Quasi-Poisson structures on moduli spaces of flat connections, entropy of simplicial Tits sets

Xin Nie

Université Paris VII - Denis Diderot

Université Pierre et Marie Curie

École Doctorale Paris Centre

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Xin Nie

Structures quasi-Poisson sur l'espace des modules de connexions plates, entropies d'ensembles de Tits simpliciaux

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dirigée par Gilles Courtois

Soutenue le 28 juin 2013 devant le jury composé de :

M. Anton Alekseev	Université de Genève	rapporteur
M. Philip Boalch	École Normale Supérieur de Paris	examinateur
M. Laurent Charles	Université Pierre et Marie Curie	examinateur
M. Gilles Courtois	Université Pierre et Marie Curie	directeur
M. Vladimir Fock	Université de Strasbourg	examinateur
M. Olivier Guichard	Université de Strasbourg	rapporteur
M. François Labourie	Université de Paris-Sud	examinateur
M. Julien Marché	Université Pierre et Marie Curie	examinateur

Institut de Mathématiques de Jussieu
175, rue du chevaleret
75 013 Paris

École doctorale Paris centre Case 188
4 place Jussieu
75 252 Paris cedex 05

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Résumé

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Cette thèse comprend deux parties. La première partie porte sur la structure quasi-Poisson sur l'espace des connexions plates. D'abord nous généralisons la formule de Goldman concernant le crochet de Poisson pour l'espace des modules de connexions plates dans le cadre quasi-Poisson. Puis nous appliquons la théorie quasi-Poisson aux espaces de configurations de drapeaux au sens large, en montrant que l'espace des modules de connexions plates avec ossatures provient d'une réduction quasi-Poisson. Ceci implique en particulier que la structure de Poisson de Fock-Goncharov coïncide avec celle d'Atiyah-Bott. Enfin, nous discutons de quantifications par déformation de variétés quasi-Poisson.

La deuxième partie traite un problème indépendant concernant la métrique de Hilbert sur les variétés projectives réelles convexes. En répondant à une question de M. Crampon, nous montrons que l'entropie volumique d'une famille à un paramètre explicite d'orbifolds projectifs convexes tend vers zero.

Quasi-Poisson structures on moduli space of flat connections, entropy of simplicial Tits sets

Abstract

This thesis consists of two parts. The first part is concerned with the quasi-Poisson structure on the space of flat connections. First we generalize Goldman's Poisson bracket for moduli space of flat connections to the quasi-Poisson setting. Then we apply the quasi-Poisson theory to configuration spaces of flags in a broad sense, showing that Fock and Goncharov's moduli space of framed flat connections arises from quasi-Poisson reduction. This in particular implies that Fock-Goncharov's Poisson structure coincides with Atiyha-Bott's. Finally, we discuss deformation quantizations of quasi-Poisson manifolds.

The second part deals with an independent problem about Hilbert metrics on convex real projective manifolds. Answering a question of M. Crampon, we show that the volume entropy of an explicit one-parameter family of convex projective orbifolds tends to zero.

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Introduction

Pour une surface compacte orientée Σ et un groupe de Lie G, on cherche à étudier l'espace

$$X_G(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma, v), G)/G$$

des classes d'équivalence de représentations du groupe fondamental (pour un choix indifférent du point base v). Pour la simplicité de l'exposé, dans cette introduction on traite $X_G(\Sigma)$ comme une variété lisse, malgré qu'il n'est prèsque jamais le cas.

L'espace $X_G(\Sigma)$ se situe à l'intersection de diverses branches des mathématiques. Citons quelques-uns d'entre eux :

- $X_{U(n)}(\Sigma)$ apparaît dans la géométrie algébrique parce que si l'on fixe une structure complexe sur Σ, il s'identifie à l'espace des classes d'équivalence de fibrés vectoriels holomorphes sur Σ [47, 18].
- Certaine sous-ensemble de $X_{\mathrm{PSL}_2\mathbb{R}}(\Sigma)$ et $X_{\mathrm{PSL}_2\mathbb{C}}(\Sigma)$ sont des objets d'étude principaux dans la géométrie hyperbolique car ils paramétrisent les structures hyperboliques sur Σ et certaines variétés hyperboliques de dimension 3 [53, 28] ; De même manière, un sous-ensemble de $X_{\mathrm{PSL}_3\mathbb{R}}(\Sigma)$ paramétrise les *structure projectives réelles convexes* sur Σ [29].
- $X_G(\Sigma)$ s'identifie à l'espace des modules de connexions plates sur les G-fibrés principaux à base Σ. Ainsi, il joue le rôle d'un espace des phases dans la théorie de Yang-Mills en dimension 2 [7].
- Lorsque G est un groupe de Lie réel semi-simple déployé, pour une composante connexe $X_G^{\mathsf{H}}(\Sigma)$ de $X_G(\Sigma)$ découverte par Hitchin [32, 33], des structrues très riches ont été révélé dans la dernière décennie par plusieurs auteurs [15, 38, 39, 25], avec des méthodes totalements différentes. Fock-Goncharov [25] et Labourie-McShane [41] proposent le nom *espace de Teichmüller généralisé (higher Teichmüller space)* pour $X_G(\Sigma)$ dans ce cas.

La première partie de ce texte porte sur l'étude de la structure de Poisson sur $X_G(\Sigma)$. Les travaux présentés ici trouvent leur origine dans une collaboration avec Yuichi Kabaya sur l'espace de Teichmüller généralisé en printemps 2012, au cours duquel nous avons besoin de savoir si cette structure de Poisson coïncide avec une autre structure introduite par Fock et Goncharov. Nous abordons ce problème en utilisant les structures quasi-Poisson. Nous renvoyons aux §0.1 et §0.2 pour une présentation plus détaillée.

La deuxième partie présente un travail effectué dans la première année de cette thèse et mis en œuvre dans [48]. Il s'agit d'une probématique très différente. Nous renvoyons à §0.3 pour une présentation de cette partie.

0.1 Motivation : structure de Poisson d'Atiyah-Bott

Au début des année 1980, Atiyah et Bott [7] ont donné une impulsion à la théorie de Yang-Mills en construisant une structure symplectique sur $X_G(\Sigma)$ lorsque $\partial\Sigma = \emptyset$ et qu'un produit scalaire invariant $(\cdot \mid \cdot)$ est prescrit sur l'algèbre de Lie $\mathfrak{g}$. Puis il est bien connu que si le bord de Σ est non-vide, cette structure symplectique se généralise en une structure de Poisson.

Malgré son importance théorique, la structure symplectique d'Atiyah-Bott est difficile à manipuler car il s'agit d'une réduction symplectique depuis la variété de dimension infinie de toutes les connexions plates. Goldman [26, 27] a jeté un nouvel éclairage à cette structure en l'interprétant par la forme d'intersection dans la cohomologie tordue de $\pi_1(\Sigma)$, qui est un objet de dimension finie. Ce point de vue lui permettait de montrer notamment une formule de crochet de Poisson pour certaines fonctions sur $X_G(\Sigma)$.

0.1.1 Formule de Goldman

Désignons par $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ le dual du produit scalaire invariant $(\cdot \mid \cdot)$. Pour une fonction $f \in C^\infty(G)$ invariante par conjugaison et un lacet α sur Σ, nous définissons la *fonction de Goldman* $f_\alpha \in C^\infty(X_G(\Sigma))$ par

$$f_\alpha([m]) = f(m([\alpha])), \quad \forall m \in \mathrm{Hom}(\pi_1(\Sigma, v), G),$$

où $[\alpha]$ désigne la classe de conjugaison dans $\pi_1(\Sigma)$ portée par α. La formule de Goldman [27] donne le crochet de Poisson pour deux telles fonctions f_α et h_β, sous l'hypothèse que α et β soient transverses :

$$\{f_\alpha, h_\beta\}(m) = \sum_{q \in \alpha \cap \beta} \varepsilon_q(\alpha, \beta) \left\langle \mathsf{d}^{\mathsf{L}} f(m(\alpha_q)) \otimes \mathsf{d}^{\mathsf{L}} h(m(\beta_q)), s \right\rangle .$$

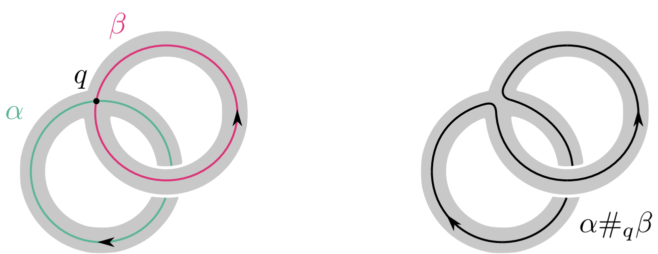
Les notations sont expliquées ci-dessous :

- $\langle \cdot, \cdot \rangle$ désigne le couplage entre $\mathfrak{g}^* \otimes \mathfrak{g}^*$ et $\mathfrak{g} \otimes \mathfrak{g}$;
- $\varepsilon_q(\alpha, \beta) = \pm 1$ est l'indice d'intersection algébrique de α et β au point q ;
- nous definissons l'application $\mathsf{d}^{\mathsf{L}} f : G \to \mathfrak{g}^*$ en ramenant chaque covecteur $\mathsf{d}_g f \in \mathsf{T}_g^* G$ à $\mathsf{T}_e^* G = \mathfrak{g}^*$ par translation à gauche. De même, on définit $\mathsf{d}^{\mathsf{R}} f$ en utilisant la translation à droite. Remarquons que l'invariance de f implique que $d^{\mathsf{L}} f = d^{\mathsf{R}} f$ est G-équivariante (par rapport à l'action par conjugaison et l'action coadjointe). Plus bas nous allons néanmoins reprendre ces notations pour f quelconque.
- α_q désigne l'élément dans $\pi_1(\Sigma, q)$ porté par α. Nous définissons $m(\alpha_q)$ en identifiant $\pi_1(\Sigma, q)$ avec $\pi_1(\Sigma, v)$ par un chemin reliant q et v. Remarquons que $m(\alpha_q)$ dépend du choix de ce chemin mais $\mathsf{d}^{\mathsf{L}} f(m(\alpha_q)) \otimes \mathsf{d}^{\mathsf{L}} h(m(\beta_q)) \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ n'en dépend pas.

En particulier, si $G = \mathrm{GL}_n\mathbb{R}$ et $\mathfrak{g} = \mathfrak{gl}_n\mathbb{R}$ est muni du produit scalaire standard $(x \mid y) = \mathrm{Tr}(xy)$, pour la fonction de trace $f = h = \mathrm{Tr}$ on obtient

$$\{\mathrm{Tr}_\alpha, \mathrm{Tr}_\beta\} = \sum_{q \in \alpha \cap \beta} \varepsilon_q(\alpha, \beta) \, \mathrm{Tr}_{\alpha \# \beta} .$$

Le lacet $\alpha \# \beta := \alpha_q \beta_q$ est dessiné ci-dessous.



Cette dernière formule suggère de définir un crochet de Lie $[\alpha, \beta] := \sum_{q \in \alpha \cap \beta} \alpha \#_q \beta$ sur l'espace vectoriel $\mathcal{L}(\Sigma)$ engendré par toutes les classes d'homotopie libre de lacets. L'algèbre de Lie ainsi obtenue est appelée l'*algèbre de Goldman*.

Une généralisation de la formule de Goldman à une classe plus large de fonctions, dite *spin network*, a également été étudiée [9, 52].

0.1.2 Structures quasi-Poisson sur l'espace des modules de connexions plates

Deux nouvelles interprétations de la structure de Poisson d'Atiyah-Bott qui n'utilisent que des constructions de dimension finie sont apparues au début des années 1990. La première, due à V. Fock et A. Rosly [24], repose sur la théorie de jauge sur réseau et utilise les r-matrices classiques sur $\mathfrak{g}$. La deuxième, due à A. Alekseev, E. Meinrenken et A. Malkin [6], est inspirée par la théorie d'application moment pour le groupe de lacet LG [46]. Cette deuxième a été élaboré par Alekseev, Meinrenken et Y. Kosmann-Schwarzbach et donne lieu à une théorie de *variété quasi-Poisson*.

David Li-Bland and Pavol Ševera ont récemment donné dans [42] une nouvelle formulation de la théorie quasi-Poisson, qui est plus générale que les versions antérieures, dont il découle notamment que la première et la deuxième approche ci-dessus sont essentiellement équivalentes. Notre exposé dans cette thèse est basé sur leur formulation.

Voici un résumé de la théorie quasi-Poisson. Supposons que Σ est à bord non-vide. Soit $V \subset \partial\Sigma$ un ensemble fini de points marqués. Rappelons que le groupoïde fundamental $\pi_1(\Sigma, V)$ consiste en toutes les classes d'homotopie de chemins orientés sur Σ reliant les points dans V. Le groupe G^V agit naturellement sur l'espace des représentations $M_G(\Sigma, V) = \mathrm{Hom}(\pi_1(\Sigma, V), G)$, et le quotient s'identifie à $X_G(\Sigma, V)$. Pour tout $\alpha \in \pi_1(\Sigma, V)$, nous désignons par $\mathrm{hol}_\alpha : M_G(\Sigma, V) \to G$ l'application d'holonomie le long de α, qui envoie $m \in M_G(\Sigma, V)$ sur $m(\alpha) \in G$.

Etant donné $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$, on peut construire de façon canonique une structure quasi-Poisson sur $M_G(\Sigma, V)$, qui induit par réduction la structure de Poisson d'Atiyah-Bott sur $X_G(\Sigma)$. Autrement dit, on a une application bilinéaire anti-symmetrique

$$\{\cdot, \cdot\} : C^\infty(M_G(\Sigma, V)) \times C^\infty(M_G(\Sigma, V)) \to C^\infty(M_G(\Sigma, V)),$$

qui satisfait la loi de Leibniz et une *identité quasi-Jacobi*

$$\{\{f_1, f_2\}, f_3\} + \{\{f_3, f_1\}, f_2\} + \{\{f_2, f_3\}, f_1\} = -\frac{1}{2}\rho_\phi(\mathsf{d}f_1, \mathsf{d}f_2, \mathsf{d}f_3),$$

où $\phi \in \bigwedge^3(\mathfrak{g}^V)$ est donné par s de façon canonique et ρ_ϕ désigne le champs de trivecteur sur $M_G(\Sigma, V)$ induit par ϕ et l'action de G^V. De plus, la restriction de $\{\cdot, \cdot\}$ aux fonctions invariantes $C^\infty(M_G(\Sigma, V))^{G^V} = C^\infty(X_G(\Sigma))$ coïncide avec le crochet de Poisson d'Atiyah-Bott.

0.1.3 Structure de Poisson de Fock-Goncharov

En généralisant la notion d'*espace de Teichmüller décoré* de R. Penner, V. Fock et A. Goncharov [25] ont introduit dans l'étude de $X_G(\Sigma)$ de nouvelles idées provenant de la théorie des algèbres amassées, sous l'hypothèse que Σ soit à bord non-vide et G soit une groupe de Lie réel semi-simple déployé. Dans le case $G = \mathrm{PSL}_n\mathbb{R}$, ils ont décrit notamment un système de coordonnéss et une structure de Poisson sur un certain revêtement fini de $X_G(\Sigma)$.

Plus précisement, soit $B \subset G$ un sous-groupe de Borel et $\mathcal{F} = G/B$ la variété de drapeaux. Prenons un ensemble fini $W \subset \partial\Sigma$ et posons $\widehat{\Sigma} = \Sigma \setminus W$. Une *connexion plate avec ossature* (*framed flat connection*) est une paire (∇, f), où ∇ est une connexion plate sur un G-fibré principal $P \to \widehat{\Sigma}$ et f est une B-réduction plate sur $\partial\Sigma \setminus W$, c'est-à-dire un choix d'une B-orbite $f(q)$ dans $P|_q$ pour tout $q \in \partial\Sigma \setminus W$, telle que ∇ translate $f(q_1)$ à $f(q_2)$ dès que q_1 et q_2 se trouvent dans la même composante de $\partial\Sigma \setminus W$.

Désignons par $\mathscr{X}_{G,\widehat{\Sigma}}$ l'espace des modules de connexions plates avec ossature. En particulier, pour le disque $\mathbb{D}$ et un sous-ensemble $W \subset \partial\mathbb{D}$ de cardinal N, $\mathscr{X}_{G,\widehat{\mathbb{D}}}$ s'identifie à l'espace des configurations de N drapeaux

$$\mathrm{Conf}_N(\mathcal{F}) = (\underbrace{\mathcal{F} \times \cdots \times \mathcal{F}}_{N})/G.$$

D'autre part, si W est vide (alors $\widehat{\Sigma} = \Sigma$), un ouvert $\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}} \subset \mathscr{X}_{G,\Sigma}$ s'identifie à un revêtement fini d'un ouvert dans $X_G(\Sigma)$.

Fock et Goncharov associent, à chaque graphe trivalent Γ aux sommets W tel que Σ rétracte sur Γ par déformation, un système de coordonnés $(X_\mathbf{i})_{\mathbf{i}\in I}$ de $\mathscr{X}_{\mathrm{PSL}_n\mathbb{R},\widehat{\Sigma}}$. Ici chaque $X_\mathbf{i} : \mathscr{X}_{\mathrm{PSL}_n\mathbb{R},\widehat{\Sigma}} \to \mathbb{R}$ est une fonction rationnelle [1] et I provient de Γ de façon combinatoire. Ils définissent ensuite une structure de Poisson $\{\cdot,\cdot\}_{FG}$ sur $\mathscr{X}_{\mathrm{PSL}_n\mathbb{R},\widehat{\Sigma}}$ en déclarant

$$\{X_\mathbf{i}, X_\mathbf{j}\}_{FG} = \epsilon_{\mathbf{ij}} X_\mathbf{i} X_\mathbf{j} \tag{1}$$

pour tout $\mathbf{i}, \mathbf{j} \in I$, où $\epsilon_{\mathbf{ij}} = 0, \pm 1$ est une constante combinatoire .

Pour des raisons techniques, nous considérons, à la place de $\mathscr{X}_{\mathrm{PSL}_n\mathbb{R},\widehat{\Sigma}}$, son revêtement fini $\mathscr{X}_{\mathrm{SL}_n\mathbb{R},\widehat{\Sigma}}$, et travaillons avec les relevés de $X_\mathbf{i}$ et $\{\cdot,\cdot\}_{FG}$ à $\mathscr{X}_{\mathrm{SL}_n\mathbb{R},\widehat{\Sigma}}$. Ceci ne change pas grand-chose. On se demande alors

Question 1. *Lorsque $W = \emptyset$, la structure de Poisson $\{\cdot,\cdot\}_{FG}$ coïncide-elles avec la structure de Poisson d'Atiyah-Bott relevée à $\mathscr{X}^{\mathrm{reg}}_{\mathrm{SL}_n\mathbb{R},\Sigma}$? Plus généralement, pour W arbitraire, peut-on obtenir $\{\cdot,\cdot\}_{FG}$ par réduction de Poisson ?*

0.1.4 Algèbre d'échange

Comme mentioné plus haut, $\mathscr{X}_{\mathrm{SL}_n\mathbb{R},\widehat{\mathbb{D}}}$ est l'espace des configurations de N drapeaux dans l'espace projective réel $\mathbb{P}^{n-1}$. En général, $\mathscr{X}_{\mathrm{SL}_n\mathbb{R},\widehat{\Sigma}}$ peut être vu comme un "espace des configurations tordues".

A travers des travaux de F. Labourie [38, 39, 40], les espaces de configurations de drapeaux interviennent également dans l'étude de $X_{\mathrm{SL}_n\mathbb{R}}(\Sigma)$ quand Σ est fermé : soit $\partial_\infty\pi_1(\Sigma)$ le bord du groupe fondamental, c'est-à-dire le cercle $\partial\mathbb{D}$ muni de l'action de $\pi_1(\Sigma)$ donnée par une hyperbolisation de Σ. Il est montré dans [38] que pour chaque représentation $m : \pi_1(\Sigma) \to \mathrm{SL}_n\mathbb{R}$ dans la composante de Hitchin $X^{\mathsf{H}}_{\mathrm{SL}_n\mathbb{R}}(\Sigma)$, il existe une application Hölder équivariante $\partial_\infty\pi_1(\Sigma) \to \mathbb{P}^{n-1}$ qui admet une courbe des drapeaux oscillants $f_m : \partial_\infty\pi_1(\Sigma) \to \mathcal{F}$. On peut donc considérer un point dans $X^{\mathsf{H}}_{\mathrm{SL}_n\mathbb{R}}(\Sigma)$ comme une configuration d'un nombre infini de drapeaux. On définit ensuite, pour chaque quadruple $x, y, z, w \in \partial_\infty\pi_1(\Sigma)$, une fonction $[x, y, z, w]$ sur $X^{\mathsf{H}}_{\mathrm{SL}_n\mathbb{R}}(\Sigma)$ dont le valeur au point $[m]$ est le birapport [2] $[f_m(x), f_m(y), f_m(z), f_m(w)]$. Les crochets de Poisson entre telles fonctions

1. Donc plus précisement $(X_\mathbf{i})_{\mathbf{i}\in I}$ est un système de coordonnés d'un ouvert à complément négligeable dans $\mathscr{X}_{\mathrm{PSL}_n\mathbb{R},\widehat{\Sigma}}$.

2. Il s'agit ici le birapport de quatre drapeaux, qui généralise le birapport classique de quatre points dans $\mathbb{P}^1$, *c.f.* §3.1.3.

sont étudiés dans [40], et donnent lieu à une algèbre de Poisson $\mathscr{Z}(\partial_\infty \pi_1(\Sigma))$ similaire à celle de Goldman, dite l'*algèbre d'échange* (*swapping algebra*).

Evidemment, la fonction $[x, y, z, w]$ peut être définie non seulement sur $X^{\mathsf{H}}_{\mathrm{SL}_n\mathbb{R}}(\Sigma)$, mais plus généralement sur n'importe lequel espace de configurations de drapeaux $\mathrm{Conf}_V(\mathcal{F}) = \mathrm{Map}(V, \mathcal{F})/G$, où V est un ensemble quelconque et $x, y, z, w \in V$. On peut aussi définir l'algèbre d'échange $\mathscr{Z}(V)$ pour n'importe lequel sous-ensemble du cercle $V \subset \partial\mathbb{D}$.

0.1.5 Quantification par déformation

Soit A une algèbre commutative associative sur $\mathbb{R}$ et $A[[\hbar]]$ l'espace des séries entières formelles à coefficient dans A et à indéterminée $\hbar$. Rappelons qu'un *star-produit* sur A est une application linéaire $\star : A \otimes A \to A[[\hbar]]$ telle que

- $\star$ s'écrit sous la forme
$$a \star b = ab + \hbar\theta_1(a, b) + \hbar^2\theta_2(a, b) + \cdots ,$$
où ab est le produit commutative de a et b dans A, tandis que chaque θ_i est une application linéaire $A \otimes A \to A$.
- le morphisme de $\mathbb{R}[[\hbar]]$-modules $A[[\hbar]] \otimes_{\mathbb{R}[[\hbar]]} A[[\hbar]] \to A[[\hbar]]$ qui étend $\star$ est un produit associatif sur $A[[\hbar]]$.

Cette dernière condition implique que $\{a, b\} = \theta_1(a, b) - \theta_1(b, a)$ définit un crochet de Poisson sur A. Le star-produit $\star$ est appelé une *quantification* de $\{\cdot, \cdot\}$.

Considérons l'algèbre tensorielle symétrique de l'algèbre de Goldman $A = S\mathcal{L}(\Sigma)$. Le crochet de Lie sur $\mathcal{L}(\Sigma)$ s'étend en un crochet de Poisson sur A de façon unique. Turaev [54] a interprété une quantification de ce crochet dans le cadre de la théorie des nœuds : il identifie $A[[\hbar]]$ au *module d'écheveau* de $\Sigma \times [0, 1]$, c'est-à-dire le $\mathbb{R}[[\hbar]]$-module engendré par les entrelacs dans $\Sigma \times [0, 1]$ modulo certaines relations, de sorte que le star-produit de deux entrelacs est obtenu par superposition du premier au-dessus du deuxième. Ph. Roche et A. Szenes [52] ont généralisé cette quantification aux *spin networks*.

D'autre part, la notion de star-produit a été généralisée dans le cadre des variétés quasi-Poisson par B. Enriquez et P. Etingof [22]. Tandis que l'associativité d'un star-produit usuel implique l'identité de Jacobi pour sa limite classique, une quantification d'un crochet quasi-Poisson doit être un star-produit *quasi-associatif* par rapport à un *associateur* $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ au sense de Drinfeld [20]. Cette quasi-associativité implique l'identité quasi-Jacobi.

Pour une classe particulière de variété quasi-Poisson, à savoir, celles qui proviennent de l'*equation de Yang-Baxeter dynamique*, les quantifications ont été bien étudiées [22, 23, 4]. D'autre part, en adaptant la méthode célèbre de M. Kontsevich [37], G. Halbout [31] a construit une quantification pour toute variété quasi-Poisson où l'action du groupe de Lie est libre.

0.2 Les principaux résultats de la Partie I

Voici une observation fondamentale qui relie la structure de Poisson de Fock-Goncharov et la théorie de jauge quasi-Poisson :

Proposition 0.1 (Sommaire de §3.2). *Soit $V \subset \partial\Sigma$ un ensemble de points marqués qui a exactement un point dans chaque composante de $\partial\Sigma \setminus W$. Alors $\mathscr{X}_{G,\widehat{\Sigma}}$ s'identifie au quotient d'une certaine sous-variété L de $M_G(\Sigma, V)$ par un certain sous-groupe $G' \subset G^V$.*

En particulier, si W intersecte chacune des composantes de $\partial\Sigma$, alors $L = M_G(\Sigma, V)$ et $G' = B^V$.

De plus, si $G = \mathrm{SL}_n\mathbb{R}$, prenons un graphe trivalent Γ comme ci-dessus, alors il y a une famille des fonctions $\Delta_{\mathbf{j}} \in C^\infty(L)$ telle que chaque $X_{\mathbf{i}}$ (relevé à L) s'écrit comme une fraction des $\Delta_{\mathbf{j}}$.

Ainsi, la question 1 ci-dessus se ramène au problème de calculer les crochets quasi-Poisson de toute paire $\Delta_{\mathbf{i}}, \Delta_{\mathbf{j}}$. Ceci nous a amené à la découverte d'une formule de Goldman quasi-Poisson.

Rappelons que Fock et Goncharov [25] ont considéré un autre espace noté par $\mathscr{A}_{G,\widehat{\Sigma}}$ et certaines fonctions qui portent la même notation $\Delta_{\mathbf{j}}$ qu'ici. En fait, les rôles joué par $M_G(\Sigma, V)$ et par $\mathscr{A}_{G,\widehat{\Sigma}}$ sont similaires, bien qu'il s'agisse de deux espace différents.

0.2.1 Formule de Goldman quasi-Poisson

Etant donnée une fonction $f \in C^\infty(G)$ et un chemin orienté α reliant les points dans V, nous posons $f_\alpha := f(\mathrm{hol}_\alpha) \in C^\infty(M_G(\Sigma))$. Nous avons mis en œuvre dans l'article [49] une généralisation de la formule de Goldman, qui calcule les crochets quasi-Poisson entre toutes les fonctions du type f_α. Citons la formule obtenue :

Théorème A (Un cas particulier du Theorem 2.6)**.** *Fixons $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ et considérons le crochet quasi-Poisson $\{\cdot,\cdot\}$ sur $C^\infty(M_G(\Sigma, V))$ par rapport à s. Supposons que α soit transverse à β, alors*

$$\{f_\alpha, h_\beta\} = \sum_{q \in \alpha\cap\beta\setminus\partial\Sigma} \varepsilon_q(\alpha,\beta) \left\langle \mathsf{d}^{\mathsf{L}} f(\mathrm{hol}_\alpha) \otimes \mathsf{d}^{\mathsf{R}} h(\mathrm{hol}_\beta), (1 \otimes \mathrm{Ad}_{\mathrm{hol}_{\beta *_q \alpha}}) s \right\rangle$$
$$+ \frac{1}{2} \sum_{i,j} \varepsilon(\alpha_i, \beta_j) \left\langle d^i f(\mathrm{hol}_\alpha) \otimes d^j h(\mathrm{hol}_\beta), s \right\rangle.$$

Expliquons les notations :

- les applications $\mathsf{d}^{\mathsf{L}} f, \mathsf{d}^{\mathsf{R}} f : G \to \mathfrak{g}^*$ sont définies comme précédemment ;
- dans la dernière somme, i et j parcourtent les deux symboles L et R ;
- α_{L} (resp. α_{R}) désigne la première (resp. la seconde) moitié de α, où on sépare α en deux à n'importe lequel point au milieu ;
- si α_i et β_j ont une extrémité commune $v \in V$, alors $\varepsilon(\alpha_i, \beta_j) = \pm 1$ est par définition l'indice d'intersection algébrique de α_i et β_j au point v ; sinon, on met $\varepsilon(\alpha_i, \beta_j) = 0$.
- $\beta *_q \alpha$ désigne le chemin qui part de l'origine de α, parcourt α jusqu'au point q, puis transfère à β et termine au point final de β.

Au cours de la rédaction de l'article mentionné ci-dessus, Anton Alekseev et Pavol Ševera nous ont communiqué deux travaux simultanés indépendants qui traitaient la même problématique : G. Massuyeau et V. Turaev [44] ont établi, avec une méthode très différente, la même formule au cas particulier où $G = \mathrm{GL}_n\mathbb{R}$ et $\#V = 1$, tandis que D. Li-Bland et P. Ševera [42] ont montré une formule plus générale que la nôtre en considérant une classe plus large de fonctions, à savoir, les *spin networks* généralisés. Cependant, comme mentionné plus haut, la théorie quasi-Poisson pour les connexions plates est essentiellement équivalente à la théorie de Fock et Rosly [24], tous ces résultats sont donc plus ou moins équivalents à une formule de Fock et Rosly [3].

Dans ce texte nous choisissons de présenter la formule de Li-Bland et Ševera (Theorem 2.6) et leur démonstration dans [42], puisqu'ils suivent la même idée que la nôtre [49] avec des arguments plus simples sur certains points clefs.

3. Cette dernière, implicite dans l'article de Fock et Rosly [24], a été explicitée par M. Audin [8].

0.2.2 Struture de Poisson de Fock-Goncharov via réduction quasi-Poisson

Alekseev, Meinrenken et Kosmann-Schwarzbach [3] ont généralisé le *Cross-section Theorem* de Guillemin et Sternberg concernant les actions hamiltoniennes dans le contexte des variétés quasi-Poisson. Cette généralisation nous permet de conclure que la sous-variété $L \subset M_G(\Sigma)$ mentionnée dans la Proposition 0.1 admet une structure quasi-Poisson. De plus, Théorème A s'adapte à L pour calculer le crochet des restrictions de f_α et f'_β à L. Nous en déduisons aussi les expressions expicites de crochets quasi-Poisson des $\Delta_{\mathbf{i}}$ puis des $X_{\mathbf{i}}$, et donnons enfin une réponse positive à la question 1.

Théorème B (Theorem 3.23)**.** *La structure de Poisson $\{\cdot,\cdot\}_{FG}$ sur $\mathscr{X}_{\mathrm{SL}_n\mathbb{R},\widehat{\Sigma}}$ coïncide avec celle donnée par la réduction quasi-Poisson depuis L. En particulier, quand $W = \emptyset$, $\{\cdot,\cdot\}_{FG}$ coïncide avec la structure de Poisson d'Atiyah-Bott relevée à $\mathscr{X}^{\mathrm{reg}}_{\mathrm{SL}_n\mathbb{R},\Sigma}$.*

0.2.3 Algèbre d'échange via réduction quasi-Poisson

Comme un sous-produit de la méthode ci-dessus, nous obtenons

Proposition 0.2 (§2.2, §3.1)**.** *Soit $V \subset \partial\mathbb{D}$ un ensemble fini, alors l'algèbre d'échange $\mathcal{Z}(V)$ est une sous-algèbre de l'algèbre quasi-Poisson $C^\infty(M_G(\mathbb{D}, V))$ de façon canonique.*

En particulier, considérons les fonctions $[x, y, z, w]$ $(x, y, z, w \in V)$ sur $\mathrm{Conf}_V(\mathcal{F}) = M_G(\mathbb{D}, V)/B^V$ mentionnées plus haut, alors les crochets de Poisson entre eux donnés l'algèbre d'échange coïncide avec ceux donné par la réduction quasi-Poisson.

Nous n'allons pas donner une preuve explicite de cette proposition car il se découle aisément de §2.2 et de §3.1 plus bas.

La réduction quasi-Poisson fournit une structure de Poisson non seulement sur l'espaces des configurations de drapeaux, mais sur une classe plus large d'espaces de configurations, *c.f.* §3.1. Pour certains d'entre eux, une structure de Poisson a été construite de manière différente. Il est naturel de comparer ces structrures. Nous en discuterons dans §3.4.

0.2.4 Quantifications de $M_G(\Sigma, V)$

Nous exhibons une quantification de la variété quasi-Poisson $M_G(\Sigma, V)$.

Théorème C (Version grossière de Proposition 4.23)**.** *Si un associateur $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ satisfait $S^{\otimes 3}(\Phi) = \Phi$ (où S désigne l'application antipode de $U(\mathfrak{g})$), alors la variété quasi-Poisson $M_G(\Sigma, V)$ admet une Φ-star-produit explicite qui dépend de certaines données initiales.*

Ce théorème est motivé par la quantification de Turaev évoqué ci-dessus. Nous espérons établir un lien analogue entre la quantification ici et la théorie des nœuds. Théorème C n'est qu'un premier pas vers cette perspective. Nous souhaitons revenir à ce problème à l'avenir.

0.3 Présentation de la Partie II

Comme mentionné au début, un sous-ensemble de $X_{\mathrm{PSL}_3\mathbb{R}}(\Sigma)$ paramétrise certaines structures projectives réelles sur Σ. On a en fait la même propriété pour les variétés fermées de dimension quelconque. La deuxième partie de ce texte est consacrée à l'étude géométrique de ces structures projectives.

0.3.1 Structures projectives convexe et ensemble de Tits

Soit $\mathbb{P}^n$ l'espace projectif réel de dimension n. Une *structure projective (réelle)* sur une variété M (ou plus généralement un *orbifold*) est un atlas dont chaque carte identifie un ouvert dans M avec un ouvert dans $\mathbb{P}^n$ et les changements de cartes sont des transformation projectives.

Une classe de structures projectives largement étudiée est donné par la construction suivante. Un ouvert $\Omega \subset \mathbb{P}^n$ est dit *proprement convexe* si Ω est un convexe borné dans une carte affine $\mathbb{R}^n \subset \mathbb{P}^n$. Soit $X = \widetilde{X}/\Pi$ un orbifold, où Π désigne un groupe agissant discrètement sur la variété $\widetilde{X}$. Une *structure projective convexe* sur X consiste en une représentation fidèle $\rho : \Pi \to \mathrm{PGL}_{n+1}\mathbb{R}$ et un ouvert proprement convex $\Omega \subset \mathbb{P}^n$ tels qu'on a un difféomorphisme ρ-equivariante $\widetilde{X} \to \Omega$. L'ouvert Ω est aussi dit un *convexe divisible*.

Il est bien connu que Ω est déterminé par ρ à dualité près. Donc nous définissons l'*espace des modules de structures projectives convexes* sur X comme le sous-ensemble dans l'espace des classes d'équivalence de représentations

$$\mathfrak{P}(X) \subset \mathrm{Hom}(\Pi, \mathrm{PGL}_{n+1}\mathbb{R})/\mathrm{PGL}_{n+1}\mathbb{R}$$

formé par tout $\rho \in \mathrm{Hom}(\Pi, \mathrm{PGL}_{n+1}\mathbb{R})$ qui vient d'une structure projective convexe. Y. Benoist [12] a montré que $\mathfrak{P}(X)$ est ouvert et fermé dans $\mathrm{Hom}(\Pi, \mathrm{PGL}_{n+1}\mathbb{R})/\mathrm{PGL}_{n+1}\mathbb{R}$. W. Goldman [29] a montré que pour une surface fermé Σ de genre $g \geq 2$, $\mathfrak{P}(X)$ est homeomorphe à $\mathbb{R}^{16g-16}$.

Dans cette thèse nous étudions le cas où X est un *orbifold de Coxeter hyperbolique simplicial* , c'est-à-dire $X = \mathbb{H}^n/\Gamma$, où $\mathbb{H}^n$ est l'espace hyperbolique de dimension n et $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ est engendré par les réflexions par rapport aux faces d'un simplexe $P \subset \mathbb{H}^n$ de sorte que P est un domaine fondamental de l'action de Γ sur $\mathbb{H}^n$. Un tel group Γ est déterminé (à conjugaison près) par un diagramme de Coxeter hyperbolique J. Désignons X par X_J si Γ vient de J.

En identifiant $\mathbb{H}^n$ avec une boule dans $\mathbb{P}^n$ et $\mathrm{Isom}(\mathbb{H}^n)$ avec le groupe des transformations projectives préservant cette boule (*i.e.*, le modèle de Beltrami-Klein de $\mathbb{H}^n$), on obtient une structure projective convexe tautologique sur X_J. J. Tits ont donné un critère assez général pour déterminer si un groupe engendré par des réflexions projectives donne lieu à un convexe divisible, donc on appelle un tel convexe un *ensemble de Tits*. Nous déduisons du critère de Tits que

$$\mathfrak{P}(X_J) \cong \begin{cases} \mathbb{R}_{>0} & \text{si } J \text{ contient une boucle,} \\ \text{un point} & \text{sinon.} \end{cases}$$

Puis nous étudions la comportement asymptotique de cette famille de structure projective.

Proposition (Proposition 5.5). *Soit P un simplexe dans $\mathbb{P}^n$. Prenons un diagramme de Coxeter hyperbolique J comme ci-dessus tel que $\mathfrak{P}(X_J) \cong \mathbb{R}_{>0}$. Alors il existe une famille à une paramètre de representations $\{\rho_t\}_{t\in\mathbb{R}_{>0}}$ de Γ dans $\mathrm{PGL}_{n+1}\mathbb{R}$ telle que*

(1) $\rho_t(\Gamma)$ est engendré par des réflexions projectives par rapport aux faces de P.

(2) L'application $\mathbb{R}_{>0} \to \mathfrak{P}(X_J)$, $t \mapsto [\rho_t]$ is bijective.

(3) Soit Ω_t l'ensemble de Tits associé à ρ_t, alors Ω_t tend vers P pour la topologie de Hausdorff, lorsque t tend vers 0 ou $+\infty$. c.f. Figure 5.2.

0.3.2 Métrique de Hilbert et entropie

Tout ouvert proprement convexe $\Omega \subset \mathbb{P}^n$ porte une métrique Finslerienne canonique d_Ω, dite la *métrique de Hilbert*. Nous renvoyons à §5.2.1 pour la définition. Si Ω est un ellipsoïde alors (Ω, d_Ω) est isométrique à l'espace hyperbolique $\mathbb{H}^n$.

Soit d_t la métrique de Hilbert sur l'ensemble de Tits Ω_t donné par la proposition précédente, au cas où J contient une boucle. Considérons la quantité suivante, dite l'*entropie* de Ω_t :

$$\delta_t = \lim_{R \to +\infty} \frac{1}{R} \log \#\Gamma x_0 \cap B_{d_t}(x_0, R),$$

où $B_{d_t}(x_0, R)$ désigne la boule à rayon R centré à $x_0 \in \Omega_t$ par rapport à la métrique d_t. Il est connu que cette dernière limite existe et ne dépend pas de x_0. De plus, une adaption d'un résultat de A. Manning [43] implique que δ_t égale à l'entropie topologique du flot géodésique de $\Omega_t/\rho_t(\Gamma)$ (un flot de billard dans le fibré tangent unitaire de P).

M. Crampon [17] a montré que l'entropie du flot géodésique d'une variété projective convexe de dimension n est majorée par $n-1$, où le maximun est atteint si et seulement si Ω est un ellipsoïde. Puis il a demandé si l'entropie admet une borne inférieure.

Le résultat principal de la deuxième partie de cette thèse donne une réponse négative :

Théorème D. *δ_t tend vers 0 lorsque t tend vers 0 ou $+\infty$.*

Rappelons que pour un espace métrique (X, d), la *systole* est l'infimum des longeures de courbes homotopiquement non-nulles. Soit $\widetilde{X}$ le revêtement universel de X. La *constante d'hyperbolicité de Gromov* est par définition le supremum des *tailles* de triangles géodésiques dans $\widetilde{X}$, où la taille $T(\Delta)$ d'une triangle Δ aux arêtes I_1, I_2, I_3 est définie par

$$T(\Delta) = \min_{x_i \in I_i} \text{diamètre}(\{x_1, x_2, x_3\})$$

La démonstration de Théorème D implique

Corollaire. *Pour toute surface fermé de genre $g \geq 2$, il existe une famille à un paramètre de structures projectives convexes dont l'entropie du flot géodésique tend vers 0, la systole et la constante d'hyperbolicité de Gromov tendent vers $+\infty$ lorsque le paramètre tend vers infini.*

Part I

On the quasi-Poisson lattice gauge theory

Chapter 1

Quasi-Poisson lattice gauge theory

In this chapter we give a self-contained presentation of Li-Bland and Ševera's version [42] of the quasi-Poisson lattice gauge theory [6, 3, 24], as well as some necessary preliminaries. §1.1 is a quick review of basic definitions and constructions about quasi-Poisson structure. After giving in §1.2 some backgrounds about flat G-connections over surfaces, we construct in §1.3 the main objet of study in the quasi-Poisson lattice gauge theory a quasi-Poisson structure on $M_G(\Sigma, V)$, and discuss its relationship with the Fock-Rosly construction. Finally we discuss in §1.4 an important aspect of quasi-Poisson theory which is omitted in §1.1, namely, moment maps and cross-section.

1.1 Quasi-Poisson manifolds

1.1.1 Schouten brackets

Let M be a smooth manifold and $C^\infty(\bigwedge^\bullet \mathsf{T}M) = \bigoplus_{k=0}^{dim M} C^\infty(\bigwedge^k \mathsf{T}M)$ be the exterior algebra of multi-vector fields on M.

The Lie bracket of vector fields

$$[\cdot,\cdot] : C^\infty(\mathsf{T}M) \otimes C^\infty(\mathsf{T}M) \to C^\infty(\mathsf{T}M)$$

extends to the *Schouten bracket*

$$[\cdot,\cdot] : C^\infty(\wedge^\bullet \mathsf{T}M) \otimes C^\infty(\wedge^\bullet \mathsf{T}M) \to C^\infty(\wedge^\bullet \mathsf{T}M),$$

which is characterized by the following properties (see *e.g.* [21] Theorem 2.1)

$$\begin{aligned}
&[C^\infty(\wedge^a \mathsf{T}M), C^\infty(\wedge^b \mathsf{T}M)] \subset C^\infty(\wedge^{a+b-1}\mathsf{T}M),\\
&[A, B] = (-1)^{(a-1)(b-1)}[B, A],\\
&[[A, B], C] + (-1)^{(a-1)(b+c-2)}[[B, C], A] + (-1)^{(c-1)(a+b-2)}[[C, A], B] = 0,\\
&[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C],\\
&[X, B] = \mathcal{L}_X B \text{ if } X \in C^\infty(\mathsf{T}M).
\end{aligned}$$

where A, B, C are multi-vector fields of degree a, b, c, respectively.

In particular, if $P \in C^\infty(\bigwedge^2 \mathsf{T}M)$, put $\{f, g\} = P(\mathsf{d}f, \mathsf{d}g)$ $(\forall f, g \in C^\infty(M))$, then

$$[P, P](\mathsf{d}f, \mathsf{d}g, \mathsf{d}h) = -2 \circlearrowleft \{\{f, g\}, h\}. \tag{1.1}$$

where $\circlearrowleft$ stands for the summation over cyclic permutations of f, g and h.

A *Poisson tensor* on M is a bivector field $P \in C^\infty(\bigwedge^2 \mathsf{T}M)$ satisfying $[P, P] = 0$. Equivalently, it is a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$, *i.e.*, a Lie bracket satisfying the Leibniz rule.

For any Lie algebra $\mathfrak{g}$, the Lie bracket also extends to an *algebraic Schouten bracket* $[\cdot,\cdot] : \bigwedge^\bullet \mathfrak{g} \otimes \bigwedge^\bullet \mathfrak{g} \to \bigwedge^\bullet \mathfrak{g}$ with similar properties.

1.1.2 Quasi-Poisson manifolds

Let $\mathfrak{g}$ be a Lie algebra. An action ρ of $\mathfrak{g}$ on M (*i.e.* a homomorphism of Lie algebras $\mathfrak{g} \to C^\infty(\mathsf{T}M)$, $x \mapsto \rho_x$) extends to a homomorphism $\bigwedge^\bullet \mathfrak{g} \to C^\infty(\bigwedge^\bullet \mathsf{T}M)$, $\xi \to \rho_\xi$ preserving Schouten brackets.

Recall that $\mathfrak{g}$ acts on $\mathfrak{g}^{\otimes k}$ in the usual way

$$x.(y_1 \otimes \cdots \otimes y_k) = [x, y_1] \otimes y_2 \otimes \cdots \otimes y_k + \cdots + y_1 \otimes \cdots \otimes y_{k-1} \otimes [x, y_k].$$

We let $(S^2\mathfrak{g})^\mathfrak{g}$ denote $\mathfrak{g}$-invariant elements in $S^2\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$. Here "invariant" means annihilated by any $x \in \mathfrak{g}$.

Let (x_i) be a basis of $\mathfrak{g}$ and (c_{ij}^k) be the structure constants of $\mathfrak{g}$ defined by $[x_i, x_j] = c_{ij}^k x_k$. Then $s \in (S^2\mathfrak{g})^\mathfrak{g}$ has the expression [1]

$$s = s^{ij} x_i \otimes x_j$$

with (s^{ij}) satisfying $s^{ij} = s^{ji}$ and $c_{kl}^i s^{lj} + c_{kl}^j s^{lj} = 0$.

There is a canonical $\mathfrak{g}$-invariant trivector $\phi \in (\bigwedge^3 \mathfrak{g})^\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ associated with s, defined by

$$\phi(\alpha, \beta, \gamma) = \frac{1}{2}\gamma([s^\sharp(\alpha), s^\sharp(\beta)]), \quad \forall \alpha, \beta, \gamma \in \mathfrak{g}^*,$$

where $s^\sharp : \mathfrak{g}^* \to (\mathfrak{g}^*)^* = \mathfrak{g}$ is the map $\alpha \mapsto s(\alpha, \cdot)$. In coordinates we have

$$\phi = \frac{1}{12} c_{pq}^i s^{pj} s^{qk} x_i \wedge x_j \wedge x_k.$$

Definition 1.1. Given a $\mathfrak{g}$-manifold M and $s \in (S^2\mathfrak{g})^\mathfrak{g}$, a *quasi-Poisson* $(\mathfrak{g}, s)$*-tensor* on M is a $\mathfrak{g}$-invariant bivector field $P \in C^\infty(\bigwedge^2 \mathsf{T}M)$ satisfying

$$[P, P] = \rho_\phi.$$

If P is a quasi-Poisson $(\mathfrak{g}, s)$-tensor, then we called the pair (M, P) a *quasi-Poisson* $(\mathfrak{g}, s)$*-manifold* and the skew-symmetric bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad \{f, g\} := P(\mathsf{d}f, \mathsf{d}g)$$

a *quasi-Poisson bracket.*

Because of Eq.(1.1), one can alternatively define a quasi-Poisson manifold as a $\mathfrak{g}$-manifold M such that $C^\infty(M)$ is equipped with a quasi-Poisson bracket, *i.e.*, a skew-symmetric bilinear map satisfying the Leibniz rule and a *quasi-Jacobi identity*

$$-2 \circlearrowleft \{\{f, g\}, h\} = \rho_\phi(\mathsf{d}f, \mathsf{d}g, \mathsf{d}h). \tag{1.2}$$

1. Here and below, we take the sum over repeated indices.

We will often encounter manifolds equipped with an action of the direct sum $\mathfrak{g}^n$ of n copies of $\mathfrak{g}$. We let

$$s^{(n)} = s \oplus \cdots \oplus s \in S^2\mathfrak{g} \oplus \cdots \oplus S^2\mathfrak{g} \subset S^2(\mathfrak{g}^n)$$

be the direct sum of n copies of s. In coordinates we have

$$s^{(n)} = \sum_{r=1}^{n} s^{ij} x_i(r) \otimes x_j(r),$$

where

$$x_i(r) = (0, \cdots, \underbrace{x_i}_{r-\text{th}}, \cdots, 0) \in \mathfrak{g}^n.$$

The canonical invariant element in $\bigwedge^3 \mathfrak{g}^n$ associated to $s^{(n)}$ is

$$\phi^{(n)} = \frac{1}{12} \sum_{r=1}^{n} c^i_{pq} s^{pj} s^{qk} x_i(r) \wedge x_j(r) \wedge x_k(r),$$

namely, the direct some of n copies of ϕ.

An obvious way of constructing new quasi-Poisson manifolds from old ones is to take the direct product. Say, if (M_1, P_1) and (M_2, P_2) are quasi-Poisson $(\mathfrak{g}, s)$-manifolds then $(M_1 \times M_2, \widetilde{P}_1 + \widetilde{P}_2)$ is a quasi-Poisson $(\mathfrak{g}^2, s^{(2)})$-manifold. Here $\widetilde{P}_1$ (resp. $\widetilde{P}_2$) is the bivector field whose restriction to each slice $M_1 \times \{m\}$ (resp. $\{m\} \times M_2$) is P_1 (resp. P_2).

In the next subsection, we shall explain a less trivial way of constructing new quasi-Poisson manifolds.

1.1.3 Fusion

Definition 1.2. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and ρ be a $\mathfrak{g}^n \oplus \mathfrak{h}$-action on a manifold M. Then the diagonal embedding

$$\mathfrak{g} \oplus \mathfrak{h} \hookrightarrow \mathfrak{g}^n \oplus \mathfrak{h}, \quad (x, y) \mapsto (x, \cdots, x, y)$$

induces an action ρ^* of $\mathfrak{g} \oplus \mathfrak{h}$ on M. We call ρ^* the *fusion* of ρ at the $\mathfrak{g}$-factors. Fusion of Lie group actions is defined in the same way.

Given $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$, we put

$$\psi = \frac{1}{2} s^{ij} x_i(2) \wedge x_j(1) \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g}).$$

One can check that ψ is independent of the basis (x_i). We also consider ψ as an element in $\bigwedge^2(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h})$ via the embedding $\mathfrak{g} \oplus \mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}$.

Definition/Proposition 1.3. *Take $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and $t \in (S^2\mathfrak{h})^{\mathfrak{h}}$. Let M be a manifold equipped with an action ρ of $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}$. Let $P \in C^\infty(\bigwedge^2 \mathsf{T}M)$ be a quasi-Poisson $(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}, s \oplus s \oplus t)$-tensor. Put*

$$P^* = P + \rho_\psi.$$

Then P^* is a quasi-Poisson $(\mathfrak{g} \oplus \mathfrak{h}, s \oplus t)$-tensor, called the fusion *of P at the two $\mathfrak{g}$-factors. Here the action ρ^* of $\mathfrak{g} \oplus \mathfrak{h}$ on M is the fusion of ρ.*

The fusion operation of quasi-Poisson tensors is not commutative in the sense that in the definition of ψ the role of the two $\mathfrak{g}$-factors are asymmetric. However, it is associative in the following sense: for a quasi-Poisson $(\mathfrak{g}^n \oplus \mathfrak{h}, s^{(n)} \oplus t)$-tensor P we can apply fusion for $n-1$ times, each time applying to two adjacent $\mathfrak{g}$-factors, and finally get a quasi-Poisson $(\mathfrak{g} \oplus \mathfrak{h}, s \oplus t)$-tensor P' (where the $\mathfrak{g} \oplus \mathfrak{h}$ action on M is the fusion of the $\mathfrak{g}^n \oplus \mathfrak{h}$-action). Then P' does not depend on where we apply fusion at each times. Indeed, P' always has the expression

$$P' = P + \rho_{\psi'}, \quad \psi' = \frac{1}{2} \sum_{k>l} s^{ij} x_i(k) \wedge x_j(l) \in \wedge^2 \mathfrak{g}^n \subset \wedge^2(\mathfrak{g}^n \oplus \mathfrak{h}). \tag{1.3}$$

Proof. We need to show $[P^*, P^*] = \rho^*_\phi$. On one hand,

$$[P^*, P^*] = [P, P] + [\rho_\psi, \rho_\psi] + 2[P, \rho_\psi] = \rho_{\phi^{(2)}} + \rho_{[\psi,\psi]} = \rho_{(\phi^{(2)}+[\psi,\psi])}$$

($[P, \rho_\psi] = 0$ because P is invariant under ρ). We have

$$\begin{aligned}\phi^{(2)} + [\psi, \psi] =& \frac{1}{12} c^i_{pq} s^{pj} s^{qk} [x_i(1) \wedge x_j(1) \wedge x_k(1) + x_i(2) \wedge x_j(2) \wedge x_k(2)] \\ &+ \frac{1}{4} c^i_{pq} s^{pj} s^{qk} (x_i(1) \wedge x_j(2) \wedge x_k(2) + x_i(1) \wedge x_j(2) \wedge x_k(2)).\end{aligned}$$

On the other hand, by definition of ρ^* we have $\rho^*_\phi = \rho_{\iota(\phi)}$, where ι is induced by the diagonal embedding $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$. We have

$$\iota(\phi) = \frac{1}{12} c^i_{pq} s^{pj} s^{qk} [x_i(1) + x_i(2)] \wedge [x_j(1) + x_j(2)] \wedge [x_k(1) + x_k(2)].$$

From these equalities we see that $\phi^{(2)} + [\psi, \psi] = \iota(\phi)$. The required equality $[P^*, P^*] = \rho^*_\phi$ follows. $\square$

Remark 1.4. As a particular instance, given quasi-Poisson $(\mathfrak{g}, s)$-manifolds (M_1, P_1) and (M_2, P_2), let $(M_1 \times M_2, P^*)$ be the quasi-Poisson $(\mathfrak{g}, s)$-manifold obtained by first performing direct product and then fusion. $(M_1 \times M_2, P^*)$ is called the *fusion product* of the two quasi-Poisson manifolds.

Example 1.5. Let $G \times G$ acts on G by $a \overset{(g,h)}{\longmapsto} gah^{-1}$. It follows from $\mathfrak{g}$-invariance of the trivector ϕ that

$$\rho_\phi = \phi^{\mathsf{L}} - \phi^{\mathsf{R}} = 0,$$

so the trivial bivector field $P = 0$ is a $(\mathfrak{g}^2, s^{(2)})$-quasi-Poisson structure for any s.

One can construct new examples from this one by taking direct product of several copies and then applying fusions. Actually, these are all the quasi-Poisson manifolds which concerns us in this thesis. The simplest one of such examples is the fusion (G, P^*), where

$$P^* = \frac{1}{2} s^{ij} x^{\mathsf{R}}_i \wedge x^{\mathsf{L}}_j \in C^\infty(\wedge^2 TG)$$

is a quasi-Poisson $(\mathfrak{g}, s)$-tensor with respect to the conjugation action of G on itself.

1.1.4 Reduction

By *reduction* we mean using a specific structure on a G-manifold M to produce a corresponding structure on the quotient $X = M/G$. When M has a quasi-Poisson structure, we expect a Poisson structure on X.

However, some caution is needed because X is not necessarily a manifold, even not necessarily Hausdorff if G is not compact. Rather than singling out a smooth part of X, we shall formally treat the algebra $C^\infty(M)^G$ of G-invariant functions on M as the "algebra of functions" of M, and sometimes denote it simply by $C^\infty(X)$. By a *Poisson structure on* X is meant a Poisson bracket on this algebra. If $M^\circ \subset M$ is a G-invariant open subset such that $X^\circ = M^\circ/G$ is a smooth manifold, then such a Poisson bracket descends to a genuine Poisson structure on X°.

The quasi-Jacobi identity (1.2) implies

Definition/Proposition 1.6. *For any quasi-Poisson* $(\mathfrak{g}, s)$*-tensor* P *on a* G*-manifold* M*, the restriction of the quasi-Poisson bracket* $\{\cdot,\cdot\}$ *to* $C^\infty(M)^G$ *is a Poisson bracket, which we call the* Poisson structure on M/G reduced from P.

This notion will be generalized in §3.1.1.

1.2 Flat connections over surfaces

1.2.1 Flat connections

We introduce in this subsection some backgrounds concerning flat connections on principal G-bundles. Details can be found, for example, in [36].

In this thesis, a *surface* always means a compact oriented surface, possibly with boundary. If the boundary is nonempty, we call it a *bordered surface*.

Let Σ be a bordered surface, G be a Lie group with Lie algebra $\mathfrak{g}$ and $\pi : P \to \Sigma$ a (left) principal G-bundle.

> To simplify notations, we assume that G is connected, so that P must be isomorphic to the trivial G-bundle [2], and we fix a trivialization $P \cong \Sigma \times G$. By definition G acts on P by right multiplication on the second factor.

A *connection* on P is a G-invariant horizontal distribution $\mathfrak{D}$. Here "distribution" means giving a linear subspace $\mathfrak{D}_p \subset T_pP$ depending smoothly on $p \in P$, and "horizontal" means that $d\pi_p(\mathfrak{D}_p) = T_{\pi(p)}\Sigma$ for any $p \in P$. Furthermore, $\mathfrak{D}$ is called a *flat connection* if it is integrable, *i.e.*, a foliation. The space of connections (resp. flat connections) on P is denoted by $\mathcal{A}$ (resp. $\mathcal{A}_{\mathit{flat}}$).

Given a connection $\mathfrak{D}$, there is a unique $\mathfrak{g}$-valued differential form $\theta \in \Omega^1(\Sigma, \mathfrak{g})$, call a *connection* 1*-form* such that $\mathfrak{D}_p$ is the kernel of the linear map

$$g^{-1}dg + \theta : T_pP \to \mathfrak{g},$$

where $g^{-1}dg \in \Omega^1(G, \mathfrak{g})$ is the left invariant Maurer-Cartan 1-form, lifted to P by the projection $P \cong \Sigma \times G \to G$. It can be shown that the connection is flat if and only if the *curvature* 2*-form*

$$F(\theta) = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(\Sigma, \mathfrak{g})$$

2. This is because principal G-bundles over Σ are classified by homotopy classes of maps $\Sigma \to BG$, but Σ is homotopic to a graph and the classifying space BG is simply connected if G is connected.

vanishes. Thus we have identifications

$$\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g}), \quad \mathcal{A}_{\mathit{flat}} = \{\theta \in \Omega^1(\Sigma, \mathfrak{g}) \mid F(\theta) = 0\}.$$

Note that these identifications depend on the choice of the trivialization $P \cong \Sigma \times G$.

A *gauge transformation* of P is a G-bundle automorphism, namely, a diffeomorphism $P \to P$ of the form

$$\Sigma \times G \cong P \to P \cong \Sigma \times G, \quad (x, s) \mapsto (x, g(x)s),$$

where $g : \Sigma \to G$ is a smooth map. Thus the group $\mathcal{G}$ of all gauge transformations is identified with the space of smooth maps $\mathrm{Map}(\Sigma, G)$.

A gauge tranformation brings a distribution to another, preserving G-invariance and horizontality, thus $\mathcal{G}$ acts on $\mathcal{A}$. This action clearly preserves $\mathcal{A}_{\mathit{flat}}$. In terms of connection 1-forms, the action is given by

$$\theta \overset{g}{\longmapsto} \mathrm{Ad}_g\, \theta - dgg^{-1}$$

for any $\theta \in \Omega^1(\Sigma, \mathfrak{g}) \cong \mathcal{A}$ and $g \in \mathcal{G} \cong \mathrm{Map}(\Sigma, G)$. Here dgg^{-1} is the pull-back of the right invariant Maurer-Cartan form by $g : \Sigma \to G$.

1.2.2 Holonomies and fundamental groupoid representations

Fix a connection $\mathfrak{D} \in \mathcal{A}$ and an oriented smooth path $\gamma \subset \Sigma$ going from a to b. Given any $p \in P|_a$, one can lift γ to a path $\widetilde{\gamma} \subset P$ which is tangent to $\mathfrak{D}$ and starts from p. Let $p' \in P|_b$ by the ending point of $\widetilde{\gamma}$. The map $p \mapsto p'$ is a bijection between the fibers $P|_a$ and $P|_b$. Moreover, it follows from G-invariance of $\mathfrak{D}$ that, under the trivialization $P \cong \Sigma \times G$, this map has the form

$$G \cong P|_a \to P|_b \cong G, \quad s \mapsto hs$$

for some $h \in G$. We call g the *holonomy* of the connection $\mathfrak{D}$ along the path γ and denote it by $h = \mathrm{hol}_\gamma(\mathfrak{D})$. It can be shown that

(1) if $\mathfrak{D}$ is flat then $\mathrm{hol}_\gamma(\mathfrak{D})$ only depends on the end-points-fixing homotopy class of γ;

(2) if $g \in \mathcal{G}$ then
$$\mathrm{hol}_\gamma(g.\mathfrak{D}) = g(b)\mathrm{hol}_\gamma(\mathfrak{D})g(a)^{-1},$$

Recall that a *groupoid* is a small category whose morphisms are all invertible. A group is considered as a groupoid with a single object. Let $V \subset \Sigma$ be a set of base points. The *fundamental groupoid* $\pi_1(\Sigma, V)$ is the groupoid whose objects are points in V, while a morphism from $u \in V$ to $v \in V$ is a homotopy class of oriented paths on Σ going from u to v. By abuse of notations, $\pi_1(\Sigma, V)$ is often understood as the set of morhpisms in this groupoid. If V is a single point, then $\pi_1(\Sigma, V)$ is just the fundamental group.

We shall write compositions in $\pi_1(\Sigma, V)$ from right to left. Namely, if $\alpha, \beta \in \pi_1(\Sigma, V)$ go from u to v and from v to w, respectively, then their composition, which goes from u to w, is denoted by $\beta\alpha$.

We call a functor between two groupoids a *homomorphism*, and call a homomorphism into a Lie group a *representation*. The main object of study in the present thesis is the space of representations
$$M_G(\Sigma, V) = \mathrm{Hom}(\pi_1(\Sigma, V), G)$$

for a non-empty finite set of boundary points $V \subset \partial\Sigma$. We call V *marked points* and the pair (Σ, V) a *marked surface* The group G^V naturally acts on $M_G(\Sigma, V)$ by

$$g.m(\gamma) = g_{\mathrm{in}(\gamma)} m(\gamma) g^{-1}_{\mathrm{out}(\gamma)}, \quad \forall g \in G^V, m \in M_G(\Sigma, V), \gamma \in \pi_1(\Sigma, V),$$

where $\mathrm{out}(\gamma)$ and $\mathrm{in}(\gamma)$are the starting and ending point of γ, respectively. Given $v \in V$, we shall denote the action of the v^{th} factor of G^V by ρ^v.

When V is a single point, $M_G(\Sigma, V)$ is a more familiar object – the representation variety of the fundamental group of Σ, where G acts on representations by conjugation.

The above two properties of holonomies shows that the holonomy data of a flat connections can be viewed as a fundamental groupoid representation. Indeed, the map

$$\mathcal{A}_{\mathit{flat}} \to \mathrm{Hom}(\pi_1(\Sigma, V), G), \quad \mathfrak{D} \mapsto (\gamma \mapsto \mathrm{hol}_\gamma(\mathfrak{D})) \tag{1.4}$$

is surjective and equivariant with respect to the restriction homomorphism

$$\mathcal{G} \to G^V \quad g \mapsto (g(v))_{v \in V}. \tag{1.5}$$

Moreover, let $\mathcal{G}_V$ denote the kernel of the homomorphism (1.5), then (1.4) induces a bijection

$$\mathcal{A}_{\mathit{flat}}/\mathcal{G}_V \xrightarrow{\sim} M_G(\Sigma, V).$$

We denote also by hol_γ the evaluation map of $M_G(\Sigma)$ at $\gamma \in \pi_1(\Sigma, V)$,

$$M_G(\Sigma, V) \to G, \quad m \mapsto m(\gamma).$$

Notice that the lift of this map to $\mathcal{A}_{\mathit{flat}}$ coincides with the holonomy $\mathrm{hol}_\gamma : \mathcal{A}_{\mathit{flat}} \to G$.

1.2.3 The Atiyah-Bott Poisson structure

We recall in this subsection a classical construction of a Poisson structure on the moduli space of flat connections

$$X_G(\Sigma) := M_G(\Sigma, V)/G^V = \mathcal{A}_{\mathit{flat}}/\mathcal{G}.$$

due to Atiyah and Bott [7] [3].

Remark 1.7. (1) $X_G(\Sigma)$ is also the space of conjugacy classes of representations of the fundamental group $\pi_1(\Sigma)$. Indeed, pick $v \in V$, then the groupoid injection $\pi_1(\Sigma, \{v\}) \to \pi_1(\Sigma, V)$ induces a map

$$M_G(\Sigma, V) \longrightarrow M_G(\Sigma, \{v\}).$$

Under the action of G^V on $M_G(\Sigma, V)$, this map is surjective and equivariant with respect to the v^{th} factor and invariant with respect to other factors. Thus

$$X_G(\Sigma) = M_G(\Sigma, \{v\})/G.$$

(2) In general, $X_G(\Sigma)$ is not a smooth manifold. Following §1.1.4, by a *Poisson structure* on $X_G(\Sigma)$ we mean a Poisson bracket on the algebra $C^\infty(X_G(\Sigma)) := C^\infty(M_G(\Sigma, V))^{G^V}$. Such a bracket descends to a genuine Poisson structure on any smooth part of $X_G(\Sigma)$.

3. Although Atiyah and Bott considered a more restrictive situation where Σ is a closed. The generalization which we present here is well known. See *e.g.* [8].

We fix an invariant scalar product $(\cdot \mid \cdot)$ on the Lie algebra $\mathfrak{g}$. The vector space $\mathcal{A} \cong \Omega^1(\Sigma, \mathfrak{g})$ carries a natural translation-invariant and $\mathcal{G}$-invariant symplectic form

$$\sigma(a, b) = \int_\Sigma (a \mid b), \quad a, b \in T_\theta \mathcal{A} \cong \Omega^1(\Sigma, \mathfrak{g}).$$

Assume that $\partial\Sigma$ has b connected components. Recall that the loop group LG is the space of smooth maps $\mathrm{Map}(S^1, G)$. Let $\mathcal{G}_{\partial\Sigma}$ denote the kernel of the restriction map

$$\mathcal{G} \cong \mathrm{Map}(\Sigma, G) \to \mathrm{Map}(\partial\Sigma, G) \cong LG^b.$$

It can be shown that the curvature

$$F : \mathcal{A} \to \Omega^2(\Sigma, \mathfrak{g}) \subset \mathrm{Lie}(\mathcal{G})^* \subset \mathrm{Lie}(\mathcal{G}_{\partial\Sigma})^*$$

is a moment map for the action of $\mathcal{G}_{\partial\Sigma}$, where the first inclusion is given by the natural pairing $\langle A, \xi \rangle = \int_\Sigma (A \mid \xi)$ between $\mathrm{Lie}(\mathcal{G}) = \Omega^0(\Sigma, \mathfrak{g})$ and $\Omega^2(\Sigma, \mathfrak{g})$.

Recall that the Marsden-Weinstein reduction (see *e.g.* [45]) says that the quotient of the zero level set of a moment map, which in is case is $\mathcal{N} = \mathcal{A}_{flat}/\mathcal{G}_{\partial\Sigma}$, carries a symplectic structure. Hence the space of functions $C^\infty(\mathcal{N})$ carries a Poisson bracket.

There is a natural projection

$$\mathcal{N} \longrightarrow M_G(\Sigma, V)$$

which is equivariant with respect to the projection of groups $LG^b = \mathcal{G}/\mathcal{G}_{\partial\Sigma} \to \mathcal{G}/\mathcal{G}_V = G^V$. Thus $C^\infty(X_G(\Sigma)) = C^\infty(M_G(\Sigma, V))^{G^V}$ is identified with the $C^\infty(\mathcal{N})^{LG^b}$, which has a Poisson bracket. This is by definition the Atiyah-Bott Poisson structure on $X_G(\Sigma)$.

Remember our assumption from the beginning of this section that G is connected. If it is not the case, then we shall let $\mathcal{A}_{flat}$ be the disjoint union $\bigsqcup_P \mathcal{A}_{flat}(P)$, where P runs over all isomorphism classes of principal G-bundles over Σ. On each $\mathcal{A}_{flat}(P)$ there is a gauge group $\mathcal{G}(P)$ acting. The above discussions carry over to this situation. For example, we have

$$M_G(\Sigma, V) = \bigsqcup_P \mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}_V(P), \quad X_G(\Sigma) = \bigsqcup_P \mathcal{A}_{flat}(P)/\mathcal{G}(P).$$

Remark 1.8. (1) To make the above discussions about infinite-dimensional manifold rigorous, one usually work with connections in a certain Sobolev class, instead of smooth ones, so as to make $\mathcal{A}$, $\mathcal{G}$, etc. into Banach manifolds. See *e.g.* [7] Section 14.

(2) The LG^b-action on the symplectic manifold $\mathcal{N}$ also admits a moment map. Applying symplectic reduction to it, one can show that a symplectic leaf of $X_G(\Sigma)$ consists of representations of the fundamental group sending each boundary loop to a prescribed conjugacy class in G. Here we pick an arbitrary conjugacy class for each boundary loop. This fact follows alternatively from quasi-Poisson theory.

Despite of its theoretical importance, the above definition of the Atiyah-Bott Poisson structure is difficult to handle in concrete applications, since it involves infinite-dimensional objects. Goldman [26, 27] gave a finite-dimensional construction in terms of twisted cohomology and used it to compute Poisson brackets of certain functions on $X_G(\Sigma)$, see §2.1.2 below. Goldman's original approach only works for closed surfaces. A generalization to bordered surfaces is given in [30].

In the 90's two alternative finite-dimensional constructions appeared: one is due to V. Fock and A. Rosly [24] and the other to A. Alekseev, E. Meinrenken and A. Malkin [6]. These constructions have the advantage of being completely explicit and calculable. The latter one is the main object of the present thesis. However, we shall explain in §3.2.6 that these two constructions actually are equivalent.

1.2.4 Combinatorics of surfaces

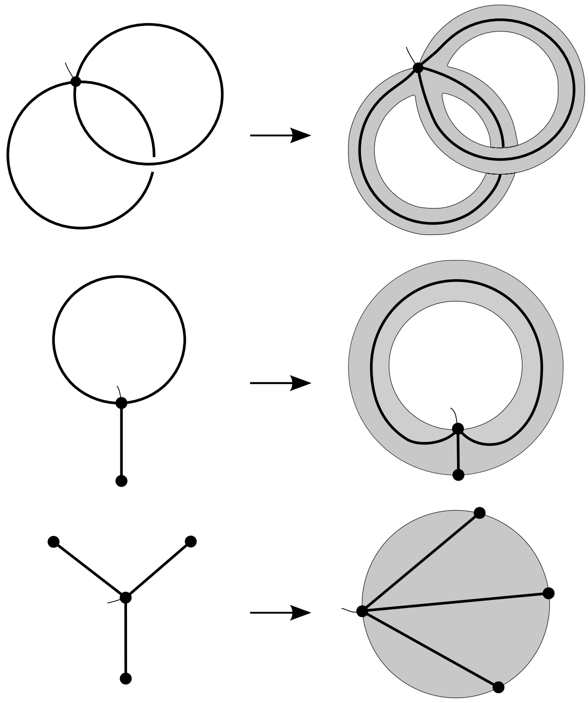
We discuss in this subsection some combinatorial facts about bordered surfaces, which are useful in the study of the representation space $M_G(\Sigma, V)$.

By a *skeleton* of a bordered surface Σ we mean an embedded graph $\Gamma \subset \Sigma$ such that the set of vertices V_Γ is contained in $\partial\Sigma$ and Σ retracts to Γ by deformation. Skeletons have the structure of *ciliated fat graph*, as we explain now.

Given a graph Γ, we shall let V_Γ and E_Γ denote the set of vertices and edges, respectively. We view each edge as consisting of two half-edges and let $\widehat{E}_\Gamma$ the set of all half-edges of Γ. Recall that a *fat graph* is a graph Γ such that at each vertex v a cyclic order is given to the set $\widehat{E}_\Gamma(v)$ of half-edges issuing from v. We usually draw a fat graph as immersed in the plane such that the cyclic order at each vertex is the counter-clockwise order.

A *ciliated fat graph* is graph such that each $\widehat{E}_\Gamma(v)$ is endowed with an order, instead of a cyclic order. We draw a ciliated fat graph an immersed fat graph in the plane with a ciliate attached to each vertex from which the ordering begins. Notice that if there is only one edge issuing from v then no ciliate is needed at v.

Given a ciliated fat graph Γ, one can fatten Γ to get a bordered surface Σ such that Γ is a skeleton of Σ, as shown in the pictures below. Precisely, we fatten each edge into a ribbon and each vertex v into a half-disk in such a way that v is on the boundary and the ciliate points outwards.



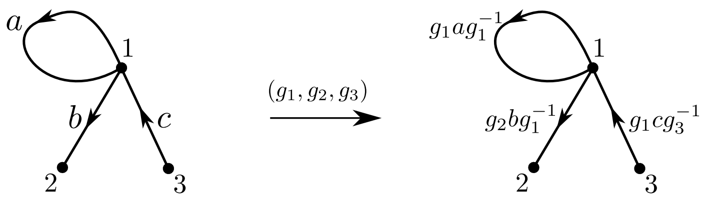
Conversely, consider a pair (Σ, Γ) where Σ is a bordered surface and Γ a skeleton. The orientation of Σ endows Γ with the structure of ciliated graph, and one can shrink Σ into a fattened Γ. Therefore, a pair (Σ, Γ) is equivalent to a ciliated fat graph.

Skeletons enable us to better understand the space of fundamental groupoid representations $M_G(\Sigma, V)$. We call Γ a skeleton of the marked surface (Σ, V) if $V_\Gamma = V$. Since (Σ, V) retracts to (Γ, V) by deformation, we have $\pi_1(\Sigma, V) = \pi_1(\Gamma, V)$. If we give each edge $e \in E_\Gamma$ an orientation, then the fundamental groupoid $\pi_1(\Gamma, V)$ is freely generated by these oriented edges, hence when defining a homomorphism from $\pi_1(\Gamma, V)$ to G, one only need to assign an arbitrary element of G to each edge. As a result, we have a bijection

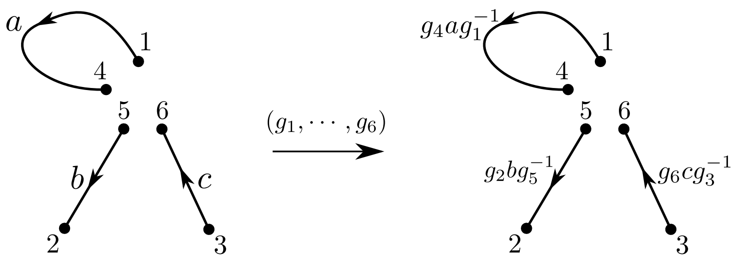
$$(\mathrm{hol}_e)_{e\in E_\Gamma} : M_G(\Sigma, V) = \mathrm{Hom}(\pi_1(\Gamma, V), G) \xrightarrow{\sim} G^{E_\Gamma}. \tag{1.6}$$

Remark 1.9. in the literature, members of $\mathrm{Hom}(\pi_1(\Gamma, V), G)$ are called *lattice gauge fields* and the G^V-action called *discrete gauge transformation.* This explains the title of this chapter.

The natural G^V-action on $M_G(\Sigma, V) \cong G^{E_\Gamma}$ can be easily read off from the graph. See the following picture. Here Γ has three edges, an element of G^{E_Γ} being an assignment of elements a, b, c in G to each edge.



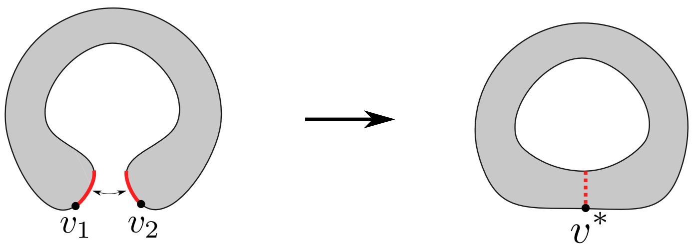
We shall interpret the G^V-action as a fusion of a finer action. We break edges of Γ apart and obtain a graph $\widetilde{\Gamma}$ which is the disjoint union of $\# E_\Gamma$ copies of segments. $\widehat{E_\Gamma}$ is viewed as the set of vertices of $\widetilde{\Gamma}$, so $G^{\widehat{E_\Gamma}}$ naturally acts on $\mathrm{Hom}(\pi_1(\widetilde{\Gamma}, \widehat{E_\Gamma}), G) \cong G^{E_\Gamma} \cong M_G(\Sigma, V)$, as pictured below. Let $\widetilde{\rho}^v$ denote the action of the subgroup $G^{\widehat{E_\Gamma}(v)} \subset G^{\widehat{E_\Gamma}}$ on $M_G(\Sigma, V)$.



Then the action ρ^v of the v^{th} factor of G^V is just the fusion of $\widetilde{\rho}^v$.

We shall now look closer into the above operation of breaking Γ down into segments.

Let (Σ, V) be a bordered surface with marked points $V \subset \partial\Sigma$. We define the *fusion* of (Σ, V) at an ordered pair of marked points (v_1, v_2) to be a new surface with marked points (Σ^*, V^*), obtained by gluing two segments of $\partial\Sigma$ together, as shown in the picture below. We require that the first (resp. second) segment issues from v_1 (resp. v_2 and runs in negative (resp. position) orientation.



Note that if we exchange the roles of v_1 and v_2, then, in general, we get a different (Σ^*, V^*). Indeed, the fusion operation for surfaces is noncommutative but associative in a similar sense as for quasi-Poisson structures.

If Γ is a skeleton of (Σ, V), then the image Γ^* of Γ in Σ^* is a skeleton of (Σ^*, V^*). As a result, the gluing map $(\Sigma, V) \to (\Sigma^*, V^*)$ induces a bijection between the space of fundamental groupoid representations. Indeed, the following diagram commutes

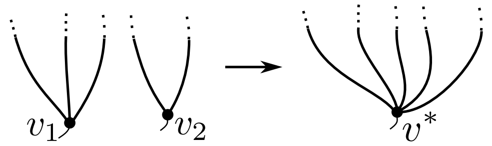
$$\begin{tikzcd} M_G(\Sigma^*, V^*) \arrow[r] \arrow[d, "(\mathrm{hol}_e)_{e\in E_{\Gamma^*}}"', "\cong"] & M_G(\Sigma, V) \arrow[d, "\cong"', "(\mathrm{hol}_e)_{e\in E_\Gamma}"] \\ G^{E_{\Gamma^*}} \arrow[r, "="] & G^{E_\Gamma} \end{tikzcd}$$

Moreover, the G^{V^*}-action on $M_G(\Sigma^*, V^*) \cong M_G(\Sigma, V)$ is obtained from the G^V-action by fusing the v_1^{th} and v_2^{th} factor.

Under the equivalence between ciliated fat graphs and surfaces with skeletons, the fusion operation

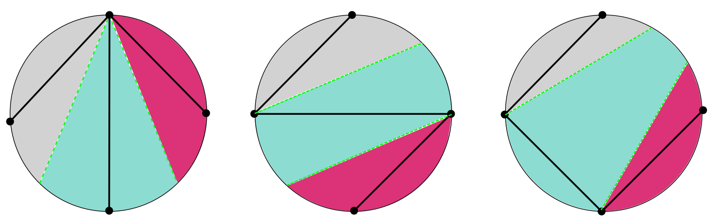
$$(\Sigma, \Gamma) \rightsquigarrow (\Sigma^*, \Gamma^*)$$

is just the operation of merging the vertices v_1 and v_2 of the two ciliated fat graphs, as shown in the picture below. This operation is clearly noncommutative but associative.



Therefore, we can think of a skeleton Γ of Σ as being built from the disjoint union $\widetilde{\Gamma}$ of segments by applying fusion repeatedly, while the ciliates on vertices of Γ serves to record the order of fusions.

Equivalently, let $\mathbf{D}$ denote the disk with two marked points, then Γ gives a way of building up (Σ, V) from a disjoint union of $\mathbf{D}$'s by fusion. For example, starting from three copies of $\mathbf{D}$ and performing fusion two times, we can get a disk with four marked points. Difference skeletons corresponds to different ways to do so, as shown in the picture below. Here the three colors stand for the three copies of $\mathbf{D}$.



1.3 The Quasi-Poisson structure on $M_G(\Sigma, V)$

1.3.1 Construction of the quasi-Poisson structure

Fix a Lie group G with Lie algebra $\mathfrak{g}$ and pick $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$. Let Σ be a bordered surface and $V \subset \partial\Sigma$ be finitely many marked points. Using the constructions in the previous subsection, we readily get

Proposition 1.10. *Given a skeleton Γ of (Σ, V), there is a canonical quasi-Poisson $(\mathfrak{g}^V, s^{(V)})$-tensor P_Γ on $M_G(\Sigma, V)$.*

Proof. The construction of P_Γ goes as follows. Since $G^{\widehat{E}_\Gamma}$-manifold $\mathrm{Hom}(\pi_1(\widetilde{\Gamma}, \widehat{E}_\Gamma), G) \cong M_G(\Sigma, V) =: M$ is the direct product of $\# E_\Gamma$ copies of the $G \times G$-manifold G, by Example 1.5, M has the trivial bivector field $P = 0$ as a quasi-Poisson $(\mathfrak{g}^{\widehat{E}_\Gamma}, s^{(\widehat{E}_\Gamma)})$-tensor. The G^V-action on M is a fusion of the $G^{\widehat{E}_\Gamma}$-action (see §1.2.4), so we get the quasi-Poisson tensor P_L as a fusion of $P = 0$ in the sense of Definition/Proposition 1.3. Note that fusion of quasi-Poisson tensors depends on orders, which, in this case, is specified by the ciliates. □

Let us exhibit an explicit expression for P_Γ. We shall first establish a notation.

Definition 1.11. Let Γ be a graph and $\widetilde{\Gamma}$ be the graph obtained by breaking apart edges of Γ, so that $G^{\widehat{E}_\Gamma}$ naturally acts on $\mathrm{Hom}(\pi_1(\widetilde{\Gamma}, \widehat{E}_\Gamma), G) \cong \mathrm{Hom}(\pi_1(\Gamma, V), G) =: M$.

For any $x \in \mathfrak{g}$ and $\mathbf{a} \in \widehat{E}_\Gamma$, we let $x^{(\mathbf{a})} \in C^\infty(\mathsf{T}M)$ denote the fundamental vector field of x induced by the action of the $\mathbf{a}^{th}$ factor of $G^{\widehat{E}_\Gamma}$.

Here is an alternative description of $x^{(\mathbf{a})}$. Choose an orientation for each edge $e \in E_\Gamma$ and use $(\mathrm{hol}_e)_{e \in E_\Gamma}$ to identify M with G^{E_Γ}. Let $e \in E_\Gamma$ be the oriented edge containing $\mathbf{a}$. If $\mathbf{a}$ is the first (resp. second) half of e, then $x^{(\mathbf{a})}$ is just the left invariant vector field x^{L} (resp. the right invariant vector field $-x^{\mathsf{R}}$) on the e^{th} factor of G^{E_Γ}.

Let (x_i) be a basis of $\mathfrak{g}$ and assume $s = s^{ij} x_i \otimes x_j$. Using the above notation and Eq.(1.3), we get

$$P_\Gamma = \frac{1}{2} \sum_{v \in V} \sum_{\mathbf{a} < \mathbf{b}} s^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{b})}, \tag{1.7}$$

where the second summation runs over half-edges $\mathbf{a}, \mathbf{b} \in \widehat{E}_\Gamma(v)$ such that $\mathbf{a} < \mathbf{b}$. Here $\widehat{E}_\Gamma(v)$ has an order "$<$" because it is a ciliated fat graph.

Theorem 1.12. *The bivector field P_Γ on $M_G(\Sigma, V)$ does not depend on the choice of the skeleton Γ.*

Thus we shall call $P = P_\Gamma$ the *canonical quasi-Poisson tensor* on $M_G(\Sigma, V)$. Notice that P depends on a choice of $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$.

Theorem 1.12 will be proved in §2.3.1 below as a consequence of Theorem 2.17.

It is instructive to verify the theorem in the following simplest example by straightforward computations.

Example 1.13. Let (Σ, V) be the disk with three marked points on the boundary and put $M = M_G(\Sigma, V)$. Let Γ, Γ' be the two skeletons as shown below, where we give each edge an orientation and let $a, b, u, v : M \to G$ be the corresponding holonomies. Let us show that $P_\Gamma = P_{\Gamma'}$.

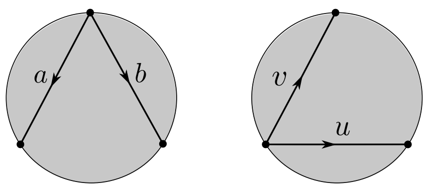


Figure 1.1: Two skeletons of the disk with three marked points.

Under the coordinates systems $(a, b) : M \xrightarrow{\sim} G \times G$, we have

$$P_\Gamma(a, b) = s^{ij} x_i^{\mathsf{L}}(a) \wedge x_j^{\mathsf{L}}(b),$$

where $x^{\mathsf{L}}(a)$ (resp. $x^{\mathsf{L}}(b)$) $\in T_{(a,b)}G \times G$ denotes the left translation of x to a (resp. b) in the first (resp. second) factor. Similarly, in the coordinates system (u, v) we have

$$P_{\Gamma'}(u, v) = s^{ij} x_i^{\mathsf{L}}(u) \wedge x_j^{\mathsf{L}}(v).$$

The coordinates change is $a = v^{-1}, b = uv^{-1}$. Some computation gives that at the point (a, b) we have

$$x_i^{\mathsf{L}}(u) = (\mathrm{Ad}_a^{-1} x_i)^{\mathsf{L}}(b), \quad x_j^{\mathsf{L}}(v) = -x_j^{\mathsf{R}}(a) - (\mathrm{Ad}_a^{-1} x_j)^{\mathsf{L}}(b).$$

Using $s^{ij} = s^{ji}$ we get

$$\begin{aligned} P_{\Gamma'}(a, b) &= s^{ij} x_i^{\mathsf{R}}(a) \wedge (\mathrm{Ad}_a x_j)^{\mathsf{L}}(b) \\ &= s^{ij} (\mathrm{Ad}_a^{-1} x_i)^{\mathsf{L}}(a) \wedge (\mathrm{Ad}_a^{-1} x_j)^{\mathsf{L}}(b) = P_{\Gamma}(a, b), \end{aligned}$$

where the last equality follows from invariance of $s \in \mathfrak{g} \otimes \mathfrak{g}$.

Remark 1.14. Extending the above computation, one can show that for general (Σ, V), if a skeleton Γ' is obtained from another one Γ by modifying only two edges in a way as above (let us call such a modification a *simple move*), then $P_{\Gamma} = P_{\Gamma'}$. One is then tempted to prove Theorem 1.12 by showing that any two skeletons of (Σ, V) are related by a sequence of simple moves. This approach, although achievable, involves lengthy arguments in order to show that for any diffeomorphism ϕ of Σ fixing $\partial\Sigma$, the skeletons Γ and $\phi(\Gamma)$ are related by simple moves – one has to find an explicit set of generators of the mapping class group and then check for each generator ϕ. The proof that we present in §2.3.1 uses completely different ideas.

The quasi-Poisson theory gives a finite-dimensional construction of the Atiyah-Bott Poisson structure in the following sense.

Theorem 1.15 (Alekseev, Malkin, Meinrenken [6])**.** *Let $(\cdot \mid \cdot)$ be an invariant scalar product on $\mathfrak{g}$ and $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ be its dual. Let P be the canonical quasi-Poisson $(\mathfrak{g}^V, s^{(V)})$-tensor on $M_G(\Sigma, V)$. Then the Poisson structure on $X_G(\Sigma) = M_G(\Sigma, V)/G^V$ reduced from P coincides with the Atiyah-Bott Poisson structure.*

We will not prove the above theorem in this thesis, but only make a few comments here about the proofs. A quick but indirect proof consist in showing that the Atiyah-Bott Poisson brackets of certain functions in $C^\infty(X_G(\Sigma)) = C^\infty(M_G(\Sigma, V))^{G^V}$, known as *spin networks*, which form a dense subset of $C^\infty(M_G(\Sigma, V))^{G^V}$, coincides with their quasi-Poisson brackets. This is possible because a formula for Atiyah-Bott brackets of spin networks is available [9, 52], while a quasi-Poisson bracket formula for them is established in Chapter 2 below. The original proof, which indicates how the quasi-Poisson structure on $M_G(\Sigma, V)$ arises, is given in [6] using the language of *quasi-Hamiltonian structures* – one shall use results from [3] to translate quasi-Hamiltonian structures to quasi-Poisson structures. However, the latter proof seems only apply to the case where V has exactly one point in each component of $\partial\Sigma$. According to David Li-Bland, a on-going joint work of A. Cabrera, M. Gualtieri and E. Meinrenken contains further clarification on this issue.

1.3.2 Equivalence with the Fock-Rosly Poisson structure

Using Poisson Lie group actions, Fock and Rosly gave in [24] a Poisson structure on $M_G(\Sigma, V)$ which also reduces to the Atiyah-Bott Poisson structure on the quotient $X_G(\Sigma)$.

The Fock-Rosly Poisson structure is of course different from the canonical quasi-Poisson structure, but it turns out to be a *twist* of the latter in the sense of Alekseev and Kosmann-Schwarzbach [5]. In this subsection we first recall some backgrounds on Poisson Lie groups and then explain this twist equivalence.

Definition/Proposition 1.16. – *A* Poisson Lie group *is a Lie group G with Poisson structure P such that the multiplication $G \times G \to G$ is a Poisson map (where $G \times G$ is endowed with the product Poisson structure).*

– *We only consider those Poisson Lie groups such that P has the form*

$$P = a^{\mathsf{R}} - a^{\mathsf{L}} \tag{1.8}$$

for some $a \in \bigwedge^2 \mathfrak{g}$. Here a^{L} and a^{R} are left and right invariant bivector fields on G which restrict to a at the origin. Notice that there are constraints on $a \in \bigwedge^2 \mathfrak{g}$ for (1.8) to be a Poisson tensor.

– *$r \in \mathfrak{g} \otimes \mathfrak{g}$ is called a* quasi-triangular classical r-matrix *if its symmetric part $s = \frac{1}{2}(r + r_{21})$ is $\mathfrak{g}$-invariant and r satisfies the following* classical Yang-Baxter equation [4]

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{CYBE}$$

– *A sufficient (but not necessary) condition for (1.8) to define a Poisson Lie group structure is that there exists $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ such that $r = a + s$ is a quasi-triangular classical r-matrix.*

– *A* Lie bialgebra *is a Lie algebra $\mathfrak{g}$ such that the dual $\mathfrak{g}^*$ is equipped with a Lie bracket $\bigwedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ whose transpose $\gamma : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ is a 1-cocycle, namely,*

$$x.\gamma(y) - y.\gamma(x) - \gamma([x, y]) = 0.$$

– *Let G be a Poisson Lie group, then its Lie algebra $\mathfrak{g}$ has a natural Lie bialgebra structure $\gamma : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ which, roughly speaking, is the differential of the Poisson tensor at the origin. In particular, if the Poisson Lie structure comes from a quasi-triangular classical r-matrix, then $\gamma(x) = x.r$ for any $x \in \mathfrak{g}$.*

– *Let M be a Poisson manifold and G be a Poisson Lie group. If an action $G \times M \to M$ is a Poisson map, then it is call a* Poisson action. *An equivalent condition is that the infinitesimal action ρ of $\mathfrak{g}$ on M satisfies*

$$\mathcal{L}_{\rho_x} P_M = \rho_{\gamma(x)}, \tag{1.9}$$

where P_M is the Poisson tensor on M and $\gamma : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$ is the natural Lie biaglebra structure.

We refer to [16] for details and proves of these statements. Now we can show the equivalence between quasi-Poisson manifolds and Poisson manifolds with Poisson Lie group actions.

4. The notations here are defined as follows. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Assuming $r = \sum_i u_i \otimes v_i \in \mathfrak{g} \otimes \mathfrak{g}$, we define $r_{12}, r_{13}, r_{23} \in U(\mathfrak{g})^{\otimes 3}$ by

$$r_{12} = \sum_i u_i \otimes v_i \otimes 1, \quad r_{13} = \sum_i u_i \otimes 1 \otimes v_i, \quad r_{23} = \sum_i 1 \otimes u_i \otimes v_i.$$

The commutator of any two of them is contained in $\mathfrak{g}^{\otimes 3}$. For example, $[r_{12}, r_{23}] = \sum_i [u_i, u_j] \otimes v_i \otimes v_j$.

Theorem 1.17 (Alekseev, Kosmann-Schwarzbach [5]). *We take $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and $a \in \bigwedge^2 \mathfrak{g}$ such that*

$$r = a + \frac{s}{2} \in \mathfrak{g} \otimes \mathfrak{g}$$

is a quasi-triangular classical r-matrix and equip G with the Poisson Lie structure (1.8). Let M be a G-manifold and $P \in C^\infty(\bigwedge^2 \mathsf{T}M)$ be a bivector field. Then P is a quasi-Poisson $(\mathfrak{g}, s)$-tensor if and only if

$$P' = P + \rho_a$$

is a Poisson tensor such that the action of the Poisson Lie group G on M is a Poisson action.

Proof. The proof of Theorem 1.17 is based on the following lemma:

Lemma 1.18. *(1) If $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and $\phi \in \bigwedge^3 \mathfrak{g}$ is the canonical trivector, then we have $\phi = -\frac{1}{2}[s_{12}, s_{23}] = \frac{1}{2}[s_{13}, s_{23}] = \frac{1}{2}[s_{12}, s_{13}]$. It follows that $\phi = \frac{1}{2}[[s, s]]$.*

(2) If $a \in \bigwedge^2 \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ then $[[a, a]] = \frac{1}{2}[a, a]$.

(3) If $r = a + s \in \mathfrak{g} \otimes \mathfrak{g}$ with $a \in \bigwedge^2 \mathfrak{g}$ and $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$, then $[[r, r]] = [[a, a]] + [[s, s]]$.

These follows from straightforward computations and we omit the proof.

The conditions that $P' = P + \rho_a$ is a Poisson tensor and the G action is a Poisson action amount to the following conditions about Schouten brackets

$$[\rho_x, P'] = \rho_{[x,r]} \quad (\forall x \in \mathfrak{g}), \qquad [P', P'] = 0.$$

Using the above lemma, one readily verifies that these conditions are equivalent to

$$[\rho_x, P] = 0 \quad (\forall x \in \mathfrak{g}), \qquad [P, P] = \rho_\phi,$$

that is, P is a quasi-Poisson $(\mathfrak{g}, s)$-tensor. □

Applying the above theorem to $M_G(\Sigma, V)$, we get a Poisson tensor P' on $M_G(\Sigma, V)$ with the property that, if we equip G with the Poisson Lie group structure (1.8), then the action of the product Poisson Lie group G^V on $M_G(\Sigma, V)$ is a Poisson action.

Let us now exhibit an explicit expression of P'. We take a basis (x_i) of $\mathfrak{g}$ and assume

$$a = a^{ij} x_i \otimes x_j = \frac{1}{2} a^{ij} x_i \wedge x_j, \quad s = s^{ij} x_i \otimes x_j$$

with $a^{ij} = -a^{ji}$, $s^{ij} = s^{ji}$. Let Γ be a skeleton of (Σ, V), then we have $\rho_a = \sum_{v \in V} \rho_a^v$, where $\rho_a^v = \frac{1}{2} \sum_{\mathbf{a},\mathbf{b} \in \widehat{E}_\Gamma(v)} a^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{b})}$. So

$$\begin{aligned}
P' = P + \rho_a &= \frac{1}{2} \sum_{v \in V} \sum_{\mathbf{a}<\mathbf{b}} s^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{b})} + \frac{1}{2} \sum_{v \in V} \sum_{\mathbf{a},\mathbf{b}} a^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{b})} \\
&= \sum_{v \in V} \left(\frac{1}{2} \sum_{\mathbf{a}<\mathbf{b}} s^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{b})} + \sum_{\mathbf{a}<\mathbf{b}} a^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{b})} + \frac{1}{2} \sum_{\mathbf{a}} a^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{a})} \right) \\
&= \sum_{v \in V} \left(\sum_{\mathbf{a}<\mathbf{b}} r^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{b})} + \frac{1}{2} \sum_{\mathbf{a}} r^{ij} x_i^{(\mathbf{a})} \wedge x_j^{(\mathbf{a})} \right).
\end{aligned}$$

Here in each summation $\mathbf{a}$ and $\mathbf{b}$ are in $\widehat{E}_\Gamma(v)$.

The Poisson structure P' is was found by Fock and Rosly [24]. Notice that P' also reduces to the Atiyah-Bott Poisson structure on the quotient $X_G(\Sigma)$, because ρ_a reduces to zero.

1.4 Lie group-valued moment maps and cross-sections

1.4.1 Lie group-valued moment maps

Recall that in the classical settings of symplectic or Poisson geometry, if (M, P) is a Poisson manifold acted upon by a Lie group G such that the P is invariant, then an equivariant map $\mu : M \to \mathfrak{g}^*$ is called a *moment map* if

$$P^\sharp(\mathsf{d}\,\langle \mu, x\rangle) = \rho_x, \quad \forall x \in \mathfrak{g}.$$

Here ρ_x is the fundamental vector field of x, and the vector bundle morphism $P^\sharp : \mathsf{T}^*M \to \mathsf{T}M$ is given by $P^\sharp(\alpha) = P(\alpha, \cdot)$ for any 1-form α. Equivalently, μ is a moment map if, for any $m \in M$, the image of $P_m \in \bigwedge^2 \mathsf{T}_m M \subset \mathsf{T}_m M \otimes \mathsf{T}_m M$ by the map

$$\mathsf{d}\mu \otimes \mathrm{id} : \mathsf{T}_m M \otimes \mathsf{T}_m M \longrightarrow \mathsf{T}_{\mu(m)}\mathfrak{g}^* \otimes \mathsf{T}_m M \cong \mathfrak{g}^* \otimes \mathsf{T}_m M$$

is the element in $\mathfrak{g}^* \otimes \mathsf{T}_m M$ corresponding to the linear map

$$\mathfrak{g} \to \mathsf{T}_m M, \quad x \mapsto \rho_x(m).$$

The quasi-Poisson version of moment map takes value in the Lie group G:

Definition 1.19. Let G and H be Lie groups and M be a $G \times H$-manifold. We fix $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and $t \in (S^2\mathfrak{h})^{\mathfrak{h}}$. Given a quasi-Poisson $(\mathfrak{g} \oplus \mathfrak{h}, s \oplus t)$-tensor P on M, a G-equivariant map $\mu : M \to G$ (where G acts on itself by conjugation) is called a *moment map* if for any $m \in M$, the image of $P_m \in \bigwedge^2 \mathsf{T}_m M$ under the linear map

$$\mu^*\theta^{\mathsf{L}} \otimes \mathrm{id} : \mathsf{T}_m M \otimes \mathsf{T}_m M \to \mathfrak{g} \otimes \mathsf{T}_m M$$

(recall that $\theta^{\mathsf{L}} \in \Omega^1(G, \mathfrak{g})$ denotes the left-invariant Maurer-Cartan form) is given by

$$\mu^*\theta^{\mathsf{L}} \otimes \mathrm{id}(P_m) = -\frac{1}{2}\left((1 + \mathrm{Ad}^{-1}_{\mu(m)}) \otimes \rho_\bullet(m)\right) s.$$

Here $\rho_\bullet(m) : \mathfrak{g} \to \mathsf{T}_m M,\ x \mapsto \rho_x(m)$ is the action of $\mathfrak{g}$ at m.

Our definition is slightly more flexible then the original one [3], where one only allows trivial H.

Remark 1.20. (1) Assume that (x_i) is a basis of $\mathfrak{g}$ and $s = s^{ij} x_i \otimes x_j$. Since $x^{\mathsf{R}}(u) = (\mathrm{Ad}_u^{-1} x)^{\mathsf{L}}(u)$ for any $u \in G$, the last condition is equivalent to the condition that the map

$$\mu_* \otimes \mathrm{id} : \mathsf{T}_m M \otimes \mathsf{T}_m M \longrightarrow \mathsf{T}_{\mu(m)} G \otimes \mathsf{T}_m M$$

sends P_m to

$$-\frac{1}{2} s^{ij} (x_i^{\mathsf{L}} + x_i^{\mathsf{R}})(\mu(m)) \otimes \rho_{x_j}(m).$$

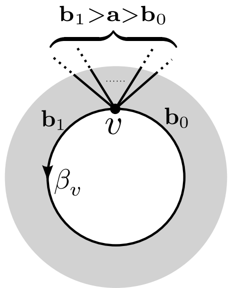
(2) Assume that $G = G_1 \times G_2$, where G_i is a Lie group with Lie algebra $\mathfrak{g}_i$, and $s = s_1 + s_2$ for $s_i \in (S^2\mathfrak{g}_i)^{\mathfrak{g}_i}$, then $\mu : M \to G$ is a moment map if and only if both factors $\mu_1 : M \to G_1$ and $\mu_2 : M \to G_2$ are moment maps.

Let Σ be a bordered surface with finitely many marked points $V \subset \partial\Sigma$. We now describe a moment map for the quasi-Poisson G^V-manifold $M_G(\Sigma, V)$. Let $V_0 \subset V$ be the set of marked points v with the property that the component of $\partial\Sigma$ containing v does not contain any other marked point

For each marked point $v \in V_0$, let β_v be the connected component of $\partial\Sigma$ containing v, oriented *against* the orientation induced from Σ, so that β_v represents an element in $\pi_1(\Sigma, V)$.

Proposition 1.21. *For each $v \in V_0$, the holonomy* $\mathrm{hol}_{\beta_v} : M_G(\Sigma, V) \to G$ *is a moment map.*

Proof. We take an oriented skeleton Γ of (Σ, V), such that β_v is an edge of Γ. So Γ looks as follows in a neighborhood of β_v. Here the half-edges $\mathbf{b_0}$ and $\mathbf{b_1}$ are, respectively, the minimal and maximal element in $\widehat{E}_\Gamma(v)$



By definition of the vector field $x^{(\mathbf{a})}$ on $M_G(\Sigma, V)$ (*c.f.* §1.3.1), we have $\mu^*\theta^{\mathsf{L}}(x^{(\mathbf{a})}) = 0$ for any $x \in \mathfrak{g}$ and any half-edge $\mathbf{a}$ other than $\mathbf{b}_0$ and $\mathbf{b}_1$, while

$$\mu^*\theta^{\mathsf{L}}(x^{(\mathbf{b_0})}(m)) = -\operatorname{Ad}_{\mu(m)}^{-1} x, \quad \mu^*\theta^{\mathsf{L}}(x^{(\mathbf{b_1})}(m)) = x, \quad \forall m \in M_G(\Sigma, V).$$

Using the expression (1.7) of the quasi-Poisson tensor P and the fact that $x^{(\mathbf{b}_0)}(m) = -(\operatorname{Ad}_{\mu(m)} x)^{(\mathbf{b_1})}(m)$, we obtain

$$\begin{aligned}
\mu^*\theta^{\mathsf{L}} \otimes \mathrm{id}(P_m) =& \mu^*\theta^{\mathsf{L}} \otimes \mathrm{id}\left(\frac{1}{2}\sum_{\mathbf{a}<\mathbf{b}} s^{ij} x_i^{(\mathbf{a})}(m) \wedge x_j^{(\mathbf{b})}(m)\right) \\
&= \frac{1}{2}\sum_{\mathbf{a}<\mathbf{b}} s^{ij}\left(\mu^*\theta^{\mathsf{L}}(x_i^{(\mathbf{a})}(m)) \otimes x_j^{(\mathbf{b})}(m) - \mu^*\theta^{\mathsf{L}}(x_j^{(\mathbf{b})}(m)) \otimes x_i^{(\mathbf{a})}(m)\right) \\
&= \frac{1}{2}\sum_{\mathbf{a}} s^{ij}\left(\mu^*\theta^{\mathsf{L}}(x_i^{(\mathbf{b}_0)}(m)) \otimes x_j^{(\mathbf{a})}(m) - \mu^*\theta^{\mathsf{L}}(x_j^{(\mathbf{b}_1)}(m)) \otimes x_i^{(\mathbf{a})}(m)\right) \\
&= -\frac{1}{2} s^{ij}(1 + \operatorname{Ad}_{\mu(m)}^{-1}) x_i \otimes \rho_{x_j}^v(m)
\end{aligned}$$

as required. Here $\mathbf{a}$ and $\mathbf{b}$ are taken over $\widehat{E}_\Gamma(v)$ in each summation, and ρ_x^v denotes the fundamental vector field of $x \in \mathfrak{g}$ induced by the action of the v^{th}-factor of G^V, which has the expression $\rho_x^v = \sum_{\mathbf{a}} x^{(\mathbf{a})}$. ☐

Remark 1.22. Li-Bland and Ševera [42] developed a more general notion of moment maps: for any marked point $v \in V$, not necessarily in V_0, they let β_v be the path in $\partial\Sigma$ which starts from the marked point v and walks against the induced orientation until the next marked point. They showed that $(\mathrm{hol}_{\beta_v})_{v\in V} : M_G(\Sigma, V) \to G^V$ is a *twisted moment map*. This notion will not be used in this thesis.

1.4.2 Quasi-Poisson cross-section theorem

The goal of this subsection is to prove a version of the *quasi-Poisson cross-section theorem* [6, 3], which states that for certain submanifold $U \subset G$ and any quasi-Poisson manifold M with moment map $\mu : M \to G$, the pre-image $L = \mu^{-1}(U)$ still has a quasi-Poisson tensor.

We shall first explain which U can occur here. Let us fix in this subsection closed subgroups $A, \widetilde{A} \subset G$ which have the same Lie algebra $\mathfrak{a} \subset \mathfrak{g}$, such that A is a normal subgroup of $\widetilde{A}$.

Definition 1.23. An open set $U \subset A$ is called a *cross-section* (with respect to the conjugation action of G) if

- U is invariant by the conjugation action of $\widetilde{A}$.
- $G.U = \{gug^{-1} \mid g \in G, u \in U\}$ is an open subset of G.
- The natural map
$$G \times_{\widetilde{A}} U \to G.U, \quad (g, u) \mapsto gug^{-1}$$
is a diffeomorphism onto the image. Here $G \times_A U$ denotes the quotient of $G \times U$ by the $\widetilde{A}$-action
$$(g, u) \overset{a \in \widetilde{A}}{\longmapsto} (ga^{-1}, aua^{-1}).$$

Example 1.24. (1) Let G be the split real form of a complex reductive group, $A = H$ be a maximal torus and $\widetilde{A} = N_G(H)$ be the normalizer. Then $U = H^{\mathrm{reg}}$, the set of loxodromic elements in H (*c.f.* §3.2.1 below for the definition), is a cross-section. One can effectively take $G = \mathrm{GL}_n\mathbb{R}$ or $\mathrm{SL}_n\mathbb{R}$ and let H (resp. H^{reg}) consist of diagonal matrices (resp. diagonal matrices without repeated eigenvalues).

One can modify this example a little by taking $\widetilde{A} = H$ and $U = \exp(\mathfrak{C}^{int}) \subset H^{\mathrm{reg}}$, where $\mathfrak{C} \subset \mathfrak{h}$ is a Weyl chamber and $\mathfrak{C}^{int}$ is its interior.

(2) When G is compact, a standard construction of cross-sections goes as follows. Let T be a maximal torus, $\mathfrak{t}$ be its Lie algebra and $\mathfrak{A} \subset \mathfrak{t}$ be a Weyl alcove. We choose a face σ of $\mathfrak{A}$, take $x \in \sigma$ and set $\widetilde{A} = A = \mathrm{Stab}_G(\exp(x)))$, which is known to be independent of the choice of x. Let $\mathfrak{A}_\sigma$ be the union of faces of $\mathfrak{A}$ whose closure contains σ. Then $U = \exp(\mathrm{Ad}_A \mathfrak{A}_\sigma)$ is a cross-section. In particular, if $\sigma = \mathfrak{A}^{int}$ then $U = \exp(\sigma)$.

A typical example for general σ is when $G = \mathrm{SU}(n)$ and A consists of block-diagonal matrices $u = \mathrm{diag}(u_1, \cdots, u_r)$, where each u_i is a unitary matrix with a certain size, while $U \subset A$ consists of those u's such that the sets of eigenvalues

$$\mathrm{eigen}(u_1),\ \mathrm{eigen}(u_2),\ \cdots,\ \mathrm{eigen}(u_r)$$

are arranged on the unit circle in a strictly clockwise manner.

Remark 1.25. The original use of cross-sections in moment map theory by Guillemin and Sternberg and its generalization to group-valued moment maps [6, 3] (*i.e.*, Theorem 1.28 below) are concerned with Example 1.24 (2); whereas for our applications in this thesis only Example 1.24 (1) will be involved.

We shall now make the additional assumption that there is a subspace $\mathfrak{b} \subset \mathfrak{g}$ which is invariant under the adjoint action of A, such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ as vector space. Note that in both of the above examples there is a standard choice of $\mathfrak{b}$, which is the orthogonal complement of $\mathfrak{a}$ with respect to a standard invariant scalar product on $\mathfrak{g}$.

Lemma 1.26. *If $U \subset A$ is a cross-section then $(\mathrm{Ad}_u - 1)|_{\mathfrak{b}}$ is invertible for any $u \in U$*

Proof. Assume by contradiction that $(\mathrm{Ad}_u - 1)|_{\mathfrak{b}}$ is not invertible, then there exists $y \in \mathfrak{b}$ such that $\mathrm{Ad}_u\, y = y$, hence $u \exp(ty) u^{-1} = \exp(ty)$ for any t. But $\exp(ty) \notin \widetilde{A}$ for t small enough, so this contradicts the fact that $\mathrm{Stab}_G(u) \subset \widetilde{A}$, which follows from the definition of cross-section. □

Lemma 1.27. *Let M be a G-manifold and $\mu : M \to G$ be an equivariant map (where G acts on itself by conjugation). If $U \subset A$ is a cross-section, then*

(1) $L = \mu^{-1}(U)$ is a smooth submanifold of M.

(2) There is an identification of quotient spaces

$$L/\widetilde{A} = \mu^{-1}(G.U)/G.$$

(3) We have a splitting of vector bundles

$$\mathsf{T}M|_L = \mathsf{T}L \oplus (L \times \mathfrak{b}). \tag{1.10}$$

Here the trivial bundle $L \times \mathfrak{b}$ is considered as a sub-bundle of $\mathsf{T}M|_L$ via the map

$$L \times \mathfrak{b} \longrightarrow \mathsf{T}M|_L, \quad (m, y) \longmapsto \rho_y(m). \tag{1.11}$$

Proof. (1) It is sufficient to show that μ is transversal to U. Let

$$\boldsymbol{v}_x = x^{\mathsf{L}} - x^{\mathsf{R}}$$

be the fundamental vector field of $x \in \mathfrak{g}$ induced by the conjugation action. Since $G.U$ is open, for any $u \in U$ we have

$$\mathsf{T}_u G = \mathsf{T}_u(G.U) = \{\boldsymbol{v}_x(u) \mid x \in \mathfrak{g}\} + \mathsf{T}_u U.$$

So the required transversality follows from equivariance of μ.

(2) It follows from the definition of cross-sections that, for any $u \in U$, the subset

$$\{g \in G \mid gug^{-1} \in U\} \subset G$$

equals $\widetilde{A}$. So equivariance of μ implies that every G-orbit in $\mu^{-1}(G.U)$ intersects L, and the intersection is a $\widetilde{A}$-orbit in L. This proves the required identification.

(3) Let us first show that the map (1.11) is an injection of vector bundles. Equivariance of μ implies that the composition of the map (1.11) with $\mu^*\theta^{\mathsf{L}} : \mathsf{T}_m M \to \mathfrak{g}$ sends (m, y) to

$$\mu^*\theta^{\mathsf{L}}(\rho_y(m)) = (1 - \mathrm{Ad}^{-1}_{\mu(m)})y \tag{1.12}$$

The injectivity results from Eq.(1.12) and Lemma 1.26.

Eq.(1.12) also implies that the sub-bundles $L \times \mathfrak{b}$ and TL are disjoint except at zero-section, because a tangent vector $w \in \mathsf{T}_m M$ ($m \in L$) belongs to $\mathsf{T}_m L$ if and only if $\mu^*\theta^{\mathsf{L}}(w) \in \mathfrak{a}$.

It remains to prove that any tangent vector $v \in \mathsf{T}_m M$ at a point $m \in L$ can be decomposed as $v = w + \rho_z(m)$ for some $z \in \mathfrak{b}$ and $w \in \mathsf{T}_m L$.

To this end, we can assume that $\mu^*\theta^{\mathsf{L}}(v) = x + y$ for some $x \in \mathfrak{a}, y \in \mathfrak{b}$. We take

$$z = (1 - \mathrm{Ad}^{-1}_{\mu(m)})^{-1} y$$

and put $w = v - \rho_z(m)$. We have

$$\mu^*\theta^{\mathsf{L}}(w) = \mu^*\theta^{\mathsf{L}}(v) - \mu^*\theta^{\mathsf{L}}(\rho_z(m)) = x + y - (1 - \mathrm{Ad}^{-1}_{\mu(m)})z = x \in \mathfrak{a},$$

which implies $w \in \mathfrak{a}$. So the required decomposition is achieved. □

Let us now show how cross-sections interplay with group-valued moment maps.

Theorem 1.28 (Quasi-Poisson Cross-Section Theorem, [3] Section 8)**.** *Let $U \subset A$ be a cross-section. Assume that $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ has a splitting*

$$s = s_{\mathfrak{a}} + s_{\mathfrak{b}} \quad (s_{\mathfrak{a}} \in \mathfrak{a} \otimes \mathfrak{a},\ s_{\mathfrak{b}} \in \mathfrak{b} \otimes \mathfrak{b}).$$

Let M be a $G \times H$-manifold and P be a quasi-Poisson $(\mathfrak{g} \oplus \mathfrak{h}, s \oplus t)$-tensor on M. If $\mu : M \to G$ is a moment map, then the submanifold $L = \mu^{-1}(U) \subset M$ satisfies

(1) The splitting (1.10) is $P|_L$-orthogonal, i.e., we have a decomposition of $P|_L \in C^\infty(\bigwedge^2 \mathsf{T}M|_L)$,

$$P|_L = P_L + P_L^\perp,$$

where $P_L \in C^\infty(\bigwedge^2 TL)$ and $P_L^\perp : L \to \bigwedge^2 \mathfrak{b}$. Moreover, $P_L^\perp$ has the expression

$$P_L^\perp(m) = -\frac{1}{2}\left(\frac{\mathrm{Ad}_{\mu(m)}+1}{\mathrm{Ad}_{\mu(m)}-1} \otimes \mathrm{id}\right) s_{\mathfrak{b}} \tag{1.13}$$

[5]*for any $m \in L$.*

(2) L is preserved by the action of $\widetilde{A}\times H$. (L, P_L) is a quasi-Poisson $(\mathfrak{a}\oplus\mathfrak{h}, s_{\mathfrak{a}}\oplus t)$-manifold and $\mu|_L : L \to A$ is a moment map.

(3) On the quotient

$$L/(\widetilde{A} \times H) \cong \mu^{-1}(G.U)/(G \times H)$$

(c.f. Lemma 1.27 (2)), the Poisson structure reduced from P_L coincides with the one reduced from the quasi-Poisson tensor P on $\mu^{-1}(G.U)$.

Proof. (1) Let $P_L^\perp$ be defined by Eq.(1.13) and put $P_L = P|_L - P_L^\perp$. We need to show that for any $m \in L$, the bivector $P_L(m) \in \bigwedge^2 \mathsf{T}_m M$ is contained in $\bigwedge^2 \mathsf{T}_m L$.

Eq.(1.12) implies that the image of $P_L^\perp(m)$ under the map $\mu^*\theta^{\mathsf{L}} \otimes \mathrm{id} : \mathsf{T}_m M \otimes \mathsf{T}_m M \to \mathfrak{g} \otimes \mathsf{T}_m M$ is

$$\begin{aligned}\mu^*\theta^{\mathsf{L}} \otimes \mathrm{id}(P_L^\perp(m)) &= -\frac{1}{2}\left((1 - \mathrm{Ad}_{\mu(m)}^{-1}) \otimes \rho_\bullet(m)\right)\left(\frac{\mathrm{Ad}_{\mu(m)}+1}{\mathrm{Ad}_{\mu(m)}-1} \otimes \mathrm{id}\right) s_{\mathfrak{b}} \\ &= -\frac{1}{2}\left((1 + \mathrm{Ad}_{\mu(m)}^{-1}) \otimes \rho_\bullet(m)\right) s_{\mathfrak{b}}.\end{aligned}$$

This equality and the definition of moment maps yield

$$\begin{aligned}\mu^*\theta^{\mathsf{L}} \otimes \mathrm{id}(P_L(m)) &= \mu^*\theta^{\mathsf{L}} \otimes \mathrm{id}(P|_L(m)) - \mu^*\theta^{\mathsf{L}} \otimes \mathrm{id}(P_L^\perp(m)) \\ &= -\frac{1}{2}\left((1 + \mathrm{Ad}_{\mu(m)}^{-1}) \otimes \rho_\bullet(m)\right) s_{\mathfrak{a}}.\end{aligned} \tag{1.14}$$

Therefore, $P_L(m)$ belongs to $\bigwedge^2 \mathsf{T}_m L$, as required.

(2) The fact that L is preserved by $\widetilde{A} \times L$ follows from equivariance of μ.

Let $\{f_1, f_2\}_L = P_L(\mathsf{d}f_1, \mathsf{d}f_2)$ $(f_1, f_2 \in C^\infty(L))$ be the bracket associated with P_L. To prove that P_L is quasi-Poisson, we need to verify that for any $m \in L$ and $f_1, f_2, f_3 \in C^\infty(L)$, we have the quasi-Jacobi identity

$$-2\{\{f_1, f_2\}_L, f_3\}_L(m) = \rho_{\phi_{\mathfrak{a}}}(m)(\mathsf{d}f_1, \mathsf{d}f_2, \mathsf{d}f_3). \tag{1.15}$$

Since $\mathsf{T}M|_L$ has the splitting (1.10), there is a neighborhood $X \subset L$ of m and a neighborhood Y of $0 \in \mathfrak{b}$ such that $(m, y) \mapsto \exp(y).m$ is a is a diffeomorphism from $X \times Y$ to a neighborhood of m in M.

Let $\widetilde{f_i} \in C^\infty(X \times Y)$ be invariant in vertical directions and restricts to f_i on X. On one hand, the splitting of $P|_L$ implies that

$$\{\{\widetilde{f}_1, \widetilde{f}_2\}, \widetilde{f}_3\}(m) = \{\{f_1, f_2\}_L, f_3\}_L(m)$$

5. The fact that the right-hand side of (1.13) belongs to $\bigwedge^2 \mathfrak{b}$ results from $\left(\mathrm{Ad}_{\mu(m)} \otimes \mathrm{Ad}_{\mu(m)}\right) s_{\mathfrak{b}} = s_{\mathfrak{b}}$.

for any $m \in L$; on the other hand, let $\phi_{\mathfrak{a}}, \phi_{\mathfrak{b}} \in \bigwedge^3 \mathfrak{g}$ denote the canonical trivectors associated to $s_{\mathfrak{a}}$ and $\mathfrak{b}$, respectively, then we have

$$-2\{\{\widetilde{f}_1, \widetilde{f}_2\}, \widetilde{f}_3\} = \rho_\phi(\mathsf{d}\widetilde{f}_1, \mathsf{d}\widetilde{f}_2, \mathsf{d}\widetilde{f}_3) = \rho_{\phi_{\mathfrak{a}}+\phi_{\mathfrak{b}}}(\mathsf{d}\widetilde{f}_1, \mathsf{d}\widetilde{f}_2, \mathsf{d}\widetilde{f}_3) = \rho_{\phi_{\mathfrak{a}}}(\mathsf{d}f_1, \mathsf{d}f_2, \mathsf{d}f_3),$$

because P is quasi-Poisson. Therefore the required Eq.(1.15) is proved.

The fact that $\mu|_L$ is a moment map results from Eq.(1.14).

(3) Part (1) implies the for any $m \in L$ and $f_1, f_2 \in C^\infty(\mu^{-1}(G.U))$ we have

$$\{f_1, f_2\}|_L(m) = \{f_1|_L, f_2|_L\}_L(m) - \frac{1}{2}\rho_{\omega(m)}(m)(\mathsf{d}f_1, \mathsf{d}f_2),$$

where the map $\omega : L \to \bigwedge^2 \mathfrak{b}$ is defined by

$$\omega(m) = \left(\frac{\mathrm{Ad}_{\mu(m)} + 1}{\mathrm{Ad}_{\mu(m)} - 1} \otimes \mathrm{id}\right) s_{\mathfrak{b}}.$$

Thus if $f_1, f_2 \in C^\infty(\mu^{-1}(G.U))^{G\times H}$ then $\{f_1, f_2\}|_L = \{f_1|_L, f_2|_L\}_L$. This is exactly what needs to be proved. $\square$

Applying Theorem 1.28 to the moment map in Proposition 1.21, we get the following conclusions. Here β_v and V_0 are defined in the paragraph preceding Proposition 1.21.

Corollary 1.29. *We fix $A, \tilde{A} \subset G$, $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ and $s = s_{\mathfrak{a}} + s_{\mathfrak{b}} \in (S^2\mathfrak{g})^{\mathfrak{g}}$ as above and let $U \subset A$ be a cross-section. Let P be the canonical quasi-Poisson $(\mathfrak{g}^V, s^{(V)})$-tensor on $M_G(\Sigma, V)$ and $\{\cdot,\cdot\}$ be the corresponding quasi-Poisson bracket.*

Given any subset V_1 of V_0, the $\widetilde{A}^{V_1} \times G^{V\setminus V_1}$-manifold

$$L = \bigcap_{v\in V_1} \mathrm{hol}_{\beta_v}^{-1}(U) \subset M_G(\Sigma, V)$$

carries a canonical quasi-Poisson $(\mathfrak{a}^{V_1} \oplus \mathfrak{g}^{V\setminus V_1}, s_{\mathfrak{a}}^{(V_1)} \oplus s^{(V\setminus V_1)})$-tensor P_L, which has the following properties

(1) The holonomies

$$\mathrm{hol}_{\beta_v}|_L : L \to A \quad (v \in V_1), \quad \mathrm{hol}_{\beta_v}|_L : L \to G \quad (v \in V \setminus V_1)$$

are moment maps.

(2) Let $\{\cdot,\cdot\}_L$ be the quasi-Poisson bracket on $C^\infty(L)$ associated with P_L, then for any $f_1, f_2 \in C^\infty(M_G(\Sigma, V))$ and $m \in L$ we have

$$\{f_1|_L, f_2|_L\}_L(m) = \{f_1, f_2\}|_L(m) + \frac{1}{2}\sum_{v\in V_1} \rho^v_{\omega_v(m)}(m)(\mathsf{d}f_1, \mathsf{d}f_2),$$

where the map $\omega_v : L \to \bigwedge^2 \mathfrak{b}$ is given by

$$\omega_v(m) = \left(\frac{\mathrm{Ad}_{\mathrm{hol}_{\beta_v}(m)} + 1}{\mathrm{Ad}_{\mathrm{hol}_{\beta_v}(m)} - 1} \otimes \mathrm{id}\right) s_{\mathfrak{b}}.$$

(3) We have an identification of quotient spaces

$$L/(\widetilde{A}^{V_1} \times G^{V\setminus V_1}) \cong \left(\bigcap_{v\in V} \mathrm{hol}_{\beta_v}^{-1}(G.U)\right)/G^V.$$

Moreover, the Poisson structure on the former quotient reduced from P_L coincides with the one on the latter reduced from P.

Chapter 2

Quasi-Poisson brackets of spin networks

The goal of this chapter is to present a formula independently found by G. Massuyeau and V. Turaev [44], the present author [49], and D. Li-Bland and P. Ševera [42], which can be seen as a generalization of Goldman's formula to the context of quasi-Poisson lattice gauge theory. The statement and proof here are due to Li-Bland and Ševera [42]. We first give in §2.1 the formula and its generalization to cross-sections without proof. Then we apply the formula to simple instances in §2.2, revealing some algebra structure analogue to Goldman's Lie algebra. Finally we give the proof of the formula in §2.3.

2.1 The quasi-Poisson bracket formula

2.1.1 Spin networks

Spin networks [9] were first introduced by Penrose as certain functions on the moduli space of flat connections $X_G(\Sigma)$. In this subsection we introduce an extension to $M_G(\Sigma, V)$ of this notion.

Definition 2.1. Let Σ be a bordered surface and $V \subset \partial\Sigma$ be finitely many marked points. A *graph diagram* on (Σ, V) is an oriented immersed graph [1] Γ on Σ with edges E_Γ and vertices V_Γ, such that a particular subset of vertices V_Γ^∂ contained in V is specified. By homotopy of Γ we mean homotopy fixing V_Γ^∂. We call $V_\Gamma^{int} = V_\Gamma \setminus V_\Gamma^\partial$ *interior vertices* (since we can move them to the interior of Σ by homotopy), and call V_Γ^∂ *boundary vertices*.

A *spin network* on (Σ, V) is a pair $[\Gamma, f]$, where Γ is a graph diagram on (Σ, V) and f is a function on G^{E_Γ}, such that f is invariant by the $G^{V_\Gamma^{int}}$. Here G^{E_Γ} is identified with $\mathrm{Hom}(\pi_1(\Gamma, V_\Gamma), G)$, on which G^{V_Γ} acts in a natural way. Such functions are called *admissible* for Γ.

A spin network $[\Gamma, f]$ gives rise to a function on $M_G(\Sigma, V)$ as follows. The groupoid homomorphism $\pi_1(\Gamma, V_\Gamma) \to \pi_1(\Sigma, V \cup V_\Gamma^{int})$ gives rise to a $G^{V_\Gamma^{int}}$-equivariant map

$$\mathrm{Hom}(\pi_1(\Sigma, V \cup V_\Gamma^{int}), G) \to \mathrm{Hom}(\pi_1(\Gamma, V_\Gamma), G) = G^{E_\Gamma}.$$

The $G^{V_\Gamma^{int}}$-invariant function f on the target lifts to a $G^{V_\Gamma^{int}}$-invariant function on the domain, which descends to the quotient

$$\mathrm{Hom}(\pi_1(\Sigma, V \cup V_\Gamma^{int}), G)/G^{V_\Gamma^{int}} = \mathrm{Hom}(\pi_1(\Sigma, V), G) = M_G(\Sigma, V).$$

1. That is, a smooth map $\psi : \Gamma \to \Sigma$, where Γ is an oriented graph, such that ψ is injective on the set of vertices, and images of different edges are transverse. We just call the image $\psi(\Gamma)$ an oriented immersed graph if there is no danger of confusion.

We denote this function on $M_G(\Sigma, V)$ also by $[\Gamma, f]$ if there is no danger of confusion.

Remark 2.2. (1) If Γ has no boundary vertex then the function $[\Gamma, f]$ is invariant by G^V, hence is a function on $X_G(\Sigma)$. This covers the original definition of spin networks on $X_G(\Sigma)$ [2].

(2) Let Γ be a skeleton, then every function on $M_G(\Sigma)$ can be considered as a spin network with graph diagram Γ.

Example 2.3. (1) Suppose that Γ has a single interior vertex, no boundary vertex, and a single edge γ (which must be a loop). Then admissible functions are functions on G which are invariant under conjugation. A spin network $[\Gamma, f] \in C^\infty(X_G(\Sigma))$ can be described as follows: let $[\gamma]$ be the conjugacy class in $\pi_1(\Sigma)$ carried by the loop γ, then the value of $[\Gamma, f]$ at $[\rho] \in X_G(\Sigma)$, where $\rho : \pi_1(\Sigma) \to G$ is a representation, is the value of f at $\rho([\gamma]) \subset G$.

(2) If Γ consists of two boundary vertices and a single edge e joining them. Then f can be any function on G and $[\Gamma, f] = f(\mathrm{hol}_e)$.

(3) If Γ is a n-pod, *i.e.*, n boundary vertices joined by edges (e_i) to a single interior vertex, then admissible functions are $f \in C^\infty(G^n)^G$ (where G acts diagonally on G^n by left-multiplication) and

$$[\Gamma, f] = f(1, \mathrm{hol}_{e_1^{-1}e_2}, \cdots, \mathrm{hol}_{e_1^{-1}e_n}).$$

We call e_i a *leg* of the n-pod, and call each boundary vertex a *foot*. Spin networks associated to tripods will be extensively used in the next chapter.

Remark 2.4. When it concerns cross-sections $L \subset M_G(\Sigma, V)$ (*c.f.* §1.4.2), we shall make a slight generalization of the definition of spin networks. Namely, assuming that $L = \bigcap_{v \in V_1} \mathrm{hol}_{\beta_v}^{-1}(U)$ (the notations here being the same as in Corollary 1.29) and that for some $v \in V_1$ the boundary loop β_v is an edge of Γ, then we shall allow the admissible function f in the spin network $[\Gamma, f]$ to be defined only on the subset $\{(g_e)_{e \in E_\Gamma} \mid g_{\beta_v} \in U\} \subset G^{E_\Gamma}$.

2.1.2 Quasi-Poisson brackets of spin networks

Definition 2.5. Two graph diagrams Γ and Γ' on (Σ, V) are said to be *transverse* if any interior intersection point $q \in \Gamma \cap \Gamma' \setminus \partial\Sigma$ is a transversal intersection point of (the interiors of) some edges $e \in E_\Gamma$ and $e' \in E_{\Gamma'}$, and moreover if $e \in E_\Gamma$ and $e' \in E_{\Gamma'}$ share a boundary vertex $v \in V$ then their tangent directions at v are distinct.

In this subsection we give the main result of this chapter, a formula (Theorem 2.6 below) which computes the quasi-Poisson bracket of two spin networks $[\Gamma, f]$ and $[\Gamma', f']$ when the graph diagrams are transverse. The result is a sum of new spin networks, whose graph diagrams are divided into two types:

- The graph diagram $\Gamma \cup \Gamma'$. The vertex (resp. edge) set of $\Gamma \cup \Gamma'$ is the union of the vertex (resp. edge) sets of Γ and Γ'.
- For each $q \in \Gamma \cap \Gamma' \setminus \partial\Sigma$, a graph diagram $\Gamma \cup_q \Gamma'$. By definition, $\Gamma \cup_q \Gamma'$ is obtained from $\Gamma \cup \Gamma'$ by adding the point q as an interior vertex.

We shall now define the admissible functions which will occur.

For each $q \in \Gamma \cap \Gamma' \setminus \partial\Sigma$, suppose $q \in e \cap e'$ for $e \in E_\Gamma$ and $e' \in E_{\Gamma'}$. Then we have $e = e_2 e_1$ and $e' = e'_2 e'_1$, where $e_1, e_2 \in E_{\Gamma \cup_q \Gamma'}$ are respectively the first half of e up to q

2. The original definition is slightly more restrictive: it requires that f comes from linear representations of G.

and the second half starting from q. We let $D_q(f,f')$ be the following function defined on $G^{E_{\Gamma\cup_q\Gamma'}}$:

$$\begin{aligned} & D_q(f,f')((g_e)_{e\in E_{\Gamma\cup_q\Gamma'}}) \\ :=& \frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} s^{ij} f\Big((g_d)_{d\in E_\Gamma\setminus\{e\}}, g_{e_2}\exp(\epsilon x_i)g_{e_1}\Big) f'\Big((g_d)_{d\in E_{\Gamma'}\setminus\{e'\}}, g_{e'_2}\exp(\delta x_j)g_{e'_1}\Big). \end{aligned} \tag{2.1}$$

Here $((g_d)_{d\in E_\Gamma\setminus\{e\}}, g_{e_2}\exp(\epsilon x_i)g_{e_1})$ is the point in G^{E_Γ} whose t^{th} component is g_t if $t\neq e$ and e^{th} component is $g_{e'_2}\exp(\epsilon x_i)g_{e_1}$.

We need to show that $D_q(f,f')$ is invariant by the action of $G^{V^{int}_{\Gamma\cup_q\Gamma'}}$. It follows from invariance of f and f' that $D_q(f,f')$ is invariant by any G-factor other than the q^{th}, while the invariance by the q^{th} factor follows from $\mathfrak{g}$-invariance of $s\in\mathfrak{g}\otimes\mathfrak{g}$.

For each pair of half-edges $\mathbf{a}\in\widehat{E}_\Gamma(v)$ and $\mathbf{a}'\in\widehat{E}_{\Gamma'}(v)$ sharing a boundary vertex $v\in V$, we define similarly a function $D_{\mathbf{a},\mathbf{a}'}(f,f')$ admissible to $\Gamma\cup\Gamma'$. Let e and e' be respectively the edges containing $\mathbf{a}$ and $\mathbf{a}'$. If both $\mathbf{a}$ and $\mathbf{a}'$ run into v, then

$$\begin{aligned} & D_{\mathbf{a},\mathbf{a}'}(f,f')((g_e)_{e\in E_{\Gamma\cup\Gamma'}}) \\ :=& \frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} s^{ij} f\Big((g_d)_{d\in E_\Gamma-\{e\}}, \exp(\epsilon x_i)g_e\Big) f'\Big((g_d)_{d\in E_{\Gamma'}-\{e'\}}, \exp(\delta x_j)g_{e'}\Big), \end{aligned} \tag{2.2}$$

whereas, if $\mathbf{a}$ and/or $\mathbf{a}'$ come out of v, we replace $\exp(\epsilon x_i)g_e$ by $g_e\exp(-\epsilon x_i)$ and/or replace $\exp(\delta x_j)g_{e'}$ by $g_{e'}\exp(-\delta x_j)$.

A neater expression of $D_{\mathbf{a},\mathbf{a}'}(f,f')$ is as follows. We define a symmetric 2-tensor field $s^{(\mathbf{a},\mathbf{a})}$ on $G^{E_\Gamma\cup E_{\Gamma'}}$ by

$$s^{(\mathbf{a},\mathbf{a})} = s^{ij}x_i^{(\mathbf{a})}\otimes x_j^{(\mathbf{a}')}$$

(*c.f.* Definition 1.11 for the notations), then we have

$$D_{\mathbf{a},\mathbf{a}'}(f,f') = s^{(\mathbf{a},\mathbf{a}')}(f,f').$$

Here f and f' are considered as functions on $G^{E_\Gamma\cup E_{\Gamma'}}$ by lifting.

Theorem 2.6. *Let $[\Gamma,f]$ and $[\Gamma',f']$ be transverse spin networks on (Σ,V). Then their quasi-Poisson bracket is*

$$\{[\Gamma,f],[\Gamma',f']\} = \sum_q \varepsilon_q(\Gamma,\Gamma')[\Gamma\cup_q\Gamma, D_q(f,f')] + \frac{1}{2}\sum_{v\in V}\sum_{\mathbf{a},\mathbf{a}'}\varepsilon(\mathbf{a},\mathbf{a}')[\Gamma\cup\Gamma', D_{\mathbf{a},\mathbf{a}'}(f,f')].$$

Here q runs over intersection points $q\in\Gamma\cap\Gamma'\setminus\partial\Sigma$, while $\mathbf{a},\mathbf{a}'$ run over half-edges $\mathbf{a}\in\widehat{E}_\Gamma(v)$, $\mathbf{a}'\in\widehat{E}_{\Gamma'}(v)$ for each $v\in V$. $\varepsilon_q(\Gamma,\Gamma')$ (resp. $\varepsilon(\mathbf{a},\mathbf{a}')$)$=\pm1$ is the oriented intersection number of Γ and Γ' at q (resp. $\mathbf{a}$ and $\mathbf{a}'$ at v).

A proof will be given in §2.3.

Remark 2.7. The theorem implies that the right-hand side of the above formula is invariant when Γ and Γ' undergo (boundary-vertex-fixing) homotopies. A manifestation of this invariance is that the *algebraic intersection number* $i(\Gamma,\Gamma')$ of Γ and Γ', defined by

$$i(\Gamma,\Gamma') = \sum_{q\in\Gamma\cap\Gamma'\setminus\partial\Sigma}\varepsilon_q(\Gamma,\Gamma') + \frac{1}{2}\sum_{v\in V}\sum_{\mathbf{a},\mathbf{a}'}\varepsilon(\mathbf{a},\mathbf{a}'),$$

is invariant under homotopy, *c.f.* Lemma 2.14 below.

Corollary 1.29 enables us to extend Theorem 2.6 immediately to cross-sections:

Corollary 2.8. *Let the notations be as in Corollary 1.29. Then we have*

$$\{[\Gamma, f], [\Gamma', f']\}_L = \sum_q \varepsilon_q(\Gamma, \Gamma')[\Gamma \cup_q \Gamma, D_q(f, f')] + \frac{1}{2}\sum_{v\in V}\sum_{\mathbf{a},\mathbf{a}'} \varepsilon(\mathbf{a}, \mathbf{a}')[\Gamma \cup \Gamma', D_{\mathbf{a},\mathbf{a}'}(f, f')]$$
$$+ \frac{1}{2}\sum_{v\in V}\sum_{\mathbf{a},\mathbf{a}'}[\Gamma \cup \Gamma' \cup \beta_v, \widetilde{D}_{\mathbf{a},\mathbf{a}'}(f, f')].$$

Here the spin networks are considered as functions on the cross-section $L \subset M_G(\Sigma, V)$. The admissible function $\widetilde{D}_{\mathbf{a},\mathbf{a}'}(f, f') \in C^\infty(G^{E_\Gamma \cup E_{\Gamma'}} \times U)$ for the graph $\Gamma \cup \Gamma' \cup \beta_v$ (see Remark 2.4) is given by

$$\widetilde{D}_{\mathbf{a},\mathbf{a}'}(f, f')((g_e)_{E_\Gamma}, (g_{e'})_{\Gamma'}, u) = \left(\left(\frac{\mathrm{Ad}_u + 1}{\mathrm{Ad}_u - 1} \otimes \mathrm{id}\right) s_{\mathfrak{b}}\right)^{(\mathbf{a},\mathbf{a}')} (f, f').$$

Remark 2.9. For spin networks in Example 2.3 (1), Theorem 2.6 is Goldman's formula [27]. The extension of Goldman's formula to spin networks without boundary vertices (*c.f.* Remark 2.2 (1)) seems well known. Thus we have well understood Poisson brackets of functions on $X_G(\Sigma)$.

Recently, three independent works emerged, all of them extending the above well known formulas to those functions on $M_G(\Sigma, V)$ which do not descent to $X_G(\Sigma)$: Massuyeau and Turaev [44] essentially proved Theorem 2.6 for spin networks from Example 2.3 (2) when $G = \mathrm{GL}_n\mathbb{R}$ and $f \in C^\infty(G)$ is a matrix entry function (*i.e.*, Corollary 2.11 below); the present author proved the theorem in [49] for the same type of spin networks and for any G and f; whereas Li-Bland and Ševera [42] prove it in full generality. The proof that we give in §2.3 follows closely the proof of Li-Bland and Ševera.

However, as pointed out in [42], since the quasi-Poisson structure on $M_G(\Sigma, V)$ is twist-equivalent to the Fock-Rosly Poisson structure (see §3.2.6), all these results are in some sense rediscoveries of a formula of Fock-Rosly [24] which computes Poisson brackets of spin networks under their Poisson structure. See also [8] for a more detailed exposition of the Fock-Rosly formula).

2.2 Algebras of Goldman, Massuyeau-Turaev and Labourie

In this section we use Theorem 2.6 to do some concrete computations in simple examples. The results could be of interest in their own right.

2.2.1 Goldman's Lie algebra

We assume that $G = \mathrm{GL}_n\mathbb{R}$, $\mathfrak{g} = \mathfrak{gl}_n\mathbb{R}$. Let $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$ be the dual of the standard invariant scalar product $(x \mid y) = \mathrm{Tr}(xy)$ on $\mathfrak{g}$. Denote by E_{kl} the $n \times n$ matrix whose (k, l)-entry is 1 and other entries are 0. We have the expression

$$s = \sum_{1\leq k,l\leq n} E_{kl} \otimes E_{lk}.$$

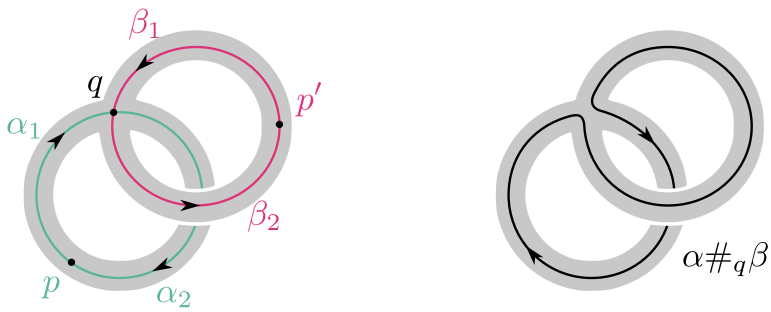
Let α be an oriented loop in the interior of Σ. We add a base point p to α so that

$$\mathrm{Tr}_\alpha := [\alpha, \mathrm{Tr}] \in C^\infty(X_G(\Sigma))$$

is a spin network as in Example 2.3(1), where $\mathrm{Tr} \in C^\infty(G)^G$ is the trace function. Take another loop β with base point p' such that α and β intersect transversally. Theorem 2.6 says that the Atiyah-Bott Poisson bracket of Tr_α and Tr_β is

$$\{\mathrm{Tr}_\alpha, \mathrm{Tr}_\beta\} = \sum_{q\in\alpha\cap\beta} \varepsilon_q(\alpha,\beta)[\alpha \cup_q \beta, D_q(\mathrm{Tr},\mathrm{Tr})],$$

where $\alpha \cup_q \beta$ is the spin network with three vertices p, p', q and four edges $\alpha_1, \alpha_2, \beta_1, \beta_2$, as shown in the picture below.



$D_q(\mathrm{Tr},\mathrm{Tr})$ is a function on G^4 given by

$$\begin{aligned} D_q(\mathrm{Tr},\mathrm{Tr})(g_{\alpha_1}, g_{\alpha_2}, g_{\beta_1}, g_{\beta_2}) &= \frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} \sum_{k,l} \mathrm{Tr}\left(g_{\alpha_2}\exp(\epsilon E_{kl})g_{\alpha_1}\right)\mathrm{Tr}\left(g_{\beta_2}\exp(\delta E_{lk})g_{\beta_1}\right) \\ &= \sum_{k,l} \mathrm{Tr}(g_{\alpha_2}E_{kl}g_{\alpha_1})\,\mathrm{Tr}(g_{\beta_2}E_{lk}g_{\beta_1}) \\ &= \sum_{k,l} (g_\alpha)_{lk}(g_\beta)_{kl} = \mathrm{Tr}(g_\alpha g_\beta). \end{aligned}$$

It is easy to see that if we let $\alpha\#_q\beta$ be the loop which runs α and β successively (see the above picture), then $[\alpha\cup_q\beta, D_q(\mathrm{Tr},\mathrm{Tr})] = \mathrm{Tr}_{\alpha\#_q\beta}$. Thus we get

Corollary 2.10 (Goldman [27])**.** *The Poisson bracket of* $\mathrm{Tr}_\alpha, \mathrm{Tr}_\beta \in C^\infty(X_G(\Sigma))$ *is*

$$\{\mathrm{Tr}_\alpha, \mathrm{Tr}_\beta\} = \sum_{q\in\alpha\cap\beta} \varepsilon_q(\alpha,\beta)\,\mathrm{Tr}_{\alpha\#_q\beta}\,.$$

This formula suggests us to define a Lie bracket $[\alpha,\beta] := \sum_{q\in\alpha\cap\beta}\varepsilon_q(\alpha,\beta)\alpha\#_q\beta$ on the vector space $\mathcal{G}$ generated by free homotopy classes of loops on Σ, so that the embedding $\mathcal{G} \to C^\infty(X_G(\Sigma))$, $\alpha \mapsto \mathrm{Tr}_\alpha$ preserves brackets. It is proved rigorously in [27] that $[\cdot,\cdot]$ is indeed a Lie bracket. $(\mathcal{G}, [\cdot,\cdot])$ is called the *Goldman Lie algebra* and is widely used in the study mapping class groups of surfaces. See *e.g.* [35].

2.2.2 Massuyeau-Turaev's quasi-Poisson algebra

We now exhibit the quasi-Poisson counterpart of Goldman's Lie algebra. Let α be an orineted path joining two marked points $v, v' \in V$ and let

$$f_{ij}(g) = g_{ij}$$

be the function which assigns to each matrix $g \in G$ its (i,j)-entry. Consider the spin network

$$\alpha_{ij} := [\alpha, f_{ij}] = (\mathrm{hol}_\alpha)_{ij}$$

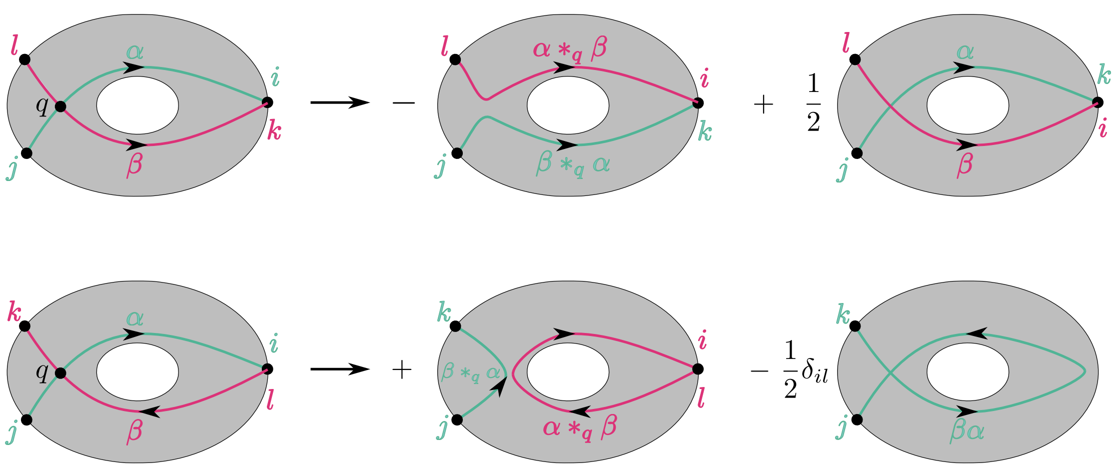
as in Example 2.3(2). The quasi-Poisson bracket $\{\alpha_{ji}, \beta_{lk}\}$ of two such spin networks can be computed in a similar way. The result is the following corollary, where the notations are defined as follows:

- $\beta *_q \alpha$ is the path which starts from the source of α, runs α up to q, then switches to β and ends at the target of β;
- $\alpha^\wedge$ and $\alpha^\vee$ are, respectively, the first and second half of α;
- $\varepsilon(\alpha^\vee, \beta^\wedge) = \pm 1$ is the oriented intersection number of the two half-edges at their common vertex (defined to be 0 if there they do not share a vertex);
- $\delta_{ij} = 0, 1$ is the Kronecker delta.

Corollary 2.11 (Massuyeau-Turaev [44], Nie [49] [3]). *The quasi-Poisson bracket of $\alpha_{ij}, \beta_{kl} \in C^\infty(M_G(\Sigma, V))$ is given by*

$$\{\alpha_{ij}, \beta_{kl}\} = \sum_{q \in \alpha \cap \beta \backslash \partial\Sigma} \varepsilon_q(\alpha, \beta)(\beta *_q \alpha)_{kj} \cdot (\alpha *_q \beta)_{il}$$
$$+ \frac{1}{2}\Big(\varepsilon(\alpha^\wedge, \beta^\wedge)\alpha_{il} \cdot \beta_{kj} + \varepsilon(\alpha^\vee, \beta^\vee)\alpha_{kj} \cdot \beta_{il} + \delta_{jk}\varepsilon(\alpha^\wedge, \beta^\vee)(\alpha\beta)_{il} + \delta_{il}\varepsilon(\alpha^\vee, \beta^\wedge)(\beta\alpha)_{kj}\Big).$$

Two typical examples are shown in the picture below. Here we compute quasi-Poisson brackets of α_{ij} and β_{kl} as pictured on the left, and the resulted is pictured on the right.



Corollary 2.11 can be interpreted in a similar way as Goldman's algebra. One should think of the symbol α_{ij} as a "labelled path", *i.e.*, a nontrivial element in $\pi_1(\Sigma, V)$ together with indices j, i attached to its source and target, respectively. Let $\mathcal{A}_n(\Sigma, V)$ be the commutative associative algebra generated by all labelled paths modulo the obvious relations

$$\sum_j \beta_{ij} \cdot \alpha_{jk} = (\beta\alpha)_{ik} \quad \text{if } \alpha \neq \beta, \qquad \sum_j (\alpha^{-1})_{ij}\alpha_{jk} = \delta_{ik}.$$

Equip $\mathcal{A}_n(\Sigma, V)$ with the obvious $(\mathfrak{gl}_n\mathbb{R})^V$-action and the bracket $\{\cdot, \cdot\}$ given by the above corollary, then $\mathcal{A}_n(\Sigma, V)$ is a quasi-Poisson $((\mathfrak{gl}_n\mathbb{R})^V, s^{(V)})$-algebra [4]. This fact can be proved by straightforward but lengthy verifications of quasi-Jacobi identities, in a similar

3. Additional technical assumptions are made in [44] and [49]: in the former paper it is assumed that V is a single point, while in the latter we assumed that V has one point on each connected component of $\partial\Sigma$.

4. A *quasi-Poisson* $(\mathfrak{g}, s)$*-algebra* is just a commutative algebra equipped with a $\mathfrak{g}$-action and a bracket $\{\cdot, \cdot\}$, satisfying the same conditions as the algebra of functions of a quasi-Poisson $\mathfrak{g}$-manifold satisfies. See Definition 4.14 below for a detailed definition.

way as Goldman did for his Lie algebra. Probably there is another proof using Van den Bergh's theory of double brackets, as Massuyeau and Turaev did in [44] [5].

Remark 2.12. (1) By definition, the natural map $\mathcal{A}_n(\Sigma, V) \to C^\infty(M_G(\Sigma, V))$ is a morphism of quasi-Poisson algebras. We conjecture that this morphism is injective.

(2) When defining $\mathcal{A}_n(\Sigma, V)$ and proving it to be quasi-Poisson, we do not need $V \subset \partial\Sigma$ to be finite – one can even take $V = \partial\Sigma$. But in this case more foundational works are needed in order to define a quasi-Poisson structure on the infinite-dimensional $M_G(\Sigma, V)$.

2.2.3 The Swapping algebra

$\mathcal{A}_n(\Sigma, V)$ has an interesting Poisson subalgebra:

Proposition 2.13. *Let $i_0, j_0 \in \{1, \cdots, n\}$ be distinct, then*

$$\{\alpha_{i_0 j_0} \mid \alpha \in \pi_1(\Sigma, V) \text{ is nontrivial } \}$$

generates a Poisson subalgebra $\mathcal{L}(\Sigma, V)$ of $\mathcal{A}_n(\Sigma, V)$.

Proof. We need to prove that the quasi-Poisson bracket of $\mathcal{A}_n(\Sigma, V)$ restricted to $\mathcal{L}(\Sigma, V)$ is a Poisson bracket, that is, satisfies the Jacobi identity.

The canonical trivector $\phi \in (\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}$ associated to $s = \sum_{k,l} E_{kl} \otimes E_{lk} \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is

$$\phi = \sum_{i,j,k} \left(E_{ij} \otimes E_{ki} \otimes E_{jk} - E_{ij} \otimes E_{jk} \otimes E_{ki}\right).$$

Note that although we write ϕ as an element of $\mathfrak{g}^{\otimes 3}$, it is indeed skew-symmetric.

Since $\mathcal{A}_n(\Sigma, V)$ is a quasi-Poisson algebra, for $\alpha_{i_0 j_0}, \beta_{i_0 j_0}, \gamma_{i_0 j_0} \in \mathcal{L}(\Sigma, V)$ we have

$$-2 \circlearrowleft \{\{\alpha_{i_0 j_0}, \beta_{i_0 j_0}\}, \gamma_{i_0 j_0}\} = \rho_{\phi^{(V)}}(\alpha_{i_0 j_0}, \beta_{i_0 j_0}, \gamma_{i_0 j_0}), \tag{2.3}$$

where ρ denotes the $\mathfrak{g}^V$-action on $\mathcal{A}_n(\Sigma, V)$. Let ρ^v be the action of the v^{th} factor. Since $\phi^{(V)}$ is the direct sum of $\#V$ copies of ϕ, in order to prove that (2.3) gives zero, we only need to show

$$\rho^v_\phi(\alpha_{i_0 j_0}, \beta_{i_0 j_0}, \gamma_{i_0 j_0}) = 0. \tag{2.4}$$

It is easy to see that the action ρ^v of $x \in \mathfrak{g}$ on $\alpha_{i_0 j_0}$ is given by

$$\rho^v_x(\alpha_{i_0 j_0}) = \begin{cases} \sum_t (\alpha_{i_0 t} x_{t j_0} - x_{i_0 t} \alpha_{t j_0}) & \text{if } \alpha \text{ starts and ends at } v, \\ \sum_t \alpha_{i_0 t} x_{t j_0} & \text{if } \alpha \text{ only starts from } v, \\ -\sum_t x_{i_0 t} \alpha_{t j_0} & \text{if } \alpha \text{ only ends at } v, \\ 0 & \text{otherwise.} \end{cases}$$

It follows immediately that the required equality (2.4) holds if at least one of the three paths α, β and γ neither start nor end at v. Case-by-case computations show that (2.4) holds as well in other situations. For example, if α, β and γ all start from v and do not

5. Massuyeau and Turaev only treated the case $\#V = 1$, where $\pi_1(\Sigma, V)$ is the fundamental group. So what need to be done here is to generalize their result to fundamental groupoids.

end at v, then

$$\begin{aligned}
&\rho^v_\phi(\alpha_{i_0j_0}, \beta_{i_0j_0}, \gamma_{i_0j_0}) \\
&= \sum_{i,j,k} \left(\rho^v_{E_{ij}}(\alpha_{i_0j_0}) \rho^v_{E_{ki}}(\beta_{i_0j_0}) \rho^v_{E_{jk}}(\gamma_{i_0j_0}) - \rho^v_{E_{ij}}(\alpha_{i_0j_0}) \rho^v_{E_{jk}}(\beta_{i_0j_0}) \rho^v_{E_{ki}}(\gamma_{i_0j_0}) \right) \\
&= \sum_{i,j,k} (\alpha_{i_0i}\delta_{jj_0}\beta_{i_0k}\delta_{ij_0}\gamma_{i_0j}\delta_{kj_0} - \alpha_{i_0i}\delta_{jj_0}\beta_{i_0j}\delta_{kj_0}\gamma_{i_0k}\delta_{ij_0}) \\
&= \alpha_{i_0j_0}\beta_{i_0j_0}\gamma_{i_0j_0} - \alpha_{i_0j_0}\beta_{i_0j_0}\gamma_{i_0j_0} = 0.
\end{aligned}$$

□

Since i_0 and j_0 are fixed, generator of $\mathcal{L}(\Sigma, V)$ are in one to one correspondence with nontrivial element of $\pi_1(\Sigma, V)$, thus $\mathcal{L}(\Sigma, V)$ can be considered as the commutative associative algebra freely generated by nontrivial elements in $\pi_1(\Sigma, V)$. The Poisson bracket is given on generators by

$$\{\alpha, \beta\} = \sum_{q \in \alpha \cap \beta \setminus \partial\Sigma} \varepsilon_q(\alpha, \beta)(\beta *_q \alpha) \cdot (\alpha *_q \beta) + \frac{1}{2}\left(\varepsilon(\alpha^\vee, \beta^\vee) + \varepsilon(\alpha^\wedge, \beta^\wedge)\right) \alpha \cdot \beta, \tag{2.5}$$

and is extended to other elements by the Leibniz rule.

For the disk $\Sigma = \mathbb{D}$, the Poisson algebra $\mathcal{L}(\mathbb{D}, V)$ was studied by Labourie [40], who call it the *swapping algebra*. To explain the name, let us note that a nontrivial element in $\pi_1(\mathbb{D}, V)$ is just an oriented chord $\overline{xy}$ going from $y \in V$ to $x \in V$, while the Poisson bracket (2.5) of two chords is

$$\{\overline{xy}, \overline{zw}\} = \varepsilon(\overline{xy}, \overline{zw})\overline{xw} \cdot \overline{zy}. \tag{2.6}$$

Here $\varepsilon(\overline{xy}, \overline{zw}) = 0, \pm 1, \pm\frac{1}{2}$ is defined in three cases respectively: it is the oriented intersection number of $\overline{xy}$ and $\overline{zw}$ if they intersect in the interior of the disk; one-half the oriented intersection number if intersect on the boundary; and 0 if $\overline{xy}$ and $\overline{zw}$ are disjoint.

$\mathcal{L}(\mathbb{D}, V)$ is related to configurations of flags in a projective space, as explained in §3.1.3 below.

2.2.4 The $\mathrm{SL}_n\mathbb{R}$ case

For our purpose later on, it is desirable to consider the structure group $\mathrm{SL}_n\mathbb{R}$ instead of $\mathrm{GL}_n\mathbb{R}$. It turns our that the above formulas adapts to this case with only a little modification.

Recall that

$$s = \sum_{kl} E_{kl} \otimes E_{lk} \in (S^2\mathfrak{gl}_n\mathbb{R})^{\mathfrak{gl}_n\mathbb{R}}$$

is the dual of the invariant scalar product $(x \mid y) = \mathrm{Tr}(xy)$ on $\mathfrak{gl}_n\mathbb{R}$. We let $s_1 \in (S^2\mathfrak{sl}_n\mathbb{R})^{\mathfrak{sl}_n\mathbb{R}}$ denote the dual of the restriction of $(\cdot \mid \cdot)$ to $\mathfrak{sl}_n\mathbb{R}$. Let $\mathrm{Pr} : \mathfrak{gl}_n\mathbb{R} \to \mathfrak{sl}_n\mathbb{R}$ be the orthogonal projection, which is given by

$$\mathrm{Pr}(x) = x - \frac{1}{n}\mathrm{Tr}(x)I.$$

Then we have

$$s_1 = (\mathrm{Pr} \otimes \mathrm{Pr})(s) = \sum_{kl} (E_{kl} - \frac{\delta_{kl}}{n} I) \otimes (E_{lk} - \frac{\delta_{kl}}{n} I) = s - \frac{1}{n} I \otimes I. \tag{2.7}$$

Let $\{\cdot,\cdot\}_{\mathrm{SL}}$ denote the canonical quasi-Poisson bracket on $C^\infty(M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V))$ with respect to s_1.

Observe that $s_0 := I \otimes I$ also belongs to $(S^2\mathfrak{gl}_n\mathbb{R})^{\mathfrak{gl}_n\mathbb{R}}$, hence there is a canonical quasi-Poisson $((\mathfrak{gl}_n\mathbb{R})^V, s_0^{(V)})$-tensor on $M_G(\Sigma, V)$. Let us consider the quasi-Poisson bracket $\{\cdot,\cdot\}_0$ associated with it.

We call a function $f : \mathrm{GL}_n\mathbb{R} \to \mathbb{R}$ *homogeneous* if $f(\lambda a) = \lambda f(a)$ for any $\lambda \in \mathbb{R}^*$.

Lemma 2.14. *Let $[\Gamma, f]$ be a spin network on (Σ, V) such that the admissible function $f \in C^\infty((\mathrm{GL}_n\mathbb{R})^{E_\Gamma})$ is homogeneous with respect to each $\mathrm{GL}_n\mathbb{R}$-factor. Let $[\Gamma', f']$ be a spin network with the same property, which is transverse to $[\Gamma, f]$. Then*

$$\{[\Gamma, f], [\Gamma', f']\}_0 = i(\Gamma, \Gamma')[\Gamma, f] \cdot [\Gamma', f'].$$

Here the algebraic intersection number $i(\Gamma, \Gamma')$ is defined in Remark 2.7.

Proof. If f and f' are homogeneous then by definition

$$[\Gamma \cup_q \Gamma', D_q(f, f')] = [\Gamma \cup \Gamma', D_{\mathbf{a},\mathbf{a}'}(f, f')] = [\Gamma, f] \cdot [\Gamma', f'].$$

So the lemma follows from Theorem 2.6. □

Corollary 2.15. *The quasi-Poisson brackets of the restrictions of $\mathrm{Tr}_\alpha, \alpha_{ij} \in C^\infty(M_{\mathrm{GL}_n\mathbb{R}}(\Sigma, V))$ to $M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V)$ are given by*

$$\{\mathrm{Tr}_\alpha, \mathrm{Tr}_\beta\}_{\mathrm{SL}} = -\frac{i(\alpha,\beta)}{n}\mathrm{Tr}_\alpha \cdot \mathrm{Tr}_\beta + \sum_{q\in\alpha\cap\beta} \varepsilon_q(\alpha,\beta)\,\mathrm{Tr}_{\alpha\#_q\beta},$$

$$\{\alpha_{ij}, \beta_{kl}\}_{\mathrm{SL}} = -\frac{i(\alpha,\beta)}{n}\alpha_{ij} \cdot \beta_{kl} + \sum_{q\in\alpha\cap\beta\setminus\partial\Sigma} \varepsilon_q(\alpha,\beta)(\beta *_q \alpha)_{kj} \cdot (\alpha *_q \beta)_{il}$$

$$+\frac{1}{2}\Big(\varepsilon(\alpha^\wedge,\beta^\wedge)\alpha_{il}\cdot\beta_{kj} + \varepsilon(\alpha^\vee,\beta^\vee)\alpha_{kj}\cdot\beta_{il} + \delta_{jk}\varepsilon(\alpha^\wedge,\beta^\vee)(\alpha\beta)_{il} + \delta_{il}\varepsilon(\alpha^\vee,\beta^\wedge)(\beta\alpha)_{kj}\Big).$$

Proof. Let $\{\cdot,\cdot\}$, $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ denote the quasi-Poisson brackets on $C^\infty(M_{\mathrm{GL}_n\mathbb{R}}(\Sigma, V))$ with respect to s, s_0 and s_1, respectively. By the definition of the quasi-Poisson tensor (1.7), we have

$$\{\varphi|_{M_{\mathrm{SL}_n\mathbb{R}}(\Sigma,V)}, \psi|_{M_{\mathrm{SL}_n\mathbb{R}}(\Sigma,V)}\}_{\mathrm{SL}} = \{\varphi,\psi\}_1|_{M_{\mathrm{SL}_n\mathbb{R}}(\Sigma,V)}$$

for any $\varphi, \psi \in C^\infty(M_{\mathrm{GL}_n\mathbb{R}}(\Sigma, V))$. On the other hand, since $s_1 = s - \frac{1}{n}s_0$, the same definition implies that

$$\{\cdot,\cdot\}_1 = \{\cdot,\cdot\} - \frac{1}{n}\{\cdot,\cdot\}_0.$$

Noting that the trace function and matrix entry functions are homogenous, the required equalities follows from Lemma 2.14 and the previously proved Corollary 2.10 and 2.11. □

2.3 Proof of Theorem 2.6

The aim of this section is to give a proof of Theorem 2.6. As a by-product, we also give a proof of Theorem 1.12 . Here we follow Li-Bland and Ševera [42] closely.

2.3.1 The homotopy intersection form

Given oriented paths α, β and a transversal intersection point $q \in \alpha \cap \beta$, we put

$$\lambda(q) = \begin{cases} 1 & \text{if } q \in \partial\Sigma, \\ 2 & \text{otherwise.} \end{cases}$$

Let α_q denote the portion of α running from the starting point up to q.

Let $\mathbb{Z}\pi_1(\Sigma, V)$ be the groupoid algebra of $\pi_1(\Sigma, V)$, namely, the free $\mathbb{Z}$-module generated by $\pi_1(\Sigma, V)$ with a multiplication defined by linearly spanning the composition law in $\pi_1(\Sigma, V)$ (if $a, b \in \pi_1(\Sigma, V)$ are not composable, we put $ab = 0 \in \mathbb{Z}\pi_1(\Sigma, V)$). If $a = \sum_i n_i a_i \in \mathbb{Z}\pi_1(\Sigma, V)$ for $n_i \in \mathbb{Z}$ and $a_i \in \pi_1(\Sigma, V)$, we put $\overline{a} := \sum_i n_i a_i^{-1}$.

Definition/Proposition 2.16. *We define the* homotopy intersection form *as a map*

$$(\cdot,\cdot) : \pi_1(\Sigma, V) \times \pi_1(\Sigma, V) \to \mathbb{Z}\pi_1(\Sigma, V)$$

$$(a, b) = \sum_{q \in \alpha\cap\beta} \lambda(q)\varepsilon_q(\alpha, \beta)[\alpha_q^{-1}\beta_q], \tag{2.8}$$

where α, β are representatives of a, b which are transverse. Then $(\cdot,\cdot)$ is well-defined and satisfies

$$(b, a) = -\overline{(a, b)}, \tag{2.9}$$

$$(ab, c) = (b, c) + b^{-1}(a, c). \tag{2.10}$$

Proof. To show that $(\cdot,\cdot)$ is well-defined, one need to verify that the right-hand side of Eq.(2.8) is invariant when α and β undergo homotopy modifications. But such modifications can be decomposed into a sequence of elementary moves shown in Figure 2.1 (see *e.g.* [27]) for a proof of this fact), and it is elementary to verify invariance under these moves. The proofs of (2.9) and (2.10) are also elementary. □

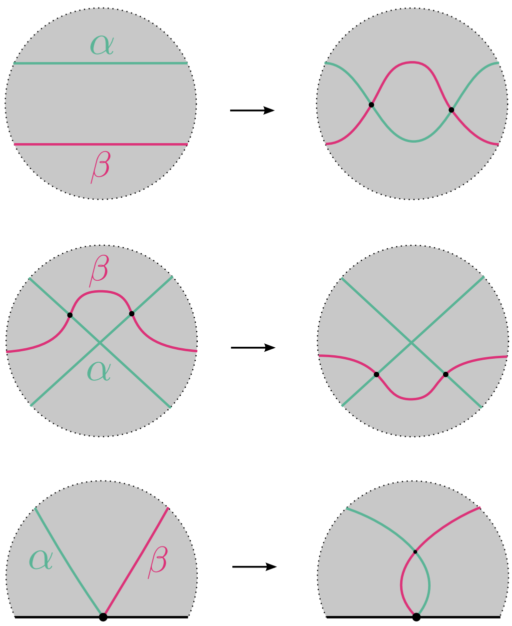


Figure 2.1: Elementary moves

Let P be a bivector field on a manifold M and $\omega, \varpi \in \Omega^1(M,\mathfrak{g}) = \Omega^1(M)\otimes\mathfrak{g}$ be $\mathfrak{g}$-valued 1-forms. The map $P(\omega,\varpi) : M \to \mathfrak{g}\otimes\mathfrak{g}$ is defined by applying P to the $\Omega^1(M)$-parts of ω, ϖ and tensor product to the $\mathfrak{g}$-parts. In other words, for any $\xi,\ \eta \in \mathfrak{g}^*$ we have

$$\langle \xi\otimes\eta, P(\omega,\varpi)\rangle = P(\langle\eta,\omega\rangle, \langle\eta,\varpi\rangle).$$

Here $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{g}^*\otimes\mathfrak{g}^*$ and $\mathfrak{g}\otimes\mathfrak{g}$ or between $\mathfrak{g}^*$ and $\mathfrak{g}$.

Theorem 2.17. *Let Γ be a skeleton of (Σ,V) and P_Γ be the quasi-Poisson $(\mathfrak{g}^V, s^{(V)})$-tensor defined in §1.3.1 Eq.(1.7). Then for any $a,b\in\pi_1(\Sigma,V)$ we have*

$$P_\Gamma(\mathrm{hol}_a^*\theta^{\mathsf{L}}, \mathrm{hol}_b^*\theta^{\mathsf{L}}) = \frac{1}{2}(\mathrm{Ad}_{\mathrm{hol}_{(a,b)}}\otimes\mathrm{id})s. \tag{2.11}$$

As a result, P_Γ is independent of the choice of Γ.

Here we extend every $m\in M_G(\Sigma,V)$ to a homomorphism of algebras $m:\mathbb{Z}\pi_1(\Sigma,V)\to\mathbb{Z}G$ and define $\mathrm{hol}_a : M_G(\Sigma,V)\to\mathbb{Z}G$ ($\forall a\in\mathbb{Z}\pi_1(\Sigma,V)$) as the evaluation map at a. The adjoint action $\mathrm{Ad}: G\to\mathrm{GL}(\mathfrak{g})$ is also extended to an algebra homomorphism $\mathrm{Ad}:\mathbb{Z}G\to\mathrm{End}(\mathfrak{g})$.

Proof. We shall first show that

– if Eq.(2.11) holds for a and b, then it also holds for b and a;

– if Eq.(2.11) holds for a and c as well as for b and c, then it also holds for ab and c.

Let $\sigma:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g}$ be the map switching the two factors. We have

$$\begin{aligned}P_\Gamma(\mathrm{hol}_b^*\theta^{\mathsf{L}}, \mathrm{hol}_a^*\theta^{\mathsf{L}}) &= -\sigma(P_\Gamma(\mathrm{hol}_a^*\theta^{\mathsf{L}}, \mathrm{hol}_b^*\theta^{\mathsf{L}})) = -\frac{1}{2}(\mathrm{id}\otimes\mathrm{Ad}_{\mathrm{hol}_{(a,b)}})s\\ &= -\frac{1}{2}(\mathrm{Ad}_{\mathrm{hol}_{\overline{(a,b)}}}\otimes\mathrm{id})s.\end{aligned}$$

So the first assertion results from Eq.(2.9). To prove the second assertion, notice that for any smooth maps λ,μ from a manifold M to G, we have

$$(\lambda\cdot\mu)^*\theta^{\mathsf{L}} = \mathrm{Ad}_\mu^{-1}\lambda^*\theta^{\mathsf{L}} + \mu^*\theta^{\mathsf{L}}.$$

Therefore

$$\begin{aligned}P_\Gamma(\mathrm{hol}_{ab}^*\theta^{\mathsf{L}}, \mathrm{hol}_c^*\theta^{\mathsf{L}}) &= P_\Gamma(\mathrm{Ad}_{\mathrm{hol}_b}^{-1}\mathrm{hol}_a^*\theta^{\mathsf{L}} + \mathrm{hol}_b^*\theta^{\mathsf{L}}, \mathrm{hol}_c^*\theta^{\mathsf{L}})\\ &= \frac{1}{2}(\mathrm{Ad}_{\mathrm{hol}_b}^{-1}\otimes\mathrm{id})(\mathrm{Ad}_{\mathrm{hol}_{(a,c)}}\otimes\mathrm{id})s + \frac{1}{2}(\mathrm{Ad}_{\mathrm{hol}_{(b,c)}}\otimes\mathrm{id})s\end{aligned}$$

So the second assertion results from Eq.(2.10).

As a result, we only need to verify Eq.(2.11) when a and b are represented by edges of Γ, since $\pi_1(\Sigma,V)$ is generated by the homotopy classes of these edges.

To this end, we assume that a is an edge of Γ equipped with an orientation, and let $\mathbf{a}_1,\mathbf{a}_2\in\widehat{E}_\Gamma$ denote the first and second half of a, respectively. For any $\mathbf{a}\in\widehat{E}_\Gamma$, we have

$$\mathrm{hol}_a^*\theta^{\mathsf{L}}(x^{(\mathbf{a})}) = \begin{cases} x & \text{if } \mathbf{a}=\mathbf{a}_1,\\ -\mathrm{Ad}_{\mathrm{hol}_a}^{-1}x & \text{if } \mathbf{a}=\mathbf{a}_2,\\ 0 & \text{otherwise.}\end{cases} \tag{2.12}$$

Put b and $\mathbf{b}_1,\mathbf{b}_2$ similarly. For any pair of half-edges $\mathbf{a},\mathbf{b}\in\widehat{E}_\Gamma$ we define

$$\delta(\mathbf{a},\mathbf{b}) = \begin{cases} 1 & \text{if } \mathbf{a},\mathbf{b}\in\widehat{E}_\Gamma(v) \text{ for some } v \text{ and } \mathbf{a}<\mathbf{b},\\ -1 & \text{if } \mathbf{a},\mathbf{b}\in\widehat{E}_\Gamma(v) \text{ for some } v \text{ and } \mathbf{a}>\mathbf{b},\\ 0 & \text{otherwise.}\end{cases}$$

The expression (1.7) of P_Γ and Eq.(2.12) yield

$$\begin{aligned}P_\Gamma(\mathrm{hol}_a^*\theta^{\mathsf{L}}, \mathrm{hol}_b^*\theta^{\mathsf{L}}) &= \frac{1}{2}s^{ij}\sum_{k,l=1,2}\delta(\mathbf{a}_k,\mathbf{b}_l)\mathrm{hol}_a^*\theta^{\mathsf{L}}(x_i^{(\mathbf{a}_k)})\otimes \mathrm{hol}_b^*\theta^{\mathsf{L}}(x_j^{(\mathbf{b}_l)})\\ &= \frac{1}{2}s^{ij}\Big[\delta(\mathbf{a}_1,\mathbf{b}_1)x_i\otimes x_j - \delta(\mathbf{a}_1,\mathbf{b}_2)x_i\otimes(\mathrm{Ad}^{-1}_{\mathrm{hol}_b}x_j)\\ &\quad - \delta(\mathbf{a}_2,\mathbf{b}_1)(\mathrm{Ad}^{-1}_{\mathrm{hol}_a}x_i)\otimes x_j + \delta(\mathbf{a}_2,\mathbf{b}_2)(\mathrm{Ad}^{-1}_{\mathrm{hol}_a}x_i)\otimes(\mathrm{Ad}^{-1}_{\mathrm{hol}_b}x_j)\Big].\end{aligned}$$

On the other hand, by definition of the homotopy intersection form,

$$\begin{aligned}\frac{1}{2}\left(\mathrm{Ad}_{\mathrm{hol}_{(a,b)}}\otimes\mathrm{id}\right)s &= \frac{1}{2}s^{ij}\Big[\varepsilon(\mathbf{a}_1,\mathbf{b}_1)x_i\otimes x_j + \varepsilon(\mathbf{a}_1,\mathbf{b}_2)x_i\otimes(\mathrm{Ad}^{-1}_{\mathrm{hol}_b}x_j)\\ &\quad + \varepsilon(\mathbf{a}_2,\mathbf{b}_1)(\mathrm{Ad}^{-1}_{\mathrm{hol}_a}x_i)\otimes x_j + \varepsilon(\mathbf{a}_2,\mathbf{b}_2)(\mathrm{Ad}^{-1}_{\mathrm{hol}_a}x_i)\otimes(\mathrm{Ad}^{-1}_{\mathrm{hol}_b}x_j)\Big].\end{aligned}$$

Thus one readily verifies Eq.(2.11) for this choice of a and b.

Finally, since $(\mathrm{hol}_e)_{e\in E_\Gamma} : M_G(\Sigma,V)\to G^{E_\Gamma}$ is a diffeomorphism, the cotangent space at any point of $M_G(\Sigma,V)$ is generated by cotangent vectors of the form $\langle hol_a^*\theta^{\mathsf{L}}, x\rangle$, where $a\in\pi_1(\Sigma,V)$. Therefore, Eq.(2.11) implies that P_Γ is independent of Γ. □

Similar reasoning as in the last paragraph of the proof yields

Corollary 2.18. *If $(\Sigma',V')\to(\Sigma,V)$ is an embedding then the induced map*

$$f : M_G(\Sigma,V)\to M_G(\Sigma',V')$$

is a quasi-Poisson map, i.e., $f^* : C^\infty(M_G(\Sigma',V'))\to C^\infty(M_G(\Sigma,V))$ preserves quasi-Poisson brackets.

2.3.2 Proof of Theorem 2.6: the case $V_\Gamma^{int} = V_{\Gamma'}^{int} = \emptyset$

Let us first assume that $V_\Gamma^{int} = V_{\Gamma'}^{int} = \emptyset$, so that each edge $e\in E_\Gamma$ and $e'\in E_{\Gamma'}$ represents an element in $\pi_1(\Sigma,V)$. Any $f\in C^\infty(G^{E_\Gamma})$ is an admissible function for Γ. Given $x\in\mathfrak{g}$ and $e\in E_\Gamma$, we let x_e denote the element of $\mathfrak{g}^{E_\Gamma}$ whose e^{th} component is x and other components are 0. We define a map $\partial_e^{\mathsf{L}} f : G^{E_\Gamma}\to\mathfrak{g}^*$ by

$$\left\langle \partial_e^{\mathsf{L}} f, x\right\rangle = x_e^{\mathsf{L}}(f),\quad \forall x\in\mathfrak{g}.$$

If $g_e : G^{E_\Gamma}\to G$ is the projection to the e^{th} factor then we have

$$df = \sum_{e\in E_\Gamma}\left\langle \partial_e^{\mathsf{L}} f, g_e^*\theta^{\mathsf{L}}\right\rangle.$$

Therefore, the differential of the spin network $[\Gamma,f] = f\big((\mathrm{hol}_e)_{e\in E_\Gamma}\big)$ is given by

$$d[\Gamma,f] = \sum_{e\in E_\Gamma}\left\langle \partial_e^{\mathsf{L}} f((\mathrm{hol}_e)_{e\in E_\Gamma}), \mathrm{hol}_e^*\theta^{\mathsf{L}}\right\rangle.$$

Theorem 2.17 yields

$$\begin{aligned}\{[\Gamma,f],[\Gamma',f']\} &= \sum_{e,e'}\left(\partial_e^{\mathsf{L}} f\otimes\partial_{e'}^{\mathsf{L}} f'\right)P(\mathrm{hol}_e^*\theta^{\mathsf{L}},\mathrm{hol}_{e'}^*\theta^{\mathsf{L}})\\ &= \sum_{e,e'}\left(\partial_e^{\mathsf{L}} f\otimes\partial_{e'}^{\mathsf{L}} f'\right)(\mathrm{Ad}_{\mathrm{hol}_{(e,e')}}\otimes\mathrm{id})s,\end{aligned}\tag{2.13}$$

where the summation runs over edges $e \in E_\Gamma$ and $e' \in E_{\Gamma'}$.

For fixed e and e', each interior intersection point $q \in e \cap e' \setminus \partial\Sigma$ contributes a term to the homotopy intersection form (e, e'), as well as each pair of half-edges $\mathbf{a} \subset e, \mathbf{a}' \subset e'$ sharing a vertex. The contribution of an interior intersection q to Eq.(2.13) is

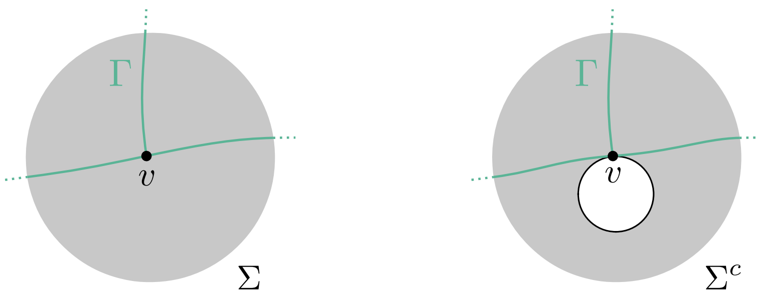
$$\begin{aligned}
&\left(\partial^{\mathsf{L}}_e f \otimes \partial^{\mathsf{L}}_{e'} f'\right)\left(\mathrm{Ad}_{e_q^{-1}e_q} \otimes \mathrm{id}\right)s \\
&= \varepsilon_q(e,e') s^{ij} \frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0} f\left((\mathrm{hol}_d)_{d\in E_\Gamma\setminus\{e\}},\ \mathrm{hol}_e \exp(\epsilon\,\mathrm{Ad}_{\mathrm{hol}_{e_q^{-1}e'_q}} x_i)\right) \\
&\qquad \cdot \frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\delta=0} f'\left((\mathrm{hol}_d)_{d\in E_{\Gamma'}\setminus\{e'\}},\ \mathrm{hol}_{e'} \exp(\epsilon x_j)\right) \\
&= \varepsilon_q(e,e') s^{ij} \frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0} f\left((\mathrm{hol}_d)_{d\in E_\Gamma\setminus\{e\}},\ \mathrm{hol}_e \mathrm{hol}_{e_q^{-1}} \exp(\epsilon x_i) \mathrm{hol}_{e_q}\right) \\
&\qquad \cdot \frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\delta=0} f'\left((\mathrm{hol}_d)_{d\in E_{\Gamma'}\setminus\{e'\}},\ \mathrm{hol}_{e'} \mathrm{hol}_{e_q'^{-1}} \exp(\epsilon x_j) \mathrm{hol}_{e'_q}\right) \\
&= \varepsilon_q(\Gamma,\Gamma')[\Gamma \cup_q \Gamma', D_q(f,f')].
\end{aligned}$$

In the second step we considered the function in question as lifted to $M_G(\Sigma, V \cup \{q\})$ and used Ad-invariance of s.

Similar computations show that the contribution of a pair of half-edges $\mathbf{a}, \mathbf{a}'$ to Eq.(2.13) is $\frac{1}{2}\varepsilon(\mathbf{a},\mathbf{a}')[\Gamma \cup \Gamma', D_{\mathbf{a},\mathbf{a}}(f,f')]$. Therefore Theorem 2.6 is proved in this case.

2.3.3 Proof of Theorem 2.6: the general case

Suppose now that Γ and Γ' are arbitrary graph diagrams transverse to each other. Let Σ^c be the surface obtained from Σ by deleting a small open disk $D_v \subset \Sigma \setminus \Gamma \cup \Gamma'$ near each $v \in V^{int}_\Gamma \cup V^{int}_{\Gamma'}$, as shown below.



There are embeddings of marked surfaces

$$(\Sigma, V) \longleftarrow (\Sigma^c, V) \longrightarrow (\Sigma^c, V \cup V^{int}_\Gamma \cup V^{int}_{\Gamma'}). \tag{2.14}$$

We view $[\Gamma, f]$ and $[\Gamma', f']$ as spin networks on (Σ^c, V) in the obvious way, and also view them as spin networks on $(\Sigma^c, V \cup V^{int}_\Gamma \cup V^{int}_{\Gamma'})$ by considering every vertex as a boundary vertex. These spin networks, as functions on the representation spaces, are lifts of each other under the maps induced by (2.14)

$$M_G(\Sigma, V) \longrightarrow M_G(\Sigma^c, V) \longleftarrow M_G(\Sigma^c, V \cup V^{int}_\Gamma \cup V^{int}_{\Gamma'}).$$

The first and second map are, respectively, injective and surjective, and by Corollary 2.18 they are both quasi-Poisson maps. Applying the previously proved $V^{int}_\Gamma = V^{int}_{\Gamma'} = \emptyset$ case to $M_G(\Sigma^c, V \cup V^{int}_\Gamma \cup V^{int}_{\Gamma'})$, we conclude that Theorem 2.6 holds in general.

Chapter 3

Configuration spaces and moduli of framed flat connections

The goal of this chapter is to give a proof of Theorem B in the introduction (*i.e.*,Theorem 3.23 below). As we will see, Fock-Goncharov's moduli space of framed flat connections generalizes both configuration space of flags and moduli spaces of surface fundamental group representations. So we shall first explain in §3.1 the basic ideas bridging quasi-Poisson theory and configuration spaces. In particular, we show how the swapping algebra computes Poisson brackets of certain functions on the configuration space of flags. In §3.2 we give a self-contained presentation of some ingredients of the Fock-Goncharov theory in order to understand the statement of Theorem 3.23 and reduce it to straightforward computations. These computations are implemented in §3.3. Finally, we discuss in §3.4 some further developments and problems.

3.1 Quasi-Poisson reduction and configuration spaces

3.1.1 Reduction with respect to a reducing subgroup

A main feature of the quasi-Poisson theory is that the reduction of the quasi-Poisson manifold $M_G(\Sigma, V)$ by the action of G^V yields the Atiyah-Bott Poisson structure on $X_G(\Sigma)$ (Theorem 1.15). Li-Bland and Ševera [42] observed that we can perform reduction with respect to a smaller group, still obtaining a Poisson quotient.

Definition 3.1. Fix $s \in (S^2\mathfrak{g})^{\mathfrak{g}}$. A subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is called *reducing* if it satisfies

$$[s^\sharp(\mathfrak{c}^\perp), s^\sharp(\mathfrak{c}^\perp)] \subset \mathfrak{c},$$

where $\mathfrak{c}^\perp = \{\alpha \in \mathfrak{g}^* \mid \alpha(\mathfrak{c}) = 0\}$. In particular, $\mathfrak{c}$ is called *coisotropic* if $s^\sharp(\mathfrak{c}^\perp) \subset \mathfrak{c}$. A closed subgroup $C \subset G$ is called reducing or coisotropic if its Lie algebra is.

Note that if a subalgebra $\mathfrak{c}$ is reducing, then so is any subalgebra containing $\mathfrak{c}$.

Proposition 3.2. *Let $C \subset G$ be a reducing subgroup. If M is a G-manifold and P is a quasi-Poisson $(\mathfrak{g}, s)$-tensor on M, then the quasi-Poisson bracket $\{\cdot,\cdot\}$ restricts to a Poisson bracket on $C^\infty(M)^C$.*

Following §1.1.4, we call this Poisson bracket "the Poisson structure on M/C reduced from P".

Proof. We need to show that $\{\cdot,\cdot\}$ satisfies the Jacobi identity when applied to functions in $C^\infty(M)^C$, or equivalently,

$$[P,P](\mathsf{d}f_1,\mathsf{d}f_2,\mathsf{d}f_3) = \rho_\phi(\mathsf{d}f_1,\mathsf{d}f_2,\mathsf{d}f_3) = 0, \quad \forall f_i \in C^\infty(M)^C.$$

We take any $m \in M$ and let $\alpha_i \in \mathfrak{g}^*$ be defined by $\alpha_i(x) = \rho_x(m)(f_i)$. Then $f_i \in C^\infty(M)^C$ means $\alpha_i \in \mathfrak{c}^\perp$. We have

$$\rho_\phi(\mathsf{d}f_1,\mathsf{d}f_2,\mathsf{d}f_3)(m) = \phi(\alpha_1,\alpha_2,\alpha_3),$$

which vanishes because of the definition of ϕ and reducing subalgebras. □

Example 3.3. (1) Let G be a complex semi-simple Lie group (or its split real form). Let s be proportional to the dual of the Killing form. Then any Borel subgroup $B \subset G$ is isotropic, hence so is any parabolic subgroups. This is easy to verify using the expression

$$s = \sum_{i,j} s_{ij} h_i \otimes h_j + \sum_{\alpha\in\Delta^+} \frac{(\alpha \mid \alpha)}{2} \left(e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha\right),$$

where we take Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, positive roots $\Delta^+ \subset \mathfrak{h}^*$ and Chevalley basis $(h_i, e_{\pm\alpha})_{1\le i\le r, \alpha\in\Delta_+}$ (r is the rank and (s_{ij}) is some symmetric $r \times r$ matrix) so that $\mathrm{Lie}(B) = \mathfrak{h} \oplus \mathrm{Span}\{e_\alpha \mid \alpha \in \Delta_+\}$.

(2) $\mathrm{SO}(n) \subset \mathrm{SL}_n\mathbb{R}$ and $\mathrm{SO}(n) \subset \mathrm{SO}(1,n)$ are reducing but not isotropic. It is likely that this generalizes to any symmetric pair.

3.1.2 Configuration spaces

Proposition 3.4. *Let $\mathbb{D}$ be the disk and V be an arbitrary subset of $\partial\mathbb{D}$. Choose a subgroup $C_v \subset G$ for each $v \in V$. Then we have a canonical identification*

$$M_G(\mathbb{D},V)/\prod_{v\in V} C_v \cong \left(\prod_{v\in V} G/C_v\right)/G.$$

On the right-hand side, we take the quotient of the diagonal G-action on the product of homogenous spaces G/C_v.

As a result, if V is finite and each C_v is reducing, then $\left(\prod_{v\in V} G/C_v\right)/G$ carries a Poisson structure.

Proof. It is sufficient to define a map $p : M_G(\mathbb{D},V) \to \prod_{v\in V} G/C_v$ such that the composition

$$\overline{p} : M_G(\mathbb{D},V) \xrightarrow{p} \prod_{v\in V} G/C_v \xrightarrow{/G} \left(\prod_{v\in V} G/C_v\right)/G \tag{3.1}$$

is surjective and each fiber of $\overline{p}$ is a $\prod_{v\in V} C_v$-orbit in $M_G(\mathbb{D},V)$.

For any $x, y \in V$, let $\overline{xy}$ denote the oriented chord going from y to x. We pick $v_0 \in V$ and define p by

$$p(m) = \left([e]_{v_0}, \left([\mathrm{hol}_{\overline{v_0 v}}(m)]_v\right)_{v\in V\setminus\{v_0\}}\right),$$

where for any $v \in V$ and $g \in G$ we let $[g]_v \in G/C_v$ denote the projection of g.

The composition (3.1) is surjective because, on one hand, any element in $\prod_{v\in V} G/C_v$ can be brought into the slice $\{[e]_{v_0}\}\times\prod_{v\in V\setminus\{v_0\}} G/C_v$ by the diagonal G-action; on the other hand, the map

$$(\mathrm{hol}_{\overline{v_0 v}})_{v\in V\setminus\{v_0\}} : M_G(\mathbb{D},V)\to G^{V\setminus\{v_0\}} \tag{3.2}$$

is bijective.

Identifying $M_G(\mathbb{D},V)$ with $G^{V\setminus\{v_0\}}$ using the map (3.2), it is easy to see that the fiber of the map (3.1) passing through any $(a_v)_{v\in V\setminus\{v_0\}}\in G^{V\setminus\{v_0\}}$ has the form

$$\{(g_0 a_v g_v^{-1})_{v\in V\setminus\{v_0\}} \mid g_0\in C_{v_0}, g_v\in C_v\}\subset G^{V\setminus\{v_0\}}.$$

This is exactly a $\prod_{v\in V} C_v$ orbit. Thus we get the required identification. Finally, one can verify that this identification is independent of the choice of v_0, hence canonical. □

If $C_v = C$ for any v, then

$$\left(\prod_{v\in V} G/C_v\right)/G = \mathrm{Map}(V,G/C)/C =: \mathrm{Conf}_V(V,G/C).$$

is a configuration space of points in the homogenous space G/C.

In this thesis we are mainly concerned with the case where $G=\mathrm{SL}_n\mathbb{R}$ and C_v is the upper-triangular subgroup B. Let $\mathcal{F}=G/B$ be the complete flag variety. Then Proposition 3.4 gives a canonical identification

$$\mathrm{Conf}_V(\mathcal{F})\cong M_{\mathrm{SL}_n\mathbb{R}}(\mathbb{D},V)/B^V.$$

Our goal is to study the Poisson structure on this space reduced from the quasi-Poisson structure of $M_{\mathrm{SL}_n\mathbb{R}}(\mathbb{D},V)$.

3.1.3 Cross ratio functions

Definition 3.5. Let $x,y\in\mathbb{P}^{n-1}$ be two points and $X,Y\subset\mathbb{P}^{n-1}$ be two hyperplanes. If $x\notin Y$ and $y\notin X$, we define their *cross ratio* as

$$[x,X,y,Y]=\frac{\langle\hat{x},\hat{X}\rangle\langle\hat{y},\hat{Y}\rangle}{\langle\hat{x},\hat{Y}\rangle\langle\hat{y},\hat{X}\rangle}\in\mathbb{R},$$

where $\hat{x},\hat{y}\in\mathbb{R}^n$ are lifts of x and y, whereas $\hat{X},\hat{Y}\in\mathbb{R}^{*n}\setminus\{0\}$ are linear forms vanishing on X and Y, respectively. $\langle\cdot,\cdot\rangle$ is the pairing of $\mathbb{R}^n$ and $\mathbb{R}^{*n}$.

It is easy it see that $[x,X,y,Y]$ does not depend on the choices of $\hat{x},\hat{y},\hat{X},\hat{Y}$, and is invariant by projective transformations. When $n=2$ we get the usual definition of cross ratio on the projective line

$$[x,X,y,Y]=\frac{(x-X)(y-Y)}{(x-Y)(y-X)}.$$

We write a flag $x\in\mathcal{F}$ as $x=(x^{(1)}\subset\cdots\subset x^{(n)})$, where $x^{(k)}$ is a $k-1$-dimensional hyperplane in $\mathbb{P}^{n-1}$. Given four flags $x,y,z,w\in\mathcal{F}$, we define their cross ratio as

$$[x,y,z,w]:=[x^{(1)},y^{(n-1)},z^{(1)},w^{(n-1)}]. \tag{3.3}$$

We shall slightly reformulate this definition. Recall that a nonzero k-vector $l \in \bigwedge^k \mathbb{R}^n$ is called *decomposable* if $l = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k$, where $\boldsymbol{v}_i \in \mathbb{R}^n$. A decomposable k-vector determines a subspace $\bar{l} = \mathrm{Span}\{\boldsymbol{v}_1, \cdots \boldsymbol{v}_k\}$ independent of the choice of the $\boldsymbol{v}_i$'s. By an *affine flag* is meant a sequence of decomposable multi-vectors

$$\lambda = (\lambda^{(1)}, \cdots, \lambda^{(n-1)}), \quad \lambda^{(k)} \in \wedge^k \mathbb{R}^n,$$

such that $\overline{\lambda}^{(k)} \subset \overline{\lambda}^{(k+1)}$. Hence there is a underlying flag $\overline{\lambda} = (\overline{\lambda}^{(1)} \subset \cdots \subset \overline{\lambda}^{(n-1)})$, and we say that λ lifts $\overline{\lambda}$.

Take $\Omega \in \bigwedge^n \mathbb{R}^{*n}$ and let X, Y, Z, W be affine flags lifting x, y, z, w, respectively. Then the cross ratio (3.3) can be rewritten as

$$[x, y, z, w] = \frac{\Omega(X^{(1)} \wedge Y^{(n-1)})\Omega(Z^{(1)} \wedge W^{(n-1)})}{\Omega(Z^{(1)} \wedge Y^{(n-1)})\Omega(X^{(1)} \wedge W^{(n-1)})}, \tag{3.4}$$

which is independent of the choices of Ω and the lifts.

Cross ratio gives rise to functions on $\mathrm{Conf}_V(\mathcal{F})$. Indeed, given $x, y, z, w \in V$, by abuse of notation, we define the *cross ratio function* as [1]

$$[x, y, z, w] : \mathrm{Map}(V, \mathcal{F}) \to \mathbb{R}, \quad [x, y, z, w](f) = [f(x), f(y), f(z), f(w)].$$

Then $[x, y, z, w]$ is invariant under projective transformations, and is considered as a function on $\mathrm{Conf}_V(\mathcal{F})$.

As in §2.2.2, for any $1 \le i, j \le n$ we define a function $\overline{xy}_{ij} \in C^\infty(M_G(\mathbb{D}, V))$ by

$$\overline{xy}_{ij}(m) = \mathrm{hol}_{\overline{xy}}(m)_{ij}, \quad \forall m \in M_G(\mathbb{D}, V).$$

(On the right-hand side, we let g_{ij} denote the (i, j)-entry of a matrix $g \in G$.)

Proposition 3.6. *The lift of the cross ratio function to $M_G(\mathbb{D}, V)$ by the quotient map $\pi : M_G(\mathbb{D}, V) \to M_G(\mathbb{D}, V)/B^V \cong \mathrm{Conf}_V(\mathcal{F})$ is given by*

$$\pi^*[x, y, z, w] = \frac{\overline{yx}_{n1} \cdot \overline{wz}_{n1}}{\overline{yz}_{n1} \cdot \overline{wx}_{n1}}.$$

Proof. Let $U \subset G$ be the group consisting of upper-triangular matrices whose diagonals are 1. Given a basis (e_i) of $\mathbb{R}^n$, we have an identification between G/U and the space $\mathcal{A}$ of all affine flags. For any $a \in G$, we let $[a] \in \mathcal{A}$ denote the affine flag corresponding to the left coset $aU \in G/U$. Put $\Omega = e_1^* \wedge \cdots \wedge e_n^*$, where (e_i^*) is the dual basis of $\mathbb{R}^{*n}$, then we have

$$\Omega([a]^{(k)} \wedge [b]^{(n-k)}) = \det(a^{(1)}, \cdots, a^{(k)}, b^{(1)}, \cdots, b^{(n-k)}),$$

where $a^{(i)}$ denotes the i^{th} column of the matrix a. In particular,

$$\Omega([e]^{(n-1)} \wedge [a]^{(1)}) = a_{n1}. \tag{3.5}$$

Let $\mathrm{Conf}_V(\mathcal{A}) := \mathrm{Map}(V, \mathcal{A})/G$ be the configuration space of affine flags. Then π factorizes through the natural projection

$$\pi_0 : \mathrm{Conf}_V(\mathcal{A}) \to \mathrm{Conf}_V(\mathcal{F}).$$

1. We allow a negligible subset on which the function does not make sense.

Eq.(3.4) implies that the function $\pi_0^*[x,y,z,w]$ on $\mathrm{Conf}_V(\mathcal{A})$ is a fraction of four functions, each of them having the form

$$\mathrm{Conf}_V(\mathcal{A}) \to \mathbb{R}, \quad f \mapsto \Omega(f(x)^{(1)} \wedge f(y)^{(n-1)}) \quad \forall f \in \mathrm{Map}(V,\mathcal{A}). \tag{3.6}$$

This expression is invariant by the G-acton on $f \in \mathrm{Map}(V,\mathcal{A})$, hence indeed gives a function on the quotient $\mathrm{Conf}_V(\mathcal{A})$.

Now we only need to show that under the quotient map

$$M_G(\mathbb{D},V) \to M_G(\mathbb{D},V)/U^V \cong \mathrm{Conf}_V(\mathcal{A}),$$

the function (3.6) lifts to $\overline{yx}_{n1} \in C^\infty(M_G(\mathbb{D},V))$. But we have the following commutative diagram

$$\begin{tikzcd}
 & M_G(\mathbb{D},V) \arrow[dl, "\Psi_y"'] \arrow[d, "/U^V"] \arrow[dr, "/B^V"] & \\
\mathrm{Map}(V,\mathcal{A}) \arrow[r, "/G"] & \mathrm{Conf}_V(\mathcal{A}) \arrow[r, "\pi_0"] & \mathrm{Conf}_V(\mathcal{F})
\end{tikzcd}$$

where Ψ_y is the map

$$M_G(\mathbb{D},V) \to \mathrm{Map}(V,\mathcal{A}), \quad m \mapsto (x \mapsto [\mathrm{hol}_{\overline{yx}}(m)]).$$

Therefore it follows from Eq.(3.5) that the map (3.6) lifts to $\overline{yx}_{n1}$, as required. □

By the discussions in §2.2.3 (reinforced with §2.2.4, since $G = \mathrm{SL}_n\mathbb{R}$), the subalgebra of $C^\infty(M_G(\mathbb{D},V))$ generated by functions of the form $\overline{xy}_{n1}$ is closed under Poisson bracket:

$$\{\overline{xy}_{n1}, \overline{zw}_{n1}\} = \varepsilon(\overline{xy}, \overline{zw}) \left(\overline{xw}_{n1} \cdot \overline{zy}_{n1} - \frac{1}{n} \overline{xy}_{n1} \cdot \overline{zw}_{n1} \right).$$

This formula and Proposition 3.6 allows us to compute Poisson brackets of any two cross ratio functions on $\mathrm{Conf}_V(\mathcal{F})$.

3.1.4 Twisted configuration spaces

In this subsection we generalize Proposition 3.4 to arbitrary bordered surface Σ. We shall first introduce some notations.

Take a subset $V \subset \partial\Sigma$ and subgroups $C_v \subset G$ ($v \in V$) as before. Let $\pi : \mathbb{D} \to \Sigma \cong \mathbb{D}/\Pi$ be the universal covering map, where $\Pi := \pi_1(\Sigma)$ is the fundamental group acting on $\mathbb{D}$ by deck transformations. Put $\widetilde{V} = \pi^{-1}(V)$. Given a representation $\varphi \in \mathrm{Hom}(\Pi, G)$, an element $(a_{\tilde{v}})$ in $\prod_{\tilde{v} \in \widetilde{V}} G/C_{\pi(\tilde{v})}$ is said to be *φ-equivariant* if

$$\varphi(\gamma).a_{\tilde{v}} = a_{\gamma.\tilde{v}}, \quad \forall \gamma \in \Pi.$$

The generalization of Proposition 3.4 to this situation is the following

Proposition 3.7. *We have a canonical identification*

$$M_G(\Sigma, V)/\prod_{v\in V} C_v = \Big\{ ((a_{\tilde{v}}), \varphi) \in \prod_{\tilde{v} \in \widetilde{V}} G/C_{\pi(\tilde{v})} \times \mathrm{Hom}(\Pi, G) \ \Big| \ (a_{\tilde{v}}) \textit{ is } \varphi\textit{-equivariant} \Big\} / G.$$

Here G acts diagonally on the product $\prod_{\tilde{v} \in \widetilde{V}} G/C_{\pi(\tilde{v})} \times \mathrm{Hom}(\Pi, G)$.

We omit the proof since it is a straightforward adaptation of the proof of Proposition 3.4.

In particular, if $C_v = C$ for any v, then Proposition 3.7 identifies $M_G(\Sigma, V)/C^V$ with

$$\left\{(f, \varphi) \in \mathrm{Map}(\widetilde{V}, G/C) \times \mathrm{Hom}(\Pi, G) \;\middle|\; f \text{ is } \varphi\text{-equivariant}\right\},$$

which is viewed as a "twisted configuration space" of points in G/C.

Example 3.8. Let Σ be the cylinder. Then for any subset $V \subset \partial\Sigma$ of order N, the pair (Σ, V) is homotopy equivalent to the pair $(\mathbb{R}/N\mathbb{Z}, \mathbb{R}/\mathbb{Z})$, so $M_G(\Sigma, V) = M_G(\mathbb{R}/N\mathbb{Z}, \mathbb{R}/\mathbb{Z})$. The twisted configuration space in this case is

$$\{((a_i)_{i\in\mathbb{Z}}, h) \mid a_i \in G/C,\ h \in G,\ h.a_{i+N} = a_i\}/G,$$

where the G-action is given by $g.((a_i)_{i\in\mathbb{Z}}, g) = ((g.a_i)_{i\in\mathbb{Z}}, ghg^{-1})$.

3.2 Fock-Goncharov Poisson structure from quasi-Poisson reduction

3.2.1 Framed connections

Let G be the split real form of a complex reductive group. Fix a maximal torus $H \subset G$ and a Borel subgroup B containing H. Let $\mathcal{F} = G/B$ denote the flag variety.

Let $\mathcal{R} \subset \mathfrak{h}^*$ be the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. Each $\alpha \in \mathcal{R}$ integrates to a character $H \to \mathbb{R}^*, \lambda \mapsto \lambda^\alpha$. An element λ in H is called *regular* (or *loxodromic*) if $\lambda^\alpha \neq 1$ for any $\alpha \in \mathcal{R}$. An element g in G is called regular if it is conjugate to a regular element in H. Let $H^{\mathrm{reg}}, G^{\mathrm{reg}}$ denote the set of regular elements in H and G, respectively.

Let Σ be compact oriented surface such that $\partial\Sigma \neq \emptyset$. We choose a finite (possibly empty) subset $W \subset \partial\Sigma$ and put $\widehat{\Sigma} = \Sigma \setminus W$. In the picture below, points in W are indicated by little circle.

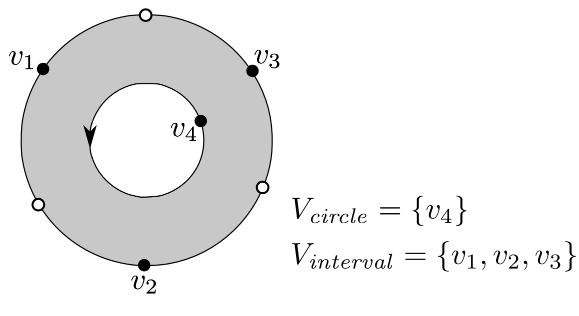


Figure 3.1: A surface with a choice of W and V.

Definition 3.9. Let $P \to \Sigma$ be a principal G-bundle. A *framed flat G-connection* on $\widehat{\Sigma}$ is a pair (∇, f), where ∇ is a flat G-connection on P and f is a ∇-invariant choice of a reduction of structure group to B on each component of $\partial\Sigma \setminus W$ (such f is called a *framing* of ∇). The space of all framed G-connections on P is denoted by $\mathcal{A}^{\mathcal{F}}_{\mathit{flat}}(P)$. We let $\mathcal{A}^{\mathcal{F}}_{\mathit{flat}} = \bigsqcup_P \mathcal{A}^{\mathcal{F}}_{\mathit{flat}}(P)$ be the disjoint union, where P runs over isomorphism classes of principal G-bundles over Σ.

Precisely, f is a choice of a B-orbit [2] in the fiber $P|_q$ for any $q \in \partial\Sigma \setminus W$, such that if q_1 and q_2 lies on the same component of $\partial\Sigma \setminus W$ then the parallel translation of ∇ carries the B-orbit in $P|_{q_1}$ to the one in $P|_{q_2}$. Clearly, to determine f we only need to pick one point on each component of $\partial\Sigma \setminus W$ and describe the B-orbits at these points.

Definition 3.10. A *circle component* (resp. *interval component*) of $\partial\Sigma\setminus W$ is a connected component of $\partial\Sigma \setminus W$ which is homeomorphic to a circle (resp. an open interval). Let $V \subset \partial\Sigma \setminus W$ be a set of marked points such that each component of $\partial\Sigma \setminus W$ contains exactly one marked point, *c.f.* Figure 3.1. Let $V_{circle}, V_{interval} \subset V$ consist of marked points on circle and interval components, respectively. For each $v \in V_{circle}$ we let β_v denote the boundary loop based at v, oriented against the induced orientation.

Fix a trivialization $P|_v \cong G$ at each $v \in V$. Then a framing of a flat connection ∇ is the same as a choice of left B-coset in each $P|_v$, such that for any marked point on circle component $v \in V_{circle}$, the chosen coset aB satisfies $\mathrm{hol}_{\beta_v}(\nabla) \in aBa^{-1}$. Thus we have an identification

$$\mathcal{A}^{\mathcal{F}}_{\mathit{flat}}(P) = \{(\nabla, (f_v)_{v\in V}) \in \mathcal{A}_{\mathit{flat}}(P) \times \mathcal{F}^V \mid \forall v \in V_{circle},\ \mathrm{hol}_{\beta_v}(\nabla) \text{ fixes the flag } f_v\} \quad (3.7)$$

Definition 3.11. The gauge group $\mathcal{G}(P)$ naturally acts on $\mathcal{A}^{\mathcal{F}}_{\mathit{flat}}(P)$ and the quotient is denoted by $\mathscr{X}_{G,\widehat{\Sigma}}(P)$. The disjoint union $\mathscr{X}_{G,\widehat{\Sigma}} = \bigsqcup_P \mathscr{X}_{G,\widehat{\Sigma}}(P)$, where P runs over isomorphism classes of principal G-bundles over Σ, is called the *moduli space of framed G-connections* over $\widehat{\Sigma}$.

Moreover, we let $\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}}(P)$ denote the $\mathcal{G}(P)$-quotient of

$$\{(\nabla, (f_v)_{v\in V}) \in \mathcal{A}^{\mathcal{F}}_{\mathit{flat}}(P) \mid \mathrm{hol}_{\beta_v} \in G^{\mathrm{reg}}, \forall v \in V_{circle}\}$$

and put $\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}} = \bigsqcup_P \mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}}(P)$

The following proposition provides our working definition of $\mathscr{X}_{G,\widehat{\Sigma}}$:

Proposition 3.12. *The subset $\bigcap_{v\in V_{circle}} \mathrm{hol}^{-1}_{\beta_v}(B)$ of the G^V-manifold $M_G(\Sigma, V)$ is invariant by B^V, and we have identifications*

$$\mathscr{X}_{G,\widehat{\Sigma}} \cong \left(\bigcap_{v\in V_{circle}} \mathrm{hol}^{-1}_{\beta_v}(B) \right) / B^V, \quad (3.8)$$

$$\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}} \cong \left(\bigcap_{v\in V_{circle}} \mathrm{hol}^{-1}_{\beta_v}(H^{\mathrm{reg}}) \right) / (B^{V_{interval}} \times H^{V_{circle}}) \quad (3.9)$$

In particular, if $V_{circle} = \emptyset$, i.e., W has at least one point on each component of $\partial\Sigma$, then $\mathscr{X}_{G,\widehat{\Sigma}} \cong M_G(\Sigma, V)/B^V$ is a twisted configuration space (c.f. §3.6).

Remark 3.13. It follows from the cross-section theory (§1.4.2) and Proposition 3.2 that the quotient (3.9) is a Poisson manifold (see §3.2.6 below for details); on the other hand, we do not know at present how to use (quasi-)Poisson theory to endow the quotient (3.8) with a Poisson structure, unless $V_{circle} = \emptyset$. This is the reason why we have to introduce the more restrictive identification (3.9) rather than working directly with (3.8).

2. Since P is a principal G-bundle, G acts on each fiber on the right.

Example 3.14. (1) Let Σ be the disk $\mathbb{D}$. Then $\partial\mathbb{D} \setminus W$ consists of finitely many open intervals and V has one point in each interval. In this case $\mathscr{X}_{G,\widehat{\mathbb{D}}}$ is a configuration space of flags:

$$\mathscr{X}_{G,\widehat{\mathbb{D}}} = M_G(\mathbb{D}, V)/B^V \cong \mathrm{Conf}_V(\mathcal{F}).$$

(2) If $W = \emptyset$, then $\widehat{\Sigma} = \Sigma$, V has exactly one point on each component of $\partial\Sigma$, and

$$\mathscr{X}^{\mathrm{reg}}_{G,\Sigma} \cong L/H^V,$$

where

$$L = \bigcap_{v\in V} \mathrm{hol}^{-1}_{\beta_v}(H^{\mathrm{reg}}).$$

We denote $M^{\mathrm{reg}}_G(\Sigma, V) = \bigcap_{v\in V} \mathrm{hol}^{-1}_{\beta_v}(G^{\mathrm{reg}})$. Then in this case $\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}}$ is a finite covering of the open subset

$$X^{\mathrm{reg}}_G(\Sigma, V) := M^{\mathrm{reg}}_G(\Sigma, V)/G^V$$

of $X_G(\Sigma) = M_G(\Sigma, V)/G^V$.

In fact, it follows from Lemma 1.27 (2) that we can identify $X^{\mathrm{reg}}_G(\Sigma, V) \cong L/N_G(H)^V$. So $\mathscr{X}^{\mathrm{reg}}_{G,\Sigma} \cong L/H^V$ is a degree $\#V \cdot \#N_G(H)/H$ covering of $X^{\mathrm{reg}}_G(\Sigma, V)$ (recall that $N_G(H)/H$ is the Weyl group and has finite order).

Proof of Proposition 3.12. For brevity we assume in this proof that G is connected, so any principal G-bundle over Σ is isomorphic to the trivial one. In the general case, one only needs to argue for each isomorphism class of principal G-bundles and take the disjoint union.

Recall that $M_G(\Sigma, V)$ is identified with the quotient of $\mathcal{A}_{\mathit{flat}}$ by the restricted gauge group

$$\mathcal{G}_V = \{g \in \mathcal{G} \mid g(v) = \mathrm{id},\ \forall v \in V\}.$$

In view of the the interpretation (3.7) of $\mathcal{A}^{\mathcal{F}}_{\mathit{flat}}$, we have

$$\mathcal{A}^{\mathcal{F}}_{\mathit{flat}}/\mathcal{G}_V = \{(m, (f_v)_{v\in V}) \in M_G(\Sigma, V) \times \mathcal{F}^V \mid \forall v \in V_{\mathit{circle}},\ \mathrm{hol}_{\beta_v}(m) \text{ fixes the flag } f_v\},$$

and $\mathscr{X}_{G,\widehat{\Sigma}} = \mathcal{A}^{\mathcal{F}}_{\mathit{flat}}/\mathcal{G}$ is the quotient of the above set by the diagonal action of $G^V = \mathcal{G}/\mathcal{G}_V$.

Let $([B], \cdots, [B]) \in \mathcal{F}^V$ be the element whose entries are all the origin $[B] \in G/B = \mathcal{F}$. Since $\mathcal{F}$ is a homogenous G-space, we can bring any element of $\mathcal{A}^{\mathcal{F}}_{\mathit{flat}}/\mathcal{G}_V$ into the slice

$$\bigcap_{v\in V_{\mathit{circle}}} \mathrm{hol}^{-1}_{\beta_v}(B) \times \{([B], \cdots, [B])\} \subset \mathcal{A}^{\mathcal{F}}_{\mathit{flat}}/\mathcal{G}_V$$

by the G^V-action. As a result, $\mathscr{X}_{G,\widehat{\Sigma}}$ is the quotient of the above subset by

$$\mathrm{Stab}_{G^V}([B], \cdots, [B]) = B^V.$$

This proves the first statement.

To prove the second statement, we put

$$L = \bigcap_{v\in V_{\mathit{circle}}} \mathrm{hol}^{-1}_{\beta_v}(H^{\mathrm{reg}}), \quad N = \bigcap_{v\in V_{\mathit{circle}}} \mathrm{hol}^{-1}_{\beta_v}(B \cap G^{\mathrm{reg}})$$

By definition and the first statement, we have $\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}} = N/B^V$. Since every B-orbit in $B \cap G^{\mathrm{reg}}$ (with respect to the conjugation action) intersects H^{reg} and we have $\mathrm{Stab}_B(h) = H$ for any $h \in H^{\mathrm{reg}}$, it follows from equivariance of the hol_{β_v}'s that every B^V-orbit in N intersects L, and the intersection is a $B^{V_{\mathit{interval}}} \times H^{V_{\mathit{circle}}}$-orbit. This establishes the required identification. $\square$

3.2.2 Triple ratio functions

In the sequel we set $G = \mathrm{SL}_n\mathbb{R}$ and let $B \subset G$ be the upper-triangular subgroup. To simplify notations, we assume $V_{circle} = \emptyset$ unless otherwise specified, so that $\mathscr{X}_{G,\widehat{\Sigma}} = M_G(\Sigma, V)/B^V$.

Fock and Goncharov [25] introduced some rational functions on $\mathscr{X}_{G,\widehat{\Sigma}}$ which provide coordinates systems on $\mathscr{X}_{G,\widehat{\Sigma}}$. These functions consists of two types: triple ratio functions and edge functions. In this subsection we give our reformulation of the definition of triple ratio functions. Edges functions will be treated in the next subsection.

Like cross ratio functions (Proposition 3.6), triple ratio functions and edge functions are fractions of certain spin networks on $M_G(\Sigma, V)$. The graph diagrams of the spin networks occurring here are tripods (see Example 2.3 (2)).

Definition 3.15. For each triple of non-negative integers p, q, r satisfying $p + q + r = n$, we define a function $\mathrm{Det}_{p,q,r} : G \times G \times G \to \mathbb{R}$ by

$$\mathrm{Det}_{p,q,r}(a, b, c) = \det(a^{(1)}, \cdots, a^{(p)}, b^{(1)}, \cdots, b^{(q)}, c^{(1)}, \cdots, c^{(r)}),$$

where for any matrix $a \in G$, we denote the i^{th} column of a by $a^{(i)}$. Note that $\mathrm{Det}_{p,q,r}$ is invariant by the diagonal action of G by left-multiplication.

Let Δ be a tripod on Σ with feet $v_1, v_2, v_3 \in V$, an interior vertex o, and legs e_1, e_2, e_3, as shown in Figure 3.2 on the left. Consider $\mathrm{Det}_{p,q,r}$ as a function on G^{E_Δ}, then it is admissible. We denote the resulting spin network by

$$\Delta_{p,q,r} := [\Delta, \mathrm{Det}_{p,q,r}] \in C^\infty(M_G(\Sigma, V)).$$

In particular, $\Delta_{n,0,0} = \Delta_{0,n,0} = \Delta_{0,0,n} \equiv 1$. By definition, the $\Delta_{p,q,r}$'s are expressed in terms of holonomies by

$$\begin{aligned}\Delta_{p,q,r} &= \mathrm{Det}_{p,q,r}(1\ , \mathrm{hol}_{e_1^{-1}e_2}, \mathrm{hol}_{e_1^{-1}e_3}) = \mathrm{Det}_{p,q,r}(\mathrm{hol}_{e_2^{-1}e_1}, 1\ , \mathrm{hol}_{e_2^{-1}e_3})\\ &= \mathrm{Det}_{p,q,r}(\mathrm{hol}_{e_3^{-1}e_2}, \mathrm{hol}_{e_3^{-1}e_1}, 1\ ).\end{aligned}$$

$\Delta_{p,q,r}$ is not invariant under B^V, but they have nice covariant properties:

Lemma 3.16. *Let $\chi_k : B \to \mathbb{R}^*$ $(1 \le k \le n)$ be the character of B whose value at $b \in B$ is the product of the first k diagonal entries of b. Then for any $(b_v)_{v\in V} \in B^V$ we have*

$$(b_v)_{v\in V}.\Delta_{p,q,r} = \chi_p(b_{v_1})\chi_q(b_{v_2})\chi_r(b_{v_3})\Delta_{p,q,r}.$$

Proof. Follows from the elementary fact that if $a, b, c \in G$ and $b_1, b_2, b_3 \in B$ then

$$\mathrm{Det}_{p,q,r}(ab_1, bb_2, cb_3) = \chi_p(b_1)\chi_q(b_2)\chi_r(b_3)\,\mathrm{Det}_{p,q,r}(a, b, c).$$

□

To better imagine the totality of all the $\Delta_{p,q,r}$'s for a fixed Δ, we fatten Δ into a triangle, and make a equilateral n-subdivision of the triangle, as shown in Figure 3.2 on the right.

Let I_Δ denote the the set of vertices of the subdivision, which are shown as black dots. As indicated in the picture, points in I_Δ are in one-one correspondence with triples of non-negative integers (p, q, r) with $p + q + r = n$. If $\mathbf{i} \in I_\Delta$ is labelled (p, q, r), we denote $\Delta_{\mathbf{i}} := \Delta_{p,q,r}$.

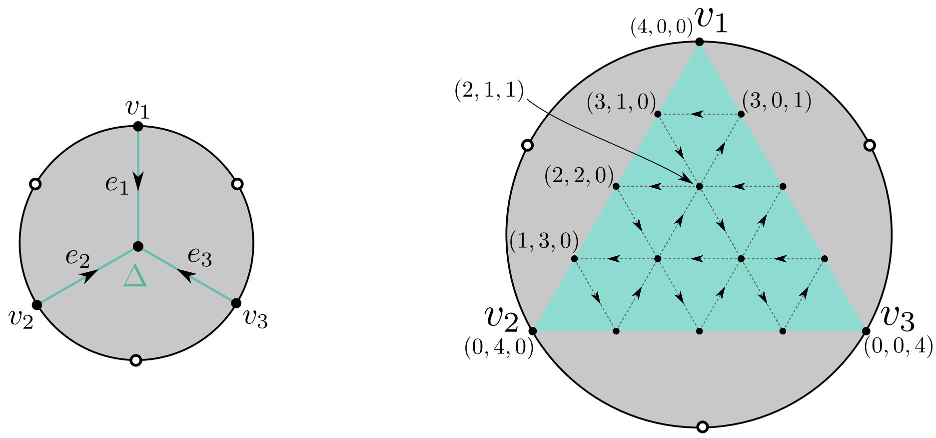


Figure 3.2: Tripod Δ, fattened Δ and the associated $\Delta_{p,q,r}$'s

We shall orient each internal edge of the equilateral n-subdivision in the way shown in Figure 3.2. For each $\mathbf{i}, \mathbf{j} \in I_\Delta$, we set

$$\epsilon_{\mathbf{ij}} = \begin{cases} 1 & \text{if there is an edge from } \mathbf{i} \text{ to } \mathbf{j}, \\ -1 & \text{if there is an edge from } \mathbf{j} \text{ to } \mathbf{i}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Definition 3.17. Let $I_\Delta^{int} \subset I_\Delta$ denote the set of internal vertices of the equilateral n-subdivision, *i.e.*, those corresponding to triples of integers (p,q,r) with $p,q,r > 0$. The *triple ratio function* $X_{\mathbf{i}}^\Delta$ at a vertex $\mathbf{i} \in I_\Delta^{int}$ is a rational function on $M_G(\Sigma, V)$ defined by

$$X_{\mathbf{i}}^\Delta = \prod_{\mathbf{j} \in I_\Delta} \Delta_{\mathbf{j}}^{\epsilon_{\mathbf{ji}}},$$

or more explicitly,

$$X_{p,q,r}^\Delta = \frac{\Delta_{p+1,q,r-1}\Delta_{p-1,q+1,r}\Delta_{p,q-1,r+1}}{\Delta_{p,q+1,r-1}\Delta_{p-1,q,r+1}\Delta_{p+1,q-1,r}}.$$

It follows from Lemma 3.16 that $X_{\mathbf{i}}$ is invariant under B^V, hence we also consider $X_{\mathbf{i}}$ as a function on the quotient

$$\mathscr{X}_{G,\widehat{\Sigma}} = M_G(\Sigma, V)/B^V,$$

where we allow a negligible subset on which $X_{\mathbf{i}}$ is not defined.

3.2.3 Edge functions

Two tripods Δ_1 and Δ_2 on (Σ, V) are said to be *adjacent* if they share two feet, and the two-leg in Δ_1 supported at these feet is homotopic to the one in Δ_2.

Let Δ_1 and Δ_2 be adjacent tripods. We fatten them into adjacent triangles, which share an edge $\boldsymbol{e}$, and make an equilateral n-subdivision on each triangle as before. See the picture below. Here $\boldsymbol{e}$ is the vertical edge in the middle.

As before, we orient each internal edge of the subdivision, and define a number $\epsilon_{\mathbf{ij}} = 0, \pm 1$ for each $\mathbf{i}, \mathbf{j} \in I_{\Delta_1} \cup I_{\Delta_2}$. We also label vertices in I_{Δ_1} and vertices in I_{Δ_2} by triples of non-negative integers (p,q,r) with $p+q+r=n$, respectively, in the way shown in the above picture.

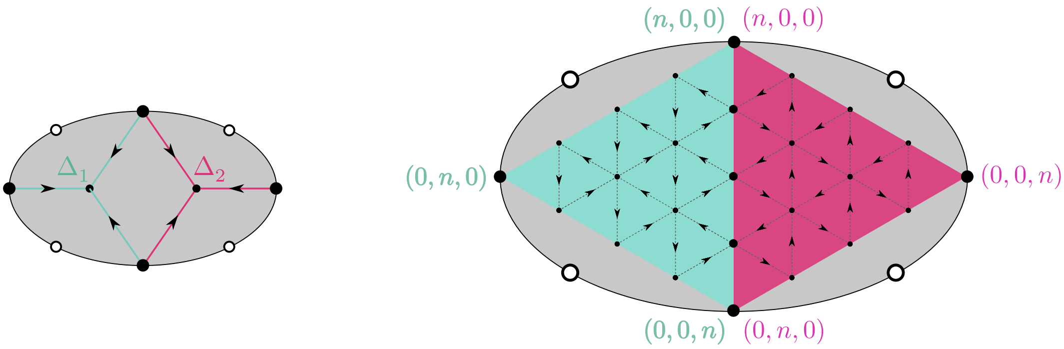


Figure 3.3: Two adjacent tripods

Definition 3.18. Let $\mathbf{i}$ be a vertex of the subdivision contained in the interior of the edge $\boldsymbol{e}$ (there are 3 such vertices in the above picture). The *edge function* $X_{\mathbf{i}}^{\Delta_1,\Delta_2}$ at $\mathbf{i}$ is a rational function on $M_G(\Sigma, V)$ given by

$$X_{\mathbf{i}}^{\Delta_1,\Delta_2} = \prod_{\mathbf{j}\in I_{\Delta_1}\cup I_{\Delta_2}} \Delta_{\mathbf{j}}^{\epsilon_{\mathbf{ji}}}.$$

Here $\Delta_{\mathbf{j}} = (\Delta_1)_{\mathbf{j}}$ (resp. $(\Delta_2)_{\mathbf{j}}$) if $\mathbf{j}$ is contained in the fattening of Δ_1 (resp. Δ_2).

By the same reason as for triple ratio functions, each edge function $X_{\mathbf{i}}^{\Delta_1,\Delta_2}$ is invariant under B^V, hence we consider it as functions on $\mathscr{X}_{G,\widehat{\Sigma}} = M_G(\Sigma,V)/B^V$, allowing a negligible subset on which $X_{\mathbf{i}}^{\Delta_1,\Delta_2}$ is not defined.

Note that each $\mathbf{i}$ in the interior of $\boldsymbol{e}$ is labelled $(k,0,l)$ in I_{Δ_1} and labelled $(k,l,0)$ in I_{Δ_2}. Here (k,l) is a pair of positive integers satisfying $k+l=n$. A more explicit expression for the edge functions is

$$X_{\mathbf{i}}^{\Delta_1,\Delta_2} = \frac{(\Delta_1)_{k-1,1,l}(\Delta_2)_{k,l-1,1}}{(\Delta_1)_{k,1,l-1}(\Delta_2)_{k-1,l,1}}.$$

Like cross ratio functions in §3.1.3, triple ratio functions and edge functions come from projective invariants of flags. In fact, given $p,q,r\in\mathbb{N}_+$ such that $p+q+r=n$, using the same notations as Eq.(3.4), the *triple ratio* of three flags $x,y,z\in\mathcal{F}$ is defined as

$$T_{p,q,r}(x,y,z) = \frac{\Omega(X^{(p+1)}\wedge Y^{(q)}\wedge Z^{(r-1)})\Omega(X^{(p-1)}\wedge Y^{(q+1)}\wedge Z^{(r)})\Omega(X^{(p)}\wedge Y^{(q-1)}\wedge Z^{(r+1)})}{\Omega(X^{(p)}\wedge Y^{(q+1)}\wedge Z^{(r-1)})\Omega(X^{(p-1)}\wedge Y^{(q)}\wedge Z^{(r+1)})\Omega(X^{(p+1)}\wedge Y^{(q-1)}\wedge Z^{(r)})},$$

which is independent of the choices of $\Omega\in\bigwedge^n\mathbb{R}^{*n}$ and the affine flags X,Y,Z lifting x,y,z, and invariant by projective transformations.

Proposition 3.19. *Let $\Sigma=\mathbb{D}$ be the disk and Δ be a tripod on $\mathbb{D}$ with feet $v_1,v_2,v_3\in V$. Then the function $X_{p,q,r}^{\Delta}$ on $M_G(\mathbb{D},V)/B^V\cong \mathrm{Conf}_V(\mathcal{F})=\mathrm{Map}(V,\mathcal{F})/G$ can be alternatively expressed as*

$$X_{p,q,r}^{\Delta}(f) = T_{p,q,r}(f(v_1),f(v_2),f(v_3)),\quad \forall f\in\mathrm{Map}(V,\mathcal{F}).$$

This statement is similar to Proposition 3.6, and the proof is essentially the same.

Similarly, given $k,l\in\mathbb{N}_+$ such that $k+l=n$, we can define a projective invariant of four flags

$$T_{k,l}(x,y,z,w) = \frac{\Omega(X^{(k-1)}\wedge Y^{(1)}\wedge Z^{(l)})\Omega(X^{(k)}\wedge Z^{(l-1)}\wedge W^{(1)})}{\Omega(X^{(k)}\wedge Y^{(1)}\wedge Z^{(l-1)})\Omega(X^{(k-1)}\wedge Z^{(l)}\wedge W^{(1)})},$$

and a similar statement as Proposition 3.19 holds. Note that the $T_{k,l}(x,y,z,w)$'s are related to the cross ratio $[x,y,z,w]$ by

$$\prod_{k+l=n} T_{k,l}(x,y,z,w) = -[y,z,w,x].$$

3.2.4 Ideal triangulation and coordinates system

Fock and Goncharov showed that one can choose some tripods on (Σ, V) such that the triple ratio functions and edge functions associated with them form a coordinates system of $\mathscr{X}_{G,\widehat{\Sigma}}$. In this subsection we briefly review this construction.

The choice of tripods here is determined by some combinatorial data – a triangulation $\mathcal{T}$ of Σ with vertex set V. It is easy to see that, by duality, giving such a triangulation is equivalent to giving a trivalent graph $\Gamma_{\mathcal{T}}$ on Σ with vertex set W such that Σ retracts to $\Gamma_{\mathcal{T}}$ by deformations. See Figure 3.4 for some examples, where each colored region represents a triangle. Note that here we need the assumption $V_{circle} = \emptyset$.

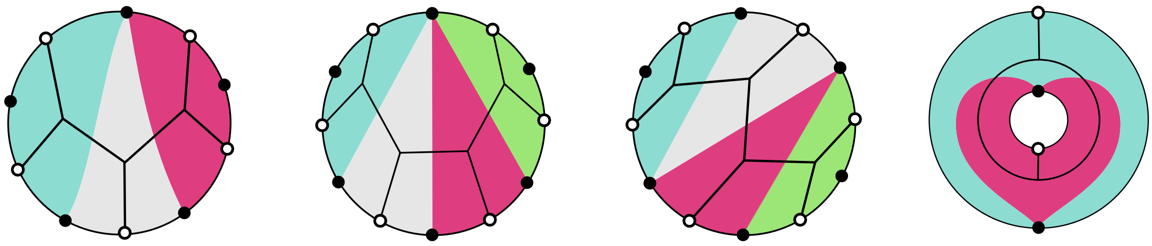
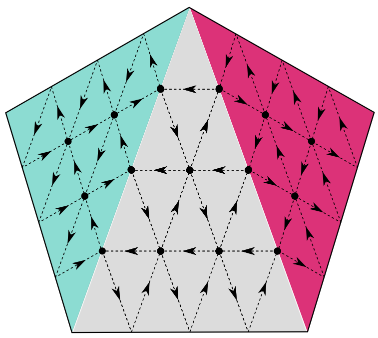


Figure 3.4: Triangulations and dual graphs.

Fix such a triangulation $\mathcal{T}$. We consider a triangle of $\mathcal{T}$ as a fattened tripod on (Σ, V). So each triangle gives rise to $\frac{(n-1)(n-2)}{2}$ triple ratio functions, and each two adjacent triangles give rise to $n-1$ edges functions. The Fock-Goncharov coordinate system just consists of all these functions.

More precisely, we make an equilateral n-subdivision of each triangle of $\mathcal{T}$ and orient the edges of the subdivision as before. *c.f.* the following picture, which corresponds to the first example in Figure 3.4.



We let $I_{\mathcal{T}}$ denote the set of vertices of the subdivision, and define $\epsilon_{\mathbf{ij}} = \pm 1$ ($\mathbf{i}, \mathbf{j} \in I_{\mathcal{T}}$) by (3.10) as before. Put $I_{\mathcal{T}}^{int} = I_{\mathcal{T}} \setminus \partial\Sigma$ (points of $I_{\mathcal{T}}^{int}$ are black dots in the above picture). Then we have a spin network $\Delta_{\mathbf{i}} \in C^\infty(M_G(\Sigma, V))$ for each $\mathbf{i} \in I_{\mathcal{T}}$, and a resulting triple ratio function or edge function

$$X_{\mathbf{i}} = \prod_{\mathbf{j} \in I_{\mathcal{T}}} \Delta_{\mathbf{j}}^{\epsilon_{\mathbf{ji}}}.$$

for each $\mathbf{i} \in I_{\mathcal{T}}^{int}$.

Theorem 3.20 (Fock-Goncharov [25]). *Let* $\mathscr{X}^{\circ}_{G,\widehat{\Sigma}} \subset \mathscr{X}_{G,\widehat{\Sigma}}$ *be an open subset on which* $X_{\mathbf{i}}$ *is defined for every* $\mathbf{i} \in I_{\mathcal{T}}^{int}$. *Then the map*

$$(X_{\mathbf{i}})_{\mathbf{i}\in I_{\mathcal{T}}^{int}} : \mathscr{X}^{\circ}_{G,\widehat{\Sigma}} \longrightarrow \mathbb{R}^{\# I_{\mathcal{T}}^{int}}$$

is a diffeomorphism to the image.

When $V_{circle} \neq \emptyset$, the same statement is true if we take a suitable triangulation $\mathcal{T}$. This will be discussed in the next subsection.

3.2.5 The $V_{circle} \neq \emptyset$ case

When $V_{circle} \neq \emptyset$, the moduli space of framed connections $\mathscr{X}_{G,\widehat{\Sigma}}$ is the quotient of

$$N = \bigcap_{v\in V_{circle}} \mathrm{hol}_{\beta_v}^{-1}(B) \subset M_G(\Sigma, V)$$

by B^V. The triple ratio functions and edges functions as defined in the same way as before and restricted to N.

However, in this case one should modify the triangulation $\mathcal{T}$ in the previous subsection in order to get a coordinates system.

Namely, rather than triangulating Σ, we shall triangulate the surface $\overline{\Sigma}$ obtained by shrinking each circle component of $\partial\Sigma \setminus W$ into a point. Let $\overline{V}$, $\overline{V}_{interval}$ and $\overline{V}_{circle}$ be, respectively, projections of V, $V_{interval}$ and V_{circle} on $\overline{\Sigma}$. So $\overline{V}_{circle}$ is in the interior of $\overline{\Sigma}$.

Any triangulation $\mathcal{T}$ of $\overline{\Sigma}$ with vertex set $\overline{V}$ is still equivalent to a trivalent graph $\Gamma_{\mathcal{T}}$ with vertex set W such that Σ retracts to $\Gamma_{\mathcal{T}}$ by deformations. *c.f.* the examples pictured below. Note that W can be empty.

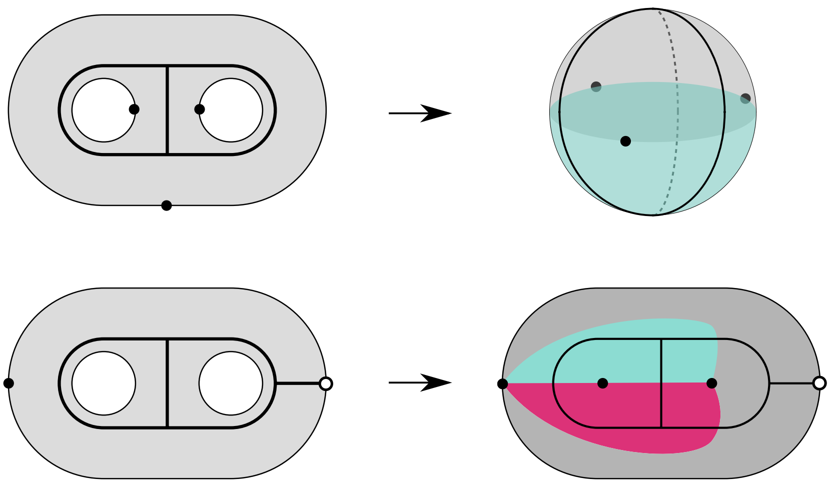


Figure 3.5: Trivalent graphs on Σ correspond to triangulations of $\overline{\Sigma}$.

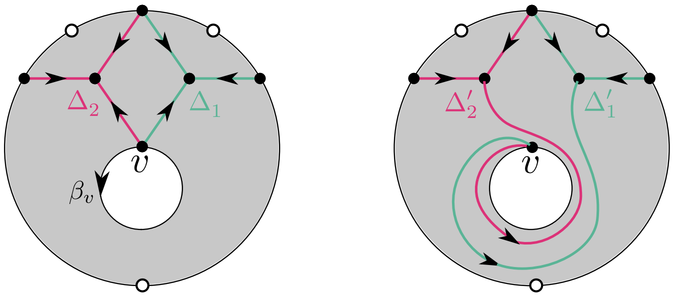
Fix such a triangulation $\mathcal{T}$. We can do the same thing to $\mathcal{T}$ as in the previous subsection, except that when defining $\Delta_{\mathbf{i}} \in C^{\infty}(M_G(\Sigma, V))$ for $\mathbf{i} \in I_{\mathcal{T}}$, we must lift the triangle in which $\mathbf{i}$ sits to a triangle (a fattened tripod) on Σ. The following lemma ensures that $\Delta_{\mathbf{i}}$ does not depend on the choice of the lift.

Lemma 3.21. *Let Δ be a tripod on (Σ, V) which has $v \in V_{circle}$ as a foot. We replace the leg e of Δ supported at v by the new leg $e\beta_v$, and denote the resulting tripod by Δ'. Then the triple ratio functions associated with Δ and Δ' coincides. Namely,*

$$X^{\Delta}_{p,q,r} = X^{\Delta'}_{p,q,r}, \quad \forall p, q, r \in \mathbb{N}_+ \text{ such that } p + q + r = n.$$

Similarly, if Δ_1, Δ_2 are adjacent tripods and Δ'_1, Δ'_2 are obtained from they by the above modification (see the picture below), then we also have equalities of edges functions

$$X^{\Delta_1,\Delta_2}_{k,l} = X^{\Delta'_1,\Delta'_2}_{k,l}, \quad \forall k, l \in \mathbb{N}_+ \text{ such that } k + l = n.$$



Proof. Let v be labelled by $(n, 0, 0)$ in the n-subdivision of fattened Δ, then for any p, q, r we have $\Delta'_{p,q,r} = \chi_p(\mathrm{hol}_{\beta_v})\Delta'_{p,q,r}$. One verifies by definition that $X^{\Delta}_{p,q,r} = X^{\Delta'}_{p,q,r}$. The seconded statement is similar.

□

Once the $\Delta_{\mathbf{i}}$'s are in hand, we define the $X_{\mathbf{i}}$'s as in the previous subsection. Theorem 3.20 still holds in this case.

3.2.6 Fock-Goncharov Poisson structure from quasi-Poisson reduction

Fock and Goncharov defined a Poisson structure on an open subset of $\mathscr{X}_{G,\widehat{\Sigma}}$ by prescribing Poisson brackets of their coordinates functions:

Definition 3.22. Let the notations be the same as in Theorem 3.20. The *Fock-Goncharov Poisson structure* on $\mathscr{X}^{\circ}_{G,\widehat{\Sigma}}$ is defined by declaring the Poisson brackets of the $X_{\mathbf{i}}$'s to be

$$\{X_{\mathbf{i}}, X_{\mathbf{j}}\} = -\epsilon_{\mathbf{ij}} X_{\mathbf{i}} X_{\mathbf{j}}, \quad \forall \mathbf{i}, \mathbf{j} \in I^{int}_{\mathcal{T}}. \tag{3.11}$$

On the other hand, the quasi-Poisson theory also provides us with a Poisson structure on an open subset of $\mathscr{X}_{G,\widehat{\Sigma}}$. Indeed, the set H^{reg} of regular elements in H is a cross-section, and we have a vector space decomposition $\mathfrak{sl}_n\mathbb{R} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ ($\mathfrak{h}^{\perp}$ consisting of matrices with zero diagonals) and a corresponding splitting $s = s_{\mathfrak{h}} + s_{\mathfrak{h}^{\perp}}$. By Corollary 1.29, the $H^{V_{circle}} \times G^{V_{interval}}$-manifold

$$L = \bigcap_{v \in V_{circle}} \mathrm{hol}^{-1}_{\beta_v}(H^{\mathrm{reg}})$$

has a canonical quasi-Poisson $(\mathfrak{h}^{V_{circle}} \oplus \mathfrak{g}^{V_{interval}}, s^{(V_{circle})}_{\mathfrak{h}} \oplus s^{(V_{interval})})$-tensor P_L. So Proposition 3.2 says that $\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}} \cong L/\left(B^{V_{interval}} \times H^{V_{circle}}\right)$ carries a Poisson structure reduced from P_L.

The main result of this section is

Theorem 3.23. *The above Poisson structure on $\mathscr{X}^{\mathrm{reg}}_{G,\widehat{\Sigma}}$ coincides with the Fock-Goncharov Poisson structure.*

In particular, when $W = \emptyset$, the Fock-Goncharov Poisson structure coincides with Atiyah-Bott's:

Corollary 3.24. *We assume $W = \emptyset$ and put $L = \bigcap_{v\in V} \mathrm{hol}^{-1}_{\beta_v}(H^{\mathrm{reg}})$, so that there is a covering map*

$$\mathscr{X}^{\mathrm{reg}}_{G,\Sigma} \cong L/H^V \to L/N_G(H)^V \cong M^{\mathrm{reg}}_G(\Sigma, V)/G^V = X^{\mathrm{reg}}_G(\Sigma)$$

(c.f. Example 3.14 (2)). Then the lift of the Atiyah-Bott Poisson structure on $X^{\mathrm{reg}}_G(\Sigma)$ coincides with the Fock-Goncharov Poisson structure on $\mathscr{X}^{\mathrm{reg}}_{G,\Sigma}$.

Proof. By Theorem 1.15 the Atiyah-Bott Poisson structure P_{AB} on $X^{\mathrm{reg}}_G(\Sigma)$ is reduced from the canonical quasi-Poisson $(\mathfrak{g}^V, s^{(V)})$-tensor on $M^{\mathrm{reg}}_G(\Sigma, V)$. By Corollary 1.29, P_{AB} can also be reduced from the quasi-Poisson $(\mathfrak{h}^V, s^{(V)}_{\mathfrak{h}})$-tensor P_L on L, hence the lift $\widetilde{P}_{\mathrm{AB}}$ of P_{AB} to $\mathscr{X}^{\mathrm{reg}}_{G,\Sigma}$ is reduced from P_L as well. So $\widetilde{P}_{\mathrm{AB}}$ is exactly the Poisson structure that we just defined on $\mathscr{X}^{\mathrm{reg}}_{G,\Sigma}$, and Theorem 3.23 says that it coincides with Fock-Goncharov's. □

To prove Theorem 3.23, we need to verify Eq.(3.11) under our Poisson structure. But we have expressed the $X_{\mathbf{i}}$'s as fractions of the spin networks $\Delta_{\mathbf{i}}$, so the proof boils down to straightforward computations with Theorem 2.6 (or more generally, Corollary 2.8, if $V_{circle} \neq \emptyset$). We display the computations in detail in §3.3.

3.3 Proof of Theorem 3.23

In this section we give a computational proof of Theorem 3.23, using the quasi-Poisson formula for spin networks established in §2.1.2.

Our goal is to to compute the Poisson bracket $\{X_{\mathbf{i}}, X_{\mathbf{j}}\}$ for each pair $\mathbf{i}, \mathbf{j} \in I^{int}_{\mathcal{T}}$. Let us first compute the quasi-Poisson bracket $\{\Delta_{\mathbf{i}}, \Delta_{\mathbf{j}}\}$, for $\mathbf{i}, \mathbf{j} \in I_{\mathcal{T}}$.

3.3.1 Poisson brackets of the $\Delta_{\mathbf{i}}$'s

Proposition 3.25. *Let Σ be a bordered surface and $V \subset \partial\Sigma$ be finitely many marked points. Let $L \subset M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V)$ be a cross-section as defined in Corollary 1.29, with respect to an arbitrary choice of V_1, and let $\{\cdot,\cdot\}$ be the quasi-Poisson bracket on $C^\infty(L)$. Let Δ be a tripod on (Σ, V). For any $p, q, r \in \mathbb{N}_+$, $p + q + r = n$, let $\Delta_{p,q,r}$ be the spin network defined in §3.2.2, considered here as a function on L by restriction. Then we have*

$$\{\Delta_{p_0,q_0,r_0}, \Delta_{p,q,r}\} = \frac{1}{2n}\omega((p_0, q_0, r_0), (p, q, r))\Delta_{p_0,q_0,r_0} \cdot \Delta_{p,q,r},$$

where

$$\omega((p_0, q_0, r_0), (p, q, r)) = \begin{cases} r_0q - q_0r & \text{if } p \geq p_0, q \leq q_0, r \leq r_0 \text{ or } p \leq p_0, q \geq q_0, r \geq r_0, \\ p_0r - r_0p & \text{if } p \geq p_0, q \leq q_0, r \geq r_0 \text{ or } p \leq p_0, q \geq q_0, r \leq r_0, \\ q_0p - p_0q & \text{if } p \geq p_0, q \geq q_0, r \leq r_0 \text{ or } p \leq p_0, q \leq q_0, r \geq r_0. \end{cases}$$

The expression for $\omega((p_0, q_0, r_0), (p, q, r))$ is more transparent from Figure 3.6 below, where we exhibit the values of $\omega((p_0, q_0, r_0), (p, q, r))$ respectively when (p, q, r) belongs to different regions.

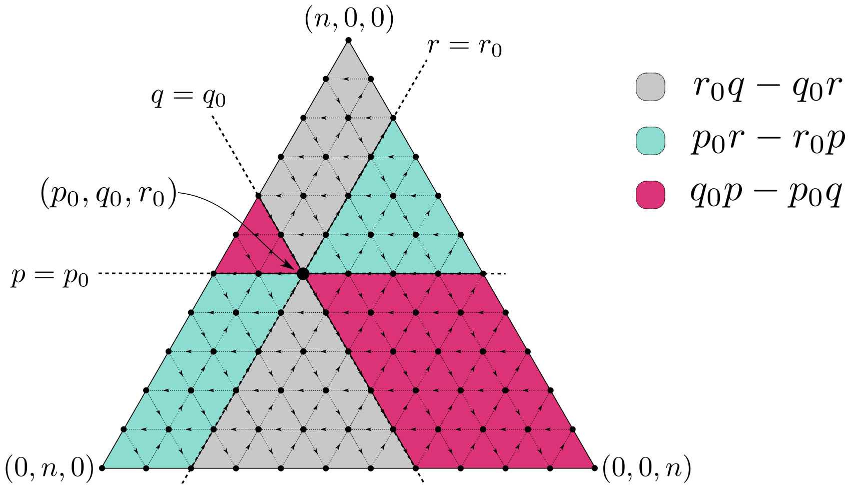


Figure 3.6: Values of $\omega((p_0,q_0,r_0),(p,q,r))$

Proof. It is just an application of the quasi-Poisson bracket formula in Corollary 2.8. We reproduce the formula here for convenience of the reader:

$$\{[\Gamma,f],[\Gamma',f']\}_L = \sum_q \varepsilon_q(\Gamma,\Gamma')[\Gamma\cup_q\Gamma, D_q(f,f')] + \frac{1}{2}\sum_{v\in V}\sum_{\mathbf{a},\mathbf{a}'}\varepsilon(\mathbf{a},\mathbf{a}')[\Gamma\cup\Gamma', D_{\mathbf{a},\mathbf{a}'}(f,f')]$$
$$+\frac{1}{2}\sum_{v\in V}\sum_{\mathbf{a},\mathbf{a}'}[\Gamma\cup\Gamma'\cup\beta_v, \widetilde{D}_{\mathbf{a},\mathbf{a}'}(f,f')].$$

Since the required equality is skew-symmetric when the roles of (p_0,q_0,r_0) and (p,q,r) are switched, we can assume $r\geq r_0$ without loss of generality.

We move Δ by homotopy to a tripod Δ' as shown in Figure 3.7, so that the two tripods are transverse and have a single interior intersection point A. The formula gives the quasi-Poisson bracket $\{\Delta_{p_0,q_0,r_0},\Delta'_{p,q,r}\}$ as a sum of three terms, which we evaluate respectively in Eq.($\star 1$), Eq.($\star 2$) and Eq.($\star 3$) below.

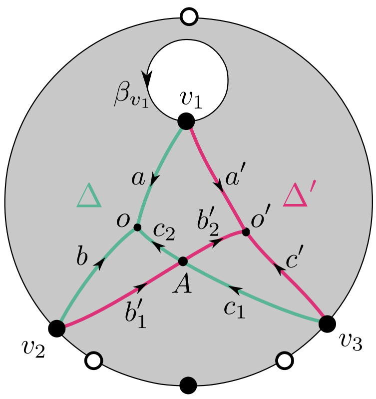


Figure 3.7: Δ and Δ'

$$\varepsilon_A(\Delta,\Delta')[\Delta\cup_A\Delta', D_A(\mathrm{Det}_{p_0,q_0,r_0},\mathrm{Det}_{p,q,r})] = \frac{r_0q}{n}\Delta_{p_0,q_0,r_0}\cdot\Delta_{p,q,r} \tag{$\star$1}$$

To prove Eq.($\star 2$), first we note that $\varepsilon_A(\Delta,\Delta')=-1$.

The graph $\Delta \cup_A \Delta'$ has eight edges. We think of an element in $G^{E_{\Delta\cup_A\Delta'}}$ as an assignment of elements $a, b, c_1, c_2, a', b_1', b_2', c' \in G$ to each edge, as shown in Figure 3.7 (here and below, we put $G = \mathrm{SL}_n\mathbb{R}$). Since the dual $s \in (S^2\mathfrak{sl}_n\mathbb{R})^{\mathfrak{sl}_n\mathbb{R}}$ of the scalar product $(x \mid y) = \mathrm{Tr}(xy)$ has the following expression (*c.f.* §2.2.4)

$$s = -\frac{1}{n} I \otimes I + \sum_{1\le k,l\le n} E_{kl} \otimes E_{lk},$$

by definition, the admissible function $\psi = D_A(\mathrm{Det}_{p_0,q_0,r_0}, \mathrm{Det}_{p,q,r}) \in C^\infty(G^{E_{\Delta\cup_A\Delta'}})$ is

$$\begin{aligned}
&\psi(a, b, c_1, c_2, a', b_1', b_2', c')\\
&= -\frac{1}{n}\frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} \mathrm{Det}_{p_0,q_0,r_0}(a, b, e^{\epsilon}c_2c_1)\cdot \mathrm{Det}_{p,q,r}(a', e^{\delta}b_2'b_1', c')\\
&\quad + \sum_{k,l}\frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} \mathrm{Det}_{p_0,q_0,r_0}(a, b, c_2\exp(\epsilon E_{kl})c_1)\cdot \mathrm{Det}_{p,q,r}(a', b_2'\exp(\delta E_{lk})b_1', c').
\end{aligned}$$

Admissibility implies that ψ is invariant by the G-action on $G^{E_{\Delta\cup_A\Delta'}}$ associated to the vertex A, so we have

$$\begin{aligned}
&\psi(a, b, c_1, c_2, a', b_1', b_2', c') = \psi(a, b, I, c_2c_1, a', c_1^{-1}b_1', b_2'c_1, c') \qquad (3.12)\\
&= -\frac{r_0 q}{n}\mathrm{Det}_{p_0,q_0,r_0}(a, b, c_2c_1)\,\mathrm{Det}_{p,q,r}(a', b_2'b_1', c')\\
&\quad + \sum_{k,l}\frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} \mathrm{Det}_{p_0,q_0,r_0}(a, b, c_2c_1\exp(\epsilon E_{kl}))\cdot \mathrm{Det}_{p,q,r}(a', b_2'c_1\exp(\delta E_{lk})c_1^{-1}b_1', c').
\end{aligned}$$

By definition, $[\Delta\cup_A\Delta', \psi]$ is a function on $M_G(\Sigma, V)$ whose lift to $M_G(\Sigma, V\cup\{A, o, o'\})$ is $\psi(a, b, c_1, c_2, a', b_1', b_2', c')$, where $a, b, c_1, \cdots : M_G(\Sigma, V\cup\{A, o, o'\}) \to G$ are now holonomies of the corresponding paths. Since Δ is homotopic to Δ', there are relations, say, $b_2'c_1 = c'$, between those holonomies. We claim that when $b_2'c_1 = c'$, the last summation in Eq.(3.12) yields zero. This implies the required equality ($\star 1$).

The claim follows from the following observation:

Lemma 3.26. *For any $a, b, c \in G$ we have*

$$\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\delta=0}\mathrm{Det}_{p,q,r}(a, c\exp(\delta E_{lk})b, c) = 0 \quad \textit{if } l \le r,$$

and

$$\frac{\mathsf{d}}{\mathsf{d}\epsilon}\Big|_{\epsilon=0}\mathrm{Det}_{p,q,r}(a, b, c\exp(\epsilon E_{kl})) = \begin{cases} \mathrm{Det}_{p,q,r}(a, b, c) & \text{if } k = l \le p,\\ \mathrm{Det}_{p,q,r}(a, b, \sigma_{kl}(c)) & \text{if } k > p \text{ and } l \le p,\\ 0 & \text{otherwise.}\end{cases} \qquad (3.13)$$

Here $\sigma_{kl}(c)$ is obtained from c by exchanging the k^{th} and l^{th} column.

Proof. To prove the first equality, we compute

$$\begin{aligned}
&\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\delta=0}\mathrm{Det}_{p,q,r}(a, c\exp(\delta E_{lk})b, c)\\
&= \det(a^{(1)}, \cdots, a^{(p)}, (cE_{lk}b)^{(1)}, (cb)^{(2)}, \cdots, (cb)^{(q)}, c^{(1)}, \cdots, c^{(r)}) + \cdots\\
&\quad + \det(a^{(1)}, \cdots, a^{(p)}, (cb)^{(1)}, \cdots, (cb)^{(q-1)}, (cE_{lk}b)^{(q)}, c^{(1)}, \cdots, c^{(r)}),
\end{aligned}$$

but $(cE_{lk}b)^{(i)}$ is a linear combination of $c^{(1)}, \cdots c^{(r)}$ if $l \leq r$, so each of the above determinants vanishes.

The second equality also follows from elementary computations and we omit the proof. □

$$\frac{1}{2}\sum_{i=1}^{3} \varepsilon(\mathbf{a}_i, \mathbf{a}_i')[\Delta \cup \Delta', D_{\mathbf{a}_i,\mathbf{a}_i'}(\mathrm{Det}_{p_0,q_0,r_0}, \mathrm{Det}_{p,q,r})] \tag{$\star 2$}$$
$$= \frac{1}{2}\left(\min(p_0, p) - \min(q_0, q) - r_0 - \frac{p_0 p}{n} + \frac{q_0 q}{n} + \frac{r_0 r}{n}\right) \Delta_{p_0,q_0,r_0} \cdot \Delta_{p,q,r}$$

To prove Eq.($\star 2$), note that $\varepsilon(\mathbf{a}_1, \mathbf{a}_1') = 1$ and $\varepsilon(\mathbf{a}_2, \mathbf{a}_2') = \varepsilon(\mathbf{a}_3, \mathbf{a}_3') = -1$, whereas the admissible function $\psi_i = D_{\mathbf{a}_i,\mathbf{a}_i'}(\mathrm{Det}_{p_0,q_0,r_0}, \mathrm{Det}_{p,q,r})$ for the graph $\Delta \cup \Delta'$ is given by

$$\begin{aligned}
&\psi_1(a, b, c, a', b', c')\\
&= -\frac{1}{n}\frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} \mathrm{Det}_{p_0,q_0,r_0}(e^{\epsilon}a, b, c) \cdot \mathrm{Det}_{p,q,r}(e^{\delta}a', b', c')\\
&\quad + \sum_{k,l} \frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} \mathrm{Det}_{p_0,q_0,r_0}(a \exp(\epsilon E_{kl}), b, c) \cdot \mathrm{Det}_{p,q,r}(a' \exp(\delta E_{lk}), b', c')\\
&= -\frac{p_0 p}{n} + \min(p_0, p)
\end{aligned}$$

(the last equality uses Eq.(3.13)), and similarly for ψ_2 and ψ_3. Hence we get Eq.($\star 2$).

$$\frac{1}{2}\sum_{i}[\Delta \cup \Delta' \cup \beta_{v_i}, \widetilde{D}_{\mathbf{a}_i,\mathbf{a}_i'}(\mathrm{Det}_{p_0,q_0,r_0}, \mathrm{Det}_{p,q,r})] = 0. \tag{$\star 3$}$$

This is a sum over those i's for which $v_i \in V_{circle}$. For any $h = \mathrm{diag}(h_1, \cdots, h_n) \in H^{\mathrm{reg}}$ we have

$$\left(\frac{\mathrm{Ad}_h + 1}{\mathrm{Ad}_h - 1} \otimes \mathrm{id}\right) s_{\mathfrak{h}^\perp} = \sum_{k \neq l} \frac{h_k + h_l}{h_k - h_l} E_{kl} \otimes E_{lk},$$

we obtain ($\star 3$) by using Eq.(3.13) again. For example, when $i = 1$, we have

$$\begin{aligned}
&\widetilde{D}_{\mathbf{a}_1,\mathbf{a}_1'}(\mathrm{Det}_{p_0,q_0,r_0}, \mathrm{Det}_{p,q,r})(a, b, c, a', b', c', h) =\\
&\frac{\mathsf{d}}{\mathsf{d}\epsilon}\frac{\mathsf{d}}{\mathsf{d}\delta}\Big|_{\epsilon=\delta=0} \sum_{k \neq l} \frac{h_k + h_l}{h_k - h_l} \mathrm{Det}_{p_0,q_0,r_0}(a \exp(\epsilon E_{kl}), b, c) \cdot \mathrm{Det}_{p,q,r}(a' \exp(\delta E_{lk}), b', c') = 0.
\end{aligned}$$

Combining Eq.($\star 1$), Eq.($\star 2$) and Eq.($\star 3$), we get

$$\begin{aligned}
&\{\Delta_{p_0,q_0,r_0}, \Delta_{p,q,r}\}\\
&= \frac{1}{2n}\Big(n\big(\min(p_0, p) - \min(q_0, q) - r_0\big) - p_0 p + q_0 q + r_0 r + 2r_0 q\Big)\Delta_{p_0,q_0,r_0} \cdot \Delta_{p,q,r}.
\end{aligned}$$

Noting $p_0 + q_0 + r_0 = p + q + r = n$, one verifies that the parenthesized terms equal $\omega((p_0, q_0, r_0), (p, q, r))$ by elementary computations. □

Proposition 3.27. *Under the hypothesis of Proposition 3.25,*

- *if Δ' is a tripod adjacent to Δ as in Figure 3.3, so that the two vertices that they share are labelled $(n, 0, 0)$ and $(0, 0, n)$ in I_Δ, respectively, and labelled $(n, 0, 0)$ and $(0, n, 0)$ in $V_{\Delta'}$, then*

$$\{\Delta_{p_0,q_0,r_0}, \Delta'_{p,q,r}\} = \frac{1}{2n}\Big(n\big(\min(p_0, p) - \min(r_0, q)\big) - \frac{p_0 p}{n} + \frac{r_0 q}{n}\Big)\Delta_{p_0,q_0,r_0} \cdot \Delta_{p,q,r}.$$

– *if Δ' is a tripod sharing with Δ a single vertex, which is labelled $(n,0,0)$ in both I_Δ and $V_{\Delta'}$, then*

$$\{\Delta_{p_0,q_0,r_0}, \Delta'_{p,q,r}\} = \frac{1}{2n}\Big(n\min(p_0,p) - \frac{p_0 p}{n}\Big)\Delta_{p_0,q_0,r_0}\cdot\Delta_{p,q,r}.$$

– *if Δ' is a tripod disjoint with Δ, then* $\{\Delta_{p_0,q_0,r_0}, \Delta'_{p,q,r}\} = 0$.

The first two statements follows from almost the same computations as in the above proof of Eq.($\star 2$) and Eq.($\star 3$). The last statement is immediate, since if two spin networks have disjoint graph diagrams then Corollary 2.8 implies that their quasi-Poisson bracket vanishes.

3.3.2 Poisson brackets of the $X_{\mathbf{i}}$'s

Using Proposition 3.25 and 3.27, we can prove Theorem 3.23 by straightforward verifications in the following cases, respectively:

(1) $\mathbf{i}$ and $\mathbf{j}$ are in the interior of some triangle $t \in \mathcal{T}$,

(2) $\mathbf{i}$ is in the interior of $t \in \mathcal{T}$, while $\mathbf{j}$ is on the boundary of t,

(3) $\mathbf{i}$ is in the interior of $t \in \mathcal{T}$, while $\mathbf{j}$ is in some triangle $t' \in \mathcal{T}$ adjacent to t,

(4) $\mathbf{i}$ is in the interior of $t \in \mathcal{T}$, while $\mathbf{j}$ is in some triangle $t' \in \mathcal{T}$ which share a single vertex with t.

(5) $\mathbf{i}$ and $\mathbf{j}$ are in two disjoint triangles $t, t' \in \mathcal{T}$.

The complexity of verifications are in a decreasing order, the last case being obvious. Here we only treat Case (1) in some detail.

Let us first show that

$$\{\log\Delta_{p,q,r}, \log X_{p',q',r'}\} = \begin{cases} 1 & \text{if } (p',q',r') = (p,q,r), \\ 0 & \text{if } (p',q',r') \neq (p,q,r). \end{cases} \tag{3.14}$$

By the definition of $X_{p,q,r}$, we have

$$\begin{aligned} \log X_{p,q,r} = &\log\Delta_{p+1,q,r-1} + \log\Delta_{p-1,q+1,r} + \log\Delta_{p,q-1,r+1} \\ &- \log\Delta_{p,q+1,r-1} - \log\Delta_{p-1,q,r+1} - \log\Delta_{p+1,q-1,r}. \end{aligned} \tag{3.15}$$

To prove the first case in (3.14), we apply Proposition 3.25 and get

$$\begin{aligned} &\{\log\Delta_{p,q,r}, \log X_{p,q,r}\} \\ &= \frac{1}{2n}\Big(\omega\big((p,q,r),(p+1,q,r-1)\big) + \omega\big((p,q,r),(p-1,q+1,r)\big) \\ &\quad + \omega\big((p,q,r),(p,q-1,r+1)\big) - \omega\big((p,q,r),(p,q+1,r-1)\big) \\ &\quad - \omega\big((p,q,r),(p-1,q,r+1)\big) - \omega\big((p,q,r),(p+1,q-1,r)\big)\Big) \\ &= \frac{1}{2n}(2p+2r+2q) = 1. \end{aligned}$$

The second case in (3.14) results from similar computations. Here one needs to compute separately for (p',q',r') belonging to each of the three regions in Figure 3.6.

It follows from Eq.(3.14) and Eq.(3.15) that

$$\{\log X_{p,q,r}, \log X_{p',q',r'}\} = \begin{cases} 1 & \text{if } (p',q',r')-(p,q,r) = (1,0,-1), (-1,1,0) \text{ or } (0,-1,1), \\ -1 & \text{if } (p',q',r')-(p,q,r) = (-1,0,1), (1,-1,0) \text{ or } (0,1,-1), \\ 0 & \text{otherwise.} \end{cases}$$

This is exactly the required equality $\{\log X_{\mathbf{i}}, \log X_{\mathbf{j}}\} = \epsilon_{\mathbf{ij}}$ in Case (1).

3.4 Further discussions

This last section of the present chapter is not detailed. We suggest some further developments of the above considerations.

3.4.1 A Poisson algebra of functions on $M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V)$

In Fock-Goncharov theory, we considered spin networks of the form $\Delta_{\mathbf{i}}$ only for tripods Δ coming from a triangulation $\mathcal{T}$ of Σ. It is natural to consider the totality of such spin networks for all tripods. The subalgebra of $C^\infty(M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V))$ generated by them is not closed under quasi-Poisson bracket, but there is a natural quasi-Poisson subalgebra $\mathcal{C}_n(\Sigma, V)$ containing it, as we explain in this subsection.

We define a *labelled n-pod* γ on (Σ, V) as a n-pod each of whose legs is labelled by an integer in $\{1, \cdots, n\}$. Overlapping feet are allowed. Labelled n-pods are considered up to homotopy.

A labelled n-pod γ gives rise to a spin network similar to $\Delta_{\mathbf{i}}$. Indeed, let $e_1, \cdots, e_n$ be the edges of γ, labelled by $p_1, \cdots, p_n$, respectively. Define $\mathrm{Det}_{p_1,\cdots,p_n} \in C^\infty((\mathrm{SL}_n\mathbb{R})^n)$ by

$$\mathrm{Det}_{p_1,\cdots,p_n}(a_1, \cdots, a_n) = \det(a_1^{(p_1)}, \cdots, a_n^{(p_n)}),$$

where $a^{(i)}$ denotes to i^{th} column of the matrix $a \in \mathrm{SL}_n\mathbb{R}$. Then $\mathrm{Det}_{p_1,\cdots,p_n}$ is an admissible functions for the n-tripod. We view the labelled n-pod γ as the spin network $[\gamma, \mathrm{Det}_{p_1,\cdots,p_n}]$.

We define $\mathcal{C}_n(\Sigma, V)$ as the subalgebra of $C^\infty(M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V))$ generated by all labelled n-pods.

$\mathcal{C}_n(\Sigma, V)$ is far from being free. There are the following relations, the first being obvious.

(1) If two edges of γ with the same label share a foot, then $\gamma = 0$

(2) Assuming $m > n$, any labelled m-pod Γ (whose edges are still labelled by $\{1, \cdots, n\}$) gives rise to a number of quadratic relations as follows, which are essentially Plücker relations from projective embedding of the Grassmannian $\mathrm{Gr}(n, m)$. We index the edges of Γ by $1, \cdots, m$ and let $\Gamma(i_1, \cdots, i_n)$ denote the labelled n-pod formed by edges with indices $1 \le i_1 < \cdots < i_n \le m$. Furthermore, for general $i_1, \cdots, i_n \in \{1, \cdots, m\}$ we put $\Gamma(i_1, \cdots, i_n) = 0$ if some of the i_k's coincide, otherwise put $\Gamma(i_1, \cdots, i_n) = \mathrm{sgn}(\sigma)\Gamma(i_{\sigma(1)}, \cdots, i_{\sigma(n)})$, where $\sigma \in \mathfrak{S}_n$ is the unique permutation such that $i_{\sigma(1)} < \cdots < i_{\sigma(n)}$. Then for any $1 \le k_0 < \cdots < k_d \le m$ and $1 \le i_1 < \cdots < i_{d-1} \le m$ we have a relation

$$\sum_{j=0}^{d} (-1)^j \Gamma(i_1, \cdots, i_{d-1}, k_j)\Gamma(k_0, \cdots, \widehat{k_j}, \cdots, k_d) = 0.$$

It is likely that these give all relations in $\mathcal{C}_n(\Sigma, V)$.

Some computations using Theorem 2.6 yield

Proposition 3.28. *$\mathcal{C}_n(\Sigma, V) \subset C^\infty(M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V))$ is closed under the quasi-Poisson bracket $\{\cdot, \cdot\}$. Moreover, the restriction of $\{\cdot, \cdot\}$ to $\mathcal{C}_n(\Sigma, V)$ is a Poisson bracket.*

Indeed, the bracket of two labelled n-pods γ and γ' can be described in a way very similar to the Massuyeau-Turaev algebra and swapping algebra. Namely, let $v_i, v_i' \in V$ $(i = 1, \cdots, n)$ denote the feet of γ and γ', respectively, then

$$\{\gamma, \gamma'\} = \sum_{q \in \gamma \cap \gamma' \setminus \partial\Sigma} \varepsilon_q(\gamma, \gamma')[\gamma, \gamma']_q + \frac{1}{2} \sum_{i,j} \varepsilon(v_i, v_j')[\gamma, \gamma']_{i,j} - \frac{1}{n} \gamma \cdot \gamma', \tag{3.16}$$

where the second summation runs over $i, j \in \{1, \cdots, n\}$ such that $v_i = v'_j$. Here $[\gamma, \gamma']_q$ denotes the product of the two labelled n-pods obtained by swapping the pair of legs intersecting at q; similarly, $[\gamma, \gamma']_{i,j}$ is the product of the two labelled n-pods obtained by swapping labels of γ and γ' at the common feet $v_i = v'_j$.

Remark 3.29. The last term in Eq.(3.16) is dispensable: replacing the coefficient $\frac{1}{n}$ by any $\lambda \in \mathbb{R}$, Eq.(3.16) still defines a Poisson bracket on $\mathcal{C}_n(\Sigma, V)$.

Certain fractions of elements in $\mathcal{C}_n(\Sigma, V)$ can be viewed as functions on twisted configuration spaces of (partial) flags. As an example, let $P \subset \mathrm{SL}_n\mathbb{R}$ be the parabolic subgroup stabilizing the point $[1 : 0 : \cdots : 0] \in \mathbb{P}^{n-1}$, so that $\mathbb{P}^{n-1} = \mathrm{SL}_n\mathbb{R}/P$, and $M_{\mathrm{SL}_n\mathbb{R}}(\Sigma, V)/P^V$ is a twisted configuration space of points in the projective space. Let $\gamma_1, \cdots, \gamma_N$ and $\gamma'_1, \cdots, \gamma'_N$ be n-pods whose edges are all labelled by 1, then the fraction

$$\frac{\gamma_1 \cdots \gamma_m}{\gamma'_1 \cdots \gamma'_m}$$

is P^V-invariant if the set of feet of $\gamma_1, \cdots, \gamma_m$, taking multiplicities into account, coincides with the set of feet of $\gamma'_1, \cdots, \gamma'_m$.

V. Ovsienko, R. Schwartz and S. Tabachnikov [51] [3] constructed a Poisson structure on the moduli space of "twisted N-gons"

$$\mathcal{P}_N = M_{\mathrm{SL}_3\mathbb{R}}(\mathbb{R}/N\mathbb{Z}, \mathbb{R}/\mathbb{Z})/P^N$$

(*c.f.* Example 3.8) in order to study a discrete dynamical system on an open part of $\mathcal{P}_N$. In fact, in a way similar to Fock and Goncharov, they defined a coordinates system $(x_i, y_i)_{i\in\mathbb{Z}/N\mathbb{Z}}$ on an open part of $\mathcal{P}_N$ and defined a Poisson structure by prescribing Poisson brackets of coordinates functions (which are also log-constant).

Let Σ be the cylinder and $V = \{v_i\}_{v\in\mathbb{Z}/N\mathbb{Z}}$ be N marked points on $\partial\Sigma$. Identifying

$$M_{\mathrm{SL}_3\mathbb{R}}(\mathbb{R}/N\mathbb{Z}, \mathbb{R}\mathbb{Z}) = M_{\mathrm{SL}_3\mathbb{R}}(\Sigma, V),$$

their coordinates admit the following expressions under our framework:

$$x_i = 1 - \frac{(i-1, i, i+1)(i-2, i+1, i+2)}{(i-2, i, i+1)(i-1, i+1, i+2)}$$

$$y_i = 1 - \frac{(i+1, i-2, i-1)(i+1, i-1, i)}{(i+1, i-2, i-1)(i+2, i-1, i)}$$

where (i, j, k) denotes the labelled tripod whose feet is v_i, v_j, v_k and each edge is labelled by 1. Thus we can compute Poisson brackets of these coordinates functions under the Poisson structure on $\mathcal{P}_N$ reduced from $M_{\mathrm{SL}_3\mathbb{R}}(\Sigma, V)$. However, computations shows that this Poisson structure does not coincides with Ovsienko-Schwartz-Tabachnikov's.

3. We would like to thank Zhe Sun for informing us of this article.

Chapter 4

Deformation quantization of $M_G(\Sigma, V)$

The goal of this chapter is to prove Theorem C in the introduction, namely, construct a star product quantizing the quasi-Poisson manifold $M_G(\Sigma, V)$. We first briefly recall in §4.1 some ingredients from the theory of quasi-Hopf algebras, due to Drinfeld [19, 20], then we discuss in §4.2 quantizations of quasi-Poisson manifolds, in particular $M_G(\Sigma, V)$.

4.1 Associators

We fix a field $\mathbf{k}$ of characteristic zero. All vector spaces and (co)algebras are defined over $\mathbf{k}$. All (co)algebras are assumed to be (co)associative and have (co-)unit unless otherwise specified. Given an algebra A and distinct integers $r_1, \cdots, r_k \in \{1, \cdots, n\}$ (where $k \leq n$), for any $\xi \in A^{\otimes k}$, we let $\xi_{r_1 \cdots r_k} \in A^{\otimes n}$ denote the image of ξ by the map $V^{\otimes k} \to V^{\otimes n}$ sending $a_1 \otimes \cdots \otimes a_k$ to $b_1 \otimes \cdots \otimes b_n$, where $b_{r_i} = a_i$ for $i = 1, \cdots, k$ and $b_r = 1$ if $r \neq r_1, \cdots, r_k$. For example, if $k = n = 3$ then $(x \otimes y \otimes z)_{231} = z \otimes x \otimes y$.

4.1.1 Quasi-bialgebras and braids

Recall that a *bialgebra* is an algebra A equipped with a map $\Delta : A \to A \otimes A$ (called a *coproduct*) satisfying

- Δ has a co-unit $\epsilon : A \to \mathbf{k}$. Namely, $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$.
- Δ is a homomorphism of algebras. Here $A \otimes A$ viewed as an algebra in the standard way. Namely, the multiplication is given by $(x \otimes y)(x' \otimes y') := xx' \otimes yy'$ and $1 \otimes 1$ is the unit.

A bialgebra A is called *coassociative* if

$$(\mathrm{id} \otimes \Delta)\Delta(x) = (\Delta \otimes \mathrm{id})\Delta(x), \quad \forall x \in A.$$

Definition 4.1. A *quasi-bialgebra* is a bialgebra A such that there exists an invertible element $\Phi \in A \otimes A \otimes A$ (called an *associator*) satisfying

$$(\mathrm{id} \otimes \Delta)\Delta(x) = \Phi \cdot (\Delta \otimes \mathrm{id})\Delta(x) \cdot \Phi^{-1}, \tag{4.1}$$

$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})\Phi = 1 \otimes 1, \tag{4.2}$$

and the *pentagon equation*

$$1 \otimes \Phi \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})\Phi \cdot \Phi \otimes 1 = (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\Phi \cdot (\Delta \otimes \mathrm{id} \otimes \mathrm{id})\Phi. \tag{Pentagon}$$

Furthermore, A is called *quasi-triangular* with *universal R-matrix* $R \in A \otimes A$ if

$$\Delta(x)_{21} = R\Delta(x)R^{-1} \quad \forall x \in A \tag{4.3}$$

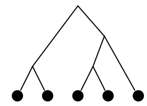
and R satisfies the following *hexagon equations*

$$(\Delta \otimes \mathrm{id})R = \Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi, \tag{Hexagon1}$$

$$(\mathrm{id} \otimes \Delta)R = \Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}\Phi^{-1}. \tag{Hexagon2}$$

To understand these definitions we need to introduce some more notions.

By a *parenthesizing* of an ordered set with n elements (or n aligned points) is meant the object shown on the left-hand side in the picture below, which is equivalent to the object on the right – a binary tree with one root and n leaves.

$(\bullet\bullet)((\bullet\bullet)\bullet) \quad \Longleftrightarrow$ 

Given a parenthesizing $\mathfrak{p}$ of an ordered set with n elements, we can iterate the coproduct Δ for $n-1$ times – with $\mathfrak{p}$ telling us where to apply Δ at each time – to get a map $\Delta^{\mathfrak{p}} : A \to A^{\otimes(n-1)}$. For example,

$$\Delta^{(\bullet\bullet)\bullet}(x) = (\Delta \otimes \mathrm{id}) \circ \Delta(x), \quad \Delta^{\bullet(\bullet\bullet)}(x) = (\Delta \otimes \mathrm{id}) \circ \Delta(x),$$

$$\Delta^{(\bullet(\bullet\bullet))\bullet}(x) = (\mathrm{id} \otimes \Delta \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta(x), \text{ etc.}$$

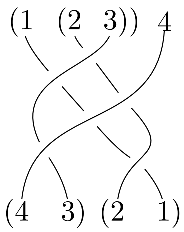
Given $n \in \mathbb{N}$, if I, J, K are disjoint subsets of $\{1, \cdots, n\}$, each endowed with a parenthesizing, we denote

$$\Phi^{I,J,K} = \Big((\Delta^I \otimes \Delta^J \otimes \Delta^K)\Phi\Big)_{I\cup J\cup K} \in A^{\otimes n},$$

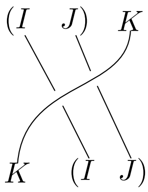
Similarly we define $R^{I,J} \in A^{\otimes n}$. For example, with this notation (Pentagon) can be written as

$$\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} = \Phi^{1,2,34}\Phi^{12,3,4}. \tag{Pentagon}$$

Definition 4.2. We define a groupoid $\mathcal{B}_n$ as follows. Objets of $\mathcal{B}_n$ are parenthesized permutations of $\{1, \cdots, n\}$. A morphsim is a braid with n strands such that the two ends of each strand are indexed by the same number, as explained by the following pictures.

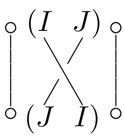


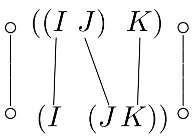
Braids are read from top to bottom. This picture presents a morphism in $\mathcal{B}_4$ from the object $(1(23))4$ to $(43)(21)$.



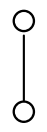
If I, J, K are disjoint parenthesized subsets of $\{1, \cdots, n\}$ and their union is $\{1, \cdots, n\}$, a picture like this presents a morphism where the upper I, J, K and the lower ones are joint by trivial braids.

The following two types of morphisms are called *elementary*



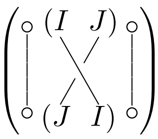
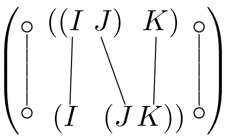
, 

.

Here  represents a (possibly empty) trivial braid whose two ends are the same parenthesized subsets of $\{1, \cdots, n\}$.

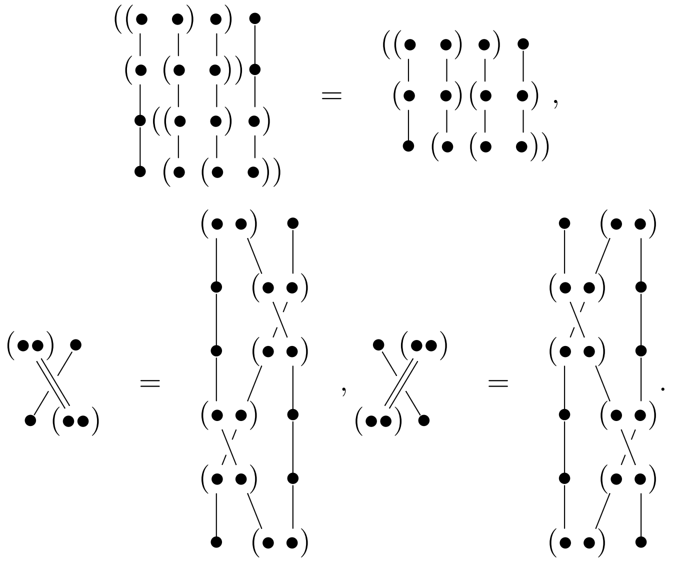
It is easy to see that elementary braids generate $\mathcal{B}_n$ in the sense that every morphism can be decomposed into elementary ones.

Proposition 4.3. *We define a homomorphism ϱ from $\mathcal{B}_n$ to the group of invertible elements in $A^{\otimes n}$ by requiring images of elementary morphisms to be*

ϱ  $= R^{I,J}$, ϱ  $= \Phi^{I,J,K}$.

Then ρ is well-defined, i.e., the image of a morphism does not depend on its decomposition into elementary ones, if and only if R and Φ satisfies (Pentagon), (Hexagon1) and (Hexagon2).

Proof of the "only if" part. (Pentagon), (Hexagon1) and (Hexagon2) are respectively given by the following generating relations



□

A prove of the "if" part can be found, *e.g.*, in [50].

4.1.2 Representations and braided monoidal categories

A representation of a quasi-bialgebra A is by definition a representation of the underlying algebra of A. Let us recall the latter notion:

Definition 4.4. Let A be an algebra. A $\mathbf{k}$-vector space M is called a *(left) A-module* or a *representation of* A if it is equipped with a linear map $\lambda : A \otimes M \to M$ such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{id}\otimes\lambda"] \arrow[d, "m\otimes\mathrm{id}"'] & A \otimes M \arrow[d, "\lambda"] \\ A \otimes M \arrow[r, "\lambda"] & M \end{tikzcd}, \quad \begin{tikzcd} \mathbf{k} \otimes M \arrow[r, "i\otimes\mathrm{id}"] \arrow[d, "\cong"'] & A \otimes M \arrow[d, "\lambda"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}.$$

We also denote $x.m := \lambda(x \otimes m)$ for $x \in A, m \in M$ (called the action of x on m). The category of all A-modules is denoted by $\mathbf{Mod}_A$.

The main point of this subsection is that when A is a quasi-triangular quasi-bialgebra, $\mathbf{Mod}_A$ has rich structures.

Definition 4.5. If A is a quasi-bialgebra, then

- the *trivial A-module* is $\mathbf{k}$ considered with the action $x.k := \epsilon(x)k$ ($\forall x \in A, k \in \mathbf{k}$).
- the *tensor product* $M \otimes N$ of A-modules M and N is the vector space tensor product $M \otimes N$ with the action $x.(m \otimes n) := \Delta(x).(m \otimes n)$.

Example 4.6. If G is a finite group and A is the group algebra $\mathbf{k}[G]$, then a A-module is the same as a G-module. If $\mathfrak{g}$ is a Lie algebra and A is the universal enveloping algebra $U\mathfrak{g}$, then a A-module is the same as a $\mathfrak{g}$-module. In these two cases, the notions of trivial A-module and tensor product of A-modules coincide with classical ones.

Since A is not necessarily coassociative, the natural vector space isomorphisms

$$(L \otimes M) \otimes N \xrightarrow{\sim} L \otimes (M \otimes N)$$

is not necessarily a A-module morphism. However, we do have a A-module morphism between them provided by the associator, as stated in the following proposition. Similarly, the permutation map

$$\sigma : M \otimes N \xrightarrow{\sim} N \otimes M, \quad m \otimes n \mapsto n \otimes m$$

is not a A-module morphism in general, but if A is quasi-triangular with universal R-matrix $R \in A \otimes A$, then there is a A-module morphism.

Proposition 4.7. *If A is a quasi-bialgebra with associator $\Phi \in A^{\otimes 3}$, then*

$$\begin{aligned} \gamma : (L \otimes M) \otimes N &\xrightarrow{\sim} L \otimes (M \otimes N) \\ a \otimes b \otimes c &\longmapsto \Phi.(a \otimes b \otimes c) \end{aligned}$$

is an isomorphism of A-modules. Furthermore, if A is quasi-triangular with universal R-matrix $R \in A \otimes A$, then

$$\begin{aligned} \beta : M \otimes N &\xrightarrow{\sim} N \otimes M \\ a \otimes b &\longmapsto \sigma(R.(a \otimes b)) \end{aligned}$$

is also an isomorphism of A-modules.

Proof. The actons of $x \in A$ on $a \otimes b \otimes c \in (L \otimes M) \otimes N$ and $\Phi.(a \otimes b \otimes c) \in L \otimes (M \otimes N)$ give respectively $(\Delta \otimes \mathrm{id})\Delta(x).(a \otimes b \otimes c)$ and $((\mathrm{id} \otimes \Delta)\Delta(x) \cdot \Phi).(a \otimes b \otimes c)$. The latter is the image of the former by γ because of the condition (4.1). Hence γ is a morphism of A-modules. γ is bijective because Φ is invertible.

Similarly, the action of $x \in A$ on $a \otimes b \in M \otimes N$ and $\sigma(R.(a \otimes b))$ are respectively $\Delta(x).(a \otimes b)$ and

$$\Delta(x).\sigma(R.(a \otimes b)) = \sigma(\Delta^{op}(x)R.(a \otimes b)) = \sigma(R\Delta(x).(a \otimes b)).$$

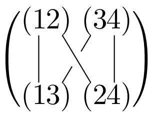
The latter is the image of the former by β. □

A *monoidal category* is roughly speaking a category $\mathcal{C}$ in which one can perform tensor product of two objects A, B to get an object $A \otimes B$. A *braided monoidal category* is then a monoidal category equipped with isomorphisms

$$\gamma : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad \beta : A \otimes B \to B \otimes A$$

for any objects A, B and C such that β and γ are functorial and satisfies some compatibility conditions which are similar to the pentagon and hexagon equations, see *e.g.* [16] for details. In particular, if A is a quasi-triangular quasi-bialgebra then $\mathbf{Mod}_A$ is a braided monoidal category.

Given n objects in a braided monoidal category $\mathcal{C}$, each parenthesized permutation of order n corresponds to a way of tensoring these objects together. For example, for $n = 4$ and objects $M_1, \cdots, M_4$, the parenthesized permutations $(12)(34)$ and $(13)(24)$ correspond to $(M_1 \otimes M_2) \otimes (M_3 \otimes M_4)$ and $(M_1 \otimes M_3) \otimes (M_2 \otimes M_4)$, respectively. A braid $\mathbf{B}$ joining two parenthesized permutations gives rise to an isomorphism $\iota(\mathbf{B})$ between the two tensor products. For example,

$\mathbf{B} =$  (4.4)

induces

$$\iota(\mathbf{B}) : (M_1 \otimes M_2) \otimes (M_3 \otimes M_4) \longrightarrow (M_1 \otimes M_3) \otimes (M_2 \otimes M_4).$$

We omit the precise definition of $\iota(\mathbf{B})$ for general braided monoidal category. For the category $\mathbf{Mod}_A$, it has the expression

$$\iota(\mathbf{B}) = \sigma_{\mathbf{B}} \circ \varrho(\mathbf{B}).$$

$\varrho : \mathcal{B}_4 \to A^{\otimes 4}$ is given in the previous subsection. An element of $A^{\otimes 4}$ is viewed here as a map from $M_1 \otimes M_2 \otimes M_3 \otimes M_4$ to itself via the A-action, and $\sigma_{\mathbf{B}}$ is the permutation given by the braid. For $\mathbf{B}$ in (4.4), $\sigma_{\mathbf{B}}$ is the permutation of 2^{nd} and 3^{rd} factors

$$\sigma_{23} : M_1 \otimes M_2 \otimes M_3 \otimes M_4 \to M_1 \otimes M_3 \otimes M_2 \otimes M_4.$$

Recall that one can define an *algebra* in a categorical way as an object A in a monoidal category together with a morphism $m : A \otimes A \to A$ which serves as multiplication. Furthermore, in a braided monoidal category one can define an *associative algebra* as an algebra such that the following diagram commutes

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[rr, "\gamma"] \arrow[dr, "m\circ(m\otimes \mathrm{id})"'] & & A \otimes (A \otimes A) \arrow[dl, "m\circ(\mathrm{id}\otimes m)"] \\
& A &
\end{tikzcd}$$

One can verify that if (A_1, m_1) and (A_2, m_2) are associative algebras in a braided monoidal category and we define a multiplication on $A_1 \otimes A_2$ as the composition

$$m : A \otimes A = (A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow{\iota(\mathbf{B})} (A_1 \otimes A_1) \otimes (A_2 \otimes A_2) \xrightarrow{m_1 \otimes m_2} A_1 \otimes A_2 = A,$$

where $\mathbf{B}$ is the braid (4.4), then (A, m) is still an associative algebra. We call (A, m) the tensor product of (A_1, m_1) and (A_2, m_2).

4.1.3 Associators in $U(\mathfrak{g})^{\otimes 3}[[\hbar]]$

Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ be the universal enveloping algebra. Fix $t \in (S^2\mathfrak{g})^{\mathfrak{g}} \subset \mathfrak{g} \otimes \mathfrak{g} \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$. In terms of $t_{12}, t_{13}, t_{23} \in U(\mathfrak{g})^{\otimes 3}$, the canonical trivector $\phi \in (\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}$ associated to t can be written as

$$\phi = -\frac{1}{2}[t_{12}, t_{23}] = \frac{1}{2}[t_{13}, t_{23}] = \frac{1}{2}[t_{12}, t_{13}].$$

We will only consider a specific type of quasi-triangular quasi-biaglebras – those of the form $(U(\mathfrak{g})[[\hbar]], \Phi, R)$, with $R = e^{\frac{\hbar t}{2}} \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ and

$$\Phi = 1 + \hbar^2 \phi_2 + O(\hbar^3) \in U(\mathfrak{g})^3[[\hbar]] \tag{4.5}$$

for some $\phi_2 \in U(\mathfrak{g})^{\otimes 3}$.

The associator Φ here can be seen as a quantization of ϕ in the sense that ϕ is given by the first approximation ϕ_2 of Φ. Precisely, we have the following proposition.

Proposition 4.8. *If $(U(\mathfrak{g})[[\hbar]], \Phi, e^{\frac{\hbar t}{2}})$ is a quasi-triangular quasi-bialgebra and Φ is given by (4.5), then ϕ_2 satisfies $\mathrm{Alt}(\phi_2) = -\frac{1}{2}\phi \in (\bigwedge^3 \mathfrak{g})^{\mathfrak{g}} \subset U(\mathfrak{g})^{\otimes 3}$. Here*

$$\mathrm{Alt}(\phi_2) := \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma)\sigma(\phi_2) \in U(\mathfrak{g})^{\otimes 3}.$$

Proof. It follows from the second-order part of (Hexagon1). □

Definition 4.9. Given $t \in (S^2\mathfrak{g})^{\mathfrak{g}}$, we call a formal power series $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ of the form (4.5) an *associator* quantizing t if $(U(\mathfrak{g})[[\hbar]], \Phi, e^{\frac{\hbar t}{2}})$ is a quasi-triangular quasi-bialgebra.

Notice that the condition (4.3) in Definition 4.1 is automatically satisfied, while the condition (4.1) means $\Phi \in (U(\mathfrak{g})^{\otimes 3}[[\hbar]])^{\mathfrak{g}}$. The bialgebra $U(\mathfrak{g})[[\hbar]]$ is itself coassociative, but we still need a nontrivial associator in order to have a nontrivial braided monoidal category $\mathbf{Mod}_{(U(\mathfrak{g})[[\hbar]], \Phi, e^{\hbar t/2})}$.

Remark 4.10. In the literature, one usually impose stronger conditions in the definition of associators, in order to gain richer structures. Namely, it is often assumed that Φ has the form $\Phi = \mathbf{\Phi}(\hbar t_{12}, \hbar t_{23})$, where we let $\mathbf{k}\langle\langle X, Y\rangle\rangle$ denote the algebra of non-commutative formal power series in the variables X, Y and require that $\mathbf{\Phi}(X, Y) \in \mathbf{k}\langle\langle X, Y\rangle\rangle$ satisfies

- $\mathbf{\Phi}(X, Y) = \exp f(X, Y)$, where $f(X, Y) \in \mathbf{k}\langle\langle X, Y\rangle\rangle$ is contained in the Lie subalgebra of $\mathbf{k}\langle\langle X, Y\rangle\rangle$ generated by X and Y.
- $\mathbf{\Phi}(X, Y)^{-1} = \mathbf{\Phi}(Y, X)$.

An in-depth study of such associators goes back to Drinfeld [20]. In particular, the first example of an associator was constructed therein, and it is shown that associators over $\mathbb{Q}$ (hence over any field $\mathbf{k}$ of characteristic 0) exit.

Let $t^{(n)} = t \oplus \cdots \oplus t \in (S^2\mathfrak{g}^n)^{\mathfrak{g}^n}$ be the direct sum of n copies of t. Given an associator $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ quantizing t, we can define an associator

$$\Phi^{(n)} \in U(\mathfrak{g}^n)^{\otimes 3}[[\hbar]] = (U(\mathfrak{g})^{\otimes n})^{\otimes 3}[[\hbar]]$$

quantizing $t^{(n)}$ as the image of $\underbrace{\Phi \otimes \cdots \otimes \Phi}_{n} \in (U(\mathfrak{g})^{\otimes 3})^{\otimes n}[[\hbar]]$ under the identification

$$\begin{aligned} (U(\mathfrak{g})^{\otimes 3})^{\otimes n}[[\hbar]] &\xrightarrow{\sim} (U(\mathfrak{g})^{\otimes n})^{\otimes 3}[[\hbar]] \\ (a_1 \otimes b_1 \otimes c_1) \otimes \cdots \otimes (a_n \otimes b_m \otimes c_n) &\longmapsto (a_1 \otimes \cdots \otimes a_n) \otimes (b_1 \otimes \cdots \otimes b_n) \otimes (c_1 \otimes \cdots \otimes c_n). \end{aligned}$$

4.2 Deformation quantization

4.2.1 Deformation quantization of Poisson manifolds

Definition 4.11. Let A be a commutative algebra. A *star product* $\star$ on A is a $\mathbf{k}[[\hbar]]$-linear associative product $\star : A[[\hbar]] \otimes_{\mathbf{k}[[\hbar]]} A[[\hbar]] \to A[[\hbar]]$, whose mod $\hbar$ reduction is the commutative product in A.

Equivalently, there is a sequence of maps $\theta_1, \theta_2, \cdots : A \otimes A \to A$ such that

$$f \star g = fg + \hbar\theta_1(f,g) + \hbar^2\theta_2(f,g) + \cdots, \quad f, g \in A, \tag{4.6}$$

and for general $f, g \in A[[\hbar]]$ the star product $f \star g$ is given by linear expansion. Associativity of $\star$ imposes constraints on the θ_i's.

Proposition 4.12. *Let $\{\cdot,\cdot\} : A \otimes A \to A$ be the skew-symmetric bilinear map defined by*

$$\{f,g\} = \frac{f \star g - g \star f}{\hbar} \mod \hbar$$

(In other words, $\{f,g\} = \theta_1(f,g) - \theta_1(g,f)$). Then $\{\cdot,\cdot\}$ is a Poisson bracket on A, and we say that the star product $\star$ quantizes *the Poisson bracket $\{\cdot,\cdot\}$.*

A more general statement will be proved later (Proposition 4.16).

Let M be a Poisson manifold, then we define a star product on M as a star product on $C^\infty(M)$ such that each θ_k in Eq.(4.6) is a bidifferential operator.

Example 4.13 (The Moyal product)**.** The following expression gives a star product on $\mathbb{R}^d$ endowed with constant Poisson structure $P = P^{ij}\partial_i\partial_j$ ($P_{ij} = -P_{ji}$), called the *Moyal product*:

$$f \star g = fg + \sum_{r\geq 1} \sum_{|I|=|J|=r} \frac{1}{r!}\left(\frac{\hbar}{2}\right)^r P^{IJ}\partial_I f \partial_J g,$$

where for any multi-index $I = (i_1, \cdots, i_r)$ and $J = (j_1, \cdots, j_r)$, we set

$$P^{IJ} = P^{i_1 j_1} \cdots P^{i_r j_r}.$$

4.2.2 Deformation quantization of quasi-Poisson manifolds

We fix from now on a Lie algebra $\mathfrak{g}$, an element t in $(S^2\mathfrak{g})^{\mathfrak{g}}$, and an associator $\Phi \in U(\mathfrak{g})^{\otimes 3}$ quantizing t. First we state an algebraic definition of quasi-Poisson brackets.

Definition 4.14. Let A be a commutative $\mathfrak{g}$-algebra[1]. A skew-symmetric bilinear map $\{\cdot,\cdot\} : \bigwedge^2 A \to A$ is called a *quasi-Poisson* $(\mathfrak{g}, t)$*-bracket* if it satisfies

- **$\mathfrak{g}$-invariance:** $x.\{a,b\} = \{x.a, b\} + \{a, x.b\}$;
- **Leibniz rule:** $\{ab, c\} = \{a,c\}b + \{b,c\}a$;
- **Quasi-Jacobi identity:** $\circlearrowleft \{\{a,b\},c\} = -\frac{1}{2}m(\phi.(a \otimes b \otimes c))$, where $\circlearrowleft$ means summations over cyclic permutations of a, b, c.

Definition 4.15. Let A be a commutative $\mathfrak{g}$-algebra. A $\mathbf{k}[[\hbar]]$-linear product

$$\star : A[[\hbar]] \otimes_{\mathbf{k}[[\hbar]]} A[[\hbar]] \longrightarrow A[[\hbar]]$$

is called a Φ*-star product* if the mod $\hbar$ reduction of $\star$ is the commutative product in A and $(A[[\hbar]], \star)$ is an associative algebra in the braided monoidal category $\mathbf{Mod}_{(U(\mathfrak{g})[[\hbar]], \Phi, e^{\hbar t/2})}$ (see §4.1.2).

1. That is, a commutative algebra on which $\mathfrak{g}$ acts by derivations

Thus $\star$ also has the expression (4.6). The latter condition amounts to

$$x \circ \star = \star \circ \Delta(x), \;\; \forall x \in \mathfrak{g}, \tag{4.7}$$
$$\star \circ (\star \otimes \mathrm{id}) = \star \circ (\mathrm{id} \otimes \star) \circ \Phi. \tag{4.8}$$

Here an element in $U(\mathfrak{g})^{\otimes k}[[\hbar]]$ is considered as a map $A^{\otimes k}[[\hbar]] \to A^{\otimes k}[[\hbar]]$ via the $\mathfrak{g}$-action on A. Notice that (4.7) just means $\star$ is $\mathfrak{g}$-invariant.

Proposition 4.16. *If $\star$ is a Φ-quasi-star product on A, then*

$$\{a, b\} := \frac{a \star b - b \star a}{\hbar} \quad \mathrm{mod}\ \hbar$$

is a quasi-Poisson $(\mathfrak{g}, t)$-bracket on A, and we say that $\star$ quantizes $\{\cdot, \cdot\}$.

When $\mathbf{k} = \mathbb{R}, \mathbb{C}$ and (M, P) is a (real or complex) quasi-Poisson G-manifold, a Φ-star product on M is by definition a Φ-star product on $C^\infty(M)$ such that each θ_k in (4.6) is a bidifferential operator.

Proof. The first-order part of Eq.(4.7) is

$$x.\theta_1(a, b) = \theta_1(x.a, b) + \theta_1(a, x.b),$$

which implies $\mathfrak{g}$-invariance of $\{\cdot, \cdot\}$.

The first and second order parts of Eq.(4.8) are respectively

$$\theta_1(a, b)c + \theta_1(ab, c) = a\theta_1(b, c) + \theta_1(a, bc), \tag{4.9}$$
$$\begin{aligned} &\theta_2(a, b)c + \theta_2(ab, c) + \theta_1(\theta_1(a, b), c) \\ &= a\theta_2(b, c) + \theta_2(a, bc) + \theta_1(a, \theta_1(b, c)) + m(\phi_2.(a \otimes b \otimes c)), \end{aligned} \tag{4.10}$$

where θ_1, θ_2 and ϕ_2 are the terms appearing in the expansions Eq.(4.6) and Eq.(4.5) of $\star$ and Φ.

The Leibniz rule of $\{\cdot, \cdot\}$ follows from Eq.(4.9):

$$\begin{aligned} \{ab, c\} &= \theta_1(ab, c) - \theta_1(c, ab) \\ &= \theta_1(ab, c) - \theta_1(a, bc) + \theta_1(a, cb) - \theta_1(ac, b) + \theta_1(ca, b) - \theta_1(c, ab) \\ &= a\theta_1(b, c) - \theta_1(a, b)c - a\theta_1(c, b) + \theta_1(a, c)b + c\theta_1(a, b) - \theta_1(c, a)b \\ &= a\{b, c\} + b\{a, c\}. \end{aligned}$$

The quasi-Jacobi identity follows from Eq.(4.10):

$$\begin{aligned} \circlearrowleft \{\{a, b\}, c\} =& \circlearrowleft (\theta_1(\theta_1(a, b), c) - \theta_1(c, \theta_1(a, b)) - \theta_1(\theta_1(b, a), c) + \theta_1(c, \theta_1(b, a))) \\ =& \circlearrowleft (\theta_1(\theta_1(a, b), c) - \theta_1(a, \theta_1(b, c)) - \theta_1(\theta_1(b, a), c) + \theta_1(b, \theta_1(a, c))) \\ =& \circlearrowleft (a\theta_2(b, c) + \theta_2(a, bc) - \theta_2(a, b)c - \theta_2(ab, c) + m(\phi_2.(a \otimes b \otimes c)) \\ & - b\theta_2(a, c) - \theta_2(b, ac) + \theta_2(b, c)c + \theta_2(ab, c)) + m(\phi_2.(b \otimes a \otimes c)) \\ =& \, m(\mathrm{Alt}(\phi_2).(a \otimes b \otimes c)) = -\frac{1}{2} m(\phi.(a \otimes b \otimes c)). \end{aligned}$$

$\square$

Remark 4.17. This definition of deformation quantization for quasi-Poisson manifolds was given by Enriquez and Etingof [22]. For a specific class of quasi-Poisson manifolds (those arising from the *classical dynamical Yang-Baxter equation*), quantizations are thoroughly studied [22, 23, 4]. Using Kontsevich's method of quantizing Poisson manifolds, Halbout [31] showed the existence of quantization for quasi-Poisson G-manifolds when the G-action is free.

In the rest of this chapter, under some assumptions on Φ, we construct an explicit $\Phi^{(V)}$-star product on $M_G(\Sigma, V)$ quantizing the canonical quasi-Poisson $t^{(V)}$-tensor. The idea of construction is the same as how the quasi-Poisson tensor itself is constructed – we start by quantizing the disk with two marked points and then show how fusion of quasi-Poisson manifolds lifts to fusion of star products.

4.2.3 Quantization of the disk with two marked points

Consider a real Lie group G as a $G \times G$-manifold via the action $\rho_{(a,b)} : g \mapsto agb^{-1}$. The induced $\mathfrak{g} \oplus \mathfrak{g}$-action is $\rho_{(x,y)}(g) = x^{\mathsf{L}}(g) - x^{\mathsf{R}}(g)$.

For any $\varphi \in \bigwedge^3 \mathfrak{g}$, the zero bivector field on G is a quasi-Poisson $(\mathfrak{g}, 0, \varphi)^2$-bivector field because $\varphi^{\mathsf{L}} - \varphi^{\mathsf{R}} = 0$. It admits a simple quantization when $\Delta = \Delta_0$, and the associator $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ is $\mathfrak{g}$-invariant and satisfies $S^{\otimes 3}(\Phi) = \Phi^{-1}$, in particular, if $\Phi = \mathbf{\Phi}(\hbar t_{1,2}, \hbar t_{2,3})$ is given by an even Lie associator $\mathbf{\Phi} \in \mathbb{R}\langle\langle X, Y\rangle\rangle$:

The idea of constructing a $\Phi^{(V)}$-star product on $M_G(\Sigma, V)$ is the same as that of constructing a

Let $\mathbf{D}$ be the disk with two marked points. Recall from §1.1.1 and §1.3.1 that $M_G(\mathbf{D}) = G$ and the G action is $\rho_{(a,b)} : g \mapsto agb^{-1}$. The trivial bivector field $P = 0$ on $M_G(\mathbf{D})$ is a quasi-Poisson $t^{(2)}$-tensor for any $t \in (S^2\mathfrak{g})^{\mathfrak{g}}$. It is natural to guess that the commutative product in $C^\infty(M)$ is a star product quantizing $P = 0$. It is indeed the case under some restrictions on Φ, as stated in the following proposition. Here $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ is the antipode map, *i.e.*, an anti-homomorphism which restricts to $-\mathrm{id}$ on $\mathfrak{g} \subset U(\mathfrak{g})$.

Proposition 4.18. *If the associator $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ satisfies $S^{\otimes 3}(\Phi) = \Phi^{-1}$, then the usual product of functions is a $\Phi^{(2)}$-quasi-star product on G.*

Remark 4.19. An associator of the form $\Phi = \mathbf{\Phi}(\hbar t_{12}, \hbar t_{23})$ as in Remark 4.10 satisfies $S^{\otimes 3}(\Phi) = \Phi^{-1}$ if and only if $\mathbf{\Phi}(X, Y)$ is *even* in the sense that $f(X, Y)$ only has terms of even degree. It is shown by Bar-Natan [10] that such $\mathbf{\Phi}(X, Y)$ exists.

A *k-multi-differential operator* on a manifold M is a linear map $\overline{D} : C^\infty(M)^{\otimes k} \to C^\infty(M)$ such that in a local chart it has the form

$$\overline{D}(f_1 \otimes \cdots \otimes f_k) = \sum_{I_1, \cdots, I_k} \lambda^{I_1, \cdots, I_k} (\partial_{I_1} f_1) \cdots (\partial_{I_k} f_k).$$

where $I_1, \cdots, I_k$ run over multi-indices and each $\lambda^{I_1, \cdots, I_k}$ is a smooth function.

Let $\mathcal{D}^k(M)$ be the space of k-multi-differential operators on M and set $\mathcal{D}(M) = \mathcal{D}^1(M)$. $\mathcal{D}^k(M)$ can be identified with the $C^\infty(M)$-module tensor product of k-copies of $\mathcal{D}(M)$. The natural map from the vector space tensor product to $C^\infty(M)$-module tensor product is

$$\begin{aligned} \mathcal{D}(M)^{\otimes k} &\longrightarrow \mathcal{D}^k(M) \\ D &\longmapsto \overline{D}, \quad \overline{D}(f_1 \otimes \cdots \otimes f_k) = m \circ D(f_1 \otimes \cdots \otimes f_k) \end{aligned}$$

where $m : C^\infty(M)^{\otimes k} \to C^\infty(M)$ is the usual product of functions.

A $\mathfrak{g}$-action on M induces an algebra homomorphism $U(\mathfrak{g})^{\otimes k} \to \mathcal{D}(M)^{\otimes k}$, hence a linear map $U(\mathfrak{g})^{\otimes k} \to \mathcal{D}^k(M)$.

When $M = G$, let $\mathbf{L}$ and $\mathbf{R}$ be the homomorphism and anti-homomorphism of algebras from $U(\mathfrak{g})$ to $\mathcal{D}(G)$ defined by

$$\mathbf{L}(x_1 \cdots x_n) = x_1^{\mathsf{L}} \cdots x_n^{\mathsf{L}}, \quad \mathbf{R}(x_1 \cdots x_n) = x_n^{\mathsf{R}} \cdots x_1^{\mathsf{R}} \quad \text{for any } x_i \in \mathfrak{g}.$$

The tensor products of k-copies of them $\mathbf{L}^{\otimes k}, \mathbf{R}^{\otimes k} : U(\mathfrak{g})^{\otimes k} \to \mathcal{D}(G)^{\otimes k}$ are also denoted by $\mathbf{L}, \mathbf{R}$ for brevity.

Clearly, the map $U(\mathfrak{g})^{\otimes k} \to \mathcal{D}(G)^{\otimes k}$ induced by the right (resp. left) action of G on itself is $\mathbf{L}$ (resp. $\mathbf{R} \circ S^{\otimes k}$). In particular, $\mathbf{L}(\xi)$ and $\mathbf{R}(S^{\otimes k}(\eta))$ commutes for any $\xi, \eta \in U(\mathfrak{g})^{\otimes k}$ since the left and right actions commutes. We also have

Lemma 4.20. *Let $A, B \in U(\mathfrak{g})^{\otimes k}$. Then $\overline{\mathbf{L}(A) \circ \mathbf{R}(B)} = \overline{\mathbf{R}(AB)}$ (resp. $\overline{\mathbf{L}(BA)}$) if A (resp. B) is $\mathfrak{g}$-invariant.*

Proof. We assume $k = 2$ to simplify notations. For general k the proof is the same.

By linearity we can assume that $A = x_1 \cdots x_m \otimes y_1 \cdots y_n$ and $B = u_1 \cdots u_p \otimes v_1 \cdots v_q$, where $x_i, y_i, u_i, v_i \in \mathfrak{g}$. For any $f, g \in C^\infty(G)$, we have

$$\mathbf{L}(A) \circ \mathbf{R}(B)(f \otimes g) = x_1^{\mathsf{L}} \cdots x_m^{\mathsf{L}} u_p^{\mathsf{R}} \cdots u_1^{\mathsf{R}}(f) \otimes y_1^{\mathsf{L}} \cdots y_n^{\mathsf{L}} v_q^{\mathsf{R}} \cdots v_1^{\mathsf{R}}(g),$$

hence for any $a \in G$ we have

$$\begin{aligned}
\overline{\mathbf{L}(A) \circ \mathbf{R}(B)}(f \otimes g)(a) =& \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \Big[f(\exp(t_1 u_1) \cdots \exp(t_p u_p) a \exp(t_1' x_1) \cdots \exp(t_m' x_m)) \\
& \cdot g(\exp(s_1 v_1) \cdots \exp(s_p v_q) a \exp(s_1' y_1) \cdots \exp(s_n' y_n)) \Big]
\end{aligned}$$

$$\begin{aligned}
=& \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \Big[f(a \exp(t_1 \operatorname{Ad}_a^{-1} u_1) \cdots \exp(t_p \operatorname{Ad}_a^{-1} u_p) \exp(t_1' x_1) \cdots \exp(t_m' x_m)) \\
& \cdot g(a \exp(s_1 \operatorname{Ad}_a^{-1} v_1) \cdots \exp(s_p \operatorname{Ad}_a^{-1} v_q) \exp(s_1' y_1) \cdots \exp(s_n' y_n)) \Big] \\
=& \overline{\mathbf{L}((\operatorname{Ad}_a B) A)}(f \otimes g)(a),
\end{aligned}$$

where "$\frac{\partial}{\partial t}|_{t=0}$" stands for

$$\frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_p} \frac{\partial}{\partial t_1'} \cdots \frac{\partial}{\partial t_m'}\Big|_{(t_1, \cdots, t_p, t_1', \cdots, t_m') = 0},$$

and similarly for "$\frac{\partial}{\partial s}|_{s=0}$".

Therefore, $\overline{\mathbf{L}(A) \circ \mathbf{R}(B)} = \overline{\mathbf{L}(BA)}$ if B is $\mathfrak{g}$-invariant. The other equality is proved by the same argument. $\square$

Proof of Proposition 4.18. The invariance (4.7) is obvious. We now prove (4.8).

It is sufficient to show that under the map $U(\mathfrak{g} \oplus \mathfrak{g})^{\otimes 3} \to \mathcal{D}^3(G)$ induced by the $\mathfrak{g} \oplus \mathfrak{g}$-action on G, the image of $\Phi^{(2)} \in U(\mathfrak{g} \oplus \mathfrak{g})^{\otimes 3}[[\hbar]]$ is trivial (*i.e.*, the product $C^\infty(G)^{\otimes 3} \to C^\infty(G)$). But this image is $\overline{\mathbf{L}(\Phi)\mathbf{R}(S^{\otimes 3}(\Phi))}$, so it follows from the above lemma that it equals $\overline{\mathbf{L}(S^{\otimes 3}(\Phi)\Phi)}$ (noting that $\Phi \in (U(\mathfrak{g})[[\hbar]])^{\mathfrak{g}}$), hence is trivial by assumption. $\square$

4.2.4 Fusion at the quantum level

The algebraic version of quasi-Poisson fusion (Defintion/Proposition 1.3) is as follows.

Definition 4.21. Let A be a $\mathfrak{g}\oplus\mathfrak{g}$-algebra and $\{\cdot,\cdot\}$ be a quasi-Poisson $(\mathfrak{g}\oplus\mathfrak{g}, t^{(2)})$-bracket on A. The *fusion* of $\{\cdot,\cdot\}$ is the quasi-Poisson $(\mathfrak{g}, t)$-bracket

$$\{a,b\}^* := \{a,b\} + \frac{1}{2} m((t_{23} - t_{41}).(a\otimes b)),$$

where $t_{23}, t_{41} \in \mathfrak{g}^{\otimes 4} \subset (U(\mathfrak{g})\otimes U(\mathfrak{g}))^{\otimes 2}$. Here A is considered as a $\mathfrak{g}$-algebra via the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}\oplus\mathfrak{g}$.

In this subsection we shall produce a Φ-star product $\star^*$ quantizing $\{\cdot,\cdot\}^*$ from a $\Phi^{(2)}$-star product quantizing $\{\cdot,\cdot\}$.

Let us first consider the simpler case where we have two quasi-Poisson $(\mathfrak{g}, t)$-algebras $(A_1, \{\cdot,\cdot\}_1)$ and $(A_2, \{\cdot,\cdot\}_2)$ and $A = A_1\otimes A_2$ is a vector space tensor product, with $\mathfrak{g}\oplus\mathfrak{g}$-action given by letting the first and second $\mathfrak{g}$-factor act on A_1 and A_2, respectively. Then the "direct product" of $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$, defined by

$$\{a_1\otimes a_2, b_1\otimes b_2\} = \{a_1,b_1\}_1 \otimes a_2b_2 + a_1b_1\otimes\{a_2,b_2\},$$

is a quasi-Poisson $(\mathfrak{g}\oplus\mathfrak{g}, t^{(2)})$-star product on A. If $\star_i$ is a Φ-star product on A_i quantizing $\{\cdot,\cdot\}_i$, it is easy to see that the composition

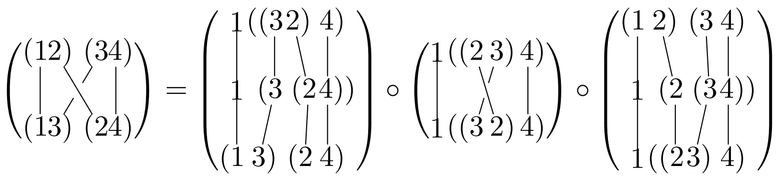
$$\star : A\otimes A = A_1\otimes A_2\otimes A_1\otimes A_2 \xrightarrow{\sigma_{23}} A_1\otimes A_1\otimes A_2\otimes A_2 \xrightarrow{\star_1\otimes\star_2} A_1\otimes A_2 = A$$

is a $\Phi^{(2)}$-star product on A. Here σ_{23} permutes the second and third factors.

One can construct a Φ-star product $\star^*$ on A quantizing $\{\cdot,\cdot\}^*$ by letting $(A[[\hbar]],\star^*)$ be the tensor product of associative algebras $(A_1[[\hbar]],\star_1)$ and $(A_2[[\hbar]],\star_2)$ in the braided monoidal category $\mathbf{Mod}_{(U(\mathfrak{g})[[\hbar]],\Phi,e^{\hbar t/2})}$, as discussed in the end of §4.1.2. Precisely, $\star^*$ is the composition

$$\star^* : A\otimes A = (A_1\otimes A_2)\otimes(A_1\otimes A_2) \xrightarrow{\iota(\mathbf{B})} (A_1\otimes A_1)\otimes(A_2\otimes A_2) \xrightarrow{\star_1\otimes\star_2} A_1\otimes A_2 = A,$$

where $\mathbf{B}$ is the braid (4.4) and $\iota(\mathbf{B}) = \sigma_{23}\circ\varrho(\mathbf{B})$. Notice that $\varrho(\mathbf{B})$ can be expressed in terms of Φ via a decomposition of $\mathbf{B}$ into elementary braids. For example, the decomposition



gives

$$\varrho(\mathbf{B}) = (\Phi^{1,3,24})^{-1}\Phi^{3,2,4}e^{\hbar t_{23}/2}(\Phi^{2,3,4})^{-1}\Phi^{1,2,34}.$$

Notice that $\star^*$ is related to $\star$ by

$$\star^* = \star\circ\varrho(\mathbf{B}). \tag{4.11}$$

In the general case where A is not necessarily a tensor product, we shall show that (4.11) defines a Φ-star product quantizing $\{\cdot,\cdot\}^*$ as well:

Proposition 4.22. *Under the settings of Definition 4.21, if $\star$ is a $\Phi^{(2)}$-star product on the $\mathfrak{g}\oplus\mathfrak{g}$-algebra A quantizing the quasi-Poisson $(\mathfrak{g},t)$-bracket $\{\cdot,\cdot\}$, then $\star^*$ given by Eq.(4.11) is a Φ-star product quantizing $\{\cdot,\cdot\}^*$.*

We shall call $\star^*$ the *fusion* of $\star$.

Proof. Let us first prove Eq.(4.7) for $\star^*$. To be more clear, we let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ denote the first and second factor of $\mathfrak{g} \oplus \mathfrak{g}$, respectively. The coproduct $\Delta^{(2)}$ of $U(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$ is

$$\begin{aligned} U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) &\longrightarrow U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \otimes U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2) \cong U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \otimes U(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \\ x \otimes y &\longmapsto \Delta^{(2)}(x \otimes y) = (\Delta(x) \otimes \Delta(y))_{1324}. \end{aligned}$$

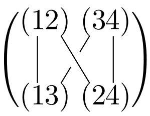
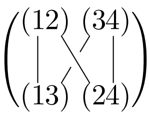
Hence Eq.(4.7) for $\star$ reads

$$(x \otimes y) \circ \star = \star \circ (\Delta(x) \otimes \Delta(y))_{1324}. \tag{4.12}$$

The action of $x \in \mathfrak{g}$ on A via the diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ is just the action of $\Delta(x) = x \otimes 1 + 1 \otimes x$ on A. Thus Eq.(4.7) for $\star^*$ amounts to

$$\Delta(x) \circ \star^* = \star^* \circ (\Delta \otimes \Delta)\Delta(x).$$

To prove this equality, we insert Eq.(4.11) and Eq.(4.12) into it, and find that it is sufficient to prove

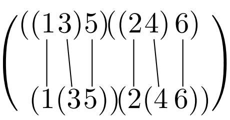
$((\Delta \otimes \Delta)\Delta(x))_{1324}$  $=$  $(\Delta \otimes \Delta)\Delta(x)$

(here an below, for any braid $\mathbf{B} \in B_n^{\mathrm{PaC}}$, we write $\varrho(\mathbf{B})$ just as $(\mathbf{B})$ for brevity). This in turn follows from condition (4.1) and (4.3) in the definition of quasi-triangular quasi-bialgebras.

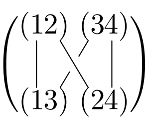
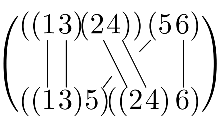
We proceed to prove Eq.(4.8) for $\star^*$. The action of $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ on $A^{\otimes 3}[[\hbar]]$ via the diagonal embedding is the action of $\Phi^{12,34,56} = (\Delta \otimes \Delta \otimes \Delta)\Phi \in U(\mathfrak{g})^{\otimes 6}[[\hbar]]$ on $A^{\otimes 6}[[\hbar]]$. Hence what we need to prove is

$$\star^* \circ (\star^* \otimes \mathrm{id}) = \star^* \circ (\mathrm{id} \otimes \star^*) \circ \Phi^{12,34,56}. \tag{4.13}$$

Recall from the end of §4.1.3 that the considered associator in $U(\mathfrak{g} \oplus \mathfrak{g})^{\otimes 3}[[\hbar]] = U(\mathfrak{g})^{\otimes 6}[[\hbar]]$ is

$\Phi^{(2)} = (\Phi \otimes \Phi)_{135246} =$  .

It is easy to see that

$(\Delta^{(2)} \otimes \mathrm{id})$  $=$  , Hence the left-hand side of Eq.(4.13) is

$$\star^* \circ (\star^* \otimes \mathrm{id}) = \star \circ \begin{pmatrix} (12)\ (34) \\ \\ (13)\ (24) \end{pmatrix} \circ (\star \otimes \mathrm{id}) \circ \begin{pmatrix} ((12)(34))\,(56) \\ \\ ((13)(24))\,(56) \end{pmatrix}$$

$$= \star \circ (\star \otimes \mathrm{id}) \circ (\Delta^{(2)} \otimes \mathrm{id}) \begin{pmatrix} (12)\ (34) \\ \\ (13)\ (24) \end{pmatrix} \circ \begin{pmatrix} ((12)(34))\,(56) \\ \\ ((13)(24))\,(56) \end{pmatrix}$$

$$= \star \circ (\mathrm{id} \otimes \star) \circ \begin{pmatrix} ((13)5)((24)\,6) \\ \\ (1(35))(2(4\,6)) \end{pmatrix} \begin{pmatrix} ((13)(24))\,(56) \\ \\ ((13)5)((24)\,6) \end{pmatrix} \begin{pmatrix} ((12)(34))\,(56) \\ \\ ((13)(24))\,(56) \end{pmatrix}$$

$$= \star \circ (\mathrm{id} \otimes \star) \circ \begin{pmatrix} ((12)(34))\,(56) \\ \\ \\ (1(35))(2(4\,6)) \end{pmatrix}.$$

Similar computations shows that the right-hand side of Eq.(4.13) is

$$\star^* \circ (\mathrm{id} \otimes \star^*) \circ \Phi^{12,34,56} = \star \circ (\mathrm{id} \otimes \star) \circ \begin{pmatrix} (12)((34)(5\,6)) \\ \\ \\ (1(35))(2(4\,6)) \end{pmatrix} \circ \Phi^{12,34,56}$$

this equals the left-hand side because we have

$$\Phi^{12,34,56} = \begin{pmatrix} ((12)(34))\,(56) \\ \\ (12)((34)(5\,6)) \end{pmatrix}.$$

$\square$

4.2.5 Quantization of $M_G(\Sigma, V)$

Let Γ be a skeleton of (Σ, V). We have seen in §1.2.4 that Γ represents a way of building up Σ from disks of two marked points by fusion. Using results from the two preceding subsections, one readily gets a Φ-star product $\star$ on $M_G(\Sigma, V)$ quantization the canonical quasi-Poisson structure (assuming $S^{\otimes 3}(\Phi) = \Phi^{-1}$). However, unlike the the quasi-Poisson tensor, $\star$ depends on the choice of a skeleton Γ of (Σ, V). Even worse, since fusion at the quantum level is not associative, $\star$ also depends on a parenthesizing of each $\widehat{E}_\Gamma(v)$.

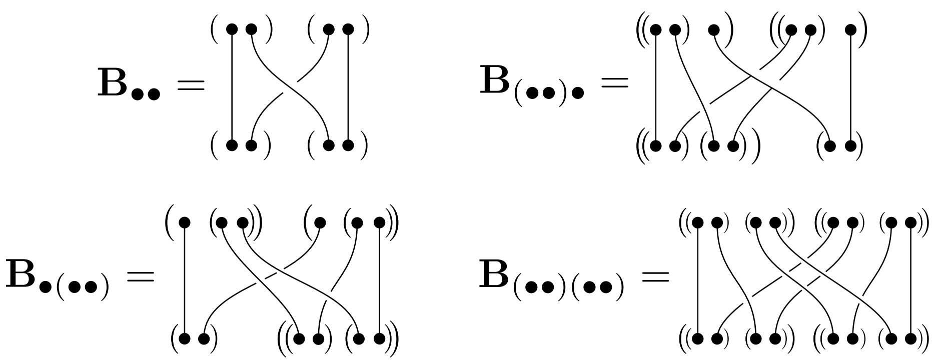
To give an explicit expression of $\star$, we define, for each parenthesizing $\mathfrak{p}$ of an ordered set with n-elements, a morphism $\mathbf{B}_\mathfrak{p}$ in the groupoid $\mathcal{B}_{2n}$. The definition of $\mathbf{B}_\mathfrak{p}$ should be clear from the following examples.

Set $n_v = \#\widehat{E}_\Gamma(v)$ for each $v \in V$. If $n_v \geq 2$, we choose a parenthesizing $\mathfrak{p}(v)$ of the ordered set $\widehat{E}_\Gamma(v)$, and get a morphism $\mathbf{B}_{\mathfrak{p}(v)}$ in $\mathcal{B}_{2n_v}$. The homomorphism $\varrho : \mathcal{B}_n \to U(\mathfrak{g})^{\otimes n}[[\hbar]]$ induces by Φ yields an element $\varrho(\mathbf{B}_{\mathfrak{p}(v)})$ in $U(\mathfrak{g})^{\otimes 2n_v}[[\hbar]]$.

Let $\widetilde{\rho}^v$ be the $G^{\widehat{E}_\Gamma(v)}$-action on $M_G(\Sigma, V)$ introduced in §1.2.4. It gives rise to a homomorphism of algebras

$$U(\mathfrak{g})^{\otimes 2n_v}[[\hbar]] \to \mathcal{D}(M_G(\Sigma, V))^{\otimes 2}[[\hbar]].$$

Let $\widetilde{\rho}^v_{\varrho(\mathbf{B}_{\mathfrak{p}(v)})}$ be the image of $\varrho(\mathbf{B}_{\mathfrak{p}(v)})$ by this map. If $n_v = 1$, then we put $\widetilde{\rho}^v_{\varrho(\mathbf{B}_{\mathfrak{p}(v)})} = 1$. Notice that the $\widetilde{\rho}^v_{\varrho(\mathbf{B}_{\mathfrak{p}(v)})}$'s commute with each other because the $\widetilde{\rho}^v$'s commute, hence we can consider their product $\prod_{v \in V} \widetilde{\rho}^v_{\varrho(\mathbf{B}_{\mathfrak{p}(v)})} \in \mathcal{D}(M_G(\Sigma, V))^{\otimes 2}[[\hbar]]$.



Proposition 4.23. *Take $t \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and an associator $\Phi \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$ quantizing t such that $S^{\otimes 3}(\Phi) = \Phi$. Given a skeleton Γ on (Σ, V) and parenthesizings $\mathfrak{p}(v)$ as above, the formal power series of bidifferential operators*

$$\star = \overline{\prod_{v \in V} \widetilde{\rho}^{v}_{\varrho(\mathbf{B}_{\mathfrak{p}(v)})}} \in \mathcal{D}^2(M_G(\Sigma, V))[[\hbar]]$$

(c.f. §4.2.3 for the notation) is a Φ-star product quantizing the canonical quasi-Poisson $(\mathfrak{g}^V, t^{(V)})$-tensor on $M_G(\Sigma, V)$.

Proof. Repeatedly using Proposition 4.22, we conclude that if $(A, \{\cdot,\cdot\})$ is a quasi-Poisson $(\mathfrak{g}^n, t^{(n)})$-algebra, $\{\cdot,\cdot\}^*$ is the fusion of $\{\cdot,\cdot\}$, which is quasi-Poisson $(\mathfrak{g}, t)$-bracket, and $\star$ is a $\Phi^{(n)}$-star product quantizing $\{\cdot,\cdot\}$, then for any parenthesizing $\mathfrak{p}$ of $\{1, \cdots, n\}$, we have a Φ-star product $\star^*$ quantizing $\{\cdot,\cdot\}^*$ given by

$$\star^* = \star \cdot \varrho(\mathbf{B}_{\mathfrak{p}}).$$

Now the proposition follows from the definition of the canonical quasi-Poisson structure on $M_G(\Sigma, V)$ and the fact established in §4.2.3 that the trivial product quantizes $M_G(\mathbf{D})$. $\square$

Part II

On the Hilbert geometry of simplicial Tits sets

Chapter 5

On the Hilbert geometry of simplicial Tits sets

This last chapter is independent of the rest the thesis. The goal here is to study the Hilbert metrics on certain specific examples of convex projective orbifolds constructed using reflection groups. We first give in §5.1 some backgrounds on reflections groups and convex projectively structure, and set up the examples which will be studied. Then we prove in §5.2 the main results, Theorem D and Corollary in the introduction.

5.1 Reflection groups and Convex projective structures

5.1.1 Convex rojective structures

Let $\mathbb{P}^n$ be the real projective space of dimension n. A (real) projective structure on a manifold (or more generally an orbifold) is an atlas which patches open sets of $\mathbb{P}^n$ together by projective transformations.

An extensively studied class of projective structures (see [13] and the references therein) comes from the following construction. We call an open subset $\Omega \subset \mathbb{P}^n$ *properly convex* if Ω is a bounded convex subset of an affine chart $\mathbb{R}^n \subset \mathbb{P}^n$. Let $X = \widetilde{X}/\Pi$ be an orbifold, where Π is a group acting discontinuously on the manifold $\widetilde{X}$. A *convex projective structure* on X consists of a faithful representation $\rho : \Pi \to \mathrm{PGL}_{n+1}\mathbb{R}$ and a properly convex open set $\Omega \subset \mathbb{P}^n$, such that there is a ρ-equivariant homeomorphism $\widetilde{X} \to \Omega$.

We shall consider convex projective structures up to projective transformations. It is well known that Ω is uniquely determined by ρ up to duality. So the *moduli space of convex projective structures* on X is defined as a subset in the space of conjugacy classes of representations

$$\mathfrak{P}(X) \subset \mathrm{Hom}(\Pi, \mathrm{PGL}_{n+1}\mathbb{R})/\mathrm{PGL}_{n+1}\mathbb{R}$$

consisting of those $\rho \in \mathrm{Hom}(\Pi, \mathrm{PGL}_{n+1}\mathbb{R})$ which arises from a convex projective structure on X. It is known that $\mathfrak{P}(X)$ is an open and closed subset [12]. When X is a orientable closed surface of genus $g \geq 2$, $\mathfrak{P}(X)$ is homeomorphic to $\mathbb{R}^{16g-16}$ [29].

5.1.2 Reflection groups

In this section we recall some well known facts about reflection groups and Tits set. *c.f.* [14, 2] for details.

Let $\mathbb{P}^n$ denote the real projective space of dimension n. A projective transformation $s \in \mathrm{PGL}_{n+1}\mathbb{R}$ is called a *reflection* if it is conjugate to $\pm\,\mathrm{diag}(-1, 1, \cdots, 1)$. The fixed point set of a reflection s is the disjoint union of a hyperplan $F \subset \mathbb{P}^n$ and a point $f \in \mathbb{P}^n$. Reflections are in one-one correspondence with pairs (f, F) with $f \notin F$.

Given a n-dimensional simplex $P \subset \mathbb{P}^n$ with face set $\{F_i\}_{i \in J}$, where $J = \{0, 1, \cdots, n\}$, we chose a reflection s_i with respect to each F_i. We are interested in the group $\Gamma \subset \mathrm{PGL}_{n+1}\mathbb{R}$ generated by $\{s_i\}_{i \in J}$. We call Γ a *simplicial reflection group*, and P the *fundamental simplex*. We will say the reflection group is *marked* if we want to keep track of the order of generators. Note that n-dimensional simplices are conjugate to each other by projective transformations. Since we do not want to distinct conjugate marked simplicial reflection groups, we can always assume that $P = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n | x_i \geq 0, \forall i\}$, whose faces are $P_i = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n | x_i = 0, x_k \geq 0, \forall k \neq i\}$.

With the above choice of fundamental simplex, a marked reflection group is determined by a $n + 1$-tuple of points $(f_0, \cdots, f_n)$ satisfying $f_i \notin P_i$. Let $f_i = [\alpha_{i0} : \cdots : \alpha_{in}]$. By a normalization, we can assume $\alpha_{ii} = 1$ for any $0 \leq i \leq n$. We record these f_i's by the matrix $\mathbf{A} = (\alpha_{ij})$ whose diagonals are 1's.

Two marked simplicial reflection groups, given by $\mathbf{A}$ and $\mathbf{A}'$ as above, are conjugate if and only if there is a projective transformation which stabilizes P and takes f_i to f_i'. This is equivalent to the existence of $\lambda_0, \cdots, \lambda_n > 0$, such that

$$\mathrm{diag}(\lambda_0, \cdots, \lambda_n)\mathbf{A}\,\mathrm{diag}(\lambda_0^{-1}, \cdots, \lambda_n^{-1}) = \mathbf{A}'.$$

this defines an equivalence relation $\mathbf{A} \sim \mathbf{A}'$. Let $\mathcal{M}^{n+1}$ be the set of $(n + 1) \times (n + 1)$-matrices whose diagonals are 1, then moduli space of marked simplicial reflection groups in $\mathbb{P}^n$ is $\mathcal{M}^{n+1}/\sim$. However, a generic point in this space generates a non-discrete group Γ. We show give conditions under which Γ is discrete.

Recall that a *Coxeter diagram* J is defined by a finite set $\mathbf{v}_J = \{0, 1, \cdots, n\}$ as set of nodes, together with integers $m_{ij} \geq 2$ associated to each non-ordered pair $i, j \in \mathbf{v}_J$, $i \neq j$ as weighted edges. We usually draw J as a graph, we shall omit weight-2 edges, only draw those with weight ≥ 3.

The Coxeter diagram $J = (\{0, 1, \cdots, n\}, \{m_{ij}\})$ yields an abstract Coxeter group

$$W_J = \langle \tau_0, \cdots, \tau_n | (\tau_i\tau_j)^{m_{ij}} = \tau_i^2 = 1, \forall i \neq j \rangle$$

The *Cartan matrix* of J, denoted by $\mathbf{C}_J$, is defined to be the symmetric matrix whose diagonal entries are 1 and the (i, j)-entry is $-\cos(\pi/m_{ij})$ if $i \neq j$.

Now we can state conditions under which $\mathbf{A} \in \mathcal{M}^{n+1}$ will give a discrete group. ([14], Theorem 1.5)

Theorem 5.1 (Tits, Vinberg). *Let $\mathbf{A} \in \mathcal{M}^{n+1}$ and $f_i \in \mathbb{P}^n$ be the point whose coordinates are given by the i-th row of $\mathbf{A}$. Let Γ be the subgroup of $\mathrm{PGL}_{n+1}\mathbb{R}$ generated by $s_0, \cdots, s_n$, where s_i is the reflection with respect to P_i and f_i.*

Then the translates γP $(\gamma \in \Gamma)$ are disjoint except at boundary if and only if there exists a Coxeter diagram $J = (\{0, 1, \cdots, n\}, \{m_{ij}\})$, such that $\mathbf{A}$ satisfies the following condition $(*_J)$:

$$(*_J): \begin{array}{l} \textit{For any distinct pair} \\ i, j \in \{0, 1, \cdots, n\}, \\ \textit{we have } \alpha_{ij} \leq 0, \textit{ and} \end{array} \begin{cases} \alpha_{ij} = \alpha_{ji} = 0 & \textit{if} \quad m_{ij} = 2 \\ \alpha_{ij}\alpha_{ji} = \cos^2(\pi/m_{ij}) & \textit{if} \quad 3 \leq m_{ij} < \infty \\ \alpha_{ij}\alpha_{ji} \geq 1 & \textit{if} \quad m_{ij} \geq \infty \end{cases}$$

When $(*_J)$ is satisfied, we have the following conclusions,

(1) $\rho : \tau_i \mapsto s_i$ $(0 \leq i \leq n)$ is an isomorphism from W_J to Γ.

(2) The set $\Omega = \cup_{\gamma \in \Gamma}\gamma P$, called the Tits set*, is either the whole $\mathbb{P}^n$ or a convex subset in some affine chart of $\mathbb{P}^n$. Γ acts discontinuously on Ω.*

(3) Ω is open if and only if the stabilizer of each vertex of P is a finite group.

Observe that if $\mathbf{A} \sim \mathbf{B}$, then $\mathbf{A}$ verifies $(*_J)$ if and only if $\mathbf{B}$ does. Let $\mathcal{M}_J \subset \mathcal{M}^{n+1}$ be the set of matrices satisfying $(*_J)$ whose diagonals are 1. Then $\mathcal{M}_J/\sim$ is the moduli space of marked reflection groups isomorphic to W_J.

We also observe that a subgroup of $\mathrm{PGL}_{n+1}\mathbb{R}$ acts discontinuously on the whole $\mathbb{P}^n$ if and only if it is a finite group. On the other hand, it is well known that W_J is a finite Coxeter group if and only if the Cartan matrix $\mathbf{C}_J$ is positively definite. So the Tits set Ω is a convex set contained in an affine chart if and only if J satisfies

(i) $\mathbf{C}_J$ is not positively definite.

Furthermore, the condition for openness in the theorem is equivalent to

(ii) Every proper principle submatrix of $\mathbf{C}_J$ is positively definite.

All the Coxeter diagrams J satisfying (i) and (ii) are completely classified, see [2]. There are two cases:

Euclidean Case: In addition to (i) and (ii), we assume $\mathbf{C}_J$ is degenerate. In this case $\mathbf{C}_J$ has corank 1, and there is a faithful representation $\rho_0 : W_J \to \mathrm{Isom}(\mathbb{E}^n)$ which realize W_J as an Euclidean simplicial reflection group. In this case, the Tits set Ω can only be either a simplex, or an affine chart.

Hyperbolic Case: In addition to (i) and (ii), we assume $\mathbf{C}_J$ is non-degenerate. Such a Coxeter diagram is called a Lannér diagram, as they are first classified by F. Lannér. In this case $\mathbf{C}_J$ has signature $(1, n)$, and there is a faithful representation $\rho_0 : W_J \to \mathrm{Isom}(\mathbb{H}^n)$ which realize W_J as a hyperbolic simplicial reflection group. We reproduce in Figure 5.1 the table of all Lannér diagrams (see [2], p.205). Note that they exist only for $n \leq 4$. In this case, since W_J is a hyperbolic group in the sense of Gromov, by the Theorem 1.1 of Benoist [11], the Tits set Ω is strictly convex, *i.e.*, $\partial\Omega$ does not contain any straight segment.

5.1.3 Moduli space of convex projective structures

In this section, we take a Lannér diagram $J = (\{0, 1, \cdots, n\}, \{m_{ij}\})$. Let $\rho_0 : W_J \to \mathrm{PGL}_{n+1}\mathbb{R}$ realizes W_J as a hyperbolic reflection group with fundamental simplex P. We will not distinguish W_J and its image $\rho_0(W_J)$. Our goal is to determine the space of convex projective structures on the orbifold $X_J = \mathbb{H}^n/W_J$.

Let $P_0, \cdots, P_n$ be the faces of P and L_i be the hyperplane of $\mathbb{P}^n$ containing P_i. Consider a faithful representation $\rho : W_J \to \mathrm{PGL}_{n+1}\mathbb{R}$ which defines a convex projective structure. There is some convex open set Ω_ρ and a homeomorphism $\Phi : \mathbb{H}^n \to \Omega_\rho$ which is ρ-equivariant, *i.e.*, $\Phi(\gamma.x) = \rho(\gamma).\Phi(x)$, for any $x \in \mathbb{H}^n$ and any $\gamma \in W_J$.

Since $\rho(\tau_i)$ has order 2, its fixed point set in $\mathbb{P}^n$ is the disjoint union of a k-dimensional subspace and a $(n-k)$-subspace. On the other hand, $\rho(\tau_i)$ fixes pointwisely $\Phi(L_i)$, a $(n-1)$-dimensional submanifold of Ω_ρ, we conclude that $k = 1$, $\Phi(L_i)$ is a $(n-1)$-subspace and $\rho(\tau_i)$ is a reflection. Therefore, $\rho(W_J)$ is a projective reflection group with fundamental simplex $\Phi(P)$.

Following the discussion in the last section, we may suppose that the fundamental simplex is $P = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n | x_i \geq 0, \forall i\}$ and the reflection group $\rho(W_J)$ is given by some matrix $\mathbf{A} \in \mathcal{M}_J$, whose i-th row is the homogenous coordinates of a fixed point f_i of $\rho(\tau_i)$. Conversely, every $\mathbf{A} \in \mathcal{M}_J$ yields a representation $\rho : W_J \to \mathrm{PGL}_{n+1}\mathbb{R}$ preserving a Tits set Ω_ρ, and thus defines an element $[\rho] \in \mathfrak{P}(X_J)$. Moreover, given two such representations ρ_1 and ρ_2 which comes from $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{M}_J$ respectively, ρ_1 and ρ_2 are conjugate if and only if $\mathbf{A}_1 \sim \mathbf{A}_2$. Therefore, we have the identification

$$\mathfrak{P}(X_J) = \mathcal{M}_J/\sim$$

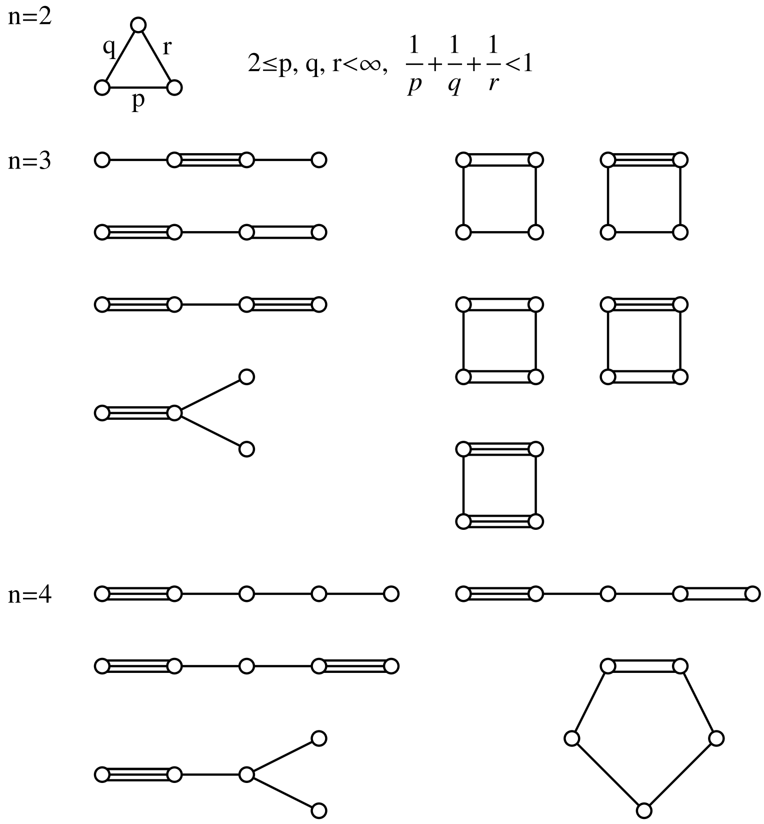


Figure 5.1: Lannér diagrams. Here single, double and triple edge stand for weight 3, 4 and 5, respectively.

Proposition 5.2.

$$\mathfrak{P}(X_J) \cong \begin{cases} \mathbb{R}_+ & \textit{if } J \textit{ has a loop,} \\ \textit{a point} & \textit{otherwise.} \end{cases}$$

Given a $n \times n$ matrix $\mathbf{A} = (a_{ij})$ and an ordered set of indices $1 \leq i_1, \cdots, i_k \leq n$ with $k \geq 1$, we call $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$ a *cyclic product of length k*.

Lemma 5.3. *Let* $\mathbf{A} = (a_{ij})$ *be a* $n \times n$ *matrix satisfying the condition:*

$$\textit{For any } i,\ a_{ii} \neq 0. \textit{ For any } i \neq j,\ a_{ij} = 0 \textit{ if and only if } a_{ji} = 0 \tag{5.1}$$

and the same hypothesis for $\mathbf{B}$. *We write* $\mathbf{A} \sim \mathbf{B}$ *if* $\mathbf{A}$ *and* $\mathbf{B}$ *are conjugate via a diagonal matrix, i.e., there are* $\lambda_1, \cdots, \lambda_n \neq 0$, *such that*

$$\operatorname{diag}(\lambda_1, \cdots, \lambda_n) \mathbf{A} \operatorname{diag}(\lambda_1^{-1}, \cdots, \lambda_n^{-1}) = \mathbf{B}.$$

Then, $\mathbf{A} \sim \mathbf{B}$ *if and only if their cyclic products with the same indices coincide, i.e., for any ordered subset* $\{i_i, \cdots, i_k\} \subset \{1, \cdots, n\}$, *we have*

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} = b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_{k-1} i_k} b_{i_k i_1}.$$

Proof. We say a matrix $\mathbf{A}$ is *reducible* if, after a reordering of basis if necessary, $\mathbf{A}$ can be put into a block-diagonal form. Otherwise $\mathbf{A}$ is said to be *irreducible.* The hypothesis on $\mathbf{A}$ and $\mathbf{B}$ implies $a_{ii} = b_{ii}$ and $a_{ij} = 0 \Leftrightarrow b_{ij} = 0$ for any $i \neq j$. Therefore, after a reordering of basis if necessary, we can assume that $\mathbf{A}$ and $\mathbf{B}$ are both block-diagonal with irreducible blocks, and the r-th block of $\mathbf{A}$ has the same size with the r-th block of $\mathbf{B}$. Clearly, $\mathbf{A}$ and $\mathbf{B}$ are conjugate via a diagonal matrix if and only if their blocks are. Thus we can assume $\mathbf{A}$ and $\mathbf{B}$ are irreducible.

We are looking for $\lambda_1, \cdots, \lambda_n$ which satisfy $\lambda_i^{-1} a_{ij} \lambda_j = b_{ij}$, or equivalently,

$$\frac{\lambda_i}{\lambda_j} = \frac{a_{ij}}{b_{ij}}, \text{ for all } i \neq j \text{ such that } b_{ij} \neq 0 \tag{5.2}$$

First, we take $\lambda_1 = 1$. Irreducibility means that, for each $i \in \{1, 2, \cdots, n\}$, there is sequence of distinct indices $1, i_1, i_2, \cdots, i_k, i$, such that a_{1i_1}, $a_{i_1 i_2}, \cdots$, $a_{i_{k-1} i_k}$, $a_{i_k i}$ are all non-zero. We should set

$$\lambda_i = \frac{\lambda_i}{\lambda_{i_k}} \frac{\lambda_{i_k}}{\lambda_{i_{k-1}}} \cdots \frac{\lambda_{i_1}}{\lambda_1} = \frac{a_{ii_k}}{b_{ii_k}} \frac{a_{i_k i_{k-1}}}{b_{i_k i_{k-1}}} \cdots \frac{a_{i_1 1}}{b_{i_1 1}} \tag{5.3}$$

this definition does not depend on the sequence of indices that we chose, since if we take another sequence $1, j_1, j_2, \cdots, j_m, i$, then the definition becomes

$$\lambda_i = \frac{a_{ii_k}}{b_{ii_k}} \frac{a_{i_k i_{k-1}}}{b_{i_k i_{k-1}}} \cdots \frac{a_{i_1 1}}{b_{i_1 1}} = \frac{b_{1j_1}}{a_{1j_1}} \frac{b_{j_1 j_2}}{a_{j_1 j_2}} \cdots \frac{b_{j_m i}}{a_{j_m i}} \tag{5.4}$$

where we used the coincidence of cyclic products $a_{ij}a_{ji} = b_{ij}b_{ji}$. Now the hypothesis implies that the right hand sides of (5.3) and (5.4) are the same. In same way, we can verify that the hypothesis implies these λ_i's satisfy (5.2). □

Proof of Proposition 5.2. If there is no loop in the Coxeter diagram of J, then for any $\mathbf{A}$ satisfying $(*_J)$, its cyclic products of length ≥ 3 are all 0, while cyclic products of length 1 are just diagonal entries, which equal 1, and cyclic products length 2 are determined by $(*_J)$. By Lemma 5.3, for any $\mathbf{A}, \mathbf{B} \in \mathcal{M}_J$, we have $\mathbf{A} \sim \mathbf{B}$.

If there is a loop in the Coxeter diagram, from Figure 5.1 we see that the whole graph is a circuit. Thus any $\mathbf{A}$ satisfying $(*_J)$ has the following form (here we set $n = 4$, for example):

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha_{01} & 0 & 0 & \alpha_{04} \\ \alpha_{10} & 1 & \alpha_{12} & 0 & 0 \\ 0 & \alpha_{21} & 1 & \alpha_{23} & 0 \\ 0 & 0 & \alpha_{32} & 1 & \alpha_{34} \\ \alpha_{40} & 0 & 0 & \alpha_{43} & 1 \end{pmatrix}$$

Again there are no choices for cyclic products of length 1 and 2. The only two non-zero cyclic products of length ≥ 3 are $\varphi(\mathbf{A}) = \alpha_{01}\alpha_{12} \cdots \alpha_{n-1,n}\alpha_{n1}$ and $\widetilde{\varphi}(\mathbf{A}) = \alpha_{10}\alpha_{21} \cdots \alpha_{n,n-1}\alpha_{1n}$. But by the condition $(*_J)$ we have

$$\varphi(\mathbf{A})\widetilde{\varphi}(\mathbf{A}) = \cos^2(\frac{\pi}{m_{01}}) \cos^2(\frac{\pi}{m_{12}}) \cdots \cos^2(\frac{\pi}{m_{n1}})$$

Therefore by Lemma 5.3, for any $\mathbf{A}, \mathbf{B} \in \mathcal{M}_J$, $\mathbf{A} \sim \mathbf{B}$ if and only if $\varphi(\mathbf{A}) = \varphi(\mathbf{B})$. This value is always positive if n is odd, and always negative if n is even. Thus the following map is a homeomorphism:

$$\begin{array}{rcl} \mathfrak{P}(X_J) = \mathcal{M}_J / \sim & \to & \mathbb{R}_+ \\ [\mathbf{A}] & \mapsto & |\varphi(\mathbf{A})| \end{array}$$

□

In order to study how the Tits set deforms when $[\mathbf{A}]$ goes to 0 or $+\infty$ in $\mathfrak{P}(X_J)$, we need the follow lemma, which bounds the Tits set by a simplex.

Lemma 5.4. *Let J be a Lannér diagram and take $\mathbf{A} \in \mathcal{M}_J$. Let $f_i \in \mathbb{P}^n$ be a point whose homogeneous coordinates are given by the i-th row of $\mathbf{A}$. Consider the representation $\rho : W_J \to \mathrm{PGL}_{n+1}\mathbb{R}$ sending τ_i to the reflection s_i fixing P_i and f_i. Then there is a simplices in $\mathbb{P}^n$ with vertices $f_0, \cdots, f_n$ which contains the Tits set Ω.*

Proof. Let L_i be the hyperplane of $\mathbb{P}^n$ passing through $f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n$. Assume by contradiction that Ω is not contained in any simplex with vertices $f_0, \cdots, f_n$. Then Ω meets some L_i. Without loss of generality, we suppose $\Omega \cap L_0 \neq \emptyset$.

L_0 is stabilized by the finite Coxeter group $\Gamma_0 = \langle s_1, \cdots, s_n \rangle$, because the reflection s_i stabilizes any hyperplane passing through f_i, and L_0 is spanned by $\{f_1, \cdots, f_n\}$. Thus Γ_0 is a finite affine transformation group of the affine chart $\mathcal{A}_0 := \mathbb{P}^n \backslash L_0 \cong \mathbb{R}^n$. It follows that Γ_0 must have a fixed point in $\mathcal{A}_0$, namely, the barycenter of an orbit.

On the other hand, let H_i denotes the hyperplane containing P_i. Since the fixed point set of s_i is $H_i \cup \{f_i\}$, the only fixed point of Γ_0 in $\mathbb{P}^n$ is $p_0 = [1 : 0 : \cdots : 0] = H_1 \cap \cdots \cap H_n$. Therefore we have proved $p_0 \notin L_0$.

We may consider the affine chart $\mathcal{A}_0$ as a linear space with origin p_0, and endow it with a Γ_0-invariant Euclidean scalar product. Γ_0 is then a finite Euclidean Coxeter group generated by n Euclidean reflections with respect to the subspaces $L_1, \cdots, L_n$. We shall remark that such a group can not preserve any convex cone except for the whole $\mathcal{A}_0$, since otherwise the barycenter p' of some non-zero orbit in the cone is a non-zero fixed point of the group, and it follows that each of $L_1, \cdots, L_n$ contains the line passing through p_0 and p', contradicting the independence of the L_i's.

Now take $C = \bigcup_{x \in \Omega \cap L_0} [p_0, x]$, the cone of $\Omega \cap L_0$ over p_0. Here $[p_0, x]$ denotes the segment in Ω joining p_0 and x. C is a Γ_0-invariant subset of Ω. $C \cap \mathcal{A}_0$ is a Γ_0-invariant convex cone of $\mathcal{A}_0$, clearly does not equals the whole $\mathcal{A}_0$. Thus the above remark concludes our contradiction argument. ☐

Proposition 5.5. *Let P be a simplex in $\mathbb{P}^n$. Let $X_J = \mathbb{H}^n/\Gamma$ be as in Proposition 5.2, such that $\mathfrak{P}(X_J) \cong \mathbb{R}_+$. Then there exists a one-parameter family of representations $\{\rho_t\}_{t \in \mathbb{R}_+}$ of Γ into $\mathrm{PGL}_{n+1}\mathbb{R}$, such that*

(1) Each $\rho_t(\Gamma)$ is generated by projective reflections with respect to faces of P.

(2) $\mathbb{R}_+ \to \mathfrak{P}(X_J)$, $t \mapsto [\rho_t]$ is bijective.

(3) Let Ω_t be the convex open subset of $\mathbb{P}^n$ associated with ρ_t. Then each Ω_t is properly convex, and Ω_t converges to P in the Hausdorff topology when t tends to 0 or $+\infty$. (See Figure 5.2)

Proof. We only consider the case $n = 3$ to simplify the notation. Let us fix a Lannér diagram J with 4 nodes which has a loop. Any $\mathbf{A} \in \mathcal{M}_J$ has the form

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha_{01} & 0 & \alpha_{03} \\ \alpha_{10} & 1 & \alpha_{12} & 0 \\ 0 & \alpha_{21} & 1 & \alpha_{23} \\ \alpha_{30} & 0 & \alpha_{32} & 1 \end{pmatrix}, \text{ where } \alpha_{ij} < 0 \text{ and } \alpha_{ij}\alpha_{ji} = \cos^2(\pi/m_{ij}).$$

We define a one-parameter family of matrices $\{\mathbf{A}_t\}_{t \in \mathbb{R}} \subset \mathcal{M}_J$ as follows,

$$\mathbf{A}_t = \begin{pmatrix} 1 & -t\cos^2(\frac{\pi}{m_{01}}) & 0 & -t^{-1} \\ -t^{-1} & 1 & -t\cos^2(\frac{\pi}{m_{12}}) & 0 \\ 0 & -t^{-1} & 1 & -t\cos^2(\frac{\pi}{m_{23}}) \\ -t\cos^2(\frac{\pi}{m_{30}}) & 0 & -t^{-1} & 1 \end{pmatrix}$$

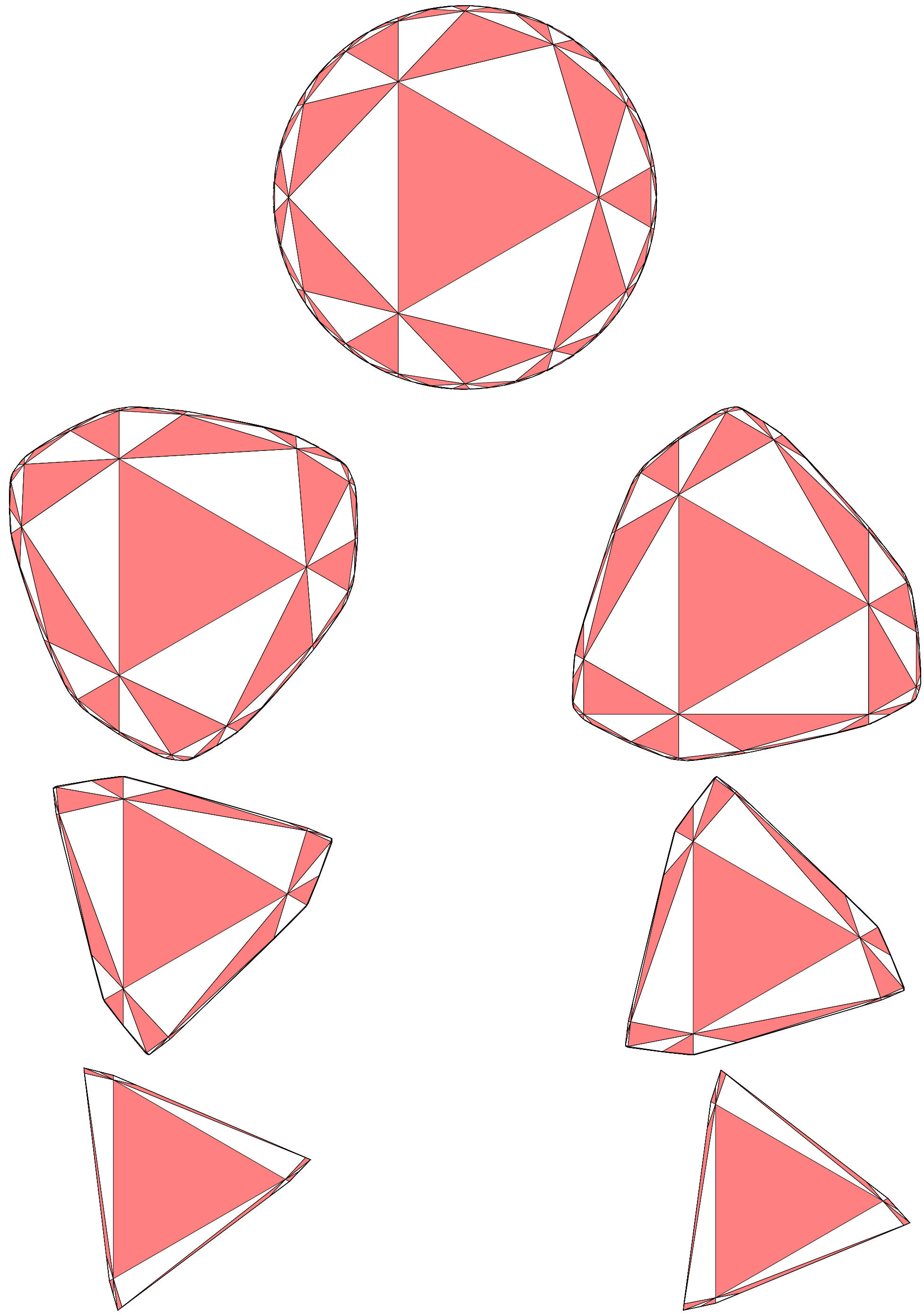


Figure 5.2: Deformation of Ω_t when t tends to 0 and to $+\infty$.

Since $|\varphi(\mathbf{A}_t)| = t^4$, by the proof of Proposition 5.2, every matrix $\mathbf{A}$ in $\mathcal{M}_J$ is $\sim$-equivalent to exactly one $\mathbf{A}_t$. $t \mapsto [\mathbf{A}_t]$ is a homeomorphism from $\mathbb{R}_+$ to $\mathcal{M}_J/\sim$.

For $i = 0, 1, 2, 3$, let $f_i(t)$ be the point in $\mathbb{P}^n$ whose homogeneous coordinates are given by the i-th row of $\mathbf{A}_t$. Define $\rho_t : W_J \to \mathrm{PGL}_{n+1}\mathbb{R}$ to be the representation sending τ_i to the reflection fixing $f_i(t)$ and the face P_i of P. Then $t \mapsto [\rho_t]$ is a homeomorphism from $\mathbb{R}$ to $\mathfrak{P}(X_J)$. Let Ω_t be the Tits set of $\rho_t(W_J)$.

$p_0 = [1:0:0:0], \cdots, p_3 = [0:0:0:1]$ are the vertices of P. Each $f_i(t)$ converges to p_{i+1} when $t \to +\infty$, and to p_{i-1} when $t \to -\infty$. Here the indices are counted mod 4. Therefore the simplex containing Ω_t given by Lemma 5.4 converges to P in the Hausdorff topology when $t \to 0$ or $+\infty$. $\square$

5.2 Entropy of the Hilbert metric on simplicial Tits sets

5.2.1 Hilbert metrics

For any properly convex open set $\Omega \subset \mathbb{P}^n$, we define the *Hilbert metric* d_Ω as follows. Take any affine chart $\mathbb{R}^n$ containing Ω. For $x, y \in \Omega$, let x_0, y_0 be the points on the boundary $\partial\Omega$ such that x_0, x, y, y_0 lie consecutively on the segment $[x_0, y_0]$, then we define

$$d_\Omega(x, y) = \frac{1}{2}\log[x_0, x, y, y_0], \text{ where } [x_0, x, y, y_0] = \frac{|x_0 - y||y_0 - x|}{|x_0 - x||y_0 - y|} \tag{5.5}$$

We refer to [11, 17] for basic properties of the Hilbert metric. In this section, we study the geometry of the Hilbert metric d_t on Ω_t (see Proposition 5.5).

Our main result concerns the metric geometry on the above family of convex sets. Recall that any properly convex open set $\Omega \subset \mathbb{P}^n$ carries a canonical Finsler metric d_Ω, called the *Hilbert metric*. If Ω is an ellipsoid, then (Ω, d_Ω) is isometric to the real hyperbolic n-space $\mathbb{H}^n$.

The one-parameter family $\{\Omega_t\}$ in Proposition 5.5 give rise to a family $\{d_t\}$ of Hilbert metrics. From Proposition 5.5 we can already deduce some easy geometric properties of this family. For example, the diameter and volume of X_J with respect to d_t tends to infinity as $t \to 0$ or ∞. The purpose of this paper is to study the following deeper quantity:

Definition 5.6. Let Γ be a group, and $(\widetilde{X}, d)$ be a metric space. The *orbit growth* of a point $x_0 \in \widetilde{X}$ with respect to an isometric discontinuous Γ-action on $\widetilde{X}$ is the number

$$\delta(\widetilde{X}, d) = \varlimsup_{R \to +\infty} \frac{1}{R} \log \#\Gamma x_0 \cap B(x_0, R)$$

where $B(x_0, R)$ denotes the ball of radius R centered at x_0.

We shall assume that the stabilizer of every point in $\widetilde{X}$ is a finite group, so that $\delta(\widetilde{X}, d)$ does not depend on the choice of x_0.

$\delta(\widetilde{X}, d)$ is also called the *volume entropy* because it equals the exponential growth rate of the volume of $B(x_0, R)$. If $(\widetilde{X}, d)$ is the universal covering of a compact non-positively curved Riemaniann manifold, A. Manning [43] proved that the topological entropy of the geodesic flow of X equals $\delta(\widetilde{X}, d)$. This result easily generalizes to geodesic flows of convex projective manifolds.

5.2.2 The main results

M. Crampon [17] proved that for any properly convex open set $\Omega \subset \mathbb{P}^n$ which covers a manifold with convex projective structure, we have

$$\delta(\Omega, d_\Omega) \leq n - 1.$$

The equality is achieved if and only if Ω is an ellipsoid. He then asked whether $\delta(\Omega, d_\Omega)$ has a lower bound.

The main result of the present part of the thesis gives a negatively answer:

Theorem 5.7. *Let X_J be a hyperbolic simplicial Coxeter orbifold with $\mathfrak{P}(X_J) \cong \mathbb{R}_+$. Let ρ_t and Ω_t be as in Proposition 5.5, and d_t be the Hilbert metric on Ω_t. Then*

$$\delta(\Omega_t, d_t) \to 0 \text{ as } t \to 0 \text{ or } +\infty.$$

The main ingredient in the proof of Theorem 5.7 is the following result

Proposition 5.8. *There exists a constant C depending only on the Coxeter diagram J, such that if A and B are two k-dimensional cells of P and $E = A \cap B$ is a $(k-1)$-dimensional cell, where $1 \leq k < n$, then for any $x \in A$, $y \in B$ and any $t \in \mathbb{R}_+$, we have*

$$Cd_t(x, y) \geq d_t(x, E) + d_t(y, E)$$

As another consequence of Proposition 5.8, we construct families of convex projective structures on surfaces which answer Crampon's problem and have some other curious properties.

Corollary 5.9. *On every oriented closed surface of genus $g \geq 2$, there exists an one-parameter family of convex projective structures such that when the parameter goes to ∞, the entropy of geodesic flow tends to 0, the systole and constant of Gromov hyperbolicity tending to $+\infty$.*

Recall that for a metrized manifold (X, d), the *systole* is defined as the infimum of lengths of homotopically non-trivial closed curves on X. Let $\widetilde{X}$ be the universal covering of X, then the *constant of Gromov hyperbolicity* is defined to be the supremum of sizes of geodesic triangles in $\widetilde{X}$ (where the size of a geodesic triangle Δ is the minimal perimeter of all geodesic triangles inscribed to Δ, see [1], Chapter 2, §3).

5.2.3 Proof of the main results based on Proposition 5.8

In subsection we shall prove Theorem 5.7 and Corollary 5.9 admitting Proposition 5.8.

Ω_t has the structure of *Coxeter complex*, *i.e.*, Ω_t is a simplicial complex whose k-cells are translates of the k-cells of P by the W_J-action. We denote the k-skeleton of Ω_t by $\Omega_t^{(k)}$, and define $d_t^{(k)}$ to be the intrinsic metric on the k-skeleton $\Omega_t^{(k)}$, *i.e.*, $d_t^{(k)}(x, y)$ equals the minimal length of piecewise segments joining x, y and lying in $\Omega_t^{(k)}$. In particular, $(\Omega_t^{(1)}, d_t^{(1)})$ is a metric graph, and $d_t^{(n)}$ is just d_t.

Proposition 5.10. *Suppose $2 \leq k \leq n$. There is a constant C, depending only on J, such that for any $t \in \mathbb{R}_+$ and any $x, y \in \Omega_t^{(k-1)}$, we have*

$$d_t^{(k)}(x, y) \leq d_t^{(k-1)}(x, y) \leq Cd_t^{(k)}(x, y)$$

As a consequence, $d_t(x, y) \leq d_t^{(1)}(x, y) \leq C'd_t(x, y)$ for some constant C' depending only on J.

Proof. The first inequality is evident from the definition.

We prove the second inequality. Let $c : [0,1] \to \Omega_t^{(k)}$ be a piecewise segment joining $x, y \in \Omega_t^{(k-1)}$ such that the length of c equals $d_t^{(k)}(x,y)$.

Let $t_0 = 0, t_1, t_2, \cdots, t_r = 1 \in [0,1]$ be such that each $c([t_{i-1}, t_i])$ lies in a single k-cell, and the $c(t_i)$'s are in $\Omega_t^{(k-1)}$. Since c is length-minimizing, each $c[t_{i-1}, t_i]$ must be a segment, whose length equals the distance between then two end points. Thus if we could prove

$$d_t^{(k-1)}(c(t_{i-1}), c(t_i)) \leq C d_t^{(k)}(c(t_{i-1}), c(t_i))$$

then we would take the sum over $1 \leq i \leq r$ and use the triangle inequality to obtain

$$d_t^{(k-1)}(x,y) \leq C d_t^{(k)}(x,y)$$

Therefore, we can assume that both x and y lie on the boundary of a k-cell. Since each k-cell is isometric to some subcell of P, it is sufficient to prove that, for any k-dimensional subcell F of P, we have for all $t \in \mathbb{R}_+$ and all $x, y \in F$

$$d_t^{(k-1)}(x,y) \leq C d_t^{(k)}(x,y) = C d_t(x,y).$$

If x, y both lie on the same $(k-1)$-dimensional subcell of F, then we have $d_t^{(k)}(x,y) = d_t^{(k-1)}(x,y)$ and there is nothing to prove. Thus, we can assume that $x \in A$ and $y \in B$, where A, B are $(k-1)$-dimensional subcells of F, such that $E = A \cap B$ is a $(k-2)$-dimensional subcell. Let $x_0, y_0 \in E$ be the nearest point to x, y in E, respectively. *i.e.*, $d_t(x, E) = d_t(x, x_0)$ and $d_t(y, E) = d_t(y, y_0)$.

The three segments $[x, x_0]$, $[x_0, y_0]$ and $[y_0, y]$ lie in $\Omega_t^{(k-1)}$, and form a piecewise segment joining x, y. Thus we have

$$d_t^{(k-1)}(x,y) \leq d_t(x, x_0) + d_t(x_0, y_0) + d_t(y_0, y) \tag{5.6}$$

by the triangle inequality, we have

$$d_t(x_0, y_0) \leq d_t(x_0, x) + d_t(x, y) + d_t(y, y_0) \tag{5.7}$$

(5.6)and (5.7) gives

$$d_t^{(k-1)}(x,y) \leq 2d_t(x, x_0) + 2d_t(y, y_0) + d_t(x,y)$$

Now we apply Proposition 5.8, and conclude that

$$d_t^{(k-1)}(x,y) \leq (2C+1) d_t(x,y)$$

this is the required inequality. □

Proof of Theorem 5.7. Note that each vertex of the simplex P lies on different orbits of W_J, so the vertex set $\Omega_t^{(0)}$ is the union of $n+1$ orbits. Hence, fixing any vertex v_0, we have the follow expression for the orbit growth h_t of the W_J-action on Ω_t:

$$h_t = \overline{\lim_{R\to\infty}} \frac{1}{n} \log \#\{v \in \Omega_t^{(0)} | d_t(v, v_0) \leq R\}$$

The entropy $h(\Omega_t^{(1)}, d_t^{(1)})$ of the metric graph $(\Omega_t^{(1)}, d_t^{(1)})$ is defined to be the exponential growth of the number of vertices in large balls, *i.e.*,

$$h(\Omega_t^{(1)}, d_t^{(1)}) = \overline{\lim_{R\to\infty}} \frac{1}{n} \log \#\{v \in \Omega_t^{(0)} | d_t^{(1)}(v, v_0) \leq R\}$$

Therefore, Proposition 5.10 implies there is a constant C' depending only on J such that

$$h(\Omega_t^{(1)}, d_t^{(1)}) \leq h_t \leq C' h(\Omega_t^{(1)}, d_t^{(1)}).$$

So it is sufficient to prove

$$h(\Omega_t^{(1)}, d_t^{(1)}) \to 0, \text{when } t \to 0 \text{ or } +\infty$$

To this end, let $l(t)$ be the minimal length of edges of P under d_t. We have seen in Proposition 5.5 that Ω_t approaches the simplex P when $t \to 0$ or $+\infty$. Using the expression of Hilbert metric (5.5) one can see the length of each edge of P tends to $+\infty$, thus $l(t) \to +\infty$.

On the other hand, a W_J-invariant geodesic metric on the graph $\Omega_t^{(1)}$ is uniquely determined by lengths of the edges of P, and monotone with respect to each of these lengths. Therefore, if we let d_1 be the metric defined by setting all edge lengths to be 1, then we have $d_t^{(1)} \geq l(t) d_1$. By definition of entropy of graphs, this gives

$$h(\Omega_t^{(1)}, d_t^{(1)}) \leq \frac{1}{l(t)} h(\Omega_t^{(1)}, d_1) \to 0$$

The proof is complete. □

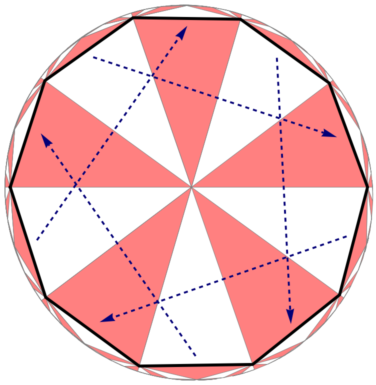
Proof of Corollary 5.9. Let Σ be a surface with genus ≥ 2. We claim that there are integers $p, q, r \geq 3$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and a subgroup Π of finite index in the (p, q, r)-triangle group $\Delta = \Delta_{p,q,r}$ such that Π acts freely on the hyperbolic plan $\mathbb{H}^2$ with quotient $\mathbb{H}^2/\Pi \cong \Sigma$. Restricting the one-parameter family of representations $\rho_t : \Delta \to \mathrm{PGL}_{n+1}\mathbb{R}$ given by Proposition 5.2 and 5.5 to Π, we obtain an one-parameter family of convex projective structures on Σ. We shall show that this family fulfils the requirements.

The entropy of geodesic flow is the orbit growth $\delta(\Omega_t, d_t)$ for the Π action, and is the same as the orbit growth for the Δ-action since $\Pi \subset \Delta$ has finite index. Thus $\delta(\Omega_t, d_t) \to 0$ by Theorem 5.7.

Proposition 5.8 implies that for any t, every triangle inscribed to the fundamental triangle P has perimeter greater than $\frac{1}{C}$ times the perimeter of P. When $t \to 0$ or $+\infty$, under the metric d_t, the length of each edge of P tends to $+\infty$, so the perimeters of all inscribed triangles tends to $+\infty$ uniformly. It follows that the constant of Gromov hyperbolicity of Ω_t tends to $+\infty$.

To show the systole goes to $+\infty$, we take a homotopically non-trivial closed curve c which is the shortest under d_t. The image of c under the orbifold covering map $\Sigma \cong \mathbb{H}^2/\Pi \to \mathbb{H}^2/\Delta \cong P$ is a closed billiard trajectory in the triangle P which hits each of the three sides. The lengths under d_t of such trajectories are bounded from below by the minimal perimeter of inscribed triangles of P. We have already seen the latter goes to $+\infty$.

Finally, we prove the claim using a constructions by hand. See the picture below. The boldfaced 10-gon consists of ten fundamental domains of $\Delta = \Delta_{5,5,5}$. We take the five elements in Δ indicated by the arrows, each of them pushing the 10-gon to an adjacent one. One can check that the group Π generated by them has the 10-gon as a fundamental domain. The quotient $\mathbb{H}^2/\Pi$ is a surface obtained by pairwise gluing edges of the 10-gon. A calculation of Euler characteristic shows $\mathbb{H}^2/\Pi$ have genus 2. Since closed surfaces of higher genus covers the surface of genus 2, by taking subgroups of Π, we conclude that all surfaces of genus ≥ 2 is the quotient of $\mathbb{H}^2$ by some subgroup of Δ, and the claim is proved. □



5.2.4 Proof of Proposition 5.8

To begin with, we need the following fact concerning simplicial and projective structures on Ω_t. Looking at Figure 5.2, we may observe that the 1-skeleton of Ω_t consists of straight lines. There is a same phenomenon in higher dimension, *i.e.*, $\Omega_t^{(k)}$ is a union of k-dimensional subspaces. Note that by "subspace" of Ω_t, we mean the intersection of a subspace of $\mathbb{P}^n$ with Ω_t. An equivalent statement of the above fact is that the k-dimensional subspace L containing a k-cell F must be an union of k-cells. This can be proved using the fact that the tangent space of a vertex in Ω_t has the structure of a finite Coxeter complex, and it is well known that the above statement holds for finite Coxeter complex (See for instance [34]). We omit the details.

First we present a proof of Proposition 5.8 for the 2-dimensional case, since the main idea is transparent in this case, while in higher dimensions we have to deal with some extra difficulties.

Proof of Proposition 5.8 for $n = 2$. We may assume

$$W_J =< \tau_1, \tau_2, \tau_3 | (\tau_1\tau_2)^p = (\tau_2\tau_3)^q = (\tau_3\tau_1)^r = \tau_1^2 = \tau_2^2 = \tau_3^2 = 1 >$$

Suppose x and y lie on the sides A and B of a triangle P in $\mathbb{P}^2$, respectively. Denote the common vertex of A and B by E. We need to prove there is a constant C depending only on p, q, r, such that $Cd_t(x, y) \geq d_t(x, E) + d_t(y, E)$ for any t.

Let us fix a t and denote $s_1 := \rho_t(\tau_1)$, $s_2 := \rho_t(\tau_2)$, which are reflections with respect to A and B, respectively. s_1s_2 is a rotation of order p.

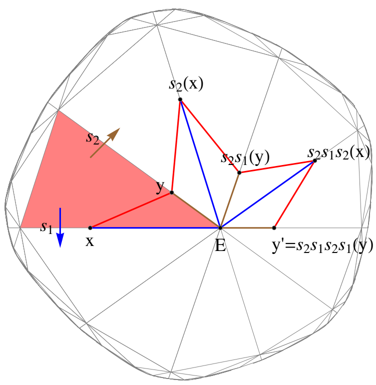


Figure 5.3: $p = 5$

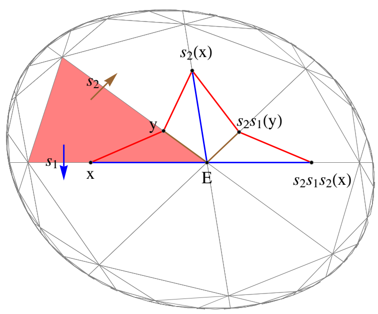


Figure 5.4: $p = 4$

When p is odd, $y' = \underbrace{s_2 s_1 s_2 \cdots s_1}_{p-1}(y)$ lies on the opposite half ray of the geodesic ray $\vec{Ex}$ (see Figure 5.3). On the other hand, the successive images of $[x,y]$ by the sequence of transformations

$$s_2,\ s_2 s_1,\ s_2 s_1 s_2,\ \cdots,\ \underbrace{s_2 s_1 s_2 \cdots s_1}_{p-1}$$

is a piecewise segment joining x and y', which consists of p pieces, each piece having the same length $d_t(x,y)$. Thus we have

$$pd_t(x,y) \geq d_t(x,y') \geq d_t(x,E)$$

When p is even, we obtain $pd_t(x,y) \geq d_t(x,E)$ in the same way (see Figure 5.4).

As for y, we have the same inequality

$$pd_t(x,y) \geq d_t(y,E)$$

Thus we may conclude that

$$2pd_t(x,y) \geq d_t(x,E) + d_t(y,E)$$

Proposition 5.8 is proved for $n = 2$. □

We introduce the following terminology. Let E be a $(k-1)$-cell of Ω_t. We say two k-cells are *E-colinear*, if they lie on the same k-dimensional subspace and their intersection is E. As we have explained in the beginning of this section, the k-dimensional subspace of Ω_t containing a k-cell A is an union of k-cells. Thus for any $(k-1)$-dimensional subcell E of A, there is an unique k-cell which is E-colinear to A.

The crucial point of the above proof is the following: let V be the k-cell E-colinear to A. Then we can connect $x \in A$ and some point of V by a curve which is piecewise isometric to $[x,y]$, and the number of pieces is determined combinatorially. We then have proved the needed inequality using the fact that the distance from x to any point of V is greater than the distance from x to E.

In higher dimensions, the situation is more delicate: the cell V which is E-colinear to A may not be a translate of A or B. In this case, we can not construct a curve piecewise isometric to $[x,y]$ going from x to V. Instead of this, we take a cell $A' = \rho_t(\gamma)A$, the translate of A by the action of some $\gamma \in W_J$, such that A' and V are contained in the same top-dimensional cell. Now we can go from x to A' along a curve piecewise isometric to $[x,y]$. To prove Proposition 5.8, we then need to show that the distance from x to A' is greater than the distance from x to E. In order to do this, we shall develop some lemmas concerning distance comparisons in Hilbert geometry.

Using the definition of Hilbert metric (5.5), it can be shown that if $\Omega \subset \mathbb{P}^n$ is a properly convex open set which is strictly convex (see the end of Section 2), then the Hilbert metric d_Ω has the following property. Let L be a subspace of arbitrary dimension of Ω and x be a point of Ω outside L. Then among all points of L, there is an unique point $x_0 \in L$ whose distance to x is minimal. We call x_0 the *projection* of x on L, and denote it by $x_0 = \mathrm{Pr}(x,L)$.

Let $L \subset \Omega$ be a hyperplane, *i.e.*, subspace of codimension one. We say that Ω have reflectional symmetry s with respect to L if $s \in \mathrm{PGL}_{n+1}\mathbb{R}$ is a reflection preserving Ω and fixing each point of L. In this case, the triangle inequality and the fact that geodesics are straight lines yields the following simple characterization of projection:

$$\mathrm{Pr}(x,L) = [x, s(x)] \cap L \tag{5.8}$$

Lemma 5.11. *Let $\Omega \subset \mathbb{P}^n$ be a properly strictly convex open set with reflectional symmetry s with respect to a hyperplane L. Then for any $x, y \in \Omega$, we have*

$$d_\Omega(\mathrm{Pr}(x, L), \mathrm{Pr}(y, L)) \leq d_\Omega(x, y)$$

In particular, if $x \in L$, then for any $y \in \Omega$ we have

$$d_\Omega(x, \mathrm{Pr}(y, L)) \leq d_\Omega(x, y)$$

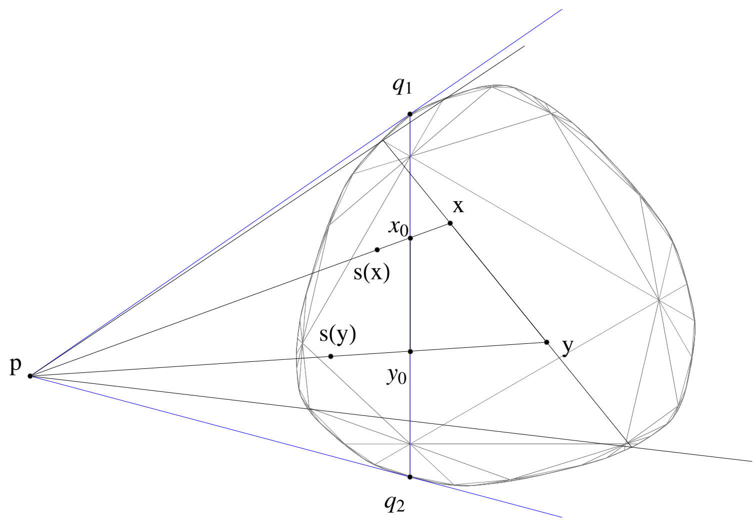


Figure 5.5: $d_\Omega(x_0, y_0) \leq d_\Omega(x, y)$

Proof. Denote $x_0 = \mathrm{Pr}(x, L)$ and $y_0 = \mathrm{Pr}(y, L)$. The reflection s has another fix point $p \in \mathbb{P}^n$ outside L. The reflection image $s(x)$ of x lies on the line $\overline{xp}$, and we have $x_0 = \overline{xp} \cap L$. The points y and y_0 has the same properties. Therefore, all the four points x, y, x_0, y_0 lie on the plane $\overline{pxy}$. This plane is invariant by s. So we may consider $\Omega_0 = \overline{pxy} \cap \Omega$ instead of Ω, and $L_0 = \overline{pxy} \cap L$ instead of L. Thus we have reduced to the two-dimensional case. See Figure 5.5

Suppose L_0 intersects $\partial\Omega_0$ at two points q_1, q_2. Since Ω_0 has reflectional symmetry with respect to L_0, the lines $\overline{pq_1}$ and $\overline{pq_2}$ are tangent to Ω. Now the inequality $d_\Omega(x_0, y_0) \leq d_\Omega(x, y)$ follows from the definition (5.5) of Hilbert metric and the following well known fact from projective geometry: given four lines l_i $(1 \leq i \leq 4)$ meeting at a point p, then for any line l intersecting l_1, l_2, l_3, l_4 consecutively at p_1, p_2, p_3, p_4, the number $[p_1, p_2, p_3, p_4]$ is a constant not depending on the choice of l.

□

Remark 5.12. Lemma 5.11 is not true without the hypothesis of symmetry.

Lemma 5.13. *Let $\Omega \subset \mathbb{P}^n$ be a properly strictly convex open set with reflectional symmetries $s_1, \cdots, s_m$ with respect to hyperplanes $L_1, L_2, \cdots, L_m$, such that $s_1, \cdots, s_m$ generates a finite group Γ. Assume $W = L_1 \cap \cdots \cap L_m$ has dimension ≥ 1 and $W \cap \Omega \neq \emptyset$. Let D be a Γ-invariant convex subset of Ω.*

Then for any point x of W outside D and any $x' \in D$, there is some point $x_0 \in D \cap W$ such that

$$d_\Omega(x, x_0) \leq d_\Omega(x, x')$$

Proof. Fix a point $x \in D$. We chose an affine chart $\mathcal{A} \subset \mathbb{P}^n$, an origin point x_0 of $\mathcal{A}$ in order to endow $\mathcal{A}$ with a linear space structure, and then an Euclidean scalar product on $\mathcal{A}$. We could make these choices so as to fulfil the following conditions:

(1) $\mathcal{A}$ contains the closure of Ω.

(2) $L_1, \cdots, L_m$ are linear subspaces of $\mathcal{A}$. In particular, the origin x_0 of $\mathcal{A}$ lies in W.

(3) Γ preserves the Euclidean scalar product.

(4) $x' \in W^\perp$, where $W^\perp$ is the orthogonal complement of W.

Our aim is to show that $x_0 \in D$ and x_0 satisfies the required inequality. We shall consider intensively the linear space $W^\perp$. Let us denote the origin x_0 simply as 0. Each of $L_i' = L_i \cap W^\perp$ is a subspace of $W^\perp$ of codimension 1, and the intersection $L_1' \cap \cdots \cap L_m' = \{0\}$. Since $D \cap W^\perp$ is Γ-invariant and convex, the barycenter of the Γ-orbit of x lies in $D \cap W^\perp$ and is fixed by Γ. But $L_1' \cap \cdots \cap L_m' = \{0\}$ implies the only fixed point of Γ in $W^\perp$ is 0, thus $x_0 \in \Omega$.

For $1 \leq i \leq m$, we define

$$C_i = \{y \in W^\perp | \angle(y, L_i') \geq \theta, \text{ or } y = 0\}$$

where $\angle(y, L_i')$ is the usual Euclidean angle. We may take θ small enough so that $C_1 \cup \cdots \cup C_m = W^\perp$. Any $y \in C_i$ verifies

$$|\operatorname{Pr}_{W^\perp}(y, L_i')| \leq |y| \cos\theta$$

where $\operatorname{Pr}_{W^\perp}(y, L_i')$ is the usual Euclidean projection of y on L_i'. Using the characterization of projection onto reflectional hyperplanes (5.8), we see that $\operatorname{Pr}_{W^\perp}(y, L_i')$ coincides with the projection $\operatorname{Pr}(y, L_i)$ in the sense of Hilbert geometry described earlier.

We construct a sequence of points $x' = y_0, y_1, y_2 \cdots \in D \cap W^\perp$ converging to 0 by recurrence as follows. Since $C_1 \cup \cdots \cup C_m = W^\perp$, there is some C_{i_k} containing y_k. Then we set $y_{k+1} = \operatorname{Pr}_{W\perp}(y_k, L_{i_k}')$. The above inequality yields

$$|y_k| \leq |y_{k-1}| \cos\theta \leq \cdots \leq |y_0| \cos^k \theta$$

Hence y_k converges to x_0 as $k \to \infty$.

As mentioned above, $y_{k+1} = \operatorname{Pr}(y_k, L_{i_k})$. Lemma 5.11 implies

$$d_\Omega(x, y_k) \leq d_\Omega(x, y_{k-1}) \leq \cdots \leq d_\Omega(x, y_0) = d_\Omega(x, x')$$

Therefore, by the continuity of d_Ω, we conclude that

$$d_\Omega(x, x_0) = \lim_{k\to\infty} d_\Omega(x, y_k) \leq d_\Omega(x, x')$$

□

Let us return to the particular convex set Ω_t. For any cell V of Ω_t, the union of all n-cells containing V is called the star-like neighborhood of V, and denoted by $\operatorname{St}(V)$. We need the following

Lemma 5.14. $\operatorname{St}(V)$ *is a convex subset of* Ω_t.

Proof. Let F be any $(n-1)$-cell lying on the boundary of $\operatorname{St}(V)$, and let L be the hyperplane containing F. L does not contain V, so V is contained in exactly one of the two closed "half spaces" of Ω_t bounded by L. Since $\operatorname{St}(V)$ is an union of n-cells containing V, using the fact that L is an union of $(n-1)$-cells, we can conclude the whole $\operatorname{St}(V)$ lie in the same half space as V.

Thus, $\operatorname{St}(V)$ is an intersection of closed half spaces, therefore convex. □

Proof of Proposition 5.8. Fix a Lannér diagram J. By definition of Lannér diagrams, a subgroup of $W_J =< \tau_0, \cdots, \tau_n | (\tau_i\tau_j)^{m_{ij}} = \tau_i^2 = 1, \forall i \neq j >$ generated by a proper subset of $\{\tau_0, \cdots, \tau_n\}$ is a finite Coxeter group. Let C be the maximum of word-length-diameters of all such subgroups. We will show $Cd_t(x, y) \geq d_t(x, E)$. Then by exchanging the roles of x and y, we have $Cd_t(x, y) \geq d_t(y, E)$, and these two inequalities give the required one.

Let V be the k-cell which is E-colinear to A. There is $\gamma \in \mathrm{Stab}_{\rho(W_J)}(E)$, such that $P' = \rho_t(\gamma)P$ is a top-dimensional cell containing V. We denote $A' := \rho_t(\gamma)A \subset P'$, $x' := \rho_t(\gamma)x \in A'$.

First we claim that there is a curve joining x and x' which is piecewise isometric to $[x, y]$, with number of pieces at most C.

Denote $s_i = \rho_t(\tau_i)$, a reflection with respect to the face P_i of P. Let $J_E \subset \{0, 1, \cdots, n\}$ be the set of indices of those P_i such that $E \subset P_i$. Then $\#J_E = n-k+1$, and $\mathrm{Stab}_{\rho(W_J)}(E)$ is generated by $\{s_i\}_{i\in J_E}$.

We can write $\rho_t(\gamma) = s_{i_1}s_{i_2}\cdots s_{i_m}$, with $i_1, \cdots, i_m \in J_E$ and $m \leq C$. Consider the sequence of segments

$$s_{i_1}([x, y]), s_{i_1}s_{i_2}([x, y]), \cdots, s_{i_1}s_{i_2}\cdots s_{i_m}([x, y]).$$

The k-cell A contains the $(k-1)$-cell E, so there is only one vertex a of A which lies outside E. Similarly B has only one vertex b outside E. Each face P_i of P must contain at least one of the two points a and b. Hence each face containing E also contains A or B. It follows that if $i \in J_E$ then s_i fixes x or y. Therefore, each segment in the above sequence shares at least one end point with the next one. So the union of these segments is connected, and we can extract a subset of these segments to form a curve joining x and $x' = \rho_t(\gamma)x$ which is piecewise isometric to $[x, y]$. The number of pieces is at most m, hence bounded by C.

Thus, we conclude

$$Cd_t(x, y) \geq d_t(x, x')$$

Next, we need to prove

$$d_t(x, x') \geq d_t(x, E) \tag{5.9}$$

We apply Lemma 5.13. Let $\mathrm{St}(V)$ be the convex compact set D in the hypothesis of Lemma 5.13, which contains x'. Let $J_A \subset \{0, 1, \cdots, n\}$ be the set of indices of those faces P_i such that $A \subset P_i$, and let L_i be the hyperplane on which P_i lies. Then $W = \cap_{i\in J_A} L_i$ is the k-dimensional subspace containing A and V. For each $i \in J_A$, since L_i contains V, the reflection s_i preserves $\mathrm{St}(V)$. Thus the hypothesis of Lemma 5.13 are verified, and we conclude that there is $x_0 \in V$ such that

$$d_t(x, x') \geq d_t(x, x_0)$$

$A \cup V$ is the intersection of $\mathrm{St}(E)$ and a k-dimensional subspace, thus must be convex. So $[x, x_0]$ intersects E at some point x_1. Clearly we have

$$d_t(x, x_0) \geq d_t(x, x_1) \geq d_t(x, E)$$

Hence we have obtained (5.9), and the proof is complete. □

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