Global discharging methods for coloring problems in graphs

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Contents

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[1]Depuis, on a quand même réussi à faire deux projets exclusivement par mail, alors je ne m'inquiète pas trop pour notre collaboration :)

[2]"Objection à la science ; le monde ne mérite pas d'être compris."

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Je continuerai à chercher.

Quelques mots de français

Ce chapitre consiste en une traduction littérale de l'introduction vers le français.

Quelques mots sur la coloration de graphes

Cette thèse explore des questions qui s'inscrivent dans le cadre de la *théorie des graphes* : on s'intéresse à des propriétés structurelles (planarité, faible densité, sous-graphes interdits) qui permettent de partitionner les éléments du graphe en peu d'ensembles sans interaction interne (par exemple la partition de sommets en ensembles indépendants, la partition d'arêtes en couplace, etc). Il y a deux dimensions à cette thèse : les problèmes qu'on étudie, et la méthode principale que l'on utilise pour les résoudre. Les problèmes de partition de graphe sont habituellement présentés en termes de *colorations*, où chaque ensemble de la partition se voit attribuer une couleur, et où les contraintes sont traduites dans ce cadre (par exemple deuw sommets adjacents ne doivent pas être coloriés pareil).

Les premiers résultats de coloration de graphe concernent principalement les graphes planaires, en terme de coloration de cartes. En essayant de colorier une carte des comtés d'Angleterre, Guthrie remarqua en 1852 que quatre couleurs suffisaient pour colorier cette carte de façon à ce que deux régions partageant une frontière reçoivent des couleurs différentes. Cela devint la Conjecture des Quatre Couleurs : tout graphe planaire est 4-coloriable (implicitement, de façon à ce que deux régions partageant une frontière reçoivent des couleurs différentes). En effet, le graphe obtenu en représentant chaque région avec un sommet et chaque frontière avec une arête entre les deux régions incidentes est planaire - en supposant qu'aucune région n'est séparée en deux morceaux ou plus. Réciproquement, tout graphe planaire correspond à une certaine carte : il y a une dualité entre cartes et graphes planaires.

Dans une tentative de s'approcher de cette conjecture, Wernicke prouva en 1905 un lemme structurel sur les graphes planaires. Dans ce but, il mit au point un outil pour prouver ce lemme par contradiction. En supposant qu'un contre-exemple existe, l'objectif est de montrer qu'un plongement planaire de ce contre-exemple ne peut satisfaire ce qu'on appelle la formule d'Euler, que tout plongement planaire satisfait. Dans ce but, il suffit de montrer qu'il y a au moins autant d'arêtes que de sommets et de faces. L'idée était d'attribuer en conséquence des poids dans le plongement (c'est à dire qu'on attribue un poids de 1 à chaque arête, et un poids de -1 à chaque sommet ou face), et de prouver qu'en définissant des règles pour redistribuer localement les poids dans le graphe (à poids total constant), on peut garantir que chaque élément (sommet, arête, face) a un poids positif. On peut immédiatement conclure : il y donc au moins autant d'arêtes que de sommets et de faces. Par conséquent, à partir d'arguments locaux on peut extraire une information globale. C'était là le premier exemple d'une *méthode de déchargement*[3]

Bien que très simple, la méthode de déchargement s'est montrée décisive pour le Théorème des Quatre Couleurs, finalement prouvé en 1976 par Appel et Haken. En réalité, elle s'est montrée extrêmement puissante dans certains contextes, en particulier lorsqu'il s'agit de problèmes de coloration dans des graphes avec une forte information structurelle (par exemple les graphes planaires ou plus généralement les graphes peu denses). Par conséquent, de nombreuses variations et astuces sont désormais venues enrichir la version de 1905.

Cependant, l'étude de la coloration de graphes a cessé depuis longtemps d'être restreinte au cadre étroit des graphes planaires. Dans l'ensemble des graphes, la question n'est plus de savoir si quatre couleurs suffisent. En effet, on peut considérer un graphe consistant en un nombre arbitraire de sommets tous adjacents les uns aux autres ; il aura besoin d'autant de couleurs qu'il contient de sommets. Pour cette raison, les seules bornes que l'on peut espérer dans le cas général seront fonctions d'autres paramètres. La première borne évidente pour le nombre de couleurs nécessaires (ce à quoi on fera désormais référence en termes de *nombre chromatique*, dénoté habituellement χ) est le nombre de sommets dans le graphe ; peu importent les adjacences dans le graphe, en coloriant chaque sommet différemment, on s'assure d'éviter que deux sommets adjacents aient la même couleur. Un algorithme naïf pour colorier le graphe est de choisir itérativement un sommet non colorié, de le colorier différemment de tous ses voisins déjà coloriés (en évitant si possible d'utiliser une nouvelle couleur), et ce jusqu'à ce que tous les sommets soient coloriés. Par conséquent, si tout sommet est adjacent à au plus, disons, 99 sommets, alors on sait immédiatement que le nombre chromatique est au plus 100. Pour un meilleur algorithme de coloration, on peut prendre soin de choisir un bon ordre sur les sommets. Par exemple, supposons que tout sommet a au plus 99 voisins, mais que l'un d'entre eux a au plus 98 voisins. Alors, si le graphe est connexe (c'est à dire en un seul morceau : on peut aller de n'importe quel sommet à n'importe quel autre en suivant les arêtes), on peut traiter les sommets par distance décroissante au sommet spécial (en tranchant arbitrairement en cas d'égalité). Tous les sommets sauf le dernier ont au moins un voisin non colorié lorsqu'ils sont traités, donc il y a au plus 98 choix de couleurs à éviter pour chaque. Dès lors, on sait que le graphe est en réalité 99-coloriable, et pas seulement 100-coloriable. En réalité, indépendamment de s'il y a un sommet de plus petit degré, on peut obtenir la même conclusion pour tous les graphes connexes à n ($n \geq 2$) sommets sauf un (si n est pair) ou deux (si n est impair).

Ceci peut être exprimé de façon plus formelle. Le degré maximum d'un graphe G, que l'on note $\Delta(G)$, est le nombre maximum de voisins qu'un sommet peut avoir. On sait de par l'algorithme naïf que $(\Delta(G) + 1)$ couleurs suffisent pour tout graphe G, et ceci peut être amélioré la plupart du temps : tout graphe connexe G est $\Delta(G)$-coloriable à moins que G soit une clique ou un cycle impair.

On peut définir une extension de la coloration de sommets, où au lieu d'avoir les mêmes k couleurs disponibles pour tout le graphe, chaque sommet a son propre ensemble de k couleurs et doit être colorié avec l'une d'entre elles. Étant donné une attribution de listes L de k couleurs à chaque sommet du graphe, on dit que le graphe

[3] Il semble que le nom provienne effectivement d'un parallèle avec les réseaux électriques [Hay77].

est L-coloriable s'il y a une coloration propre telle que tout sommet est colorié depuis sa liste attribuée. Si le graphe est L-coloriable pour toute telle attribution, alors on dit que le graphe est *k-choisissable*. De façon peut-être contre-intuitive, il n'est pas plus facile de colorier un graphe où les couleurs disponibles pour des sommets voisins peuvent différer (voir la Figure 0.1).

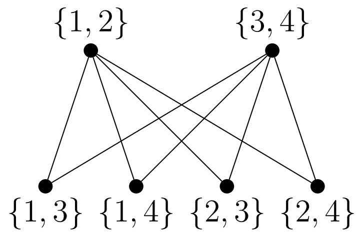


Figure 0.1: Un exemple de graphe 2-coloriable qui n'est pas 2-choisissable.

L'exemple de la Figure 0.1 peut en réalité être généralisé de façon à montrer que pour tout k, un graphe 2-coloriable n'est pas nécessairement k-choisissable. Cependant, il y a effectivement des graphes pour lesquels calculer le nombre chromatique par liste est aisé. Par exemple, tous les cycles pairs sont 2-choisissables.

D'autres variations de colorations peuvent être introduites. Par exemple, on peut essayer de colorier les sommets avec la contrainte supplémentaire que deux sommets avec un voisin commun ne doivent pas recevoir la même couleur. C'est alors une *coloration de carré*, et le nombre chromatique correspondant est noté χ^2. De façon similaire, on peut essayer de colorier les arêtes du graphe, en lieu et place de ses sommets, avec la condition que deux arêtes incidentes à un même sommet doivent recevoir des couleurs différentes. C'est alors une *coloration d'arêtes*. Ces deux colorations peuvent être naturellement étendues aux listes. Dans cette thèse, nous cherchons des conditions suffisantes pour que des graphes soient coloriés au carré ou que leurs arêtes soient coloriées par liste avec peu de couleurs.

Vue d'ensemble

Le Chapitre 1 contient une introduction progressive aux graphes et à la coloration, des définitions et notations impliquées dans cette thèse , ainsi qu'une brève présentation de la méthode de déchargement. Cette thèse suit trois axes principaux. Dans le Chapitre 2, on donne un aperçu illustré des outils de déchargement qui sont utilisés dans ces travaux : des méthodes élégantes que nous appliquons, et des astuces pratiques que nous développons. On essaye en particulier de donner une intution des méthodes de déchargement dites globales. Dans ce but, on considère une coloration exotique, à savoir la "coloration d'arêtes voisin-distinguante", où l'on recherche une coloration d'arêtes propre, avec la condition supplémentaire que deux voisins ne doivent pas être incidents au même ensemble de couleurs. En plus de son histoire propre, cette coloration présente la caractéristique intéressante de bien se comporter par rapport à toutes les méthodes et outils désirés, ce qui en fait un candidat naturel pour l'illustration.

On traite ensuite de deux cas particulier ; la coloration d'arêtes par liste (Chapitre 3) et la coloration de carré (Chapitre 4). Dans le cadre de la coloration d'arêtes par liste, on prouve notamment que la List Coloring Conjecture faible est vraie pour les graphes planaires de degré maximum 8 (c'est-à-dire qu'ils sont arête 9-choisissables), ce qui améliore un résultat de Borodin de 1990. Bien que la preuve complète soit longue de 24 pages, par rapport à seulement deux dans le cas de degré maximum au moins 9, elle repose sur une seule idée décisive, qui consiste en un argument de recoloration pour contourner les problèmes de réduction.

Dans le cas de la coloration de carré, l'objectif est de colorier le carré du graphe, ou, de façon équivalente, de colorier à distance 2 le graphe : on recherche une coloration propre des sommets qui satisfait la condition supplémentaire que deux sommets avec un voisin commun ne peuvent recevoir la même couleur. La maille est la longueur d'un plus petit cycle dans le graphe. On s'intéresse principalement aux conditions suffisantes sur la densité d'un graphe pour qu'il soit coloriable avec le nombre minimum de couleurs, ou presque. On généralise la plupart des résultats existants sur les graphes planaires de maille donnée en remplaçant la condition sur la maille par une condition plus générale sur le degré moyen maximum (*mad*) et en abandonnant l'hypothèse de planarité.

De plus, le fait que la maille soit un paramètre discret contrairement au mad signifie qu'on peut obtenir une information plus fine avec le mad. En particulier, on sait que le carré des graphes planaires de maille au moins 7 et de degré maximum suffisamment grand peut être colorié avec le nombre minimum de couleurs, alors que cela peut être faux pour le carré de graphes planaires de maille 6. Les graphes planaires de maille au moins 6 ont un mad strictement inférieur à 3. À l'aide d'un argument de déchargement global, on peut en réalité prouver que l'hypothèse de planarité est, une fois de plus, superflue, et que cette propriété est en fait vraie dès lors que le degré maximum est suffisamment grand par rapport à la différence entre 3 et le mad.

Introduction

A few words on graph coloring

This thesis explores subjects in the field of *graph theory*: it studies structural properties (planarity, sparsity, forbidden subgraphs) that help partition elements of the graph into few sets with no interaction within (e.g. vertex partition into stable sets, edge partition into matchings, etc). There are two dimensions to this thesis: the problems we study, and the main method that we use to solve them. Partition problems in graph are usually discussed in terms of *colorings*, where each set in the partition is assigned a color, and the constraints are translated in that setting (e.g. two adjacent vertices should not be colored the same).

The first results about graph coloring deal mainly with planar graphs in the form of the coloring of maps. While trying to color a map of the counties of England, Guthrie noted in 1852 that four colors were sufficient to color the map so that no regions sharing a common border received the same color. It became known as the Four Color Conjecture, that every planar graph is 4-colorable (implicitely: so that no two adjacent vertices receive the same color). Indeed, the graph obtained by representing each region with a vertex and each border with an edge between the two incident regions is planar - assuming no region is split in two parts or more. Reciprocally, every planar graph corresponds to some map: there is a duality between maps and planar graphs.

In an attempt to get closer to the conjecture, Wernicke proved in 1905 a structural lemma on planar graphs. For this purpose, he introduced a tool to prove the lemma by contradiction. Assuming a counter-example exists, the goal is to prove that a planar embedding of this graph cannot satisfy the so-called Euler's formula, which every planar embedding satisfies. To that purpose, it suffices to prove that there are at least as many edges as there are vertices and faces. The idea was to assign weights accordingly in the embedding (that is, we assign a weight of 1 for every edge, -1 for every vertex and face), and prove that by defining rules to redistribute the weight locally in the graph (with constant total weight), we can ensure that every element (vertex, edge, face) has a non-negative weight. The conclusion immediately follows: there are at least as many edges as there are vertices and faces. Consequently, from local counting arguments we can derive a global fact. This was the first example of a *discharging method*[4]. Albeit very simple, the discharging method proved to be decisive toward the Four Color Theorem, eventually proved in 1976 by Appel and Haken. In fact, it proved itself extremely powerful in some settings, particularly when it comes to coloring problems in graphs with strong structural information (e.g. planar graphs or

[4]Apparently the name does indeed stem from a comparison with electrical networks [Hay77].

more generally sparse graphs). Consequently, many variations and tricks have now enriched the 1905 version.

The study of graph coloring has however long stopped being restricted to the narrow field of planar graphs. In the class of all graphs, the question is no longer whether four colors suffice. Indeed, we can consider a graph made of any amount of vertices all adjacent to each other: it will need just as many colors as it has vertices. Therefore, the only bounds we can hope for in the case of general graphs will be functions of other parameters. The first obvious bound for the number of colors required (which we will refer to from now as the *chromatic number*, usually denoted by χ) is the number of vertices in the graph: no matter the adjacencies in the graph, by coloring each vertex in a different color, we ensure that no two adjacent vertices will receive the same color. A naive algorithm to color the graph is to pick any vertex, color it differently from all its already colored neighbors (avoiding to introduce a new color if possible), and repeat until the whole graph is colored. Consequently, if no vertex is adjacent to more than, say, 99 vertices, then we immediately know that its chromatic number is at most 100. For a better coloring algorithm, we can choose a good order on the vertices. For example, assume that no vertex is adjacent to more than 99 neighbors, but we know that there is one which is adjacent to at most 98. Then, if the graph is connected (i.e. in one piece: we can go from any vertex to any other) then we can pick vertices by decreasing distance to that special vertex (breaking ties arbitrarily). All vertices except for the last one have a neighbor which is not colored yet, so at most 98 colors to avoid. Therefore, we know that the graph is actually 99-colorable, and not merely 100-colorable. In fact, regardless of whether there is a vertex of smaller degree, we can reach the same conclusion for every connected graph on n ($n \geq 2$) vertices except for one (if n is even) or two (if n is odd).

We can express it more formally as follows. The maximum degree of a graph G, denoted $\Delta(G)$, is the maximum number of neighbors a vertex can have. We know from the naive algorithm that $(\Delta(G)+1)$ colors suffice for every graph G, and this can be improved most of the time: every graph G is $\Delta(G)$-colorable unless G is a clique or an odd cycle.

We can define an extension of vertex coloring, where instead of having the same k colors available for the whole graph, every vertex has its own set of k colors and has to be colored from it. Given a list assignment L of k colors to each vertex of the graph, we say that the graph is L-colorable if there is a proper coloring such that every vertex is colored from its assigned list. If the graph is L-colorable for every such assignment, then we say the graph is *k-choosable*.Perhaps counter-intuitively, it is not easier to color a graph where the colors available for neighbors may be different (see Figure 0.2).

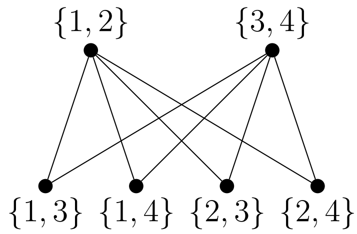


Figure 0.2: An example of a 2-colorable graph which is not 2-choosable.

The example of Figure 0.2 can in fact be generalized so as to show that for every k, a 2-colorable graph may not be k-choosable. However, there are indeed some graphs for which computing the list chromatic number is easy. For example, all even cycles are 2-choosable.

Other coloring variants can be introduced. For example, we can try to color the vertices with the additional constraint that two vertices with a common neighbor may not receive the same color. This is a *square coloring*, and the corresponding chromatic number is denoted by χ^2. Similarly, we can try to color the edges of the graph, instead of the vertices, with the condition that two edges incident to a same vertex must receive different colors. This is an *edge coloring*. Both colorings can naturally be extended to lists. In this thesis, we seek sufficient conditions for graphs to be square colored or list edge colored with few colors.

Global picture

Chapter 1 contains a gentle introduction to graphs and colorings, some definitions and notation involved in this thesis, as well as a brief presentation of the discharging method. This PhD thesis follows three main axes. In Chapter 2, we give an illustrated overview of the discharging tools that are used for this work: nice methods that we apply, and handy tricks that we develop. We try in particular to give an intuition of global discharging arguments. For that purpose, we consider an exotic kind of coloring, namely "adjacent vertex distinguishing edge coloring", where we seek a proper edge coloring with the extra property that no two neighbors are incident with the same set of colors. Beside its own history, the interesting characteristic of that coloring is that it behaves nicely with regard to all desired methods and tools, which makes it a natural candidate for illustration.

We then discuss two special cases: list edge coloring (Chapter 3) and square coloring (Chapter 4). In the realm of list edge coloring, we most notably prove that the weak List Coloring Conjecture is true for planar graphs of maximum degree 8 (i.e. that they are edge 9-choosable), thus improving over a result of Borodin from 1990. Though the full proof is 24 pages long, compared to a lean two to solve the case of maximum degree at least 9, it rests upon a single decisive idea, which lies in a recoloring argument to get around reducibility problems.

In the case of square coloring, the goal is to color the square of the graph, or, equivalently, to 2-distance color the graph: we look for a proper coloring of a graph that satisfies the additional condition that the same color cannot be assigned to two vertices with a common neighbor. We are mainly interested in sufficient conditions on the sparsity of a graph for it to be colorable with the minimum number of colors or close to it. We generalize most of the existing results on planar graphs with given girth by supplanting the condition on the girth with a more general condition on the maximum average degree (mad) and dropping the planarity hypothesis.

Also, the fact that the girth is a discrete parameter contrary to the mad means that we can obtain more refined information with mad. In particular, it was known that the square of planar graphs with girth at least 7 and sufficiently large maximum degree can be colored with the minimum number of colors, while this can be false for squares of planar graphs with girth 6. It holds that all planar graphs with girth at least 6 have mad less than 3. With a global discharging argument, we can actually prove that planarity is, again, unnecessary, and that this property is in fact true as soon as the maximum degree is sufficiently larger than a function of the gap between 3 and the mad.

Chapter 1

Preliminaries

In this chapter, we recall all the definitions and notation used in the thesis. Section 1.1 contains an illustrated introduction to the notions of graphs and colorings. Section 1.2 contains the additional definitions needed, presented in a colder way. In Section 1.3, we briefly introduce the idea of a discharging argument.

1.1 Basic introduction to graphs and coloring

Graphs

Informally, a graph is a set of points, with lines connecting some pairs of them (see Figure 1.1).

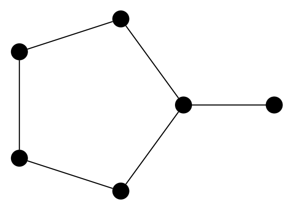
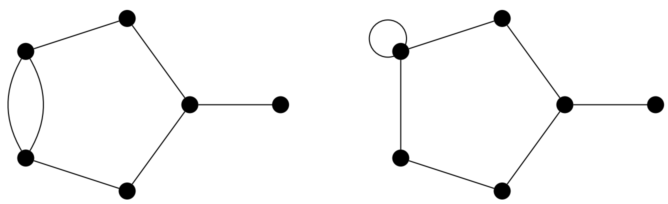


Figure 1.1: This is a graph.

We refer to the points as *vertices*, and to the lines as *edges*. There cannot be two edges connecting the same pair of two vertices (see Figure 1.2a), nor can there be an edge connecting a vertex to itself (see Figure 1.2b).



(a) (b)

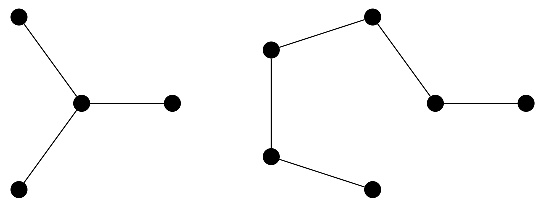
Figure 1.2: These are not graphs.

Let us formalize the previous notions. For any set X, we denote by $\binom{X}{2}$ the set of 2-element subsets of X.

Definition 1.1. *A* graph *G is an ordered pair (V, E) of finite sets, with $E \subseteq \binom{V}{2}$. The set V is the set of* vertices *and E the set of* edges*.*

Note that E induces a symmetric and irreflexive binary relation over V, the *adjacency relation*. Given an edge e, since $e \in \binom{V}{2}$ there are two distinct vertices x, y such that $e = \{x, y\}$, which we refer to as the *endpoints* of e. By abuse of notation, given an edge e with endpoints x and y, we write equivalently $e = (x, y)$ or $e = (y, x)$ instead of $e = \{x, y\}$. If two vertices x, y are adjacent, we say that y is a *neighbor* of x, and that the edge (x, y) is *incident* to x and to y. Given a vertex x, we define its degree $d(x)$ to be its number of neighbors.

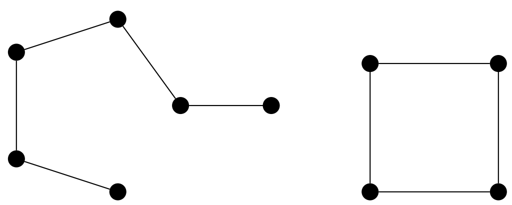
Given a graph $G = (V, E)$ and a subset $X \subseteq V$ of vertices, we can define the *subgraph induced by X in G*, denoted by $G[X]$, as the graph $G' = (X, E')$ with $E' = E \cap \binom{X}{2}$. More generally, we say a graph H is an *induced subgraph* of G if it is the subgraph induced in G by some subset of vertices (see Figure 1.3). Very informally, it corresponds to erasing from the pictures all vertices in $V \setminus X$ and thus all incident edges too.



(a) (b)

Figure 1.3: The graph on the left is an induced subgraph of the graph in Figure 1.1, while the one on the right is not.

We can relax a bit the definition of induced subgraphs. Given a graph $G = (V, E)$, $H = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$ (see Figure 1.4). Very informally, a subgraph is what we can obtain by erasing from the picture some vertices and their incident edges, and/or some edges. We denote this property by $H \subseteq G$. We say that H is a *proper* subgraph of G if H is not G itself.



(a) (b)

Figure 1.4: The graph on the left is a subgraph of the graph in Figure 1.1, while the one on the right is not.

Given a graph $G = (V, E)$, let R^* be the transitive closure of the adjacency relation in G. If R^* is total (i.e. for every two vertices $x, y \in V$, we have xR^*y), then we say the graph G is *connected*. Less formally, given two vertices $u, v \in V$, we say that there is a *path* between u and v if there is a sequence $(u = w_0, w_1, \dots, w_p = v)$ of vertices such that each w_i is adjacent to w_{i+1}. The graph G is connected if there is path between any two vertices in V. In other words, the graph is in one piece, i.e. if we build a physical representation of the graph where vertices are wooden rings with actual strings between neighbors, then it suffices to take hold of just one ring for the whole set of rings to follow. Not all graphs are connected, so we say that an induced subgraph H of G is a *connected component* of G if it is a maximal connected induced subgraph of G. Note that the decomposition of G into connected components is unique. We denote $\#cc(G)$ the number of connected components of G.

We draw a graph on paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints (see Figure 1.1). The way the vertices and edges are drawn are considered irrelevant: all that matters is the number of vertices and the adjacency relation between them.

However, the way the graph can be drawn is not irrelevant. We say that two graphs are *isomorphic* if they can be drawn in the same way. More formally, two graphs are isomorphic if there exists a bijection between their respective sets of vertices that preserves the adjacency relation. In this work, we do not make a distinction between two isomorphic graphs. In particular, any drawing of a graph is enough to reconstruct the graph itself.

Sparse graphs

A *planar embedding* of a graph is a drawing of it such that no two edges cross. Since no two edges cross, in a planar embedding of a connected graph, the plane is thus divided into regions delimited by an alternating sequence of vertices and edges. In other words, for every point M in the plane that stays clear of the points/vertices and of the curves/edges, if there are at least 3 vertices in the graph, there exists some $p \geq 3$ such that the region of the plane to which M belongs is delimited by some $(v_1, e_1, v_2, \dots, v_p, e_p)$ such that each e_i corresponds to (v_i, v_{i+1}) (indices taken modulo p). Since we consider E as a set of unordered pairs of vertices, some of the information here is unnecessary. It suffices to say that the region of the plane to which M belongs is delimited by $(v_1, v_2, \dots, v_p)$. This is a *face* of the planar embedding. Therefore, a planar embedding can be defined as the triple (V, E, F), where F is the set of faces.

Definition 1.2. *A* planar *graph is a graph that admits a planar embedding, i.e. can be drawn on the plane so that no two edges cross.*

In fact, we know that a planar graph admits in particular a planar embedding where every edge is drawn straight [Fáry48]. The graph depicted in Figure 1.5 is planar, because it is isomorphic to the graph in Figure 1.1 and thus admits a planar embedding.

Not all graphs are planar. For example, the graph K_5 on five vertices with all possible edges between them is not planar (see Figure 1.6). A graph on n vertices with all possible edges between them is called a *complete graph*, and is referred to as K_n.

The embedding presented in Figure 1.6 is only one possible embedding of the graph: how can we argue that not a single embedding of the graph K_5 is planar?

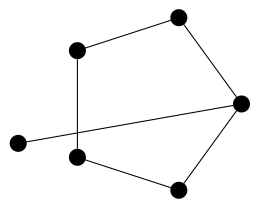


Figure 1.5: A non-planar embedding of a planar graph.

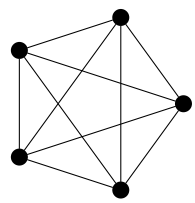


Figure 1.6: A non-planar graph.

A first partial answer is linked to the properties of a planar embedding.

Lemma 1.1 (Euler's formula)**.** *For every planar embedding $\mathcal{M} = (V, E, F)$ of a connected graph, it holds that*

$$|V| - |E| + |F| = 2$$

If the embedded graph $G = (V, E)$ is not connected, we plug in the number of connected components as $|V| - |E| + |F| = 1 + \#cc(G)$. However, when trying to decide whether a graph is planar, it suffices to consider each connected component independently. The same will go for all problems we study in this thesis, so the connected case, as presented in Lemma 1.1, is enough. There is a straightforward proof of the formula by recurrence, but we present later in this chapter a proof using a discharging argument (see Section 1.3).

We will use Lemma 1.1 to argue that the graph in Figure 1.6 admits no planar embedding. Assume for contradiction that it does, that there is a planar embedding $\mathcal{M} = (V, E, F)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \binom{V}{2}$. Then $|V| = 5$ and $|E| = \binom{5}{2} = 10$, therefore, by Lemma 1.1, $|F| = 7$. However, note that in any planar embedding, each edge belongs to at most two faces, while each face contains at least three edges. Therefore, $|F| \leq \frac{2|E|}{3} < 7$, a contradiction. We can similarly argue that the graph $K_{3,3}$ in Figure 1.7 is not planar either. A graph on $n + m$ vertices with all possible edges between the n first vertices and the m last ones, and no other edge in the graph, is called a *complete bipartite* graph and is referred to as $K_{n,m}$.

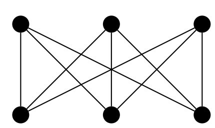


Figure 1.7: $K_{3,3}$ is not planar.

In fact, the distinction between planar and non-planar graphs lies precisely in these two graphs. *Contracting* an edge (u, v) in a graph G means deleting the edge (u, v) and replacing the two vertices u and v with a new vertex w with the same neighbors (i.e. w is adjacent to a vertex z in the new graph iff u or v was adjacent to z in G). A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting some edges. This notion of minor helps us characterize planar graphs, as follows.

Theorem 1.2. *[Wag37] A graph G is planar iff neither K_5 nor $K_{3,3}$ is a minor of G.*

The argument that we used to prove that K_5 is not a planar graph can be generalized. Indeed, we argued that for every planar embedding (V, E, F), it holds that $|F| \leq \frac{2|E|}{3}$. If we plug this back in Euler's formula, we obtain that $2 = |V| - |E| + |F| \leq |V| - \frac{|E|}{3}$. Therefore, in every planar graph, $3|V| > |E|$. We can reformulate this in terms of *average degree*. The average degree of a graph $G = (V, E)$, denoted $\mathrm{ad}(G)$, is the sum of the degrees of its vertices, averaged by the number of vertices. By the so-called handshake lemma, it is also equal to twice the number of edges over the number of vertices.

$$\mathrm{ad}(G) = \frac{\sum\limits_{v \in V} d(v)}{|V|} = \frac{2|E|}{|V|}$$

Therefore, in a planar graph G, we have $\mathrm{ad}(G) < 6$. However, the average degree is not a very tell-tale parameter on a graph. Indeed, consider a graph with on one side 10 vertices all adjacent to each other, and on the other side 90 vertices with no neighbor at all. We know that a subgraph of it has average degree 9 and is thus not planar. However, the whole graph has average degree 4.5, which does not contradict Euler's formula. A better parameter is the *maximum average degree*, denoted $\mathrm{mad}(G)$, which is the maximum average degree of a subgraph of G.

$$\mathrm{mad}(G) = \max_{H \subseteq G} \mathrm{ad}(H)$$

The parameter $\mathrm{mad}(G)$ can be efficiently computed by translating the question into a flow problem on the right graph [Coh10]. Since every subgraph of a planar graph is itself a planar graph, it holds that $\mathrm{mad}(G) < 6$ for every planar graph. However, by the fact that $\mathrm{mad}(K_5) = 4$, this is not a characterization of planar graphs.

Note that there exist non-planar graphs with mad arbitrarily close to 2 (from above), as can be seen by considering K_5 where edges (u, v) are repeatedly substituted with a vertex of degree 2 adjacent to both u and v (there are exactly five vertices of degree 4, and overwhelmingly many vertices of degree 2). However, we can prove that every graph G with $\mathrm{mad}(G) \leq 2$ is planar. Indeed, assume by contradiction that some graphs with $\mathrm{mad} \leq 2$ are not planar, and take $G = (V, E)$ to be one with a minimum number of vertices. Then every proper induced subgraph H of G still satisfies $\mathrm{mad}(H) \leq 2$, and has less vertices than G, thus is planar. Consequently, the graph G is connected. If there is a vertex u in G that is of degree 1, let $H = G[V \setminus \{u\}]$, and v the neighbor of u in G. By assumption, H is planar. Given a planar embedding of H, we can add u close to v and thus obtain a planar embedding of G, a contradiction. Now, if there is no vertex of degree 1 in G, since G is connected and non-planar, every vertex must be of degree at least 2. However, the average degree is at most 2, so every vertex must be of degree exactly 2. Then G is a cycle, which is a planar graph (see Figure 1.8), a contradiction. A *cycle* is a connected graph where every vertex is of degree 2 (see Figure 1.8). A cycle can be written as $(v_1, v_2, \ldots, v_p)$ (we then denote it C_p) where p is the *length*, i.e. the number of vertices in the cycle, and each v_i is adjacent exactly to v_{i-1} and v_{i+1} (subscripts taken modulo p).

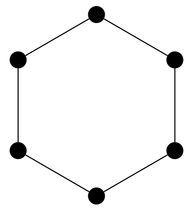


Figure 1.8: The cycle on six vertices.

Consequently, as far as the maximum average degree is concerned, we determined the best possible sufficient condition for a graph to be planar. Proving anything stronger without involving any other parameter than the mad would be impossible. We address similar questions in harder settings along this thesis. However, there is more to the mad in planar graphs than the above remark that $\mathrm{mad}(G) < 6$ for any planar graph G.

To argue that K_5 is not planar, we introduced the inequality that $|F| \leq \frac{2|E|}{3}$. However, this inequality is not enough to argue that $K_{3,3}$ is not planar either. Here, the number of vertices is 6, the number of edges is 9, so the number of faces in a potential planar embedding could be 5 without contradiction. However, we can note that there are no three vertices in $K_{3,3}$ all adjacent to each other (i.e. $K_{3,3}$ contains no *triangle*). Consequently, in a potential planar embedding every face would contain at least four edges, thus the previous inequality becomes $|F| \leq \frac{|E|}{2}$, and we can conclude that $K_{3,3}$ is not planar. Again, we can plug the inequality back in Euler's formula, and say that in any planar embedding (V, E, F) of a graph with no triangle, we must have $2 = |V| - |E| + |F| \leq |V| - \frac{|E|}{2}$. Therefore, in every such planar graph, $2|V| > |E|$, and $\mathrm{mad}(G) < 4$.

More generally, given a graph G, we define its *girth* $g(G)$ as the length of a smallest cycle which is a subgraph of G. If no subgraph of G is a cycle, we set $g(G) = +\infty$. The previous two formulas (when $g(G) = 3$ or $g(G) = 4$) can be generalized as Lemma 1.3 when G is planar.

Lemma 1.3 (Folklore). *Every planar graph G satisfies* $(\mathrm{mad}(G) - 2)(g(G) - 2) < 4$.

Consequently, any planar graph with finite girth $g(G)$ satisfies $\mathrm{mad}(G) < \frac{2g(G)}{g(G)-2}$, and every planar graph with infinite girth satisfies $\mathrm{mad}(G) < 2$. The correspondence for small girth is presented in Table 1.1.

Vertex coloring

Let us now introduce the notion of *coloring*. Informally, a coloring of a graph is an assignment of one color to each vertex such that no two neighbors have the same color.

If G is planar and $g(G) \geq$	3	4	5	6	7	8	9	10	11	12	13	...	$+\infty$
Then $\mathrm{mad}(G) <$	6	4	$\frac{10}{3}$	3	$\frac{14}{5}$	$\frac{8}{3}$	$\frac{18}{7}$	$\frac{5}{2}$	$\frac{22}{9}$	$\frac{12}{5}$	$\frac{26}{11}$	...	2

Table 1.1: Girth/mad correspondence when G is a planar graph.

Definition 1.3. *Given a graph $G = (V, E)$ and an integer k, a function $c : V \to \{1, \ldots, k\}$ is a* proper k-coloring *of G if $c(u) \neq c(v)$ for every edge (u, v).*

All along this thesis, we often drop the adjective "proper", as all the colorings we consider are proper. If a graph admits a k-coloring, we say it is *k-colorable*. It is easy to find a coloring of a graph: it suffices to color each vertex in a different way. However, the question is to minimize the number of colors used on the whole graph, or at least to bound it reasonably. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G admits a k-coloring.

Computing the chromatic number of a graph is an NP-complete problem [GJ79]. In fact, with no additional information on the input graph, it is even NP-hard to decide whether a given graph admits a 3-coloring, even for a planar graph where every vertex has at most 4 neighbors [Dai80]. However, if one is willing to compromise on the number of colors, it is often possible to efficiently find a coloring with fewer colors than the total number of vertices in a graph. The compromise has to be generous, for it is NP-complete even to approximate the chromatic number of a graph within a bounded factor [Zuc06]. We use the notion of greedy coloring, as follows. Given a graph $G = (V, E)$ and an ordering $\mathcal{O} = (x_1, \ldots, x_n)$ on V, in the *greedy coloring* of G relative to $\mathcal{O}$, every x_i has the smallest color that does not appear on $N(x_i) \cap \{x_1, \ldots, x_{i-1}\}$.

Let the *maximum degree* $\Delta(G)$ of a graph G be the maximum degree of a vertex in G. By considering the greedy coloring of G relative to any order, we obtain that $\chi(G) \leq \Delta(G) + 1$. This bound is obviously reached for complete graphs (for any n, we have $\chi(K_n) = n$ and $\Delta(K_n) = n - 1$), but also for odd cycles (see Figure 1.9).

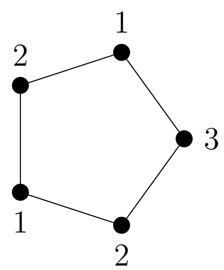


Figure 1.9: The cycle on five vertices, C_5, satisfies $\Delta(C_5) = 2$ and $\chi(C_5) = 3$.

In fact, we can prove that these are the only two cases where the bound is tight.

Theorem 1.4 (Brooks' theorem [Bro41])**.** *Every connected graph G satisfies $\chi(G) \leq \Delta(G)$ unless G is a clique or an odd cycle.*

There are many independent proofs of Theorem 1.4, a nice sample of which can be found in a recent survey by Cranston and Rabern [CR14]. Interestingly, one of them involves nothing else than considering a greedy coloring relative to a right order on the vertices. Given a graph G, the size of a maximum clique of G, denoted $\omega(G)$, is the largest integer p such that K_p is a subgraph of G (a clique of G is a subgraph of G which is a complete graph). Computing a maximum clique in a graph is NP-complete, and the size of a maximum clique is a lower-bound of the chromatic number. However, the chromatic number cannot in general be bounded from above by a function of the size of a maximum clique. In fact, there are graphs with arbitrarily large girth and arbitrarily large chromatic number [Erd59].

The notion of coloring can be extended. Instead of having the same set of k colors for the whole graph, we can wonder what happens when each vertex has its own list of k colors to be colored from. Given a list assignment $L : V \to \mathcal{P}(\mathbb{N})$, we say a graph $G = (V, E)$ is *L-colorable* if there is a coloring $c : V \to \mathbb{N}$ such that (i) $c(u) \neq c(v)$ whenever (u, v) is an edge, and (ii) $c(u) \in L(u)$ for every vertex u. For any integer k, a graph $G = (V, E)$ is *k-choosable* if G is L-colorable for every list assignment L of at least k colors to each vertex.

Again, we try to minimize the number of colors used in each list. The *choice number* $\chi_\ell(G)$ of a graph G is the smallest integer k such that G is k-choosable.

A k-choosable graph is L-colorable for any list assignment of k colors to each vertex, so it is in particular k-colorable. Any graph is L-colorable for every list assignment L of disjoint non-empty sets of colors to each vertex. Therefore, it might be tempting to believe that the hardest case is when all the lists are the same and there is the most possible conflicts between neighbors. In other words, that every k-colorable graph is necessarily k-choosable too. However, the hardest case is in between, as $K_{2,4}$ is 2-colorable (each side gets one color) but not 2-choosable (see Figure 1.10).

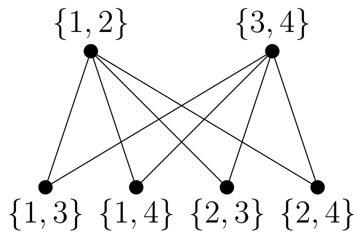


Figure 1.10: An example of a 2-colorable graph which is not 2-choosable.

In fact, we can generalize the example of $K_{2,4}$ by considering any K_{n,n^n}, which is a 2-colorable but not n-choosable graph. Therefore, the parameter $\chi_\ell(G) - \chi(G)$ can be arbitrarily large, as stated in Theorem 1.5.

Theorem 1.5 (Folklore). *The parameter $\chi_\ell - \chi$ is not bounded on the class of all graphs.*

There are also positive results to obtain. Theorem 1.4 still holds in the case of list coloring.

Theorem 1.6 ([Viz76]). *Every graph G satisfies $\chi_\ell(G) \leq \Delta(G)$ unless G is a clique or an odd cycle.*

As a direct corollary of Theorem 1.6, even cycles are 2-choosable, which will actually be a useful fact all along this thesis.

Lemma 1.7 (Folklore)**.** *Even cycles are* 2*-choosable.*

The above result can be strongly generalized, as was done by Erdős et al. [ERT79]:

Lemma 1.8 ([ERT79])**.** *If G is a* 2*-connected graph that is neither a clique nor an odd cycle, and L is a list assignment on the vertices of G such that $\forall u \in V(G), |L(u)| \geq d(u)$, then G is L-colorable.*

In this thesis, we are interested in positive results where the choice number is the same as, or not much bigger than, the chromatic number. We consider in particular two coloring variants.

Edge coloring

The first variant consists in coloring edges instead of vertices. Given a graph $G = (V, E)$ and an integer k, a function $c : E \to \{1, \dots, k\}$ is an *edge k-coloring* of G if $c(u, v) \neq c(v, w)$ for every two incident edges (u, v) and (v, w). Again, we try to minimize the number of colors used. The *chromatic index* $\chi'(G)$ of a graph G is the smallest integer k such that G admits an edge k-coloring.

Note that $\chi'(G) \geq \Delta(G)$, as the edges incident to a vertex of degree d need to receive d different colors. Note also that $\chi'(G) \leq 2\Delta(G)-1$, as no edge is incident to more than $2\Delta(G) - 2$ other edges. As a direct consequence, the inapproximability result about vertex coloring (that approximating the chromatic number within a constant factor is NP-hard) cannot be translated to edge coloring. A much stronger result actually holds:

Theorem 1.9 (Vizing's theorem [Viz64])**.** *Every graph G satisfies $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.*

The proof that $\chi'(G) \leq \Delta(G) + 1$ is constructive, and we can thus easily edge color any graph with just one more color than is optimal. However, it is still NP-complete to decide whether $\chi'(G) = \Delta(G)$ (G is said to be *Class 1*) or $\chi'(G) = \Delta(G) + 1$ (G is said to be *Class 2*) [Hol81].

We define the list extension of edge coloring similarly as the list extension of vertex coloring. Given a list assignment $L : E \to \mathcal{P}(\mathbb{N})$, we say a graph $G = (V, E)$ is *edge L-colorable* if there is a coloring $c : E \to \mathbb{N}$ such that (i) $c(u, v) \neq c(v, w)$ whenever (u, v) and (v, w) are edges, and (ii) $c(u, v) \in L(u, v)$ for every edge (u, v). For any integer k, a graph $G = (V, E)$ is *edge k-choosable* if G is edge L-colorable for every list assignment L of at least k colors to each edge. The *choice index* $\chi'_\ell(G)$ of a graph G is the smallest integer such that G is edge k-choosable.

Note that the previous trivial bounds still hold here: every graph G satisfies $\Delta(G) \leq \chi'_\ell(G) \leq 2\Delta(G) - 1$ (see Section 3.1 for more information). We finish by stating here a theorem on the list edge coloring of multigraphs which will also be put in context in Section 3.1, but which will prove useful in all the other chapters. A graph with no odd cycle is a *bipartite* graph. A *multigraph* is a graph where there may have more than one edge between two vertices, and there may be an edge whose two endpoints are the same (that is, a *loop*). The two drawings in Figure 1.2 are not graphs, but are multigraphs. Most definitions have a natural extension to multigraphs. A bipartite multigraph is a multigraph with no loop, such that the underlying graph is bipartite.

Theorem 1.10. *[BKW97] For every bipartite multigraph $G = (V, E)$ and every edge list assignment $L : E \to \mathcal{P}(\mathbb{N})$ such that $|L(u, v)| \geq max(d(u), d(v))$, the multigraph G is edge L-colorable.*

Square coloring

The second variant consists in coloring the square of the graph, i.e. properly coloring the vertices with the extra condition that two vertices with a common neighbor must receive different colors. Given a graph $G = (V, E)$ and an integer k, a function $c : V \to \{1, \ldots, k\}$ is a *square k-coloring* of G if $c(u) \neq c(v)$ for every edge (u, v) and $c(u) \neq c(v)$ for every two vertices u, v with a common neighbor. We denote by $\chi^2(G)$ the smallest integer k such that G admits a square k-coloring.

Note that $\chi^2(G) \geq \Delta(G)+1$, as the vertices adjacent to a vertex u of degree d need to receive pairwise different colors (they have a common neighbor), each different from the color received by u (they are all adjacent to u), which makes for $d + 1$ different colors on u and its neighborhood. Note also that $\chi^2(G) \leq \Delta(G)^2 + 1$, as each vertex has at most $\Delta(G)$ neighbors, each of which has at most $\Delta(G) - 1$ other neighbors.

The list extension goes with no surprise. Given a list assignment $L : V \to \mathcal{P}(\mathbb{N})$, we say a graph $G = (V, E)$ is *square L-colorable* if there is a coloring $c : V \to \mathbb{N}$ such that (i) $c(u) \neq c(v)$ whenever (u, v) is an edge, (ii) $c(u) \neq c(v)$ whenever u and v have a common neighbor, and (iii) $c(u) \in L(u)$ for every vertex u. For any integer k, a graph $G = (V, E)$ is *square k-choosable* if G is square L-colorable for every list assignment L of at least k colors to each vertex. We denote by $\chi^2_\ell(G)$ of a graph G the smallest integer such that G is square k-choosable.

The term *"2-distance"* can also be used instead of "square" in the above definitions. We investigate more deeply this notion in Section 4.1.

1.2 Definitions and notation

We introduce the additional definitions and notation needed in this thesis. We split them in two parts: first the standard notions, and then the more specific ones.

Standard notions

Let us first extend a bit our basic notions, so as to make it a bit easier to manipulate graphs. We drop subscripts when there is no ambiguity about the graph considered.

Given a graph $G = (V, E)$, a vertex $v \in V$ and a set $A \subseteq V$, the *neighborhood* of A, denoted by $N_G(A)$, is the set of vertices of $V \setminus A$ adjacent to a vertex of A. The *degree* of v in the subset A, denoted by $d_A(v)$ is its number of neighbors in G that belong to A. The latter case is an abuse of notation, which requires that there is no ambiguity about the graph considered. The notion of adjacency, incidency, and neighbors can be naturally extended to multigraphs. The *degree* of v in a multigraph $G = (V, E, \mu)$, denoted by $d_G(v)$ is its number of incident edges (loops are counted twice).

We now consider how two vertices relate in a graph. We denote P_{p+1} the graph on $p + 1$ vertices which contains a path of length p and no other edge. Given two vertices $v, w \in V$, the *distance* between v and w, denoted by $d(v, w)$, is the length of a shortest path between them. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between two vertices in G (which is infinite if some pair is such that there is no path between them).

Let us now look at selected parts of graph. Given two sets $V' \subseteq V$ and $E' \subseteq E$, we define $G \setminus (V' \cup E')$ (the graph G where the elements of V' and E' have been deleted), as the subgraph of G obtained from $G[V \setminus V'] = (V'', E'')$ by considering $(V'', E'' \setminus E')$. A connected graph G is *2-connected* for every $u \in V$, the graph $G \setminus \{u\}$ is connected.

We look into the number of neighbors of each vertex in a graph. Given a graph $G = (V, E)$, we define the *minimum degree* of G, denoted by $\delta(G)$, as the minimum degree of a vertex in G. Given an integer d, the graph G is *d-regular* if $\Delta(G) = \delta(G)$, i.e. every vertex in G is of degree d. The graph G is *d-degenerate* if every subgraph H of G satisfies $\delta(H) \leq d$ (in other words, there is a total order on the vertices of the graph such that every vertex has at most d neighbors after himself in that order).

We consider cycles in a graph. An *odd cycle* is a cycle of odd length, and an *even cycle* is a cycle of even length. We also call C_3 a *triangle*. Given a graph $G = (V, E)$ and two cycles C and C' in G, we say that C and C' are *adjacent* if they share an edge, and *incident* if they share a vertex.

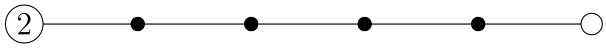
We now go into special graph classes. A *forest* is a graph with no cycle as a subgraph. A *tree* is a connected forest. A *cactus* is a graph whose every 2-connected subgraph is a cycle. A *star* is a graph of diameter 2, with no triangle. A *subcubic* graph is a graph with no vertex of degree more than 3.

We consider some transformations of a graph. Given a graph $G = (V, E)$, we define a 1-*subdivision* of G as the graph obtained from G by replacing some edge $(u, v) \in E$ with a vertex w adjacent only to u and v. We define a *subdivision* of G as a graph obtained from G by iteratively (any number of times, from 0 to arbitrarily many) considering 1-subdivisions of G. We define the *square* of G, denoted by G^2, as the graph (V, E') obtained from G by adding an edge between any two vertices with a common neighbor in G. More formally, $E' = E \cup \{(v, w) | \exists z \in V, (v, z), (w, z) \in E\}$. In other words, $E' = \{(v, w) | 1 \leq d_G(v, w) \leq 2\}$. More generally, given an integer k, we define the k^{th} *power* of G, denoted by G^k, as the graph (V, E') obtained from G by adding an edge between any two distinct vertices at distance at most k in G. More formally, $E' = \{(v, w) | 1 \leq d_G(v, w) \leq k\}$.

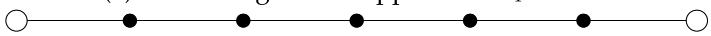
We finish with some definitions in the case of a planar embedding $\mathcal{M} = (V, E, F)$. A vertex v or an edge $e = (v, w)$ is *incident* to a face f if v appears on the boundary of f, or the vertices v and w appear consecutively on the boundary of f, respectively. The *degree* of a face, denoted $d(f)$, is the length of the corresponding cycle.

Specific notions

All along this thesis, in the figures, when drawing a part of a graph (or *configuration*), we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. Note that the white vertices may coincide with other vertices (black or white), provided that this does not contradict the existing edges (a white vertex may only coincide with a vertex that is depicted as neither a neighbor nor a neighbor of neighbor). When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label $'i'$ means "exactly i neighbors", the label $'i^+'$ (resp. $'i^-'$) means that it has at least (resp. at most) i neighbors. There is a distinction between a black vertex with i depicted neighbors, and a white vertex with a label $'i'$ inside. Examples of this can be found in Figure 1.11.



(a) This configuration appears in C_4.



(b) This configuration does not appear in C_4.

Figure 1.11: Examples of configurations which appear or do not appear in the cycle on four vertices C_4.

The first definitions are about an approach opposite to that of graph subdivision: we seek to view paths with all internal vertices of degree 2 as we would a single edge. In a graph $G = (V, E)$, a *chain* is a set of vertices of degree 2 that form an induced path. The *length* of a chain is its number of vertices. The *endpoints* of a chain C are the two vertices in $N(C)$. A chain of p vertices is also referred to as a p*-link*, whose endpoints are thus p*-linked*.

All along this thesis, we consider graph properties whose study can be reduced to connected graphs. For example, when trying to decide whether a graph is 3-colorable, it suffices to consider each connected component independently. Therefore, we rarely even mention connectivity. When we consider a graph, we always assume implicitly that it is connected, and if it is not, we implicitly consider each of its connected components independently.

Given a theorem, a *minimal counter-example* is a graph that does not satisfy the theorem, but such that every smaller graph satisfies it. Unless specified otherwise, a smaller graph is one with less vertices, or one with as many vertices but less edges. In other words, the default order on graphs is the lexicographic order on $(|V(G)|, |E(G)|)$.

1.3 What is a discharging argument?

First, we would like to emphasize that there is no formal definition of what a discharging method is. A discharging proof usually follows this outline:

1. Assume we have a set S of various elements that interact in a given way (it could be the vertices, edges and faces in some planar embedding, or the n integers from 1 to n, etc), and that we want to compute a function f of S that can be expressed, for some function ω, as $f(S) = \sum_{a \in S} \omega(a)$.

2. Assign to each element a of S a weight of $\omega(a)$.

3. Design discharging rules in order to reorganize the weight along S while maintaining a constant total weight.

4. Compute, for each $a \in S$, the new weight $\omega'(a)$.

5. Observe that $\sum_{a \in S} \omega'(a)$ is easier to compute, and derive the value $f(S)$.

Mostly, we say that a proof is based on a discharging method when it relies on the idea of local counting arguments in order to derive a global formula. In particular, in a discharging proof, the local counting arguments are usually presented in terms of *discharging*. The motivation behind that is to make the calculation easier. By defining formal rules about who gives what weight to who, we ensure that the total weight is constant, and in particular that no weight is lost or counted twice.

In a way, we could say that Gauss' idea[1] about how to compute the sum of all integers from 1 to n is a primitive example of a discharging method. Here, $S = \{1, 2, \ldots, n\}$, and we want to compute $N = \sum_{a \in S} a$. Therefore, we assign to each integer its own value as a weight. Then we design a single discharging rule, that every integer a gives half its initial weight to the element $n + 1 - a$ (see Figure 1.12). Now, all the elements have the same[2] weight of $\frac{n+1}{2}$. It is now extremely easy to derive N as $\sum_{a \in S} a = N = n \times \frac{n+1}{2}$.

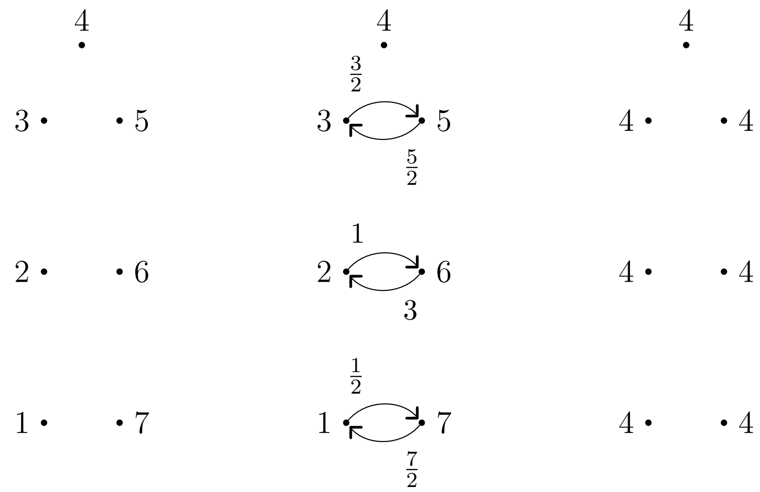


Figure 1.12: An illustration of Gauss' argument for $n = 7$ from a discharging perspective, from left to right.

Another nice, simple illustration of what a discharging method can be is a graphical proof of Euler's formula in planar graphs. The proof was initially presented in the more general setting of polyhedra [Thu78]. We insist that the proof sketched here is purely for illustrational purpose, and has no pretention to rigor. Here, we have a planar embedding $\mathcal{M} = (V, E, F)$ of our favorite planar graph, and we want to compute the exact value of $|V| - |E| + |F|$, so we set $S = V \cup E \cup F$. We can pick a notion of right and left in the embedding, and without loss of generality assume that $\mathcal{M}$ is such that no edge is perfectly horizontal. We assign to each vertex and each face (including the outer face) a weight of 1, and to each edge a weight of -1. Again, we define a single discharging rule, that each vertex and each edge gives all its weight to the face immediately to its right (see Figure 1.13): this is well-defined since no edge is perfectly horizontal. All vertices and edges have a final weight of 0,

[1] Or so the legend has it... [Hay06]

[2] The use of the word "discharging" makes all the more sense here: at the end the total weight is uniformly distributed, thus reaching equilibrium.

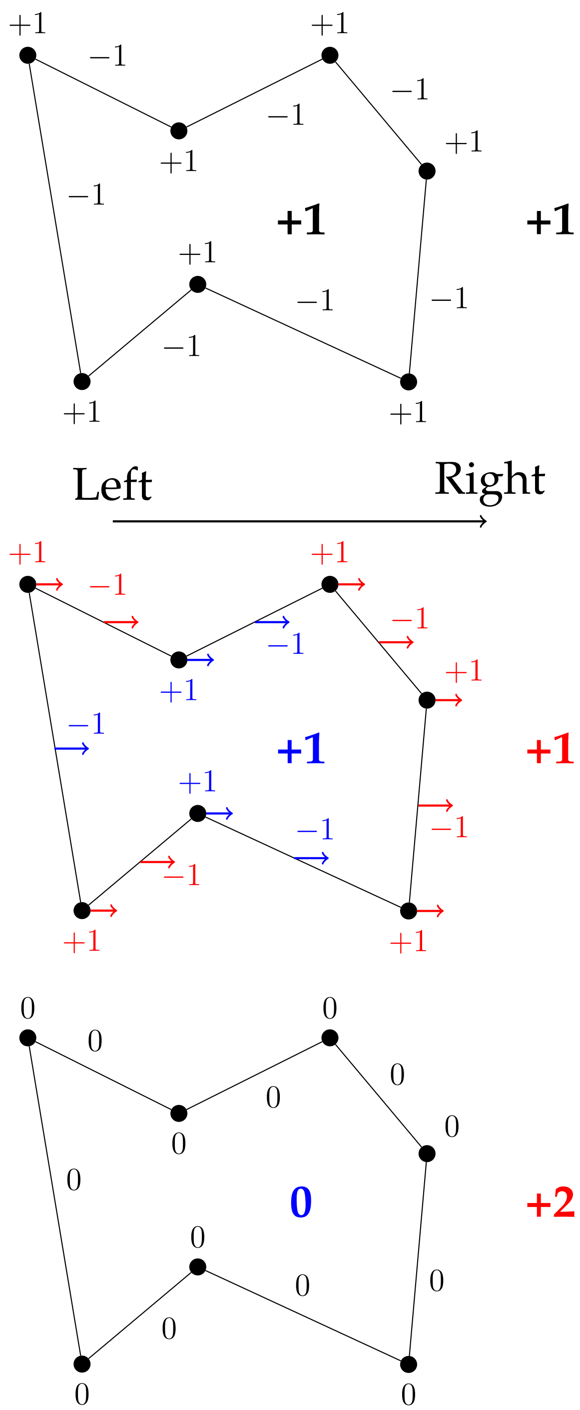


Figure 1.13: An illustration of the proof of Euler's formula from a discharging perspective, from top to bottom. Here we consider a simple graph with just two faces.

so we can concentrate on faces. No matter how bizarre the embedding can be, the vertices and edges immediately to the left of an inner face necessarily form a sequence of alternating vertices and edges, whose two endpoints are vertices who give their weight to another face. Thus, the number of vertices immediately to the left of an inner face will always be one less than the number of edges. It follows that the final weight of any inner face is 0. We can then safely disregard them, and look only at the final weight of the outer face. The same analysis stands in that case, except that there may be more than one sequence (if the graph is not connected, one for each connected component), and that the two endpoints of each sequence don't give their weight to another face. Consequently, each sequence has one more vertex than edges, and contributes a weight of 1 to the outer face. For $\#cc$ the number of connected components, the final weight of the outer face is thus $1 + \#cc$, hence the conclusion that $|V| - |E| + |F| = 1 + \#cc$, as stated in the connected case in Lemma 1.1.

For example, let us consider the same setting as in the above paragraph, up until the discharging rule. The initial weights are thus of 1 for every vertex or face, and of -1 for every edge. Here, we set a rule that every edge shares all its weight equally among its two incident vertices and two incident faces (if the edge is twice incident to the same face, the face still receives two parts of the share). It follows immediately that every vertex v has a final weight of $1 - \frac{d(v)}{4}$, and similarly for each face. Consequently, if we multiply by 4 to get rid of fractions and combine with the previous remarks, we get that

$$\sum_{v \in V}(4 - d(v)) + \sum_{f \in F}(4 - d(f)) = 4 + 4\#cc$$

Instead of sharing equally between incident vertices and incident faces, we can choose to unbalance it in favor of vertices or faces. In fact, we can choose any ratio, and thus obtain for any $a, b \in \mathbb{N}$ that

$$\sum_{v \in V}(2(a+b) - a \times d(v)) + \sum_{f \in F}(2(a+b) - bd(f)) = 2 \times (1 + \#cc) \times (a+b)$$

For example, if we take $a = 1$ and $b = 2$, we derive

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -6 \times (1 + \#cc) < 0 \tag{1.1}$$

The simple action of assigning to each vertex and each face a good weight, whose total on the graph we know to be negative, can already be informative. Here, let us set that each vertex is assigned a weight of $d(v) - 6$, and each face a weight of $2d(f) - 6$: we know the total to be negative by (1.1). Since every face of a simple planar graph has degree at least 3, no face has a negative initial weight. Therefore, at least one vertex must have a negative weight. In other words, every planar graph contains a vertex of degree at most 5. We call this an *unavoidable configuration* (a planar graph cannot avoid containing it).

We can use this fact to prove the following lemma.

Lemma 1.11 (Folklore). *Every planar graph is 6-colorable.*

Proof. Assume for contradiction that some planar graphs are not 6-colorable. Among them, take G to be one with the minimum number of vertices. If G contains a vertex u of degree at most 5, we can 6-color $G \setminus \{u\}$ by minimality. Then u has at most 5 neighbors, so there is one of the six colors that does not appear on any neighbor of

u, and we color u with it. Therefore, we can extend the 6-coloring of $G \setminus \{u\}$ to G, a contradiction. For this reason, that configuration (the graph contains a vertex of degree at most 5) is called *reducible*.

The graph G is thus a planar graph that does not contain a vertex of degree at most 5. However, we argued that this is an unavoidable configuration for planar graphs, a contradiction. Consequently, no counter-example exists and every planar graph is 6-colorable. □

This argument can be pushed a bit further to obtain that every planar graph is 5-colorable. Here no discharging at all is involved, just a good weight assignment to the vertices. Introducing even a tiny amount of discharging strengthens a lot the conclusion. We already mentioned that Wernicke proved the first structural lemma involving discharging, let us state the result here.

Lemma 1.12. *[Wer04] Every planar graph G contains a vertex of degree at most 4 or a vertex of degree 5 adjacent to a vertex of degree 5 or 6.*

Proof. By contradiction. Assume that G is a planar graph whose every vertex has degree at least 5, and such that the neighbors of every vertex of degree 5 are all of degree at least 7. Let $\mathcal{M} = (V, E, F)$ be a planar embedding of G. For simplicity, we only present the proof when $\mathcal{M}$ is a triangulation. We could argue that it is sufficient to prove that case, or adapt the proof to deal with the case where $\mathcal{M}$ is not a triangulation. However, our goal is merely to give an idea of the proof.

We assign an initial weight ω of $\omega(v) = d(v) - 6$ to each vertex $v \in V$, and $\omega(f) = 2d(f) - 6$ to each face $f \in F$. We know by (1.1) that the total weight of the graph is negative. We try to redistribute the weight along the graph in such a way that every vertex and every face has a non-negative final weight, a contradiction. Note that since $\mathcal{M}$ is a triangulation, every face f has an initial weight of exactly 0, so we can concentrate on the vertices. Only vertices of degree 5 have a negative weight, and every vertex of degree at least 7 has a positive weight.

We define a single rule R_1: for every vertex v of degree at least 7,

- Rule R_1 is when v is adjacent to three vertices u_1, u_2, u_3, such that (v, u_1, u_2) and (v, u_2, u_3) are faces, and $d(u_2) = 5$. Then v gives $\frac{1}{5}$ to u_2 (see Figure 1.14).

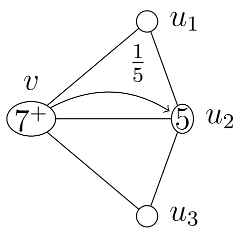


Figure 1.14: The discharging rule R_1.

Let ω' be the final weight assignment on the graph, after application of Rule R_1. Let us argue that $\omega'(v) \geq 0$ for every vertex $v \in V$ and $\omega'(f) \geq 0$ for every face $f \in F$. The second holds immediately because $\omega(f) = \omega'(f)$ for every face $f \in F$.

Let u be a vertex of degree 5. By assumption, all five neighbors of u are of degree at least 7, and all give incident faces are triangles. Therefore, Rule R_1 applies fives times, and the vertex u has an initial weight of -1, receives $5 \times \frac{1}{5}$, and thus has a non-negative final weight.

Let u be a vertex of degree 6. By definition, Rule R_1 does not apply, and $\omega'(u) = \omega(u) = 0$, so u has a non-negative final weight.

Let u be a vertex of degree at least 7. By definition and since all incident faces are triangles, Rule R_1 applies for each neighbor of degree 5. By assumption, the vertex u cannot belong to a triangle with two vertices of degree 5. Therefore, at most $\frac{d(u)}{2}$ neighbors of u may be of degree 5. The vertex u has an initial weight of $d(u) - 6 \geq 1$, gives at most $\frac{d(u)}{2} \times \frac{1}{5}$ away, thus has a final weight of $\frac{9}{10}d(u) - 6$, which is non-negative since $d(u) \geq 7$.

□

Lemma 1.12 is a proof that these two configurations (a vertex of degree at most 4, a vertex of degree 5 adjacent to a vertex of degree 5 or 6) are unavoidable for a planar graph. Lemma 1.12 can in fact be used to obtain the fastest possible algorithm to 5-color a planar graph.

When a coloring theorem is said to be proved through a discharging argument, it usually means that its proof has the following outline:

Assume for contradiction that the theorem is not true, and consider G a minimal counter-example (for some order $\prec$ on graphs).

1. Prove that there are some configurations $\{C_1, C_2, \ldots, C_p\}$ that G cannot contain (these are *reducible configurations*). Typically, argue that if G contains a configuration C_i, then there exists a smaller graph G' that is a counter-example to the theorem, a contradiction to the minimality of G.

2. Use a discharging method to prove the structural lemma that every graph satisfying the theorem hypotheses must contain one of $\{C_1, C_2, \ldots, C_p\}$ (this is an *unavoidable set of configurations*).

In the proof of Lemma 1.12, the discharging rules are very simple. In particular, the charge is only sent to a vertex at a bounded distance (here only to neighbors). The discharging method is based on the idea of using local counting arguments in order to derive a global formula. However, it could happen that there is a sub-structure of unbounded size on which the sum of weights is easy to compute, actually easier than through local arguments. Then, it might be interesting to design discharging rules that can send some charge arbitrarily far away. We then say that the discharging argument is *global*. This variant on the discharging method was only introduced in 2007 by Borodin, Ivanova and Kostochka [BIK07]. When there is a single global discharging rule, we typically call it R_g (where 'g' stands for global). For the design of a global discharging rule, we are often interested in using the sub-structure without having to make everything explicit. To that purpose, it can be convenient to use the notion of *common pot*, where the sub-structure contains vertices which receive from that common pot, and vertices that give to it. To keep some information on the total weight, we need to control somehow what happens inside the common pot. Since we usually try to show that everything has a non-negative final weight, we are usually satisfied with checking that the value of the common pot at the end is non-negative, i.e. no weight was created. In that case we say the global discharging rule is *valid*. Sometimes, the global discharging rule can be designed in such a way that the weight actually does not travel arbitrarily far, but the design still depends on a sub-structure of unbounded size (thus the proof is not made of purely local arguments). We then say that the discharging argument is *semi-global*.

By increasing the number of reducible configurations to over 600 and designing involved discharging rules, the bound in Lemma 1.11 can be further lowered to 4. In the same spirit as Lemma 1.12, significant research effort has been devoted to studying unavoidable sets in planar graphs, as a source of interest regardless of whether the configurations in these sets are reducible for some coloring problems. For example, it can be proved that in a planar graph of minimum degree 4, there must be, simultaneously, a triangle, a cycle of length 5 and a cycle of length 6 [FJMŠ02]. More often, the goal is to prove that in a planar graph with no small vertex (i.e. no vertex of degree less than 3, 4 or 5), there is necessarily a configuration of bounded degree, which is sought as large as possible. Note that if a planar graph is allowed to have vertices of degree 2, then absolutely nothing of the kind can be said, since it suffices to artificially increase the degree of all the other vertices by adding many parallel vertices of degree 2. Lemma 1.12 states that in a planar graph of minimum degree 5, there is a configuration of two adjacent vertices of small degree (the sum of the degrees is at most 11). This bound of 11 can be proved to be optimal. The same question was studied for planar graphs of minimum degree 3 or 4. Also, what about the minimal sum of the degrees of the vertices in a triangle? In a (not necessarily induced) path of three vertices? In a (not necessarily induced) cycle of length four? These questions were largely studied, but are not the topic of this thesis. We refer the reader to two nice surveys [Bor13, JV13], and from now on focus on unavoidable sets that were designed with a particular coloring problem in mind.

Chapter 2

Illustrated Discharging Methods

In this chapter, we illustrate some aspects of the discharging method. Unless specified otherwise, the ideas presented here appeared in a joint work with Nicolas Bousquet and Hervé Hocquard [BBH13].

The presentation is necessarily biased from personal experience of discharging. However, this bias is made necessary by the fact that a full survey of various discharging arguments would hardly fit in a single chapter. We refer the reader to the nice recent guide to discharging by Cranston and West [CW13] for an overview of discharging methods used for coloring purpose. The discharging method also proves itself useful outside the area of coloring, for example in combinatorial geometry as illustrated in a paper by Radoičić and Tóth [RT08], but we largely disregard such applications here. In a way, some amortized analysis proofs of algorithm complexity can be said to be of the same kin as discharging proofs. This is however not the topic of this chapter.

As introduced in Section 1.3, a discharging proof of a coloring theorem is almost always presented as a final product, which the authors present out of the blue in the form of a set of reducible configurations and a set of discharging rules, with the appropriate arguments about their correctness. Here we are interested in the process behind it. We consider a problem, and strive step by step to obtain the best results for it through discharging arguments. We start with standard, local arguments (see Section 2.1) then move on to more exotic global arguments (see Section 2.2), before making some more general remarks (see Section 2.3). The goal of this chapter is to give some intuition on the discharging method: when possible we refrain from being too formal. On the contrary, the proofs in Chapters 3 and 4 are presented in a more classical way.

We will from now on concentrate on a variant of coloring which is well-adapted for the illustration of various discharging methods. An *Adjacent Vertex-Distinguishing edge k-coloring* (*AVD k-coloring*) is a proper edge k-coloring such that no two neighbors are incident with the same set of colors. More formally, for any edge coloring $c : E \to \mathbb{N}$ and any vertex $u \in V$, we set $\phi_c(u) = \{c(uv)|v \in N(u)\}$. In that setting, an AVD k-coloring is a proper edge coloring $c : E \to \{1, \dots, k\}$ such that for any edge $uv \in E$, $\phi_c(u) \neq \phi_c(v)$. Note that there is no way to AVD color the graph K_2, as no coloring of its single edge can distinguish the two vertices. In order to avoid dealing with special cases everytime we reduce a configuration, we extend the definition so that any edge coloring of K_2 is considered to be an acceptable AVD 1-coloring. Note that this only influence results for graphs with a connected component isomorphic to K_2. We define for every graph G the AVD chromatic index χ'_{avd} as the smallest integer such that G is AVD k-colorable.

We do not strive for a full bibliography around this problem, but we present shortly some selected facts. Since an AVD coloring is a proper edge coloring, every graph G satisfies $\chi'_{avd}(G) \geq \Delta(G)$. In addition, every graph G with two adjacent vertices of degree $\Delta(G)$ satisfies $\chi'_{avd}(G) \geq \Delta(G)+1$. Zhang *et al.* [ZLW02] completely determined χ'_{avd} for paths, cycles, trees, complete graphs, and complete bipartite graphs. They noted that a cycle of length five requires five colors, but conjectured that it is the only graph with such a gap between $\chi'_{avd}(G)$ and $\Delta(G)$. Note that the problem of AVD coloring is unusual in this that an AVD k-coloring of a graph does not necessarily induce an AVD k-coloring of its subgraphs, as illustrated in Figure 2.1.

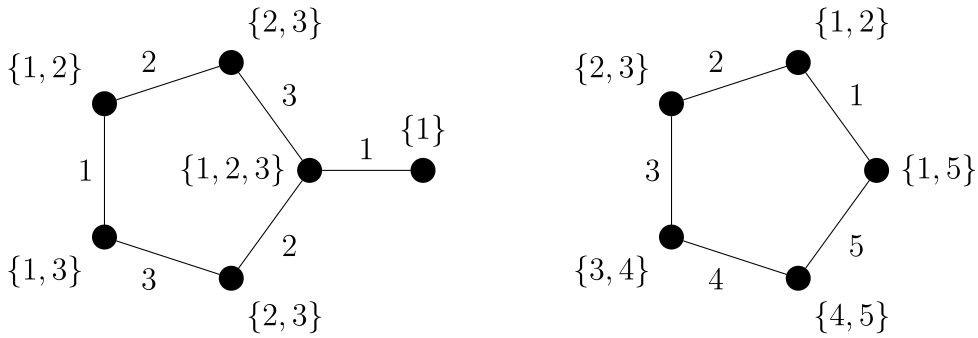


Figure 2.1: An example of a graph (left) which admits an AVD 3-coloring, while it admits an induced subgraph (right) which requires 5 colors.

Conjecture 2.1. *[ZLW02] Every graph G on at least* 6 *vertices satisfies* $\chi'_{avd}(G) \leq \Delta(G)+2$.

Balister *et al.* [BGLS07] proved Conjecture 2.1 for graphs with $\Delta(G) = 3$ and for bipartite graphs.
For edge coloring, Theorem 1.9 ensures that the chromatic index of a graph is either $\Delta(G)$ or $\Delta(G)+1$. The classification of graphs depending on this received considerable interest (for instance [SZ01]). For AVD coloring, Conjecture 2.1 would imply that the AVD chromatic index of a graph can only have three values: $\Delta(G)$, $\Delta(G)+1$ or $\Delta(G)+2$. When considering a given graph class that allows two vertices of maximum degree to be adjacent, there are only two possible upper bounds: $\Delta(G)+1$ or $\Delta(G)+2$. Similarly, the classification of graph classes depending on this received subsequent interest, and we consider here sufficient conditions with regards to $\Delta(G)$ and $\mathrm{mad}(G)$ for a graph G to be AVD $(\Delta(G)+1)$-colorable.

2.1 Local arguments

Let $m \in \mathbb{R}^+$ and $D \in \mathbb{N}$. We seek a theorem of the form "Every graph G with $\mathrm{mad}(G) < m$ and $\Delta(G) \geq D$ is AVD $(\Delta(G)+1)$-colorable". If we consider a minimal counter-example G to that theorem, we obtain that every proper subgraph H of G such that

$\mathrm{mad}(H) < m$ and $\Delta(H) \geq D$ is AVD $(\Delta(H) + 1)$-colorable. The first condition is always true, as $\mathrm{mad}(H) \leq \mathrm{mad}(G)$ by definition. However, we can easily imagine that $\Delta(H)$ may become smaller than D, and then we cannot assume anything about the AVD colorability of H. To avoid that tricky situation, we reformulate the theorem so that the hypotheses are hereditarily satisfied: "For every integer $k \geq D$, every graph G with $\mathrm{mad}(G) < m$ and $\Delta(G) \leq k$ is AVD $(k+1)$-colorable". Note that the new statement is only stronger than the previous one.

We follow the usual outline of a discharging proof, and first look for configurations that cannot appear in a minimal counter-example. Keep in mind that our goal is to prove that the existence of a minimal counter-example is a contradiction, by showing that it cannot satisfy the theorem hypotheses. Here, we will try to prove that a minimal counter-example has large average degree. In other words, we want to argue that there cannot be too large a proportion of small vertices in the graph.

Let $k \geq D$, and let G be a minimal graph with $\Delta(G) \leq k$ that is not AVD $(k+1)$-colorable. Our goal is to prove that $\mathrm{mad}(G) \geq m$. Indeed, if every minimal graph G with $\Delta(G) \leq k$ that is not AVD $(k+1)$-colorable satisfies $\mathrm{mad}(G) \geq m$, then every graph G with $\Delta(G) \leq k$ and $\mathrm{mad}(G) < m$ is AVD $(k+1)$-colorable. We will proceed step-by-step and try to obtain the best possible lower bounds on $\mathrm{mad}(G)$, i.e. the largest possible value for m such that $\mathrm{mad}(G) \geq m$.

Note that, by the existence of C_5 which requires 5 colors, we cannot hope for any result with $D < 4$ with the stronger version of our theorem. We thus assume from now on $D \geq 4$. In particular, if G contains at most five edges, we can color each with a different color, thus obtaining an AVD $(k+1)$-coloring. Therefore we can assume that G contains at least six edges.

One reducible configuration, no discharging rule

We can prove that a vertex with a neighbor of degree 1 must have many other neighbors.

Lemma 2.2. *G cannot contain a vertex u adjacent to at least one vertex of degree 1 and at most $\frac{k}{2}$ vertices of degree > 1 (see Figure 2.3).*

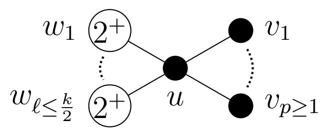


Figure 2.2: The configuration of Lemma 2.2.

Proof. Assume for contradiction that G contains a vertex u with $p \geq 1$ neighbors $v_1, \ldots, v_p$ of degree 1 and $\ell \leq \frac{k}{2}$ vertices of degree ≥ 2. Color by minimality $G \setminus \{v_1, \ldots, v_p\}$. We try to color the edges (u, v_i) so as to extend the AVD $(k+1)$-coloring to the whole graph. For the coloring to be proper, every edge (u, v_i) must avoid the ℓ colors which already appear in the neighborhood of u. That leaves $k+1-\ell$ possible colors. We now try to find a coloring which distinguishes u from its neighbors of same degree, which are at most ℓ. There are $\binom{k+1-\ell}{p}$ different ways of coloring the edges (u, v_i). There are at most ℓ sets of colors which are equal to $\phi_c(w)$ for some neighbor w of u. Therefore, if $\binom{k+1-\ell}{p} \geq \ell + 1$, then the coloring can be extended to G, a contradiction. Since $k - \ell \geq p \geq 1$, it suffices to have $k + 1 - \ell \geq \ell + 1$, hence the conclusion. $\square$

Note that Lemma 2.2 implies that no vertex of degree at most $\frac{k}{2} + 1$ can have a neighbor of degree 1. Is Lemma 2.2 sufficient to obtain a theorem of the desired form? It is enough for $m = 2$.

Proposition 2.3. *The graph G satisfies* $\mathrm{ad}(G) \geq 2$.

Proof. Assume for contradiction that $\mathrm{ad}(G) < 2$. A connected graph $H = (V, E)$ with a cycle satisfies $|E| \geq |V|$, hence such a graph satisfies $\mathrm{ad}(H) \geq 2$. Therefore, since G is connected by assumption, the graph G contains no cycle. Then G is a tree. We remove the leaves of the tree, and take a leaf of the resulting tree (which is non-empty since G contains more than just one edge). This vertex has at least one neighbor of degree 1 (it was not a leaf of G), and at most one neighbor of degree more than 1 (it is a leaf of the resulting tree). By Lemma 2.2, this is not possible. $\square$

In our pursuit of a largest possible m, we therefore set $m = 2 + a$, with $a \in \mathbb{R}^+$. Note that Lemma 2.2 is not sufficient to prove the theorem with any $a > 0$, as a long cycle has maximum average degree equal to its average degree of 2, and Lemma 2.2 does not apply.

Two configurations, two discharging rules

We can prove that there cannot be a long chain of vertices of degree 2, as was done in previous works.

Lemma 2.4 ([HM13])**.** *G cannot contain a chain of length 3 (see Figure 2.3).*

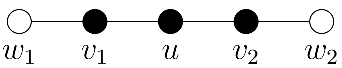


Figure 2.3: The configuration of Lemma 2.4.

Proof. Assume for contradiction that G contains a vertex u with two neighbors v_1 and v_2, $d(u) = d(v_1) = d(v_2) = 2$. Let w_1 and w_2 be the other neighbors of v_1 and v_2, respectively (see Figure 2.3). Color by minimality $G \setminus \{u\}$. For the coloring to be proper, every edge (u, v_i) must avoid the color that appears on (v_i, w_i). If w_1 is of degree 2, note that the single edge incident to w_1 and not to v_1 is different from (u, v_2) by assumption (otherwise G would be a C_3 and thus have less than 6 edges).

We choose for (u, v_1) a color that is distinct from the colors of (v_1, w_1), (v_2, w_2), and from that of the single edge incident to w_1 and not to v_1 if $d(w_1) = 2$ (this last requirement to ensure we distinguish v_1 and w_1). Then we pick for (u, v_2) a color that is distinct from the colors of (u, v_1), (v_2, w_2), (v_1, w_1), and from that of the single edge incident to w_2 and not to v_2 if $d(w_2) = 2$. Since $D \geq 4$, there are at least 5 colors and such choices are possible. Now, the vertices v_1 and w_1 are indeed distinguished, either by the fact that $d(w_1) \neq 2$ or by the color choice of (u, v_1). Similarly, u is distinguished from v_1 by choice of (u, v_2). We symmetrically reach the same conclusions for v_2 and w_2, and u and v_2, respectively.

We thus exhibited an AVD $(k+1)$-coloring of G, a contradiction. □

Are Lemmas 2.2 and 2.4 sufficient to obtain a theorem of the desired form with $a > 0$? We assign to each vertex u a weight of $d(u) - 2 - a$. We strive for a discharging procedure that leaves a non-negative weight on all vertices at the end. This will guarantee us that $\mathrm{ad}(G) \geq 2 + a$ and thus $\mathrm{mad}(G) \geq 2 + a$. Vertices of degree 1 have a negative weight of $-1 - a$, and a single neighbor. Vertices of degree 2 have a negative weight of $-a$, no neighbor of degree 1 (by Lemma 2.2) and at least one neighbor of degree more than 2 (by Lemma 2.4). Note that vertices of degree at least 3 have positive weight, and we will try to discharge the weight from them to the vertices of degree 1 or 2. Since we do not have any other piece of information about the graph, every vertex of degree at least 3 must be able to provide $1 + a$ to each of its neighbors of degree 1, and a to each of its neighbors of degree 2. By assuming $D \geq 4$ (remember that $k \geq D$), we can ensure by Lemma 2.2 that no vertex of degree 3 will have a neighbor of degree 1. However, there may be vertices of degree 3 only adjacent to vertices of degree 2 whose other neighbor is of degree 2. Therefore, a must satisfy $3 - 2 - a \geq 3 \times a$, i.e. $a \leq \frac{1}{4}$. This is in fact sufficient as soon as $D \geq 6$, and we obtain the following proposition.

Proposition 2.5. *If $D \geq 6$, the graph G satisfies* $\mathrm{ad}(G) \geq 2 + \frac{1}{4}$.

Proof. We reformulate the above arguments in a more formal way. Let us assign to each vertex u in G a weight of $\omega(u) = d(u) - 2 - \frac{1}{4}$. We design two discharging rules R_1 and R_2 (see Figure 2.4): the first states that every vertex u adjacent to a vertex v of degree 1 gives a charge of $1 + \frac{1}{4}$ to v, the second states that every vertex u adjacent to a vertex v of degree 2 gives a charge of $\frac{1}{4}$ to v. We apply R_1 and R_2 on G with the initial weight assignment. Let ω' be the resulting weight assignment on the vertices of G. Our goal is to prove that $\omega'(u) \geq 0$ for every vertex u of G.

Figure 2.4: The discharging rules R_1 (left) and R_2 (right) of Proposition 2.5.

Let u be a vertex of G. We consider different cases depending on the degree of u. Note that by Lemma 2.2, a vertex with a neighbor of degree 1 must have at least 4 neighbors of degree at least 2, and thus be itself of degree at least 5.

- *Assume $d(u) = 1$.*
  Then $\omega(u) = -1 - \frac{1}{4}$. As already noted, the neighbor of u is not of degree 1 or 2. Therefore, the vertex u gives nothing and receives $1 + \frac{1}{4}$ by R_1. Consequently, the vertex u has a non-negative final weight.

- *Assume $d(u) = 2$.*
  Then $\omega(u) = -\frac{1}{4}$. If u has a neighbor of degree 2, then they both give $\frac{1}{4}$ to the

other, so it cancels out and we can pretend it doesn't happen. No neighbor of u is of degree 1, and u may not have both neighbors of degree 2 by Lemma 2.4. Therefore, the vertex u gives nothing and receives at least $\frac{1}{4}$ by R_2: it has a non-negative final weight.

- *Assume* $d(u) = 3$.
  Then $\omega(u) = \frac{3}{4}$. All the neighbors of u are of degree at least 2. The vertex u gives at most $3 \times \frac{1}{4}$ by R_2 and has a non-negative final weight.

- *Assume* $4 \leq d(u) \leq 7$.
  Then $\omega(u) = d(u) - 2 - \frac{1}{4}$. The vertex u has more than $\frac{k}{2}$, thus at least 4, neighbors of degree at least 2. Therefore, it gives at most $(d(u) - 4) \times (1 + \frac{1}{4}) + 4 \times \frac{1}{4} = d(u) \times \frac{5}{4} - 4$ by rules R_1 and R_2. It has thus a non-negative final weight, as $d(u) - 2 - \frac{1}{4} \geq d(u) \times \frac{5}{4} - 4$ when $d(u) \leq 7$.

- *Assume* $d(u) \geq 8$.
  Then $\omega(u) = d(u) - 2 - \frac{1}{4}$. The vertex u has more than $\frac{k}{2}$, thus at least $\frac{k+1}{2}$, neighbors of degree at least 2. Therefore, it gives at most $(d(u) - \frac{k+1}{2}) \times (1 + \frac{1}{4}) + \frac{k+1}{2} \times \frac{1}{4} = d(u) \times \frac{5}{4} - \frac{k+1}{2}$ by Rules R_1 and R_2. Let us argue that it has a non-negative final weight: we must have $d(u) - 2 - \frac{1}{4} \geq d(u) \times \frac{5}{4} - \frac{k+1}{2}$, i.e. $\frac{k}{2} - \frac{7}{4} \geq d(u) \times \frac{1}{4}$. Since $d(u) \leq k$, it suffices to have $\frac{k}{4} \geq \frac{7}{4}$, i.e. $k \geq 7$, which holds since $k \geq d(u) \geq 8$. Therefore, the vertex u has a non-negative final weight.

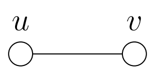
Every vertex in G has a non-negative final weight, thus $\mathrm{ad}(G) - 2 - \frac{1}{4} = \sum_{v \in V} \omega(v) = \sum_{v \in V} \omega'(v) \geq 0$, hence the conclusion. $\square$

From now on we omit the formal case analysis: all relevant information is already contained in the forbidden configurations, rule definitions, and analysis of the bounds on a.

Three reducible configurations, two discharging rules

As noted before, Lemma 2.2 implies that no vertex of degree less than $\frac{k}{2} + 1$ can have a neighbor of degree 1. One can strengthen that result.

Lemma 2.6 ([HM13]). *G cannot contain two adjacent vertices u and v with $d(u) \neq d(v)$ and $d(u) + d(v) \leq \frac{k}{2} + 2$ (see Figure 2.5).*



$$d(u) \neq d(v)$$

$$d(u) + d(v) \leq \frac{k}{2} + 2$$

Figure 2.5: The configuration of Lemma 2.6.

Proof. Assume for contradiction that G contains two adjacent vertices u and v with $d(u) \neq d(v)$ and $d(u) + d(v) \leq \frac{k}{2} + 2$. The proof here is much more direct than that of Lemma 2.2. Color by minimality $G \setminus \{(u, v)\}$. Since $d(u) \neq d(v)$, no effort is required to distinguish u and v. There are initially $k + 1$ colors available for the edge (u, v), we possibly remove as many as $(d(u) - 1) + (d(v) - 1)$ colors to enforce the propriety of the coloring. Now, if there are at least $(d(u) - 1) + (d(v) - 1) + 1$ choices of colors for (u, v), we know that there will be at least one that will distinguish u and v from their respective neighbors. This holds since $k + 1 - 2(d(u) + d(v) - 2) \geq 1$, hence the conclusion. $\square$

Now, Lemma 2.6 tells us that a vertex u with $3 \leq d(u) \leq \frac{k}{2}$ has no neighbor of degree 1, nor any of degree 2. As argued before, Lemmas 2.2 and 2.4 guarantee that every vertex of degree 1 or 2 has a neighbor of degree at least 3 (assuming $D \geq 4$). Therefore, the only constraint is that vertices of degree at least $\frac{k}{2} + 1$ must afford to give weight to all their neighbors of degree 1 or 2 ($1 + a$ to each neighbor of degree 1, and a to each neighbor of degree 2). Let us consider the worst case scenario for the neighborhood of a vertex u with $d(u) \geq \frac{k}{2} + 1$. The worst neighbors are those of degree 1. By Lemma 2.2, the vertex u cannot have more than $d(u) - \frac{k}{2} - 1$ neighbors of degree 1. Assume it has just as many. Its other neighbors are of degree at least 2, and the worst case is when they are all of degree 2. Consequently, a must satisfy $d(u) - 2 - a \geq d(u) \times a + (d(u) - \frac{k}{2} - 1) \times 1$ for every $d(u) \geq \frac{k}{2} + 1$. In other words, the constant a must satisfy $\frac{k}{2} - 1 \geq a \times (d(u) + 1)$ for every $d(u) \geq \frac{k}{2} + 1$. The strongest constraint comes from $d(u) = k$, so it suffices to have $a \leq \frac{\frac{k}{2} - 1}{k+1} = \frac{1}{2} - \frac{3}{2(k+1)}$. We can take a arbitrarily close to $\frac{1}{2}$, which results in the following.

Proposition 2.7. *For every $\epsilon > 0$, if d is large enough, then* $\mathrm{ad}(G) \geq 2 + \frac{1}{2} - \epsilon$.

Shifting the density argument to a subgraph of G

There is a simple trick that drastically improves Proposition 2.7. When we try to prove that no minimal counter-example can exist, we assume one exists, and prove that its average degree is at least $2+a$, a contradiction to the theorem hypothesis that the maximum average degree is less than $2 + a$. It could happen that the considered minimal counter-example has low average degree, but still contains a dense subgraph.

Let us sketch the consequences here. Let H be the graph obtained from G by deleting all vertices of degree 1, as was done in [HM13]. Again, we assign to every vertex u of H a weight of $d_H(u) - 2 - a$. By Lemma 2.2, the vertices of degree at most $\frac{k}{2}$ in H have the same degree in G. Therefore, by Lemma 2.6, vertices of degree at most $\frac{k}{2}$ in H have no neighbor of degree 2 in H. We can again concentrate on large vertices: a must satisfy $d(u) - 2 - a \geq d(u) \times a$ for every $d(u) \geq \frac{k}{2} + 1$ (by the same argumentation as for Proposition 2.7). The strongest constraint comes from $d(u) = \frac{k+1}{2}$, i.e. $\frac{k+1}{2} - 2 - a \geq \frac{k+1}{2} \times a$. In other words, we can take $a = 1 - \frac{6}{k+3}$, and thus a arbitrarily close to 1. Therefore, simply by shifting the density argument to a well-chosen subgraph of the minimal counter-example, we move up from Proposition 2.7 to the following.

Proposition 2.8. *For every $\epsilon > 0$, if d is large enough, then* $\mathrm{mad}(G) \geq 3 - \epsilon$.

Shifting the notion of minimality

There are other tricks that prove useful. For example, the considered order on graphs is traditionally the subgraph order ($H \prec G$ iff H is a proper subgraph of G). Sometimes, it is interesting to transform the graph more subtly than just by considering a subgraph. For this to be possible, we need to pick the right notion of minimality.

In our case, let us consider the lexicographic order on the sequence of the number of vertices of given degree in the graph, sorted by decreasing order. Now, if G contains a vertex u with exactly two neighbors v_1 and v_2, then G is larger than the graph obtained from G by replacing u with two vertices of degree 1, one adjacent to v_1 and the other to v_2. We call such a transformation $G \otimes \{u\}$. We can generalize it to a vertex u of any degree $p \geq 2$. In that case, $G \otimes \{u\}$ corresponds to the graph obtained from G by replacing u with p vertices of degree 1, each adjacent to a different neighbor of u in G. We can further generalize this notion to any set S of q vertices $u_1, \dots, u_q$ of degree at least 2. We set $G \otimes S$ to be the graph obtained from G by deleting the edges in $G[S]$ then successively considering when $d(u_i) \geq 2$ the operation $\otimes\{u_i\}$, for $1 \leq i \leq q$ (see Figure 2.6). Note that the order on the vertices of S has no influence on $G \otimes S$.

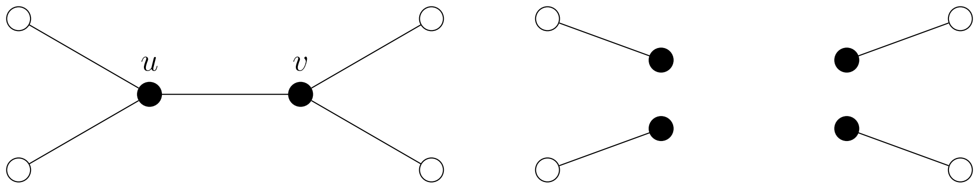


Figure 2.6: An example of the effect of $\otimes$: here we consider two adjacent vertices u, v of degree 3 (left), and the result of $\otimes\{u, v\}$ (right).

We do not define $G \otimes \{v\}$ when v is a vertex of degree 1, for that would result in the very same graph. As S consists only of vertices of degree at least 2, it holds that $G \otimes S \prec G$ for non-empty S.

We suddenly obtain much more information on our minimal counter-example, see Lemma 2.9 (note that a proper subgraph of G is still smaller than G in the new order, which makes the previous lemmas still valid here). Note also that $\mathrm{mad}(G \otimes S) \leq \mathrm{mad}(G)$, which is in fact crucial. When considering a proper subgraph, or the result of a $\otimes$ operation on G, this is obvious and could even be omitted. However, some graph transformations, no matter how tempting (like contracting every edge between two vertices of degree 2), are made impossible by the fact that the resulting graph, though smaller, may have a larger maximum average degree and not satisfy the induction hypotheses. That constraint has to be kept in mind. On the contrary, when considering planar graphs, edge contractions are extremely convenient (multiple edges beware), while some transformations used in the setting of bounded average degree graphs might result in a non-planar graph.

Let us first make some remarks about chains of degree 2. By Lemma 2.4, in G, all maximal chains of vertices of degree 2 are of length either 1 or 2 (i.e. contain exactly one vertex of degree 2 or exactly two). Note that from an AVD point of view, a chain of length two and a chain of length one behave the same, in the sense that coloring one is equivalent to coloring the other. More formally, the graph obtained from G by contracting an edge between two vertices of degree 2 or subdividing an edge incident to a vertex of degree 2 with no neighbor of degree 2 is AVD $(k+1)$-colorable iff G is. Indeed, $D \geq 4$, and a maximal chain does not, by definition, have any neighbor of degree 2. Consider two maximal chains of degree 2, (u_1, v_1, v_2, u_2) and (u_1, v, u_2) (internal vertices of degree 2). When restricting the problem to the chain, the only constraint for the second chain is that the colors of (u_1, v) and (u_2, v) differ. The vertex v has no neighbor of degree 2, and thus does not need to be distinguished. The main constraint for the first chain is that v_1 is distinguished from v_2, i.e. that the colors of (u_1, v_1) and (u_2, v_2) differ. Then the coloring has to be proper, which means we need to find a color for (v_1, v_2) that differs from those of (u_1, v_1) and (u_2, v_2). However, $D \geq 4$, so such a color is available regardless of the situation, and that constraint is insignificant. Therefore, from now on, we do not make separate cases for both. We are now ready for the following lemma.

Lemma 2.9. *G cannot contain a vertex u adjacent to both a vertex of degree 1 and a vertex of degree 2 (see Figure 2.7).*

Figure 2.7: The configuration of Lemma 2.9.

Proof. Assume for contradiction that G contains a vertex u adjacent to both a vertex v_1 of degree 1, and a vertex v_2 of degree 2. Let C be the maximal chain of vertices of degree 2 to which v_2 belongs, with w its other endpoint (remember $|C| \leq 2$ by Lemma 2.4). Color by minimality $G \otimes \{C\}$. Now, by the above paragraph, the only constraint on (u, v_2) is that its color should differ from that of the edge between w and C. Assume that both edges are of the same color. Then we swap the colors of (u, v_2) and of (u, v_1). The set of colors incident to u remains the same, and v_1 has no neighbor to be distinguished from, but now the color of (u, v_2) differs from that at the other end of the chain, and we can safely extend the AVD $(k+1)$-coloring to G (if $|C| = 2$, we pick a proper color for the internal edge, which is possible since it has two constraints and there are at least 5 colors). $\square$

Lemma 2.9 is very convenient in the sense that, in the discharging argument, the only two troublesome neighbors are those of degree 1 and those of degree 2, which abide by different rules. This lemma guarantees us that we never have to deal with both at the same time. If a vertex has neighbors which require charge from it, then either they are all of degree 1 or all of degree 2. If we follow the same path as before, by considering the graph G where all vertices of degree 1 have been removed, we miss some of essential information gained through Lemma 2.9. The information loss might grow in the way if we are to design other lemmas similar to the last. When we shift the density argument to a proper subgraph H of the graph G on which we have structural information, it often happens that, in the discharging analysis, we have to make double considerations about the degree of the vertex in H, in G (also about the nature of its neighbors in G that do not appear in H, and about the degree in G

of its neighbors in H). In a sense, while we consider H for the analysis, the actual information we need is in G. Let us consider a trick around this issue in the following section.

Ghost vertices

We introduce a way of combining the power of the previous approach (considering only a subgraph for density measures, as used for Proposition 2.8) and the information of the initial one (no vertex deletion, all information stays in the graph, as used for Proposition 2.7). The trick is to design discharging arguments that use the structure of the initial graph to prove that the considered subgraph has a high average degree.

The approach used for Proposition 2.8 was:

Dense subgraph method

- Let $V_1 \subsetneq V$, and consider $G[V_1]$.
- Every vertex u in $G[V_1]$ has an initial weight of $d_{V_1}(u) - m$.
- Can we discharge in $G[V_1]$ in such a way that every vertex in V_1 has a non-negative weight?
- If yes, then we have $\sum_{u \in V_1}(d_{V_1}(u) - m) \geq 0$, thus $\mathrm{mad}(G) \geq \mathrm{ad}(G[V_1]) \geq m$.

The new approach is:

Ghost vertices method

- Let $V_1 \cup V_2$ be a partition of V.
- Every vertex u in G has an initial weight of $d(u) - m$.
- Can we discharge in G in such a way that :
    1. Every vertex in V_1 has a non-negative weight,
    2. Every vertex u in V_2 has a final weight of at least $d(u) - m + d_{V_1}(u)$?
- If yes, then for ω' the new weight assignment, we have $\sum_{v \in V_2}(d(v) - m + d_{V_1}(v)) \leq \sum_{v \in V_2} \omega'(v)$, as well as $\sum_{v \in V} \omega(v) = \sum_{v \in V} \omega'(v)$ and $\sum_{v \in V_1} \omega'(v) \geq 0$. Therefore,

$$\begin{aligned}
\sum_{v \in V_1}(d_{V_1}(v) - m) &\geq \sum_{v \in V_1}(d_{V_1}(v) - m) + \sum_{v \in V_2}(d(v) - m + d_{V_1}(v)) - \sum_{v \in V_2} \omega'(v) \\
&\geq \sum_{v \in V_1}(d_{V_1}(v) - m) + |E(V_1, V_2)| + \sum_{v \in V_2}(d(v) - m) - \sum_{v \in V_2} \omega'(v) \\
&\geq \sum_{v \in V_1}(d(v) - m) + \sum_{v \in V_2}(d(v) - m) - \sum_{v \in V_2} \omega'(v) \\
&\geq \sum_{v \in V} \omega(v) - \sum_{v \in V_2} \omega'(v) \\
&\geq \sum_{v \in V_1} \omega'(v) \\
&\geq 0
\end{aligned}$$

We can conclude that $\mathrm{mad}(G) \geq \mathrm{ad}(G[V_1]) \geq m$.

In other words, the vertices in V_2 can be seen but, in a way, do not contribute to the sum analysis (the meaning of their final weight is essentially "this vertex has no positive contribution on the total weight of the rest of the graph"). This particularity leads us to informally refer to them as *ghost vertices*. Any result proved using ghost vertices can be proved, albeit more tediously perhaps, when deleting them completely from the graph. However, they can simplify the presentation of the discharging analysis, and this is the point of their introduction.

Similarly, the idea of ghost vertices can be translated to planar graphs, where not all vertices are mapped in the mapping considered (not all vertices have shadows). However, in that case, considering a proper subgraph cannot help: a graph may have low average degree and a subgraph of high average degree, but if a subgraph is not planar, then neither was the initial graph. The only point of introducing ghost vertices in the planar case is to reduce the number of discharging rules. That trick appears in a joint work with Jakub Przybyło on a variant of AVD coloring on planar graphs, which we do not present here.

In our problem, we consider V_2 (the set of ghost vertices) to be the set of vertices of degree 1 in G, and again assign to each vertex u in G an initial weight of $d(u) - 2 - a$. Vertices of degree 1 have a weight of $-1-a$, instead of needing an extra charge of $1+a$, they only need an extra charge of 1 in the latter approach. The bound on a does not change, as it still needs to be smaller than 1 but can be arbitrarily close to it (just as in the proof of Proposition 2.8, a vertex may be such that all its neighbors are of degree 2). However, this allows for a more refined bound on the corresponding d, but we do not dwell on this.

Despite this progress, we still are not able to prove anything for the class of graphs with $\mathrm{mad} < 3$ in general (no ϵ inserted). We turn to a powerful variant of the discharging method, where Lemma 2.9 and the notion of ghost vertices will turn out to be useful.

2.2 Global arguments

We claimed that the discharging method is based on the idea of local counting arguments in order to derive a global formula. However, it could happen that there is a sub-structure of unbounded size on which the sum of weights is easy to compute, actually easier than through local arguments. Therefore, we sometimes mix local and global arguments in the discharging process. This variant on the dicharging method depends heavily on the emergence of a good sub-structure.

Alternating cycles

For example, in G, we can note that there cannot be two vertices of degree 2 with the same two neighbors. Indeed, if there are two such vertices u_1 and u_2 with the same two neighbors v_1 and v_2 (note that $d(v_1), d(v_2) \geq 3$ by Lemma 2.4), we color $G \otimes \{u_1, u_2\}$, and consider the corresponding coloring of G (which is not necessarily an AVD coloring). Then, we switch if necessary the colors of (u_1, v_1) and (u_2, v_1) so that the coloring is proper, which results in an AVD $(k+1)$-coloring of G. In fact, this possibility to switch the colors of two incident edges can be interpreted in terms of list coloring: here the goal was to list color a cycle on 4 vertices, where the list of colors assigned to the pair (u_i, v_j) is that of the colors of (u_1, v_j) and (u_2, v_j).

We generalize this idea by setting H the multigraph with vertex set $V_H = \{v \in V \mid d(v) \geq \frac{k+1}{2}\}$, and with edge set E_H the pairs of vertices in V_H that are the endpoints in G of a (non-empty) chain of vertices of degree 2. By Lemma 2.6, the set E_H is in bijection with the set of maximal chains of vertices of degree 2 in G. The previous remark implies that there is no digon in H. We can similarly prove that there is no loop, nor any triangle, etc. In fact, we can go further and claim that H is a forest.

Indeed, assume that there is a cycle C in H. Color by minimality $G \otimes \{\text{Corresponding maximal chains of } C\}$ (see Figure 2.8 for a case where C is a triangle). Similarly as before, it all boils down to L-coloring an even cycle, where L is an assignment of two colors to each edge. By Lemma 1.7 (which obviously still stands in the case of edge coloring), the even cycle is L-colorable, and G is AVD $(k+1)$-colorable. Consequently, H is a forest.

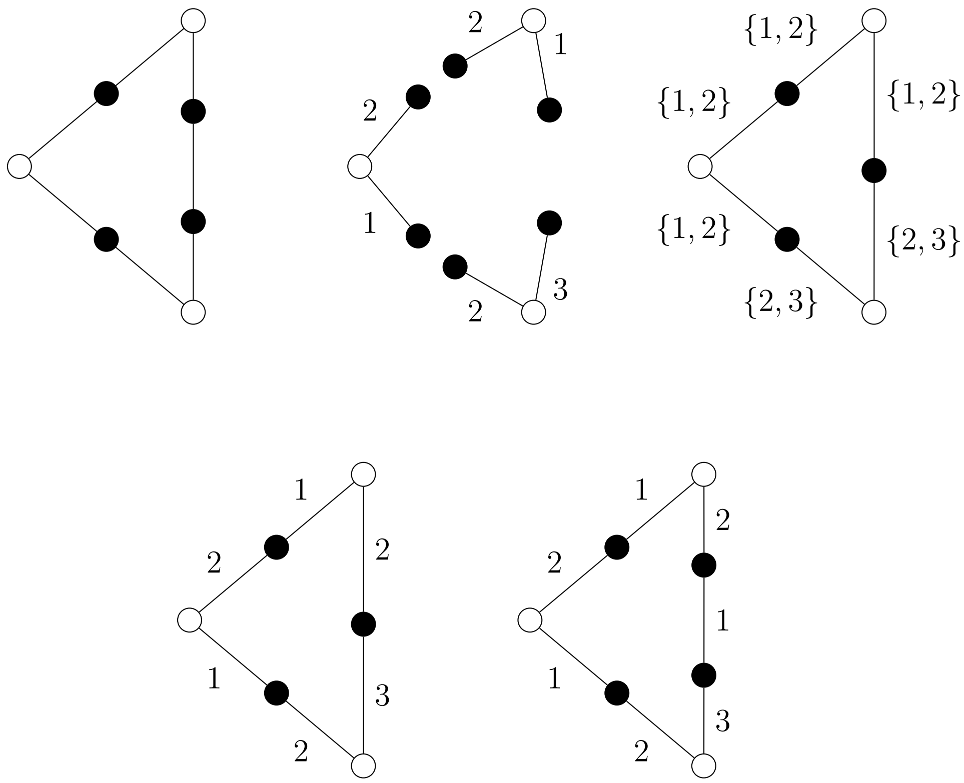


Figure 2.8: There is no triangle in H.

Since H is a forest, every connected component contains more vertices than edges. In other words, there are at least as many vertices of degree $\frac{k}{2}+1$ adjacent to a vertex of degree 2 than maximal chains of vertices of degree 2. We can make use of that property, as follows. We assign to each vertex u a weight of $d(u) - 2 - a$, and design two discharging rules. As before, we take the vertices of degree 1 to be ghost vertices, and thus set that every vertex of degree at least 2 gives a charge of 1 to every neighbor of degree 1. The second rule is what we call a *global discharging rule*, in the sense that the charge might travel arbitrarily far away. Every vertex of degree at least $\frac{k+1}{2}$ adjacent to a vertex of degree 2 gives a charge of $2a$ to a common pot, while every vertex of degree 2 receives a from the common pot. We already argued that the common pot will have a non-negative final value, which we refer to as the global discharging rule being *valid* (i.e. not used for spontaneous charge generation).

Assume $D \geq 9$. Lemma 2.2 ensures that vertices with a neighbor of degree 1 have at least 5 neighbors of degree at least 2, and no neighbor of degree 2 by Lemma 2.9. Therefore, if a vertex u has a neighbor of degree 1, then only the first discharging rule applies. The vertex u has a non-negative final weight as long as $d(u) - 2 - a \geq (d(u) - 5) \times 1$, i.e. $a \leq 3$. By Lemma 2.6, only vertices of degree at least 5 may have a neighbor of degree 2, and if they do, by Lemma 2.9, they do not have any neighbor of degree 1. Therefore, if a vertex u has a neighbor of degree 2, then $d(u) \geq 5$ and only the second discharging rule applies. The vertex u has a non-negative final weight as long as $d(u) - 2 - a \geq 2 \times a$, i.e. $a \leq 1$. It follows that every vertex of degree 1 has a final weight of -1, and every other vertex a non-negative weight, hence the following conclusion:

Proposition 2.10. *If* $D \geq 9$, *then* $\mathrm{mad}(G) \geq 3$.

We defined a global discharging rule as a rule that may allow some charge to travel arbitrarily far. One may note that we could avoid that situation here. Indeed, since we know H to be a forest, we could merely pick a vertex of degree 1 in H, assign it the maximal chain which corresponds to its incident edge, and remove it. By repeating the operation until H is a stable set, evey maximal chain of vertices of degree 2 is assigned to a neighbor of it. We can then transform the global discharging rule to make it state instead that every vertex to which a chain is assigned distributes a charge of 2 equally among the vertices in the chain. Then no charge moves further away than to a neighbor of neighbor. According to our definition, this should mean that the discharging proof is not global anymore. However, since the discharging rules have to take into account a structure of unbounded size in order to be defined (here, you need the full knowledge of a tree H to be able to decide to which edge a vertex in the middle gives weight to), we cannot say the method is purely local neither. When a global discharging proof can be written so that no charge travels arbitrarily far away, we say that the proof is semi-global.

Consequently, this simple trick of using well-known coloring results (e.g. every even cycle is 2-choosable) so as to reduce large structures with too many small vertices enables us to reach the threshold of 3. Purely local arguments seemed too weak for such a conclusion. However, in this very special case, it turns out that a purely local argument, when combined with a right order on graphs, can be even more powerful. Indeed, by considering a yet more refined order on graphs, we can prove that there is no vertex with two neighbors of degree 2 in G. This, combined with a slightly technical argument to improve Lemma 2.6, yields a proof using purely local discharging arguments that every graph G with $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 4$ is AVD $(\Delta(G) + 1)$-colorable. This is optimal, as shows the graph in Figure 2.9. However, this is a purely ad hoc proof which has very little chance of being of interest in other settings, contrary to the proof presented above, so we do not go into details here. A similar global discharging argument based on Lemma 1.7 can for example be found in [CH10]. We do not dwell on the topic, but rather question if the threshold of 3, once close, now reached, can be broken.

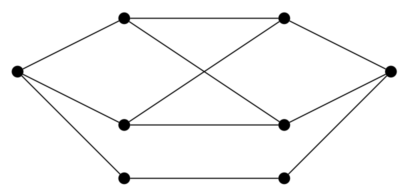


Figure 2.9: A graph G with $\Delta(G) = 3$ and $\mathrm{mad}(G) = \frac{11}{4} < 3$ such that $\chi'_{avd}(G) = 5$.

Beyond alternating cycles

In order to reach the threshold of 3, we simply used the classical result that an even cycle is 2-choosable, a result which proved decisive in other discharging proofs, e.g. in the case of list edge coloring [CH10]. There are stronger list coloring results than the fact that even cycles are 2-choosable, and the solution again came from the area of list edge coloring [BKW97]. The following method is due to Woodall [Woo10] who first proposed an alternative presentation of a theorem in [BKW97] in terms of discharging, when the initial paper offered a less tell-tale sequence of equations.

We present here its transposition in the setting of AVD coloring, and choose simplicity over any improvement on the lower bound on the maximum degree.

We start with an easy structural lemma on bipartite multigraphs.

Lemma 2.11. *Let H be a bipartite multigraph with vertex set $V(H)$ bipartitioned into $A \cup B$, with $A \neq \emptyset$. For $\alpha > 0$, if for every non-empty subset $B' \subseteq B$ and $A' = N(B') \subseteq A$, there exists a vertex $u \in A'$ with $d_{B'}(u) < \alpha$, then $\alpha|A| > |B|$.*

Proof. By induction on $|B|$. If $|B| < \alpha$, since $|A| \geq 1$, the conclusion holds. If $|B| \geq \alpha$, there exists $u \in A$ with $d(u) < \alpha$. We apply the induction hypothesis to the graph $H \setminus (\{u\} \cup N(u))$. It follows that $\alpha(|A| - 1) > |B| - \alpha$, hence the result. $\square$

Similarly as for Lemma 2.6, we can prove that no small vertex has more than one small neighbor.

Lemma 2.12. *G cannot contain a vertex v adjacent to two vertices u and w with $d(u) = d(v) = d(w)$ and $d(u) \leq \frac{k}{4} + 1$.*

Proof. We color by minimality $G \setminus \{(u, v), (v, w)\}$. Each edge of (u, v) and (v, w) has at most $(d(u) - 1) + (d(u) - 2)$ colors to avoid for the coloring to be proper, and an additional $(d(u) - 1)$ colors to avoid conflicts (each neighbor of u or w might create a conflict). Therefore, each of the two edges has at least $\frac{k}{4} + 2$ available colors. Now we only have to pick the right colors among those, so that there is no conflict between v and its neighbors. We color the edge (v, w) in such a way that there can be no conflict between u and v (this is possible as $\frac{k}{4} + 2 \geq 2$). Now we know that v cannot be in conflict with u, the edge (u, v) is the only one left uncolored and it still has $\frac{k}{4} + 1$ possible choices if we disregard the conflicts between v and its neighbors. The vertex

v has at most $\frac{k}{4}$ neighbors with whom to be in conflict, and none of them is u. We only need to remove an additional $\frac{k}{4}$ colors from the colors available for (u, v), and we color it with one of the remaining colors (there is at least one). We thus obtain an AVD $(k+1)$-coloring of G, a contradiction. $\square$

We consider vertices of degree at most m to be *small*, and set D such that $m \leq \frac{k}{8}$.

By Lemma 2.6, all the neighbors of a small vertex u must either be of degree at least $\frac{3k}{8}$ or of degree exactly $d(u)$. By Lemma 2.12, the vertex u has at most one neighbor of degree $d(u)$, and thus has at least $d(u) - 1$ neighbors of degree at least $\frac{3k}{8}$. Let S be the set of vertices of degree at least 2 and at most m in G. Note that by the above remark, every vertex of S is of degree 0 or 1 in $G[S]$.

Let G' be the multigraph obtained from G by contracting any edge in $G[S]$ (we keep the multiple edges thus created, if any), and S' the set of vertices corresponding to S in G'. Note that no vertex in S' is of degree less than 2 or more than $2m$ in G'. Let B be the set of vertices of degree at least $\frac{3k}{8}$ in G. By the above remark, all the neighbors of a vertex in S are either in B or in S. Note that $B \cap S = \emptyset$ since $\frac{3k}{8} > m$. Let H be the bipartite multigraph obtained from $G'[B \cup S']$ by deleting the edges in $G'[B]$. We try to prove that $|S'| < 2m|B|$.

Assume it is not the case, and $|S'| \geq 2m|B|$.

We claim that there is a non-empty set $S'' \subsetneq S'$ such that the subgraph H'' of H obtained by considering $H[S'' \cup N(S'')]$ is such that every vertex in $B'' = N(S'')$ has degree at least $2m$ in H''. Indeed, otherwise, in every subgraph $S'' \subsetneq S'$, there is a vertex in $N(S'')$ of degree less than $2m$ in $H[S'' \cup N(S'')]$, thus Lemma 2.11 holds and $|S'| < 2m|B|$, a contradiction.

Color by minimality $G \otimes S''$. As already noted in the case of alternating cycles, this corresponds to a list of $d_{S''}(v)$ colors assigned to each edge incident to $v \in B''$ and to a vertex in S''. Similarly, this corresponds to an edge list assignment of H'' such that $\forall u \in B'', \forall v \in S''$, if (u, v) is an edge then $|L(u, v)| = d_{H''}(u) \geq 2m \geq d_{H''}(v)$. Consequently, Theorem 1.10 applies, and H'' can be colored. To obtain an AVD $(k+1)$-coloring, it remains to color any edge between two vertices of degree at most m. They are already distinguished since they were contracted, so their incident sets of colors will be disjoint except for the color of their common edge. Since $m \leq \frac{k}{8}$, it suffices to take any color beside the $\frac{k}{4}$ that may already appear around one of the two vertices. Hence the conclusion that $|S'| < 2m|B|$.

We consider G, and assign to each vertex u a weight of $d(u) - m$. We introduce a global discharging rule, that every vertex in B gives a charge of m^2 to a common pot, from which every vertex u in S draws a charge of $d(u)$ if $d(u) \leq \frac{m}{2}$, and $m - d(u)$ if $d(u) \geq \frac{m}{2}$. Note that every small vertex draws from the common pot a charge of at most $\frac{m}{2}$. Therefore, the vertices in S' draw a total weight of at most $\frac{m}{2} \times |S'|$ from the common pot, while the vertices in B give a total weight of $m^2 \times |B|$ to the common pot. We know that $|S'| < 2m|B|$, so $\frac{m}{2} \times |S'| < m^2 \times |B|$, and the global rule is hence valid. We reintroduce the now usual rule that every vertex of degree at least $\frac{k}{2}$ gives a charge of 1 to every neighbor of degree 1. We take all vertices of degree at most $\frac{m}{2}$ to be ghosts. The two rules above ensure that every ghost vertex gives nothing and receives a weight equal to its degree, and that each small vertex that is not a ghost gives nothing and receives a weight which is just enough to get to a non-negative final weight. For large enough D, we can ensure that the vertices in B can afford to give to the common pot and to their neighbors of degree 1 (remember from Lemma 2.2 that they make for at most half the neighbors) without getting to a negative final weight. The vertices that are neither small nor in B have a final weight equal to their initial weight, which was positive by choice of "small". Therefore, the following holds.

Proposition 2.13. *If D is large enough, then* $\mathrm{mad}(G) \geq m$.

In other words, there is no threshold on the maximum average degree.

Theorem 2.14. *For every $m \in \mathbb{R}^+$, if G is a graph with $\mathrm{mad}(G) < m$ and $\Delta(G)$ large enough with regard to m, then G is AVD $(\Delta(G)+1)$-colorable.*

2.3 Some remarks on the discharging method

The proof sketch behind a coloring theorem obtained with a discharging argument (with the corresponding unavoidable set $\{C_1, \ldots, C_p\}$ of reducible configurations) can equivalently be seen as proof by induction (The empty graph can be colored, any graph obtained from a colorable graph by the reverse operation to the reduction when detecting a configuration C_i is colorable), where some space is dedicated to proving that the induction is complete (every graph can be built from the empty graph exclusively by the previous operations). The reducibility proofs could also be seen as parts of a pre-processing algorithm, which is proved by the discharging argument to always output the empty graph. Note that the reducible configurations detection can be sped up by assigning weight, applying the discharging rules, and looking around elements with a negative weight. When the discharging argument is purely local, this will very often grant a linear or quadratic coloring algorithm.

Note that a global discharging argument relies heavily on the sub-structure that is used. We need both to have a coloring argument to deal with it when present, and a discharging argument to say that its absence has implications on the average degree. The structure of an even cycle is massively used, but Theorem 1.10 is also extremely powerful. However, there is no reason to restrain oneself from using more exotic structures, like cacti with the help of Brooks' theorem (see the proof of Theorem 4.19). The main drawback to global discharging proofs is that it can significantly increase the running time of the coloring algorithm (if the global sub-structure is too hard to detect), and sometimes destroy entirely the initial hope of a coloring algorithm (if we appeal to an external, non-constructive coloring result).

Despite this warning, we should emphasize that most discharging proofs are constructive and immediately yield a polynomial coloring algorithm. We started this chapter by trying to define discharging proofs in opposition to non-discharging proofs. It could be noted that every single discharging proof can be translated in the realm of linear programming, with no discharging involved. This gives hope for at least partial automatization of the discharging proving (or checking) process.

It seems reasonable to believe that, in the last proof that we sketched, the strong global discharging argument cannot be replaced with local ones. However, can we find a natural example of a global discharging proof on planar graphs which cannot be transformed into one with no unbounded structure involved?

Chapter 3

Edge Coloring

In this chapter we consider the problem of list edge coloring. It includes personal work [Bon13], as well as joint work with Benjamin Lévêque and Alexandre Pinlou [BLP14d].

3.1 An overview of edge coloring

Edge coloring is a subcase of vertex coloring, as coloring the edges of a graph G is equivalent to coloring the vertices of the corresponding line graph $L(G)$, defined as follows. The *line graph* $L(G)$ has vertex set $E(G)$, and two vertices of $L(G)$ are adjacent iff their corresponding edges are incident in G. The class of line graphs is a strong restriction of the whole class of graphs. Indeed, an edge (u, v) is incident only to edges that have u or v as an endpoint. All the edges with endpoint u are incident to each other, and symmetrically for v. Consequently, in a line graph, there cannot be a vertex with three neighbors pairwise non-adjacent. There are other induced subgraphs that cannot appear in a line graph, and Beineke proved that we can actually obtain a characterization of the class of line graphs this way.

Theorem 3.1 ([Bei70])**.** *The class of line graphs is exactly the class of graphs that do not contain any of the nine graphs depicted in Figure 3.1 as an induced subgraph.*

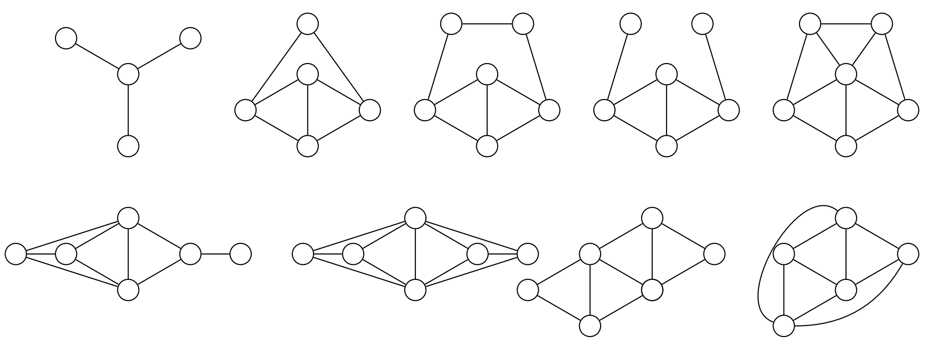


Figure 3.1: The nine forbidden induced subgraphs in a line graph.

The restriction is in fact so strong that, contrary to what happens in the general case where the choice number can be arbitrarily larger than the chromatic number (see Section 1.1), it is conjectured that, in this special case, there can be no difference at all between the two.

Conjecture 3.2 (List Coloring Conjecture)**.** *Every graph G satisfies $\chi'(G) = \chi'_\ell(G)$.*

The conjecture was suggested independently by Vizing in 1975, Albertson and Collins in 1981, and Bollobás and Harris in 1985. The latters were the only group to publish it [BH85] (see [JT96] for a full survey). Little is known toward this conjecture, but at least $\chi'_\ell(G)$ cannot be arbitrarily larger than $\chi'(G)$:

Theorem 3.3 ([Kah96])**.** *On the set of all graphs, it holds that $\chi'_\ell(G) = \chi'(G) \cdot (1 + o(1))$ as $\Delta(G) \to +\infty$.*

Let us now make some easy observations on edge coloring. Note that $\Delta(G) \leq \chi'(G) \leq \chi'_\ell(G) \leq 2\Delta(G) - 1$. The first part follows from the observation that given a vertex u in a graph G, all the edges incident to u are incident to each other, and need different colors. If a graph G is edge k-choosable, then it is edge L-colorable for every list assignment L of k colors to each edge. In particular, the graph G is edge k-colorable, which settles the middle inequality. For the last inequality, we can simply use a greedy algorithm that colors all the edges one after the other with the smallest available color. In the worst case, the edge to color is incident to two vertices of degree $\Delta(G)$ whose all other incident edges are already colored. Consequently, there is a color available for the edge, besides the $2 \times (\Delta(G) - 1)$ colors possibly used on the incident edges. We say that a graph G is *minimally edge-choosable* if it is edge $\Delta(G)$-choosable.

We do not have much information about Class 1 graphs in general (recall that a graph G is Class 1 if it is edge $\Delta(G)$-colorable). A natural option toward Conjecture 3.2 is therefore to restrict the problem to a graph class which was already proved to be Class 1. The hope is that the same structural properties that were helpful for standard edge coloring will turn out to remain helpful when lists are involved.

This hope proved to be justified in the case of bipartite graphs for example, as proved by Galvin through a kernel argument [Gal95].

Theorem 3.4 ([Gal95])**.** *Every bipartite multigraph is minimally edge-choosable.*

Using a similar argument, Borodin, Kostochka and Woodall managed to strengthen the previous theorem. It takes into account local maximal degrees instead of the maximum degree of the graph (see Theorem 1.10). We will mention its implications later in this chapter.

Note that Conjecture 3.2 can be generalized to lists of vectors instead of lists of colors. In that setting, each edge is assigned a linearly independent family of k vectors, and we try to color them in such a way that the set of vectors incident to each vertex is a linearly independent family. We denote by $\chi'_v(G)$ the minimum number of (linearly independent) vectors to assign to each edge to ensure that G can be colored. In 1989, Rota conjectured that in the complete bipartite graphs $K_{n,n}$, a choice of n vectors on each edge is enough, provided some restrictions on the choice of lists. This is called Rota's Bases Conjecture, though it was initially not presented in terms of graph coloring [HR94]. Only partial results are known toward this so far (see e.g. [AK14] for a weaker version or [Dri97, Dri98] for particular values of n). However, we are not aware of any indication that it should not be true for every graph G with no restrictions on the choice of lists (replacing n with $\chi'(G)$), hence the following question: is it true that every graph G satisfies $\chi'_v(G) = \chi'(G)$?

Another direction of research is to find lower bounds on the average degree of a critical Class 2 graph (that is, a Class 2 graph whose proper subgraphs are Class 1). The best known bound so far is that of Theorem 3.5.

Theorem 3.5 ([Woo07]). *Every critical Class 2 graph G satisfies* $\mathrm{ad}(G) \geq \frac{2\Delta(G)}{3}$.

As a direct corollary, it holds that graphs with smaller maximum average degree must be Class 1, as follows.

Corollary 3.6. *Every graph G with* $\mathrm{mad}(G) < \frac{2\Delta(G)}{3}$ *is edge* $\Delta(G)$*-colorable.*

It is even conjectured that Theorem 3.5 is true with $\Delta(G)-1$ instead of $\frac{2\Delta(G)}{3}$ [Viz65, SZ01, SZ02]. However, we do not know yet how to obtain such a linear dependency in the case of list coloring. So far, the best and only result of the kind is as follows:

Theorem 3.7 ([BKW97]). *Every graph G with* $\mathrm{mad}(G) < \sqrt{2\Delta(G)}$ *is edge* $\Delta(G)$*-choosable.*

The proof method behind Theorem 3.7 is similar to the one presented in Chapter 2 (see the proof of Theorem 2.14) as a global discharging argument coupled with a powerful coloring result (namely Theorem 1.10). The idea is actually useful outside the field of list edge coloring. As illustrated in Chapters 2 and 4, the same proof method can be successfully applied to other colorings.

In the case of planar graphs, Vizing [Viz65] proved that $\chi'(G) = \Delta(G)$ for every planar graph G with $\Delta(G) \geq 8$. He gave examples of planar graphs with $\Delta(G) = 4$ or 5 that are not Class 1, and conjectured that no such graph exists for $\Delta(G) = 6$ or 7. This remains open for $\Delta(G) = 6$, but the case $\Delta(G) = 7$ was solved through a discharging argument.

Theorem 3.8 ([SZ01]). *Every planar graph G with* $\Delta(G) \geq 7$ *is edge* $\Delta(G)$*-colorable.*

However, the best known result for list coloring is that when $\Delta(G) \geq 12$, a planar graph G is minimally edge-choosable [BKW97].

Theorem 3.9 ([BKW97]). *Every planar graph G with* $\Delta(G) \geq 12$ *is edge* $\Delta(G)$*-choosable.*

A lot of incremental results were proved about planar graphs with restrictions on cycles. Two cycles are said to be *incident* if they share at least a vertex, and *adjacent* if they share at least an edge.

For example, Cranston [Cra09] proved that, when adjacent triangles do not appear in the graph, $\Delta(G) \geq 9$ suffices to prove that G is edge $\Delta(G)$-choosable. This is also true when $\Delta(G) \geq 8$ and there is either no 5-cycle or no 6-cycle [MWY09].

It is also known that planar graphs with $\Delta(G) \geq 7$ and no C_4 [HLC06] or no two adjacent $C_{\leq 4}$ [LMW13], or $\Delta(G) \geq 8$ and no triangle adjacent to a C_4 [LX11] are minimally edge-choosable. We strengthened these last results by proving that:

Theorem 3.10. *Every planar graph G with* $\Delta(G) \geq 7$ *and no triangle adjacent to a* C_4 *satisfies* $\chi'_\ell(G) = \Delta(G)$.

The proof, which can be found in Section 3.2, relies on a global discharging argument. The global character of the proof aims at ensuring that vertices of degree 2 receive extra weight, and stems from two global structures. One is a tree-like structure based on the fact that there is no alternating cycle of vertices of degree 2. The other, more atypical, is a fan-like structure composed of consecutive faces of degree 4 incident to a same vertex, with a vertex of degree 3 opposite to it. The claim is that if one end of the fan is a vertex of degree 2, then by spreading out the fan as much as possible, the other end of the fan will afford to give some extra weight to the vertex of degree 2.

As mentioned earlier, an easier case of the List Coloring Conjecture is when χ' reaches the lower bound Δ: otherwise it is harder to prove that χ'_ℓ is actually exactly the same. Fortunately, we also have a strong upper-bound on the χ', as $\chi'(G) \leq \Delta(G)+1$ by Theorem 1.9.

Consequently, a weaker form of Conjecture 3.2 (called the *weak List Coloring Conjecture* (weak LCC), or alternatively *Vizing's conjecture*) is the following:

Conjecture 3.11 (weak List Coloring Conjecture). *Every graph G satisfies $\chi'_\ell(G) \leq \Delta(G)+1$.*

Borodin [Bor90] proved in 1990 that the weak List Coloring Conjecture is true for planar graphs of maximum degree at least 9.

Theorem 3.12 ([Bor90]). *Every planar graph G with $\Delta(G) \geq 9$ is edge $(\Delta(G)+1)$-choosable.*

His proof was later simplified by Cohen and Havet [CH10] into a short and elegant global discharging proof. Again, a lot of partial positive results appeared on planar graphs with restrictions on cycles. Here we prove the following:

Theorem 3.13. *Every planar graph G with $\Delta(G) \leq 8$ is edge 9-choosable.*

This solves the weak List Coloring Conjecture for planar graphs with maximum degree 8, mentioned in a recent survey of Borodin [Bor13] as Problem 5.9. Unfortunately, the proof is rather long and mostly made of case analyses, as presented in Section 3.3. Here, global discharging did not appear to be useful, and the configurations and arguments are purely local. In fact, the decisive idea in the proof lies on recoloring arguments using directed graphs (see Claims 3, 4 and 6 of Section 3.3 for occurrences of it in the proof). Roughly, we prove that, when trying to reduce a given configuration, we can extend the partial coloring to the whole graph unless the remaining colors for each edge satisfy a very specific property. We note that any recoloring of a particular set of (already colored) edges will break that property. We consider a larger set of edges, that we try to recolor so as to recolor in particular one of the desired edges. We therefore represent the constraints with a directed graph with the larger set of edges as a vertex set. We add a directed edge from a vertex to another when the edge corresponding to the former could be colored with the color of the latter (in case the latter was recolored too, that is). Then it suffices to study the out-degrees and prove that there is either a color permutation which contains one of the desired edges, or that a new color can be introduced on one edge and the change propagated to a desired edge.

This new trick allows us to deal with configurations that would not yield under usual techniques, and thus to lower the bound in Theorem 3.12 as follows.

Corollary 3.14. *Every planar graph with $\Delta(G) \geq 8$ is edge $(\Delta(G) + 1)$-choosable.*

Though this simple argument does not seem to be enough to prove Conjecture 3.11 for $\Delta = 7$, it might be interesting to try to improve similarly Theorem 3.9.

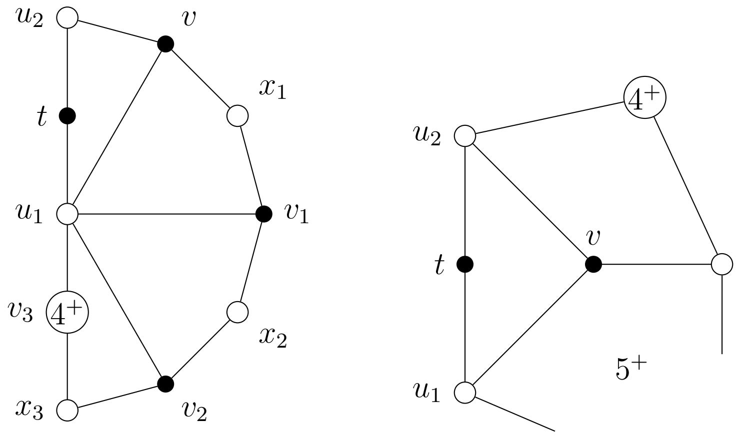
Since it is known that planar graphs with $\Delta(G) \leq 4$ are edge $(\Delta(G) + 1)$-choosable [JMS99, Viz76], the remaining cases for the weak List Coloring Conjecture restricted to planar graphs are when $\Delta(G) \in \{5, 6, 7\}$. There is no obvious order on the complexity of these three cases. If other problems are any indication, the case $\Delta(G) = 5$ will be solved through an ad hoc coloring of the entire graph by breaking it into appropriate substructures (e.g. two subcubic forests, or a spanning collection of cycles and a subcubic graph), while the case $\Delta(G) = 7$ will be solved through a (possibly computer-assisted) discharging argument. As for $\Delta(G) = 6$, it will be a hybrid case, the last one to yield, because none of the two approaches preferred for higher and lower cases apply here. Of course, this is mere speculation. However, note that the case $\Delta(G) = 6$ is the only open case left for standard edge coloring with Δ colors, as well as for other similar problems which we do not dwell on.

For an informal hint of why the discharging method is harder to use when $\Delta(G) \leq 6$ and edges are involved (not that it is impossible), let us recall from Chapter 1 the usual proof sketch. We consider by contradiction a minimal counter-example, and first prove that some configurations are reducible. Usually, such configurations correspond to ruling out the cases where there are too many small vertices in one place, or where there is a small vertex with no large vertex around. The notions of "small" and "large" usually depend on the number of colors available. However, when $\Delta(G) \leq 6$, and the number of colors is say 7, while edges have to be colored, the notion of "small" can barely include vertices of degree 5, let alone vertices of degree 6. Then, we cannot rule out the case of a planar triangulation where most vertices are of degree 6, and the rest of degree 5. Assume we assign to each vertex a weight of its degree minus 6 and to each face a weight of twice its degree minus 6, and try to prove that the total weight cannot be negative (looking for a contradiction to Euler's formula). Then no element has a positive weight, while vertices of degree 5 have a negative one: the situation is hopeless even if we manage to relax it to some small degree. In our case, the current techniques that we use to reduce configurations for the list edge coloring problem are simply not enough.

3.2 Every planar graph G with $\Delta(G) \geq 7$ and no triangle adjacent to a C_4 is edge $\Delta(G)$-choosable

In this section, we prove Theorem 3.10 that a planar graph G with $\Delta(G) \geq 7$ and no triangle adjacent to a cycle of length 4 is edge $(\Delta(G))$-choosable.

Let $k \geq 7$. Given a planar embedding of a graph and a face $f = (u, v, w, x)$, we say w is the vertex *opposite* to u in f. If there is a face (t, u_1, v, u_2) with $d(t) = 2$ and $d(v) = 3$, we say a neighbor w of u_1 is the (v, u_1)-*support of* t if the sequence $(t, v, v_1, v_2, \ldots, v_{p-1}, v_p = w)$ of consecutive neighbors of u_1 contains only vertices of degree 3 except for t, and any two consecutive neighbors of the sequence are part of the boundary of a face of degree 4 that contains u_1, while the edge (u_1, w) belongs to a face of degree at least 5, or to a face of degree 4 with a vertex of degree at least 4 opposite to w (see Figure 3.2). Given t, v and u_1, at most one vertex can satisfy this property. Note that v can itself be the (v, u_1)-support of t, and that it can even be also the (v, u_2)-support of t. Note that, by definition, if w is the (v, u_1)-support of t, then the edge (u_1, w) is incident, on one side, to either a face of degree at least 5, or to a face of degree 4 where the vertex opposite to w is of degree ≥ 4, and, on the other side, to a face of degree 4 where the vertex opposite to w is of degree 3. Consequently, a vertex cannot be support more than twice, as a support vertex is of degree 3.



The vertex v_2 is the (v, u_1)-support of t.

The vertex v is both the (v, u_1)-support and the (v, u_2)-support of t.

Figure 3.2: Examples of supports.

Forbidden Configurations

A *constraint* of an element $e \in E$ is an already colored element of E that is adjacent to e.

We define configurations **(**C_1**)** to **(**C_7**)** (see Figure 3.3). Configurations **(**C_1**)**, **(**C_4**)** and **(**C_7**)** are standard. Configurations **(**C_2**)** and **(**C_3**)** follow from the theorem statement. Configuration **(**C_5**)** appears in [CKŠ07], and we introduce Configuration **(**C_6**)**.

- **(**C_1**)** is an edge (u, v) with $d(u) + d(v) \leq k + 1$ and $d(u) \leq \lfloor \frac{k}{2} \rfloor$.
- **(**C_2**)** is a cycle (u, v, w, x) such that (u, w) is a chord.
- **(**C_3**)** is a cycle (u, v, w, x, y) such that (w, y) is a chord.
- **(**C_4**)** is a cycle $(u_1, v_1, ..., u_p, v_p, u_1), p \geq 2$ where $\forall i, d(v_i) = 2$.
- **(**C_5**)** is a vertex v_1 with $d(v_1) = 2$ such that, for u and x_1 its two neighbors, there is a path $(v_1, x_1, v_2, \ldots, v_p, x_p, v_{p+1})$ $(p \geq 1)$ such that $\forall i, v_i$ is adjacent to u, with $\forall\, 2 \leq i \leq p, d(v_i) = 3$, and $d(v_{p+1}) = 2$.

- **(C_6)** is a vertex v_1 with $d(v_1) = 2$ such that, for u and x_1 its two neighbors, there is a cycle $(x_1, v_2, x_2, \ldots, x_{p-1}, v_p)$ such that $\forall i, v_i$ is adjacent to u, and $\forall i \geq 2, d(v_i) = 3$.
- **(C_7)** is a vertex u with $d(u) = 4$ that has at least two neighbors u_1 and u_2 with $d(u_1) = d(u_2) = 4$.

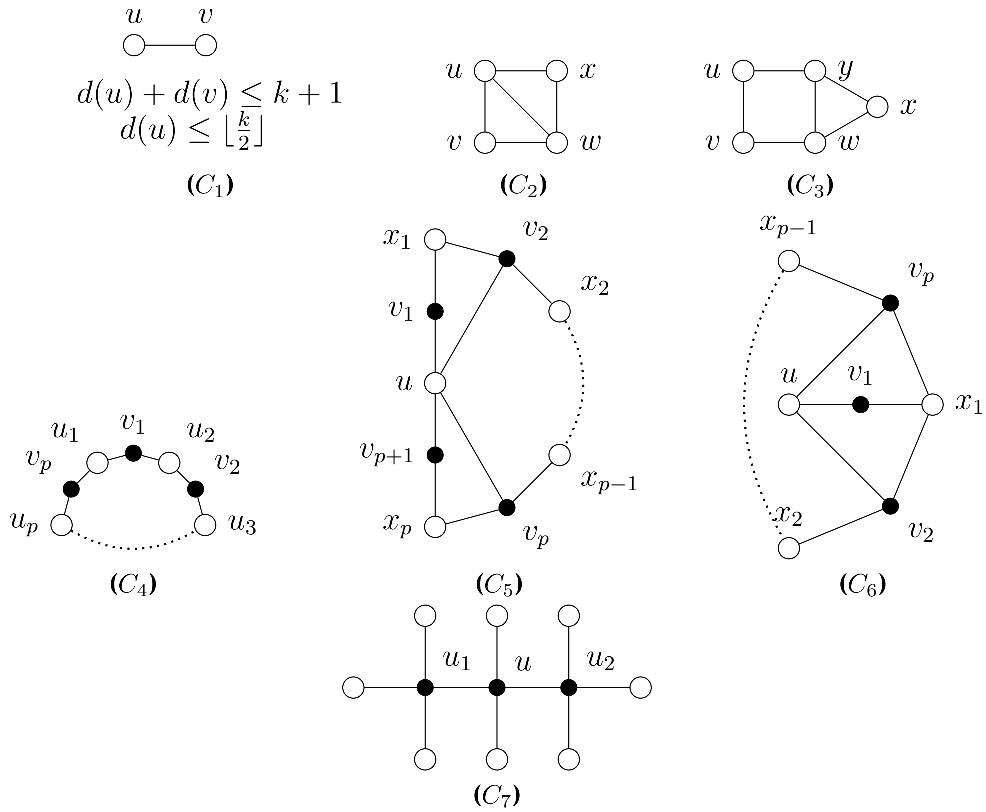


Figure 3.3: Forbidden configurations.

Lemma 3.15. *If G is a minimal planar graph such that $\Delta(G) \leq k$, no triangle is adjacent to a cycle of length four, and $\chi''_\ell(G) > k + 1$ or $\chi'_\ell(G) > k$, then G cannot contain any of Configurations (C_1) to (C_7).*

Proof.

Claim 1. *G cannot contain (C_1).*

Proof. Using the minimality of G, we color $G \setminus \{(u, v)\}$. Since $\Delta(G) \leq k$, and $d(u) + d(v) \leq k + 1$, the edge (u, v) has at most $k - 1$ constraints. There are k colors, so we can color (u, v), thus extending the coloring of $G \setminus \{(u, v)\}$ to G. □

Claim 2. *G cannot contain (C_2).*

Proof. The triangle (u, v, w) shares two edges with the cycle (u, v, w, x) of length 4. □

Claim 3. *G cannot contain (C_3).*

Proof. The triangle (u, x, y) shares an edge with the cycle (u, v, w, x) of length 4. $\square$

Claim 4. *G cannot contain (C_4).*

Proof. Using the minimality of G, we color $G \setminus \{v_i\}_{1 \leq i \leq p}$. Every edge (u_i, v_i) or (v_i, u_{i+1}) (subscript taken modulo p) has at most $k-2$ constraints, so there are at least 2 colors available for each of them. Since even cycles are 2-choosable, we can color the (u_i, v_i)'s and (v_i, u_{i+1})'s. Then we can extend the coloring of $G \setminus \{v_i\}_{1 \leq i \leq p}$ to G. $\square$

Claim 5. *G cannot contain (C_5).*

Proof. Let $L : E \to \mathcal{P}(\mathbb{N})$ be a color assignment such that $\forall a \in E, |L(a)| \geq k$ and such that G is not L-colorable. Using the minimality of G, we L-color $G \setminus \{v_i | 1 \leq i \leq p+1\}$. We denote by $L'(e)$ the remaining available colors for every edge e that is not colored yet.

Every edge e incident to u and not colored yet has at most $d(u)-(p+1) \leq k-(p+1)$ constraints, thus $|L'(e)| \geq p+1$. Every edge e that is not incident to u and is not colored yet has at most $k-2$ constraints, thus $|L'(e)| \geq 2$. We consider the worst case, i.e. that these inequalities are actually equalities.

We first consider the case where $L'(v_1, x_1) \not\subset L'(u, v_1)$ or $L'(v_{p+1}, x_p) \not\subset L'(u, v_{p+1})$. Consider w.l.o.g. $L'(v_1, x_1) \not\subset L'(u, v_1)$. Color (v_1, x_1) with a color that does not belong to $L'(u, v_1)$, and color arbitrarily $(x_1, v_2), \ldots, (x_p, v_{p+1})$, successively. Then at least $p-1$ colors remain for each (u, v_i) with $2 \leq i \leq p$, while p colors remain for (u, v_{p+1}) and $p+1$ for (u, v_1) by assumption. We color arbitrarily $(u, v_2), \ldots, (u, v_{p+1})$, in that order, and finally (u, v_1): then G is L-colorable, a contradiction. Thus we can assume from now on that $L'(v_1, x_1) \subset L'(u, v_1)$ and $L'(v_{p+1}, x_p) \subset L'(u, v_{p+1})$. We prove the following.

1. *We can color $\{(u, v_i), (v_i, x_i), (x_i, v_{i+1}) | 1 \leq i \leq p\} \setminus \{u, v_1\}$ in such a way that for L'' the list assignment of remaining available colors for the edges uncolored yet (here (u, v_1) and (u, v_{p+1})), we have $L''(u, v_1) \neq L''(u, v_{p+1})$ if $|L''(u, v_1)| = |L''(u, v_{p+1}| = 1$.*

Proof. We consider two cases depending on whether $L'(u, v_1) = L'(u, v_{p+1})$.

- *Assume $L'(u, v_1) \neq L'(u, v_{p+1})$.*
  Let a be a color in $L'(u, v_1) \setminus L'(u, v_{p+1})$. Color (v_1, x_1) with a color other than a, then color successively $(x_1, v_2), \ldots, (x_p, v_{p+1}), (u, v_2), \ldots, (u, v_p)$. Now $|L''(u, v_1)| \geq 1$, $|L''(u, v_{p+1})| \geq 1$ and $L''(u, v_1) \neq L''(u, v_{p+1})$ if $|L''(u, v_1)| = |L''(u, v_{p+1})| = 1$. Indeed, if $|L''(u, v_{p+1})| = 1$ then, since $a \notin L'(u, v_{p+1})$, the color a does not appear on the edges $(x_1, v_2), \ldots, (x_p, v_{p+1}), (u, v_2), \ldots, (u, v_p)$. Together with the fact that (v_1, x_1) was purposely not colored with a and these are the only uncolored edges around (u, v_1), we have that $a \in L''(u, v_1) \setminus L''(u, v_{p+1})$.

- *Assume $L'(u, v_1) = L'(u, v_{p+1})$.*
  We color $(v_1, x_1), (x_1, v_2), \ldots, (x_p, v_{p+1})$ as though it were a cycle (i.e. (v_1, x_1) and (x_p, v_{p+1}) have to receive different colors): it is possible since even cycles are 2-choosable. Then we color arbitrarily the (u, v_i)'s with $2 \leq i \leq p$. It follows that $L''(u, v_1) \neq L''(u, v_{p+1})$ if $|L''(u, v_1)| = |L''(u, v_{p+1})| = 1$. Indeed, $L'(u, v_1) = L'(u, v_{p+1})$, and for S the set of colors on the edges $(u, v_2), \ldots, (u, v_p)$, for α and β the colors of (v_1, x_1) and (v_{p+1}), we have $L''(u, v_1) = L'(u, v_1) \setminus (S \cup \alpha)$ and

$L''(u, v_{p+1}) = L'(u, v_1) \setminus (S \cup \beta)$. If $|L''(u, v_1)| = |L''(u, v_{p+1})| = 1$, then $\{\alpha, \beta\} \cap S = \emptyset$. Since $\alpha \neq \beta$, this implies $L''(u, v_1) \neq L''(u, v_{p+1})$.

$\Diamond$

By (1), we color $\{(u, v_i), (v_i, x_i), (x_i, v_{i+1}) | 1 \leq i \leq p\} \setminus \{u, v_1\}$ in such a way that, for L'' the list of remaining available colors for (u, v_1) and (u, v_{p+1}), we have $L''(u, v_1) \neq L''(u, v_{p+1})$ if $|L''(u, v_1)| = |L''(u, v_{p+1}| = 1$. We color arbitrarily (u, v_1) and (u, v_{p+1}), starting with the one with fewest available colors if any, thus extending the L-coloring of $G \setminus \{v_i | 1 \leq i \leq p+1\}$ to an L-coloring of G, a contradiction. $\square$

We first prove an intermediary claim which will be instrumental in the proof of Claim 7.

Claim 6. *Let Γ be $K_{2,3}$, and y (resp. z) be a vertex of degree 3 (resp. 2) in Γ. The graph Γ is L_1-edge-colorable for any list assignment L_1 of 3 colors to each of the two edges incident to z and 2 colors to each of the other edges, where the two edges incident to y but not to z do not receive the two same colors.*

Proof. We denote a, b, c the three edges incident to y, where c is the edge (y, z), and d (resp. e, f) the other edge incident to c (resp. b, a). We have $|L_1(a)| = |L_1(b)| = |L_1(e)| = |L_1(f)| = 2$ and $|L_1(c)| = |L_1(d)| = 3$, with $L_1(a) \neq L_1(b)$. We consider different cases depending on the list intersections. In the first three cases, we do not use the fact $L(a) \neq L(b)$, which allows us to consider in these cases the problem to be symmetric w.r.t. (a, b, c) and (f, e, d).

- *Assume $L_1(a) \cap L_1(e) \neq L_1(b) \cap L_1(f)$.*
  W.l.o.g., assume that $(L_1(a) \cap L_1(e)) \setminus (L_1(b) \cap L_1(f)) \neq \emptyset$ and take an element α of it. Color a and e with α. One of b and f still has 2 colors available. Assume w.l.o.g. it is b. Then we color successively f, d, c and b.

- *Assume $L_1(a) \cap L_1(e) = L_1(b) \cap L_1(f)$ and $L_1(a) \cup L_1(e) \neq L_1(b) \cup L_1(f)$.*
  Then assume w.l.o.g. $L_1(a) \subsetneq L_1(b) \cup L_1(f)$. Color a with $\alpha \notin (L_1(b) \cup L_1(f))$. Then either $L_1(e) = L_1(f)$ and we color d with $\beta \notin L(f)$, then color successively c, b, e and f. Or $L_1(f) \neq L_1(e)$: we color f with a color not in $L_1(e)$, and we can color (b, c, d, e) since even cycles are 2-choosable.

- *Assume $L_1(a) \cap L_1(e) = L_1(b) \cap L_1(f)$, $L_1(a) \cup L_1(e) = L_1(b) \cup L_1(f)$ and $L_1(a) \cup L_1(b) \nsubseteq L_1(c)$ or $L_1(e) \cup L_1(f) \nsubseteq L_1(d)$.*
  Then assume w.l.o.g. $L_1(a) \cup L_1(b) \nsubseteq L_1(c)$ and there is $\alpha \in L_1(a) \setminus L_1(c)$. Color a with α. If $\alpha \notin L_1(b)$, color f, e, b, d and c, and similarly if $\alpha \notin L_1(f)$: color b, e, f, d and c. If $\alpha \in L_1(b) \cap L_1(f)$, then $\alpha \in L_1(e)$ by assumption. Then we color e with α, and color successively b, f, d and c.

- *Assume $L_1(a) \cap L_1(e) = L_1(b) \cap L_1(f)$, $L_1(a) \cup L_1(e) = L_1(b) \cup L_1(f)$, $L_1(a) \cup L_1(b) \subseteq L_1(c)$ and $L_1(e) \cup L_1(f) \subseteq L_1(d)$.*
  Then we must have $L_1(a) = \{1, 2\}$, $L_1(b) = \{1, 3\}$ and $L_1(c) = \{1, 2, 3\}$, and for some $\alpha \notin \{2, 3\}$, $L_1(f) = \{\alpha, 2\}$, $L_1(e) = \{\alpha, 3\}$ and $L_1(d) = \{\alpha, 2, 3\}$. Then we color a with 1, b with 3, c with 2, d with 3, e with α and f with 2.

Thus Γ is L_1-colorable. $\square$

Claim 7. *G cannot contain (C_6).*

Proof. Let $L : V \cup E \to \mathcal{P}(\mathbb{N})$ (resp. $E \to \mathcal{P}(\mathbb{N})$) be a color assignment such that $\forall a \in V \cup E, |L(a)| \geq k+1$ (resp. $\forall a \in E, |L(a)| \geq k$) and such that G is not L-colorable. Using the minimality of G, we L-color $G \setminus \{v_i | 1 \leq i \leq p\}$. Note that $d(v_i) \leq 3 < \frac{k}{2}$ for all $1 \leq i \leq p$, so coloring the edges is enough. We denote by $L'(e)$ the remaining available colors for every edge e that is not colored yet.

Every edge e incident to u and not colored yet has at most $d(u)-p+1$ (resp. $d(u)-p$) constraints, thus $|L'(e)| \geq p$. Every edge e incident to x_1 and not colored yet has at most $d(x_1) - 3 + 1$ (resp. $d(x_1) - 3$) constraints, thus $|L'(e)| \geq 3$. Every edge e that is not incident to u nor x_1 and is not colored yet has at most $k-2+1$ (resp. $k-2$) constraints, thus $|L'(e)| \geq 2$. In the worst case, these inequalities are actually equalities. We first prove the following two claims.

1. *We can color $\{(u, v_i), (v_i, x_i), (x_i, v_{i+1}) | 2 \leq i \leq p-1\} \setminus \{u, v_2\}$ in such a way that for L'' the list assignment of remaining available colors for the edges uncolored yet, we have $L''(x_1, v_2) \neq L''(x_1, v_p)$ if $|L''(x_1, v_2)| = |L''(x_1, v_p)| = 2$.*

Proof. We consider two cases depending on whether $L'(x_1, v_2) = L'(x_1, v_p)$.

- *Assume $L'(x_1, v_2) \neq L'(x_1, v_p)$.*
  Let $a \in L'(x_1, v_2) \setminus L'(x_1, v_p)$, and color (v_2, x_2) with a color distinct from a. Color successively $(x_2, v_3), \ldots, (x_{p-1}, v_p)$, then $(u, v_3), \ldots, (u, v_{p-1})$. Now $a \in L''(x_1, v_2) \setminus L''(x_1, v_p)$ unless $|L''(x_1, v_p)| \geq 3$.

- *Assume $L'(x_1, v_2) = L'(x_1, v_p)$.*
  We color $(v_2, x_2), (x_2, v_3), \ldots, (x_{p-1}, v_p)$ as though it were a cycle (i.e. (v_2, x_2) and (x_{p-1}, v_p) have to receive different colors): it is possible since even cycles are 2-choosable. Then we color arbitrarily the (u, v_i)'s with $3 \leq i \leq p-1$. It follows that $L''(x_1, v_2) \neq L''(x_1, v_p)$ if $|L''(x_1, v_2)| = |L''(x_1, v_p)| = 2$.

$\diamondsuit$

By (1), we color $\{(u, v_i), (v_i, x_i), (x_i, v_{i+1}) | 2 \leq i \leq p-1\} \setminus \{u, v_2\}$ in such a way that for L'' the list assignment of remaining available colors for the edges uncolored yet, we have $L''(x_1, v_2) \neq L''(x_1, v_p)$ if $|L''(x_1, v_2)| = |L''(x_1, v_p)| = 2$. Then, we can assume $|L''(x_1, v_2)| = |L''(x_1, v_p)| = |L''(u, v_2)| = |L''(u, v_p)| = 2$, $|L''(u, v_1)| = |L''(v_1, x_1)| = 3$ and $L''(x_1, v_2) \neq L''(x_1, v_p)$. Then we color G by Claim 6. $\square$

Claim 8. *G cannot contain (C_7).*

Proof. We color $G \setminus (u, u_1)$, it has at most 6 adjacent edges, and at least 7 colors in its list, so we can color it. $\square$

$\square$

Discharging rules

Given a planar map, we design discharging rules $R_{1.1}$, $R_{1.2}$, $R_{1.3}$, $R_{1.4}$, $R_{2.1}$, $R_{2.2}$, $R_{3.1}$, $R_{3.2}$, R_4 and R_g (see Figure 3.4). We also use a so-called *common pot* which is empty at the beginning, receives weight from some vertices and gives weight to some others.

Rules on faces:

For any face f of degree at least 4,

- Rule R_1 is when f is incident to a vertex u of degree $d(u) \leq 3$.
  - Rule $R_{1.1}$ is when $d(f) = 4$, and for v the vertex incident to f that is not consecutive to u on the boundary of f, we have $d(v) \leq 3$. Then f gives 1 to u.
  - Rule $R_{1.2}$ is when $d(f) = 4$, and for v the vertex incident to f that is not consecutive to u on the boundary of f, we have $d(v) \geq 4$. Then f gives $\frac{3}{2}$ to u.
  - Rule $R_{1.3}$ is when $d(f) \geq 5$ and $d(u) = 3$ or $d(u) = 2$ and the two neighbors of u are not adjacent. Then f gives $\frac{3}{2}$ to u.
  - Rule $R_{1.4}$ is when $d(f) \geq 6$ and $d(u) = 2$ such that its two neighbors are adjacent. Then f gives $\frac{5}{2}$ to u.
- Rule R_2 is when f is incident to a vertex u of degree $4 \leq d(u) \leq 5$.
  - Rule $R_{2.1}$ is when $d(f) = 4$ or $d(u) = 5$. Then f gives $\frac{1}{2}$ to u.
  - Rule $R_{2.2}$ is when $d(f) \geq 5$ and $d(u) = 4$. Then f gives 1 to u.
- Rule R_3 is when f contains an edge such that there is a vertex u of degree $d(u) = 2$ that is adjacent to its two endpoints. Then f gives $\frac{1}{2}$ to u.

Note that if a vertex u appears more than once on the boundary of f, the rules are applied as many times as u appears on the boundary.

Rules on vertices:

- Rule R_4 states that for any quadruple (x, u, u_1, v) such that x is the (v, u_1)-support of u, x gives $\frac{1}{4}$ to u. (Note that R_4 can be applied twice for the same x and u if there are two different such quadruples involving them).
- Rule R_g states that for any vertex x of degree k, x gives 1 to the common pot, and every vertex of degree 2 draws 1 from it.

Lemma 3.16. *A graph G with $\Delta(G) \leq k$ that does not contain Configurations* (C_1) *to* (C_7) *is not planar.*

Proof. Assume for contradiction that G is planar. Then it admits an embedding in the plane with no crossing edges. We attribute to each vertex u a weight of $d(u) - 6$, and to each face a weight of $2d(f) - 6$, and apply discharging rules R_1, R_2, R_3, R_4 and R_g.

Since Configurations (C_1) and (C_4) do not appear, the subgraph induced in G by the edges incident to a vertex of a degree 2 is a forest, both its neighbors are of degree

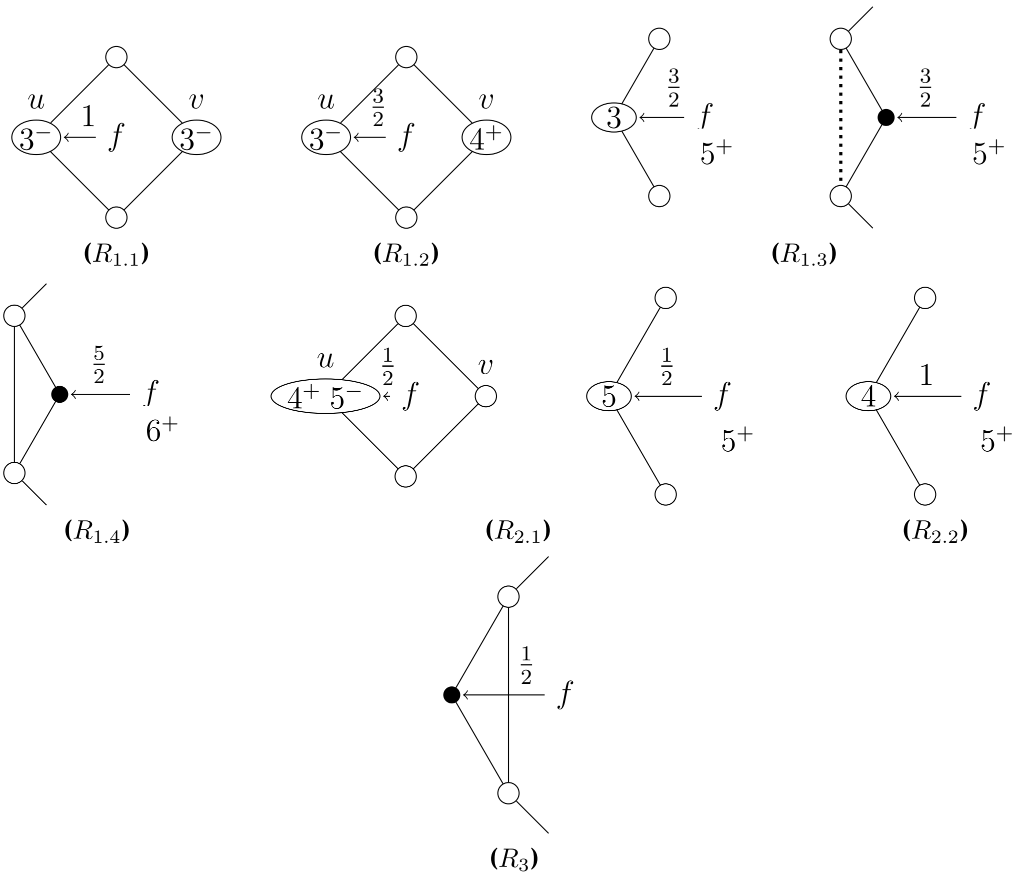


Figure 3.4: Discharging rules R_1, R_2 and R_3.

k. Thus there are at least as many vertices of degree k as there are vertices of degree 2, so R_g is valid: the common pot does not distribute more weight than it receives.

We first prove the following useful lemma:

Lemma 3.17. *In G, every vertex v_0 with $d(v_0) = 2$ that belongs to a face $f_0 = (u_1, v_1, u_2, v_0)$ with $d(v_1) = 3$ admits a (v_1, u_1)-support and a (v_1, u_2)-support.*

Proof. Assume by contradiction that v_0 has no (v_1, u_1)-support. Let $(f_0, f_1, \ldots, f_p)$ be a maximal sequence of distinct faces of degree 4 where $f_i = (u_1, v_{i+1}, x_i, v_i)$ (here $x_0 = u_2$) and $d(v_{i+1}) \leq 3$. Note that $d(v_{i+1}) = 3$ for every $0 \leq i \leq p$ since Configurations **(**C_5**)** and **(**C_6**)** do not appear. Let f' be the other face to which the edge (u_1, v_{p+1}) belongs. We have $d(f') \geq 4$ since G does not contain Configuration **(**C_3**)**. By the contradiction assumption, we have $f' = (u_1, v_{p+1}, x_{p+1}, v_{p+2})$ with $d(v_{p+2}) \leq 3$ as v_{p+1} would otherwise be a (v_1, u_1)-support of v_0. Since p was chosen to be maximal, we must have $f' = f_0$, a contradiction with the fact that Configuration **(**C_6**)** does not appear in G. □

We show that all the vertices have a weight of at least 0 in the end.

Let u be a vertex of G. Since Configuration **(**C_1**)** does not appear, $d(u) \geq 2$. We consider different cases depending on the value of $d(u)$.

1. $d(u) = 2$.
   We consider two cases depending on whether u is incident to a triangle.

   a) *Assume u belongs to a triangle (u, v, w).*
      Let f_1 and f_2 be the two faces adjacent to (u, v, w), where f_1 is the face incident to u. Then, in order to avoid Configurations **(**C_2**)** and **(**C_3**)**, we must have $d(f_1) \geq 6$ and $d(f_2) \geq 5$. So, by Rules $R_{1.4}$, R_3 and R_g, u receives $\frac{5}{2}$ from f_1, $\frac{1}{2}$ from f_2 and 1 from the common pot. So u has an initial weight of -4, gives nothing and receives 4, so it has a non-negative final weight.

   b) *Otherwise, let f_1 and f_2 be the two faces to which u belongs, with $d(f_1), d(f_2) \geq 4$.*
      For each $f_i \in \{f_1, f_2\}$, we have three cases:

      i. *Either $f_i = (u, u_1, v, u_2)$, with $d(v) \leq 3$.*
         Then $d(v) = 3$ since Configuration **(**C_4**)** does not appear, and by Lemma 3.17, u has a (v, u_1)-support, and a (v, u_2)-support. Thus, by Rules $R_{1.1}$ and R_4, u receives 1 from f_i and $\frac{1}{4}$ from each of its $(v, \_)$-supports, so u receives $\frac{3}{2}$ on the side of f_i.

      ii. *Or $f_i = (u, u_1, v, u_2)$, with $d(v) \geq 4$.*
          Then, by Rule $R_{1.2}$, u receives $\frac{3}{2}$ on the side of f_i.

      iii. *Or $d(f_i) \geq 5$.*
           Then, by Rule $R_{1.3}$, u receives $\frac{3}{2}$ on the side of f_i.

      So u receives $2 \times \frac{3}{2}$ from f_1 and f_2, and it receives 1 from the common pot: u has an initial weight of -4, gives nothing and receives 4, so it has a non-negative final weight.

2. $d(u) = 3$.
   We consider three cases depending on the faces u is incident to.

   a) *Assume u belongs to a triangle (u, v, w).*
      Let f_1 and f_2 be the two other faces that are incident to u. To avoid Configurations **(**C_2**)** and **(**C_3**)**, we must have $d(f_1), d(f_2) \geq 5$. So u gives nothing as it cannot be a support. By Rule $R_{1.3}$, u receives $2 \times \frac{3}{2}$, has an initial weight of -3 and gives nothing, so it has a non-negative final weight.

   b) *Assume u belongs to three faces $f_1 = (u, u_1, v_1, u_2)$, $f_2 = (u, u_2, v_2, u_3)$ and $f_3 = (u, u_3, v_3, u_1)$, with $d(v_1), d(v_2), d(v_3) \leq 3$.*
      Then u cannot be a support so it gives nothing. Vertex u has an initial weight of -3, gives nothing, and receives 3×1 by Rule $R_{1.1}$, so it has a non-negative final weight.

   c) *Otherwise, u belongs to a face f_1 such that either $d(f_1) \geq 5$ or $f_1 = (u, u_1, v_1, u_2)$ with $d(v_1) \geq 4$.*
      Then u has an initial weight of -3, gives at most $2 \times \frac{1}{4}$ by R_4 as a vertex cannot be support more than twice, and receives at least $\frac{3}{2} + 2 \times 1$ by Rule R_1, so it has a non-negative final weight.

3. $d(u) = 4$.
   We consider two cases depending on whether u is incident to a triangle.

a) *Assume u is incident to a triangle.*
Then, since Configurations **(**C_2**)** and **(**C_3**)** do not appear, u is incident to two faces f_1 and f_2 such that $d(f_1)$, $d(f_2) \geq 5$. So u has an initial weight of -2, gives nothing, and receives at least 2×1 by Rule $R_{2.2}$, so it has a non-negative final weight.

b) *Otherwise, u is incident to at least 4 faces of degree at least 4.*
Then u has an initial weight of -2, gives nothing, and receives at least $4 \times \frac{1}{2}$ by Rule R_2, so it has a non-negative final weight.

4. $d(u) = 5$.
Since Configurations **(**C_2**)** and **(**C_3**)** do not appear, u is incident to (at least three and in particular) two faces f_1 and f_2 such that $d(f_1), d(f_2) \geq 4$. So u has an initial weight of -1, gives nothing, and receives at least $2 \times \frac{1}{2}$ by Rule $R_{2.1}$, so it has a non-negative final weight.

5. $6 \leq d(u) \leq k - 1$.
Vertex u has a non-negative initial weight, gives nothing, receives nothing, so it has a non-negative final weight.

6. $d(u) = k$.
Then u has an initial weight of at least 1, gives 1 to the common pot according to R_g and no other rule applies, so it has a non-negative final weight.

So all the vertices have a non-negative final weight after application of the discharging rules. Let us now prove that the same holds for the faces.

Let f be a face of G. We consider different cases depending on the value of $d(f)$. Since Configuration **(**C_3**)** does not appear, f cannot give weight according to R_3 if $d(f) \leq 4$. Note also that since Configuration **(**C_1**)** does not appear in G, R_3 can only be applied if the two endpoints of the edge are of degree k.

1. $d(f) = 3$.
Then f has an initial weight of 0, gives nothing, receives nothing, so it has a non-negative final weight.

2. $d(f) = 4$.
Assume $f = (u, v, w, x)$, where u has the minimum degree. We consider two cases depending on $d(u)$.

   a) $d(u) \leq 3$.
   Then, since Configuration **(**C_1**)** does not appear, $d(v), d(x) \geq 6$, and f gives nothing to them. Face f has an initial weight of 2. It gives at most 2×1 to u and w by Rule $R_{1.1}$, or at most $\frac{3}{2} + \frac{1}{2}$ to u and w by Rules $R_{1.2}$ and $R_{2.1}$ (depending on whether $d(w) \leq 3$). So it has a non-negative final weight.

   b) $d(u) \geq 4$.
   Then f has an initial weight of 2, gives at most $4 \times \frac{1}{2}$ to u, v, w and x by Rule $R_{2.1}$, so it has a non-negative final weight.

3. $d(f) = 5$.
   We take $f = (u, v, w, x, y)$, where u has minimum degree, and $d(w) \leq d(x)$. Face f has an initial weight of 4. We consider different cases depending on $d(u)$.

   a) $d(u) \leq 3$.
      Then, since Configuration **(**C_1**)** does not appear in G, $d(v), d(y) \geq 6$. We are in one of the following three cases.

      i. $d(w) \leq 3$.
         Then, since Configuration **(**C_1**)** does not appear in G, $d(x) \geq 6$. So, f gives $\frac{3}{2}$ both to u and w by Rule $R_{1.3}$, and may give $\frac{1}{2}$ to a vertex of degree 2 adjacent to both x and y, by Rule R_3. So f has an initial weight of 4, gives at most $\frac{7}{2}$, and has a non-negative final weight.

      ii. $4 \leq d(w) \leq 5$.
          Then f gives $\frac{3}{2}$ to u by Rule $R_{1.3}$, at most 1 to w by Rule R_2, and may give 1 to x by Rule R_2 or $\frac{1}{2}$ to a vertex of degree 2 adjacent to both x and y by Rule R_3. So f has an initial weight of 4, gives at most $\frac{7}{2}$, and has a non-negative final weight.

      iii. $d(w) \geq 6$.
           Then f gives $\frac{3}{2}$ to u by Rule $R_{1.3}$, and may give $\frac{1}{2}$ to a vertex of degree 2 adjacent to both v and w, both w and x, or both x and y, respectively, by Rule R_3. So f has an initial weight of 4, gives at most $\frac{3}{2} + 3 \times \frac{1}{2} = 3$, and has a non-negative final weight.

   b) $d(u) \geq 4$.
      Then, since Configuration **(**C_7**)** does not appear in G, there are at most 3 vertices of degree 4 in f. So f has an initial weight of 4, gives at most $3 \times 1 + 2 \times \frac{1}{2} = 4$, by Rules $R_{2.2}$, $R_{2.1}$ and R_3, and has a non-negative final weight.

4. $d(f) = 6$.
   Face f has an initial weight of 6, so it must not give more than 6 away. Since Configuration **(**C_3**)** does not appear in G, $R_{1.4}$ cannot apply more than once. We consider four cases depending on the number N of vertices of degree at most 3 on the boundary of f. Note that $N \leq 3$ since Configuration **(**C_1**)** does not appear in G.

   - If $N = 0$, then by Rules R_2 and R_3, f gives at most $d(f) \times 1 \leq 6$ away.
   - If $N = 1$, then since Configuration **(**C_1**)** does not appear in G, f is incident to at most two vertices of degree 4. Thus, by Rule R_1, f gives at most $\frac{5}{2}$ to its only neighbor of degree at most 3, and by Rules R_2 and R_3, f gives at most $4 \times \frac{1}{2}$ extra weight. So f gives at most $\frac{5}{2} + 3 \leq 6$ away.
   - If $N = 2$, then since Configuration **(**C_1**)** does not appear in G, f is incident to at most one neighbor of degree 4. Thus, by Rule R_1 and since $R_{1.4}$ is applied at most once, f gives at most $\frac{5}{2} + \frac{3}{2}$ to its two incident vertices of degree at most 3, and by Rules R_2 and R_3, f gives at most 1 extra weight. So f gives at most $4 + 1 \leq 6$ away.
   - Otherwise, $N = 3$. Since Configuration **(**C_1**)** does not appear in G, f is incident to no vertex of degree 4, and R_3 cannot be applied. Thus, by Rule

R_1 and since $R_{1.4}$ is applied at most once, f gives at most $\frac{5}{2} + 2 \times \frac{3}{2}$ to its three incident vertices of degree at most 3, and neither R_2 nor R_3 apply. So f gives at most $\frac{11}{2} \leq 6$ away.

5. $d(f) \geq 7$.
   In the worst case, f gives $\frac{5}{2} \times \lfloor \frac{d(f)}{2} \rfloor$ by $R_{1.4}$, and it may give an additional $\frac{1}{2}$ by R_3 if $d(f)$ is odd, so f has a non-negative final weight. It can easily be checked, as follows.

   a) If $d(f) = 7$.
      Then $2d(f) - 6 - (3 \times \frac{5}{2} + \frac{1}{2}) = 0 \geq 0$
   b) If $d(f) = 8$.
      Then $2d(f) - 6 - (\frac{d(f)}{2} \times \frac{5}{2}) = \frac{3}{4}d(f) - 6 \geq 0$
   c) Otherwise, $d(f) \geq 9$.
      Then $2d(f) - 6 - (\frac{d(f)}{2} \times \frac{5}{2} + \frac{1}{2}) = \frac{3}{4}d(f) - \frac{13}{2} \geq 0$.

Consequently, after application of the discharging rules, every vertex and every face of G has a non-negative weight, $\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) \geq 0$. Therefore, G is not planar. □

Conclusion

Proof of Theorem 3.10

Let Γ be a planar graph with no triangle adjacent to a cycle of length four, such that $\Delta(\Gamma) \geq 7$, and Γ is not list edge $\Delta(\Gamma)$-choosable (resp. list total $(\Delta(\Gamma) + 1)$-choosable). Graph Γ has a subgraph G that is a minimal graph such that G is not list edge $\Delta(\Gamma)$-choosable (resp. list total $(\Delta(\Gamma) + 1)$-choosable). We set $k = \Delta(\Gamma) \geq 7$. As $\Delta(G) \leq \Delta(\Gamma) = k$, by Lemma 3.15, graph G cannot contain **(**C_1**)** to **(**C_7**)**. Lemma 3.16 implies that G is not planar, thus Γ is not planar, a contradiction. □

3.3 Every planar graph G with $\Delta(G) \geq 8$ is edge $(\Delta(G) + 1)$-choosable

In this section, we prove Theorem 3.13 that planar graphs with maximum degree at most 8 are edge 9-choosable. Combined with Theorem 3.12, this implies Corollary 3.14 that every planar graph G with maximum degree $\Delta(G) \geq 8$ is edge $(\Delta(G) + 1)$-chosable.

Terminology and notation

For a given planar embedding, a vertex v is a *weak* neighbor of a vertex u when the two faces adjacent to the edge (u, v) are triangles (see Figure 3.5). A vertex v is a *semi-weak* neighbor of a vertex u when one of the two faces adjacent to the edge (u, v) is a triangle and the other is a cycle of length four (see Figure 3.6).

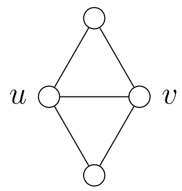


Figure 3.5: Vertex v is a weak neighbor of u.

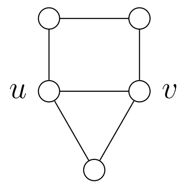
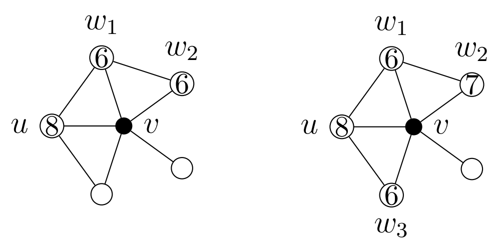


Figure 3.6: Vertex v is a semi-weak neighbor of u.

For any vertex u, we define special types of weak neighbors of degree 5 of u, as follows. The notation comes from E for ”Eight” (when $d(u) = 8$) and S for ”Seven” (when $d(u) = 7$). The index corresponds to the discharging rules (introduced in Section 3.3). Consider a weak neighbor v of degree 5 of u.

- Vertex v is an E_2*-neighbor* of u with $d(u) = 8$ when one of the two following conditions is satisfied (see Figure 3.7):
  - There are two vertices w_1 and w_2 with $d(w_1) = d(w_2) = 6$ such that (u, v, w_1) and (v, w_1, w_2) are faces (see Figure 3.7a).
  - There are three vertices w_1, w_2 and w_3 with $d(w_1) = d(w_3) = 6$ and $d(w_2) = 7$ such that (u, v, w_1), (v, w_1, w_2) and (u, v, w_3) are faces (see Figure 3.7b).



(a) (b)

Figure 3.7: Vertex v is an E_2-neighbor of u.

- Vertex v is an E_3*-neighbor* of u with $d(u) = 8$ when v is not an E_2-neighbor of u, and there is a vertex w with $d(w) \leq 7$ such that (u, v, w) is a face (see Figure 3.8).
- Vertex v is an E_4*-neighbor* of u with $d(u) = 8$ when v is not an E_2 nor an E_3-neighbor of u (see Figure 3.9). That is, when the third vertices of the two faces containing the edge (u, v) are both of degree 8.

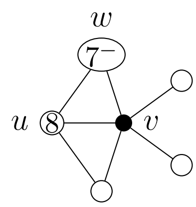


Figure 3.8: Vertex v is an E_3-neighbor of u.

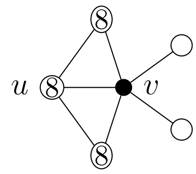


Figure 3.9: Vertex v is an E_4-neighbor of u.

- Vertex v is an S_2-*neighbor* of u with $d(u) = 7$ when there are two vertices w_1 and w_2 with $d(w_1) = d(w_2) = 6$ such that (u, v, w_1) and (u, v, w_2) are faces (see Figure 3.10).

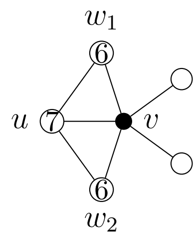


Figure 3.10: Vertex v is an S_2-neighbor of u.

- Vertex v is an S_3-*neighbor* of u with $d(u) = 7$ when v is not an S_2-neighbor of u, and v has four neighbors w_1, w_2, w_3 and w_4 such that (u, v, w_1) and (u, v, w_4) are faces, and one of the following two conditions is satisfied:
  - (v, w_1, w_2), (v, w_2, w_3) and (v, w_3, w_4) are faces, and $d(w_1) = d(w_4) = 7$ and $d(w_2) = d(w_3) = 6$ (see Figure 3.11a).
  - $d(w_4) = d(w_2) = 6$ and either $d(w_1) = 7$ (see Figure 3.11b) or $d(w_3) = 7$ (see Figure 3.11c). Note that there is no constraint on the order of w_2 and w_3 in the embedding.

- Vertex v is an S_4-*neighbor* of u with $d(u) = 7$ when v is not an S_2- nor S_3- neighbor of u, and either there is a vertex w with $d(w) \leq 7$ such that (u, v, w) is a face (see Figure 3.12a), or v is adjacent to two vertices w_1 and w_2 (both distinct from u) such that $d(w_1) = 6$ and $d(w_2) = 7$ (see Figure 3.12b).

Note that if a weak neighbor v of u has none of the previous types, then u and v must satisfy one of the following four hypotheses:

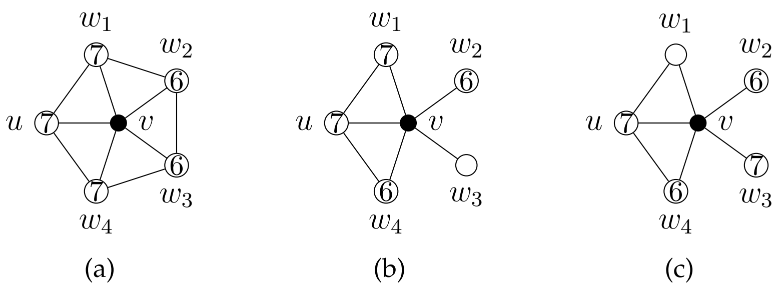


Figure 3.11: Vertex v is an S_3-neighbor of u.

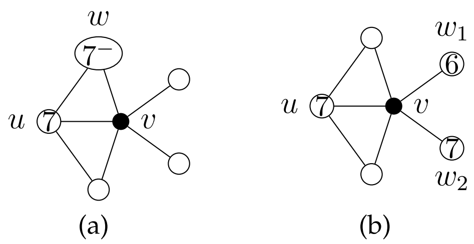


Figure 3.12: Vertex v is an S_4-neighbor of u.

- $d(v) \neq 5$
- $d(u) \notin \{7; 8\}$
- $d(u) = 7$, the third vertex of each of the two triangles adjacent to (u, v) is of degree 8, and the two other neighbors of v are either both of degree 6 or both of degree at least 7.

Forbidden Configurations

We define configurations **(**C_1**)** to **(**C_{11}**)** (see Figure 3.13).

- **(**C_1**)** is an edge (u, v) with $d(u) + d(v) \leq 10$.
- **(**C_2**)** is a cycle (u, v, w, x) such that $d(u) = d(w) = 3$.
- **(**C_3**)** is a vertex u with $d(u) = 8$ that has three neighbors v_1, v_2 and v_3 such that v_1 and v_2 are weak neighbors of u, with $d(v_1) = d(v_2) = 3$ and $d(v_3) \leq 5$.
- **(**C_4**)** is a vertex u with $d(u) = 8$ that has four neighbors v_1, v_2, v_3 and v_4 such that v_1 is a weak neighbor of u and v_2 is a semi-weak neighbor of u, with $d(v_1) = d(v_2) = 3$, $d(v_3) \leq 5$ and $d(v_4) \leq 5$.
- **(**C_5**)** is a vertex u with $d(u) = 8$ that has four weak neighbors v_1, v_2, v_3 and v_4 with $d(v_1) = 3$, $d(v_2) = d(v_3) = 4$ and $d(v_4) \leq 5$.
- **(**C_6**)** is a vertex u with $d(u) = 8$ that has five neighbors v_1, v_2, v_3, v_4 and v_5 such that v_1 is a weak neighbor of u with $d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) \leq 5$, $d(v_4) \leq 5$ and $d(v_5) \leq 7$.

- **(C_7)** is a vertex u with $d(u) = 8$ that has four weak neighbors v_1, v_2, v_3 and v_4, such that $d(v_1) = 3$, vertex v_2 is an E_2-neighbor of u, $d(v_3) \leq 5$ and $d(v_4) \leq 5$.

- **(C_8)** is a vertex u with $d(u) = 7$ that has three neighbors v, w and x such that w is adjacent to v and x, $d(w) = 6$, $d(v) = d(x) = 5$, and there is a vertex y of degree 6, distinct from w, that is adjacent to x.

- **(C_9)** is a vertex u with $d(u) = 7$ that has three weak neighbors v_1, v_2 and v_3 such that $d(v_1) = d(v_2) = 4$ and either v_3 is an S_2, S_3 or S_4-neighbor, or $d(v_3) = 4$.

- **(C_{10})** is a vertex u with $d(u) = 7$ that has three neighbors v_1, v_2 and v_3 such that $d(v_1) = 4$, vertex v_2 is an S_3-neighbor of u and $d(v_3) \leq 5$.

- **(C_{11})** is a vertex u with $d(u) = 5$ that has three neighbors v, w and x such that w is adjacent to v and x, and $d(v) = d(w) = d(x) = 6$.

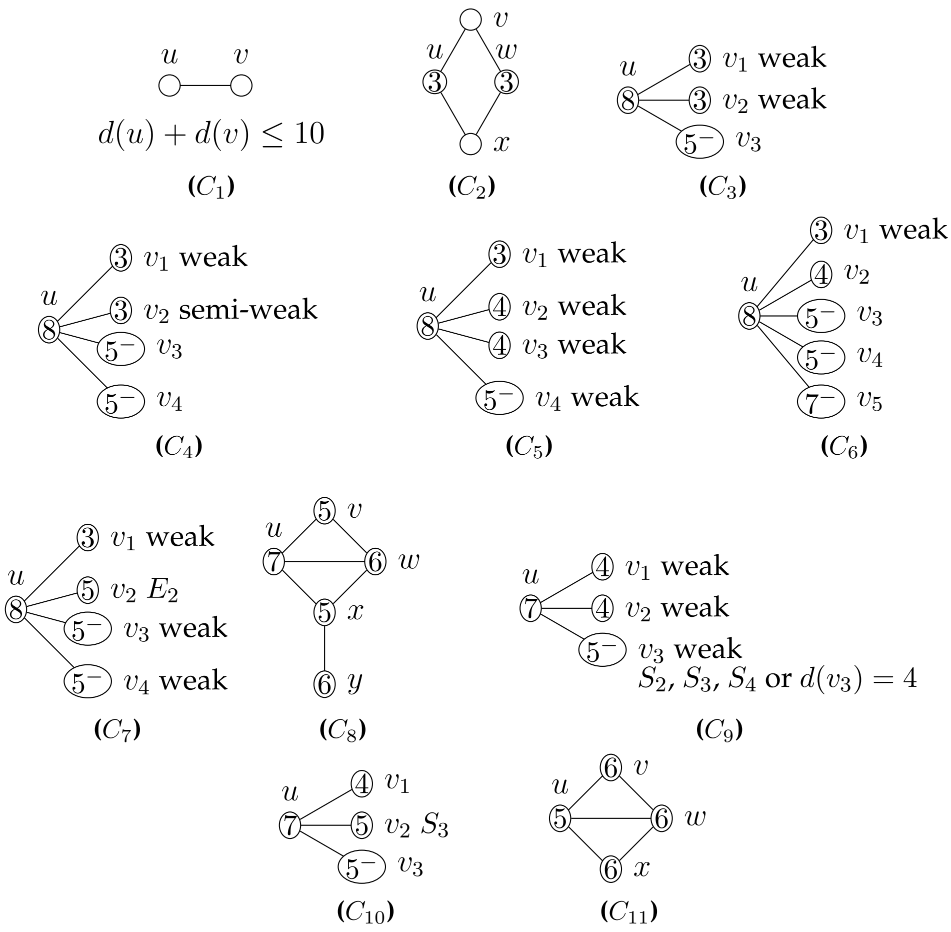


Figure 3.13: Forbidden configurations.

We first introduce the two following useful lemmas.

Lemma 3.18. *Let G be the graph with five edges (a, b, c, d, e) such that (b, c, d, e) forms a cycle and a is incident only to b and e (see Figure 3.14). Let $L : \{a, b, c, d, e\} \to \mathcal{P}(\mathbb{N})$ a list* *assignment of at least two colors on every edge, where either $|L(b)| \geq 3$ or $L(b) \neq L(a)$. The graph G is L-edge-colorable.*

Proof. We consider w.l.o.g. the worst case, i.e. $|L(a)| = |L(c)| = |L(d)| = |L(e)| = 2$. We consider two cases depending on whether $L(c) \cap L(e) = \emptyset$.

- $L(c) \cap L(e) \neq \emptyset$.
  Then let $\alpha \in L(c) \cap L(e)$. We color c and e in α. Since $|L(b)| \geq 3$ or $L(a) \neq L(b)$, we can color a and b. We color d.

- $L(c) \cap L(e) = \emptyset$.
  Then let $\alpha \in L(b) \setminus L(a)$. We color b in α, and consider two cases depending on whether $\alpha \notin L(c)$ or $\alpha \notin L(e)$.

  - $\alpha \notin L(c)$.
    Then we color successively e, a, d and c.
  - $\alpha \notin L(e)$.
    Then we color successively c, d, e and a.

□

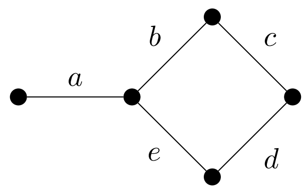


Figure 3.14: The graph of Lemma 3.18.

Lemma 3.19. *Let G be the star on three edges (a, b, c). Let $L : \{a, b, c\} \to \mathcal{P}(\mathbb{N})$ a list assignment such that $|L(a)| \geq 2$, $|L(b)| \geq 2$, $|L(c)| \geq 2$. The graph G is L-edge-colorable unless $L(a)$, $L(b)$ and $L(c)$ are all equal and of cardinality 2.*

Proof. Assume we do not have $L(a) = L(b) = L(c)$ with $|L(a)| = 2$. We assume without loss of generality that $|L(a)| \geq 3$ or that $|L(a)| = |L(b)| = |L(c)| = 2$ with $L(a) \neq L(b)$ and $L(a) \neq L(c)$. We color c in a color that is not available for a if possible, in an arbitrary color otherwise. We color successively b and a. □

Lemma 3.20. *If G is a minimal planar graph with $\Delta(G) \leq 8$ such that $\chi'_\ell(G) > 9$, then G does not contain any of Configurations* **(**C_1**)** *to* **(**C_{11}**)**.

Proof. Let L be a list assignment on the edges of G with $|L(e)| \geq 9$ for every edge e of G. We prove that if G contains any of Configurations **(**C_1**)** to **(**C_{11}**)**, then there is a subgraph H of G, that can be L-edge-colored by minimality, and whose L-edge-coloring is extendable to G, a contradiction.

A *constraint* of an edge $e \in E$ is an already colored edge that is incident to e. In the following, we denote generically $\hat{e}$ the list of available colors for an edge e at the moment it is used: the list is implicitely modified as incident edges are colored. Proving that the L-edge-coloring of H can be extended to G is equivalent to proving that the graph induced by the edges that are not colored yet is L'-colorable, where $L'(e) = \hat{e}$ for every edge e. We sometimes *delete* edges. Deleting an edge means that no matter the coloring of the other uncolored edges, there will still be a free color for it (for example, when the edge has more colors available than uncolored incident edges). Thus the deleted edge is implicitely colored after the remaining uncolored edges.

We use the same notations as in the definition of Configurations (C_1) to (C_{11}) (see Figure 3.13).

Claim 1. *G cannot contain (C_1).*

Proof. Using the minimality of G, we color $G \setminus \{(u, v)\}$. Since $d(u)+d(v) \leq 10$, the edge (u, v) has at most $10 - 2$ constraints. There are 9 colors, so we can color (u, v), thus extending the coloring to G. □

Claim 2. *G cannot contain (C_2).*

Proof. Using the minimality of G, we color $G \setminus \{(u, v), (v, w), (w, x), (x, u)\}$. Since $\Delta(G) \leq 8$ and $d(u) = d(w) = 3$, every uncolored edge has at most $8 - 2 + 1$ constraints. There are 9 colors, so every uncolored edge has at least two available colors, and they form a cycle of length four. We can thus apply Lemma 1.7 to extend the coloring to G. □

Claim 3. *G cannot contain (C_3).*

Proof. By Claim 2, vertices v_1 and v_2 have no common neighbor other than u. By Claim 1, for $i \in \{1, 2, 3\}$, vertex v_i is adjacent only to vertices of degree at least 6. So the v_i's are pairwise non-adjacent. We name the edges according to Figure 3.15.

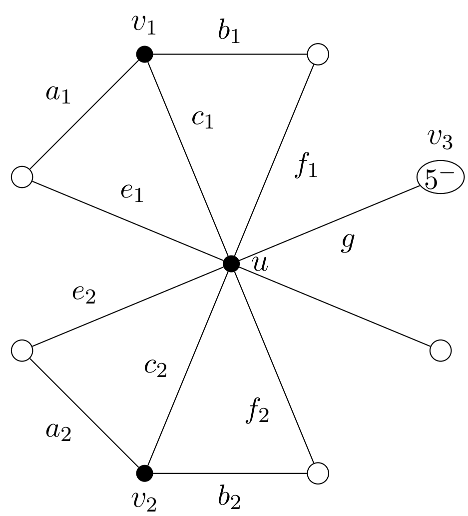


Figure 3.15: Notations of Claim 3

By minimality of G, we color $G \setminus \{v_1, v_2\}$. Since there are 9 colors and every vertex is of degree at most 8, we have $|\hat{a_1}|, |\hat{a_2}|, |\hat{b_1}|, |\hat{b_2}| \geq 2$ and $|\hat{c_1}|, |\hat{c_2}| \geq 3$. We first prove the following.

1. *If $\hat{a_1} = \hat{b_1}$, $\hat{a_2} = \hat{b_2}$, and $|\hat{a_1}| = |\hat{b_1}| = |\hat{a_2}| = |\hat{b_2}| = 2$. Then we can recolor $G \setminus \{v_1, v_2\}$ so that the hypothesis is not satisfied anymore.*

Proof. For $i \in \{1, 2\}$, let $\hat{a_i} = \hat{b_i} = \{\alpha_i, \beta_i\}$. For $i \in \{1, 2\}$, let γ_i and δ_i be the color of e_i and f_i, respectively. Note that $\gamma_i \in L(a_i)$ and $\delta_i \in L(b_i)$ since $|\hat{a_i}| = |\hat{b_i}| = 2$. Note that for a given $i \in \{1, 2\}$, the colors $\alpha_i, \beta_i, \gamma_i$ and δ_i are all different.

We claim that any recoloring of $\{e_1, f_1, e_2, f_2, g\}$ such that the color of at least one of $\{e_1, f_1, e_2, f_2\}$ has been changed breaks the hypothesis of (1). Indeed, assume w.l.o.g. that the color of e_1 can be changed while recoloring only edges of $\{e_1, f_1, e_2, f_2, g\}$, and consider such a coloring. We have $\gamma_1 \in \hat{a_1}$ since $\gamma_1 \in L(a_1)$ and the only edge of $\{e_1, f_1, e_2, f_2, g\}$ that is incident to a_1 is e_1, which is not colored in γ_1 anymore. We have $\gamma_1 \notin \hat{b_1}$ since $\gamma_1 \notin \{\alpha_1, \beta_1, \delta_1\}$ and the only edge of $\{e_1, f_1, e_2, f_2, g\}$ that is incident to b_1 is f_1, which was colored in δ_1. Thus $\hat{a_1} \neq \hat{b_1}$, and the hypothesis of (1) is broken.

We prove now that there exists such a recoloring. Aside from the constraints derived from $\{e_1, e_2, f_1, f_2, g\}$, each edge e_i or f_i has at most $(8-2)+(8-7) = 7$ constraints, and g has at most $(5-1)+(8-7) = 5$ constraints. Let L' be the list assignment of the colors available for those edges, when ignoring the constraints derived from $\{e_1, e_2, f_1, f_2, g\}$. Note that $|L'(e_i)|, |L'(f_i)| \geq 2$ and $|L'(g)| \geq 4$. Let us build the directed graph D whose vertex set is $V(D) = \{e_1, e_2, f_1, f_2, g\}$ and where for any two distinct $u, v \in V(D)$, there is an edge from u to v if the color of u belongs to $L'(v)$. We consider two cases depending on whether there is a cycle in D.

- *There is a cycle in D.*
  Then we recolor accordingly the edges in G (for any edge from u to v in the cycle, v takes the initial color of u, which belongs by definition to $L'(v)$). Since a cycle contains at least two vertices, at least one of $\{e_1, e_2, f_1, f_2\}$ has been recolored.

- *There is no cycle in D.*
  Then some vertex has in-degree 0. We consider two cases depending on whether some e_i or f_i has in-degree 0.

  - *Some e_i or f_i has in-degree 0.*
    Then it can be recolored without conflict (i.e. without recoloring the other vertices of D).

  - *Every e_i and f_i has in-degree at least 1.*
    Then g has in-degree 0. So g can be recolored without conflict. Since there is no cycle in D and every e_i and f_i is of in-degree at least 1, there is necessarily an edge from g to some e_i or f_i, which can now be recolored without conflict.

$\Diamond$

By (1), we can assume that we have a coloring of $G \setminus \{v_1, v_2\}$ that does not satisfy the hypothesis of (1). W.l.o.g., we consider the case where $\hat{a_1} \neq \hat{b_1}$ or $|\hat{a_1}| \geq 3$. We color a_2, b_2 and c_2. Then $|\hat{c_1}| \geq 2$ and $\hat{a_1}$ and $\hat{b_1}$ have not been modified. So we apply Lemma 3.19 to the edges incident to v_1. $\square$

Claim 4. *G cannot contain (C_4).*

Proof. We prove Claim 4 similarly as Claim 3. By Claim 2, vertices v_1 and v_2 have no common neighbor other than u. By Claim 1, for $i \in \{1, 2, 3, 4\}$, vertex v_i is adjacent only to vertices of degree at least 6. So the v_i's are pairwise non-adjacent. We name the edges according to Figure 3.16. Note that among the edges named here, the edge b_2 is incident only to a_2 and c_2.

By minimality of G, we color $G \setminus \{v_1, v_2\}$. Since there are 9 colors and every vertex is of degree at most 8, we have $|\hat{a_1}|, |\hat{a_2}|, |\hat{b_1}|, |\hat{b_2}| \geq 2$ and $|\hat{c_1}|, |\hat{c_2}| \geq 3$. We proceed as for Claim 3 and prove the following.

1. *If $\hat{a_1} = \hat{b_1}$, $\hat{a_2} = \hat{b_2}$, and $|\hat{a_1}| = |\hat{b_1}| = |\hat{a_2}| = |\hat{b_2}| = 2$. Then we can recolor $G \setminus \{v_1, v_2\}$ so that the hypothesis is not satisfied anymore.*

Proof. For $i \in \{1, 2\}$, let $\hat{a_i} = \hat{b_i} = \{\alpha_i, \beta_i\}$. For $i \in \{1, 2\}$, let γ_i be the color of e_i. Let δ_1 be the color of f_1. Note that $\gamma_i \in L(a_i)$ since $|\hat{a_i}| = 2$. Similarly, $\delta_1 \in L(b_1)$. Note that for a given $i \in \{1, 2\}$, the colors $\alpha_i, \beta_i, \gamma_i$ (and δ_1 if $i = 1$) are all different.

We claim that any recoloring of $\{e_1, f_1, e_2, g_1, g_2\}$ such that the color of at least one of $\{e_1, f_1, e_2\}$ has been changed breaks the hypothesis of (1). Indeed, assume that the color of e_1 can be changed while recoloring only edges of $\{e_1, f_1, e_2, g_1, g_2\}$, and consider such a coloring. (The cases where the color of f_1 or e_2 can be changed are similar). We have $\gamma_1 \in \hat{a_1}$ since $\gamma_1 \in L(a_1)$ and the only edge of $\{e_1, f_1, e_2, g_1, g_2\}$ that is incident to a_1 is e_1, which is not colored in γ_1 anymore. We have $\gamma_1 \notin \hat{b_1}$ since $\gamma_1 \notin \{\alpha_1, \beta_1, \delta_1\}$ and the only edge of $\{e_1, f_1, e_2, f_2, g\}$ that is incident to b_1 is f_1, which was colored in δ_1. Thus $\hat{a_1} \neq \hat{b_1}$.

We prove now that there exists such a recoloring. Aside from the constraints derived from $\{e_1, e_2, f_1, g_1, g_2\}$, each edge e_1, e_2 and f_1 has at most $(8 - 2) + (8 - 7) = 7$ constraints, and each g_i has at most $4 + 1 = 5$ constraints. Let L' be the list assignment of the colors available for those edges, when ignoring the constraints derived from $\{e_1, e_2, f_1, g_1, g_2\}$. Note that $|L'(e_i)|, |L'(f_1)| \geq 2$, and $|L'(g_i)| \geq 4$. We consider w.l.o.g. the worst case, i.e. $|L'(e_1)| = |L'(e_2)| = |L'(f_1)| = 2$. Let us build the directed graph D whose vertex set is $V(D) = \{e_1, e_2, f_1, g_1, g_2\}$ and where there is an edge from u to v if the color of u belongs to $L'(v)$.

First note that if there is an edge from some g_i to some $v \in \{e_1, e_2, f_1\}$, then there are all edges from $\{e_1, e_2, f_1\} \setminus \{v\}$ to g_i. Indeed, if vertex g_i has in-degree at most 2, we recolor g_i and recolor v into the former color of g_i. So we assume vertex g_i has in-degree at least 3. If there is an edge from v to g_i, we exchange the colors of v and g_i. Thus there are all possible edges from $\{e_1, e_2, f_1\} \setminus \{v\}$ to g_1.

If some e_i or f_i has in-degree 0, se can recolor it without conflict. So we can assume that all of e_1, e_2 and f_1 have in-degree at least 1. If there is no edge from $\{g_1, g_2\}$ to $\{e_1, e_2, f_1\}$, then there is a directed cycle in $\{e_1, e_2, f_1\}$, and we recolor accordingly the edges in G. So there is at least an edge from $\{g_1, g_2\}$ to $\{e_1, e_2, f_1\}$. We consider w.l.o.g. that there is an edge from g_1 to e_1. By the previous remark, there is an edge from e_2 and f_1 to g_1. Both e_2 and f_1 have in-degree at least 1. If there is an edge from e_2 to f_1

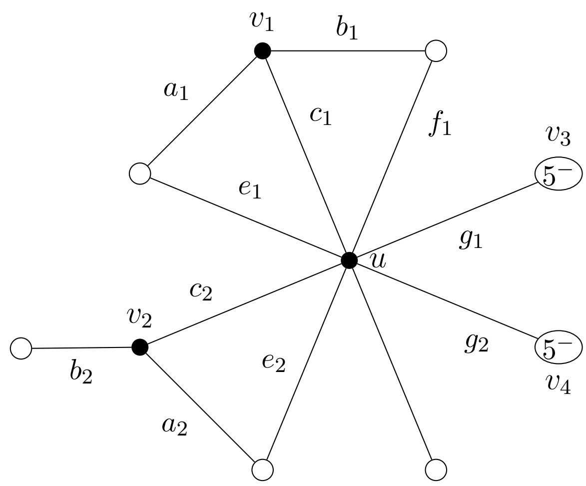


Figure 3.16: Notations of Claim 4

and an edge from f_1 to e_2, we exchange their colors. So we assume w.l.o.g. that there is an edge from $\{e_1, g_1, g_2\}$ to e_2. If there is an edge from e_1 to e_2, there is a directed cycle on $\{e_1, e_2, g_1\}$, and we recolor accordingly the edges in G. If there is an edge from g_1 to e_2, we exchange the colors of g_1 and e_2. If there is an edge from g_2 to e_2, then by the previous remark, there is an edge from e_1 to g_2. Thus there is a directed cycle on $\{e_1, g_2, e_2, g_1\}$ and we recolor accordingly the edges in G. ◊

By (1), we can assume that we have a coloring of $G \setminus \{v_1, v_2\}$ that does not satisfy the hypothesis of (1). W.l.o.g., we consider the case where $\hat{a_1} \neq \hat{b_1}$ or $|\hat{a_1}| \geq 3$. We color a_2, b_2, c_2 and apply Lemma 3.19 to the edges incident to v_1. □

Claim 5. *G cannot contain (C_5).*

Proof. By Claim 1, no two v_i are adjacent. Since every v_i is a weak neighbor of u, and $d(u) = 8$, the neighborhood of u forms a cycle (see Figure 3.17). We consider two cases depending on whether there is a vertex x such that v_2, x and v_3 appear consecutively around u.

- *There is a vertex x such that v_2, x and v_3 appear consecutively around u.*
  We consider without loss of generality that the neighbors of u are, clockwise, v_1, w_1, v_2, w_2, v_3, w_3, v_4 and w_4. We name the edges according to Figure 3.17a. Note that the edges l and o are distinct. By minimality of G, we color $G \setminus \{a, \ldots, r\}$.

  Without loss of generality, we consider the worst case, i.e. $|\hat{l}| = |\hat{o}| = |\hat{q}| = |\hat{r}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{f}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = |\hat{n}| = |\hat{p}| = 4$, $|\hat{g}| = 7$, and $|\hat{a}| = |\hat{c}| = |\hat{e}| = 9$. We consider two cases depending on whether $\hat{i} = \hat{j}$ and $\hat{i} \cap \hat{h} \neq \emptyset$.

  - *$\hat{i} \neq \hat{j}$ or $\hat{i} \cap \hat{h} = \emptyset$.*
    If $\hat{i} \neq \hat{j}$, we color i in a color that does not belong to $\hat{j}$. Otherwise $\hat{h} \cap \hat{i} = \emptyset$ and i can be deleted. In any case $|\hat{j}| = 4$ and j has exactly 3 uncolored incident edges, so we can delete it. Then $|\hat{a}| \geq 8$ and a has 7 uncolored

incident edges, so we can delete it. Since $|\hat{b}| + |\hat{l}| > |\hat{k}|$, there exists a color $\alpha \in (\hat{b} \cap \hat{l}) \cup ((\hat{b} \cup \hat{l}) \setminus \hat{k})$. Note that b and l are not incident. We color b and l in α if possible, in an arbitrary color otherwise. If $\alpha \in \hat{b} \cap \hat{l}$, then b and l are colored in α and $|\hat{k}| \geq 3$. If $\alpha \in (\hat{b} \cup \hat{l}) \setminus \hat{k}$, then at least one of b and l is colored in α and $|\hat{k}| \geq 3$. So we can delete k. Then, successively, c, m, e, n, o, p, g, d, f, h, q and r can be deleted.

- *$\hat{i} = \hat{j}$ and $\hat{i} \cap \hat{h} \neq \emptyset$.*
  Then let $\alpha \in \hat{j} \cap \hat{h}$. Note that j and h are not incident. We color j and h in α. Since i (resp. a) is incident to both j and h, we can successively delete i and a. We color successively r and q. Without loss of generality, we consider the worst case, i.e. $|\hat{f}| = |\hat{l}| = |\hat{o}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{k}| = |\hat{p}| = 3$, $|\hat{g}| = |\hat{m}| = |\hat{n}| = 4$, and $|\hat{c}| = |\hat{e}| = 8$. We consider three cases depending on whether $\hat{f} \cap \hat{n} = \emptyset$ and $\hat{p} \setminus \hat{o} \subset \hat{n}$.

  * $\hat{f} \cap \hat{n} \neq \emptyset$.
    Then let $\beta \in \hat{f} \cap \hat{n}$. We color f and n in β. We delete successively e, p, o, c, and g. we color l. We apply Lemma 1.7 on (b, k, m, d).
  * $\hat{p} \setminus \hat{o} \not\subset \hat{n}$.
    Then let $\beta \in \hat{p} \setminus (\hat{o} \cup \hat{n})$. We color p in β. We color f. Since $|\hat{m}| + |\hat{o}| > |\hat{n}|$, there exists $\gamma \in (\hat{m} \cap \hat{o}) \cup ((\hat{m} \cup \hat{o}) \setminus \hat{n})$. If $\gamma \in \hat{d}$ or $\gamma \notin \hat{m}$, we color d and o in γ if possible, in an arbitrary color otherwise. We delete successively n, e, c, m, l, k, g and b. If $\gamma \notin \hat{d}$ and $\gamma \in \hat{m}$, we color m and o in γ if possible, in an arbitrary color otherwise. Note that $|\hat{d}| \geq 2$. We delete successively e, c, g, d, b, k and l.
  * $\hat{f} \cap \hat{n} = \emptyset$ *and* $\hat{p} \setminus \hat{o} \subset \hat{n}$.
    Then let $\beta \in \hat{p} \setminus \hat{o}$. We color p in β. By assumption, $\beta \notin \hat{f} \cup \hat{o}$. We color n in a color that does not belong to o. We delete successively o, e, c and g. We color l, and apply Lemma 3.18 on (f, b, k, m, d).

- *There is no vertex x such that v_2, x and v_3 appear consecutively around u.*
  We consider without loss of generality that the neighbors of u are, clockwise, v_1, w_1, v_2, w_2, v_4, w_3, v_3 and w_4. We name the edges according to Figure 3.17b. By minimality of G, we color $G \setminus \{a, \dots, r\}$.

  Without loss of generality, we consider the worst case, i.e. $|\hat{l}| = |\hat{n}| = |\hat{o}| = |\hat{q}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{f}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = |\hat{p}| = |\hat{r}| = 4$, $|\hat{e}| = 7$, and $|\hat{a}| = |\hat{c}| = |\hat{g}| = 9$. We consider two cases depending on whether $\hat{i} = \hat{j}$ and $\hat{i} \cap \hat{h} \neq \emptyset$.

  - *$\hat{i} \neq \hat{j}$ or $\hat{i} \cap \hat{h} = \emptyset$.*
    If $\hat{i} \neq \hat{j}$, we color i in a color that does not belong to $\hat{j}$. Otherwise $\hat{i} \cap \hat{h} = \emptyset$, we can delete i. In both cases, we can delete successively j and a.

    We consider three cases depending on whether $\hat{q} \cap \hat{h} = \emptyset$ and $\hat{q} \subset \hat{r}$.

    * $\hat{q} \cap \hat{h} \neq \emptyset$.
      Then let $\alpha \in \hat{q} \cap \hat{h}$. We color q and h in α. We delete successively g, c, e, r, p, k, m, l, b, d, n, p and f.

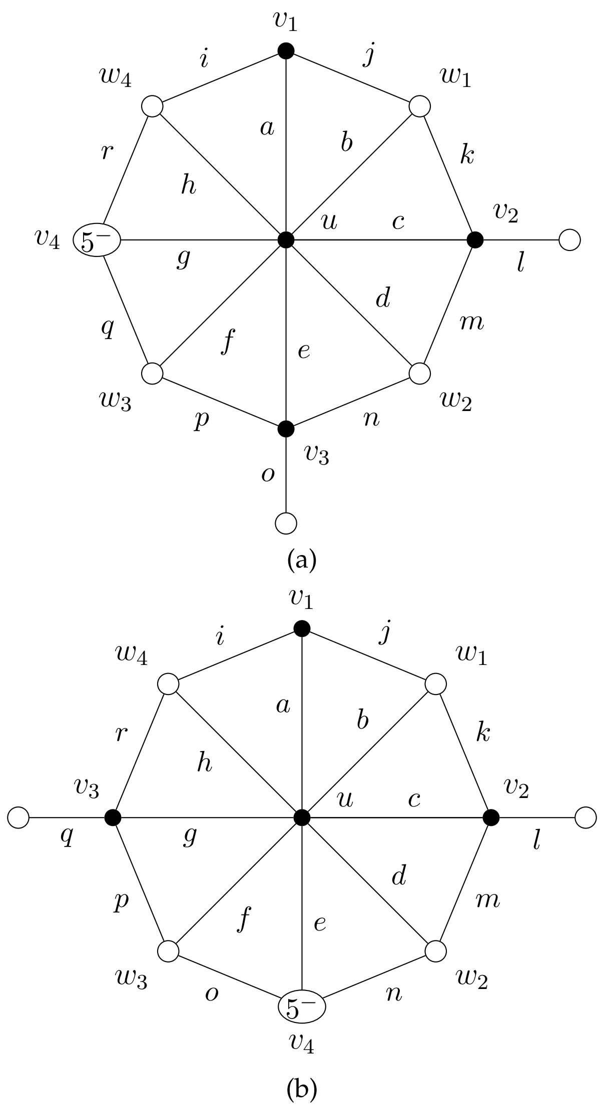


Figure 3.17: Notations of Claim 5

- * $\hat{q} \not\subset \hat{r}$.
  Then let $\alpha \in \hat{q} \setminus \hat{r}$. We color q in α. Since $|\hat{h}| + |\hat{p}| > |\hat{r}|$, there exists a color $\beta \in (\hat{h} \cap \hat{p}) \cup ((\hat{h} \cup \hat{p}) \setminus \hat{r})$. We color h and p in β if possible, in an arbitrary color otherwise. We delete successively r, g, c, e, k, l, m, b, d, f, n and o.
- * $\hat{q} \cap \hat{h} = \emptyset$ *and* $\hat{q} \subset \hat{r}$.
  Then let $\alpha \in \hat{q}$. By assumption, $\alpha \in \hat{r} \setminus \hat{h}$. We color r in α, and color q. Since $|\hat{b}| + |\hat{l}| > |\hat{k}|$, there exists a color $\beta \in (\hat{b} \cap \hat{l}) \cup ((\hat{b} \cup \hat{l}) \setminus \hat{k})$. We color b and l in β if possible, in an arbitrary color otherwise. We delete successively k, c, g, e, and m. We color h in such a way that afterwards, $|\hat{f}| \geq 3$ or $\hat{f} \neq \hat{p}$. Then we apply Lemma 3.18 on (p, f, d, n, o).

– $\hat{i} = \hat{j}$ *and* $\hat{i} \cap \hat{h} \neq \emptyset$.
Since $\hat{i} = \hat{j}$, there exists $\alpha \in \hat{j} \cap \hat{h}$, we color j and h in α, and delete i and a.

When we say that we color $q|r$ in a color α, it means that we color q in α if possible, otherwise we color r in α.

Let $C = \hat{c}$ and $G = \hat{g}$. If $\hat{q} \cup \hat{r} \not\subset \hat{g}$, we consider $\alpha \in (\hat{q} \cup \hat{r}) \setminus \hat{g}$, and color $q|r$ in α. Assume that $\hat{q} \cup \hat{r} \subset \hat{g}$. Note that $|((\hat{q} \cup \hat{r}) \cap \hat{c}) \cup (\hat{c} \setminus \hat{g})| \geq |\hat{q} \cup \hat{r}| \geq 3$, and that $|\hat{l}| = 2$. We consider $\alpha \in (((\hat{q} \cup \hat{r}) \cap \hat{c}) \cup (\hat{c} \setminus \hat{g})) \setminus \hat{l}$. We color $q|r$ in α if possible, in an arbitrary color otherwise.

Note that since q and r have the same incidencies in the resulting graph, and since $|\hat{r}| \geq |\hat{q}| - 1$, the identity of the edge that is colored has no impact, and we can consider w.l.o.g. that q is colored and r remains uncolored. We remove color α from $\hat{k}$ and $\hat{m}$. We consider w.l.o.g. the worst case, i.e. $|\hat{k}| = |\hat{l}| = |\hat{n}| = |\hat{o}| = |\hat{r}| = 2$, $|\hat{b}| = |\hat{d}| = |\hat{f}| = |\hat{m}| = |\hat{p}| = 3$, $|\hat{e}| = 6$, $|\hat{g}| = 7$ and $|\hat{c}| = 8$.

We consider two cases depending on whether $\hat{k} = \hat{l}$.

* $\hat{k} = \hat{l}$. Then we color m in a color that does not belong to $\hat{l}$. We color successively n, o, d, f, b, k and l.
* $\hat{k} \neq \hat{l}$. Then we color l in a color that does not belong to $\hat{k}$. If $\hat{m} = \hat{n}$, then we color d in a color that does not belong to $\hat{m}$, and apply Lemma 1.7 on (b, k, m, n, o, f). If $\hat{m} \neq \hat{n}$, then we color m in a color that does not belong to $\hat{n}$, we color k and we apply Lemma 3.18 on (b, f, o, n, d).

We then color p, q and e. We claim that $\hat{c} \neq \hat{g}$ if $|\hat{c}| = |\hat{g}| = 1$. Indeed, assume $|\hat{c}| = |\hat{g}| = 1$. Then, all the edges incident to g are colored differently, and their colors belong to G. We consider two cases depending on whether q is colored in α.

* *Edge q is colored in α.* Then $\alpha \in G$, which implies $\alpha \in C$ by choice of α. Since the edges incident to g are all colored differently and q is colored in α, none of $\{b, d, e, f, h\}$ is colored in α. By construction, none of $\{k, m\}$ is colored in α. By choice of α, l is not colored in α. Thus $\alpha \in \hat{c}$ and $\alpha \notin \hat{g}$, so $\hat{c} \neq \hat{g}$.
* *Edge q is not colored in α.* Then, by choice of α, we have $\alpha \in C \setminus G$. Since the colors of the edges incident to g all belong to G, none of $\{b, d, e, f, h\}$ is colored in α. By construction, none of $\{k, m\}$ is colored in α. By choice of α, l is not colored in α. Thus $\alpha \in \hat{c}$ and $\alpha \notin \hat{g}$, so $\hat{c} \neq \hat{g}$.

Note that $|\hat{c}| \geq 1$ and $|\hat{g}| \geq 1$. If $|\hat{c}| = |\hat{g}| = 1$, then $\hat{c} \neq \hat{g}$, so we color c and g independently. If not, assume w.l.o.g. that $|\hat{c}| \geq 2$, and color successively g and c.

□

Claim 6. *G cannot contain (C_6).*

Proof. We prove Claim 6 similarly as Claim 3. By Claim 1, for $i \in \{1, 2, 3, 4, 5\}$, vertex v_i is adjacent only to vertices of degree at least $11 - d(v_i)$. We name the edges according to Figure 3.18. By minimality of G, we color $G \setminus \{v_1\}$. Since there are 9 colors and every vertex is of degree at most 8, we have $|\hat{a}|, |\hat{b}|, |\hat{c}| \geq 2$. We proceed as for Claim 3 and prove the following.

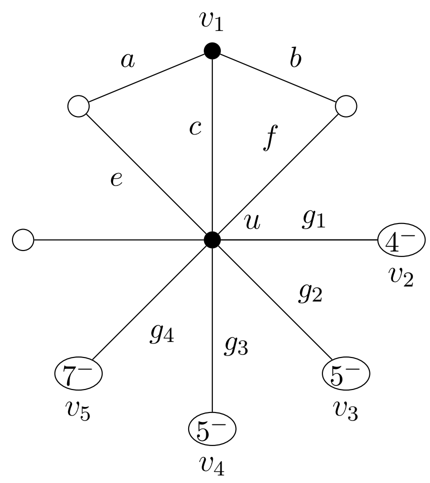


Figure 3.18: Notations of Claim 6

1. *If $\hat{a} = \hat{b}$ and $|\hat{a}| = |\hat{b}| = 2$. Then we can recolor $G \setminus \{v_1\}$ so that the hypothesis is not satisfied anymore.*

Proof. Let $\hat{a} = \hat{b} = \{\alpha, \beta\}$. Let γ be the color of e and δ the color of f. Note that $\gamma \in L(a)$ and $\delta \in L(b)$ since $|\hat{a}| = |\hat{b}| = 2$. Note also that α, β, γ and δ are all different.

We claim that any recoloring of $\{e, f, g_1, g_2, g_3, g_4\}$ such that the color of at least one of e, f has been changed breaks the hypothesis of (1). Indeed, assume w.l.o.g. that the color of e can be changed while recoloring only edges of $\{e, f, g_1, g_2, g_3, g_4\}$, and consider such a coloring. We have $\gamma \in \hat{a}$ since $\gamma \in L(a)$ and the only edge of $\{e, f, g_1, g_2, g_3, g_4\}$ that is incident to a is e, which is not colored in γ anymore. We have $\gamma \notin \hat{b}$ since $\gamma \notin \{\alpha, \beta, \delta\}$ and the only edge of $\{e, f, g_1, g_2, g_3, g_4\}$ that is incident to b is f, which was colored in δ. Thus $\hat{a} \neq \hat{b}$.

We prove that there exists such a recoloring. Aside from the constraints derived from $\{e, f, g_1, g_2, g_3, g_4\}$, both e and f have at most 7 constraints, edge g_1 has at most 4 constraints, edges g_2 and g_3 have at most 5 constraints, and g_4 has at most 7 constraints. Let L' be the list assignment of the colors available for those edges, when ignoring the constraints derived from $\{e, f, g_1, g_2, g_3, g_4\}$. Note that $|L'(e)|, |L'(f)|, |L'(g_4)| \geq 2$, $|L'(g_2)|, |L'(g_3)| \geq 4$ and $|L'(g_1)| \geq 5$. Let us build the directed graph D whose vertex set is $V(D) = \{e, f, g_1, g_2, g_3, g_4\}$ and where there is an edge from u to v if the color of u belongs to $L'(v)$. Let D_1 be the graph obtained from D by removing any vertex v such that there is no directed path from v to e. Let D_2 be the graph obtained from D by removing any vertex v such that there is no directed path from v to f. If $e \in D_2$ and $f \in D_1$, then there is a directed path from e to f and a directed path from f to e. So there exists a directed cycle that contains e, which we recolor accordingly. So we can assume that $e \notin D_2$ or $f \notin D_1$. We consider w.l.o.g. the case $f \notin D_1$. We consider four cases depending on the structure of D_1.

- $V(D_1) = \{e\}$. Then we recolor e without conflict.

- $|V(D_1)| \geq 2$, *and some vertex $v \neq e$ has in-degree at most $L'(v) - 2$.* Then we recolor v, and recolor accordingly the path from v to e.

- $|V(D_1)| \geq 2$, *and there is an edge from* e *to a vertex* v. Then by definition of D_1, there is a directed cycle that contains e, which we recolor accordingly.

- $|V(D_1)| \geq 2$, *every vertex* $v \neq e$ *has in-degree at least* $L'(v) - 1$, *and* e *has out-degree* 0. Since $f \notin D_1$, we have $\{g_1, g_2, g_3, g_4\} \cap D_1 \neq \emptyset$. Let j be the minimum i such that $g_i \in D_1$. Vertex g_j has in-degree at least $L'(g_j) - 1 \geq |V(D) \setminus \{f, g_1, \ldots, g_j\}|$, and there is no edge from e to g_j, a contradiction.

$\Diamond$

By (1), we can assume that we have a coloring of $G \setminus \{v_1\}$ that does not satisfy the hypothesis of (1). We apply Lemma 3.19 to the edges incident to v_1. $\square$

Claim 7. *G cannot contain (C_7).*

Proof. By Claim 1, no two v_i are adjacent, nor is v_1 adjacent to a vertex of degree at most 7. Since every v_i is a weak neighbor of u, and $d(u) = 8$, the neighborhood of u forms a cycle (see Figure 3.19). We consider two cases depending on whether there is a vertex x such that v_2, x and v_3 appear consecutively around u.

- *There is a vertex x such that v_3, x and v_4 appear consecutively around u.*
  W.l.o.g. the neighbors of u are, clockwise, v_1, x_1, v_2, x_2, v_3, x_3, v_4, x_4. Since v_2 is an E_2-neighbor of u and $d(x_1) = 8$, we have $d(x_2) = 6$ and there is a vertex y of degree 6 such that (x_2, v_2, y) is a face. We name the edges according to Figure 3.19a. By minimality, we color $G \setminus \{a, \ldots, s\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{n}| = |\hat{o}| = |\hat{p}| = |\hat{q}| = 2$, $|\hat{s}| = 3$, $|\hat{b}| = |\hat{f}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{k}| = 4$, $|\hat{m}| = |\hat{r}| = 5$, $|\hat{d}| = |\hat{e}| = |\hat{g}| = |\hat{l}| = 7$, and $|\hat{a}| = |\hat{c}| = 9$. Note that the edges k and s are not incident. Since $|\hat{k}| + |\hat{s}| > |\hat{r}|$, there exists $\alpha \in (\hat{k} \cap \hat{s}) \cup ((\hat{k} \cup \hat{s}) \setminus \hat{r})$. We color k and s in α if possible, in an arbitrary color otherwise. We can delete successively r, l and m. We color q. Note that the edges p and j are not incident. Since $|\hat{p}| + |\hat{j}| > |\hat{i}|$, there exists $\beta \in (\hat{p} \cap \hat{j}) \cup ((\hat{p} \cup \hat{j}) \setminus \hat{i})$. Thus we color p and j in β if possible, in an arbitrary color otherwise. We delete successively i, a, c, e, g, d, h, b, f, n and o.

- *There is no vertex x such that v_2, x and v_3 appear consecutively around u.*
  W.l.o.g. the neighbors of u are, clockwise, v_1, x_1, v_3, x_2, v_2, x_3, v_4, x_4, with $d(x_2) \geq d(x_3)$. We consider two cases depending on whether $d(x_2) = 6$.

  - $d(x_2) = 6$.
    W.l.o.g., since v_2 is an E_2-vertex, there is a vertex y of degree 6 or 7 such that (y, v_2, x_3) is a face. We name the edges according to Figure 3.19b. By minimality, we color $G \setminus \{a, \ldots, s\}$. W.l.o.g., we consider the worst case, i.e. $|\hat{k}| = |\hat{p}| = |\hat{q}| = |\hat{s}| = 2$, $|\hat{b}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{l}| = |\hat{r}| = 4$, $|\hat{o}| = 5$, $|\hat{d}| = |\hat{m}| = 6$, $|\hat{c}| = |\hat{f}| = |\hat{g}| = |\hat{n}| = 7$, and $|\hat{a}| = |\hat{e}| = 9$.
    Note that the edges r and l cannot be incident. Since $|\hat{r}| + |\hat{l}| > |\hat{m}|$, there exists $\alpha \in (\hat{r} \cap \hat{l}) \cup ((\hat{r} \cup \hat{l}) \setminus \hat{m})$. We color r and l in α if possible, in an arbitrary color otherwise. We can delete successively m, n and o. We color q, s and k successively. Note that p and j cannot be incident. Since $|\hat{p}| + |\hat{j}| > |\hat{i}|$, there

exists $\beta \in (\hat{p} \cap \hat{j}) \cup ((\hat{p} \cup \hat{j}) \setminus \hat{i})$. Thus we color p and j in β if possible, in an arbitrary color otherwise. We delete successively i, a, e, g, f, c, d, h and b.

- $d(x_2) \geq 7$.
  Since v_2 is an E_2-vertex, we have $d(x_3) = 6$ and there is a vertex y of degree 6 such that (y, v_2, x_3) is a triangle. We name the edges according to Figure 3.19c. By minimality, we color $G \setminus \{a, \ldots, s\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{k}| = |\hat{l}| = |\hat{p}| = |\hat{q}| = 2$, $|\hat{s}| = 3$, $|\hat{b}| = |\hat{d}| = |\hat{h}| = |\hat{i}| = |\hat{j}| = |\hat{m}| = 4$, $|\hat{o}| = |\hat{r}| = 5$, $|\hat{c}| = |\hat{f}| = |\hat{g}| = |\hat{n}| = 7$, and $|\hat{a}| = |\hat{e}| = 9$. Note that since G is a simple graph, the edges s and m can-

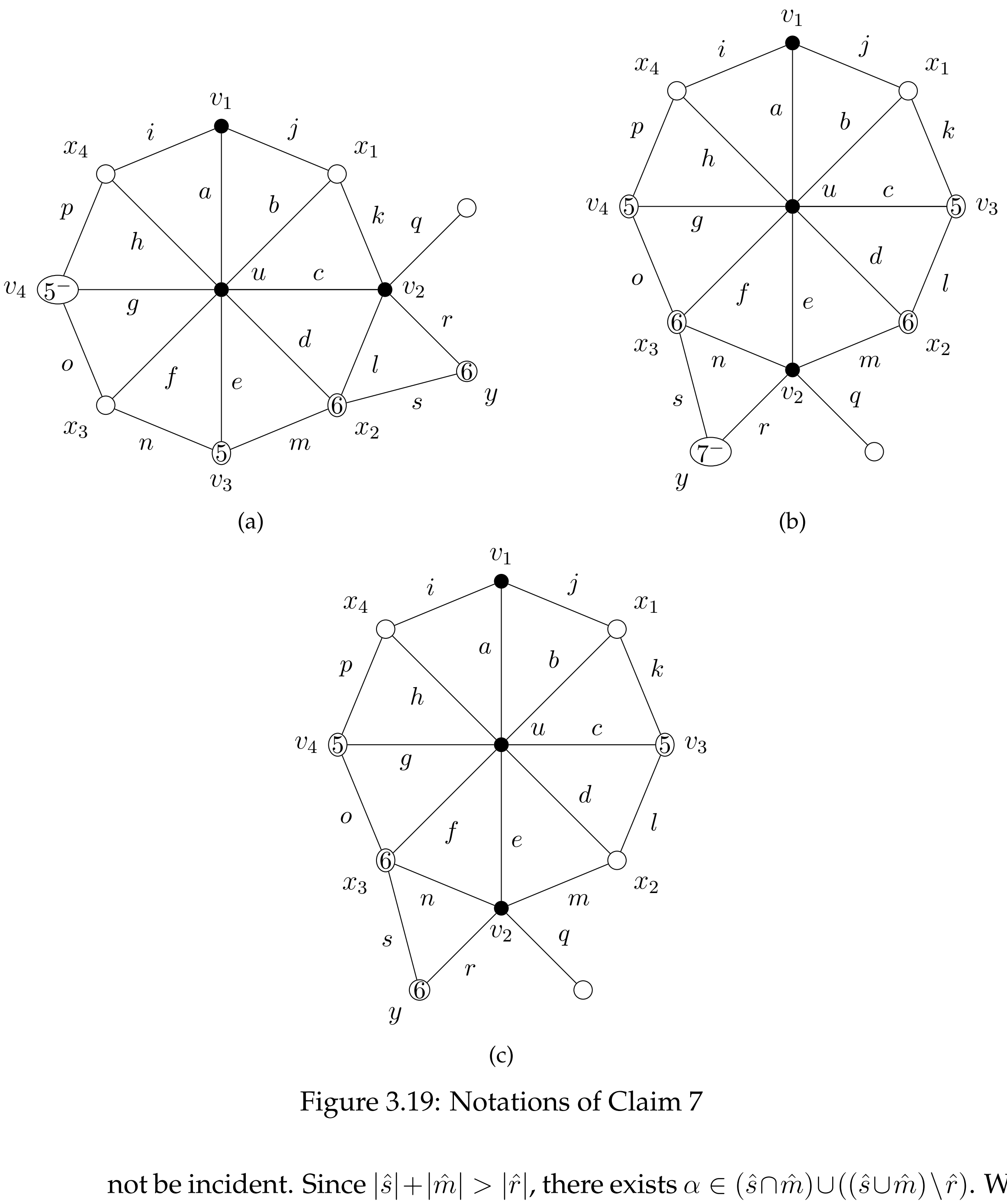


Figure 3.19: Notations of Claim 7

not be incident. Since $|\hat{s}| + |\hat{m}| > |\hat{r}|$, there exists $\alpha \in (\hat{s} \cap \hat{m}) \cup ((\hat{s} \cup \hat{m}) \setminus \hat{r})$. We color m and s in α if possible, in an arbitrary color otherwise. We can delete successively r, n and o. We color successively q, l and k. Note that p and j cannot be incident. Since $|\hat{p}| + |\hat{j}| > |\hat{i}|$, there exists $\beta \in (\hat{p} \cap \hat{j}) \cup ((\hat{p} \cup \hat{j}) \setminus \hat{i})$. Thus we color p and j in β if possible, in an arbitrary color otherwise. We delete successively i, a, e, f, g, c, h, b and d.

□

Claim 8. *G cannot contain (C_8).*

Proof. Not that since G is simple and $d(y) \neq d(v)$, all the vertices named here are distinct. We name the edges according to Figure 3.20. By minimality, we color $G \setminus \{a, \ldots, f\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{a}| = |\hat{d}| = |\hat{f}| = 2$, $|\hat{c}| = |\hat{e}| = 3$ and $|\hat{b}| = 4$.

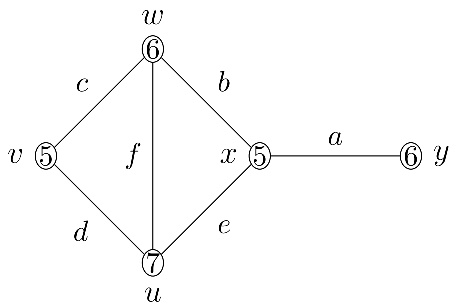


Figure 3.20: Notations of Claim 8

We consider two cases depending on whether $\hat{f} = \hat{d}$.

- $\hat{f} \neq \hat{d}$. We color f in a color that does not belong to $\hat{d}$. We apply Lemma 3.18 on (a, b, c, d, e).
- $\hat{f} = \hat{d}$. We color e and c in a color that does not belong to $\hat{f}$. We color successively a, b, f and d.

□

Claim 9. *G cannot contain (C_9).*

Proof. Note that by Claim 1, $\{v_1, v_2, v_3\}$ forms a stable set. We consider two cases depending on whether there are two weak neighbors w_1 and w_2 of u with $d(w_1) = d(w_2) = 4$ and a vertex x, such that (w_1, x, u) and (w_2, x, u) are faces.

- *There are two weak neighbors w_1 and w_2 of u with $d(w_1) = d(w_2) = 4$ and a vertex x, such that (w_1, x, u) and (w_2, x, u) are faces.*
  We assume w.l.o.g. that the neighborhood of u is, clockwise, y_1, v_1, x, v_2, y_2, v_3 and z. We are in one of the following three cases: either $d(z) \leq 7$, or $d(y_2) \leq 7$, or $d(z) = d(y_2) = 8$.

– $d(z) \leq 7$.
We name the edges according to Figure 3.21a. By minimality, we color $G \setminus \{a, \ldots, o\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{l}| = |\hat{n}| = |\hat{o}| = 2$, $|\hat{a}| = |\hat{h}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{g}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = 4$, $|\hat{f}| = 7$ and $|\hat{b}| = |\hat{d}| = 9$. We first prove the following.

1. *We can color $a, c, e, f, g, h, i, j, k, l, m, n$ and o in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$.*

Proof. Let $B = \hat{b}$ and $D = \hat{d}$. If $\hat{b} = \hat{d}$, then we consider $\alpha \in \hat{i}$, and color i in α. If $\hat{b} \neq \hat{d}$, then we consider $\alpha \in \hat{d} \setminus \hat{b}$, and color i arbitrarily. We remove color α from $\hat{k}$, $\hat{l}$ and $\hat{m}$. We color l. We consider two cases depending on whether $\hat{m} = \hat{n}$.

* $\hat{n} \neq \hat{m}$.
Then we color m in a color that does not belong to $\hat{n}$. We color k. Since $|\hat{h}| + |\hat{c}| > |\hat{a}|$, there exists $\beta \in (\hat{h} \cap \hat{c}) \cup ((\hat{h} \cup \hat{c}) \setminus \hat{a})$. We color h and c in β if possible. We color successively j, and h or c if not colored already. We apply Lemma 3.18 on (a, g, o, n, e). We color f.
* $\hat{n} = \hat{m}$.
Since $|\hat{a}| + |\hat{o}| > |\hat{g}|$, there exists $\beta \in (\hat{a} \cap \hat{o}) \cup ((\hat{a} \cup \hat{o}) \setminus \hat{g})$. If $\beta \in \hat{e} \setminus \hat{m}$ or $\beta \notin \hat{a}$, we color e and o in β if possible, in an arbitrary color otherwise ($\notin \hat{m}$ in the case of e). We color n, m and k. We delete g, and we apply Lemma 1.7 on (h, j, c, a). If $\beta \notin \hat{e} \setminus \hat{m}$ and $\beta \in \hat{a}$, we color a and o in β if possible, in an arbitrary color otherwise. Note that a is colored in β, and that β does not belong to $\hat{e}$ or belongs to $\hat{m}$, in which case one of $\{m, n\}$ will be colored in β. We color successively n, m, k, h, j, c, and e. We color g, and f.

Assume $|\hat{b}| = |\hat{d}| = 1$. Then the colors of the edges incident to b are all different and belong to B. We consider two cases depending on whether $B = D$.

* $B = D$. Since i is colored in α, no edge in $\{a, c, e, g\}$ is colored in α, and $\alpha \in D$. By construction, none of $\{k, l, m\}$ is colored in α. Thus $\alpha \in \hat{d}$ and $\alpha \notin \hat{b}$, so $\hat{b} \neq \hat{d}$.
* $B \neq D$. Since $\alpha \notin B$, no edge in $\{a, c, e, g\}$ is colored in α. By construction, none of $\{k, l, m\}$ is colored in α. Thus $\alpha \in \hat{d}$ and $\alpha \notin \hat{b}$, so $\hat{b} \neq \hat{d}$.

$\Diamond$

By (1), we color $a, c, e, f, g, h, i, j, k, l, m, n$ and o in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. We color b and d.

– $d(y_2) \leq 7$.
We name the edges according to Figure 3.21b. By minimality, we color $G \setminus \{a, \ldots, m\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{g}| = |\hat{i}| = |\hat{l}| = 2$, $|\hat{a}| = |\hat{h}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = 4$, $|\hat{f}| = 5$ and $|\hat{b}| = |\hat{d}| = 9$. We first prove the following.

2. *We can color $a, c, e, f, g, h, i, j, k, l$ and m in such a way that, afterwards, $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$.*

Proof. If $\hat{b} = \hat{d}$, then we consider $\alpha \in \hat{l}$, and color l in α. If $\hat{b} \neq \hat{d}$, then we consider $\alpha \in \hat{b} \setminus \hat{d}$, and color l arbitrarily. We remove color α from $\hat{h}$, $\hat{i}$ and $\hat{j}$. We color successively $i, h, j, a, g, c, k, e, m$ and f. By the same analysis as in the previous case, $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. ◊

By (2), we color $a, c, e, f, g, h, i, j, k, l$ and m in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. We color b and d.

- $d(z) = d(y_2) = 8$.
Then either v_3 is a weak neighbor of u of degree 4, or v_3 is a weak neighbor of u of degree 5 adjacent to a vertex of degree 6. We will deal with the two cases at once. We consider that v_3 is of degree 5 in both cases, by adding a neighbor of degree 6 to v_3 if it is of degree 4: a proper coloring of this graph will yield a proper coloring of the initial graph. We name the edges according to Figure 3.21b.
By minimality, we color $G \setminus \{a, \ldots, q\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{l}| = |\hat{p}| = 2$, $|\hat{a}| = |\hat{g}| = |\hat{h}| = |\hat{o}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{j}| = |\hat{k}| = |\hat{m}| = |\hat{n}| = |\hat{q}| = 4$ and $|\hat{b}| = |\hat{d}| = |\hat{f}| = 9$. We first prove the following.

3. *We can color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that, afterwards, $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$.*

Proof. If $\hat{b} = \hat{d}$, then we consider $\alpha \in \hat{l}$, and color i in α. If $\hat{b} \neq \hat{d}$, then we consider $\alpha \in \hat{d} \setminus \hat{b}$, and color i arbitrarily. We remove color α from $\hat{k}$, $\hat{l}$ and $\hat{m}$.
We color l. Since $|\hat{g}| + |\hat{p}| > |\hat{o}|$, there exists $\beta \in (\hat{g} \cap \hat{p}) \cup ((\hat{g} \cup \hat{p}) \setminus \hat{o})$. We color g and p in β if possible, in an arbitrary color otherwise. We color m so that $\hat{e} \neq \hat{a}$ if $|\hat{a}| = |\hat{e}| = 2$, which is possible as $|\hat{m}| \geq 2$. We color k, and we apply Lemma 3.18 on (e, a, h, j, c). We color n, o, q and f.
By the same analysis as in the two previous cases, we have $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. ◊

By (3), we color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that $\hat{b} \neq \hat{d}$ if $|\hat{b}| = |\hat{d}| = 1$. We color b and d.

- *There are no two weak neighbors w_1 and w_2 of u with $d(w_1) = d(w_2) = 4$ for which there exists a vertex x such that (w_1, x, u) and (w_2, x, u) are faces.*
Then v_3 must be a vertex of degree 5. W.l.o.g., the neighborhood of u is, clockwise, y_1, v_1, y_2, v_3, y_3, v_2, y_4. We consider two cases depending on whether $d(y_2) = d(y_3) = 8$.

  - $d(y_2) \leq 7$ *or* $d(y_3) \leq 7$.
  Consider w.l.o.g. that $d(y_2) \leq 7$. We name the edges according to Figure 3.21d. By minimality, we color $G \setminus \{a, \ldots, o\}$. Without loss

of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{l}| = |\hat{n}| = 2$, $|\hat{a}| = |\hat{g}| = |\hat{h}| = |\hat{k}| = |\hat{o}| = 3$, $|\hat{e}| = |\hat{m}| = 4$, $|\hat{c}| = |\hat{j}| = 5$, $|\hat{d}| = 7$ and $|\hat{b}| = |\hat{f}| = 9$.

Note that the edges k and h are not incident. Since $|\hat{k}| + |\hat{h}| > |\hat{j}|$, there exists $\alpha \in (\hat{k} \cap \hat{h}) \cup ((\hat{k} \cup \hat{h}) \setminus \hat{j})$. We color k and h in α if possible, in an arbitrary color otherwise. We can delete successively j, b, i, f, d and c. We color a, l and n. We apply Lemma 1.7 on (e, m, o, g).

– $d(y_2) = d(y_3) = 8$.
Then v_3 must be a weak neighbor of degree 5 whose two other neighbors are of degree 6 and 7, respectively. We name the edges according to Figure 3.21e. By minimality, we color $G \setminus \{a, \ldots, q\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{i}| = |\hat{n}| = 2$, $|\hat{a}| = |\hat{g}| = |\hat{h}| = |\hat{o}| = |\hat{q}| = 3$, $|\hat{c}| = |\hat{e}| = |\hat{j}| = |\hat{k}| = |\hat{l}| = |\hat{m}| = |\hat{p}| = 4$, and $|\hat{b}| = |\hat{d}| = |\hat{f}| = 9$. We first prove the following.

4. *We can color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that, afterwards, $\hat{b} \neq \hat{f}$ if $|\hat{b}| = |\hat{f}| = 1$.*

Proof. If $\hat{b} = \hat{f}$, then we consider $\alpha \in \hat{n}$, and color n in α. If $\hat{b} \neq \hat{f}$, then we consider $\alpha \in \hat{b} \setminus \hat{d}$, and color n arbitrarily. We remove color α from $\hat{h}$, $\hat{i}$ and $\hat{j}$. We color successively $i, h, j, k, c, a, g, e, o, m, l, q, p$ and d. By the same analysis as in the previous cases, we have $\hat{b} \neq \hat{f}$ if $|\hat{b}| = |\hat{f}| = 1$. $\diamondsuit$

By (4), we color $a, c, e, f, g, h, i, j, k, l, m, n, o, p$ and q in such a way that $\hat{b} \neq \hat{f}$ if $|\hat{b}| = |\hat{f}| = 1$. We color b and f.

□

Claim 10. *G cannot contain (C_{10}).*

Proof. We consider two cases depending on whether v_2 and u have a common neighbor of degree 6.

- *Vertices v_2 and u have a common neighbor y of degree* 6.
  By definition of an S_3-neighbor, vertex v_2 has two other neighbors of degree 7 and 6, respectively. We name the edges according to Figure 3.22a. Since the graph is simple, there is no $1 \leq i \leq 3$ such that the edges e and c_i are incident.

  By minimality, we color $G \setminus \{a, b_1, b_2, c_1, c_2, c_3, d, e\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{b_1}| = |\hat{c_1}| = |\hat{e}| = 2$, $|\hat{b_2}| = |\hat{c_2}| = 3$, $|\hat{c_3}| = 4$, $|\hat{d}| = 5$, and $|\hat{a}| = 6$.

  Since $|\hat{e}| + |\hat{c_3}| > |\hat{d}|$, there exists $\alpha \in (\hat{e} \cap \hat{c_3}) \cup ((\hat{e} \cup \hat{c_3}) \setminus \hat{d})$. If $\alpha \in \hat{c_3}$, let i be the minimum integer such that $\alpha \in \hat{c_i}$. If $\alpha \notin \hat{c_3}$, then $\alpha \in \hat{e} \setminus (\hat{d} \cup \hat{c_3})$, let i be 1. We color e and c_i in α if possible, in an arbitrary color otherwise (by choice of i, if c_i is not colored in α then $i = 1$). We delete d. If $i \neq 3$, edge c_3 is not colored and we delete it. Then, if $i \neq 2$, edge c_2 is not colored, and either $i = 3$ and c_3 is colored in α (which was not an available color for c_2 by choice of i), or $i = 1$ and c_3 has been

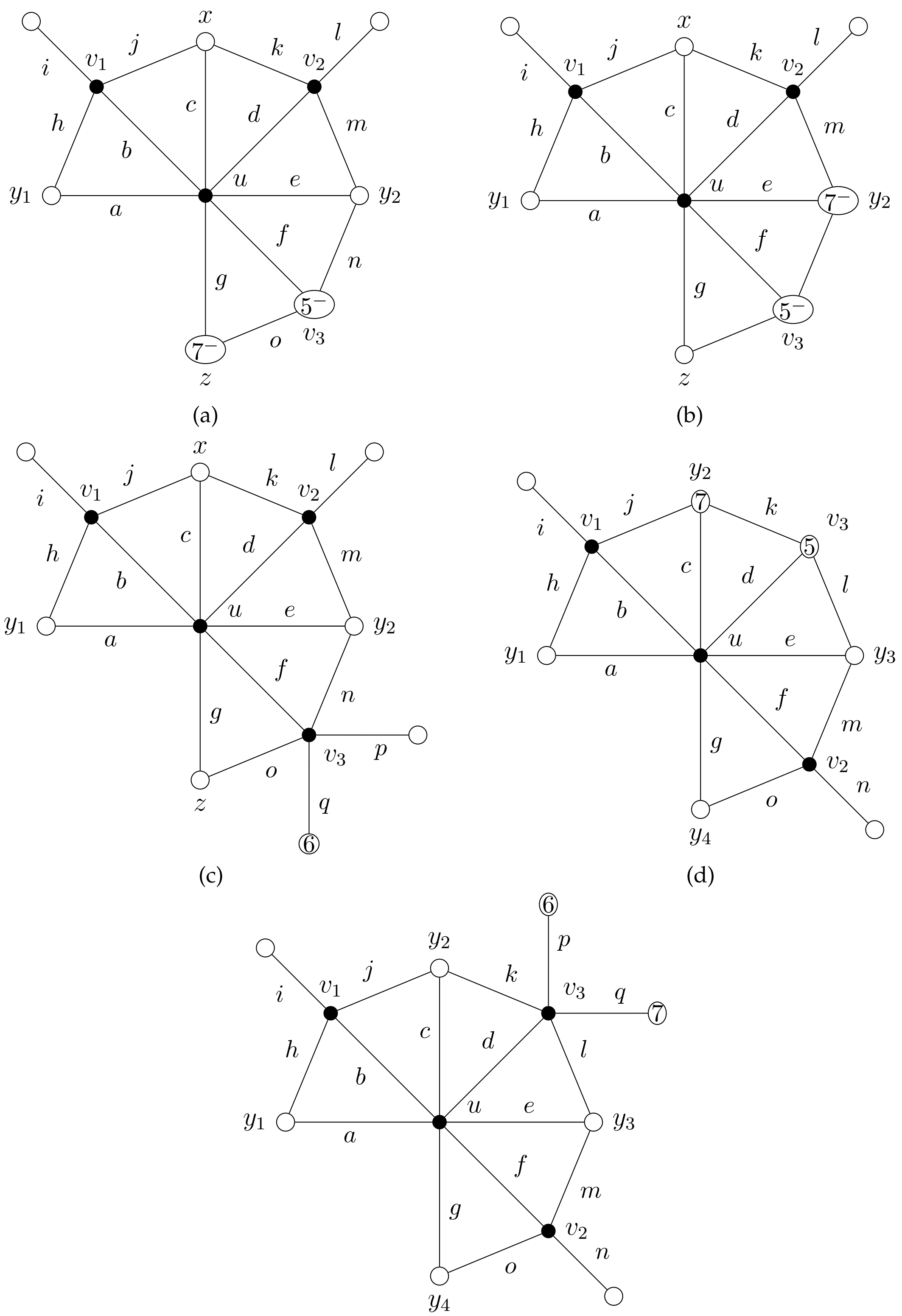


Figure 3.21: Notations of Claim 9

deleted; In both cases, we can delete c_2. Then, if $i \neq 1$, edge c_1 is not colored, and the edges c_2 and c_3 are deleted or colored in α (which was not an available color for c_1 by choice of i), so we can delete c_1. We delete successively a, b_2, b_1.

- *Vertices v_2 and u have no common neighbor of degree 6.*
  Then, by definition of an S_3-vertex, the neighborhood of v_2 is, clockwise, (u, y_1, z_1, z_2, y_2), with $d(y_1) = d(y_2) = 7$ and $d(z_1) = d(z_2) = 6$. We name the edges according to Figure 3.22b. By minimality, we color $G \setminus \{a, \ldots, k\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{f}| = |\hat{g}| = |\hat{h}| = |\hat{j}| = 2$, $|\hat{i}| = |\hat{k}| = 3$, $|\hat{b}| = |\hat{e}| = 5$, and $|\hat{a}| = |\hat{c}| = |\hat{d}| = 6$. We first prove the following.

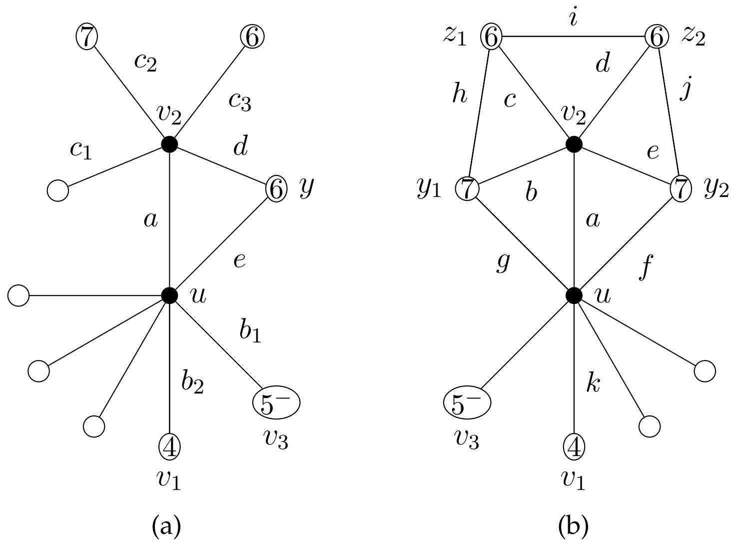


Figure 3.22: Notations of Claim 10

1. *We can color f, g, h, i, j and k in such a way that, afterwards, $\hat{c} \neq \hat{d}$ if $|\hat{c}| = |\hat{d}| = 4$.*

Proof. We consider two cases depending on whether $\hat{c} = \hat{d}$.

- $\hat{c} = \hat{d}$. We apply Lemma 1.7 on (f, g, h, j) by considering that f and j are incident so they receive different colors. We color i and k. The new constraints of c are i and h, and the new constraints of d are i and j. Since h and j receive distinct colors, we have $|\hat{c}| \geq 5$ or $|\hat{d}| \geq 5$ or $\hat{c} \neq \hat{d}$.
- $\hat{c} \neq \hat{d}$. Let $\alpha \in \hat{c}, \notin \hat{d}$. We color h in a color other than α. We color g, f, j, i and k successively. Thus, either $|\hat{d}| \geq 5$ or $\alpha \in \hat{c}$ so $\hat{c} \neq \hat{d}$.

$\lozenge$

By (1), we color f, g, h, i, j and k in such a way that $\hat{c} \neq \hat{d}$ if $|\hat{c}| = |\hat{d}| = 4$. We color a, b and e. Either $|\hat{c}| \geq 2$ (resp. $|\hat{d}| \geq 2$), and we color d and c (resp. c and d). Or $|\hat{c}| = |\hat{d}| = 1$ and $\hat{c} \neq \hat{d}$, we color d and c independently.

$\square$

Claim 11. *G cannot contain (C_{11}).*

Proof. We name the edges according to Figure 3.23. By minimality, we color $G \setminus \{a, \ldots, e\}$. Without loss of generality, we consider the worst case, i.e. $|\hat{d}| = |\hat{e}| = 2$, $|\hat{a}| = |\hat{c}| = 3$ and $|\hat{b}| = 4$. We consider two cases depending on whether $\hat{e} \subset \hat{b}$.

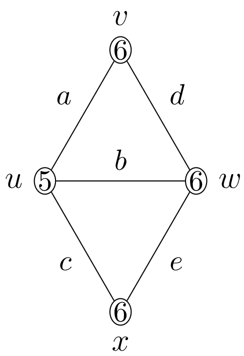


Figure 3.23: Notations of Claim 11

- $\hat{e} \not\subset \hat{b}$. Then we color e in a color that does not belong to $\hat{b}$. We can delete successively b, a, c and d.
- $\hat{e} \subset \hat{b}$. Then, since $|\hat{c}| + |\hat{d}| > |\hat{b}|$ and $\hat{d} \subset \hat{b}$, there exists $\alpha \in (\hat{c} \cap \hat{d}) \cup ((\hat{c} \cup \hat{d}) \setminus \hat{b})$. We color c and d in α if possible, in an arbitrary color otherwise. Note that since $\hat{e} \subset \hat{b}$, we have $|\hat{e}| \geq 1$ in both cases. We delete successively b, e and a.

$\square$

Lemma 3.20 holds by Claims 1 to 11. $\square$

Discharging rules

We design discharging rules R_1, R_2, ..., R_{11} (see Figure 3.24):

For any face f of degree at least 4,

- Rule R_1 is when $d(f) = 4$ and f is incident to a vertex v of degree $d(v) \leq 5$. Then f gives 1 to v.
- Rule R_2 is when $d(f) \geq 5$ and f is incident to a vertex v of degree $d(v) \leq 5$. Then f gives 2 to v.

For any vertex u of degree at least 7,

- Rule R_3 is when u has a weak neighbor v of degree 3. Then u gives 1 to v.
- Rule R_4 is when u has a semi-weak neighbor v of degree 3. Then u gives $\frac{1}{2}$ to v.
- Rule R_5 is when u has a weak neighbor v of degree 4. Then u gives $\frac{1}{2}$ to v.

For any vertex u of degree 8,

- Rule R_6 is when u has an E_2-neighbor v. Then u gives $\frac{1}{2}$ to v.
- Rule R_7 is when u has an E_3-neighbor v. Then u gives $\frac{1}{3}$ to v.
- Rule R_8 is when u has an E_4-neighbor v. Then u gives $\frac{1}{4}$ to v.

For any vertex u of degree 7,

- Rule R_9 is when u has an S_2-neighbor v. Then u gives $\frac{1}{2}$ to v.
- Rule R_{10} is when u has an S_3-neighbor v. Then u gives $\frac{1}{3}$ to v.
- Rule R_{11} is when u has an S_4-neighbor v. Then u gives $\frac{1}{4}$ to v.

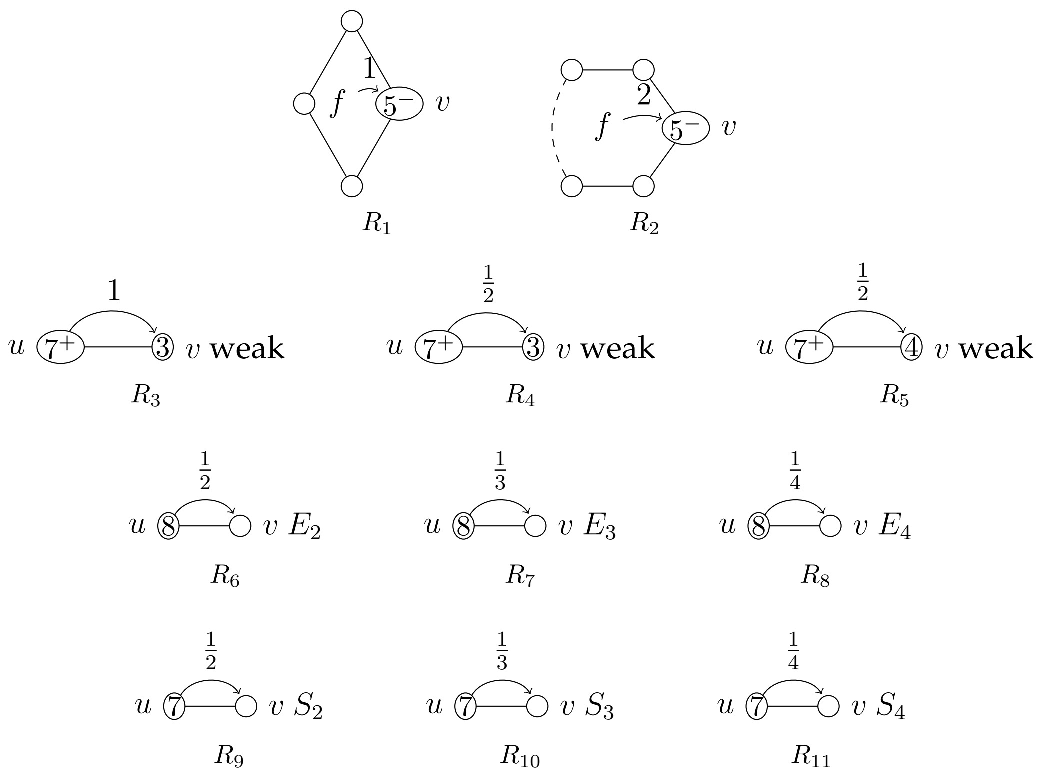


Figure 3.24: Discharging rules.

Note that according to these rules, only vertices of degree at most 5 receive weight, and only faces of degree at least 4 and vertices of degree at least 7 give weight. Note that the notation E_i and S_i corresponds to the fact that a vertex u gives a weight of $\frac{1}{i}$ to every E_i- or S_i-neighbor.

Lemma 3.21. *A planar graph G with $\Delta(G) \leq 8$ that does not contain Configurations* (C_1) *to* (C_{11}) *is a stable set.*

Proof. We can assume without loss of generality that G is connected (if it is not, we simply consider a connected component of G, as it satisfies the same hypothesis). Assume by contradiction that G is not a single vertex. Thus G is connected and contains at least one edge. According to Configuration **(**C_1**)**, every vertex x of G satisfies $d(x) \geq 3$. We consider a planar embedding of G.
We attribute to each vertex u a weight of $d(u) - 6$, and to each face a weight of $2d(f) - 6$. We apply discharging rules R_1, R_2, ..., R_{11}. We show that all the faces and vertices have a weight of at least 0 in the end.

Note that the degree of a face is the number of vertices on its boundary, while walking through a facial walk (i.e. some vertices are counted with multiplicity). The discharging rules on the faces also apply with multiplicity: R_1 and R_2 apply to each vertex of degree at most 5 incident to f as many times as it appears on the boundary of f.

Let f be a face in G. By Configuration (C_1), no two vertices of degree at most 5 are adjacent. Thus f is incident to at most $\lfloor \frac{d(f)}{2} \rfloor$ vertices of degree ≤ 5. We consider four cases depending on $d(f)$.

1. $d(f) = 3$. Then f has an initial weight of 0 and gives nothing, so it has a final weight of at least 0.
2. $d(f) = 4$. Face f is incident to at most 2 vertices of degree ≤ 5. So f has an initial weight of 2 and gives at most two times 1 according to R_1. Thus f has a final weight of at least $2 - 2 \times 1 \geq 0$.
3. $d(f) = 5$. Face f is incident to at most 2 vertices of degree ≤ 5. So f has an initial weight of 4 and gives at most two times 2 according to R_2. Thus f has a final weight of at least $4 - 2 \times 2 \geq 0$.
4. $d(f) \geq 6$. Face f is incident to at most $\lfloor \frac{d(f)}{2} \rfloor \leq \frac{d(f)}{2}$ vertices of degree ≤ 5. So f has an initial weight of $2 \times d(f) - 6$ and gives at most $\frac{d(f)}{2}$ times 2 according to R_2. Thus f has a final weight of at least $2 \times d(f) - 6 - 2 \times \frac{d(f)}{2} = d(f) - 6 \geq 0$.

So all the faces have a final weight of at least 0 after application of the discharging rules. Let us now prove that the same holds for the vertices.

Let x be a vertex of G. We consider different cases corresponding to the value of $d(x)$.

1. $d(x) = 3$. Vertex x has an initial weight of -3. We show that it receives at least 3, thus has a non-negative final weight. By Configuration (C_1), the three neighbors of x are of degree 8. We consider four cases depending on the degrees of the three faces f_1, f_2 and f_3 incident to x. We assume $d(f_1) \geq d(f_2) \geq d(f_3)$. Let u_1, u_2 and u_3 be the three neighbors of u, where for every $i \in \{1, 2, 3\}$, the edge (x, u_i) belongs to f_{i-1} and f_i (subscripts taken modulo 3).

   a) $d(f_1) \geq 5$ *and* $d(f_2) \geq 4$.
      So x receives 2 from f_1 by R_2, and at least 1 from f_2 by R_1 or R_2.

b) $d(f_1) = d(f_2) = d(f_3) = 4$.
So x receives 1 from each f_i by R_1.

c) $d(f_1) = d(f_2) = 4$ *and* $d(f_3) = 3$.
So x receives 1 from both f_1 and f_2 by R_1. Besides, x is a semi-weak neighbor of u_1 and u_3, so x receives $\frac{1}{2}$ from u_1 and u_2 by R_4.

d) $d(f_1) \geq 5$ *and* $d(f_2) = d(f_3) = 3$.
So x receives 2 from f_1 by R_2. Vertex x is a weak neighbor of u_3, so x receives 1 from u_3 by R_3.

e) $d(f_1) = 4$, *and* $d(f_2) = d(f_3) = 3$.
So x receives 1 from f_1 by R_1. Besides, x is a weak neighbor of u_3 and a semi-weak neighbor of u_1 and u_2, so x receives 1 from u_3 by R_3, and $\frac{1}{2}$ from both u_1 and u_2 by R_4.

f) $d(f_1) = d(f_2) = d(f_3) = 3$.
Then x is a weak neighbor of u_1, u_2 and u_3, so x receives 1 from u_1, u_2 and u_3 by R_3.

2. $d(x) = 4$. Vertex x has an initial weight of -2. We show that it receives at least 2, thus has a non-negative final weight. By Configuration (C_1), the four neighbors u_1, u_2, u_3 and u_4 of x are of degree at least 7. We consider three cases depending on how many triangles are incident to x.

   a) *Vertex x is incident to at most 2 triangles.*
   Then x is incident to at least two faces f_1 and f_2 with $d(f_1), d(f_2) \geq 4$. So x receives at least 1 from both f_1 and f_2 by R_1 or R_2.

   b) *Vertex x is incident to exactly 3 triangles $(x, u_1, u_2), (x, u_2, u_3)$ and (x, u_3, u_4).*
   Then x is incident to a face f_1 with $d(f_1) \geq 4$. So x receives at least 1 from f_1 by R_1 or R_2. Besides, x is a weak neighbor of u_2 and u_3, so x receives $\frac{1}{2}$ from both u_2 and u_3 by R_5.

   c) *Vertex x is incident to 4 triangles.*
   Then x is a weak neighbor of u_1, u_2, u_3 and u_4, so x receives $\frac{1}{2}$ from u_1, u_2, u_3 and u_4 by R_5.

3. $d(x) = 5$. Vertex x has an initial weight of -1. We show that it receives at least 1, thus has a non-negative final weight. By Configuration (C_1), the five consecutive neighbors u_1, u_2, u_3, u_4 and u_5 of x are of degree at least 6.

   In the case where x is incident to a face f with $d(f) \geq 4$, vertex x receives at least 1 from f by R_1 or R_2. So we can assume that x is incident to five triangles (x, u_1, u_2), (x, u_2, u_3), (x, u_3, u_4), (x, u_4, u_5) and (x, u_5, u_1). We consider four cases depending on the number of vertices of degree 6 incident to x.

   a) *Vertex x has at least three neighbors of degree 6.*
   By Configuration (C_{11}), they cannot appear consecutively around x, so they are exactly three. Without loss of generality, we assume $d(u_1) = d(u_2) = d(u_4) = 6$, hence $d(u_3), d(u_5) \geq 7$. Then x is an E_2- or S_2-neighbor of u_3 and u_5, so it receives $\frac{1}{2}$ from both u_3 and u_5 by R_6 or R_9.

b) *Vertex x has exactly two neighbors of degree* 6.
We consider two cases depending on whether these vertices of degree 6 appear consecutively around x.

i. *Vertex x has two consecutive neighbors of degree* 6.
We can assume w.l.o.g. that $d(u_1) = d(u_2) = 6$, and that $d(u_3) \geq d(u_5)$. We consider three cases depending on $d(u_3)$ and $d(u_5)$.

A. $d(u_3) = d(u_5) = 8$.
Then x is an E_2-neighbor of u_3 and u_5, so x receives $\frac{1}{2}$ from both u_3 and u_5 by R_6.

B. $d(u_3) = 8$, $d(u_5) = 7$.
Then x is an E_2-neighbor of u_3, an S_3- or S_4-neighbor of u_5 (depending on the degree of u_4), and an S_4- or E_3-neighbor of u_4, so x receives $\frac{1}{2}$ from u_3 by R_6, and at least $\frac{1}{4}$ from both u_4 and u_5 by R_7, R_{10} or R_{11}.

C. $d(u_3) = d(u_5) = 7$.
Then x is an S_3-neighbor of u_3 and u_5, and an S_3- or E_3-neighbor of u_4, so x receives $\frac{1}{3}$ from u_3, u_4 and u_5 by R_7 or R_{10}.

ii. *Vertex x has no two consecutive neighbors of degree* 6.
We can assume without loss of generality that $d(u_1) = d(u_4) = 6$ and that $d(u_2) \geq d(u_3)$. We consider two cases depending on $d(u_3)$.

A. $d(u_3) = 8$.
Then $d(u_2) = 8$. Vertex x is an E_3- or S_2-neighbor of u_5, and an E_3-neighbor of u_2 and u_3, so x receives at least $\frac{1}{3}$ from u_2, u_3 and u_5 by R_7 or R_9.

B. $d(u_3) = 7$.
Then x is an E_2- or S_2-neighbor of u_5, and an S_3-, S_4- or E_3-neighbor of u_2 and u_3, so x receives $\frac{1}{2}$ from u_5 by R_6 or R_9, and at least $\frac{1}{4}$ from u_2 and u_3 by R_7, R_{10} or R_{11}.

c) *Vertex x has exactly one neighbor of degree* 6.
We can assume without loss of generality that $d(u_1) = 6$, and $d(u_2) \geq d(u_5)$ or $d(u_3) \geq d(u_4)$ if $d(u_2) = d(u_5)$. We consider three cases depending on $d(u_5)$ and $d(u_3)$.

i. $d(u_5) = 8$ *and* $d(u_3) = d(u_4)$.
Then x is an E_3-neighbor of u_2 and u_5, so it receives $\frac{1}{3}$ from both by R_7. Besides, since $d(u_3) = d(u_4)$, vertex x is an S_4- or E_4-neighbor of u_3 and u_4, so it receives $\frac{1}{4}$ from both by R_8 or R_{11}.

ii. $d(u_5) = 8$ *and* $d(u_3) \neq d(u_4)$.
Then $d(u_2) = d(u_3) = 8$ and $d(u_4) = 7$. Vertex x is an E_3-neighbor of u_2, u_3 and u_5, so it receives $\frac{1}{3}$ from each by R_7.

iii. $d(u_5) = 7$.
Then vertex x is an E_3-, E_4- or S_4-neighbor of every u_i for $i \in \{2, 3, 4, 5\}$, so it receives at least $\frac{1}{4}$ from each by R_7, R_8 or R_{11}.

d) *Vertex x has no neighbor of degree* 6.
We consider three cases depending on the degrees of the u_i's.

i. *Vertex x has at least 4 neighbors of degree 8.*
Then x is an E_3- or E_4-neighbor of each of them, so it receives at least $\frac{1}{4}$ from each by R_7 or R_8.

ii. *Vertex x has two consecutive neighbors of degree 7.*
We consider w.l.o.g. that $d(u_1) = d(u_2) = 7$. Then x is an S_4-neighbor of u_1 and u_2, so it receives at least $\frac{1}{4}$ from each by R_{11}. Vertex x is also an S_4- or E_3-neighbor of u_3 and u_5, so it receives at least $\frac{1}{4}$ from each by R_7 or R_{11}.

iii. *Vertex x has at most 3 neighbors of degree 8, and has no two consecutive neighbors of degree 7.*
Since x is only adjacent to vertices of degree 7 or 8, we consider w.l.o.g. that $d(u_1) = d(u_3) = 7$, and $d(u_2) = d(u_4) = d(u_5) = 8$. Then x is an E_3-neighbor of u_2, u_4 and u_5, so it receives $\frac{1}{3}$ from each by R_7.

4. $d(x) = 6$. Vertex x has an initial weight of 0, gives nothing away, and has a final weight of at least 0.

5. $d(x) = 7$. Vertex x has an initial weight of 1. We show that it gives at most 1, thus has a non-negative final weight. By Configuration (C_1), the neighbors of x have degree at least 4, and x has at most 3 weak neighbors of degree at most 5. We consider four cases depending on the weak neighbors of x.

   a) *Vertex x has an S_2-neighbor v.*
   Let $v, w_1, w_2, w_3, w_4, w_5$ and w_6 be the consecutive neighbors of x. By definition of an S_2-neighbor, $d(w_1) = d(w_6) = 6$. By Configuration (C_8), if w_2 (resp. w_5) is a weak neighbor of x, then $d(w_2) > 5$ (resp. $d(w_5) > 5$). Assume w.l.o.g. that $d(w_3) \geq d(w_4)$. Then by Configuration (C_1), if w_3 and w_4 are adjacent then $d(w_3) > 5$. Thus x has at most two weak neighbors of degree at most 5: v and possibly w_4. Besides, $d(v), d(w_4) > 3$. By Rules R_5, R_9, R_{10} and R_{11}, vertex x gives at most $\frac{1}{2}$ to each.

   b) *Vertex x has at least two weak neighbors of degree 4.*
   By Configuration (C_9), x is adjacent to no other weak neighbor of degree 4, and no S_2, S_3 or S_4-neighbor. Thus x gives $\frac{1}{2}$ to each of the two weak neighbors of degree 4 by R_5.

   c) *Vertex x has exactly one weak neighbor v of degree 4 and no S_2-neighbor.*
   If x has an S_3-neighbor v_2, then by Configuration (C_{10}), it has no other neighbor of degree at most 5. Thus x gives $\frac{1}{2}$ to v by R_5, $\frac{1}{3}$ to v_2 by R_{10}.
   If x has no S_3-neighbor, then x has at most two other weak neighbors v_1 and v_2 of degree at most 5, which are of degree 5 by assumption. So x gives $\frac{1}{2}$ to v by R_5, $\frac{1}{4}$ to v_1 and v_2 by R_{11}.

   d) *Vertex x has no weak neighbor of degree 4, and no S_2-neighbor.*
   Vertex x has at most three weak neighbors v_1, v_2 and v_3 of degree at most 5, which are of degree 5 by assumption. So x gives at most $\frac{1}{3}$ to each by R_{10} or R_{11}.

6. $d(x) = 8$. Vertex x has an initial weight of 2. We show that it gives at most 2, thus has a non-negative final weight. By Configurations (C_1) and (C_2), vertex x has at

most 4 neighbors that are either semi-weak with degree 3 or weak with degree at most 5. We consider eight cases depending on the neighborhood of x.

a) *Vertex x has at least two weak neighbors v_1 and v_2 of degree 3.*
Then by Configuration (C_3), vertex x has exactly two neighbors of degree at most 5. Thus x gives 1 to v_1 and v_2 by R_3.

b) *Vertex x has exactly one weak neighbor v_1 of degree 3, and at least one semi-weak neighbor v_2 of degree 3.*
Then by Configuration (C_4), vertex x has at most one other neighbor v_3 of degree at most 5. By assumption, vertex v_3 is not a weak neighbor of x of degree 3, so x gives at most $\frac{1}{2}$ to v_3 by R_4, R_5, R_6, R_7 or R_8. Vertex x gives 1 to v_1 by R_3, and $\frac{1}{2}$ to v_2 by R_4.

c) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, and at least two weak neighbors v_2 and v_3 of degree 4.*
Then, by Configuration (C_5), vertex x has no other weak neighbor of degree at most 5. By assumption, it has no semi-weak neighbor of degree 3. So x gives 1 to v_1 by R_3, $\frac{1}{2}$ to v_2 and v_3 by R_5.

d) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, exactly one weak neighbor v_2 of degree 4, and at least one E_2- or E_3-neighbor v_3.*
By definition of E_2- and E_3-neighbor, vertices x and v_3 have a common neighbor v_4 of degree at most 7, which by Configuration (C_1) has degree 6 or 7. Then, by Configuration (C_6), vertex x has no other neighbor of degree at most 5. So x gives 1 to v_1 by R_3, $\frac{1}{2}$ to v_2 by R_5, at most $\frac{1}{2}$ to v_3 by R_6 or R_7.

e) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, exactly one weak neighbor v_2 of degree 4, and no E_2- or E_3-neighbor.*
Then x has at most two other weak neighbors v_3 and v_4 of degree at most 5, which are by assumption E_4-neighbors. So x gives 1 to v_1 by R_3, $\frac{1}{2}$ to v_2 by R_5, $\frac{1}{4}$ to v_3 and v_4 by R_8.

f) *Vertex x has exactly one weak neighbor v_1 of degree 3, no semi-weak neighbor of degree 3, no weak neighbor v_2 of degree 4, and at least an E_2-neighbor v_2.*
Then by Configuration (C_7), vertex x has at most one other weak neighbor v_3 of degree at most 5, which is by assumption of degree 5. So x gives 1 to v_1 by R_3, at most $\frac{1}{2}$ to v_2 and v_3 by R_6, R_7 or R_8.

g) *Vertex x has exactly one weak neighbor v_1 of degree 3, no weak neighbor v_2 of degree 4, no semi-weak neighbor of degree 3, and no E_2-neighbor.*
Then x has at most three other weak neighbors v_2, v_3 and v_4 of degree at most 5, which are by assumption of degree 5. Vertex x has no E_2-neighbor, so they are E_3 or E_4-neighbors of x. So x gives 1 to v_1 by R_3, at most $\frac{1}{3}$ to v_2, v_3 and v_4 by R_7 or R_8.

h) *Vertex x has no weak neighbor of degree 3.*
Then x has at most four neighbors v_1, v_2, v_3 and v_4 of degree at most 5 that are either weak with degree at least 4 or semi-weak with degree 3. So x gives at most $\frac{1}{2}$ to each by R_4, R_5, R_6, R_7 or R_8.

Consequently, after application of the discharging rules, every vertex and every face of G has a non-negative weight, $6|E|-6|V|-6|F| = (2|E|-6|V|)+(4|E|-6|F|) = \sum_{v\in V}(d(v)-6)+\sum_{f\in F}(2d(f)-6) \geq 0$, a contradiction to Euler's Formula. $\square$

Conclusion

Proof of Theorem 3.13

Let G be a minimal planar graph with $\Delta(G) \leq 8$ such that G is not 9-edge-choosable. By Lemma 3.20, graph G cannot contain **(**C_1**)** to **(**C_{11}**)**. Lemma 3.21 implies that G is a stable set, thus 9-edge-choosable, a contradiction. $\square$

Chapter 4

Square coloring

In this chapter we consider the problem of square coloring. This is based on joint works with Benjamin Lévêque and Alexandre Pinlou [BLP14a, BLP14b, BLP14c], and with Nicolas Bousquet [BB14a].

4.1 An overview of square coloring

Square coloring is a subcase of vertex coloring, as square coloring a graph G is equivalent to coloring its square G^2. Contrary to what happens with line graphs, we cannot characterize squares in terms of forbidden subgraphs. There is no forbidden induced subgraph in the class of square graphs. Indeed, consider any graph H. Take an edge subdivision H' of H. Its square H'^2 belongs by definition to the class of square graphs, and contains H as an induced subgraph. Beside, the path on three vertices is not a square graph. In particular, the class of square graphs is not closed under induced subgraphs.

However, a conjecture similar to Conjecture 3.2 was still formulated in the case of square graphs.

Conjecture 4.1 ([KW01]). *Every graph G satisfies $\chi(G^2) = \chi_\ell(G^2)$.*

This turns out not to be the case: not only is the equality not always true, but the difference can be arbitrarily large [KP14b], even in the case of bipartite graphs [KP14a]. Zhu asked whether this could not be satisfied at least for higher powers of graphs, but it is, again, not true.

Theorem 4.2 ([KKP13, KPRY14]). *There exists $c > 0$, such that for any $k \in \mathbb{N}^*$, there is a graph family $(G_{n,k})_{n\in\mathbb{N}}$ with $\chi_\ell(G^k_{n,k}) \geq c \cdot log(\chi(G^k_{n,k})) \cdot \chi(G^k_{n,k})$ and unbounded $\chi_\ell(G^k_{n,k})$.*

This leads to another question, about the relative size of the gap.

Question 4.3 ([KPRY14]). *Is it true that, for every $k \geq 2$, we have $\chi_\ell(G^k) = o(\chi(G^k)^2)$ for every G?*

Kosar *et al.* further ask in [KPRY14] whether, for every $k \geq 2$, there exists c_k such that $\chi_\ell(G^k) \leq c_k \cdot log(\chi(G^k)) \cdot \chi(G^k)$, and whether the dependency in k can be dropped (i.e. c_k replaced with c). The fact that $\chi_\ell(G^k)$ is bounded with a function of $\chi(G^k)$

follows from comparing each with $\Delta(G)$ (Remember that no such function exists for $k = 1$ by Theorem 1.5). We now detail how $\chi_\ell(G^k)$ and $\chi(G^k)$ relate with $\Delta(G)$ for each $k \geq 2$.

Let u be a vertex of degree $\Delta(G)$. Now, in G^2 we have a clique $N(u) \cup \{u\}$ of size $\Delta(G) + 1$. It then follows that $\chi(G^2) \geq \Delta(G) + 1$. For every $k \geq 2$, we have $\chi(G^k) \geq \chi(G^2)$. However, we cannot in general provide a better lower bound than $\Delta(G) + 1$ even for very large k. We can observe this by considering stars, though no larger connected graph would achieve it. On the other side, we can note that G^2 has maximum degree at most $\Delta(G)^2 = \Delta(G) + \Delta(G) \times (\Delta(G) - 1)$. More generally, it holds that G^k has maximum degree at most $D_{k,\Delta(G)}$, with

$$D_{a,b} = b \times \sum_{i=1}^{a} (b-1)^{i-1}$$

(note that $D_{2,\Delta(G)} = \Delta(G)^2$). These maxima are reached e.g. for a tree with a root u, every vertex at distance less than k from u of degree $\Delta(G)$, and every vertex at distance k from u of degree 1. In G^k, the vertex u is adjacent to all the other vertices in the tree, and is therefore of degree $D_{k,\Delta(G)}$.

From Theorem 1.4, we can derive that G^k can be colored with $D_{k,\Delta(G)}$ colors unless G^k is a clique on $D_{k,\Delta(G)} + 1$ vertices or an odd cycle. Note that a connected square graph on at least three vertices contains a triangle: this is obviously still true for higher powers. Consequently, G^k cannot be an odd cycle other than a triangle. Let us now consider the case when it is a clique on $D_{k,\Delta(G)} + 1$ vertices.

For G^k to be a clique, the graph G must have diameter at most k. A d-regular graph with diameter at most k cannot have more than $D_{k,d} + 1$ vertices. Furthermore, a graph G of diameter at most k with a vertex of degree at most $\Delta(G) - 1$ must have less than $D_{k,\Delta(G)} + 1$ vertices. Therefore, it all boils down to the existence of d-regular graphs of diameter k with $D_{k,d} + 1$ vertices. Such a graph is called a *Moore graph*. Moore graphs have been extensively studied, see most notably [MŠ05] for a full survey. In particular, we know that Moore graphs can only exist for specific values of (k, d). Namely when $k = 1$ (all cliques are regular graphs of diameter 1), $d = 2$ (all cycles are 2-regular graphs) or $k = 2$ and $d \in \{3, 7, 57\}$ (the Moore graph for $(2, 3)$ is the famous Petersen graph) [HS60]. In fact, we do not know whether there actually is a Moore graph for $(2, 57)$. However, for each couple (k, d), we know there cannot be two corresponding Moore graphs.

We from now on restrict the study to graphs with maximum degree at least 3, and to $k \geq 2$. In other words, we rule out paths and cycles. It follows from the previous observations that G^k is $D_{k,\Delta(G)}$-colorable unless $k = 2$ and G is a Moore graph.

Cranston and Kim [CK08] were the first to question the optimality of that upper bound.

Conjecture 4.4 ([CK08]). *Every graph G with $\Delta(G) \geq 3$ satisfies $\chi(G^2) \leq \Delta(G)^2 - 1$ unless G is a Moore graph.*

They proved Conjecture 4.4 for $\Delta = 3$, namely that the square of every subcubic graph is 8-colorable, except for the Petersen graph whose square requires 10 colors. The bound of 8 is optimal since the Petersen graph minus an edge requires 8 colors.

Cranston and Rabern [CR13] later solved it for every $\Delta(G) \geq 4$, thus settling Conjecture 4.4. However, there is no reason why it should be true for squares and not for higher powers, as conjectured by Miao and Fan [MF12].

Conjecture 4.5 ([MF12]). *For every $k \geq 2$, every graph G with $\Delta(G) \geq 3$ satisfies $\chi(G^k) \leq D_{k,\Delta(G)} - 1$ unless $k = 2$ and G is one of the corresponding Moore graphs.*

Conjecture 4.5 is already known to be true for $k = 2$, and we prove this for $k \geq 3$ with a short argument, thus settling Conjecture 4.5.

Theorem 4.6. *For every $k \geq 3$, every graph G with $\Delta(G) \geq 3$ satisfies $\chi(G^k) \leq D_{k,\Delta(G)} - 1$.*

The strategy is to provide a good order on the vertices and conclude with a greedy coloring on it. The idea is very roughly as follows. If there is a short cycle (shorter than $2k$), then any vertex on it has relatively few neighbors in G^k, and we can order the vertices by decreasing distance to two given adjacent vertices on the cycle. If there are two intersecting cycles of length $2k$, we can conclude similarly by considering two vertices, one on the intersection and the other a neighbor of the former on one cycle. Then, if the graph has diameter at least $k+1$, we consider two vertices at distance $k+1$, find another pair of respective neighbors also at distance $k+1$ of each other, give each of the two pairs the same color, and conclude again by considering two internal vertices of a $(k+1)$-path between the initial two vertices. Finally, if the graph has diameter at most k, theory related to Moore graphs guarantees us that it has at most $D_{k,\Delta(G)} - 1$ vertices [Dam73], hence the conclusion. Even though the above argument does skip a few details, the actual proof, which is presented in Section 4.2 is not much more involved. This, combined with the fact that the proof for $k = 2$ is a lot deeper, leads us to believe that the bound of Theorem 4.6 is probably not the optimal bound for higher powers. A reasonable first step toward improving the upper bound would be to try to gain one more color per additional graph power, as follows.

Conjecture 4.7. *For every $k \geq 2$, the k^{th} power of every graph G with $\Delta(G) \geq 3$ is $(D_{k,\Delta(G)} + 1 - k)$-colorable except for a finite set of graphs.*

All the upper bounds presented here still hold in the case of list coloring. After looking at the upper bound to $\chi(G^k)$, we now study the lower bound, and more precisely sufficient conditions for it to be reached. As mentioned earlier, we cannot get any interesting lower bound on the chromatic number of k^{th} powers of graphs for $k \geq 3$. Therefore, we now restrict the study to squares. We know that the square of every graph G requires at least $\Delta(G) + 1$ colors. We are looking for sufficient conditions for G^2 to be $(\Delta(G) + 1)$-colorable.

A first direction is to look into planar graphs with sufficiently large girth:

Conjecture 4.8 ([WL03]). *There exists an integer M such that every planar graph G such that $g(G) \geq 5$ and $\Delta(\mathrm{G}) \geq M$ satisfies $\chi^2(G) = \Delta(\mathrm{G}) + 1$.*

Conjecture 4.8 was proved in [BGI$^+$04, BIN04, DKNŠ08, DKNŠ09] to be true for $g(G) \geq 7$ and false for $g(G) \in \{5, 6\}$. More precisely, the following is known.

Theorem 4.9 ([BGI$^+$04, Iva10]).

(1) *[BGI+04] There exist planar graphs G with $g(G) = 6$ such that $\chi^2(G) > \Delta(G) + 1$ for arbitrarily large $\Delta(G)$.*

(2) *[Iva10] Every planar graph with $g(G) \geq 7$ and $\Delta(G) \geq 16$ satisfies $\chi^2(G) = \Delta(G) + 1$.*

Theorem 4.9 is in fact still true in the case of list coloring. This is immediate in the case of (1), and (2) was proved in that setting. Conjecture 4.8 is completely solved, but we can try to obtain stronger statements. We know from Lemma 1.3 (see Chapter 1 for greater details) that every planar graph G with $g(G) \geq 6$ (resp. 7) satisfies $\mathrm{mad}(G) < 3$ (resp. $\frac{14}{5}$). Therefore, proving, for example, that every graph G with $\mathrm{mad}(G) < \frac{14}{5}$ and sufficiently large $\Delta(G)$ satisfies $\chi^2_\ell(G) = \Delta(G) + 1$ would be a generalization of Theorem 4.9.(2). Thanks to the fact that the maximum average degree is a real value, we can also ask for the exact threshold.

Question 4.10. *What is the supremum M such that any graph G with $\mathrm{mad}(G) < M$ and large enough $\Delta(G)$ (depending only on $\mathrm{mad}(G)$) satisfies $\chi^2_\ell = \Delta(G) + 1$?*

We know that $M \leq 3$ from the family of graphs of Theorem 4.9.(1). Another witness family, also introduced in [BGI+04], is presented in Figure 4.1 (note that $g = 5$ so it is not an alternative family for Theorem 4.9.(1)).

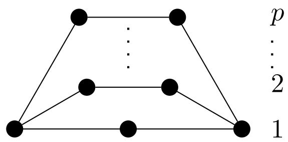


Figure 4.1: A graph G_p with $\Delta(G_p) = p$, $\mathrm{mad}(G_p) = 3 - \frac{5}{2p+1}$ and $\chi^2(G_p) = \Delta(G_p) + 2$.

The question of the value of M is also, indirectly, a question of whether the planarity and/or the girth truly are decisive for square coloring. In other words, can we at least match what is already known for planar graphs, i.e. obtain $M \geq \frac{14}{5}$ with M being a maximum when $M = \frac{14}{5}$? If we can, the exact value of M ($M = \frac{14}{5}$ being a maximum, or $\frac{14}{5} < M \leq 3$) would also yield extra information as to the case of planar graphs with given girth. Interestingly, we can obtain $M = 3$. In a way, this means that Conjecture 4.8 for $g(G) = 6$ is only barely false.

Theorem 4.11. *For any small enough $\epsilon > 0$, every graph G with $\mathrm{mad}(G) < 3 - \epsilon$ and $\Delta(G) \geq \frac{3}{\epsilon^2}$ satisfies $\chi^2_\ell(G) = \Delta(G) + 1$.*

The idea of the proof, which is fully presented in Section 4.3 is relatively simple, building on the elegant proof of Borodin Kostochka Woodall [BKW97] already mentioned in Chapters 1 and 3. It therefore relies on a global discharging argument. After reducing simple configurations in a minimal counter-example, we identify a global structure whose reduction can be boiled down to a list edge coloring problem. When this structure is too dense, the corresponding list edge coloring problem can be tackled with Theorem 1.10. We then apply local discharging rules, and note that only elements of the structure may still have deficient weight afterwards. We then conclude as in [BKW97] using a global discharging rule that compensates for it.

Now that Question 4.10 is solved, we can ask about possible generalizations of it, e.g. ask what happens when we allow an additional constant number of colors. A similar question was already asked in the setting of planar graphs [WL03], and we generalize it here.

Question 4.12. *What is the supremum N such that any graph G with $\mathrm{mad}(G) < N$ and large enough $\Delta(G)$ (depending only on $\mathrm{mad}(G)$) satisfies $\chi^2_\ell(G) = \Delta(G) + \mathcal{O}(1)$?*

Again, we can precisely answer that question, with $N = 4$.

Theorem 4.13. *For any small enough $\epsilon > 0$, every graph G with $\mathrm{mad}(G) < 4 - \epsilon$ satisfies $\chi^2_\ell(G) \leq \Delta(G) + \frac{40}{\epsilon}$.*

To prove Theorem 4.13, we use a very short local discharging argument, with two forbidden configurations and two reduction rules, as presented in Section 4.4. Note that we did not make use of a lower bound on $\Delta(G)$, which would probably strengthen the conclusion but was unnecessary for this purely theoretical result that $N \geq 4$. It holds that $N \leq 4$ due to a family of graphs called Shannon's triangle (see Figure 4.2), where each graph G_p has $\mathrm{mad}(G_p) < 4$ and requires almost $\frac{3\Delta(G_p)}{2}$ colors. Note that this family of graphs happens to be planar of girth 4, which proves the result to be tight even in the case of planar graphs with a lower bound on the girth.

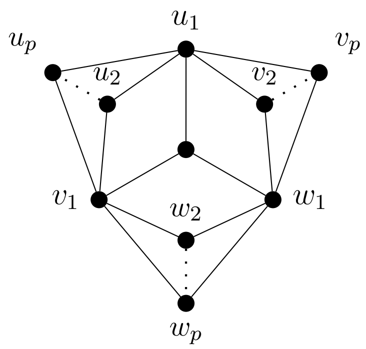


Figure 4.2: A graph G_p with $\Delta(G_p) = 2p - 1$, $\mathrm{mad}(G_p) = 4 - \frac{10}{3p+1}$ and $\chi^2(G) = 3p - 2$.

As a sidenote, we can also wonder about the exact progression of the mad supremum for an increasing number of additional colors. More formally, what is, for any $C \geq 2$, the supremum $M(C)$ such that any graph G with $\mathrm{mad}(G) < M(C)$ and sufficiently large $\Delta(G)$ (depending only on $\mathrm{mad}(G)$) satisfies $\chi^2_\ell(G) \leq \Delta(\mathrm{G}) + C$?

We know that $M(1) = M = 3$, and that $\lim_{C\to\infty} M(C) = N = 4$. Charpentier [Cha14] generalized the family of graphs presented in Figure 4.1 to obtain for each C a family of graphs which are of maximum average degree less than $\frac{4C+2}{C+1}$, of increasing maximum degree, and whose square requires $\Delta + C + 1$ colors to be colored (see Figure 4.3). Consequently, for every C, we have $M(C) \leq \frac{4C+2}{C+1}$. For $C = 2$, this gives an upper-bound of $\frac{10}{3}$, which is incidently also the upper-bound on the mad of planar graphs with girth at least 5.

This result, and the fact that $\frac{4C+2}{C+1}$ equals $M(C)$ when $C = 1$ and when C tends to infinite, raise the following question.

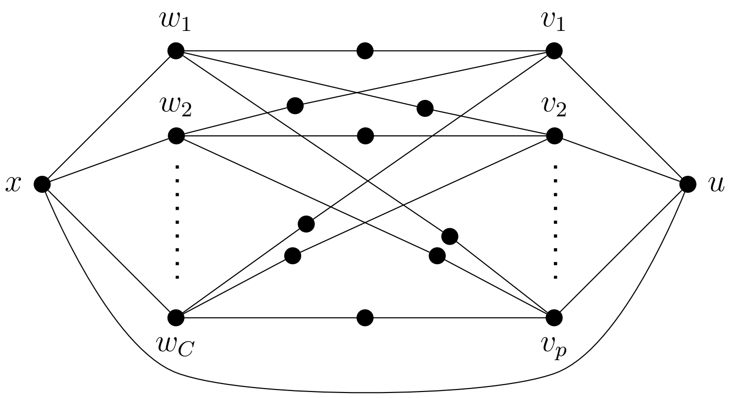


Figure 4.3: For $p \geq C$, a graph $G_{p,C}$ with $\Delta(G_{p,C}) = p + 1$, $\mathrm{mad}(G_{p,C}) = \frac{(2C+1)(2p+1)+1}{(C+1)(p+1)+1}$ and $\chi^2(G_{p,C}) = p + C + 2$.

Question 4.14. *Is it true that $M(C) = \frac{4C+2}{C+1}$ for any $C \geq 1$?*

We believe that the proof of Theorem 4.11 could be adapted to answer positively to Question 4.14 for other small values of C, but it would of course be more exciting to search for a more general answer (i.e. a proof or counter-example for any large enough C).

Now, what can we say about the lower-bounds on $\Delta(G)$? For any $p \in \mathbb{N}^*$, the square of the cycle C_{3p+1} requires at least 4 colors. Therefore, without lower bounds on $\Delta(G)$, even in the case of planar graphs with extremely large girth, we cannot immediately reach the conclusion that $\Delta(G) + 1$ colors are enough. We can then try to find, for every appropriate pair (g, C) (resp. (m, C)), the best possible d such that every planar graph with girth at least g (resp. every graph with mad smaller than m) and Δ at least d satisfies $\chi^2_\ell(G) \leq \Delta(G) + C$. For $C = 1$, the best results proved in the setting of planar graphs are as follows.

Theorem 4.15 ([Iva10])**.** *If G is a planar graph, then $\chi^2_\ell(G) = \Delta(\mathrm{G}) + 1$ in each of the following cases:*

(1) $\Delta(\mathrm{G}) \geq 5$ *and* $g(G) \geq 12$

(2) $\Delta(\mathrm{G}) \geq 6$ *and* $g(G) \geq 10$

(3) $\Delta(\mathrm{G}) \geq 10$ *and* $g(G) \geq 8$

(4) $\Delta(\mathrm{G}) \geq 16$ *and* $g(G) \geq 7$

In the setting of general graphs with an upper-bound on the maximum average degree, we prove the following.

Theorem 4.16. *For any graph G, $\chi^2_\ell(G) = \Delta(\mathrm{G}) + 1$ in each of the following cases:*

(1) $\Delta(\mathrm{G}) \geq 5$ *and* $\mathrm{mad}(G) < \frac{12}{5}$

(2) $\Delta(\mathrm{G}) \geq 6$ *and* $\mathrm{mad}(G) < \frac{5}{2}$

(3) $\Delta(\mathrm{G}) \geq 8$ *and* $\mathrm{mad}(G) < \frac{18}{7}$

We prove this using a global discharging argument along a tree-like structure (see Section 4.5). Independently, Cranston and Skrekovski [CŠ14] proved a more general version of Theorem 4.16 through very similar arguments.

Theorem 4.17 ([CŠ14]). *If G is a graph with $\Delta(G) \geq 6$ and $\mathrm{mad}(G) < 2 + \frac{4\Delta(G)-8}{5\Delta(G)+2}$, then $\chi^2_\ell(G) = \Delta(G) + 1$.*

They also prove that conclusion for $\Delta(G) = 5$ and $\mathrm{mad}(G) < \frac{70}{29}$ [CŠ14], which strengthens Theorem 4.16.

Note that Theorem 4.17 matches Theorem 4.16 for $\Delta(G) \in \{6, 8\}$, while the upper-bound on $\mathrm{mad}(G)$ tends to $\frac{14}{5}$ when $\Delta(G)$ is large.

These theorems, once transposed to planar graphs with a lower-bound on the girth, yields the following.

Corollary 4.18. *If G is a planar graph, then $\chi^2_\ell(G) = \Delta(\mathrm{G}) + 1$ in each of the following cases:*

(1) $\Delta(\mathrm{G}) \geq 5$ *and* $g(G) \geq 12$

(2) $\Delta(\mathrm{G}) \geq 6$ *and* $g(G) \geq 10$

(3) $\Delta(\mathrm{G}) \geq 8$ *and* $g(G) \geq 9$

(4) $\Delta(\mathrm{G}) \geq 14$ *and* $g(G) \geq 8$

Corollary 4.18 matches Theorem 4.15 for $g(G) \geq 12, 10$ and improves it for $g(G) \geq 9$. However, the bounds for $g(G) \in \{7, 8\}$ are yet to be matched.

As to larger values of C, we can in particular ask about the first missing case when $C = 1$, i.e. planar graphs with girth at least 6. Bu and Zhu [BZ12] proved that, with no constraint on $\Delta(G)$, we can prove that $C = 5$ is enough (i.e. every planar graph G with $g(G) \geq 6$ satisfies $\chi^2_\ell(G) \leq \Delta(G) + 5$). In the specific case of $C = 2$, it was first proved that $\Delta(G) \geq 8821$ was enough [DKNŠ08], which was later improved to $\Delta(G) \geq 24$ in the case of list square coloring [BI09b, BI09c] and $\Delta(G) \geq 18$ in the case of standard square coloring [BI09a]. We strengthen this as follows.

Theorem 4.19. *Every graph G with $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 17$ satisfies $\chi^2_\ell \leq \Delta(G) + 1$.*

Despite the fact that Theorem 4.19 is purely incremental, the proof, as presented in Section 4.6 is somewhat interesting from a technical point of view. Indeed, it relies on a global discharging argument which does not simply use the degeneracy of some structure (usually a forest). The structure we consider is a special kind of cactus where each edge may have multiplicity 1 or 2. Instead of using the fact that cacti are 2-degenerate, thus 4-degenerate when we take multiplicity into account, we use the special properties of our cactus to show that it contains in fact at most twice as many edges as it does vertices. In the proofs where degeneracy is crucial, the decisive coloring result behind the global structure are usually either Lemma 1.7 that even cycles are 2-choosable or more generally Theorem 1.10. Here, the decisive coloring result behind our cactus is Lemma 1.8 (see Chapter 1). This is to our knowledge the first occurence of it, and it might be useful for other problems.

In the setting of planar graphs with no restriction on the girth, the ultimate goal is the following long-standing conjecture, which we know to be, if true, optimal, as Wegner [Weg77] provided graphs that show the bounds to be tight for every value of $\Delta(\mathrm{G})$.

Conjecture 4.20 ([Weg77]). *If G is a planar graph, then:*

- $\chi^2(G) \leq 7$ *if* $\Delta(\mathrm{G}) = 3$
- $\chi^2(G) \leq \Delta(\mathrm{G}) + 5$ *if* $4 \leq \Delta(\mathrm{G}) \leq 7$

- $\chi^2(G) \leq \lfloor \frac{3\,\Delta(\mathrm{G})}{2} \rfloor + 1$ *if* $\Delta(\mathrm{G}) \geq 8$

From the previouly mentioned result of Bu and Zhu [BZ12], it follows that Conjecture 4.20 holds when we assume the girth to be at least 6. However, in the general setting, the conjecture is widely open for every value of $\Delta(\mathrm{G})$. Similarly as Conjecture 3.2, we know at least that the order of magnitude is correct, i.e. $\chi^2(G) \leq \lfloor \frac{3\,\Delta(\mathrm{G})}{2} \rfloor + o(\Delta(\mathrm{G}))$ for large $\Delta(\mathrm{G})$ [HHMR07].

The results in this chapter seem to support the idea that it is relevant to try to relax the planarity hypothesis when studying the 2-distance colorability of sparse graphs, despite the fact that some bounds are yet to be matched. However, one might note that none of them deal with planar graphs as a whole. This is no surprise considering that Wegner's conjecture quickly appears to be false when relaxing the hypothesis to graphs with $\mathrm{mad} < 6$ or even $\leq \frac{14}{5}$. Simple examples of this, presented by Cranston and Kim [CK08], are the Petersen graph, which is 3-regular and whose square requires 10 colors, or the Petersen graph minus an edge, whose mad is lowered to $\frac{14}{5}$ but whose square still needs 8 colors. Note also that the Petersen graph has girth 5, which means that planarity is the decisive information here.

We even believe that the relaxation from planar graphs with bounded girth to graphs with bounded maximum average degree might in fact be relevant only for planar graph with girth at least 6. A strong evidence for this would perhaps be to prove the following conjecture from Dvořák, Král, Nejedlý and Škrekovski [DKNŠ08].

Conjecture 4.21. *[DKNŠ08] Every planar graph G with girth at least 5 and sufficiently large $\Delta(G)$ satisfies $\chi^2(G) \leq \Delta(G) + 2$.*

We know from the family of graphs in Figure 4.3 that no such theorem for graphs with $\mathrm{mad} < \frac{10}{3}$ can hold. Therefore, if Conjecture 4.21 holds, then we have an example of a result on planar graphs with girth 5 whose relaxation to graphs with bounded maximum average degree is false. Actually, even in the case of planar graphs with large girth, it might be that a difference appears when these theorems are improved to their optimal values, which are yet to be determined. We have no reason yet to believe that Conjecture 4.1 could not partially hold here.

Question 4.22. *Does there exist $p \geq 5$, $d \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, such that every planar graph G with $g(G) \geq p$ and $\Delta(G) \geq d$ satisfies $\chi^2(G) \leq f(\Delta(G))$, but some do not satisfy $\chi^2_\ell(G) \leq f(\Delta(G))$?*

Question 4.23. *Does there exist $m \leq \frac{10}{3}$, $d \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, such that every graph G with $\mathrm{mad}(G) < m$ and $\Delta(G) \geq d$ satisfies $\chi^2(G) \leq f(\Delta(G))$, but some do not satisfy $\chi^2_\ell(G) \leq f(\Delta(G))$?*

4.2 A Brooks-like theorem on powers of graphs

In this section, we prove Theorem 4.6 that for every $k \geq 3$, every graph G with $\Delta(G) \geq 3$ satisfies $\chi(G^k) \leq D_{k,\Delta(G)} - 1$.

Let $k \geq 3$. Let G be a graph, of maximum degree $\Delta \geq 3$. Let $M = D_{k,\Delta}$. We prove that G^k is $M - 1$ colorable. Note that $M \geq 21$ as $\Delta \geq 3$.

We will need the following lemma, which is essentially an easy adaptation of existing results [CK08, MF12].

Lemma 4.24. *If G satisfies any of the following:*

1. *G contains a vertex of degree smaller than Δ.*

2. *G contains a cycle shorter than $2k$.*

3. *G contains two intersecting cycles of length $2k$.*

4. *$\mathrm{diam}(G) \leq k$.*

Then G^k is $(M-1)$-choosable.

Proof. In each of the first three cases, the proof consists in a greedy coloring relative to a well-chosen order on the vertices, and nothing else.

- Assume G contains a vertex v with $d(v) \leq \Delta - 1$. Since G is connected, the distance to v is well-defined. Order the vertices by decreasing order to v, breaking ties arbitrarily. Proceed with a greedy algorithm on the ordering. Every vertex x at distance at least two from v has at least two neighbors which are not constraints (indeed the vertices on a shortest path from x to v are considered after the vertex x in the order), so x can be colored. For every vertex w which is a neighbor of v, since $k \geq 2$, the degree of w in G^k is at most $M-1$. Moreover, the vertex v is considered after the vertex w in the order, so the vertex w can be colored. Since $k \geq 2$, $\Delta \geq 3$ and $d(v) \leq \Delta - 1$, the degree of v in G^k is at most $M - \Delta < M - 1$, so v is also colored.

- Assume G contains a cycle C of length at most $2k-1$. Let v and w be two adjacent vertices on C. Since C is of length at most $2k-1$, the degree of v and w in G^k is less than $M-1$. Then we greedily color the vertices by decreasing distance to $\{v, w\}$ (breaking ties arbitrarily) and ending with v and w.

- Assume G contains a vertex v belonging to two cycles of length $2k$. Let w be a neighbor of v on one cycle of length $2k$. Vertex v has degree at most $M-2$ in G^k, and w at most $M-1$. Then we greedily color the vertices by decreasing distance to $\{v, w\}$ (breaking ties arbitrarily and ending with w and then v).

- Assume $\mathrm{diam}(G) \leq k$. Then G contains at most $M+1$ vertices, and G^k is a clique. The graph G contains at most $M-1$ vertices [Dam73], and is thus $(M-1)$-choosable.

The result follows. $\square$

By Lemma 4.24 we can assume from now on that G is Δ-regular, with $g(G) \geq 2k$ (i.e. the length of a shortest cycle is at least $2k$), that the cycles of length $2k$ in G are disjoint, and that $\mathrm{diam}(G) \geq k+1$.

Lemma 4.25. *The graph G contains two vertices x_1 and y_1 at distance $k+1$ from each other, with two neighbors x_2, y_2 (respectively) at distance at least $k+1$ from each other.*

Proof. Since $\mathrm{diam}(G) \geq k+1$, G contains two vertices x_1 and y_1 at distance $k+1$ from each other. Let us prove that x_1 has a neighbor x_2 and y_1 a neighbor y_2 such that x_2 and y_2 are at distance at least $k+1$ from each other. Assume for contradiction that each of the Δ neighbors of x_1 are at distance at most k from each of the Δ neighbors of y_1. Let z be a neighbor of x_1. Only $\Delta - 1$ neighbors of z can be part of a path of length at most k between z and a neighbor of y_1, as x_1 is itself at distance at least k from all the neighbors of y_1. Therefore there is a neighbor z' of z that belongs to two paths of length at most k between z and two different neighbors of y_1. Since y_1 is at distance at least k from all the neighbors of x_1, it does not belong to these two paths and this yields a cycle C of length at most $2k$ containing y_1. The cycle C is actually of length $2k$ and contains z', as z' is the endpoint of two different paths of length at most k to y_1 and there is no cycle of length less than $2k$ by assumption. Consequently, y_1 and z' are diametrically opposite on C. Let w be another neighbor of x_1. By the same argument, a neighbor w' of w belongs to a cycle C' of length $2k$ that contains y_1, and w' is diametrically opposite to y_1 in C'. Then C and C' intersect on y_1, which by Lemma 4.24 implies that C and C' are actually the same cycle. Thus w' and z' are actually the same vertex. Now, (w', w, x_1, z) is a cycle of length 4, a contradiction to Lemma 4.24 and the fact that $k \geq 3$. □

We now describe an algorithm to list color G^k. Let L be a list assignment of $M-1$ colors to each vertex.

At any step of the algorithm, the number of *constraints* of a vertex v is the number of colors in $L(v)$ that appear on (already colored) vertices at distance at most k from v in G. Similarly, the number of constraints *implied* on a vertex v by a set S of vertices is the number of colors in $L(v)$ that appear on vertices of S. Note that the number of constraints on a vertex v is bounded by its degree in G^k, and that this upper bound is lowered by 1 if two neighbors of v in G^k have the same color or if a neighbor of v in G^k either is not colored or its color does not belong to $L(v)$.

We consider four vertices x_1, x_2, y_1 and y_2 obtained from Lemma 4.25. Let P be a path of length $k+1$ between x_1 and y_1. Note that by definition of x_2, y_2, at most one of them is on P. Let v be a vertex at distance at least two on P from both x_1 and y_1 (such a vertex exists since P has length at least 4), and let w be a neighbor of v on P distinct from x_2 and y_2. Observe that v is at distance at most k from all of x_1, x_2, y_1, y_2 and w is at distance at most k from x_1, y_1. Our goal is to find an ordering of the vertices of G such that every vertex that is considered for coloring has at most $M-2$ constraints. The order we choose is x_1, y_1, x_2, y_2, followed by all other vertices in decreasing distance to $\{v, w\}$ (the distance to a set is the minimum of the distance to an element of the set). Ties are broken arbitrarily, with the exception that we ensure that the order ends with w and then v. The algorithm is as follows: we choose a good coloring (described below) of $\{x_1, y_1, x_2, y_2\}$, then proceed with a greedy algorithm on the rest of the ordering.

Let us first prove that every vertex $u \notin \{x_1, x_2, y_1, y_2, v, w\}$ is colored at the end of the coloring algorithm (whatever choices we made for x_1, x_2, y_1, y_2). Let us prove that u has at most $M-2$ constraints, i.e. u can be colored since $|L(u)| = M-1$:

- If u is at distance at most k from both v and w, then both v and w are adjacent to u in G^k. Since they are after u in the order, the result holds.

- If u is at distance at least $k+1$ from v or w, let Q be a shortest path from u to $\{v, w\}$. Assume w.l.o.g. that Q is a shortest path from u to v. Let z_1, z_2 and z_3 be the three vertices consecutive to u in Q. These vertices exist since $d(u,v) > k \geq 3$. If $\{z_1, z_2, z_3\} \cap \{x_1, x_2, y_1, y_2\}$ has size at most one, then at least two of $\{z_1, z_2, z_3\}$ are after u in the order, hence the result.
  Otherwise, at least two of $\{z_1, z_2, z_3\}$ are in $\{x_1, x_2, y_1, y_2\}$. Since $d(x_1, y_1) = k+1$, if $x_1 \in \{z_1, z_2, z_3\}$ then none of y_1, y_2 is in this set. The same holds for x_2.
  We may assume w.l.o.g. that the intersection is exactly x_1, x_2. Let w_1 be a neighbor of z_2 distinct from z_1 and z_3. Note that w_1 is neither y_1 nor y_2. Moreover $d(w_1, v) < d(u, v)$ since Q is a shortest path. So w_1 appears after u in the order and $d(w_1, u) \leq k$. Two vertices at distance at most three from u are after u in the order, so u has at most $|M| - 2$ constraints.

Now, let us argue that there is a coloring of $\{x_1, x_2, y_1, y_2\}$ that ensures that v and w will be colored.

In standard vertex coloring (i.e. without involving lists), we set x_1 and y_1 to color 1, and x_2 and y_2 to color 2: then vertices v and w each have at most $M-2$ colors appearing on their neighborhood in G^k. So they each have at most $M-2$ constraints and then both v and w are colored at the end of the greedy coloring.

Since we are considering list coloring, the procedure is slightly more complicated, though the idea remains the same. We want to make sure that the coloring of $\{x_1, y_1\}$ implies at most one constraint on both v and w, and the coloring of $\{x_2, y_2\}$ implies at most one constraint on v. Thus when we consider w, it has one less constraint by $\{x_1, y_1\}$ and one less by v (since w is before v in the order), and then v has two less constraints by $\{x_1, y_1, x_2, y_2\}$.

If $L(x_1) \cap L(y_1) \neq \emptyset$, we color both x_1 and y_1 with the same color. If $L(x_1) \cap L(y_1) = \emptyset$, assume that one of x_1 and y_1 can be colored with a color that belongs neither to $L(v)$ nor to $L(w)$: then we color it accordingly, and the other is then colored arbitrarily. Assume now that it is not the case, i.e. $L(x_1) \cap L(y_1) = \emptyset$ and $L(x_1) \cup L(y_1) \subseteq L(v) \cup L(w)$. Then $L(x_1) \cup L(y_1) = L(v) \cup L(w)$, and we color one of x_1 and y_1 with a color that does not belong to $L(v)$ and the other with a color that does not belong to $L(w)$. Note that in all cases, the colors of $\{x_1, y_1\}$ imply at most one constraint on each of v and w.

The vertices x_2 and y_2 are colored similarly. If $|L(x_2) \cap L(y_2)| \geq 3$, we color both x_2 and y_2 with the same color that differs from the colors used on x_1 and y_1. If $|L(x_2) \cap L(y_2)| \leq 2$ and there is a color in $L(x_2) \cup L(y_2)$ that does not belong to $L(v)$ nor is used on x_1 and y_1, then we color x_2 or y_2 accordingly and color the other arbitrarily (in a color other than those of x_1 and y_1). Assume now that we are in the remaining case, i.e. $|L(x_2) \cap L(y_2)| \leq 2$ and $L(x_2) \cup L(y_2) \subseteq L(v)$. This is impossible since $|L(x_2) \cup L(y_2)| \geq 2 \times |L(v)| - 2 > |L(v)|$. In all possible cases, the colors of $\{x_2, y_2\}$ imply at most one constraint on v.

This completes the proof of Theorem 4.6.

4.3 The threshold for $(\Delta + 1)$-coloring squares of sparse graphs is 3

In this section, we prove Theorem 4.11 that there exists a function f such that for any $\epsilon > 0$, every graph G with $\mathrm{mad}(G) < 3 - \epsilon$ and $\Delta(G) \geq f(\epsilon)$ satisfies $\chi^2_\ell(G) = \Delta(G) + 1$. In the following, we try to simplify the proof rather than improve the function f.

For technical reasons, we will have to consider $\epsilon \leq \frac{1}{20}$. For $\epsilon > \frac{1}{20}$, it suffices to set $f(\epsilon) = f(\frac{1}{20})$. Indeed, if $\epsilon > \frac{1}{20}$, then for every graph with $\mathrm{mad}(G) < 3 - \epsilon$ and $\Delta(G) \geq f(\epsilon)$, we have in particular $\mathrm{mad}(G) < 3 - \frac{1}{20}$ and $\Delta(G) \geq f(\frac{1}{20})$, thus the conclusion holds. From now on, we consider $\epsilon \leq \frac{1}{20}$.

Let $f : \epsilon \mapsto \frac{3}{\epsilon^2}$. Assume by contradiction that there exists a constant $\frac{1}{20} \geq \epsilon > 0$ and a graph Γ with $\mathrm{mad}(\Gamma) < 3 - \epsilon$ and $\Delta(\Gamma) \geq f(\epsilon)$ that satisfies $\chi^2_\ell(\Gamma) > \Delta(\Gamma) + 1$. There is a *minimal* subgraph G of Γ such that $\chi^2_\ell(G) > \Delta(\Gamma) + 1$, in the sense that the square of every proper subgraph of G is list $(\Delta(\Gamma) + 1)$-colorable. For $k = \Delta(\Gamma)$, the graph G satisfies $\Delta(G) \leq k$ and $\chi^2_\ell(G) > k + 1$, while the square of all its proper subgraphs are list $(k + 1)$-colorable. We aim at proving that $\mathrm{mad}(G) \geq 3 - \epsilon$, a contradiction to the fact that G is a subgraph of Γ with $\mathrm{mad}(\Gamma) < 3 - \epsilon$.

Let $M = \frac{6}{\epsilon}$. Note that since $\epsilon \leq \frac{1}{20}$, we have $k = \Delta(\Gamma) \geq f(\epsilon) = \frac{3}{\epsilon^2} \geq \frac{18}{\epsilon} = 3 \times M$.

Forbidden Configurations

We define configurations **(**C_1**)** to **(**C_3**)** (see Figure 4.4).

- **(**C_1**)** is a vertex u of degree 0 or 1.
- **(**C_2**)** is a vertex w_1 of degree at most $k - 1$ that is 2-linked (through w_1-u_1-u_2-w_2) to a vertex w_2 of degree at most $k - 2$.
- **(**C_3**)** is a vertex u with $3 \leq d(u) \leq M$ that is 1-linked (through u-v_i-w_i) to $(d(u) - 2)$ vertices $(w_i)_{1 \leq i \leq d(u)-2}$ of degree at most M, and such that the sum of the degrees of its two other neighbors x and y is at most $k - M + 2$.

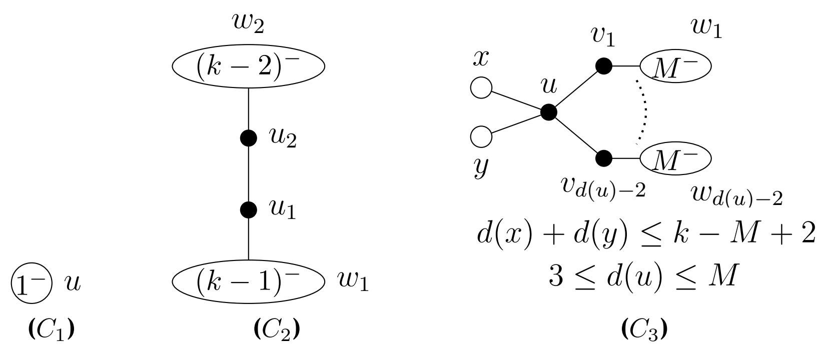


Figure 4.4: Forbidden configurations for Theorem 4.11.

Lemma 4.26. *Graph G cannot contain any of Configurations* **(**C_1**)** *to* **(**C_3**)**.

Proof. We assume G contains a configuration, apply the minimality to color a subgraph of G, and prove this coloring can be extended to the whole graph, a contradiction.

Claim 1. *G cannot contain (C_1).*

Proof. Using the minimality of G, we color $G \setminus \{u\}$. Since $\Delta(G) \leq k$, and $d(u) \leq 1$, vertex u has at most k constraints. There are $k+1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G. □

Claim 2. *G cannot contain (C_2).*

Proof. Using the minimality of G, we color $G \setminus \{u_1, u_2\}$. The vertex u_1 has at most $|\{w_2\}| + d(w_1) \leq 1 + (k-1) \leq k$ constraints. Hence we can color u_1. Then u_2 has at most $|\{w_1, u_1\}| + d(w_2) \leq 2 + (k-2) \leq k$ constraints, so we can extend the coloring of $G \setminus \{u_1, u_2\}$ to G. □

Claim 3. *G cannot contain (C_3).*

Proof. Using the minimality of G, we color $G \setminus \{v_1, \cdots, v_{d(u)-2}\}$. We did not delete u in order to obtain a coloring where x and y receive different colors, but u might have the same color as some w_i, so it needs to be recolored. The vertex u has at most $M - 2 + d(x) + d(y) \leq k$ constraints, hence we can recolor u. Then every v_i has at most $M + M \leq k$ constraints, so we can extend the coloring of $G \setminus \{v_1, \cdots, v_{d(u)-2}\}$ to G. □

This concludes the proof of Lemma 4.26. □

Global structure

We define three sets V_1, V_2 and T that will outline some global structure on G. We build step-by-step the set V_1 as follows.

Any vertex u of degree at most $M-1$ belongs to V_1 if it has $d(u)-1$ neighbors $v_1, \ldots, v_{d(u)-1}$ of degree 2 whose other neighbors $w_1, \ldots, w_{d(u)-1}$ are of degree at most $M-1$, and at most one of $\{w_1, \ldots, w_{d(u)-1}\}$ does not belong to V_1.

Thus, at first, the only vertices in V_1 are those of degree 2 which are adjacent to a vertex of degree 2 whose other neighbor is of degree at most $M-1$. Note that the set is well-defined as a vertex that satisfies at some point the requirements to be in V_1 will always satisfy them, and the order in which vertices are declared to be in V_1 has absolutely no influence on the set V_1 as it is when no more vertex can be added (equivalently, when all the vertices satisfying the requirements are already in V_1).

As for V_2, any vertex u of degree at most $M-1$ belongs to V_2 if it has $d(u)-1$ neighbors $v_1, \ldots, v_{d(u)-1}$ of degree 2 whose other neighbors $w_1, \ldots, w_{d(u)-1}$ are of degree at most $M-1$, and all of $\{w_1, \ldots, w_{d(u)-1}\}$ belong to V_1. Note that V_2 is a subset of V_1.

We define T as the set of vertices of degree 2 whose both neighbors are in V_1. See Figure 4.5 for examples of vertices in V_1, V_2 or T. In the figures, we denote by a label V_1 (resp. V_2, T) the fact that a vertex belongs to V_1 (resp. V_2, T). Similarly, we denote by a label $\overline{V}_1$ a vertex that does not belong to V_1. Since $V_2 \subset V_1$, we omit the label V_1 on vertices labelled V_2.

Lemma 4.27. *The vertices of V_1 satisfy the following:*

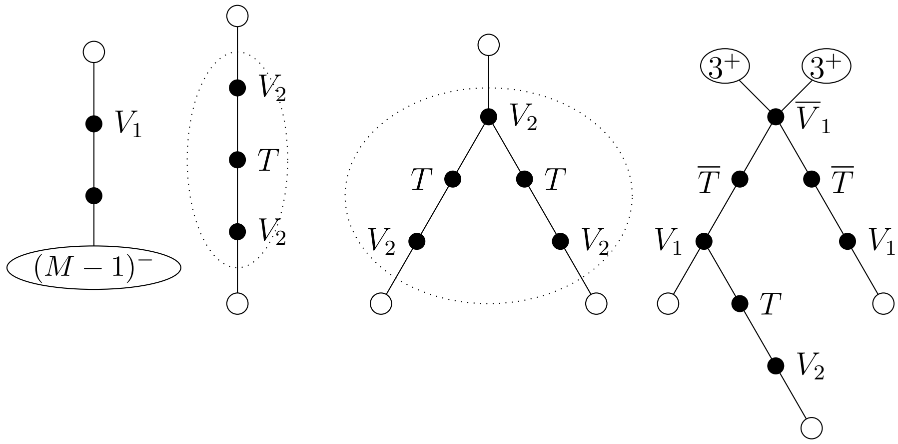


Figure 4.5: Examples of vertices in V_1, V_2 or T.

- *Every vertex of V_1 has exactly one neighbor of degree at least $k - M$.*
- *The set V_1 is a stable set.*
- *The sets V_1 and T are disjoint.*

Proof. Assume by contradiction that a vertex u of V_1 has no neighbor of degree at least $k - M$. Then u is adjacent to $d(u) - 1$ vertices $v_1, \ldots, v_{d(u)-1}$ of degree 2 whose other neighbors are of degree at most $M - 1$, and to another vertex w of degree at most $k - M - 1$. We consider two cases depending on whether $d(u) = 2$.

- If $d(u) = 2$, then the other neighbor of v_1 is a vertex of degree at most $M-1 \leq k-1$ that is 2-linked to w, which is a vertex of degree at most $k - M - 1 \leq k - 2$. By Claim 2 in Lemma 4.26, Configuration **(**C_2**)** is not contained in G, a contradiction.
- If $d(u) \geq 3$, then u is a vertex with $3 \leq d(u) \leq M-1 \leq M$ that is 1-linked (through v_i, for $1 \leq i \leq d(u) - 2$) to $d(u) - 2$ vertices of degree at most $M - 1 \leq M$, and such that the sum of the degrees of its two other neighbors w and $v_{d(u)-1}$ is at most $k - M - 1 + 2 \leq k - M + 2$. By Claim 3 in Lemma 4.26, Configuration **(**C_3**)** is not contained in G, a contradiction.

Therefore every vertex u of V_1 has a neighbor of degree at least $k - M$. By definition of V_1, all the other neighbors of u are of degree 2. Thus u has a unique neighbor of degree at most $k - M$.

Since $k \geq 2M$ then $k - M > M - 1$ and vertex u has no neighbor v of degree $3 \leq d(v) \leq M - 1$. Consequently, two vertices u, v of V_1 that are adjacent must both be of degree 2. By definition of V_1, the other neighbors of u and v must be of degree at most $M-1$, a contradiction. It follows that V_1 is a stable set in G and thus $T \cap V_1 = \emptyset$. $\square$

Any connected component C of $G[V_1 \cup T]$ is a *weak component of G* if every vertex belongs to V_2 or T (in other words, if no vertex of C belongs to V_1 and not to V_2). The only apparent weak components on Figure 4.5 are encircled. The *strength* of a component of $G[V_1 \cup T]$ is the number of vertices of V_1 it contains. Let C_w be the set of weak components of G of strength less than $\frac{1}{\epsilon}$. Let S_w be the set of vertices of V_2 that belong to an element of C_w. Let U be the set of vertices of degree at least $k - M$ with a neighbor in S_w.

Lemma 4.28. *The graph G satisfies $|C_w| \leq \frac{1}{\epsilon} \times |\{v \in V \mid d_G(v) \geq k - M\}|$.*

Proof. Assume by contradiction that $|C_w| > \frac{1}{\epsilon} \times |\{v \in V \mid d(v) \geq k - M\}|$.

Recall that by Lemma 4.27, every vertex of $S_w \subseteq V_1$ has a unique neighbor in U. Let D be the bipartite multigraph whose vertex set is $V(D) = U \cup C_w$, and whose edge set is in bijection with S_w: for every element $v \in S_w$, we add an edge (u, w), where u is the element of U adjacent to v and w is the element of C_w to which v belongs.

For $A = \{v \in V \mid d(v) \geq k - M\}$, $B = C_w$ and $\alpha = \frac{1}{\epsilon}$, we have $|B| > \alpha|A|$. So by Lemma 2.11, there is a subset C'_w of C_w such that, for U' the neighbors of C'_w in U, the subgraph D' induced in D by $C'_w \cup U'$ satisfies $\forall u \in U'$, $d_{D'}(u) \geq \frac{1}{\epsilon}$.

Let S'_w (resp. T') be the set of vertices of S_w (resp. T) that belong to an element of C'_w.

We color by minimality $G \setminus (S'_w \cup T')$. Note that every vertex v of S'_w, belonging to V_2, is adjacent to exactly one vertex u of degree at least $k - M$, and that all its other neighbors $v_1, \ldots, v_{d(u)-1}$ are vertices of T whose other neighbors $w_1, \ldots, w_{d(u)-1}$ are in V_1. Since the element C of $C'_w \subseteq C_w$ to which v belongs is a connected component of $G[V_1 \cup T]$, all the v_i's and w_i's belong to $C \in C'_w$. Consequently, for every i, we have $v_i \in T'$ and $w_i \in S'_w$. Thus v has at most $k + 1 - d_{D'}(u)$ constraints, hence v has at least $d_{D'}(u)$ colors available. To color the vertices of S'_w, it is sufficient to list-color the edges of D', where every edge is assigned the same list of colors as the vertex of S'_w it is in bijection with.

By definition of C_w and since $C'_w \subseteq C_w$, every element of C'_w contains at most $\frac{1}{\epsilon}$ vertices of V_2, so it has degree at most $\frac{1}{\epsilon}$ in D thus in D'. Moreover, every vertex of U' has degree at least $\frac{1}{\epsilon}$ in D'. Thus for every edge (u, v) of D', with $u \in U$ and $v \in C'_w$, we have $\max(d_{D'}(u), d_{D'}(v)) = d_{D'}(u)$. So D' is a bipartite multigraph whose every edge has a list assignment of size at least $\max(d_{D'}(u), d_{D'}(v))$. We apply Theorem 1.10 to color the vertices of S'_w.

It then remains to color the vertices of T'. These are vertices of degree 2 whose both neighbors are in S'_w. But all the vertices of S'_w are of degree at most M. So the vertices of T' have at most $2 \times M \leq k$ constraints, and we can color the vertices of T', a contradiction. □

Discharging rules

We introduce four discharging rules R_1, R_2, R_3 and R_g ('g' stands for 'global'), as follows (see Figure 4.6). We will use them in the case where the initial weight of a vertex v is $d(v) - 3 + \epsilon$. The weight of a subset of vertices is the sum of the weights of the vertices it contains. During the discharging process, a subset of vertices (here, a weak component) may receive some charge: the question of which vertices in that subset actually receive this charge is of no importance. Indeed, we later consider only the weight of the component, and do not care for the details inside.

Here each connected component of $G[T \cup V_1]$ (and in particular each weak component of G) behaves as a single entity. For any vertex x,

- Rule R_1 is when $d(x) = 2$ and its two neighbors a and b are such that $d(a) = 2$ and $d(b) \geq M$, and the other neighbor c of a is not in V_1. Then x gives $\frac{\epsilon}{2}$ to a.

- Rule R_2 is when $3 \leq d(x) \leq M - 1$ and $x \notin V_1$. If x has a neighbor a of degree 2 whose other neighbor is y,
  - Rule $R_{2.1}$ is when $d(y) = 2$. Then x gives $1 - \frac{3\epsilon}{2}$ to a.
  - Rule $R_{2.2}$ is when $3 \leq d(y) < M$. If $y \notin V_1$, then x gives $\frac{1-\epsilon}{2}$ to a. If $y \in V_1$, then x gives $1 - \epsilon$ to a.

- Rule R_3 is when $M \leq d(x)$. Then x gives $1 - \frac{\epsilon}{2}$ to each of its neighbors.

- Rule R_g states that every vertex of degree at least $k - M$ gives an additional $\frac{1}{\epsilon}$ to an initially empty common pot, and every weak component of G of strength less than $\frac{1}{\epsilon}$ receives 1 from this pot.

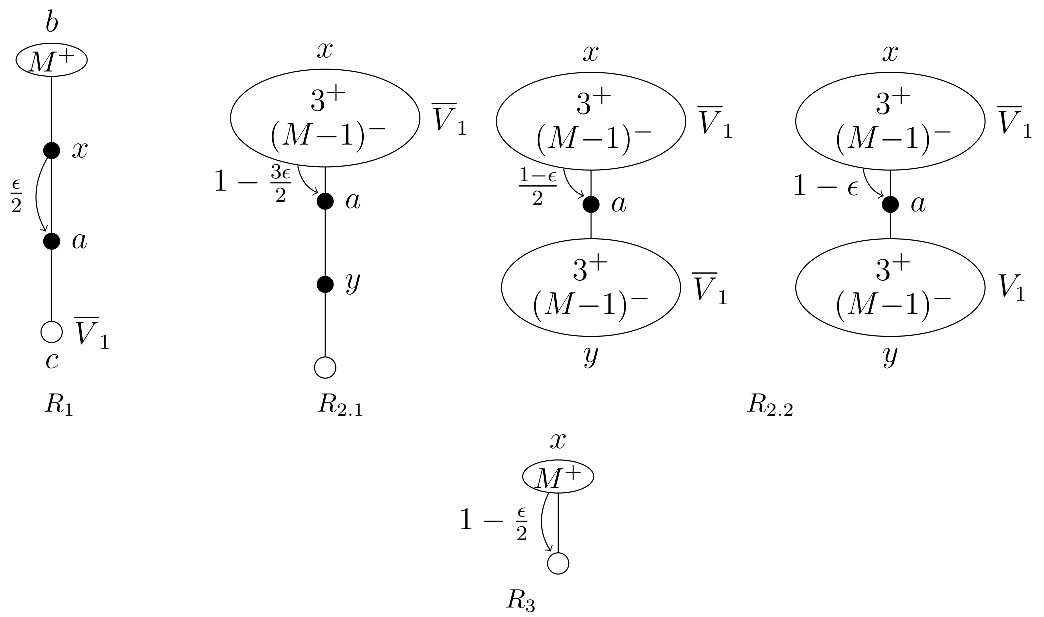


Figure 4.6: Discharging rules R_1, R_2, and R_3 for Theorem 4.11.

We use these discharging rules to prove the following lemma:

Lemma 4.29. *Graph G satisfies* $\mathrm{mad}(G) \geq 3 - \epsilon$.

Proof. We attribute to each vertex v a weight equal to $d(v)-3+\epsilon$, and apply discharging rules R_1, R_2, R_3 and R_g. We show that all the vertices of $G \setminus (T \cup V_1)$ have a non-negative weight in the end, and that each connected component of $G[T \cup V_1]$ has a non-negative total weight.

By Lemma 4.28, the common pot has a non-negative value, and Rule R_g is valid.

Let x be a vertex of $G \setminus (T \cup V_1)$. By Configuration **(**C_1**)**, we have $d(x) \geq 2$.

1. $d(x) = 2$.
   The vertex x has an initial weight of $-1 + \epsilon$. We prove that it receives at least $1 - \epsilon$. Let u_1 and u_2 be its two neighbors. We consider two cases depending on whether one of them is of degree at least M.

   a) *$d(u_1) \geq M$ or $d(u_2) \geq M$.*
      Consider w.l.o.g. that $d(u_1) \geq M$. By R_3, vertex u_1 gives $1 - \frac{\epsilon}{2}$ to x. The vertex x gives at most $\frac{\epsilon}{2}$ to u_2 by R_1. So x receives at least $1 - \epsilon$.

   b) *$d(u_1) < M$ and $d(u_2) < M$.*
      Assume that u_1 or u_2 is of degree 2. Consider w.l.o.g. that $d(u_1) = 2$. Then u_1 belongs to V_1 by definition, and the other neighbor of u_1 is of degree at least M. Since $u_1 \in V_1$ and $x \notin T$, then $u_2 \notin V_1$ and we have $M \geq d(u_2) \geq 3$. By R_1 and $R_{2.1}$, vertex u_1 gives $\frac{\epsilon}{2}$ to x, and u_2 gives $1 - \frac{3\epsilon}{2}$. So x receives $1 - \epsilon$ and gives no weight away. If both u_1 and u_2 have degree at least three, then since $x \notin T$, at most one of u_1 and u_2 is in V_1 and $R_{2.2}$ applies. So vertices u_1 and u_2 give a total of $1 - \epsilon$ to x, and x gives no weight away.

2. $3 \leq d(x) \leq M - 1$.
   The vertex x has an initial weight of $d(x) - 3 + \epsilon \geq \epsilon$. Let $u_1, \ldots, u_q$ denote its neighbors of degree 2 whose other neighbor is of degree at most $M - 1$, where $u_1, \ldots, u_p$ denote its neighbors of degree 2 whose other neighbor belongs to V_1 (note that p may be equal to 0 when x has no such neighbor, and that q may be equal to p). We consider two cases depending on q.

   a) $q \leq d(x) - 3$.
      Then x gives at most $(d(x) - 3) \times (1 - \epsilon) \leq d(x) - 3 + \epsilon$ by R_2.

   b) $q \geq d(x) - 2$.
      Then, by Configuration **(**C_3**)**, vertex x has a neighbor v with $d(v) \geq \frac{k-M+2}{2} \geq M$ (recall that $k \geq 3 \times M$). By Rule R_3, vertex x receives $1 - \frac{\epsilon}{2}$ from v. We consider two cases depending on p.

      i. $p \leq d(x) - 3$. By Rule R_2, x gives at most $(d(x) - 3) \times (1 - \epsilon) + 2 \times \frac{1-\epsilon}{2} \leq d(x) - 3 + \epsilon + (1 - \frac{\epsilon}{2})$.

      ii. $p \geq d(x) - 2$. Since $x \notin V_1$, we have $p = q = d(x) - 2$. By Rule R_2, x gives at most $(d(x) - 2) \times (1 - \epsilon) \leq d(x) - 3 + \epsilon + (1 - \frac{\epsilon}{2})$.

3. $M \leq d(x) \leq k - M - 1$.
   By Rule R_3, vertex x gives at most $d(x) \times (1 - \frac{\epsilon}{2})$. Since $M = \frac{6}{\epsilon}$, we have $d(x) \times \frac{\epsilon}{2} \geq 3 \geq 3 - \epsilon$, so x has a non-negative final weight.

4. $k - M \leq d(x)$.
   By Rules R_3 and R_g, vertex x gives at most $\frac{1}{\epsilon} + d(x) \times (1 - \frac{\epsilon}{2})$. Since $k \geq \frac{3}{\epsilon^2}$, $M = \frac{6}{\epsilon}$ and $\epsilon \leq \frac{1}{20}$, we have $d(x) \times \frac{\epsilon}{2} \geq (\frac{3}{\epsilon^2} - \frac{6}{\epsilon}) \times \frac{\epsilon}{2} = \frac{3}{2\epsilon} - 3 \geq \frac{1}{\epsilon} + 10 - 3 \geq \frac{1}{\epsilon} + 3 - \epsilon$, so x has a non-negative final weight.

Therefore, every vertex of $G \setminus (T \cup V_1)$ has a non-negative final weight. It remains to consider vertices of $G[T \cup V_1]$. Let C be a connected component of $G[T \cup V_1]$. Let s be the strength of C. Note that $s \geq 1$.

If $s = 1$, then C consists of a single vertex u of degree 2 in G and that is adjacent to a vertex v of degree at least $k - M$ and 1-linked to a vertex w of degree less than M. Thus, by R_1 and R_3, vertex u has an initial weight of $-1+\epsilon$, receives $1-\frac{\epsilon}{2}$ from v, and gives $\frac{\epsilon}{2}$ to its neighbor of degree 2: its final weight is 0. We assume from now on that $s \geq 2$.

No vertex of C gives weight. Indeed, a vertex of C can only send charge according to R_1 since all the other rules are for vertices of degree at least M or vertices of degree at least 3 but not in V_1. If a vertex x of C sends some charge according to R_1, then $d(x) = 2$ and its two neighbors a and b are such that $d(b) \geq M$ and $d(a) = 2$, where the other neighbor c of a is not in V_1. Since $d(b) \geq M$, we have $b \notin V_1$ and $x \notin T$, so $x \in V_1$. Then, since V_1 is a stable set by Lemma 4.27, we have $a \notin V_1$, and $s = 1$, a contradiction with our assumption.

We denote by $N(C)$ the set of vertices that do not belong to C but are adjacent to a vertex in C. Since every vertex in $C \cap V_1$ has a neighbor of degree at least $k - M$ (thus not in C), and the vertices in $C \cap V_1$ that are not in V_2 have a neighbor of degree $2 < k - M$ that is not in C, we have $\sum_{v \in V_1 \cap C} d_{N(C)}(v) \geq s + |C \cap (V_1 \setminus V_2)|$. Also, every vertex u in $C \cap V_1$ receives $1 - \frac{\epsilon}{2}$ from its neighbor of degree at least $k - M$. Thus the weight W of C (without taking R_g into account) is as follows.

$$
\begin{aligned}
W &\geq \sum_{v \in C} (d(v) - 3 + \epsilon) + s \times (1 - \frac{\epsilon}{2}) \\
&\geq \sum_{v \in T \cap C} (d(v) - 3 + \epsilon) + \sum_{v \in V_1 \cap C} (d(v) - 3 + \epsilon) + s \times (1 - \frac{\epsilon}{2}) \\
&\geq \sum_{v \in T \cap C} (-1 + \epsilon) + \sum_{v \in V_1 \cap C} d_C(v) + \sum_{v \in V_1 \cap C} d_{N(C)}(v) + \sum_{v \in V_1 \cap C} (-3 + \epsilon) + s \times (1 - \frac{\epsilon}{2}) \\
&\geq \sum_{v \in T \cap C} (-1 + \epsilon) + \sum_{v \in V_1 \cap C} d_C(v) + (s + |C \cap (V_1 \setminus V_2)|) + s \times (-3 + \epsilon) + s \times (1 - \frac{\epsilon}{2})
\end{aligned}
$$

Remember that the vertex set of C is the union of $V_1 \cap C$ and $T \cap C$, which are stable sets. Also, the two neighbors of a vertex in T belong to C, so $\sum_{v \in V_1 \cap C} d_C(v) = \sum_{v \in T \cap C} d_C(v) = 2|T \cap C|$. Since C is a connected component, we have $|T \cap C| \geq |V \cap C| - 1 = s - 1$. Then,

$$
\begin{aligned}
W &\geq |T \cap C| \times (-1 + \epsilon) + 2|T \cap C| + |N(C) \setminus U| + s \times (-1 + \frac{\epsilon}{2}) \\
&\geq (-1 - \epsilon + \frac{3\epsilon s}{2}) + |N(C) \setminus U|
\end{aligned}
$$

We consider three cases depending on whether C is weak and $s < \frac{1}{\epsilon}$.

1. *C is a weak component of G and $s < \frac{1}{\epsilon}$.*
   By R_g, component C receives an extra weight of 1. Thus, its final weight is $1 + W \geq 1 + (-1 - \epsilon + \frac{3\epsilon s}{2}) = -\epsilon + \frac{3\epsilon s}{2} > 0$.

2. *C is a weak component of G and $s \geq \frac{1}{\epsilon}$.*
   Then the final weight of C is $W \geq -1 - \epsilon + \frac{3\epsilon s}{2} \geq -1 - \epsilon + \frac{3\epsilon}{2 \times \epsilon} \geq 0$.

3. *C is not a weak component of G.*
   There is at least a vertex v in $(V_1 \cap C) \setminus V_2$. Then the final weight of C is $W \geq (-1 - \epsilon + \frac{3\epsilon s}{2}) + 1 \geq 0$.

Consequently, after application of the discharging rules, every vertex v of $G \setminus \{V_1 \cup T\}$ has a non-negative final weight, and every connected component C of $G[V_1 \cup T]$ has a non-negative final total weight, meaning that $\sum_{v \in G}(d(v) - 3 + \epsilon) \geq 0$. Therefore, $\mathrm{mad}(G) \geq 3 - \epsilon$. This completes the proof of Lemma 4.29, and thus of Theorem 4.11.

□

4.4 The threshold for $(\Delta + \mathcal{O}(1))$-coloring squares of sparse graphs is 4

In this section, we prove Theorem 4.13 that there exists a function h such that for any $\epsilon > 0$, every graph G with $\mathrm{mad}(G) < 4 - \epsilon$ satisfies $\chi^2_\ell(G) \leq \Delta(G) + h(\epsilon)$.

Let $1 > \epsilon > 0$, let $M = \frac{8}{\epsilon} - 2$, and $h(\epsilon) = 5M - 6$.

Note that $M - (4 - \epsilon) = M \times (1 - \frac{\epsilon}{2})$, and that $h(\epsilon) \geq 2M + 3$.

Again, we choose to present a simple proof despite the fact that it means the function h is probably not as good as possible. However it is still optimal up to a constant factor as the graph family presented in Figure 4.2 shows that it could not be less than $\frac{2}{\epsilon}$. Indeed, the family $(G_p)_{p \in \mathbb{N}^*}$ satisfies $\chi^2_\ell(G_p) \geq \chi^2(G_p) \geq 3p = \Delta(G_p) + \frac{2}{4 - \mathrm{mad}(G_p)}$.

We prove by contradiction that every graph G with $mad(G) < 4 - \epsilon$ admits a 2-distance $(\Delta(\mathrm{G}) + h(\epsilon))$-list-coloring.

We call *weak* a vertex of degree 2 or 3 that has at most one neighbor of degree M^+. In the figures, the label $'w'$ means the vertex is weak.

Forbidden Configurations

We define Configurations **(C_1)** and **(C_2)** (see Figure 4.7). Configuration **(C_1)** is a vertex u of degree 1. Configuration **(C_2)** is a vertex u of degree M^- that has a weak neighbor x, and at most 3 neighbors of degree 4^+, among which at most one is of degree M^+.

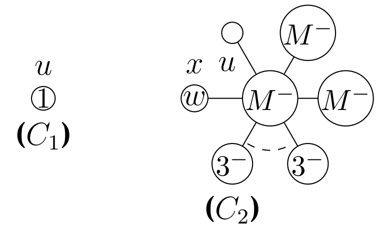


Figure 4.7: Forbidden configurations for Theorem 4.13.

Lemma 4.30. *If G is a minimal graph such that G admits no list 2-distance $(\Delta(G) + h(\epsilon))$-coloring, then G cannot contain Configurations* **(**C_1**)** *nor* **(**C_2**)***.*

Proof. **(**C_1**)** We color $G \setminus \{u\}$ using the minimality of G. Vertex u has at most Δ(G) constraints, so there is a free color for u, a contradiction.

(C_2**)** We remove the (u, x) edge, and use the minimality of G to color the resulting graph. We recolor u (at most Δ(G) $+2M + 3(M - 3) + 2 = \Delta$(G) $+5M - 7$ constraints), and x (at most Δ(G) $+M + M$ constraints), so we can transform the coloring of $G \setminus \{(u, x)\}$ into a coloring of G.

□

Discharging rules

Let R_1, R_2 be two discharging rules (see Figure 4.8). Discharging rule R_1 states that a vertex of degree at least M gives $1 - \frac{\epsilon}{2}$ to each of its neighbors. Discharging rule R_2 states that a vertex of degree less than M gives $1 - \frac{\epsilon}{2}$ to each of its weak neighbors.

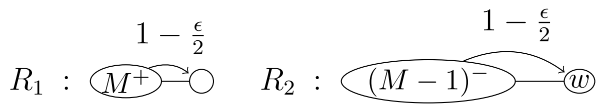


Figure 4.8: Discharging rules R_1, R_2 for Theorem 4.13.

We use these discharging rules to prove the following lemma.

Lemma 4.31. *A graph G that does not contain Configurations* **(**C_1**)** *or* **(**C_2**)** *satisfies* $\mathrm{mad}(G) \geq 4 - \epsilon$.

Proof. We attribute to each vertex a weight equal to its degree, and apply the two discharging rules R_1, R_2. We show that each vertex of G has a weight of at least $4 - \epsilon$ at the end of the discharging.

Let u be a vertex of G. Since Configuration **(**C_1**)** is forbidden, we have $d(u) \geq 2$. We make a case analysis whether u gives some wieght away or not.

- u gives some weight away.
  - If $d(u) \geq M$, **(**R_1**)** is applied, and by definition of M, vertex u gives $1 - \frac{\epsilon}{2}$ to each of its neighbors and still has a weight of at least $4 - \epsilon$.
  - If $d(u) < M$, **(**R_2**)** is applied and u has a weak neighbor x. Since **(**C_2**)** is forbidden, u is in one of these two situations:
    * u has at least two neighbors of degree M^+. According to R_1, they each give $1 - \frac{\epsilon}{2}$ to u. Then u has at most $d - 2$ weak neighbors, and $d(u) - (4 - \epsilon) + 2(1 - \frac{\epsilon}{2}) \geq (d(u) - 2)(1 - \frac{\epsilon}{2})$, so u has a weight of at least $4 - \epsilon$ after application of the discharging rules.

        * u has at least four neighbors of degree 4^+. So u has at most $d-4$ weak neighbors, and $d(u)-(4-\epsilon) \geq (d(u)-4)(1-\frac{\epsilon}{2})$, hence u has a weight of at least $4-\epsilon$ after application of the discharging rules.

- u gives no weight away.
  - $d(u) \geq 4$. Then u still has a weight of at least $4-\epsilon$ after application of the discharging rules.
  - u is a weak vertex. Then, according to **(C_2)**, it can't be adjacent to another weak vertex, so it gives nothing away and receives $1-\frac{\epsilon}{2}$ from each of its neighbors. After application of the discharging rules, it has a weight at least $2+2\times(1-\frac{\epsilon}{2}) = 4-\epsilon$
  - $d(u) \leq 3$ and u is not weak. Then, u has at least two neighbors of degree at least M, so u receives at least $2\times(1-\frac{\epsilon}{2})$. It had initially a weight of at least 2 and gave nothing away, meaning that it has a weight of at least $4-\epsilon$ after application of the discharging rules.

Consequently, after application of the discharging rules, every vertex in G has a weight of at least $4-\epsilon$ after application of the discharging rules, meaning that $\sum_{v\in V} d(v) \geq \sum_{v\in V}(4-\epsilon)$. Therefore, $\mathrm{mad}(G) \geq 4-\epsilon$. □

Conclusion

Proof of Theorem 4.13

We prove by contradiction that $\forall 1 > \epsilon > 0$, every graph G with $\mathrm{mad}(G) < 4-\epsilon$ satisfies $\chi^2_\ell(G) \leq \Delta(\mathrm{G}) + h(\epsilon)$. Let G be a minimal graph with $\mathrm{mad}(G) < 4-\epsilon$ that does not admit a list 2-distance $(\Delta(G)+h(\epsilon))$-coloring. Graph G is also a minimal graph that does not admit a list 2-distance $\Delta(G)+h(\epsilon)$-coloring. By Lemma 4.30, G cannot contain Configurations **(C_1)** nor **(C_2)**. Lemma 4.31 implies that $\mathrm{mad}(G) \geq 4-\epsilon$, a contradiction. □

4.5 $(\Delta+1)$-coloring squares of very sparse graphs

We prove Theorem 4.16 that every graph G with $\Delta(\mathrm{G}) \geq 5$ (resp. 6, 8) and $\mathrm{mad}(G) < \frac{12}{5}$ (resp. $\frac{5}{2}$, $\frac{18}{7}$) satisfies $\chi^2_\ell(G) = \Delta(\mathrm{G}) + 1$.

Let k be a constant integer, $k \geq 5$.

Forbidden Configurations

We define configurations **(C_1)** to **(C_8)** (see Figure 4.9).

- **(C_1)** is a vertex u of degree 0 or 1.

- **(C_2)** is a vertex w_1 of degree at most $k-1$ that is 2-linked (through a path $w_1 - v_1 - v_2 - w_2$) to a vertex w_2 of degree at most $k-2$.

- **(C_3)** is a vertex u of degree 3 that is 1-linked (through a path $u - v_1 - w_1$) to a vertex w_1 of degree at most $k-2$, 1-linked (through a path $u - v_2 - w_2$) to a vertex w_2 of degree at most $k-3$, and whose third neighbour v_3 is of degree at most $k-2$.

- **(C_4)** is a set of vertices $\{a_i\}_{0 \le i \le p-1}$, $p \ge 3$, such that $\forall i$ (i taken modulo p), a_i is 3-linked (through a path $a_i - b_{2i} - c_i - b_{2i+1} - a_{i+1}$) to a_{i+1}.

- **(C_5)** is a vertex u of degree 3 that is 1-linked (through two paths $u - v_1 - w_1$ and $u - v_2 - w_2$) to two vertices w_1 and w_2 of degree at most $k-2$, and whose third neighbour v_3 is of degree at most $k-4$.

- **(C_6)** is a vertex u of degree 4 that is 2-linked (through two paths $u - v_1 - w_1 - x_1$ and $u - v_2 - w_2 - x_2$) to two vertices x_1 et x_2, 1-linked (through a path $u - v_3 - w_3$) to a vertex w_3 of degree at most $k-2$, and whose fourth neighbour v_4 is of degree at most $k-3$.

- **(C_7)** is a vertex u of degree 3 that is 2-linked (through a path $u - v_1 - w_1 - x_1$) to a vertex x_1, and such that the sum of the degrees of its two other neighbours is at most $k-1$.

- **(C_8)** is a vertex u of degree 5 that is 2-linked (through a path $u - v_i - w_i - x_i$, $i \in \{1, 2, .., 5\}$) to five vertices x_1,..,x_5.

Lemma 4.32. *If G is a minimal graph such that $\Delta(G) \le k$ and G admits no list 2-distance $(k+1)$-coloring, and if $i \le k$, then G cannot contain Configuration **(C_i)**.*

Proof. We assume G contains the configuration, apply the minimality to color a subgraph of G, and prove this coloring can be extended to the whole graph, a contradiction.

Claim 1. *G cannot contain **(C_1)***

Proof. Using the minimality of G, we color $G \setminus \{u\}$. Since $\Delta(G) \le k$, and $d(u) \le 1$, vertex u has at most k constraints. There are $k+1$ colors, so the coloring of $G \setminus \{u\}$ can be extended to G. □

Claim 2. *G cannot contain **(C_2)***

Proof. Using the minimality of G, we color $G \setminus \{v_1, v_2\}$. Vertex v_1 has at most $|\{w_2\}| + d(w_1) \le 1 + (k-1) \le k$ constraints. Hence we can color v_1. Then v_2 has at most $|\{w_1, v_1\}| + d(w_2) \le 2 + (k-2) \le k$ constraints, so we can extend the coloring of $G \setminus \{v_1, v_2\}$ to G. □

Claim 3. *G cannot contain **(C_3)***

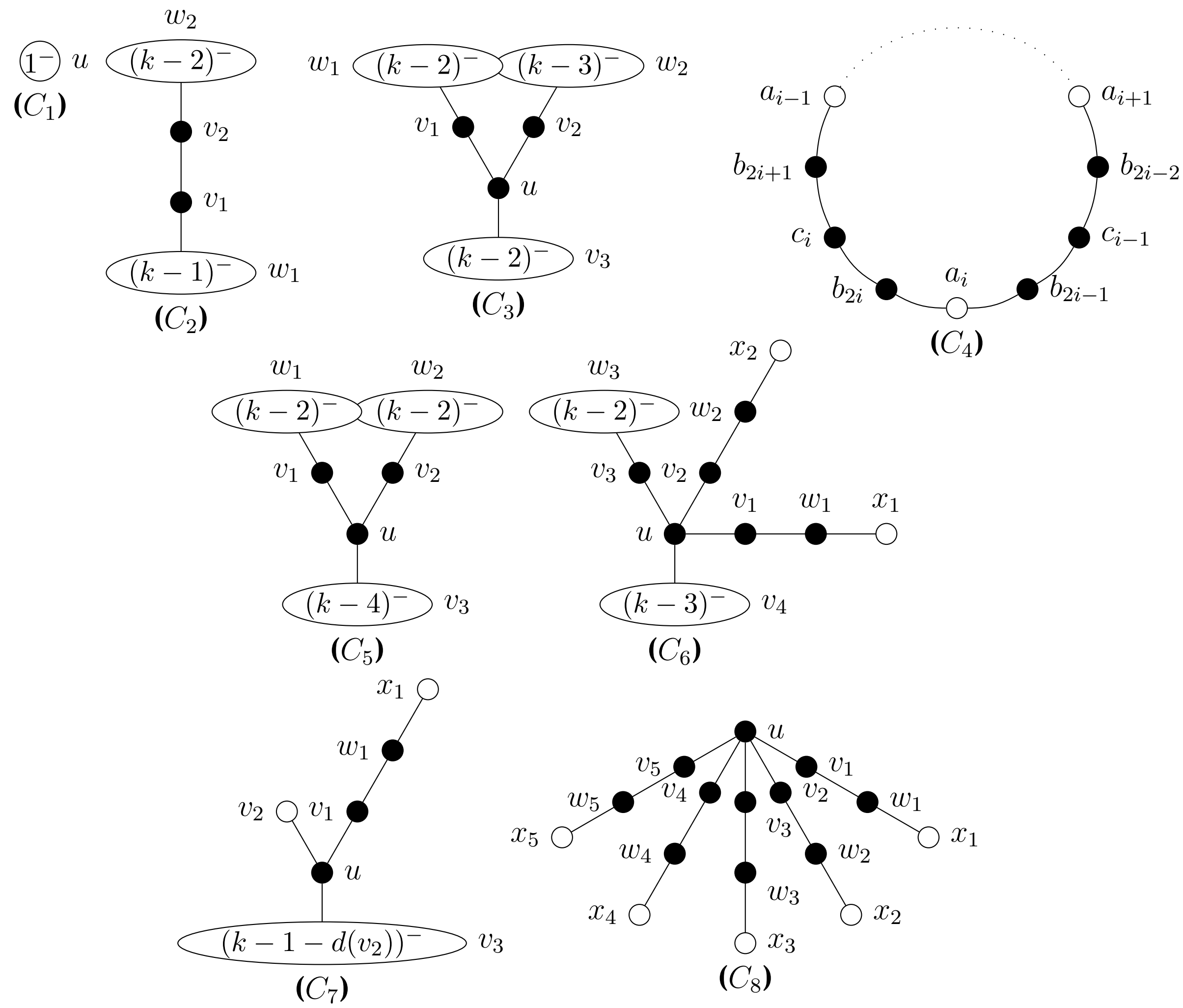


Figure 4.9: Forbidden configurations for Theorem 4.16.

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2\}$. Vertex u has at most $k-2+1+1 \leq k$ constraints. Hence we can color u. Then we colour v_1 (at most $k-2+2 \leq k$ constraints), and v_2 (at most $k-3+3 \leq k$ constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2\}$ to G. $\square$

Claim 4. *G cannot contain (C_4)*

Proof. Using the minimality of G, we color $G \setminus \{b_1, \ldots, b_{2p-1}, c_1, \ldots, c_p\}$. For every j, b_j has at most $k-2$ constraints, hence it has at least 2 colors available. So coloring the set $\{b_1, \ldots, b_{2p-1}\}$ is equivalent to list 2-coloring an even cycle. Then every c_i has at most $4 \leq k$ constraints, so we can extend the coloring of $G \setminus \{b_1, \ldots, b_{2p-1}, c_1, \ldots, c_p\}$ to G. $\square$

Claim 5. *G cannot contain (C_5)*

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2\}$. We can color successively v_1 (at most $k-2+1 \leq k$ constraints), v_2 (at most $k-2+2 \leq k$ constraints), and u (at most $k-4+2+2 \leq k$ constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2\}$ to G. $\square$

Claim 6. *If $k \geq 6$, G cannot contain (C_6)*

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2, v_3\}$. Vertex u has at most $k - 3 + 1 + 1 + 1 \leq k$ constraints. Hence we can color u. Then we colour v_3 (at most $k - 2 + 2 \leq k$ constraints), v_2 (at most $5 \leq k$ constraints) and v_2 (at most $6 \leq k$ constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2\}$ to G. □

Claim 7. *If $k \geq 7$, G cannot contain (C_7)*

Proof. Using the minimality of G, we color $G \setminus \{v_1\}$. We recolor u (at most $k-1+1 \leq k$ constraints), then we color v_1 (at most $5 \leq k$ constraints), so we can transform the coloring of $G \setminus \{v_1\}$ into a coloring of G. □

Claim 8. *If $k \geq 8$, G cannot contain (C_8)*

Proof. Using the minimality of G, we color $G \setminus \{u, v_1, v_2, v_3, v_4, v_5\}$. We color vertices u, v_1, v_2, v_3, v_4, v_5 (each has at most 7 constraints), so we can extend the coloring of $G \setminus \{u, v_1, v_2, v_3, v_4, v_5\}$ to G. □

This concludes the proof of Lemma 4.32.

□

Discharging rules

Let α, β ($1 > \beta \geq \alpha > 0$), M ($k - 1 \geq M \geq 3$) be parameters that we will assign later. Let R_1, R_2, R_3 and R_g be four discharging rules (see Figure 4.10): for any vertex x of degree at least 3,

- Rule R_1 is when $3 \leq d(x) \leq M$.
  - If x has a neighbor a of degree 2 whose other neighbor is y,
    * Rule $R_{1.1}$ is when $d(y) = 2$. Then x gives $2\alpha - \beta$ to a.
    * Rule $R_{1.2}$ is when $3 \leq d(y) \leq M$. Then x gives $\frac{\alpha}{2}$ to a.
  - Rule $R_{1.3}$ is when $d(x) \geq 4$ and x has a neighbour a of degree 3. Then x gives $2(\beta - \alpha)$ to a.
- Rule R_2 is when $M < d(x) < k - 1$. Then x gives α to each of its neighbors.
- Rule R_3 is when $k - 1 \leq d(x)$. Let a be a neighbor of x.
  - Rule $R_{3.1}$ is when $d(a) = 2$. Then, for y the other neighbor of a, x gives α to a and $\beta - \alpha$ to y.
  - Rule $R_{3.2}$ is when $d(a) \geq 3$. Then x gives β to a.
- Rule R_g states that every vertex of degree k gives an additional $3\alpha - 2\beta$ to a common pot, and every vertex of degree 2 which is adjacent to two vertices of degree 2 receives $3\alpha - 2\beta$ from this pot.

We use these discharging rules to prove the following lemma:

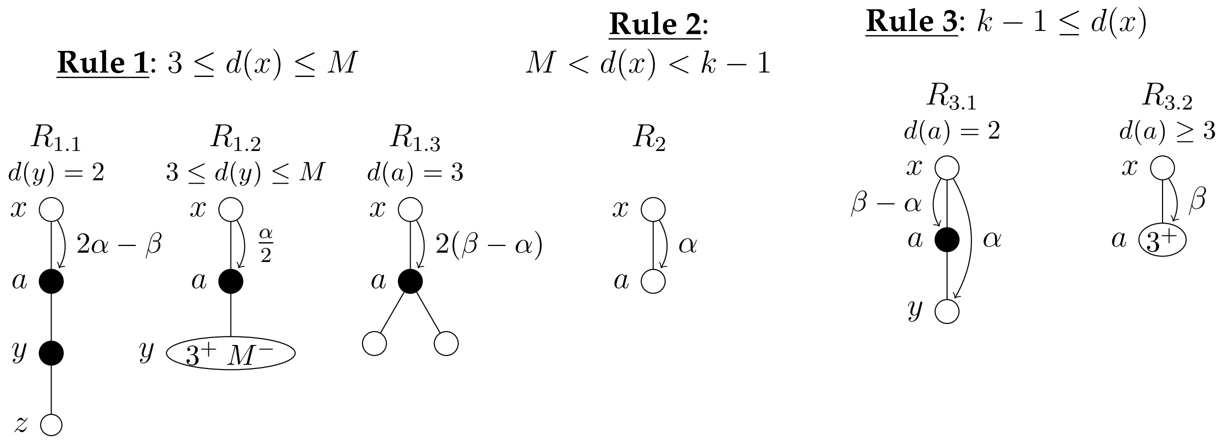


Figure 4.10: Discharging rules R_1, R_2, and R_3 for Theorem 4.16.

Lemma 4.33. *A graph G with $\Delta(G) \leq k$ that does not contain Configuration (C_i) for $i \leq k$ satisfies $\mathrm{mad}(G) \geq 2+\alpha$, where $\alpha = \frac{2}{5}$ if $k=5$, $\frac{1}{2}$ if $7 \geq k \geq 6$, $\frac{4}{7}$ if $k \geq 8$.*

Proof. If $k \leq 7$, we choose $M = k-2$ and $\beta = \alpha$. If $k \geq 8$, we choose $M = k-3$ and $\beta = \alpha + \frac{1}{21}$.

We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2, R_3 and R_g. We show that all the vertices have a weight of at least $2+\alpha$ in the end.

Since (C_4) is forbidden, if we consider the structure A induced in G by the paths $a_1, \ldots, a_5$ where $d(a_2) = d(a_3) = d(a_4) = 2$ ((C_2) implies that $d(a_1) = d(a_5) = k$), A is a forest. This means that in G, there are less vertices of degree 2 adjacent to two vertices of degree 2 than vertices of degree k: hence Rule R_g is valid.

- There are no vertices of degree 0 or 1.
- Let s be a maximal path of vertices of degree 2 (maximal in the sense that it does not admit a vertex of degree 2 as a neighbor; every vertex of degree 2 belongs to such a path as Configuration (C_2) is forbidden). According to the discharging rules, a vertex of degree 2 never gives away weight. We prove that it receives at least α. There are three cases depending on the size of s (s can't be of size greater than 3 due to Configuration (C_2)):
  - $|s| = 1$. Let a be the only vertex in s.
    * a has a neighbor x of degree at least $M+1$: then it receives at least α from it, according to Rule R_2 or R_3.
    * a has two neighbors x_1 and x_2 of degree at most M: then it receives $\frac{\alpha}{2}$ from each, according to Rule $R_{1.2}$.
  - $|s| = 2$. Let a and b be the vertices of s, and x (resp. y) the other neighbor of a (resp. b), with $d(x) \geq d(y)$. Due to Configuration (C_2), $d(x) \geq k-1$. Then a receives α from x (Rule $R_{3.1}$), and b receives $\beta-\alpha$ from x (Rule $R_{3.1}$), and at least $2\alpha-\beta$ from y (Rules $R_{1.1}$, R_2 and $R_{3.1}$).

- $|s| = 3$. Due to Configuration **(**C_2**)**, for $a_2 - a_3 - a_4$ the vertices of s and a_1 (resp. a_5) the other neighbor of a_2 (resp. a_4), $d(a_1) = d(a_5) = k$. Then Rules $R_{3.1}$ and R_g apply: a_2 (resp. a_4) receives α from a_1 (resp. a_5), and a_3 receives $\beta - \alpha$ from a_1 and a_5, and $3\alpha - 2\beta$ from R_g.

- Let x be a vertex with $d(x) = 3$. We prove that x loses a weight of at most $1 - \alpha$.
  - If x is adjacent to a vertex of degree 2 whose other neighbour is also of degree 2.
    - x has no other neighbour of degree 2 whose other neighbour is of degree at most M. Then, according to Rule $R_{1.1}$, it gives $2\alpha - \beta$, which is less than $1-\alpha$ if $k \leq 7$, as $\alpha = \beta \leq \frac{1}{2}$. If $k \geq 8$, then according to Configuration **(**C_7**)**, x has a neighbour of degree at least 4, hence $R_{1.3}$, R_2 or R_3 applies and x receives at least $2(\beta - \alpha)$, and $1 - \alpha + 2(\beta - \alpha) \geq 2\alpha - \beta$ when $\alpha = \frac{4}{7}$ and $\beta = \frac{4}{7} + \frac{1}{21}$.
    - x has a second neighbour of degree 2 whose other neighbour is of degree at most M. Then, according to Configuration **(**C_3**)**, the third neighbour of x is of degree at least $k-1$. Then, x receives at least β (Rule $R_{3.2}$), gives at most $2\times(2\alpha-\beta)$ (Rules $R_{1.1}$ and $R_{1.2}$), and $1-\alpha+\beta \geq 2\times(2\alpha-\beta)$.
  - If x is adjacent to two vertices of degree 2 whose other neighbor is of degree at least 3 and at most M, then we have three cases:
    - $k = 5$, $\alpha = \frac{2}{5}$. Vertex x gives at most $3 \times \frac{\alpha}{2}$ (Rule $R_{1.2}$), but $1 - \alpha \geq 3 \times \frac{\alpha}{2}$ as $\alpha = \frac{2}{5}$.
    - $7 \geq k \geq 6$, $\alpha = \frac{1}{2}$. According to Configuration **(**C_5**)**, the third neighbour of x is of degree at least 3, so, according to Rule $R_{1.2}$, x gives $2\times\frac{\alpha}{2} \leq 1-\alpha$ as $\alpha = \frac{1}{2}$.
    - $k \geq 8$, $\alpha = \frac{4}{7}$. Then $M = k - 3$. According to Configuration **(**C_3**)**, the third neighbor of x is of degree at least $k - 1$, hence x receives β (Rule $R_{3.2}$) and gives at most $2 \times \frac{\alpha}{2}$ (Rule $R_{1.2}$) away, so it loses nothing.
  - If x is adjacent to exactly one vertex of degree 2 whose other neighbor y is of degree at least 3 and at most 13, then x gives at most $\frac{\alpha}{2}$ (Rule $R_{1.2}$), which is possible as $1 - \alpha \geq \frac{\alpha}{2}$.
  - If x is adjacent to no vertex of degree 2 whose other neighbor is of degree at most 13, then it gives nothing away.

- Let x be a vertex with $d(x) = 4$, in the case where $4 \leq M$. Then $k \geq 6$ and $\alpha \geq \frac{1}{2}$. We are in one of the following two cases.
  - If x is 2-linked to two vertices, and 1-linked to a vertex of degree at most M, then, according to Configuration **(**C_6**)**, its other neighbour y is of degree at least $k - 2$. According to Rules $R_{1.1}$ and $R_{1.2}$, x gives nothing to y, and gives at most $2\alpha - \beta$ to its other three neighbours. If $k \leq 7$ and $\alpha \leq \frac{1}{2}$, on the whole x gives at most $3(2\alpha - \beta) \leq 2 - \alpha$. If $k \geq 8$, $M = k - 3$, hence x receives at least α from y, and $3(2\alpha - \beta) \leq 2 - \alpha + \alpha$.
  - If not, according to R_1, x gives at most $3 \times \frac{\alpha}{2} + 2\alpha - \beta$ or $2 \times (2\alpha - \beta) + 2 \times (2\beta - 2\alpha)$, and in both cases, it has a weight of at least $2 + \alpha$ at the end.

- Let x be a vertex with $d(x) = 5$, in the case where $5 \leq M$. Then $k \geq 7$. If $k = 7$ and $\alpha = \frac{1}{2}$, then it gives at most $5 \times (2\alpha - \beta) \leq 3 - \alpha$. If $k \geq 8$ and $\alpha = \frac{4}{7}$, Configuration **(**C_8**)** states that x cannot be 2-linked to 5 vertices, hence it gives at most $4 \times (2\alpha - \beta) + \frac{\alpha}{2} \leq 3 - \alpha$.
- Let x be a vertex with $6 \leq d(x) \leq M$. It gives at most $d(x) \times (2\alpha - \beta)$ away, which means it has at least a weight of $2 + \alpha$ at the end since $d(x) \geq 6$.
- Let x be a vertex with $M < d(x) < k - 1$. It gives at most $d(x) \times \alpha$ away (R_2), which means it has at least a weight of $2 + \alpha$ at the end since $d(x) > M$.
- Let x be a vertex with $k - 1 \leq d(x) < k$. It gives at most $d(x) \times \beta$ away (R_3), which means it has at least a weight of $2 + \alpha$ in the end since $d(x) \geq k - 1$.
- Let x be a vertex with $d(x) = k$. It gives at most $d(x) \times \beta + 3\alpha - 2\beta$ away (R_3 and R_g), which means it has at least a weight of $2 + \alpha$ in the end since $d(x) = k$.

Consequently, after application of the discharging rules, every vertex v of G has a weight of at least $2 + \alpha$, meaning that $\sum_{v \in G} d(v) \geq \sum_{v \in G}(2 + \alpha)$. Therefore, $\mathrm{mad}(G) \geq 2 + \alpha$.

□

Conclusion

Proof of Theorem 4.16

We prove by contradiction that $\forall k \geq 5$, every graph G with $\Delta(G) \leq k$ and $\mathrm{mad}(G) < 2 + \alpha$, where $\alpha = \frac{2}{5}$ if $k = 5$, $\frac{1}{2}$ if $7 \geq k \geq 6$, $\frac{4}{7}$ if $k \geq 8$, satisfies $\chi^2_\ell(G) \leq k + 1$. Let G be a minimal graph such that $\Delta(G) \leq k$, $\mathrm{mad}(G) < 2 + \alpha$ and G does not admit a list 2-distance $(k+1)$-coloring. Graph G is also a minimal graph such that $\Delta(G) \leq k$ and G does not admit a list 2-distance $(k+1)$-coloring (all its proper subgraphs satisfy $\Delta \leq k$ and $\mathrm{mad} < 2 + \alpha$, so they admit a list 2-distance $(k+1)$-coloring). By Lemma 4.32, graph G cannot contain Configuration **(**C_i**)** if $i \leq k$. Lemma 4.33 implies that $\mathrm{mad}(G) \geq 2 + \alpha$, a contradiction. □

4.6 $(\Delta + 2)$-coloring squares of sparse graphs

In this section we prove Theorem 4.19 that a graph G with $\Delta(G) \geq 17$ and $\mathrm{mad}(G) < 3$ satisfies $\chi^2_\ell(G) \leq \Delta(G) + 2$.

Notation and Terminology

A vertex x is *weak* when it is of degree 3 and is 1-linked to two vertices of degree at most 14, or twice 1-linked to a vertex of degree at most 14 (see Figure 4.11). A weak vertex is represented with a w label inside ($\overline{w}$ if it is not weak).

A vertex x is *support* when it is either (see Figure 4.12):

Type (S_1): a vertex of degree 2 adjacent to another vertex of degree 2;

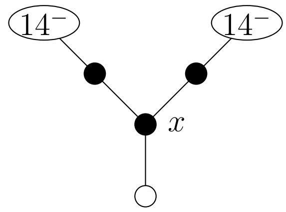


Figure 4.11: A weak vertex x.

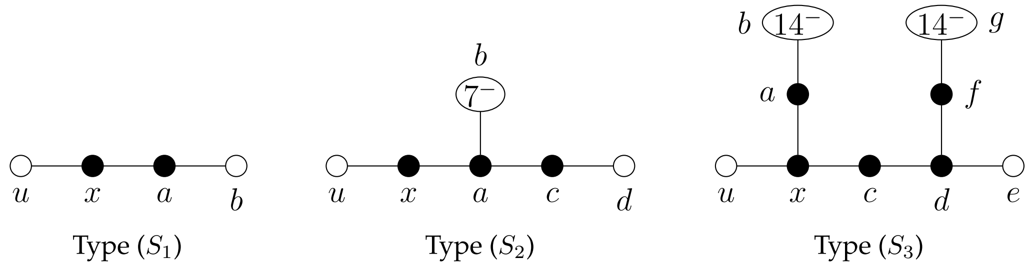


Figure 4.12: Support vertices x.

Type (S_2): a vertex of degree 2 that is adjacent to a vertex of degree 3 which is adjacent to another vertex of degree 2 and to a vertex of degree at most 7;

Type (S_3): a weak vertex 1-linked to another weak vertex.

A vertex is *positive* when it is of degree at least 4 and is adjacent to a support vertex. A vertex u is *locked* if it has two neighbors v_1 and v_2, where v_1 and v_2 are both 1-linked to the same two vertices w_1 and w_2 that have a common neighbor, and $d(v_1) = d(v_2) = d(w_1) = d(w_2) = 3$ (see Figure 4.13). This configuration is called a *lock*.

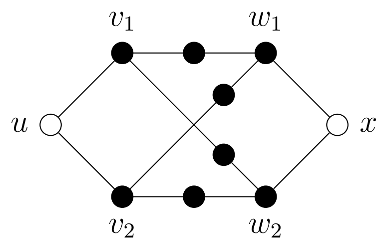


Figure 4.13: A locked vertex u.

Forbidden Configurations

Here, k is a constant integer at least 17 and G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k+2)$-list-coloring.

We define configurations (C_1) to (C_{11}) (see Figures 4.14, 4.15 and 4.16). Note that configurations similar to Configurations (C_1), (C_2) and (C_4) already existed in the litterature, for example in [DKNŠ08].

- (C_1) is a vertex u with $d(u) \leq 1$

- (C_2) is a vertex u with $d(u) = 2$ that has two neighbors v, w and u is 1-linked through v to a vertex of degree at most $k-1$.

- (C_3) is a vertex u with $d(u) = 3$ that has three neighbors v, w, x with $d(w)+d(x) \leq k-1$, and u is 1-linked through v to a vertex of degree at most $k-1$.

- (C_4) is a vertex u with $d(u) = 3$ that has three neighbors v, w, x with $d(w)+d(x) \leq k-1$, and v has exactly three neighbors u, y, z with $d(z) \leq 7$ and $d(y) = 2$.

- (C_5) is a vertex u with $d(u) = 3$ that has three neighbors v, w, x with $d(x) \leq k-1$ and u is 1-linked through v (resp. through w) to a vertex of degree at most 14. (Note that u is a weak vertex.)

- (C_6) is a vertex u with $d(u) = 4$ that has four neighbors v, w, x, y with $d(w) \leq 7$, $d(x) \leq 3$, $d(y) \leq 3$, and u is 1-linked through v to a vertex of degree at most 14.

- (C_7) is a vertex u with $d(u) = 4$ that has four neighbors v, w, x, y with $d(x)+d(y) \leq k-1$ and u is 1-linked through v (resp. through w) to a vertex of degree at most 14.

- (C_8) is a vertex u with $d(u) = 5$ that has five neighbors v, w, x, y, z with $d(w) \leq 7$, $d(x) \leq 3$, $d(y) \leq 3$, $d(z) = 2$, and u is 1-linked through v to a vertex of degree at most 7.

- (C_9) is a vertex u with $d(u) = 6$ that has six neighbors v, w, x, y, z, t with $d(w) \leq 7$, $d(x) \leq 3$, $d(y) \leq 3$, $d(z) = 2$, $d(t) = 2$, and u is 1-linked through v to a vertex of degree at most 7.

- (C_{10}) is a vertex u with $d(u) = 7$ that has seven neighbors $v, w_1, \ldots, w_6$ with $d(v) \leq 7$ and u is 1-linked through w_i, $1 \leq i \leq 6$, to a vertex of degree at most 3.

- (C_{11}) is a vertex u with $d(u) = k$ that has three neighbors v, w, x with x is a support vertex, v, w are both 1-linked to a same vertex y of degree 3, and v (resp. w) is 1-linked to a vertex of degree at most 14 distinct from y. (Note that v, w are weak vertices.)

Lemma 4.34. *G does not contain Configurations (C_1) to (C_{11}).*

Proof. Given a partial 2-distance list-coloring of G, a *constraint* of a vertex u is any color appearing on a vertex at distance at most 2 from u in G.

Notation refers to Figures 4.14, 4.15 and 4.16.

Claim 1. *G does not contain (C_1).*

Proof. Suppose by contradiction that G contains (C_1). Using the minimality of G, we color $G \setminus \{u\}$. Since $\Delta(G) \leq k$, and $d(u) \leq 1$, vertex u has at most k constraints (one for its neighbor and at most $k-1$ for the vertices at distance 2 from u). There are $k+2$ colors available in the list of u, so the coloring of $G \setminus \{u\}$ can be extended to G, a contradiction. □

Claim 2. *G does not contain (C_2).*

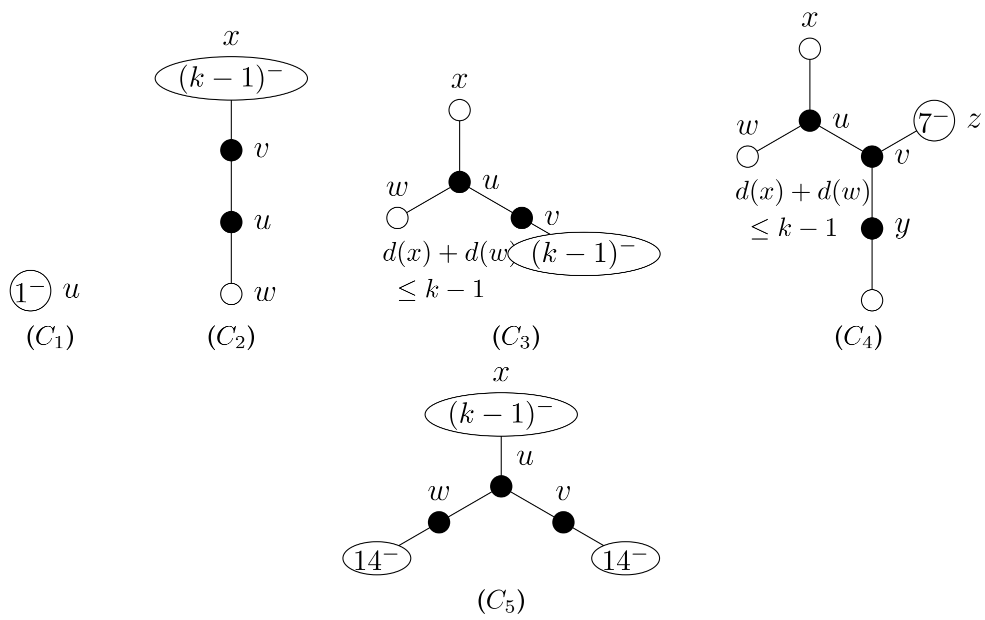


Figure 4.14: Forbidden configurations (C_1) to (C_5).

Proof. Suppose by contradiction that G contains (C_2). Using the minimality of G, we color $G \setminus \{u, v\}$. Vertex u has at most $k+1$ constraints. Hence we can color u. Then v has at most $k-1+2 = k+1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. $\square$

Claim 3. *G does not contain (C_3).*

Proof. Suppose by contradiction that G contains (C_3). Using the minimality of G, we color $G \setminus \{v\}$. Because of u, vertices w and x have different colors. We discolor u. Vertex v has at most $k-1+2 = k+1$ constraints. Hence we can color v. Vertex u has at most $d(w)+d(x)+2 \leq k+1$ constraints. Hence we can color u. So we can extend the coloring to G, a contradiction. $\square$

Claim 4. *G does not contain (C_4).*

Proof. Suppose by contradiction that G contains (C_4). Let e be the edge uv. Using the minimality of G, we color $G \setminus \{e\}$. We discolor u and v. Vertex u has at most $d(w)+d(x)+2 \leq k+1$ constraints. Hence we can color u. Vertex v has at most $7+3+2 \leq k+1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. $\square$

Claim 5. *G does not contain (C_5).*

Proof. Suppose by contradiction that G contains (C_5). Using the minimality of G, we color $G \setminus \{u, v, w\}$. Vertex u has at most $k-1+2 = k+1$ constraints. Hence we can color u. Vertices v and w have at most $14+3 \leq k+1$ constraints respectively. Hence we can color v and w. So we can extend the coloring to G, a contradiction. $\square$

Claim 6. *G does not contain (C_6).*

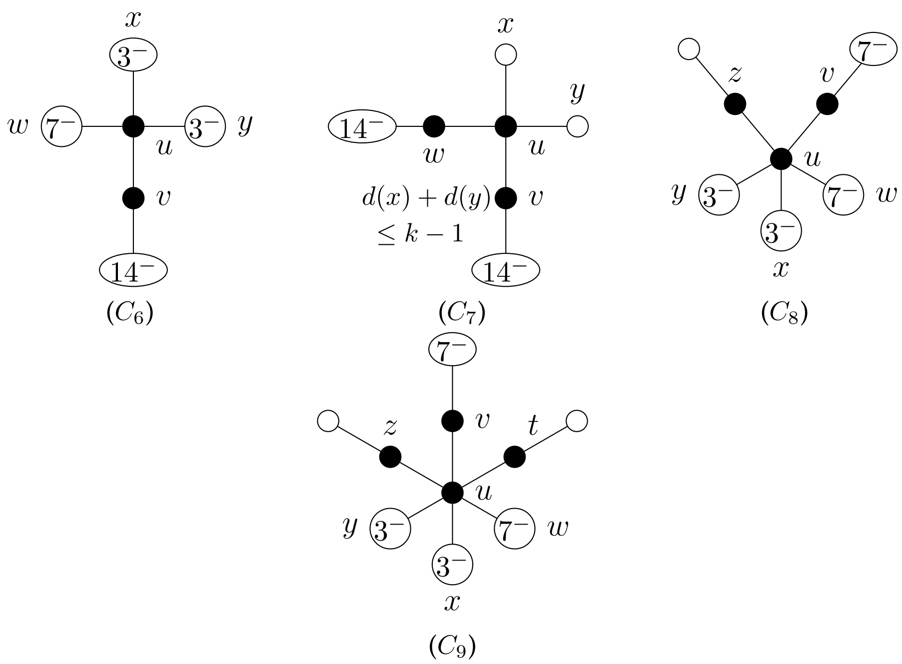


Figure 4.15: Forbidden configurations (C_6) to (C_9).

Proof. Suppose by contradiction that G contains (C_6). Using the minimality of G, we color $G \setminus \{v\}$. We discolor u. Vertex v has at most $14 + 3 \le k + 1$ constraints. Hence we can color v. Vertex u has at most $2 + 3 + 3 + 7 \le k + 1$ constraints. Hence we can color u. So we can extend the coloring to G, a contradiction. □

Claim 7. *G does not contain (C_7).*

Proof. Suppose by contradiction that G contains (C_7). Using the minimality of G, we color $G \setminus \{v, w\}$. We discolor u. Vertex u has at most $d(x) + d(y) + 2 \le k + 1$ constraints. Hence we can color u. Vertices v and w have at most $14 + 4 \le k + 1$ constraints respectively. Hence we can color v and w. So we can extend the coloring to G, a contradiction. □

Claim 8. *G does not contain (C_8).*

Proof. Suppose by contradiction that G contains (C_8). Using the minimality of G, we color $G \setminus \{v\}$. We discolor u. Vertex u has at most $7 + 3 + 3 + 2 + 1 \le k + 1$ constraints. Hence we can color u. Vertex v has at most $7 + 5 \le k + 1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. □

Claim 9. *G does not contain (C_9).*

Proof. Suppose by contradiction that G contains (C_9). Using the minimality of G, we color $G \setminus \{v\}$. We discolor u. Vertex u has at most $7+3+3+2+2+1 \le k+1$ constraints. Hence we can color u. Vertex v has at most $7 + 6 \le k + 1$ constraints. Hence we can color v. So we can extend the coloring to G, a contradiction. □

Claim 10. *G does not contain (C_{10}).*

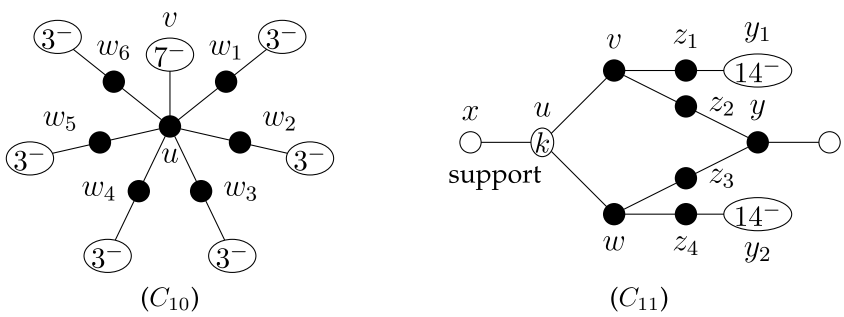


Figure 4.16: Forbidden configurations (C_{10}) and (C_{11}).

Proof. Suppose by contradiction that G contains (C_{10}). Using the minimality of G, we color $G \setminus \{u, w_1, \ldots, w_6\}$. Vertex u has at most $7 + 6 \leq k + 1$ constraints. Hence we can color v. Each vertex w_i has at most $3 + 7 \leq k + 1$ constraints. Hence we can color $w_1, ..., w_6$. So we can extend the coloring to G, a contradiction. □

Claim 11. *G does not contain (C_{11}).*

Proof. Suppose by contradiction that G contains (C_{11}). Since x is a support vertex, and u is of degree k, it is of Type (S_1), (S_2) or (S_3) of support vertices with the notation of Figure 4.12. Note that some vertices may coincide between Figure 4.12 and Figure 4.16.

We define a set of vertices A as follows:

$$A = \begin{cases} \{a\} & \text{if } x \text{ is of Type } (S_1) \\ \{a, c\} & \text{if } x \text{ is of Type } (S_2) \\ \{a, c\} & \text{if } x \text{ is of Type } (S_3) \end{cases}$$

Using the minimality of G, we color $G \setminus (\{v, w, x, y, z_1, \ldots, z_4\} \cup A)$. If x is of Type (S_1) (resp. (S_2)), a (resp. c) has at most $k+1$ constraints. Hence we can color a (resp. c). For the three types (S_i), x has at most $k - 3 + 1 + 2 = k$ constraints, thus it has at least 2 available colors. Vertex y has at most k constraints, thus it has at least 2 available colors. Both v and w have at most $k - 3 + 1 + 1 \leq k - 1$ constraints, so they have at least 3 available colors in their list.

We now explain how to color v, w, x, y (other uncolored vertices will be colored later). Suppose x and y can be assigned the same color, then both v and w have at least 2 available colors and thus can be colored.

Suppose the lists of available colors of x and y are disjoint. We color v with a color not appearing in the list of x. Then we color y that has $k + 1$ constraints. (Vertex x has still at least 2 available colors.) Then we color w that has $k + 1$ constraints and finally x.

Now we assume that we cannot assign the same color to x and y and that their lists of available colors are not disjoint. This means that x and y are either adjacent or have a common neighbor. So some vertices coincide between Figure 4.12 and Figure 4.16. The different cases where x and y are either adjacent or have a common neighbor are the following:

(S_1) – $b = y$

(S_2) – $b = y$

– $a = y$ and w.l.o.g $b = z_2$, $c = z_3$ and $d = w$.

(S_3) – $b = y$

– $d = y$, and w.l.o.g. $f = z_2$, $g = v$ and $e = z_3$.

In all these cases, y has at most 1 constraint. So we can color x, v, w, y, in this order as they all have at most $k + 1$ constraints when they are colored.

If x is of Type (S_2) (resp. (S_3)), vertex a (resp vertices a, c) has at most 11 constraints (resp. 17, 6), so we can color them. The vertices z_i have at most $17 \leq k + 1$, so we can color them. Thus the coloring have been extended to G, a contradiction. □

□

Structure of support vertices

Let $H(G)$ be the subgraph of G induced by the edges incident to at least a support vertex. We prove several properties of support vertices and of the graph $H(G)$.

Lemma 4.35. *Each positive vertex is of degree k and each support vertex is adjacent to exactly one positive vertex.*

Proof. By Lemma 4.34, G does not contain Configurations (C_2), (C_3) and (C_5). So a support vertex is adjacent to a vertex of degree k (Configurations (C_2), (C_3) and (C_5) correspond respectively to support vertices of Type (S_1), (S_2) and (S_3)). By definition, a support vertex has at most one neighbor of degree at least 4, thus it is adjacent to exactly one vertex of degree at least 4 and this vertex has in fact degree k. So all the positive vertices are of degree k and a support vertex is adjacent to exactly one positive vertex. □

Lemma 4.36. *Each cycle of $H(G)$ with an odd number of support vertices contains a subpath $s_1v_1s_2v_2s_3$ where s_1, s_2, s_3 are support vertices of type (S_3) and v_1, v_2 are vertices of degree 2.*

Proof. Let C be a cycle of $H(G)$ with an odd number of support vertices. Cycle C does not contain just one support vertex, as all its edges have to be adjacent to a support vertex (there is no loop or multiple edge in $H(G)$). So C contains at least three support vertices.

Suppose that C contains no positive vertices. Then it contains no support vertices of type (S_1) or (S_2) as such vertices are of degree 2, so all their neighbors would be on C, and they are adjacent to a positive vertex by Lemma 4.35. So C contains only support vertices of type (S_3). Let s_1, s_2, s_3 be three support vertices of C appearing consecutively along C. A support vertex of Type (S_3) is of degree 3, adjacent to two vertices of degree 2 and to a positive vertex. So the neighbors of s_i on C are vertices of degree 2 that are not support vertices. As $H(G)$ contains only edges incident to support vertices, there exist v_1, v_2 of degree 2 such that $s_1v_1s_2v_2s_3$ is a subpath of C.

Suppose now that C contains some positive vertices. Let $p_1, \ldots, p_\ell$ be the set of positive vertices of C appearing in this order along C while walking in a chosen direction (subscripts are understood modulo ℓ). Let Q_i, $1 \leq i \leq \ell$, be the subpath of C between

p_i and p_{i+1} (in the same chosen direction along C). (Note that if $\ell = 1$, then $Q_1 = C$ is not really a subpath.) As C contains an odd number of support vertices, there exists i such that Q_i contains an odd number of support vertices. If Q_i contains just one support vertex v, then Q_i has length 2, since $H(G)$ contains only edges incident to support vertices. So v is adjacent to two different positive vertices (or has a multiple edge if $\ell = 1$), a contradiction to Lemma 4.35. So Q_i contains at least 3 support vertices. Let s_1, s_2, s_3 be three support vertices of Q_i appearing consecutively along Q_i.

If one of the s_i is of Type (S_1), let x be such a vertex. With the notation of Figure 4.12, vertex x is of degree 2, so its two neighbors u, a are on C, with u a positive vertex and a a support vertex of Type (S_1). Then vertex a is of degree 2 so its neighbor b distinct from x is also on C. Vertex b is positive so Q_i is the path u, x, a, b and contains just two support vertices, a contradiction.

If one of the s_i is of Type (S_2), let x be such a vertex. With the notation of Figure 4.12, vertex x is of degree 2, so its two neighbors u, a are on C, with u a positive vertex and a a vertex of degree 3. Vertex a is not adjacent to vertices of degree k so by Lemma 4.35, it is not a support vertex. Let c' be the neighbor of a on C that is distinct from x. As all the edges of $H(G)$ are incident to support vertices, c' is a support vertex. Since c' is adjacent to a vertex of degree 3 it is a support vertex of Type (S_2) and can play the role of c of Figure 4.12. Then c is of degree 2 and its neighbor on C distinct from a is a positive vertex d. So Q_i is the path u, x, a, c, d and contains just two support vertices, a contradiction.

So s_1, s_2, s_3 are all of Type (S_3). A support vertex of Type (S_3) is of degree 3, adjacent to two vertices of degree 2 and to a positive vertex. So the neighbors of s_2 on C are vertices v_1, v_2 of degree 2 that are not support vertices. As $H(G)$ contains only edges incident to support vertices, we can assume w.l.o.g. that $s_1 v_1 s_2 v_2 s_3$ is a subpath of C. □

Lemma 4.37. *$H(G)$ does not contain a 2-connected subgraph of size at least three with exactly two support vertices.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that has exactly two support vertices $S = \{s_1, s_2\}$. We color by minimality $G \setminus (S \cup \{v \in N_G(S) | d_G(v) \leq 3\})$. (Note that by Lemma 4.35, the set $\{v \in N_G(S) | d_G(v) \leq 3\}$ corresponds to vertex a of Figure 4.12 if the support vertex is of Type (S_1) or (S_2) and to vertices a, c if the support vertex is of Type (S_3).)

We first show how to color S. For that purpose we consider three cases corresponding to the type of s_1.

- s_1 *is of Type (S_1).* Then s_1 is of degree 2, has a positive neighbor u and a support neighbor a of Type (S_1). As s_1 is of degree 2, both its neighbors are in C. So a is a support vertex of C, thus $a = s_2$. Let v be the neighbor of s_2 of degree k. Since C contains no other support vertex and is 2-connected, we must have $u = v$. Then u has two neighbors s_1, s_2 that are not colored, so s_1 and s_2 have at most k constraints, and we can color them.

- s_1 *is of Type (S_2).* Then s_1 is of degree 2, has a positive neighbor u and another neighbor a of degree 3. Vertex a is not a support vertex by Lemma 4.35 since it has no neighbor of degree k. As s_1 is of degree 2, all its neighbors are in C. Vertices u and a are in C that is 2-connected so they have at least two neighbors in C. Since they are not support vertices, all their neighbors in C are support vertices. So both u and a are adjacent to s_2. Vertex s_2 is support, it is adjacent to a that is of degree 3, so s_2 is of Type (S_2). Then u is of degree k, has two neighbors s_1, s_2 that are not colored, so s_1 and s_2 have at most k constraints, and we can color them.

- s_1 *is of Type* (S_3)*.* Then s_1 is of degree 3, has a positive neighbor u and two other neighbors w, w' of degree 2. Vertices w, w' are not support vertices by Lemma 4.35 since they have no neighbor of degree k. As s_1 is of degree 3, two of u, w, w' are in C. Let Y be the neighbors of s_1 in C. We can assume by symmetry that either $\{v, w\} \subseteq Y$ or $\{w, w'\} \subseteq Y$. Vertices of Y are in C that is 2-connected so they have at least two neighbors in C. Since they are not support vertices, all their neighbors in C are support vertices. So all the vertices of Y are adjacent to s_2. Vertex s_2 is a support vertex, it is adjacent to w that is non support and of degree 2, so s_2 is of Type (S_3). In both cases ($\{v, w\} \subseteq Y$ or $\{w, w'\} \subseteq Y$), vertices s_1 and s_2 have at most k constraints, and we can color them.

Every vertex of $\{v \in N_G(S) | d_G(v) \leq 3\}$ has at most 17 constraints, hence we can extend the coloring to the whole graph, a contradiction. □

Lemma 4.38. *Every 2-connected subgraph of $H(G)$ that contains exactly three support vertices is a cycle.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that has exactly three support vertices $S = \{s_1, s_2, s_3\}$ and that is not a cycle.

Suppose by contradiction that C contains no cycle C' with $S \subseteq C' \subseteq C$. As C is 2-connected, by Menger's Theorem there exist two internally vertex-disjoint paths Q, Q' between s_1, s_2. Let C'' be the cycle $Q \cup Q'$. By assumption C'' does not contain s_3. So it contains just two support vertices, a contradiction to Lemma 4.37. So C contains a cycle C' with $S \subseteq C' \subseteq C$.

By Lemma 4.36, cycle C' contains a subpath $x_1v_1x_2v_2x_3$ where x_1, x_2, x_3 are support vertices of Type (S_3) and v_1, v_2 are vertices of degree 2. As C contains just three support vertices, we have $S = \{x_1, x_2, x_3\}$. Vertices x_1, x_3 are support vertices of Type (S_3), they are of degree 3 and only adjacent to positive vertices and to vertices of degree 2 so they are not adjacent. The graph $H(G)$ contains only edges incident to support vertices, so there exists a vertex y of C' adjacent to x_1, x_3, and $x_1v_1x_2v_2x_3y$ is the cycle C'. If C' has some chords in $H(G)$, then $H(G)$ contains a cycle with two support vertices only, a contradiction to Lemma 4.37. So C' is an induced cycle of $H(G)$ and so C' has strictly less vertices than C. Let y' be a vertex of C distinct from $x_1, v_1, x_2, v_2, x_3, y$. Vertex y' is not a support vertex, C is 2-connected and $H(G)$ contains only edges incident to support vertices, so y' is adjacent to at least two vertices in S. Then $H(G)$ contains a cycle with two support vertices only, a contradiction to Lemma 4.37. □

Lemma 4.39. *Every 2-connected subgraph of $H(G)$ of size at least three is either a cycle with an odd number of support vertices or a subgraph of a lock of $H(G)$.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that is not a cycle with an odd number of support vertices nor a subgraph of a lock of $H(G)$. Let $S = \{s_1, \ldots, s_p\}$ be the support vertices of C. By Lemma 4.37, $p \geq 3$. Let $\mathcal{S}$ be the graph with $V(\mathcal{S}) = S$ where there is an edge between s_i and s_j if and only if they are adjacent or have a common neighbor in G.

Claim 12. *$\mathcal{S}$ is not a clique of size at least four.*

Proof. Suppose, by contradiction that $\mathcal{S}$ is a clique with $p \geq 4$.

Given a support vertex x, we say that a support vertex x', distinct from x, satisfies the property P_x if it is either adjacent to x in G or has a non-positive common neighbor with x in G. At most two vertices can satisfy P_x (vertex a of Figure 4.12 if x is of Type (S_1), vertices b, c if x is of Type (S_2), vertices b, d if x is of Type (S_3)). Note that if x' satisfies P_x, then x satisfies $P_{x'}$.

We claim that there exist two support vertices in S that do not have a positive common neighbor in G. Suppose by contradiction, that every pair of vertices of S has a positive common neighbor. By Lemma 4.35, every support vertex has at most one positive neighbor, so all the vertices of S are adjacent to the same positive vertex v. As C is 2-connected, there is a path Q in $C \setminus \{v\}$ between s_1, s_2. Let s_i be the first support vertex, distinct from s_1, appearing along Q while starting from s_1 (maybe $i = 2$ if there is no support vertex in the interior of Q). Let Q' be the subpath of Q between s_1 and s_i (maybe $Q = Q'$). Then $Q' \cup \{v\}$ forms a 2-connected subgraph of size ≥ 3 with exactly two support vertices, a contradiction to Lemma 4.37. So there exist two support vertices x, x' in S that do not have a positive common neighbor in G. Since $\mathcal{S}$ is a clique, vertices x, x' are adjacent or have a common non-positive neighbor, so x satisfies $P_{x'}$ (and x' satisfies P_x).

Suppose there exists a support vertex $y \in S$ that does not satisfy P_x nor $P_{x'}$. Since $\mathcal{S}$ is a clique, vertex y has a common positive neighbor z with x and z' with x'. Since x and x' have no positive common neighbor, z and z' are distinct. Thus y has two positive neighbors, a contradiction. So every vertex of $S \setminus \{x, x'\}$ satisfies either P_x or $P_{x'}$. If two vertices y, y' of $S \setminus \{x, x'\}$ satisfy P_x, then at least three vertices, x', y, y' satisfy P_x, a contradiction. So there is at most one vertex of $S \setminus \{x, x'\}$ satisfying P_x and similarly at most one satisfying $P_{x'}$. So $p \leq 4$ and we can assume, w.l.o.g., that $S = \{x, x', y, y'\}$, where vertex y satisfies P_x and not $P_{x'}$ and vertex y' satisfies $P_{x'}$ and not P_x. Thus x has a common positive neighbor z with y' and x' has a common positive neighbor z' with y. Since x, x' do not have a common positive neighbor, z and z' are distinct. Vertices y, y' have at most one positive neighbor, thus, they do not have a common positive neighbor. Since $\mathcal{S}$ is a clique, y satisfies $P_{y'}$. Let $(y_1, y_2, y_3, y_4) = (x, x', y', y)$ (subscript are understood modulo 4).

Suppose there exists $i \in \{1, 2, 3, 4\}$ such that y_i, y_{i+1} are adjacent in G. Two support vertices can be adjacent only if they are of Type (S_1). So y_i, y_{i+1} are of Type (S_1) and of degree two. Then y_i is only adjacent to y_{i+1} and to a positive vertex in $\{z, z'\}$. If y_i is adjacent to y_{i-1}, then $y_{i-1} = y_{i+1}$, a contradiction. If y_i is not adjacent to y_{i-1}, then y_{i+1} is a common neighbor of y_i and y_{i-1}. Since y_{i+1} is of degree two and has a positive neighbor, $y_i = y_{i-1}$, a contradiction. So y_i, y_{i+1} are not adjacent in G for any $1 \leq i \leq 4$. Let w_i be a non-positive common neighbor of y_i, y_{i+1}.

Suppose there exists $i \in \{1,2,3,4\}$ such that $d(y_i) = 2$. Then $w_i = w_{i-1}$. So $\{y_{i-1}, y_i, y_{i+1}\} \subseteq N(w_i)$, and w_i is not positive, so $d(w_i) = 3$. Two support vertices can have a common neighbor of degree 3 only if they are both of degree two (Type (S_2)). So $d(y_{i-1}) = d(y_i) = d(y_{i+1}) = 2$. Since y_{i+1} is of degree two and has a positive neighbor, $w_i = w_{i+1}$, so $y_{i+2} \in N(w_i)$, a contradiction. So $d(y_i) \geq 3$ for any $1 \leq i \leq 4$.

Then all the y_i are of Type (S_3), they are of degree three and their non-positive neighbors are of degree two. Thus $d(w_i) = 2$ for any $1 \leq i \leq 4$. So $y_1, \ldots, y_4, w_1, \ldots, w_4, z, z'$ induce a lock. So all the edges incident to $S = \{y_1, \ldots, y_4\} = \{s_1, \ldots, s_4\}$ belong to a lock, contradicting the definition of C. □

By Lemma 4.37, the graph $\mathcal{S}$ is not an edge. If $\mathcal{S}$ is a triangle, then C contains exactly three support vertices and, by Lemma 4.38, it is a cycle with an odd number of support vertices, a contradiction. So $\mathcal{S}$ is not a triangle. By Claim 12, $\mathcal{S}$ is not a clique of size at least 4. So finally, $\mathcal{S}$ is not a clique.

Suppose, by contradiction, that $\mathcal{S}$ is an odd cycle with ≥ 5 vertices. Then C is a 2-connected graph that is not a cycle, so it contains a vertex v with at least 3 neighbors in C. If v is not a support vertex, then it has at least 3 support neighbors in C that form a triangle in $\mathcal{S}$, a contradiction. So v is a support vertex. Then either v has three neighbors in $\mathcal{S}$, a contradiction to $\mathcal{S}$ being a cycle, or C contains a cycle with two support vertices, a contradiction to Lemma 4.37. So $\mathcal{S}$ is not an odd cycle.

Suppose, by contradiction, that $\mathcal{S}$ is not 2-connected. Then there exist three support vertices s, s', s'' of S such that s', s'' appears in two different connected components of $\mathcal{S} \setminus \{s\}$. As C is 2-connected, there exists a path Q between s', s'' in $C \setminus \{s\}$. This path Q is composed only of edges incident to support vertices so in $\mathcal{S} \setminus \{s\}$ it corresponds to a path between s', s'', a contradiction. So $\mathcal{S}$ is 2-connected.

We now consider the graph G, we color by minimality $G \setminus (S \cup \{v \in N_G(S) | d_G(v) \leq 3\})$. We show how to color S. In the three Types (S_j), the number of constraints on a support vertex s_i of Type (S_j) is at most $k+2$ minus the number of its neighbors in $\mathcal{S}$. So the number of available colors of a support vertex is at least its degree in $\mathcal{S}$. Now Lemma 1.8 can be applied to $\mathcal{S}$, which is not a clique, not an odd cycle and 2-connected. So we can color S. Every vertex of $\{v \in N_G(S) | d_G(v) \leq 3\}$ has at most 17 constraints, hence we can extend the coloring to the whole graph, a contradiction. □

Lemma 4.40. *Every connected component of $H(G)$ is either a cactus where each cycle has an odd number of support vertices or a lock.*

Proof. All the edges of a lock are incident to support vertices of type (S_3) so all the edges of a lock of G appear in $H(G)$. The only vertices of a lock that can have neighbors outside a lock are locked vertices (vertices u and x on Figure 4.13). By Lemma 4.34, graph G does not contain Configuration (C_{11}), so a locked vertex is incident to only two support vertices, the two support vertices of a lock. A lock is a connected component of $H(G)$.

Let C be a connected components of $H(G)$ that is not a lock. By Lemma 4.39, each 2-connected subgraph of C is a cycle with an odd number of support vertices. So C is a cactus where each cycle of C has an odd number of support vertices. □

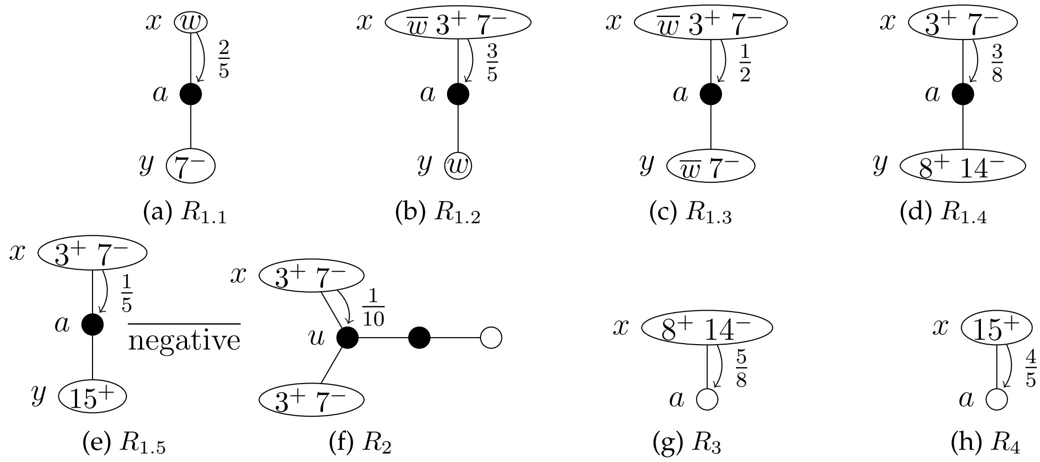


Figure 4.17: Discharging rules $R_{1.i}$, R_2, R_3, and R_4

Discharging rules

A *negative* vertex is a support vertex of type (S_1) or (S_2) or a vertex of degree 2 adjacent to two support vertices of type (S_3). In this case we say that the negative vertex is of type (N_1), (N_2) or (N_3) respectively.

Each vertex has an initial weight (later defined). The discharging rules $R_{1.1}$, $R_{1.2}$, $R_{1.3}$, $R_{1.4}$, $R_{1.5}$, R_2, R_3, R_4 and R_g (see Figure 4.17) defined below explain how vertices will receive and/or give weight. We also use a so-called *common pot* which is empty at the beginning, receives weight from some vertices and gives weight to some others. For any vertex x of degree at least 3,

- Rule R_1 is when $3 \leq d(x) \leq 7$, and x is 1-linked (with a path $x - a - y$) to a vertex y.
  - Rule $R_{1.1}$ is when x is weak with $d(y) \leq 7$. Then x gives $\frac{2}{5}$ to a.
  - Rule $R_{1.2}$ is when x is not weak and y is weak. Then x gives $\frac{3}{5}$ to a
  - Rule $R_{1.3}$ is when x and y are not weak, with $d(y) \leq 7$. Then x gives $\frac{1}{2}$ to a.
  - Rule $R_{1.4}$ is when $8 \leq d(y) \leq 14$. Then x gives $\frac{3}{8}$ to a.
  - Rule $R_{1.5}$ is when $15 \leq d(y)$ and a is not negative. Then x gives $\frac{1}{5}$ to a.
- Rule R_2 is when $3 \leq d(x) \leq 7$ and x is adjacent to a vertex u of degree 3 that is adjacent to a vertex of degree 2 and a vertex of degree at most 7. Then x gives $\frac{1}{10}$ to u.
- Rule R_3 is when $8 \leq d(x) \leq 14$. Then x gives $\frac{5}{8}$ to each of its neighbors.
- Rule R_4 is when $15 \leq d(x)$. Then x gives $\frac{4}{5}$ to each of its neighbors.
- Rule R_g states that each positive vertex gives $\frac{2}{5}$ to a common pot, and that each negative vertex receives $\frac{1}{5}$ from the common pot.

Lemma 4.41. *The common pot has non-negative value after applying R_g.*

Proof. Given a set of vertices X, let $n(X)$ be its number of negative vertices and $p(X)$ its number of positive vertices. To prove that the common pot has positive value after applying R_g, we show that each connected component C of $H(G)$ satisfies $p(C) \geq \left\lceil \frac{n(C)}{2} \right\rceil$.

Let C be a connected component of $H(G)$. By Lemma 4.40, C is either a cactus where each cycle has an odd number of support vertices or a lock. If C is a lock, then $n(C) = 4$ and $p(C) = 2$, so we are done. So we can assume that C is a cactus where each cycle has an odd number of support vertices.

Claim 13. *Every connected subgraph C' of C, whose pendant vertices are positive vertices, whose support vertices are adjacent to their positive neighbor in C' and whose negative vertices of Type (N_3) are adjacent to their two neighbors in C', satisfies $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.*

Proof. Suppose by contradiction that this is false. Let C' be a connected subgraph of C of minimum number of vertices, whose pendant vertices are positive vertices, whose support vertices are adjacent to their positive neighbor in C', and such that $p(C') < \left\lceil \frac{n(C')}{2} \right\rceil$. The graph C' is a connected subgraph of a cactus so it is also a cactus.

Suppose first that C' contains a pendant vertex u. Let x be the neighbor of the positive vertex u in C'. As $H(G)$ contains only edges incident to support vertices, x is a support vertex. So it is not positive and thus is not a pendant vertex of C'. So x has at least two neighbors in C'. We consider different cases according to the Type of x and its number of neighbors in C'.

- *x is of Type (S_1).* Then let a be the neighbor of x distinct from u. We have $a \in C'$ and a is a support vertex of Type (S_1). The positive neighbor b of a is in C' by assumption. Let C'' be the graph $C' \setminus \{u, x, a\}$. We have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C since u, x, a is a subpath of C' where u is pendant and x, a are of degree 2. All the pendant vertices of C'' are positive since the only new possible pendant vertex is b. All the support vertices of C'' are adjacent to their positive neighbor in C'' since the only positive vertex that has been removed is u and its support neighbor x has also been removed. All the negative vertices of Type (N_3) are adjacent to their two neighbors in C' as no support vertex of Type (S_3) has been removed. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') = p(C'') + 1 \geq \left\lceil \frac{n(C'')+2}{2} \right\rceil = \left\lceil \frac{n(C')}{2} \right\rceil$.

- *x is of Type (S_2).* Then let a be the neighbor of x distinct from u. We have $a \in C'$ and a is of degree 3. Let b, c be the neighbors of a distinct from x. Since a is not positive, it is not a pendant vertex of C', so at least one of b, c is in C'. We assume w.l.o.g. that c is in C'. As $H(G)$ contains only edges incident to support vertices, vertex c is a support vertex of Type (S_2). We consider two cases depending on whether a has its three neighbors in C' or not.

  If $b \in C'$, then let C'' be the graph $C' \setminus \{u, x\}$. We have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in C'' and all negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

If $b \notin C'$, then let C'' be the graph $C' \setminus \{u, x, a, c\}$. We have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in C'' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

- *x is of Type (S_3) and has two neighbors in C'.* Then let c be the neighbor of x distinct from u that is in C'. Vertex c is of degree 2, it is not positive, so its neighbor d, distinct from x, is in C'. As $H(G)$ contains only edges incident to support vertices and c is not a support vertex, vertex d is a support vertex and so of Type (S_3). Let e, f be the neighbors of d distinct from c where e is a positive vertex and f is a vertex of degree 2. Vertex e is the positive neighbor of d so it is in C' by assumption. We consider two cases corresponding to whether d has its three neighbors in C' or not. If $f \in C'$, then let C'' be the graph $C' \setminus \{u, x, c\}$. If $f \notin C'$, then let C'' be the graph $C' \setminus \{u, x, c, d\}$. In both cases, we have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$. The graph C'' is a connected subgraph of C, all its pendant vertices are positive, all its support vertices are adjacent to their positive neighbor in C'' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality, we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.
- *x is of Type (S_3) and has three neighbors in C'.* Then let a, c be the neighbors of x distinct from u. We have a, c in C'. Vertex a (resp. c) is of degree 2, it is not positive, so its neighbor b (resp. d) is in C'. As $H(G)$ contains only edges incident to support vertices and a and c are not support vertices, vertices b and d are support vertices and thus of Type (S_3). The positive neighbor h of b (resp. e of d) is in C', by assumption. We consider several cases corresponding to whether b and d have their three neighbors in C' or not. If b and d both have their three neighbors in C', then let C'' be the graph $C' \setminus \{u, x, c, a\}$. If b has its three neighbors in C' but not d, then let C'' be the graph $C' \setminus \{u, x, c, a, d\}$. If d has its three neighbors in C' but not b, then let C'' be the graph $C' \setminus \{u, x, c, a, b\}$. If none of b and d has its three neighbors in C', then let C'' be the graph $C' \setminus \{u, x, c, a, b, d\}$. In the four cases we have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. The graph C'' is not necessarily connected but it is composed of one or two connected subgraphs of C whose all pendant vertices are positive, all support vertices are adjacent to their positive neighbor in C'' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality (on each component of C''), we have $p(C'') \geq \left\lceil \frac{n(C'')}{2} \right\rceil$, and so $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$.

Now we can assume that C' contains no pendant vertex. Suppose that C' is a single vertex v. Then v is not support as all support vertices have their positive neighbor in

C' and v is not negative of Type (N_3) as negative vertices of Type (N_3) have their two neighbors in C'. So v is not negative and $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil = 0$. Now we can assume that C' is not a single vertex. The graph C' is a cactus, not a single vertex, contains no pendant vertex, so it contains a cycle C'', of size ≥ 3, such that $C''' = C' \setminus C''$ is connected (note that we may have $C' = C''$ and C''' empty). Cycle C'' is a cycle of C so it has an odd number of support vertices by Lemma 4.40. Let S be the set of support vertices of C'', with $s = |S|$. By Lemma 4.36, cycle C'' contains a subpath $s_1v_1s_2v_2s_3$ where s_1, s_2, s_3 are support vertices of Type (S_3) and v_1, v_2 are vertices of degree 2. By assumption, the positive vertex z that is adjacent to s_2 is in C'. It is not in C'' as there is no chord in C''. So the only vertex of C'' that has some neighbors in $C' \setminus C''$ is s_2. So all the positive vertices that are adjacent to $S \setminus \{s_2\}$ are vertices of C' and thus of C''. A positive vertex of C'' has at most two support neighbors in C'' so $p(C'') \geq \lceil \frac{s-1}{2} \rceil$. A support vertex of Type (S_1) or (S_2) is a negative vertex of Type (N_1) or (N_2). A negative vertex of Type (N_3) of C'' is of degree 2 and so has its two neighbors on C'' and this two neighbors are support vertices of Type (S_3). So the number of negative vertices of C'' is at most the number of support vertices of C'' and strictly less if C'' contains a vertex of Type (N_3). Vertex v_1 is of Type (N_3), so $s > n(C'')$ and so $p(C'') \geq \lceil \frac{s-1}{2} \rceil \geq \left\lceil \frac{n(C'')}{2} \right\rceil$. The graph C''' is a connected subgraph of C whose all pendant vertices are positive, all support vertices are adjacent to their positive neighbor in C''' and all its negative vertices of Type (N_3) are adjacent to their two neighbors in C'. So by minimality we have $p(C''') \geq \left\lceil \frac{n(C''')}{2} \right\rceil$. So finally, $p(C') = p(C'') + p(C''') \geq \left\lceil \frac{n(C''')}{2} \right\rceil + \left\lceil \frac{n(C'')}{2} \right\rceil \geq \left\lceil \frac{n(C'')+n(C''')}{2} \right\rceil = \left\lceil \frac{n(C')}{2} \right\rceil$.

□

Let C' be the graph obtained from C by removing all pendant vertices that are not positive vertices. We claim that C' is a connected subgraph of C, whose pendant vertices are positive vertices, whose support vertices have their positive neighbor in C', whose negative vertices of Type (N_3) are adjacent to their two neighbors in C' and such that $n(C') = n(C)$. As C is connected and only pendant vertices have been removed from C, the graph C' is also connected. All support and negative vertices are of degree 2 or 3 and have all their incident edges in $H(G)$ and thus in C, so there is no pendant vertex of C that is a support or a negative vertex. So no support or negative vertex has been removed from C and $n(C') = n(C)$. A pendant vertex of C that has been removed is not positive, not support, not negative but incident to a support, so it is necessarily a degree 2 vertex a incident to a support vertex x of Type (S_3) (with notations of Figure 4.12). When a is removed from C, this does not create any new pendant vertex as x has degree 2 after the removal. All pendant vertices that are not positive are removed from C, no new pendant vertices are created, thus in C all pendant vertices are positive. No positive vertex has been removed and each support vertex is adjacent to its positive neighbor in $H(G)$, so support vertices of C' are adjacent to their positive neighbor in C'. No support vertex has been removed and each negative vertex of Type (N_3) is adjacent to its support neighbors of Type (S_3) in $H(G)$, so negative vertices of Type (N_3) of C' are adjacent to their two neighbors in C'. By Claim 13 applied to C', we have $p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil$. So $p(C) = p(C') \geq \left\lceil \frac{n(C')}{2} \right\rceil =$

$\left\lceil \frac{n(C)}{2} \right\rceil$ and we are done. $\square$

We now use the discharging rules to prove the following:

Lemma 4.42. $\mathrm{mad}(G) \geq 3$.

Proof. We attribute to each vertex a weight equal to its degree, and apply discharging rules R_1, R_2, R_3, R_4 and R_g. The common pot is empty at the beginning and, by Lemma 4.41, it has non-negative value after applying R_g. We show that all the vertices have a weight of at least 3 at the end.

Let u be a vertex of G. By Lemma 4.34, graph G does not contain Configurations (C_1) to (C_{11}). According to Configuration (C_1), we have $d(u) \geq 2$. We now consider different cases corresponding to the value of $d(u)$.

1. $d(u) = 2$.

   So u has an initial weight of 2 and gives nothing. We show that it receives at least 1, so it has a final weight of at least 3.

   a) *Assume u is adjacent to a vertex u_2 of degree 2.*
   Then u is a negative vertex of Type (N_1) and receives $\frac{1}{5}$ from the common pot by R_g. According to Configuration (C_2), vertex u is adjacent to a vertex v with $d(v) = k$. Since $k \geq 17$, according to R_4, vertex v gives $\frac{4}{5}$ to u.

   b) *Assume both neighbors v_1 and v_2 of u are of degree at least 3.*
   Vertex u is not a negative vertex of Type (N_1) since it has no neighbor of degree 2.

      i. *u has two weak neighbors*
      Then u is a negative vertex of Type (N_3). It receives $\frac{1}{5}$ from the common pot by R_g and $\frac{2}{5}$ from each of its two neigbors by $R_{1.1}$.

      ii. *u has one weak neighbor w and one non-weak neighbor v*

         A. $3 \leq d(v) \leq 7$
         Vertex u receives $\frac{3}{5}$ from v by $R_{1.2}$ and $\frac{2}{5}$ from w by $R_{1.1}$.

         B. $8 \leq d(v) \leq 14$
         Vertex u receives $\frac{5}{8}$ from v by R_3 and $\frac{3}{8}$ from w by $R_{1.4}$.

         C. $15 \leq d(v)$
         Vertex w is weak and v has degree at least 15, so one can check that u is not negative of Type (N_1) or (N_3). According to Configuration (C_3), it is not negative of Type (N_2). So u is not negative and it receives $\frac{1}{5}$ from w by $R_{1.5}$ and $\frac{4}{5}$ from v by R_4.

      iii. *u has two non-weak neighbors v, v'*

         A. $3 \leq d(v) \leq 7$ *and* $3 \leq d(v') \leq 7$
         Vertex u receives $\frac{1}{2}$ from each neighbor by $R_{1.3}$.

         B. $3 \leq d(v) \leq 7$ *and* $8 \leq d(v') \leq 14$
         Vertex u receives $\frac{5}{8}$ from v' by R_3 and $\frac{3}{8}$ from v by $R_{1.4}$.

C. $3 \le d(v) \le 7$ *and* $15 \le d(v')$
If u is negative, it receives $\frac{1}{5}$ from the common pot by R_g. If u is non-negative, it receives $\frac{1}{5}$ from v by $R_{1.5}$. In both cases, it receives $\frac{4}{5}$ from v' by R_4.

D. $8 \le d(v)$ *and* $8 \le d(v')$
Vertex u receives at least $\frac{5}{8}$ from each neighbor by R_3 or R_4 .

2. $d(u) = 3$.
So u has an initial weight of 3. We show that it has a final weight of at least 3.

a) *Assume u has three neighbors y_1, y_2 and y_3 of degree 2.*
Let z_i, $1 \le i \le 3$, be the neighbors of y_i distinct from u. According to Configuration (C_3), $d(z_1) = d(z_2) = d(z_3) = k$. So y_1, y_2 and y_3 are negative vertices of Type (N_2). So no rule applies to u.

b) *Assume u has exactly two neighbors y_1 and y_2 of degree 2.*
Let z_i, $1 \le i \le 2$, be the neighbors of y_i distinct from u. Let x be the third neighbor of u, $d(x) \ge 3$. According to Configuration (C_3), we are in one of the two following cases:

i. $d(x) \ge k - 2$.
Vertex x gives $\frac{4}{5}$ to u by R_4 and u gives nothing to x.

A. *Assume vertex u is weak.*
Since u is weak, $d(y_i) \le 14$, so vertex u gives at most $\frac{2}{5}$ to each of y_1, y_2 by $R_{1.1}$ or $R_{1.4}$.

B. *Assume vertex u is not weak.*
Then, w.l.o.g., $d(z_1) \ge 15$. So vertex u gives at most $\frac{1}{5}$ to y_1 by $R_{1.5}$. Vertex u gives at most $\frac{3}{5}$ to y_2 by $R_{1.2}$, $R_{1.3}$, $R_{1.4}$ or $R_{1.5}$.

ii. $d(z_1) = d(z_2) = k$.

A. $d(x) \le 7$.
According to Configuration (C_4), vertex u gives nothing to x by R_2. Vertices y_1 and y_2 are negative (of Type (N_2)) and u gives nothing to y_1, y_2.

B. $d(x) \ge 8$.
Vertex u gives $\frac{1}{5}$ to y_1 and y_2 by $R_{1.5}$. Vertex x gives at least $\frac{5}{8}$ to u by R_3 or R_4.

c) *Assume u has exactly one neighbor y of degree 2*
Let z be the neighbor of y distinct from u. Let w and x be the other neighbors of u, where $d(w) \ge d(x) \ge 3$. We consider three cases according to the value of $d(w)$.

i. $15 \le d(w)$.
Then, vertex u gives at most $\frac{3}{5}$ to y by $R_{1.i}$, $1 \le i \le 5$. Vertex u gives at most $\frac{1}{10}$ to x by R_2. Vertex w gives $\frac{4}{5}$ to u by R_4.

ii. $8 \le d(w) \le 14$.
According to Configuration (C_4), vertex u gives nothing to x by R_2. Vertex u gives at most $\frac{3}{5}$ to y by $R_{1.i}$, $1 \le i \le 5$. Vertex w gives $\frac{5}{8}$ to u by R_3.

iii. $d(w) \leq 7$.
According to Configuration (C_4), vertex u gives nothing to x and w by R_2. According to Configuration (C_3), we have $d(z) = k$. Vertex u gives $\frac{1}{5}$ to y by $R_{1.5}$. Both w and x give $\frac{1}{10}$ to u by R_2.

d) *Assume all the neighbors of u have degree at least 3 and at most 7.*
According to Configuration (C_4), vertex u gives nothing to its neighbors by R_2.

e) *Assume u has no neighbor of degree 2 and at least a neighbor v of degree at least 8.*
Vertex v gives at least $\frac{5}{8}$ to u by R_3 or R_4. Vertex u gives at most $\frac{1}{10}$ to each of its other neighbors by R_2.

3. $d(u) = 4$.
So u has an initial weight of 4. We show that it has a final weight of at least 3.

a) *Assume u has at least three neighbors y_1, y_2 and y_3 of degree 2*
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq d(z_2) \geq d(z_3)$. Let x be the neighbor of u distinct from y_1, y_2 and y_3. We consider three cases depending on $d(z_2)$ and $d(z_3)$.

i. $d(z_2) \leq 14$.
According to Configuration (C_7), we have $d(x) \geq k-2$. Vertex u gives at most $3 \times \frac{3}{5}$ by $R_{1.i}$, $1 \leq i \leq 5$. Vertex x gives $\frac{4}{5}$ to u by R_4.

ii. $d(z_2) \geq 15$ and $d(z_3) \leq 14$.
According to Configuration (C_6), we have $d(x) \geq 8$. Vertex u gives at most $\frac{1}{5}$ to each of y_1, y_2 by $R_{1.5}$. Vertex u gives at most $\frac{3}{5}$ to y_3 by $R_{1.i}$.

iii. $d(z_3) \geq 15$.
Vertex u gives at most $\frac{1}{5}$ to each of its neighbors by $R_{1.5}$.

b) *Assume u has exactly two neighbors y_1 and y_2 of degree 2*
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq d(z_2)$. Let w and x the neighbors of u distinct from y_1, y_2. We assume that $d(w) \geq d(x) \geq 3$. We consider two cases depending on $d(z_1)$.

i. $d(z_1) \leq 14$.
According to Configuration (C_7), we have $d(w) \geq 9$. Vertex u gives at most $\frac{3}{5}$ to each of y_1, y_2 by $R_{1.i}$, and at most $\frac{1}{10}$ to x by R_2. Vertex x gives at least $\frac{5}{8}$ to u by R_3 or R_4.

ii. $d(z_1) \geq 15$.
Vertex u gives at most $\frac{1}{5}$ to y_1 by $R_{1.6}$, at most $\frac{3}{5}$ to y_2 by $R_{1.i}$, and at most $\frac{1}{10}$ to each of w, x by R_2.

c) *Assume u has at most one neighbor of degree 2.*
Vertex u gives at most $3 \times \frac{1}{10}$ by R_2, and at most $\frac{3}{5}$ by $R_{1.i}$.

4. $d(u) = 5$.
So u has an initial weight of 5. We show that it has a final weight of at least 3.

a) *Assume u has at least four neighbors y_1, y_2, y_3 and y_4 of degree 2*
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq d(z_2) \geq$

$d(z_3) \geq d(z_4)$. Let x be the neighbor of u distinct from the y_i's. We consider two cases depending on $d(z_4)$.

i. $d(z_4) \leq 7$.
According to Configuration (C_8), we have $d(x) \geq 8$. Vertex u gives at most $\frac{3}{5}$ to each of y_i by $R_{1.i}$. Vertex x gives at least $\frac{5}{8}$ to u by R_3 or R_4.

ii. $d(z_4) \geq 8$.
Vertex u gives at most $5 \times \frac{3}{8}$ on total its neighbors y_i's and x by $R_{1.4}$ or $R_{1.5}$.

b) *Assume u has at most three neighbors of degree* 2.
Vertex u gives at most $3 \times \frac{3}{5}$ by $R_{1.i}$, and at most $2 \times \frac{1}{10}$ by R_2.

5. $d(u) = 6$.
So u has an initial weight of 6. We show that it has a final weight of at least 3.

a) *Assume u has at least five neighbors $y_1, \ldots, y_5$, of degree* 2
Let z_i be the neighbors of y_i distinct from u. We assume that $d(z_1) \geq \cdots \geq d(z_5)$. Let x be the neighbors of u distinct from y_i's. According to Configuration (C_9), we are in one of the following two cases.

i. $d(z_5) \geq 8$.
Vertex u gives at most $6 \times \frac{3}{8}$ on total to its neighbors by $R_{1.4}$ or $R_{1.5}$.

ii. $d(x) \geq 8$.
Vertex u gives at most $5 \times \frac{3}{5}$ on total to the y_i's.

b) *Assume u has at most four neighbors of degree* 2.
Vertex u gives at most $4 \times \frac{3}{5}$ by $R_{1.i}$, and at most $2 \times \frac{1}{10}$ by R_2.

6. $d(u) = 7$.
So u has an initial weight of 7. We show that it has a final weight of at least 3.

a) *Assume u has at least six neighbors of degree 2 adjacent to vertices of degree at most* 3.
According to Configuration (C_{10}), vertex u has a neighbor v of degree at least 8. Vertex u gives at most $6 \times \frac{3}{5}$ by $R_{1.i}$.

b) *Assume u has at most five neighbors of degree 2 adjacent to vertices of degree at most* 3.
Vertex u gives at most $5 \times \frac{3}{5}$ by $R_{1.i}$, and at most $2 \times \frac{1}{2}$.

7. $8 \leq d(u) \leq 14$.
Then Rule R_3 applies to every neighbor of u, and $d(u) - (d(u) \times \frac{5}{8}) \geq 3$.

8. $15 \leq d(u) < k$.
Then Rule R_4 applies to every neighbor of u, and $d(u) - (d(u) \times \frac{4}{5}) \geq 3$.

9. $d(u) = k$.
Then Rule R_4 applies to every neighbor of u and R_g applies to u. We have $k \geq 17$ so $k - (k \times \frac{4}{5} + \frac{2}{5}) \geq 3$.

Consequently, after application of the discharging rules, every vertex v of G has a weight of at least 3, meaning that $\sum_{v \in G} d(v) \geq \sum_{v \in G} 3 = 3|V|$. Therefore, $\mathrm{mad}(G) \geq 3$. $\square$

Finally, k is a constant integer greater than 17 and G is a minimal graph such that $\Delta(G) \leq k$ and G admits no 2-distance $(k+2)$-list-coloring. By Lemma 4.42, we have $\mathrm{mad}(G) \geq 3$. So Theorem 4.19 is true.

Chapter 5

Conclusion

In this thesis, we were interested in coloring results through discharging procedures.

We mainly worked on two problems: list edge coloring planar graphs, and list square coloring sparse graphs. In the first problem, we proved most notably that planar graphs with maximum degree 8 are list edge 9-choosable, thus solving a 1990 question by Borodin. In the second problem, we most notably determined the exact threshold on the maximum average degree of a graph that allows for its square to be minimally choosable if the maximum degree is large enough, thus generalizing previous results on planar graphs. Along the way, we increased ever so slightly the fauna around discharging methods, and now hope that they will grow and breed, thus becoming an ever more powerful species.

Many open questions were mentioned or raised throughout this work. We recall some of our favorites here.

Conjecture 5.1. *Every planar graph G with $\Delta(G) \leq 11$ is edge 11-choosable.*

Conjecture 5.2. *For every $k \geq 2$, the k^{th} power of every graph G is $(D_{k,\Delta(G)}+1-k)$-colorable except for a finite set of graphs.*

Question 5.3. *Is it true that for every $C \in \mathbb{N}^*$, and $\epsilon > 0$, there exists some d such that every graph G with $\mathrm{mad}(G) < \frac{4C+2}{C+1} - \epsilon$ and $\Delta(G) \geq d$ is square $(\Delta(G)+C)$-choosable?*

We only mentioned above the tractable questions, i.e. those that seem within reach. When removing that constraint, the conjecture which is arguably the most beautiful within the scope of this thesis is the List Coloring Conjecture.

Conjecture 5.4 (List Coloring Conjecture)**.** *Every graph G satisfies $\chi'(G) = \chi'_\ell(G)$.*

In this thesis, we sometimes sought extremal results, i.e. general results that do not make any attempt at optimality, but merely try to probe for possible thresholds. Here, given some coloring rules (square coloring, list edge coloring, AVD coloring...), an extremal result can be one of the kind "for which values of m can you prove that the class of graphs with maximum average degree less than m, and large enough maximum degree, is colorable with few colors?". In some situations, like for square coloring a graph G with $\Delta(G)+1$ colors, a small threshold appears, and the goal is then to determine its exact value. In other situations (list edge coloring, AVD coloring), it appears that there is no threshold. Even though the corresponding lower bound on the maximum degree may be quite provokingly large, its very existence already adds some perspective for future research directions.

In the realm of discharging, there are many developments to look for. We hope for more global discharging arguments backboned with an exotic sub-structure. We hope for an efficient automatization of the discharging process, which would alleviate the pains of both the authors and the readers in some technical analyses - as well as immediately open a whole new world of possibilies. It might also be of interest to look into the potential technique (a new discharging method recently introduced by Kostochka and Yancey), or into future or existing applications of the discharging method outside the coloring field.

Publications

This thesis was backboned with 7 publications [Bon13, BB14a, BLP14a, BLP14b, BLP14c, BLP14d, BBH13], four of them published in international journals ([BB14a, BLP14a, BLP14b, BLP14c]), one in minor revision for SIAM Journal of Discrete Mathematics ([Bon13]), one submitted ([BLP14d]), and one whose long version is in preparation but whose short version already appears in conference proceedings ([BBH13]).

However, during the preparation of this thesis, we worked on various other projects, which we chose not to include in this manuscript. We shortly present below the 7 resulting papers that are available online at this moment.

- About graph recoloring (given two k-colorings of a graph, can we slowly transform one into the other by recoloring one vertex at a time and maintaining a proper k-coloring?), we first proved with Johnson, Lignos, Patel and Paulusma [BJL$^+$14] that chordal graphs and similar classes could be recolored in quadratic time as soon as the number of colors outnumbered the chromatic number. With Bousquet [BB14b], we generalized this result in particular to bounded treewidth graphs and distance-hereditary graphs. We also considered together another reconfiguration problem, namely independent set reconfiguration [BB14c], on the class of cographs. The corresponding proofs rely on good decompositions of the considered graph classes.

- About a variant on AVD coloring where we consider sums instead of sets, we proved with Przybyło [BP14] that planar graphs of sufficiently large maximum degree were colorable with the minimum number of colors. The proof consists in a discharging argument.

- About the Erdős-Hajnal Conjecture, we proved with Bousquet and Thomassé [BBT14] that the class of graphs with no long induced cycle nor long induced complement of a cycle is such that every graph inside contains a large clique or a large stable set. The proof consists in a short argument using external extraction theorems on the graph or its complement to obtain a large stable set.

- About the structure of graphs with large chromatic number, we proved with Charbit and Thomassé [BCT14] that graphs with no induced cycle of length $3k$

for any k have bounded chromatic number. The proof consists in a gradual destruction of a hypothetical graph with large chromatic number and no induced cycle of length $0 \bmod 3$.

- About FPT kernels, we proved with Kowalik [BK14] that planar Feedback Vertex Set (how many vertices do we need to remove, for the graph to become a forest?), when parameterized with the size of the solution (is removing k vertices enough?), admits a $13k$-kernel. A short version of this paper was accepted to IPEC'2014. The proof consists in explicit reduction rules, whose efficiency is proved through a region decomposition technique.

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Abstract

This thesis falls within graph theory, and deals more precisely with graph coloring problems. In this thesis, we use and develop the discharging method, a counting argument that makes strong advantage of the graph structure. This method is decisive in the proof of the Four Color Theorem. We first give an illustrated overview of the discharging tools that are used for this work: nice methods that we apply, and handy tricks that we develop. In particular, we present the main ideas in a global discharging argument. In the realm of list edge coloring, we most notably prove that the weak List Coloring Conjecture is true for planar graphs of maximum degree 8 (i.e. that they are edge 9-choosable), thus improving over a result of Borodin from 1990. We finally present our results about square coloring, where the goal is to color the vertices in such a way that two vertices that are adjacent or have a common neighbor receive different colors. We look in particular into sufficient conditions on the density of a graph (i.e. the maximum average degree of a subgraph) for its square to be colorable with few colors.

Keywords *Graphs, coloring, discharging method, planar, maximum average degree, list edge coloring, square coloring*

Résumé

Cette thèse s'inscrit dans le cadre de la théorie des graphes, et porte plus particulièrement sur des problèmes de coloration de graphes. Dans cette thèse, nous nous intéressons à l'utilisation et au développement de la méthode de déchargement, un argument de comptage qui exploite fortement la structure du graphe. Cette méthode est décisive dans la preuve du Théorème des Quatre Couleurs. Nous donnons d'abord une vue d'ensemble des outils de déchargement que nous utilisons dans ce travail, entre les méthodes élégantes mises en application, et les astuces développées. Nous présentons en particulier l'idée essentielle d'une preuve par déchargement global. Dans le cadre de la coloration d'arêtes par liste, nous résolvons la Conjecture de Coloration par Liste faible dans le cas des graphes planaires de degré maximum 8, en prouvant qu'on peut colorier par liste les arêtes de ces derniers avec 9 couleurs seulement. Ceci améliore un résultat de Borodin de 1990. Enfin, nous présentons nos résultats dans le cadre de la coloration de carrés, où il s'agit de colorier les sommets sans qu'il y ait deux sommets adjacents ou avec un voisin commun qui soient de la même couleur. On s'intéresse en particulier à des conditions suffisantes sur la densité du graphe (c-à-d le degré moyen maximum d'un sous-graphe) pour qu'on puisse colorier son carré avec peu de couleurs.

Mots-clefs *Graphes, coloration, méthode de déchargement, planaire, degré moyen maximum, coloration d'arêtes par liste, coloration du carré*

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