I \to \overline{\mathbb{C}}$ rel $F_{r+\delta}^{-1}(\overline{\mathcal{U}}_\delta)$ such that

- $H_1(\cdot, 0) = \phi_{\delta,1}$.
- $H_1(\cdot, 1) =: \phi_{\delta,2}$.
- $H_1(z, t) = \phi_{\delta,1}(z)$ if $z \in F^{-1}(\overline{\mathcal{U}}_\delta)$ and $0 \leq t \leq 1$.
- $\phi_{\delta,1} \circ F_{r+\delta} = f_{r+\delta} \circ \phi_{\delta,2}$ on $\overline{\mathbb{C}}$ and $\phi_{\delta,2}(F_{r+\delta}^{-1}(\overline{\mathcal{U}}_\delta)) = f_{r+\delta}^{-1}(\overline{\mathcal{U}'}_\delta)$.

Repeating this argument we get quasi-conformal map $\phi_{\delta,n}$ and isotopies H_n between

$\overline{\mathbb{C}}$ and $\overline{\mathbb{C}}$ rel. $F_{r+\delta}^{-n}(\overline{\mathcal{U}}_\delta) \supseteq \overline{\mathcal{U}}_\delta$, for $n \geq 0$, such that

(1) $H_n(\cdot, 0) = \phi_{\delta,n}$.

(2) $H_n(\cdot, 1) = \phi_{\delta,n+1}$.

(3) $H_n(z,t) = \phi_{\delta,n}(z)$ if $z \in F^{-n}(\overline{\mathcal{U}}_\delta)$ and $0 \leq t \leq 1$.

(4) $\phi_{\delta,n} \circ F_{r+\delta} = f_{r+\delta} \circ \phi_{\delta,n+1}$ on $\overline{\mathbb{C}}$ and $\phi_{\delta,n+1}(F_{r+\delta}^{-n}(\overline{\mathcal{U}}_\delta)) = f_{r+\delta}^{-n}(\overline{\mathcal{U}'}_\delta)$.

Step IV. Apply Thurston Algorithm to obtain $\{\eta_{\delta,n}\}_{n\geq 0}$ and polynomials $\{f_{\delta,n}\}_{n\geq 0}$.

Let $\eta_{\delta,0} = \text{id}$. Then $\eta_{\delta,0} \circ F_{r+\delta}$ defines a complex structure on $\overline{\mathbb{C}}$ by pulling back the standard complex structure on $\overline{\mathbb{C}}$. The uniformiaztion theorem guarantees the existence of an unique homeomorphism $\eta_{\delta,1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, normalized to fix z_1, z_2, ∞ with z_i close to ∞, such that $f_{\delta,0} := \eta_{\delta,0} \circ F_{r+\delta} \circ \eta_{\delta,1}^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is holomorphic. Moreover $\eta_{\delta,1}$ is quasi-conformal and holomorphic except on W_δ, because $F_{r+\delta}$ is quasi-regular and holomorphic on $\overline{\mathbb{C}} \setminus W_\delta$.

Recursively, there exist quasi-conformal map $\eta_{\delta,n}$ and polynomial $f_{\delta,n}$, for $n \geq 0$, such that

(1) $\eta_{\delta,n} \circ F_{r+\delta} = f_{\delta,n} \circ \eta_{\delta,n+1}$ on $\overline{\mathbb{C}}$,

(2) $\eta_{\delta,n+1}$ is holomorphic on $\overline{\mathbb{C}} \setminus \bigcup_{0\leq i\leq n} F_{r+\delta}^{-n}(W_\delta) \supseteq \overline{\mathbb{C}} \setminus \bigcup_{i\geq 0} f_r^{-i}(W_\delta)$,

(3) $\eta_{\delta,n}$ fixes z_1, z_2 and ∞.

Step V. The uniformly convergence of $f_{\delta,n}$ and $\psi_{\delta,n}$.

Let $z'_{\delta,i} := \phi_{\delta,n}(z_i)$ $i \in \{1,2\}$. By Step V (3), $z'_{\delta,i}$ are independent of n. Set

$$\psi_{\delta,n} := \eta_{\delta,n} \circ \phi_{\delta,n}^{-1} \tag{4.39}$$

to be quasi-conformal map. Then, combining with Step V (3) and Step VI (1), we have

$$\psi_{\delta,n} \circ f_{r+\delta} = f_{\delta,n} \circ \psi_{\delta,n+1}. \tag{4.40}$$

Suppose $\psi_{\delta,0}$ be K_δ-quasiconformal. Choose V to be a neighborhood of ∞ such that $\psi_{\delta,0}$ is holomorphic on V and $f_{r+\delta}(V) \subseteq V$. By (4.40), $\psi_{\delta,n}$ is K_δ-quasiconformal and $\psi_{\delta,n}$ is holomorphic on $\bigcup_{0\leq i\leq n} f_{r+\delta}^{-i}(V)$. It follows that $\{\psi_{\delta,n}\}$ is a normal family.

For any sequence $\{\psi_{\delta,k_i}\}$ there is a subsequence locally uniform converging on $\mathbb{C}$ to a K_δ-quasiconformal map $\psi_\delta : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Moreover, ψ_δ is holomorphic on $\bigcup_{i\geq 0} f_{r+\delta}^{-i}(V)$, which is the Fatou set. Thus ψ_δ is holomorphic on $\overline{\mathbb{C}}$ since the measure of the Julia set of f_r is zero. Combining with the conditions that $\psi_\delta(z'_{\delta,i}) = z_i$ $i \in \{1,2\}$ and $\psi_\delta(\infty) = \infty$, we know that ψ_δ is the affine map

$$\psi_\delta : z \mapsto \alpha_\delta z + \beta_\delta \tag{4.41}$$

which sending $z'_{\delta,i}$ to z_i. Since ψ_δ is independent of any locally uniform converging subsequence in $\{\psi_{\delta,n}\}$, we know that the entire sequence $\{\psi_{\delta,n}\}$ locally uniform converges on

$\mathbb{C}$ to ψ_δ.

Applying the same method to sequences $\{1/\psi_{\delta,n}(1/z)\}$ and $\{\psi_{\delta,n}^{-1}\}$, we know that $\{\psi_{\delta,n}\}$ and $\{\psi_{\delta,n}^{-1}\}$ uniformly converge on $\overline{\mathbb{C}}$ equipped with spherical metric.

It follows that the thurston sequence $\{f_{\delta,n}\}$ uniformly converges to a polynomial $\widehat{f}_\delta$ on $\overline{\mathbb{C}}$, that is, for any $\epsilon > 0$, there exists $N := N(\delta, \epsilon)$ such that

$$\sup_{n\geq N, z\in\overline{\mathbb{C}}} \text{dist } (f_{\delta,n}(z), \widehat{f}_\delta(z)) \leq \epsilon \tag{4.42}$$

Moreover, $\widehat{f}_\delta$ and $f_{r+\delta}$ conjugate by the affine map ψ_δ by (4.40), that is,

$$\widehat{f}_\delta \circ \psi_\delta(z) = \psi_\delta \circ f_{r+\delta}. \tag{4.43}$$

We denote by

$$\widehat{f}_\delta(z) = a_{\delta,d} z^d + a_{\delta,d-1} z^{d-1} + \cdots + a_{\delta,1} z + a_{\delta,0}.$$

Step VI. Estimate the distance between f_r and $\widehat{f}_\delta$.

Let $\mathcal{V}_r := X_{f_r}(C_0)$, where C_0 is sufficient large number such that $\mathcal{V}_r \subseteq \mathcal{V}_\delta$, for any $|\delta| \leq \delta_0$, and $z_i \in \mathcal{V}_r$. By Step I(1) and Step IV(1), we know that $\eta_{\delta,n}$ is holomorphic on $\mathcal{V}_r$. The theorem 4.5 guarantees the following crucial distortion,

$$\sup_{|\delta|\leq\tau, n\geq 0, z\in\mathcal{V}_r} \text{dist}(\eta_{\delta,n}(z), z) \to 0 \quad \text{as} \quad \tau \to 0, \tag{4.44}$$

Thus there exists a neighborhood $\mathcal{V}_r'$ of ∞ such that $\mathcal{V}_r' \subseteq \eta_{\delta,n}(\mathcal{V}_r)$, for any $|\delta| \leq \delta_0$ and $n \geq 0$, and

$$\sup_{|\delta|\leq\tau, n\geq 0, z\in\mathcal{V}_r'} \text{dist } (\eta_{\delta,n}^{-1}(z), z) \to 0 \quad \text{as} \quad \tau \to 0, \tag{4.45}$$

Since

$$\begin{aligned}\text{dist } (f_{\delta,n}(z), f_r(z)) &= \text{dist } (\eta_{\delta,n} \circ f_r \circ \eta_{\delta,n+1}^{-1}(z), f_r(z)) \\ &\leq \text{dist } (\eta_{\delta,n} \circ f_r \circ \eta_{\delta,n+1}^{-1}(z), f_r \circ \eta_{\delta,n+1}^{-1}(z)) + \text{dist } (f_r \circ \eta_{\delta,n+1}^{-1}(z), f_r(z)).\end{aligned}$$

Combining with (4.44) and (4.45), we have for any $\epsilon > 0$, there exist $\tau := \tau(\epsilon)$ such that

$$\sup_{|\delta|\leq\tau, n\geq 0, z\in\mathcal{V}_r'} \text{dist } (f_{\delta,n}(z), f_r(z)) \leq \epsilon. \tag{4.46}$$

By (4.42) and (4.46), for any $\epsilon > 0$, there exist $\tau := \tau(\epsilon) \leq \delta_0$ such that

$$\begin{aligned}\sup_{|\delta|\leq\tau, z\in\mathcal{V}_r'} \text{dist } (\widehat{f}_\delta(z), f_r(z)) &\leq \sup_{|\delta|\leq\tau, z\in\mathcal{V}_r'} \text{dist } (\widehat{f}_\delta(z), f_{\delta,N(\delta)}(z)) + \text{dist } (f_{\delta,N(\delta)}(z), f_r(z)) \\ &\leq 2\epsilon, \end{aligned} \tag{4.47}$$

where $N(\delta) = N(\delta, \epsilon)$ is defined in (4.42). By Lemma 4.4, $\{\widehat{f}_\delta\}$ uniformly converges to f_r on $\overline{\mathbb{C}}$. Therefore

$$a_{\delta,d} \to 1, a_{\delta,d-1} \to 0 \quad \text{as} \quad \delta \to 0. \tag{4.48}$$

Step VII. Estimate the distance between $\widehat{f}_\delta$ and $f_{r+\delta}$.

By Step VI and (4.44), the sequence $\{\eta_{\delta,n}|_{\mathcal{V}_r}\}_{n\geq 0}$ is holomorphic and locally uniformly bounded on $\mathcal{V}_r \setminus \{\infty\}$. Thus it is a normal family and the entire sequence converges to a holomorphic map $\eta_\delta|_{\mathcal{V}_r}$, i.e., for any $\epsilon > 0$, there exists $N = N(\delta, \epsilon)$ such that for $n \geq N$

$$\sup_{n\geq N, z\in\mathcal{V}_r} \text{dist } (\eta_\delta(z), \eta_{\delta,n}(z)) \leq \epsilon. \tag{4.49}$$

The new sequence $\{\eta_\delta|_{\mathcal{V}_r}\}$ converges to id on $\mathcal{V}_r$ as well. Because, by (4.44) and (4.49), for any $\epsilon > 0$, there exists $\tau = \tau(\epsilon)$ such that

$$\begin{aligned}\sup_{|\delta|\leq\tau, z\in\mathcal{V}_r} \text{dist } (\eta_\delta(z), z) &\leq \sup_{|\delta|\leq\tau, z\in\mathcal{V}_r} \text{dist } (\eta_\delta(z), \eta_{\delta,N(\delta)}(z)) + \text{dist } (\eta_{\delta,N(\delta)}(z), z) \\ &\leq 2\epsilon \end{aligned} \tag{4.50}$$

where $N(\delta) = N(\delta, \epsilon)$ is defined in (4.49).

Note that $\phi'_{\delta,n}(\infty) = 1$ in Step II. Then (4.39), (4.41), (4.49), (4.50), Weierstrass convergence theorem and the chain rule give

$$\begin{aligned}\alpha_\delta = \psi'_\delta(\infty) &= \lim_{n\to\infty} \psi'_{\delta,n}(\infty) \\ &= \lim_{n\to\infty} \eta'_{\delta,n}(\infty) \cdot (\phi^{-1}_{\delta,n})'(\infty) \\ &= \lim_{n\to\infty} \eta'_{\delta,n}(\infty) = \eta'_\delta(\infty) \to 1 \qquad \text{as} \quad \delta \to 0. \end{aligned} \tag{4.51}$$

Now consider β_δ. By (4.43), (4.48) and (4.51), a simple computation implies

$$\beta_\delta = \frac{a_{\delta,d-1} \cdot a_\delta}{d \cdot a_{\delta,d}} \to 0 \quad \text{as} \quad \delta \to 0$$

Thus both ψ_δ and ψ_δ^{-1} uniformly converge to identity on $\overline{\mathbb{C}}$. Since $f_{r+\delta}$ conjugates $\widehat{f}_\delta$ by ψ_δ in (4.43), we have

$$\sup_{|\delta|\leq\tau, z\in\overline{\mathbb{C}}} \text{dist } (f_{r+\delta}(z), \widehat{f}_\delta(z)) \to 0 \quad \text{as} \quad \tau \to 0.$$

Therefore, by Lemma 4.4 and (4.47), we have

$$\sup_{|\delta|\leq\tau, z\in\overline{\mathbb{C}}} \text{dist } (f_r(z), f_{r+\delta}(z)) \to 0 \quad \text{as} \quad \tau \to 0.$$

This completes the proof. □

Proof of Theorem 4.1. By theorem 4.6, the map P is well-defined and one-to-one. The continuity at (Θ, r) follows nearly the same as Theorem 4.7. We only need to change slightly in Step I. The surgery ξ_δ sends v_i to $\Psi^{-1}(e^{d(r+\delta)}e^{2\pi i\theta'_{v_i}}) \in W'_{\delta,v_i}$, as Θ' is close enough to Θ and $|\delta|$ is sufficiently small. We omit the proof. $\square$

Proof of Theorem 4.2. We know that J_f is locally connected [Yin99]. If $R_\Theta(t)$ lands at f, Theorem 1 in [Ki05] implies that the external rays of f with arguments in Θ_i land at a common point c_i which must be critical. By the unlinked property of critical portrait, we have $c_i \neq c_j$ if $i \neq j$. Thus the local degree of f at critical point c_i is $\#\Theta_i$. Therefore, Θ is a critical portrait of f.

For the sufficiency, we adopt exactly the same method as Theorem 4.7. Indeed, in dynamic plane $R(\theta'_i)$ with $\theta'_i := \sigma_d(\Theta_i)$ lands at critical value v_i. We can construct F_δ, $\delta > 0$, exactly the same in Step I of Theorem 4.7. Both of the distance dist (f, F_δ) and dist (F_δ, f_δ) converge to 0 as $\delta \to 0$. Thus the parameter ray $R_\Theta(t)$ lands at f. We omit the details. $\square$

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Chapter 5

On the dynamics of a family of generated renormalization transformations

5.1 Introduction

The statistical mechanical models on hierarchical lattices have attracted many interests recently since they exhibit a deep connection between their limiting sets of the zeros of the partition functions and the Julia sets of rational maps in complex dynamics [BL, DSI, Qi, QL, QYG]. The well-known Yang-Lee theorem in statistical mechanics shows that the zeros of the partition function is dense in a line for many magnetic materials in a complex magnetic field plane. This means that the complex singularities of the free energy lie on this line, where the free energy is the logarithm of the partition function [YL]. By the works of Fisher and others [Fi], it was generally believed that the zeros of the partition function condense to some simple curve.

Until 1983, Derrida et al. showed that the zeros of the partition function condense to the Julia set of the renormalization transformation of so-called *standard hierarchical lattices* [DSI]. They proved that the singularities of the free energy lie on the Julia set of the rational map

$$z \mapsto \left(\frac{z^2+\lambda-1}{2z+\lambda-2}\right)^2. \tag{5.1}$$

This means that the distribution of the singularities of the free energy is not as simple as one desired. Henceforth, a lot of works related on the Julia sets of this renormalization transformation appeared [AY, BL, Ga, HL, Os, Qi, QL, QYG, WQYQG]. For the ideas formulated in renormalization transformation in statistical mechanics, see [Wi].

Recently, Qiao considered the generalized diamond hierarchical Potts model and proved that the family of rational maps

$$U_{mn\lambda}(z)=\left(\frac{(z+\lambda-1)^m+(\lambda-1)(z-1)^m}{(z+\lambda-1)^m-(z-1)^m}\right)^n \tag{5.2}$$

are actually the renormalization transformation of the *generalized diamond hierarchical Potts model* [Qi, Theorem 1.1], where $m,n\geq 2$ are both integers and $\lambda\in\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$ is a complex parameter. The standard diamond lattice ($m=n=2$) and the *diamond-like lattice* ($m=2$ and $n\in\mathbb{N}$) are the special cases of (5.2).

In this chapter, we will consider the case for $d:=m=n\geq 2$. For simplicity, we use $U_{d\lambda}$ to denote $U_{dd\lambda}$ in (5.2). We not only study the properties of the Julia sets of $U_{d\lambda}$, but also consider the connectivity of the non-escaping locus of the parameter space of this renormalization transformation.

If $\lambda=0$, then $U_{d\lambda}$ degenerates to a parabolic polynomial $U_{d0}(z)=(\frac{z+d-1}{d})^d$ whose Julia set is a Jordan curve. For the connectivity of the Julia sets of $U_{d\lambda}$, we have following Theorem.

Theorem 5.1. *The Julia set of $U_{d\lambda}$ is always connected for every $d\geq 2$ and $\lambda\in\mathbb{C}^*$.*

Note that Qiao and Li proved that the Julia set of $U_{d\lambda}$ is connected for $d=2$ and $\lambda\in\mathbb{R}$ [QL]. We would like to remark that if $m\neq n$, then there exists parameter $\lambda\in\mathbb{C}^*$ such the Julia set of $U_{mn\lambda}$ defined in (5.2) is disconnected (see [Qi, Figure 3.1] for example).

The *Mandelbrot set* of quadratic polynomials $f_c(z)=z^2+c$ is defined by

$$M=\{c\in\mathbb{C}: f_c^{\circ n}(0)\not\to\infty \text{ as } n\to\infty\}.$$

Douady and Hubbard showed that M is connected [DH]. For higher degree polynomials with only one critical point, there are associated *Multibrot sets*. For rational maps, one way to study the parameter space is to consider the *connectedness locus*, which consists of all parameters such the corresponding Julia set is connected. However, the connectedness locus makes no sense in our case since every Julia set is connected.

For $\lambda\neq 0$, then 1 and ∞ are two superattracting fixed points of $U_{d\lambda}$. The *non-escaping locus* $\mathcal{M}_d$ associated to this family is defined by

$$\mathcal{M}_d=\{\lambda\in\mathbb{C}^*: U_{d\lambda}^{\circ n}(0)\not\to 1 \text{ and } U_{d\lambda}^{\circ n}(0)\not\to\infty \text{ as } n\to\infty\}\cup\{0\}. \tag{5.3}$$

Obviously, "non-escaping" here means the collection of those parameters such that the orbit of 0 cannot be attracted by 1 and ∞. Note that 0 is a critical value of $U_{d\lambda}$.

The non-escaping locus $\mathcal{M}_d$ can be identified as the complex plane cutting out infinitely many simply connected domains, which will be called "capture domains" later (see Figure 2 and Proposition 5.1). There exist many small copies of the Mandelbrot set M in $\mathcal{M}_d$ which correspond to the renormalizable parameters.

For the connectivity of Mandelbrot set $\mathcal{M}_d$, Wang et al. proved that $\mathcal{M}_2$ is connected in [WQYQG, Theorem 1.1]. We now generate this result to all $\mathcal{M}_d$, where $d \geq 2$.

Theorem 5.2. *The non-escaping locus $\mathcal{M}_d$ is connected for $d \geq 2$.*

The proof of the connectivity of $\mathcal{M}_2$ in [WQYQG] is based on constructing Riemann mapping from the capture domain to the unit disk $\mathbb{D}$, which is tediously long. Here, we give a proof of Theorem 5.2 by using the methods of Teichmüller theory of the rational maps which was developed in [McS]. The proof is largely simplified and there are lots of additional results. For example, we show that the Julia set of $U_{d\lambda}$ is a quasicircle if and only if λ lies in the unbounded capture domain $\mathcal{H}_0$ (Proposition 5.2) and each bounded capture domain contains exactly one *center* (Theorem 5.10).

If λ is large enough, then the Julia set of $U_{d\lambda}$ is a quasicircle (see Proposition 5.2). Hu and Lin observed that these circles becomes more and more "circular" as λ tends to ∞ in the case of $d = 2$ [HL]. In [Ga], Gao proved the Hausdorff dimension of the Julia set of $U_{2n\lambda}$ tends to 1 for every $n \geq 2$, which gave an affirmative answer of Hu and Lin proposed in 1989. In this chapter, we consider the asymptotic formula of the Hausdorff dimension of the Julia set $J_{d\lambda}$ of $U_{d\lambda}$ as the parameter λ tends to ∞.

Theorem 5.3. *Let $d \geq 2$. For large λ such that $J_{d\lambda}$ is a quasicircle, the Hausdorff dimension of $J_{d\lambda}$ is given by*

$$\dim_H(J_{d\lambda}) = 1 + \frac{1}{4\log d}|\lambda|^{-\frac{2}{d+1}} + \mathcal{O}(\lambda^{-\frac{3}{d+1}}). \tag{5.4}$$

Theorem 5.3 is a generation of [Os] in which the asymptotic formula of the Hausdorff dimension of $J_{2\lambda}$ was calculated.

This chapter is organized as follows. In Section 5.2, we analyse the location of the critical points of $U_{d\lambda}$ and show that the Julia set of $U_{d\lambda}$ is always connected and prove Theorem 5.1. In section 5.4, we show that the parameter plane of $U_{d\lambda}$ can be decomposed into the non-escaping locus $\mathcal{M}_d$ union infinitely many capture domains. In section 5.5, we give a complete classification of the quasiconformal conjugacy classes of $U_{d\lambda}$. In section 5.6, we show that each bounded capture domain is simply connected and the unique unbounded capture domain is homeomorphic to the punctured disk and prove Theorem 5.2. We will prove the asymptotic formula (5.4) of Theorem 5.3 in section 5.7 but leave the complicated calculations to the last section as an appendix.

5.2 The location of critical points and the connected Julia sets

Firstly, we give a splitting principle for $U_{d\lambda}$. This principle is not exist if one considers $U_{mn\lambda}$ with $m \neq n$. This is the reason why we set $m = n$ in this paper. For every $\lambda \in \mathbb{C}^*$, it is straightforward to verify that $U_{d\lambda} = T_{d\lambda} \circ T_{d\lambda}$, where

$$U_{d\lambda}(z) = \left(\frac{(z+\lambda-1)^d + (\lambda-1)(z-1)^d}{(z+\lambda-1)^d - (z-1)^d} \right)^d \text{ and } T_{d\lambda}(z) = \left(\frac{z+\lambda-1}{z-1} \right)^d. \tag{5.5}$$

A direct calculation shows that the set of all critical points of $T_{d\lambda}$ is $\{1, 1-\lambda\}$, and both with multiplicity $d-1$. Note that

$$U_{d\lambda}^{-1}(\infty) = T_{d\lambda}^{-1}(1) = \bigcup_{k=0}^{d-1} \{\xi_k\} \text{ and } U_{d\lambda}^{-1}(0) = T_{d\lambda}^{-1}(1-\lambda) = \bigcup_{k=0}^{d-1} \{\omega_k\}, \tag{5.6}$$

where

$$\xi_k = \frac{e^{\frac{2k\pi i}{d}} + \lambda - 1}{e^{\frac{2k\pi i}{d}} - 1} \text{ and } \omega_k = \frac{(1-\lambda)^{\frac{1}{d}} e^{\frac{2k\pi i}{d}} + \lambda - 1}{(1-\lambda)^{\frac{1}{d}} e^{\frac{2k\pi i}{d}} - 1}. \tag{5.7}$$

It follows that ξ_k and ω_k are critical points of $U_{d\lambda}$ with multiplicity $d-1$, where $0 \le k \le d-1$. In particular, $\xi_0 = \infty$. Therefore, the set of all critical points of $U_{d\lambda}$ is

$$\mathrm{Crit}(U_{d\lambda}) = \{1, 1-\lambda, \infty\} \cup \bigcup_{k=1}^{d-1} \{\xi_k\} \cup \bigcup_{k=0}^{d-1} \{\omega_k\}. \tag{5.8}$$

Since $T_{d\lambda}(1) = \infty$, $T_{d\lambda}(\infty) = 1$ and $1, \infty$ are both critical points of $U_{d\lambda}$, it means that there exist two fixed immediate superattracting basins $\mathcal{A}_{d\lambda}(1)$ and $\mathcal{A}_{d\lambda}(\infty)$ of $U_{d\lambda}$ with centers 1 and ∞ respectively. Under the iteration of $T_{d\lambda}$, we have the following forward orbits:

$$\xi_k \mapsto 1 \mapsto \infty \mapsto 1 \mapsto \infty \mapsto \cdots \quad \text{and} \quad \omega_k \mapsto 1-\lambda \mapsto 0 \mapsto (1-\lambda)^d \mapsto \cdots \tag{5.9}$$

for every $0 \le k \le d-1$. Since the dynamical behaviors are determined by the critical forward orbits essentially, we only need to focus on the *free* critical orbit of $1-\lambda$ (or equivalently, the forward orbit of 0) under the iteration of $T_{d\lambda}$ or $U_{d\lambda}$. This is the reason why we define the non-escaping locus $\mathcal{M}_d$ as in (5.3).

Lemma 5.1. *Let U and V be two domains on $\overline{\mathbb{C}}$ and assume that V is simply connected. If $f : U \to V$ is a branched covering with only one critical value in V (counted without multiplicity), then U is also simply connected.*

Proof. Let v be the unique critical value lying in V. Consider the unramified covering $f : U \setminus f^{-1}(v) \to V \setminus \{v\}$. Since $V \setminus \{v\}$ is an annulus with Euler characteristic 0, it follows that $U \setminus f^{-1}(v)$ is also an annulus by the Riemann-Hurwitz formula. This means that U is a topological disk, which is simply connected as desired. □

In order to prove a rational map has connected Julia set, one often needs to exclude the existence of Herman ring. The following lemma was proved in [Ya].

Lemma 5.2 ([Ya, Corollary 3.2]). *The renormalization transformation $U_{d\lambda}$ has no Herman ring.*

The proof of Lemma 5.2 relies on the quasiconformal surgery and the arguments are divided into two cases: Herman ring with period 1 and period at least two. However, the prove idea is different from [Mi2, Appendix A].

Theorem 5.4. *The Julia set of $T_{d\lambda}$ is always connected for every $d \geq 2$ and $\lambda \in \mathbb{C}^*$.*

Proof. The proof idea is more or less similar to the case of quadratic rational maps in [Mi1, Lemma 8.2]. Note that the Julia set is connected if and only if each Fatou component is simply connected. By Sullivan's classification of the periodic Fatou components, every periodic Fatou component of $T_{d\lambda}$ is either a Siegel disk, a Herman ring, or an immediate basin for some attracting or parabolic point. By Lemma 5.2, it is known $T_{d\lambda}$ has no Herman ring.

By [Mi1, Lemma 8.1], we know that if all the critical values of a rational map are contained in a single component of the Fatou set, then the Julia set is totally disconnected. However, the Julia set $J_{d\lambda}$ cannot be totally disconnected since $T_{d\lambda}$ has a superattracting periodic orbit of period 2. Therefore, the critical points 1 and $1-\lambda$ lie in different Fatou components and each Fatou component of $T_{d\lambda}$ contains at most one critical value (∞ or 0 by (5.9)).

Now we prove each Fatou component of $T_{d\lambda}$ is simply connected. Firstly, we assume that every periodic Fatou component of $T_{d\lambda}$ is simply connected. Note that the periodic orbit $1 \leftrightarrow \infty$ is superattracting. There leaves only one critical point $1-\lambda$ needing to consider. According to Lemma 5.1, the preimage of a simply connected region under a branched covering with only one critical value is again simply connected. This means every Fatou component of $T_{d\lambda}$ is simply connected by induction.

Then suppose that there exists a periodic Fatou component U of $T_{d\lambda}$ which is not simply connected and the period is $p \geq 1$. This means that U is an attracting basin or a parabolic basin since $T_{d\lambda}$ has no Herman ring. Let z_0 be the attracting periodic point in U or parabolic periodic point on ∂U. We use V to denote a simply connected neighborhood or a simply connected petal of z_0 such that $T_{d\lambda}^{\circ p}(V) \subset V$ according to U is attracting or parabolic. Let V_k be the component of $T_{d\lambda}^{-kp}(V)$ containing V. Then $U = \bigcup_{k\geq 0} V_k$ and $V_{k+1} \mapsto T_{d\lambda}(V_{k+1}) \mapsto \cdots \mapsto T_{d\lambda}^{\circ p-1}(V_{k+1}) \mapsto V_k$ is a successive branched covering under $T_{d\lambda}$ with at most one critical value in each codomain since each Fatou component of $T_{d\lambda}$ contains at most one critical value. Suppose V_{k_0} is simply connected (at least $k_0 = 0$ is satisfied). By Lemma 5.1, we know that $T_{d\lambda}^{\circ p-1}(V_{k_0+1}), \cdots, T_{d\lambda}(V_{k_0+1}), V_{k_0+1}$ are all simply connected since V_{k_0} is also. Inductively, it follows that each V_k is simply connected and hence U is also simply connected. This contradicts the assumption that U is not simply connected.

Therefore, in any case, the Julia set of $T_{d\lambda}$ is always connected. This ends the proofs of Theorems 5.4 and 5.1. $\square$

5.3 The Julia set cannot be a Sierpińsk carpet

In this section, we will prove that if the parameter λ lies on the real axis, then the Julia set of $U_{d\lambda}$ can never be a Sierpińsk carpet by showing there always exist two Fatou components of $U_{d\lambda}$ whose boundaries are intersecting to each other.

Lemma 5.3. *For every $d \geq 2$ and $\lambda \in \mathbb{R}$, there exist two Fatou components V_1, V_2 of $U_{d\lambda}$ such that $\overline{V}_1 \cap \overline{V}_2 \neq \emptyset$.*

Proof. If $\lambda = 0$, then $U_{d\lambda}$ degenerates to a parabolic polynomial $U_{d0}(z) = (\frac{z+d-1}{d})^d$ whose Julia set J_{d0} is a Jordan curve. Let $V_1 = \mathcal{A}_{d\lambda}(1)$ and $V_2 = \mathcal{A}_{d\lambda}(\infty)$ be the immediate superattracting basins of 1 and ∞ respectively. We have $\overline{V}_1 \cap \overline{V}_2 = J_{d0} \neq \emptyset$.

In the following, we assume that $\lambda \in \mathbb{R} \setminus \{0\}$. The dynamics of $U_{d\lambda}$ will be restricted on the real axis and the arguments will be divided into several cases. Let $x \in \mathbb{R}$, by a direct calculation, we have

$$U'_{d\lambda}(x) = \frac{d^2\lambda^2(x-1)^{d-1}(x+\lambda-1)^{d-1}((x+\lambda-1)^d + (\lambda-1)(x-1)^d)^{d-1}}{((x+\lambda-1)^d - (x-1)^d)^{d+1}}. \tag{5.10}$$

(1) Let $\lambda > 0$. If $x \geq 1$, we have $x-1 \geq 0$, $x+\lambda-1 > 0$, $(x+\lambda-1)^d + (\lambda-1)(x-1)^d > 0$ and $(x+\lambda-1)^d - (x-1)^d > 0$. This means that $U'_{d\lambda}(x) \geq 0$ and $U_{d\lambda}$ is increasing on $[1, +\infty)$. Moreover, $U'_{d\lambda}(x) = 0$ if and only if $x = 1$. We claim that there exists at least one fixed point of $U_{d\lambda}$ lying in $(1, +\infty)$. Otherwise, we then have $1 < U_{d\lambda}(x) < x$ for every $x > 1$ since $U_{d\lambda}(1) = 1$ and $U'_{d\lambda}(1) = 0$. This means that the interval $(1, +\infty)$ is contained in the attracting basin of 1, which is a contradiction since ∞ is a superattracting fixed point of $U_{d\lambda}$.

Let $1 = x_0 < x_1 < \cdots < x_n < +\infty$ be the collection of all the fixed points of $U_{d\lambda}$ lying in $[1, +\infty)$, where $n \geq 1$. It is easy to see $U_{d\lambda}(x) > x$ if $x > x_n$. In particular, we have $(x_n, +\infty) \subset \mathcal{A}_{d\lambda}(\infty)$. Note that $U'_{d\lambda}(x_n) \geq 1$. If $U'_{d\lambda}(x_n) = 1$, then x_n is a parabolic fixed point of $U_{d\lambda}$ and $\mathcal{A}_{d\lambda}(x_n)$ contains a small interval on the left of x_n, where $\mathcal{A}_{d\lambda}(x_n)$ is the immediate parabolic basin of x_n. Let $V_1 = \mathcal{A}_{d\lambda}(x_n)$ and $V_2 = \mathcal{A}_{d\lambda}(\infty)$. We have $x_n \in \overline{V}_1 \cap \overline{V}_2$. If $U'_{d\lambda}(x_n) > 1$, then x_n is a repelling fixed point of $U_{d\lambda}$ and x_{n-1} is an (or parabolic) attracting fixed point of $U_{d\lambda}$. Moreover, $[x_{n-1}, x_n) \subset \mathcal{A}_{d\lambda}(x_{n-1})$, where $\mathcal{A}_{d\lambda}(x_{n-1})$ is the immediate attracting (or parabolic) basin of x_{n-1}. Let $V_1 = \mathcal{A}_{d\lambda}(x_{n-1})$ and $V_2 = \mathcal{A}_{d\lambda}(\infty)$. We have $x_n \in \overline{V}_1 \cap \overline{V}_2$.

(2) Let $\lambda < 0$. If $0 \leq x \leq 1$, then $x - 1 \leq 0$ and $x + \lambda - 1 < 0$. If $d \geq 2$ is even, then $(x+\lambda-1)^d + (\lambda-1)(x-1)^d > 0$, $(x+\lambda-1)^d - (x-1)^d > 0$ and $U'_{d\lambda}(x) \geq 0$. If $d \geq 2$ is odd, then $U'_{d\lambda}(x) \geq 0$. This means that $U_{d\lambda}$ is increasing on $[0, 1]$ for every

$d \geq 2$. Moreover, $U'_{d\lambda}(x) = 0$ if and only if $x = 1$. By a straightforward calculation, we have $0 < U_{d\lambda}(0) < 1$. Now we divide the arguments into two cases.

If there exists no fixed point of $U_{d\lambda}$ in $(0, 1)$, then we have $0 < x < U_{d\lambda}(x) < 1$ for every $0 < x < 1$. This means that 0 lies in the immediate attracting basin of 1. By Lemma 5.4(5), we know that $J_{d\lambda}$ is a quasicircle. In particular, $\overline{\mathcal{A}_{d\lambda}(1)} \cap \overline{\mathcal{A}_{d\lambda}(\infty)} = J_{d\lambda} \neq \emptyset$. If there exists at least one fixed point of $U_{d\lambda}$ in $(0, 1)$, we denote all of them by $0 < x_1 < \cdots < x_n < 1$, where $n \geq 1$. By a completely similar argument as the case $\lambda > 0$, one can show that the fixed point x_n is contained in the boundaries of two different Fatou components. Therefore, the proof is complete. □

Theorem 5.5. *For every $d \geq 2$ and $\lambda \in \mathbb{R}$, the Julia set $J_{d\lambda}$ is not a Sierpińsk carpet.*

Proof. Note that if $J_{d\lambda}$ is a Sierpiński carpet, then the closure of any two Fatou components of $U_{d\lambda}$ cannot be intersecting to each other. But this contradicts Lemma 5.3. The proofs of Theorems 5.5 and **??** are finished. □

By computer experiments, it is shown that $\overline{\mathcal{A}_{d\lambda}(1)} \cap \overline{\mathcal{A}_{d\lambda}(\infty)} = \{z_0\}$ for $\lambda \in \mathbb{C}$, where z_0 is a repelling fixed point of $U_{d\lambda}$. Therefore, the Julia set $J_{d\lambda}$ can never be a Sierpińsk carpet for any $\lambda \in \mathbb{C}$ (see Figures 5.1 and 5.2).

5.4 Decomposition of the parameter space

In this section, we divide the parameter space of $T_{d\lambda}$ into the non-escaping locus $\mathcal{M}_d$ union countably many capture domains. Recall that $\mathcal{A}_{d\lambda}(1)$ and $\mathcal{A}_{d\lambda}(\infty)$ are the immediate superattracting basins of 1 and ∞ respectively.

Lemma 5.4. *For each $\lambda \in \mathbb{C}^*$, the following conditions are equivalent:*

(1) *The Julia set $J_{d\lambda}$ of $T_{d\lambda}$ is a quasicircle;* (2) $\xi_k \in \mathcal{A}_{d\lambda}(\infty)$ *for all* $0 \leq k \leq d-1$; (3) $\omega_k \in \mathcal{A}_{d\lambda}(1)$ *for all* $0 \leq k \leq d-1$; (4) $1 - \lambda \in \mathcal{A}_{d\lambda}(\infty)$; (5) $0 \in \mathcal{A}_{d\lambda}(1)$.

In particular, $\omega_k \in \mathcal{A}_{d\lambda}(1)$ if and only if $\omega_l \in \mathcal{A}_{d\lambda}(1)$, where $0 \leq k, l \leq d-1$.

Proof. We first prove (1) ⇒ (2)(3)(4)(5). If $J_{d\lambda}$ is a quasicircle, the Fatou set of $T_{d\lambda}$ consists of two simply connected Fatou components $A_{d\lambda}(1)$ and $A_{d\lambda}(\infty)$ whose common boundary is $J_{d\lambda}$. Since $T_{d\lambda}$ permutes 1 and ∞, by (5.9), it follows that (2) holds and $\{\omega_1, \cdots, \omega_d\}$ lies in a single Fatou component. Applying the Riemann-Hurwitz formula to $U_{d\lambda} : \mathcal{A}_{d\lambda}(\infty) \to \mathcal{A}_{d\lambda}(\infty)$, it follows that $\{\omega_1, \cdots, \omega_d, 0\} \subset \mathcal{A}_{d\lambda}(1)$ and $1-\lambda \in \mathcal{A}_{d\lambda}(\infty)$. Therefore, (3)(4)(5) hold.

By (5.9), we have (3) ⇒ (4) ⇒ (5). Now we prove (5) ⇒ (1). Suppose that $0 \in \mathcal{A}_{d\lambda}(1)$. By (5.6), we have $U_{d\lambda}^{-1}(0) = \bigcup_{k=0}^{d-1}\{\omega_k\}$. Since $U_{d\lambda}(\mathcal{A}_{d\lambda}(1)) = \mathcal{A}_{d\lambda}(1)$, there exists some k_0 such that $\omega_{k_0} \in \mathcal{A}_{d\lambda}(1)$ and hence $1 - \lambda \in \mathcal{A}_{d\lambda}(\infty)$. Note that $T_{d\lambda} : \mathcal{A}_{d\lambda}(1) \to \mathcal{A}_{d\lambda}(\infty)$ is d to 1. We claim that $\omega_k \in \mathcal{A}_{d\lambda}(1)$ for every $0 \leq k \leq d-1$. In fact, if not, then $1 - \lambda$ has at least $d+1$ preimages under $T_{d\lambda}$ (counted with multiplicity, d in $\mathcal{A}_{d\lambda}(1)$ and at least one elsewhere), which is impossible. The same argument also shows that $\omega_k \in \mathcal{A}_{d\lambda}(1)$ if and only if $\omega_l \in \mathcal{A}_{d\lambda}(1)$, where $0 \le k, l \le d-1$. Then, $\mathcal{A}_{d\lambda}(1)$ contains critical points $\{\omega_1, \cdots, \omega_d, 1\}$ of $U_{d\lambda}$. This means that $\mathcal{A}_{d\lambda}(1)$ is completely invariant under $U_{d\lambda}$.

Since $1-\lambda \in \mathcal{A}_{d\lambda}(\infty)$, it means that $T_{d\lambda} : \mathcal{A}_{d\lambda}(\infty) \to \mathcal{A}_{d\lambda}(1)$ is d to 1. Therefore, $\xi_k \in \mathcal{A}_{d\lambda}(\infty)$ for every $1 \le k \le d-1$ since $\xi_0 = \infty \in \mathcal{A}_{d\lambda}(\infty)$ and $T_{d\lambda}(\xi_k) = 1$. Moreover, $\mathcal{A}_{d\lambda}(\infty)$ contains critical points $\{\xi_1, \cdots, \xi_d, 1-\lambda\}$ of $U_{d\lambda}$. This means that $\mathcal{A}_{d\lambda}(\infty)$ is also completely invariant under $U_{d\lambda}$. Therefore, $J_{d\lambda}$ is a quasicircle since $T_{d\lambda}$ is hyperbolic and $T_{d\lambda}$ has exactly two Fatou components. This ends the proof of (5) $\Rightarrow$ (1).

To finish, we prove (2) $\Rightarrow$ (4). If $\xi_k \in \mathcal{A}_{d\lambda}(\infty)$ for all $0 \le k \le d-1$, then $T_{d\lambda} : \mathcal{A}_{d\lambda}(\infty) \to \mathcal{A}_{d\lambda}(1)$ is d to 1. This means that $1-\lambda \in \mathcal{A}_{d\lambda}$ by Riemann-Hurwitz formula. The proof is complete. $\square$

Lemma 5.5. *For every $\lambda \in \mathbb{C}^*$, we have $0 \notin \mathcal{A}_{d\lambda}(\infty)$ and $1-\lambda \notin \mathcal{A}_{d\lambda}(1)$.*

Proof. If $0 \in \mathcal{A}_{d\lambda}(\infty)$, then $1-\lambda \in \mathcal{A}_{d\lambda}(1)$ by (5.9). Note that 1 lies also in $\mathcal{A}_{d\lambda}(1)$. This means that $T_{d\lambda}$ has $2d-1$ preimages in $\mathcal{A}_{d\lambda}(1)$ for each point in $\mathcal{A}_{d\lambda}(\infty)$ by Riemann-Hurwitz formula, which is a contradiction. Moreover, $0 \notin \mathcal{A}_{d\lambda}(\infty)$ means $1-\lambda \notin \mathcal{A}_{d\lambda}(1)$ by (5.9). $\square$

Since 1 and ∞ are always periodic with period 2 under $T_{d\lambda}$, the *non-escaping locus* $\mathcal{M}_d$ associated to $T_{d\lambda}$ can be defined as

$$\mathcal{M}_d = \{\lambda \in \mathbb{C}^* : T_{d\lambda}^{\circ 2n}(0) \not\to 1 \text{ and } T_{d\lambda}^{\circ 2n+1}(0) \not\to 1 \text{ as } n \to \infty\} \cup \{0\}. \tag{5.11}$$

Definition 5.1. *Define $\mathcal{H}_0 := \{\lambda \in \mathbb{C}^* : 0 \in \mathcal{A}_{d\lambda}(1)\}$. For every $n \ge 1$, define*

$$\mathcal{H}_n := \{\lambda \in \mathbb{C}^* : T_{d\lambda}^{\circ n}(0) \in \mathcal{A}_{d\lambda}(1) \text{ and } T_{d\lambda}^{\circ n-1}(0) \notin \mathcal{A}_{d\lambda}(\infty)\}. \tag{5.12}$$

Each component of $\mathcal{H}_n$ is called a capture domain *of* depth n, *where $n \ge 0$.*

Proposition 5.1. *The parameter space of $T_{d\lambda}$ has the following decomposition:*

$$\mathbb{C} = \mathcal{M}_d \sqcup (\bigsqcup_{n \ge 0} \mathcal{H}_n). \tag{5.13}$$

Proof. By definitions of the non-escaping locus and $\mathcal{H}_n$, we have $\mathcal{M}_d \cap (\bigcup_{n\ge 0} \mathcal{H}_n) = \emptyset$. We need to show that two capture domains with different depths are disjoint and each $\lambda \in \mathbb{C} \setminus \mathcal{M}$ belongs to $\mathcal{H}_n$ for some $n \ge 0$. First, suppose that $\lambda \in \mathcal{H}_m \cap \mathcal{H}_n$ for $m \ne n$. Without loss of generality, assume that $m > n \ge 0$. By Definition 5.1, we have $T_{d\lambda}^{\circ n}(0) \in \mathcal{A}_{d\lambda}(1)$ and $T_{d\lambda}^{\circ m-1}(0) \notin \mathcal{A}_{d\lambda}(\infty)$. This means that $T_{d\lambda}^{\circ m-1}(0) \in \mathcal{A}_{d\lambda}(1)$ and hence $T_{d\lambda}^{\circ m}(0) \in \mathcal{A}_{d\lambda}(\infty)$, which contradicts $T_{d\lambda}^{\circ m}(0) \in \mathcal{A}_{d\lambda}(1)$. Therefore $\mathcal{H}_m \cap \mathcal{H}_n = \emptyset$ for $m \ne n$.

By (5.11), if $\lambda \notin \mathcal{M}_d$, there exists a minimal $k \geq 0$ such that $T_{d\lambda}^{\circ k}(0) \in \mathcal{A}_{d\lambda}(1)$. If $k=0$, then $\lambda \in \mathcal{H}_0$. If $k=1$, then $T_{d\lambda}(0) \in \mathcal{A}_{d\lambda}(1)$. Lemma 5.5 asserts that $0 \notin \mathcal{A}_{d\lambda}(\infty)$. Therefore, $\lambda \in \mathcal{H}_1$ in this case. If $k \geq 2$, we claim that $T_{d\lambda}^{\circ k-1}(0) \notin \mathcal{A}_{d\lambda}(\infty)$. In fact, if not, we have $T_{d\lambda}^{\circ k-2}(0) \in \mathcal{A}_{d\lambda}(1)$. This contradicts the choice of the integer k. So we have $\lambda \in \mathcal{H}_k$ in this case. The proof is complete. □

See Figure 2 for the non-escaping loci $\mathcal{M}_2$ and $\mathcal{M}_3$. There some capture domains are also clearly visible (blank regions).

5.5 Quasiconformal conjugacy classes

Let $\mathcal{R}_d$ be the collection of all $T_{d\lambda}$, where $\lambda \in \mathbb{C}^*$. In this section, we give a complete characterization of the quasiconformal conjugacy classes in $\mathcal{R}_d$.

Definition 5.2. *Let Λ be a complex manifold. A* holomorphic family *of rational maps* parameterized *by Λ is a holomorphic map $f_\lambda : \Lambda \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f_\lambda(z)$ is a rational map for fixed $\lambda \in \Lambda$ and depends holomorphically on $\lambda \in \Lambda$ for fixed $z \in \overline{\mathbb{C}}$.*

The parameter $\lambda \in \Lambda$ is called a *J-stable* parameter of a holomorphic family of rational maps f_λ if the total number of attracting cycles of f_λ is constant in a neighborhood of λ.

Theorem 5.6. *The boundary $\partial\mathcal{M}_d$ is the set of parameters such that $T_{d\lambda}$ are not J-stable in $\mathcal{R}_d$.*

Proof. By [Mc, Theorem 4.2], $T_{d\lambda_0}$ is J-stable if and only if both critical sequences $\{T_{d\lambda}^{\circ k}(1-\lambda)\}_{k\geq 0}$ and $\{T_{d\lambda}^{\circ k}(1)\}_{k\geq 0}$ are normal for λ in a neighborhood of λ_0. Since $\{T_{d\lambda}^{\circ k}(1)\}_{n\geq 0}$ lies in a finite orbit $1 \leftrightarrow \infty$, we only need to consider the orbit of $1-\lambda$. If $\lambda_0 \in \mathcal{H}_n$ for some $n \geq 0$, the orbit of $1-\lambda_0$ will be attracted by the cycle $1 \leftrightarrow \infty$. For λ close to λ_0, the orbit of $1-\lambda$ still converges to the cycle $1 \leftrightarrow \infty$. By Montel's theorem, $\{T_{d\lambda}^{\circ k}(1-\lambda)\}_{k\geq 0}$ is normal at λ_0. Similarly, $\{T_{d\lambda}^{\circ k}(1-\lambda)\}_{k\geq 0}$ is normal at each point in the interior of $\mathcal{M}_d$ since $\{T_{d\lambda}^{\circ k}(1-\lambda)\}_{k\geq 0}$ is disjoint with the attracting basin of $1 \leftrightarrow \infty$. This means that $T_{d\lambda}$ is J-stable in $\mathbb{C} \setminus \partial\mathcal{M}_d$.

On the other hand, if $\lambda_0 \in \partial\mathcal{M}_d$, then $\{T_{d\lambda_0}^{\circ k}(1-\lambda)\}_{k\geq 0}$ omits the attracting basin of $1 \leftrightarrow \infty$. However, there are arbitrary small perturbation of λ_0 such that $\{T_{d\lambda}^{\circ k}(1-\lambda)\}_{k\geq 0}$ converges to the cycle $1 \leftrightarrow \infty$. This means that $T_{d\lambda}$ is not J-stable on $\partial\mathcal{M}_d$. □

Corollary 5.1. *Let W be a component in the interior of $\mathcal{M}_d$. If there exists $\lambda_0 \in W$ such that $1-\lambda_0$ converges to an attracting cycle, then every $\lambda \in W$ also has this property.*

Proof. By Theorem 5.6, every $T_{d\lambda} \in W$ is J-stable. This means that there exists a small neighborhood of λ such the number of attracting cycles is constant. Since $1-\lambda_0$ converges to an attracting cycle, this means that the constant is 2. The corollary follows. □

In the case of Corollary 5.1, W is called a *hyperbolic* component. Otherwise, W is called a *queer* component. It was generally believed that queer components do not exist. But if they do, then every $T_{d\lambda}$ admits an invariant line field on its Julia set and the Julia set has positive Lebesgue area. See Figures 5.1 and 5.2 for various Julia sets of $J_{d\lambda}$.

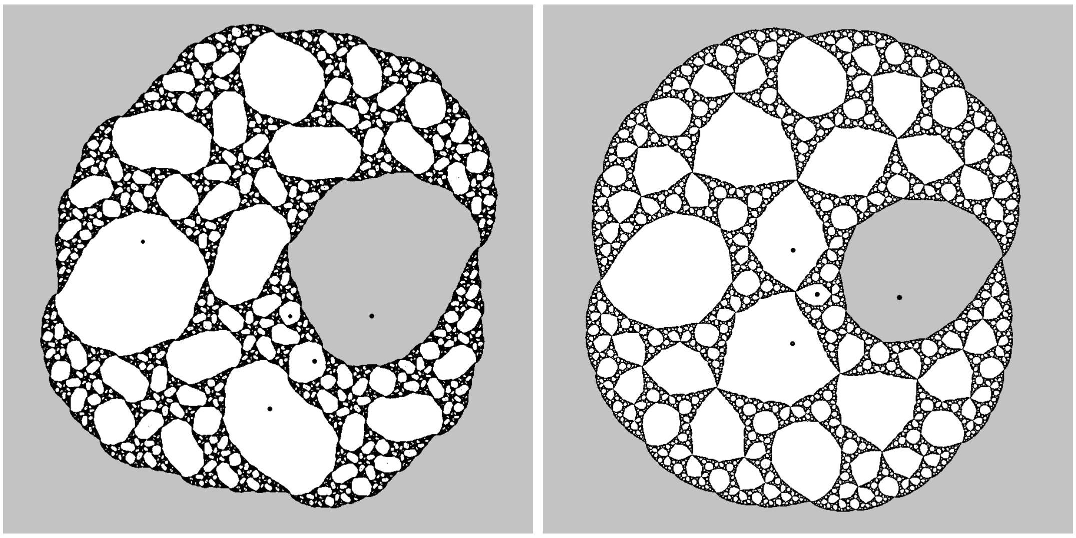

Figure 5.1: Julia sets of $T_{2\lambda}$ with $\lambda_1 \approx 1.319448 + 1.633170i$ and $\lambda_2 \approx 1.5 + 0.866025i$. The critical orbit $1 \leftrightarrow \infty$ captures the critical orbit $1-\lambda_1 \mapsto 0 \mapsto a \mapsto b \mapsto 1$ and disjoint with the critical orbit $1-\lambda_2 \mapsto 0 \mapsto c \mapsto 1-\lambda_2$.

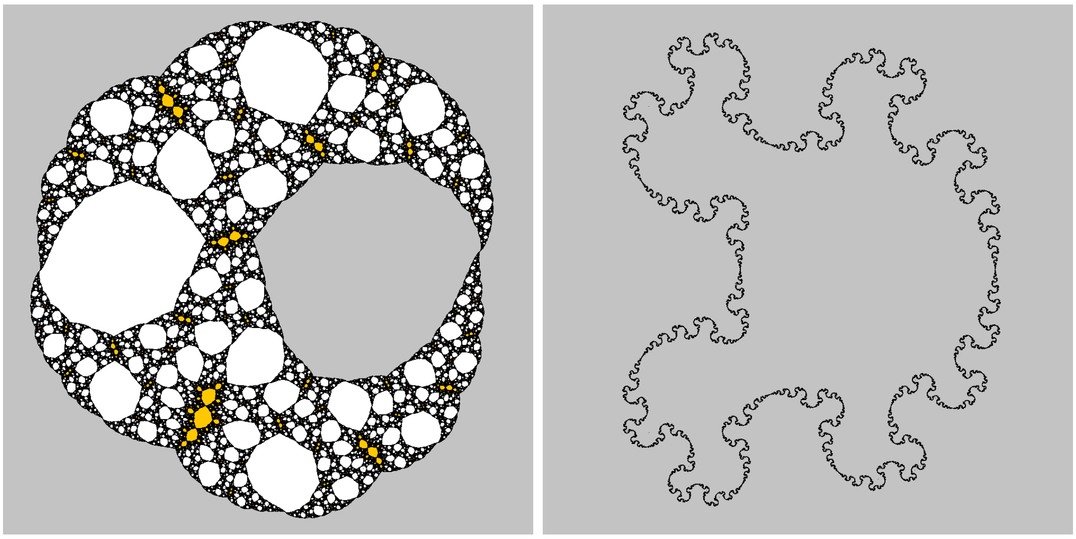

Figure 5.2: Julia sets of $T_{2\lambda}$ with $\lambda_3 \approx 2.046736+1.589069i$ and $\lambda_4 = 4.0$. $T_{2\lambda_3}$ has a Siegel disk with periodic 4 and $J_{2\lambda_4}$ is a quasicircle.

Now we state a theorem of parameterization of quasiconformal conjugacy classes.

Theorem 5.7. *Let $T_{d\lambda_0}, T_{d\lambda_1} \in \mathcal{R}_d$ be two different maps and let $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a K -quasiconformal homeomorphism which conjugates $T_{d\lambda_0}$ to $T_{d\lambda_1}$ such that $\varphi(\lambda_0) = \lambda_1$. Then there exists a holomorphic map $t \mapsto \lambda_t$ from an open disk $\mathbb{D}(0,r)$ $(r > 1)$ into $\mathbb{C}^*$*

which maps 0 *to* λ_0 *and* 1 *to* λ_1*, such that for every* $t \in \mathbb{D}(0,r)$*,* $T_{d\lambda_0}$ *is conjugate to* $T_{d\lambda_t}$ *by a* K_t *-quasiconformal mapping* $\varphi_t : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$*. Moreover,* $K_t \to 1$ *as* $t \to 0$*.*

The idea of the proof of Theorem 5.7 is standard in holomorphic dynamics. One can refer [Za, Theorem 5.1] for a proof in the similar situation. As an immediate corollary, we have

Corollary 5.2. *Quasiconformal conjugacy classes in* $\mathcal{R}_d$ *are either single points or open and connected. In particular, the conjugacy classes on* $\partial\mathcal{M}_d$ *are single points.*

A holomorphic family of rational maps $f_\lambda : \Lambda \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is *quasiconformally constant* if f_{λ_1} and f_{λ_2} are quasiconformally conjugate for any λ_1 and λ_2 in the same component of Λ. We call the family f_λ has *constant critical orbit relations* if any coincidence $f_\lambda^{\circ n}(c_1) = f_\lambda^{\circ m}(c_2)$ between the forward orbits of two critical points c_1 and c_2 of f_λ persists under perturbation of λ. The following theorem was proved in [McS, Theorem 2.7].

Theorem 5.8 ([McS])**.** *A holomorphic family* f_λ *of rational maps with constant critical orbit relations is quasiconformally constant.*

Proposition 5.2. *The Julia set* $J_{d\lambda}$ *of* $T_{d\lambda}$ *is a quasicircle if and only if* $\lambda \in \mathcal{H}_0$*. Moreover,* $\mathcal{H}_0$ *is unbounded and connected.*

A more precise characterization of the structure of $\mathcal{H}_0$ will be given in Theorem 5.10.

Proof. By the definition of $\mathcal{H}_0$ and Lemma 5.4, it follows that if $\lambda \in \mathcal{H}_0$, then $J_{d\lambda}$ is a quasicircle. Conversely, if $J_{d\lambda}$ is a quasicircle, then $1 - \lambda \in \mathcal{A}_{d\lambda}(\infty)$. This means that $T_{d\lambda}$ and $T_{d\lambda_0}$ have the same critical orbit relations, where $\lambda_0 \in \mathcal{H}_0$. By Theorem 5.8, $T_{d\lambda}$ and $T_{d\lambda_0}$ are quasiconformally conjugate to each other. By Corollary 5.2, it follows that $\lambda \in \mathcal{H}_0$ and $\mathcal{H}_0$ is connected.

To finish, we only need to show that $\mathcal{H}_0$ is unbounded. Let $\alpha = \lambda^{-\frac{1}{d+1}}$ and $\varphi_\alpha(z) = \alpha^d(z-1)$ be a linear transformation. By a straightforward calculation, we have

$$f_\alpha(z) := \varphi_\alpha \circ T_{d\lambda} \circ \varphi_\alpha^{-1} = \sum_{i=0}^{d-1} \frac{C_d^i \alpha^i}{z^{d-i}} = \frac{1}{z^d} + \frac{C_d^1 \alpha}{z^{d-1}} + \cdots + \frac{C_d^1 \alpha^{d-1}}{z}.$$

If $\alpha \neq 0$ is small enough, then the Julia set of f_α is a quasicircle since the Julia set of $z \mapsto 1/z^d$ is the unit circle. This means that $J_{d\lambda}$ is a quasicircle if λ is large enough. $\square$

By definition, the parameter $\lambda \in \bigcup_{n \geq 0} \mathcal{H}_n$ if and only if the critical orbit $1 - \lambda \mapsto 0 \mapsto (1-\lambda)^d \mapsto \cdots$ tends to the attracting periodic cycle $1 \mapsto \infty \mapsto 1$. A point λ is called a *center* of a hyperbolic component $W \subset \mathcal{M}_d$ if the critical point $1 - \lambda$ is periodic. On the other hand, λ is called a *center* of a capture domain of $\bigcup_{n \geq 1} \mathcal{H}_n$ if the critical point $1 - \lambda$ is eventually mapped to 1.

Lemma 5.6. *Every hyperbolic component in $\mathcal{M}_d$ and capture domain in $\mathcal{H}_n$ has a center, where $n \geq 1$. Meanwhile, $\mathcal{H}_0$ has no center.*

It will be proved in next section that every hyperbolic component in $\mathcal{M}_d$ and capture domain in $\mathcal{H}_n$ has exactly one center, where $n \geq 1$ (Theorem 5.10).

Proof. Let W be a hyperbolic component in $\mathcal{M}_d$. For every $\lambda \in W$, let $m(\lambda)$ be the multiplier of the attracting periodic orbit of $T_{d\lambda}$ other than $1 \leftrightarrow \infty$. It can be checked as in [Do, Theorem 4, p. 46] and [**?**, Theorem 2.1, p. 134] that the multiplier mapping $\lambda \mapsto m(\lambda)$ defined from W to $\mathbb{D}$ is proper and holomorphic. This means that W has at least one center.

Let W be a component of $\mathcal{H}_n$, where $n \geq 1$. Then for every $\lambda \in W$, $T_{d\lambda}^{\circ n}(0) \in \mathcal{A}_{d\lambda}(1)$ and n is the smallest integer satisfying this property. Let $\psi_\lambda : \mathcal{A}_{d\lambda}(1) \to \mathbb{D}$ be the unique Böttcher map define on the immediate basin of 1 such that $\psi_\lambda \circ U_{d\lambda} = (\psi_\lambda(z))^d$, $\psi_\lambda(1) = 0$ and $\psi_\lambda'(1) = 1$. By the definition of ψ_λ, it follows that ψ_λ depends holomorphically on $\lambda \in W$. Define a map $m : W \to \mathbb{D}$ by $m(\lambda) = \psi_\lambda(T_{d\lambda}^{\circ n}(0))$. It is clearly that m is holomorphic. We then prove m is proper. Let $\lambda_k \in W$ be a sequence converging to $\lambda \in \partial W$ as $n \to \infty$. Suppose that there exists a subsequence of λ_k, denote also by λ_k, such that $m(\lambda_k)$ converges to an interior point $w \in \mathbb{D}$. Since the family of univalent mappings $\{\psi_{\lambda_k}^{-1} : \mathbb{D} \to \mathbb{C}\}$ is normal, we can suppose that $\psi_{\lambda_k}^{-1} \to \psi^{-1}$ locally uniformly on $\mathbb{D}$. So $\psi^{-1}(\mathbb{D}) \subset \mathcal{A}_{d\lambda}(1)$. This means that $\psi^{-1}(w) = \lim_{k\to\infty} \psi_{\lambda_k}^{-1}(m(\lambda_k)) = \lim_{k\to\infty} T_{d\lambda_k}^{\circ n}(0) = T_{d\lambda}^{\circ n}(0) \in \mathcal{A}_{d\lambda}(1)$. Hence $T_{d\lambda}$ is hyperbolic. This is a contradiction since $\lambda \in \partial W$.

Finally, by the definition of $\mathcal{H}_0$ and Lemma 5.4, $\mathcal{A}_{d\lambda}(1)$ contains only one critical point 1 (counted without multiplicity). Note that $\mathcal{A}_{d\lambda}(1)$ lies in a superattracting periodic Fatou component and $T_{d\lambda}(1-\lambda) = 0 \neq 1$, it follows that the orbit of $1-\lambda$ is disjoint with the orbit $1 \leftrightarrow \infty$. The proof is complete. □

Now we give a complete characterization of the quasiconformal conjugacy classes in $\mathcal{R}_d$.

Theorem 5.9. *Quasiconformal conjugacy classes in $\mathcal{R}_d$ can be listed as follows:*

(1) *Hyperbolic components in the interior of $\mathcal{M}_d$ with the center removed.*
(2) *Capture components of $\mathcal{H}_n$ with the center (if any) removed, where $n \geq 0$.*
(3) *Centers of hyperbolic or capture domains.*
(4) *Queer components in the interior of $\mathcal{M}_d$.*
(5) *Single points on the boundary of $\mathcal{M}_d$.*

Proof. By Corollary 5.2, the five cases stated in the theorem are disjoint to each other and (4)(5) are indeed quasiconformal conjugacy classes. (1)(2) are quasiconformal conjugacy classes by Theorem 5.8. As every queer component is a conjugacy class, one can get a proof in [Za, Theorem 3.4] by a word for word analysis. □

5.6 Simply connectivity of the capture domains

In this section, we prove that the non-escaping locus $\mathcal{M}_d$ is connected. This amounts to showing that $\mathcal{H}_0$ is homeomorphic to the punctured disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ and each of the component of $\mathcal{H}_n$ is homeomorphic to the unit disk for $n \geq 1$.

One way to do this is to follow the standard way of Douady-Hubbard's parameterization of the hyperbolic components of the quadratic Mandelbrot set [Do]. This method was developed by Roesch to study the parameter space of the cubic Newton maps [Ro1, Ro2] and Qiu, Roesch, Wang and Yin to study the parameter space of the McMullen maps [QRWY]. Moreover, this parameterized method was generated and then used in the proof of $\mathcal{M}_2$ is connected [WQYQG, Theorem 1.1].

However, to prove $\mathcal{H}_0$ is homeomorphic to the punctured disk $\mathbb{D}^*$ and each of the component of $\mathcal{H}_n$ is homeomorphic to the unit disk for $n \geq 1$, it would be much easier to use the methods of Teichmüller theory of the rational maps which was developed in [McS] (in which, a different proof of the connectivity of the Mandelbrot set was given).

We first recall some definitions in [McS]. By definition, the *Teichmüller space* $\mathrm{Teich}(T_{d\lambda})$ of $T_{d\lambda}$ consists of all pairs $(T_{d\lambda'}, [\varphi])$, where $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasiconformal mapping which conjugates $T_{d\lambda'}$ to $T_{d\lambda}$. Here $[\varphi]$ means the isotopy class of φ. The *modular group* $\mathrm{Mod}(T_{d\lambda})$ is the group of isotopy classes of quasiconformal homeomorphism commuting with $T_{d\lambda}$. The modular group $\mathrm{Mod}(T_{d\lambda})$ acts on the Teichmüller space $\mathrm{Teich}(T_{d\lambda})$ properly discontinuously by $[\psi](T_{d\lambda'}, [\varphi]) = (T_{d\lambda'}, [\psi \circ \varphi])$. The *moduli space* of $T_{d\lambda}$ is defined as the quotient $\mathrm{Teich}(T_{d\lambda})/\mathrm{Mod}(T_{d\lambda})$, which is isomorphic to the quasiconformal conjugacy class of $T_{d\lambda}$.

Moreover, one can define the Teichmüller space $\mathrm{Teich}(U, T_{d\lambda})$ on an open set U which is invariant under $T_{d\lambda}$. The set $\mathrm{Teich}(U, T_{d\lambda})$ consists of all the triples $(V, T_{d\lambda'}, [\varphi])$, where V is open and invariant under $T_{d\lambda'}$, and the quasiconformal mapping $\varphi : V \to U$ conjugates $T_{d\lambda'}$ to $T_{d\lambda}$. Here $[\varphi]$ denotes the isotopy class of φ relative ideal boundary of V.

Theorem 5.10. *Each component of $\mathcal{H}_n$ is homeomorphic to $\mathbb{D}$ and contains exactly one center, where $n \geq 1$. Moreover, $\mathcal{H}_0$ is homeomorphic to the punctured disk $\mathbb{D}^*$.*

Proof. Let W be a component of $\mathcal{H}_n$ with all centers removed. Then the forward orbit of $1-\lambda$ under $T_{d\lambda}$ is infinite for $\lambda \in W$. By Theorem 5.9, W denotes a single quasiconformal conjugacy class.

For any basepoint $\lambda \in W$, it follows that the critical point $1-\lambda$ belongs to the attracting basin of the cycle $1 \mapsto \infty \mapsto 1$. In particular, $T_{d\lambda}^{\circ n}(0) \in \mathcal{A}_{d\lambda}(1)$ and $T_{d\lambda}^{\circ n}(0) \neq 1$. Define the Green function on $\mathcal{A}_{d\lambda}(1)$ by

$$G_{d\lambda}(z) = -\lim_{k\to\infty} d^{-k} \log |U_{d\lambda}^{\circ k}(z) - 1|, \text{ where } z \in \mathcal{A}_{d\lambda}(1).$$

Note that $G_{d\lambda}$ can be extended to the Fatou set of $T_{d\lambda}$ by pulling back.

Let γ be the equipotential of $G_{d\lambda}$ passing through the critical point $1-\lambda$ (In particular, it is homeomorphic to the figure 8 if $d=2$). Define

$$\widehat{J}_{d\lambda} := J_{d\lambda} \cup \bigcup_{n\in\mathbb{Z}} T_{d\lambda}^{\circ n}(\gamma \cup \{0\}).$$

Then $\widehat{J}_{d\lambda}$ is the closure of the grand orbits of all periodic points and critical points of $T_{d\lambda}$. The complement $U := \overline{\mathbb{C}} \setminus \widehat{J}_{d\lambda}$ consists of countably many annuli with finite modulus which lie in a same grand orbit. By [McS, Theorem 6.2], we have

$$\mathrm{Teich}(T_{d\lambda}) \simeq \mathrm{Teich}(U, T_{d\lambda}) \times M_1(J_{d\lambda}, T_{d\lambda}),$$

where $M_1(J_{d\lambda}, T_{d\lambda})$ denotes the unit ball in the space of all $T_{d\lambda}$-invariant Beltrami differentials supported on $J_{d\lambda}$. Note that every hyperbolic rational map carries no invariant line fields on the Julia set, it follows that $M_1(J_{d\lambda}, T_{d\lambda})$ is trivial since $T_{d\lambda}$ is hyperbolic when $\lambda \in W \subset \mathcal{H}_n$.

Since W denotes a single quasiconformal conjugacy class, we have

$$W \simeq \mathrm{Teich}(T_{d\lambda})/\mathrm{Mod}(T_{d\lambda}) \simeq \mathrm{Teich}(U, T_{d\lambda})/\mathrm{Mod}(T_{d\lambda}) \simeq \mathbb{H}/\mathrm{Mod}(T_{d\lambda})$$

by [McS, Theorem 6.1]. Note that every quasiconformal self-conjugacy ψ of $T_{d\lambda}$ fixes the grand orbits of the critical points 1 and $1-\lambda$ and hence fixes the boundaries of each annulus of U. Moreover, ψ is the identity on $J_{d\lambda}$. Therefore, $[\psi] \in \mathrm{Mod}(T_{d\lambda})$ is identity on $\widehat{J}_{d\lambda}$ and it is possibly a power of a Dehn twist in the annuli of U. This means that $\mathrm{Mod}(T_{d\lambda})$ is a subgroup of $\mathbb{Z}$.

By Lemma 5.6, each W cannot be simply connected is a component of $\mathcal{H}_n$ for $n \geq 1$. On the other hand, W is not simply connected if $W = \mathcal{H}_0$ by Proposition 5.2. So $\mathrm{Mod}(T_{d\lambda}) = \mathbb{Z}$. This means that W is homeomorphic to a punctured disk. This means that each W contains exactly only one center if $W \neq \mathcal{H}_0$. The proof is complete. □

Proof of Theorem 5.2. This is a direct corollary of Proposition 5.1 and Theorem 5.10. □

5.7 Proof of the asymptotic formula

By Proposition 5.2, if the parameter λ lies in the unbounded capture domain $\mathcal{H}_0$, then the Julia set $J_{d\lambda}$ is a quasicircle. In this case, $J_{d\lambda}$ moves holomorphically in $\mathcal{H}_0$ and its Hausdorff dimension depends real analytically on λ by a classic result of Ruelle. The following Theorem 5.11 is a weak version of [Ru, Corollary 6].

Theorem 5.11. *Let $f_\lambda : \Lambda \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a holomorphic family of hyperbolic rational maps parameterized by Λ, where Λ is a complex manifold. Then the Hausdorff dimension of the*

Julia set of f_λ depends real analytically on $\lambda \in \Lambda$.

Let Ω be a closed subset of $\mathbb{R}^n$. A map $S : \Omega \to \Omega$ is called a *contraction* on Ω if there exists a real number $c \in (0,1)$ such that $|S(x) - S(y)| \le c|x - y|$ for all $x, y \in \Omega$. A finite family of contractions $\{S_1, S_2, \cdots, S_m\}$ defined on $\Omega \subset \mathbb{R}^n$, with $m \ge 2$, is called an *iterated function system* or IFS in short.

To compute the Hausdorff dimension of $J_{d\lambda}$ with $\lambda \in \mathcal{H}_0$, we need the following result (see [Fa, Theorem 9.1, Propositions 9.6 and 9.7]).

Theorem 5.12 ([Fa]). *Let $\{S_1, \ldots, S_m\}$ be an IFS on a closed set $\Omega \subset \mathbb{R}^n$ such that $|S_i(x) - S_i(y)| \le c_i|x - y|$ with $0 < c_i < 1$. Then:*

(1) *There exists a unique non-empty compact set J such that $J = \bigcup_{i=1}^m S_i(J)$.*

(2) *The Hausdorff dimension $\dim_H(J)$ of J satisfies $\dim_H(J) \le s$, where $\sum_{i=1}^m c_i^s = 1$.*

(3) *If we require further $|S_i(x) - S_i(y)| \ge b_i|x - y|$ for $0 < b_i < 1$, then $\dim_H(J) \ge s'$, where $\sum_{i=1}^m b_i^{s'} = 1$.*

The non-empty compact set J appeared in Theorem 5.12(1) is called the *attractor* of the IFS $\{S_1, \ldots, S_m\}$.

Let f be a rational map with degree at least two. We use $\mathrm{Fix}(f)$ to denote the set of all the fixed points in the Julia set of f.

Lemma 5.7. *Let f be a hyperbolic rational map whose Julia set J is a quasicircle. Then the Hausdorff dimension $D := \dim_H(J)$ of J is determined by*[1] *$\lim_{n\to\infty} A_n(D) = 1$, where*

$$A_n(D) = \sum_{z \in \mathrm{Fix}(f^{\circ n})} |(f^{\circ n})'(z)|^{-D}. \tag{5.14}$$

The notation $\mathrm{Fix}(f^{\circ n})$ in (5.14) denotes the collection of all the repelling periodic points of f with period n (the period is not necessary the smallest). The Julia set of a hyperbolic rational map can be seen as the limit of a sequence of IFS which are defined in terms of the inverse branches of the iterations of the rational map.

Proof. Let $d \ge 2$ be the degree of f. Since f is hyperbolic and the Julia set J of f is a quasicircle, there exist a pair of closed annular neighborhoods W_1, W_2 of J and a quasiconformal mapping $\phi : W_1 \to \mathbb{A}_\varepsilon$, such that ϕ conjugates $f : W_1 \to W_2$ to $z \mapsto z^d$ or $z \mapsto z^{-d}$, where $\mathbb{A}_\varepsilon := \{z : 1 - \varepsilon \le |z| \le 1 + \varepsilon\}$ is a closed annular neighborhood of the unit circle and $\varepsilon > 0$ is small enough. Without loss of generality, we only consider the first case since the completely similar argument can be applied to the second one.

In order to define IFS, it is more convenient to lift J and f under the exponential map. Hence we assume further that J separates 0 and ∞. Define a curve $\gamma := \phi^{-1}([(1 - \varepsilon)^d, (1 + \varepsilon)^d]) \subset W_2$. Fix a component of $\exp^{-1}(W_2 \setminus \gamma)$ and denote it by U. Then U is

[1]The statement and the proof of this lemma were not correct in the previous version. We would like to thank Peter Haïssinsky for pointing out to us. See [WBKS, §4, (4.2)] for the same statement.

topologically a strip and $\exp : U \to W_2 \setminus \gamma$ is conformal in the interior of U, whose inverse is denoted by $\log : W_2 \setminus \gamma \to U$ (see Figure 5.3).

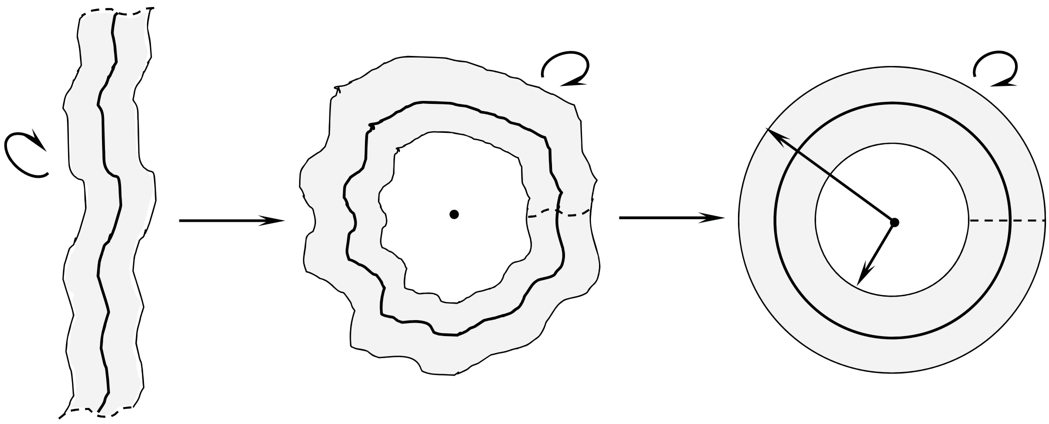

Figure 5.3: Sketch illustration of the construction of the IFS.

Let V_n be the unique component of $f^{-n}(W_2)$ containing J. For each $n \geq 1$, the map $f^{\circ n} : V_n \to W_2$ has d^n inverse branches, say $T_1, \cdots, T_{d^n}$, each maps $W_2 \setminus \gamma$ onto a half open quadrilateral such that their images are arranged in anticlockwise order one by one. Let $S_i := \log \circ T_i \circ \exp$ be the map defined in U, where $1 \leq i \leq d^n$. It is easy to see each S_i is conformal in the interior of U and can be conformally extended to an open neighborhood of $\overline{U}$.

By the definition, $\{S_1, \cdots, S_{d^n}\}$ is an IFS defined on $\overline{U}$ for large n since $f^{\circ n}$ is strictly expanding on W_1 in the Euclidean metric if n is large[2]. For the convenient of the argument, we assume that $f^{\circ n}$ is expanding in the Euclidean metric for all $n \geq 1$. The attractor J' of $\{S_1, \cdots, S_{d^n}\}$ is a closed set satisfying $J = \exp(J')$. Moreover, $J \setminus \{z_1\}$ is the conformal image of J' with two ends removed, where $z_1 \in J \cap \gamma$ is a fixed point of f. This means that the Hausdorff dimensions of J' and J satisfy $\dim_H(J') = \dim_H(J)$.

Let $F_n|_U := \bigsqcup_{i=1}^{d^n} S_i^{-1}|_{S_i(U)}$ be the lift of $f^{\circ n}$ under exp. Then each $S_i(U)$ contains exactly one fixed point $\zeta_i \in J'$ of F_n in its interior for $1 < i < d^n$ and on its boundary for $i = 1$ and d^n.

By Koebe distortion theorem there exist two sequences of numbers $0 < A_n \leq 1 \leq B_n$, such that

$$\frac{A_n}{|F_n'(\zeta_i)|} \leq \frac{|S_i(x) - S_i(y)|}{|x - y|} \leq \frac{B_n}{|F_n'(\zeta_i)|}, \ \forall\, 1 \leq i \leq d^n, \ x, y \in \overline{U}, \tag{5.15}$$

and $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = 1$. See also [Mi1, Theorem 2.7].

By Theorem 5.12, the Hausdorff dimension $D = \dim_H(J') = \dim_H(J)$ satisfies $0 \leq$

[2]The map f is expanding in the hyperbolic metric in a neighborhood of J.

$s_{n,1} \leq D \leq s_{n,2} \leq 2$, where

$$\sum_{i=1}^{d^n} \left(\frac{A_n}{|F_n'(\zeta_i)|} \right)^{s_{n,1}} = 1 \text{ and } \sum_{i=1}^{d^n} \left(\frac{B_n}{|F_n'(\zeta_i)|} \right)^{s_{n,2}} = 1.$$

Then, we have

$$\frac{1}{B_n^{s_{n,2}}} \leq \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^{s_{n,2}}} \leq \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^{D}} \leq \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^{s_{n,1}}} = \frac{1}{A_n^{s_{n,1}}}. \tag{5.16}$$

The $d^n - 1$ fixed points of $f^{\circ n}$ in the Julia set J are $\{z_i = \exp(\zeta_i) : 1 \leq i < d^n\}$. In particular, $z_1 = \exp(\zeta_1) = \exp(\zeta_{d^n})$. Since F_n is conformally conjugate to $f^{\circ n}$ in the interior of each $S_i(U)$, we have $F_n'(\zeta_i) = (f^{\circ n})'(z_i)$ for $1 \leq i < d^n$. Therefore, by (5.16), we have

$$\sum_{z \in \mathrm{Fix}(f^{\circ n})} \frac{1}{|(f^{\circ n})'(z)|^D} = \sum_{i=1}^{d^n - 1} \frac{1}{|(f^{\circ n})'(z_i)|^D} = \sum_{i=1}^{d^n} \frac{1}{|F_n'(\zeta_i)|^D} - \frac{1}{|F_n'(\zeta_{d^n})|^D} \to 1 \text{ as } n \to \infty$$

since $\lim_{n\to\infty} A_n^{s_{n,1}} = \lim_{n\to\infty} B_n^{s_{n,2}} = 1$ and $\lim_{n\to\infty} |F_n'(\zeta_{d^n})| = +\infty$. The proof is completed. $\square$

As the parameter λ tends to ∞, the diameter of the Julia set $J_{d\lambda}$ of $T_{d\lambda}$ becomes larger and larger in the Euclidean metric and the shape of $J_{d\lambda}$ becomes more and more circular (see Figure 5.4). Therefore, one can make a scaling of $J_{d\lambda}$ (or equivalently, make a conjugate), such the new Julia set converges to the unit circle.

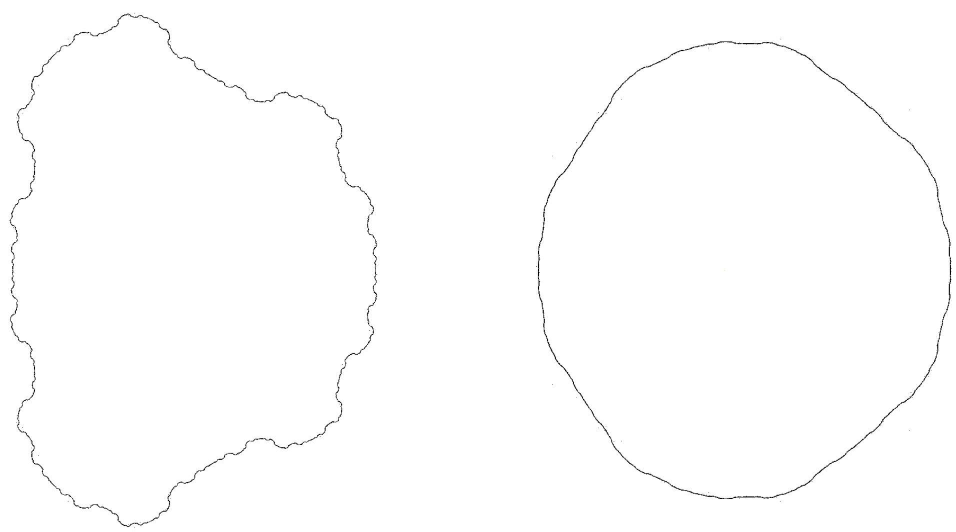

Figure 5.4: The Julia sets of $T_{2\lambda}$, both are quasicircles, where $\lambda = 30$ and 1000, respectively. It can be seen that the Julia set becomes more circular as the parameter λ becomes more larger (compare the right picture in Figure 5.2). Figure ranges: $[-10, 16] \times [-13, 13]$ and $[-125, 125] \times [-125, 125]$.

Specifically, define

$$J^*_{d\lambda} = \{\lambda^{-\frac{d}{d+1}}(z-1) : z \in J_{d\lambda}\}. \tag{5.17}$$

The following Lemma 5.8 has been proved in [Qi, Theorem 4.3] as a special case.

Lemma 5.8. *The scaled Julia set $J^*_{d\lambda}$ converges to the unit circle in the Hausdorff topology as λ tends to ∞ and the Hausdorff dimension of $J_{d\lambda}$ tends to 1 as λ tends to ∞.*

Although Lemma 5.8 is significant, however, we want to know further about the asymptotic formula of of the Hausdorff dimension of $J_{d\lambda}$ as λ tends to ∞. In order to calculate the Hausdorff dimension of $J_{d\lambda}$, we do some setting first.

Recall that in Proposition 5.2, $\alpha = \lambda^{-\frac{1}{d+1}}$. Then $\lambda\alpha^d = \alpha^{-1}$. Let $\varphi_\alpha(z) = \alpha^d(z-1)$ be the linear transformation as before. We define a new rational map with parameter α as

$$f_\alpha(z) := \varphi_\alpha \circ T_{d\lambda} \circ \varphi_\alpha^{-1} = \sum_{i=0}^{d-1} \frac{C_d^i\,\alpha^i}{z^{d-i}} = \frac{1}{z^d} + \frac{C_d^1\alpha}{z^{d-1}} + \cdots + \frac{C_d^1\alpha^{d-1}}{z}. \tag{5.18}$$

This means that there exists a small $\varepsilon > 0$ such that $f_\alpha : \mathbb{D}_\varepsilon \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a holomorphic family of hyperbolic rational maps parameterized by $\mathbb{D}_\varepsilon$, where $\mathbb{D}_\varepsilon := \{z : |z| < \varepsilon\}$. Note that the Hausdorff dimension is invariant under a conformal isomorphism. This means that we only need to calculate the Hausdorff dimension of the Julia set J_α of f_α with $\alpha \in \mathbb{D}_\varepsilon$ since $\dim_H(J_\alpha) = \dim_H(J_{d\lambda})$. We would like to remark that $J_\alpha = J^*_{d\lambda}$.

Let E be a subset of $\overline{\mathbb{C}}$ and (Λ, λ_0) a connected complex manifold with basepoint λ_0. A family of maps $h_\lambda : E \to \overline{\mathbb{C}}$ is called a *holomorphic motion* of E parameterized by Λ and with base point λ_0 if: (1) For each $\lambda \in \Lambda$, h_λ is injective on E; (2) For each $z \in E$, $h_\lambda(z)$ is a holomorphic function of $\lambda \in \Lambda$; and (3) h_{λ_0} is identity on E (see [Ly], [MSS] or [Mc, Chap. 4]).

Proof of Theorem 5.3. By (5.18), it follows that the Julia set J_α is the unit circle if $\alpha = 0$. For $z \in J_0 = \mathbb{T}$, we have $f_0(z) = z^{-d}$. Note that f_α is a holomorphic family of hyperbolic rational maps with parameter $\alpha \in \mathbb{D}_\varepsilon$. There exists a holomorphic motion $\phi_\alpha : J_0 \to \overline{\mathbb{C}}$ of J_0 parameterized by $\mathbb{D}_\varepsilon$ and with base point 0 such that $\phi_\alpha(J_0) = J_\alpha$ and

$$f_\alpha \circ \phi_\alpha(z) = \phi_\alpha \circ f_0(z) = \phi_\alpha(z^{-d}) \tag{5.19}$$

for all $z \in J_0$, see [Mc, Chap. 4]. Since every point on J_0 moves holomorphically, we can write $\phi_\alpha(z)$ in power series of α as

$$\phi_\alpha(z) = z\,(1 + u_1(z)\alpha + u_2(z)\alpha^2 + \mathcal{O}(\alpha^3)), \tag{5.20}$$

where $z \in J_0$.

In the following, we adopt the notation $q := -d$ since the negative sign is boring in the expressions during the calculation. Meantime, we assume that $d \geq 3$ first. If α is small enough, we can expand f_α in (5.18) in power series of α as

$$f_\alpha(z) = z^q - qz^{q+1}\alpha + \frac{q(q+1)}{2}z^{q+2}\alpha^2 + \mathcal{O}(\alpha^3). \tag{5.21}$$

Substituting (5.20) and (5.21) into (5.19), then comparing the terms to the second order in α, we obtain the following equations:

$$\begin{aligned} u_1(z^q) - qu_1(z) &= -qz, & (5.22)\\ u_2(z^q) - qu_2(z) &= \frac{q(q-1)}{2}u_1^2(z) - q(q+1)zu_1(z) + \frac{q(q+1)}{2}z^2. & (5.23)\end{aligned}$$

For each non-zero integer $l \in \mathbb{Z}$, the functional equation

$$u(z^q) - qu(z) = -qz^l \tag{5.24}$$

has the formal solution

$$u(z) = \sum_{k=0}^{+\infty} \frac{z^{lq^k}}{q^k}. \tag{5.25}$$

Note that the solution (5.25) is convergent if $|z| = 1$. This means that the solution of (5.22) is

$$u_1(z) = \sum_{k=0}^{+\infty} \frac{z^{q^k}}{q^k}. \tag{5.26}$$

Therefore, the equation (5.23) can be reduced to

$$u_2(z^q) - qu_2(z) = -q\left((q+1)\sum_{l=0}^{+\infty}\frac{z^{q^l+1}}{q^l} - \frac{q-1}{2}\left(\sum_{l=0}^{+\infty}\frac{z^{q^l}}{q^l}\right)^2 - \frac{q+1}{2}z^2\right). \tag{5.27}$$

By (5.24) and (5.25), the solution of u_2 is

$$u_2(z) = \sum_{k=0}^{+\infty}\left((q+1)\sum_{l=0}^{+\infty}\frac{z^{q^{l+k}+q^k}}{q^{l+k}} - \frac{(q-1)}{2q^k}\left(\sum_{l=0}^{+\infty}\frac{z^{q^{l+k}}}{q^l}\right)^2 - \frac{(q+1)}{2q^k}z^{2q^k}\right). \tag{5.28}$$

For each $n \geq 1$, the collection of the fixed points of $f_\alpha^{\circ n}$ on the Julia set J_α forms the finite set

$$\mathrm{Fix}(f_\alpha^{\circ n}) = \left\{\phi_\alpha(e^{2\pi i t_j}) : t_j = \frac{j}{q^n - 1}, 1 \leq j \leq |q^n - 1|\right\}. \tag{5.29}$$

By (5.19) and the chain rule, we have $(f_\alpha^{\circ n})'(\phi_\alpha(e^{2\pi i t_j})) = \prod_{m=0}^{n-1} f'_\alpha(\phi_\alpha(e^{2\pi i q^m t_j}))$. The calculation in Appendix (§5.8) shows that for every $D > 0$ and all sufficiently large n, the following holds:

$$\frac{1}{|q^n-1|}\sum_{j=1}^{|q^n-1|}\prod_{m=0}^{n-1}\left|f'_\alpha(\phi_\alpha(e^{2\pi i q^m t_j}))\right|^{-D} = |q|^{-nD}\left(1 + \frac{D^2 n}{4}|\alpha|^2 + \mathcal{O}(\alpha^3)\right). \tag{5.30}$$

Let $D_\alpha := \dim_H(J_\alpha)$ be the Hausdorff dimension of J_α. One can write the corresponding (5.14) of f_α in Lemma 5.7 as

$$A_n(D_\alpha) = |q^n - 1|\,|q|^{-nD_\alpha}\left(1 + \frac{D_\alpha^2 n}{4}|\alpha|^2 + \mathcal{O}(\alpha^3)\right). \tag{5.31}$$

Fix some large n, when α is small enough, (5.31) is equivalent to

$$\lim_{n\to\infty} \exp\left(n\left(\frac{D_\alpha^2}{4}|\alpha|^2 - (D_\alpha - 1)\log|q|\right) + \mathcal{O}(\alpha^3)\right) = 1. \tag{5.32}$$

By Theorem 5.11 and Lemma 5.8, D_α depends real analytically on α in a small neighborhood of the origin and $D_0 = 1$. This means that in a small neighborhood of 0, the Hausdorff dimension of J_α can be written as

$$D_\alpha = 1 + a_{10}\alpha + a_{01}\overline{\alpha} + a_{20}\alpha^2 + a_{02}\overline{\alpha}^2 + a_{11}|\alpha|^2 + \mathcal{O}(\alpha^3). \tag{5.33}$$

Substituting (5.33) into (5.32) and comparing the corresponding coefficients, we have

$$a_{10} = a_{01} = a_{20} = a_{02} = 0 \text{ and } a_{11} = 1/(4\log|q|). \tag{5.34}$$

This means that

$$D_\alpha = 1 + \frac{|\alpha|^2}{4\log|q|} + \mathcal{O}(\alpha^3). \tag{5.35}$$

Note that $q = -d$ and $\alpha = \lambda^{-\frac{1}{d+1}}$. This ends the proof of Theorem 5.3 in the case of $d \geq 3$.

If $d = 2$, then (5.21) can be written as $f_\alpha(z) = z^q - qz^{q+1}\alpha$. Following the calculation process of $d \geq 3$ and carefully omitting some corresponding terms, it can be checked that Theorem 5.3 still holds for $d = 2$. The proof is complete. □

5.8 Appendix

This section will devote to proving (5.30). From (5.21), we have

$$f'_\alpha(z) = qz^{q-1} - q(q+1)z^q\alpha + \frac{q(q+1)(q+2)}{2}z^{q+1}\alpha^2 + \mathcal{O}(\alpha^3). \tag{5.36}$$

Substituting (5.20) into (5.36), we have

$$\begin{aligned} f'_\alpha(\phi_\alpha(z)) = {} & qz^{q-1} + qz^{q-1}[(q-1)u_1(z) - (q+1)z]\,\alpha + qz^{q-1}\left[\frac{(q+1)(q+2)}{2}z^2\right.\\ & \left. + \frac{(q-1)(q-2)}{2}u_1^2(z) - q(q+1)zu_1(z) + (q-1)u_2(z)\right]\alpha^2 + \mathcal{O}(\alpha^3). \end{aligned} \tag{5.37}$$

Define $\sigma := \sigma(t) = e^{2\pi i t} \in \mathbb{T}$. Then $\sigma\overline{\sigma} = 1$. For $0 \le m \le n-1$, by (5.37), we have

$$\begin{aligned} &|f'_\alpha(\phi_\alpha(\sigma^{q^m}))|^2 = f'_\alpha(\phi_\alpha(\sigma^{q^m}))\,\overline{f'_\alpha(\phi_\alpha(\sigma^{q^m}))} \\ &= q^2 + A_m\alpha + \overline{A}_m\overline{\alpha} + A_m\overline{A}_m|\alpha|^2/q^2 + B_m\alpha^2 + \overline{B}_m\overline{\alpha}^2 + \mathcal{O}(\alpha^3), \end{aligned} \tag{5.38}$$

where

$$A_m = q^2(q-1)\,u_1(\sigma^{q^m}) - q^2(q+1)\,\sigma^{q^m} \tag{5.39}$$

and

$$\begin{aligned} B_m = \frac{q^2(q+1)(q+2)}{2}\sigma^{2q^m} + \frac{q^2(q-1)(q-2)}{2}u_1^2(\sigma^{q^m}) \\ - q^3(q+1)\sigma^{q^m}u_1(\sigma^{q^m}) + q^2(q-1)u_2(\sigma^{q^m}). \end{aligned} \tag{5.40}$$

For every $D > 0$, by (5.38), we have

$$\begin{aligned} &\prod_{m=0}^{n-1} |f'_\alpha(\phi_\alpha(\sigma^{q^m}))|^{-D} = \prod_{m=0}^{n-1} (|f'_\alpha(\phi_\alpha(\sigma^{q^m}))|^2)^{-\frac{D}{2}} \\ &= |q|^{-nD}\prod_{m=0}^{n-1}\left(1 + \frac{A_m\alpha + \overline{A}_m\overline{\alpha} + B_m\alpha^2 + \overline{B}_m\overline{\alpha}^2}{q^2} + \frac{A_m\overline{A}_m|\alpha|^2}{q^4} + \mathcal{O}(\alpha^3)\right)^{-\frac{D}{2}} \\ &= |q|^{-nD} - \frac{D}{2}|q|^{-nD-2}\sum_{m=0}^{n-1}\left(A_m\alpha + \overline{A}_m\overline{\alpha} + B_m\alpha^2 + \overline{B}_m\overline{\alpha}^2\right) \\ &- \frac{D}{2}|q|^{-nD-4}\left(\sum_{0\le m_1<m_2\le n-1}(A_{m_1}A_{m_2}\alpha^2 + \overline{A}_{m_1}\overline{A}_{m_2}\overline{\alpha}^2) + \sum_{0\le m_1,m_2\le n-1} A_{m_1}\overline{A}_{m_2}|\alpha|^2\right) \\ &+ \frac{D(D+2)}{8}|q|^{-nD-4}\left(\sum_{m=0}^{n-1}(A_m\alpha + \overline{A}_m\overline{\alpha})\right)^2 + \mathcal{O}(\alpha^3). \end{aligned} \tag{5.41}$$

Lemma 5.9. *Let $m, m_1, m_2 \in \mathbb{N}$. If $n \ge 1$, then:*
(1) *$q^m \not\equiv 0 \bmod q^n - 1$.*
(2) *$q^{m_1} + q^{m_2} \not\equiv 0 \bmod q^n - 1$.*
(3) *$q^{m_1} - q^{m_2} \equiv 0 \bmod q^n - 1$ if and only if $m_1 - m_2 = kn$ for some $k \in \mathbb{Z}$.*

Proof. Since $(q, q^n - 1) = 1$, it means that $(q^m, q^n - 1) = 1$ for $m \ge 0$. Then (1) follows.

To prove (2), it suffices to show that $q^m + 1 \not\equiv 0 \bmod q^n - 1$ for $m \ge 0$ since $q^n - 1$ is relative prime to $q^{m'}$ for $m' \ge 0$ by (1). Set $m = kn + r$, where $k \ge 0$ and $0 \le r \le n-1$. We have

$$q^m + 1 = q^{kn+r} - q^r + q^r + 1 \equiv q^r + 1 \not\equiv 0 \bmod q^n - 1$$

since $0 < |q^r + 1| < |q^n - 1|$.

The proof of (3) is similar to that of (2). Since $q^n - 1$ is relative prime to $q^{m'}$ for $m' \ge 0$, we need to find out the condition on m such that $q^m - 1 \equiv 0 \bmod q^n - 1$ for fixed

$n \geq 1$. Set $m = kn + r$, where $k \geq 0$ and $0 \leq r \leq n-1$. We have

$$q^m - 1 = q^{kn+r} - q^r + q^r - 1 \equiv q^r - 1 \bmod q^n - 1.$$

This means that $q^m - 1 \equiv 0 \bmod q^n - 1$ if and only if $r = 0$ since $|q^r - 1| < |q^n - 1|$. $\square$

Following [WBKS, § 2], it is convenient to introduce the *average notation*

$$\langle G(t)\rangle_n := \frac{1}{|q^n - 1|} \sum_{j=1}^{|q^n-1|} G(t_j), \tag{5.42}$$

where G is a continuous function defined on the interval $[0, 1)$ and $t_j = j/(q^n - 1)$ is defined in (5.29).

In order to prove (5.30), we only need to prove for every $D > 0$ and sufficiently large n, the following holds

$$\left\langle \prod_{m=0}^{n-1} \left|f'_\alpha(\phi_\alpha(\sigma^{q^m}))\right|^{-D} \right\rangle_n = |q|^{-nD} \left(1 + \frac{D^2 n}{4}|\alpha|^2 + \mathcal{O}(\alpha^3)\right). \tag{5.43}$$

For each $n \geq 1$ and any $k \in \mathbb{Z}$, it is straightforward to verify the average in (5.42) has the following useful property:

$$\langle \sigma^k \rangle_n = \langle e^{2\pi i k t}\rangle_n = \begin{cases} 1 & \text{if } k \equiv 0 \bmod q^n - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.44}$$

Lemma 5.10. *For $0 \leq m, m_1, m_2 \leq n-1$, we have $\langle \sigma^{q^m}\rangle_n = 0$, $\langle u_1(\sigma^{q^m})\rangle_n = 0$, $\langle \sigma^{q^{m_1}+q^{m_2}}\rangle_n = 0$, $\langle \sigma^{q^{m_1}} u_1(\sigma^{q^{m_2}})\rangle_n = 0$, $\langle u_1(\sigma^{q^{m_1}}) u_1(\sigma^{q^{m_2}})\rangle_n = 0$ and $\langle u_2(\sigma^{q^m})\rangle_n = 0$.*

Proof. By (5.26) and (5.28), the average property (5.44) and Lemma 5.9(1)(2), the equations stated in the Lemma can be verified directly. $\square$

As an immediate corollary of Lemma 5.10, from (5.39) and (5.40), we have

Corollary 5.3. $\langle A_m\rangle_n = \left\langle \overline{A}_m\right\rangle_n = 0, \langle B_m\rangle_n = \left\langle \overline{B}_m\right\rangle_n = 0, \langle A_{m_1} A_{m_2}\rangle_n = \left\langle \overline{A}_{m_1}\overline{A}_{m_2}\right\rangle_n = 0$ *for* $0 \leq m, m_1, m_2 \leq n-1$.

By (5.41) and Corollary 5.3, we have

$$\left\langle \prod_{m=0}^{n-1} |f'_\alpha(\phi_\alpha(\sigma^{q^m}))|^{-D} \right\rangle_n = |q|^{-nD} \left(1 + \frac{D^2}{4}|q|^{-4} \sum_{0 \leq m_1, m_2 \leq n-1} \langle A_{m_1}\overline{A}_{m_2}\rangle_n |\alpha|^2 \right) + \mathcal{O}(\alpha^3). \tag{5.45}$$

By (5.39) and (5.40), we have

$$\begin{aligned} \left\langle A_{m_1}\overline{A}_{m_2}\right\rangle_n = {} & q^4(q-1)^2 \langle u_1(\sigma^{q^{m_1}})\overline{u_1(\sigma^{q^{m_2}})}\rangle_n + q^4(q+1)^2 \langle \sigma^{q^{m_1} - q^{m_2}}\rangle_n \\ & - q^4(q^2-1)\langle u_1(\sigma^{q^{m_1}})\sigma^{-q^{m_2}} + \overline{u_1(\sigma^{q^{m_2}})}\sigma^{q^{m_1}}\rangle_n. \end{aligned} \tag{5.46}$$

Since $0 \leq m_1, m_2 \leq n-1$, it follows that $m_1 - m_2 = kn$ for $k \in \mathbb{Z}$ if and only if $m_1 = m_2$. By Lemma 5.9(3), we have

$$\left\langle \sigma^{q^{m_1}-q^{m_2}} \right\rangle_n = \begin{cases} 1 & \text{if } m_1 = m_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.47}$$

This means that

$$\sum_{0 \leq m_1, m_2 \leq n-1} \langle \sigma^{q^{m_1}-q^{m_2}} \rangle_n = n. \tag{5.48}$$

Similarly, by Lemma 5.9(3), we have

$$\begin{aligned} \left\langle u_1(\sigma^{q^{m_1}})\sigma^{-q^{m_2}} \right\rangle_n &= \sum_{k=0}^{+\infty} \frac{\langle \sigma^{q^{k+m_1}-q^{m_2}} \rangle_n}{q^k} \\ = &\begin{cases} \sum_{k=0}^{+\infty} \frac{1}{q^{n-(m_1-m_2)+kn}} = \frac{q^{m_1-m_2}}{q^n-1} & \text{if } m_1 > m_2, \\ \sum_{k=0}^{+\infty} \frac{1}{q^{m_2-m_1+kn}} = \frac{q^{n-(m_2-m_1)}}{q^n-1} & \text{if } m_1 \leq m_2. \end{cases} \end{aligned} \tag{5.49}$$

This means that

$$\begin{aligned} &\sum_{0 \leq m_1, m_2 \leq n-1} \langle u_1(\sigma^{q^{m_1}})\sigma^{-q^{m_2}} \rangle_n = \sum_{0 \leq m_2 < m_1 \leq n-1} \frac{q^{m_1-m_2}}{q^n-1} + \sum_{0 \leq m_1 \leq m_2 \leq n-1} \frac{q^{n-(m_2-m_1)}}{q^n-1} \\ &= \frac{n}{q^n-1}(q + q^2 + \cdots + q^n) = \frac{nq}{q-1}. \end{aligned} \tag{5.50}$$

Moreover, by Lemma 5.9(3), we have

$$\begin{aligned} \left\langle u_1(\sigma^{q^{m_1}})\overline{u_1(\sigma^{q^{m_2}})} \right\rangle_n &= \sum_{k_1=0}^{+\infty} \sum_{k_2=0}^{+\infty} \frac{\langle \sigma^{q^{k_1+m_1}-q^{k_2+m_2}} \rangle_n}{q^{k_1+k_2}} \\ = &\begin{cases} \left(\frac{1}{q^{m_1-m_2}} + \frac{1}{q^{n-(m_1-m_2)}}\right)\frac{q^{2+n}}{(q^2-1)(q^n-1)} & \text{if } m_1 > m_2, \\ \left(\frac{1}{q^{m_2-m_1}} + \frac{1}{q^{n-(m_2-m_1)}}\right)\frac{q^{2+n}}{(q^2-1)(q^n-1)} & \text{if } m_1 \leq m_2. \end{cases} \end{aligned} \tag{5.51}$$

This means that (similar to the reduction process of (5.50))

$$\sum_{0 \leq m_1, m_2 \leq n-1} \langle u_1(\sigma^{q^{m_1}})\overline{u_1(\sigma^{q^{m_2}})} \rangle_n = \frac{nq^2}{(q-1)^2}. \tag{5.52}$$

By substituting (5.48), (5.50) and (5.52) into (5.46), we have

$$\sum_{0 \leq m_1, m_2 \leq n-1} \langle A_{m_1} \overline{A}_{m_2} \rangle_n = nq^4. \tag{5.53}$$

By (5.45) and (5.53), it follows that (5.43) holds. The proof of (5.30) is completed.

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PÔLE DE RECHERCHE ET D'ENSEIGNEMENT SUPÉRIEUR

université angers

Thèse de Doctorat

Jinsong ZENG

Rigidité quasi-symétrique, tapis de Julia et le débarquement de dynamique resp. paramètres rayons

Quasisymmetric rigidity, carpet Julia sets and the landing of dynamic resp. parameter rays

Résumé

Cette thèse est constituée de cinq parties distinctes.

La première partie est consacrée au problème de rigidité quasi-symétrique associé à un nouveau modèle de tapis de Sierpinski, qui ne sont pas quasi-symétriquement équivalent aux tapis de Sierpinski usuels.

La seconde partie est une discussion portant sur la géométrie quasi-symétrique des ensembles de tapis de Julia, incluant en outre le quasi-cercle uniforme, ainsi que certaines propriétés de séparation uniforme.

Lors de la troisième partie, nous déterminerons une condition permettant de savoir quand deux rayons externes d'un polynôme tendent vers un même point. Comme application, nous montrerons également la monotonie de l'entropie associée à une famille de polynômes quadratiques.

La quatrième partie est inspirée du travail récent de Cui Guizhen et Tan Lei. En utilisant des outils classiques (module d'anneau et chirurgie quasi-conforme), nous étudierons la convergence de certains rayons en campagne locus espace des paramètres.

Enfin, la dernière partie pore sur la famille des transformations de renormalisations générées. Plus précisément, cette partie abordera la connexité de ces ensembles de Julia, et le lieu de confinement dans l'espace des paramètres, ainsi que la formule asymptotique de la dimension d'Hausdorff des ensembles de Julia.

Mots clés

rigidité quasisymétrique, sierpinski usuels, tapis de Julia, rayons externes, rayons paramètres, transformations de renormalisations générées.

Abstract

The thesis consists of five parts.

The first part is concerned with the quasisymmetric rigidity of a new Sierpinski carpet, which are not quasisymmetrically equivalent to the standard Sierpinski carpets.

The second part discusses the quasisymmetrically geometry of the carpet Julia sets, including the uniformly quasicircle and uniformly separated properties.

The third part is to determine when two external rays of a polynomial land at the same point. As an application, we also show the monotonicity of core-entropy on a family of quadratic polynomials.

In the fourth part, following Cui and Tan's work, we use the classic tools modulus of annulus and quasi-conformal surgery to study the landing of some parameter rays in shift locus parameter space.

The last part discusses a family of generated renormalization transformations. Specifically, it is on the connectivity of its Julia sets and the non-escaping locus in its parameter space, the asymptotic formula of the Hausdorff dimention of the Julia sets.

Key Words

Quasisymmetric rigidity, sierpinski carpets, carpet Julia sets, external rays, parameter rays, generated renormalization transformations.