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إلى الغاليين على قلبي: أمي وأبي، أعجز
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Chapitre 1

Introduction Générale

Cette thèse présente un ensemble de résultats théoriques relatifs à un problème couplé de deux équations de types Cahn-Hilliard et Allen-Cahn. Avant de parler de ces deux types d'équations et de leur couplage, on décrit brièvement deux méthodes de modélisation pour l'évolution des microstructures : la méthode des champs de phase et, en opposition, la méthode classique aussi connue sous le nom de la méthode d'interface raide "sharp interface". Ces méthodes sont à l'origine des équations de type Cahn-Hilliard et Allen-Cahn.

Dans la méthode classique, les régions séparant les domaines structuraux sont traitées comme des interfaces raides, pour lesquelles une certaine régularité est habituellement exigée. Il est alors nécessaire d'avoir des équations d'évolution associées aux variables physiques convenables dans chaque région mais aussi des équations d'évolution aux interfaces. Bien que cette méthodologie soit efficace en dimension 1, il devient difficile de l'appliquer en dimension 2 ou 3. En effet, ces interfaces ont des géométries compliquées séparant les différentes microstructures qui apparaissent souvent durant des processus de mûrissement appelés mûrissement d'Ostwald.

Par contre, ces dernières années, la méthode des champs de phase est devenue assez importante dans la modélisation de différents types d'évolution des microstructures. Elle se base sur une description d'interfaces raides développée par van der Waals il y a plus d'un siècle [185] et par Cahn et Hilliard d'une manière indépendante il y a presque cinquante ans [33].
L'idée principale de cette méthode est de remplacer le traitement macroscopique singulier de l'interface raide par une description régularisée. La singularité vient souvent d'une discontinuité de surface pour quelques variables. Dans ce but, on introduit un ou plusieurs champs auxiliaires nommés champs de phase, variant régulièrement à travers les interfaces.

La méthode des champs de phase permet de prévoir l'évolution des morphologies arbitraires et de microstructures compliquées sans suivre explicitement la position des interfaces. En plus, les modèles des champs de phase décrivent les phénomènes microstructurals à l'échelle mésoscopique et contiennent les descriptions convenables des interfaces raides, comme par exemple une limite particulière. Les modèles des champs de phase ont été utilisés dans une grande variété de problèmes ; quelques applications notables de cette méthodologie sont : simulation de

la diffusion limitée de la croissance cristalline [49, 115, 109], les instabilités induites par des contraintes dans les solides [111], la convection de Maragoni [26], les facteurs d'évolution (dynamics) de goutelettes et des vésicules [14], les mélanges polymériques [178] et le phénomène de mûrissement d'Ostwald [74].

Les modèles des champs de phase peuvent être associés à des variables conservées ou non conservées ; le premier cas entraîne des équations de type Cahn-Hilliard non linéaires, tandis que le deuxième cas engendre en général des équations de type Allen-Cahn non linéaires.

1.1 Sur l'Equation de Cahn-Hilliard

L'équation de Cahn-Hilliard est une équation aux dérivées partielles parabolique du quatrième ordre, introduite en 1958 par Cahn et Hilliard, [33]. Elle représente un paramètre d'ordre conservé et joue un important rôle dans la science des matériaux. Elle décrit des caractéristiques qualitatives des systèmes de deux phases en séparation, en ayant des hypothèses de l'isotropie et de température constante. C'est une équation d'évolution en temps de la concentration d'un matériau. Elle apparaît principalement quand un alliage binaire est refroidi convenablement. Le matériau homogène devient hétérogène et les nucléiques apparaissent : C'est ce qu'on appelle décomposition spinodale qui se déroule en première étape. Dans la deuxième étape, les microstructures grossissent mais plus lentement que dans le processus de la décomposition spinodale. Ce phénomène est essentiel pour étudier la force et la résistance du matériau. Pour plus de détails, le lecteur est invité à consulter les références [32, 33, 116, 124, 133, 134, 156, 157].

L'équation de Cahn-Hilliard a été mise en place pour décrire la séparation de phase avec conservation de masse. Dans le cas d'une mobilité constante avec énergie libre non singulière, le comportement asymptotique est donné par un système avec des conditions aux bords libres connu sous le nom de problème de Mullins-Sekerka [39, 165].

Cependant, l'utilisation de l'équation de Cahn-Hilliard a été élargie à divers domaines de la chimie, de la physiques, de la biologie et au secteur de l'ingénierie comme les copolymères diblocs, la retouche d'image, les flux des fluides multiphases, les microstructures avec des inhomogénités élastiques, la simulation de la croissance tumorale, l'optimisation de topologie, la dynamique des populations, la théorie de la formation de galaxies, et même les caractéristiques des anneaux de Saturne ! C'est pourquoi, il est important de comprendre le rôle fondamental de cette équation dans chaque type de modélisation, à ce propos le lecteur pourra consulter [114]. Pour voir les applications, le lecteur peut regarder [6, 8, 17, 22, 45, 48, 82, 100, 105, 106, 118, 190, 196].

L'équation de base de la modélisation mathématique et des modèles dynamiques est l'équation de Cahn-Hilliard suivante :

$$\frac{\partial \alpha}{\partial t}(x,t) = \nabla.[m(\alpha(x,t))\nabla\mu(x,t)], \ x \in \Omega, \ t > 0, \tag{1.1}$$

$$\mu(x,t) = F'(\alpha(x,t)) - \epsilon^2 \Delta\alpha(x,t), \tag{1.2}$$

$$\frac{\partial \alpha}{\partial t}(x,t) = \frac{\partial \mu}{\partial t}(x,t) = 0, \ x \in \Gamma, \tag{1.3}$$

où $\Omega \subset \mathbb{R}^N$, $(N = 1, 2, \text{ or } 3)$ est un domaine borné de frontière Γ et $\alpha(x,t) = \alpha_r(x,t) - \alpha_s(x,t)$ est un champ de phase représentant la différence entre les fractions de mole des champs r et s.

La fonction $F(\alpha)$ est un potentiel à double puits d'un système homogène de composition $\alpha(x,t)$, la fonction $\mu(x,t)$ est un potentiel chimique, la fonction $m(\alpha)$ est une mobilité positive, et le paramètre ϵ est une constante positive reliée à l'épaisseur de l'interface.
A l'origine, la mobilité était de la forme $m(\alpha) = m_0(1-\alpha)(1+\alpha)$, elle dépendait de la concentration et m_0 est une constante.

Il existe plusieurs choix de $F(\alpha)$ qui peuvent être considérés.
Les formes les plus utilisées sont

$$F(\alpha) = 0.25(\alpha^2 - 1)^2 \tag{1.4}$$

et

$$F_{log}(\alpha) = \frac{\theta}{2}[(1+\alpha)\log(1+\alpha) + (1-\alpha)\log(1-\alpha)] + \frac{\theta_c}{2}(1+\alpha)(1-\alpha), \tag{1.5}$$

où $\alpha \in (-1, 1)$, θ est la température et θ_c est une température critique.

On remarque qu'au voisinage de $\alpha = 0$, le potentiel $F_{log}(\alpha)$ se réduit à l'approximation classique de $F(\alpha)$ qui est un polynôme du degré 4 (1.4).

La fonction F qu'on appelle aussi l'énergie libre homogène est souvent régulière pour simplifier l'analyse mathématique. La forme (1.4) décrit deux phases stables ($s = \pm 1$). Il est important de trouver des solutions qui prennent leurs valeurs dans l'ensemble $[-1, 1]$ qui est physiquement significatif. Malheureusement, le principe du maximum ne s'applique pas dans notre cas puisque l'équation de Cahn-Hilliard est du quatrième ordre. Une solution à ce problème est de considérer des potentiels qui vérifient les conditions suivantes :

$$F : [-1, 1] \to \mathbb{R}, \tag{1.6}$$

$$F \in C^2, \tag{1.7}$$

$$F'(s) \geq -c_0, \ c_0 > 0, s \in \mathbb{R}, \tag{1.8}$$

$$\lim_{s \to \pm 1} f(s) = \pm\infty, \text{ et } \lim_{s \to \pm 1} f'(s) = +\infty. \tag{1.9}$$

On peut retrouver l'équation de Cahn-Hilliard à partir de l'énergie libre de Ginzburg-Landau

$$E(\alpha) = \int_\Omega (F(\alpha) + \frac{\epsilon^2}{2}|\nabla \alpha|^2) dx. \tag{1.10}$$

Le potentiel chimique μ est alors déduit de (1.10) par :

$$\mu = \frac{\partial E}{\partial \alpha} = F'(\alpha) - \epsilon^2 \Delta \alpha.$$

Alors, par la loi de conservation de masse, on obtient l'équation de Cahn-Hilliard

$$\frac{\partial \alpha}{\partial t} = -\nabla.\Phi,$$

où $\Phi = -m\nabla\mu$ est le flux correspondant.
On pourra consulter [126] pour plus de détails sur l'origine physique, mathématique et numérique de l'équation Cahn-Hilliard binaire.

Pour une fonction F régulière sur $\mathbb{R}$, Elliott et Zheng ont démontré l'existence de solutions globales dans L^2 [69]. Prüs et al. ont démontré que le problème était bien-posé globalement avec des solutions fortes sous des conditions aux bords dynamiques [166]. Par la suite, Prüs et Wilke ont démontré les mêmes résultats mais dans un cas plus général : l'équation de Cahn-Hilliard non-isothermes avec des potentiels de croissance polynomiale [167].

Un nombre important d'auteurs comme Chill, Hoffman, Rybka, Prüs, Wilke, Wu et Zheng ont démontré des résultats de convergence de solution à des états stables dans le cas des potentiels réguliers [43, 103, 167, 192].
Elliott et Luckhaus ont démontré des résultats d'existence et d'unicité de la solution dans le cas d'un mélange de plusieurs composants [64]. Debussche et Dettori ont donné une autre démonstration dans le cas d'un mélange de deux composants, ils ont trouvé un attracteur global et ils ont estimé sa dimension [53]. Des propriétés supplémentaires des attracteurs ont été étudiées par Dupaix [58], mais aussi par Miranville et Zelik [150].

Comme nous l'avons vu, il existe une large littérature étudiant l'équation de Cahn-Hilliard. Mais la plupart des références disponibles traitent de cas où le domaine est borné et le modèle est sujet à des conditions de bords convenables [46, 71, 130, 182].

Cependant, il y a quelques auteurs qui ont traité le problème dans $\mathbb{R}^N$ ou dans des domaines non bornés. Par exemple, Bricmont et al. ont étudié en dimension une, la stabilité d'une solution particulière croissante $tanh(\frac{x}{2})$, où des estimations de stabilité et de décroissance en temps ont été établies pour des perturbations des données initiales [29]. Liu et al. ont étudié la stabilité dans $\mathbb{R}^N$ des solutions constantes en supposant des conditions de croissance locales sur le terme non linéaire et certaine régularité sur la donnée initiale [131]. La stabilité est obtenue dans $L^\infty(\mathbb{R}^N)$ et la décroissance en temps est obtenue dans des $L^p(\mathbb{R}^N)$. Un résultat de décroissance plus fort est trouvé plus tard pour $N \geq 3$ [57].
L'équation de Cahn-Hilliard est naturellement dissipative dans H^{-1} dans le sens où la solution semi-groupe est bien définie dans H^{-1} et possède une boule absorbante dans cet espace.

En outre, beaucoup de modifications sur l'équation de Cahn-Hilliard initiale ont été proposées et étudiées pour donner des modèles complets du point de vue physique. Par exemple : le modèle visqueux de Cahn-Hilliard [25, 37, 55, 68, 144, 159], un modèle de Cahn-Hilliard-Navier-Stokes-Fourier [101], un modèle avec des termes d'inertie [94] et un modéle avec l'effet d'anisotropie [135]. De plus, différentes perturbations de cette équation ont été analysées en dimension 1 [197, 198] et des perturbations stochastiques comme les équations de Cahn-Hilliard-

Cook peuvent être également trouvées [23, 24]. Le système de Cahn-Hilliard avec élasticité linéaire a été aussi étudié [19, 20, 34, 81, 164].
D'autre part, Heida et al. ont fourni une base thermodynamique à la modélisation mathématique pour le développement des modèles de champs de phase et des généralisations des équations de Cahn-Hilliard [101].

Chercher des solutions numériques de l'équation de Cahn-Hilliard n'était pas souvent facile à cause de trois difficultés principales : la présence de ϵ dans l'équation, la non linéarité du système et la différence des échelles de temps à chaque étape de l'évolution de la concentration. Alors, une résolution numérique du problème demande une relation entre les échelles numériques, particulièrement, entre la taille du maillage, la taille du pas et la longeur de l'interaction. Il existe maintenant une littérature suffisante qui donne des approximations numériques précises, voir [28, 50, 73, 92, 93, 113, 139, 179, 191, 183].

Finalement, des généralisations des modèles de Cahn-Hilliard ont été étudiées, où un nombre arbitraire de phases peut être considéré, consulter [9, 10, 128, 125, 186, 191].

1.2 Sur l'Equation d'Allen-Cahn

L'équation d'Allen-Cahn est une équation aux dérivées partielles parabolique du deuxième ordre, introduite en 1979 dans [5] par Allen et Cahn où ils ont traité le mouvement des grains sur la frontière dans des solides cristallins. Dans ce travail, les auteurs parlent des frontières d'antiphase courbée "APB : curved antiphase boundaries" en les traitant comme des surfaces avec des propriétés géométriques comme l'aire et la courbure, des propriétés thermodynamiques et chimiques comme l'excès de l'énergie libre par unité d'aire et d'adsorption et des propriétés cinétiques comme leur vitesse sous l'effet d'une force agissante. Depuis cet article, les équations d'Allen-Cahn sont devenues un modèle prototype pour les phases de transition isothermes.

L'équation d'Allen-Cahn représente une variable non conservée. C'est pourquoi elle est utilisée pour la modélisation de la ségrégation et de la précipitation dans les solides ou dans des situations plus générales où il existe une réorganisation des maillages de cristaux.
Elle décrit le mouvement macroscopique des frontières de phase entraîné par la tension de surface. Elle sert encore comme un modèle de transition de phase dans des alliages métalliques binaires. De plus, elle apparaît comme une équation de diffusion-convection dans la dynamique des fluides numériques ou comme un problème de réaction-diffusion dans la science des matériaux. Elle a été aussi utilisée pour modéliser le débit moyen de courbure et la segmentation d'images [15, 77, 104].

L'équation d'Allen-Cahn joue encore un rôle dans l'étude des ondes stationnaires pour l'équation de Schrödinger non-linéaire et a été beaucoup étudiée. Elle donne un cadre solide pour la description mathématique des problèmes de frontière libre en ce qui concerne les transitions de phase.

La solution d'équation d'Allen-Cahn subit un phénomène qu'on appelle métastabilité [132] : elle présente des zones plates proches des valeurs séparées par les interfaces qui peuvent disparaître à long terme. Une liste de références concernant les applications de cette équation dans le domaine d'ingénierie se trouve dans [176].

En 1993, des travaux ont été effectués sur les équations d'Allen-Cahn anisotropiques qui ont été initialement proposées par Mcfadden et al. [137], puis développées par Eliott et al. en 1996, où l'existence de la solution globale a été établie [65]. En 1996 et 1997, Eliott et Schätzle ont démontré la convergence de la solution vers un flux anisotropique de courbure moyenne en se basant sur des estimations qui dépendent de la dérivée de la densité de la tension superficielle [66, 67].
En 2007, Ninomiya a examiné des solutions d'ondes mobiles cylindriquement symétriques [155]. Un an plus tard, Alfaro et al. ont trouvé un approche analytique qui étudie le mouvement de l'interface et de la génération des phases [3].

L'équation d'Allen-Cahn peut être considérée comme le flot du gradient de la même énergie libre utilisée pour la dérivation de l'équation de Cahn-Hilliard dans L^2.

$$E(\alpha) = \int_\Omega (F(\alpha) + \frac{\epsilon^2}{2}|\nabla\alpha|^2)dx,$$

où $F(\alpha) = \frac{1}{4\epsilon^2}(\alpha^2 - 1)^2$.

Cette dérivation de l'équation d'Allen-Cahn est la méthode classique. Elle consiste à définir la dérivée par rapport au temps du paramètre d'ordre considéré égale à une fonction convenable du gradient de la dérivée variationnelle de l'énergie libre de Ginzbaurg-Landau par rapport au paramétre d'ordre considéré, ce qui donne l'équation Allen-Cahn. La plupart du temps, cette fonction est prise linéaire. Une récente étude est faite pour voir si les autres possibilités sont meilleures, en se basant sur un exemple concret étudié dans [2]. Dans une étude faite en 2017, on trouve une nouvelle dérivation des équations d'Allen-Cahn et Cahn-Hilliard basée sur l'énergie libre fonctionnelle généralisée [191].

Quand le paramètre ϵ tend vers zéro, la solution de l'équation d'Allen-Cahn doit se comporter comme une fonction définie par morceaux avec les valeurs ± 1 dans une des deux régions massives qui sont séparées par une couche d'épaisseur interfaciale diffusive . Ce comportement limite est connu sous l'appelation "the sharp interface limit". Le lecteur peut trouver des analyses formelles et plus rigoureuses dans la littérature [70, 104, 153]. Pour des petites valeurs de ϵ, dans la couche interfaciale, la solution reste régulière mais elle développe un gradient spatial important. De telles couches bougent avec le temps. Un portrait de α à travers les couches interfaciales se trouve dans [162].
L'équation d'Allen-Cahn, comme l'équation de Cahn-Hilliard est naturellement dissipative, mais dans l'espace L^2.

En outre, un nombre important d'études a porté sur les solutions numériques et analytiques de l'équation d'Allen-Cahn car elles ont de nombreuses applications (voir [38]).
Ce calcul numérique est souvent basé sur des schémas explicites conduisants à des problèmes importants de stabilité qui limitent les intervalles de temps en fonction de la taille du pas de maillage. Une des difficultés principales dans les études numériques est de surmonter la couche fine. Pour plus de discussions sur ce sujet, on pourra consulter [12], [77]-[79]. En plus, des solutions numériques efficaces ont été établies pour une meilleure compréhension des facteurs de l'équation d'Allen-Cahn. Le lecteur peut se rapporter aux articles : [15, 44, 51, 40, 79, 80, 129, 136, 187].

On note aussi que la solution analytique de l'équation d'Allen-Cahn avec des conditions de bord non-périodiques est difficile à obtenir ce qui a entraîné des recherches additionnelles des solutions numériques (voir par exemple [4, 13, 76, 110, 132]).

L'équation d'Allen-Cahn et ses formes modifiées ont été appliquées dans plusieurs problèmes : le mouvement cristallin des interfaces en premier lieu, analyse d'images, mouvement de la courbure moyenne, flux de fluides à deux phases, la croissance de cristaux, la segmentation d'images et autres [5, 15, 16, 41, 56, 70, 77, 86, 104, 112, 163, 189, 195]. On note aussi que le système d'Allen-Cahn avec élasticité linéaire est étudiée dans [21].

Enfin, des études sur les modèles d'Allen-Cahn à phases multiples ont été développées [117, 127, 191] . Leur cohérence avec les modèles à phase binaire peut facilement être démontrée. Par ailleurs, des équations d'Allen-Cahn à valeurs vectorielles ont été examinées, voir par exemple [31], [83]-[85] et [117].

1.3 Sur le Couplage des Equations Allen-Cahn/Cahn-Hilliard

Il existe différentes façons de coupler les équations d'Allen-Cahn et de Cahn-Hilliard. Cependant, on s'interesse dans ce travail au modèle introduit dans [35].
Ce système d'Allen-Cahn/Cahn-Hilliard (AC/CH) a été proposé par Cahn et Novick-Cohen en 1994 comme une extension du modèle classique de Cahn-Hilliard en séparation de phase d'un mélange binaire sous refroidissement. Le mélange peut représenter par exemple des alliages binaires, des populations biologiques, des liquides à deux composants et autres. Ses propriétés mathématiques ont été étudiées dans [11, 158] et les références associées.

Une des plus importantes applications de ce couplage est la cristallographie. En séparation de phase, quand on travaille avec des alliages, la cristallographie distingue plusieurs structures bien ordonnées. Ces structures consistent non pas en des domaines séparés occupés par les deux composants exclusivement (ce qui consiste la première approche de comprendre la séparation de phase) mais aussi par quelques distributions régulières des atomes de deux composants sur des réseaux cristallisés. Le processus est décrit par l'évolution des deux variables u et v comme suit :

$$\frac{\partial u}{\partial t} = h^2\Delta(f(u+v) + f(u-v) - h^2\Delta u),$$
$$\frac{\partial v}{\partial t} = -f(u+v) + f(u-v) - \alpha v + h^2\Delta v.$$

La fonction u est la concentration de l'un des composants, la fonction v exprime une certaine mesure d'ordre entre les atomes des composants et h est le paramètre de la tension de surface. Le paramètre α représente la localisation du système dans le maillage de la phase et la fonction f représente l'énergie d'interaction entre les deux composants. La fonction f est la non-linéarité du système.

Ce système d'équations décrit un flot de gradients dans $H^{-1} \times L^2$ de l'énergie libre

$$J(u,v) = \int_\Omega \Big(F(u+v) + F(u-v) + \frac{\alpha}{2}v^2 + \frac{1}{2}h^2(|\nabla u|^2 + |\nabla v|^2)\Big)dx,$$

où $f = F'$.

En particulier, ce système se réduit à l'équation d'Allen-Cahn quand $u = \frac{1}{2}$ et se réduit à l'équation de Cahn-Hilliard quand $v = 0$.

Deux méthodes phénoménologiques entraînant des équations couplées de Cahn-Hilliard et Allen-Cahn ont été explorées dans [35]. Ces approches décrivent la dynamique des systèmes qui subissent une séparation de phase du premier ordre puis des transitions d'ordre-désordre.

Dans la première méthode, une limite quasicontinuum de l'énergie libre est prise en compte et l'évolution du système est ensuite supposée donnée par un flot de gradients. Tandis que, dans la deuxième, un ensemble discret de flot de gradients (d'équations d'évolution) est trouvé en premier et ensuite une limite quasicontinuum est prise en compte. Les méthodes du continuum ont été employées en 1983 et 1986 par Krzanowski et Allen [120]-[123] et en 1990 et 1998 par Eguchi et al. [63, 180].

Dans le travail décrit dans [35], le contexte d'un alliage binaire Fe-Al sur un maillage ayant une forme cubique (BCC : body cubic center) est bien détaillé où l'importance du choix des variables est démontrée. Il faut que ces variables s'adaptent avec les variations de la concentration moyenne ainsi que la structure sous-jacente ordonée des phases possibles coexistantes. C'est le seul cas où les deux approches conduisent approximativement à une même description du continuum. Le résultat est décrit par une équation de type Cahn-Hilliard couplée avec une équation de type Allen-Cahn.

Le système généralisé est défini avec une mobilité $Q(u,v)$ qui n'est pas constante. Il a été traité dans [52] par exemple où la forme de $Q(u,v)$ permet de faire disparaître cette mobilité aux phases pures. On note que la dérivation du système dans [35] peut être étendue à une mobilité non-constante.

Le système AC/CH, par construction, est désigné pour décrire des processus d'ordre-désordre et de séparation de phase simultanés. Il peut être considéré comme un modèle d'interface diffus pour le frittage des petits grains et le rainurage des frontières de grains des films polycristalline [158].

Brochet et al. ont commencé l'étude mathématique de notre système en 1994 : elles ont démontré que le problème avec des conditions au bord de type Neumann homogène est bien posé. Elles ont aussi démontré l'existence d'un attracteur global et d'un attracteur exponentiel [30].
En 1999, Passo et al. ont analysé le même modèle couplé AC/CH avec une mobilité dégénérée et une énergie libre qui contient des termes logarithmiques [52], ils ont démontré l'existence des solutions faibles avec des conditions au bord de type Neuman.
Entre 1996 et 2005, Cahn, Novick-Cohen et Peres Hari ont développé des résultats asymptotiques géométriques de notre modèle. Ils ont démontré que le comportement asymptotique fait intervenir le mouvement par courbure moyenne en dimension 2 et 3 [36, 158, 160, 161].

En 2002, Gokieli a considéré le modèle avec des contraintes sur les valeurs des inconnus. Elle a démontré l'existence et l'unicité de la solution et elle a étudié le comportement asymptotique pour une forme donnée de l'entropie [88].
Après, Gokieli et Ito ont démontré des résultats d'existence, d'unicité et l'existence d'un attracteur global compact dand $L^2(\Omega)^2$ pour le système dynamique associé, pour deux exemples différents de l'entropie [89].
Gokieli et Marcinkowski ont aussi présenté un schéma numérique implicite basé sur l'élément fini pour résoudre le système AC/CH et elles ont prouvé la convergence des solutions numériques vers la solution exacte [90]. Ce travail a enfin été amélioré en 2010, par l'analyse d'un solveur - une méthode itérative effective pour chaque étape du schéma numérique implicite et les auteurs ont aussi démontré la convergence de cette méthode [91]. De plus, en 2012, une méthode NKS (Newton-Krylov-Schwarz) pour la solution implicite de notre système AC/CH a été proposée [194].

Boussinot et al. ont par ailleurs présenté un système couplé d'Allen-Cahn/Cahn-Hilliard en 2010. Ce système indique des modèles de champ de phase 2D et 3D pour l'évolution des microstructures sous la présence d'un réseau inadapté et avec des constantes élastiques non homogènes [27].
Plus tard, en 2014, un système couplé de type AC/CH avec élasticité linéaire dans des dimensions $D \leq 3$ a été étudié. Ce modèle donne une base pour la généralisation des modèles d'interface diffus isothérmiques déjà existants [20].

Enfin, plusieurs méthodes numériques ont été utilisées pour résoudre des systèmes couplés de type AC/CH [91, 138, 177, 184], [186]-[191] et [193].

1.4 Plan de la thèse

On présente maintenant la structure de ce manuscrit qui regroupe les résultats des articles [147]-[149] :

1. Dans le premier article, [148], on considère le modèle suivant des équations couplées de Cahn-Hilliard/Allen-Cahn :

$$(P)\ \begin{cases} \frac{\partial u}{\partial t} &= h^2\Delta(f(u+v)+f(u-v)-h^2\Delta u) \\ \frac{\partial v}{\partial t} &= -f(u+v)+f(u-v)-\alpha v+h^2\Delta v \end{cases}$$

Le problème est posé dans un domaine $\Omega \subset \mathbb{R}^N$ ($N = 1, 2$, or 3) ayant une frontière régulière Γ. La fonction u est la concentration d'un des deux composants du mélange, qui est une quantité conservée, la fonction v est un paramètre d'ordre, le paramètre (positif) h représente l'espacement du maillage, le paramètre α correspond à la localisation du système dans le diagramme de phase (il peut-être positif ou négatif) et le terme non linéaire f est la dérivée d'un potentiel à double puits. C'est une fonction polynomiale de degré impair quelconque. Le modèle a été proposé par J. W. Cahn et A. Novick-Cohen en 1994 dans [35].

On associe ce problème à des conditions du bord de type Dirichlet :

$$u = v = 0 \text{ sur } \Gamma,$$

et aux conditions initiales suivantes

$$u_{|t=0} = u_0, v_{|t=0} = v_0.$$

La fonction f satisfait

$$f \in C^2,\ f(0) = 0,$$

$$f'(s) \geq -c_0, c_0 > 0, s \in \mathbb{R},$$

$$f(s)s \geq c_1F(s) - c_2,\ F(s) \geq -c_3,\ c_1 > 0,\ c_2,\ c_3 \geq 0,\ s \in \mathbb{R},$$

où $F(s) = \int_0^s f(\xi)d\xi$.

Dans ce travail, on améliore les résultats de [30]. On les démontre pour un polynôme de degré trois quelconque. On prouve l'existence et l'unicité d'une solution faible globale dans $L^\infty(0,T;H^2(\Omega))$, le résultat est illustré par :

Théorème 1.4.1. *On prend $T > 0$ et on suppose que $(u_0, v_0) \in H^2(\Omega)^2$. Alors, (P) admet une solution unique (u, v) sachant que $(u, v) \in L^\infty(0,T;H^2(\Omega)^2)$ et $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^\infty(0,T;H^{-1}(\Omega)\times L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)^2)$.*

Ce théorème nous permet de définir un semi-groupe continu. On démontre alors que ce semi-groupe est dissipatif.

Théorème 1.4.2. *Le semigroupe associé à (P) est dissipatif dans $H^2(\Omega)^2$, i.e., il possède un ensemble borné absorbant $\mathcal{B}_0$ dans $H^2(\Omega)^2$.*

Le résultat le plus important du papier est l'existence d'un attracteur exponentiel. C'est un ensemble compact, robuste aux perturbations en temps, et il attire la solution du problème à une grande vitesse (exponentielle). Par conséquence, on aura l'existence d'un attracteur global de dimension fractale finie.

Théorème 1.4.3. *Le semigroupe S(t) possède un attracteur exponentiel $\mathcal{M} \subset \mathcal{B}_0$, i.e.,*
(i) $\mathcal{M}$ est compact dans $H^1(\Omega) \times H^1(\Omega)$;
(ii) $\mathcal{M}$ est invariant positivement, $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall\, t \geq 0$;
(iii) $\mathcal{M}$ a une dimension fractale finie dans $H^1(\Omega) \times H^1(\Omega)$;
(iv) $\mathcal{M}$ attire exponentiellement les ensembles bornés de $H^2(\Omega)^2$: $\forall\, B \subset H^2(\Omega)^2$ borné, $dist_{H^1(\Omega)\times H^1(\Omega)}(S(t)B, \mathcal{M}) \leq Q(\|B\|_{H^2(\Omega)^2})e^{-ct}$, $c > 0$, $t \geq 0$, où la constante c est indépendente de B et $dist_{H^1(\Omega)\times H^1(\Omega)}$ indique la semidistance Hausdroff entre les ensembles définis par

$$dist_{H^1(\Omega)\times H^1(\Omega)}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_{H^1(\Omega)\times H^1(\Omega)}.$$

La dimension finie est assez importante, elle nous permet de dire que malgré la dimension infinie de l'espace de phase, on peut situer la solution dans un espace de dimension finie.

2. Dans le deuxième article, [149], avec le même système (P), on considère au lieu du potentiel polynomial un qui est logarithmique :

$$f(s) = (F'(s)) = -\theta_c s + \frac{\theta}{2} \ln\Big(\frac{1+s}{1-s}\Big),$$

où

$$F(s) = \frac{\theta_c}{2}(1 - s^2) + \frac{\theta}{2}\Big[(1-s) \ln\Big(\frac{1-s}{2}\Big) + (1+s) \ln\Big(\frac{1+s}{2}\Big)\Big],$$

θ = température, θ_c= la température critique, et $s \in (-1, 1)$.

En utilisant le développement limité, on approche notre f qui est $C^1(-1, 1)$ et singulière par une suite de fonctions $(f_N)_{n\in\mathbb{N}}$ qui est $C^1(\mathbb{R})$:

$$f_N(s) = \begin{cases} f(-1 + \frac{1}{N}) + f'(-1 + \frac{1}{N})\,(s + 1 - \frac{1}{N}), & s < -1 + \frac{1}{N}, \\ f(s), & |s| \leq 1 - \frac{1}{N}, \\ f(1 - \frac{1}{N}) + f'(1 - \frac{1}{N})(s - 1 + \frac{1}{N}), & s > 1 - \frac{1}{N}. \end{cases}$$

Chaque fonction f_N satisfait

$$f_N' \geq -c_0$$

et, en prenant $F_N(s) = \int_0^s f_N(r)dr$, on a :

$$-c_3 \leq F_N(s) \leq c_4\, f_N(s)s + c_5, \;\; c_4 > 0, \;\; c_3, c_5 \geq 0, \;\; s \in \mathbb{R},$$

$$f_N(s)s \geq c_6\, |f_N(s)| - c_7, \;\; c_6 > 0, \;\; c_7 \geq 0, \;\; s \in \mathbb{R},$$

où les constantes $c_i, \;\; i = 3, ..., 7$, sont indépendentes de N.

On associe le problème aux conditions initiales et aux conditions du bord suivantes :

$$u_N = \Delta u_N = v_N = 0 \text{ sur } \Gamma,$$

$$u_{N|t=0} = u_0, \;\; v_{N|t=0} = v_0.$$

On démontre tout d'abord l'existence et l'unicité de la solution faible dans $C_w([0,T]; H_0^1(\Omega)^2) \cap L^2(0,T; H^2(\Omega)^2) \cap L^\infty(0,T; H_0^1(\Omega)^2)$.

Théorème 1.4.4. *On suppose que* $(u_0, v_0) \in H^1(\Omega)^2$, $\|u_0 + v_0\|_{L^\infty(\Omega)} < 1$, *et* $\|u_0 - v_0\|_{L^\infty(\Omega)} < 1$. *Alors, le système (P) possède une solution unique (faible) sachant que,* $\forall \;\; T > 0$,

$$(u, v) \in C_w([0,T]; H_0^1(\Omega)^2) \cap \;\; L^2(0,T; H^2(\Omega)^2) \cap L^\infty(0,T; H_0^1(\Omega)^2)$$

et

$$(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(0,T; H^{-1}(\Omega) \times L^2(\Omega)),$$

où w est utilisé pour la topologie faible, et

$$\frac{d}{dt}((u,q))_{-1} + ((\nabla u, \nabla q)) + ((f(u+v) + f(u-v), q)) = 0,$$

$$\frac{d}{dt}((v,q)) + ((\nabla v, \nabla q)) + ((f(u+v) - f(u-v), q)) = 0,$$

$$u(0) = u_0, \;\; v(0) = v_0,$$

p.p. $t \in [0,T], \;\; \forall \; q \in C_c^\infty(\Omega)$.

La difficulté étant dans le passage à la limite du problème approché surtout sur le terme non linéaire f. Cette difficulté est surmontée par la possibilité de borner f_N dans $L^1(0,T;\Omega)$ et $L^2(0,T;\Omega)$.

Ensuite, on définit le semi-groupe $S(t)$ associé à l'opérateur différentiel et on démontre sa dissipativité. Puis, on démontre l'existence d'un attracteur global. Ces résultats sont annoncés comme suit :

Théorème 1.4.5. *La famille d'opérateurs associée au système (P) est dissipative dans* $H^1(\Omega)^2$ *alors elle possède un ensemble borné absorbant* $\mathcal{B}_1 \subset H^1(\Omega)^2$, *i.e.* $\forall\ B \subset \Phi_1$, $\exists\ t_0 = t_0(B)$ *tel que* $t \geq t_0$ *ce qui implique* $S(t)B \subset \mathcal{B}_1$,

Φ_1 étant l'espace suivant :

$$\Phi_1 = \Phi \cap \{(u, v) \in L^\infty(\Omega)^2;\ \|u + v\|_{L^\infty(\Omega)} < 1 \text{ et } \|u - v\|_{L^\infty(\Omega)} < 1\},$$

où

$$\Phi = \{(u, v) \in H^1(\Omega)^2;\ |u + v| < 1 \text{ et } |u - v| < 1 \text{ p.p.}\},$$

Théorème 1.4.6. *Le semigroupe S(t) possède l'attracteur global* $\mathcal{A}$ *sur* Φ *(i.e.* $\mathcal{A}$ *est compact dans* $H^{-1}(\Omega) \times L^2(\Omega)$, *borné dans* Φ, *invariant et attire les images de tous les ensembles bornés de* Φ *par rapport à la topologie de* $H^{-1}(\Omega) \times L^2(\Omega)$*).*

A la fin, on démontre la propriété stricte de la séparation stricte en dimension une qui a une importance physique remarquable.

Théorème 1.4.7. *Supposons que* $N = 1$ *et que* f *satisfait la condition*

$$\lim_{s \to \pm 1} f(s) = \pm\infty, \text{ et } \lim_{s \to \pm 1} f'(s) = +\infty.$$

Alors, il existe deux constantes positives δ, δ' *et* $T > 0$ *sachant que*

$$\|(u + v)(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\ 2 \leq t \leq T,$$

et

$$\|(u - v)(t)\|_{L^\infty(\Omega)} \leq 1 - \delta', \quad \forall\ 2 \leq t \leq T.$$

3. Dans le troisième article, [147], on couple le système (P) avec la température et on divise le travail en deux parties :

   Dans la première partie, la température suit la loi classique de Fourrier présentée par l'équation

   $$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial v}{\partial t}.$$

   La fonction f satisfait

   $$f \in C^2,\ f(0) = 0,$$

   $$f'(s) \geq -c_0, c_0 > 0, s \in \mathbb{R},$$

   $$f(s)s \geq cF(s) - c',\ F(s) \geq -c'',\ c > 0,\ c',\ c'' \geq 0,\ s \in \mathbb{R},$$

où $F(s) = \int_0^s f(\xi)d\xi$.

Les conditions aux bords sont les conditions de type Dirichlet. Le système est alors :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \tag{1.11}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = \theta, \tag{1.12}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial v}{\partial t}, \tag{1.13}$$

$$u = v = \theta = 0 \text{ sur } \Gamma, \tag{1.14}$$

$$u_{|t=0} = u_0, v_{|t=0} = v_0, \theta_{|t=0} = \theta_0, \tag{1.15}$$

où Ω est un domaine borné de $\mathbb{R}^N$ ($N = 1, 2$, ou 3) avec une frontiére régulière Γ.

On commence par démontrer l'existence et l'unicité de la solution dans $L^\infty_{loc}(\mathbb{R}^+; (H^2(\Omega)^3)$ cette fois sur deux théorèmes.

Théorème 1.4.8. *Supposons que* $(u_0, v_0, \theta_0) \in H^1_0(\Omega)^3$. *Alors, (1.11)-(1.15) possède au moins une solution* (u, v, θ) *tel que*
$(u, v, \theta) \in L^\infty(\mathbb{R}^+; H^1_0(\Omega)^2 \times L^2(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\Omega)^3)$, $\theta \in L^\infty(\mathbb{R}^+; L^2(\Omega) \cap H^1_0(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\Omega))$ *et* $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \theta}{\partial t}) \in L^2(\mathbb{R}^+; H^{-1} \times L^2(\Omega)^2)$.

Théorème 1.4.9. *Prenons* (u, v, θ) *une solution de (1.11)-(1.15) avec données initiales* (u_0, v_0, θ_0) *obtenus dans Théorème 1.4.8. Si* $(u_0, v_0, \theta_0) \in (H^2(\Omega) \cap H^1_0(\Omega))^3$, *alors il existe une solution unique* $(u, v, \theta) \in L^\infty_{loc}(\mathbb{R}^+; (H^2(\Omega)^3)$ *et*
$(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \theta}{\partial t}) \in L^\infty_{loc}(\mathbb{R}^+; H^{-1}(\Omega) \times L^2(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; H^1_0(\Omega)^2) \times L^2_{loc}(\mathbb{R}^+; H^1_0(\Omega))$.

Ensuite, on définit un semi-groupe qui est continu et dissipatif et on construit une famille d'attracteurs exponentiels.
On obtient alors un attracteur global de dimension fractale finie.

Ces résultats sont donnés par les théorèmes suivants :

Théorème 1.4.10. *Le semigroupe* $S(t)$ *associé au (1.11)-(1.15) possède un ensemble borné aborbant* $\mathcal{B}_0$ *dans* $\Phi = (H^2(\Omega) \cap H^1_0(\Omega))^3$ *tel que, pour tout ensemble borné* $B \subset \Phi$, *il existe* $t_0 = t_0(B) \geq 0$ *sachant que* $t \geq t_0$ *implique* $S(t)B \subset \mathcal{B}_0$. *C'est pourquoi* $S(t)$ *est un semigroupe.*

Théorème 1.4.11. *Le semigroupe $S(t)$ possède un attracteur exponentiel $\mathcal{M} \subset \mathcal{B}_0$, i.e.,*
(i) $\mathcal{M}$ est compact dans $H^1(\Omega)^2 \times L^2(\Omega)$;
(ii) $\mathcal{M}$ est invariant positivement : $S(t)\mathcal{M} \subset \mathcal{M},\ \forall\, t \geq 0$;
(iii) $\mathcal{M}$ a une dimension fractale finie dans $H^1(\Omega)^2 \times L^2(\Omega)$;
(iv) $\mathcal{M}$ attire exponentiellement les ensembles bornés de Φ : $\forall\, B \subset \Phi$ borné, $dist_{H^1(\Omega)^2\times L^2(\Omega)}(S(t)B, \mathcal{M}) \leq Q(\|B\|_\Phi)e^{-ct}, c > 0,\ t \geq 0$, où la constante c est indépendente de B et $dist_{H^1(\Omega)^2\times L^2(\Omega)}$ indique la semidistance Hausdroff entre les ensembles définis par

$$dist_{H^1(\Omega)^2\times L^2(\Omega)}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_{H^1(\Omega)^2\times L^2(\Omega)}.$$

La loi de Fourier n'est pas vraiment réaliste puisqu'elle suppose que les signaux thermiques se propagent avec une vitesse infinie, ce qui contradicte le paradoxe de la conduction thermique [47]. C'est pourquoi, on refait la même étude avec une loi plus concrète.

Dans la deuxième partie, la température suit une loi thermodynamique de type III, donnée par l'équation :

$$\frac{\partial^2\alpha}{\partial t^2} - k^* \frac{\partial}{\partial t}\Delta\alpha - k\Delta\alpha = -\frac{\partial u}{\partial t},$$

où α est la température et k une constante positive.

f est C^2 et satisfait

$$-c_0 \leq F(s) \leq f(s)s,\ c_o \geq 0,\ s \in \mathbb{R},$$

où

$$F(s) = \int_0^s f(\tau)d\tau.$$

On suppose aussi que

$$f(0) = 0,\ \ f'(s) \geq c_6,\ \ s \in \mathbb{R}.$$

Les conditions initiales et les conditions aux bords sont les mêmes que dans la première partie et le système est alors :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u + v) + f(u - v)) = 0, \tag{1.16}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u + v) - f(u - v) = \frac{\partial \alpha}{\partial t}, \tag{1.17}$$

$$\frac{\partial^2\alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -\frac{\partial v}{\partial t}, \tag{1.18}$$

$$u = v = \alpha = 0 \text{ sur } \Gamma, \tag{1.19}$$

$$u_{|t=0} = u_0,\ v_{|t=0} = v_0,\ \alpha_{|t=0} = \alpha_0,\ \frac{\partial\alpha}{\partial t}_{|t=0} = \alpha_1, \tag{1.20}$$

où Ω est un domaine borné de $\mathbb{R}^N$ (N = 1, 2, ou 3) avec une frontière régulière Γ.

Pareillement, on démontre l'existence et l'unicité de la solution dans $L^\infty(\mathbb{R}^+; \mathrm{H}_0^1(\Omega)^3) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^2(\Omega)^3)$, la dissipativité du semi-groupe. Enfin pour démontrer l'existence d'un attracteur exponentiel, le système est divisé en somme de deux systèmes due à la difficulté engendrée par la troisième équation.

D'une manière plus explicite, les résultats sont :

Théorème 1.4.12. *Pour tout* $(u_0, v_0, \alpha_0, \alpha_1) \in (H^2(\Omega)^3 \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, *(1.16)-(1.20) possède une solution unique* $(u, v, \alpha, \frac{\partial \alpha}{\partial t})$ *tel que* $(u, v, \alpha) \in L^\infty(\mathbb{R}^+; \mathrm{H}_0^1(\Omega)^3) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^2(\Omega)^3)$, $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^\infty(\mathbb{R}^+; \mathrm{H}^{-1}(\Omega) \times \mathrm{L}^2(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}_0^1(\Omega)^2)$ *et* $\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; \mathrm{H}_0^1(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}^2(\Omega)) \cap \mathrm{L}^2(\mathbb{R}^+; \mathrm{H}_0^1(\Omega)) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{H}_0^1(\Omega))$.

Théorème 1.4.13. *Le semigroupe associé au (1.16)-(1.20) possède un ensemble borné absorbant* $\mathcal{B}_1$ *dans* $\Phi' = (H^2(\Omega)^3 \times H_0^1(\Omega))$.

Théorème 1.4.14. *Le semigroupe* $S(t)$ *possède un attracteur exponential* $\mathcal{M}' \subset \mathcal{B}_1$, *i.e.,*
(i) $\mathcal{M}'$ *est compact dans* $H^1(\Omega)^3 \times L^2(\Omega)$ *;*
(ii) $\mathcal{M}'$ *est invariant positivement,* $S(t)\mathcal{M} \subset \mathcal{M}', \ \forall\, t \geq 0$ *;*
(iii) $\mathcal{M}'$ *a une dimension fractale finie dans* $H^1(\Omega)^3 \times L^2(\Omega)$ *;*
(iv) $\mathcal{M}'$ *attire exponentiellement les ensembles bornés de* Φ' *:* $\forall\, B \subset H_0^1(\Omega)^3$ *borné,* $dist_{H^1(\Omega)^3 \times L^2(\Omega)}(S(t)B, \mathcal{M}') \leq Q(\|B\|_{H^1(\Omega)^3 \times L^2(\Omega)}) e^{-ct}, c > 0, \ t \geq 0$, *où la constante c est indépendente de B et* $dist_{H^1(\Omega)^3 \times L^2(\Omega)}$ *indique la semidistance Hausdroff entre les ensembles définis par*

$$dist_{H^1(\Omega)^3 \times L^2(\Omega)}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{H^1(\Omega)^3 \times L^2(\Omega)}.$$

Chapitre 2

Asymptotic Behavior of a Model for Order-Disorder and Phase Separation

Comportement Asymptotique d'un Modèle d'Ordre-Désordre et Séparation de Phase

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Asymptotic Behavior of a Model for Order-Disorder and Phase Separation

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Abstract : Our aim in this article is to study the long time behavior, in terms of finite-dimensional attractors, of a coupled Allen–Cahn/Cahn–Hilliard system. In particular, we prove the existence of an exponential attractor and, as a consequence, the existence of the global attractor with finite fractal dimension.

Keywords : Allen-Cahn/Cahn-Hilliard system, global attractor, exponential attractor, fractal dimension.

2.1 Introduction

J. Cahn and A. Novick-Cohen introduced in [35] the following phase-field system :

$$\frac{\partial u}{\partial t} = h^2\Delta(f(u+v) + f(u-v) - h^2\Delta u),$$

$$\frac{\partial v}{\partial t} = -f(u+v) + f(u-v) - \alpha v + h^2\Delta v,$$

to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point. These authors explored two phenomenological approaches leading to systems of coupled Allen–Cahn/Cahn–Hilliard (AC/CH) equations (see [35] for more details).

Another important application of coupled (AC/CH) equations is the following. Under appropriate compositional conditions, ordering can be induced in a previously homogeneous material.

If the composition differs slightly from these conditions, the excess composition can emerge as droplets along the boundaries between the ordered regions. This phenomena can be modeled by a coupled (AC/CH) system with degenerate mobilities. In similar applications, surface diffusion coupled with motion by mean curvature appears quite naturally. There are additional effects which are often neglected and which arguably should be included. However, the coupled motion, by itself, is not overly well understood and it was thus reasonable to isolate it and study it, even given its limitations (see [54]).

Here, u denotes the concentration of one of the components and is a conserved quantity, while v is an order parameter. Furthermore, h is a (positive) parameter which represents the lattice spacing and the parameter α reflects the location of the system within the phase diagram and may be either positive or negative. We further note that the system is a gradient flow in $\left(H^1\right)' \times L^2$ for the free energy

$$J(u,v) = \int_\Omega \left\{ G(u+v) + G(u-v) + \frac{\alpha}{2}v^2 + \frac{1}{2}h^2(|\nabla u|^2 + |\nabla v|^2) \right\} dx,$$

where $f = G'$.

These equations, endowed with Neumann boundary conditions, have been studied in [30] by A. Novick-Cohen, D. Brochet, and D. Hilhorst who proved the well-posedness and the existence of maximal attractors and inertial sets (i.e., exponential attractors) for the usual cubic nonlinear term $f(s) = s^3 - \beta s$ in three space dimensions.

Our main aim in this paper is to improve these results. In particular, taking initial conditions in H^2 allows us to prove the existence of exponential attractors (and, thus, of the finite-dimensional global attractor) for a large class of nonlinear terms containing polynomials of arbitrary odd degree with a strictly positive leading coefficient in three space dimensions.

A similar system, with a non-constant mobility, was treated by R. Dal Passo, L. Giacomelli, and A. Novick-Cohen in [52] who proved the existence of weak solutions for the Neumann problem for a degenerate parabolic system consisting of a fourth-order and a second-order equations with singular lower-order terms in one space dimension. In addition, asymptotics for a similar system with a non-constant mobility were developed as a diffuse interface model for simultaneous order-disorder and phase separation in [158]. There, A. Novick-Cohen focused on motion in the plane. This framework yields both sharp interface and diffuse interface models of sintering of small grains and thermal grains boundary grooving in polycrystalline films. This work was extended in [160], where the authors studied the partial wetting case, and their analysis accounts for motion in three dimensions.

We also mention that numerical methods to solve coupled (AC/CH) systems were studied in, e.g., [138, 177, 184, 188, 193]. Furthermore, a NKS method for the implicit solution of a coupled (AC/CH) system was proposed in [194].

In this work, we take $h = 1$ and $\alpha = 0$ and obtain the following system :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = 0, \tag{2.2}$$

$$u = \Delta u = v = 0 \text{ on } \Gamma, \tag{2.3}$$

$$u_{|t=0} = u_0, v_{|t=0} = v_0, \tag{2.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$ ($N = 1, 2$, or 3) with smooth boundary Γ.

As far as the nonlinear term is concerned, we make the following assumptions :

$$f \text{ is of class } C^2,\ f(0) = 0, \tag{2.5}$$

$$f'(s) \geq -c_0, c_0 > 0, s \in \mathbb{R}, \tag{2.6}$$

$$f(s)s \geq c_1 F(s) - c_2,\ F(s) \geq -c_3,\ c_1 > 0,\ c_2,\ c_3 \geq 0,\ s \in \mathbb{R}, \tag{2.7}$$

where $F(s) = \int_0^s f(\xi)d\xi$.

Remark 2.1.1. *In particular, these assumptions are satisfied by polynomials of degree* $2p + 1$ *with a strictly positive leading coefficient,* $p \geq 1$.

Throughout this work, the same letter c (and, sometimes, c' and c'') denotes constants which may change from line to line, or even in a same line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may change from line to line, or even in a same line.

2.2 A Priori Estimates

In this section, we will establish a number of important inequalities that will be used later in the proof of the existence of the solution and the existence of finite-dimensional attractors.

In what follows, the Poincaré, Hölder and Young inequalities are extensively used, without further referring to them. We rewrite (2.1) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u+v) + f(u-v) = 0. \tag{2.8}$$

We multiply (2.8) by u, integrate over Ω and have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1}^2 + \|\nabla u\|^2 + ((f(u+v) + f(u-v), u)) = 0, \tag{2.9}$$

where $\|\,.\,\|_{-1}^2 = \|(-\Delta)^{-\frac{1}{2}}\,.\,\|$, $\|\,.\,\|$ being the usual L^2-norm, with associated scalar product $((.\,,.))$.

We then multiply (2.2) by v and have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\nabla v\|^2 + ((f(u+v) - f(u-v), v)) = 0. \tag{2.10}$$

The sum of (2.9) and (2.10) gives

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 + ((f(u+v), u+v)) + ((f(u-v), u-v)) = 0, \tag{2.11}$$

which yields, owing to (2.7),

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 + c\int_\Omega \Big(F(u+v) + F(u-v)\Big)dx \le c', \tag{2.12}$$

$$c, c' \ge 0.$$

We multiply (2.8) by $\frac{\partial u}{\partial t}$ to obtain

$$\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + ((f(u+v) + f(u-v), \frac{\partial u}{\partial t})) = 0. \tag{2.13}$$

We now multiply (2.2) by $\frac{\partial v}{\partial t}$ and find

$$\left\|\frac{\partial v}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + ((f(u+v) - f(u-v), \frac{\partial v}{\partial t})) = 0. \tag{2.14}$$

Then, we sum (2.13) and (2.14) and have

$$\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + ((f(u+v), \frac{\partial}{\partial t}(u+v))) + ((f(u-v), \frac{\partial}{\partial t}(u-v))) = 0, \tag{2.15}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2 + \|\nabla v\|^2 + 2\int_\Omega F(u+v)dx + 2\int_\Omega F(u-v)dx\Big) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 = 0. \tag{2.16}$$

Summing (2.12) and (2.16), we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|_{-1}^2 + \|v\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + 2\int_\Omega \Big(F(u+v) + F(u-v)\Big)dx\Big) + \|\nabla u\|^2 + \|\nabla v\|^2 + \int_\Omega \Big(F(u+v) + F(u-v)dx\Big) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 \le c', c' \ge 0.$$

Let $E_1 = \|u\|_{-1}^2 + \|v\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + 2\int_\Omega \Big(F(u+v) + F(u-v)\Big)dx$.

Then the previous inequality is equivalent to

$$\frac{1}{2}\frac{d}{dt}E_1 + cE_1 + c'\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right) \leq c'', \tag{2.17}$$

whence $\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 \in L^1(0,T)$.

In particular, we deduce from (2.17) the dissipative estimate

$$\begin{aligned} E_1(t) \leq ce^{-c't}\Big(&\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \\ &\int_\Omega \Big(F(u_0+v_0) + F(u_0-v_0)\Big)dx\Big) + c'', c' > 0, t \geq 0. \end{aligned} \tag{2.18}$$

Furthermore, for every $r > 0$,

$$\begin{aligned} \int_t^{t+r}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2\right)d\tau \leq ce^{-c't}\Big(&\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 \\ &+ \int_\Omega \Big(F(u_0+v_0) + F(u_0-v_0)\Big)dx\Big) + c''(r), c' > 0, t \geq 0. \end{aligned} \tag{2.19}$$

We multiply (2.1) by $\frac{\partial u}{\partial t}$. It follows from the continuity of f and the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$ that

$$\frac{d}{dt}\|\Delta u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \leq Q(\|u\|_{H^2(\Omega)} + \|v\|_{H^2(\Omega)}). \tag{2.20}$$

We can obtain a similar inequality for v by multiplying (2.2) by $(-\Delta)\frac{\partial v}{\partial t}$, yielding

$$\frac{d}{dt}\|\Delta v\|^2 + \left\|\frac{\partial \nabla v}{\partial t}\right\|^2 \leq Q(\|u\|_{H^2(\Omega)} + \|v\|_{H^2(\Omega)}). \tag{2.21}$$

Summing then (2.20) and (2.21), we obtain

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\Delta v\|^2) + \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial \nabla v}{\partial t}\right\|^2 \leq Q(\|u\|_{H^2} + \|v\|_{H^2}). \tag{2.22}$$

In particular, setting

$$y = \|\Delta u\|^2 + \|\Delta v\|^2,$$

we deduce from (2.22) a differential inequality of the form

$$y' \leq Q(y). \tag{2.23}$$

Let z be the solution of the ordinary differential equation

$$z' = Q(z), \text{ with } z(0) = y(0). \tag{2.24}$$

It follows from the comparison principle that there exists
$T_0 = T_0(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)})$ belonging to, say, $(0, \frac{1}{2})$ such that

$$y(t) \le z(t), \ t \in [0, T_0], \tag{2.25}$$

whence

$$\|u(t)\|^2_{H^2} + \|v(t)\|^2_{H^2} \le Q(\|u_0\|_{H^2}, \|v_0\|_{H^2}), \ t \le T_0. \tag{2.26}$$

Therefore, u and v a priori belong to $L^\infty(0, T_0; H^2(\Omega))$.

We now need to prove that u and v are in $L^\infty(T_0, T; H^2(\Omega))$, so that we can obtain the regularity $u, v \in L^\infty(0, T; H^2(\Omega))$. To do so, we differentiate equations (2.2) and (2.8) with respect to time,

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) + f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{2.27}$$

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial t} - \Delta\frac{\partial v}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) + f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{2.28}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \text{ on } \Gamma. \tag{2.29}$$

We start by multiplying (2.27) by $t\frac{\partial u}{\partial t}$ and (2.28) by $t\frac{\partial v}{\partial t}$ to obtain, summing the two resulting equalities, then using (2.6) and an interpolation inequality :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(t\Big\|\frac{\partial u}{\partial t}\Big\|^2_{-1} + t\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + ct\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + t\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \\ \le \frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|^2_{-1} + \frac{1}{2}\Big\|\frac{\partial v}{\partial t}\Big\|^2 + c't\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{-1} + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big).\end{aligned} \tag{2.30}$$

We deduce from (2.19), (2.30), and Gronwall's lemma that

$$\Big\|\frac{\partial u}{\partial t}(t)\Big\|^2_{-1} + \Big\|\frac{\partial v}{\partial t}(t)\Big\|^2 \le \frac{1}{t}Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}), \ t \le T_0. \tag{2.31}$$

Summing (2.27) times $\frac{\partial u}{\partial t}$ and (2.28) times $\frac{\partial v}{\partial t}$ gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{-1} + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + \Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 + \\ ((f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}), \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t})) + ((f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}), \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t})) = 0,\end{aligned} \tag{2.32}$$

which yields, owing to (2.6),

$$\frac{1}{2}\frac{d}{dt}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2_{-1} + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + \Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \le c\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big). \tag{2.33}$$

Employing the interpolation inequality

$$\left\|\frac{\partial u}{\partial t}\right\|^2 \leq c'\left\|\frac{\partial u}{\partial t}\right\|_{-1}\left\|\nabla\frac{\partial u}{\partial t}\right\|,\ c'>0,$$

and then Young's inequality, we deduce that

$$\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}\right\|^2\right)+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+2\left\|\nabla\frac{\partial v}{\partial t}\right\|^2\leq c\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}\right\|^2\right). \tag{2.34}$$

Using now (2.19) and Gronwall's lemma, we obtain

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}(t)\right\|^2\leq e^{ct}Q\left(\left\|\frac{\partial u}{\partial t}(T_0)\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}(T_0)\right\|^2\right),\quad t\geq T_0, \tag{2.35}$$

where

$$Q=Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)}).$$

Hence, owing to (2.31),

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}(t)\right\|^2\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)}),\quad t\geq T_0. \tag{2.36}$$

We now rewrite (2.8) in the elliptic form

$$-\Delta u+f(u+v)+f(u-v)=h_u(t),\ u=0\text{ on }\Gamma, \tag{2.37}$$

for $t\geq T_0$ fixed, where

$$h_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t} \tag{2.38}$$

satisfies, owing to (2.35),

$$\|h_u(t)\|\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)}),\ t\geq T_0. \tag{2.39}$$

Furthermore, we rewrite (2.2) in the elliptic form

$$-\Delta v+f(u+v)-f(u-v)=h_v(t),\ v=0\text{ on }\Gamma, \tag{2.40}$$

for $t\geq T_0$ fixed, where

$$h_v(t)=-\frac{\partial v}{\partial t} \tag{2.41}$$

satisfies, owing to (2.36),

$$\|h_v(t)\|\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)}),\ t\geq T_0. \tag{2.42}$$

We multiply (2.37) by u and (2.40) by v and we sum the result.

Then, noting that $f(s)s \geq -c, c \geq 0$, we obtain

$$\|\nabla u\|^2 + \|\nabla v\|^2 \leq c(\|h_u(t)\|^2 + \|h_v(t)\|^2) + c'. \tag{2.43}$$

Multiplying now (2.37) by $-\Delta u$ and (2.40) by $-\Delta v$, summing the two resulting equations, and since $f'(s) \geq -c_0$, we have

$$\|\Delta u(t)\|^2 + \|\Delta v(t)\|^2 \leq c(\|h_u(t)\|^2 + \|h_v(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2),\ t \geq T_0. \tag{2.44}$$

We thus deduce from (2.39) and (2.42)-(2.44) that

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}),\ t \geq T_0. \tag{2.45}$$

Combining (2.26) and (2.45), we obtain

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}),\ t \geq 0. \tag{2.46}$$

We deduce that u and v are now in $L^\infty(0, T; H^2(\Omega))$, $T > 0$.

We now wish to find a dissipative estimate that is stronger than (2.18) :

We multiply (2.8) by $-\Delta u$ and (2.2) by $-\Delta v$ and sum the two resulting equations to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2) + \|\Delta u\|^2 + \|\Delta v\|^2 + \\ &((f'(u+v)\nabla(u+v), \nabla(u+v))) + ((f'(u-v)\nabla(u-v), \nabla(u-v))) = 0, \end{aligned} \tag{2.47}$$

which gives, owing to (2.6),

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2) + \|\Delta u\|^2 + \|\Delta v\|^2 \leq c(\|\nabla u\|^2 + \|\nabla v\|^2). \tag{2.48}$$

Furthermore,

$$\|u\|^2_{H^1(\Omega)} \leq c\|u\|\ \|u\|_{H^2(\Omega)}, \leq \frac{1}{2}\|\Delta u\|^2 + c\|u\|^2,\ c > 0.$$

Therefore,

$$\frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2) + c(\|\Delta u\|^2 + \|\Delta v\|^2) \leq c'(\|u\|^2 + \|\nabla v\|^2). \tag{2.49}$$

Using now Gronwall's lemma, we obtain

$$\|u(t)\|^2 + \|\nabla v(t)\|^2 \leq e^{c't}(\|u_0\|^2 + \|\nabla v_0\|^2). \tag{2.50}$$

It follows from (2.18) and (2.49) that

$$\int_0^1 (\|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)})dt \leq c\Big(\|u_0\|^2 + \|v_0\|^2_{H^1(\Omega)} + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c'. \tag{2.51}$$

Hence, there exists $T \in (0,1)$ such that

$$\|u(T)\|^2_{H^2(\Omega)} + \|v(T)\|^2_{H^2(\Omega)} \leq c\Big(\|u_0\|^2 + \|v_0\|^2_{H^1(\Omega)} + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c'. \tag{2.52}$$

Repeating the estimates leading to (2.46), but starting from $t = T$ instead of $t = 0$, we have the smoothing property

$$\|u(1)\|^2_{H^2(\Omega)} + \|v(1)\|^2_{H^2(\Omega)} \leq Q\Big(\|u_0\|^2 + \|v_0\|^2_{H^1(\Omega)} + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c'. \tag{2.53}$$

Since our equations are autonomous, we can make a translation in time and, repeating the estimates leading to (2.53), we have

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq Q\Big(&\|u(t-1)\|^2 + \|v(t-1)\|^2_{H^1(\Omega)} \\ &+ \int_\Omega [F(u(t-1)+v(t-1)) + F(u(t-1)-v(t-1))]dx\Big) + c',\ t \geq 1,\end{aligned} \tag{2.54}$$

which yields, owing to (2.18),

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq e^{-ct}Q\Big(&\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} \\ &+ \int_\Omega (F(u_0+v_0) + F(u_0-v_0))dx\Big) + c',\ t \geq 1.\end{aligned} \tag{2.55}$$

Combining the above estimate with (2.46) from 0 to 1, we obtain the dissipative estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}) + c',\ t \geq 0. \tag{2.56}$$

2.3 Existence and Uniqueness of Solutions

Theorem 2.3.1. *Let $T > 0$ be given and assume that $(u_0, v_0) \in (H^2(\Omega) \cap H^1_0(\Omega))^2$. Then, (2.1)-(2.2) possesses a unique solution (u, v) such that $(u, v) \in L^\infty(0, T; H^2(\Omega)^2)$ and $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^\infty(0, T; H^{-1}(\Omega) \times L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)^2)$.*

Proof.

a) Existence
The proof of existence (as well as the above (and the subsequent) a priori estimates) are based, e.g., on a classical Galerkin scheme.

Let A denote the minus Laplace operator associated with Dirichlet boundary conditions. This operator is a bounded, selfadjoint and strictly positive operator with compact inverse from $H^1_0(\Omega)$ onto $H^{-1}(\Omega)$. There is a set of eigenvectors $\{\varphi_i, i \geq 1\}$ for this operator that is regular

as much as needed, associated with the eigenfunctions $0 < \lambda_1 \leq \lambda_2 \leq .....$ such that it is orthonormal relative to the inner product in $L^2(\Omega)$ and orthogonal relative to the one in $H_0^1(\Omega)$.

Setting $V_m = span\{\varphi_1, ..., \varphi_m\}$, we consider the following approximating problem, written in functional form :

$$\frac{du_m}{dt} + A^2 u_m + A(f(u_m + v_m) + f(u_m - v_m)) = 0,$$

$$\frac{dv_m}{dt} + Av_m + f(u_m + v_m) - f(u_m - v_m) = 0,$$

together with suitable initial conditions, namely,

$$u_m(0) = P_m u_0, v_m(0) = P_m v_0,$$

where P_m is the orthogonal projector from $L^2(\Omega)$ onto V_m (for the L^2-metric).

This is equivalent to the following :

$$\frac{d}{dt}((u_m, \phi)) - ((\Delta u_m, \Delta\phi)) + ((\nabla f(u_m + v_m), \nabla\phi)) - ((\nabla f(u_m - v_m), \nabla\phi)) = 0 \ ,$$

$$\frac{d}{dt}((v_m, \omega)) + ((\nabla v_m, \nabla\omega)) + ((f(u_m + v_m), \omega)) - ((f(u_m - v_m), \omega)) = 0 \ ,$$

$\forall \phi, \omega \in V_m$, together with the above initial conditions. The proof of existence of a local (in time) solution to the approximating problem is standard (indeed, one has to solve a continuous system of ODEs).

Furthermore, we can write the equivalent of the previous estimates (with u and v replaced by u_m and v_m, respectively ; this is now fully justified and no longer formal). Then (2.16) yields that this solution is actually global. Finally, the passage to the limit is based on classical (Aubin-Lions type) compactness results. Indeed, we have, in particular, u_m bounded in $L^\infty(0, T; H^2(\Omega))$ and $\frac{du_m}{dt}$ bounded in $L^\infty(0, T; H^{-1}(\Omega))$, independently of m, which yields that (at least for a subsequence which we do not relabel) u_m converges strongly to, say, u in $C([0, T]; H^{2-\delta}(\Omega))$, $\forall\ \delta > 0$.

In addition, v_m is bounded in $L^\infty(0, T; H_0^1(\Omega))$ and $\frac{dv_m}{dt}$ is bounded in $L^\infty(0, T; L^2(\Omega))$, independently of m, which also yields the strong convergence of v_m to, say, v in $C([0, T]; H^{1-\delta}(\Omega))$, $\forall\ \delta > 0$. We also note that it follows from (2.45) that $(u, v) \in L^\infty(0, T; H^2(\Omega)^2)$, from (2.31) and (2.36) that $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^\infty(0, T; H^{-1}(\Omega) \times L^2(\Omega))$ and from (2.34) that $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(0, T; H_0^1(\Omega)^2)$, whence $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^\infty(0, T; H^{-1}(\Omega) \times L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)^2)$. Besides, our solution is weakly continuous with respect to time, i.e., it belongs to $C_w([0, T]; H^2(\Omega)^2)$.

b) Uniqueness
Let (u_1, v_1) and (u_2, v_2) be two solutions with initial data $(u_{0,1}, v_{0,1})$ and $(u_{0,2}, v_{0,2})$, respectively.

We set $(u, v) = (u_1 - u_2, v_1 - v_2)$ and $(u_0, v_0) = (u_{0,1} - u_{0,2}, v_{0,1} - v_{0,2})$ and have

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u_1 + v_1) - f(u_2 + v_2) + f(u_1 - v_1) - f(u_2 - v_2) = 0, \tag{2.57}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u_1 + v_1) - f(u_2 + v_2) - f(u_1 - v_1) + f(u_2 - v_2) = 0, \tag{2.58}$$

$$u = v = 0 \text{ on } \Gamma, \tag{2.59}$$

$$u_{|t=0} = u_0, v_{|t=0} = v_0. \tag{2.60}$$

We multiply (2.57) by $\frac{\partial u}{\partial t}$ and (2.58) by $\frac{\partial v}{\partial t}$ and obtain, summing the two resulting equalities,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \left(\left(f(u_1 + v_1) - f(u_2 + v_2), \frac{\partial}{\partial t}(u + v)\right)\right) \\ + \left(\left(f(u_1 - v_1) - f(u_2 - v_2), \frac{\partial}{\partial t}(u - v)\right)\right) = 0.\end{aligned} \tag{2.61}$$

Furthermore,

$$\begin{aligned}&\left(\left(f(u_1 + v_1) - f(u_2 + v_2), \frac{\partial}{\partial t}(u + v)\right)\right) \\ &\qquad = \left|\left(\left((-\Delta)^{\frac{1}{2}}(f(u_1 + v_1) - f(u_2 + v_2)), (-\Delta)^{\frac{-1}{2}}\frac{\partial}{\partial t}(u + v)\right)\right)\right| \\ &\qquad \le c\left\|\frac{\partial}{\partial t}(u + v)\right\|_{-1} \|\nabla(f(u_1 + v_1) - f(u_2 + v_2))\|,\ c > 0,\end{aligned} \tag{2.62}$$

and similarly

$$\begin{aligned}&\left(\left(f(u_1 - v_1) - f(u_2 - v_2), \frac{\partial}{\partial t}(u - v)\right)\right) \\ &\qquad \le c'\left\|\frac{\partial}{\partial t}(u - v)\right\|_{-1} \|\nabla(f(u_1 - v_1) - f(u_2 - v_2))\|,\ c' > 0.\end{aligned} \tag{2.63}$$

Therefore,

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|^2 + \|\nabla v\|^2) + c\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + c\left\|\frac{\partial v}{\partial t}\right\|^2 \\ &\qquad \le c\|\nabla(f(u_1 + v_1) - f(u_2 + v_2))\|^2 + c\|\nabla(f(u_1 - v_1) - f(u_2 - v_2))\|^2.\end{aligned} \tag{2.64}$$

We can see that, owing to (2.56),

$$\begin{aligned}
&\|\nabla(f(u_1+v_1)-f(u_2+v_2))\| \\
&=\|\nabla\Big(\int_0^1 f'(u_1+v_1+s(u_2+v_2-u_1-v_1))ds(u-v)\Big)\| \\
&\leq\|\int_0^1 f'(u_1+v_1+s(u_2+v_2-u_1-v_1))ds\nabla(u-v)\| \\
&\quad+\|(u-v)\int_0^1 f''(u_1+v_1+s(u_2+v_2-u_1-v_1))(\nabla(u_1+v_1) \\
&\quad+s\nabla(u_2+v_2-u_1-v_1))ds\| \\
&\leq Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|v_{0,1}\|_{H^2(\Omega)},\|v_{0,2}\|_{H^2(\Omega)})(\|\nabla(u-v)\| \\
&\quad+\|\,|u-v|\,|\nabla(u_1+v_1)|\,\|+\|\,|u-v|\,|\nabla(u_2+v_2)|\,\|) \\
&\leq Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|v_{0,1}\|_{H^2(\Omega)},\|v_{0,2}\|_{H^2(\Omega)})(\|\nabla u\|+\|\nabla v\|).
\end{aligned} \tag{2.65}$$

In the same way,

$$\begin{aligned}
&\|\nabla(f(u_1-v_1)-f(u_2-v_2))\| \\
&\quad\leq Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|v_{0,1}\|_{H^2(\Omega)},\|v_{0,2}\|_{H^2(\Omega)})(\|\nabla u\|+\|\nabla v\|).
\end{aligned} \tag{2.66}$$

We deduce from (2.61)-(2.66) that

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|^2+\|\nabla v\|^2)+c\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+c\Big\|\frac{\partial v}{\partial t}\Big\|^2 \\
&\quad\leq Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|v_{0,1}\|_{H^2(\Omega)},\|v_{0,2}\|_{H^2(\Omega)})(\|\nabla u\|^2+\|\nabla v\|^2).
\end{aligned}$$

Now using Gronwall's lemma, we obtain

$$\|\nabla u(t)\|^2+\|\nabla v(t)\|^2\leq e^{Qt}(\|\nabla u_0\|^2+\|\nabla v_0\|^2), \tag{2.67}$$

whence the uniqueness (taking $(u_0,v_0)=(0,0)$), as well as the continuous dependence with respect to the initial data.

□

2.4 Existence of Finite-Dimensional Attractors

Theorem 2.4.1. *The semigroup associated with (2.1)-(2.2) is dissipative in $H^2(\Omega)^2$, i.e., it possesses a bounded absorbing set $\mathcal{B}_0$ in $H^2(\Omega)^2$.*

Proof.

We can define the semigroup

$$\begin{aligned}
S(t):H^2(\Omega)^2&\longrightarrow H^2(\Omega)^2\\
(u_0,v_0)&\longrightarrow(u(t),v(t)),
\end{aligned}$$

where (u,v) is the unique solution to our system. The result then follows from (2.56). □

Remark 2.4.1. *It is easy to see that we can assume, without loss of generality, that $\mathcal{B}_0$ is positively invariant by $S(t)$, i.e., $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$.*

Theorem 2.4.2. *The semigroup S(t) possesses an exponential attractor $\mathcal{M} \subset \mathcal{B}_0$, i.e.,*
(i) $\mathcal{M}$ is compact in $H^1(\Omega) \times H^1(\Omega)$;
(ii) $\mathcal{M}$ is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall\ t \geq 0$;
(iii) $\mathcal{M}$ has a finite fractal dimension in $H^1(\Omega) \times H^1(\Omega)$;
(iv) $\mathcal{M}$ attracts exponentially fast the bounded subsets of $H^2(\Omega)^2$:
$\forall\ B \subset H^2(\Omega)^2$ bounded, $dist_{H^1(\Omega)\times H^1(\Omega)}(S(t)B, \mathcal{M}) \leq Q(\|B\|_{H^2(\Omega)^2})e^{-ct}$,
$c > 0$, $t \geq 0$, where the constant c is independent of B and $dist_{H^1(\Omega)\times H^1(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$dist_{H^1(\Omega)\times H^1(\Omega)}(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_{H^1(\Omega)\times H^1(\Omega)}.$$

Proof.

Let (u_1, v_1) and (u_2, v_2) be two solutions to our system with initial data $(u_{0,1}, v_{0,1})$ and $(u_{0,2}, v_{0,2})$, respectively. We note that it is sufficient here to take the initial data in the bounded absorbing set $\mathcal{B}_0$ constructed in the previous section.
We set $(u, v) = (u_1, v_1) - (u_2, v_2)$ and $(u_0, v_0) = (u_{0,1}, v_{0,1}) - (u_{0,2}, v_{0,2})$ and have

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u_1 + v_1) - f(u_2 + v_2) + f(u_1 - v_1) - f(u_2 - v_2)) = 0, \tag{2.68}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u_1 + v_1) - f(u_2 + v_2) - f(u_1 - v_1) + f(u_2 - v_2) = 0, \tag{2.69}$$

$$u = v = 0 \text{ on } \Gamma, \tag{2.70}$$

$$u_{|t=0} = u_0, v_{|t=0} = v_0. \tag{2.71}$$

We recall the inequality (2.67),

$$\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 \leq e^{Qt}(\|\nabla u_0\|^2 + \|\nabla v_0\|^2). \tag{2.72}$$

We then multiply (2.68) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and (2.69) by v and have, summing the two resulting equations,

$$\begin{aligned}\frac{d}{dt}(\|\nabla u\|^2 + \|v\|^2) + c\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + c\|\nabla v\|^2 \leq c\|\nabla(f(u_1 + v_1) - f(u_2 + v_2))\|^2 \\ + c\|\nabla(f(u_1 - v_1) - f(u_2 - v_2))\|^2.\end{aligned} \tag{2.73}$$

We recall inequality (2.65),

$$\begin{aligned}&\|\nabla(f(u_1 + v_1) - f(u_2 + v_2))\| \\ &\quad \leq Q(\|u_{0,1}\|_{H^2(\Omega)}, \|u_{0,2}\|_{H^2(\Omega)}, \|v_{0,1}\|_{H^2(\Omega)}, \|v_{0,2}\|_{H^2(\Omega)})(\|\nabla u\| + \|\nabla v\|).\end{aligned} \tag{2.74}$$

The same can be done to obtain

$$\begin{aligned}&\|\nabla(f(u_1-v_1)-f(u_2-v_2))\|\\&\quad\le Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|v_{0,1}\|_{H^2(\Omega)},\|v_{0,2}\|_{H^2(\Omega)})(\|\nabla u\|+\|\nabla v\|).\end{aligned}\tag{2.75}$$

We deduce from (2.73)-(2.75) that

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|^2+\|v\|^2)+c\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+c\|\nabla v\|^2\\&\quad\le Q(\|u_{0,1}\|_{H^2(\Omega)},\|u_{0,2}\|_{H^2(\Omega)},\|v_{0,1}\|_{H^2(\Omega)},\|v_{0,2}\|_{H^2(\Omega)})(\|\nabla u\|^2+\|\nabla v\|^2).\end{aligned}\tag{2.76}$$

It follows from (2.76) that

$$\int_0^t\Big(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\|\nabla v\|^2\Big)dx\le ce^{c't}(\|\nabla u_0\|^2+\|\nabla v_0\|^2),\ t\ge 0,\tag{2.77}$$

where the constants only depend on $\mathcal{B}_0$.

We then differentiate (2.68) with respect to time and have

$$\begin{aligned}&(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t}-\Delta\frac{\partial u}{\partial t}+f'(u_1+v_1)\frac{\partial}{\partial t}(u+v)\\&\quad+\Big(f'(u_1+v_1)-f'(u_2+v_2)\Big)\frac{\partial}{\partial t}(u_2+v_2)+f'(u_1-v_1)\frac{\partial}{\partial t}(u-v)\\&\quad+\Big(f'(u_1-v_1)-f'(u_2-v_2)\Big)\frac{\partial}{\partial t}(u_2-v_2)=0.\end{aligned}\tag{2.78}$$

We multiply (2.78) by $(t-T_0)\frac{\partial u}{\partial t}$, where T_0 is the same as mentioned before, and we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2\Big)+(t-T_0)\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\le c(t-T_0)\left\|\frac{\partial u}{\partial t}\right\|^2+c(t-T_0)\left\|\frac{\partial v}{\partial t}\right\|^2\\&\quad+(t-T_0)\int_\Omega\Big(|f'(u_1+v_1)-f'(u_2+v_2)|\left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|\Big)dx\\&\quad+(t-T_0)\int_\Omega\Big(|f'(u_1-v_1)-f'(u_2-v_2)|\left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|\Big)dx+\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2.\end{aligned}$$

Noting that

$$\begin{aligned}&\int_\Omega|f'(u_1+v_1)-f'(u_2+v_2)|\left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|dx\\&\qquad\le c\int_\Omega|u+v|\left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial}{\partial t}(u_2+v_2)\right|dx\\&\qquad\le c\|\nabla(u+v)\|\left\|\nabla\frac{\partial u}{\partial t}\right\|\left\|\frac{\partial}{\partial t}(u_2+v_2)\right\|,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega|f'(u_1-v_1)-f'(u_2-v_2)|\left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|dx\\&\qquad\le c\int_\Omega|u-v|\left|\frac{\partial u}{\partial t}\right|\left|\frac{\partial}{\partial t}(u_2-v_2)\right|dx\\&\qquad\le c\|\nabla(u-v)\|\left\|\nabla\frac{\partial u}{\partial t}\right\|\left\|\frac{\partial}{\partial t}(u_2-v_2)\right\|,\end{aligned}$$

we obtain, owing to a proper interpolation inequality,

$$\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2\Big)+(t-T_0)\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 \le c(t-T_0)(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2)$$
$$+c'(t-T_0)\Big\|\frac{\partial}{\partial t}(u_2+v_2)\Big\|^2\ \|\nabla(u+v)\|^2 \tag{2.79}$$
$$+c'(t-T_0)\Big\|\frac{\partial}{\partial t}(u_2-v_2)\Big\|^2\ \|\nabla(u-v)\|^2+\frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2.$$

It follows from (2.19), (2.34), and (2.36) (for $(u,v)=(u_1,v_1)$ and then $(u,v)=(u_2,v_2)$)that

$$\int_{T_0}^t\Big(\Big\|\frac{\partial u_2}{\partial t}\Big\|^2+\Big\|\frac{\partial v_2}{\partial t}\Big\|^2\Big)d\tau \le ce^{c't},\ t\ge T_0, \tag{2.80}$$

where the constants only depend on $\mathcal{B}_0$.

Applying Gronwall's lemma on (2.79) over (T_0,t) (note that $T_0<1$) and owing to (2.72), (2.77), and (2.80) we obtain

$$\Big\|\frac{\partial u}{\partial t}(t)\Big\|_{-1}^2 \le ce^{c't}(\|u_0\|_{H^1(\Omega)}^2+\|v_0\|_{H^1(\Omega)}^2),\ t\ge 1, \tag{2.81}$$

where the constants only depend on $\mathcal{B}_0$.

We now rewrite (2.68) in the form

$$-\Delta u=\tilde{h}_u(t),\ u=0 \text{ on } \Gamma, \tag{2.82}$$

for $t\ge 1$ fixed, where

$$\tilde{h}_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t}-\Big(f(u_1+v_1)-f(u_2+v_2)\Big)-\Big(f(u_1-v_1)-f(u_2-v_2)\Big), \tag{2.83}$$

satisfies, owing to (2.72) and (2.81)

$$\|\tilde{h}_u(t)\|^2 \le ce^{c't}(\|u_0\|_{H^1(\Omega)}^2+\|v_0\|_{H^1(\Omega)}^2),\ t\ge 1, \tag{2.84}$$

where the constants only depend on $\mathcal{B}_0$.

Multiplying then (2.82) by $-\Delta u$, we have

$$\|\Delta u(t)\| \le \|\tilde{h}_u(t)\|,\ t\ge 1,$$

whence

$$\|u(t)\|^2_{H^2(\Omega)} \le ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)}),\ t \ge 1, \tag{2.85}$$

where the constants only depend on $\mathcal{B}_0$.

Next, we differentiate (2.69) with respect to time and have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial v}{\partial t} - \Delta\frac{\partial v}{\partial t} + f'(u_1+v_1)\frac{\partial}{\partial t}(u+v) + \big(f'(u_1+v_1) - f'(u_2+v_2)\big)\frac{\partial}{\partial t}(u_2+v_2)\\ + f'(u_1-v_1)\frac{\partial}{\partial t}(v-u) + \big(f'(u_2-v_2) - f'(u_1-v_1)\big)\frac{\partial}{\partial t}(u_2-v_2) = 0.\end{aligned} \tag{2.86}$$

We multiply (2.86) by $(t-T_0)\frac{\partial v}{\partial t}$, where T_0 is the same as above, and obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)\\ &\quad + (t-T_0)\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \le (t-T_0)\int_\Omega\Big(|f'(u_1+v_1)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big)dx\\ &\quad + (t-T_0)\int_\Omega\Big(|f'(u_1+v_1) - f'(u_2+v_2)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u_2+v_2)\Big|\Big)dx\\ &\quad + (t-T_0)\int_\Omega\Big(|f'(u_2-v_2) - f'(u_1+v_1)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u_2-v_2)\Big|\Big)dx\\ &\quad + (t-T_0)\int_\Omega\Big(|f'(u_1-v_1)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u-v)\Big|\Big)dx + \frac{1}{2}\Big\|\frac{\partial v}{\partial t}\Big\|^2.\end{aligned}$$

The terms

$$\int_\Omega\Big(|f'(u_1+v_1) - f'(u_2+v_2)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u_2+v_2)\Big|\Big)dx$$

and

$$\int_\Omega\Big(|f'(u_2-v_2) - f'(u_1+v_1)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u_2-v_2)\Big|\Big)dx$$

can be estimated as above.

Furthermore,

$$\begin{aligned}
&\int_\Omega \Big(|f'(u_1+v_1)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big)dx\\
&\quad=\int_\Omega \Big(|f'(u_1+v_1)-f'(0)+f'(0)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big)dx\\
&\quad\le\int_\Omega \Big(|f'(u_1+v_1)-f'(0)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big)dx+\int_\Omega \Big(|f'(0)|\,\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big)dx\\
&\quad\le c(\|u_1\|_{L^\infty(\Omega)}+\|v_1\|_{L^\infty(\Omega)})\int_\Omega \Big(\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big)dx+c\int_\Omega \Big(\Big|\frac{\partial v}{\partial t}\Big|\,\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big)dx\\
&\quad\le c(\|u_1\|_{H^2(\Omega)}+\|v_1\|_{H^2(\Omega)})\Big(\Big\|\frac{\partial v}{\partial t}\Big\|\,\Big\|\frac{\partial}{\partial t}(u+v)\Big\|\Big)+c\Big(\Big\|\frac{\partial v}{\partial t}\Big\|\,\Big\|\frac{\partial}{\partial t}(u+v)\Big\|\Big)\\
&\quad\le c\Big(\Big\|\frac{\partial v}{\partial t}\Big\|\,\Big\|\frac{\partial}{\partial t}(u+v)\Big\|\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)+(t-T_0)\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 &\le c(t-T_0)\Big\|\frac{\partial v}{\partial t}\Big\|^2+c(t-T_0)\Big\|\frac{\partial u}{\partial t}\Big\|^2\\
&\quad+c(t-T_0)\Big\|\frac{\partial}{\partial t}(u_2+v_2)\Big\|^2\|\nabla(u+v)\|^2+\frac{1}{2}\Big\|\frac{\partial v}{\partial t}\Big\|^2\\
&\quad+c(t-T_0)\Big\|\frac{\partial}{\partial t}(u_2-v_2)\Big\|^2\|\nabla(u-v)\|^2.
\end{aligned}\tag{2.87}$$

We now apply Gronwall's lemma to (2.87) and find, owing to (2.72), (2.74), and (2.80),

$$\Big\|\frac{\partial v}{\partial t}(t)\Big\|^2\le ce^{c't}(\|\nabla u_0\|^2+\|\nabla v_0\|^2),\ t\ge T_0,\tag{2.88}$$

where the constants only depend on $\mathcal{B}_0$.

In the same way, we rewrite (2.69) in the form

$$-\Delta v=\tilde{h}_v(t),\ v=0\text{ on }\Gamma,\tag{2.89}$$

for $t\ge 1$ fixed, where

$$\tilde{h}_v(t)=-\frac{\partial v}{\partial t}-\Big(f(u_1+v_1)-f(u_2+v_2)\Big)-\Big(f(u_2-v_2)-f(u_1-v_1)\Big),\tag{2.90}$$

satisfies, owing to (2.72) and (2.88),

$$\|\tilde{h}_v(t)\|^2\le ce^{c't}(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}),\ t\ge 1,\tag{2.91}$$

where the constants only depend on $\mathcal{B}_0$.

Multiplying (2.89) by $-\Delta v$, we have

$$\|\Delta v(t)\|\le\|\tilde{h}_v(t)\|,\ t\ge 1,$$

whence

$$\|v(t)\|^2_{H^2(\Omega)} \leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)}),\ t \geq 1, \tag{2.92}$$

where the constants only depend on $\mathcal{B}_0$.

We deduce from (2.85) and (2.92) that

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)}),\ t \geq 1, \tag{2.93}$$

where the constants only depend on $\mathcal{B}_0$.

Finally, we derive a Hölder (both with respect to space and time) estimate. Actually, owing to (2.72), it suffices to prove the Hölder continuity with respect to time.

We have

$$\begin{aligned}
&\|u(t_1) - u(t_2)\|_{H^1(\Omega)} + \|v(t_1) - v(t_2)\|_{H^1(\Omega)} \\
&\qquad \leq c\|\nabla\Big(u(t_1) - u(t_2)\Big)\| + c\|\nabla\Big(v(t_1) - v(t_2)\Big)\| \\
&\qquad \leq c\Big(\left\|\int_{t_1}^{t_2} \nabla\frac{\partial u}{\partial t}d\tau\right\| + \left\|\int_{t_1}^{t_2} \nabla\frac{\partial v}{\partial t}d\tau\right\|\Big) \\
&\qquad \leq c|t_1 - t_2|^{\frac{1}{2}} \left|\int_{t_1}^{t_2} \Big(\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial v}{\partial t}\right\|^2\Big)d\tau\right|^{\frac{1}{2}}.
\end{aligned} \tag{2.94}$$

We note that it follows from (2.19), (2.34), and (2.36) that

$$\left|\int_{t_1}^{t_2} \Big(\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial v}{\partial t}\right\|^2\Big)d\tau\right| \leq c, \tag{2.95}$$

where c only depends on $\mathcal{B}_0$ and $T \geq T_0$ such that $t_1,\ t_2 \in [T_0, T]$. Actually, replacing $\mathcal{B}_0$ by $S(1)\mathcal{B}_0$, we can assume that $T_0 = 0$.

Finally, we have

$$\|u(t_1) - u(t_2)\|_{H^1(\Omega)} + \|v(t_1) - v(t_2)\|_{H^1(\Omega)} \leq c|t_1 - t_2|^{\frac{1}{2}}, \tag{2.96}$$

where c only depends on $\mathcal{B}_0$ and T is a positive number such that $t_1,\ t_2 \in [0, T]$ and the result follows from (2.72), (2.93), and (2.96) (see [59, 60]). □

Moreover, we can deduce from Theorem 2.4.2 and standard results the

Corollary 1. *The semigroup $S(t)$ possesses the finite-dimensional global attractor $\mathcal{A} \subset \mathcal{B}_0$.*

Remark 2.4.2. *We recall that the global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A} = \mathcal{A},\ \forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity ; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system. Furthermore, the finite dimensionality means, roughly speaking, that, even though the initial phase space is infinite dimensional, the reduced dynamics is, in some proper sense, finite dimensional and can be described by a finite number of parameters. We refer the reader to [7, 151, 182] for more details and discussions on this.*

Remark 2.4.3. *Compared to the global attractor, an exponential attractor is expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories to the global attractor may be slow and it is very difficult, if not impossible, to estimate this rate of attraction with respect to the physical parameters of the problem in general. As a consequence, global attractors may change drastically under small perturbations. We refer the reader to [59, 151] for discussions on this subject.*

Chapitre 3

On the Cahn-Hilliard/Allen-Cahn Equations with Singular Potentials

Sur les Équations Cahn-Hilliard/Allen-Cahn avec des Potentiels Singuliers

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On the Cahn-Hilliard/Allen-Cahn Equations with Singular Potentials

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Abstract : The purpose of this work is to prove the existence and uniqueness of the solution for a Cahn-Hilliard/Allen-Cahn system with singular potentials (and, in particular, the thermodynamically relevant logarithmic potentials). We also prove the existence of the global attractor. Finally, we show further regularity results and we prove a strict separation property (from the pure states) in one space dimension.

Keywords : Allen-Cahn/Cahn-Hilliard equations, logarithmic potential, global attractor, strict separation property.

3.1 Introduction

The following Cahn-Hilliard/Allen-Cahn model was introduced by J. Cahn and A. Novick-Cohen in [35] :

$$\frac{\partial u}{\partial t} = h^2\Delta(f(u+v) + f(u-v) - h^2\Delta u),$$

$$\frac{\partial v}{\partial t} = -f(u+v) + f(u-v) - \alpha v + h^2\Delta v,$$

to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point. These authors explored two phenomenological approaches leading to systems of coupled Allen-Cahn/Cahn-Hilliard (AC/CH) equations (see [35] for more details).

Here, u denotes the concentration of one of the components and is a conserved quantity, while v is an order parameter. Furthermore, h is a (positive) parameter which represents the lattice spacing, the parameter α reflects the location of the system within the phase diagram and may be

either positive or negative, and the nonlinear term f is the derivative of a double-well potential F. We further note that the system is a gradient flow in $\left(H^1\right)' \times L^2$ for the free energy

$$J(u,v) = \int_\Omega \left\{ F(u+v) + F(u-v) + \frac{\alpha}{2}v^2 + \frac{1}{2}h^2(|\nabla u|^2 + |\nabla v|^2) \right\} dx, \text{ where } f = F'.$$

These equations, endowed with Neumann boundary conditions, have been studied in [30] by A. Novick-Cohen, D. Brochet, and D. Hilhorst who proved the well-posedness and the existence of maximal attractors and inertial sets (i.e., exponential attractors) for the usual cubic nonlinear term $f(s) = s^3 - \beta s$ in three space dimensions. These results were improved in [148], taking initial conditions in $H^2(\Omega)$, which allowed to prove the existence of exponential attractors (and, thus, of the finite-dimensional global attractor) for a large class of nonlinear terms containing polynomials of arbitrary odd degree with a strictly positive leading coefficient in three space dimensions.

A similar system, with a non-constant mobility, was treated in [52] where the existence of weak solutions for the Neumann problem for a degenerate parabolic system consisting of a fourth-order and a second-order equations with singular lower-order terms in one space dimension was proved. In addition, asymptotics for a similar system with a non-constant mobility, proposed as a diffuse interface model for simultaneous order-disorder and phase separation, was studied in [158]. There, A. Novick-Cohen focused on motion in the plane. This framework yields both sharp interface and diffuse interface models of sintering of small grains and thermal grains boundary grooving in polycrystalline films. This work was extended in [160], where the authors studied the partial wetting case, and their analysis accounts for motion in three space dimensions.

We also mention that numerical methods to solve coupled (AC/CH) systems were studied in, e.g. [35, 138, 177, 184, 188, 193]. Furthermore, a NKS method for the implicit solution of a coupled (AC/CH) system was proposed in [194].

In this work, we take $h = 1$ and $\alpha = 0$ and obtain the following system :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \tag{3.1}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = 0, \tag{3.2}$$

$$u = \Delta u = v = 0 \text{ on } \Gamma, \tag{3.3}$$

$$u_{|t=0} = u_0,\ v_{|t=0} = v_0, \tag{3.4}$$

where Ω is a bounded domain of $\mathbb{R}^N$ ($N = 1, 2$, or 3) with smooth boundary Γ.

In the Cahn-Hilliard theory, a thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model (see [150, 151]) :

$$F(s) = \frac{\theta_c}{2}(1-s^2) + \frac{\theta}{2}\Big[(1-s)\ln\Big(\frac{1-s}{2}\Big) + (1+s)\ln\Big(\frac{1+s}{2}\Big)\Big],\ s \in (-1,1),\ \ 0 < \theta < \theta_c. \tag{3.5}$$

Furthermore,

$$f(s) = (F'(s)) = -\theta_c s + \frac{\theta}{2}\ln\Big(\frac{1+s}{1-s}\Big). \tag{3.6}$$

The logarithmic potential F is very often approximated by regular ones, typically,

$$F(s) = \frac{1}{4}(s^2-1)^2, \tag{3.7}$$

leading to the following cubic nonlinear term

$$f(s) = s^3 - s. \tag{3.8}$$

Note, however, that such an approximation is reasonable when the quench is shallow, i.e. when the absolute temperature θ is close to the critical one θ_c. Also note that the nonlinear term f (3.6) leads to essential difficulties, due to the fact that we need to prove that the order parameter remains in in the physically relevant interval $(-1, 1)$ (see, e.g. [42, 150, 152]).

In this article, we first prove the existence of weak solutions to the (AC/CH) equations with singular nonlinear terms. To do so, we approximate the singular nonlinear terms by regular ones and prove the convergence of the solutions to the approximated problems to that to the limit singular one. Then, we prove the uniqueness of the solution, which allows us to define the corresponding semigroup and prove the existence of the global attractor. We finally prove some higher-order regularity results which lead to a strict separation property in one space dimension.

3.2 Setting of the Problem

As far as the nonlinear term f is concerned, we assume more generally that

$$f \in C^1(-1,1),\ f(0) = 0, \tag{3.9}$$

$$\lim_{s\to\pm1} f(s) = \pm\infty, \text{ and } \lim_{s\to\pm1} f'(s) = +\infty. \tag{3.10}$$

In particular, it follows from these assumptions that

$$f' \geq -c_0,\ c_0 \geq 0, \tag{3.11}$$

and

$$-c_1 \leq F(s) \leq f(s)s + c_2,\ c_1,\ c_2 \geq 0,\ s \in (-1,1), \tag{3.12}$$

where $F(s) = \int_0^s f(r)dr$ (in particular, in order to obtain the right-hand side of (3.12), we can study the variations of the function $s \longmapsto f(s)s - F(s) + \frac{c_0}{2}s^2$, whose derivative has, owing to (3.11), the sign of s).

Remark 3.2.1. *In particular, the thermodynamically relevant logarithmic functions (3.6) satisfy the above assumptions.*

Next, we define, for $N \in \mathbb{N}$, the approximated function $f_N \in C^1(\mathbb{R})$ by

$$f_N(s) = \begin{cases} f(-1+\frac{1}{N}) + f'(-1+\frac{1}{N})\,(s+1-\frac{1}{N}), & s < -1+\frac{1}{N}, \\ f(s), & |s| \leq 1-\frac{1}{N}, \\ f(1-\frac{1}{N}) + f'(1-\frac{1}{N})(s-1+\frac{1}{N}), & s > 1-\frac{1}{N}. \end{cases}$$

We thus have

$$f_N' \geq -c_0 \tag{3.13}$$

and, setting $F_N(s) = \int_0^s f_N(r)dr$,

$$-c_3 \leq F_N(s) \leq c_4\, f_N(s)s + c_5, \quad c_4 > 0, \quad c_3, c_5 \geq 0, \quad s \in \mathbb{R}, \tag{3.14}$$

$$f_N(s)s \geq c_6\,|f_N(s)| - c_7, \quad c_6 > 0, \quad c_7 \geq 0, \quad s \in \mathbb{R}, \tag{3.15}$$

where the constants $c_i,\ i = 3, ..., 7$, are independent of N (see [152]).

Then, we introduce the approximated problem :

$$\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta(f_N(u_N + v_N) + f_N(u_N - v_N)) = 0, \tag{3.16}$$

$$\frac{\partial v_N}{\partial t} - \Delta v_N + f_N(u_N + v_N) - f_N(u_N - v_N) = 0, \tag{3.17}$$

$$u_N = \Delta u_N = v_N = 0 \text{ on } \Gamma, \tag{3.18}$$

$$u_{N|t=0} = u_0,\ v_{N|t=0} = v_0. \tag{3.19}$$

3.3 A Priori Estimates

Our aim in this section is to derive a priori estimates for the solutions u_N and v_N to (3.16)-(3.19). These a priori estimates are independent of N and are formal, i.e. we assume that u_N and v_N are as smooth as needed. In particular, we will derive a priori estimates which are independent of N on $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$ in $L^2((0,T)\times\Omega),\ T > 0$. These a priori estimates allow us to obtain the existence of a solution to (3.16)-(3.19) by implementation of a Galerkin approximation. This will also allow us to pass to the limit $N \to +\infty$ in the approximated system (3.16)-(3.19).

We start by assuming that :

$$\|u_0 + v_0\|_{L^\infty(\Omega)} \leq 1 - \delta, \tag{3.20}$$

and

$$\|u_0 - v_0\|_{L^\infty(\Omega)} \leq 1 - \delta', \ \delta, \delta' \in (0,1), \tag{3.21}$$

where δ and δ' are fixed positive constants.

We now write equation (3.16) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u_N}{\partial t} - \Delta u_N + f_N(u_N + v_N) + f_N(u_N - v_N) = 0. \tag{3.22}$$

We multiply (3.22) by u_N, (3.17) by v_N and sum the resulting equalities to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u_N\|_{-1}^2 + \|v_N\|^2) + \|\nabla u_N\|^2 + \|\nabla v_N\|^2 \\ + ((f_N(u_N + v_N), u_N + v_N)) + ((f_N(u_N - v_N), u_N - v_N)) = 0,\end{aligned} \tag{3.23}$$

where $((\, . \, , \, . \,))$ and $\|.\|$ are the usual L^2-scalar product and associated norm. Furthermore, $\|.\|_{-1}$ is the H^{-1}norm. More generally, $\|.\|_X$ denotes the norm in the Banach space X.

Using (3.15), the previous equation gives

$$\frac{1}{2}\frac{d}{dt}(\|u_N\|_{-1}^2 + \|v_N\|^2) + \|\nabla u_N\|^2 + \|\nabla v_N\|^2 + \|f_N(u_N + v_N)\|_{L^1(\Omega)} + \|f_N(u_N - v_N)\|_{L^1(\Omega)} \leq c'. \tag{3.24}$$

Then, we multiply (3.22) by $\frac{\partial u_N}{\partial t}$ and (3.17) by $\frac{\partial v_N}{\partial t}$. We sum the resulting equalities and obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u_N\|^2 + \|\nabla v_N\|^2) + \left\|\frac{\partial u_N}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v_N}{\partial t}\right\|^2 \\ + ((f_N(u_N + v_N), \frac{\partial}{\partial t}(u_N + v_N))) + ((f_N(u_N - v_N), \frac{\partial}{\partial t}(u_N - v_N))) = 0,\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u_N\|^2 + \|\nabla v_N\|^2 + 2\int_\Omega F_N(u_N + v_N)dx + 2\int_\Omega F_N(u_N - v_N)dx\Big) \\ + \left\|\frac{\partial u_N}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v_N}{\partial t}\right\|^2 = 0.\end{aligned} \tag{3.25}$$

Owing to (3.14), equation (3.23) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u_N\|_{-1}^2 + \|v_N\|^2) + \|\nabla u_N\|^2 + \|\nabla v_N\|^2 \\ + c\int_\Omega \Big(F_N(u_N + v_N) + F_N(u_N - v_N)\Big)dx \leq c', \ c > 0, c' \geq 0.\end{aligned} \tag{3.26}$$

The sum of (3.24), (3.25), and (3.26) gives

$$\begin{aligned}\frac{d}{dt}E_N + c\Big(E_N + \|f_N(u_N + v_N)\|_{L^1(\Omega)} + \|f_N(u_N - v_N)\|_{L^1(\Omega)} \\ + \left\|\frac{\partial u_N}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v_N}{\partial t}\right\|^2\Big) \leq c', \ c > 0, \ c' \geq 0,\end{aligned} \tag{3.27}$$

where $E_N = \|u_N\|_{-1}^2 + \|v_N\|^2 + \|\nabla u_N\|^2 + \|\nabla v_N\|^2 + 2\int_\Omega \Big(F_N(u_N + v_N) + F_N(u_N - v_N)\Big)dx$.

We note that (3.27) and Gronwall's lemma imply the dissipative estimate

$$E_N(t) \le e^{-ct}E_N(0) + c',\ c > 0,\ t \ge 0. \tag{3.28}$$

Integrating now (3.27) with respect to time, we have

$$\begin{aligned}\|f_N(u_N + v_N)\|_{L^1((0,T)\times\Omega)} + \|f_N(u_N - v_N)\|_{L^1((0,T)\times\Omega)} \le ce^{-c't}\Big(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 \\ + c\int_\Omega \Big(F_N(u_0 + v_0) + F_N(u_0 - v_0)\Big)dx\Big) + c''.\end{aligned} \tag{3.29}$$

Furthermore, for every $r > 0$,

$$\begin{aligned}\int_t^{t+r}\Big(\Big\|\frac{\partial u_N}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v_N}{\partial t}\Big\|^2\Big)d\tau \le ce^{-c't}\Big(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 \\ + c\int_\Omega \Big(F_N(u_0 + v_0) + F_N(u_0 - v_0)\Big)dx\Big) + c''(r),\ c' > 0,\ t \ge 0.\end{aligned} \tag{3.30}$$

Also note that $\int_\Omega \Big(F_N(u_0+v_0)+F_N(u_0-v_0)\Big)dx \le c$, since $\|u_0+v_0\|_{L^\infty} \le 1-\delta$ and $\|u_0-v_0\|_{L^\infty} \le 1-\delta'$.

Therefore, (3.29) yields

$$\begin{aligned}\|f_N(u_N + v_N)\|_{L^1((0,T)\times\Omega)} + \|f_N(u_N - v_N)\|_{L^1((0,T)\times\Omega)} \\ \le ce^{-c't}\Big(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + c''\Big).\end{aligned} \tag{3.31}$$

We have thus found an estimate on the L^1 norm of $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$.

We now need to derive an estimate of its L^2 norm. Intergrating (3.27) over $(0, t)$, we obtain

$$\int_0^t \Big(\Big\|\frac{\partial u_N}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v_N}{\partial t}\Big\|^2\Big)dt \le c'e^{-ct}(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + c''). \tag{3.32}$$

We next multiply (3.22) by $-\Delta u_N$, (3.17) by $-\Delta v_N$, and sum the resulting equations to find :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u_N\|^2 + \|\nabla v_N\|^2) + \|\Delta u_N\|^2 + \|\Delta v_N\|^2 + ((f'(u_N + v_N)\nabla(u_N + v_N), \nabla(u_N + v_N)\ )) \\ + ((f'(u_N - v_N)\nabla(u_N - v_N), \nabla(u_N - v_N)\ )) = 0,\end{aligned} \tag{3.33}$$

which gives, owing to (3.13),

$$\frac{1}{2}\frac{d}{dt}(\|u_N\|^2 + \|\nabla v_N\|^2) + \|\Delta u_N\|^2 + \|\Delta v_N\|^2 \le c(\|\nabla u_N\|^2 + \|\nabla v_N\|^2). \tag{3.34}$$

Furthermore,

$$\|u_N\|_{H^1(\Omega)}^2 \le c\|u_N\|\ \|u_N\|_{H^2(\Omega)}, \le \frac{1}{2}\|\Delta u_N\|^2 + c\|u_N\|^2,\ c > 0.$$

Therefore,

$$\frac{d}{dt}(\|u_N\|^2 + \|\nabla v_N\|^2) + c(\|\Delta u_N\|^2 + \|\Delta v_N\|^2) \leq c'(\|u_N\|^2 + \|\nabla v_N\|^2). \tag{3.35}$$

Using now Gronwall's lemma, we obtain

$$\|u_N(t)\|^2 + \|\nabla v_N(t)\|^2 \leq e^{c't}(\|u_0\|^2 + \|\nabla v_0\|^2). \tag{3.36}$$

It follows from (3.35) that

$$\int_0^t (\|\Delta u_N\|^2 + \|\Delta v_N\|^2)dt \leq ce^{c't}(\|u_0\|^2 + \|\nabla v_0\|^2). \tag{3.37}$$

Hence,

$$\|u_N\|^2_{L^2(0,T;H^2(\Omega))} + \|v_N\|^2_{L^2(0,T;H^2(\Omega))} \leq ce^{c't}(\|u_0\|^2 + \|\nabla v_0\|^2). \tag{3.38}$$

We now rewrite (3.22) and (3.17) as

$$f_N(u_N + v_N) + f_N(u_N - v_N) = -(-\Delta)^{-1}\frac{\partial u_N}{\partial t} + \Delta u_N, \tag{3.39}$$

$$f_N(u_N + v_N) - f_N(u_N - v_N) = -\frac{\partial v_N}{\partial t} + \Delta v_N. \tag{3.40}$$

We sum (3.39) and (3.40), then subtract them, to have

$$2f_N(u_N + v_N) = -(-\Delta)^{-1}\frac{\partial u_N}{\partial t} + \Delta u_N - \frac{\partial v_N}{\partial t} + \Delta v_N,$$

$$2f_N(u_N - v_N) = -(-\Delta)^{-1}\frac{\partial u_N}{\partial t} + \Delta u_N + \frac{\partial v_N}{\partial t} - \Delta v_N.$$

Therefore, using (3.30) and (3.38), we obtain

$$\|f_N(u_N + v_N)\|_{L^2(\Omega)} + \|f_N(u_N - v_N)\|_{L^2(\Omega)} \leq c'e^{-ct}\Big(\|u_0\|^2_{-1} + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + c''\Big). \tag{3.41}$$

Integrating (3.41) with respect to time, we have

$$\begin{aligned}\|f_N(u_N + v_N)\|_{L^2((0,T)\times\Omega)} + \|f_N(u_N - v_N)\|_{L^2((0,T)\times\Omega)} \\ \leq ce^{-c't}\Big(\|u_0\|^2_{-1} + \|v_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + c''\Big).\end{aligned} \tag{3.42}$$

We can then write (3.28) as

$$\begin{aligned}\|u_N(t)\|^2_{-1} + \|v_N(t)\|^2 + \|\nabla u_N(t)\|^2 + \|\nabla v_N(t)\|^2 \leq e^{-ct}\Big(\|u_0\|^2_{-1} + \|v_0\|^2 + \|\nabla u_0\|^2 + \\ \|\nabla v_0\|^2 + 2\int_\Omega \Big(F_N(u_0 + v_0) + F_N(u_0 - v_0)\Big)dx\Big) + c'.\end{aligned}$$

Recalling that $\int_\Omega \Big(F_N(u_0 + v_0) + F_N(u_0 - v_0)\Big)dx \leq c$, because, noting that if N is large enough, $F_N(s_0) = F(s_0)$.

Then, owing to (3.20) and (3.21), the previous relation becomes

$$\|u_N(t)\|^2_{-1} + \|v_N(t)\|^2 + \|\nabla u_N(t)\|^2 + \|\nabla v_N(t)\|^2 \leq ce^{-c't}(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + c'') + c'''. \tag{3.43}$$

3.4 Existence and Uniqueness of Solutions

Theorem 3.4.1. *We assume that u_0 and v_0 are given such that $(u_0, v_0) \in H^1(\Omega)^2$, $\|u_0 + v_0\|_{L^\infty(\Omega)} < 1$, and $\|u_0 - v_0\|_{L^\infty(\Omega)} < 1$. Then, (3.1)-(3.4) possesses a unique (weak) solution (u, v) such that, $\forall\ T > 0$,*

$$(u, v) \in C_w([0, T]; H_0^1(\Omega)^2) \cap\ L^2(0, T; H^2(\Omega)^2) \cap L^\infty(0, T; H_0^1(\Omega)^2)$$

and

$$(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(0, T; H^{-1}(\Omega) \times L^2(\Omega)),$$

where the subscript w stands for the weak topology, and

$$\frac{d}{dt}((u, q))_{-1} + ((\nabla u, \nabla q)) + ((f(u + v) + f(u - v), q)) = 0,$$

$$\frac{d}{dt}((v, q)) + ((\nabla v, \nabla q)) + ((f(u + v) - f(u - v), q)) = 0,$$

$$u(0) = u_0,\ v(0) = v_0,$$

a.e. $t \in [0, T]$, $\forall\ q \in C_c^\infty(\Omega)$.

Proof.

a) Existence
We consider the solution (u_N, v_N) to the approximated problem (3.16)-(3.19) (as already mentionned in the previous section, the proof of existence of such a solution having the above regularity can be obtained by a standard Galerkin scheme).

Furthermore, since the estimates derived in the previous section are independent of N, this solution converges, up to a subsequence which we do not relabel, to a limit function (u, v) in the following sense :
$u_N \longrightarrow u \in L^2(0, T; H^2(\Omega))$ weakly owing to (3.38),
$\frac{\partial u_N}{\partial t} \longrightarrow \frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$ weakly owing to (3.32),
$v_N \longrightarrow v \in L^2(0, T; H^2(\Omega))$ weakly owing to (3.38),
$\frac{\partial v_N}{\partial t} \longrightarrow \frac{\partial v}{\partial t} \in L^2(0, T; L^2(\Omega))$ weakly owing to(3.32).
Then we use classical Aubin-Lions compactness results for the strong convergence.

The only difficulty, when passing to the limit, is to pass to the limit in the nonlinear terms containing f_N. First, it follows from (3.31) that $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$ are bounded, independently of N, in $L^1((0, T) \times \Omega)$. Then, it follows from the explicit expression of f_N that

$$meas(F_{N,M}) \leq \mu(\frac{1}{N}),\ N \leq M,$$

where

$$F_{N,M} = \{(t,x) \in (0,T) \times \Omega,\ |u_M(t,x) + v_M(t,x)| > 1 - \frac{1}{N}\}$$

and

$$\mu(s) = \frac{c}{min(|f(1-s)|, |f(s-1)|)}.$$

Here, the constant c is independent of N and M. Note that there holds

$$\begin{aligned} \int_0^T \int_\Omega |f_M(u_M + v_M)| dx\, dt &\geq \int_{F_{N,M}} |f_M(u_M + v_M)| dx\, dt \\ &\geq c' meas(F_{N,M}) \frac{1}{\mu(\frac{1}{N})}, \end{aligned} \tag{3.44}$$

where the constant c' is independent of N and M.

We can pass to the limit $M \longrightarrow \infty$ (employing Fatou's lemma on (3.44)) and then $N \longrightarrow \infty$ (noting that $\lim_{s \longrightarrow 0} \mu(s) = 0$) to find

$$meas\{(t,x) \in (0,T) \times \Omega, |u(t,x) + v(t,x)| \geq 1\} = 0,$$

so that

$$-1 < u(t,x) + v(t,x) < 1 \text{ a.e. } (t,x). \tag{3.45}$$

In the same way, we can prove that

$$-1 < u(t,x) - v(t,x) < 1 \text{ a.e. } (t,x). \tag{3.46}$$

Next, it follows from the above almost everywhere convergence of u_N and v_N, from (3.45), (3.46), and the explicit expression of f_N that

$$f_N(u_N + v_N) \longrightarrow f(u+v)\ a.e.\ (t,x) \in (0,T) \times \Omega \tag{3.47}$$

and

$$f_N(u_N - v_N) \longrightarrow f(u-v)\ a.e.\ (t,x) \in (0,T) \times \Omega. \tag{3.48}$$

Finally, since, owing to (3.42), $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$ are bounded, independently of N, in $L^2((0,T) \times \Omega)$, it follows from (3.47) and (3.48) that $f_N(u_N + v_N) \longrightarrow f(u+v)$ and $f_N(u_N - v_N) \longrightarrow f(u-v)$ in $L^2((0,T) \times \Omega)$ weakly, which finishes the proof of the passage to the limit (the weak continuity property follows from Strauss's lemma, see, e.g. [181]).
We also note that it follows from (3.43) that $(u,v) \in L^\infty(0,T; H_0^1(\Omega)^2)$.

b) Uniqueness
Let (u_1, v_1) and (u_2, v_2) be two solutions to (3.1)-(3.4) with initial data $(u_{0,1}, v_{0,1})$ and $(u_{0,2}, v_{0,2})$, respectively. We set $(u,v) = (u_1 - u_2, v_1 - v_2)$ and $(u_0, v_0) = (u_{0,1} - u_{0,2}, v_{0,1} - v_{0,2})$ and have

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u_1 + v_1) - f(u_2 + v_2) + f(u_1 - v_1) - f(u_2 - v_2) = 0, \tag{3.49}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u_1 + v_1) - f(u_2 + v_2) - f(u_1 - v_1) + f(u_2 - v_2) = 0, \tag{3.50}$$

$$u = \Delta u = v = 0 \text{ on } \Gamma, \tag{3.51}$$

$$u_{|t=0} = u_0, \ v_{|t=0} = v_0. \tag{3.52}$$

We multiply (3.49) by u, (3.50) by v, and sum to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 + ((f(u_1 + v_1) - f(u_2 + v_2), u + v)) \\ + ((f(u_1 - v_1) - f(u_2 - v_2), u - v)) = 0.\end{aligned} \tag{3.53}$$

Let $p = u_1 + v_1$, $q = u_2 + v_2$, $h = u_1 - v_1$, and $l = u_2 - v_2$.

We have

$$\begin{aligned}((f(u_1 + v_1) - f(u_2 + v_2), u + v)) &= ((f(p) - f(q), p - q)) \\ &= ((f'(\xi)(p - q), p - q)) \\ &\geq -c_0\|p - q\|^2 \\ &\geq -c_0\|u + v\|^2,\end{aligned} \tag{3.54}$$

and

$$\begin{aligned}((f(u_1 - v_1) - f(u_2 - v_2), u - v)) &= ((f(h) - f(l), h - l)) \\ &\geq -c_0\|u - v\|^2.\end{aligned} \tag{3.55}$$

Hence, using the previous inequalities, equation (3.53) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 \\ &\qquad = -((f(u_1 + v_1) - f(u_2 + v_2), u + v)) \\ &\qquad - ((f(u_1 - v_1) - f(u_2 - v_2), u - v)) \\ &\qquad \leq c_0\|u + v\|^2 + c_0\|u - v\|^2 \\ &\qquad \leq c(\|u\|^2 + \|v\|^2).\end{aligned} \tag{3.56}$$

Employing the interpolation inequality

$$\|u\|^2 \leq c\ \|u\|_{-1}\|\nabla u\| \leq \frac{1}{2}\|\nabla u\|^2 + c\|u\|_{-1}^2,$$

we deduce that

$$\frac{d}{dt}(\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 \leq c'(\|u\|_{-1}^2 + \|v\|^2). \tag{3.57}$$

It finally follows from Gronwall's lemma that

$$(\|u\|_{-1}^2 + \|v\|^2) \leq e^{c't}(\|u_0\|_{-1}^2 + \|v_0\|^2), \tag{3.58}$$

hence the uniqueness (taking $(u_0, v_0) = (0, 0)$), as well as the continuous dependence with respect to the initial data in $H^{-1}(\Omega) \times L^2(\Omega)$. □

3.5 Existence of the Global Attractor

It follows from Theorem 3.4.1 that we can define the continuous family of operators

$$S(t) : \Phi_1 \longrightarrow \Phi$$

$$(u_0, v_0) \longmapsto (u(t), v(t)),$$

where

$$\Phi = \{(u, v) \in H^1(\Omega)^2;\ |u + v| < 1 \text{ and } |u - v| < 1 \text{ a.e.}\}$$

and

$$\Phi_1 = \Phi \cap \{(u, v) \in L^\infty(\Omega)^2;\ \|u + v\|_{L^\infty(\Omega)} < 1 \text{ and } \|u - v\|_{L^\infty(\Omega)} < 1\}.$$

Theorem 3.5.1. *The family of operators associated with (3.1)-(3.2) is dissipative in $H^1(\Omega)^2$ in the sense that it possesses a bounded absorbing set $\mathcal{B}_1 \subset H^1(\Omega)^2$, i.e. $\forall\ B \subset \Phi_1$ bounded, $\exists\ t_0 = t_0(B)$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_1$.*

The dissipativity of $S(t)$ and the existence of a bounded absorbing set in $H^1(\Omega)^2$ immediately follow from (3.43).

We now assume that

$$\lim_{s \to \pm 1} F(s) = c, \tag{3.59}$$

where c is a constant (note that this holds for the thermodynamically relevant logarithmic potentials). Then, $S(t)$ as defined above is a semigroup now (i.e. $S(0) = I$ (identity operator) and $S(t + \tau) = S(t) \circ S(\tau),\ t,\ \tau \geq 0$).

As a consequence of Theorem 3.5.1 and of (3.59), it follows from standard results (see, e.g. [7, 140, 151, 182]) that we have the following theorem.

Theorem 3.5.2. *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ on Φ (i.e. $\mathcal{A}$ is compact in $H^{-1}(\Omega) \times L^2(\Omega)$, bounded in Φ, invariant and attracts the images of all bounded subsets of Φ with respect to the topology of $H^{-1}(\Omega) \times L^2(\Omega)$).*

Remark 3.5.1. *In order to prove that one has the global attractor and, in particular, the attraction property in the natural topology of the phase space Φ, one would need additional regularity on the solutions or the strict separation property from the singular values ± 1 which we are not able to prove, except in one space dimension (see Section 3.6 below). We can also note that it follows from (3.58) that we can extend, in a unique way and by continuity, the semigroup $S(t)$ to the closure of Φ in the $H^{-1}(\Omega) \times L^2(\Omega)$-topology, namely to*

$$\overline{\Phi} = \{(u, v) \in L^\infty(\Omega)^2;\ \|u + v\|_{L^\infty(\Omega)} \leq 1 \text{ and } \|u - v\|_{L^\infty(\Omega)} \leq 1\}.$$

The corresponding semigroup again possesses the global attractor which is precisely $\mathcal{A}$.

3.6 Further Regularity Results

In what follows, we set $V = H_0^1(\Omega)$. We also denote by V' its dual space and by $\|.\|_{V'}$ its norm.

We can decompose the singular potential F as

$$F(x) = S(x) + \frac{\theta_c}{2}(1 - x^2),$$

with

$$\lim_{x\to -1} S'(x) = -\infty,\ \lim_{x\to 1} S'(x) = +\infty,\ S''(x) \geq \theta > 0,\ \forall x \in (-1, 1), \tag{3.60}$$

and we let

$$\theta_c - \theta = \alpha > 0.$$

We also require that S satisfies

$$|S''(x)| \leq e^{c|S'(x)|+c},\ \forall x \in (-1, 1), \tag{3.61}$$

for some positive constant c, and S is convex.

We mention below a Trudinger-Moser type inequality (see, e.g., [154]) which is needed later on.

Lemma 1. *Let Ω be a bounded smooth domain of $\mathbb{R}^2$. Then, there exists a positive constant c such that*

$$\int_\Omega e^{|u|}dx \leq ce^{c\|u\|_V^2},\ \forall\ u \in V. \tag{3.62}$$

Let us now define the free energy functional

$$\mathcal{E}(u + v) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla v\|^2 + 2\int_\Omega F(u + v)dx + 2\int_\Omega F(u - v)dx.$$

Theorem 3.6.1. *Let u_0 and v_0 be in V such that $F(u_0 + v_0)$ and $F(u_0 - v_0) \in L^1(\Omega)$. Then, there exists a unique solution $(u, v) \in C([0, T]; H_w^{-1}(\Omega) \times L_w^2(\Omega))$ which fulfills the estimate*

$$\mathcal{E}((u + v)(t)) + \int_t^{t+1} (\|\nabla\mu(s)\|^2 + \|v_t(s)\|^2)ds \leq \mathcal{E}(u_0 + v_0),\ \forall\ t \geq 0. \tag{3.63}$$

Proof.

The existence and uniqueness of the solution can be proved in the same way as in section 3.4. Therefore, we confine ourselves only to the proof of (3.63). We rewrite equation (3.1) in the form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u + v) + f(u - v) = 0. \tag{3.64}$$

We start by differentiating equations (3.64) and (3.2) with respect to time to find

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) + f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{3.65}$$

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial t} - \Delta\frac{\partial v}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) - f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{3.66}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \text{ on } \Gamma. \tag{3.67}$$

Summing (3.65) times $t\frac{\partial u}{\partial t}$ and (3.66) times $t\frac{\partial v}{\partial t}$, then using (3.11), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(t\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + t\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + t\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + t\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \\ \leq \frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \frac{1}{2}\Big\|\frac{\partial v}{\partial t}\Big\|^2 + c't\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big).\end{aligned}$$

Employing the interpolation inequality

$$\begin{aligned}\Big\|\frac{\partial u}{\partial t}\Big\|^2 &\leq c'\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}\Big\|\nabla\frac{\partial u}{\partial t}\Big\| \\ &\leq c'\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \frac{1}{2}\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2,\end{aligned}$$

we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(t\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + t\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + ct\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + t\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \\ \leq \frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \frac{1}{2}\Big\|\frac{\partial v}{\partial t}\Big\|_{-1}^2 + c't\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big).\end{aligned} \tag{3.68}$$

We deduce from (3.30), (3.31) (which hold when $N \to +\infty$), (3.68), and Gronwall's lemma that

$$\Big\|\frac{\partial u}{\partial t}(t)\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}(t)\Big\|^2 \leq \frac{1}{t}Q(\|u_0\|_{H^1(\Omega)}, \|v_0\|_{H^1(\Omega)}),\ t \in (0,1]. \tag{3.69}$$

Summing then (3.65) times $\frac{\partial u}{\partial t}$ to (3.66) times $\frac{\partial v}{\partial t}$ and using (3.11) and an interpolation inequality, we have

$$\frac{d}{dt}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + \Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + 2\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \leq c\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big). \tag{3.70}$$

Fianlly, we conclude from (3.69), (3.70), and Gronwall's lemma that

$$\Big\|\frac{\partial u}{\partial t}(t)\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}(t)\Big\|^2 \leq e^{ct}Q(\|u_0\|_{H^1(\Omega)}, \|v_0\|_{H^1(\Omega)}),\ t \geq 1. \tag{3.71}$$

where Q denotes a monotone increasing function.

Integrating now (3.70) between t and $t+1$ and using (3.71), we find

$$\int_t^{t+1} \left(\left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial v}{\partial t} \right\|^2 \right) dt \leq e^{ct} Q(\|u_0\|_{H_0^1(\Omega)}, \|v_0\|_{H_0^1(\Omega)}), \tag{3.72}$$

for $t \geq 1$. Therefore, $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(t, t+1; H_0^1(\Omega)^2)$.

Then, we can rewrite equations (3.1) and (3.2) as

$$u_t = \Delta \mu, \tag{3.73}$$

where

$$\mu = -\Delta u + F'(u+v) + F'(u-v),$$

and

$$v_t + F'(u+v) - F'(u-v) = \Delta v. \tag{3.74}$$

Equations (3.73) and (3.74) are equivalent to

$$((u_t, q)) + ((\nabla \mu, \nabla q)) = 0, \quad \forall\, q \in H_0^1(\Omega), \tag{3.75}$$

and

$$\begin{aligned} ((v_t, q')) + ((F'(u+v), q')) - ((F'(u-v), q')) \\ + ((\nabla v, \nabla q')) = 0, \quad \forall\, q' \in H_0^1(\Omega). \end{aligned} \tag{3.76}$$

Using equations (3.75)-(3.76) and the standard chain rule in $L^2(0,T;V) \cap H^1(0,T;V')$, we get

$$\begin{aligned} ((u_t, \mu)) + ((\nabla \mu, \nabla \mu)) + ((v_t, v_t)) + ((F'(u+v), v_t)) \\ - ((F'(u-v), v_t)) + ((\nabla v, \nabla v_t)) = 0, \end{aligned}$$

which gives the energy equality

$$\frac{d}{dt} \mathcal{E}(u+v) + \|\nabla \mu\|^2 + \|v_t\|^2 = 0.$$

It finally follows from the Gronwall lemma that

$$\mathcal{E}((u+v)(t)) + \int_t^{t+1} (\|\nabla \mu(s)\|^2 + \|v_t(s)\|^2) ds \leq \mathcal{E}(u_0 + v_0), \ \forall\, t \geq 0.$$

□

In what follows, according to (3.55), the generic positive constant c may also depend on the initial energy $\mathcal{E}(u_0 + v_0)$. In particular, we will use

$$\mathcal{E}((u+v)(t)) + \int_t^{t+1} (\|\nabla \mu(s)\|^2 + \|v_t(s)\|^2) ds \leq c, \ \forall\, t \geq 0. \tag{3.77}$$

Theorem 3.6.2. *Let the assumptions of Theorem 3.6.1 holds. Then there exists a positive constant c such that*

$$\|\mu\|_{L^\infty(1,t;V)} + \|v_t\|_{L^\infty(1,t;L^2(\Omega))} \leq c, \quad \forall\, t \geq 1, \tag{3.78}$$

and

$$\begin{aligned} \|u_t + v_t\|_{L^\infty(1,t;V')} + \|u_t - v_t\|_{L^\infty(1,t;V')} + \|u_t\|_{L^2(t,t+1;V)} + \|v_t\|_{L^2(t,t+1;V)} \\ \leq c, \quad \forall\, t \geq 1. \end{aligned} \tag{3.79}$$

Proof.

We start by differentiating equation (3.74) with respect to time, which yields

$$\frac{\partial}{\partial t} v_t + \frac{\partial}{\partial t}[F'(u+v) - F'(u-v)] = \Delta v_t. \tag{3.80}$$

Testing (3.73) by μ_t and (3.80) by v_t, then summing the resulting equations, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\mu\|^2 + \|v_t\|^2) + \|\nabla v_t\|^2 + ((u_t, -\Delta u_t)) \\ &\quad + ((F''(u+v)(u_t+v_t), u_t+v_t)) + ((F''(u-v)(u_t-v_t), u_t-v_t)) = 0. \end{aligned}$$

We observe that

$$\begin{aligned} ((u_t, -\Delta u_t)) + ((F''(u_t+v_t)(u_t+v_t), u_t+v_t)) + ((F''(u_t-v_t)(u_t-v_t), u_t-v_t)) \\ \geq \|\nabla u_t\|^2 - \alpha\|u_t+v_t\|^2 - \alpha\|u_t-v_t\|^2 \\ \geq \frac{1}{2}\|\nabla u_t\|^2 - c\|\nabla v_t\|^2 - c\|u_t+v_t\|_{V'}^2 - c\|u_t-v_t\|_{V'}^2. \end{aligned} \tag{3.81}$$

Accordingly, setting

$$\psi(t) = \frac{1}{2}\|\nabla\mu(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2,$$

we end up with the differential inequality

$$\frac{d}{dt}\psi + \frac{1}{2}\|\nabla u_t\|^2 + c\|\nabla v_t\|^2 \leq c\|u_t+v_t\|_{V'}^2 + c\|u_t-v_t\|_{V'}^2. \tag{3.82}$$

Using (3.77), the definition of $\mathcal{E}$, and (3.12) we can deduce that

$$\int_t^{t+1} \psi(s)ds \leq c, \quad c \geq 0. \tag{3.83}$$

Furthermore, using (3.72),

$$\int_t^{t+1} \|u_t+v_t\|_{V'}^2 ds \leq c \tag{3.84}$$

and

$$\int_t^{t+1} \|u_t-v_t\|_{V'}^2 ds \leq c. \tag{3.85}$$

Therefore, the uniform Gronwall lemma leads to

$$\psi(t) \leq c, \quad \forall\, t \geq 1.$$

In particular, we have the bound

$$\|\mu\|_{L^\infty(1,t;V)} + \|v_t\|_{L^\infty(1,t;L^2(\Omega))} \leq c, \quad \forall\, t \geq 1,$$

which in turn gives

$$\|u_t\|_{L^\infty(1,t;V')} + \|v_t\|_{L^\infty(1,t;V')} \leq c, \quad \forall\, t \geq 1.$$

The desired conclusion (3.79) follows from an integration in time of (3.82) on $(t, t+1)$, $t \geq 1$, combined with the previous inequality. □

Remark 3.6.1. *The proof of Theorem 3.6.1 is formal, but it can be justified within a Galerkin scheme as in the proof of Theorem 3.4.1. More precisely, all the computations can be rigorously performed within the Galerkin scheme. Given that F'' is controlled from below, the estimates turn out to be independent of the approximation parameter and a final passage to the limit gives the result.*

Lemma 2. *Let $N = 2$ and u_0 and $v_0 \in V$ be such that $F(u_0 + v_0)$ and $F(u_0 - v_0) \in L^1(\Omega)$. Then, for any $p \geq 1$, there exist two positive constants c and c' depending on p such that*

$$\|S''(u+v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c(p), \quad \forall\, t \geq 1, \tag{3.86}$$

and

$$\|S''(u-v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c'(p), \quad \forall\, t \geq 1. \tag{3.87}$$

Proof.

Equations (3.1) and (3.2) can be written in the equivalent form

$$-\Delta u + S'(u+v) + S'(u-v) = \tilde{\mu}, \tag{3.88}$$

$$-\Delta v + S'(u+v) - S'(u-v) = \tilde{\mu'}, \tag{3.89}$$

where

$$\tilde{\mu} = -(-\Delta)^{-1}\frac{\partial u}{\partial t} + 2\theta_c u$$

and

$$\tilde{\mu'} = -\frac{\partial v}{\partial t} + 2\theta_c v.$$

Combining (3.88) and (3.89), we obtain

$$-\Delta(u+v) + 2S'(u+v) = \tilde{\mu} + \tilde{\mu'} \tag{3.90}$$

and

$$-\Delta(u-v)+2S'(u-v)=\tilde{\mu}-\tilde{\mu}'. \tag{3.91}$$

For any $L>0$, we consider

$$g_1=S'(u+v)e^{L|S'(u+v)|}$$

and

$$g_2=S'(u-v)e^{L|S'(u-v)|}.$$

We observe that

$$\nabla g_1=S''(u+v)[1+L|S'(u+v)|]e^{L|S'(u+v)|}\nabla(u+v).$$

Then we consider equation (3.90) and test it with g_1, which yields

$$\int_\Omega \nabla(u+v)\nabla(u+v)S''(u+v)[1+L|S'(u+v)|]e^{L|S'(u+v)|}dx$$
$$+2\int_\Omega S'(u+v)^2e^{L|S'(u+v)|}dx=\int_\Omega \tilde{\mu}g_1dx+\int_\Omega \tilde{\mu}'g_1dx.$$

The first term on the left-hand side is nonnegative (S'' is convex). Therefore, the previous relation is equivalent to

$$2\int_\Omega S'(u+v)^2e^{L|S'(u+v)|}dx\le\int_\Omega \tilde{\mu}g_1dx+\int_\Omega \tilde{\mu}'g_1dx.$$

Now, the right-hand side can be controlled by means of a generalized Young's inequality (see [1], Section 8.2) as follows :

$$\int_\Omega \tilde{\mu}S'(u+v)e^{L|S'(u+v)|}dx\le\int_\Omega |\tilde{\mu}||S'(u+v)|e^{L|S'(u+v)|}dx$$
$$\le\frac{1}{2}\int_\Omega S'(u+v)^2e^{L|S'(u+v)|}dx+\int_\Omega e^{c(L)\tilde{\mu}}dx+c.$$

In the same way,

$$\int_\Omega \tilde{\mu}'g_1dx\le\frac{1}{2}\int_\Omega S'(u+v)^2e^{L|S'(u+v)|}dx+\int_\Omega e^{c(L)\tilde{\mu}'}dx+c.$$

Using Lemma 1, we end up with

$$2\int_\Omega S'(u+v)^2e^{L|S'(u+v)|}dx\le\int_\Omega S'(u+v)^2e^{L|S'(u+v)|}dx+2c,$$

whence

$$\int_\Omega S'(u+v)^2e^{L|S'(u+v)|}dx\le c, \tag{3.92}$$

where c depends on L. On account of (3.61), we observe that, for any $p\ge1$,

$$|S''(x)|^p\le e^{pc}(c+|S'(x)|^2e^{pc|S'(x)|}),\quad\forall\, x\in(-1,1). \tag{3.93}$$

Combining finally (3.61) and (3.93) and taking $L = pc$, we deduce that

$$\|S''(u+v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c(p), \quad \forall\, t \geq 1,$$

In the same way, if we consider equation (3.91) and test it with g_2, we can easily obtain

$$\|S''(u-v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c(p), \quad \forall\, t \geq 1.$$

□

Lemma 3. *Let the assumptions of Lemma 2 hold. Then, there exists a constant c such that*

$$\begin{aligned}\|u_t\|_{L^\infty(2,t;L^2(\Omega))} + \|u_t\|_{L^2(t,t+1,H^2(\Omega))} + \|v_t\|_{L^\infty(2,t;H^1(\Omega))} \\ + \|v_t\|_{L^2(t,t+1;H^2(\Omega))} \leq c, \quad \forall\, t \geq 2.\end{aligned} \tag{3.94}$$

Proof.

Differentiating equations (3.73) and (3.74) with respect to time, we obtain

$$\frac{\partial}{\partial t} u_t = \Delta \mu_t \tag{3.95}$$

and

$$\frac{\partial}{\partial t} v_t + \frac{\partial}{\partial t}[S'(u+v) - S'(u-v)] - 2\theta_c v_t = \Delta v_t. \tag{3.96}$$

Multiplying (3.95) by u_t and (3.96) by $-\Delta v_t$ and summing up, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u_t\|^2 + \|\nabla v_t\|^2) + \|\Delta v_t\|^2 = 2\theta_c \|\nabla v_t\|^2 \\ + \int_\Omega \frac{\partial}{\partial t}[S'(u+v) - S'(u-v)]\Delta v_t dx + ((\Delta \mu_t, u_t)).\end{aligned} \tag{3.97}$$

Note that

$$((\Delta \mu_t, u_t)) = -((\Delta u_t, \Delta u_t)) - 2((\theta_c \Delta u_t, u_t)) + ((\frac{\partial}{\partial t}[S'(u+v) + S'(u-v)], \Delta u_t)).$$

Moreover, using Hölder and Young's inequalities and Lemma 2, we have

$$\begin{aligned}\int_\Omega S''(u+v)(u_t+v_t)\Delta v_t dx &\leq \|S''(u+v)\|_{L^3(\Omega)} \|u_t+v_t\|_{L^6(\Omega)}\| \|\Delta v_t\|_{L^2(\Omega)} \\ &\leq \frac{3c(p)^2}{4}\|u_t+v_t\|^2_{L^6(\Omega)} + \frac{1}{4}\|\Delta v_t\|^2\end{aligned} \tag{3.98}$$

and similarly

$$\begin{aligned}\int_\Omega S''(u-v)(u_t-v_t)\Delta v_t dx &\leq \|S''(u-v)\|_{L^3(\Omega)} \|u_t-v_t\|_{L^6(\Omega)}\| \|\Delta v_t\|_{L^2(\Omega)} \\ &\leq \frac{3c(p)^2}{4}\|u_t-v_t\|^2_{L^6(\Omega)} + \frac{1}{4}\|\Delta v_t\|^2,\end{aligned} \tag{3.99}$$

$$\begin{aligned}\int_\Omega S''(u+v)(u_t+v_t)\Delta u_t dx &\le \|S''(u+v)\|_{L^3(\Omega)}\,\|u_t+v_t\|_{L^6(\Omega)}\|\,\|\Delta v_t\|_{L^2(\Omega)}\\ &\le \frac{3c(p)^2}{4}\|u_t+v_t\|^2_{L^6(\Omega)}+\frac{1}{4}\|\Delta u_t\|^2,\end{aligned} \tag{3.100}$$

$$\begin{aligned}\int_\Omega S''(u-v)(u_t-v_t)\Delta u_t dx &\le \|S''(u-v)\|_{L^3(\Omega)}\,\|u_t-v_t\|_{L^6(\Omega)}\|\,\|\Delta v_t\|_{L^2(\Omega)}\\ &\le \frac{3c(p)^2}{4}\|u_t-v_t\|^2_{L^6(\Omega)}+\frac{1}{4}\|\Delta u_t\|^2.\end{aligned} \tag{3.101}$$

Therefore, equation (3.97) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u_t\|^2+\|\nabla v_t\|^2)+\|\Delta v_t\|^2+c\|\Delta u_t\|^2 &\le c\|u_t+v_t\|^2_{L^6(\Omega)}\\ &\quad + c\|u_t-v_t\|^2_{L^6(\Omega)}+\frac{1}{2}\|\Delta v_t\|^2+\frac{1}{2}\|\Delta u_t\|^2,\end{aligned} \tag{3.102}$$

where c depends on p.

Then, owing to the continuous embedding $H^1(\Omega)\subset L^6(\Omega)$, we have

$$\frac{1}{2}\frac{d}{dt}(\|u_t\|^2+\|\nabla v_t\|^2)+c\|\Delta v_t\|^2+c\|\Delta u_t\|^2\le c\|\nabla v_t\|^2+c\|\nabla u_t\|^2.$$

Furthermore,

$$\begin{aligned}\|u_t\|^2_{H^1(\Omega)} &\le c\|u_t\|\,\|u_t\|_{H^2(\Omega)}\\ &\le c\|u_t\|^2+c\|\Delta u_t\|^2,\quad c>0.\end{aligned}$$

Hence

$$\frac{1}{2}\frac{d}{dt}(\|u_t\|^2+\|\nabla v_t\|^2)+c\|\Delta v_t\|^2+c\|\Delta u_t\|^2\le c(\|u_t\|^2+\|\nabla v_t\|^2). \tag{3.103}$$

Using (3.78), (3.79), and the uniform Gronwall lemma to (3.103), we obtain the desired result. □

Remark 3.6.2. *The proof of Lemma 3 is carried out with the solution itself since (3.86) and (3.87) cannot be guaranteed within a Galerkin scheme.*

Theorem 3.6.3. *Let $N=1$ and let f satisfy assumption (3.10). Then, there exist two positive constants δ and δ' and $T>0$ such that*

$$\|(u+v)(t)\|_{L^\infty(\Omega)}\le 1-\delta,\quad \forall\ 2\le t\le T, \tag{3.104}$$

and

$$\|(u-v)(t)\|_{L^\infty(\Omega)}\le 1-\delta',\quad \forall\ 2\le t\le T. \tag{3.105}$$

Proof.

Since we are in one space dimension, we have the Sobolev embedding $H^1(\Omega)\subset C(\overline{\Omega})$. Therefore, it follows from (3.77) and (3.79) that

$$\|\tilde{\mu}\|_{L^\infty(2,t;L^\infty(\Omega))}\le c,\quad \forall\ 2\le t\le T.$$

Testing (3.90) by $|S'(u+v)|^{p-2}S'(u+v)$, we get

$$(p-1)\int_\Omega |S'(u+v)|^{p-2}S''(u+v)|\nabla(u+v)|^2dx + 2\|S'(u+v)\|^p_{L^p(\Omega)}$$
$$= \int_\Omega \tilde{\mu}|S'(u+v)|^{p-2}S'(u+v)dx - \int_\Omega v_t|S'(u+v)|^{p-2}S'(u+v)dx$$
$$+ 2\theta_c\int_\Omega v|S'(u+v)|^{p-2}S'(u+v)dx.$$

The first term on the left-hand side is nonnegative, so that an application of the Hölder inequality yields

$$\|S'(u+v)\|_{L^p(\Omega)} \le c(\|\tilde{\mu}\|_{L^p(\Omega)} + \|v_t\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}).$$

Recalling equation (3.35) which holds when $N \to +\infty$, we have

$$\frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2) + c(\|\Delta u\|^2 + \|\Delta v\|^2) \le c'(\|u\|^2 + \|\nabla v\|^2). \tag{3.106}$$

Integrating (3.36) (which also holds when $N \to +\infty$) yields

$$\int_t^{t+1} (\|u\|^2 + \|\nabla v\|^2)d\tau \le c,\ t \ge 0. \tag{3.107}$$

Then, applying the uniform Gronwall's lemma to (3.106) and using (3.107), we find

$$\|u(t)\|^2 + \|\nabla v(t)\|^2 \le c,\ t \ge 1. \tag{3.108}$$

Hence, using again the continuous embedding $H^1(\Omega) \subset C(\overline{\Omega})$, integrating in time from t to $t+1$ and using (3.78) and (3.108), we have

$$\|S'(u+v)\|_{L^p(\Omega\times(t,t+1))} \le c,\quad \forall\ 2 \le t \le T,$$

where c is independent of p and t.

Applying Theorem 2.14 in [1], we obtain

$$\|S'(u+v)\|_{L^\infty(\Omega\times(t,t+1))} \le c,\quad \forall\ 2 \le t \le T.$$

This implies that there exists $\delta > 0$ such that

$$\|u+v\|_{L^\infty(\Omega\times(t,t+1))} \le 1-\delta,\quad \forall\ 2 \le t \le T.$$

Since $u+v \in L^\infty(0,t;L^\infty(\Omega))$ for all $t \ge 0$, we also infer that

$$\|u+v\|_{L^\infty(2,t;L^\infty(\Omega))} \le 1-\delta,\quad \forall\ 2 \le t \le T.$$

Finally, we deduce (3.104) from the continuity in time.

In the same way, we can obtain the second inequality (3.105) using equation (3.91) instead of (3.90). □

An immediate consequence is the following

Corollary 2. *Let the assumptions of Theorem 3.6.3 hold. Then, there exists a positive constant C such that*

$$\|u(t)\|_{H^4(\Omega)} \leq C, \quad \forall\, t \geq 3.$$

Corollary 3. *It also follows from Theorem 3.6.3 that u and v are strictly separated from the pure states, i.e. there exist two positive constants γ and γ' such that*

$$|u(t,x)| \leq 1-\gamma, \quad \forall\, (x,t) \in \overline{\Omega} \times (2,T),$$

and

$$|v(t,x)| \leq 1-\gamma', \quad \forall\, (x,t) \in \overline{\Omega} \times (2,T).$$

Remark 3.6.3. *We can note that the strict separation property in two space dimensions can be obtained from the estimates of section 3.6 provided that we can prove that $\|v\|_{L^\infty} \leq c$ and $\|v_t\|_{L^\infty} \leq c'$, where c and c' are two constants which are independent of p (see [87] ; see also [141, 150]). Such estimates seem difficult to derive, due to the coupling, and will be addressed elsewhere.*

Chapitre 4

Asymptotic Behavior of a Cahn-Hilliard/Allen-Cahn System with Temperature

Comportement Asymptotique d'un systéme de Cahn-Hilliard/Allen-Cahn en Tenant Compte de la Température

Ce chapitre est constitué d'un article terminé récemment.

Asymptotic Behavior of a Cahn-Hilliard/Allen-Cahn System with Temperature

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Abstract : The main goal of this paper is to study the asymptotic behavior of a coupled Cahn-Hilliard/Allen-Cahn system with temperature. The work is divided into two parts : In the first part, the heat equation is based on the usual Fourrier law. In the second one, it's based on the type III heat conduction law. In both parts, we prove the existence of exponential attractors and, therefore, of finite-dimensional global attractors.

Keywords : Allen-Cahn/Cahn-Hilliard equations, Fourrier law, type III heat conduction, asymptotic behavior of the solutions.

4.1 Introduction

J. Cahn and A. Novick-Cohen introduced, in [35], the following system :

$$\frac{\partial u}{\partial t} = h^2\Delta(f(u+v) + f(u-v) - h^2\Delta u),$$

$$\frac{\partial v}{\partial t} = -f(u+v) + f(u-v) - \alpha v + h^2\Delta v,$$

where u is the concentration of one of the components and it is a conserved quantity, v is an order parameter, h is a (positive) parameter which represents the lattice spacing, α is a parameter that reflects the location of the system within the phase diagram (it may be either positive or negative), and the nonlinear term f is the derivative of a double-well potential F.

The system models simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point.
We further note that it is a gradient flow in $\left(H^1\right)' \times L^2$ for the free energy

$$J(u,v) = \int_\Omega \left\{ F(u+v) + F(u-v) + \frac{\alpha}{2}v^2 + \frac{1}{2}h^2(|\nabla u|^2 + |\nabla v|^2) \right\} dx,$$

These equations , endowed with Neumann boundary conditions, have been studied in [30] by A. Novick-Cohen, D. Brochet, and D. Hilhorst who proved the well-posedness and the existence of maximal attractors and inertial sets (i.e., exponential attractors) for the usual cubic nonlinear term $f(s) = s^3 - \beta s$ in three space dimensions.
These results were improved in [148] : taking initial conditions in $H^2(\Omega)$ allowed the authors to prove the existence of exponential attractors (and, thus, of the finite-dimensional global attractor) for a large class of nonlinear terms containing polynomials of arbitrary odd degree with a strictly positive leading coefficient in three space dimensions. This model has been also studied in [149], where an exponential attractor for singular potentials was found, and by consequence a global attractor of finite dimension.

A similar system, with a non-constant mobility, was treated in [52] where the authors proved the existence of weak solutions for the Neumann problem for a degenerate parabolic system consisting of a fourth-order and a second-order equations with singular lower-order terms in one space dimension. In addition, asymptotics for a similar system with a non-constant mobility, proposed as a diffuse interface model for simultaneous order-disorder and phase separation, was studied in [158]. This work was extended in [160], where the authors studied the partial wetting case, and their analysis accounts for motion in three space dimensions.

We also mention that numerical methods to solve coupled AC/CH systems were studied in, e.g. [138, 177, 184, 188, 193]. Furthermore, a NKS method for the implicit solution of a coupled AC/CH system was proposed in [194].

In this work, we study two systems of three simultaneous equations : a Cahn-Hilliard equation, an Allen-Cahn equation and a heat equation. In the first part of the paper, the heat equation is based on the usual Fourier law. We were able to find exponential attractors then a global attractor associated to the system. In the second part, the heat equation is based on the type III law of thermoelasticity. There, to find exponential attractors, we were obliged to decompose the system into the sum of two systems to overcome the difficulties created by a second-order derivative term.
Not much work has been done in similar cases. However, in a recent paper, [119], the authors studied a coupled AC/CH system with temperature and long-time oscillating properties were found.

Throughout this paper, the same letter c (and, sometimes, c', c'', and c''') denotes constants which may change from line to line, or even in a same line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may change from line to line, or even in the same line.

4.2 Part I : The Usual Fourier Law

In this part, as mentioned before, the generalized heat equation is based on the classical Fourier law for heat conduction. Indeed, we can rewrite this equation as

$$\frac{\partial H}{\partial t} = -\text{div } q,$$

where q is the thermal flux vector and, assuming the Fourrier law

$$q = -\nabla\theta.$$

The function H is the enthalpy defined by

$$H = v + \theta,$$

and θ is the relative temperature.

4.2.1 Setting of the Problem

We take $h = 1$ and $\alpha = 0$ for simplicity, and consider what follows

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \tag{4.1}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = \theta, \tag{4.2}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial v}{\partial t}, \tag{4.3}$$

$$u = \Delta u = v = \theta = 0 \text{ on } \Gamma, \tag{4.4}$$

$$u_{|t=0} = u_0, v_{|t=0} = v_0, \theta_{|t=0} = \theta_0, \tag{4.5}$$

where Ω is a bounded domain of $\mathbb{R}^N$ ($N = 1, 2$, or 3) with smooth boundary Γ.

As far as the nonlinear term is concerned, we make the following assumptions :

$$f \text{ is of class } C^2,\ f(0) = 0, \tag{4.6}$$

$$f'(s) \geq -c, c > 0, s \in \mathbb{R}, \tag{4.7}$$

$$f(s)s \geq cF(s) - c',\ F(s) \geq -c'',\ c > 0,\ c',\ c'' \geq 0,\ s \in \mathbb{R}, \tag{4.8}$$

where $F(s) = \int_0^s f(\xi)d\xi$.

We denote by $\|.\|$ the usual L^2-norm, $((.,.))$ its associated scalar product, $\|.\|_{-1} = \|(-\Delta)^{\frac{-1}{2}}\|$, and $\|.\|_X$ is the norm in the Banach space X.

4.2.2 Existence and Uniqueness of Solutions

We rewrite equation (4.1) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u+v) + f(u-v) = 0. \tag{4.9}$$

We multiply (4.9) by $\frac{\partial u}{\partial t}$, (4.2) by $\frac{\partial v}{\partial t}$, and (4.3) by θ and then integrate by parts over Ω, we obtain

$$\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + ((f(u+v)+f(u-v), \frac{\partial u}{\partial t})) = 0, \tag{4.10}$$

$$\left\|\frac{\partial v}{\partial t}\right\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + ((f(u+v)-f(u-v), \frac{\partial v}{\partial t})) = ((\theta, \frac{\partial v}{\partial t})), \tag{4.11}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \|\nabla\theta\|^2 = ((-\frac{\partial v}{\partial t}, \theta)). \tag{4.12}$$

Summing (4.10), (4.11), and (4.12), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\theta\|^2 + 2\int_\Omega [F(u+v)+F(u-v)]dx\Big) + \|\nabla\theta\|^2 \\ + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 = 0.\end{aligned} \tag{4.13}$$

Based on (4.13), we have the following result

Theorem 4.2.1. *Assume that* $(u_0, v_0, \theta_0) \in H_0^1(\Omega)^3$. *Then, (4.1)-(4.5) possesses at least one solution* (u, v, θ) *such that* $(u, v, \theta) \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)^2 \times L^2(\Omega)) \cap L^2_{loc}(\mathbb{R}^+; H^2(\Omega)^3)$, $\theta \in L^\infty(\mathbb{R}^+; L^2(\Omega) \cap H_0^1(\Omega))$ *and* $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \theta}{\partial t}) \in L^2(\mathbb{R}^+; H^{-1} \times L^2(\Omega)^2)$.

Proof.

The proof of existence (as well as the above and the subsequent a priori estimates) are based, e.g. on a classical Galerkin scheme. Let A denote the minus Laplace operator associated with Dirichlet boundary conditions. This operator is a bounded, selfadjoint and strictly positive operator with compact inverse from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. There is a set of eigenvectors $\{\phi_i,\ i \geq 1\}$ for this operator, associated with the eigenfunctions $0 < \lambda_1 \leq \lambda_2 \leq ...$, such that it is orthonormal relative to the inner product in $L^2(\Omega)$ and orthogonal relative to the one in $H_0^1(\Omega)$. Setting $V_m = S\,pan\{\phi_1, ..., \phi_m\}$, we consider the following approximating problem, written in the fuctional form :

$$\frac{du_m}{dt} + A^2 u_m + A(f(u_m+v_m) + f(u_m-v_m)) = 0, \tag{4.14}$$

$$\frac{dv_m}{dt} + Av_m + f(u_m+v_m) - f(u_m-v_m) = \theta_m, \tag{4.15}$$

$$\frac{d\theta_m}{dt} + A\theta_m = -\frac{dv_m}{dt}, \tag{4.16}$$

together with suitable initial conditions, namely,

$$u_{m|t=0} = P_m u_0,\ v_{m|t=0} = P_m v_0,\ \theta_{m|t=0} = P_m \theta_0,$$

where P_m is the orthogonal projector from $L^2(\Omega)$ onto V_m (for the L^2-metric).

This is equivalent to the following problem :

$$\frac{d}{dt}((u_m, p)) - ((\Delta u_m, \Delta p)) + ((\nabla f(u_m + v_m), \nabla p)) + ((\nabla f(u_m - v_m), \nabla p)) = 0,$$

$$\frac{d}{dt}((v_m, q)) + ((\nabla v_m, \nabla q)) + ((f(u_m + v_m), q)) - ((f(u_m - v_m), q)) = ((\theta_m, q)),$$

$$\frac{d}{dt}((\theta_m, r)) + ((\nabla \theta_m, \nabla r)) = -\frac{d}{dt}((v_m, r)),$$

$\forall\ p, q, r \in V_m$, together with the above initial conditions. The proof of existence of a local (in time) solution to the approximating problem is standard (indeed, one has to solve a continuous system of ODEs).

Furthermore, we can write the equivalent of the previous and the subsequent estimates (with u, v and θ replaced by u_m, v_m, and θ_m respectively) ; this is now fully justified and no longer formal. Then we can deduce from (4.13) that this solution is actually global. And, the passage to the limit is based on classical (Aubin-Lions type) compactness results. Indeed, we have, in particular, u_m bounded in $L^\infty(0, T; H_0^1(\Omega))$ and $\frac{du_m}{dt}$ bounded in $L^2(0, T; H^{-1}(\Omega))$, independently of m, which yields that (at least for a subsequence which we do not relabel) u_m converges strongly to, say, u in $C([0, T]; H^{1-\delta}(\Omega))$, $\forall\ \delta > 0$. In addition, v_m is bounded in $L^\infty(0, T; H_0^1(\Omega))$ and $\frac{dv_m}{dt}$ is bounded in $L^2(0, T; L^2(\Omega))$, independently of m, which also yields the strong convergence of v_m to, say, v in $C([0, T]; H^{1-\delta}(\Omega))$, $\forall\ \delta > 0$.

We also note that it follows from (4.13) that $(u, v) \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)^2)$, $\theta \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(\mathbb{R}^+; H_0^1(\Omega))$ and that $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(\mathbb{R}^+; H^{-1}(\Omega) \times L^2(\Omega))$.

We now multiply (4.9) by $-\Delta u$, (4.2) by $-\Delta v$, (4.3) by $-\Delta\theta$, and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + ((f'(u + v)\nabla(u + v), \nabla u)) + ((f'(u - v)\nabla(u - v), \nabla u)) = 0, \tag{4.17}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \|\Delta v\|^2 + ((f'(u + v)\nabla(u + v), \nabla v)) - ((f'(u - v)\nabla(u - v), \nabla v)) \\ = ((\theta, -\Delta v)),\end{aligned} \tag{4.18}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|^2 + \|\Delta\theta\|^2 = ((\frac{\partial v}{\partial t}, \Delta\theta)). \tag{4.19}$$

Summing (4.17), (4.18), and (4.19) and using (4.7), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u\|^2+\|\nabla v\|^2+\|\nabla\theta\|^2)+\|\Delta u\|^2+c'(\|\Delta v\|^2+\|\Delta\theta\|^2)\\ \leq c(\|\nabla u\|^2+\|\nabla v\|^2)+c\left\|\frac{\partial v}{\partial t}\right\|^2.\end{aligned} \tag{4.20}$$

Hence, we can deduce from (4.20) that $(u, v, \theta) \in L^2_{loc}(\mathbb{R}^+; H^2(\Omega)^3)$.

We finally multiply (4.3) by $\frac{\partial\theta}{\partial t}$ and get

$$\frac{d}{dt}\|\nabla\theta\|^2+\left\|\frac{\partial\theta}{\partial t}\right\|^2 \leq \left\|\frac{\partial v}{\partial t}\right\|^2. \tag{4.21}$$

Then $\frac{\partial\theta}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega))$ which finishes the proof of the regularity of the solution. □

We develop now some formal estimates needed for this and the following sections :
We start by multiplying (4.9) by u, (4.2) by v, sum them together and use (4.8) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u\|^2_{-1}+\|v\|^2)+\|\nabla u\|^2+c\|\nabla v\|^2+c'\int_\Omega[F(u+v)+F(u-v)]dx\\ \leq c''+c\|\theta\|^2.\end{aligned} \tag{4.22}$$

Then, we sum (4.13) and ϵ(4.22), where ϵ is small enough so that

$$\begin{aligned}&\frac{d}{dt}\Big(\epsilon\|u\|^2_{-1}+\epsilon\|v\|^2+\|\nabla u\|^2+\|\nabla v\|^2+\|\theta\|^2+2\int_\Omega[F(u+v)+F(u-v)]dx\Big)\\ &\quad+c\Big(\epsilon\|\nabla u\|^2+\|\nabla v\|^2+\|\nabla\theta\|^2+\int_\Omega[F(u+v)+F(u-v)]dx\Big)\\ &\quad+\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}+\left\|\frac{\partial v}{\partial t}\right\|^2\leq c''.\end{aligned} \tag{4.23}$$

We set

$$E_1=\epsilon\|u\|^2_{-1}+\epsilon\|v\|^2+\|\nabla u\|^2+\|\nabla v\|^2+\|\theta\|^2+2\int_\Omega[F(u+v)+F(u-v)]dx.$$

Whence, we deduce from (4.23) the inequality

$$\frac{dE_1}{dt}+c\Big(E_1+\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}+\left\|\frac{\partial v}{\partial t}\right\|^2+\|\nabla\theta\|^2\Big)\leq c'. \tag{4.24}$$

Next, we add ϵ'(4.21) and (4.24), we obtain

$$\frac{dE_2}{dt} + c\Big(E_2 + \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2 + \Big\|\frac{\partial \theta}{\partial t}\Big\|^2\Big) \leq c', \tag{4.25}$$

where $E_2 = E_1 + \epsilon'\|\nabla\theta\|^2$.

Applying Gronwall's lemma to (4.25) yields

$$E_2(t) \leq e^{-ct}E_2(0) + c', \quad \forall\, t \geq 0, \quad c > 0. \tag{4.26}$$

We can deduce from (4.25) and (4.26) that, for every $r > 0$,

$$\begin{aligned}\int_t^{t+r} \Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2 + \Big\|\frac{\partial \theta}{\partial t}\Big\|^2\Big)d\tau \leq ce^{-c't}\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\theta_0\|^2 \\ + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c''(r).\end{aligned} \tag{4.27}$$

We now differentiate equations (4.2) and (4.9) with respect to time, we obtain

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) + f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{4.28}$$

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial t} - \Delta\frac{\partial v}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) - f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = \frac{\partial \theta}{\partial t}, \tag{4.29}$$

with

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial \theta}{\partial t} = 0 \text{ on } \Gamma. \tag{4.30}$$

We then multiply (4.28) by $t\frac{\partial u}{\partial t}$, (4.29) by $t\frac{\partial v}{\partial t}$ and we sum the resulting equations, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(t\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + t\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + ct\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + t\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 + \\ &\quad((f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}), \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t})) + ((f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}), \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t})) \\ &\quad = t\Big(\Big(\frac{\partial \theta}{\partial t}, \frac{\partial v}{\partial t}\Big)\Big) + \frac{1}{2}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big),\end{aligned}$$

which yields, owing to (4.7),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(t\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + t\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big) + ct\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2 + t\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 \\ &\quad \leq c't\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2 + \Big\|\frac{\partial \theta}{\partial t}\Big\|^2\Big) + \frac{1}{2}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2\Big).\end{aligned}$$

Employing the interpolation inequality

$$\Big\|\frac{\partial u}{\partial t}\Big\|^2 \leq c'\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}\Big\|\nabla\frac{\partial u}{\partial t}\Big\|, \ c' > 0,$$

and the Young's inequality, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(t\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+t\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)+ct\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+t\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2\\ \leq c't\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2+\Big\|\frac{\partial\theta}{\partial t}\Big\|^2\Big)+\frac{1}{2}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big).\end{aligned} \tag{4.31}$$

We now apply Gronwall's lemma and use (4.27) to obtain

$$\begin{gathered}\Big\|\frac{\partial u}{\partial t}(t)\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}(t)\Big\|^2\leq\frac{1}{t}e^{c't}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\\ t\in(0,T_0].\end{gathered} \tag{4.32}$$

Moreover, summing (4.28) times $\frac{\partial u}{\partial t}$ and (4.29) times $\frac{\partial v}{\partial t}$, then using (4.7) and an interpolation inequality, we find

$$\frac{d}{dt}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)+\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2\leq c\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)+c'\Big\|\frac{\partial\theta}{\partial t}\Big\|^2. \tag{4.33}$$

Applying Gronwall's lemma and using (4.27) and (4.32), we obtain

$$\begin{gathered}\Big\|\frac{\partial u}{\partial t}(t)\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}(t)\Big\|^2\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\\ t\geq T_0.\end{gathered} \tag{4.34}$$

Furthermore, we multiply (4.9) by $-\Delta\frac{\partial u}{\partial t}$, (4.2) by $-\Delta\frac{\partial v}{\partial t}$, and (4.3) by $-\Delta\theta$ and sum the resulting equations. Then using (4.6) and the continuous embedding $H^2(\Omega)\subset C(\overline{\Omega})$, we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}(\|\Delta u\|^2+\|\Delta v\|^2+\|\nabla\theta\|^2)+c(\|\nabla\frac{\partial v}{\partial t}\|^2+\Big\|\frac{\partial u}{\partial t}\Big\|^2+\|\Delta\theta\|^2)\\ \leq Q(\|u\|_{H^2(\Omega)},\|v\|_{H^2(\Omega)}).\end{gathered} \tag{4.35}$$

Setting $y=\|\Delta u\|^2+\|\Delta v\|^2+\|\nabla\theta\|^2$, we deduce from (4.35) a differential inequality of the form

$$y'\leq Q(y).$$

Let z be the solution of the ordinary differential equation $z'=Q(z)$ with $z(0)=y(0)$.

It follows from the comparison principle that there exists
$T_0=T_0(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)})$ belonging to, say, $(0,\frac{1}{2})$ such that

$$y(t)\leq z(t),t\in[0,T_0],$$

whence

$$\begin{gathered}\|u(t)\|_{H^2(\Omega)}^2+\|v(t)\|_{H^2(\Omega)}^2+\|\theta(t)\|_{H^1(\Omega)}^2\leq Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\\ t\leq T_0.\end{gathered} \tag{4.36}$$

We now rewrite equations (4.2) and (4.9) in the following forms

$$-\Delta u + f(u+v) + f(u-v) = h_u(t), \tag{4.37}$$

$$-\Delta v + f(u+v) - f(u-v) = h_v(t), \tag{4.38}$$

$$\text{with} \quad u = \Delta u = v = \theta = 0 \text{ on } \Gamma, \tag{4.39}$$

for $t \geq T_0$ fixed, where

$$h_u(t) = -(-\Delta)^{-1}\frac{\partial u}{\partial t},$$

and

$$h_v(t) = -\frac{\partial v}{\partial t} + \theta,$$

satisfy, owing to (4.26) and (4.34)

$$\|h_u(t)\| \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}), \quad t \geq T_0, \tag{4.40}$$

and

$$\|h_v(t)\| \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}), \quad t \geq T_0. \tag{4.41}$$

We multiply (4.37) by u and (4.38) by v and we sum the result.
Noting then that $f(s)s \geq -c,\ c \geq 0$, we obtain

$$\|\nabla u\|^2 + \|\nabla v\|^2 \leq c(\|h_u(t)\|^2 + \|h_v(t)\|^2) + c'. \tag{4.42}$$

Next, we multiply (4.37) by $-\Delta u$ and (4.38) by $-\Delta v$, we sum the resulting equations and obtain, using (4.7),

$$\|\Delta u(t)\|^2 + \|\Delta v(t)\|^2 \leq c(\|h_u(t)\|^2 + \|h_v(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2). \tag{4.43}$$

We thus deduce from (4.40)-(4.43) that

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}), \quad t \geq T_0. \tag{4.44}$$

And therefore,

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}), \quad t \geq 0. \tag{4.45}$$

Finally, we can say that (u, v) belongs a priori to $L^\infty(0, T; H^2(\Omega)^2)$, $T > 0$.

Plus, we note from (4.19) that we have

$$\frac{d}{dt}\|\nabla\theta\|^2 + \|\Delta\theta\|^2 \leq \left\|\frac{\partial v}{\partial t}\right\|^2. \tag{4.46}$$

Integrating (4.46) between T_0 and t, then using (4.27) and (4.36), we obtain

$$\|\nabla\theta(t)\|^2 \leq e^{ct} Q(\|u_0\|_{H^2}, \|v_0\|_{H^2}, \|\theta_0\|_{H^1}). \tag{4.47}$$

Combining (4.36) with (4.47), we get

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}), \quad t \geq 0. \tag{4.48}$$

Multiplying then (4.3) by $-\Delta\frac{\partial\theta}{\partial t}$, we obtain

$$\frac{d}{dt}\|\Delta\theta\|^2 + \|\nabla\frac{\partial\theta}{\partial t}\|^2 \leq \|\nabla\frac{\partial v}{\partial t}\|^2. \tag{4.49}$$

Hence $\theta \in L^\infty(0,T;H^2(\Omega))$ and $\frac{\partial\theta}{\partial t} \in L^2(0,T;H^1_0(\Omega))$ a priori.

Our goal now is to find a dissipative estimate :
We start by recalling the inequality (4.20)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2 + \|\nabla\theta\|^2) + \|\Delta u\|^2 + c(\|\Delta v\|^2 + \|\Delta\theta\|^2) \\ \leq c(\|\nabla u\|^2 + \|\nabla v\|^2) + c''\left\|\frac{\partial v}{\partial t}\right\|^2.\end{aligned} \tag{4.50}$$

Using the interpolation inequality

$$\|u\|^2_{H^1(\Omega)} \leq c\|u\|\,\|u\|_{H^2(\Omega)}, \quad c \geq 0,$$

and then the Young's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2 + \|\nabla\theta\|^2) + c(\|\Delta u\|^2 + \|\Delta v\|^2 + \|\Delta\theta\|^2) \\ \leq c'(\|u\|^2 + \|\nabla v\|^2) + c\left\|\frac{\partial v}{\partial t}\right\|^2.\end{aligned} \tag{4.51}$$

Applying now Gronwall's lemma and using(4.27), we find

$$\begin{aligned}\|u(t)\|^2 + \|\nabla v(t)\|^2 + \|\nabla\theta(t)\|^2 \leq e^{c't} Q\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\theta_0\|^2 \\ + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big).\end{aligned} \tag{4.52}$$

In addition, it follows from (4.51) after using (4.27) and (4.52) that

$$\begin{aligned}\int_0^1 (\|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Omega)} + \|\theta\|^2_{H^2(\Omega)})dt \leq Q\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\theta_0\|^2 \\ + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big).\end{aligned} \tag{4.53}$$

Hence, there exists $T \in (0,1)$ such that

$$\begin{aligned}\|u(T)\|^2_{H^2(\Omega)} + \|v(T)\|^2_{H^2(\Omega)} + \|\theta(T)\|^2_{H^2(\Omega)} \leq Q\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\theta_0\|^2 \\ + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big).\end{aligned} \tag{4.54}$$

Repeating the estimates leading to (4.48), but starting from $t = T$ instead of $t = 0$, we have

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \leq e^{ct} Q\Big(\|u(T)\|^2_{H^2(\Omega)} + \|v(T)\|^2_{H^2(\Omega)} + \|\theta(T)\|^2_{H^1(\Omega)}\Big).$$

Then using (4.54), we obtain

$$\|u(1)\|^2_{H^2(\Omega)} + \|v(1)\|^2_{H^2(\Omega)} + \|\theta(1)\|^2_{H^1(\Omega)} \leq Q\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla \theta_0\|^2 + \int_\Omega [F(u_0 + v_0) + F(u_0 - v_0)]dx\Big). \tag{4.55}$$

We now repeat the estimates leading to (4.55), and since our equations are autonomous, we can make a translation in time. We obtain, for $t \geq 1$,

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \leq Q\Big(\|\nabla u(t-1)\|^2 + \|\nabla v(t-1)\|^2 + \|\nabla \theta(t-1)\|^2 + \int_\Omega [F(u(t-1) + v(t-1)) + F(u(t-1) - u(t-1))]dx\Big), \tag{4.56}$$

which yields, owing to (4.26),

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \leq e^{-ct} Q\Big(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2_{H^1(\Omega)} + \int_\Omega [F(u_0 + v_0) + F(u_0 - v_0)]dx\Big) + c', \ c > 0, \ \ t \geq 1. \tag{4.57}$$

Combining the above estimate with (4.48) from 0 to 1, we obtain the dissipative estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \leq e^{-ct} Q\Big(\|u_0\|^2_{H^2(\Omega)} + \|v_0\|^2_{H^2(\Omega)} + \|\theta_0\|^2_{H^1(\Omega)}\Big) + c', \ c > 0, \ \ t \geq 0. \tag{4.58}$$

We then recall the equation (4.49)

$$\frac{d}{dt}\|\Delta\theta\|^2 + \left\|\nabla \frac{\partial \theta}{\partial t}\right\|^2 \leq \left\|\nabla \frac{\partial v}{\partial t}\right\|^2. \tag{4.59}$$

Noting that it follows from (4.27), and (4.33) that

$$\int_t^{t+r} \left(\left\|\nabla \frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla \frac{\partial v}{\partial t}\right\|^2\right)dx \leq e^{-ct} Q\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla \theta_0\|^2 + \int_\Omega [F(u_0 + v_0) + F(u_0 - v_0)]dx\Big) + c'(r), \ c > 0, \ r > 0, \tag{4.60}$$

and from (4.46), (4.57) and (4.60) that

$$\int_t^{t+r} \|\Delta\theta\|^2 dx \leq e^{-ct} Q\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\theta_0\|^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c'(r),\ c>0,\ \ r>0. \tag{4.61}$$

We deduce from (4.59)-(4.61) and, the uniform's Gronwall lemma, (see, e.g. [182]), that

$$\|\theta(t)\|^2_{H^2(\Omega)} \leq e^{-ct} Q\Big(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\theta_0\|^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c'(r),\ c>0,\ r>0. \tag{4.62}$$

Collecting (4.58) and (4.62), we obtain the dissipative estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^2(\Omega)} \leq e^{-ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c'(r),\ c>0,\ r>0. \tag{4.63}$$

Theorem 4.2.2. *Let (u,v,θ) be the solution to the problem with initial data (u_0,v_0,θ_0) obtained in Theorem 2.1. If $(u_0,v_0,\theta_0) \in (H^2(\Omega)\cap H_0^1(\Omega))^2 \times H_0^1(\Omega)$, then there exists a unique solution $(u,v,\theta) \in L^\infty_{loc}(\mathbb{R}^+; (H^2(\Omega)^3)$, and $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \theta}{\partial t}) \in L^\infty_{loc}(\mathbb{R}^+; H^{-1}(\Omega)\times L^2(\Omega)^2) \cap L^2_{loc}(\mathbb{R}^+; H_0^1(\Omega)^3)$.*

Proof.

The proof of the existence (as well as the above a priori estimates) are based on a classical Galerkin scheme as in the previous section and mainly on the estimations (4.48) and (4.49).

We now turn our attention to prove the uniqueness :
We consider two solutions (u_1,v_1,θ_1) and (u_2,v_2,θ_2) to the problem with initial data $(u_{0,1},v_{0,1},\theta_{0,1})$ and $(u_{0,2},v_{0,2},\theta_{0,2})$ respectively.
We set $(u,v,\theta) = (u_1,v_1,\theta_1) - (u_2,v_2,\theta_2)$ and $(u_0,v_0,\theta_0) = (u_{0,1},v_{0,1},\theta_{0,1}) - (u_{0,2},v_{0,2},\theta_{0,2})$.
We then have the following system

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u_1+v_1) - f(u_2+v_2) + f(u_1-v_1) - f(u_2-v_2) = 0, \tag{4.64}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u_1+v_1) - f(u_2+v_2) - f(u_1-v_1) + f(u_2-v_2) = \theta, \tag{4.65}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial v}{\partial t}, \tag{4.66}$$

$$\text{with}\quad u = \Delta u = v = \theta = 0 \text{ on } \Gamma, \tag{4.67}$$

$$u_{|t=0} = u_0,\ v_{|t=0} = v_0,\ \theta_{|t=0} = \theta_0. \tag{4.68}$$

We multiply (4.64) by $\frac{\partial u}{\partial t}$, (4.65) by $\frac{\partial v}{\partial t}$, and (4.66) by θ. We then sum the result to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2+\|\nabla v\|^2+\|\theta\|^2\Big)+\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}\right\|^2+\|\nabla\theta\|^2\\ +((f(u_1+v_1)-f(u_2+v_2),\frac{\partial}{\partial t}(u+v)))\\ +((f(u_1-v_1)-f(u_2-v_2),\frac{\partial}{\partial t}(u-v)))=0.\end{aligned}\tag{4.69}$$

Furthermore,

$$\begin{aligned}&\Big(\Big(f(u_1+v_1)-f(u_2+v_2),\frac{\partial}{\partial t}(u+v)\Big)\Big)\\ &\quad=\left|\left(\left((-\Delta)^{\frac{1}{2}}(f(u_1+v_1)-f(u_2+v_2)),(-\Delta)^{\frac{-1}{2}}\frac{\partial}{\partial t}(u+v)\right)\right)\right|\\ &\quad\le c\left\|\frac{\partial}{\partial t}(u+v)\right\|_{-1}\|\nabla(f(u_1+v_1)-f(u_2+v_2))\|,\ c>0,\end{aligned}\tag{4.70}$$

and similarly

$$\begin{aligned}&\Big(\Big(f(u_1-v_1)-f(u_2-v_2),\frac{\partial}{\partial t}(u-v)\Big)\Big)\\ &\quad\le c'\left\|\frac{\partial}{\partial t}(u-v)\right\|_{-1}\|\nabla(f(u_1-v_1)-f(u_2-v_2))\|,\ c'>0.\end{aligned}\tag{4.71}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2+\|\nabla v\|^2+\|\theta\|^2\Big)+c\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+c'\left\|\frac{\partial v}{\partial t}\right\|^2+\|\nabla\theta\|^2\\ &\quad\le c(\|\nabla(f(u_1+v_1)-f(u_2+v_2))\|^2+\|\nabla(f(u_1-v_1)-f(u_2-v_2))\|^2).\end{aligned}\tag{4.72}$$

Owing to (4.58), we can see that

$$\begin{aligned}&\|\nabla(f(u_1+v_1)-f(u_2+v_2))\|\\ &=\|\nabla\Big(\int_0^1 f'(u_1+v_1+s(u_2+v_2-u_1-v_1))ds(u-v)\Big)\|\\ &\le\|\int_0^1 f'(u_1+v_1+s(u_2+v_2-u_1-v_1))ds\nabla(u-v)\|\\ &\quad+\|(u-v)\int_0^1 f''(u_1+v_1+s(u_2+v_2-u_1-v_1))(\nabla(u_1+v_1)\\ &\quad+s\nabla(u_2+v_2-u_1-v_1))ds\|\\ &\le Q(\|\nabla(u-v)\|+\|\,|u-v|\,|\nabla(u_1+v_1)|\,\|+\|\,|u-v|\,|\nabla(u_2+v_2)|\,\|)\\ &\le Q(\|\nabla u\|+\|\nabla v\|).\end{aligned}\tag{4.73}$$

In the same way,

$$\|\nabla(f(u_1-v_1)-f(u_2-v_2))\|\le Q(\|\nabla u\|+\|\nabla v\|),\tag{4.74}$$

where

$$Q = Q(\|u_{0,1}\|_{H^2(\Omega)}, \|u_{0,2}\|_{H^2(\Omega)}, \|v_{0,1}\|_{H^2(\Omega)}, \|v_{0,2}\|_{H^2(\Omega)}, \|\theta_{0,1}\|_{H^2(\Omega)}, \|\theta_{0,2}\|_{H^2(\Omega)}).$$

We deduce from (4.69)-(4.74) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\theta\|^2\Big) + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \|\nabla\theta\|^2 \\ &\qquad \leq Q(\|\nabla u\|^2 + \|\nabla v\|^2). \end{aligned} \tag{4.75}$$

Now using Gronwall's lemma, we obtain

$$\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + \|\theta(t)\|^2 \leq e^{Qt}(\|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\theta_0\|^2), \tag{4.76}$$

whence the uniqueness (taking $(u_0, v_0, \theta_0) = (0, 0, 0)$), as well as the continuous dependence with respect to the initial data.

□

4.2.3 Global and Exponential attractors

We set $\Phi = (H^2(\Omega) \cap H_0^1(\Omega))^3$.
Note that it follows from Theorem 2.2 that we can define the semigroup

$$S(t) : \Phi \longrightarrow \Phi$$

$$(u_0, v_0, \theta_0) \longrightarrow (u(t), v(t), \theta(t)),$$

where (u, v, θ) is the unique solution to our system.

Theorem 4.2.3. *The semigroup $S(t)$ associated with (4.1)-(4.5) possesses a bounded absorbing set $\mathcal{B}_0$ in Φ such that, for every bounded set $B \subset \Phi$, there exists $t_0 = t_0(B) \geq 0$ such that $t \geq t_0$ implies $S(t)B \subset \mathcal{B}_0$. Therefore, $S(t)$ is a dissipative semigroup.*

It comes directly from (4.26) and (4.63).

Remark 4.2.1. *We can assume, without loss of generality, that $\mathcal{B}_0$ is positively invariant by $S(t)$, i.e., $S(t)\mathcal{B}_0 \subset \mathcal{B}_0, \forall\, t \geq 0$.*

Theorem 4.2.4. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M} \subset \mathcal{B}_0$, i.e.,*
(i) $\mathcal{M}$ is compact in $H^1(\Omega)^2 \times L^2(\Omega)$;
(ii) $\mathcal{M}$ is positively invariant, which means that $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall\, t \geq 0$;
(iii) $\mathcal{M}$ has a finite fractal dimension in $H^1(\Omega)^2 \times L^2(\Omega)$;
(iv) $\mathcal{M}$ attracts exponentially fast the bounded subsets of Φ : $\forall\, B \subset \Phi$ bounded,
$dist_{H^1(\Omega)^2 \times L^2(\Omega)}(S(t)B, \mathcal{M}) \leq Q(\|B\|_\Phi)e^{-ct}, c > 0,\ t \geq 0$, where the constant c is independent of B and $dist_{H^1(\Omega)^2 \times L^2(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$dist_{H^1(\Omega)^2 \times L^2(\Omega)}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{H^1(\Omega)^2 \times L^2(\Omega)}.$$

Proof.

Here, we assume that the initial conditions are in the bounded absorbing set $\mathcal{B}_0$. To complete the proof, we need an asymptotic smoothing property on the difference of two solutions, a Hölder estimate with respect to space and time, and a compactness estimate of the solution. These are the key tools to construct exponential attractors (see [60]-[62], [72, 150, 151]).

The Hölder estimate is as follows

$$\begin{aligned}
&\|u(t_1)-u(t_2)\|_{H^1(\Omega)}+\|v(t_1)-v(t_2)\|_{H^1(\Omega)}+\|\theta(t_1)-\theta(t_2)\|_{L^2(\Omega)}\\
&\qquad\le c(\|\nabla(u(t_1)-u(t_2))\|+\|\nabla(v(t_1)-v(t_2))\|+\|(\theta(t_1)-\theta(t_2))\|)\\
&\qquad\le c\Big(\Big\|\int_{t_1}^{t_2}\nabla\frac{\partial u}{\partial t}d\tau\Big\|+\Big\|\int_{t_1}^{t_2}\nabla\frac{\partial v}{\partial t}d\tau\Big\|+\Big\|\int_{t_1}^{t_2}\frac{\partial\theta}{\partial t}d\tau\Big\|\Big)\\
&\qquad\le c|t_1-t_2|^{\frac{1}{2}}\Big|\int_{t_1}^{t_2}\Big(\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2+\Big\|\frac{\partial\theta}{\partial t}\Big\|^2\Big)d\tau\Big|^{\frac{1}{2}}.
\end{aligned}$$

We note that it follows from (4.27), (4.32), (4.33) and the fact that the initial conditions are in a bounded absorbing set that

$$\int_{t_1}^{t_2}\Big(\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2+\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2\Big)d\tau\le c, \tag{4.77}$$

where c only depends on $\mathcal{B}_0$ and $T\ge T_0$ such that $t_1,\ t_2\ \in[T_0,T]$.

Moreover, it follows from (4.75) and (4.76) that

$$\int_0^t\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2+\|\nabla\theta\|^2\Big)dx\le ce^{Qt}(\|\nabla u_0\|^2+\|\nabla v_0\|^2+\|\theta_0\|^2)\le c'', \tag{4.78}$$

where c'' only depends on $\mathcal{B}_0$.

Plus, it follows from (4.21), (4.52), and (4.78) that

$$\int_{t_1}^{t_2}\Big\|\frac{\partial\theta}{\partial t}\Big\|^2dx\le c, \tag{4.79}$$

where c only depends on $\mathcal{B}_0$.

Therefore, we have

$$\|u(t_1)-u(t_2)\|_{H^1(\Omega)}+\|v(t_1)-v(t_2)\|_{H^1(\Omega)}+\|\theta(t_1)-\theta(t_2)\|_{L^2(\Omega)}\le c|t_1-t_2|^{\frac{1}{2}}, \tag{4.80}$$

where c only depends on $\mathcal{B}_0$, and $t_1,t_2\ \in\ [T_0,T]$, where $T\in\mathbb{R}^+$.

We now want to find a compactness estimate :

First, we differentiate (4.64) and (4.65) with respect to time, we multiply the resulting equations by $(t-T_0)\frac{\partial u}{\partial t}$ and $(t-T_0)\frac{\partial v}{\partial t}$ respectively, where T_0 is the same as before and we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+(t-T_0)\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)+(t-T_0)\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2\\
&\quad+(t-T_0)\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2\leq c(t-T_0)\Big(\Big\|\frac{\partial u}{\partial t}\Big\|^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)\\
&\quad+\frac{1}{2}\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\frac{1}{2}\Big\|\frac{\partial v}{\partial t}\Big\|^2-(t-T_0)\Big(\Big(\frac{\partial\theta}{\partial t},\frac{\partial v}{\partial t}\Big)\Big)\\
&\quad+(t-T_0)\int_\Omega\Big(|f'(u_1+v_1)-f'(u_2+v_2)|\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big|\frac{\partial}{\partial t}(u_2+v_2)\Big|\Big)dx\\
&\quad+(t-T_0)\int_\Omega\Big(|f'(u_1-v_1)-f'(u_2-v_2)|\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big|\frac{\partial}{\partial t}(u_2-v_2)\Big|\Big)dx.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_\Omega\Big(|f'(u_1+v_1)-f'(u_2+v_2)|\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big|\frac{\partial}{\partial t}(u_2+v_2)\Big|\Big)dx\\
&\qquad\leq c\int_\Omega\Big(|u+v|\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big|\frac{\partial}{\partial t}(u_2+v_2)\Big|\Big)dx\\
&\qquad\leq c\|\nabla(u+v)\|\Big\|\nabla\frac{\partial}{\partial t}(u+v)\Big\|\,\Big\|\frac{\partial}{\partial t}(u_2+v_2)\Big\|,
\end{aligned}\tag{4.81}$$

using Hölder inequality and the continuous embeddings $H^1(\Omega)\subset L^3(\Omega)$ and $H^1(\Omega)\subset L^6(\Omega)$.

Similarly,

$$\begin{aligned}
&\int_\Omega\Big(|f'(u_1-v_1)-f'(u_2-v_2)|\Big|\frac{\partial}{\partial t}(u+v)\Big|\Big|\frac{\partial}{\partial t}(u_2-v_2)\Big|\Big)dx\\
&\qquad\leq c\|\nabla(u-v)\|\Big\|\nabla\frac{\partial}{\partial t}(u+v)\Big\|\,\Big\|\frac{\partial}{\partial t}(u_2-v_2)\Big\|.
\end{aligned}\tag{4.82}$$

Owing then to a proper interpolation inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+(t-T_0)\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)+c(t-T_0)\Big(\Big\|\nabla\frac{\partial u}{\partial t}\Big\|^2\\
&\quad+\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2\Big)\leq c(t-T_0)\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2+\Big\|\frac{\partial\theta}{\partial t}\Big\|^2\Big)\\
&\quad+\frac{1}{2}\Big(\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2\Big)+c'(t-T_0)\|\nabla(u+v)\|^2\Big\|\frac{\partial}{\partial t}(u_2+v_2)\Big\|^2\\
&\quad+c'(t-T_0)\|\nabla(u-v)\|^2\Big\|\frac{\partial}{\partial t}(u_2-v_2)\Big\|^2.
\end{aligned}\tag{4.83}$$

Noting also that it follows from (4.27), (4.33), and (4.34) that

$$\int_{T_0}^{t}\left(\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\left\|\nabla\frac{\partial v}{\partial t}\right\|^2\right)dx\le ce^{c't},\ t\ \ge\ T_0,$$

where the constants only depend on $\mathcal{B}_0$. Hence we deduce that

$$\int_{T_0}^{t}\left(\left\|\frac{\partial u_2}{\partial t}\right\|^2+\left\|\frac{\partial v_2}{\partial t}\right\|^2\right)dx\le ce^{c't},\ t\ \ge\ T_0, \tag{4.84}$$

for $(u,v)=(u_2,v_2)$ and the constants only depend on $\mathcal{B}_0$.

Furthermore, applying Gronwall's lemma on (4.83) over (T_0,t) and owing to (4.76), (4.78), (4.79) and (4.84), we obtain

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}(t)\right\|^2\le ce^{c't}(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}+\|\theta_0\|^2),\quad t\ge 1, \tag{4.85}$$

where the constants only depend on $\mathcal{B}_0$.

Next, we rewrite equations (4.64) and (4.65) in the following forms

$$-\Delta u=\tilde{h}_u(t), \tag{4.86}$$

and

$$-\Delta v=\tilde{h}_v(t), \tag{4.87}$$

where $u=\Delta u=v=\theta=0$ on Γ, for $t\ge 1$ fixed,

and

$$\tilde{h}_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t}-\Big(f(u_1+v_1)-f(u_2+v_2)\Big)-\Big(f(u_1-v_1)-f(u_2-v_2)\Big), \tag{4.88}$$

$$\tilde{h}_v(t)=-\frac{\partial v}{\partial t}-\Big(f(u_1+v_1)-f(u_2+v_2)\Big)-\Big(f(u_2-v_2)-f(u_1-v_1)\Big)+\theta, \tag{4.89}$$

satisfy, owing to (4.76) and (4.85),

$$\|\tilde{h}_u(t)\|^2\le ce^{c't}(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}+\|\theta_0\|^2),\ t\ge 1, \tag{4.90}$$

and

$$\|\tilde{h}_v(t)\|^2\le ce^{c't}(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}+\|\theta_0\|^2),\ t\ge 1, \tag{4.91}$$

where the constants only depend on $\mathcal{B}_0$.

Multiplying now (4.86) by $-\Delta u$ and (4.87) by $-\Delta v$, we obtain

$$\|\Delta u(t)\|\le\|\tilde{h}_u(t)\|,\ t\ge 1,$$

and

$$\|\Delta v(t)\| \leq \|\tilde{h}_v(t)\|, \ t \geq 1,$$

whence

$$\begin{gathered}\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2), \\ t \geq 1,\end{gathered} \tag{4.92}$$

where the constants only depend on $\mathcal{B}_0$.

We also multiply (4.66) by $-(t - T_0)\Delta\theta$ and find

$$\frac{d}{dt}((t - T_0)\|\nabla\theta\|^2) + (t - T_0)\|\Delta\theta\|^2 \leq c(t - T_0)\Big(\Big\|\nabla\frac{\partial v}{\partial t}\Big\|^2 + \|\nabla\theta\|^2\Big). \tag{4.93}$$

We find combining (4.83) and (4.93) then applying Gronwall's lemma (applied over (T_0, t) ; note that $T_0 \leq 1$) and using (4.76)-(4.79), and (4.84), we find

$$\begin{gathered}\Big\|\frac{\partial u}{\partial t}(t)\Big\|^2_{-1} + \Big\|\frac{\partial v}{\partial t}(t)\Big\|^2 + \|\nabla\theta(t)\|^2 \leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2), \\ t \geq 1,\end{gathered} \tag{4.94}$$

where the constants only depend on $\mathcal{B}_0$.

At the end, we can see from (4.92) and (4.94), that

$$\begin{gathered}\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} + \|\theta_0\|^2), \\ t \geq 1,\end{gathered} \tag{4.95}$$

And the result follows from (4.76), (4.80), and (4.95). □

Moreover, we can deduce from Theorem 4.1 and standard results the

Corollary 4. *The semigroup $S(t)$ possesses the finite dimensional global attractor $\mathcal{A} \subset \mathcal{B}_0$.*

Remark 4.2.2. *The global attractor $\mathcal{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathcal{A} = \mathcal{A}, \ \forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity ; that's why, it's important in the study of the asymptotic behavior of the system. Furthermore, the finite dimensionality means, roughly speaking, that, even though the initial phase space is infinite dimensional, the reduced dynamics can be described by a finite number of parameters. We refer the reader to [7, 151, 182] for more details and discussions on this.*

Remark 4.2.3. *An exponential attractor is expected to be more robust than a global attractor under perturbations. And that's because the rate of attraction of the global attractor is slow and it's very hard to estimate it with respect to the physical parameters of the problem in general. Therefore, global attractors may change drastically under small perturbations. However, the rate of attraction of exponential attractors is considerably fast (an exponential rate) and that gives them some kind of resilience. We refer the reader to [59] and [151] for more details on this subject.*

4.3 Part II : The Type III Law

The classical Fourier law presented in the previous section has one essential drawback, that is, it predicts that thermal signals propagate with an infinite speed, which violates causality (the so-called 'paradox of heat conduction', see, e.g. [47]). That's why, several modifications of the classical Fourrier law have been proposed in the literature to correct this unrealistic feature, leading to a second order in time equation for the temperature.
In particular, in [146], the authors considered

$$\left(1+\eta\frac{\partial}{\partial t}\right)q=-\nabla\theta,\ \eta>0,$$

assuming the classical Fourrier law

$$q=-\nabla\theta,$$

which leads to

$$\eta\frac{\partial^2\theta}{\partial t^2}+\frac{\partial\theta}{\partial t}-\Delta\theta=-\eta\frac{\partial^2 v}{\partial t^2}-\frac{\partial v}{\partial t},$$

(see also [107, 108] the Maxwell-Cattaneo law).

On the other side, Green and Naghdi proposed in [96]-[99] an alternative treatment for a thermomechanical theory of deformable media which presents an entropy balance rather than the usual entropy inequality. However, if we restrict our attention to the heat conduction, we recall that three different theories, labelled as type I, type II and type III, were proposed. In particular, the Fourrier law is found when type I is linearized. The linearized versions of the two other theories are described by the following constitutive equations (knowing that we are going to study only the type III in what follows)

$$q=-k\nabla\alpha,\ k>0,\ \text{(Type II)} \tag{4.96}$$

and

$$q=-k\nabla\alpha-k^*\nabla\theta,\ k,\ k^*>0,\ \text{(Type III)}, \tag{4.97}$$

where

$$\alpha(t)=\int_{t_0}^{t}\theta(\tau)d\tau+\alpha_0,\ \ \left(\theta=\frac{\partial\alpha}{\partial t}\right)$$

is called the thermal displacement variable. These theories were well studied in the recent years and, particularly, a special interest was given to the qualitative study of the solutions (see e.g. [168]-[173] for studies concerned with linear thermoelastic theories). In addition, non-linear acceleration waves have been studied for types II and III non-linear thermoelasticity [174] and fluids without energy dissipation [175].

Adding equations (4.96) and (4.97) to the equation

$$\frac{\partial H}{\partial t}+\text{div}\ q=0, \tag{4.98}$$

we obtain the following equations

$$\frac{\partial^2 \alpha}{\partial t^2} - k\Delta\alpha = -\frac{\partial v}{\partial t}$$

for type II and

$$\frac{\partial^2 \alpha}{\partial t^2} - k^* \frac{\partial}{\partial t}\Delta\alpha - k\Delta\alpha = -\frac{\partial v}{\partial t}$$

for type III.

4.3.1 Setting of the New Problem

We consider the following initial and boundary value problem (for simplicity, we take $k = k^* = 1$) :

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \tag{4.99}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = \frac{\partial \alpha}{\partial t}, \tag{4.100}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -\frac{\partial v}{\partial t}, \tag{4.101}$$

$$u = \Delta u = v = \alpha = 0 \text{ on } \Gamma, \tag{4.102}$$

$$u_{|t=0} = u_0,\ v_{|t=0} = v_0,\ \alpha_{|t=0} = \alpha_0,\ \frac{\partial \alpha}{\partial t}\Big|_{t=0} = \alpha_1, \tag{4.103}$$

where Ω is a bounded domain of $\mathbb{R}^N$ ($N = 1, 2$, or 3) with smooth boundary Γ.

We assume that f is of class C^2 and satisfies

$$-c \le F(s) \le f(s)s,\ c \ge 0,\ s \in \mathbb{R}, \tag{4.104}$$

where

$$F(s) = \int_0^s f(\tau)d\tau.$$

We also assume that

$$f(0) = 0,\ \ f'(s) \ge -c',\ \ s \in \mathbb{R},\ c' \ge 0. \tag{4.105}$$

4.3.2 A Priori Estimates

First, we rewrite (4.99) in the equivalent form

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u+v) + f(u-v) = 0 \tag{4.106}$$

We multiply (4.106) by $\frac{\partial u}{\partial t}$, (4.100) by $\frac{\partial v}{\partial t}$ and have, summing the results,

$$\begin{aligned}\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \frac{d}{dt}\Big(\frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla v\|^2 + \int_\Omega [F(u+v)+F(u-v)]dx\Big)\\ = \Big(\Big(\frac{\partial \alpha}{\partial t}, \frac{\partial v}{\partial t}\Big)\Big).\end{aligned} \tag{4.107}$$

We then multiply (4.101) by $\frac{\partial \alpha}{\partial t}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \|\nabla\alpha\|^2\Big) + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 = -\Big(\Big(\frac{\partial \alpha}{\partial t}, \frac{\partial v}{\partial t}\Big)\Big). \tag{4.108}$$

Thus, summing (4.107) and (4.108), we find

$$\begin{aligned}\frac{d}{dt}\Big(\|\nabla u\|^2 + \|\nabla v\|^2 + 2\int_\Omega [F(u+v)+F(u-v)]dx + \left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \|\nabla\alpha\|^2\Big)\\ + 2\Big(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2\Big) = 0.\end{aligned} \tag{4.109}$$

We now multiply (4.106) by u, (4.100) by v and sum the results to have, owing to (4.104),

$$\frac{d}{dt}(\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 + \int_\Omega [F(u+v)+F(u-v)]dx \le c\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2. \tag{4.110}$$

Then, we multiply (4.101) by α and obtain

$$\frac{d}{dt}\Big(\|\nabla\alpha\|^2 + 2(\frac{\partial \alpha}{\partial t}, \alpha)\Big) + c\|\nabla\alpha\|^2 \le c'\left\|\frac{\partial v}{\partial t}\right\|^2 + c''\left\|\frac{\partial \alpha}{\partial t}\right\|^2. \tag{4.111}$$

We sum (4.109), ϵ_1(4.110) and ϵ_2(4.111), where ϵ_1 and $\epsilon_2 > 0$ are chosen small enough so that

$$\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \epsilon_2\Big(\|\nabla\alpha\|^2 + 2(\frac{\partial \alpha}{\partial t}, \alpha)\Big) \ge c\Big(\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \|\nabla\alpha\|^2\Big),\ c > 0, \tag{4.112}$$

$$2 - \epsilon_1 c - \epsilon_2 c' > 0, \tag{4.113}$$

$$2 - \epsilon_2 c'' > 0, \tag{4.114}$$

and have an inequality of the form

$$\frac{dE_3}{dt} + E_3 + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2 + \|\nabla\alpha\|^2 \le 0, \tag{4.115}$$

where

$$E_3 = \epsilon_1\|u\|_{-1}^2 + \epsilon_1\|v\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + 2\int_\Omega [F(u+v) + F(u-v)]dx + (1+\epsilon_2)\|\nabla\alpha\|^2 + 2\epsilon_2((\frac{\partial\alpha}{\partial t}, \alpha)) + \left\|\frac{\partial\alpha}{\partial t}\right\|^2,$$

satisfies

$$E_3 \geq c\Big(\|\nabla u\|^2 + \|\nabla v\|^2 + \int_\Omega [F(u+v) + F(u-v)]dx + \|\nabla\alpha\|^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2\Big) - c', \quad c > 0. \tag{4.116}$$

In particular, we deduce from (4.115) the dissipative estimate

$$E_3(t) \leq ce^{-c't}\Big(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\alpha_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\alpha_0\|^2 + \|\alpha_1\|^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big), \ c' > 0, \ t \geq 0. \tag{4.117}$$

Furthermore, for every $r > 0$,

$$\int_t^{t+r}\Big(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + \left\|\frac{\partial v}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2 + \|\nabla\alpha\|^2\Big)dx \leq ce^{-c't}\Big(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\alpha_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla\alpha_0\|^2 + \|\alpha_1\|^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big) + c''(r), \quad c' > 0, \ t \geq 0. \tag{4.118}$$

Next, We multiply (4.99) by $\frac{\partial u}{\partial t}$, (4.100) by $-\Delta\frac{\partial v}{\partial t}$, and (4.101) by $-\Delta\frac{\partial\alpha}{\partial t}$, we obtain

$$\frac{d}{dt}\|\Delta u\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \leq Q(\|u\|_{H^2(\Omega)}^2 + \|v\|_{H^2(\Omega)}^2), \tag{4.119}$$

$$\frac{d}{dt}\|\Delta v\|^2 + \left\|\nabla\frac{\partial v}{\partial t}\right\|^2 \leq Q(\|u\|_{H^2(\Omega)}^2 + \|v\|_{H^2(\Omega)}^2) + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2, \tag{4.120}$$

and

$$\frac{d}{dt}\Big(\|\Delta\alpha\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\Big) + c\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2 \leq \left\|\nabla\frac{\partial v}{\partial t}\right\|^2, \tag{4.121}$$

Summing (4.119), (4.120), and (4.121) yields

$$\frac{d}{dt}\Big(\|\Delta u\|^2 + \|\Delta v\|^2 + \|\Delta\alpha\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\Big) + \left\|\frac{\partial u}{\partial t}\right\|^2 + c\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2 \leq Q(\|u\|_{H^2(\Omega)}^2 + \|v\|_{H^2(\Omega)}^2). \tag{4.122}$$

In particular, we set

$$y = \|\Delta u\|^2 + \|\Delta v\|^2 + \|\Delta\alpha\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2.$$

Thus, we deduce from (4.122) a differential inequality of the form

$$y' \le Q(y).$$

Let z be the solution of the ordinary differential equation

$$z' = Q(z),$$

with $z(0) = y(0)$. It follows from the comparison principle that there exists $T_0 = T_0(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)})$ belonging to, say, $(0, \frac{1}{2})$ such that

$$y(t) \le z(t), \quad t \in [0, T_0],$$

whence

$$\begin{aligned} \|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\alpha(t)\|^2_{H^2(\Omega)} + \left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2_{H^1(\Omega)} \\ \le Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}), \quad t \le T_0. \end{aligned} \tag{4.123}$$

Therefore $(u, v, \alpha, \frac{\partial \alpha}{\partial t}) \in L^\infty(0, T_0; H^2(\Omega)^3 \times H^1(\Omega))$ a priori.

We now differentiate (4.100) and (4.106) with respect to time to find, owing to (4.101)

$$(-\Delta)^{-1}\frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) + f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = 0, \tag{4.124}$$

$$\frac{\partial}{\partial t}\frac{\partial v}{\partial t} - \Delta\frac{\partial v}{\partial t} + f'(u+v)(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t}) - f'(u-v)(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}) = -\frac{\partial v}{\partial t} + \Delta\frac{\partial \alpha}{\partial t} + \Delta\alpha, \tag{4.125}$$

$$\text{with} \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial \alpha}{\partial t} = 0 \text{ on } \Gamma. \tag{4.126}$$

Multiplying then (4.124) by $t\frac{\partial u}{\partial t}$, (4.125) by $t\frac{\partial v}{\partial t}$, and (4.101) by $t\frac{\partial \alpha}{\partial t}$ to obtain, summing the three resulting inequalities and using (4.105) and an interpolation inequality,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + t\left\|\frac{\partial v}{\partial t}\right\|^2 + t\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + t\|\nabla\alpha\|^2\right) + ct\left(\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial v}{\partial t}\right\|^2\right. \\ &\left. + \left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2\right) \le \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + \frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|^2 + \frac{1}{2}\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + \frac{1}{2}\|\nabla\alpha\|^2 \\ &+ c'\left(t\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + t\left\|\frac{\partial v}{\partial t}\right\|^2 + t\left\|\frac{\partial \alpha}{\partial t}\right\|^2 + t\|\nabla\alpha\|^2\right). \end{aligned} \tag{4.127}$$

We apply Gronwall's lemma to (4.127) and we use (4.118), we get

$$\begin{aligned} &\left\|\frac{\partial u}{\partial t}(t)\right\|^2_{-1} + \left\|\frac{\partial v}{\partial t}(t)\right\|^2 + \left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2 + \|\nabla\alpha(t)\|^2 \le \frac{1}{t}Q\Big(\|u_0\|^2_{H^1(\Omega)} \\ &+ \|v_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big), \\ &t \in (0, T_0]. \end{aligned} \tag{4.128}$$

Summing (4.124) times $\frac{\partial u}{\partial t}$, (4.125) times $\frac{\partial v}{\partial t}$, and (4.101) times $\frac{\partial \alpha}{\partial t}$ and interpolating yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}\right\|^2+\left\|\frac{\partial \alpha}{\partial t}\right\|+\|\nabla\alpha\|^2\right)+c\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\\ +\left\|\nabla\frac{\partial v}{\partial t}\right\|^2+c'\left\|\nabla\frac{\partial \alpha}{\partial t}\right\|^2\leq c''\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}\right\|^2+\left\|\frac{\partial \alpha}{\partial t}\right\|^2+\|\nabla\alpha\|^2\right).\end{aligned} \tag{4.129}$$

Applying then Gronwall's lemma to (4.129) from t to T_0 and using (4.128), we obtain

$$\begin{aligned}&\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}(t)\right\|^2+\left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2+\|\nabla\alpha(t)\|^2\leq e^{ct}Q\Big(\|u_0\|_{H^1(\Omega)}^2\\ &+\|v_0\|_{H^1(\Omega)}^2+\|\alpha_0\|_{H^1(\Omega)}^2+\|\alpha_1\|^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big),\\ &t\geq T_0.\end{aligned} \tag{4.130}$$

Combining (4.128) and (4.130) yields

$$\begin{aligned}&\left\|\frac{\partial u}{\partial t}(t)\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}(t)\right\|^2+\left\|\frac{\partial \alpha}{\partial t}(t)\right\|^2+\|\nabla\alpha(t)\|^2\leq e^{ct}Q\Big(\|u_0\|_{H^1(\Omega)}^2\\ &+\|v_0\|_{H^1(\Omega)}^2+\|\alpha_0\|_{H^1(\Omega)}^2+\|\alpha_1\|^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big),\\ &t>0.\end{aligned} \tag{4.131}$$

We now rewrite (4.100) and (4.106) in the following forms

$$-\Delta u+f(u+v)+f(u-v)=h_u(t), \tag{4.132}$$

$$-\Delta v+f(u+v)-f(u-v)=h_v(t), \tag{4.133}$$

where $u=\Delta u=v=0$ on $\Gamma,\ t\geq T_0$,

and

$$h_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t} \tag{4.134}$$

$$h_v(t)=\frac{\partial \alpha}{\partial t}-\frac{\partial v}{\partial t}, \tag{4.135}$$

satisfy, owing to (4.130),

$$\|h_u(t)\|\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^1(\Omega)},\|\alpha_1\|),\ t\geq T_0, \tag{4.136}$$

and

$$\|h_v(t)\|\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^1(\Omega)},\|\alpha_1\|),\ t\geq T_0. \tag{4.137}$$

Next, we multiply (4.132) by u, (4.133) by v and sum the result. Then, noting that $f(s)s\geq -c,\ c\geq 0$, we obtain

$$\|\nabla u\|^2+\|\nabla v\|^2\leq c(\|h_u(t)\|^2+\|h_v(t)\|^2)+c'. \tag{4.138}$$

Multiplying now (4.132) by $-\Delta u$ and (4.133) by $-\Delta v$, summing the result, then noting that $f'(s) \geq -c'$, we obtain

$$\|\Delta u(t)\|^2 + \|\Delta v(t)\|^2 \leq c(\|h_u(t)\|^2 + \|h_v(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2). \tag{4.139}$$

We finally deduce that

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} \leq ce^{c't} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^1(\Omega)},\\ \|\alpha_1\|), \quad t \geq T_0.\end{aligned} \tag{4.140}$$

Moreover, we multiply (4.101) by $-\Delta\frac{\partial\alpha}{\partial t}$ and find

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta\alpha\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right) + c\left\|\Delta\frac{\partial\alpha}{\partial t}\right\|^2 \leq \left\|\nabla\frac{\partial v}{\partial t}\right\|^2. \tag{4.141}$$

Integrating now (4.129) between t and T_0 and using (4.130), we obtain

$$\begin{aligned}\int_t^{T_0} \left(\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial v}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right)dx \leq e^{ct} Q\Big(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)}\\ + \|\alpha_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big), \ t \geq T_0.\end{aligned} \tag{4.142}$$

Applying then Gronwall's lemma to (4.141) and using (4.142), we find

$$\begin{aligned}\|\Delta\alpha(t)\|^2 + \left\|\nabla\frac{\partial\alpha}{\partial t}(t)\right\|^2 \leq ce^{c't} Q\Big(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^1(\Omega)}\\ + \|\alpha_1\|^2 + \int_\Omega [F(u_0+v_0) + F(u_0-v_0)]dx\Big), \ t \geq T_0.\end{aligned} \tag{4.143}$$

Thus, we deduce from (4.123), (4.140), and (4.143) that

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)} + \|v(t)\|^2_{H^2(\Omega)} + \|\alpha(t)\|^2_{H^2(\Omega)} + \left\|\frac{\partial\alpha}{\partial t}(t)\right\|^2_{H^1(\Omega)}\\ \leq e^{ct} Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)}, \|\alpha_1\|_{H^1(\Omega)}), \quad t \geq 0.\end{aligned} \tag{4.144}$$

4.3.3 Dissipative Estimate

We start by multiplying (4.106) by $-\Delta u$, (4.100) by $-\Delta v$, and (4.101) by $\frac{\partial\alpha}{\partial t}$, we obtain, after summing the result and using (4.105),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|u\|^2 + \|\nabla v\|^2 + \|\nabla\alpha\|^2 + \left\|\frac{\partial\alpha}{\partial t}\right\|^2\right) + \|\Delta u\|^2 + c\|\Delta v\|^2 + c'\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\\ \leq c''(\|\nabla u\|^2 + \|\nabla v\|^2) + \left\|\frac{\partial v}{\partial t}\right\|^2.\end{aligned} \tag{4.145}$$

Using an interpolation inequality and Young's inequality, we have

$$\|u\|^2_{H^1(\Omega)} \leq c\|u\| \, \|u\|_{H^2(\Omega)} \leq \frac{1}{2}\|\Delta u\|^2 + c\|u\|^2, \ c \geq 0.$$

Therefore

$$\begin{aligned}\frac{d}{dt}\Big(\|u\|^2+\|\nabla v\|^2+\|\nabla\alpha\|^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2\Big)+c\Big(\|\Delta u\|^2+\|\Delta v\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2\Big)\\ \leq c'(\|u\|^2+\|\nabla v\|^2)+\Big\|\frac{\partial v}{\partial t}\Big\|^2.\end{aligned}\tag{4.146}$$

It follows from (4.146), using (4.118) and (4.131), that

$$\begin{aligned}\int_0^1(\|u\|^2_{H^2(\Omega)}+\|v\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2_{H^1(\Omega)})dt\leq Q\Big(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}\\ +\|\alpha_0\|^2_{H^1(\Omega)}+\|\alpha_1\|^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big).\end{aligned}\tag{4.147}$$

Hence, there exists $T\in(0,1)$ such that

$$\begin{aligned}\|u(T)\|^2_{H^2(\Omega)}+\|v(T)\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}(T)\Big\|^2_{H^1(\Omega)}\leq Q\Big(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}\\ +\|\alpha_0\|^2_{H^1(\Omega)}+\|\alpha_1\|^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big).\end{aligned}\tag{4.148}$$

Repeating now the estimates leading to (4.144) from $t=T$ instead of $t=0$, we have

$$\begin{aligned}\|u(1)\|^2_{H^2(\Omega)}+\|v(1)\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}(1)\Big\|^2_{H^1(\Omega)}\leq Q\Big(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}\\ +\|\alpha_0\|^2_{H^1(\Omega)}+\|\alpha_1\|^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big).\end{aligned}\tag{4.149}$$

Similarly, repeating the estimates leading to (4.149) also, we have, for $t\geq 1$,

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)}+\|v(t)\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}(t)\Big\|^2_{H^1(\Omega)}\leq Q\Big(\|u(t-1)\|^2_{H^1(\Omega)}+\|v(t-1)\|^2_{H^1(\Omega)}\\ +\|\alpha(t-1)\|^2_{H^1(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}(t-1)\Big\|^2+\int_\Omega[F(u(t-1)+v(t-1))\\ +F(u(t-1)-v(t-1))]dx\Big).\end{aligned}\tag{4.150}$$

Owing to (4.117), the above estimate yields

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)}+\|v(t)\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}(t)\Big\|^2_{H^1(\Omega)}\leq e^{-ct}Q\Big(\|u_0\|^2_{-1}+\|v_0\|^2+\|\nabla u_0\|^2+\|\nabla v_0\|^2\\ +\|\nabla\alpha_0\|^2+\int_\Omega[F(u_0+v_0)+F(u_0-v_0)]dx\Big),\ c>0,\ t\geq 1.\end{aligned}\tag{4.151}$$

Combining (4.151) with (4.144) for $t=0$, we get

$$\begin{aligned}\|u(t)\|^2_{H^2(\Omega)}+\|v(t)\|^2_{H^2(\Omega)}+\Big\|\frac{\partial\alpha}{\partial t}(t)\Big\|^2_{H^1(\Omega)}\leq e^{-ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)},\|\alpha_1\|_{H^1(\Omega)}),\\ c>0,\ t\geq 0.\end{aligned}\tag{4.152}$$

Moreover, we note that it follows from (4.122) and (4.152) that

$$\frac{d}{dt}\Big(\|\Delta u\|^2+\|\Delta v\|^2+\|\Delta\alpha\|^2+\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2\Big)+\Big\|\frac{\partial u}{\partial t}\Big\|^2+\Big\|\Delta\frac{\partial\alpha}{\partial t}\Big\|^2 \leq e^{-ct}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)},\|\alpha_1\|_{H^1(\Omega)}). \tag{4.153}$$

Multiplying then (4.101) by $-\Delta\alpha$, we have

$$\frac{d}{dt}\Big(\|\Delta\alpha\|^2+2((\Delta\alpha,\frac{\partial\alpha}{\partial t}))\Big)+c\|\Delta\alpha\|^2\leq c\Big\|\frac{\partial v}{\partial t}\Big\|^2+2\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2. \tag{4.154}$$

We now sum (4.115) and ϵ_3(4.146), where $\epsilon_3>0$ is small enough so that

$$\frac{dE_4}{dt}+c(E_4+\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2)+c'\|\Delta u\|^2+\epsilon_3\|\Delta v\|^2+c''\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2\leq 0, \tag{4.155}$$

where

$$E_4=E_3+\epsilon_3(\|u\|^2+\|\nabla v\|^2+c\|\nabla\alpha\|^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|^2).$$

We also sum (4.153), ϵ_4(4.154), and (4.155), where $\epsilon_4>0$ is small enough so that

$$\Big\|\frac{\partial\alpha}{\partial t}\Big\|_{H^2(\Omega)}+\epsilon_4(\|\Delta\alpha\|^2+2((\Delta\alpha,\frac{\partial\alpha}{\partial t})))\geq c(\Big\|\frac{\partial\alpha}{\partial t}\Big\|_{H^2(\Omega)}+\|\Delta\alpha\|^2),$$

and

$$\frac{dE_5}{dt}+c\Big(E_5+\Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2+\Big\|\frac{\partial v}{\partial t}\Big\|^2+\Big\|\frac{\partial\alpha}{\partial t}\Big\|_{H^2(\Omega)}^2\Big)\leq e^{-c't}Q(\|u_0\|_{H^2(\Omega)},\|v_0\|_{H^2(\Omega)}, \|\alpha_0\|_{H^2(\Omega)},\|\alpha_1\|_{H^1(\Omega)}),\quad c>0,\ t\geq 0, \tag{4.156}$$

where

$$E_5=E_4+\|\Delta u\|^2+\|\Delta v\|^2+(1+\epsilon_4)\|\Delta\alpha\|^2+\epsilon_4((\Delta\alpha,\frac{\partial\alpha}{\partial t}))+\Big\|\nabla\frac{\partial\alpha}{\partial t}\Big\|^2.$$

We finally deduce from (4.156) the dissipative estimate

$$\|u(t)\|_{H^2(\Omega)}^2+\|v(t)\|_{H^2(\Omega)}^2+\|\alpha(t)\|_{H^2(\Omega)}^2+\Big\|\frac{\partial\alpha}{\partial t}(t)\Big\|_{H^1(\Omega)}^2\leq e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|v_0\|_{H^2(\Omega)},\|\alpha_0\|_{H^2(\Omega)},\|\alpha_1\|_{H^1(\Omega)}),\ c>0,\ t\geq 0. \tag{4.157}$$

4.3.4 Global and Exponential Attractors

Theorem 4.3.1. *For every* $(u_0,v_0,\alpha_0,\alpha_1)\in(H^2(\Omega)\cap H_0^1(\Omega))^3\times H_0^1(\Omega)$, *(4.99)-(4.103) possesses a unique solution* $(u,v,\alpha,\frac{\partial\alpha}{\partial t})$ *such that* $(u,v,\alpha)\in L^\infty(\mathbb{R}^+;\mathrm{H}_0^1(\Omega)^3)$ $\cap\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{H}^2(\Omega)^3)$, $(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t})\in L^\infty(\mathbb{R}^+;\mathrm{H}^{-1}(\Omega)\times\mathrm{L}^2(\Omega))\cap\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^+;\mathrm{H}_0^1(\Omega)^2)$ *and* $\frac{\partial\alpha}{\partial t}\in L^\infty(\mathbb{R}^+;\mathrm{H}_0^1(\Omega))\cap$ $\mathrm{L}^2(\mathbb{R}^+;\mathrm{H}^2(\Omega))$.

Proof.

The proof of existence is based on the a priori estimates derived in the previous section and on, e.g., a standard Galerkin scheme similar to the proof of Theorem 2.1 based mainly on (4.109) and (4.144). Therefore, we will only be proving the uniqueness.

Let $(u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha}{\partial t}^{(1)})$ and $(u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha}{\partial t}^{(2)})$ be two solutions of (4.99)-(4.103) with initial data $(u_0^{(1)}, v_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)})$ and $(u_0^{(2)}, v_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)})$ respectively.

We set

$$(u, v, \alpha) = (u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha}{\partial t}^{(1)}) - (u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha}{\partial t}^{(2)}),$$

and

$$(u_0, v_0, \alpha_0) = (u_0^{(1)}, v_0^{(1)}, \alpha_0^{(1)}, \alpha_1^{(1)}) - (u_0^{(2)}, v_0^{(2)}, \alpha_0^{(2)}, \alpha_1^{(2)}).$$

Hence (u, v, α) satisfy

$$(-\Delta)^{-1}\frac{\partial u}{\partial t} - \Delta u + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) + f(u^{(1)} - v^{(1)}) - f(u^{(2)} - v^{(2)}) = 0, \tag{4.158}$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) - f(u^{(1)} - v^{(1)}) + f(u^{(2)} - v^{(2)}) = \frac{\partial \alpha}{\partial t}, \tag{4.159}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial v}{\partial t}, \tag{4.160}$$

$$u = \Delta u = v = \alpha = 0 \text{ on } \Gamma, \tag{4.161}$$

$$u_{|t=0} = u_0,\ v_{|t=0} = v_0,\ \alpha_{|t=0} = \alpha_0,\ \frac{\partial \alpha}{\partial t}\Big|_{t=0} = \alpha_1. \tag{4.162}$$

First, we multiply (4.158) by $\frac{\partial u}{\partial t}$, (4.159) by $\frac{\partial v}{\partial t}$, (4.160) by $\frac{\partial \alpha}{\partial t}$, and we sum the resulting equations to obtain, using the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla \alpha\|^2 + \Big\|\frac{\partial \alpha}{\partial t}\Big\|^2\Big) + \Big\|\frac{\partial u}{\partial t}\Big\|_{-1}^2 + \Big\|\frac{\partial v}{\partial t}\Big\|^2 \\ &\quad + \Big\|\nabla \frac{\partial \alpha}{\partial t}\Big\|^2 \leq Q\Big(\|u_0^{(1)}\|_{H^2(\Omega)}, \|u_0^{(2)}\|_{H^2(\Omega)}, \|v_0^{(1)}\|_{H^2(\Omega)}, \|v_0^{(2)}\|_{H^2(\Omega)}, \\ &\|\alpha_0^{(1)}\|_{H^2(\Omega)}, \|\alpha_0^{(2)}\|_{H^2(\Omega)}, \|\alpha_1^{(1)}\|_{H^1(\Omega)}, \|\alpha_1^{(2)}\|_{H^1(\Omega)}\Big). \end{aligned} \tag{4.163}$$

It thus follows from (4.163) and Gronwall's lemma that

$$\begin{aligned} \|u(t)\|_{H^1(\Omega)}^2 + \|v(t)\|_{H^1(\Omega)}^2 + \|\alpha(t)\|_{H^1(\Omega)}^2 + \Big\|\frac{\partial \alpha}{\partial t}(t)\Big\|^2 \leq e^{Qt}\Big(\|u_0\|_{H^1(\Omega)}^2 \\ + \|v_0\|_{H^1(\Omega)}^2 + \|\alpha_0\|_{H^1(\Omega)}^2 + \|\alpha_1\|^2\Big),\ t \geq 0, \end{aligned} \tag{4.164}$$

hence the uniqueness, as well as the continuity (with respect to the $H_0^1(\Omega)^3 \times L^2(\Omega)$-norm) with respect to the initial data. □

It follows from the previous theorem that we can define the family of solving operators

$$S(t) : \Phi' \longrightarrow \Phi'$$

$$(u_0, v_0, \alpha_0, \alpha_1) \longrightarrow (u(t), v(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t)), \ t \geq 0,$$

where $(u, v, \alpha, \frac{\partial \alpha}{\partial t})$ is the unique solution to our system and $\Phi' = (H^2(\Omega)^3 \times H_0^1(\Omega))$.

Furthermore, this family of solving operators forms a continuous semigroup (for the $H^1(\Omega)^3 \times L^2(\Omega)$-topology), i.e. $S(0) = Id$ and $S(t+\tau) = S(t) \circ S(\tau), \forall\, t, \tau \geq 0$.

Theorem 4.3.2. *The semigroup associated with (4.99)-(4.103) possesses a bounded absorbing set $\mathcal{B}_1$ in Φ'.*

The result follows directly from (4.157).

We will be searching now for exponential attractors :

The classical methods of constructing an exponential attractor do not seem to work here because the term $\frac{\partial^2 \alpha}{\partial t^2}$ creates difficulties in the calculations. Therefore, as in [143] and [144], we decompose the solution $(u, v, \alpha, \frac{\partial \alpha}{\partial t})$ to our system with initial data $(u_0, v_0, \alpha_0, \alpha_1)$ into the sums

$$u(t) = w_1(t) + w_2(t), \tag{4.165}$$

$$v(t) = q_1(t) + q_2(t), \tag{4.166}$$

and

$$\alpha(t) = r_1(t) + r_2(t), \tag{4.167}$$

where w_1, q_1, and r_1 are solutions to

$$(-\Delta)^{-1} \frac{\partial w_1}{\partial t} - \Delta w_1 = 0, \tag{4.168}$$

$$\frac{\partial q_1}{\partial t} - \Delta q_1 = \frac{\partial r_1}{\partial t}, \tag{4.169}$$

$$\frac{\partial^2 r_1}{\partial t^2} - \Delta \frac{\partial r_1}{\partial t} - \Delta r_1 = -\frac{\partial q_1}{\partial t}, \tag{4.170}$$

$$w_1 = \Delta w_1 = q_1 = r_1 = 0 \text{ on } \Gamma, \tag{4.171}$$

$$w_{1|t=0} = u_0, \ q_{1|t=0} = v_0, \ r_{1|t=0} = \alpha_0, \ \frac{\partial r_1}{\partial t}\Big|_{t=0} = \alpha_1. \tag{4.172}$$

and w_2, q_2, and r_2 are solutions to

$$(-\Delta)^{-1} \frac{\partial w_2}{\partial t} - \Delta w_2 + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) + f(u^{(1)} - v^{(1)}) - f(u^{(2)} - v^{(2)}) = 0, \tag{4.173}$$

$$\frac{\partial q_2}{\partial t} - \Delta q_2 + f(u^{(1)} + v^{(1)}) - f(u^{(2)} - v^{(2)}) - f(u^{(1)} - v^{(1)}) + f(u^{(2)} - v^{(2)}) = \frac{\partial r_2}{\partial t}, \tag{4.174}$$

$$\frac{\partial^2 r_2}{\partial t^2} - \Delta \frac{\partial r_2}{\partial t} - \Delta r_2 = -\frac{\partial q_2}{\partial t}, \tag{4.175}$$

$$w_2 = \Delta w_2 = q_2 = r_2 = 0 \text{ on } \Gamma, \tag{4.176}$$

$$w_{2|t=0} = 0,\ q_{2|t=0} = 0,\ r_{2|t=0} = 0,\ \frac{\partial r_2}{\partial t}\Big|_{t=0} = 0. \tag{4.177}$$

Repeating the same calculations leading to (4.156), but considering now (4.168)-(4.172), where $f = 0$, we obtain

$$\frac{dE_6}{dt} + cE_6 \leq 0,\ c \geq 0, \tag{4.178}$$

where

$$\begin{aligned} E_6 &= \|\Delta w_1\|^2 + \|\Delta q_1\|^2 + (1+\epsilon_4)\|\Delta r_1\|^2 + \epsilon_1\|w_1\|_{-1}^2 + \epsilon_1\|q_1\|^2 + \|\nabla w_1\|^2 \\ &+ (1+\epsilon_3)\|\nabla q_1\|^2 + (1+\epsilon_2+c\epsilon_3)\|\nabla r_1\|^2 + 2\epsilon_2((\frac{\partial r_1}{\partial t}, r_1)) + (1+\epsilon_3)\left\|\frac{\partial r_1}{\partial t}\right\|^2 \\ &+ ((\Delta r_1, \frac{\partial r_1}{\partial t}))| + \epsilon_3\|w_1\|^2 + \left\|\nabla \frac{\partial r_1}{\partial t}\right\|^2. \end{aligned} \tag{4.179}$$

Using Gronwall's lemma on (4.178), we obtain

$$E_6(t) \leq e^{-ct} E_6(0). \tag{4.180}$$

On the other side,

$$\begin{aligned} &\|u(t_1) - u(t_2)\|_{H^1(\Omega)} + \|v(t_1) - v(t_2)\|_{H^1(\Omega)} + \|\alpha(t_1) - \alpha(t_2)\|_{H^1(\Omega)} \\ &+ \left\|\frac{\partial \alpha}{\partial t}(t_1) - \frac{\partial \alpha}{\partial t}(t_2)\right\| \leq c\Big(\|\nabla(u(t_1) - u(t_2))\| + \|\nabla(v(t_1) - v(t_2))\| \\ &+ \|\nabla(\alpha(t_1) - \alpha(t_2))\| + \left\|\frac{\partial \alpha}{\partial t}(t_1) - \frac{\partial \alpha}{\partial t}(t_2)\right\|\Big) \\ &\leq c\Big(\left\|\int_{t_1}^{t_2} \nabla \frac{\partial u}{\partial t} d\tau\right\| + \left\|\int_{t_1}^{t_2} \nabla \frac{\partial v}{\partial t} d\tau\right\| + \left\|\int_{t_1}^{t_2} \nabla \frac{\partial \alpha}{\partial t} d\tau\right\| + \left\|\int_{t_1}^{t_2} \frac{\partial^2 \alpha}{\partial t^2} d\tau\right\|\Big) \\ &\leq c|t_1 - t_2|^{\frac{1}{2}} \left|\int_{t_1}^{t_2} \Big(\left\|\nabla \frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla \frac{\partial v}{\partial t}\right\|^2 + \left\|\nabla \frac{\partial \alpha}{\partial t}\right\|^2 + \left\|\frac{\partial^2 \alpha}{\partial t^2}\right\|^2\Big) d\tau\right|^{\frac{1}{2}}. \end{aligned} \tag{4.181}$$

And it follows from (4.118), (4.129), (4.130) and the fact that the initial data are in a bounded absorbing set, that

$$\int_{t_1}^{t_2} \Big(\left\|\nabla \frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla \frac{\partial v}{\partial t}\right\|^2 + \left\|\nabla \frac{\partial \alpha}{\partial t}\right\|\Big)^2 \leq c, \tag{4.182}$$

where c only depends on $\mathcal{B}_1$ and $T \geq T_0$ such that $t_1, t_2 \in [T_0, T]$..

Moreover, looking at the equation (4.160), we can see that

$$\left\|\frac{\partial^2 \alpha}{\partial t^2}\right\| \leq \left\|\Delta \frac{\partial \alpha}{\partial t}\right\| + \left\|\frac{\partial v}{\partial t}\right\| + \|\Delta \alpha\|. \tag{4.183}$$

Thus, it follows from (4.141), (4.144), (4.182), and (4.183) that

$$\int_{t_1}^{t_2} \left\| \frac{\partial^2 \alpha}{\partial t^2} \right\| \leq c, \tag{4.184}$$

where c only depends on $\mathcal{B}_1$.

Whence, we have

$$\begin{aligned} \|u(t_1) - u(t_2)\|_{H^1(\Omega)} + \|v(t_1) - v(t_2)\|_{H^1(\Omega)} + \|\alpha(t_1) - \alpha(t_2)\|_{H^1(\Omega)} \\ + \left\| \frac{\partial \alpha}{\partial t}(t_1) - \frac{\partial \alpha}{\partial t}(t_2) \right\| \leq c' |t_1 - t_2|^{\frac{1}{2}}. \end{aligned} \tag{4.185}$$

It's important to note that w_2, q_2, and r_2 verify the same system as u, v, and α so they satisfy the same estimations found in the previous sections.

Furthermore, we differentiate (4.173) and (4.174) with respect to time and we multiply the resulting equations by $(t - T_0)\frac{\partial w_2}{\partial t}$ and $(t - T_0)\frac{\partial q_2}{\partial t}$, respectively. Then, we sum the result with $(t - T_0)\frac{\partial r_2}{\partial t}$ times (4.175), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left((t - T_0)\left\| \frac{\partial w_2}{\partial t} \right\|_{-1}^2 + (t - T_0)\left\| \frac{\partial q_2}{\partial t} \right\|^2 + (t - T_0)\left\| \frac{\partial r_2}{\partial t} \right\|^2 + (t - T_0)\|\nabla r_2\|^2\right) \\ &\quad + c(t - T_0)\left(\left\| \nabla \frac{\partial w_2}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial q_2}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial r_2}{\partial t} \right\|^2\right) \\ &\quad \leq c'(t - T_0)\left(\left\| \frac{\partial w_2}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial q_2}{\partial t} \right\|^2 + \|\nabla r_2\|^2\right) + c''(t - T_0)\left(\left\| \frac{\partial u}{\partial t} \right\|^2 \right. \\ &\quad \left. + \left\| \frac{\partial v}{\partial t} \right\|^2\right) + \frac{1}{2}\left(\left\| \frac{\partial w_2}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial q_2}{\partial t} \right\|^2 + \left\| \frac{\partial r_2}{\partial t} \right\|^2 + \|\nabla r_2\|^2\right) \\ &\quad + c'''(t - T_0)\|\nabla(u + v)\|\left\| \nabla \frac{\partial w_2}{\partial t} \right\| \left\| \frac{\partial}{\partial t}(u_2 + v_2) \right\| \\ &\quad + c'''(t - T_0)\|\nabla(u + v)\|\left\| \nabla \frac{\partial q_2}{\partial t} \right\| \left\| \frac{\partial}{\partial t}(u_2 + v_2) \right\| \\ &\quad + c'''(t - T_0)\|\nabla(u - v)\|\left\| \nabla \frac{\partial w_2}{\partial t} \right\| \left\| \frac{\partial}{\partial t}(u_2 - v_2) \right\| \\ &\quad + c'''(t - T_0)\|\nabla(u - v)\|\left\| \nabla \frac{\partial q_2}{\partial t} \right\| \left\| \frac{\partial}{\partial t}(u_2 - v_2) \right\|, \end{aligned} \tag{4.186}$$

similarly to (4.83).

It follows from (4.129), (4.130) that

$$\int_{T_0}^{t} \left(\left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial v}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2\right) \leq c e^{c' t}, \ t \geq T_0, \tag{4.187}$$

where the constants only depend on $\mathcal{B}_1$.

We can thereby deduce that

$$\int_{T_0}^{t}\left(\left\|\frac{\partial u_2}{\partial t}\right\|^2+\left\|\frac{\partial v_2}{\partial t}\right\|^2\right)dx\leq ce^{c't},\ t\geq T_0, \tag{4.188}$$

for $(u,v)=(u_2,v_2)$ and the constants only depend on $\mathcal{B}_1$.

It also follows from (4.163) and (4.164) that

$$\begin{aligned}\int_0^t\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial v}{\partial t}\right\|^2+\left\|\nabla\frac{\partial\alpha}{\partial t}\right\|^2\right)dx\leq c(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}\\ +\|\alpha_0\|^2_{H^1(\Omega)}+\|\alpha_1\|^2),\ t\geq 0.\end{aligned} \tag{4.189}$$

Applying Gronwall's lemma to (4.186) over (T_0,t) and owing to (4.164)and (4.187)-(4.189), we obtain

$$\begin{aligned}\left\|\frac{\partial w_2}{\partial t}(t)\right\|_{-1}^2+\left\|\frac{\partial q_2}{\partial t}(t)\right\|^2+\left\|\frac{\partial r_2}{\partial t}(t)\right\|^2+\|\nabla r_2(t)\|^2\leq ce^{c't}(\|u_0\|^2_{H^1(\Omega)}+\\ \|v_0\|^2_{H^1(\Omega)}+\|\alpha_0\|^2_{H^1(\Omega)}+\|\alpha_1\|^2),\ t\geq 1.\end{aligned} \tag{4.190}$$

where the constants only depend on $\mathcal{B}_1$.

We now rewrite equations (4.173) and (4.174) in the following forms

$$-\Delta w_2=\tilde{h}_{w_2}(t), \tag{4.191}$$

and

$$-\Delta q_2=\tilde{h}_{q_2}(t), \tag{4.192}$$

where $w_2=\Delta w_2=q_2=0$ on Γ, for $t\geq 1$ fixed, and

$$\tilde{h}_{w_2}(t)=-(-\Delta)^{-1}\frac{\partial w_2}{\partial t}-\left(f(u^{(1)}+v^{(1)})-f(u^{(2)}+v^{(2)})\right)-\left(f(u^{(1)}-v^{(1)})-f(u^{(2)}-v^{(2)})\right), \tag{4.193}$$

$$\tilde{h}_{q_2}(t)=-\frac{\partial q_2}{\partial t}+\frac{\partial r_2}{\partial t}-\left(f(u^{(1)}+v^{(1)})-f(u^{(2)}+v^{(2)})\right)+\left(f(u^{(1)}-v^{(1)})-f(u^{(2)}-v^{(2)})\right), \tag{4.194}$$

satisfy, owing to (4.164) and (4.190),

$$\|\tilde{h}_{w_2}(t)\|^2\leq ce^{Qt}(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}+\|\alpha_0\|^2_{H^1(\Omega)}+\|\alpha_1\|^2),\ t\geq 1, \tag{4.195}$$

and

$$\|\tilde{h}_{q_2}(t)\|^2\leq ce^{Qt}(\|u_0\|^2_{H^1(\Omega)}+\|v_0\|^2_{H^1(\Omega)}+\|\alpha_0\|^2_{H^1(\Omega)}+\|\alpha_1\|^2),\ t\geq 1. \tag{4.196}$$

where the constants only depend on $\mathcal{B}_1$.

Multiplying now (4.191) by $-\Delta w_2$ and (4.192) by $-\Delta q_2$, we obtain

$$\|\Delta w_2(t)\|\leq\|\tilde{h}_{w_2}(t)\|,\ t\geq 1,$$

and

$$\|\Delta q_2(t)\| \leq \|\tilde{h}_{q_2}(t)\|,\ t \geq 1.$$

Whence

$$\begin{aligned}\|w_2(t)\|^2_{H^2(\Omega)} + \|q_2(t)\|^2_{H^2(\Omega)} \leq ce^{Qt}(\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^1(\Omega)} \\ + \|\alpha_1\|^2),\ t \geq 1.\end{aligned} \tag{4.197}$$

Next, we multiply (4.175) by $-(t-T_0)\Delta\frac{\partial r_2}{\partial t}$ and find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big((t-T_0)\Big\|\nabla\frac{\partial r_2}{\partial t}\Big\|^2 + (t-T_0)\|\Delta r_2\|^2\Big) + (t-T_0)\Big\|\Delta\frac{\partial r_2}{\partial t}\Big\|^2 \\ &\quad\leq c(t-T_0)\Big\|\nabla\frac{\partial r_2}{\partial t}\Big\|^2 + c(t-T_0)\Big\|\nabla\frac{\partial q_2}{\partial t}\Big\|^2 + \frac{1}{2}\Big\|\nabla\frac{\partial r_2}{\partial t}\Big\|^2 + \frac{1}{2}\|\Delta r_2\|^2.\end{aligned} \tag{4.198}$$

Then applying Gronwall's lemma over (T_0, t) and using (4.143) and (4.187), we obtain

$$\Big\|\nabla\frac{\partial r_2}{\partial t}(t)\Big\|^2 + \|\Delta r_2(t)\|^2 \leq ce^{c't}. \tag{4.199}$$

Finally, summing (4.197) and (4.199), we obtain

$$\begin{aligned}\|w_2(t)\|^2_{H^2(\Omega)} + \|q_2(t)\|^2_{H^2(\Omega)} + \|r_2(t)\|^2_{H^2(\Omega)} + \Big\|\frac{\partial r_2}{\partial t}(t)\Big\|^2_{H^1(\Omega)} \leq ce^{Qt}(\|u_0\|^2_{H^1(\Omega)} \\ + \|v_0\|^2_{H^1(\Omega)} + \|\alpha_0\|^2_{H^1(\Omega)} + \|\alpha_1\|^2),\ t \geq 1,\end{aligned} \tag{4.200}$$

where the constants only depend on $\mathcal{B}_1$.

And the existence of exponential attractors follows from ((4.164)), (4.180), (4.185), and (4.200) (see [59]- [60]). Therefore, we have

Theorem 4.3.3. *The semigroup $S(t)$ possesses an exponential attractor $\mathcal{M}' \subset \mathcal{B}_1$, i.e.,*
(i) $\mathcal{M}'$ is compact in $H^1(\Omega)^3 \times L^2(\Omega)$;
(ii) $\mathcal{M}'$ is positively invariant, $S(t)\mathcal{M} \subset \mathcal{M}',\ \forall\, t \geq 0$;
(iii) $\mathcal{M}'$ has a finite fractal dimension in $H^1(\Omega)^3 \times L^2(\Omega)$;
(iv) $\mathcal{M}'$ attracts exponentially fast the bounded subsets of Φ' :

$\forall\, B \subset \Phi'$ bounded, $dist_{H^1(\Omega)^3\times L^2(\Omega)}(S(t)B, \mathcal{M}') \leq Q(\|B\|_{\Phi'})e^{-ct}$,
$c > 0,\ t \geq 0$, where the constant c is independent of B and $dist_{H^1(\Omega)^3\times L^2(\Omega)}$ denotes the Hausdorff semidistance between sets defined by

$$dist_{H^1(\Omega)^3\times L^2(\Omega)}(A, B) = \sup_{a\in A}\inf_{b\in B}\|a-b\|_{H^1(\Omega)^3\times L^2(\Omega)}.$$

Consequently, we deduce from standard results the

Corollary 5. *The semigroup $S(t)$ possesses the finite dimensional global attractor $\mathcal{A}'' \subset \mathcal{B}_1$.*

Chapitre 5

Conclusion et Perspectives

L'objectif de cette thèse a été d'étudier le comportement asymptotique d'un système couplé d'une équation de Cahn-Hilliard avec une équation d'Allen-Cahn.
On considère dans un premier temps un terme non linéaire régulier. En particulier, on a démontré l'existence et l'unicité de la solution puis l'existence d'un attracteur exponentiel et par conséquence l'existence d'un attracteur global de dimension fractale finie.
Ensuite, la deuxième étude du système était faite avec un terme singulier (logarithmique en particulier). Ici aussi, on a pu démontrer l'existence et l'unicité de la solution qui ont permi de définir un semigroupe à lequel on a trouvé un ensemble borné absorbant. Puis, un attracteur global est trouvé. Enfin, on a établi des résultats de régularité supplémentaire qui ont aidé à démontrer la séparation stricte en dimension une.
La troisième partie de la thèse était consacrée à l'étude du même système mais couplé cette fois avec une troisième équation : l'équation de la chaleur.
Cette partie est divisée elle même en deux :
Dans la première partie, l'équation de la chaleur suit la loi classique de Fourrier. À ce niveau là, on a démontré l'existence et l'unicité de la solution pour pouvoir trouver un attracteur exponentiel et par conséquence un attracteur global de dimension finie.
Dans la deuxième partie, l'équation de la chaleur suit une loi thermodynamique de type III. Dans ce cas, on a aussi réussi à trouver un attracteur global de dimension finie à l'aide d'un attracteur exponentiel construit après pouvoir démontrer en premier lieu l'existence et l'unicité de la solution du système associé.

Pour aller plus loin, notamment en vue de l'étude asymptotique, l'analyse numérique et des simulations des modèles considérés peuvent être envisagées.
Dans le cas du potentiel logarithmique, des estimations additionnelles peuvent être trouvées pour aboutir à une séparation stricte en dimension deux. En dimension trois, cette question reste ouverte.
Concernant l'équation de la chaleur, on peut étudier le cas où cette équation est gouvernée par la loi de Maxwell-Cattanéo ou d'autres lois de type thermomécaniques.
De même, il est intéressant d'étudier le système posé avec des conditions de bord de type Neuman et des conditions de bord dynamiques.

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