Quantitative recurrence properties in infinite measure

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Contents

Resumé

Dans cette thèse, nous étudions les propriétés quantitative de récurrence de certains systèmes dynamique préservant une mesure infinie. Nous nous intéressons au premier temps de retour des orbites d'un système dynamique dans un petit voisinage de leur points de départ. Tout d'abord, nous commençons par considérer un modèle jouet probabilistique pour éclairer la stratégie de nos preuves. On s'intéresse particulièrement au cas où la mesure est infinie, plus précisément, nous considérons les $\mathbb{Z}$-extensions des sous-shift de type fini. Nous étudions le comportement asymptotique du premier temps de retour au voisinage de l'origine, et nous établissons des résultats de type de convergence presque partout, et aussi de convergence en loi par rapport à toute mesure de probabilité absolument continue par rapport à la mesure infinie. Dans ce travail, nous nous également intéressons à d'autres système dynamiques. Nous considérons un flot Axiome A $\{g_t\}_{t\in\mathbb{R}}$ sur une variété riemannienne M munie d'une mesure $\sigma-$finie μ. Nous supposerons que la mesure μ est une mesure d'équilibre pour $\{g_t\}_{t\in\mathbb{R}}$. Afin d'établir nos résultats, nous introduisons des notions de dynamique hyperbolique. En particulier, nous considérons la section de Markov qui a été introduite par Bowen et Ratner.

***Mots-clés*:** Temps de retour, Récurrence quantitatives, Théorème de Limite Locale, Flot Axiome A, Systm̀es dynamiques, Sous-shift de type fini.

Abstract

In this thesis, we study the quantitative recurrence properties of some dynamical systems preserving an infinite measure. We are interested in the first return time of the orbits of a dynamical system into a small neighborhood of their starting points. First, we start by considering a toy probabilistic model to clarify the strategy of our proofs. Our interest is when the measure is indeed infinite, more precisely we consider the $\mathbb{Z}-$extensions of subshifts of finite type. We study the asymptotic behavior of the first return time near the origin, and we establish results of an almost everywhere convergence kind, and a convergence in distribution with respect to any probability measure absolutely continuous with respect to the infinite measure. In this work, we are also interested in another dynamical systems. We consider an Axiom A flow $\{g_t\}_t$ on a Riemannian manifold M endowed with a $\sigma-$

finite measure μ. We will assume that the measure μ is an equilibrium measure for $\{g_t\}_{t\in\mathbb{R}}$. In order to establish our results, we introduce notions from hyperbolic dynamics. In particular, we consider the Markov section which was constructed by Bowen and Ratner.

***Key words*:** Return time, Quantitative recurrence, Local Limit Theorem, Axiom A flow, Dynamical systems, Subshift of finite type.

Introduction

Les propriétés de récurrence quantitative des systèmes dynamiques préservant une mesure de probabilité ont été étudiées par de nombreux auteurs depuis les travaux de Hirata [28], Boshernitzan [14]. Nous mentionnons le travail de Callot et Glavez, qui sont à l'origine de ces idées de récurrence dans [20]. Certaines propriétés sont définies en estimant le premier temps de retour d'un système dynamique dans un petit voisinage de son point de départ. Des résultats dans ce contexte ont été décrits dans [46], citons les travaux dans cette situation [2, 48]. Et des transformations préservant de la mesure similaires aux décalages de Markov dans [50]. Cette question a été moins étudiée dans le contexte des systémes dynamiques préservant une mesure infinie. Dans [15], Bressaud et Zweimüller ont établi les premiers résultats de la récurrence quantitative pour les applications affines par morceaux de l'intervalle avec une mesure infinie. Le cas de $\mathbb{Z}^2$-extension de sous-shift mélangeant de type fini ont été étudiés dans [38]. Des résultats ont également été établis pour des marches aléatoires sur la ligne [39], pour des billards dans le plan [37] et pour des applications de Markov récurrentes nulles dans [40].

Nous considérons des systèmes dynamiques conservatifs, c'est-à-dire les systèmes dynamiques pour lesquels la conclusion du théorème de Poincaré est satisfaite. Nous savons donc que le système va revenir proche de sa position initiale. Il est naturel d'étudier les instants de visite successives à un ensemble fixe, en particulier pour un point fixe x et $\epsilon > 0$, nous nous intéresserons à la première fois que l'orbite reviendra dans le ϵ-voisinage de x. C'est ce qu'on appelle le premier temps de retour, et le sujet principal de cette thèse est d'étudier les comportement de ces temps de retour quand $\epsilon \to 0$. De nombreux travaux ont étudié le comportement des temps de retour, nous mentionnons par exemple [3], [8], [16], [18], [19], [18], [24],[29], [31], [35], [45], [47], [30]. Les résultats souhaités sont de nature stochastique (convergence presque partout, convergence en loi par rapport à toute mesure de probabilité absolument continue par rapport à la mesure invariante infinie, etc.). Des résultats de ce type ont été obtenus dans quelques cas par X. Bressaud, S. Galatolo, D.-H. Kim, K. Park, F. Pène, B. Saussol and R. Zweimüller.

Un système dynamique préservant la mesure est donné par $(X, \mathcal{B}, \mu, T)$, où:

- $(X, \mathcal{B})$ est un ensemble mesurable,
- μ est une mesure positive finie où σ-finie ,
- $T : X \to X$ est une transformation mesurable préservant la mesure μ (i.e. $\mu(T^{-1}A) = \mu(A)$, pour tout $A \in \mathcal{B}$).

Nous nous intéressons au cas où μ est σ-finie. Nous supposons que X est muni de certaine métrique d_X et que $\mathcal{B}$ contient les boules ouvertes $B(x, r)$ de X. Nous nous intéressons à la première fois que l'orbite revient proche de sa position initiale. Pour tout $y \in X$, nous définissons le premier temps de retour τ_ϵ de l'orbite de y dans la boule $B(y, \epsilon)$,

$$\tau_\epsilon(y) := \inf\{n \geq 1 : T^n(y) \in B(y, \epsilon)\} \in \mathbb{N} \cup \{+\infty\}.$$

L'objectif de cette thèse est d'étudier la récurrence quantitative des systèmes dynamiques preservant une mesure infinie. Les résultats principaux de cette thèse ont été établis en considérant d'abord un modèle jouet probabiliste. Nous avons également prouvé ces résultats dans le cas de $\mathbb{Z}-$extension du sous-shift de type fini. De plus, nous établissons des résultats dans le cas de $\mathbb{Z}-$ extension d'un flot Axiom A.

Quelques résultats antérieurs sur la récurrence quantitative

Comme nous l'avons vu, notre intérêt est d'étudier les propriétés quantitatives de récurrence des systèmes dynamiques. Selon le Théorème de Recurrence de Poincaré, presque chaque orbite d'un système dynamique revient près de son point de départ. Une question naturelle est la suivante: Quel est le temps nécessaire pour qu'une orbite d'un système dynamique revient au voisinage de son point de départ? De nombreux résultats ont été établis concernant le cas où la mesure est finie.

Des résultats existants pour les propriétés de récurrence quantitative en mesure infinie sont peu nombreux et récents. Ils sont basés sur des arguments spécifiques qui nécessitent beaucoup de choses à faire. Péne et Saussol [38] ont abordé cette question dans un processus étendu deux-dimensionnel, où ils ont étudié le comportement quantitatif des temps de retour et le lient à la dimension du processus. Ils sont établi leur résultat principal dans le cas de $\mathbb{Z}^2$-extension du sous-shift de type fini. Si τ_ϵ définit ce premier temps de retour, d étant la dimension de Hausdorff de la mesure ν, ils ont prouvé que

$$\lim_{\epsilon\to 0} \frac{\log\log \tau_\epsilon}{-\log \epsilon} = d \quad a.s.$$

De plus, ils ont prouvé que la suit de variables aléatoires $\nu(B_\epsilon(.))\log\tau_\epsilon(.)$ converge en loi, quand $\epsilon \to 0$ à une variable aléatoire avec fonction de distribution de la densité $t \mapsto \frac{\beta t}{1+\beta t} 1_{(0,+\infty)}(t)$.

Il y a un travail récent pour Rechberger et Zweimüller [43] où ils ont étudié la convergence des distributions de retour et de temps de frappe des petits ensembles lorsque la mesure de ces ensembles tends vers à 0 dans des systèmes dynamiques ergodiques récurrents préservant une mesure infinie.

En ce qui concerne le cas en temps continu, des études ont été établies pour la récurrence des flots hyperboliques, nous mentionnons certains des travaux effectués: [10], [5], [6], [7] Pesin et Sadovskaya [41] ont développé l'analyse multifractale des flots Axiom A conformes. L'outil principal est le formalisme thermodynamique des flots hyperboliques par Bowen et Ruelle [13]. Des exemples dans ce contexte comprennent flots d'Anosov et en particulier les flots géodésiques sur des surfaces lisses compactes de courbure négative. Barreira et Saussol [9] ont établi l'analyse multifractale des flots hyperboliques et des flots suspendus sur des sous-shift de type fini. Ils ont prouvé que pour une mesure ν dans l'état d'équilibre d'un potentiel Hölder, le temps de retour pour ν-presque tout point y dans $B(y, r)$ se comporte comme r^{-d+1}, où d est la dimension de Hausdorff de la mesure ν.
Rousseau a étudié la récurrence de Poincaré pour les flots et l'observation des flots [44], où le résultat établi a été appliqué à le flot géodésique sur une variété lisse de courbure strictement négative. En définissant le temps de retour du flot

$$\bar{\tau}_\epsilon = \inf\{t > 1 : g_t(y) \in B(y, \epsilon)\},$$

Soit g_t un flot d'Anosov. Si ν est un état d'équilibre d'un potentiel H ölder, alors pour $\nu-$presque tous les points $y \in M$,

$$\frac{\log \bar{\tau}_\epsilon(y)}{-\log \epsilon} = d - 1.$$

Présentation du travail effectué

Dans ce travail, nous avons utilisé les propriétés de la théorie ergodique, de la probabilité et de la géométrie. Les modèles pris en compte seront récurrents ergodiques, ce qui garantit que les trajectoires visitent infiniment souvent n'importe quel voisinage de n'importe quel point. Dans les trois premiers chapitres, nous parlons du premier résultat obtenu, à savoir un article publié sous le titre ' textit Quantitative recurrence of some dynamical systems preserving an infinite measure in dimension one'. Ensuite, dans les chapitres 5 et 6, nous parlons des résultats de $\mathbb{Z}-$extensions de flot Axiome A, qui sont les premiers résultats de ce type, où nous rencontrons un nouveau niveau de difficultés lorsqu'il s'agit des flots. Le théorème limite local avec la vitesse prouvée au chapitre 2 est suffisant pour traiter les $\mathbb{Z}$-extensions des flots Axiome A, du fait qu'elles sont visualisées comme des $\mathbb{Z}-$extensions des flots suspendus, mais la difficulté principale est de voir comment réduire l'analyse du flot vers l'application de Poincaré.

Dans chapitre un, nous commençons par considérer le modèle probabiliste jouet conçu pour donner une idée des résultats dans le cas général et pour mettre en évidence la stratégie de nos preuves. Ce modèle est donné par (Y_n, S_n), où S_n est la marche aléatoire simple symétrique sur $\mathbb{Z}$ et Y_n est une suite de variables aléatoires indépendantes, avec une distribution uniforme sur $(0, 1)^d$ et où Y_n et S_n sont indépendants. Le processus concerné dans ce modèle traite du fait qu'il existe une grande échelle où nous définissons un temps de retour R_n et une petite échelle, où nous définissons un temps de retour T_ϵ. Nous étudions le comportement asymptotique de R_n et T_ϵ, à partir duquel nous prouvons la convergence ponctuelle du taux de récurrence et le relions à la dimension du processus. Et après nous montrons la convergence en loi du temps de retour redimensionné.

Dans le deuxième chapitre, nous avons montré un résultat concernant le théorème limite local avec la vitesse pour le sous-shift de type fini, où nous donnons un terme d'erreur plus précis pour tenir compte du cas unidimensionnel ($\mathbb{Z}$-extension du sous-shift de type fini). Tout d'abord, c'est local dans le sens où nous examinons la probabilité que $S_n\varphi = 0$. Nous avions besoin de prouver un tel LTT, où nous examinions la probabilité conditionnée par le fait que nous partions d'un ensemble A pour atterrir sur un ensemble B. Dans ce chapitre, nous avons commencé par donner un bref aperçu des outils théoriques spectraux dont nous aurons besoin. Plus précisément, nous donnons une analyse spectrale de l'opérateur de Perron-Frobenius, qui sert à pr'eciser la vitesse de convergence du th'eorème de la limite locale.

Dans le troisième chapitre, nous présentons un exemple classique de systèmes dynamiques préservant une mesure infinie qui est donnée par le $\mathbb{Z}-$extension d'un système dynamique préservant de probabilité. Étant donné un système dynamique préservant de probabilité $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ et une fonction mesurable $\varphi : \bar{X} \to \mathbb{Z}$, nous construisons the $\mathbb{Z}$-extension $(X, \mathcal{B}, \mu, T)$ de $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ en posant $X := \bar{X} \times \mathbb{Z}$, $\mathcal{B} := \bar{\mathcal{B}} \otimes \mathcal{P}(\mathbb{Z})$, $\mu := \nu \otimes \sum_{l\in\mathbb{Z}} \delta_l$ et $T(x, l) = (\bar{T}(x), l + \varphi(x))$. On muni X avec la métrique de produit donée par $d_X((x, l),(x^{'}, l^{'})) := \max\{d_{\bar{X}}(x, x^{'}), \mid l - l^{'} \mid\}$. Par conséquent $T^n(x, l) = (\bar{T}^n x, l + S_n\varphi(x))$, où $S_n\varphi$ est la somme ergodique $S_n\varphi := \sum_{k=0}^{n-1} \varphi \circ \bar{T}^k$. Donc, pour ϵ assez petit,

$$T^n(x, l) \in B((x, l), \epsilon) \Longleftrightarrow \bar{T}^n(x) \in B_{\bar{X}}(x, \epsilon) \text{ and } S_n\varphi(x) = 0.$$

Nos résultats principaux concernent le cas où $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ est un sous-shift mélangeant de type fini (voir section 3 pour une définition précise), qui sont des systèmes dynamiques classiques utilisés pour modéliser une large classe de systèmes dynamiques tels comme les flots géodésiques en courbure négative, etc.
Considérons $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ un sous-shift mélangeant de type fini et ν une mesure de Gibbs associée à un potentiel Hölder continu. En plus, nous avons une fonction hölderienne continue ν-centrée φ. Ensuite, nous obtenons le résultat suivant:

Theorem 3.1.1

$$\lim_{\epsilon\to 0} \frac{\log \tau_\epsilon}{\log \epsilon} = -2d, \tag{0.0.1}$$

μ-presque partout, où d est la dimension de Hausdorff de ν.

De plus, la convergence suivante tient en loi par rapport à toute mesure de probabilité absolument continue par rapport à μ:

Theorem 3.1.2
$$\mu(B(.,\epsilon))\sqrt{\tau_\epsilon(.)} \underset{\epsilon\to 0}{\longrightarrow} \frac{\mathcal{E}}{|\mathcal{N}|}, \tag{0.0.2}$$
où $\mathcal{E}$ and $\mathcal{N}$ sont deux variables aléatoires indépendantes avec une distribution exponentielle de la moyenne 1 et de la distribution normale standard respectivement.
En gros, la stratégie de notre preuve est qu'il y a une grande échelle (correspondant à $S_n\varphi(x)$) et une petite échelle (correspondant à $\bar{T}^n(x)$), qui se comportent indépendamment asymptotiquement. L'idée sera d'utiliser la méthode de perturbation d'opérateur pour établir un théorème de limite local précis afin d'établir les résultats mentionnés ci-dessus.

Dans le quatrième chapitre, nous introduisons toutes les notions nécessaires et les résultats de la dynamique hyperbolique. Nous considérons une variété Riemannienne M. Nous donnons la définition d'un flot Axiom A et d'un ensemble hyperbolique pour un flot. En particulier, nous considérons la section de Markov construite par Bowen [11] et Ratner [42] pour un ensemble hyperbolique localement maximal.
Nous définissons donc la section Poincaré pour le flot g_t noté par X, et la fonction de hauteur $R : X \to (0,\infty)$ telle que $R(x) = \min\{s > 0 : g_s x \in X\}$. On donne alors une notion de flot suspendu $\psi = \{\psi_t\}_{t\in\mathbb{R}}$ sur une transformation $T : X \to X$, avec la fonction de hauteur R. Nous considérons l'espace
$$X_R = \{(x,s) \in X \times \mathbb{R} : 0 \le s \le R(x)\},$$
où les points $(x,R(x))$ et $(T(x),0)$ sont identifiés pour chaque $x \in X$.
Un autre point est de décrire comment une section de Markov pour un ensemble hyperbolique donne lieu à une dynamique symbolique. Nous considérons un ensemble Σ_A avec l'application de décalage σ (voir au section 4.4). Et ensuite nous donnons une définition de la fonction de codage $\chi : \Sigma_A \to X$. Et ainsi de la même manière, nous définissons le flot suspendu symbolique $\mathcal{S} = \{\mathcal{S}_t\}_{t\in\mathbb{R}}$ over $\sigma_{|\Sigma_A}$. Nous introduisons une section définissant une mesure d'équilibre d'un flot. En particulier, nous expliquons l'existence d'une mesure d'équilibre unique pour une fonction hölderienne H pour le flot suspendu symbolique $\mathcal{S} = \{\mathcal{S}_t\}_{t\in\mathbb{R}}$.
Dans la dernière section de ce chapitre, nous travaillons sur l'établissement de certaines propriétés sur les boules et le codage. Dans certaines conditions prises sur les longueurs de courbes stables et instables, une boule donnée contient et est contenue dans un cylindre. Cela sert à étudier le comportement asymptotique de

notre temps de retour au cylindre et à le déduire ensuite du temps de retour à une boule. Et pourtant, prouver la convergence presque sûre au chapitre 5 (5.1.1) et la convergence de la distribution au chapitre 6 (6.0.5).

Dans le chapitre 5, nous montrons la convergence ponctuelle du taux de récurrence vers la dimension dans le cas d'un temps continu. Comme précédemment, nos travaux concernent le cas où la mesure est infinie. Plus précisément, nous considérons une variété Riemannienne $\tilde{M}$ de dimension 3, muni d'une mesure $\sigma-$finie $\tilde{\mu}$, et $(\tilde{g}_t)_{t\in\mathbb{R}}$ un flot sur $\tilde{M}$ préservant la mesure $\tilde{\mu}$. Nous définissons Γ comme un groupe infini d'isométries de $\tilde{M}$. Alors on suppose que $M = \tilde{M}/\Gamma$ est une variété compacte, et après on définit un flot $(g_t)_{t\in\mathbb{R}}$ sur M , (voir la section 5.1 pour une description plus précise de cette $\mathbb{Z}-$extension). Soit μ la mesure définie sur M à partir de la mesure $\tilde{\mu}$ en passant par le quotient. Nous supposons que μ est une mesure d'équilibre pour g_t et que $(M, (g_t)_t)$ est un flot Axiom A.
Nous nous intéressons au premier temps de retour du flot $\tilde{g}_t$ dans un $\epsilon-$voisinage de son point de départ. Ainsi, pour tout $x \in \tilde{M}$, nous définissons le temps de retour du flot $\tilde{g}_t$ par:

$$\tau_\epsilon(x) := \inf\ \{t > 1 : \tilde{g}_t(x) \in B(x, \epsilon)\},$$

où $B(x, \epsilon)$ est la boule de centre x et rayon ϵ. Comme dans les chapitres précédents, nous souhaitons étudier le comportement asymptotique de τ_ϵ quand $\epsilon \to 0$. Ainsi, nous prouvons le résultat suivant:
Theorem 5.1.1 *Soit $(\tilde{M}, \{\tilde{g}\}_t, \tilde{\mu})$ un flot satisfaisant à toutes les hypothèses ci-dessus, pour $\tilde{\mu}$- presque tous les points $x \in \tilde{M}$,*

$$\lim_{\epsilon\to 0} \frac{\log\sqrt{\tau_\epsilon}}{-\log\epsilon} = \frac{h}{L^u} + \frac{h}{L^s}.$$

Enfin, au chapitre 6, nous montrons la convergence en loi de $\mathbb{Z}$-extension du flot Axiom A. Soit $\tilde{y} \in \tilde{M}$. Nous décrivons l'existence d'une mesure $\tilde{\nu}_0$ sur un disque centré sur $\tilde{y}$, qui est transversal au flot et orthogonal à $\tilde{y}$. Nous l'appelons mesure transversale au flot à $\tilde{y}$. Nous prouvons le résultat suivant:

Theorem 6.0.5 *La suite des variables aléatoires $\tilde{\nu}_0(B(.,\epsilon))^2\tau_\epsilon(.)$ converge en loi, par rapport à toute mesure de probabilité absolument continue par rapport à $\tilde{\mu}$, quand $\epsilon \to 0$ à $\sigma^2_{flow}\frac{\mathcal{E}^2}{\mathcal{N}^2}$, où $\mathcal{E}$ et $\mathcal{N}$ sont des variables aléatoires indépendantes, $\mathcal{E}$ ayant une distribution exponentielle de moyenne 1 et $\mathcal{N}$ ayant une distribution Gaussienne standard.*

Introduction

The quantitative recurrence properties of dynamical systems preserving a probability measure have been studied by many authors since the work of Hirata [28], Boshernitzan [14]. We mention the work of Callot and Glavez who were from the originators of these recurrence ideas in [20]. Some properties are defined by estimating the first return time of a dynamical system into a small neighbourhood of its starting point. Results in this concern have been described in [46], let us mention works in this situation [2, 48]. This question has been less investigated in the context of dynamical systems preserving an infinite measure. In [15], Bressaud and Zweimüller have established first results of quantitative recurrence for piecewise affine maps of the interval with infinite measure. The case of $\mathbb{Z}^2$-extension of mixing subshifts of finite type has been investigated in [38]. Results have been also established for random walks on the line [39], for billiards in the plane [37] and for null-recurrent Markov maps in [40], in addition to measure preserving transformations similar to Markov shifts in [50].

We consider conservative dynamical systems, that is dynamical systems for which the conclusion of the Poincaré theorem is satisfied. Thus we know that the system will return back close to its initial position. It's natural to study the successive visit times to a fixed set, in particular for a fixed point x and $\epsilon > 0$, we will be interested in the first time the orbit comes back to the ϵ-neighbourhood of x. That's what we call the first return time, and the main subject of this thesis is to study the behavior of these return times as $\epsilon \to 0$. There have been a lot of works which studied the behavior of return times, we mention for example [3], [8], [16], [18], [19], [18], [24],[29], [31], [35], [45], [47], [30]. The desired results are of stochastic nature (almost everywhere convergence, convergence in distribution with respect to any probability measure absolutely continuous with respect to the infinite invariant measure, etc.). Results of this kind have been obtained in few cases by X. Bressaud, S. Galatolo, D.-H. Kim, K. Park, F. Pène, B. Saussol and R. Zweimüller.

A measure-preserving dynamical system is given by $(X, \mathcal{B}, \mu, T)$, where:

- $(X, \mathcal{B})$ is a measurable set,
- μ is a finite or σ-finite positive measure,
- $T : X \to X$ is a measurable transformation preserving the measure μ (i.e. $\mu(T^{-1}A) = \mu(A)$, for every $A \in \mathcal{B}$).

We are interested in the case where μ is σ-finite. We assume that X is endowed with some metric d_X and that $\mathcal{B}$ contains the open balls $B(x, r)$ of X. Our interest is in the first time the orbit comes back close to its initial position. For every $y \in X$, we define the first return time τ_ϵ of the orbit of y in the ball $B(y, \epsilon)$ as:

$$\tau_\epsilon(y) := \inf\{n \geq 1 : T^n(y) \in B(y, \epsilon)\} \in \mathbb{N} \cup \{+\infty\}.$$

The goal of this thesis is to study the quantitative recurrence of dynamical systems preserving an infinite measure. The main results of this thesis were established by first considering a probabilistic toy model. We also proved these results in the case of $\mathbb{Z}-$extension of subshift of finite type. Moreover, we establish results in the case of $\mathbb{Z}-$extension of an Axiom A flow.

Some previous results on quantitative recurrence

As we have seen, our interest is to study the quantitative recurrence properties of dynamical systems. The Poincaré Recurrence Theorem states that almost every orbit in a dynamical system returns close to its starting point. A natural question is: What is the time needed for an orbit of a dynamical system to return back to the neighborhood of its starting point? There are many results which have been established concerning the case where the measure is finite.

Existing results in quantitative recurrence properties in infinite measure are few and recent. They are based on specific arguments which require a lot of things to be carried out. Pène et Saussol [38] have addressed this question in some two-dimensional extended process, where they have studied the quantitative behavior of return times and relate it to the dimension of the process. They established their main result in the case of $\mathbb{Z}^2-$extension of subshift of finite type. If τ_ϵ defines this first return time, d being the Hausdorff dimension of the measure ν, they have proved that

$$\lim_{\epsilon\to 0}\frac{\log\log\tau_\epsilon}{-\log\epsilon}=d \quad a.s.$$

Moreover, they proved that the sequence of random variables $\nu(B_\epsilon(.))\log\tau_\epsilon(.)$ converges in distribution as $\epsilon\to 0$ to a random variable with distribution function of density $t\mapsto\frac{\beta t}{1+\beta t}1_{(0,+\infty)}(t)$.

There is a recent work for Rechberger and Zweimüller [43] where they studied the convergence of return and hitting-time distributions of small sets when the measure of these sets goes to 0 in recurrent ergodic dynamical systems preserving an infinite measure.

Concerning the case in continuous time, studies have been established for the recurrence of hyperbolic flows, we mention some of the works done: [10], [5], [6], [7]. Pesin and Sadovskaya [41] developed the multifractal analysis of conformal Axiom A flows, The main tool is the thermodynamic formalism for hyperbolic flows by Bowen and Ruelle [13]. Examples in this concern include Anosov flows and in particular geodesic flows on compact smooth surfaces of negative curvature. Barreira and Saussol [9] have established the multifractal analysis of hyperbolic flows and of suspension flows over subshifts of finite type. They have proved that for a measure ν in the equilibrium state of a Hölder potential, the return time for ν-almost very point y in $B(y,r)$ behaves like r^{-d+1}, where d is the Hausdorff dimension of the measure ν.
Rousseau has studied Poincaré recurrence for flows and observation of flows [44], where the result established have been applied to the geodesic flow on smooth manifold of strictly negative curvature. Defining the return time of the flow

$$\bar{\tau}_\epsilon=\inf\{t>1: g_t(y)\in B(y,\epsilon)\},$$

Let g_t be an Anosov flow. If ν is an equilibrium state of an Hölder potential. Then for $\nu-$ almost every point $y\in M$,

$$\frac{\log\bar{\tau}_\epsilon(y)}{-\log\epsilon}=d-1.$$

Presentation of the work done

In this work, we used properties of the ergodic theory, of probability and of geometry. The models taken in consideration will be recurrent ergodic, which ensures that trajectories visit infinitely often any neighbourhood of any point. In the first three chapters, we talk about the first result obtained which is a paper published under the title '*Quantitative recurrence of some dynamical systems preserving an infinite measure in dimension one*'. Then, in chapters 5 and 6, we talk about the results on $\mathbb{Z}-$extensions of Axiom A flow, which are the first results of this type, where we encounter a new level of difficulties when dealing with the flows. The local limit theorem with speed proved in chapter 2 is enough to deal with $\mathbb{Z}$-extensions of Axiom A flows as well due to viewing them as $\mathbb{Z}-$extensions of suspension flows, but the main difficulty is to see how to reduce the analysis of the flow to the Poincaré map.

In chapter one, we start by considering the toy probabilistic model designed to give the hint of the results in the general case and to clarify the strategy of our proofs. This model is given by (Y_n, S_n), where S_n is the simple symmetric walk on $\mathbb{Z}$ and Y_n is a sequence of independent random variables, with uniform distribution on $(0,1)^d$ and where Y_n and S_n are independent. The process involved in this model deals with the fact that there is a large scale where we define a return time R_n and a small scale, where we define a return time T_ϵ. We study the asymptotic behavior of R_n and T_ϵ, from which we prove the pointwise convergence of the recurrence rate and relate it to the dimension of the process. We prove then the convergence in distribution of the rescaled return time.

In the second chapter, we showed a result concerning the local limit theorem with speed for subshift of finite type, where we give a more precise error term to accommodate the one-dimensional case ($\mathbb{Z}$-extension of subshift of finite type). First it is local in the sense we are looking at the probability that $S_n\varphi = 0$. We needed to prove such an LLT, where we were looking at the probability conditioned to the fact that we are starting from a set A and landing on a set B. In this chapter, we started by giving a brief overview of the spectral theory tools we will need. More precisely, we give a spectral analysis of the Perron-Frobenius operator, which serves in precising the speed of convergence in the local limit theorem.

In the third chapter, we present a classical example of dynamical systems preserving an infinite measure which is given by the $\mathbb{Z}-$extension of a probability-preserving dynamical systems. Given a probability-preserving dynamical system $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ and a measurable function $\varphi : \bar{X} \to \mathbb{Z}$, we construct the $\mathbb{Z}$-extension $(X, \mathcal{B}, \mu, T)$ of $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ by setting $X := \bar{X} \times \mathbb{Z}$, $\mathcal{B} := \bar{\mathcal{B}} \otimes \mathcal{P}(\mathbb{Z})$, $\mu := \nu \otimes \sum_{l \in \mathbb{Z}} \delta_l$ and $T(x, l) = (\bar{T}(x), l + \varphi(x))$. We endow X with the product metric given by $d_X((x, l), (x', l')) := \max\{d_{\bar{X}}(x, x'), | l - l' |\}$. Hence $T^n(x, l) = (\bar{T}^n x, l + S_n\varphi(x))$, where $S_n\varphi$ is the ergodic sum $S_n\varphi := \sum_{k=0}^{n-1} \varphi \circ \bar{T}^k$. Therefore, for ϵ small enough,

$$T^n(x, l) \in B((x, l), \epsilon) \Longleftrightarrow \bar{T}^n(x) \in B_{\bar{X}}(x, \epsilon) \text{ and } S_n\varphi(x) = 0.$$

Our main results concern the case when $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ is a mixing subshift of finite type (see Section 3 for precise definition), which are classical dynamical systems used to model a wide class of dynamical systems such as geodesic flows in negative curvature, etc.
Consider $(\bar{X}, \bar{\mathcal{B}}, \nu, \bar{T})$ a mixing subshift of finite type and ν a Gibbs measure associated to a Hölder continuous potential. Moreover we have a ν-centered Hölder continuous function φ. Then we get the following result:

Theorem 3.1.1

$$\lim_{\epsilon \to 0} \frac{\log \tau_\epsilon}{\log \epsilon} = -2d, \tag{0.0.3}$$

μ-almost everywhere, where d is the Hausdorff dimension of ν.

Moreover the following convergence holds in distribution with respect to any probability measure absolutely continuous with respect to μ:

Theorem 3.1.2

$$\mu(B(., \epsilon))\sqrt{\tau_\epsilon(.)} \underset{\epsilon \to 0}{\longrightarrow} \frac{\mathcal{E}}{|\mathcal{N}|}, \tag{0.0.4}$$

where $\mathcal{E}$ and $\mathcal{N}$ are two independent random variables with respective exponential distribution of mean 1 and standard normal distribution.

Roughly speaking the strategy of our proof is that there is a large scale (corresponding to $S_n\varphi(x)$) and a small scale (corresponding to $\bar{T}^n(x)$), which behave independently asymptotically. The idea will be to use the operator perturbation method to establish a precised local limit theorem in order to establish the above mentioned results.

In the fourth chapter, we introduce all the necessary notions and results from hyperbolic dynamics. We consider a Riemannian manifold M. We give the definition of an Axiom A flow and of a hyperbolic set for a flow. In particular, we consider the Markov section constructed by Bowen [11] and Ratner [42] for a locally maximal hyperbolic set.
Thus we define the Poincaré section for the flow g_t denoted by X, and the height function $R : X \to (0, \infty)$ such that $R(x) = \min\{s > 0 : g_s x \in X\}$. Then we give a notion of a suspension flow $\psi = \{\psi_t\}_{t\in\mathbb{R}}$ over a transformation map $T : X \to X$, with height function R. We consider the space

$$X_R = \{(x, s) \in X \times \mathbb{R} : 0 \leq s \leq R(x)\},$$

where the points $(x, R(x))$ and $(T(x), 0)$ are identified for each $x \in X$.
Another point is to describe how a Markov section for a hyperbolic set gives rise to a symbolic dynamics. We consider a set Σ_A together with the shift map σ (see Section 4.4). And then we give a definition of the coding map $\chi : \Sigma_A \to X$. And thus in the same way we define the symbolic suspension flow $\mathcal{S} = \{\mathcal{S}_t\}_{t\in\mathbb{R}}$ over $\sigma_{|\Sigma_A}$. We introduce a section defining an equilibrium measure of a flow. In particular, we explain the existence of a unique equilibrium measure for an Hölder function H for the symbolic suspension flow $\mathcal{S} = \{\mathcal{S}_t\}_{t\in\mathbb{R}}$. In the last section of this chapter, we work on establishing some properties on balls and coding. Under some conditions taken on the lengths of stable and unstable curves, a given ball contains and is contained in a cylinder. This serves in studying the asymptotic behavior of our return time to a cylinder and then deducing it for the return time to a ball. And yet proving the almost sure convergence result in Chapter 5 (5.1.1), and the convergence in distribution in chapter 6 (6.0.5).

In chapter 5, we prove the pointwise convergence of the recurrence rate to the dimension in the case of a continuous time. As previously, our work concern the case where the measure is infinite. More precisely, we consider a Riemannian manifold $\tilde{M}$ of dimension 3, endowed with a $\sigma-$finite measure $\tilde{\mu}$, and $(\tilde{g}_t)_{t\in\mathbb{R}}$ a flow on $\tilde{M}$ preserving the measure $\tilde{\mu}$. We set Γ to be an infinite group of isometries of $\tilde{M}$. Then we suppose that $M = \tilde{M}/\Gamma$ is a compact manifold, and later we define a flow $(g_t)_{t\in\mathbb{R}}$ on M, (see section 5.1 for more precise description of this $\mathbb{Z}-$extension). Let μ be the measure defined on M from the measure $\tilde{\mu}$ by passing through the quotient. We assume that μ is an equilibrium measure for g_t, and that $(M, (g_t)_t)$ is an Axiom A flow.

We are interested in the first return time the flow $\tilde{g}_t$ returns to an $\epsilon-$neighborhood of its starting point. Thus for any $x \in \tilde{M}$, we define the return time of the flow $\tilde{g}_t$, by:

$$\tau_\epsilon(x) := \inf \{t > 1 : \tilde{g}_t(x) \in B(x, \epsilon)\},$$

where $B(x, \epsilon)$ is the ball of center x and radius ϵ. As in the preceding chapters, we want to study the asymptotic behavior of τ_ϵ as $\epsilon \to 0$. Thus we prove the following result:

Theorem 5.1.1 *Let $(\tilde{M}, \{\tilde{g}\}_t, \tilde{\mu})$ be a flow satisfying all the hypotheses above, for $\tilde{\mu}$-almost every point $x \in \tilde{M}$,*

$$\lim_{\epsilon \to 0} \frac{\log \sqrt{\tau_\epsilon}}{-\log \epsilon} = \frac{h}{L^u} + \frac{h}{L^s}.$$

Finally, in Chapter 6, we prove the convergence in distribution for $\mathbb{Z}-$extension of Axiom A flow. Let $\tilde{y} \in \tilde{M}$. We describe the existence of a measure $\tilde{\nu}_0$ on a disk centered at $\tilde{y}$, which is transversal to the flow, and orthogonal to it at $\tilde{y}$. We call it measure transversal to the flow at $\tilde{y}$. We prove the following result:

Theorem 6.0.1 *The sequence of random variables $\tilde{\nu}_0(B(., \epsilon))^2 \tau_\epsilon(.)$ converges in distribution, with respect to any probability measure absolutely continuous with respect to $\tilde{\mu}$, as $\epsilon \to 0$ to $\sigma^2_{flow} \frac{\mathcal{E}^2}{\mathcal{N}^2}$, where $\mathcal{E}$ and $\mathcal{N}$ are independent random variables, $\mathcal{E}$ having an exponential distribution of mean 1 and $\mathcal{N}$ having a standard Gaussian distribution.*

Chapter 1

Recurrence in a Probabilistic Toy Model

In this chapter, we consider a toy probabilistic model to clarify the strategy of our proofs. The process involved in this model deals with the fact that there's a large scale where we define a return time R_n, and a small scale where we define a return time T_ϵ. We study the asymptotic behavior of R_n and T_ϵ, from which we prove the pointwise convergence of the recurrence rate and relate it to the dimension of the process. In the third section we prove the convergence in distribution of the rescaled return time.

Let $d \in \mathbb{N}$. Given a real random process $(M_n)_{n\geq 0}$ with values in $\mathbb{R}\times]0,1[^{d-1}$, we define for every $\epsilon > 0$, the return time τ_ϵ in the open ball $B(M_0, \epsilon)$ of radius ϵ centered at M_0 (for some metric associated to a norm $|\cdot|$ on $\mathbb{R}^d$) as follows:

$$\tau_\epsilon = \inf\{n \geq 1 : |M_n - M_0| < \epsilon\}$$

Observe that τ_ϵ corresponds to the first return time of $(M_n)_n$ in the ϵ-neighborhood $B(M_0, \epsilon)$ of the initial value M_0.

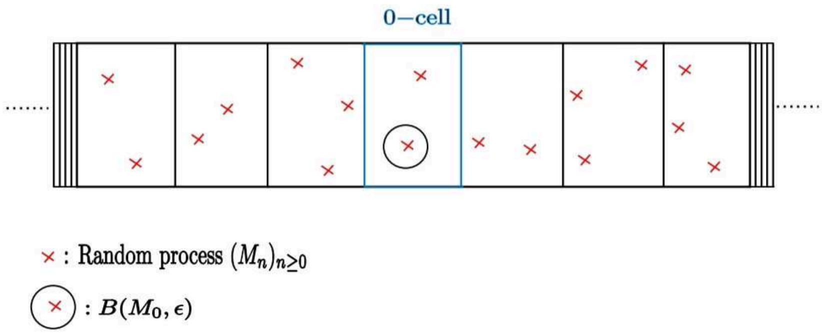
1.1 Description of the model and statements of the results.

Here the random process $(M_n)_{n\geq 0}$ is given by $M_n = (S_n, 0) + Y_n$ where $(S_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ are independent. $(Y_n)_{n\geq 0}$ is a sequence of independent random variables

uniformly distributed on $(0,1)^d$ and $(S_n)_{n\geq 0}$ is the simple symmetric random walk on $\mathbb{Z}$.
The simple random walk $(S_n)_{n\geq 0}$ is given by $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$ and $(X_k)_k$ is a sequence of independent random variables such that:

$$\mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = 1/2.$$

See the following figure representing the random process, considering the case where $d = 2$ for example.



We want to study the asymptotic behavior, as ϵ goes to 0, of the return time of $(M_n)_{n\geq 0}$ in the ϵ-neighborhood $B(M_0, \epsilon)$ of the initial value M_0. We will prove the following results:

Theorem 1.1.1. *Almost surely, $\frac{\log \tau_\epsilon}{-\log \epsilon}$ converges to $2d$ as ϵ goes to 0.*

Observe that there exists $c > 0$ such that for every $x \in (0,1)^d$, $Leb(B(x,\epsilon)) \leq c\epsilon^d$ and $Leb(B(x,\epsilon)) = c\epsilon^d$ for ϵ small enough, where c is the Lebesgue measure of the unit ball of $\mathbb{R}^d$. For this constant $c > 0$, we have the following result:

Theorem 1.1.2. *The sequence of random variables $((c\epsilon^d)\sqrt{\tau}_\epsilon)_\epsilon$ converges in distribution to $\frac{\mathcal{E}}{|\mathcal{N}|}$, where $\mathcal{E}$ and $\mathcal{N}$ are two independent random variables , $\mathcal{E}$ having an exponential distribution of mean 1 and $\mathcal{N}$ having a standard Gaussian distribution.*

1.2 Proof of the pointwise convergence of the recurrence rate to the dimension.

We suppose that the initial point M_0 is in $(0,1)^d$. Let $\epsilon > 0$ so small that $B(M_0, \epsilon)$ is contained in $(0,1)^d$, then $Leb(B(x,\epsilon)) = c\epsilon^d$.

Definition. Setting $R_0 = 0$, let us first define $R_1 = \min\{m \geq 1 : S_m = 0\}$. We then define by induction, for any $p \geq 0$ the p^{th} return time R_p of $(M_n)_{n\geq 0}$ in $(0,1)^d$, by

$$R_{p+1} := \inf\big\{m > R_p : S_m = 0\big\}.$$

Definition. We define the return time T_ϵ of the sequence of random variables $(Y_{R_n})_{n\geq 0}$ in the $\epsilon-$neighborhood $B(Y_0, \epsilon)$, by:

$$T_\epsilon := \min\{l \geq 1 : Y_{R_l} \in B(Y_0, \epsilon)\}.$$

Thus, we have the following relation

$$\tau_\epsilon = R_{T_\epsilon}. \tag{1.2.1}$$

We will study the asymptotic behavior of the random variables R_n and T_ϵ and use the relation (1.2.1) to prove Theorem 1.1.1.

1.2.1 Behavior of the random variable R_n.

Lemma 1.2.1. $\sum_{n\geq 0} \mathbb{P}(S_{2n} = 0)s^{2n} = \frac{1}{\sqrt{1-s^2}}$ *and* $\mathbb{P}(S_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}}$.

Proof. Recall that $(X_i)_i$ is a sequence of independent random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. Note that $\frac{(X_i+1)}{2}$ is equal to 1 if $X_i = 1$, and is equal to 0 if $X_i = -1$. Hence $\frac{(X_i+1)}{2}$ is a random variable of Bernoulli distribution of parameter $\frac{1}{2}$.
And, the random variable B_n, which is given by:

$$B_n = \sum_{i=1}^{n} \frac{(X_i+1)}{2} = \frac{S_n + n}{2}.$$

has a binomial distribution $B(n, \frac{1}{2})$. Hence,

$$\begin{aligned}
\mathbb{P}(S_{2n} = 0) &= \mathbb{P}(B_{2n} = n) \\
&= \frac{2n!}{n!n!}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^n \\
&= 2^{-2n}\frac{(2n)!}{n!n!}.
\end{aligned}$$

Using the Stirling formula, we have $n! \sim \sqrt{2\pi n} n^n e^{-n}$, thus we get

$$\mathbb{P}(S_{2n} = 0) \sim 2^{-2n}\sqrt{4\pi n}2^{2n}e^{-2n}\frac{n^{2n}}{(\sqrt{2\pi n}n^n e^{-n})^2} = \frac{1}{\sqrt{\pi n}}.$$

And in turn, $\forall s \in [0,1]$, we get

$$\sum_{n\geq 0}\mathbb{P}(S_{2n} = 0)s^{2n} = \sum_{n\geq 0}\frac{2^{-2n}(2n)!s^{2n}}{n!n!} = \frac{1}{\sqrt{1-s^2}}.$$

□

For all $k \geq 0$, we observe that the sequence $(S_{2k+m} - S_{2k})_{m\geq 1}$ has the same distribution as $(S_m)_{m\geq 1}$. We have the following proposition:

Proposition 1.2.2. : *There exists $C > 0$ such that:*

$$\mathbb{P}(R_1 > s) \sim \frac{C}{\sqrt{s}}, \quad \textit{as } s \to \infty \tag{1.2.2}$$

Proof.

$$\begin{aligned}
1 &= \sum_{k=0}^{n}\mathbb{P}(S_k = 0, S_{k+1} \neq 0, S_{k+2} \neq 0, ..., S_n \neq 0) \\
&= \sum_{k=0}^{n}\mathbb{P}(S_k = 0)\mathbb{P}(S_{k+1} - S_k \neq 0, S_{k+2} - S_k \neq 0, ..., S_n - S_k \neq 0) \\
&= \sum_{k=0}^{n}\mathbb{P}(S_k = 0)\mathbb{P}(S_1 \neq 0, ..., S_{n-k} \neq 0) \\
&= \sum_{k=0}^{n}\mathbb{P}(S_k = 0)\mathbb{P}(R_1 > n-k).
\end{aligned}$$

Then, for all $0 < s < 1$, we have

$$\frac{1}{1-s} = \sum_{n\geq 0}s^n = \left(\sum_{k\geq 0}\mathbb{P}(S_k = 0)s^k\right)\left(\sum_{m\geq 0}\mathbb{P}(R_1 > m)s^m\right)$$

therefore, due to Lemma 1.2.1

$$\sum_{m\geq 0}\mathbb{P}(R_1 > m)s^m = \frac{1}{1-s}\sqrt{1-s^2} = \sqrt{\frac{1+s}{1-s}} \underset{s\to 1^-}{\longrightarrow} \frac{\sqrt{2}}{\sqrt{1-s}}.$$

Hence, using the Tauberian Theorem [23], and since $m \mapsto \mathbb{P}(R_1 > m)$ is decreasing, we get

$$\mathbb{P}(R_1 > n) \sim \frac{\sqrt{2}}{\Gamma(1/2)}.\frac{1}{\sqrt{n}} \quad \textit{as } n \to \infty.$$

□

We observe that R_p is the $p-$th return time of the random walk S_n to the origin, thus we have the following proposition

Proposition 1.2.3. *The delays between successive return times $R_p - R_{p-1}$, are independent and identically distributed. For all $s > 0$, we have*

$$\mathbb{P}(R_p - R_{p-1} > s) = \mathbb{P}(R_1 > s), \quad \forall s > 0.$$

Proof. It's enough to prove that for every $n \geq 1$, $\forall l_1, l_2, ..., l_n \geq 0$, the following property holds:

$$\mathbb{P}(R_1 = l_1, ..., R_n - R_{n-1} = l_n) = \prod_{i=1}^{n} \mathbb{P}(R_1 = l_i).$$

We will prove this property by induction. Thus we start by considering the case $n = 1$ which is trivial, because $\forall l_1 \geq 0$, we have:

$$\mathbb{P}(R_1 = l_1) = \prod_{i=1}^{1} \mathbb{P}(R_1 = l_i).$$

Let $n \geq 1$, assume that the property is true for n, and let us prove that it is true for $n + 1$,

$$\mathbb{P}(R_1 = l_1, R_2 - R_1 = l_2, .., R_{n+1} - R_n = l_{n+1}) \text{ is equal to} \tag{1.2.3}$$

$$\begin{aligned}\mathbb{P}\Big(\quad & S_1 \neq 0, ..., S_{l_1-1} \neq 0, S_{l_1} = 0;\\ & S_{l_1+1} \neq 0, ..., S_{l_1+l_2-1} \neq 0, S_{l_1+l_2} = 0; ...;\\ & S_{l_1+...+l_{n-1}+1} \neq 0, ..., S_{l_1+...+l_n-1} \neq 0, S_{l_1+...l_{n-1}+l_n} = 0;\\ & S_{l_1+...+l_n+1} - S_{l_1+...+l_n} \neq 0, ..., S_{l_1+...+l_{n+1}} - S_{l_1+...+l_n} = 0\Big). \end{aligned} \tag{1.2.4}$$

Now using the independence of $(S_1, ..., S_{l_1+...+l_n})$ and $(S_{l_1+...+l_n+1} - S_{l_1+...+l_n}, ..., S_{l_1+...+l_{n+1}} - S_{l_1+...+l_n})$, the later formula (1.2.4) is equal to

$$\begin{aligned} \mathbb{P}\Big(\quad & S_1 \neq 0, ..., S_{l_1-1} \neq 0, S_{l_1} = 0; \\ & S_{l_1+1} \neq 0, ..., S_{l_1+l_2-1} \neq 0, S_{l_1+l_2} = 0; ...; \\ & S_{l_1+...+l_{n-1}+1} \neq 0, ..., S_{l_1+...+l_n-1} \neq 0, S_{l_1+...l_{n-1}+l_n} = 0\Big) \\ \times\mathbb{P}\Big(\quad & S_{l_1+...+l_n+1} - S_{l_1+...+l_n} \neq 0, ..., S_{l_1+...+l_{n+1}} - S_{l_1+...+l_n} = 0\Big). \end{aligned} \tag{1.2.5}$$

Using the recurrence hypothesis, the first probability in (1.2.5) is equal to $\prod_{i=1}^{n} \mathbb{P}(R_1 = l_i)$. And since $(S_{l_1+...+l_n+1} - S_{l_1+...+l_n}, ..., S_{l_1+...+l_{n+1}} - S_{l_1+...+l_n})$ has the same distribution as $(S_1, ..., S_{ll_n+1})$, the the second probability in (1.2.5) is equal to $\mathbb{P}(R_1 = l_{n+1})$. Hence we obtain that the probability in (1.2.3) is equal to

$$\prod_{i=1}^{n+1} \mathbb{P}(R_1 = l_i).$$

It follows that R_1, $R_2 - R_1$, ... , $R_n - R_{n-1}$ are independent and identically distributed. And consequently, we get, $\forall s > 0$

$$\mathbb{P}(R_p - R_{p-1} > s) = \mathbb{P}(R_1 > s), \qquad \forall p \geq 0.$$

□

Remark 1.2.4. Let $a_i \geq 0$, and $p < 1$, we have:

$$\sum_{i=1}^{n} a_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}.$$

Considering the process in the large scale, we start by studying the asymptotic behavior of the random variable R_n, which is illustrated in the following lemma:

Lemma 1.2.5. *Almost surely, $\frac{\log\sqrt{R_n}}{\log n}$ converges to 1 as n goes to ∞.*

Proof. It suffices to prove that for any $0 < \alpha < 1$, provided n is sufficiently large, the following inequalities hold almost surely:

$$n^{1-\alpha} \leq \sqrt{R_n} \leq n^{1+\alpha}. \tag{1.2.6}$$

Let $\alpha \in (0,1)$, observe that whenever $\sqrt{R_n} \leq n^{1-\alpha}$, we have $\forall p \leq n$, $\sqrt{R_p - R_{p-1}} \leq n^{1-\alpha}$. Then by independence and using proposition 1.2.3, we have

$$\begin{aligned} \mathbb{P}(\sqrt{R_n} \leq n^{1-\alpha}) &\leq \mathbb{P}(\forall p \leq n, \sqrt{R_p - R_{p-1}} \leq n^{1-\alpha}) \\ &\leq \mathbb{P}(\sqrt{R_1} \leq n^{1-\alpha})^n. \end{aligned}$$

According to the asymptotic formula given in Proposition 1.2.2,there exists $C > 0$ such that, for every n large enough

$$\mathbb{P}(\sqrt{R_1} \leq n^{1-\alpha})^n \leq \left(1 - \frac{C}{2n^{1-\alpha}}\right)^n \leq \exp\left(-C\frac{n^\alpha}{2}\right).$$

Thus the first inequality of (1.2.6) follows from the Borel Cantelli lemma, since $\sum_{n\geq 1} \mathbb{P}(\sqrt{R_n} \leq n^{1-\alpha}) \leq \sum_{n\geq 1} \exp\left(-C\frac{n^\alpha}{2}\right) < +\infty$.

Using again Proposition 1.2.2, there exists $C' > 0$ such that we have $P\left(R_1^{\frac{1}{2+\epsilon}} > s\right) \leq \frac{C'}{\sqrt{s^{2+\epsilon}}} = \frac{C'}{s^{1+\frac{\epsilon}{2}}}$. And hence we get

$$\mathbb{E}\left(R_1^{\frac{1}{2+2\alpha}}\right) = \int_0^\infty \mathbb{P}\left(R_1^{\frac{1}{2+2\alpha}} > s\right) ds \leq 1 + \int_1^\infty \frac{C'}{s^{1+\alpha}} < \infty$$

Note that the random variables R_n can be written as follows

$$\begin{aligned} R_n &= R_1 + R_2 - R_1 + \ldots\ldots + R_n - R_{n-1} \\ &= U_1 + U_2 + \ldots + U_n \\ &= \sum_{i=1}^n U_i. \end{aligned}$$

where $U_i = R_i - R_{i-1}$ and due to Proposition 1.2.3, $(U_i)_i$ are i.id. Now using the Remark 1.2.4, applied with $p = \frac{1}{2+2\alpha}$, we get

$$\begin{aligned} R_n &= \sum_{i=1}^n U_i \\ &\leq \left(\sum_{i=1}^n U_i^{\frac{1}{2+2\alpha}}\right)^{2+2\alpha} \\ &= n^{2+2\alpha}\left(\frac{1}{n}\sum_{i=1}^n U_i^{\frac{1}{2+2\alpha}}\right)^{2+2\alpha}, \end{aligned}$$

but $\frac{1}{n}\sum_{i=1}^n U_i^{\frac{1}{2+2\alpha}}$ converges almost surely to $\mathbb{E}\left(R_1^{\frac{1}{2+2\alpha}}\right)$, which is finite, from which we get that $R_n = O(n^{2+2\alpha})$ almost surely. From this we get the second inequality of (1.2.6). $\square$

1.2.2 Behavior of the random variable T_ϵ

In this section, we consider the process in the small scale. Thus we study the asymptotic behavior of T_ϵ, and we prove the following lemma

Lemma 1.2.6. *Almost surely,* $\frac{\log T_\epsilon}{-\log \epsilon} \to d$*, as* $\epsilon \to 0$

Proof. Given Y_0, the random variable T_ϵ has a geometric distribution with parameter $\lambda_\epsilon := \lambda(B(Y_0, \epsilon))$.
For any $\alpha \in (0,1)$, a simple decomposition gives:

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{\log T_\epsilon}{-\log \epsilon} - d\right| > \alpha\right) &= \mathbb{P}\left(\frac{\log T_\epsilon}{-\log(\epsilon)} - d < -\alpha \text{ or } \frac{\log T_\epsilon}{\log \epsilon} - d > \alpha\right) \\
&= \mathbb{P}\left(\log T_\epsilon < (d-\alpha)\log(\epsilon^{-1}) \text{ or } \log T_\epsilon > (d+\alpha)\log(\epsilon^{-1})\right) \\
&= \mathbb{P}\left(\log T_\epsilon < \log(\epsilon^{-d+\alpha}) \text{ or } \log T_\epsilon > \log(\epsilon^{-d-\alpha})\right) \\
&= \mathbb{P}\left(T_\epsilon > (\epsilon^{-d-\alpha})) + \mathbb{P}(T_\epsilon < (\epsilon^{-d+\alpha})\right).
\end{aligned}$$

The first term is handled by the Markov inequality :

$$\begin{aligned}
\mathbb{P}(T_\epsilon > (\epsilon^{-d-\alpha}) \mid Y_0) &\leq \frac{\mathbb{E}(T_\epsilon \mid Y_0)}{\epsilon^{-d-\alpha}} \\
&\leq \frac{(\lambda_\epsilon)^{-1}}{\epsilon^{-d-\alpha}} = \epsilon^\alpha \frac{\epsilon^d}{\lambda_\epsilon},
\end{aligned}$$

knowing that for ϵ small enough, $\lambda_\epsilon = c\epsilon^d$ as soon as $d(Y_0, \partial(]0,1[^d)) > \epsilon$.

$$\begin{aligned}
\mathbb{P}(T_\epsilon > (\epsilon^{-d-\alpha})|Y_0) &= \mathbb{E}\left[\mathbb{P}(T_\epsilon > (\epsilon^{-d-\alpha}) \mid Y_0)1_{\{\lambda_\epsilon = c\epsilon^d\}}\right] \\
&+ \mathbb{E}\left[\mathbb{P}(T_\epsilon > (\epsilon^{-d-\alpha}) \mid Y_0)1_{\{\lambda_\epsilon \neq c\epsilon^d\}}\right] \\
&\leq \mathbb{E}\left[\frac{\epsilon^\alpha}{c} \times 1\right] + \mathbb{E}\left[1 \times 1_{\{d(Y_0, \partial(]0,1[^d)) < \epsilon\}}\right] \\
&\leq \frac{\epsilon^\alpha}{c} + \mathbb{P}(d(Y_0, \partial(]0,1[^d)) < \epsilon) \\
&\leq \frac{\epsilon^\alpha}{c} + O(\epsilon) \\
&= O(\epsilon^\alpha) \qquad (\text{ since } \alpha < 1).
\end{aligned}$$

While the second term is handled using the geometric distribution:

$$\begin{aligned}
\mathbb{P}(T_\epsilon < \epsilon^{-d+\alpha}) &= \sum_{i=1}^{\epsilon^{-d+\alpha}} \mathbb{P}(T_\epsilon = i) \\
&= \sum_{i=1}^{\epsilon^{-d+\alpha}} \lambda_\epsilon (1-\lambda_\epsilon)^{i-1} \\
&= \lambda_\epsilon \sum_{i=0}^{\epsilon^{-d+\alpha}-1} (1-\lambda_\epsilon)^i \\
&= \lambda_\epsilon \frac{1-(1-\lambda_\epsilon)^{\epsilon^{-d+\alpha}}}{1-(1-\lambda_\epsilon)} \\
&= 1-(1-\lambda_\epsilon)^{\epsilon^{-d+\alpha}} \\
&\leq 1-(1-c\epsilon^d)^{\epsilon^{-d+\alpha}} \\
&\leq 1-\exp[\epsilon^{-d+\alpha}\log(1-c\epsilon^d)] \\
&\leq -\epsilon^{-d+\alpha}\log(1-c\epsilon^d) \\
&= O(\epsilon^\alpha).
\end{aligned}$$

Moreover, we get that $\mathbb{P}\left(\left|\frac{\log T_\epsilon}{-\log\epsilon} - d\right| > \alpha\right) = O(\epsilon^\alpha)$, as $\epsilon \to 0$.
Let us set $\epsilon_n := n^{\frac{-2}{\alpha}}$. Since T_ϵ is monotone in ϵ, we get

$$\sum_{n\geq 1} \mathbb{P}\left(\left|\frac{\log T_{\epsilon_n}}{-\log\epsilon_n} - d\right| > \alpha\right) < +\infty.$$

Then, using the Borel Cantelli Lemma, we have almost surely $\frac{\log T_{\epsilon_n}}{-\log\epsilon_n} \to d$ as $n \to +\infty$.
We have $\lim\limits_{n\to+\infty} \epsilon_n = 0$ and $\lim\limits_{n\to+\infty} \frac{\epsilon_n}{\epsilon_{n+1}} = 1$, thus $(\epsilon_n)_{n\geq 1}$ is a decreasing sequence of real numbers. Now, given $\epsilon > 0$, note that if n is such that $\epsilon_{n+1} \leq \epsilon \leq \epsilon_n$, we get

$$\frac{\log T_{\epsilon_n}}{-\log\epsilon_{n+1}} \leq \frac{\log T_\epsilon}{-\log\epsilon} \leq \frac{\log T_{\epsilon_{n+1}}}{-\log\epsilon_n} \tag{1.2.7}$$

Moreover,

$$\frac{\log\epsilon_{n+1}}{\log\epsilon_n} = \frac{\log\left(\frac{\epsilon_{n+1}}{\epsilon_n}\right) + \log\epsilon_n}{\log\epsilon_n} \sim 1.$$

The right hand side of (1.2.7) is equal to $\frac{\log T_{\epsilon_{n+1}}}{-\log\epsilon_{n+1}} \times \frac{\log\epsilon_{n+1}}{\log\epsilon_n}$ which goes to $d \times 1$ as n goes to ∞.

And the left hand side of (1.2.7) is equal to $\frac{\log T_{\epsilon_n}}{-\log \epsilon_n} \times \frac{\log \epsilon_n}{\log \epsilon_{n+1}}$ which goes to $d \times 1$ as n goes to ∞.

$\square$

Proof of Theorem 1.1.1. The proof of the theorem directly follows using the two previous lemmas 1.2.5 and 1.2.6, since:

$$\frac{\log \sqrt{\tau_\epsilon}}{-\log \epsilon} = \frac{\log \sqrt{R_{T_\epsilon}}}{\log T_\epsilon} \frac{\log T_\epsilon}{-\log \epsilon} \to 1 \times d = d \quad \text{a.s.}$$

Hence, we get:

$$\frac{\log \tau_\epsilon}{-\log \epsilon} \to 2d \text{ as } \epsilon \to 0 \quad \text{a.s.}$$

$\square$

1.3 Proof of the convergence in distribution of the rescaled return time.

Lemma 1.3.1. *The moment generating function of* $|\mathcal{N}|^{-2}$ *is* $\mathbb{E}\left[e^{-t|\mathcal{N}|^{-2}}\right] = e^{-\sqrt{2t}}$, $\forall t \geq 0$, *where* $\mathcal{N}$ *is standard Gaussian random variable.*

Proof. Let $t \geq 0$, we set $G(t) = H(t^2)$, where $H(t)$ is defined as follows

$$H(t) = \mathbb{E}\left[e^{-t|\mathcal{N}|^{-2}}\right] = \int_{\mathbb{R}} e^{-\frac{t}{x^2}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

Then $G(t)$ and $G^{'}(t)$ have the following formulas:

$$G(t) = 2\int_0^{+\infty} e^{-\frac{t^2}{x^2}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{t^2}{x^2}} e^{-\frac{x^2}{2}} dx$$

and

$$G'(t) = \frac{-2\sqrt{2}}{\pi} \int_0^{+\infty} \frac{t}{x^2} e^{-\frac{t^2}{x^2}} e^{-\frac{x^2}{2}} dx.$$

Making a change of variable $y = \frac{t\sqrt{2}}{x}$ in $G^{'}(t)$, we get

$$\begin{aligned} G'(t) &= \frac{-2\sqrt{2}}{\pi} \left(-\int_0^{+\infty} \frac{y^2}{2t} e^{-\frac{y^2}{2}} e^{-\frac{t^2}{y^2}} \left(\frac{-\sqrt{2}t}{y^2} \right) dy \right) \\ &= \frac{-2}{\sqrt{\pi}} \int_0^{+\infty} e^{-\frac{y^2}{2}} e^{-\frac{t^2}{y^2}} dy = -\sqrt{2}G(t), \end{aligned}$$

from which we get that $G(t) = ae^{-\sqrt{2}t}$, where a is a constant, but $G(0) = 1$ so $a = 1$. Therefore, we end up by

$$G(t) = e^{-\sqrt{2}t} \text{ and } H(t) = G(\sqrt{t}) = e^{-\sqrt{2t}}$$

□

Proposition 1.3.2. *The sequence of random variables $(\frac{R_n}{n^2})_n$ converges in distribution to $|\mathcal{N}|^{-2}$ where $\mathcal{N}$ is a standard Gaussian random variable.*

The proof of Proposition is below. We have the following lemma:

Lemma 1.3.3. *For all $s > 0$, we have $\mathbb{E}\left[s^{R_1}\right] = 1 - \sqrt{1 - s^2}$.*

Proof. Let $n \geq 1$
If we have $S_{2n} = 0$, we consider the last visit time $2k$ to position zero before $2n$.

$$\{S_{2n} = 0\} = \bigsqcup_{k=0}^{n-1} \{S_{2k} = 0, \forall m = 1, ..., 2n - 2k - 1, S_{2k+m} \neq 0, S_{2n} = 0\},$$

thus calculating the probabilities of these quantities, we get

$$\begin{aligned}
\mathbb{P}(S_{2n} = 0) &= \sum_{k=0}^{n-1} \mathbb{P}\Bigg(S_{2k} = 0, \forall m \in \{1, ..., 2n - 2k - 1\}, \\
&\qquad\qquad S_{2k+m} - S_{2k} \neq 0, S_{2k+(2n-2k)} - S_{2k} = 0\Bigg) \\
&= \sum_{k=0}^{n-1} \mathbb{P}(S_{2k} = 0)\mathbb{P}(\forall m \in \{1, ..., 2n - 2k - 1\}, S_m \neq 0, S_{2n-2k} = 0) \\
&= \sum_{k=0}^{n-1} \mathbb{P}(S_{2k} = 0)\mathbb{P}(R_1 = 2n - 2k),
\end{aligned}$$

where we use the independence of S_{2k} and of $(S_{2k+1} - S_{2k}, ..., S_{2n} - S_{2k})$ and the fact that $(S_{2k+1} - S_{2k}, ..., S_{2n} - S_{2k})$ has the same distribution as $(S_1, ..., S_{2n-2k})$. Let us set $a_{2n} = \mathbb{P}(S_{2n} = 0)$ and $b_{2n} = \mathbb{P}(R_1 = 2n)$. Observing that $b_0 = \mathbb{P}(R_1 = 0) = 0$, we have the following

$$\begin{aligned}
\left(\sum_{n\geq 0} a_{2n}s^{2n}\right)\left(\sum_{m\geq 1} b_{2m}s^{2m}\right) &= \left(\sum_{n\geq 0} a_{2n}s^{2n}\right)\left(\sum_{m\geq 0} b_{2m}s^{2m}\right) \\
&= \sum_{n\geq 0}\sum_{k=0}^{n} \mathbb{P}(S_{2k} = 0)\mathbb{P}(R_1 = 2(n-k))s^{2n} \\
&= \sum_{n\geq 1} \mathbb{P}(S_{2n} = 0)s^{2n} = \sum_{n\geq 1} a_{2n}s^{2n}.
\end{aligned}$$

Hence, for all $s > 0$ we deduce

$$\begin{aligned}
\mathbb{E}(s^{R_1}) &= \sum_{n\geq 1} b_{2n}s^{2n} \\
&= \sum_{n\geq 1} \mathbb{P}(R_1 = 2n)s^{2n} \\
&= \frac{\sum_{n\geq 1}\mathbb{P}(S_{2n}=0)s^{2n}}{\sum_{n\geq 0}\mathbb{P}(S_{2n}=0)s^{2n}} \\
&= \frac{\sum_{n\geq 0}\mathbb{P}(S_{2n}=0)s^{2n} - 1}{\sum_{n\geq 0}\mathbb{P}(S_{2n}=0)s^{2n}} \\
&= \frac{\frac{1}{\sqrt{1-s^2}} - 1}{\frac{1}{\sqrt{1-s^2}}} = 1 - \sqrt{1-s^2}.
\end{aligned}$$

□

Proof of Proposition 1.3. Knowing that $R_1, (R_2 - R_1), ..., (R_n - R_{n-1})$ are i.i.d., and using Lemma 1.3.3, we have

$$\begin{aligned}
\mathbb{E}[e^{-\frac{t}{n^2}R_n}] &= \mathbb{E}\left[e^{-\frac{t}{n^2}(R_1+(R_2-R_1)+...+(R_n-R_{n-1}))}\right] \\
&= (\mathbb{E}[e^{-\frac{t}{n^2}R_1}])^n \\
&= \left(1-\sqrt{1-e^{-2\frac{t}{n^2}}}\right)^n, \quad \forall t \geq 0.
\end{aligned}$$

Hence $\forall t \geq 0$, and due to Lemma 1.3.1, we have $\lim_{n\to\infty} E\left[e^{-\frac{t}{n^2}R_n}\right] = \lim_{n\to\infty}[1-\frac{\sqrt{2t}}{n}]^n = e^{-\sqrt{2t}} = \mathbb{E}[e^{-t|\mathcal{N}|^{-2}}]$, which proves that $(\frac{R_n}{n^2})_n$ converges in distribution to $|\mathcal{N}|^{-2}$. □

Lemma 1.3.4. *$(\lambda_\epsilon T_\epsilon)_\epsilon$ converges in distribution to an exponential random variable $\mathcal{E}$ of mean 1, (that is $\mathbb{P}(\mathcal{E} \leq t) = 1 - e^{-t}$ for every $t \geq 0$).*

Proof. Given Y_0, T_ϵ has a geometric distribution of parameter $\lambda_\epsilon = \lambda(B(Y_0, \epsilon))$, thus

$$\forall k \geq 1, \quad \mathbb{P}(T_\epsilon = k) = \lambda_\epsilon(1-\lambda_\epsilon)^{k-1}.$$

Let $t > 0$, we have

$$\begin{aligned}\mathbb{P}(\lambda_\epsilon T_\epsilon \le t \mid Y_0) &= \sum_{n=1}^{\left\lfloor \frac{t}{\lambda_\epsilon}\right\rfloor} \mathbb{P}(T_\epsilon = n \mid Y_0) \\ &= \sum_{n=1}^{\left\lfloor \frac{t}{\lambda_\epsilon}\right\rfloor} \lambda_\epsilon (1-\lambda_\epsilon)^{n-1} \\ &= 1 - (1-\lambda_\epsilon)^{\left\lfloor \frac{t}{\lambda_\epsilon}\right\rfloor} \\ &= 1 - \exp\left(\left\lfloor \frac{t}{\lambda_\epsilon}\right\rfloor \log(1-\lambda_\epsilon)\right),\end{aligned}$$

but $\frac{1}{\lambda_\epsilon} \to \infty$ as $\epsilon \to 0$. Thus, since $\left\lfloor \frac{t}{\lambda_\epsilon}\right\rfloor \sim \frac{t}{\lambda_\epsilon}$ and $\log(1-\lambda_\epsilon) \sim \lambda_\epsilon$, we get $\lim_{\epsilon\to 0} \left\lfloor \frac{t}{\lambda_\epsilon}\right\rfloor \log(1-\lambda_\epsilon) = t$ almost surely. Hence, for $\mathcal{E}$ a random variable which follows $\exp(1)$, we get

$$\forall t > 0, \quad \lim_{\epsilon\to 0} \mathbb{P}(\lambda_\epsilon T_\epsilon \le t \mid Y_0) = 1 - e^{-t} = \mathbb{P}(\mathcal{E} \le t), \quad a.s.,$$

which holds also for $t = 0$. Therefore, using the Lebesgue Dominated Convergence Theorem:

$$\forall t \ge 0, \quad \mathbb{P}(\lambda_\epsilon T_\epsilon \le t) = \mathbb{E}[\mathbb{P}(\lambda_\epsilon T_\epsilon \le t \mid Y_0)] \underset{\epsilon\to 0}{\longrightarrow} \mathbb{P}(\mathcal{E} \le t).$$

□

Proof of Theorem 1.1.2. Due to Proposition 1.3 and Lemma 1.3.4, we have that $\left(\frac{R_n}{n^2}\right)_{n\ge 0}$ and $(\lambda_\epsilon T_\epsilon)_\epsilon$ converge in distribution to $|\mathcal{N}|^{-2}$ and $\mathcal{E}$ respectively.
Let us prove that the family of couples $\left(\lambda_\epsilon T_\epsilon, \frac{R_{T_\epsilon}}{T_\epsilon}\right)_{\epsilon>0}$ converges in distribution, as $\epsilon \to 0$, to $(\mathcal{E}, |\mathcal{N}|^{-2})$, where $\mathcal{E}$ and $\mathcal{N}$ are assumed to be as above and independent.
Let $s > 0$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}\mathbb{P} &\left(\lambda_\epsilon T_\epsilon > s, \frac{R_{T_\epsilon}}{T_\epsilon^2} > t\right) - \mathbb{P}\left(\lambda_\epsilon T_\epsilon > s\right)\mathbb{P}\left(|\mathcal{N}|^{-2} > t\right) \\ &\le \sum_{n > \frac{s}{c\epsilon^d}} \mathbb{P}\left(T_\epsilon = n, \frac{R_n}{n^2} > t\right) - \mathbb{P}(T_\epsilon = n)\mathbb{P}(|\mathcal{N}|^{-2} > t) \\ &= \sum_{n > \frac{s}{c\epsilon^d}} \mathbb{P}(T_\epsilon = n)\left(\mathbb{P}\left(\frac{R_n}{n^2} > t\right) - \mathbb{P}(|\mathcal{N}|^{-2} > t)\right) \\ &= \mathbb{E}\left[\sum_{n > \frac{s}{c\epsilon^d}} \lambda_\epsilon (1-\lambda_\epsilon)^{n-1}\left(\mathbb{P}\left(\frac{R_n}{n^2} > t\right) - \mathbb{P}(|\mathcal{N}|^{-2} > t)\right)\right].\end{aligned}$$

Hence, $|\mathbb{P}(\lambda_\epsilon T_\epsilon > s, \frac{R_{T_\epsilon^2}}{T_\epsilon} > t) - \mathbb{P}(\lambda_\epsilon T_\epsilon > s)\mathbb{P}(|\mathcal{N}|^{-2} > t)| \leq \sup_{n > \frac{s}{c\epsilon^d}} |\mathbb{P}(\frac{R_n}{n^2} > t) - \mathbb{P}(|\mathcal{N}|^{-2} > t)|$ which goes to 0 as $\epsilon \to 0$, due to Proposition 1.3. Moreover, we have $\mathbb{P}(\lambda_\epsilon T_\epsilon > s) \to \mathbb{P}(\mathcal{E} > s)$ as $\epsilon \to 0$, hence we get

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\lambda_\epsilon T_\epsilon > s, \frac{R_{T_\epsilon^2}}{T_\epsilon} > t\right) - \mathbb{P}(\mathcal{E} > s, |\mathcal{N}|^{-2} > t) = 0.$$

Now, if $s \leq 0$, we have $\mathbb{P}(\lambda_\epsilon T_\epsilon > 0, \frac{R_{T_\epsilon^2}}{T_\epsilon^2} > t) - \mathbb{P}(\mathcal{E} > 0, |\mathcal{N}|^{-2} > t)$ is equal to the following series of equalities:

$$\begin{aligned}
\mathbb{P}\left(\frac{R_{T_\epsilon^2}}{T_\epsilon^2} > t\right) - \mathbb{P}(|\mathcal{N}|^{-2} > t) &= \mathbb{E}\left[\sum_{n\geq 1} \lambda_\epsilon (1-\lambda_\epsilon)^{n-1} \mathbb{P}\left(\frac{R_n}{n^2} > t\right) - \mathbb{P}(|\mathcal{N}|^{-2} > t)\right] \\
&= \mathbb{E}\left[\sum_{n\geq 1} \lambda_\epsilon (1-\lambda_\epsilon)^{n-1} \left(\mathbb{P}\left(\frac{R_n}{n^2} > t\right) - \mathbb{P}(|\mathcal{N}|^{-2} > t)\right)\right].
\end{aligned}$$

Let $\alpha > 0$ and $n_0 \in \mathbb{N}$ be such that $\forall n \geq n_0$, we have $\left|\mathbb{P}\left(\frac{R_n}{n^2} > t\right) - \mathbb{P}(|\mathcal{N}|^{-2} > t)\right| < \alpha$. Then, we get

$$\begin{aligned}
\left|\mathbb{P}\left(\lambda_\epsilon T_\epsilon > 0, \frac{R_{T_\epsilon^2}}{T_\epsilon^2} > t\right) - \mathbb{P}(\mathcal{E} > 0, |\mathcal{N}|^{-2} > t)\right| &\leq \mathbb{E}\Bigg[\sum_{n=1}^{n_0} \lambda_\epsilon (1-\lambda_\epsilon)^{n-1} \\
&\qquad + \sum_{n\geq n_0} \lambda_\epsilon (1-\lambda_\epsilon)^{n-1}(\alpha)\Bigg] \\
&\leq \mathbb{E}\left[\lambda_\epsilon n_0 + \alpha\right] \leq c\epsilon^d n_0 + \alpha.
\end{aligned}$$

Hence, for all $\alpha > 0$

$$\limsup_{\epsilon \to 0} \left|\mathbb{P}\left(\lambda_\epsilon T_\epsilon > 0, \frac{R_{T_\epsilon}}{T_\epsilon^2} > t\right) - \mathbb{P}(\mathcal{E} > 0, |\mathcal{N}|^{-2} > t)\right| \leq \alpha.$$

Thus, we conclude that:

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\lambda_\epsilon T_\epsilon > 0, \frac{R_{T_\epsilon}}{T_\epsilon^2} > t\right) - \mathbb{P}(\mathcal{E} > 0, |\mathcal{N}|^{-2} > t) = 0.$$

And by this, we proved that:

$$\left(\lambda_\epsilon T_\epsilon, \frac{R_{T_\epsilon}}{T_\epsilon}\right)_{\epsilon > 0} \xrightarrow[\epsilon \to 0]{dist.} (\mathcal{E}, |\mathcal{N}|^{-2}) \tag{1.3.1}$$

Now, using the relation $\tau_\epsilon = R_{T_\epsilon}$, we get

$$(c\epsilon^d)^2\tau_\epsilon = \left(\frac{c\epsilon^d}{\lambda_\epsilon}\right)^2 \lambda_\epsilon^2 R_{T_\epsilon} = \left(\frac{c\epsilon^d}{\lambda_\epsilon}\right)^2 \lambda_\epsilon^2 T_\epsilon^2 \frac{R_{T_\epsilon}}{T_\epsilon^2}$$

and since $(x, y) \mapsto x^2 y$ is continuous and using the convergence in distribution in (1.3.1), we get

$$\lambda_\epsilon^2 T_\epsilon^2 \frac{R_{T_\epsilon}}{T_\epsilon^2} \overset{dist.}{\to} \mathcal{E}|\mathcal{N}|^{-2}, \qquad \text{as } \epsilon \to 0.$$

Moreover, we have $\left(\frac{c\epsilon^d}{\lambda_\epsilon}\right)^2$ converges almost surely to 1, hence using Slutzky's Lemma, we deduce that

$$(c\epsilon^d)^2\tau_\epsilon \overset{dist.}{\to} \mathcal{E}|\mathcal{N}|^{-2}, \qquad \text{as } \epsilon \to 0.$$

□

Chapter 2

Local Limit Theorem with speed for subshift of finite type

Let us fix a finite set $\mathcal{A}$ ($\#A \geq 2$) called the alphabet. Let us consider a matrix M indexed by $\mathcal{A} \times \mathcal{A}$ with 0-1 entries. We suppose that there exists a positive integer n_0 such that each entry of M^{n_0} is non zero. We define the set of the allowed sequences Σ as follows

$$\Sigma := \{w := (w_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} : \forall n \in \mathbb{Z}, M(w_n, w_{n+1}) = 1\}$$

We endow Σ with the metric d given by

$$d(w, w^{'}) := e^{-m},$$

where m is the greatest integer such that $w_i = w_i^{'}$ whenever $|i| < m$.
We define the shift $\theta : \Sigma \to \Sigma$ by $\theta((w_n)_{n\in\mathbb{Z}}) = (w_{n+1})_{n\in\mathbb{Z}}$.

Definition. Let $q, q^{'}$ be two positive integers, and $a_{-q}, ...a_0, ...a_q$ be a finite symbol sequence. We define a $(q, q^{'})$-cylinder by

$$C_{a_{-q},a_{q^{'}}} = \{(w_n)_{n\in\mathbb{Z}} \in \Sigma : w_{-q} = a_{-q}, ..., w_{q^{'}} = a_{q^{'}}\}.$$

We will denote a $(q, q^{'})-$cylinder containing a point $x \in \Sigma$ by $C_{q,q^{'}}(x)$.

Proposition 2.0.5. *Let ν be a Gibbs measure associated to a Hölder continuous potential h. There exists $K_G \geq 1$ such that for every $C_{q,q^{'}}$ $(q, q^{'})-$cylinder, for every $x \in C_{q,q^{'}}$, we have*

$$\frac{1}{K_G} \leq \frac{\nu(C_{q,q^{'}}(x))}{\exp\left(\sum_{k=-q}^{q^{'}} \varphi \circ T^k(x)\right)} \leq K_G. \tag{2.0.1}$$

In all what follows in this chapter and the chapter that follows, ν will be a Gibbs measure associated to a Hölder potential h.

Definition. For any Hölder function $f : \Sigma \to \mathbb{R}$, such that $\int f d\nu = 0$, we define its ergodic sum by

$$S_n f = \sum_{l=0}^{n-1} f \circ \theta^l$$

Definition. Let us denote by σ_f^2 the asymptotic variance of a function f:

$$\sigma_f^2 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[(S_n f)^2]$$

Note that this limit is well defined when f is Hölder and it's given by $\sigma_f^2 \sum_{n\in\mathbb{Z}} \mathbb{E}(f.f\circ \theta^k)$. Moreover, let σ_f^2 the asymptotic variance of f under the measure ν. Recall that σ_f^2 vanishes if and only if f is cohomologous to a constant. This is true for every Hölder function f, and in this case ν is the unique measure of maximal entropy.
Let us consider a Hölder continuous function $\varphi : \Sigma \to \mathbb{Z}$, such that $\int \varphi d\nu = 0$. We can easily prove that there exists a positive integer m_0 such that the function φ is constant on each m_0-cylinders.

Definition. We say that φ is non-arithmetic if for any $u \in [-\pi; \pi]$ the only solutions (λ, g), with $\lambda \in \mathbb{C}$ and $g : \Sigma \to \mathbb{C}$ measurable with $|g| = 1$, of the functional equation

$$g \circ \sigma \bar{g} = \lambda e^{iu.\varphi} \tag{2.0.2}$$

is the trivial one $\lambda = 1$, $u = 0$ and $g = const$.

From now on, we assume that φ is non-arithmetic. In particular, $\sigma_\varphi^2 \neq 0$. This will be discussed in Chapter 3, where we study the case of $\mathbb{Z}-$extension of subshift of finite type.

2.1 Spectral Analysis of the Perron-Frobenius operator.

We define the one-sided shift

$$\hat{\Sigma} := \{w := (w_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} : \forall n \in \mathbb{N}, M(w_n, w_{n+1}) = 1\}.$$

As we did for Σ, we endow $\hat{\Sigma}$ with the metric $\hat{d}$ defined by

$$\hat{d}((w_n)_{n\geq 0},(w_n^{'})_{n\geq 0}) := e^{-\hat{r}(w,w^{'})}$$

with $\hat{r}((w_n)_{n\geq 0},(w_n^{'})_{n\geq 0}) = \inf\{m \geq 0 : w_m \neq w_m^{'}\}$. Moreover, the shift $\hat{\theta}$ is the restriction on $\hat{\Sigma}$ of the one-sided shift defined by

$$\hat{\theta}((w_n)_{n\geq 0}) = (w_{n+1})_{n\geq 0}.$$

Remark 2.1.1. For any Hölder function $\tilde{f}$ defined on $\hat{\Sigma}$, $\tilde{f} : \hat{\Sigma} \to \mathbb{R}$, such that $\int \tilde{f} d\hat{\nu} = 0$, we denote its ergodic sum by

$$\hat{S}_n \tilde{f} = \sum_{l=0}^{n-1} \tilde{f} \circ \hat{\theta}^l.$$

Let us define the canonical projection $\Pi : \Sigma \to \hat{\Sigma}$ defined by $\Pi((w_n)_{n\in\mathbb{Z}}) = (w_n)_{n\geq 0}$. Let $\hat{\nu}$ be the image probability measure (on $\hat{\Sigma}$) of ν by Π. Let $\omega \in \Sigma$. Since φ is constant on the m_0-cylinder, $\varphi \circ \theta^{m_0}\omega$ depends only on $\Pi\omega$, then there exists a function $\psi : \hat{\Sigma} \to \mathbb{Z}$ such that $\psi \circ \Pi = \varphi \circ \theta^{m_0}$.
Let us denote by $P : L^2(\hat{\nu}) \to L^2(\hat{\nu})$ the Perron-Frobenius operator such that:

$$\forall f, g \in L^2(\hat{\nu}), \int_{\hat{\Sigma}} Pf(x)g(x)d\hat{\nu}(x) = \int_{\hat{\Sigma}} f(x)g \circ \hat{\theta}(x)d\hat{\nu}(x).$$

Let $\eta \in]0;1[$. Let us denote by $\mathcal{B}$ the set of bounded η-Hölder continuous function $g : \hat{\Sigma} \to \mathbb{C}$ endowed with the usual Hölder norm :

$$||g||_{\mathcal{B}} := ||g||_\infty + \sup_{x\neq y} \frac{|g(y)-g(x)|}{\hat{d}(x,y)^\eta}.$$

We denote by $\mathcal{B}^*$ the topological dual of $\mathcal{B}$. For all $u \in \mathbb{R}$, we consider the operator P_u defined on $(\mathcal{B}, ||.||_{\mathcal{B}})$ by :

$$P_u(f) := P(e^{iu\psi} f).$$

Note that the hypothesis of non-arithmeticity of φ is equivalent to the following one on ψ: for any $u \in [-\pi;\pi]\backslash\{0\}$, the operator P_u has no eigenvalue on the unit circle.
We will use the method introduced by Nagaev in [33][34], adapted by Guivarc'h and Hardy [25] and extended by Hennion and Hervé in [27]. It is based on the family of operators $(P_u)_u$ and their spectral properties expressed in the two next propositions.

Proposition 2.1.2. *(Uniform Contraction). For all $\beta > 0$, there exist $\alpha \in (0;1)$ and $C > 0$ such that, for all $u \in [-\pi;\pi]\backslash[-\beta;\beta]$ and all integers $n \geq 0$, for all $f \in \mathcal{B}$, we have:*

$$||P_u^n(f)||_{\mathcal{B}} \leq C\alpha^n ||f||_{\mathcal{B}}. \tag{2.1.1}$$

This proposition follows from the fact the the spectral radius is smaller than 1 for $u \neq 0$, thanks to Lemma 4.3 in [1]. In addition, since P is a quasicompact operator on $\mathcal{B}$ and since $u \mapsto P_u$ is a regular perturbation of $P_0 = P$, we have:

Proposition 2.1.3. *(Perturbation Result). There exist $\alpha > 0, \beta > 0, C > 0, c_1 > 0, \theta \in]0;1[$ such that: there exists $u \mapsto \lambda_u$ belonging to $C^3([-\beta;\beta] \to \mathbb{C})$, there exists $u \mapsto v_u$ belonging to $C^3([-\beta;\beta] \to \mathcal{B})$, there exists $u \mapsto \phi_u$ belonging to $C^3([-\beta;\beta] \to \mathcal{B}^*)$ such that, for all $u \in [-\beta;\beta]$, for all $f \in \mathcal{B}$ and for all $n \geq 0$, we have the decomposition:*

$$P_u^n(f) = \lambda_u^n \phi_u(f) v_u + N_u^n(f),$$

and the following properties

1. *$||N_u^n(f)||_{\mathcal{B}} \leq C\alpha^n ||f||_{\mathcal{B}}$,*
2. *$|\lambda_u| \leq e^{-c_1 u^2}$ and $c_1 u^2 \leq \sigma_\varphi^2 u^2$,*
3. *with initial values : $v_0 = 1, \phi_0 = \hat{\nu}, \lambda'_{u=0} = 0$ and $\lambda''_{u=0} = -\sigma_\varphi^2$.*

Lemma 2.1.4. *There exists $C_\alpha > 0$ such that, for every non-negative integers q, q', such that $q \geq m_0$ and $q' \leq k$, and for every $(q+q')$-cylinder $\hat{A}$ of $\hat{\Sigma}$, we have:*

$$\forall u \in [-\pi, \pi], \quad ||P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})||_{\mathcal{B}} \leq C_\alpha \hat{\nu}(\hat{A}) \tag{2.1.2}$$

Proof.

$$\begin{aligned}
P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})(y) &= P_u^{q'} P^{q-m_0}(1_{\hat{A}})(y) \\
&= P^{q'} \left(e^{iu\hat{S}_{q'}\psi} P^{q-m_0}(1_{\hat{A}}) \right)(y) \\
&= \sum_{z:\hat{\theta}^{q'} z = y} e^{S_{q'} h(z)} e^{iu\hat{S}_{q'}\psi(z)} \left(P^{q-m_0}(1_{\hat{A}}) \right)(z) \\
&= \sum_{w:\hat{\theta}^{q+q'-m_0} w = y} e^{S_{q-m_0} h(w)} 1_{\hat{A}}(w) e^{\hat{S}_{q'} h \circ \hat{\theta}^{q-m_0}(w)} e^{iu\hat{S}_{q'}\psi \circ \hat{\theta}^{q-m_0}(w)} \\
&= \sum_{w:\hat{\theta}^{q+q'-m_0} w = y} e^{S_{q+q'-m_0} h(w)} 1_{\hat{A}}(w) e^{iu\hat{S}_{q'}\psi \circ \hat{\theta}^{q'-m_0}(w)}.
\end{aligned}$$

Thus,

$$P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})(y) = 1_{[\hat{\theta}^{q-m_0}\hat{A}]}(y) e^{S_{q+q'-m_0} h(w_y)} e^{iu\hat{S}_{q'}\psi \circ \hat{\theta}^{q-m_0}(w_y)},$$

where w_y is the element of $\hat{A}$ such that $\hat{\theta}^{q+q'-m_0} w_y = y$.
We first compute the following norm $||.||_\infty$ using Proposition 2.0.5, with the notation K_G introduced therein, we have:

$$\begin{aligned}
||P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})||_\infty &\leq ||P^{q+q'}(1_{\hat{A}} \circ \hat{\theta}^{m_0})||_\infty = ||P^{q+q'-m_0}(1_{\hat{A}})||_\infty \\
&\leq \sup_{x \in \hat{\Sigma}} e^{S_{q+q'-m_0} h(w_x)} \\
&= \sup_{x \in \hat{\Sigma}} e^{S_{q+q'} h(w_x) - \sum_{k=q+q'-m_0}^{q+q'-1} h \circ \hat{\theta}^k(w_x)} \\
&\leq \sup_{x \in \hat{\Sigma}} e^{S_{q+q'} h(w_x)} e^{m_0 ||h||_\infty} \\
&\leq e^{m_0||h||_\infty} K_G \hat{\nu}(\hat{A}).
\end{aligned}$$

Setting $c_G := e^{m_0||h||_\infty} K_G$, we conclude that

$$||P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})||_\infty \leq c_G \hat{\nu}(\hat{A}) \tag{2.1.3}$$

Now we compute the Hölder norm. Let x, $y \in \hat{\Sigma}$, $x \neq y$. Then $\hat{d}(x, y) = e^{-n}$ for some $n \geq 0$, we have the following two cases:

- 1^{st} case: if $n > m_0$, we have

$$x \in \hat{\theta}^{q+q'-m_0}\hat{A} \Leftrightarrow y \in \hat{\theta}^{q+q'-m_0}\hat{A}$$

  - either $x \notin \hat{\theta}^{q+q'-m_0}\hat{A}$, then $y \notin \hat{\theta}^{q+q'-m_0}\hat{A}$, and hence

$$|P_u^{q'} P^{q-m_0}(1_{\hat{A}}(y)) - P_u^{q'} P^{q-m_0}(1_{\hat{A}}(x))| = 0.$$

  - or $x, y \in \hat{\theta}^{q+q'-m_0}\hat{A}$, $\hat{d}(w_x, w_y) = q + q' - m_0 + n$,

$$\begin{aligned}
|P_u^q P^{q-m_0}(1_{\hat{A}}(y)) - P_u^{q'} P^{q-m_0}(1_{\hat{A}}(x))| &= |e^{S_{q+q'-m_0} h(w_y) + iu\hat{S}_{q'}\psi \circ \hat{\theta}^{q-m_0}(w_y)} \\
&\quad - e^{S_{q+q'-m_0} h(w_x) + iu\hat{S}_{q'}\psi \circ \hat{\theta}^{q-m_0}(w_x)}|
\end{aligned}$$

Let us denote for simplicity $F_{h,\psi} = e^{S_{q+q'-m_0}h(.)-iu\hat{S}_{q'}\psi\circ\hat{\theta}^{q-m_0}(.)}$. We have

$$\begin{aligned}
\left|S_{q+q'-m_0}h(w_y) - S_{q+q'-m_0}h(w_x)\right| &= \left|\sum_{i=0}^{q+q'-m_0-1} h\circ\hat{\theta}^i(w_y) - h\circ\hat{\theta}^i(w_x)\right| \\
&\leq \sum_{i=0}^{q+q'-m_0-1} |h|_\alpha e^{-\alpha(q+q'-m_0+n-i)} \\
&\leq \left(\sum_{i=0}^{q+q'-m_0-1} e^{-\alpha(q+q'-m_0-i)}\right) e^{-\alpha n}|h|_\alpha \\
&\leq \left(\sum_{j\geq 1} e^{-\alpha j}\right) e^{-\alpha n}|h|_\alpha \\
&\leq c|h|_\alpha \hat{d}^\alpha(x,y), \qquad (2.1.4)
\end{aligned}$$

where c is a constant such that $\sum_{j\geq 1} e^{-\alpha j} \leq c < \infty$.

$$\begin{aligned}
\left|\hat{S}_{q'}\psi(\hat{\theta}^{q-m_0}(w_y)) - \hat{S}_{q'}\psi(\hat{\theta}^{q-m_0}(w_x))\right| &= \left|\sum_{i=0}^{q'-1} \psi(\hat{\theta}^{q-m_0+i}(w_y)) - \psi(\hat{\theta}^{q-m_0+i}(w_x))\right| \\
&\leq \sum_{i=0}^{q-1} |\psi|_\alpha e^{-\alpha(q+q'-m_0+n-q+m_0-i)} \\
&\leq |\psi|_\alpha e^{-\alpha n}\sum_{j\geq 1} e^{-\alpha j} \\
&\leq c|\psi|_\alpha \hat{d}^\alpha(x,y). \qquad (2.1.5)
\end{aligned}$$

Thus, using the equations (2.1.3),(2.1.4) and (2.1.5), we get:

$$\begin{aligned}
|P_u^{q'}P^{q-m_0}(1_{\hat{A}}(y)) - P_u^{q'}P^{q-m_0}(1_{\hat{A}}(x))| &= |\ e^{F_{h,\psi}(w_y)} - e^{F_{h,\psi}(w_x)}\ | \\
&\leq \max\left(e^{F_{h,\psi}(.)}\right)\ |\ F_{h,\psi}(w_y) - F_{h,\psi}(w_x)\ | \\
&\leq e^{c(|h|_\alpha+|\psi|_\alpha)\hat{d}^\alpha(x,y)} c(|h|_\alpha + |\psi|_\alpha) \\
&\times\ \hat{d}^\alpha(x,y)||P^{q+q'}(1_{\hat{A}}\circ\hat{\theta}^{m_0})||_\infty \\
&\leq e^{c(|h|_\alpha+|\psi|_\alpha)} c(|h|_\alpha + |\psi|_\alpha) \times \hat{d}^\alpha(x,y)c_G\hat{\nu}(\hat{A})
\end{aligned}$$

- 2^{nd} case: Whenever $n \leq m_0$, if $x \in \hat{\theta}^{q+q'-m_0}\hat{A}$, then $y \notin \hat{\theta}^{q+q'-m_0}\hat{A}$, thus using (2.1.3)

$$\begin{aligned} |P_u^{q'} P^{q-m_0}(1_{\hat{A}})(y) - P_u^{q'} P^{q-m_0}(1_{\hat{A}})(x)| &\leq \sup_{\hat{\Sigma}} |P_u^{q'} P^{q-m_0}(1_{\hat{A}})(.)| \\ &\leq ||P_u^{q'} P^{q-m_0}(1_{\hat{A}})||_\infty \\ &\leq c_G \hat{\nu}(\hat{A}) \leq c_G \hat{\nu}(\hat{A}) e^{\alpha n} e^{-\alpha n} \\ &\leq c_G \hat{\nu}(\hat{A}) e^{\alpha m_0} \hat{d}^\alpha(x,y) \end{aligned}$$

Thus, we get, $\forall n \geq 0$:

$$|P_u^{q'} P^{q-m_0}(1_{\hat{A}})|_\alpha \leq c_G \max\left(e^{\alpha m_0}, e^{c(|h|_\alpha + |\psi|_\alpha)} c(|h|_\alpha + |\psi|_\alpha)\right) \hat{\nu}(\hat{A})$$

Combining this with (2.1.3), we obtain

$$||P_u^{q'} P^q (1_{\hat{A}} \circ \hat{\theta}^{m_0})||_{\mathcal{B}} \leq C_\alpha \hat{\nu}(\hat{A})$$

where $C_\alpha = K_G(1 + \max\left(e^{\alpha m_0}, e^{c(|h|_\alpha + |\psi|_\alpha)} c(|h|_\alpha + |\psi|_\alpha)\right))$. □

2.2 Proof of the Local Limit Theorem

Next proposition is very essential in this work. It may be viewed as a doubly local version of the central limit theorem: first, it is local in the sense that we are looking at the probability that $S_n\varphi = 0$ while the classical central limit theorem is only concerned with the probability that $|S_n\varphi| \leq \epsilon\sqrt{n}$; second, it is local in the sense that we are looking at this probability conditioned to the fact that we are starting from a set A and landing on a set B.

Proposition 2.2.1. *There exist real numbers $C_1 > 0$ and $\gamma > 0$ such that, for all integers n, q, q', k such that $n - 2k \geq m_0$ and $m_0 < q \leq k$, for all (q, q')-cylinders A of Σ and all measurable subset B of $\hat{\Sigma}$, we have:*

$$\left| \nu\left(A \cap \{S_n\varphi = 0\} \cap \theta^{-n}(\theta^k(\Pi^{-1}(B)))\right) - \frac{\nu(A)\hat{\nu}(B)}{\sqrt{2\pi}\sqrt{n-k}\sigma_\varphi} \right| \leq C_1 \frac{\hat{\nu}(B) k \nu(A)}{n - 2k}.$$

Proof. We want to estimate the measure of the set $Q = A \cap \{S_n\varphi = 0\} \cap \theta^{-n}(\theta^k \Pi^{-1}(B))$. Since A is a (q, q')-cylinder, $\theta^{-q} A = \Pi^{-1}\hat{A}$ for the $q + q'$-cylinder set $\hat{A} = \Pi\theta^{-q} A$.

Next, since $\varphi \circ \theta^{m_0} = \psi \circ \Pi$ and using the semi-conjugacy $\hat{\theta} \circ \Pi = \Pi \circ \theta$, we have the identity:

$$\{S_n \varphi \circ \theta^{m_0} = 0\} = \{\hat{S}_n \psi \circ \Pi = 0\}.$$

Thus, now we have

$$\begin{aligned} \theta^{-q-m_0} Q &= \theta^{-q-m_0}(A \cap \{S_n \varphi = 0\} \cap \theta^{-n}(\theta^k \Pi^{-1}(B))) \\ &= \Pi^{-1} \left(\hat{\theta}^{-m_0} \hat{A} \cap \{\hat{S}_n \psi \circ \hat{\theta}^q = 0\} \cap \hat{\theta}^{-n-q+(k-m_0)}(B) \right). \end{aligned}$$

Since ψ is integer-valued, the relation $1_{\{k=0\}} = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{iu.k} du$, for any $k \in \mathbb{Z}$, gives

$$\begin{aligned} 1_{\theta^{-q-m_0} Q} &= \left(1_{\hat{A}} \circ \hat{\theta}^{m_0} . 1_B \circ \hat{\theta}^{q+n-(k-m_0)} . 1_{\{\hat{S}_n \psi \circ \hat{\theta}^q = 0\}} \right) \circ \Pi \\ &= \left(1_{\hat{A}} \circ \hat{\theta}^{m_0} . 1_B \circ \hat{\theta}^{q+n-(k-m_0)} . \frac{1}{2\pi} \int e^{iu.\hat{S}_n \psi \circ \hat{\theta}^q} du \right) \circ \Pi. \end{aligned}$$

By integrating both sides with respect to ν and by using the θ-invariance of ν, we get :

$$\begin{aligned} \nu(Q) &= \nu \left(\theta^{-q-m_0} Q \right) \\ &= \mathbb{E}_\nu \left[\left(1_{\hat{A}} \circ \hat{\theta}^{m_0} . 1_B \circ \hat{\theta}^{q+n-(k-m_0)} . \frac{1}{2\pi} \int e^{iu.\hat{S}_n \psi \circ \hat{\theta}^q} du \right) \circ \Pi \right] \\ &= \mathbb{E}_{\hat{\nu}} \left[1_{\hat{A}} \circ \hat{\theta}^{m_0} . 1_B \circ \hat{\theta}^{q+n-(k-m_0)} . \frac{1}{2\pi} \int e^{iu.\hat{S}_n \psi \circ \hat{\theta}^q} du \right] \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]} \mathbb{E}_{\hat{\nu}} \left(1_{\hat{A}} \circ \hat{\theta}^{m_0} . 1_B \circ \hat{\theta}^{q+n-(k-m_0)} e^{iu.\hat{S}_n \psi \circ \hat{\theta}^q} \right) du. \end{aligned}$$

We then estimate the expectation $a(u) = \mathbb{E}_{\hat{\nu}}(...)$. Using the fact that the Perron-Frobenius P is the dual of $\hat{\theta}$, we have

- $P^q(g \times f \circ \hat{\theta}^q) = P^q(g) \times f$
- $\int P^a f d\hat{\nu} = \int f d\hat{\nu}$, for all $a \in \mathbb{Z}$
- $P_u^n(f) = P^n(e^{iu.\hat{S}_n \psi} f)$
- $P_u^n \left(f \times g \circ \hat{\theta}^n \right) = P_u^n(f) \times g.$

Hence we get

$$\begin{aligned}
a(u) &= \mathbb{E}_{\hat\nu}\left(1_{\hat A}\circ\hat\theta^{m_0}.1_B\circ\hat\theta^{q+n-(k-m_0)}e^{iu.\hat S_n\psi\circ\hat\theta^q}\right)\\
&= \mathbb{E}_{\hat\nu}\left(P^q(1_{\hat A}\circ\hat\theta^{m_0})\exp(iu.\hat S_n\psi)1_B\circ\hat\theta^{n-(k-m_0)}\right)\\
&= \mathbb{E}_{\hat\nu}\left(P^n\left(P^q(1_{\hat A}\circ\hat\theta^{m_0})\exp(iu.\hat S_n\psi)1_B\circ\hat\theta^{n-(k-m_0)}\right)\right)\\
&= \mathbb{E}_{\hat\nu}\left(P_u^n\left(P^q(1_{\hat A}\circ\hat\theta^{m_0})1_B\circ\hat\theta^{n-(k-m_0)}\right)\right)\\
&= \mathbb{E}_{\hat\nu}\left(P_u^{k-m_0}(1_BP_u^{n-(k-m_0)}P^q(1_{\hat A}\circ\hat\theta^{m_0}))\right).
\end{aligned}$$

Let us denote for simplicity $l=n-(k-m_0)-q$. We first show that for large u, the quantity $a(u)$ is negligible. Using the contraction inequality given in Proposition 2.1.2 applied to $P_u^l(1)$, the fact that $||P_u^{q'}P^q(1_{\hat A}\circ\theta^{m_0})||_{\mathcal B}\le C_\alpha\nu(A)$ from Lemma 2.1.4, and the fact that $\left|\mathbb{E}\left(P_u^{k-m_0}(1_Bg)\right)\right|\le\hat\nu(B)||g||_{\mathcal B}$ we get whenever $u\notin[-\beta,\beta]$,

$$\begin{aligned}
|a(u)| &= \left|\mathbb{E}_{\hat\nu}\left(P_u^{k-m_0}(1_BP_u^{n-(k-m_0)-q'}P_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0}))\right)\right|\\
&\le \hat\nu(B)||P_u^lP_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0})||_{\mathcal B}\\
&\le \hat\nu(B)C\alpha^l||P_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0})||_{\mathcal B}\\
&\le \hat\nu(B)C\alpha^lC_\alpha\nu(A)\\
&= O\left(\hat\nu(B)\nu(A)\right).
\end{aligned}$$

We then estimate the main term, coming from small values of u. The decomposition given in Proposition 2.1.3 gives for any $u\in[-\beta,\beta]$:

$$\begin{aligned}
a(u) &= \mathbb{E}_{\hat\nu}\left(P_u^{k-m_0}(1_BP_u^lP_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0}))\right)\\
&= \mathbb{E}_{\hat\nu}\left(P_u^{k-m_0}(1_B\lambda_u^l\varphi_u(P_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0})))v_u+N_u^l(P_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0}))\right)\\
&= \lambda_u^l\varphi_u(P_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0}))\mathbb{E}_{\hat\nu}\left(P_u^{k-m_0}(1_Bv_u)\right)+\mathbb{E}_{\hat\nu}\left(P_u^{k-m_0}(1_BN_u^l(P_u^{q'}P^q(1_{\hat A}\circ\hat\theta^{m_0}))\right)\\
&= a_1(u)+a_2(u).
\end{aligned}$$

Notice that the second term is, by inequality (1) in Proposition 2.1.3, of order

$$a_2(u)=O(\hat\nu(B)\alpha^l\nu(A)). \tag{2.2.1}$$

because,

$$\begin{aligned}
|a_2(u)| &= \left|\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B N_u^l(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0})))\right)\right| \\
&\leq \mathbb{E}_{\hat{\nu}}\left(|P_u^{k-m_0}(1_B N_u^l(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0})))|\right) \\
&\leq \mathbb{E}_{\hat{\nu}}\left(P^{k-m_0}(|1_B N_u^l(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0}))|)\right) \\
&\leq \mathbb{E}_{\hat{\nu}}\left(|1_B N_u^l(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0}))|\right) \\
&\leq \hat{\nu}(B)||N_u^l(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0}))||_{\mathcal{B}} \\
&\leq C\alpha^l\hat{\nu}(B)||P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0})||_{\mathcal{B}} \\
&\leq C\alpha^l\hat{\nu}(B)C_\alpha\nu(A) \\
&= O(\hat{\nu}(B)\alpha^l\nu(A)).
\end{aligned}$$

Moreover, $u \mapsto v_u$ and $u \mapsto \varphi_u$ are C^1-regular with $v_0 = 1$ and $\varphi_0 = \hat{\nu}$, then:

$$v_u = v_0 + O(u) = 1 + O(u) \quad \in \mathcal{B}.$$

and

$$\varphi_u = \varphi_0 + O(u) = \hat{\nu} + O(u) \in \mathcal{B}^*.$$

hence, the first term has the estimate :

$$\begin{aligned}
a_1(u) &= \lambda_u^l\varphi_u(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0}))\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B v_u)\right) \\
&= \lambda_u^l\hat{\nu}(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0}))\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B v_u)\right) \\
&\quad +\lambda_u^l O\left(u||P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0})||_{\mathcal{B}}\right)\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B v_u)\right) \\
&= \lambda_u^l\hat{\nu}(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0}))\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B)\right) \\
&\quad +\lambda_u^l\hat{\nu}(P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0}))\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B O(u))\right) \\
&\quad +\lambda_u^l O\left(u||P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0})||_{\mathcal{B}}\right)\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B)\right) \\
&\quad +\lambda_u^l O\left(u||P_u^{q'}P^q(1_{\hat{A}}\circ\hat{\theta}^{m_0})||_{\mathcal{B}}\right)\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B O(u))\right) \\
&= a_1^1(u) + a_1^2(u) + a_1^3(u) + a_1^4(u).
\end{aligned}$$

We remind that $a_1^1(u)$ will be the principal term. Thus considering the other terms, we have

- $$\begin{aligned} |a_1^2(u)| &= |\lambda_u^l \hat{\nu}(P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0}))\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B O(u))\right)| \\ &\leq \lambda_u^l ||P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})||_{\mathcal{B}} \mathbb{E}_{\hat{\nu}}\left(|e^{iuS_{k-m_0}\psi} 1_B O(u)|\right) \\ &\leq \lambda_u^l \hat{\nu}(\hat{A})\hat{\nu}(B)O(|u|) \\ &= O(\hat{\nu}(\lambda_u^l \hat{A})\hat{\nu}(B)|u|). \end{aligned}$$

- $$\begin{aligned} |a_1^3(u)| &= \lambda_u^l O(|u|)||P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})||_{\mathcal{B}} \mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B)\right) \\ &= \lambda_u^l O(|u|)||(P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0}))||_{\mathcal{B}} |\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B)\right)| \\ &\leq \lambda_u^l \hat{\nu}(\hat{A})O(|u|)\hat{\nu}(B) \\ &= O\left(\lambda_u^l \nu(A)|u|\hat{\nu}(B)\right). \end{aligned}$$

- $$\begin{aligned} |a_1^4(u)| &= \lambda_u^l O(|u|)||P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0})||_{\mathcal{B}} \mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0} O(1_B u))\right) \\ &= \lambda_u^l O(|u|)||(P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0}))||_{\mathcal{B}} |\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0} O(1_B u))\right)| \\ &\leq \lambda_u^l O(|u|)\hat{\nu}(B)\hat{\nu}(\hat{A}) \\ &= O(\lambda_u^l |u|\hat{\nu}(B)\nu(A)) \end{aligned}$$

Now if we reintroduce the unperturbed Perron-Frobenius operator P in P_u, we get

$$\begin{aligned} \left|\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B)\right) - \hat{\nu}(B)\right| &= \left|\mathbb{E}_{\hat{\nu}}\left(P^{k-m_0}(e^{iu.\hat{S}_{k-m_0}\psi} 1_B)\right) - \hat{\nu}(P^{k-m_0} 1_B)\right| \\ &= |\mathbb{E}_{\hat{\nu}}\left(P^{k-m_0}(e^{iu.\hat{S}_{k-m_0}\psi} - 1)1_B\right)| \\ &\leq ||e^{iu.\hat{S}_{k-m_0}\psi} - 1||_\infty ||1_B||_{L^1(\hat{\nu})} \\ &\leq ||u.S_{k-m_0}\psi||_\infty \hat{\nu}(B) \\ &\leq |u|.(k-m_0)||\psi||_\infty \hat{\nu}(B). \end{aligned}$$

This gives us an estimation of the principal term $a_1^1(u)$. Therefore we can estimate the first term $a_1(u)$:

$$\begin{aligned}
a_1(u) &= \lambda_u^l \hat{\nu}(P_u^{q'} P^q(1_{\hat{A}} \circ \hat{\theta}^{m_0}))\mathbb{E}_{\hat{\nu}}\left(P_u^{k-m_0}(1_B)\right) + O(\lambda_u^l |u| \hat{\nu}(B)\hat{\nu}(\hat{A})) \\
&= \lambda_u^l \left[\hat{\nu}(\hat{A}) + O(\min(1, |u|q')\hat{\nu}(\hat{A}))\right]\left[\hat{\nu}(B) + O\left(\min(1, |u|(k-m_0))\hat{\nu}(B)\right)\right] \\
&\quad + O(\lambda_u^l |u| \hat{\nu}(B)\nu(A)) \\
&= \lambda_u^l \hat{\nu}(\hat{A})\hat{\nu}(B)\left(1 + O(|u|q')\right)(1 + O(|\ u\ |\ (k-m_0)) + O(\lambda_u^l |u| \hat{\nu}(B)\nu(A)) \\
&= \lambda_u^l \hat{\nu}(\hat{A})\hat{\nu}(B) + O(\lambda_u^l |u| \hat{\nu}(B)\hat{\nu}(A)k) + O(\lambda_u^l |u| \hat{\nu}(B)\nu(A)) \\
&= \lambda_u^l \hat{\nu}(\hat{A})\hat{\nu}(B) + O(\lambda_u^l |u| \hat{\nu}(B)k\nu(A)).
\end{aligned}$$

In addition, the intermediate value theorem yields, using C^3 smoothness of λ_u and Proposition 2.1.3 (the bounds 1 and initial value 3)

$$\begin{aligned}
|\lambda_u^l - e^{-\frac{l}{2}\sigma_\varphi^2 u^2}| &\leq l \max(|\lambda_u|, e^{-\frac{1}{2}\sigma_\varphi^2 u^2})^{l-1} |\lambda_u - e^{-\frac{1}{2}\sigma_\varphi^2 u^2}| \\
&\leq l(e^{-c_1|u|^2})^{l-1} |\lambda_u - e^{-\frac{1}{2}\sigma_\varphi^2 u^2}| \\
&= l e^{-c_1 l|u|^2} e^{c_1|u|^2} |(1 + 0 - \frac{1}{2}\sigma_\varphi^2 u^2 + O(u^3)) - (1 - \frac{1}{2}\sigma_\varphi^2 u^2 + O(u^4))| \\
&= l e^{-c_1 l|u|^2} e^{c_1|u|^2} O(|u|^3) \\
&\leq C_0 \left(l|u|^2 e^{-c_1 l|u|^2}\right) e^{c_1|u|^2}|u| \\
&\leq C_0 \left(2e^{-\frac{c_1}{2} l|u|^2}\right) e^{c_1|u|^2}|u| \\
&= O(e^{-c_2 l|u|^2}|u|), \qquad \text{for the constant } c_2 = c_1/2.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
a_1(u) &= e^{-\frac{l}{2}\sigma_\varphi^2 u^2}\hat{\nu}(\hat{A})\hat{\nu}(B) + O(\hat{\nu}(\hat{A})\hat{\nu}(B)e^{-c_2 l|u|^2}|u|) + O(\lambda_u^l |u| k \hat{\nu}(\hat{A})) \\
&= e^{-\frac{l}{2}\sigma_\varphi^2 u^2}\hat{\nu}(\hat{A})\hat{\nu}(B) + O(\hat{\nu}(B)\hat{\nu}(\hat{A})e^{-c_2 l|u|^2}|u|) \\
&\quad + O(e^{-c_1 u^2}\hat{\nu}(B)|u|k\hat{\nu}(\hat{A})) \\
&= e^{-\frac{l}{2}\sigma_\varphi^2 u^2}\hat{\nu}(\hat{A})\hat{\nu}(B) + O(\hat{\nu}(B)e^{-c_2 l|u|^2}|u|) \\
&\quad + O(e^{-c_1 u^2}\hat{\nu}(B)|u|k\nu(A)) \\
&= e^{-\frac{l}{2}\sigma_\varphi^2 u^2}\nu(A)\hat{\nu}(B) + O(e^{-c_2 l|u|^2}|u|\hat{\nu}(B)k\nu(A)).
\end{aligned}$$

and so

$$a_1(u) = e^{-\frac{l}{2}\sigma_\varphi^2 u^2}\nu(A)\hat{\nu}(B) + O(e^{-c_2 l|u|^2}|u|\hat{\nu}(B)k\nu(A)).$$

By the classical change of variable $v = u\sqrt{l}$ and the Gaussian integral, one can easily see that:

$$\begin{aligned}
\int_{[-\beta,\beta]} e^{-\frac{l}{2}\sigma_\varphi^2 u^2} du &= \frac{1}{\sqrt{l}} \int_{[-\beta\sqrt{l},\beta\sqrt{l}]} e^{-\frac{1}{2}\sigma_\varphi^2 v^2} dv \\
&= \frac{1}{\sqrt{l}} \int_{\mathbb{R}} e^{-\frac{1}{2}\sigma_\varphi^2 v^2} dv + O(l^{-1}) \\
&= \frac{\sqrt{2\pi}}{\sqrt{l}\sigma_\varphi} + O\left(\frac{1}{l}\right)
\end{aligned}$$

Proceeding similarly with the error term one gets as well:

$$\begin{aligned}
\int_{[-\beta,\beta]} |u| e^{-c_2 l |u|^2} du &= \frac{1}{l} \int_{[-\beta\sqrt{l},\beta\sqrt{l}]} |v| e^{-c_2 |v|^2} dv \\
&= O\left(\frac{1}{l}\right).
\end{aligned}$$

Combining these two computations gives, by integration of the approximation of $a_1(u)$ obtained above, that:

$$\begin{aligned}
\int_{[-\beta,\beta]} a_1(u) du &= \frac{\sqrt{2\pi}}{\sqrt{l\sigma_\varphi^2}} \nu(A)\hat{\nu}(B) + O\left(\frac{1}{l}\nu(A)\hat{\nu}(B)\right) \\
&= \frac{\sqrt{2\pi}}{\sqrt{l\sigma_\varphi^2}} \nu(A)\hat{\nu}(B) + O\left(\frac{\hat{\nu}(B)k\nu(A)}{l}\right).
\end{aligned}$$

From this main estimate and (2.2.1) and (2.2), and since $l = n-k+m_0-q \geq n-2k$ (as $q \leq k$ and $m_0 \geq 0$) it follows immediately that:

$$\begin{aligned}
\nu(Q) &= \frac{1}{2\pi} \int_{[-\pi,\pi]} a(u) du \\
&= \frac{1}{2\pi} \int_{[-\beta,\beta]} a(u) du + \frac{1}{2\pi} \int_{[-\pi,\pi]|[-\beta,\beta]} a(u) du \\
&= \frac{\sqrt{2\pi}}{2\pi} \frac{1}{\sqrt{l\sigma_\varphi^2}} \hat{\nu}(A)\hat{\nu}(B) + O\left(\frac{\hat{\nu}(B)k\nu(A)}{l}\right) + O(\hat{\nu}(B)\alpha^l \nu(A)) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{l\sigma_\varphi^2}} \hat{\nu}(A)\hat{\nu}(B) + O\left(\frac{\hat{\nu}(B)k\nu(A)}{l}\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n-k}\sigma_\varphi} \hat{\nu}(A)\hat{\nu}(B) + O\left(\frac{\hat{\nu}(B)k\nu(A)}{n-2k}\right).
\end{aligned}$$

□

Chapter 3

Recurrence for $\mathbb{Z}$-extension of subshift of finite type.

In this chapter, we adapt the same notations and definitions introduced in Chapter 2. We recall that (Σ, θ, ν) is a subshift of finite type, θ is the shift map, and ν is a Gibbs measure.

3.1 Description of the $\mathbb{Z}$-extension of a mixing subshift and statement of the results

Recall that for any function $f : \Sigma \to \mathbb{R}$, we denote by $S_n f = \sum_{l=0}^{n-1} f \circ \theta^l$ its ergodic sum. Let us consider a Hölder continuous function $\varphi : \Sigma \to \mathbb{Z}$, such that $\int \varphi d\nu = 0$. We define the $\mathbb{Z}$-extension F of the shift θ by

$$\begin{aligned} F : \Sigma \times \mathbb{Z} &\to \Sigma \times \mathbb{Z} \\ (x, m) &\to (\theta x, m + \varphi(x)). \end{aligned}$$

We want to know the time needed for a typical orbit starting at $(x, m) \in \Sigma \times \mathbb{Z}$ to return ϵ-close to the initial point after iterations of the map F. By the translation invariance we can assume that the orbit starts in the cell $m = 0$. More precisely, let

$$\tau_\epsilon(x) = \inf\{n \geq 1 : F^n(x, 0) \in B(x, \epsilon) \times \{0\}\}.$$

Observe that $F^n(x, m) = (\theta^n x, m + S_n\varphi(x))$, thus

$$\tau_\epsilon(x) = \inf\{n \geq 1 : S_n\varphi(x) = 0 \text{ and } d(\theta^n x, x) < \epsilon\}.$$

We are interested in dynamical systems preserving an infinite measure which are conservative and ergodic. Since our study corresponds to the $\mathbb{Z}$-extension of a subshift of finite type, we assume that φ is not cohomologous to a constant.
We reinforce this by the non-arithmeticity hypothesis on φ (see Definition 2). The fact that there is no non constant g satisfying (2.0.2) for $\lambda = 1$ ensures that φ is not a coboundary and so that $\sigma_\varphi^2 \neq 0$. The fact that there exists (λ, g) satisfying (2.0.2) with $\lambda \neq 1$ should mean that the range of $S_n\varphi$ is essentially contained in a sub-lattice of $\mathbb{Z}$; in this case we could just work on a sub-lattice and apply our result to the new $\mathbb{Z}$-extension. We emphasize that this non-arithmeticity condition is equivalent to the fact that all the circle extensions T_u defined by $T_u(x,t) = (\theta(x), t + u.\varphi(x))$ are weakly mixing for $u \in [-\pi;\pi]\backslash\{0\}$.
In this context we establish results of almost sure convergence and convergence in distribution for τ_ϵ, which are more difficult versions of Theorem 1.1.1 and Theorem 1.1.2, where we consider the ϵ-return time to a cylinder and apply the local limit theorem proved in chapter 2. We prove the following:

Theorem 3.1.1. *The sequence of random variables $\frac{\log\sqrt{\tau_\epsilon}}{-\log\epsilon}$ converges almost surely as $\epsilon \to 0$ to the Hausdorff dimension d of the measure ν.*

Theorem 3.1.2. *The sequence of random variables $\nu((B_\epsilon(.))\sqrt{\tau_\epsilon(.)}$ converges in distribution as $\epsilon \to 0$ to $\sigma_\varphi\frac{\mathcal{E}}{|\mathcal{N}|}$, where $\mathcal{E}$ and $\mathcal{N}$ are independent random variables, $\mathcal{E}$ having an exponential distribution of mean 1 and $\mathcal{N}$ having a standard Gaussian distribution.*

Corollary 3.1.3. *If the measure ν is not the measure of maximal entropy, then the sequence of random variables $\frac{\log\sqrt{\tau_\epsilon}+d\log\epsilon}{\sqrt{-\log\epsilon}}$ converges in distribution as $\epsilon \to 0$ to a centered Gaussian random variable of variance $2\sigma_h^2$.*

3.2 Proof of the pointwise convergence of the recurrence rate to the dimension

In this section we prove Theorem 3.1.1.

Let us denote by $G_n(\epsilon)$ the set of points for which n is an ϵ-return :

$$G_n(\epsilon) := \{x \in \Sigma : S_n\varphi(x) = 0 \text{ and } d(\theta^n(x), x) < \epsilon\}.$$

Let us consider the first return time in an ϵ-neighborhood of a starting point $x \in \Sigma$:

$$\begin{aligned} \tau_\epsilon(x) &:= \inf\{m \geq 1 : S_m\varphi(x) = 0 \text{ and } d(\theta^m(x), x) < \epsilon\} \\ &= \inf\{m \geq 1 : x \in G_m(\epsilon)\}. \end{aligned}$$

Proof of Theorem 3.1.1. Let us denote by $\mathcal{C}_k$ the set of k-cylinders of Σ. For any $\delta > 0$ denote by $\mathcal{C}_k^\delta \subset \mathcal{C}_k$ the set of cylinders $C \in \mathcal{C}_k$ such that $\nu(C) \in (e^{-(d+\delta)k}, e^{-(d-\delta)k})$. For any $x \in \Sigma$, let $C_k(x) \in \mathcal{C}_k$ be the k-cylinder which contains x. Since d is twice the entropy of the ergodic measure ν, by the Shannon-McMillan-Breiman theorem, the set $K_N^\delta = \{x \in \Sigma : \forall k \geq N, C_k(x) \in \mathcal{C}_k^\delta\}$ has a measure $\nu(K_N^\delta) > 1 - \delta$ provided N is sufficiently large.

- First, let us prove that, almost surely :

$$\liminf_{\epsilon \to 0} \frac{\log \sqrt{\tau_\epsilon}}{-\log \epsilon} \geq d.$$

let $\alpha > \frac{1}{d}$ and $0 < \delta < d - \frac{1}{\alpha}$. Let us take $\epsilon_n := n^{-\frac{\alpha}{2}}$ and $k_n := \lceil -\log \epsilon_n \rceil$ so that $e^{-k_n} \sim \epsilon_n$. Since $\theta^{-k_n}(C) = \Pi^{-1}(\hat{C})$, where $\hat{C}$ is a $2k_n$- cylinder of $\hat{\Sigma}$, according to Proposition 2.2.1, whenever $k_n \geq N$, we have :

$$\begin{aligned} \nu(K_N^\delta \cap G_n(\epsilon_n)) &= \nu\left(\{x \in K_N^\delta : S_n\varphi(x) = 0 \text{ and } \theta^n(x) \in C_{k_n}(x)\}\right) \\ &= \sum_{C \in \mathcal{C}_{k_n}^\delta} \nu(C \cap \{S_n\varphi = 0\} \cap \theta^{-n}(C)) \\ &= \sum_{C \in \mathcal{C}_{k_n}^\delta} \nu\left(C \cap \{S_n\varphi = 0\} \cap \theta^{-n}\theta^{k_n}(\theta^{-k_n}(C))\right) \\ &= \sum_{C \in \mathcal{C}_{k_n}^\delta} \frac{\nu(C)\nu(C)}{\sqrt{2\pi}\sigma_\varphi\sqrt{n-k_n}} + O\left(\frac{\nu(C)k_n\nu(C)}{n-2k_n}\right) \\ &= O\left(\sum_{C \in \mathcal{C}_{k_n}^\delta} \frac{\nu(C)^2}{\sqrt{n}}\right). \end{aligned}$$

The last equality is because: $\epsilon_n = n^{-\frac{\alpha}{2}}$ which implies that $n = \frac{1}{\epsilon_n^{\frac{2}{\alpha}}}$. But $k_n \sim -\log \epsilon_n$, and so $k_n \ll n$. This ensures that

$$\frac{k_n}{n-2k_n} = O\left(\frac{k_n}{n}\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

Moreover, by definition of $C_{k_n}^{\delta}$, for every $C \in C_{k_n}^{\delta}$

$$\nu(C) \leq e^{-(d-\delta)k_n} \simeq \epsilon_n^{d-\delta} = n^{-2\alpha(d-\delta)},$$

from which it follows that

$$\begin{aligned}
\nu(K_N^{\delta} \cap G_n(\epsilon_n)) &= O\left(\sum_{C \in \mathcal{C}_{k_n}^{\delta}} \frac{\nu(C)^2}{\sqrt{n}}\right) \\
&= O\left(\sum_{C \in \mathcal{C}_{k_n}^{\delta}} \frac{\nu(C)}{\sqrt{n}} n^{-\frac{\alpha(d-\delta)}{2}}\right) \\
&= O\left(\frac{1}{n^{(1+\alpha(d-\delta))/2}}\right),
\end{aligned}$$

but $\alpha(d-\delta) > 1$ and so $\frac{1+\alpha(d-\delta)}{2} > 1$, then $\sum_n \nu(K_N^{\delta} \cap G_n(\epsilon_n)) < \infty$. Hence, by the Borel-Cantelli lemma, for a.e. $x \in K_N^{\delta}$, there exists N_x such that, for every $n \geq N_x$, $F^n(x,0) \notin B(x,\epsilon) \times \{0\}$. Consider such a point x. observe that $\min\limits_{n \leq N_x} (d(x,\theta^n x) + |S_n x|) > 0$, otherwise x would be a periodic point of F which would contradict the previous fact. Therefore, for n large enough ($n \geq N_x$ and n such that $\epsilon_n < 1$ and $\epsilon_n < \min\limits_{n \leq N_x} (d(x,\theta^n x) + |S_n \varphi(x)|)$), we have $\tau_{\epsilon n}(x) > n$. But this implies that

$$\frac{\log \sqrt{\tau_{\epsilon_n}}}{-\log \epsilon_n} > \frac{\log n}{-2\log \epsilon_n} = \frac{1}{\alpha}$$

Therefore we get

$$\liminf_{n\to\infty} \frac{\log \sqrt{\tau_{\epsilon_n}}}{-\log \epsilon_n} \geq \frac{1}{\alpha} \quad a.e.,$$

which proves the the lower bound on the lim inf, since $(\epsilon_n)_n$ decreases to zero and $\liminf\limits_{n\to\infty} \frac{\epsilon_n}{\epsilon_{n+1}} = 1$.

- Second, let us prove that almost surely :

$$\limsup_{\epsilon\to 0} \frac{\log \sqrt{\tau_{\epsilon}}}{-\log \epsilon} \leq d.$$

Let $0 < \alpha < \frac{1}{d}$ and $\delta > 0$ such that $1 - \alpha d - \alpha\delta > 0$. Let us take $\epsilon_n := n^{-\frac{\alpha}{2}}$ and $k_n := \lceil -\log \epsilon \rceil$. Observe that $e^{-k_n} \sim \epsilon_n$. For all $l = 1, ..., n$, we define:

$$A_l(\epsilon) := G_l(\epsilon) \cap \theta^{-l}\{\tau_{\epsilon} > n - l\}$$

Let us take $L_n := \lceil n^{\frac{a}{2}} \rceil$, with $a > 0$. The sets $A_l(\epsilon)$ are pairewise disjoint thus:

$$1 \geq \sum_{l=0}^{n} \nu(A_l(\epsilon_n)) \geq \sum_{l=L_n}^{n} \sum_{C \in \mathcal{C}_{k_n}^{\delta}} \nu(C \cap A_l(\epsilon_n)).$$

According to Proposition (2.2.1), we have :

$$\begin{aligned}
\nu(C \cap A_l(\epsilon_n)) &= \nu(C \cap \{S_l\varphi = 0\} \cap \theta^{-l}(C \cap \{\tau_{\epsilon_n} > n - l\})) \\
&= \nu(C \cap \{S_l\varphi = 0\} \cap \theta^{-l}\theta^{k}(\theta^{-k}(C \cap \{\tau_{\epsilon_n} > n - l\}))) \\
&= \frac{\nu(C)\nu(C \cap \{\tau_{\epsilon_n} > n - l\})}{\sqrt{2\pi}\sigma_\varphi\sqrt{l - k_n}} + O(\frac{\nu(C \cap \{\tau_{\epsilon_n} > n - l\})k_n\nu(C)}{l - 2k_n}) \\
&= \left(\frac{\nu(C)}{\sqrt{2\pi}\sigma_\varphi} + O\left(\frac{k_n\nu(C)}{\sqrt{l - k_n}}\right)\right)\frac{1}{\sqrt{l - k_n}}\nu(C \cap \{\tau_{\epsilon_n} > n - l\}) \\
&\geq c\epsilon_n^{d+\delta}\frac{1}{\sqrt{l - k_n}}\nu(C \cap \{\tau_{\epsilon_n} > n - l\}).
\end{aligned}$$

For every $l = L_n, ..., n$ and for any $C \in \mathcal{C}_{k_n}^{\delta}$ provided $k_n \geq N$; indeed the error is negligible since $k_n \ll \sqrt{L_n - k_n}$. This chain of inequalities gives

$$\nu\left(K_N^{\delta} \cap \{\tau_{\epsilon_n} > n\}\right) \leq \sum_{C \in \mathcal{C}_{k_n}^{\delta}} \nu\left(C \cap \{\tau_{\epsilon_n} > n\}\right)$$

but, for every $C \in \mathcal{C}_{k_n}^{\delta}$

$$\begin{aligned}
\sum_{l=L_n}^{n} \nu(C \cap A_l(\epsilon_n)) &\geq \sum_{l=L_n}^{n} c\epsilon_n^{d+\delta}\frac{\nu(C \cap \{\tau_{\epsilon_n} > n\})}{\sqrt{l - k_n}} \\
&= c\epsilon_n^{d+\delta}\nu(C \cap \{\tau_{\epsilon_n} > n\})\sum_{l=L_n}^{n}\frac{1}{\sqrt{l - k_n}} \\
&\geq c\epsilon_n^{d+\delta}\nu(C \cap \{\tau_{\epsilon_n} > n\})\left(\sqrt{n - k_n} - \sqrt{L_n - k_n}\right)
\end{aligned}$$

for every n large enough, so we get:

$$1 \geq \sum_{C \in \mathcal{C}_{k_n}^{\delta}} \sum_{l=L_n}^{n} \nu(C \cap A_l(\epsilon_n)) \geq \sum_{C \in \mathcal{C}_{k_n}^{\delta}} c\epsilon_n^{d+\delta}\nu(C \cap \{\tau_{\epsilon_n} > n\})\left(\sqrt{n - k_n} - \sqrt{L_n - k_n}\right)$$

hence

$$\begin{aligned}
\sum_{C \in \mathcal{C}_{k_n}^{\delta}} c\epsilon_n^{d+\delta}\nu(C \cap \{\tau_{\epsilon_n} > n\}) &\leq (c\epsilon_n^{d+\delta})^{-1}\frac{1}{\sqrt{n - k_n} - \sqrt{L_n - k_n}} \\
&= O\left(\frac{1}{n^{\frac{1-\alpha(d+\delta)}{2}}}\right).
\end{aligned}$$

Now let us take $n_p := p^{\frac{4}{1-\alpha d-\alpha\delta}}$. We have:

$$\sum_{p\geq 1}\nu(K_N^\delta\cap\{\tau_{\epsilon_{n_p}}>n_p\}) = \sum_{p\geq 1} O\left(\frac{1}{n_p^{\frac{1-\alpha(d+\delta)}{2}}}\right) = \sum_{p\geq 1} O\left(\frac{1}{p^2}\right) < +\infty.$$

Hence, by the Borel Cantelli lemma, almost surely $x\in K_N^\delta$, we have, for every p large enough, $\tau_{\varepsilon_p}\leq n_p$ and so $\frac{\log\sqrt{\tau_{\epsilon_{n_p}}}}{-\log\epsilon_{n_p}}\leq\frac{1}{\alpha}$. This implies that

$$\limsup_{n\to+\infty}\frac{\log\sqrt{\tau_{\epsilon_n}}}{-\log\epsilon_n}\leq\frac{1}{\alpha},$$

which gives the estimate lim sup since $(\epsilon_{n_p})_p$ decreases to 0 and since $\lim_{p\to+\infty}\frac{\epsilon_{n_p}}{\epsilon_{n_{p+1}}} = 1$.

□

3.3 Fluctuations of the rescaled return time.

Here, we prove Theorem 3.1.2 and its Corollary 3.1.3.
Throughout this subsection, we adapt the general strategy of [39, 40]. Recall that $C_k(x)=\{y\in\Sigma : d(x,y)<e^{-k}\}$. Let $R_k(y)=\min\{n\geq 1:\theta^n(y)\in C_k(y)\}$ denotes the first return time of a point y into its k-cylinder $C_k(y)$, or equivalently the first repetition time of the first k symbols of y. There have been a lot of studies on this quantity, among all the results we will use the following.

Proposition 3.3.1. *(Hirata[28]) For ν-almost every point $x\in\Sigma$, the return times into the cylinders $C_k(x)$ are asymptotically exponentially distributed in the sense that, for a.e. x*

$$\lim_{k\to+\infty}\nu_{C_k(x)}\left(R_k(.)>\frac{t}{\nu(C_k(x))}\right)=e^{-t},\quad\forall t\geq 0.$$

Note that this convergence is uniform in t.

Lemma 3.3.2. *The family of distributions of the random variables $\left(\nu(C_k(x))\sqrt{\tau_{e^{-k}}}|C_k(x)\right)_{k\geq 0}$ is tight.*

Hence it will be enough to prove that the advertised limit law is the only possible accumulation point of our destination. We hence abbreviate

$$X_k := \nu(C_k(x))\sqrt{\tau_{e^{-k}}}.$$

Lemma 3.3.3. *Let x be such that* $\lim_{k\to+\infty}\nu_{C_k(x)}\left(R_k(.) > \frac{t}{\nu(C_k(x))}\right) = e^{-t}$*, for all* $t>0$*. Then, for all* $t>0$*, we have :*

$$\lim_{k\to+\infty}\nu\left(\tau_{e^{-k}} > \left(\frac{t}{\nu(C_k(x))}\right)^2 \Big| C_k(x)\right) \le \frac{1}{1+\beta t},$$

with $\beta := \frac{1}{\sqrt{2\pi}\sigma_\varphi}$.

Proof. Let $k \ge m_0$ and n be some integers. We make a partition of a cylinder $C_k(x)$ according to the value $l \le n$ of the last passage in the time interval $0, ..., n$ of the orbit of $(x,0)$ by the map F into $C_k(x)\times\{0\}$. This gives the following equality

$$\nu(C_k(x)) = \sum_{l=0}^{n}\nu\left(C_k(x)\cap\{S_l=0\}\cap\theta^{-l}(C_k(x)\cap\{\tau_{e^{-k}} > n-l\})\right). \qquad (3.3.1)$$

Let $n_k = \left(\frac{t}{\nu(C_k(x))}\right)^2$. First we claim that :

$$\limsup_{k\to\infty}\nu(\{\tau_{e^{-k}} > n_k\} \mid C_k(x)) \le \frac{1}{1+\beta t}$$

According to the decomposition (3.3.1) and to Proposition 2.2.1, there exists $c_1>0$ such that we have :

$$\begin{aligned}
\nu(C_k(x)) &\ge \nu(C_k(x)\cap\{\tau_{e^{-k}} > n_k\}) + \sum_{l=2k+1}^{n_k}\frac{\beta\nu(C_k(x))\nu(C_k(x)\cap\{\tau_{e^{-k}} > n_k - l\})}{\sqrt{l-k}} \\
&\quad -c_1\sum_{l=2k+1}^{n_k}\frac{k\nu(C_k(x))\nu(C_k(x)\cap\{\tau_{e^{-k}} > n_k-l\})}{l-2k} \\
&\ge \nu(C_k(x)\cap\{\tau_{e^{-k}} > n_k\})\left(1+\beta\nu(C_k(x))\sum_{l=2k+1}^{n_k}\frac{1}{\sqrt{l-k}}\right) \\
&\quad -c_1\nu(C_k(x))k\nu(C_k(x))\sum_{l=2k+1}^{n_k}\frac{1}{l-2k}
\end{aligned}$$

it follows that,

$$\frac{\nu(C_k(x)\cap\{\tau_{e^{-k}} > n_k\})}{\nu(C_k(x))} \le \frac{1+c_1k\nu(C_k(x))\sum_{l=2k+1}^{l=n_k}\frac{1}{l-2k}}{1+\beta\nu(C_k(x))\sum_{l=2k+1}^{l=n_k}\frac{1}{\sqrt{l-k}}}$$

but,

$$
\begin{aligned}
1+\beta\nu(C_k(x))\sum_{l=2k+1}^{l=n_k}\frac{1}{\sqrt{l-k}} &\sim 1+\beta\nu(C_k(x))(\sqrt{n_k-k}-\sqrt{L_k-k}) \\
&\sim 1+\beta\nu(C_k(x))(\sqrt{n_k}) \\
&= 1+\beta\nu(C_k(x))\left(\frac{t}{\nu(C_k(x))}\right) \\
&= 1+\beta t.
\end{aligned}
$$

and

$$
k\nu(C_k(x))\sum_{l=2k+1}^{l=n_k}\frac{1}{l-2k} \sim c_1 k\nu(C_k(x))\log\left(\frac{n_k}{k+1}\right) \ll 1
$$

Hence, we get :

$$
\limsup_{k\to\infty}\nu(\{\tau_{e^{-k}}>n_k\}\mid C_k(x)) \leq \frac{1}{1+\beta t}
$$

□

Lemma 3.3.4. *Suppose that the conditional distributions of the subsequence $X_{k_p}|C_{k_p}(x)$ converge to the law of some random variable X. Then it satisfies the integral equation:*

$$
1=\mathbb{P}(X>t)+\beta t\int_0^1\frac{\mathbb{P}(X>t\sqrt{1-u})}{\sqrt{u}}du \quad \forall t>0.
$$

Proof.

- (i)Let $t>0$, we write $f(t):=\mathbb{P}(X>t)$, and first prove that

$$
\forall t>0 \quad 1\geq f(t)+\beta t\int_0^1 u^{-1/2}f(t(1-u)^{1/2})du.
$$

Recall that $n_k=\left(\frac{t}{\nu(C_k(x))}\right)^2$. According to the decomposition (3.3.1) and to Proposition 2.2.1, there exists $c>0$ such that we have:

$$
\begin{aligned}
\nu(C_k(x)) &\geq \nu(C_k(x),\tau_{e^{-k}}>n_k)+\beta\sum_{l=2k+1}^{n_k}\frac{\nu(C_k)\nu(C_k(x),\tau_{e^{-k}}>n_k-l)}{\sqrt{l-k}} \\
&\quad -c\sum_{l=2k+1}^{n_k}\frac{k\nu(C_k(x))\nu(C_k(x))}{l-2k},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
1 &\geq \nu(\tau_{e^{-k}}>n_k\mid C_k(x))+\beta\nu(C_k(x))\sum_{l=2k+1}^{n_k}\frac{\nu(\tau_{e^{-k}}>n_k-l\mid C_k(x))}{\sqrt{l-k}} \\
&\quad -c\sum_{l=2k+1}^{n_k}\frac{k\nu(C_k(x))}{l-2k}.
\end{aligned}
$$

Next, by monotonicity,

$$\begin{aligned}
B_{n_k} &:= \sum_{l=2k+1}^{n_k} \frac{\nu(\tau_{e^{-k}} > n_k - l \mid C_k(x))}{\sqrt{l-k}} \\
&\geq \sum_{l=\lfloor \frac{n_k}{N} \rfloor}^{N\lfloor \frac{n_k}{N} \rfloor} \frac{\nu(\tau_{e^{-k}} > n_k - l \mid C_k(x))}{\sqrt{l-k}} \\
&= \sum_{r=1}^{N-1} \sum_{l=0}^{\lfloor \frac{n_k}{N} \rfloor} \frac{\nu(\tau_{e^{-k}} > n_k - l - (r\lfloor n_k/N \rfloor) \mid C_k(x))}{\sqrt{l + r\lfloor n_k/N \rfloor}}.
\end{aligned}$$

We notice that the term:

$$\begin{aligned}
\nu(\tau_{e^{-k}} > n_k - l - (r\lfloor n_k/N \rfloor) \mid C_k(x)) &\geq \nu(\tau_{e^{-k}} > n_k - (r\lfloor n_k/N \rfloor) \mid C_k(x)) \\
&\geq \nu(\tau_{e^{-k}} > (1 - r/N)n_k + r \mid C_k(x)).
\end{aligned}$$

Now, we evaluate this sum:

$$\begin{aligned}
\sum_{l=0}^{\lfloor \frac{n_k}{N} \rfloor} \frac{1}{\sqrt{l + r\lfloor n_k/N \rfloor}} &\geq \int_{r\lfloor n_k/N \rfloor}^{(r+1)\lfloor n_k/N \rfloor + 1} \frac{dx}{\sqrt{x}} \\
&\geq 2\sqrt{\left\lfloor \frac{n_k}{N} \right\rfloor + r\left\lfloor \frac{n_K}{N} \right\rfloor + 1} - \sqrt{r\left\lfloor \frac{n_k}{N} \right\rfloor} \\
&\geq 2\sqrt{\left\lfloor \frac{n_k}{N} \right\rfloor}\left(\sqrt{r + 1 + \frac{N}{n_k}} - \sqrt{r}\right) \\
&\geq \sqrt{\left\lfloor \frac{n_k}{N} \right\rfloor} \frac{1 + \frac{N}{n_k}}{\sqrt{r + 1 + \frac{N}{n_k}}},
\end{aligned}$$

where we used the fact that $\sqrt{r+s} - \sqrt{r} = \frac{s}{\sqrt{r+s}+\sqrt{r}} \geq \frac{s}{2\sqrt{r+s}}$. Combining these formulas, we get

$$B_{n_k} = \sum_{r=1}^{N-1} \sqrt{\left\lfloor \frac{n_k}{N} \right\rfloor} \frac{1 + N/n_k}{\sqrt{r + 1 + N/n_k}} \nu(\tau_{e^{-k}} > (1 - r/N)n_k + r \mid C_k(x)).$$

The second term is equal to

$$\begin{aligned}
\nu\left(\tau_{e^{-k}} > (1 - r/N)n_k + r \mid C_k(x)\right) &= \nu\left(\tau_{e^{-k}} > (1 - r/N)(t/\nu(C_k(x)))^2 \mid C_k(x) + r\right) \\
&= \mathbb{P}\left(X_k^2 > ((1 - r/N)t^2 + r(\nu(C_k(x))^2 \mid C_k(x)\right).
\end{aligned}$$

But, since $(X_{k_p}|C_{k_p}(x))$ converges in distribution to X, $(X_{k_p}^2-r(\nu(C_{k_p}(x))^2|C_{k_p(x)})$ converges in distribution to X^2 and so

$$\nu(X_{k_p}^2>((1-r/N)t^2+r(\nu(C_k(x))^2|C_{k_p}(x))\underset{k_p\to\infty}{\longrightarrow}P(X^2>t^2(1-r/N))=f(t\sqrt{1-r/N}).$$

As a consequence, we see that

$$\begin{aligned}
\liminf_{p\to\infty} \quad & B_{n_{k_p}} \\
\geq \quad & \liminf_{p\to\infty}\nu(C_{k_p}(x))\sqrt{n_{k_p}}\sum_{r=1}^{N-1}\nu(X_{k_p}^2>((1-r/N)t^2+r(\nu(C_k(x))^2|C_{k_p}(x))\sqrt{1/N}\sqrt{1/r} \\
\geq \quad & \liminf_{p\to\infty}\nu(C_{k_p(x)})\frac{t}{\nu(C_{k_p}(x))}\frac{1}{N}\sum_{r=1}^{N-1}\frac{\nu(X_{k_p}^2>((1-r/N)t^2+r(\nu(C_k(x))^2|C_{k_p}(x))}{\sqrt{(r+1)/N}} \\
\geq \quad & \frac{t}{N}\sum_{r=1}^{N-1}\frac{\mathbb{P}\left(X>t\sqrt{1-r/N}\right)}{\sqrt{(r+1)/N}} \\
\geq \quad & t\int_0^1\frac{f\left(t\sqrt{1-u}\right)}{\sqrt{u}}du.
\end{aligned}$$

Note that through the proof of Lemma 3.3.3, it has been proved that:

$$\lim_{k_p\to\infty}\sum_{l=2k+1}^{n_{k_p}}\frac{k_p\nu(C_{k_p}(x))}{l-2k_p}=0$$

Combining all these asymptotic estimates and taking the limit when $k_p\to\infty$, we end up with the desired inequality:

$$1\geq f(t)+\beta t\int_0^1\frac{f\left(t\sqrt{1-u}\right)}{\sqrt{u}}du.$$

- (ii)The converse inequality is proved analogously. Starting from the formula (3.3.1), we have

$$\begin{aligned}
\sum_{l=1}^{m_k} \quad & \{S_l=0\}\cap\theta^{-l}(C_k(x)\cap\{\tau_{e^{-k}}>m_k-l\})|C_k(x)) \\
= \quad & \nu\left(\bigcup_{l=1}^{m_k}\{S_l=0\}\cap\theta^{-l}(C_k(x)\cap\{\tau_{e^{-k}}>m_k-l\})|C_k(x)\right) \\
\leq \quad & \nu(\{\tau_{e^{-k}}\leq m_k\}|C_k(x)),
\end{aligned}$$

then using Proposition 2.2.1, there exists $c^{'} > 0$, such that

$$\begin{aligned}1 \quad &\leq \quad \nu(\tau_{e^{-k}} \leq m_k|C_k(x)) + \nu(\tau_{e^{-k}} > n_k \mid C_k(x)) \\ &+\beta\nu(C_k(x))\sum_{l=m_k}^{n_k}\frac{\nu(\tau_{e^{-k}} > n_k - l \mid C_k(x))}{\sqrt{l-k}} + c^{'}\sum_{l=m_k}^{n_k}\frac{k\nu(C_k(x))}{l-2k}\end{aligned}$$

where $m_k = o\left(\frac{1}{\nu(C_k(x))}\right)$. Using Proposition 3.3.1,

$$\nu(\tau_{e^{-k}} \leq m_k|C_k(x)) \leq \nu(R_k \leq m_k|C_k(x)).$$

But $(\nu(C_k(x))R_k|C_k(x))_k$ converges in distribution, so that $(R_k/m_k|C_k(x))_k$ converges in distribution and in probability to $+\infty$, which ensures that

$$\nu(\tau_{e^{-k}} \leq m_k|C_k(x)) \overset{k\to\infty}{\longrightarrow} 0.$$

But,

$$\begin{aligned}\sum_{l=m_k}^{\lfloor n_k/N\rfloor}\frac{\nu(\tau_{e^{-k}} > n_k - l \mid C_k(x))}{\sqrt{l-k}} \quad &\leq \quad \sum_{l=m_k}^{\lfloor n_k/N\rfloor}\frac{1}{\sqrt{l-k}} \\ &\leq \quad \int_{m_k-1}^{\lfloor n_k/N\rfloor}\frac{dx}{\sqrt{x}} \\ &\leq \quad 2\left(\sqrt{\left\lfloor\frac{n_k}{N}\right\rfloor - k} - \sqrt{m_k-k-1}\right) \\ &\leq \quad 2\sqrt{\frac{n_k}{N}} \leq \frac{2t}{\nu(C_k(x))\sqrt{N}}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}1 \quad &\leq \quad \nu(\tau_{e^{-k}} > n_k \mid C_k(x)) + \beta\nu(C_k(x))\sum_{l=\lfloor\frac{n_k}{N}\rfloor}^{n_k}\frac{\nu(\tau_{e^{-k}} > n_k - l \mid C_k(x))}{\sqrt{l-k}} \\ &+ \quad c^{'}\sum_{l=m_k}^{n_k}\frac{k\nu(C_k(x))}{l-k} + o(\nu(C_k(x))) + \beta\frac{2t}{\sqrt{N}}.\end{aligned}$$

Note that whenever $l \geq \frac{n_k}{N}$,

$$\frac{1}{\sqrt{l-k}} = \frac{1}{\sqrt{l}}(1+\epsilon_k^{(N)}),$$

moreover, we have

$$\begin{aligned}\sum_{l=N\lfloor n_k/N\rfloor}^{n_k} \frac{\nu(\tau_{e^{-k}} > n_k - l \mid C_k(x))}{\sqrt{l}} &\leq \sum_{l=N\lfloor n_k/N\rfloor}^{n_k} \frac{1}{\sqrt{l}} \\ &\leq \frac{1}{\sqrt{n_k - N}} \sum_{l=N\lfloor n_k/N\rfloor}^{n_k} 1 \\ &\leq N \frac{1}{\sqrt{n_k - N}}.\end{aligned}$$

Hence, we get the following

$$\sum_{l=\lfloor \frac{n_k}{N}\rfloor}^{n_k} \frac{\nu\left(\tau_{e^{-k}} > n_k - l \mid C_k(x)\right)}{\sqrt{l-k}} \leq (1+\epsilon_k) \left(\sum_{l=\lfloor n_k/N\rfloor}^{N\lfloor n_k/N\rfloor} \frac{\nu(\tau_{e^{-k}} > n_k - l \mid C_k(x))}{\sqrt{l}} + N \frac{1}{\sqrt{n_k - N}} \right).$$

Now, similarly as we did in the first inequality

$$\begin{aligned}B'_{n_k} &:= \sum_{l=\lfloor n_k/N\rfloor}^{N\lfloor n_k/N\rfloor} \frac{\nu\left(\tau_{e^{-k}} > n_k - l \mid C_k(x)\right)}{\sqrt{l}} \\ &= \sum_{r=1}^{N-1} \sum_{l=r\lfloor \frac{n_k}{N}\rfloor}^{(r+1)\lfloor \frac{n_k}{N}\rfloor - 1} \frac{\nu\left(\tau_{e^{-k}} > n_k - l \mid C_k(x)\right)}{\sqrt{l}} \\ &\leq \sum_{r=1}^{N-1} \sum_{l=0}^{\lfloor \frac{n_k}{N}\rfloor - 1} \frac{\nu\left(\tau_{e^{-k}} > n_k - ((r+1)\lfloor n_k/N\rfloor) \mid C_k(x)\right)}{\sqrt{l + r\lfloor n_k/N\rfloor}}.\end{aligned}$$

But, we have

$$\begin{aligned}\sum_{l=0}^{\lfloor \frac{n_k}{N}\rfloor - 1} \frac{1}{\sqrt{l + r\lfloor n_k/N\rfloor}} &\leq \int_{r\lfloor n_k/N\rfloor - 1}^{(r+1)\lfloor n_k/N\rfloor - 1} \frac{dx}{\sqrt{x}} \\ &= 2\sqrt{\left\lfloor \frac{n_k}{N}\right\rfloor} \left(\sqrt{r + 1 - \frac{1}{\lfloor n_k/N\rfloor}} - \sqrt{r - \frac{1}{\lfloor n_k/N\rfloor}} \right) \\ &\leq \sqrt{\left\lfloor \frac{n_k}{N}\right\rfloor} \frac{1}{\sqrt{r}},\end{aligned}$$

from which it follows that

$$B'_{n_k} \leq \sum_{r=1}^{N-1} \sqrt{\left\lfloor \frac{n_k}{N}\right\rfloor} \frac{1}{\sqrt{r}} \nu\left(\tau_{e^{-k}} > (1 - (r+1)/N) n_k \mid C_k(x)\right).$$

Now, applying lim sup when $p \to \infty$, then

$$\limsup_{p\to\infty} \nu(C_{k_p}(x)) \sum_{l=\lfloor n_{k_p}/N \rfloor}^{N\lfloor n_{k_p}/N \rfloor} \frac{\nu\left(\tau_{e^{-k_p}} > n_{k_p} - l \mid C_{k_p}(x)\right)}{\sqrt{l}} \leq \beta t \int_0^1 \frac{f(t\sqrt{1-u})}{\sqrt{u}} du.$$

Taking the limit when $k_p \to \infty$, and combining all these estimates, we get the second inequality:

$$1 \leq f(t) + \beta t \int_0^1 \frac{f(t\sqrt{1-u})}{\sqrt{u}} du.$$

□

Corollary 3.3.5. *The conditional distributions of the X_{k_p} converge to a random variable X iff the conditional distributions of the $X^2_{k_p}$ converge to X^2. The later then satisfies*

$$1 = \mathbb{P}(X^2 > t) + \beta \int_0^t \frac{\mathbb{P}(X^2 > t - v)}{\sqrt{v}} dv, \qquad \forall t > 0.$$

Lemma 3.3.6. *Let W be a random walk variable with values in $[0, \infty[$ satisfying*

$$\mathbb{P}(W \leq t) = \beta \int_0^t \frac{\mathbb{P}(W > t - v)}{\sqrt{v}} dv, \qquad \forall t > 0,$$

then

$$\mathbb{E}\left[e^{-sW}\right] = \frac{1}{1 + \sqrt{2s}\sigma_\varphi}, \qquad \forall s > 0.$$

In particular, the distribution of W coincides with that of $\sigma^2_\varphi \dfrac{\mathcal{E}^2}{|\mathcal{N}|^2}$, where the independent variables $\mathcal{E}$ and $\mathcal{N}$ are the exponential distribution of mean 1 and the standard Gaussian distribution respectively.

Proof. Let $s > 0$. We have

$$\begin{aligned}
\mathbb{E}[e^{-sW}] &= \int_0^\infty \mathbb{P}(e^{-sW} \geq u) du \\
&= \int_0^\infty \mathbb{P}\left(W \leq -\frac{\log(u)}{s}\right) du \\
&= \int_0^\infty \mathbb{P}(W \leq v) s e^{-sv} dv,
\end{aligned}$$

where we used the change of variable $v = \frac{-\log u}{s}$ and that W is with values in $[0,\infty[$. Hence, using Fubini's theorem, for any $s>0$, we find

$$\begin{aligned}
\mathbb{E}[e^{-sW}] &= \int_0^\infty \left[\beta \int_0^v \frac{\mathbb{P}(W \geq v-w)}{\sqrt{w}} dw\right] se^{-sv} dv \\
&= \int_0^\infty \frac{1}{\sqrt{w}} \left[\beta \int_w^\infty \mathbb{P}(W \geq v-w) se^{-sv} dv\right] dw \\
&= \int_0^\infty \frac{e^{-sw}}{\sqrt{w}} \left[\beta \int_0^\infty \mathbb{P}(W \geq z) se^{-sz} dz\right] dw \\
&= \beta \int_0^\infty \frac{e^{-sw}}{\sqrt{w}} \left[1 - \int_0^\infty \mathbb{P}(W \leq z) se^{-sz} dz\right] dw \\
&= \beta \int_0^\infty \frac{e^{-sw}}{\sqrt{w}} dw. \left[1 - \mathbb{E}[e^{-sW}]\right],
\end{aligned}$$

and our claim about the Laplace transform of W follows since

$$\begin{aligned}
\int_0^\infty \frac{e^{-sw}}{\sqrt{w}} dw &= \int_0^\infty \frac{e^{-\frac{1}{2}v^2}}{v/\sqrt{2s}} \frac{2v}{2s} dv \\
&= \int_0^\infty e^{-\frac{1}{2}v^2} \frac{\sqrt{2}}{\sqrt{s}} dv \\
&= \frac{\sqrt{2}}{\sqrt{s}} \sqrt{2\pi}. \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}v^2} \right) \\
&= \frac{\sqrt{\pi}}{\sqrt{s}}.
\end{aligned}$$

Since $\beta = \frac{1}{\sqrt{2\pi}}$, hence, we end up with

$$\mathbb{E}[e^{-sW}] = \frac{1}{1+\sigma_\varphi \sqrt{2s}}.$$

On the other hand, since $\mathcal{E}$ and $\mathcal{N}$ are independent, using Lemma 1.3.1

$$\begin{aligned}
\mathbb{E}\left[e^{-s\sigma_\varphi^2 \mathcal{E}^2/\mathcal{N}^2}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-s\sigma_\varphi^2 \mathcal{E}^2/\mathcal{N}^2} | \mathcal{E}\right]\right] \\
&= \mathbb{E}\left[\exp\left(-\sqrt{2s\sigma_\varphi^2 \mathcal{E}^2}\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\sqrt{2s}\sigma_\varphi \mathcal{E}\right)\right] \\
&= \frac{1}{1+\sqrt{2s}\sigma_\varphi},
\end{aligned}$$

since $\mathbb{E}[e^{-s\mathcal{E}}] = \frac{1}{1+s}$. Therefore W has the same Laplace transform of $\sigma_\varphi^2 \frac{\mathcal{E}^2}{\mathcal{N}^2}$. $\square$

Proof of Theorem 3.1.1. According to Lemma 3.3.3, the family of distributions of X_k is tight. By Lemmas 3.3.4, 3.3.5 and 3.3.6, the law of $\sigma_\varphi \dfrac{\mathcal{E}}{|\mathcal{N}|}$ is the only possible accumulation point of $\left(\nu(C_k(x))\sqrt{\tau_{e^{-k}}}\right)_{k\geq 0}$. □

Proof of Corollary 3.1.3. Let us set:

$$Y_k := \frac{\log\sqrt{\tau_{e^{-k}}}(.) - kd}{\sqrt{k}}.$$

We have the case that ν is a Gibbs measure with a non degenerate Hölder potential h. There is a constant $c_h > 0$ such that $\log\nu(C_k(x)) = \sum_{j=-k}^{k} h\circ\sigma^j(x)$. This Birkhoff sum follows a central limit theorem (e.g. [12]), which implies that:

$$\frac{\log\nu(C_k(.)) + kd}{\sqrt{k}} \xrightarrow{dist} \mathcal{N}(0, 2\sigma_h^2).$$

Observe that Y_k has the following decomposition:

$$Y_k = \frac{\log\left(\nu(C_k(.))\sqrt{\tau_{e^{-k}}(.)}\right)}{\sqrt{k}} - \frac{\log\nu(C_k(.)) + kd}{\sqrt{k}}.$$

Hence, by Slutsky's lemma, it will be enough to prove that the first term of Y_k converges in distribution to 0, which is true due to Theorem 3.1.2. □

Chapter 4

Properties of Axiom A flows

In this chapter we recall the necessary notions and definitions of an Axiom A flow. We start by defining a hyperbolic set for a flow. We introduce the definition of a Markov section which is a classical method of studying the symbolic representation of a dynamical system. These were developed by Bowen [11] and Ratner [42]. Then we represent the special flow over a subshift, where we define the Poincaré section and the coding map. We refer to Barreira [4] for a general reference of the definitions and properties. Proceeding in this chapter, we work on establishing some properties concerning balls and coding. For an $x \in X$, and $r > 0$, under some conditions taken on the dilatation and the contraction in the unstable and stable direction of the flow respectively, we showed that the ball $B(x, r)$ contains and is contained in a cylinder. Proving this property, serves in the sense that the asymptotic behavior of the return time to a ball will be studied through considering the return time to a cylinder. And that was the strategy followed in proving the almost sure convergence result in Chapter 5, and for the convergence in distribution result in chapter 6.

4.1 Definition of Axiom A Flows

Let M be a Riemannian manifold of dimension 3. First we give in general, the formal definition of a flow:

Definition. A C^1 flow $(g_t)_{t\in\mathbb{R}}$ on a smooth manifold M is a map

$$\begin{array}{rcl} \mathbb{R} \times M & \to & M \\ (t, x) & \mapsto & g_t(x) \end{array}$$

such that, for all $x \in X$ and real numbers s and t,

- $g_0(x) = x$,
- $g_t \circ g_s(x) = g_{t+s}(x)$,
- and that the map $(t, x) \mapsto g_t(x)$ is of class C^1.

We introduce the notion of a hyperbolic set. Thus, first we need to define a non-wandering point:

Definition. A point $x \in M$ is said to be non-wandering if for every open neighborhood U of x, there exists $t > 0$ such that $g_t(U) \cap U \neq \emptyset$. Set $\Omega(g_t)$ the set of non-wandering points of g_t.

Definition. A compact g_t-invariant set $\Lambda \subset M$ is said to be a hyperbolic set for g_t if there exists a continuous splitting of the tangent space

$$T_\Lambda M = E^s \oplus E^u \oplus E^0,$$

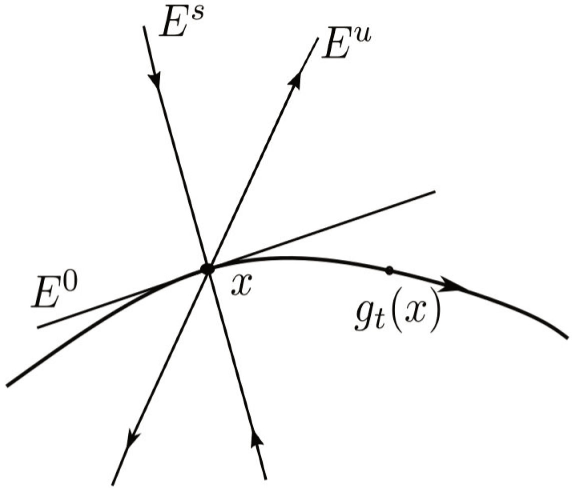
and constants $c > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$ and $t \in \mathbb{R}$:

1. the vector $\frac{dg_t(x)}{dt} |_{t=0}$ generates $E^0(x)$, which is the flow direction;
2. $d_x g_t E^s(x) = E^s(g_t(x))$ and $d_x g_t E^u(x) = E^u(g_t(x))$;
3. for all $t > 0$,
$$||d_x g_t v|| \leq c\lambda^t ||v|| \text{ for } v \in E^s(x),$$
$$||d_x g_{-t} v|| \leq c\lambda^t ||v|| \text{ for } v \in E^u(x).$$

The subspaces $E^s(x)$ and $E^u(x)$ are called the stable and unstable subspaces at x. A consequence of the definition is that they depend continuously on the point and are invariant.

Proposition 4.1.1. *Let Λ be a hyperbolic set for $(g_t)_{t\in\mathbb{R}}$. For each $x \in \Lambda$, there exist stable and unstable local manifolds at the point x, $W^s_{loc}(x)$ and $W^u_{loc}(x)$. They have the following properties:*

1. *$x \in W^s_{loc}(x)$ and $x \in W^u_{loc}(x)$;*



2. $T_x W^s_{loc}(x) = E^s(x)$ *and* $T_x W^u_{loc}(x) = E^u(x)$*;*

3. for each $t > 0$, $g_t(W^s_{loc}(x)) \subset W^s_{loc}(g_t(x))$, $g_{-t}(W^u_{loc}(x)) \subset W^u_{loc}(g_{-t}(x))$*;*

4. there exists $\kappa > 0$ *and* $\mu > 0$ *such that for each* $t > 0$ *we have*

$$d(g_t(x), g_t(y)) \leq \kappa e^{-\mu t} d(x, y) \text{ for } y \in W^s_{loc}(x), \tag{4.1.1}$$

and

$$d(g_{-t}(x), g_{-t}(y)) \leq \kappa e^{-\mu t} d(x, y) \text{ for } y \in W^u_{loc}(x),$$

where d *here is the distance induced in* M *by the Riemannian metric.*

Definition. A hyperbolic set Λ is said to be locally maximal (with respect to a flow g_t) if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} g_t(U)$$

Let Λ be a locally maximal hyperbolic set. For every $\epsilon > 0$ there is $\delta > 0$ such that for any x, y in Λ with $d(x, y) \leq \delta$ there exists a unique $t = t(x, y) \in [-\epsilon, \epsilon]$, for which the following intersection:

$$[x, y] := W^s_{loc}(g_t(x)) \cap W^u_{loc}(y)$$

consists of a single point in Λ; moreover, the map $(x, y) \mapsto t(x, y)$ is continuous. We define the global stable and unstable manifolds at a point $x \in \Lambda$ by:

$$W^{(s)}(x) = \bigcup_{t \geq 0} g_{-t}\left(W^{(s)}_{loc}(g_t(x))\right) \ , W^{(u)}(x) = \bigcup_{t \geq 0} g_t\left(W^{(u)}_{loc}(g_{-t}(x))\right).$$

They can be characterized as follows:

$$W^s(x) = \{y \in \Lambda : d(g_t(x), g_t(y)) \to 0 \text{ as } t \to \infty\},$$

$$W^u(x) = \{y \in \Lambda : d(g_{-t}(x), g_{-t}(y)) \to 0 \text{ as } t \to \infty\}.$$

Definition. A flow g_t is said to be an Axiom A flow if the set of its non-wandering points is hyperbolic.

We suppose from now on that the flow $(g_t)_{t\in\mathbb{R}}$ is Axiom A and topologically mixing.

4.2 Markov Sections

We introduce the notion of a Markov system which was developed by Bowen [11] and Ratner [42]. Let Λ be a maximal hyperbolic set with respect to the flow g_t. A decomposition of this set provides an analysis of its dynamics, which is the construction of the Markov collection.

Given a point $x \in \Lambda$, consider a small compact disk $D \subset M$ containing x of co-dimension one which is transversal to the flow g_t. This disk is a local section of the flow, i.e., there exists $\tau > 0$ such that the map $(x,t) \to g_t(x)$ is a diffeomorphism of the direct product $D \times [-\tau, \tau]$ onto a neighborhood $U_\tau(D)$:

$$\begin{array}{rcl} \varphi_D : D \times [0,\tau] & \to & \varphi(D \times [-\tau,\tau]) \subset M \\ (x,t) & \to & y = g_t(x) \end{array}$$

The function φ_D is Lipschitz. We set $k_{\varphi_D} := \max\limits_D(Lip\ \varphi_D, Lip\ \varphi_D^{-1})$.

Definition. The projection $P_D : U_\tau(D) \to D$ is a differentiable map, defined such that $P_D(z) := g_{-s}(z)$, where $s > 0$ is the minimum value such that $g_t(z) \in D$. The projection of the ball $B(y_0, r)$, where $y_0 \in M$, on D is

$$P_D(B(y_0,r)) := \{x \in D : \exists s < R(x) : g_s(x) \in B(y_0,r)\}.$$

Definition. Consider now a closed set $\Pi \subset \Lambda \cap D$ which doesn't intersect the boundary ∂D. For any two points y, $z \in \Pi$, we set

$$\{y,z\} := P_D([y,z]). \tag{4.2.1}$$

Definition. The set Π is said to be a rectangle if $\Pi = \overline{\text{int}\Pi}$ (where the interior is considered with respect to the induced topology of $\Lambda \cap D$) and $\{y, z\} \in \Pi$ for any $y, z \in \Pi$.

Proposition 4.2.1. *([4, 26]) Let Π be a rectangle. Suppose that the flow g_t is conformal on Λ. The maps $x \mapsto E^s(x) \oplus E^0(x)$ and $x \mapsto E^u(x) \oplus E^0(x)$ are Lipschitz. This implies that there exists $c_L > 0$ such that*

$$\Pi \times \Pi \ni (x, y) \mapsto x, y \in \Pi$$

is a Lipschitz map of Lipschitz constant $c_L > 0$ and with Lipschitz inverse.

Let Π be a rectangle, then for every $x \in \Pi$, we set:

$$W^s_{loc}(x, \Pi) = \{\{x, y\} : y \in \Pi\} = \Pi \cap P_D \left(U_\tau(D) \cap W^s_{loc}(x)\right),$$
$$W^u_{loc}(x, \Pi) = \{\{x, y\} : y \in \Pi\} = \Pi \cap P_D \left(U_\tau(D) \cap W^u_{loc}(x)\right).$$

4.3 Representation by a special flow over a subshift

Now we consider a collection of rectangles $\Pi_1, ..., \Pi_n$ (each contained in some open disk transversal to the flow) such that

$$\Pi_i \cap \Pi_j = \partial\Pi_i \cap \partial\Pi_j \text{ for } i \neq j.$$

Set $X = \bigcup_{i=1}^n \Pi_i$. We assume that there exists $\beta_0 > 0$ such that:

1. $\Lambda = \bigcup_{t\in[0,\beta_0]} g_t(X)$;

2. for each $i \neq j$, for every $t \in [0, \beta_0]$, at least:

$$g_t(\Pi_i) \cap \Pi_j = \emptyset \text{ or } g_t(\Pi_j) \cap \Pi_i = \emptyset$$

Definition. The set X is a Poincaré section for the flow g_t. For every $x \in X$, one can find the smallest positive number $R(x)$ such that $g_{R(x)}(x) \in X$. We define thus

- the height function $R : X \to (0, \infty)$ by

$$R(x) := \min\{s > 0 : g_s x \in X\}, \tag{4.3.1}$$

- the Poincaré map $T : \Lambda \longrightarrow X$ by

$$T(x) = g_{R(x)}(x). \tag{4.3.2}$$

Furthermore, the map R is Hölder continuous on each domain of continuity, and

$$0 < \inf\{R(x) : x \in X\} \leq \sup\{R(x) : x \in \Lambda\} < \infty. \tag{4.3.3}$$

Definition. A Markov collection is a finite collection of rectangles $\Pi_1, ..., \Pi_n$ satisfying, for every $x \in \mathrm{int}(\Pi_i) \cap \mathrm{int}T^{-1}(\Pi_j)$:

$$T\left(\mathrm{int}(W^s_{loc}(x) \cap \Pi_i)\right) \subset \mathrm{int}\left(W^s_{loc}(T(x)) \cap \Pi_j\right)$$

and for every $x \in \mathrm{int}(\Pi_i) \cap \mathrm{int}T(\Pi_k)$:

$$T^{-1}\left(\mathrm{int}(W^u_{loc}(x) \cap \Pi_i)\right) \subset \mathrm{int}\left(W^u_{loc}(T^{-1}(x)) \cap \Pi_k\right).$$

It follows from the work of Bowen and Ratner [11] and [42] that any locally maximal hyperbolic set Λ has a Markov collection of arbitrary small diameter. Given a rectangle Π_i, we call the set

$$R_i = \bigcup_{x \in \Pi_i} \bigcup_{0 \leq t \leq R(x)} g_t(x) \subset \Lambda \tag{4.3.4}$$

a Markov set. Note that $R_i = \overline{\mathrm{int}R_i}$ and $\mathrm{int}R_i \cap \mathrm{int}R_j = \emptyset$ for $i \neq j$.

4.4 Suspension Flow

We introduce the notion of suspension flow. Let $T : X \to X$ be the transformation map defined as in (4.3.2) restricted on X, and let R be the height function in (4.3.1). Consider the space

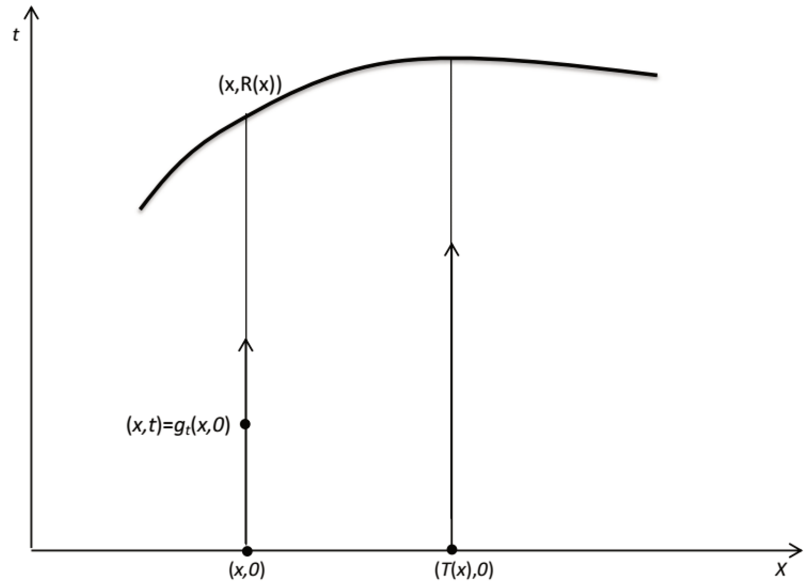
$$X_R = \{(x, s) \in X \times \mathbb{R} : 0 \leq s \leq R(x)\},$$

where the points $(x, R(x))$ and $(T(x), 0)$ are identified for each $x \in X$.

Definition. The suspension flow over T, with height function R is the flow $\psi = \{\psi_t\}_{t\in\mathbb{R}}$ in X_R, given by

$$\psi_t(x,s) = (T^n x, s + t - S_n R(x)),$$

where n is the unique non negative integer such that $0 \leq s+t-S_nR(x) < R(T^n x)$, i.e. $S_nR(x) \leq s+t < S_{n+1}R(x)$.



Using a Markov collection of a hyperbolic set Λ, we can obtain a symbolic representation of the flow g_t. Let Λ be a locally maximal hyperbolic set for the flow g_t, and given $\Pi_1, .., \Pi_n$ the associated Markov collection. Consider the $n \times n$ matrix A with entries

$$a_{ij} = \begin{cases} 1 \text{ if } \mathrm{int}T(\Pi_i) \cap \mathrm{int}\Pi_j \neq \emptyset, \\ 0 \text{ otherwise,} \end{cases}$$

where T is the Poincaré map. We also consider the set Σ_A given by:

$$\Sigma_A = \{\omega = (\omega_n)_{n\in\mathbb{Z}} : \forall n \in \mathbb{Z}, a_{\omega_n\omega_{n+1}} = 1\},$$

together with the shift map $\sigma : \Sigma_A \to \Sigma_A$ defined by $\sigma((\omega_n)_{n\in\mathbb{Z}}) = (\omega_{n+1})_{n\in\mathbb{Z}}$. Since the flow is topologically mixing, the matrix A is primitive aperiodic, i.e. there exists $n_0 \in \mathbb{N}$ such that $A^{n_0} > 0$.
Let $q, q^{'}$ be two positive integers, and $a_{-q}, .., a_0, .., a_q$ be a finite symbol sequence, we recall the definition of a $(q, q^{'})$-cylinder

$$C_{q,q'} = \{(\omega)_{n\in\mathbb{Z}} \in \Sigma_A : \omega_{-q} = a_{-q}, ..., \omega_{q'} = a_{q'}\}.$$

We denote by $C_{q,q'}(\omega)$ a $(q, q^{'})$-cylinder containing a point $\omega \in \Sigma_A$.

Definition. We define a coding map $\chi : \Sigma_A \to X$ by:

$$\chi(\omega) = \bigcap_n \overline{\bigcap_{|k|\leq n} T^{-k}(\mathrm{int}\Pi_{\omega_k})}$$

One can easily verify that the following diagram

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \downarrow{\chi} & & \downarrow{\chi} \\ X & \xrightarrow{T} & X \end{array}$$

is commutative i.e.

$$\chi \circ \sigma = T \circ \chi. \tag{4.4.1}$$

As at the beginning of this section, we define in the same way the suspension flow $\mathcal{S} = \{\mathcal{S}_t\}_{t\in\mathbb{R}}$ over $\sigma|_{\Sigma_A}$ with the Hölder height function $r = R \circ \chi$. We extend χ to finite-to-one onto map $\chi : \Delta \to \Lambda$ by $\chi(\omega, s) = (g_s \circ \chi)(\omega)$, for $(\omega, s) \in \Delta$. where $\Delta := \{(\omega, s) \in \Sigma_A \times \mathbb{R} : 0 \leq s \leq r(\omega)\}$. Due to (4.4.1), for every $t \in \mathbb{R}$ the following diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\mathcal{S}_t} & \Delta \\ \downarrow{\chi} & & \downarrow{\chi} \\ \Lambda & \xrightarrow{g_t} & \Lambda \end{array}$$

commutes i.e. $\chi \circ \mathcal{S}_t = g_t \circ \chi$, expressing that the flows g_t and $\mathcal{S}_t$ are conjugated.

4.5 Equilibrium Measures

4.5.1 Equilibrium measures for the flows

Let φ be a continuous function on M. For every $\epsilon > 0$ and $s > 0$ a set $E \subset M$ is called (ϵ, t) -separated if $x, y \in E$, $x \neq y$ implies that $d(g_\tau(x), g_\tau(y)) > \epsilon$ for some $\tau \in [0, s]$. Set

$$Z_s(\{g_t\}_t, \varphi, \epsilon) = \sup \left\{ \sum_{x \in E} \exp \int_0^t \varphi(f^\tau(x)) d\tau \right\} \tag{4.5.1}$$

where the supremum is taken over all (ϵ, s)-separated sets $E \subset X$. Define

$$P_M(\{g_t\}_t, \varphi, \epsilon) = \limsup_{s \to \infty} \frac{1}{t} \log Z_s(\{g_t\}_t, \varphi, \epsilon),$$

$$P_M(\{g_t\}_t, \varphi) = \lim_{\epsilon \to 0} P_M(\{g_t\}_t, \varphi, \epsilon). \tag{4.5.2}$$

Definition. We call $P_M(\{g_t\}_t, \varphi)$ the **topological pressure** of the function φ on M (with respect to the flow $\{g_t\}_t$). One can show that

$$P_M(\{g_t\}_t, \varphi) = P_M(g_1, \varphi_1)$$

where g_1 is a time-one map and $\varphi_1 = \int_0^1 \varphi(g_t(x)) dt$. Moreover, one can express the variational principal for the topological pressure in the case of flows as follows

$$P_M(\{g_t\}_t, \varphi) = \sup_{\mu \in \mathcal{M}(g_t)} \left(h_\mu(g_1) + \int_M \varphi_1 d\mu \right),$$

where $\mathcal{M}(g_t)$ is the set of all g_t-invariant Borel probability measures on M. Note that for any such measure μ $\int \varphi_1 d\mu = \int \varphi d\mu$.

Definition. A measure $\mu \in \mathcal{M}(g_t)$ is called an equilibrium measure for the function φ if

$$P_M(\{g_t\}_t, \varphi) = h_\mu(g_1) + \int_M \varphi_1 d\mu = h_\mu(g_1) + \int_M \varphi d\mu$$

.

4.5.2 Equilibrium measures for symbolic suspension flows

There is a canonical identification between the space of invariant measures of the symbolic suspension flows and of the subshift of finite type. Namely, for any invariant measure ν on Σ_A, the measure

$$\lambda_\nu = \frac{(\nu \times Leb)|\Delta}{\int r d\nu} \tag{4.5.3}$$

is a probability measure on Δ, which is invariant by $\{\mathcal{S}_t\}$, where Leb is the Lebesgue measure on $\mathbb{R}$.
Let $H : \Delta \to \mathbb{R}$ be a continuous function. Set

$$h_0(\omega) = \int_0^{r(\omega)} H\left(g_t(\chi(\omega))\right) dt, \quad h(\omega) = h_0(\omega) - cr(\omega), \tag{4.5.4}$$

where $c := P_\Delta(\mathcal{S}, H)$ is the topological pressure of the function H on Δ with respect to the symbolic suspension flow $\mathcal{S}$. $P_{\Sigma_A}(\sigma, h) = 0$, since $P_\Delta(\mathcal{S}, H)$ is the unique real number such that $P_{\Sigma_A}(\sigma, h_0 - cr(\omega)) = 0$ (see [36]).

The following statement describes equilibrium measures for symbolic suspension flows.

Proposition 4.5.1. *Assume that the function h is Hölder continuous on Σ_A with respect to the d_β metric for some $\beta > 1$. Then*

1. *there exists a unique equilibrium measure μ_H for the function H for the symbolic suspension flow $\mathcal{S} = \{\mathcal{S}_t\}$; the measure μ_H is ergodic and positive on open sets,*

2. *$\mu_H = \lambda_{\nu_h}$ where ν_h is the unique equilibrium for the function h and the measure λ_{ν_h} is defined by (4.5.3).*

Proposition 4.5.2. *Let Λ be a locally maximal hyperbolic set for the flow g_t and $H : \Lambda \to \mathbb{R}$ a Hölder continuous function. Then there exists a unique equilibrium measure μ_H corresponding to H. Moreover, the measure μ_H is ergodic and positive on open sets.*

From now on, we fix an equilibrium measure μ associated to a fixed Hölder continuous potential H. Let us also fix a measure ν associated to an Hölder continuous potential h.

4.6 Balls and Coding

We denote by Σ_A^+ the set of (one-sided) sequences $(\omega_n)_{n\geq 0}$ such that:

$$(\omega_n)_{n\geq 0} = (w_n)_{n\geq 0} \text{ for some } (w_n)_{n\in\mathbb{Z}} \in \Sigma_A,$$

and by Σ_A^- the set of (one-sided) sequences $(\omega_n)_{n\leq 0}$ such that:

$$(\omega_n)_{n\leq 0} = (w_n)_{n\leq 0} \text{ for some } (w_n)_{n\in\mathbb{Z}} \in \Sigma_A.$$

We also consider the shift maps $\sigma^+ : \Sigma_A^+ \to \Sigma_A^+$ and $\sigma^- : \Sigma_A^- \to \Sigma_A^-$ defined by

$$\sigma^+((\omega_n)_{n\geq 0}) = (\omega_{n+1})_{n\geq 0} \text{ and } \sigma^-((\omega_n)_{n\leq 0}) = (\omega_{n-1})_{n\leq 0}$$

Now we describe how we use symbolic dynamics to characterize distinct points in a stable or unstable manifold. Given $x \in \Lambda$, take $\omega \in \Sigma_A$ such that $\chi(\omega) = x$. Let $\Pi(x)$ be a rectangle of the Markov collection containing x. Let $\chi^+ : \Sigma_A \to \Sigma_A^+$ and $\chi^- : \Sigma_A \to \Sigma_A^-$ be the projection maps, defined by:

$$\chi^+((\omega_n)_{n\in\mathbb{Z}}) = (\omega_n)_{n\geq 0} \text{ and } \chi^-((\omega_n)_{n\in\mathbb{Z}}) = (\omega_n)_{n\leq 0}.$$

For each $\omega^{'} \in \Sigma_A$, we have

$$\chi(\omega^{'}) \in W^u_{loc}(x) \cap R(x) \text{ whenever } \chi^-(\omega^{'}) = \chi^-(\omega),$$

and

$$\chi(\omega^{'}) \in W^s_{loc}(x) \cap R(x) \text{ whenever } \chi^+(\omega^{'}) = \chi^+(\omega).$$

Therefore, writing $\omega = (\omega_n)_{n\in\mathbb{Z}}$, the set $W^u_{loc}(x) \cap R(x)$ can be identified with the cylinder set:

$$C^+_{\omega_0} = \{(a_n)_{n\geq 0} \in \Sigma_A^+ : a_0 = \omega_0\} \subset \Sigma_A^+ \tag{4.6.1}$$

and the set $W^s_{loc}(x) \cap R(x)$ can be identified with the cylinder set:

$$C^-_{\omega_0} = \{(a_n)_{n\leq 0} \in \Sigma_A^- : a_0 = \omega_0\} \subset \Sigma_A^-. \tag{4.6.2}$$

Definition. We define the measure ν^u on Σ_A^+ such that for any cylinder C_{i_0,i_n}, we have

$$\nu^u(\chi^+(C_{i_0,i_n})) = \nu(C_{i_0,i_n}).$$

Similarly, we define the measure ν^s on Σ_A^- such that for any cylinder C_{i_{-n},i_0}, we have

$$\nu^s(\chi^-(C_{i_{-n},i_0})) = \nu(C_{i_{-n},i_0}).$$

Definition. Given $x \in \Lambda$, we define functions $a^{(u)}(x)$ and $a^{(s)}(x)$ by

$$a^{(u)}(x) = \lim_{t \to 0} \frac{\log ||dg_t|_{E^u(x)}||}{t}, \tag{4.6.3}$$

$$a^{(s)}(x) = \lim_{t \to 0} \frac{\log ||dg_t|_{E^s(x)}||}{t}.$$

Since the subspaces $E^u(x)$ and $E^s(x)$ depend Hölder continuously on x the functions $a^{(u)}(x)$ and $a^{(s)}(x)$ are also Hölder continuous. Note that there exist constants $\bar{c}$ and $\bar{c'}$ such that $a^{(u)}(x) > \bar{c} > 0$ and $a^{(s)}(x) < \bar{c'} < 0$ for every $x \in \Lambda$. For any $x \in \Lambda$ and $t \in \mathbb{R}$, we have:

$$||dg_t(v)|| = ||v|| \exp \int_0^t a^{(u)}(g_\tau(x))d\tau \quad \text{for any } v \in E^u(x), \tag{4.6.4}$$

and

$$||dg_t(w)|| = ||w|| \exp \int_0^t a^{(s)}(g_\tau(x))d\tau \quad \text{for any } w \in E^s(x). \tag{4.6.5}$$

Let $\tilde{a}^{(u)}$ and $\tilde{a}^{(s)}$ be the pull back of the functions $a^{(u)}$ and $a^{(s)}$ by the coding map χ, defined by

$$\tilde{a}^{(u)}(\omega, t) := a^{(u)}(g_t(\chi(\omega))) \text{ and } \tilde{a}^{(s)}(\omega, t) := a^{(s)}(g_t(\chi(\omega))). \tag{4.6.6}$$

Let also $\mathbf{a}^{(u)}$ and $\mathbf{a}^{(s)}$ be the Hölder continuous function on Σ_A defined by

$$\mathbf{a}^{(u)}(\omega) = \exp \int_0^{r(\omega)} \tilde{a}^{(u)}(\omega, t)dt, \quad \mathbf{a}^{(s)}(\omega) = \exp \int_0^{r(\omega)} \tilde{a}^{(s)}(\omega, t)dt. \tag{4.6.7}$$

These are respectively the dilatation in the unstable direction and the contraction in the stable direction of the flow $(g_t)_{t \in \mathbb{R}}$ between two consecutive passages in the Poincaré section (these two passages are coded by ω and $\sigma(\omega)$ respectively): This what the following lemma says

Lemma 4.6.1. *For any $\omega \in \Sigma_A$ such that $x = \chi(\omega)$, we have*

$$\boldsymbol{a}^{(u)}(\omega) = |D_x T_{|E^u}| \text{ and } \boldsymbol{a}^{(s)}(\omega) = |D_x T_{|E^s}|$$

This comes directly from (4.6.4) and (4.6.5) applied to $x \in X$ and $t = R(x)$

Proof. Using the definition of $a^u(x)$ in (4.6.3), then for a small t we have $|D_x g_t|E^u| \simeq e^{ta^u(x)}$, and since $a^u(.)$ is continuous, we get:

$$\begin{aligned}
\mathbf{a}^{(u)}(\omega) &= \exp \int_0^{r(\omega)} a^{(u)}(g_s(x))ds \\
&= \prod_{i=1}^{m} \exp \int_{s_i}^{s_i+t_i} a^{(u)}(g_s(x))ds = \prod_{i=1}^{m} \exp\left(t_i a^{(u)}(g_{s_i}(x))\right) \\
&= \prod_{i=1}^{m} |D_{g_{s_i(x)}} g_{t_i}{}_{|E^u}| = |D_x g_{t|E^u}| = |D_x T_{|E^u}|.
\end{aligned}$$

□

As we will work with the Poincaré section, these quantities will naturally appear in our calculations.

Proposition 4.6.2. *There exist $c > 0$ and $\alpha_1 > 0$, such that for any positive integer k and any 2-sided k-cylinder C_{ω_{-k},ω_k},*

$$\textit{diam}\ \chi\left(C_{\omega_{-k},\omega_k}\right) \leq ce^{-\alpha_1 k}$$

The following lemma is a result of regularity of $\mathbf{a}^{(u)}$ and of $\mathbf{a}^{(s)}$ which will be very useful later.

Lemma 4.6.3. *There exist $\alpha > 0$, $\alpha' > 0$, $c^u_{\mathbf{a}} > 0$ and $c^s_{\mathbf{a}} > 0$ such that, for any ω, ω' such that $d(\omega, \omega') \leq e^{-k}$,*

$$\boldsymbol{a}^{(u)}(\omega) \leq \boldsymbol{a}^{(u)}(\omega')e^{c^u_{\mathbf{a}}}e^{-\alpha k}, \quad \boldsymbol{a}^{(s)}(\omega) \leq \boldsymbol{a}^{(s)}(\omega')e^{c^s_{\mathbf{a}}}e^{-\alpha' k} \tag{4.6.8}$$

Proof. we have

$$\frac{\mathbf{a}^{(u)}(\omega)}{\mathbf{a}^{(u)}(\omega')} = \exp\left(\int_0^{r(\omega)} \tilde{a}^{(u)}(\omega, t)dt - \int_0^{r(\omega')} \tilde{a}^{(u)}(\omega', t)dt\right).$$

The function $\tilde{a}^{(u)}(., t)$ is Hölder, indeed, due to Proposition 4.6.2, then $d(\chi(\omega), \chi(\omega')) \leq ce^{-\alpha_1 k}$, and since $a^{(u)}$ is Hölder then there exist $c_a > 0$ and $\alpha_2 > 0$ such that we have:

$$\begin{aligned}
|\tilde{a}^{(u)}(\omega) - \tilde{a}^{(u)}(\omega')| &= |a^{(u)}(\chi(\omega)) - a^{(u)}(\chi(\omega'))| \\
&\leq c_a d(\chi(\omega), \chi(\omega'))^{\alpha_2} \\
&\leq c_a ce^{-\alpha_1\alpha_2 k}.
\end{aligned}$$

Moreover, the height function r is also Hölder, then there is $c_r > 0$ and $\alpha_3 > 0$ such that

$$\begin{aligned}\frac{\mathbf{a}^{(u)}(\omega)}{\mathbf{a}^{(u)}(\omega^{'})} &= \exp\left(\int_0^{r(\omega)} \tilde{a}^{(u)}(\omega,t)dt - \int_0^{r(\omega)} \tilde{a}^{(u)}(\omega^{'},t)dt \right. \\ &\qquad \left. + \int_0^{r(\omega)} \tilde{a}^{(u)}(\omega^{'},t)dt - \int_0^{r(\omega^{'})} \tilde{a}^{(u)}(\omega^{'},t)dt\right) \\ &\leq \exp\left(r(\omega)c_a c e^{-\alpha_1\alpha_2 k} + \sup|\tilde{a}^u(\omega^{'},t)|c_r d(\omega,\omega^{'})^{\alpha_3}\right) \\ &\leq c_{\mathbf{a}} e^{-\alpha k},\end{aligned}$$

where $c_{\mathbf{a}} = \max(r_{\max} c c_a, \sup|\tilde{a}^u(.,t)|c_r)$ and $\alpha = \min(\alpha_1\alpha_2, \alpha_3)$. We prove the second inequality of (4.6.8) analogously. $\square$

Definition. Let $x \in \Lambda$ such that $x = \chi(\omega)$, where $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Sigma_A$. Set $l_n^u(\omega)$ which represent the minimal unstable length between x and the extremities of $T^{-n}(W^u_{\Pi_n}(T^n(x)))$ and $l_m^s(\omega)$ representing the minimal stable length between x and $T^m(W^s_{\Pi_{-m}}(T^{-m}(x)))$. We write

$$W^u_{\Pi_n}(T^n(x)) = W^{u+}_{\Pi_n}(T^n(x)) \sqcup W^{u-}_{\Pi_n}(T^n(x)),$$

where $W^{u+}_{\Pi_n}(T^n(x))$ and $W^{u-}_{\Pi_n}(T^n(x))$ are two curves starting at $T^n(x)$ and ending at one of the extremities of $W^u_{\Pi_n}(T^n(x))$. We have

$$l_n^u(\omega) = \min\{l_n^{u+}(\omega), l_n^{u-}(\omega)\},$$

where $l_n^{u+}(\omega)$ and $l_n^{u-}(\omega)$ are the lengths of $T^{-n}(W^{u+}_{\Pi_n}(T^n(x)))$ and $T^{-n}(W^{u-}_{\Pi_n}(T^n(x)))$ respectively. We also define $l_m^s(\omega)$, by

$$l_m^s(\omega) = \min\{l_m^{s+}(\omega), l_m^{s-}(\omega)\},$$

where $l_m^{s+}(\omega)$ and $l_m^{s-}(\omega)$ are the lengths of $T^m(W^{s+}_{\Pi_m}(T^{-m}(x)))$ and $T^m(W^{s-}_{\Pi_m}(T^{-m}(x)))$ respectively.

Proposition 4.6.4. *There exists $C > 0$ such that for all $x \in X$, $x = \chi(\omega)$, where $\omega := (\omega_n)_{n\in\mathbb{Z}} \in \Sigma_A$, for all $n \geq 0$ and all $m \geq 0$, we have the following property*

$$\frac{1}{C}\prod_{k=0}^{n-1} \boldsymbol{a}^u(\sigma^k\omega)^{-1} \leq l_n^{u+}(\omega) + l_n^{u-}(\omega) \leq C\prod_{k=0}^{n-1} \boldsymbol{a}^u(\sigma^k\omega)^{-1}$$

and

$$\frac{1}{C}\prod_{k=-m}^{-1} \boldsymbol{a}^s(\sigma^{-k}\omega) \leq l_m^{s+}(\omega) + l_m^{s-}(\omega) \leq C\prod_{k=-m}^{-1} \boldsymbol{a}^s(\sigma^{-k}\omega)$$

Proof. Consider $W^u(T^n x)$ the unstable manifold at $T^n x$. We consider the local unstable manifold $W^u_{\Pi_{\omega_n}}(T^n x)$. Denote by $W := W^u_{\Pi_{\omega_n}}(T^n x)$ and $\gamma := T^{-n}\left(W^u_{\Pi_{\omega_n}}(T^n x)\right)$. γ is a curve on M, $\gamma : [a,b] \to M$, such that $|\dot{\gamma}(s)| = 1$, therefore $\ell(\gamma) = b - a$. First let us compute

$$\ell(W) = \int_a^b |(D(T^n \circ \gamma(s)))| ds.$$

Let $\omega_s' \in \Sigma_A$ such that $\gamma(s) = \chi(\omega_s')$, then we have $T^i(\gamma(s)) = \chi(\sigma^i \omega_s')$, according to (4.4.1). Then using the chain rule and Lemma 4.6.1

$$\begin{aligned}
\ell(W) &= \int_a^b |D_{T^{n-1}\gamma(s)} T(T^{n-1}(\gamma(s)))|.|D_{T^{n-2}\gamma(s)} T(T^{n-2}(\gamma(s)))|...|D_{\gamma(s)} T(\gamma(s))|.|\dot{\gamma}(s)| ds \\
&= \int_a^b \mathbf{a}^{(u)}(\sigma^{(n-1)}(\omega_s')).\mathbf{a}^{(u)}(\sigma^{(n-2)}(\omega_s'))...\mathbf{a}^{(u)}(\sigma(\omega_s')).\mathbf{a}^{(u)}(\omega_s') ds \\
&= \int_a^b \prod_{i=0}^{n-1} \mathbf{a}^{(u)}(\sigma^i \omega_s').
\end{aligned}$$

For all $s \in [a,b]$, we have $\omega_s' \in C_{\omega_0,\omega_n}$ and $\gamma(s) \in \chi(C_{\omega_0,\omega_n})$ then $\sigma^i \omega_s' \in \sigma^i(C_{\omega_0,\omega_n})$, and hence we get:

$$d(\sigma^i \omega, \sigma^i \omega_s') \leq e^{-(n-i)}. \tag{4.6.9}$$

Then using Lemma 4.6.3, it follows that for all $s, r \in (a,b)$

$$\begin{aligned}
\prod_{i=0}^{n-1} \mathbf{a}^u(\sigma^i(\omega_s')) &\leq \prod_{i=0}^{n-1} \mathbf{a}^u(\sigma^i(\omega_r')) \exp c_{\mathbf{a}} e^{-\alpha(n-i)} \\
&\leq \prod_{i=0}^{n-1} \mathbf{a}^u(\sigma^i(\omega_r')) \exp c_{\mathbf{a}} \sum_{i=0}^{n-1} e^{-\alpha(n-i)},
\end{aligned}$$

but $\sum_{i=0}^{n-1} e^{-\alpha(n-i)} \leq \frac{1}{1-e^{-\alpha i}}$, then setting the constant $C := \exp\left(\frac{c_a}{1-e^{-\alpha}}\right)$, we get

$$\prod_{i=0}^{n-1} \mathbf{a}^u(\sigma^i(\omega_s')) \leq C \prod_{i=0}^{n-1} \mathbf{a}^u(\sigma^i(\omega_r')) \tag{4.6.10}$$

Applying (4.6.10), with $\omega_r' = \omega$, we get an upper bound of the product depending on s by something which doesn't depend on s, we obtain

$$\ell(W) = \int_a^b \prod_{i=0}^{n-1} \mathbf{a}^u(\sigma^i(\omega_s')) ds \leq C \prod_{i=0}^{n-1} \mathbf{a}^u(\sigma^i(\omega)) \int_a^b ds = C\ell(V) \prod_{i=0}^{n-1} \mathbf{a}^u(\omega),$$

now applying (4.6.10), with $\omega_s^{'} = \omega$, we get this time a lower bound

$$C^{-1}\ell(V)\prod_{i=0}^{n-1}\mathbf{a}^u(\sigma^i(\omega)) \leq \int_a^b \prod_{i=0}^{n-1}\mathbf{a}^u(\sigma^i(\omega_r^{'}))dr = \ell(W).$$

There exists $C^{'} > 1$ such that the length of W is uniformly bounded from below by $\frac{1}{C'}$ and from above by $C^{'}$

$$\frac{1}{CC'}\prod_{i=0}^{n-1}\left(\mathbf{a}^u(\sigma^i(\omega))\right)^{-1} \leq l_n^u(\omega) \leq CC^{'}\prod_{i=0}^{n-1}\left(\mathbf{a}^u(\sigma^i(\omega))\right)^{-1}.$$

The proof for $l_m^s(\omega)$ follows exactly the same scheme up to replacing T by T^{-1}. □

Lemma 4.6.5. *Let $x_1, x_2 \in X$, $r_1, r_2 > 0$, $c_L > 0$ be the Lipschitz constant in Proposition 4.2.1. For all $m, n \geq 0$, $\omega = (\omega_i)_{i\in\mathbb{Z}} \in \Sigma_A$. Consider the stable and unstable balls such that:*

$$B^s(x_1, r_1) \subset \chi(C_{\omega_0,\omega_n}) \text{ and } B^u(x_2, r_2) \subset \chi(C_{\omega_{-m},\omega_0}).$$

There exists $x \in X$ such that we have

$$B\left(x, \frac{\min(r_1, r_2)}{c_L}\right) \subset B^s(x, r_1) \times B^u(x, r_2) \subset \chi(C_{\omega_{-m},\omega_n})$$

Proof. Set $x = \{x_1, x_2\}$ and let $z \in B\left(x, \frac{\min(r_1,r_2)}{c_L}\right)$. Set $z = z^s, z^u$, we have $z^s \in W^s(z)$ and $z^u \in W^u(z)$. Due to Proposition 4.2.1, there exists $c_L > 0$ such that

$$d^s(x_1, z^s) \leq c_L\frac{\min(r_1, r_2)}{c_L} \leq r_1 \quad \text{and} \quad d^u(x_2, z^u) \leq r_2,$$

and hence we get

$$z \in B^s(x, r_1) \times B^u(x, r_2) \subset \chi(C_{\omega_{-m},\omega_0}).$$

□

Lemma 4.6.6. *Let $x \in X$, $r > 0$ and $c_L > 0$ be the Lipschitz constant in Proposition 4.2.1. Consider the $(q, q^{'})$-cylinder $C_{q,q^{'}}(x)$ where q and $q^{'}$ are minimal such that $c_L l_q^s(x) \leq \frac{r}{2}$ and $c_L l_{q^{'}}^u(x) \leq \frac{r}{2}$, then we have*

$$\chi(C_{q,q^{'}}) \subset B(x, r).$$

Proof. Let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ such that $x = \chi(\omega)$. Suppose that $x \in \mathrm{int}\Pi_{\omega_0}$, where we consider $\Pi_{\omega_0}, \Pi_{\omega_1}, ..., \Pi_{\omega_n}$ the Markov collection for g_t on Λ. Each rectangle Π_{ω_i} is contained in a smooth disk D_i and has the product structure $\{.,.\}$ as in (4.2.1). Let z and $z^{'}$ two points in Π_{ω_i} such that

$$z = \{x_1, x_2\} \text{ and } z^{'} = \{x_1^{'}, x_2^{'}\},$$

then due to Proposition 4.2.1 we get

$$\begin{aligned}
\mathrm{diam}(\chi(C_{-q,q^{'}})) &= \max_{z,z'\in\bigcap_{i=-q}^{q^{'}} T^{-i}(\Pi_{\omega_i})} d(z,z^{'}) \\
&= \max_{x_1,x_2,x_1^{'},x_2^{'}\in\Pi_i} d\left(\{x_1,x_2\},\{x_1^{'},x_2^{'}\}\right) \\
&\leq c_L d\left((x_1,x_2),(x_1^{'},x_2^{'})\right) \\
&\leq c_L\left(d(x_1,x_1^{'}) + d(x_2,x_2^{'})\right) \\
&\leq c_L\left(l_q^s(\omega) + l_{q^{'}}^u(\omega)\right) \leq r
\end{aligned}$$

which proves the lemma due to the conditions considered on q and $q^{'}$. □

Lemma 4.6.7. *For almost everywhere $x \in X$, any $r > 0$ small enough, and for all m and n maximal such that $c_L\ r \leq \min(l^u_{\delta_n,n}(x), l^s_{\delta_m,m}(x))$, the (m,n)-cylinder $C_{m,n}(x)$ is such that*

$$B(x,r) \subset \chi(C_{m,n}(x))$$

Proof. Let $\delta > 0$. Given $x \in X$, we set $W^u_\delta(T^n x)$ the local unstable manifold at $T^n x$ of length δ. Let $p \geq 0$, set $\delta_n = |n^{-p}| > 0$, we claim that

$$\sum_{n\in\mathbb{Z}} \chi_*\nu(x : d(T^n x : \partial^u\Pi_{\omega_n}) < \delta_n) < \infty,$$

Due to the Borel Cantelli Lemma, for $\chi_*\nu$-a.e. $x \in X$, there exists $n(x)$ such that $\forall |n| > n(x)$, we have $d(T^n x, \partial^u\Pi_{\omega_n}) > \delta_n$. Then $\Lambda \cap W^u_{\delta_n}(T^n x) \subset \Pi_{\omega_n}$, and hence we get

$$B^u(x, l^u_{\delta_n,n}(x)) \subset T^{-n}(W^u_{\delta_n}(T^n x)). \tag{4.6.11}$$

Due to the Markov property, the inclusion $\Lambda \cap T^{-k}(W^u_{\delta_n}(T^n x)) \subset \Pi_{w_{n-k}}$ holds for every $k = 0,..,n$, and hence $\Lambda \cap T^{-n}(W^u_{\delta_n}(T^n x)) \subset \chi(C_{\omega_0,\omega_n})$. Then with (4.6.11), we get

$$\Lambda \cap B^u(x, l^u_{\delta_n,n}(x)) \subset \chi(C_{\omega_0,\omega_n}) \tag{4.6.12}$$

Similarly with respect to the stable direction, $\forall m > n(x)$, we have $\Lambda \cap W^s_{\delta_m}(T^{-m}x) \subset \Pi_{\omega_{-m}}$ and we also get that

$$\Lambda \cap B^s(x, l^s_{\delta_m,m}(x)) \subset \chi(C_{\omega_{-m},\omega_0}) \tag{4.6.13}$$

Let $y \in \Lambda$ be very close to x ($y \in B(x,r)$), then $y \in \Pi_{\omega_0}$. Set $z = \{y, x\} \in W^s(y) \cap \Pi_{\omega_0}$.
Using Proposition 4.2.1, we have $z \in B^u(x, c_L r)$. Moreover due to [4] then $z \in \Lambda$.
Using (4.6.12) and since $c_L\ r \le l^u_{\delta_n,n}(x)$, this gives that $z \in \chi(C_{\omega_0,\omega_n})$.
We have that $z \in W^s(y)$, y and z are in the same rectangle Π_{ω_0}. Then due to the Markov property, as $z \in \chi(C_{\omega_0,\omega_n})$ then also is y, giving

$$B(x,r) \subset \chi(C_{\omega_0,\omega_n}).$$

In the same way, considering the stable direction, using (4.6.13) and since $c_L\ r \le l^s_{\delta_m,m}(x)$, we get

$$B(x,r) \subset \chi(C_{\omega_{-m},\omega_0}),$$

finishing the proof.

$\square$

Chapter 5

Pointwise convergence of the recurrence rate to the dimension

5.1 Description of the $\mathbb{Z}$-extension

Consider a Riemannian manifold $\tilde{M}$ endowed with a $\sigma-$finite measure $\tilde{\mu}$. Let $\tilde{g}_t : \tilde{M} \to \tilde{M}$ be a flow on $\tilde{M}$ preserving the measure $\tilde{\mu}$.

Let $I : \tilde{M} \circlearrowleft$ be an isometry of $\tilde{M}$ such that: $\Gamma = \{I^n, n \in \mathbb{Z}\}$ is an infinite group of isometries ($i.e. I^n = I^m \Rightarrow n = m$) and such that I preserves $\tilde{\mu}$. We suppose that:

- $M = \tilde{M}/\Gamma = \{\Gamma.\tilde{x}, \tilde{x} \in \tilde{M}\}$ is a compact manifold.
- $\tilde{g}_t(I.\tilde{x}) = I.\tilde{g}_t(\tilde{x})$

This ensures that we can define a flow $(g_t)_t$ on M by:

$$g_t(\Gamma\tilde{x}) = \Gamma\tilde{g}_t(\tilde{x}).$$

Moreover, we assume that $(M, (g_t)_t)$ is an Axiom A flow and that the measure μ defined on M from the measure $\tilde{\mu}$ by passing through the quotient, is an equilibrium measure for $(g_t)_t$. By construction of μ, for every $A \in \mathcal{B}(\tilde{M})$ such that $A \cap \bigcup_{k\in\mathbb{Z}}(I^k A) = \emptyset$, we have

$$\mu(\{\Gamma.x, x \in A\}) = \tilde{\mu}(A)$$

Our interest is to study the time needed for the flow $\tilde{g}_t$ to return back to an ϵ-neighborhood of its starting point.

Definition. For any $x \in \tilde{M}$, we define the return time of the flow $\tilde{g}_t$:

$$\tau_\epsilon(x) := \inf\{t > 1 : \tilde{g}_t(x) \in B(x, \epsilon)\},$$

where $B(x, \epsilon)$ is the ball of center x and radius ϵ.

We will prove the following result:

Theorem 5.1.1. *Let $(\tilde{M}, \{\tilde{g}_t\}, \tilde{\mu})$ be a flow satisfying all the hypothesis above. Then for $\tilde{\mu}$-almost every point $x \in \tilde{M}$,*

$$\lim_{\epsilon \to 0} \frac{\log \sqrt{\tau_\epsilon}}{-\log \epsilon} = \frac{h}{L^u} + \frac{h}{L^s}.$$

Let $M_0 \subset \tilde{M}$ be a fundamental domain which is closed and connected, such that

$$\bar{M_0} = \overline{\mathrm{int} M_0} \text{ and } \tilde{\mu}(\partial M_0) = 0.$$

Furthermore, we assume that M_0 is such that $\#\{k \in \mathbb{Z} : dist(M_0, I^k M_0) \leq \tilde{r}_{inj}\} < \infty$, where we set $\tilde{r}_{inj}$ to be the radius of injectivity of $\tilde{M}$. We define the constant $M_L > 0$ by:

$$M_L := \sup\{|k| : dist(M_0, I^k M_0) \leq \tilde{r}_{inj}\} < \infty. \tag{5.1.1}$$

We take the notations and notions from chapter 4 for $(M, (g_t)_t, \mu)$. For every $\ell \in \mathbb{Z}$, we call ℓ-cell the set $I^\ell M_0$.

Definition. Let $y \in M$ and $t > 0$.
We define $\varphi_t(y)$ as the index of the cell containing $\tilde{g}_t(\tilde{y}_0)$ where $\tilde{y}_0$ is the representant of y in M_0, that is $\varphi_t(y)$ is the unique integer such that $\tilde{g}_t(\tilde{y}_0)$ is in $I^{\varphi_t(y)} M_0$.
We define $n_t(y)$ as the number of times the $(g_s(y))_{s \in [0,t]}$ traverses X.

Definition. We define $\varphi : X \to \mathbb{Z}$ to be the unique integer such that $\tilde{g}_{R(x)}(\tilde{x}_0)$ is in $I^{\varphi(x)} M_0$.

Proposition 5.1.2. *The function φ is Hölder.*

Proof. We have that M_L defined in (5.1.1) is finite. This implies that $\inf_{x \in \tilde{M}, n \in \mathbb{Z}} d(x, I^n x) \geq \epsilon_0' > 0$.
Let us take a Poincaré section introduced in chapter 4 such that $\epsilon_0 < \epsilon_0'$ and

strictly less than $\tilde{r}_{inj}$. Adapting M_0 we can suppose that the set of disks $\mathcal{D}$ of the Poincaré section are the projections of disks contained in M_0, by replacing M_0 by

$$\left(M_0 \cup \bigcup_{\tilde{D}\in\mathcal{D}:\tilde{D}\cap\partial M_0} \tilde{D}^{[\tilde{\epsilon}_0]} \right) \setminus \bigcup_{\tilde{D}\in\mathcal{D}:\tilde{D}\cap\partial M_0} \bigcup_{n\neq 0} I^n(\tilde{D}^{[\tilde{\epsilon}_0]})$$

with $\tilde{\epsilon}_0$ satisfying:

- $\tilde{\mu}\left(\partial\left(\tilde{D}^{[\tilde{\epsilon}_0]}\right)\right) = 0$,
- the projections $\Gamma.y \,:\, y \in \tilde{D}^{[\tilde{\epsilon}_0]}$ are pairwise disjoint.

Therefore, with this modification φ is constant on each $D_i \cap T^{-1}D_j$ for $i \neq j$. □

Definition. We define $\phi : X_R \to \mathbb{R}$, by:

$$\phi(x,u) = \frac{\varphi(x)}{R(x)}, \quad x \in X, u \in [0, R(x)[.$$

Lemma 5.1.3. *$\int_0^t \phi \circ \psi_s ds$ satisfies the Central Limit Theorem with the variance $\sigma_\phi^2 := \frac{\sigma_\varphi^2}{\int_X R d\nu}$, where $\{\psi_s\}_{s\in\mathbb{R}}$ is the suspension flow defined in 4.4. That is, $\frac{1}{\sqrt{t}}\int_0^t \phi \circ \psi_s ds$ converges in distribution with respect to the measure μ as $t \to +\infty$ to a centered Gaussian random variable with variance σ_ϕ^2.*

Proof. We have φ and R satisfies the CLT using [17] and $\int_0^{R(x)} \phi(x,u)du = \frac{\varphi(x)}{R(x)}R(x) = \varphi(x)$, then we deduce using Theorem 1.1 in [32] that ϕ satisfies the CLT with variance $\frac{\sigma_\phi^2}{\int_X R d\nu}$. □

Lemma 5.1.4. *For every $(x,u) \in X_R$, there exists a real number $N_L > 0$, such that for all $t > 0$ and $\ell \in \mathbb{Z}$, we have*

$$|\int_0^t \phi(\psi_s(x,u))ds + \varphi_{t+u}(x)| \leq 2\|\varphi\|_\infty + N_L \tag{5.1.2}$$

Proof. Let $x \in X$ and $0 < u < R(x)$, we have, for all $t > 0$:

$$\begin{aligned}\int_0^t \phi(\psi_s(x,u))ds &= \int_0^{R(x)-u} \frac{\varphi(x)}{R(x)}ds + \sum_{k=1}^{n_t(g_u(x))-1} \int_{S_k R(x)-u}^{S_{k+1}R(x)-u} \phi(\psi_s(x,u))ds \\ &\quad + \int_{S_{n_t(g_u(x))}R(x)-u}^{t} \phi(\psi_s(x,u))ds. \end{aligned} \tag{5.1.3}$$

Note that whenever $S_kR(x) - u \leq s \leq S_{k+1}R(x) - u$, we have

$$\phi(\psi_s(x,u)) = \phi(\psi_{s+u}(x,0)) = \phi(T^k x, t_1), \text{ where } t_1 \in [0, R(T^k(x))]$$

from which we get whenever $S_kR(x) \leq s + u \leq S_{k+1}R(x)$, that

$$\sum_{k=1}^{n_t(g_u(x))-1} \int_{S_kR(x)}^{S_{k+1}R(x)} \frac{\varphi(T^k x)}{R(T^k x)} ds = \frac{\varphi(T^k x)}{R(T^k x)} R(T^k x) = \varphi(T^k x).$$

Thus the following term in equality (5.1.3) is equal to:

$$\sum_{k=1}^{n_t(g_u(x))-1} \int_{S_kR(x)-u}^{S_{k+1}R(x)-u} \phi(\psi_s(x,u))ds = \sum_{k=1}^{n_t(g_u(x))-1} \varphi(T^k x) = S_{n_t(g_u(x))}\varphi(x).$$

Substituting the previous term in (5.1.3), and using the definition of the function ϕ, we get

$$\int_0^t \phi(\psi_s(x,u))ds = S_{n_t(g_u(x))}\varphi(x) - \int_{-u}^0 \frac{\varphi(x)}{R(x)} ds + \int_{S_{n_t(g_u(x))}R(x)-u}^t \frac{\varphi(T^{n_t(g_u(x))}(x))}{R(T^{n_t(g_u(x))}(x))} ds,$$

Note that whenever $t < S_{n_t(g_u(x))+1}R(x) - u$, the following term in the previous equation is bounded as follows:

$$\int_{S_{n_t(g_u(x))}R(x)-u}^t \frac{\varphi(T^{n_t(g_u(x))}(x))}{R(T^{n_t(g_u(x))}(x))} ds \leq \frac{\|\varphi\|_\infty}{R(T^{n_t(g_u(x))}(x))} R(T^{n_t(g_u(x))}(x)) = \|\varphi\|_\infty.$$

Let us set the constant $N_L = \sup_{x \in M} \sup_{t \leq R(x)} |\varphi_{t+u}(x) - S_{n_t(g_u(x))}\varphi(x)|$. Combining all the previous assumptions together, we get

$$\left| \int_0^t \phi(\psi_s(x,u))ds + \varphi_{t+u}(x) \right| \leq 2\|\varphi\|_\infty + N_L$$

□

We aim to deduce the statistical properties of φ_t from the corresponding statistical properties of ϕ on the base dynamics. Thus we have the following proposition

Proposition 5.1.5. *φ_t satisfies the Central Limit Theorem with the variance $\sigma^2_{flow} := \frac{\sigma^2_\varphi}{\int_X R d\nu}$. That is, $\frac{1}{\sqrt{t}}\varphi_t$ converges in distribution with respect to the measure μ as $t \to +\infty$ to a centered Gaussian random variable with variance σ^2_{flow}.*

Proof. We set $\epsilon_t := \frac{2\|\varphi\|_\infty}{\sqrt{t}} + \frac{N_L}{\sqrt{t}}$. Now using (5.1.2) and dividing by $\sqrt{t}$, for all $t > 0$, we get

$$\left| \frac{\varphi_{t+u}(x)}{\sqrt{t}} - \frac{\int_0^t \phi(\psi_s(x,u))ds}{\sqrt{t}} \right| \leq \epsilon_t$$

When t goes to infinity, the term ϵ_t goes to zero. Then using Lemma 5.1.3, we deduce the CLT for φ_t. □

5.2 Proof of the almost sure convergence Theorem

We want to know the time needed for the flow $\tilde{g}_t$ starting from a point $y \in \tilde{M}$ to return back into an ϵ-neighborhood of this point, say $B(y, \epsilon)$. Setting $y = g_s(x)$, the point y corresponds to the coordinates (x, s, n_0), where x is a point on the Poincaré section X in an initial cell n_0. Projecting on D, let $k = \sup_D\{\text{constant Lipschitz of } P_D\}$, we can observe that:

$$B_D(x, \frac{\epsilon}{k}) \subset P_D(B_M(y, \epsilon)) \subset B_D(x, \epsilon k) \tag{5.2.1}$$

Definition. Let $x \in X$, such that $y = g_s(x)$, We define the first return time of the transformation $\tilde{T}$ into $P_D(B(g_s(x), \epsilon))$ in the cell n_0 by:

$$w_\epsilon(x) = \min\{k \geq 1 : T^k(x) \in P_D(B(g_s(x), \epsilon)) \text{ and } \sum_{j=0}^{k-1} \varphi \circ T^j(x) = 0\}$$

Lemma 5.2.1. *Let $y \in M$ and $x \in X$, such that $y = (x, s)$, where $s < R(x)$. The return time of the flow g_t and that of the transformation T are related in a manner such that, for almost every $y = (x, s)$*

$$\tau_\epsilon(y) \sim w_\epsilon(x) \int_X R d\nu. \tag{5.2.2}$$

Proof. Recall that $\tau_\epsilon(y) = \inf\{t > 0 : \tilde{g}_t(y) \in B(y, \epsilon)\}$, we have

$$\sum_{k=0}^{w_\epsilon(x)-1} R \circ T^k(x) \leq \tau_\epsilon(y) + s < \sum_{k=0}^{w_\epsilon(x)} R \circ T^k(x),$$

from which we deduce that

$$\tau_\epsilon(y) + s = \sum_{k=0}^{w_\epsilon(x)-1} R \circ T^k(x) + s^{'}, \tag{5.2.3}$$

where $0 \leq s^{'} = s^{'}_\epsilon(x) < R \circ T^{w_\epsilon(x)} \leq \|R\|_\infty$.

The space (X, T, ν) is a probability space and $T : X \to X$ is an ergodic measure-preserving transformation, then using the Birkhoff's Ergodic Theorem, for $\nu-$a.e. $x \in X$, we have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} R(T^k(x)) = \int_X R d\nu. \tag{5.2.4}$$

The height function R is bounded from below and above by $R_{\min}$ and $R_{\max}$. In addition, we have $|\tau_\epsilon(y) - \sum_{k=0}^{w_\epsilon(x)-1} R \circ T^k| = |s - s^{'}| \leq R_{\max}$, hence using (5.2.4)

$$w_\epsilon(x) \int_X R d\nu \underset{\epsilon\to 0}{\sim} \sum_{k=0}^{w_\epsilon(x)-1} R \circ T^k - R_{\max} \leq \tau_\epsilon(y) \leq \sum_{k=0}^{w_\epsilon(x)-1} R \circ T^k + R_{\max} \underset{\epsilon\to 0}{\sim} w_\epsilon(x) \int_X R d\nu,$$

because $w_\epsilon \to +\infty$ ν-a.s., where $y \in \tilde{M}$ is associated to the coordinates (x, s), then as $\epsilon \to 0$, we have

$$\tau_\epsilon(y) \sim w_\epsilon(x) \int_X R d\nu. \tag{5.2.5}$$

□

Due to the latter relation (5.2.5), it will be enough to study the behavior of $w_\epsilon(x)$ and conclude the results for $\tau_\epsilon(y)$.

Remark 5.2.2. Hence, proving our results for $\tau_\epsilon(.)$ will be by studying the asymptotic behavior of $w_\epsilon(.)$. But due to (5.2.1) and to the Lemmas 4.6.6 and 4.6.7, we will start by studying the asymptotic behavior of the first return time into a cylinder, from which we can conclude the behavior of $w_\epsilon(.)$.

Thus we consider the following definition

Definition. The first return time in a $(q, q^{'})$-cylinder of a starting point $\omega \in \Sigma_A$ is defined by:

$$\tau_{q,q^{'}}(\omega) := \min\{m \geq 1 : S_m\varphi(\omega) = 0 \text{ and } \sigma^m(\omega) \in C_{q,q^{'}}(\omega)\}$$

Let us denote by $G_n(q,q^{'})$ the set of points for which n is a return time to the cylinder $C_{q,q'}(\omega)$:

$$G_n(q,q^{'}) := \{\omega \in \Sigma_A : S_n\varphi(\omega) = 0 \text{ and } \sigma^n(\omega) \in C_{q,q'}(\omega)\}.$$

Thus, we have:

$$\tau_{q,q'}(\omega) = \min\{m \geq 1 : \omega \in G_m(q,q^{'})\}.$$

Lemma 5.2.3. *Let $a, b \geq 0$ such that $(a+b)h > \frac{1}{2}$. Set $q_n := \lceil a \log n \rceil$, $q_n^{'} := \lceil b \log n \rceil$, then almost surely, we have:*

$$\liminf_{n\to\infty} \frac{\log \sqrt{\tau_{q_n,q_n^{'}}}}{q_n + q_n^{'}} \geq \frac{1}{2(a+b)} \tag{5.2.6}$$

Proof. Let us denote by $\mathcal{C}_{q,q'}$ the set of $(q,q^{'})$-cylinders of Σ. Let $\delta > 0$ such that $\frac{1}{2} + (a+b)h > 1 + \delta(a+b)$, denote by $\mathcal{C}^{\delta}_{q,q'} \subset \mathcal{C}_{q,q'}$ the set of cylinders $C \in \mathcal{C}_{q,q'}$ such that $\nu(C) \in (e^{-(h+\delta)(q+q')}, e^{-(h-\delta)(q+q')})$. For any $x \in \Sigma$, recall that $C_{q,q'}(x) \in \mathcal{C}_{q,q'}$ is the (q,q')-cylinder which contains x. By the Shannon-McMillan-Breiman theorem, the set $K^{\delta}_N = \{x \in \Sigma : \forall q, q' \geq N, C_{q,q'}(x) \in \mathcal{C}^{\delta}_{q,q'}\}$ has a measure $\nu(K^{\delta}_N) > 1-\delta$ provided N is sufficiently large. According to Proposition 2.2.1, whenever $n \geq N$, we have :

$$\begin{aligned}
\nu(K^{\delta}_N \cap G_n(q_n,q_n^{'})) &= \nu\left(\{x \in K^{\delta}_N : S_n\varphi(x) = 0 \text{ and } \theta^n(x) \in C_{q_n,q_n^{'}}(x)\}\right) \\
&= \sum_{C \in \mathcal{C}^{\delta}_{q_n,q_n^{'}}} \nu(C \cap \{S_n\varphi = 0\} \cap \theta^{-n}(C)) \\
&= \sum_{C \in \mathcal{C}^{\delta}_{q_n,q_n^{'}}} \nu\left(C \cap \{S_n\varphi = 0\} \cap \theta^{-n}\theta^{q_n}(\theta^{-q_n}(C))\right) \\
&= \sum_{C \in \mathcal{C}^{\delta}_{q_n,q_n^{'}}} \frac{\nu(C)\nu(C)}{\sqrt{2\pi}\sigma_\varphi\sqrt{n-2q_n}} + O\left(\frac{\nu(C)q_n\nu(C)}{n-q_n}\right) \\
&= \sum_{C \in \mathcal{C}^{\delta}_{q_n,q_n^{'}}} \frac{\nu(C)^2}{\sqrt{2\pi}\sigma_\varphi\sqrt{n}} + O\left(\frac{\nu(C)q_n\nu(C)}{n}\right).
\end{aligned}$$

Hence it follows that:

$$\begin{aligned}
\nu(K^{\delta}_N \cap G_n(q_n,q_n^{'})) &= O\left(\sum_{C \in \mathcal{C}^{\delta}_{q_n,q_n^{'}}} \frac{\nu(C)^2}{\sqrt{n}}\right) \\
&= O\left(\frac{1}{n^{\frac{1}{2}+(a+b)(h-\delta)}}\right),
\end{aligned}$$

but $\frac{1}{2} + (a+b)(h-\delta) > 1$ then $\sum_n \nu(K_N^\delta \cap G_n(q_n, q_n^{'})) < \infty$. Hence by the Borel Cantelli argument, for a.e. $x \in K_N^\delta$, if n is large enough, we get $\tau_{q_n,q_n'} > n$, proceeding similarly as at the end of the second item of section 3.2.
Moreover, we have the following implications

$$\begin{aligned}\tau_{q_n,q_n'} > n \quad &\Rightarrow \quad (a+b)\log\sqrt{\tau_{q_n,q_n'}} > \frac{1}{2}\log n^{a+b} = \frac{1}{2}(q_n + q_n^{'}) \\ &\Rightarrow \quad \frac{\log\sqrt{\tau_{q_n,q_n'}}}{q_n + q_n^{'}} > \frac{1}{2(a+b)}.\end{aligned}$$

This implies that:

$$\liminf_{n\to\infty} \frac{\log\sqrt{\tau_{q_n,q_n'}}}{q_n + q_n^{'}} \geq \frac{1}{2(a+b)} \quad a.e.,$$

□

Lemma 5.2.4. *let $a, b \geq 0$ such that $2(a+b) < \frac{1}{h}$, then taking $q_n := \lceil a\log n\rceil$, $q_n^{'} := \lceil b\log n\rceil$, we have almost surely:*

$$\limsup_{n\to\infty} \frac{\log\sqrt{\tau_{q_n,q_n'}}}{q_n + q_n^{'}} \leq \frac{1}{2(a+b)}$$

Proof. We keep the same notations of the proof of the previous lemma with $\delta > 0$ such that $\frac{1}{2} + (a+b)h < 1 + \delta(a+b)$. For all $l = 1, ..., n$, we define:

$$A_l(q_n, q_n^{'}) := G_l(q_n, q_n^{'}) \cap \theta^{-l}\{\tau_{q_n,q_n'} > n - l\}$$

Let us take $L_n := \lceil n^{a^{'}}\rceil$, with $a^{'} > 2(a+b)(d+\delta-\gamma)$. The sets $A_l(q_n, q_n^{'})$ are pairwise disjoint thus:

$$1 = \sum_{l=0}^{n} \nu(A_l(q_n, q_n^{'})) \geq \sum_{l=L_n}^{n} \sum_{C\in\mathcal{C}^\delta_{q_n,q_n^{'}}} \nu(C \cap A_l(q_n, q_n^{'})).$$

Note that $C \cap \{\tau_{q_n,q_n'} > n - l\}\}$ is a union of $C_{q_n,n-l+q_n'}$-cylinders, then using Proposition 2.2.1, we get

$$\begin{aligned}\nu(C \cap A_l(q_n, q_n^{'})) &= \nu(C \cap \{S_l\varphi = 0\} \cap \theta^{-l}(C \cap \{\tau_{q_n,q_n'} > n-l\})) \\ &= \nu(C \cap \{S_l\varphi = 0\} \cap \theta^{-l}\theta^{q_n}(\theta^{-q_n}(C \cap \{\tau_{q_n,q_n'} > n-l\}))) \\ &= \frac{\nu(C)\nu(C \cap \{\tau_{q_n,q_n'} > n-l\})}{\sqrt{2\pi}\sigma_\varphi\sqrt{l-q_n}} + O\left(\frac{\nu(C \cap \{\tau_{q_n,q_n'} > n-l\})q_n\nu(C)}{l-q_n}\right) \\ &= \left(\frac{\nu(C)}{\sqrt{2\pi}\sigma_\varphi} + O\left(\frac{q_n\nu(C)}{\sqrt{l-2q_n}}\right)\right)\frac{1}{\sqrt{l-q_n}}\nu(C \cap \{\tau_{q_n,q_n'} > n-l\}) \\ &\geq cn^{-(a+b)(d+\delta)}\frac{1}{\sqrt{l}}\nu(C \cap \{\tau_{q_n,q_n'} > n-l\}),\end{aligned}$$

for any $C \in \mathcal{C}^{\delta}_{q_n}$ provided $q_n \geq N$ and $l \geq L_n$. We have

$$\nu\left(K^{\delta}_N \cap \{\tau_{q_n,q'_n} > n\}\right) \leq \sum_{C \in \mathcal{C}^{\delta}_{q_n,q'_n}} \nu\left(C \cap \{\tau_{q_n,q'_n} > n\}\right)$$

but,

$$\begin{aligned}\sum_{l=L_n}^{n} \nu(C \cap A_l(q_n, q'_n)) &\geq cn^{-(a+b)(d+\delta)}\nu(C \cap \{\tau_{q_n,q'_n} > n\}) \sum_{l=L_n}^{n} \frac{1}{\sqrt{l}} \\ &\simeq cn^{-(a+b)(d+\delta)}\nu(C \cap \{\tau_{q_n,q'_n} > n\}) \left(\sqrt{n} - \sqrt{L_n}\right)\end{aligned}$$

so we get:

$$1 \geq \sum_{C \in \mathcal{C}^{\delta}_{q_n,q'_n}} \sum_{l=L_n}^{n} \nu(C \cap A_l(q_n, q'_n)) \geq \sum_{C \in \mathcal{C}^{\delta}_{q_n+q'_n}} n^{-(a+b)(d+\delta)}\nu(C \cap \{\tau_{q_n,q'_n} > n\}) \left(\sqrt{n} - \sqrt{L_n}\right)$$

hence,

$$\sum_{C \in \mathcal{C}^{\delta}_{k_n}} \nu(C \cap \{\tau_{q_n,q'_n} > n\}) = O\left(\frac{1}{n^{\frac{1}{2}-(a+b)(d+\delta)}}\right).$$

Now let us take $n_p := p^{-\frac{4}{1-2(a+b)(d+\delta)}}$. We have:

$$\sum_{p \geq 1} \nu(K^{\delta}_N \cap \{\tau_{q_{n_p},q'_{n_p}} > n_p\}) = \sum_{p \geq 1} O\left(\frac{1}{n_p^{\frac{1}{2}-(a+b)(d+\delta)}}\right) = \sum_{p \geq 1} O\left(\frac{1}{p^2}\right) < +\infty.$$

Hence, by the Borel Cantelli lemma, for ν-almost everywhere $x \in K^{\delta}_N$, for every p large enough, $\tau_{q_{n_p},q'_{n_p}} \leq n_p$. Moreover we have the following implications:

$$\tau_{q_{n_p},q'_{n_p}} \leq n_p \Rightarrow \frac{\log \sqrt{\tau_{q_{n_p},q'_{n_p}}}}{q_{n_p}, q'_{n_p}} \leq \frac{1}{2(a+b)},$$

which implies that

$$\limsup_{n \to +\infty} \frac{\log \sqrt{\tau_{q_n,q'_n}}}{q_n, q'_n} \leq \frac{1}{2(a+b)}.$$

$\square$

Proof of Theorem 5.1.1. Fix an $\epsilon > 0$. We consider a point x in M which is coded by an $\omega \in \Sigma_A$ such that $x = \chi(\omega)$. We control the ball $B(x, \epsilon)$ using the Lemmas

4.6.6 and 4.6.7 as follows:

Then as $\chi(C_{q_\epsilon}, q'_\epsilon(x)) \subset B(x,\epsilon) \subset \chi(C_{p_\epsilon,p'_\epsilon}(x))$, we get:

$$\tau_{p_\epsilon,p'_\epsilon} < \tau_\epsilon < \tau_{q_\epsilon,q'_\epsilon}. \tag{5.2.7}$$

Now q_ϵ, q'_ϵ, p_ϵ and p'_ϵ are well defined using l^s_n and l^u_n. Back to the definitions of l^s_n and l^u_n, using Proposition 4.6.4 and the formulas (4.6.6) and (4.6.7), the two are written as an exponential of an ergodic sum:

$$\begin{aligned}
l^u_n(\omega) &\sim \prod_{k=0}^{n-1} \mathbf{a}^u(\sigma^k\omega)^{-1} \\
&= \prod_{k=0}^{n-1} \left(\exp \int_0^{r(\omega)} \tilde{a}^{(u)}(\sigma^k\omega, t)dt\right)^{-1} \\
&= \prod_{k=0}^{n-1} \exp\left(-\int_0^{r(\omega)} a^{(u)}(g_t(\chi(\sigma^k\omega)))dt\right) \\
&= \exp\left(-\sum_{k=0}^{n-1}\int_0^{r(\omega)} a^{(u)}(g_t(\chi(\sigma^k\omega)))dt\right).
\end{aligned}$$

and

$$\begin{aligned}
l^s_m(\omega) &\sim \prod_{k=-1}^{-m} \mathbf{a}^s(\sigma^{-k}\omega) \\
&= \prod_{k=-m}^{-1} \left(\exp \int_0^{r(\omega)} \tilde{a}^{(s)}(\sigma^{-k}\omega, t)dt\right) \\
&= \prod_{k=-m}^{-1} \exp\left(\int_0^{r(\omega)} a^{(s)}(g_t(\chi(\sigma^{-k}\omega)))dt\right) \\
&= \exp\left(\sum_{k=0}^{m-1}\int_0^{r(\omega)} a^{(s)}(g_t(\chi(\sigma^{k+1}\omega)))dt\right).
\end{aligned}$$

Then $\frac{\log l^s_n}{n}$ and $\frac{\log l^u_n}{n}$ converge almost everywhere on Σ_A. We set L_s and L_u to be their limits respectively.

Let $\eta \in (0,1)$. There exists $n_0 > 0$ such that for all $n \geq n_0$

$$L_s - \eta < \frac{\log l^s_n}{n} < L_s + \eta \text{ and } L_u - \eta < \log\frac{l^u_n}{n} < L_u + \eta.$$

On the other hand, q_ϵ, $q_\epsilon^{'}$, p_ϵ, $p_\epsilon^{'}$ verify the following:

$$\log l^s_{q_\epsilon} < \log(\epsilon/2c_L) < \log l^s_{q_\epsilon - 1}, \quad \log l^u_{q_\epsilon^{'}} < \log(\epsilon/2c_L) < \log l^u_{q_\epsilon^{'}-1}$$

and

$$\log l^s_{p_\epsilon} < \log(\epsilon/2c_L) < \log l^s_{p_\epsilon - 1}, \quad \log l^u_{p_\epsilon^{'}} < \log(\epsilon/2c_L) < \log l^u_{p_\epsilon^{'}-1},$$

from which we get

$$L_s q_\epsilon \sim \log l^s_{q_\epsilon} \sim \log \epsilon, \quad L_u q_\epsilon^{'} \sim \log l^u_{q_\epsilon^{'}} \sim \log \epsilon$$

and

$$L_s p_\epsilon \sim \log l^s_{p_\epsilon} \sim \log \epsilon, \quad L_u p_\epsilon^{'} \sim \log l^u_{p_\epsilon^{'}} \sim \log \epsilon$$

Then we have

$$q_\epsilon, p_\epsilon \sim \frac{\log \epsilon}{L_s} \text{ and } q_\epsilon^{'}, p_\epsilon^{'} \sim \frac{\log \epsilon}{L_u},$$

Let $\eta \in (0,1)$. For $\epsilon > 0$, large enough, we have

$$q_\epsilon < \frac{\log \epsilon}{(L_s(1-\eta))} = \frac{(1-\eta)(\log \epsilon^{(1-\eta)^{-2}})}{L_s},$$

and

$$q_\epsilon^{'} < \frac{\log \epsilon}{(L_u(1-\eta))} = \frac{(1-\eta)(\log \epsilon^{(1-\eta)^{-2}})}{L_u}$$

We set $a_\eta = \frac{(1-\eta)}{L_s}$, $b_\eta = \frac{(1-\eta)}{L_u}$ and $N_{\eta,\epsilon} := \lceil \epsilon^{(1-\eta)^{-2}} \rceil$. We have $a_\eta + b_\eta < \frac{1}{L_s} + \frac{1}{L_u}$, then using Lemma 5.2.4 (applied to $a_\eta, b_\eta, N_{\eta,\epsilon}$), we have almost surely

$$\limsup_{\epsilon \to 0} \frac{\log \sqrt{\tau_{q_\epsilon, q_\epsilon^{'}}}}{q_\epsilon + q_\epsilon^{'}} \leq \frac{1}{2(a_\eta + b_\eta)},$$

but, $q_\epsilon + q_\epsilon^{'} < \frac{|\log \epsilon|}{(1-\eta)} \left(\frac{1}{L_s} + \frac{1}{L_u} \right)$ and $a_\eta + b_\eta = (1-\eta)\left(\frac{1}{L_s} + \frac{1}{L_u} \right)$, then we obtain almost surely

$$\limsup_{\epsilon \to 0} \frac{(1-\eta)\log \sqrt{\tau_{q_\epsilon, q_\epsilon^{'}}}}{|\log \epsilon| \left(\frac{1}{L_s} + \frac{1}{L_u} \right)} \leq \limsup_{\epsilon \to 0} \frac{\log \sqrt{\tau_{q_\epsilon, q_\epsilon^{'}}}}{q_\epsilon + q_\epsilon^{'}} \leq \frac{1}{2(1-\eta)\left(\frac{1}{L_s} + \frac{1}{L_u} \right)},$$

Hence we have almost surely

$$\limsup_{\epsilon \to 0} \frac{(1-\eta)\log \sqrt{\tau_{q_\epsilon, q_\epsilon^{'}}}}{|\log \epsilon|} \leq \frac{1}{2(1-\eta)},$$

On the other hand, for all $\epsilon > 0$ large enough, we have

$$p_\epsilon > \frac{\log \epsilon}{(L_s(1+\eta))} = \frac{(1+\eta)(\log \epsilon^{(1+\eta)^{-2}})}{L_s},$$

and

$$p_\epsilon^{'} > \frac{\log \epsilon}{(L_u(1+\eta))} = \frac{(1+\eta)(\log \epsilon^{(1+\eta)^{-2}})}{L_u}.$$

We set $a_\eta = \frac{(1+\eta)}{L_s}$, $b_\eta = \frac{(1+\eta)}{L_u}$ and $N_{\eta,\epsilon} := \lceil \epsilon^{(1+\eta)^{-2}} \rceil$. We have $a_\eta + b_\eta > \frac{1}{L_s} + \frac{1}{L_u}$, then using Lemma 5.2.4 (applied to $a_\eta, b_\eta, N_{\eta,\epsilon}$), we have almost surely

$$\liminf_{\epsilon \to 0} \frac{\log \sqrt{\tau_{p_\epsilon, p_\epsilon^{'}}}}{p_\epsilon + p_\epsilon^{'}} \geq \frac{1}{2(a_\eta + b_\eta)},$$

but, $p_\epsilon + p_\epsilon^{'} > \frac{|\log \epsilon|}{(1+\eta)} \left(\frac{1}{L_s} + \frac{1}{L_u} \right)$ and $a_\eta + b_\eta = (1+\eta) \left(\frac{1}{L_s} + \frac{1}{L_u} \right)$, then we obtain almost surely

$$\liminf_{\epsilon \to 0} \frac{(1+\eta) \log \sqrt{\tau_{p_\epsilon, p_\epsilon^{'}}}}{|\log \epsilon| \left(\frac{1}{L_s} + \frac{1}{L_u} \right)} \geq \liminf_{\epsilon \to 0} \frac{\log \sqrt{\tau_{p_\epsilon, p_\epsilon^{'}}}}{p_\epsilon + p_\epsilon^{'}} \geq \frac{1}{2(1+\eta) \left(\frac{1}{L_s} + \frac{1}{L_u} \right)},$$

Hence we have almost surely

$$\liminf_{\epsilon \to 0} \frac{(1+\eta) \log \sqrt{\tau_{p_\epsilon, p_\epsilon^{'}}}}{|\log \epsilon|} \geq \frac{1}{2(1+\eta)},$$

Therefore for all $\eta \in (0,1)$, using (5.2.7), we have the following two formulas, for almost surely

$$\limsup_{\epsilon \to 0} \frac{(1-\eta) \log \sqrt{\tau_\epsilon}}{|\log \epsilon|} \leq \limsup_{\epsilon \to 0} \frac{(1-\eta) \log \sqrt{\tau_{q_\epsilon, q_\epsilon^{'}}}}{|\log \epsilon|} \leq \frac{1}{2(1-\eta)},$$

$$\liminf_{\epsilon \to 0} \frac{(1+\eta) \log \sqrt{\tau_\epsilon}}{|\log \epsilon|} \geq \liminf_{\epsilon \to 0} \frac{(1+\eta) \log \sqrt{\tau_{p_\epsilon, p_\epsilon^{'}}}}{|\log \epsilon|} \geq \frac{1}{2(1+\eta)},$$

from which we get that

$$\lim_{\epsilon \to 0} \frac{\log \sqrt{\tau_\epsilon}}{|\log \epsilon|} = \frac{1}{2}, \qquad \text{almost surely.}$$

□

Chapter 6

Convergence in distribution for $\mathbb{Z}$-extension of Axiom A flow

In this chapter we consider the same hypothesis and notations introduced in Chapters 4 and 5.
Let $\tilde{y} \in \tilde{M}$, and $\tilde{D}_0 = D_0(\tilde{y})$ a disk centered on $\tilde{y}$ and which is transversal to the flow, and orthogonal to it at $\tilde{y}$.
We will show a result of convergence in distribution for τ_ϵ. In the normalization appears a transversal measure $\tilde{\nu}_0$ on $\tilde{D}_0$ defined by:

$$\tilde{\nu}_0(A) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \tilde{\mu} \left(\bigcup_{-\epsilon < s < \epsilon} \tilde{g}_s(\tilde{D}_0 \cap A) \right), \quad \forall A \subset \tilde{D}_0. \tag{6.0.1}$$

The existence of this measure is proved later in Lemma 6.2.12. We call it measure transversal to the flow at $\tilde{y}$.

Theorem 6.0.5. *The sequence of random variables $\tilde{\nu}_0(B(.,\epsilon))^2 \tau_\epsilon(.)$ converges in distribution, with respect to any probability measure absolutely continuous with respect to $\tilde{\mu}$, as $\epsilon \to 0$ to $\sigma^2_{flow} \frac{\mathcal{E}^2}{\mathcal{N}^2}$, where $\mathcal{E}$ and $\mathcal{N}$ are independent random variables, $\mathcal{E}$ having an exponential distribution of mean 1 and $\mathcal{N}$ having a standard Gaussian distribution.*

6.1 Construction of the partition

Given $\epsilon > 0$, we fix $\theta_\epsilon = \frac{\epsilon}{|\log \epsilon|}$. We want to construct a partition $\mathcal{D}_\epsilon$ of Σ_A of diameter $< \theta_\epsilon$. Let $c_g > 0$ the Lipschitz constant of $(x, s) \mapsto g_s(x)$ on $X \times [0, \beta_0]$,

$c_L > 0$ the Lipschitz constant of the map in Proposition 4.2.1, and the constant $C > 0$ in Proposition 1.1.6. Let $\omega = (\omega_n)_{n\in\mathbb{Z}} \in \Sigma_A$, we set the following:

$$n_1(\omega) = n_1(\omega,\epsilon) := \min\{n \geq 1 \text{ s.t. } \sup_{\substack{\omega' \in \Sigma_A \\ \omega'_0=\omega_0,..,\omega'_n=\omega_n}} \prod_{k=0}^{n} \mathbf{a}^{(u)}(\sigma^k\omega')^{-1} < \frac{\theta_\epsilon}{2Cc_gc_l}\},$$

and

$$n_2(\omega) = n_2(\omega,\epsilon) := \min\{n \geq 1 \text{ s.t. } \sup_{\substack{\omega' \in \Sigma_A \\ \omega'_{-1}=\omega_{-1},..,\omega'_{-n}=\omega_{-n}}} \prod_{k=-n}^{-1} \mathbf{a}^{(s)}(\sigma^{-k}\omega') < \frac{\theta_\epsilon}{2Cc_gc_l}\}.$$

We set $M_\epsilon := 2 \max_{\omega\in\Sigma_A} n_1(\omega) + 1$.

Definition. Let $\epsilon > 0$. We set $Q_\epsilon := \sup_{\omega\in\Sigma_A} n_2(\omega)$ and define $M_\epsilon := 2Q_\epsilon + 1$.

Lemma 6.1.1. *For all $\omega \in \Sigma_A$, the numbers $n_1(\omega)$ and $n_2(\omega)$ are well defined and finite.*

Proof. First, the function $a^{(u)}$ is positive and Hölder on Λ which is a compact set, thus $\exists\, m_u > 0$ such that:

$$a^{(u)}(x) > m_u, \quad \forall x \in \Lambda.$$

We know that the height function r is bounded from above and below such that $0 < r_{min} < r(\omega') < r_{max} < \infty$, $\forall\omega' \in \Sigma_A$, then using the definition of $\mathbf{a}^{(u)}$, we get:

$$\mathbf{a}^{(u)}(\omega') > \exp(m_u r_{min}), \quad \forall\omega' \in \Sigma_A.$$

Now going back to the definition of $n_1(\omega)$, $\forall n \geq 1$, there exists $\rho_u < 1$ such that

$$\sup_{\omega'\in\Sigma_A} \prod_{k=0}^{n-1} \mathbf{a}^{(u)}(\sigma^k\omega')^{-1} < (\exp(m_u r_{min}))^{-n} = \rho_u^n.$$

from which we conclude that, $n_1(\omega) \leq \left\lceil \frac{\log\left(\frac{\theta_\epsilon}{2Cc_gc_l}\right)}{\log\rho_u} \right\rceil := \rho_{u,\epsilon}$. Indeed every $n \geq \rho_{u,\epsilon}$ satisfies $\sup_{\substack{\omega'\in\Sigma_A \\ \omega'_0=\omega_0,..,\omega'_n=\omega_n}} \prod_{k=0}^{n} \mathbf{a}^{(u)}(\sigma^k\omega')^{-1} < \frac{\theta_\epsilon}{2Cc_gc_l}\}$. This gives that $n_1(\omega)$ is finite.

Similarly for $n_2(\omega)$, the function $a^{(s)}$ is negative and Hölder on the compact set Λ, then there exists $m_s < 0$ such that $\forall x \in \Lambda$, $a^s(x) < m_s$. Using definition of $\mathbf{a}^s$, $\forall \omega' \in \sigma_A$, $\mathbf{a}^s(\omega') < \exp(m_s r_{max})$, getting that $\forall n \geq 1$, $\exists\ \rho_s < 1$ such that

$$\sup_{\omega' \in \Sigma_A} \prod_{k=0}^{n-1} \mathbf{a}^s(\sigma^k \omega') < (\exp m_s r_{max})^n = \rho_s^n,$$

we have $\rho_s < 1$ since $m_s < 0$. Consequently there exists $\rho_{s,\epsilon} > 0$, such that $n_2(\omega) \leq \left\lceil \frac{\log\left(\frac{\theta_\epsilon}{2Cc_g c_l}\right)}{\log \rho_s} \right\rceil = \rho_{s,\epsilon}$, showing that $n_2(\omega)$ is also finite. □

Proposition 6.1.2. *For all $\omega \in \Sigma_A$, the family of cylinders $D_\epsilon(\omega) = C_{-n_2(\omega),n_1(\omega)}$ forms a finite partition $\mathcal{D}_\epsilon = \{D_{\epsilon,i}, i = 1, ..., N_\epsilon\}$ of Σ_A.*

Proof. Let $\omega'' \in \Sigma_A$ such that $\omega'' \in D_\epsilon(\omega)$, then we have $\omega_{-n_2(\omega)} = \omega''_{-n_2(\omega)}, .., \omega_0 = \omega''_0, .., \omega_{n_1(\omega)} = \omega''_{n_1(\omega)}$, from which we have

$$\sup_{\substack{\omega' \in \Sigma_A \\ \omega'_0 = \omega''_0, .., \omega'_n = \omega''_n}} \prod_{k=0}^{n} \mathbf{a}^{(u)}(\sigma^k \omega')^{-1} = \sup_{\substack{\omega' \in \Sigma_A \\ \omega'_0 = \omega_0, .., \omega'_n = \omega_n}} \prod_{k=0}^{n} \mathbf{a}^{(u)}(\sigma^k \omega')^{-1}$$

and

$$\sup_{\substack{\omega' \in \Sigma_A \\ \omega'_{-1} = \omega''_{-1}, .., \omega'_{-n} = \omega''_{-n}}} \prod_{k=-n}^{-1} (\mathbf{a})^s(\sigma^{-k} \omega') = \sup_{\substack{\omega' \in \Sigma_A \\ \omega'_{-1} = \omega_{-1}, .., \omega'_{-n} = \omega_{-n}}} \prod_{k=-n}^{-1} \mathbf{a}^{(s)}(\sigma^{-k} \omega')$$

hence $n_1(\omega'') = n_1(\omega)$ and $n_2(\omega) = n_2(\omega'')$ and therefore $D_\epsilon(\omega) = D_\epsilon(\omega'')$. □

Definition. For $D_{\epsilon,i}$ an element of the partition $\mathcal{D}_\epsilon$, χ the coding map and $(g_t)_{t \in \mathbb{R}}$ the flow, we define the family of sets $(\mathcal{P}_{\epsilon,i,j})_{\epsilon,i,j}$, such that:

i. $\mathcal{P}_{\epsilon,i,j} = \{g_s(\chi(x)), (x, s) \in D_{\epsilon,i} \times [\frac{j\theta_\epsilon}{c_g}, \frac{(j+1)\theta_\epsilon}{c_g}[\}$.

ii. Let $y_{\epsilon,i,j} \in \mathcal{P}_{\epsilon,i,j}$, there exists $\omega_{\epsilon,i} \in D_{\epsilon,i}$ the smallest element in the lexicographic order such that $y_{\epsilon,i,j} = \chi(\omega_{\epsilon,i}, j\theta_\epsilon)$.

Lemma 6.1.3. *$\forall \epsilon > 0$, $i = 1, ..., N_\epsilon$ and $j \geq 0$, the family $(\mathcal{P}_{\epsilon,i,j})_{\epsilon,i,j}$ satisfies*

$$diam(\mathcal{P}_{\epsilon,i,j}) < \theta_\epsilon.$$

Proof. Let $\omega, \omega^{'} \in \Sigma_A$, we have $d(g_s(\chi(\omega), g_{s^{'}}(\chi(\omega^{'})))) \leq c_g \max(|s-s^{'}|, d(\chi(\omega), \chi(\omega^{'})))$ but due to (1.1.25), there exists $c_L > 0$ such that

$$diam(\chi(D_{\epsilon,i})) \leq c_L(l^s_{n_2}(\omega_{\epsilon,i}) + l^u_{n_1}(\omega_{\epsilon,i})) \leq c_L(\frac{\theta_\epsilon}{2c_g c_L} + \frac{\theta_\epsilon}{2c_g c_L}) < \frac{\theta_\epsilon}{c_g},$$

hence we get $d(g_s(\chi(\omega), g_{s^{'}}(\chi(\omega^{'})))) < c_g(\frac{\theta_\epsilon}{2c_g}, \frac{\theta_\epsilon}{2c_g}) < \theta_\epsilon$. □

Lemma 6.1.4. *There exists $K > 0$ such that for all $\epsilon > 0$ and every $i = 1, ..., N_\epsilon$, there exists $\omega_i := \omega_{i,\epsilon} \in \Sigma_A$ such that*

$$B\left(\chi(\omega_i), \frac{\theta_\epsilon}{K}\right) \subset \chi(D_{\epsilon,i}).$$

Proof. For all $\omega \in \Sigma_A$, set $q = n_1(\omega)$, and $q^{'} = n_2(\omega)$. Using the definition of $n_1(\omega)$, then there exists $\omega^{'}$ such that

$$\prod_{k=0}^{n_1(\omega)} \mathbf{a}^u(\sigma^k(\omega^{'}))^{-1} < \frac{\theta_\epsilon}{2Cc_g c_L} \leq \prod_{k=0}^{n_1(\omega)-1} \mathbf{a}^u(\sigma^k(\omega^{'}))^{-1},$$

from which we get, using the definition of l^u_q that there exists $c_1 = \frac{\mathbf{a}^u(\sigma^k(\omega^{'}))^{-1}}{2Cc_g c_L} > 0$, such that

$$l^u_q(\omega^{'}) \geq c_1\theta_\epsilon,$$

hence there exists $\omega^u \in W^u_q(\omega^{'})$ such that

$$B^u\left(\omega^u, \frac{c_1\theta_\epsilon}{3}\right) \subset W^u_q(\omega^{'}).$$

Similarly, using the definition of $n_2(\omega)$ and that of $l^s_{q^{'}}$, there exists $\omega^3 \in \Sigma_A$, $c_2 > 0$ such that $l^s_{q^{'}}(\omega^3) \geq c_2\theta_\epsilon$ from which in turn, we have that there exists $\omega^s \in W^s_{q^{'}}(\omega^3)$ such that

$$B^s\left(\omega^s, \frac{c_2\theta_\epsilon}{3}\right) \subset W^s_{q^{'}}(\omega).$$

Let $k > \frac{3c_L}{\min(c_1,c_2)}$, then using Lemma 4.6.5, there exists $\omega_i \in \Sigma_A$ such that $\chi(\omega_i) = \{\omega^s, \omega^u\}$ and

$$B\left(\chi(\omega_i), \frac{\theta_\epsilon}{k}\right) \subset C_{-q,q^{'}}.$$

□

Definition. Set $\mathcal{P}_\epsilon = \{\mathcal{P}_{\epsilon,i,j} : i = 1, ..., N_\epsilon; j \geq 0, D_{\epsilon,i} \times [j\theta_\epsilon, (j+1)\theta_\epsilon[\subset \Delta\}$. We will consider the elements of $\mathcal{P}_\epsilon$ which intersect the boundary of Δ, thus we define the set $\bar{\mathcal{P}}_\epsilon$:

$$\bar{\mathcal{P}}_\epsilon = \{\mathcal{P}_{\epsilon,i,j} : i = 1, ..., N_\epsilon; j \geq 0, D_{\epsilon,i} \times [j\theta_\epsilon, (j+1)\theta_\epsilon[\cap \partial\Delta \neq \emptyset\}.$$

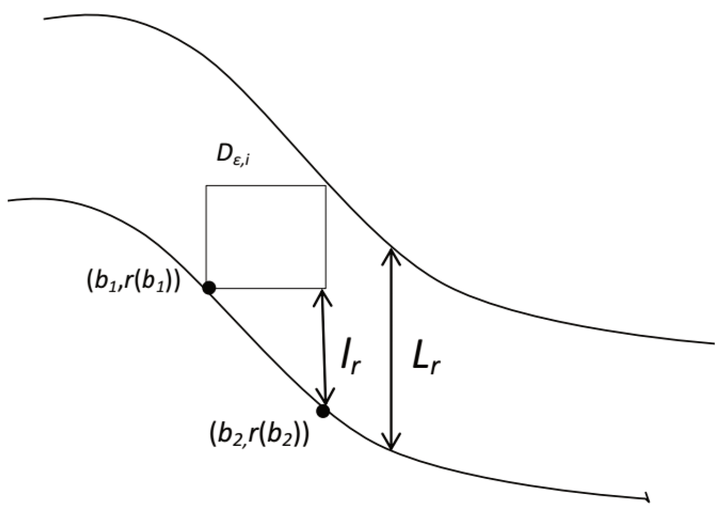
We will show that the measure of the set $\bar{\mathcal{P}}_\epsilon$ can be neglected in our study. This is verified in the following lemma

Lemma 6.1.5. *The ν-measure of the set $\bar{\mathcal{P}}_\epsilon$ is $O(\theta_\epsilon)$.*

Proof. Consider the figure , and let $D_{\epsilon,i}$ be a cylinder in $\bar{\mathcal{P}}_\epsilon$, i.e. which intersects the boundary of Δ. Consider the lengths L_r and l_r as indicated in the figure. To estimate the measure of $\bar{\mathcal{P}}_\epsilon$, we need to estimate L_r. Knowing that $diam(D_{\epsilon,i}) < \theta_\epsilon$, then $L_r < \theta_\epsilon + l_r$. Let $b_1, b_2 \in \Sigma_A$, such that $(b_1, r(b_1))$ and $(b_2, r(b_2))$ are as located in the figure, where r is the height function, and c_r its corresponding Hölder coefficient. There exists $c > 0$ such that

$$l_r \leq cd(r(b_1), r(b_2)) \leq cc_r d(b_1, b_2) < cc_r\theta_\epsilon$$

Thus we get $L_r < (1 + cc_r)\theta_\epsilon$. $\square$



We denote by $A_{\epsilon-\theta_\epsilon}(y_{\epsilon,i,j})$ and $A_{\epsilon+\theta_\epsilon}(y_{\epsilon,i,j})$ the projections of the balls $B(y_{\epsilon,i,j}, \epsilon-\theta_\epsilon)$ and $B(y_{\epsilon,i,j}, \epsilon + \theta_\epsilon)$ on Σ_A respectively, we set $A_\epsilon^- = \cup_{k\in I} D_{\epsilon,k}$ and $A_\epsilon^+ = \cup_{l\in J} D_{\epsilon,l}$, where $I = \{k : D_{\epsilon,k} \subset A_{\epsilon-\theta_\epsilon}(y_{\epsilon,i,j})\}$ and $J = \{l : D_{\epsilon,l} \cap A_{\epsilon+\theta_\epsilon}(y_{\epsilon,i,j}) \neq \emptyset\}$. Then we have the following inclusions:

$$A_\epsilon^-(y_{\epsilon,i,j}) \subset A_{\epsilon-\theta_\epsilon}(y_{\epsilon,i,j}) \subset A_\epsilon(y_{\epsilon,i,j}) \subset A_{\epsilon+\theta_\epsilon}(y_{\epsilon,i,j}) \subset A_\epsilon^+(y_{\epsilon,i,j}). \tag{6.1.1}$$

Let $y \in M$, if $\epsilon > 0$ is small enough, $y \in \bigcup_{P\in\mathcal{P}_\epsilon} P$, there exists i, j such that $y \in \mathcal{P}_{\epsilon,i,j}$. Consider the ball $B(y,\epsilon)$, and let $y_{\epsilon,i,j} \in \mathcal{P}_{\epsilon,i,j}$, we have:

$$B(y_{\epsilon,i,j}, \epsilon - \theta_\epsilon) \subset B(y,\epsilon) \subset B(y_{\epsilon,i,j}, \epsilon + \theta_\epsilon). \tag{6.1.2}$$

This comes from the fact that the diameter of $\mathcal{P}_{\epsilon,i,j}$ is bounded by θ_ϵ. Furthermore, after projecting on Σ_A, let $x_{\epsilon,i} \in X$ such that $x_{\epsilon,i} = \chi(\omega_{\epsilon,i})$, we note that there exists α_1, $\alpha_2 > 0$, such that:

$$\nu(\chi^{-1}(B(x_{\epsilon,i}, \alpha_1\epsilon))) \leq \nu(A_\epsilon(y_{\epsilon,i,j})) \leq \nu(\chi^{-1}(B(x_{\epsilon,i}, \alpha_2\epsilon))). \tag{6.1.3}$$

In what follows, we consider the notations $A_\epsilon^-(y) := A_\epsilon^-(y_{\epsilon,i,j})$, $A_\epsilon^+(y) := A_\epsilon^+(y_{\epsilon,i,j})$ and $D_\epsilon(x) := D_{\epsilon,i}$, where x is the point of X such that $y = g_s(x)$ for some $s \in [0, R(x)[$.

Proposition 6.1.6. *Recall that a measure ν on a metric space is called **Federer** if there exists a constant $K_F > 0$ such that for any point x and any $r > 0$,*

$$\nu(B(x,2r)) \leq K_F \nu(B(x,r)).$$

Remark 6.1.7. Due to Theorem 5.1 in [41], the measure ν is Federer. We note that there exists a constant $c_F > 0$ such that for all $m \geq 0$,

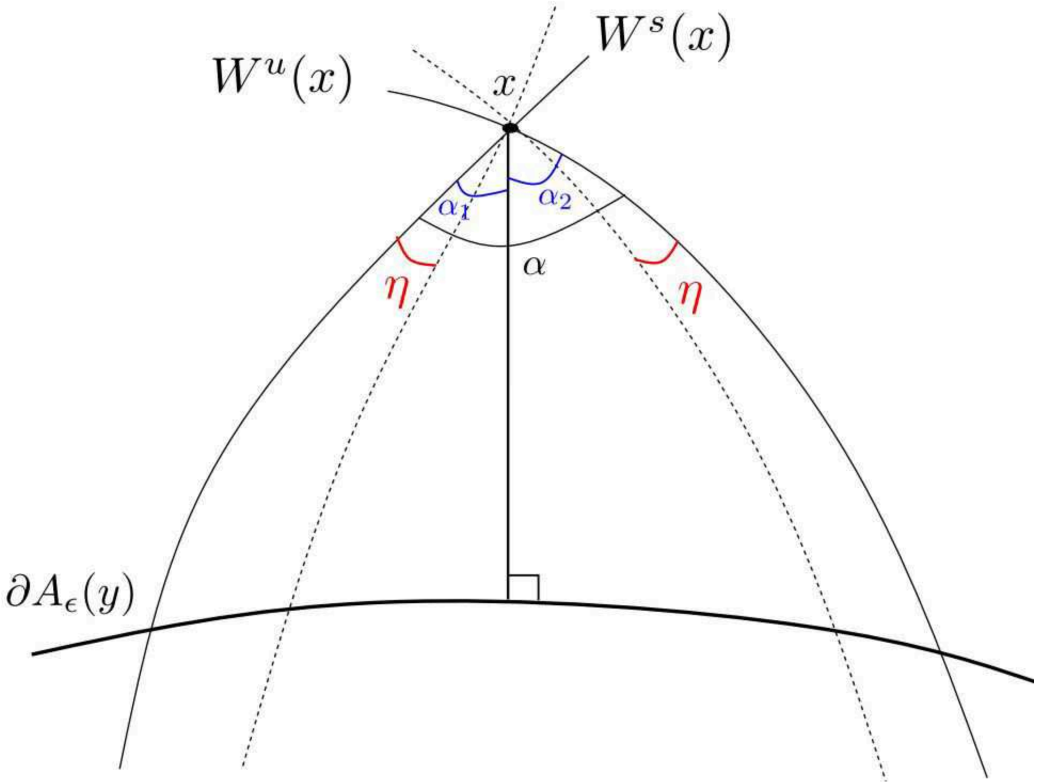
$$\nu(B(x,2^m r)) \leq c_F^m \nu(B(x,r)). \tag{6.1.4}$$

6.2 Proof of the convergence in distribution

Lemma 6.2.1. *Let $x \in (\partial A_\epsilon(y))^{[\delta]}$, then there exists $z \in \partial A_\epsilon(y)$, there exists $\bar{c}$ such that $d(x,z) \leq \bar{c}\delta$ and $z \in W^u(x)$ or $W^s(x)$.*

Proof. Let $x \in (\partial A_\epsilon(y))^{[\delta]}$. The stable and unstable manifolds are transversal, let $\alpha > 0$ be the angle between $W^s(x)$ and $W^u(x)$. Let $\eta > 0$ be the angle of variation of the stable manifold and that of the unstable manifold at the point x.
Consider $\bar{\alpha} > 0$ such that $\pi = \alpha + \bar{\alpha}$, there exists α_0, such that $\bar{\alpha} \geq \alpha_0$. We have $\alpha = \alpha_1 + \alpha_2 - 2\eta$ as in the figure. And therefore we have

$$\frac{\pi - \alpha_0}{2} \geq \frac{\alpha_1 + \alpha_2 - 2\eta}{2}$$



We know that the distance from x to $\partial A_\epsilon(y)$ is less than or equal δ. Now we want to know the minimum distance from x to $\partial A_\epsilon(y)$ along either $W^s(x)$ or $W^u(x)$, thus we have

$$\min\left(\frac{\delta}{\cos\alpha_1}, \frac{\delta}{\cos\alpha_2}\right) \leq \frac{\delta}{\min(\cos\alpha_1, \cos\alpha_2)} = \frac{\delta}{\cos(\min(\alpha_1, \alpha_2))},$$

on the other hand we notice that

$$\min(\alpha_1, \alpha_2) \leq \frac{\alpha_1 + \alpha_2}{2} \leq \frac{\pi - (\alpha_0 - 2\eta)}{2}.$$

which in turn gives that

$$\cos(\min(\alpha_1, \alpha_2)) \geq \cos\left(\frac{\alpha_1 + \alpha_2}{2}\right) = \sin\left(\frac{\alpha_0 - 2\eta}{2}\right).$$

Setting $\bar{c}^{-1} = \sin\left(\frac{\alpha_0 - 2\eta}{2}\right) > 0$, we get

$$\min\left(\frac{\delta}{\cos\alpha_1}, \frac{\delta}{\cos\alpha_2}\right) \leq \frac{\delta}{\sin\left(\frac{\alpha_0 - 2\eta}{2}\right)}.$$

Thus there exists $z \in W^s(x)$ or $z \in W^u(x)$, there exists $\bar{c} > 0$ such that $d(x, z) \leq \bar{c}\delta$. $\square$

Lemma 6.2.2. *For all $\epsilon > 0$, the ν-measures of the sets $A_\epsilon^+(y)$ and $A_\epsilon^-(y)$ are equivalent.*

Proof. We know that the cylinders of $\mathcal{D}_\epsilon$ are of diameter less than θ_ϵ, and the projection map on Σ_A is Lipschitz with constant k_L. Set $\delta = (k_L + 1)\theta_\epsilon$ and let k_0 be the smallest positive number satisfying $k_L\epsilon + \delta + \epsilon \leq k_0\epsilon$. By the coding map χ, we have on the Poincaré section X that:

$$\begin{aligned} \chi(A_\epsilon^+(y)) \backslash \chi(A_\epsilon^-(y)) &= (\chi(A_\epsilon(y))^{[(k_L+1)\theta_\epsilon]} \cap (\chi(A_\epsilon(y)^C)^{[(k_L+1)\theta_\epsilon]} \\ &= (\partial\chi(A_\epsilon(y)))^{[\delta]}, \end{aligned}$$

thus we need to prove that the measure of the δ-neighborhood of $A_\epsilon(y)$ is negligible with respect to the measure of $A_\epsilon(y)$. Indeed, let $m_F > 0$ be such that $k_0 \leq 2^{m_F}\alpha_1$, then using the Federer property (Remark 6.1.7), and the inclusion in (6.1.3), there exists $c_F > 0$ such that:

$$\nu(\chi^{-1}(B(x, k_0\epsilon))) \leq c_F^{m_F}\nu(\chi^{-1}(B(x, \alpha_1\epsilon))) \leq c_F^{m_F}\nu(A_\epsilon(x)), \tag{6.2.1}$$

hence it's sufficient to prove that $\nu\left((\partial A_\epsilon(y))^{[\delta]}\right)$ is negligible with respect to $\nu\left(\chi^{-1}(B(x, k_0\epsilon))\right)$.

Now, due to Lemma 6.2.1, we will work on two parts of $\partial A_\epsilon(y)^{[\delta]}$:

Let x_0 be a point in D. Let $x \in W^s(x_0)$, we set $I^u(x) := W^u_{loc}(x) \cap (\partial A_\epsilon(y))^{[\delta]}$, such that $\forall \xi_x \in I^u(x)$, there exists $z \in \partial A_\epsilon$ as in Lemma 6.2.1 and where $z \in W^u(\xi_x)$. On the other hand, we set $I^s(x) := W^s_{loc}(x) \cap (\partial A_\epsilon(y))^{[\delta]}$, such that $\forall \xi_x' \in I^s(x)$, there exists $z' \in \partial A_\epsilon$ as in Lemma 6.2.1 and where $z' \in W^s(\xi_x')$.

Hence, we have $\partial A_\epsilon(y)^{[\delta]} = \mathcal{I}^u \cup \mathcal{I}^s$, where

$$\mathcal{I}^u := \bigcup_{x \in W^s(x_0)} I^u(x) \text{ and } \mathcal{I}^s := \bigcup_{x \in W^s(x_0)} I^s(x)$$

our aim is to estimate the measure of $\mathcal{I}^u$ with respect to that of $B(x, k_0\epsilon)$. We will start working on $\nu^u(\pi^+(\chi^{-1}(I^u)))$ and then after integrating over $\Sigma_A^- \times \Sigma_A^+$, we conclude the estimation of $\nu(\chi^{-1}(\mathcal{I}^u))$.

Let q_i be the greatest integer such that $l^u_{q_i} > 3c\delta$, then we have:

$$I^u(x) \subset \bigcup_{i=1}^{2} \chi(C_{-\infty, q_i}(\omega_i)) \subset B(x, k_0\epsilon),$$

and so $\nu^u(\pi^+(\chi^{-1}(I^u))) \leq \sum_{i=1}^{2} \nu^u(C_{0,q_i}(\pi^+(\omega_i)))$. Knowing that $W^u(x) = \chi\left(C_{-\infty,0}(\omega)\right) = \bigcap_{i=1}^{\infty} \chi\left(C_{-i,0}(\omega)\right)$, let p_i be the smallest integer such that $l^u_{p_i}(\omega_i) \leq \frac{k_0\epsilon}{2}$, then using Lemma 4.6.6, we have

$$\chi\left(C_{-\infty,p_i}(\omega)\right) \subset W^u(x) \cap B(x, k_0\epsilon), \tag{6.2.2}$$

but $\nu^u(C_{0,p_i}(\pi^+(\omega_i))) \leq \nu^u(\pi^+(\chi^{-1}(B(x,k_0\epsilon) \cap W^u(x))))$, thus using Proposition 2.0.1 and that $\max \varphi < -\gamma$, we get:

$$\begin{aligned}
\frac{\nu^u(\pi^+(\chi^{-1}(I^u)))}{\nu^u(\pi^+(\chi^{-1}(B(x,k_0\epsilon) \cap W^u(x))))} &\leq \sum_{i=1}^{2} \frac{\nu^u(C_{0,q_i}(\pi^+(\omega_i)))}{\nu^u(C_{0,p_i}(\pi^+(\omega_i)))} \\
&\leq \sum_{i=1}^{2} k_\varphi^2 \exp\left(\sum_{k=p_i+1}^{q_i} \varphi \circ \sigma^k(\omega_i)\right) \\
&\leq \sum_{i=1}^{2} k_\varphi^2 \exp(\gamma(q_i - p_i)) \\
&\leq \eta(\epsilon),
\end{aligned}$$

where $\eta(\epsilon) \to 0$ as ϵ goes to 0. Set $\tilde{I}^u = \{(\omega^-, \omega^+) : \exists \omega = (\omega_n)_{n\in\mathbb{Z}} \in \Sigma \text{ s.t.} \chi(\omega) \in I^u; \omega^- = (\omega_n)_{n\leq 0} \text{ and } \omega^+ = (\omega_n)_{n\geq 0}\}$, and in same way we define the set $\tilde{B}(x, k_0\epsilon)$. We have the following:

$$\begin{aligned}
\nu^u \otimes \nu^s(\chi^{-1}(\mathcal{I}^u)) &= \int_{\Sigma_A^-} \left(\int_{\Sigma_A^+} 1_{\tilde{I}^u(\omega^-,\omega^+)} d\nu^u(\omega^+)\right) d\nu^s(\omega^-) \\
&= \int_{\Sigma_A^-} \nu^u(\{\omega^+ : (\omega^-,\omega^+) \in \tilde{I}^u\}) d\nu^s(\omega^-) \\
&\leq \eta(\epsilon) \int_{\Sigma_A^-} \nu^u(\{\omega^+ : (\omega^-,\omega^+) \in \tilde{B}(x,k_0\epsilon)\}) d\nu^s(\omega^-) \\
&\leq \eta(\epsilon) \nu^u \otimes \nu^s(\chi^{-1}(B(x,k_0\epsilon)))
\end{aligned}$$

On the other hand, the measure ν is absolutely continuous with respect to the product measure $\nu^u \otimes \nu^s$ with density function h. Then using the previous inequality and Proposition 2.0.1, we conclude that:

$$\begin{aligned}
\nu(\chi^{-1}(\mathcal{I}^u)) &= \int_{\Sigma} 1_{\chi^{-1}(\mathcal{I}^u)} h(\omega) d(\nu^u \otimes \nu^s)(\omega) \\
&\leq \kappa_2 \nu^u \otimes \nu^s(\chi^{-1}(\mathcal{I}^u)) \\
&\leq \kappa_2 \eta(\epsilon) \nu^u \otimes \nu^s(\chi^{-1}(B(x,k_0\epsilon))) \\
&\leq \tilde{\eta}(\epsilon) \nu(B(x,k_0\epsilon)).
\end{aligned}$$

where $\tilde{\eta}(\epsilon) = \frac{\kappa_2}{\kappa_1}\eta(\epsilon)$.
Proceeding with the same strategy above, there exists $\bar{\eta}(\epsilon)$ such that $\nu(\chi^{-1}(\mathcal{I}^s)) \leq \bar{\eta}(\epsilon)\nu(B(x, k_0\epsilon))$ and this finishes the proof since $\tilde{\eta}(\epsilon)$ and $\bar{\eta}(\epsilon)$ goes to 0 as $\epsilon \to 0$. $\square$

Recall 6.2.3. *We recall the first return time of the flow $\tilde{g}_t$ to $B(y, \epsilon)$ for $y \in \tilde{M}$:*

$$\tau_\epsilon(y) = \inf\{t > 1, \tilde{g}_t(y) \in B(y, \epsilon)\}. \tag{6.2.3}$$

Definition. Let us consider the function $\varphi : X \to \mathbb{Z}$ introduced at the beginning of Chapter 5. Let us denote by σ_φ^2 the asymptotic variance of φ. We consider the $\mathbb{Z}-$extension $\tilde{T}$ of T by φ:

$$\begin{aligned} \tilde{T} : X \times \mathbb{Z} &\to X \times \mathbb{Z} \\ (x, m) &\to (Tx, m + \varphi(x)). \end{aligned}$$

Definition. We define the first return time to $A_\epsilon(y)$ in Σ_A, by

$$\begin{aligned} w_{A_\epsilon(y)}(x) &:= \inf\{n \geq 1, \tilde{T}^n(x, 0) \in A_\epsilon(y) \times \{0\}\} \\ &= \inf\{n \geq 1, S_n\varphi(x) = 0 \text{ and } \sigma^n x \in A_\epsilon(y)\}. \end{aligned}$$

Remark 6.2.4. We will start by studying the asymptotic behavior of $w_{A_\epsilon(y)}(x)$ to deduce that of $\tau_\epsilon(y)$. we will need the following definition:

Definition. We denote by $R_{A_\epsilon^+(y)}(x) = \min\{n \geq 1 : \sigma^n(x) \in A_\epsilon^+(y)\}$ the first return time of a point x into $A_\epsilon^+(y)$.

Proposition 6.2.5. *For every $y \in M$, every $\epsilon > 0$, consider $n_\epsilon = n_\epsilon(t, y) :=$ and M_ϵ and Q_ϵ as defined in Section 6.1. Let D be a set satisfying either ($D \in \mathcal{D}_\epsilon$ and $D \subset A_\epsilon^+(y)$) or $D = A_\epsilon^+(y)$, we have:*

a. $1 \geq \nu(w_{A_\epsilon^+(y)} > n_\epsilon | D) + \frac{1}{\sqrt{2\pi}\sigma_\varphi} \sum_{r=M_\epsilon}^{n_\epsilon} \frac{\nu(A_\epsilon^+(y) \cap \{w_{A_\epsilon^+(y)} > n_\epsilon - r\})}{\sqrt{r - Q_\epsilon}} - o(1).$

b. $1 \leq \nu(w_{A_\epsilon^+(y)} > n_\epsilon | D) + \frac{1}{\sqrt{2\pi}\sigma_\varphi} \sum_{r=M_\epsilon}^{n_\epsilon} \frac{\nu(A_\epsilon^+(y) \cap \{w_{A_\epsilon^+(y)} > n_\epsilon - r\})}{\sqrt{r - q - l}} + \nu(R_{A_\epsilon^+(y)} \leq M_\epsilon | D) + o(1).$

Proof. We use the same method by Dvoretzky and Erdös in [21], We make a partition of the cylinder D according to the last passage $r \leq n$ in the time interval $0, ..., n$ of the orbit of $(x, 0)$ by the map $\tilde{T}$ into $A_\epsilon^+(y) \times \{0\}$. This can be seen as follows.
For simplicity, we will denote by A_ϵ^+ and A_ϵ^- instead of $A_\epsilon^+(y)$ and $A_\epsilon^-(y)$ respectively.

$$\nu(D) = \sum_{r=0}^{n} \nu\left(x \in D; S_r\varphi(x) = 0\ ,\ \sigma^r(x) \in A_\epsilon^+; \forall l = r, ..., n, S_l\varphi(x) \neq 0 \text{ or } \sigma^l(x) \notin A_\epsilon^+\right), \tag{6.2.4}$$

Knowing that $S_r\varphi(x) = 0$, then

$$\{\forall l = r, .., n, S_l\varphi(x) - S_r\varphi(x) \neq 0 \text{ or } \sigma^l(x) \notin A_\epsilon^+\}$$

is equal to

$$\{\forall k = 1, .., n-r, S_{k+r}\varphi(x) - S_r\varphi(x) \neq 0 \text{ or } \sigma^{k+r}(x) \notin A_\epsilon^+\},$$

but $S_{k+r}\varphi(x) - S_r\varphi(x) = \sum_{l=0}^{k-1} \varphi \circ \sigma^{l+r}(x) = S_k\varphi(\sigma^r x)$, it follows that $\nu(D)$ is equal to

$$\sum_{r=0}^{n} \nu\left(x \in D; S_r(x) = 0, \sigma^r(x) \in A_\epsilon^+; \sigma^{-r}\{z \in \Sigma_A : \forall k = 1, .., n-r, S_k\varphi(z) \neq 0, \sigma^k(z) \notin A_\epsilon^+\}\right),$$

from which we get

$$\begin{aligned} \nu(D) &= \sum_{r=0}^{n} \nu\left(D \cap \{S_r = 0\} \cap \theta^{-r}\left(A_\epsilon^+ \cap \{w_{A_\epsilon^+}(x) > n-r\}\right)\right) \\ &\geq \nu\left(D \cap A_\epsilon^+ \cap \{w_{A_\epsilon^+}(x) > n\}\right) \\ &\quad + \sum_{r=2q+1}^{n} \nu\left(D \cap \{S_r = 0\} \cap \theta^{-r}\left(A_\epsilon^+\right) \cap \{w_{A_\epsilon^+}(x) > n-r\}\right). \end{aligned}$$

We have D a $(q, q')-$cylinder with $q \leq Q_\epsilon$, and A_ϵ^+ is a union of $(Q_\epsilon, q^")-$cylinders for some $q^" > 0$. Let $E = A_\epsilon^+(y) \cap \{w_{A_\epsilon^+} > n\}$ which is the set of points of $A_\epsilon^+(y)$ which doesn't return to A_ϵ^+ before n, that is $E = A_\epsilon^+ \cap \bigcap_{s=1}^{n} \left(\{S_s\varphi \neq 0\} \cup \sigma^{-s}(\Sigma_A \setminus A_\epsilon^+)\right)$. we know that φ is constant on m_0-cylinders, then $\{S_s\varphi \neq 0\}$ is a union of $(m_0, m_0 - s + 1)$-cylinders, and $\sigma^{-s}(\Sigma_A \setminus A_\epsilon^+) = \Sigma_A \setminus \sigma^{-s}(A_\epsilon^+)$ is a union of $(q + s, q' + s')$-cylinders. Then we get that E is a union of $Q_\epsilon, q^"$-cylinders, for some $q^{"} > 0$. Hence E can be written as $\sigma^q(\Pi^{-1}B)$, where B is a subset of Σ_A^+. Now, let $n = n_\epsilon$, with all these conditions, we apply the Local Limit Theorem in Proposition 2.2.1, thus there is $C_1 > 0$ such that:

$$\begin{aligned}\nu(D) &\geq \nu\left(D \cap \{w_{A_\epsilon^+}(x) > n_\epsilon\}\right) + \sum_{r=M_\epsilon}^{n_\epsilon} \frac{\nu(D)\nu(A_\epsilon^+ \cap \{w_{A_\epsilon^+}(x) > n_\epsilon - r\})}{\sqrt{2\pi}\sqrt{r-(Q_\epsilon+l)}\sigma_\varphi} \\ &- C_1 \sum_{r=M_\epsilon}^{n_\epsilon} \frac{\nu(A_\epsilon^+ \cap \{w_{A_\epsilon^+}(x) > n_\epsilon - r\}) q\nu(D)}{r - 2Q_\epsilon}.\end{aligned}$$

There is $C_3 > 0$ such that the error term is controlled by

$$\sum_{r=M_\epsilon}^{n_\epsilon} \frac{\nu(A_\epsilon^+) q\nu(D)}{r - 2Q_\epsilon} \leq C_3 q \log n_\epsilon \nu(D)\nu(A_\epsilon^+) \tag{6.2.5}$$

And hence we get

$$\begin{aligned}\nu(D) &\geq \nu\left(D \cap \{w_{A_\epsilon^+}(x) > n_\epsilon\}\right) + \nu(D) \sum_{r=M_\epsilon}^{n_\epsilon} \frac{\nu(A_\epsilon^+ \cap \{w_{A_\epsilon^+}(x) > n_\epsilon - r\}}{\sqrt{2\pi}\sqrt{r - Q_\epsilon}\sigma_\varphi} \\ &-o(\nu(D)),\end{aligned}$$

Dividing by $\nu(D)$ yields (a) for $D \in \mathcal{D}_\epsilon$ such that $D \subset A_\epsilon^+(y)$.

We prove inequality (b) following the same strategy. We set $n_\epsilon = \left(\frac{t}{\nu(A_\epsilon^+)(y)}\right)^2$, using the same decomposition as in (6.2.4), we have

$$\nu(D) = \sum_{r=0}^{n_\epsilon} \nu\left(D \cap \{S_r = 0\} \cap \sigma^{-r}\left(A_\epsilon^+(y) \cap \{w_{A_\epsilon^+(y)}(x) > n_\epsilon - r\}\right)\right) := \sum_{r=0}^{n_\epsilon} G_{r,\epsilon}(x)$$

For $r = 0$, the first term is equal to $\nu\left(D \cap \{w_{A_\epsilon^+(y)}(x) > n_\epsilon\}\right)$. We then compute the sum of terms between 1 and M_ϵ

$$\begin{aligned}\sum_{r=1}^{M_\epsilon} \nu &\left(D \cap \{S_l = 0\} \cap \sigma^{-r}\left(A_\epsilon^+(y) \cap \{w_{A_\epsilon^+(y)} > n_\epsilon - r\}\right)\right) \\ &= \nu\left(\bigcup_{r=1}^{M_\epsilon} D \cap \{S_l = 0\} \cap \sigma^{-r}\left(A_\epsilon^+(y) \cap \{w_{A_\epsilon^+(y)} > n_\epsilon - r\}\right)\right) \\ &\leq \nu(D \cap \{R_{A_\epsilon^+(y)} \leq M_\epsilon\}).\end{aligned}$$

Then using the local limit theorem in Proposition 2.2.1, there exists $C_4 > 0$ such that we have the following:

$$\begin{aligned}\sum_{r=M_\epsilon}^{n_\epsilon} G_{r,\epsilon}(x) &\leq \frac{1}{\sqrt{2\pi}\sigma_\varphi}\sum_{r=M_\epsilon}^{n_\epsilon}\frac{\nu(D)\nu(A_\epsilon^+\cap\{w_{A_\epsilon^+}>n_\epsilon-r\})}{\sqrt{r-q-l}}\\ &\quad +C_4\frac{\nu(A_\epsilon^+\cap\{w_{A_\epsilon^+}>n_\epsilon-r\})q\nu(D)}{r-2Q_\epsilon}.\end{aligned}$$

Similarly as in (6.2.5), the error term is controlled by $o(\nu(D))$. Combining these assumption together, we obtain

$$\begin{aligned}\nu(D) &\leq o(\nu(D))+\nu(D\cap\{w_{A_\epsilon^+}>n_\epsilon\})+\nu(R_{A_\epsilon^+}\leq M_\epsilon|D)\nu(D)\\ &+ \frac{1}{\sqrt{2\pi}\sigma_\varphi}\nu(D)\nu(A_\epsilon^+\cap\{w_{A_\epsilon^+}>n_\epsilon\})\left(\sum_{r=M_\epsilon}^{n_\epsilon}\frac{1}{\sqrt{r-q}}\right)\end{aligned}\tag{6.2.6}$$

Dividing by $\nu(D)$ yields the desired inequality. □

Corollary 6.2.6. *Under the hypothesis of Proposition 6.2.5, where n_ϵ, M_ϵ and D are considered similarly, we have:*

a. $1+o(1) = \nu(w_{A_\epsilon^+(y)} > n_\epsilon|D) + \frac{1}{\sqrt{2\pi}\sigma_\varphi}\sum_{r=M_\epsilon}^{n_\epsilon}\frac{\nu(A_\epsilon^+(y)\cap\{w_{A_\epsilon^+(y)}>n_\epsilon-r\})}{\sqrt{r-q}} + O\left(\nu(R_{A_\epsilon^+(y)}\leq M_\epsilon|D)\right)$

b. $1+o(1) = \nu(w_{A_\epsilon^+(y)} > n_\epsilon|A_\epsilon^+(y)) + \frac{1}{\sqrt{2\pi}\sigma_\varphi}\sum_{M_\epsilon}^{n_\epsilon}\frac{\nu(A_\epsilon^+(y)\cap\{w_{A_\epsilon^+(y)}>n_\epsilon-r\})}{\sqrt{r-q}} + O\left(\nu(R_{A_\epsilon^+(y)}\leq M_\epsilon|A_\epsilon^+(y))\right)$.

Lemma 6.2.7. *From the previous Corollary, we have*

$$\begin{aligned}\nu(w_{A_\epsilon^+(y)}>n_\epsilon|D) &= \nu(w_{A_\epsilon^+(y)}>n_\epsilon|A_\epsilon^+(y))+O\left(\nu(R_{A_\epsilon^+(y)}\leq M_\epsilon|A_\epsilon^+(y))\right)\\ &\quad +O\left(\nu(R_{A_\epsilon^+(y)}\leq M_\epsilon|D)\right)+o(1).\end{aligned}$$

Thus we will study the convergence in distribution of $w_{A_\epsilon^+(y)}$ with respect to $\nu(.|A_\epsilon^+(y))$ and the we will deduce that with respect to $\nu(.|D)$. So, first we have the following proposition:

Proposition 6.2.8. *The family of distributions of $\left(\nu(A_\epsilon^+(y))^2 w_{A_\epsilon^+(y)}\right)_{\epsilon>0}$ is tight with respect to the family of probability measures $(\nu(.|A_\epsilon^+(y)))_{\epsilon>0}$.*

Proof. Let $t > 0$, using inequality (a) of Proposition 6.2.5 with $n_\epsilon = \left(\frac{t}{\nu(A_\epsilon^+(y))}\right)^2$, we have

$$\begin{aligned} 1 &\geq \nu(w_{A_\epsilon^+(y)} > n_\epsilon | A_\epsilon^+(y)) + \frac{1}{\sqrt{2\pi}\sigma_\varphi} \sum_{M_\epsilon}^{n_\epsilon} \frac{\nu(A_\epsilon^+(y) \cap \{w_{A_\epsilon^+(y)} > n_\epsilon - r\})}{\sqrt{r-q}} - o(1) \\ &\geq \nu(w_{A_\epsilon^+(y)} > n_\epsilon | A_\epsilon^+(y)) + \frac{1}{\sqrt{2\pi}\sigma_\varphi} \nu(A_\epsilon^+(y) \cap \{w_{A_\epsilon^+(y)} > n_\epsilon\}) \sum_{M_\epsilon}^{n_\epsilon} \frac{1}{\sqrt{r-q}} - o(1), \end{aligned}$$

from which it follows that

$$\forall t > 0 \quad \nu\left(w_{A_\epsilon^+(y)} > \left(\frac{t}{\nu(A_\epsilon^+(y))}\right)^2 \mid A_\epsilon^+(y)\right) \leq \frac{1 + o(1)}{1 + \frac{1}{\sqrt{2\pi}\sigma_\varphi}\nu(A_\epsilon^+(y))\sum_{M_\epsilon}^{n_\epsilon}\frac{1}{\sqrt{r-q}}}, \tag{6.2.7}$$

but $\sum_{M_\epsilon}^{n_\epsilon} \frac{1}{\sqrt{r-q}} = 2(\sqrt{n_\epsilon} - \sqrt{2q-1}) \geq \sqrt{n_\epsilon} \geq \frac{t}{\nu(A_\epsilon^+(y))}$, then we get

$$\limsup_{\epsilon \to 0} \nu\left(\nu(A_\epsilon^+(y))^2 w_{A_\epsilon^+(y)} > t^2 \mid A_\epsilon^+(y)\right) \leq \frac{1}{1+t}, \tag{6.2.8}$$

which proves the tightness. □

Theorem 6.2.9 ([44]). *Define $\bar{\tau}_\epsilon = \inf\{t > 1 : g_t(y) \in B(y, \epsilon)\}$. For μ-almost every $y \in M$*

$$\lim_{\epsilon \to 0} \frac{\log \bar{\tau}_\epsilon(y)}{-\log \epsilon} = d - 1.$$

Consider the following Lemma, where we will define a family of sets D_ϵ, and which we will apply later in our proofs with $D_\epsilon = A_\epsilon(y)$ and $D_\epsilon \in \mathcal{D}_\epsilon$.

Lemma 6.2.10. *For ν-almost every $x \in \Sigma_A$, for all $k_0 > 0, m_F > 0, c_F > 0$, Let $(D_\epsilon)_\epsilon$ be a family of sets containing x such that $D_\epsilon \subset B(x, k_0\epsilon)$ and $\nu(B(x, diam(D_\epsilon))) \leq c_F^{m_F}\nu(D_\epsilon)$. For M_ϵ defined as in Section 6.1, we have*

$$\lim_{\epsilon \to 0} \nu(R_{B(x,k_0\epsilon)} \leq M_\epsilon | D_\epsilon) = 0.$$

Proof. Let $\alpha > 0, k_0 > 0, m_F > 0, c_F > 0$. Choose some $a \in (0, \alpha)$ and set for some $\epsilon_0 > 0$,

$$F_a = \{x \in \Sigma_A : \forall \epsilon \leq \epsilon_0, M_\epsilon^{-1} R_{B(x,2k_0\epsilon)}(x) > a\}$$

We will show that $\nu(F_a) \to 1$ as $\epsilon_0 \to 0$. First, let us prove that $\nu-$a.e. $x \in \Sigma_A$,

$$\lim_{\epsilon \to 0} M_\epsilon^{-1} R_{B(x,2k_0\epsilon)}(x) = +\infty. \tag{6.2.9}$$

An intermediate step to do that is to prove that

$$\lim_{\epsilon\to 0} \frac{\log R_{B(x,2k_0\epsilon)}}{-\log \epsilon} = d-1, \quad \nu - a.e. \tag{6.2.10}$$

Let $r_{\min} > 0$ and $r_{\max} > 0$ the lower and upper bounds of the height function r. Using (5.2.5), for $\bar{\mu}-$almost every $y = (x, s) \in \Delta$, we have

$$\bar{\tau}_\epsilon(y)\frac{1}{r_{\max}} \leq R_{B(x,2k_0\epsilon)}(x) \leq \bar{\tau}_\epsilon(y)\frac{1}{r_{\min}}, \tag{6.2.11}$$

which gives

$$\frac{\log r_{\max}}{\log \epsilon} + \frac{\log \bar{\tau}_\epsilon(y)}{-\log \epsilon} \leq \frac{\log R_{B(x,2k_0\epsilon)}(x)}{-\log \epsilon} \leq \frac{\log r_{\min}}{\log \epsilon} + \frac{\log \bar{\tau}_\epsilon(y)}{-\log \epsilon},$$

using Theorem 6.2.9, it follows that $\lim\limits_{\epsilon\to 0} \frac{\log R_{B(x,2k_0\epsilon)}}{-\log \epsilon} = d-1$, $\nu-$ almost everywhere.
Now we go back to show (6.2.9). Let $0 < \delta_0 < d-1$, we have the following:

$$\log(M_\epsilon^{-1} R_{B(x,2k_0\epsilon)}(x)) = \log(M_\epsilon^{-1}\epsilon^{-\delta_0}) + \log(\epsilon^{\delta_0} R_{B(x,2k_0\epsilon)}(x)) \tag{6.2.12}$$

Note that the first term of (6.2.12) goes to $+\infty$, as ϵ goes to 0. This is true because using Lemma 6.1.1 and the definition of M_ϵ, there exists $\rho_0 > 0$ such that $M_\epsilon < 2^{\frac{|\log\left(\frac{\theta_\epsilon}{2Cc_g c_l}\right)|}{|\log \rho_u|}} + 1 \leq \rho_0 |\log \epsilon|$, which is negligible with respect to $\epsilon^{-\delta_0}$.
Now, considering the second term, we observe that

$$\frac{\log(\epsilon^{\delta_0} R_{B(x,2k_0\epsilon)}(x))}{-\log \epsilon} = -\delta_0 + \frac{\log R_{B(x,2k_0\epsilon)}(x)}{-\log \epsilon},$$

from which we get $\log(\epsilon^{\delta_0} R_{B(x,2k_0\epsilon)}(x)) \underset{\epsilon\to 0}{\to} (-\delta_0 + d - 1)(+\infty)$. And therefore (6.2.9) is proved.
Taking $\epsilon_0 \to 0$, by Egoroff's Theorem, $M_\epsilon^{-1} R_{B(x,2k_0\epsilon)}$ converges uniformly to $+\infty$ on F_a, and $\nu(F_a^C) \to 0$.
Let $\epsilon < \epsilon_0$. if $x' \in B(x, k_0\epsilon)$ and $R_{B(x,k_0\epsilon)}(x') \leq M_\epsilon$, we have $R_{B(x,2k_0\epsilon)}(x') \leq M_\epsilon$ as well, hence $x' \in F_a^C$. Therefore

$$D_\epsilon \cap \{x' \in \Sigma_A, R_{B(x,k_0\epsilon)}(x') \leq M_\epsilon\} \subset D_\epsilon \cap F_a^C$$

Then for any density point x of the set F_a relative to the Lebesgue basis given by $(B(., \delta))_\delta$. This means that $\frac{\nu(F_a \cap B(x,\delta))}{\nu(B(x,\delta))} \to 1$, as $\delta \to 0$, and this true for almost every point x. Therefore, we get

$$
\begin{aligned}
\nu(R_{B(x,k_0\epsilon)} \leq M_\epsilon | D_\epsilon) &\leq \nu(F_a^C | D_\epsilon) \\
&= \frac{\nu(F_a^C \cap D_\epsilon)}{\nu(D_\epsilon)} \\
&\leq \frac{\nu(F_a^C \cap B(x, diam(D_\epsilon)))}{\nu(D_\epsilon)} \\
&\leq \nu(F_a^C | B(x, diam(D_\epsilon))) \frac{\nu(B(x, diam(D_\epsilon)))}{\nu(D_\epsilon)},
\end{aligned}
$$

which converges to 0 as $\epsilon \to 0$, since $\lim\limits_{\epsilon\to 0} \nu(F_a^C | B(x, diam(D_\epsilon))) = 0$ and $\frac{\nu(B(x,diam(D_\epsilon))}{\nu(D_\epsilon)}$ is bounded by $c_F^{m_F}$. □

Proposition 6.2.11. *For almost every $y \in M$,*

$$\lim_{\epsilon\to 0} \nu\left((\nu(A_\epsilon^+(y))^2 w_{A_\epsilon^+(y)} > t | A_\epsilon^+(y)\right) = \mathbb{P}\left(\sigma_\varphi^2 \frac{\mathcal{E}^2}{\mathcal{N}^2} > t\right), \quad \forall t > 0,$$

and

$$\lim_{\epsilon\to 0} \nu\left((\nu(A_\epsilon^-(y))^2 w_{A_\epsilon^-(y)} > t | A_\epsilon^-(y)\right) = \mathbb{P}\left(\sigma_\varphi^2 \frac{\mathcal{E}^2}{\mathcal{N}^2} > t\right), \quad \forall t > 0,$$

where $\mathcal{E}$ and $\mathcal{N}$ are independent random variables, $\mathcal{E}$ follows the exponential distribution of mean 1 and $\mathcal{N}$ having the standard normal distribution.

Proof. After proving the tightness in Proposition 6.2.8, then it will be enough to prove that the advertised limit is the only possible accumulation point of the distribution. Let $(\epsilon_p)_{p\geq 1}$ be a positive sequence with $\lim\limits_{p\to\infty} \epsilon_p = 0$ and such that, for all $t > 0$ the conditional distributions of the $(\nu(A_{\epsilon_p}^+)\sqrt{w_{A_{\epsilon_p}^+}} | A_{\epsilon_p}^+)_{p\geq 0}$ converges to the law of some random variable X. Then using Lemma 3.3.4 in Chapter 3, and due to Lemma 6.2.10, X satisfies the integral equation:

$$1 = \mathbb{P}(X > t) + t\frac{1}{\sqrt{2\pi}\sigma_\varphi} \int_0^1 \frac{\mathbb{P}(X > t\sqrt{1-u})}{\sqrt{u}} du \quad \forall t > 0. \tag{6.2.13}$$

Then the conclusion follows as in Theorem 3.1.2, the distribution of X coincides with that of $\sigma_\varphi^2 \frac{\mathcal{E}^2}{\mathcal{N}^2}$, where the independent variables $\mathcal{E}$ and $\mathcal{N}$ are the exponential distribution of mean 1 and the standard Gaussian distribution respectively. □

Let $\tilde{\mu} = \nu \otimes Leb \otimes \sum_k \delta_k$ be the measure on $\tilde{M}$. Set $\bar{\mu} := \nu \otimes Leb$ the measure restricted on the 0-cell and let μ be a probability measure absolutely continuous with respect to μ with a density function h. Set $H(.) := \sum_{k\in\mathbb{Z}} h(., k)$, by $\mathbb{Z}-$periodicity, the distribution of τ_ϵ under μ is the same as under the probability measure absolutely continuous with respect to $\bar{\mu}$ with density H. We have $y = (x, s) \in \bigcup_{P\in\mathcal{P}_\epsilon} P$, and $\exists i, j$ such that $y \in \mathcal{P}_{\epsilon,i,j}$. Let $y_{\epsilon,i,j} \in \mathcal{P}_{\epsilon,i,j}$, then:

$$H(y_{\epsilon,i,j}) - w(H, \theta_\epsilon) \leq H(y) \leq H(y_{\epsilon,i,j}) + \omega(H, \theta_\epsilon), \quad y \in \mathcal{P}_{\epsilon,i,j}$$

then we have,

$$\min_{y\in\mathcal{P}_{\epsilon,i,j}} H(y) \geq H(y_{\epsilon,i,j}) - \omega(H, \theta_\epsilon), \tag{6.2.14}$$

and

$$\max_{y\in\mathcal{P}_{\epsilon,i,j}} H(y) \leq H(y_{\epsilon,i,j}) - \omega(H, \theta_\epsilon), \tag{6.2.15}$$

Now we are ready to proof the main result of the chapter. For simplicity we will denote by $A^{\overset{-}{+}}_{\epsilon,i,j} := A^{\overset{-}{+}}_\epsilon(y_{\epsilon,i,j})$.

Proof of Theorem 6.0.5. For all $\epsilon > 0$, $\forall t > 0$ we set

$$\begin{aligned} E_{\epsilon,t} &:= (H\bar{\mu})\big(y = (x, s, 0) : \nu(A_\epsilon(y))^2\tau_\epsilon(y) > t\big) \\ &= \sum_{i,j} \int_{\mathcal{P}_{\epsilon,i,j}} 1_{\{y:\nu(A_{\epsilon,i,j})^2\tau_\epsilon(y)>t\}} H(x, s) d\nu(x) dx \end{aligned} \tag{6.2.16}$$

First, due to the inclusions in (6.1.1), then

$$\nu(A^-_{\epsilon,i,j})^2 w_{A^+_{\epsilon,i,j}}(x) \leq \nu(A_{\epsilon,i,j})^2 w_{A_{\epsilon,i,j}}(x) \leq \nu(A^+_{\epsilon,i,j})^2 w_{A^-_{\epsilon,i,j}}(x), \tag{6.2.17}$$

we also have from (5.2.5) that there exists $\zeta(\epsilon) > 0$ such that

$$(1 - \zeta(\epsilon))\tau_\epsilon \leq w_{A_{\epsilon,i,j}} \int_X R d\nu \leq (1 + \zeta(\epsilon))\tau_\epsilon \tag{6.2.18}$$

For these reasons, and using the inequality (6.2.15), we get:

$$\begin{aligned}
E_{\epsilon,t} &\leq \sum_{i,j}\int_{\mathcal{P}_{\epsilon,i,j}} 1_{\{x:\nu(A_{\epsilon,i,j})^2 w_{A_{\epsilon,i,j}}(x)>\frac{t(1-\zeta(\epsilon))}{\int Rd\nu}\}} \max_{y\in\mathcal{P}_{\epsilon,i,j}} H(y)d\nu(x)ds \\
&\leq \sum_{i,j}\int_{\mathcal{P}_{\epsilon,i,j}} 1_{\{x:\nu(A^+_{\epsilon,i,j})^2 w_{A^-_{\epsilon,i,j}}(x)>\frac{t(1-\zeta(\epsilon))}{\int Rd\nu}\}} H(y_{\epsilon,i,j})d\nu(x)ds + \sum_{i,j}\int_{\mathcal{P}_{\epsilon,i,j}} \omega(H,\theta_\epsilon)d\nu ds \\
&\leq \sum_{i,j} H(y_{\epsilon,i,j})\theta_\epsilon\nu\left(\{x\in D_{\epsilon,i}:\nu(A^+_{\epsilon,i,j})^2 w_{A^-_{\epsilon,i,j}}(x) > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}\}\right) \\
&\quad + \sum_{i,j}\omega(H,\theta_\epsilon)\nu(D_{\epsilon,i})\theta_\epsilon \\
&\leq \sum_{i,j} H(y_{\epsilon,i,j})\nu(D_{\epsilon,i})\theta_\epsilon\nu\left(\nu(A^+_{\epsilon,i,j})^2 w_{A^-_{\epsilon,i,j}}(x) > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}|D_{\epsilon,i}\right) + \omega(H,\theta_\epsilon)
\end{aligned}$$

For all $y\in\mathcal{P}_{\epsilon,i,j}$, we have $A^-_\epsilon(y) = A^-_{\epsilon,i,j}$, $A^+_\epsilon(y) = A^+_{\epsilon,i,j}$, and $D_{\epsilon,i} = D_\epsilon(x)$. Then $\forall t>0$

$$\begin{aligned}
E_{\epsilon,t} &\leq \sum_{i,j} H(y_{\epsilon,i,j})\int_{P_{i,j}}\nu\left(\nu(A^+_\epsilon(y))^2 w_{A^-_\epsilon(y)} > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}|D_\epsilon(x)\right)d\bar{\mu}(y) + \omega(H,\theta_\epsilon) \\
&\leq \int_M H(y)\nu\left(\nu(A^+_\epsilon(y))^2 w_{A^-_\epsilon(y)} > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}|D_\epsilon(x)\right)d\bar{\mu}(y) + 2\omega(H,\theta_\epsilon) \quad (6.2.19)
\end{aligned}$$

Now we have $\nu(A^-_\epsilon(y))$ and $\nu(A^+_\epsilon(y))$ are equivalent (Lemma 6.2.2), then there exists $\alpha(\epsilon)>0$ such that $\alpha(\epsilon)\to 0$ as $\epsilon\to 0$ and

$$\nu(A^-_\epsilon(y))(1-\alpha(\epsilon)) \leq \nu(A^+_\epsilon(y)) \leq \nu(A^-_\epsilon(y))(1+\alpha(\epsilon)), \qquad (6.2.20)$$

then using (6.2.20) and due to Lemma 6.2.7, we get

$$\begin{aligned}
\nu\left(\nu(A^+_\epsilon(y)^2 w_{A^-_\epsilon(y)} > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}|D_\epsilon(x)\right) &\leq \nu\left((1+\alpha(\epsilon))^2\nu(A^-_\epsilon(y))^2 w_{A^-_\epsilon(y)} > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}|D_\epsilon(x)\right) \\
&\leq \nu\left((1+\alpha(\epsilon))^2\nu(A^-_\epsilon(y))^2 w_{A^-_\epsilon(y)} > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}|A^-_\epsilon(y)\right) \\
&\quad + \nu(R_{A^-_\epsilon(y)}\leq M_\epsilon|A^-_\epsilon(y)) - \nu(R_{A^-_\epsilon(y)}\leq M_\epsilon|D_\epsilon(x)) \\
&\quad + o(1).
\end{aligned}$$

Set $Q^-_\epsilon = \frac{(1+\alpha(\epsilon))^2}{1-\zeta(\epsilon)}$, we have $\lim_{\epsilon\to 0} Q^-_\epsilon = 1$, then using Proposition 6.2.11 and due to Slutsky's Lemma, we get the following convergence in distribution:

$$\lim_{\epsilon\to 0}\nu\left((1+\alpha(\epsilon))^2\nu(A^-_\epsilon(y))^2 w_{A^-_\epsilon(y)} > \frac{t(1-\zeta(\epsilon))}{\int Rd\nu}|A^-_\epsilon(y)\right) = \mathbb{P}\left(\sigma^2_\varphi\frac{\mathcal{E}^2}{\mathcal{N}^2} > t\right), \quad \forall t>0$$

Moreover, due to Lemma 6.2.10, as $A_\epsilon^-(y)$ and $D_\epsilon(x)$ are contained in $B(x, k_0\epsilon)$, we have $\lim_{\epsilon\to 0}\nu(R_{A_\epsilon^-(y)} \le M_\epsilon | A_\epsilon^-(y)) = 0$, and $\lim_{\epsilon\to 0}\nu(R_{A_\epsilon^-(y)} \le M_\epsilon | D_\epsilon(x)) = 0$. Moreover H is uniformly continuous then $\lim_{\epsilon\to 0}\omega(H,\nu_\epsilon) = 0$. Thus using the Lebesgue's dominated convergence theorem in 6.2.19, we end up having

$$\lim_{\epsilon\to 0} E_{\epsilon,t} \le \mathbb{P}\left((\sigma_\varphi^2\int R d\nu)\frac{\mathcal{E}^2}{\mathcal{N}^2} > t\right), \quad \forall t > 0. \tag{6.2.21}$$

We do analogously the second inequality, thus using (6.2.14) and the left hand side inequality in (6.2.17), and again due to (6.2.18) there exists $\zeta(\epsilon) > 0$ such that

$$\begin{aligned} E_{\epsilon,t}(y) &\ge \sum_{i,j}\int_{\mathcal{P}_{\epsilon,i,j}} 1_{\{x:\nu(A_{\epsilon,i,j})^2 w_{A_{\epsilon,i,j}}(x) > \frac{t(1-\zeta(\epsilon))}{\int R d\nu}\}} \min_{y\in\mathcal{P}_{\epsilon,i,j}} H(y) d\nu(x) ds \\ &\ge \sum_{i,j} H(y_{\epsilon,i,j})\nu(D_{\epsilon,i})\theta_\epsilon \nu\left(\nu(A_{\epsilon,i,j}^-)^2 w_{A_{\epsilon,i,j}^+}(x) > \frac{t(1-\zeta(\epsilon))}{\int R d\nu}|D_{\epsilon,i}\right) + \omega(H,\theta_\epsilon) \end{aligned}$$

Similarly, due to Lemma 6.2.7, and using (6.2.20), there exists $\alpha(\epsilon) > 0$ such that

$$\begin{aligned} \nu\left(\nu(A_\epsilon^-(y))^2 w_{A_\epsilon^+} > \frac{t(1-\zeta(\epsilon))}{\int R d\nu}|D_\epsilon(x)\right) \ &\ge \ \nu\left(\frac{\nu(A_\epsilon^+(y))^2}{(1+\alpha(\epsilon))^2} w_{A_\epsilon^+} > \frac{t(1-\zeta(\epsilon))}{\int R d\nu}|A_\epsilon^+(y)\right) \\ &\quad + \nu(R_{A_\epsilon^+(y)} \le M_\epsilon | A_\epsilon^+(y)) - \nu(R_{A_\epsilon^+(y)} \le M_\epsilon | D_\epsilon(x)) \\ &\quad + o(1). \end{aligned}$$

Hence setting $Q_\epsilon^+ = \frac{1}{(1+\alpha(\epsilon))^2(1-\zeta(\epsilon))}$, whose limit is 1 as $\epsilon \to 0$, then using Proposition 6.2.11 and due to Slutsky's Lemma, we get:

$$\lim_{\epsilon\to 0}\nu\left(\frac{\nu(A_\epsilon^+)^2}{(1+\alpha(\epsilon))^2} w_{A_\epsilon^+(y)} > \frac{t(1-\zeta(\epsilon))}{\int R d\nu}|A_\epsilon^+(y)\right) = \mathbb{P}\left(\frac{\sigma_\varphi^2\mathcal{E}^2}{\mathcal{N}^2} > t\right), \quad \forall t > 0$$

For the same reasons we used to prove (6.2.21), we will obtain:

$$\lim_{\epsilon\to 0} E_{\epsilon,t} \ge \mathbb{P}\left((\sigma_\varphi^2\int R d\nu)\frac{\mathcal{E}^2}{\mathcal{N}^2} > t\right), \quad \forall t > 0,$$

from which we conclude the convergence in law of $\nu(A_\epsilon)^2\tau_\epsilon$,

$$\lim_{\epsilon\to 0}\mu(\nu(A_\epsilon(.))^2\tau_\epsilon(.) > t) = \mathbb{P}\left((\sigma_\varphi^2\int R d\nu)\frac{\mathcal{E}^2}{\mathcal{N}^2} > t\right), \quad \forall t > 0.$$

The conclusion follows from Lemma 6.2.12 below. □

Lemma 6.2.12. *For $\tilde{\nu}$ almost every $\tilde{y} \in \tilde{M}$, if $\tilde{D}_0$ is small enough, the measure $\tilde{\nu}_0$ given by (6.0.1) is well defined and*

$$\frac{\nu(A_\epsilon(y))}{\int_X R d\nu} \text{ is equivalent to } \tilde{\nu}_0(B(\tilde{y}, \epsilon)),$$

with $y := \Gamma(\tilde{y}) \in M$.

Proof. Let $\tilde{y} \in \tilde{M}$. Note that since we are interested in the asymptotic behavior of $\tilde{\nu}_0(B(\tilde{y}, \epsilon))$ when ϵ tends to 0, we can replace $\tilde{D}_0(\tilde{y})$ by a smaller disk if we want. If the disk $\tilde{D}_0(\tilde{y})$ and if ϵ are small enough, the image measure $\tilde{\mu}(\underset{-\epsilon<s<\epsilon}{\cup} \tilde{g}_s(\tilde{D}_0(\tilde{y}) \cap \cdot)$ by the projection of $\tilde{M}$ to M is equal to the measure $\mu(\underset{-\epsilon<s<\epsilon}{\cup} g_s(D_0(y) \cap \cdot)$ where $D_0 = D_0(y)$ is the disk around y obtained by the projection of the disk $\tilde{D}_0 = \tilde{D}_0(\tilde{y})$. So now we have to show that, for ν-almost every y in M the measure ν_0 defined on $D_0(y)$ by

$$\nu_0(A) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mu\left(\bigcup_{-\epsilon<s<\epsilon} g_s(D_0 \cap A) \right), \quad \forall A \subset D_0.$$

is well defined and that $\nu_0(B(y, \epsilon))$ is equivalent, when ϵ tends to 0, to $\nu(A_\epsilon(y))/\int_X R d\nu$. We assume $y \in M \setminus X$ and we assume that y is represented by a couple $(x, s) \in X_R$ with $x \notin \partial X$. Let us call D the connected component of X containing the point x. For ϵ small enough, $D_0(y)$ small enough, $\bigcup_{-\epsilon<s<\epsilon} g_s(D_0 \cdot)$ is contained in $\{(x', s') \in X_R \ : \ x \in D\}$.

In X_R, the set D_0 is represented by a curve $\{(x', h(x)) : x' \in \Pi_X(D_0)\}$ with h of class C^1 of uniformly bounded derivative, and such that $Dh(x) = 0$ (the tangent plane is horizontal above the point x) and we have, for every measurable set $A \subset D_0(y)$:

$$\mu(\bigcup_{-\epsilon<s<\epsilon} g_s(D_0(y) \cap A) = \frac{2\epsilon \nu(\chi^{-1}(\Pi_X(D_0(y) \cap A)))}{\int_X R \, d\nu}$$

This assures the existence of ν_0 and the fact that:

$$\nu_0(.) = \frac{\nu(\chi^{-1}(\Pi_X(. \cap D_0)))}{\int_X R d\nu}.$$

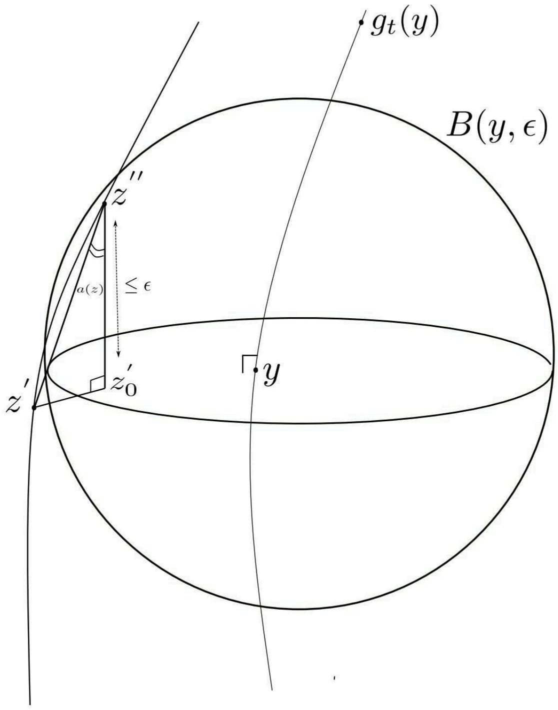
We have to prove that $\nu(A_\epsilon(.))$ is equivalent to $\nu(\chi^{-1}(\Pi_X(B(., \epsilon) \cap D_0)))$. We observe that $D_0 \cap B(y, \epsilon) \subset B(y, \epsilon)$. Therefore, we have

$$\nu(A_\epsilon(y)) = \nu(\chi^{-1}(\Pi_X(B(y, \epsilon)))) \geq \nu(\chi^{-1}(\Pi(D_0 \cap B(y, \epsilon)))).$$

We need to prove that $\nu(A_\epsilon(y) \setminus (\chi^{-1}(\Pi_X(D_0 \cap B(y, \epsilon))))$ is negligible with respect to $\nu(A_\epsilon(y))$.
We recall that $(x, s) \mapsto g_s(x)$ is a C^1-diffeomorphism of X_R on its image which contains the ball $B(y, \epsilon)$, assuming ϵ is sufficiently small.

Let $z \in A_\epsilon(y) \setminus \chi^{-1}(\Pi_X(D_0 \cap B(y, \epsilon)))$. Let us consider the point $z' = g_{h(z)}(\chi(z)) \in D_0 \setminus B(y, \epsilon)$. The diameter of $A_\epsilon(y)$ is bounded by $C_y\epsilon$ and g_t is C^1. Then $d(z', y) \leq C'_y\epsilon$.

Since $z \in A_\epsilon(y)$, there exists $s \in (-h(z), R(z) - h(z))$ such that $z" = g_s(z') \in B(y, \epsilon)$. Since we have to compose by the exponential map, we are placed in a local chart containing y and in this chart, the picture is the following:



The angle $a(z)$ between the direction of the flow at y and the vectors of extremities z' and $z"$ is bounded by $c_g C'_y \epsilon$. So the distance between z' and the orthogonal projection $z"_0$ of $z"$ on the plane containing D_0 is less than or equal $\epsilon tan(c_g C'_y \epsilon) \leq 2c_g C'_y \epsilon^2$. Then $d(\chi(z), \Pi_X(z"_0)) < K\epsilon^2$ (for some $K > 0$). But $z"_0$ is in $D_0 \cap B(y, \epsilon)$. We have thus shown that $A_\epsilon \setminus \Pi_X(\chi^{-1}(D_0 \cap B(y, \epsilon))$ is contained in $\chi^{-1}\left((\Pi_X(D_0 \cap B(y, \epsilon)))^{[K\epsilon^2]}\right)$. Now the ν-measure of this set is negligible with respect to $\nu(A_\epsilon)$, (see the proof of Lemma 6.2.2).

□

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Titre : Propriétés Quantitative de Récurrence en Mesure Infinie

Mots clés : Temps de retour, Récurrence quantitatives, Théorème de Limite Locale, Flot Axiome A, Systèmes dynamiques, Sous-shift de type fini.

Résumé : Dans cette thèse, nous étudions les propriétés quantitative de récurrence de certains systèmes dynamique préservant une mesure infinie. Nous nous intéressons au premier temps de retour des orbites d'un système dynamique dans un petit voisinage de leur points de départ. Tout d'abord, nous commençons par considérer un modèle jouet probabilistique pour éclairer la stratégie de nos preuves. On s'intéresse particulièrement au cas où la mesure est infinie, plus précisément, nous considérons les $\mathbb{Z}$ -extensions des sous-shift de type fini. Nous étudions le comportement asymptotique du premier temps de retour au voisinage de l'origine, et nous établissons des résultats de type de convergence presque partout, et aussi de convergence en loi par rapport à toute mesure de probabilité absolument continue par rapport à la mesure infinie. Dans ce travail, nous nous également intéressons à d'autres système dynamiques. Nous considérons un flot Axiome A $(g_t)_t$ sur une variété riemannienne $\mathbb{M}$ munie d'une mesure σ -finie μ. Nous supposerons que la mesure μ est une mesure d'équilibre pour $(g_t)_t$. Afin d'établir nos résultats, nous introduisons des notions de dynamique hyperbolique. En particulier, nous considérons la section de Markov qui a été introduite par Bowen et Ratner.

Title : Quantitative Recurrence Properties in Infinite Measure

Keywords : Return time, Quantitative recurrence, Local Limit Theorem, Axiom A flow, Dynamical systems, Subshift of finite type.

Abstract : In this thesis, we study the quantitative recurrence properties of some dynamical systems preserving an infinite measure. We are interested in the first return time of the orbits of a dynamical system into a small neighborhood of their starting points. First, we start by considering a toy probabilistic model to clarify the strategy of our proofs. Our interest is when the measure is indeed infinite, more precisely we consider the $\mathbb{Z}$-extensions of subshifts of finite type. We study the asymptotic behavior of the first return time near the origin, and we establish results of an almost everywhere convergence kind, and a convergence in distribution with respect to any probability measure absolutely continuous with respect to the infinite measure. In this work, we are also interested in another dynamical systems. We consider an Axiom A flow $(g_t)_t$ on a Riemannian manifold $\mathbb{M}$ endowed with a σ-finite measure μ. We will assume that the measure μ is an equilibrium measure for $(g_t)_t$. In order to establish our results, we introduce notions from hyperbolic dynamics. In particular, we consider the Markov section which was constructed by Bowen and Ratner.