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Mécanique statistique des champs gaussiens

Statistical mechanics of Gaussian fields

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Abstract

In this thesis, we study the level sets smooth Gaussian fields, or random smooth functions. Several directions are explored, some linked to spectral theory, some to statistical mechanics.
The first object of focus is a family of Gaussian fields on compact Riemannian manifolds defined as linear combinations of eigenfunctions of the Laplacian with independent Gaussian weights. In special cases, this family specializes to the band-limited ensemble which has received a lot of attention in recent years, but also to the cut-off Gaussian Free Field, which is the projection of the Gaussian Free Field on the first eigenspaces of the Laplacian. We study the covariance function of these fields, the expected number of connected components of their zero set, and, in the case of the cut-off Gaussian Free Field, derive a precise large deviation estimate on the event that the field is positive on a fixed set when the energy cut-off tends to infinity.
Next, we study percolation of excursion sets of stationary fields on the plane using techniques from Bernoulli precolation. We first derive a mixing bound for the topology of nodal sets of planar Gaussian fields. Then, we prove a sharp phase transition result for the Bargmann-Fock random field.

Résumé

Dans cette thèse, on étudie les lignes de niveaux de champs gaussiens lisses, ou fonctions aléatoires lisses. Plusieures directions sont explorées, certaines en lien avec la géométrie spectrale, d'autres avec la mécanique statistique.
On s'intéresse d'abord à une famille de champs gaussiens sur des variétés Riemannienne compactes définis comme des combinaisons linéaires de fonctions propres du Laplacien avec des poids Gaussiens indépendants. Dans des cas particuliers, cette famille donne l'ensemble à bande limitée qui a beaucoup été étudié ces dernières années, mais aussi le Champ Libre Gaussien coupé en fréquence, qui es la projection du Champ Libre Gaussien sur les premiers espaces propres du Laplacien. On étudie la fonction de covariance de ces champs, le nombre moyen de composantes connexes de leur lieu d'annulation, et, dans le cas du Champ Libre Gaussien coupé en fréquence, on obtient une estimation précise de grandes déviations pour la probabilité que le champ soit positif sur un ensemble fixe lorsque le seuil de coupure en énergie tend vers l'infini.
Puis, on étudie la percolation des sur-niveaux de champs stationnaires sur le plan en utilisant des techniques de percolation de Bernoulli. On démontre d'abord une estimation de mélange pour la topologie des lignes de niveaux de champs gaussiens planaires. Puis, on démontre un résultat de transition de phase soudaine pour le champ de Bargmann-Fock.

Keywords: Gaussian fields, Random topology, Spectral asymptotics, Percolation, Gaussian Free Field

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Chapter 1

Introduction

This thesis is about the topology of level sets and excursion sets of smooth Gaussian fields. It contains results about some local quantities, such as the number of connected components of a given level set, but also global quantities, such as the existence of unbounded connected components. When asked to define a Gaussian field, a probabilist is likely to answer that it is a random process $(\phi_x)_{x\in X}$ over some space X such that for any $x_1, \dots, x_k \in X$, the law of $(\phi_{x_1}, \dots, \phi_{x_k})$ is Gaussian. But one can also think of a smooth Gaussian field as a random smooth function f on a manifold X such that for any points $x_1, \dots, x_k \in X$, the random vector $(f(x_1), \dots, f(x_k))$ has a Gaussian distribution. The level sets of f are just the sets $\mathcal{N}_s = f^{-1}(s)$ for a given $s \in \mathbb{R}$, while the excursion sets are just the $\mathcal{D}_s = f^{-1}([s, +\infty[)$. If f is such that for each x, the vector $(f(x), d_x f)$ is non-degenerate, then, for each $s \in \mathbb{R}$, the level set $\mathcal{N}_s$ will be a.s. smooth and $\mathcal{D}_s$ will be a closed subset with smooth boundary $\mathcal{N}_s$. One can study these random shapes from many different angles.

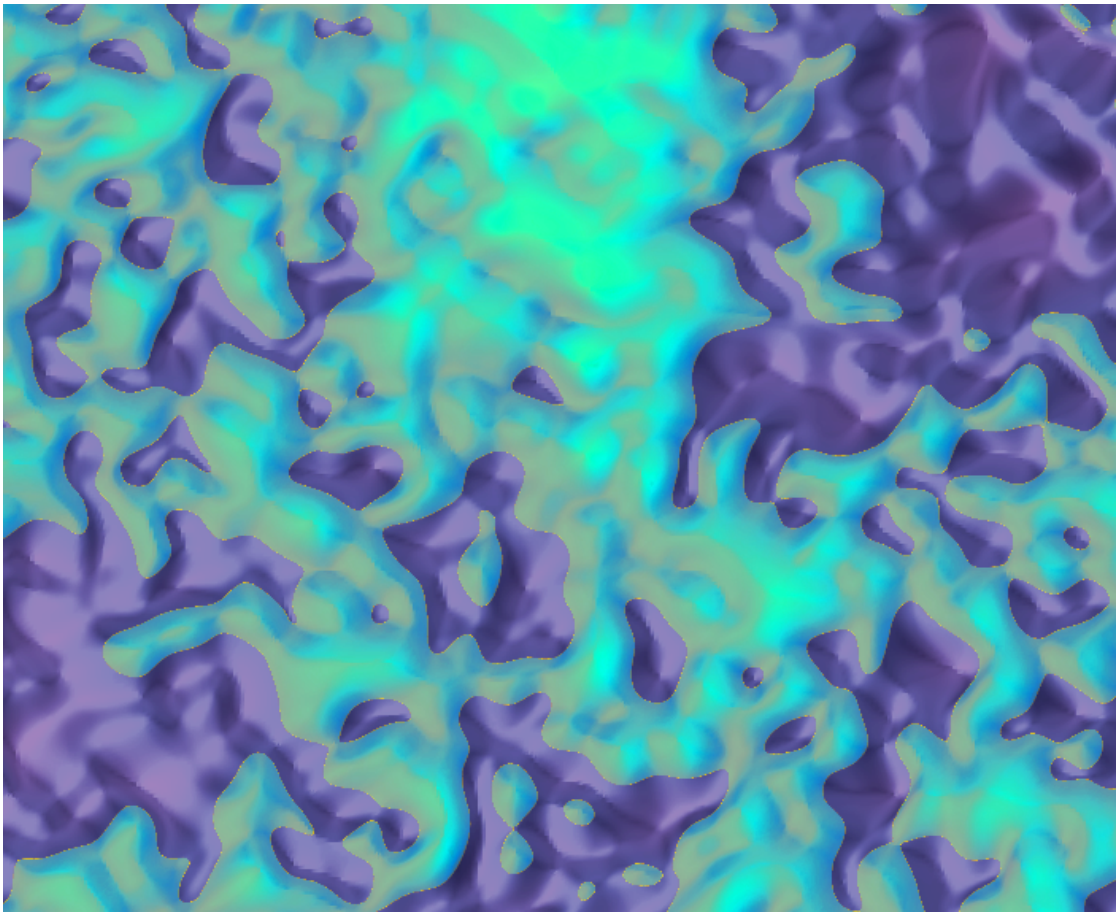


Figure 1.1: Here, in dark blue is a detail the graph of a random function on the flat torus (more precisely, it is the cut-off Gaussian Free Field, defined in Subsection 1.2.2 below). The graph is superimposed with a flat translucent light blue surface at level 0. The region $\mathcal{D}_0$ is where the graph is above the blue surface while $\mathcal{N}_0$ is the interface between $\mathcal{D}_0$ and the region where the graph is below the blue surface. What is the typical number of connected components of $\mathcal{D}_0$? How curved is the interface $\mathcal{N}_0$? What is the typical maximum height of $\mathcal{D}_0$?

A word about the format:
This thesis is organized as follows. The present chapter contains a general introduction to the topic with a compilation of previous results. Once the stage is set, we present each of the results we obtained with a proof sketch. Sections containing new results are indicated by the symbol † in the title. The following chapters of the manuscript are reproductions of preprints and articles in which the results are stated more formally and fully proved. Here is a more detailed summary of the introduction, for readers that are familiar with the subject matter:

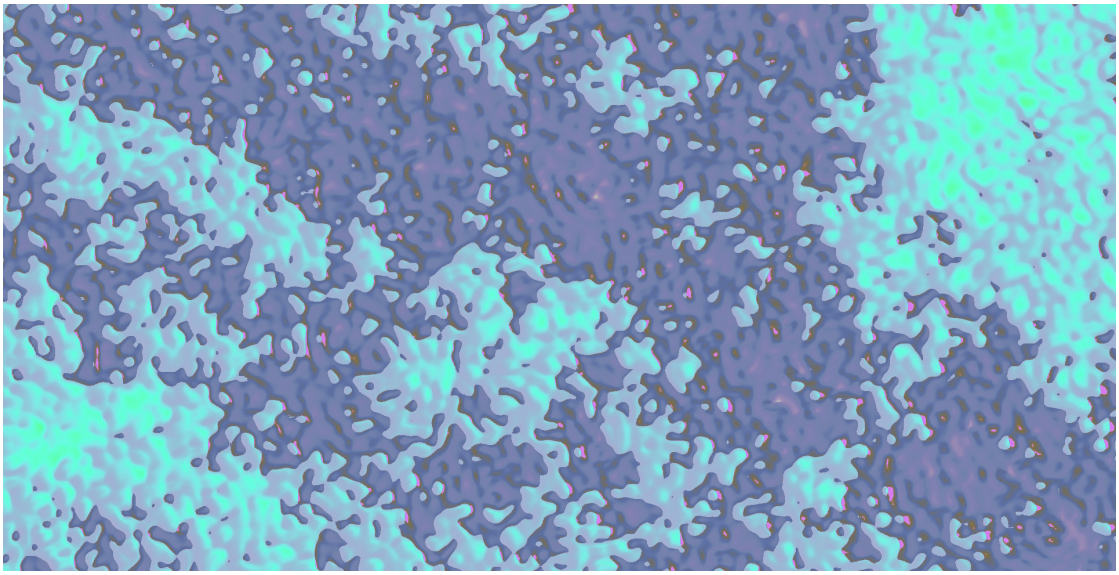


Figure 1.2: Taking a step back and flattening the graph a little bit, the large connected components start to stand out. How likely is it that such connected components exist? Are there large areas untouched by $\mathcal{N}_0$?

- In Section 1.1 we define Gaussian fields first as a random linear form on a Hilbert space and second using the Kolmogorov extension theorem. Then, we define the covariance function and the spectral measure for stationary fields. Finally, we present the conditioning formula, Bulinskaya's lemma and the Kac-Rice formula.

- In Section 1.2 we discuss the results we obtained concerning random sums of eigenfunctions of the Laplacian and the Gaussian Free Field. We start with some spectral theory prerequisites. Then, we define band-limited random functions, random spherical harmonics and random monochromatic waves. We also introduce some new models, called cut-off fractional Gaussian fields. Next, we give a panorama of classical techniques used to study the topology of nodal sets of smooth Gaussian fields. This is followed by a presentation of the Gaussian Free Field (or GFF). Indeed, it is the limit of a particular cut-off fractional Gaussian field, which we call the cut-off Gaussian Free Field (or CGFF). We then present some results concerning these objects. First, we compute the main term in the asymptotic expansion of the covariance function of cut-off fractional Gaussian fields when the cut-off tends to infinity using a result from [Hör68]. Second, there is an application to the study of the topology of the nodal set of these fields using a result from [NS16] and ideas from [GW17] and [GW16b]. Third, there is a large deviation result concerning the nodal set of the CGFF in the spirit of [BDG01]. Lastly, there is a lower bound on the density of connected components of the monochromatic random wave obtained in collaboration with Maxime Ingremeau.

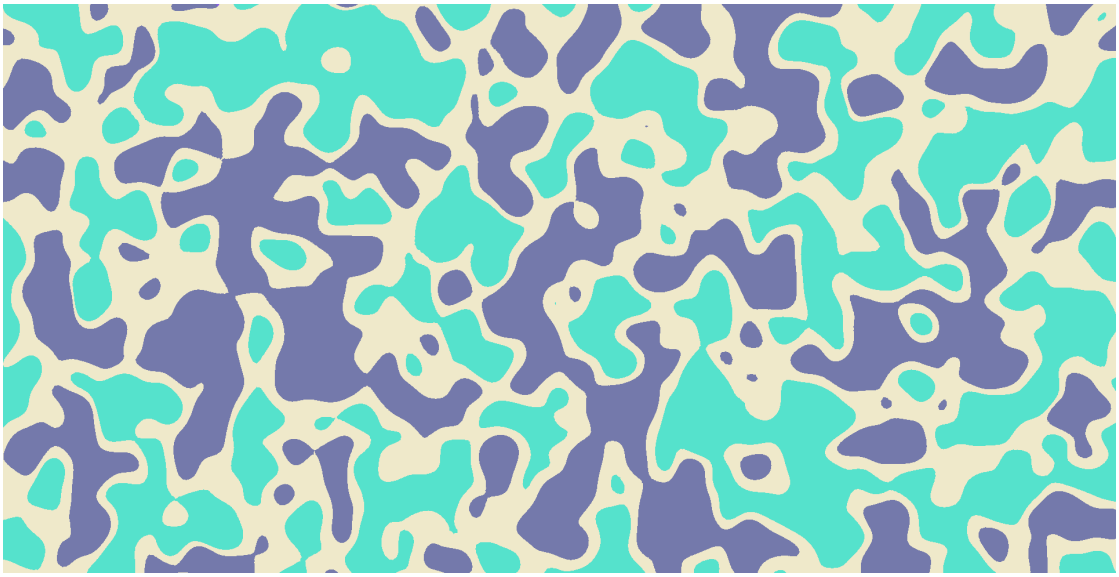


Figure 1.3: Changing the height of the threshold will shift the interface and maybe even modify its topology. Here, the dark blue region represents the sub-level set $\{f \leq -1\}$, the light turquoise region represents the super-level set $\{f \geq 1\}$. In this picture, it seems like the connectivity properties are governed mostly by the remaining, pale yellow region, $\{-1 < f < 1\}$.

- In Section 1.3, we discuss percolation of excursion sets of Gaussian fields. We start with a discussion of Bernoulli percolation with some emphasis on influences, the KKL theorem and the Harris-Kesten theorem. We then define percolation for Gaussian fields and present previous results. Finally, we discuss previous strategies to obtain decorrelation inequalities for Gaussian fields. The results presented are the fruit of joint work with Hugo Vanneuville. The first is a decorrelation inequality for crossing and component counting events for planar Gaussian fields. Using this inequality, we generalize the RSW estimate from [BG17a] and [BM18] and we prove a lower concentration result about the number of connected components of the zero set around its expectation in the setting of [NS16]. The second result is a phase transition result concerning the excursion sets of the Bargmann-Fock field. We prove that a.s. for any $p > 0$, the set $\{f \geq -p\}$ has a unique unbounded connected component, while for $p \leq 0$ it a.s. has only bounded connected components. We also prove that for $p > 0$, the left-right crossing probability for the rectangle $[0, 2R] \times [0, R]$ converges to one exponentially fast in R as $R \to +\infty$.

1.1 Generalities on Gaussian fields

This section is a brief introduction to Gaussian fields. We start by defining Gaussian fields as stochastic processes and present their characterization in terms of the covariance function. Next, we introduce the reproducing kernel Hilbert space associated to a random field. This provides an alternate construction of Gaussian fields that allows us to define generalized Gaussian fields. Our main reference for these two first subsections is [Jan97]. Next we focus on stationary Gaussian fields and discuss spectral measures. We finish off by presenting three classical tools used when studying smooth Gaussian fields.

1.1.1 Gaussian fields, covariance functions and Hilbert spaces

A **centered Gaussian field** over a smooth manifold M and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable mapping $f : \Omega \times M \to \mathbb{R}$ (where M and $\mathbb{R}$ are equipped with their respective Borel σ-algebras) $(x, \omega) \mapsto f_\omega(x)$ such that for each $x_1, \dots, x_k \in M$, the random variable $(f(x_1), \dots, f(x_k))$ is centered and Gaussian. From now on, unless otherwise stated, all the Gaussian fields we consider will be centered. Any Gaussian field defines a random measurable function[1] $x \mapsto f(x)$. If f is a Gaussian field such that the corresponding random function is a.s. continuous, then it is entirely determined by its finite-dimensional marginals. We will (somewhat incorrectly) say in this case that f is **a.s. continuous** (and use similar terminology for other degrees of regularity). But finite dimensional Gaussian vectors are entirely determined by their covariance matrices. Thus, for any a.s. continuous Gaussian field f, the law of f is entirely determined by the function $(x, y) \mapsto \mathbb{E}[f(x)f(y)]$. This function is such that, for each $x_1, \dots, x_k \in M$, the matrix $(K(x_i, x_j))_{1 \le i,j \le k}$ is symmetric non-negative. It is called the **covariance function** (or just the covariance) of the field f. Conversely, we can reconstruct the field from its covariance, and the regularity of the field can be (partly) deduced from the regularity of its covariance:

Theorem 1.1.1 (Kolmogorov's theorem, see Appendix A of [NS16])**.** *Let f be a Gaussian field defined on a smooth manifold M with covariance K. Let $k \in \mathbb{N}$ and assume that the derivatives of K up to order $k+1$ in each variable are well defined and continuous*[2]*. Then, f is almost surely of class C^k. Conversely, if f is almost surely of class C^k, then the derivatives of K up to order k in each variable exist and are continuous. Moreover, for any differential operators P and Q of order at most k acting on M,*

$$\mathbb{E}\left[Pf(x)Qf(y)\right] = (P \otimes Q)K(x, y)\,.$$

Let us illustrate these constructions with an example.

[1] It is a random variable if we equip the space of measurable mappings $M \to \mathbb{R}$ with the σ-algebra generated by cylinder sets.

[2] The regularity assumptions are not optimal in the present statement or in [NS16] but they are sufficient for our purposes.

Example. *Let $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\{0,\dots,N\}}$ be independent with law $\mathcal{N}(0,1)$. Then, the following random trigonometric polynomial*

$$f_m(t) = a_0 + \sqrt{2}\left(\sum_{k=1}^{m} a_k \cos(kt) + b_k \sin(kt)\right)$$

defines an a.s. C^∞ Gaussian field and its covariance K_m that maps any $(s,t) \in S^1 \times S^1$ to

$$\mathbb{E}[f_m(s)f_m(t)] = 1 + 2\left(\sum_{k=0}^{m} \cos(ks)\cos(kt) + \sin(ks)\sin(kt)\right) = \frac{\sin\left(\frac{1}{2}(2m+1)(s-t)\right)}{\sin\left(\frac{1}{2}(s-t)\right)}\,.$$

It is the unique a.s. continuous Gaussian field with covariance function K_m. By Theorem 1.1.1, we have

$$\mathbb{E}[f_m(t)f_m'(t)] = \left(id \otimes \frac{d}{dt}\right) K_m(t,t) = \frac{d}{dt}\Big|_{t=0} \frac{\sin\left(\frac{1}{2}(2m+1)t\right)}{\sin\left(\frac{1}{2}t\right)} = 0$$

for parity reasons. In particular, for each $t \in S^1$, $f_m(t)$ is independent from $f_m'(t)$.

1.1.2 Reproducing kernel Hilbert spaces and generalized Gaussian fields

Let f be an a.s. continuous Gaussian field on a smooth manifold M. Consider the space of linear combinations of the random variables $f(x)$ for $x \in M$. This is a subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and we can equip it with the L^2 scalar product. Let $H \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ be its closure with respect to the topology induced by the scalar product. Since L^2 limits of centered Gaussian random variables are also centered Gaussian, H contains only centered Gaussian random variables. Each element $\xi \in H$ defines a function $h_\xi : M \mapsto \mathbb{R}$ as follows. For each $x \in M$, $h_\xi = \mathbb{E}[\xi f(x)]$. The space $\Gamma(H)$ of functions h_ξ for $\xi \in H$ is called the **reproducing kernel Hilbert space** for f. By pushing forward the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ scalar product via the map $\xi \mapsto h_\xi$, we obtain a Hilbert space structure $(\Gamma(H), \langle\cdot,\cdot\rangle)$ on $\Gamma(H)$. Note that for each $x \in M$, $K(x,\cdot) = h_{f(x)}$ so $K(x,\cdot) \in H$ and for each $h_\xi \in \Gamma(H)$, $\langle h, K(x,\cdot)\rangle = \mathbb{E}[\xi f(x)] = h_\xi(x)$. In particular, evaluation mappings are continuous linear functionals on $\Gamma(H)$. Conversely, given $(\Gamma, \langle\cdot,\cdot\rangle_\Gamma)$ a Hilbert space of functions on M on which the evaluation mappings $h \mapsto h(x)$ for all $x \in M$ are continuous, there exists a unique symmetric map $K : M \times M \to \mathbb{R}$ such that for each $(x_1, \dots, x_k) \in \mathbb{R}$, the matrix $(K(x_i, x_j))_{1\le i,j\le k}$ is non-negative definite, such that for each $x \in M$, $K(x,\cdot) \in \Gamma$ and for each $h \in \Gamma$, $\langle K(x,\cdot), h\rangle_\Gamma = h(x)$ (see Appendix F of [Jan97] for more details). The map K is called the **reproducing kernel** of Γ. Consider $(\psi_k)_{k\in\mathbb{N}}$ a Hilbert basis for Γ. Then, for each $x \in M$ and each $k \in \mathbb{N}$, $\langle K(x,\cdot), \psi_k\rangle_\Gamma = \psi_k(x)$. In particular, $\sum_{k\in\mathbb{N}} \psi_k(x)^2 = \langle K(x,\cdot), K(x,\cdot)\rangle_\Gamma = K(x,x) < \infty$ so

$$K(x,\cdot) = \sum_{k\in\mathbb{N}} \psi_k(x)\psi_k\,.$$

Moreover, if $(\xi_k)_{k\in\mathbb{N}}$ is a sequence of independent random variables with law $\mathcal{N}(0,1)$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, then, for each $x\in M$, the sum

$$f(x) := \sum_{k\in\mathbb{N}} \xi_k \psi_k(x)$$

converges in L^2 to a centered Gaussian with variance $K(x,x)$. More generally, the function $f = \sum_{k\in\mathbb{N}} \xi_k\psi_k$ defines a Gaussian field on M with covariance K, and with reproducing kernel Hilbert space Γ (see Chapter 8 of [Jan97] for more details). We call f the **Gaussian field associated to Γ**.

Remark 1.1.2. A useful feature of the reproducing kernel Hilbert space perspective is that for any orthogonal decomposition $\Gamma_1 \oplus \Gamma_2$ of Γ, there exists a Hilbert basis of Γ that can be split into Hilbert bases of Γ_1 and Γ_2. Consequently, the series defining f can be split as an independent sum $f_1 + f_2$ where for $i\in\{1,2\}$, f_i is a Gaussian field associated to Γ_i.

Example. *The reproducing kernel Hilbert space for the random trigonometric polynomial $f_m(t) = a_0 + \sqrt{2}\left(\sum_{k=1}^m a_k\cos(kt) + b_k\sin(kt)\right)$ from Example 1.1.1 is the space of real trigonometric polynomials of degree at most m equipped with the following scalar product:*

$$(P,Q) \mapsto \frac{1}{2\pi}\int_0^{2\pi} P(t)Q(t)dt\,.$$

We can split f_m into the sum of even and odd parts. Since sines and cosines are L^2-orthogonal, these two parts are independent and we get a decomposition of f_m into two indepedent components

$$f_m(t) = \left[a_0 + \sqrt{2}\sum_{k=1}^m a_k\cos(kt)\right] + \left[\sqrt{2}\sum_{k=1}^m b_k\sin(kt)\right]$$

as in Remark 1.1.2.

Now, given any $h\in\Gamma$, the series $\sum_{k\in\mathbb{N}}\langle\psi_k,h\rangle$ converges in $L^2(\Gamma)$ so the random variable

$$\sum_{k\in\mathbb{N}} \xi_k\langle\psi_k,h\rangle$$

converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$ to a Gaussian random variable. Its variance is $\langle h,h\rangle_\Gamma$. With a slight abuse of notation, we denote this random variable by $\langle f,h\rangle$ and see f as a random (unbounded) linear form on H. In particular, if $C_c^\infty(M)\subset H$, we can see f as a random "distribution"[3] on M. It is easy to see that the map $h\mapsto\langle f,h\rangle$ defines an isometry from Γ into $L^2(\Omega,\mathcal{F},\mathbb{P})$. Its image is a space of Gaussian random variables H, and it entirely determines f since for each $x\in M$, it contains $\langle f, K(x,\cdot)\rangle = f(x)$. This allows us to generalize the definition of Gaussian field to the case where Γ is small enough for the

[3] In general, it may not be a weakly continuous linear form on $C_c^\infty(M)$.

evaluation mappings to be continuous. Indeed, we can define a **generalized Gaussian field** associated to Γ to be an isometry from Γ to a close subspace $H \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ made up of Gaussian random variables. The isometry condition implies that two generalized Gaussian fields associated to Γ will have the same law so we will sometimes speak of "the (generalized) Gaussian field associated to Γ" to refer to any such Gaussian field.

Example. *Let $U \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary. By the Poincaré inequality, the bilinear form $(u, v) \mapsto \int_U \nabla u \nabla v$ defines a scalar product on $C_c^\infty(U)$. Let $H_0^1(U)$ be the completion of $C_c^\infty(U)$ in the induced topology. If $n \geq 2$, the elements in $H_0^1(U)$ need not be well defined pointwise as functions on U. The generalized Gaussian field on $H_0^1(U)$ is called the Gaussian Free Field on $H_0^1(U)$. For each $u \in C_c^\infty(U)$, $\langle f, u \rangle$ is a centered Gaussian random variable with variance $\|\nabla u\|^2_{L^2(U)}$. The Gaussian Free Field will be discussed in more detail in Subsection 1.2.4.*

1.1.3 Stationary fields and the spectral measure

A Gaussian field f defined on $\mathbb{R}^n$ is called **stationary** if for any $v \in \mathbb{R}^n$, $f(\cdot + v)$ has the same law as f. Stationary fields arise naturally from geometry if the scalar product defining the Gaussian field is translation invariant. It is also natural to study such fields in statistical mechanics where one is concerned with large scale properties of a stationary field. This usually involves a scaling parameter which is only meaningful if the typical variation length of the field is the same everywhere. If f is stationary and $v \in \mathbb{R}^n$, for any $x, y \in \mathbb{R}^n$, $K(x + v, y + v) = K(x, y)$. Thus, K is really a function of the difference[4] $y - x$. That is, there exists a function $\kappa : \mathbb{R}^n \to \mathbb{R}$ such that $K(x, y) = \kappa(y - x)$. When dealing with stationary fields we will usually use the term **covariance function** to denote κ instead of K. Bochner's theorem provides a straightforward characterization of covariance functions κ among real valued functions of $\mathbb{R}^n$:

Theorem 1.1.3 (Bochner's theorem, Theorem 2.1.3 of [CI13])**.** *A continuous function κ on $\mathbb{R}^n$ is a covariance function of a Gaussian field on $\mathbb{R}^n$ if and only if it is the Fourier transform of a finite symmetric (non-negative) measure ρ. That is,*

$$\kappa(x) = \hat{\rho}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} d\rho(\xi) \, .$$

It is usually much easier to check whether ρ is a finite symmetric measure than whether κ is a covariance. Let ρ be a finite (non-negative) symmetric measure on $\mathbb{R}^n$. Then, the Fourier transform of ρ is a covariance function and thus determines a Gaussian field f. Moreover, ρ is entirely determined by f. We call ρ the **spectral measure** of f.

Remark 1.1.4. Bochner's theorem has an analog on compact abelian groups which is easy to prove. As an example, let f be a stationary Gaussian field on S^1 with covariance κ. Then, Bochner's theorem just says that the Fourier modes of f must be independent.

[4]The argument of Example 1.1.1 holds for any such covariance function. In particular, for any stationary C^1 Gaussian field f, for any $x \in \mathbb{R}^n$, $f(x)$ is independent of $d_x f$.

Let us check that this is indeed the case. For each $n \in \mathbb{Z}$, let $c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} f(\theta)d\theta$. Fix $n, m \in \mathbb{Z}$. Then,

$$\mathbb{E}\left[c_n(f)\overline{c_m(f)}\right] = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} e^{in\theta - im\varphi}\kappa(\theta-\varphi)d\theta d\varphi\,.$$

Replacing φ by $\tau = \varphi - \theta$, we get

$$\frac{1}{2\pi}\int_0^{2\pi}\kappa(\tau)\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\theta - im\tau}d\theta d\tau = \overline{c_m(\kappa)}\delta_{n,m}\,.$$

In particular, $c_m(\kappa) \geq 0$ and the Fourier modes of f are independent Gaussians whose variances are the Fourier modes of κ.

The spectral measure is useful to decompose the field into independent sums as in the Hilbert space decomposition in Remark 1.1.2. Indeed consider some decomposition $\rho = \rho_1 + \rho_2$ of ρ into a sum of two positive symmetric measures on $\mathbb{R}^n$. Then, if f_1 and f_2 are independent Gaussian fields with spectral measures ρ_1 and ρ_2 respectively, $f_1 + f_2$ has the same law as f.

1.1.4 Basic properties of smooth Gaussian fields

In this subsection we present three basic tools often used when studying smooth Gaussian fields.

- **Conditioning**:
  Conditioning a random field with respect to its value at a point often leaves a field with an unknown law which makes it hard to handle. Gaussian fields have the advantage of behaving well under conditioning.

  Lemma 1.1.5 (see Proposition 1.2 of [AW09])**.** *Let $(\xi, \zeta) = (\xi_1, \dots, \xi_N, \zeta_1, \dots, \zeta_M)$ be a Gaussian vector with mean $(\mu, \nu) = (\mu_1, \dots, \mu_N, \nu_1, \dots, \nu_M)$ and covariance $\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}$ where Σ_{11} is the covariance of ξ and Σ_{22} that of ζ. Assume that Σ_{22} is non-degenerate. Then, the law of ξ conditioned on ζ is that of a Gaussian field with covariance*

  $$\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

  and mean

  $$\mu + \Sigma_{12}\Sigma_{22}^{-1}(\zeta - \nu)\,.$$

  In particular, the covariance is deterministic. Moreover, given the explicit dependence on ζ in the mean, for each bounded continuous function $\varphi : \mathbb{R}^N \to \mathbb{R}$, $\mathbb{E}[\varphi(\xi)\,|\,\zeta]$ will depend continuously on ζ and so will be defined pointwise as a function in ζ. Thus, for each $\zeta_0 \in \mathbb{R}^M$, we can consider the mean of $\varphi(\xi)$ conditionally on $\zeta = \zeta_0$ and denote it by $\mathbb{E}[\varphi(\xi) \mid \zeta = \zeta_0]$. This result extends naturally to Gaussian fields thanks to Kolmogorov's extension theorem (Theorem 1.1 of [AW09]).

One can condition a smooth Gaussian field f, not only with respect to a finite set of values, but also to values of its derivatives at fixed points, provided what one is conditioning against forms a non-degenerate Gaussian vector. For example, if f is a (centered) Gaussian field on $\mathbb{R}^n$ with covariance K such that $\partial_1 f(0)$ has positive variance, then f conditioned on $\partial_1 f(0) = 0$ is still a (centered) Gaussian field with covariance

$$\tilde{K}(x,y) = K(x,y) - \frac{(id \otimes \partial_1) K(x,0) (\partial_1 \otimes id) K(0,y)}{(\partial_1 \otimes \partial_1) K(0,0)} .$$

- **Probabilistic transversality**:
  Intuitively, in a generic situation, two smooth planar curves should intersect on a locally finite set of points, and two curves in $\mathbb{R}^3$ should not intersect at all. Similarly, two surfaces in $\mathbb{R}^5$ should not intersect in a generic situation. Transversality theory (see Chapter 4 of [GG73]) provides a powerful interpretation of this intuition in terms of dense subsets for the C^∞ topology in spaces of mappings. However, it seems also natural that in the case of a Gaussian field, whose graph is a random smooth manifold, this intuition should find some interpretation in terms of almost sure events. This is indeed the case, and one can deal with most situations of this kind using the following lemma:

  Lemma 1.1.6 (see Lemma 11.2.10 of [AT07]). *Let $T \subset X$ be a compact subset of a smooth manifold X. Assume that T has Hausdorff dimension n and let $f : X \to \mathbb{R}^{n+1}$ be a smooth Gaussian field, a.s. C^1. Assume that for any $x \in X$, the covariance matrix of the Gaussian vector $f(x)$ is non-degenerate. Then, for each $y \in \mathbb{R}^{n+1}$, a.s, $y \notin f(T)$.*

  As an example, let $f : \mathbb{R} \to \mathbb{R}$ be a stationary Gaussian field that is a.s. C^2 and assume that $f(0)$ and $f'(0)$ both have unit variance. Then, $g = (f, f')$ is a.s. C^1 and the covariance of $g(x)$ is I_2 so Lemma 1.1.6 applies and for any $k \in \mathbb{N}$ a.s. g does not vanish on $[k, k+1]$. Consequently f has a.s. no critical points on $\mathbb{R}$ with critical value 0. Applying the same reassoning in local charts and in higher dimension yields:

  Lemma 1.1.7 (Bulinskaya's lemma). *Let f be a real valued a.s. C^2 Gaussian field on a smooth manifold M. Assume that for each $x \in M$, the covariance of the Gaussian vector $(f(x), d_x f)$ is non-degenerate. Then, a.s., 0 is a regular value of f. In particular, $f^{-1}(0)$ is smooth.*

  Of course, there is nothing special about the level set 0. The lemma is just as true for other level sets.

- **The Kac-Rice formula(s)**:
  Consider f a Gaussian field on $\mathbb{R}^n$ and for each $t \in \mathbb{R}$, set $\mathcal{D}_t = \{x \in \mathbb{R}^n \ : \ f(x) \geq$

$t\}$. Fix $U \subset \mathbb{R}^n$ a bounded open subset. What can we say about the volume of $\mathcal{D}_t \cap U$? Firstly, for each $k \in \mathbb{N}$, $k \geq 1$,

$$\mathrm{Vol}\left(\mathcal{D}_t \cap U\right)^k = \int_{U^k} \mathbb{1}_{[f(x_1)\geq t]} \dots \mathbb{1}_{[f(x_k)\geq t]} dx_1 \dots dx_k \,. \tag{1.1.1}$$

Taking expectations, by Fubini's theorem,

$$\mathbb{E}\left[\mathrm{Vol}\left(\mathcal{D}_t \cap U\right)^k\right] = \int_{U^k} \mathbb{P}\left[f(x_1) \geq t, \dots, f(x_k) \geq t\right] dx_1 \dots dx_k \,.$$

The integrand now depends explicitely on the finite dimensional marginals of f so it is more tractable than the whole set $\mathcal{D}_t \cap U$ itself. In particular, if f is stationary and $f(0)$ has unit variance,

$$\mathbb{E}\left[\mathrm{Vol}\left(\mathcal{D}_t \cap U\right)\right] = \left(1 - \Phi(t)\right) \mathrm{Vol}(U)$$

where $\Phi(t) = \frac{1}{2\pi} \int_t^{+\infty} e^{-s^2/2} ds$. While higher moments are less explicit, the integral expression gives some information in various asymptotic regimes like $t \gg 1$, $\mathrm{Vol}(U) \gg 1$ or $k \gg 1$. But what about the set $f^{-1}(t)$? This set is more tricky to study since it is a hypersurface so its size should be measured in terms of the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$. However, Equation (1.1.1) can be replaced by the co-area formula (see for instance Theorem 13.4.2 of [BZ88]). Taking expectations leads to the following Lemma:

Lemma 1.1.8 (The Kac-Rice formula, see for instance Theorem 6.8 of [AW09])**.** *Let f be an a.s. C^2 Gaussian field on $\mathbb{R}^n$ such that for each $x \in \mathbb{R}^n$, $K(x,x) > 0$. Let $U \subset \mathbb{R}^n$ a bounded open subset. Then,*

$$\mathbb{E}\left[\mathcal{H}^{n-1}\left(\{x \in U \ : \ f(x) = t\}\right)\right] = \int_U \mathbb{E}\left[|d_x f| \mid f(x) = t\right] \frac{e^{-\frac{t^2}{2K(x,x)}}}{\sqrt{2\pi K(x,x)}} dx$$

where $|\cdot|$ denotes the Euclidean norm.

The $|d_x f|$ term is a Jacobian coming from the coarea formula (see Theorem 13.4.2 of [BZ88]) and the Gaussian term with the denominator is just the density of $f(x)$ at t. The Kac-Rice formula admits many generalizations: to higher moments, where, as in the toy model $\mathcal{D}_t \cap U$, the integral becomes a multiple integral, to higher dimensions and codimensions were f is vector valued and one considers sets of the form $f(x) \in A$ where A has a certain Hausdorff dimension, to integrals against a measure absolutely continuous with respect to the Lebesgue measure and to manifolds (using partitions of unity). An important example is counting the number of critical points of f. This is just $\mathcal{H}^0\left(\{x \ : \ d_x f = 0\}\right)$ and admits a corresponding Kac-Rice formula (see [Nic15]). We will not provide more general statements here but Lemma 1.1.8 can be thought of as a token for a wide array of similar formulas for computing integral quantities depending on the level sets of f.

1.2 Random sums of eigenfunctions and the Gaussian Free Field

In this section, we will present the results from Chapters 2, 3 and 4, which correspond to [Riv18b], [Riv18a] and [Riv17] respectively.

Just as a stationary field on $\mathbb{R}^n$ is determined by its spectral measure, one can define a Gaussian field on a compact Riemannian manifold by specifying its distribution along different eigenmodes of the manifold, that is, the joint law of its L^2 scalar products against the different eigenfunctions of the Laplacian. By Bochner's theorem, the Fourier modes of a field on the circle must be independent for it to be stationary (see Remark 1.1.4). The natural analog, here would be to require that coefficients in the eigenmode decomposition of the field to be independent gaussian, in which case the field is determined by the sequence of their variances. The behavior these fields could reflect interesting properties of Laplace eigenfunctions and eigenvalues. We will start with a brief discussion of Laplace eigenfunctions and eigenvalues (see Subsection 1.2.1). We will then present several examples families of Gaussian fields defined by taking random linear combinations of Laplace eigenfunctions in various ways (see Subsection 1.2.2). Next, we will discuss a few classical arguments used when studying the topology of the level sets of these fields (see Subsection 1.2.3. One particular family of Gaussian fields defined here behaves like the Gaussian Free Field which shows up in statistical mechanics. The following subsection (Subsection 1.2.4) will be a short introduction to the the Gaussian Free Field. We will provide some motivation, define several variants of the Gaussian Free Field and state some results that inspired the work described here. Lastly, in Subsections 1.2.5, 1.2.6, 1.2.7 and 1.2.8, we will present the author's contributions to this topic.

1.2.1 Spectral theory of elliptic operators

Let (M, μ) be a closed manifold of dimension n equipped with a smooth positive density μ and a differential operator A of positive order m acting on M. Thus, in any local chart U, there exists a family $(a_\alpha)_{\alpha\in\mathbb{N}^d,|\alpha|\leq m}$ of smooth functions on U such that for each $u \in C_c^\infty(U)$ and each $x \in U$,

$$Au(x) = \sum_{|\alpha|\leq m} a_\alpha(x)(-i\partial)^\alpha u(x)\,.$$

Assume that A is **elliptic** in the sense that its **principal symbol** $\sigma_A(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$, defined for $(x,\xi) \in T^*U \simeq U \times \mathbb{R}^n$ takes positive values as long as $\xi \neq 0$, and that A is symmetric on $L^2(M,\mu)$. For example, if $M = \mathbb{R}^n$ and $A = \Delta = -\partial_1^2 - \cdots - \partial_n^2$ is the Laplace operator, then, $\sigma_A(x,\xi) = |\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$. We could also take for example $A = \partial_1^4 + \ldots \partial_n^4 - 2\partial_1^2 + \cdots - 2\partial_n^2$, in which case $\sigma_A(x,\xi) = \xi_1^4 + \cdots + \xi_n^4$. The following classical result is a combination of the spectral theorem for compact self-adjoint operators, of the compact Sobolev embeddings and of elliptic regularity:

Theorem 1.2.1. *There exists an unbmounded, non-decreasing sequence $(\lambda_k)_{k\geq 0}$ of real numbers, and a sequence $(\psi_k)_{k\geq 0}$ of smooth functions on M such that*

- *The sequence $(\psi_k)_{k\geq 0}$ is a Hilbert basis in $L^2(M,\mu)$.*
- *For each $k \geq 0$, $A\psi_k = \lambda_k\psi_k$. That is, the ψ_k are eigenfunctions of A with eigenvalue λ_k.*

The chief example is the following: if g is a Riemannian metric on M, $\mu = |dV_g|$ is the corresponding volume density and $A = \Delta$ the Laplace operator (with the convention $\Delta = -\mathrm{div}\,(\nabla\cdot)$), then the theorem applies. In this particular case, the distribution of the $(\lambda_k)_{k\geq 0}$ and the geometric properties of the $(\psi_k)_{k\geq 0}$ are of great importance to physics and mathematics since they appear naturally in the wave equation, the heat equation and the Schrödinger equation. On the one hand, their behavior is constrained by the eigenvalue equation they satisfy, which leads to some universal results, and on the other hand, their appearance in these physical equations intertwines them with the geometry of (M,g) in mysterious ways. Here are a few emblematic results that illustrate these principles.

- The equation $\Delta\psi_k = \lambda_k\psi_k$ satisfied by ψ_k implies that the **nodal set** $Z_k = \psi_k^{-1}(0)$ of ψ_k is a smooth hypersurface up to a set of zero $(n-1)$-Hausdorff measure. Courant's nodal domain theorem restricts the topology of $M \setminus Z_k$ (and therefore of Z_k).

  Theorem 1.2.2 (Courant, see Chapter 6, paragraph 6 of [CH89])**.** *For each $k \in \mathbb{N}$, the complement of Z_k has at most $k+1$ connected components.*

  The estimate is sharp (up to a constant factor) and there are examples of sequences of eigenfunctions for which $M\setminus Z_k$ has a bounded number of connected components (see for instance [Ste25], [Lew77] or [EJN07]).

- Yau's conjecture concerns the order of growth in λ_k of $\mathcal{H}^{n-1}(Z_k)$.

  Conjecture 1.2.3 (Yau, Problem 74 of [Yau82])**.** *For each Riemannian manifold (M,g) there exist two constants $C_1 = C_1(M,g) > 0$ and $C_2 = C_2(M,g) < +\infty$ depending only on the area of M such that for each $k \in \mathbb{N}$,*

  $$C_1\lambda_k^{1/2} \leq \mathcal{H}^{n-1}(Z_k) \leq C_2\lambda_k^{1/2}\,.$$

  So far, the conjecture has been confirmed in the case of real analytic manifolds [DF88], the lower bound has been established [Log18a] and the best known upper bound is [Log18b].

- Hörmander's local Weyl law is an assertion about the Schwartz kernel of the L^2 spectral projector onto the space spanned by the ψ_k for which λ_k belong to some

prescribed interval. It actually holds in the general case when A is an elliptic operator as described above. More precisely, for $L \in \mathbb{R}$, let $E_L(x,y)$ be the Schwartz kernel of the $L^2(M,\mu)$ orthogonal projector onto the space V_L spanned by the eigenfunctions ψ_k for which $\lambda_k \leq L$, also known as **the spectral function** of A. In other words, $E_L : M \times M \to \mathbb{R}$ is characterized by the property that, for each $u \in C^\infty(M,\mu)$, the following integral is well defined

$$\int_M E_L(\cdot,y)u(y)d\mu(y)$$

and is equal to the L^2-orthogonal projection of u onto the space V_L spanned by the ψ_k such that $\lambda_k \leq L$. By definition, the sequence $(E_L)_{L\geq 0}$ completely characterizes the eigenfunctions and eigenvalues of the Laplacian. This orthogonal projector was studied by Hörmander in [Hör68], with later refinements by many authors (see [Vas84, Xu04, GW17]). In particular, we have the following result:

Theorem 1.2.4 (Hörmander[5], Theorem 5.1 of [Hör68])**.** *Consider local coordinates in M such that μ agrees with the Lebesgue measure in these coordinates. Let σ_A be the principal symbol of A in these coordinates. Then, for each compact set $\Omega \subset \mathbb{R}^n$, $\alpha,\beta \in \mathbb{N}^n$ and each $R \in [0,+\infty[$ we have, uniformly for $w \in \Omega$, $|h| \leq R$ and $L \geq 1$,*

$$\partial^\alpha \otimes \partial^\beta E_L\left(w, w + L^{-n/m}h\right) = L^{(n+|\alpha|+|\beta|)/m}\frac{1}{(2\pi)^n}\int_{\sigma_A(w,\xi)\leq 1}(i\xi)^\alpha(-i\xi)^\beta e^{-i\langle h,\xi\rangle}d\xi + O\left(L^{(n+|\alpha|+|\beta|-1)/m}\right).$$

In particular, the map $h \mapsto L^{-n/m}E_L\left(w, w+L^{-n/m}h\right)$ converges as $L \to +\infty$, uniformly for $w \in \Omega$, in the C^∞-topology of the ball $\{h \in \mathbb{R}^n \ : \ |h| \leq R\}$, to the function

$$\frac{1}{(2\pi)^n}\int_{\sigma_A(w,\xi)\leq 1} e^{-i\langle h,\xi\rangle}d\xi\,.$$

Moreover, considering different measure preserving local charts for each variable of E_L, for each compact subset W of $M \times M \setminus \{(x,x) \ : \ x \in M\}$ and each $\alpha,\beta \in \mathbb{N}^n$, uniformly for $(x,y) \in W$,

$$\partial^\alpha \otimes \partial^\beta E_L(x,y) = O\left(L^{(n+|\alpha|+|\beta|-1)/m}\right).$$

In the case of the Laplacian, under some dynamical assumptions on the geodesic flow of (M,g), the O may be replaced by a o so we can recover the leading term for $E_L - E_{L-cL^{1/2}}$ which is the Schwartz kernel for the orthogonal projector onto the

[5]The actual statement given by Hörmander and used in the proof describes the function E_L on a macroscopic neighborhood of the diagonal, though the full statement involves a couple of long definitions so we do not make it here.

space generated by eigenfunctions ψ_k such that $\lambda_k \in]L - cL^{1/2}, L]$. This was done originally in [Saf88] on the diagonal, and recently completed in [CH15, CH15] (see Theorem 1 of [CH15]).

- The **quantum ergodicity** principle asserts that when the geodesic flow of (M, g) is ergodic, eigenvalues should equidistribute in the following sense.

  Theorem 1.2.5 (Quantum ergodicity, [Shn74], [Zel87], [Col85])**.** *Assume that the geodesic flow of (M, g) is ergodic. Then, there exists a sequence $(k_l)_{l\in\mathbb{N}}$ of density one*[6] *such that for each $U \subset M$ open subset,*

  $$\int_U |\psi_{k_l}(x)|^2 |dV_g|(M) \xrightarrow[l\to+\infty]{} \frac{\mathrm{Vol}_g(U)}{\mathrm{Vol}_g(M)}\,.$$

  More generally, for any self-adjoint pseudo-differential operator P of order 0 on M, with principal symbol $\sigma_P \in C^\infty(T^*M)$,

  $$\int_{S^*M} \overline{\psi_{k_l}}(x) P\psi_{k_l}(x) |dV_g|(x) \xrightarrow[l\to+\infty]{} \frac{\int_{S^*M} \sigma_P(x,\xi) d\mu_{\mathcal{L}}(x,\xi)}{\int_{S^*M} d\mu_{\mathcal{L}}(x,\xi)}$$

  where $d\mu_{\mathcal{L}}$ is the Liouville measure on the unit cotangent bundle[7] *S^*M.*

  One might wonder whether one could take $k_l = l$ for each l. This phenomenon is called quantum unique ergodicity. As it turns out, there are a few counterexamples, but Rudnick and Sarnak have conjectured that it holds whenever (M, g) has negative sectional curvature (here are a few results in that direction (see for instance [Lin06],[Ana08] and [AKN09])).

- Quantum ergodicity asserts that on a manifold (M, g) with a "chaotic" geodesic flow, most high frequency eingenvalues should equidistribute. In [Ber77], Berry conjectured that their fluctuations should be Gaussian, and more precisely, that they should locally behave like monochromatic random waves (which we will define in the next section). This statement is somewhat enigmatic since eigenfunctions at a given frequency are deterministic. Recently, Ingremeau suggested a possible interpretation in [Ing17]. In any case, this conjecture has motivated a lot of research concerning monochromatic random waves and smooth Gaussian fields in general.

1.2.2 Some Gaussian fields related to the Laplacian

Starting from Laplace eigenfunctions, one can define various Gaussian fields, whose covariances functions have a certain spectral interpretation. Here are some examples of such fields.

[6] We say that a sequence $(n_k)_{k\in\mathbb{N}}$ has density one if $\frac{\mathrm{Card}\{k\in\mathbb{N} \,:\, n_k \leq N\}}{N} \xrightarrow[T\to+\infty]{} 1$.

[7] The Liouville measure is just the product of the measure $|dV_g|$ on the manifold with the hypersurface measure on the fibers of S^*M. It is invariant by the geodesic flow (see Proposition 5.3.6 of [KH95])

- **Monochromatic random waves**:
  As will be apparent later, this model is archetypical of what "choosing a random eigenfunction with eigenvalue λ on a compact manifold" should be like locally at scale $\lambda^{-1/2}$. Let ρ be the uniform probability measure on the unit sphere in $\mathbb{R}^n$. The random monochromatic wave is a Gaussian field f with spectral measure ρ. Thus, for each $x, y \in \mathbb{R}^n$,

$$\mathbb{E}\left[f(x)f(y)\right] = \int_{S^{n-1}} e^{i\langle x-y,\omega\rangle} d\rho(\omega)\,.$$

  Since for each $\omega \in S^{n-1}$, $\Delta_x(e^{i\langle x-y,\omega\rangle}) = e^{i\langle x-y,\omega\rangle}$, $\mathbb{E}[(\Delta f(x) - f(x))^2] = 0$ so a.s., we have $\Delta f = f$. Thus, f is a random (generalized) Laplace eigenfunction in $\mathbb{R}^n$.

- **Random spherical harmonics**:
  To actually choose a random fixed frequency eigenfunction on a compact manifold, Laplace eigenvalues must have some multiplicity (otherwise the function will be deterministic up to a constant factor). The following model is among the simplest to have this property. On $M = S^2$ with the standard metric, the eigenvalues of the Laplacian are of the form $l(l+1)$ where $l \in \mathbb{N}$. The eigenspace E_l associated to $l(l+1)$ has dimension $2l+1$. Equip E_l with the L^2 scalar product on S^2. A random spherical harmonic is a random function f_l in E_l chosen with the standard Gaussian measure and normalized so that its value at any point has unit variance. The covariance function for random spherical harmonics is defined as follows. For each $l \in \mathbb{N}$, let $P_l = \frac{1}{2^l l!}\frac{d^l}{dX^l}(X^2-1)^l$ be the l-th Legendre polynomial. Given $x, y \in S^2$,

$$\mathbb{E}[f_l(x)f_l(y)] = P_l(\langle x, y\rangle)$$

  where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product. Another way of building f_l is to take $(Y_{l,m})_m$ an L^2-orthonormal basis of S^2 and $(\xi_{l,m})$ a family of independent centered Gaussians with unit variance and to set

$$f_l = \frac{1}{\sqrt{2l+1}} \sum_m \xi_{l,m} Y_{l,m}\,.$$

- **Band-limited random function**:
  This time, take (M^n, μ) a closed manifold with a smooth positive density μ and A an elliptic differential operator of order m, symmetric with respect to the $L^2(M,\mu)$ scalar product. In general, eigenfunctions will be simple so it does not make sense to take a random eigenfunction. Instead, we can look at random sums of eigenfunctions with eigenvalues in a given interval. More precisely, given $I \subset \mathbb{R}$ an interval with a finite upper bound, let V_I be the linear span of the eigenfunctions ψ_k of A such that $\lambda_k \in I$. Then, this space is finite dimensional. To define a band limited-random function f_I endow it with the L^2 scalar product on (M,μ) one can choose a function at random in V_I with the standard Gaussian measure. As before, a standard way of building f_I is as a linear combination of the elements

of an orthonormal basis of V_I. Here, we can take $(\xi_k)_k$ a sequence of independent centered Gaussians with unit variance and set

$$f_I = \sum_{\lambda_k \in I} \xi_k \psi_k \, .$$

In particular, for each $x, y \in M$,

$$\mathbb{E}[f_I(x) f_I(y)] = \sum_{\lambda_k \in I} \psi_k(x) \psi_k(y) =: e_I(x,y)$$

where e_I is the Schwartz kernel for the L^2-orthogonal projector onto V_I. For $I =]-\infty, L]$ with $L \to +\infty$, $e_{]-\infty,L]} = E_L$ which was described in Theorem 1.2.4. Recall that Theorem 1.2.4 implies that the covariance function E_L of $f_{]-\infty,L]}$ converges as $L \to +\infty$ in C^∞ at scale $L^{-1/m}$ around any point $w \in \Omega$ to an explicit function after renormalization, at a rate uniform in w. This means that the field $\left(L^{-n/2m} f_{]-\infty,L]}\left(w + L^{-1/m} x\right)\right)_x$ (read in the same charts as E_L), will converge in C^∞, at a rate uniform in $w \in \Omega$ to a smooth field on $\mathbb{R}^n$ with covariance

$$(x,y) \mapsto \frac{1}{(2\pi)^n} \int_{\sigma_A(w,\xi) \leq 1} e^{i\langle x-y, \xi \rangle} d\xi \, .$$

We can also choose other kinds of intervals by taking advantage of the above theorem. For instance, for each $t \in [0,1[$ and each $L > 0$ $f_{]tL,L]}$ will have covariance $e_{]tL,L]} = E_L - E_{tL}$, which satisfies

$$e_{]tL,L]}\left(w + L^{-1/m} x, w + L^{-1/m} y\right) = \frac{1}{(2\pi)^n} \int_{t \leq \sigma_A(w,\xi) \leq 1} e^{i\langle x-y, \xi \rangle} d\xi L^{n/m} + O\left(L^{(n-1)/m}\right) .$$

- **Cut-off fractional Gaussian fields**:
  In the previous example, the reproducing kernel Hilbert space of the function $f_{]-\infty,L]}$ is the space V_L generated by eigenfunctions ψ_k such that $\lambda_k \leq L$ and equipped with the $L^2(M,\mu)$ scalar product (see Subsection 1.1.2). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which it is defined. For each $u \in L^2(M,\mu)$, $(\langle f_{]-\infty,L]}, u \rangle_{L^2})_{L \in \mathbb{R}}$ is a sequence of Gaussian random variables that converges in L^2 as $L \to +\infty$ to a centered Gaussian random variable with variance $\|u\|^2_{L^2}$. Thus, the map $u \mapsto \langle f_{]-\infty,L]}, u \rangle_{L^2}$ from $L^2(M,\mu)$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ converges weakly as $L \to +\infty$ to an isometry. This isometry is an instance of the generalized Gaussian field associated to $L^2(M,\mu)$, also known as the L^2 **white noise** on M. Thus, $f_{]-\infty,L]}$ is a smooth approximation of the L^2 white noise on M. But the white noise is part of a natural family of Gaussian fields indexed by $s \in \mathbb{R}$, which are just the generalized Gaussian fields associated to the Sobolev spaces $H^s(M)$ or $H^s_0(M)$ (the subscript 0 stands for zero-mean on each connected component). The regularity of these Gaussian fields depends on the value of s (see [CI13]). We will now extend the previous construction to define smooth approximations of the fields on $H^s_0(M)$.

The case $H^s(M)$ is similar but the construction is slightly less natural. For each $s \in \mathbb{R}$ and each $L > 0$, set

$$f_{s,L} = \sum_{0<\lambda_k \leq L} \frac{\xi_k}{\lambda_k^{s/2}} \psi_k$$

whose covariance is

$$K_L^s(x,y) = \sum_{0<\lambda_k \leq L} \frac{1}{\lambda_k^s} \psi_k(x)\psi_k(y)\,.$$

We call this the cut-off fractional Gaussian field because of the eigenvalue cut-off $\lambda_k \leq L$ and the possibly fractional exponent $-s/2$, so the field f typically converges in fractional Sobolev spaces. It is also a reference to the fields studied in [CI13]. The results of this section pertain to cut-off fractional Gaussian fields.

- **Monochromatic waves on compact manifolds**:
  Consider the setting of Theorem 1.2.4 where A is the Laplacian. As mentioned above, under some dynamical assumptions on the geodesic flow, the O in the remainder term in the asymptotics of E_L may be replaced by a o. Using this estimate, in [CH15, CH18], Canzani and Hanin study the field

  $$f_{[L-cL^{1/2},L]} = \sum_{L-L^{1/2} \leq \lambda_k \leq L} \xi_k \psi_k$$

  for some fixed $c > 0$. This is in a sense as close as we can get to choosing a random fixed frequency Laplace eigenfunction on a general compact manifold. They prove in particular that, in measure preserving local charts

  $$\lim_{L\to+\infty} L^{(1-n)/2}\mathbb{E}\left[f_{[L-cL^{1/2},L]}\left(L^{-1/2}x\right) f_{[L-cL^{1/2},L]}\left(L^{-1/2}y\right)\right] = \frac{1}{(2\pi)^n}\int_{S^{n-1}} e^{i\langle x-y,\omega\rangle} d\rho(\omega)$$

  where $d\rho$ is the surface area measure on the sphere. In other words, as L goes to infinity, the field $L^{(1-n)/4} f_{[L-cL^{1/2},L]}\left(L^{-1/2}\cdot\right)$ converges in law to a monochromatic random wave.

1.2.3 Topology of nodal sets of Gaussian fields

The study of roots of random polynomials dates back to the work of Kac [Kac43]. As the subject developed, people started studying the nodal set of multivariate polynomials and general random fields. The books [AW09] and [AT07] are a testimony of the many developments in the topic. Over the past twenty years, however, many new ideas coming from geometry and analysis have changed the field (see for example [SZ99], [NS09], [GW11], [GW16a], [GW16b], [NS16]). In this subsection, we will review some of the techniques used to describe the topology of the zero set of Gaussian fields. Throughout the discussion, we will consider f a smooth centered Gaussian field with covariance K on a closed Riemannian manifold (M, g) of dimension n, and will occasionally assume that f is non-degenerate in various ways. The symbol Z_f will denote the nodal set $f^{-1}(0)$ of f and $\beta_0(Z_f)$ will denote the number of connected components of Z_f.

- The first tool is **the Kac-Rice formula** mentioned earlier (see Lemma 1.1.8). This formula has many variants which give integral expressions for expectations of critical point or zero set densities of f. Here is an application first used in [Nic15]. First of all, by Lemma 1.1.7, if for each $x \in M$, the Gaussian vector $(f(x), d_x f)$ is non-degenerate, Z_f is a.s. smooth. Let $\mathrm{Crit}(f)$ be the set of critical points of f. Then, it follows from either Theorem 2.2 of [Nic15] or from Theorem 6.8 of [AW09] applied in local charts that

$$\mathbb{E}\left[\mathrm{Card}\left(\mathrm{Crit}(f)\right)\right] = \int_M \mathbb{E}\left[|\det(\mathrm{H}_x(f))| \ \middle| \ d_x f = 0\right] \gamma_{d_x f}(0) |dV_g|(x) \, .$$

Here, $\gamma_{d_x f}$ is the density of $d_x f$ in $(T^*_x M, g_x^{-1})$. The importance of this formula comes from Morse theory. Indeed, since f is Morse, the number of connected components of Z_f is no greater than the number of critical points x of f such that $f(x) > 0$. But by symmetry, the expected number of positive critical points is half the expected number of critical points. In particular,

$$\mathbb{E}\left[\beta_0(Z_f)\right] \leq \frac{1}{2} \int_M \mathbb{E}\left[|\det(\mathrm{H}_x(f))| \ \middle| \ d_x f = 0\right] \gamma_{d_x f}(0) |dV_g|(x) \, . \tag{1.2.1}$$

Note that while the left-hand side is a topological quantity, the right-hand side is an integral quantity. One only needs to know the law of the two-jet of f at each point. The conditional expectation can then be explicitely computed in terms of derivatives the covariance K at (x, x) (see Lemma 1.1.5) and the expected absolute value of the determinant of a $n \times n$ Gaussian random matrix. Using this argument, in [Nic15], Nicolaescu proves the following theorem:

Theorem 1.2.6 (Theorem 1.1 of [Nic15])**.** *Let (M, g) a closed Riemannian manifold of dimension $n > 1$. For each $L > 0$ f_L be the band-limited random function on (M, g) associated to the Laplacian Δ (as defined in Subsection 1.2.2) acting on $L^2(M, |dV_g|)$ and associated to the frequency interval $[0, L]$. Then, there exists a constant $0 < C_n < +\infty$ depending only on n such that as $L \to +\infty$,*

$$\mathbb{E}\left[\mathrm{Card}(\mathrm{Crit}(f_L))\right] \sim C_n \mathrm{Vol}_g(M) L^{n/2} \, .$$

In particular,

$$\limsup_{L \to +\infty} L^{-n/2} \mathbb{E}\left[\beta_0(Z_{f_L})\right] \leq C_n \mathrm{Vol}_g(M) \, .$$

This approach has many variants. For instance, one can instead fix $p : M \to \mathbb{R}$ a Morse function and compute the expected number of critical points of given index of $p_{|Z_f}$ using another version of the Kac-Rice formula. Using the Morse inequalities, this calculation gives upper bounds on the Betti numbers of Z_f as well as an integral formula for the expected Euler characteristic of Z_f. This technique was introduced in [GW17]. If one is interested specifically in the Euler characteristic, one can use the Gauss-Bonnet theorem to express the Euler characteristic as an integral over Z_L and apply the Kac-Rice formula to this integral. This was used in [Let16a].

- While critical points are useful to provide upper bounds on the topological complexity of Z_f, they cannot possibly provide lower bounds. Indeed, one could imagine that Z_f is an extremely corrugated connected hypersurface of M. This would produce many critical points and make the absolute curvature explode while leaving the topology unchanged. To produce lower bounds, the main technique used so far is **the barrier method**, which is more "hands-on". More precisely, one fixes an open domain $D \subset M$ and finds a way to decompose f as an independent sum

$$f = \xi h + f_0 \tag{1.2.2}$$

where ξ is Gaussian, h is deterministic and vanishes transversally along some hypersurface $\Sigma \subset D$ and f_0 is a smooth Gaussian field. Given this data, if f_0 is C^1-small and ξ is large enough, then, f will vanish along a hypersurface isotopic to Σ in D. This gives lower bounds for the probability that f has a topologically interesting zero set in D. In particular, if $D_1, \dots, D_k$ are disjoint open domains of M on which one has managed to implement this method, then, for each $i \in \{1, \dots, k\}$, one obtains $a_i > 0$ such that $\mathbb{P}[Z_f \text{ has a connected component inside } D_i] \geq a_i$, so that:

$$\mathbb{E}\left[\beta_0(Z_f)\right] \geq \sum_{i=1}^{k} a_i \,.$$

Of course it is not always possible to obtain a decomposition such as (1.2.2). Assume that the measure of f gives mass to some function space V where one can find $h \in V$ with the required properties. Then, the orthogonal decomposition principle described in Remark 1.1.2 yields the desired decomposition. This approach was popularized by [NS09] and later used in [GW14], [LL15], [GW16b] and [SW16] among other places. For instance, using this method, in [GW16b], Gayet and Welschinger prove the following result:

Theorem 1.2.7 ([GW16b])**.** *Let (M, μ) be a closed manifold with a smooth positive density and let A be an elliptic pseudo-differential operator of order $m > 0$ on M. Let g be an auxiliary Riemannian metric on M such that $|dV_g| = \mu$. Let $(f_L)_{L>0}$ be the sequence of band-limited functions associated to A on (M, μ) on the intervals $[0, L]$ for $L > 0$. Then, for each compact hypersurface Σ of $\mathbb{R}^n$ there exist constants $c = c(\Sigma, A, M) > 0$ and $0 < R = R(\Sigma, A, M) < +\infty$ such that for all large enough values of L, for all $x \in M$, the probability that there exists a connected component of Z_{f_L} diffeomorphic to Σ included inside the geodesic ball of radius $L^{-1/m}R$ centered at x is at least c. In particular,*

$$\liminf_{L \to +\infty} L^{-n/m} \mathbb{E}\left[\beta_0(Z_{f_L})\right] > 0 \,.$$

Note that when $A = \Delta$, $m = 2$ so Theorems 1.2.6 and 1.2.7 together prove that for the model of 1.2.6 $\mathbb{E}\left[\beta_0(Z_{f_L})\right] \asymp L^{n/2}$. A variant of this approach, already implicitely present in [GW16b], is the spectral measure decomposition. If one

can approximate f in local charts by a stationary field $\tilde{f}$, then one can use the decomposition described in Subsection 1.1.3 to construct an expression close to (1.2.2) (see for instance [NS16] or [FFN17]) .

- The last approach we will mention is the method of **local averages**. It is particularly useful to prove concentration results around the mean for topological and metric properties of the zero set of sequences of Gaussian fields such as those presented in Subsection 1.2.2. It was first used in [NS09] and later in [NS16] and [SW16] . This approach proceeds in two steps. To begin with, one has to prove that there are not too many large connected components. This is done using the Kac-Rice formula: let $R > 0$ be smaller than the cut-locus of M and cover M with $\Theta\left(R^{-n}\mathrm{Vol}_g(M)\right)$ geodesic balls of radius R. Any connected component of diameter larger than $2R$ must intersect one of the spheres binding these balls and thus create a connected component of the zero set of f restricted to this sphere. We can then apply (1.2.1) to obtain an upper bound for the number of such connected components. To understand why this upper bound could be useful, let us assume that f typically oscillates at scale $0 < \frac{1}{r} \ll 1$, so that, uniformly for $x \in M$, in orthonormal coordinates around x, for each $i \in \mathbb{N}$ and each multiindex $\alpha \in \mathbb{N}^n$, $\mathrm{Var}\left(\partial_i \partial^\alpha f(x)\right) \asymp r^{-2}\,\mathrm{Var}(\partial^\alpha f(x))$. Then, the right hand side of (1.2.1) will be of order $R^{n-1} \times \mathrm{Vol}_g(M) R^{-n} \times r^{1-n} = \mathrm{Vol}_g(M) r^{n-1}/R$. Indeed, the total surface of the R-spheres is of order $R^{n-1} \times \mathrm{Vol}_g(M) R^{-n}$ and the integrand is a quotient of a homogeneous polynomial of degree $n-1$ in the second derivatives of the field by a homogeneous function of degree $(n-1)/2$ in the covariance of the first derivatives of the field. Leaving aside the possibility of cancellation in these integrals, as well as the effect of the conditioning, the integrand should be of order $1/r^{n-1}$ in r. On the other hand, the barrier method tells us that the total number of connected components should be of order r^{-n}. Thus, if we choose R so that $r^{1-n}/R \ll r^{-n}$, or equivalently $R \gg r$, then the number of components of diameter larger than R will be a negligible fraction of the total number of components. This allows one to estimate $\beta_0(Z_f)$ by the integral of a local quantity: Let $\beta_0(Z_f, R)$ be the number of connected components of Z_f of diameter at most R. We have established that for $R \gg r$, this was a good approximation of $\beta_0(Z_f)$. Also, for each $x \in M$, let $\beta_0^x(Z_f, R, s)$ be the number of connected components of diameter at most R included inside the ball of radius s centered at x. Then, for each $\varepsilon > 0$,

$$\int_M \frac{\mathbb{E}\left[\beta_0^x(Z_f, R, R)\right]}{\mathrm{Vol}_g(B(x,R))} |dV_g|(x) \leq \mathbb{E}\left[\beta_0(Z_f, R)\right] \leq \int_M \frac{\mathbb{E}\left[\beta_0^x(Z_f, R, 2R)\right]}{\mathrm{Vol}_g(B(x,R))} |dV_g|(x)\,. \tag{1.2.3}$$

Many of the models presented in Subsection 1.2.2, have the property that they vary at a natural small scale r and that when blown up to that scale, they converge to a universal model. In that setting, (1.2.3) already suggests that $\mathbb{E}\left[\beta_0(Z_f)\right]$ will be equivalent to some universal constant times $\mathrm{Vol}_g(M) \times r^n$. In [NS16], Nazarov and Sodin go one step further. Indeed, in the limit, most examples from Subsection 1.2.2 actually converge to stationary fields. In that setting, using ergodicity

arguments, they show that $\beta_0(Z_f)$ converges in L^1. More precisely, they prove the following theorem.

Theorem 1.2.8 (Theorem 1.1 of [NS16]). *Let f be a stationary Gaussian field on $\mathbb{R}^n$ with spectral measure ρ. Assume that ρ satisfies the following conditions:*

- *The measure ρ has finite fourth moments:*

$$\int_{\mathbb{R}^n} |\xi|^4 d\rho(\xi) < +\infty\,.$$

- *The measure ρ has no atoms.*
- *The measure ρ is not supported on a linear hyperplane.*

Under these three conditions, there exists a constant $\nu = \nu(\rho) \geq 0$ such that, a.s. and in L^1, as $R \to +\infty$,

$$N_R \sim \nu \times \mathrm{Vol}\,[B(0,R)]\ . \tag{1.2.4}$$

Moreover, assume there exists a finite, compactly supported Hermitian measure μ with $\mathrm{spt}(\mu) \subset \mathrm{spt}(\rho)$ and a bounded domain $D \subset \mathbb{R}^d$ such that $\mathcal{F}(\mu)_{|\partial D} < 0$ and $\mathcal{F}(\mu)(x_0) > 0$ for some $x_0 \in D$. Then $\nu > 0$.

From this theorem, using the local average method, they prove in a general setting of parametric ensembles $(f_L)_L$ with stationary local limits such as those presented in Subsection 1.2.2, that $\beta_0(Z_{f_L})$ asymptotically concentrates around its mean in the L^1 sense.

1.2.4 The Gaussian Free Field

In some exceptional situations, the fractional Gaussian field converges in the sense of distributions to the Gaussian Free Field (or GFF). In these cases, some of the properties of the GFF are shared by the fractional Gaussian field. The Gaussian Free Field appears naturally when trying to model a random interface between two fluids unaffected by any exterior forces. The idea is to see this interface as the graph of a function $f : U \to \mathbb{R}$, where U is some bounded open subset of $\mathbb{R}^n$ with smooth boundary. The Gaussian Free Field should be a random system with Hamiltonian $H(f) = \int_U |\nabla_x f|^2 dx$. The corresponding Gibbs measure should therefore be proportional to $e^{-\frac{1}{2}H(f)}df$ where df should be some kind of Lebesgue measure on the space of functions. This of course is badly defined, but can be interpreted as follows. First, notice that he Hamiltonian is a non-negative quadratic functional. Restricted to $C_c^\infty(U)$ it is actually positive so f can be thought of as a random distribution on U such that for each $u \in C_c^\infty(U)$, $\langle \nabla f, \nabla u \rangle$ is centered Gaussian with variance $H(u)$. More formally, a version of the **Gaussian Free Field** (a.k.a the GFF) on U with Dirichlet boundary conditions is (as in Example 1.1.2), a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a subspace $H \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ containing

only centered Gaussian random variables, and an isometry $H_0^1(U) \to H$, where $H_0^1(U)$ is the closure of $C_c^\infty(U)$ with respect to the scalar product $\langle u, v\rangle = \int_U \langle \nabla u, \nabla v\rangle$. The same construction works on a closed connected[8] Riemannian manifold (M, g) replacing $C_c^\infty(U)$ by the space $C_0^\infty(M)$ of smooth functions with zero mean on M. More concretely, if $(\lambda_k)_k$ and $(\psi_k)_{k\geq 0}$ are the eigenvalues and eigenfunctions of the Laplacian on (M, g) (as in Subsection 1.2.1), then, by Stoke's formula (and given our sign convention for Δ), for each $k, l \in \mathbb{N}$, $k, l > 0$,

$$\begin{aligned}\int_M \left(\nabla \lambda_k^{-1/2}\psi_k, \nabla \lambda_l^{-1/2}\psi_l\right) |dV_g| &= \lambda_k^{-1/2}\lambda_l^{-1/2}\int_M \psi_k \Delta\psi_l |dV_g| \\ &= (\lambda_l/\lambda_k)^{-1/2}\int_M \psi_k\psi_l |dV_g| \\ &= \delta_{kl}\,.\end{aligned}$$

Thus, the family $(\lambda_k^{-1/2}\psi_k)_{\lambda_k>0}$ forms a Hilbert basis for $H_0^1(M)$. In particular, if $(\xi_k)_k$ is a sequence of i.i.d. $\mathcal{N}(0, 1)$ random variables, the following series converges a.s. in the space of distributions to the Gaussian Free Field:

$$\sum_{\lambda_k>0} \frac{\xi_k}{\sqrt{\lambda_k}}\psi_k\,.$$

The covariance function of the GFF, while a priori not well defined as a function, can be defined as a distribution, acting on $C_c^\infty(U \times U)$. Consider the case where $U \subset \mathbb{R}^n$ is an open bounded subset with smooth boundary (the case of a closed manifold is of course very similar). Then, the covariance of the GFF should be some symmetric distribution $G \in \mathcal{D}'(U \times U)$ such for each $u_1, u_2 \in C_c^\infty(U)$,

$$\int_U \int_U G(x, y)u_1(x)u_2(y)dxdy = \mathbb{E}\left[\langle f, u_1\rangle\langle f, u_2\rangle\right]\,.$$

Taking u_1 and u_2 of the form $u_i = \Delta v_i$ for some $v_1, v_2 \in C_c^\infty(U)$, by definition of f,

$$\mathbb{E}\left[\langle f, \Delta v_1\rangle\langle f, \Delta v_2\rangle\right] = \langle v_1, \Delta v_2\rangle$$

so

$$\int_U \int_U G(x, y)\Delta v_1(x)\Delta v_2(y)dxdy = \int_U v_1(x)\Delta v_2(x)dx\,.$$

By Stoke's formula, applied along the integral in y to the left-hand side, we get $\Delta G(x, \cdot) = \delta_x$. Therefore, G is the Green function of the Laplace operator on U. This function has explicit expressions for different shapes of U but its degree of regularity depends only on the dimension (see for instance (5.10) of [Joh50]):

[8]The definition works as well for non-connected manifolds but we must impose that functions have zero mean on each connected component. Moreover, the full theory can be recovered by taking independent GFFs on each connected component of the manifold.

- In dimension one, G is continuous.
- In dimension $n \geq 3$, the Green function is singular on the diagonal and $G(x,y) \asymp |x-y|^{2-n}$ as $|x-y| \to 0$.
- In dimension two,
$$G(x,y) = -\log(|x-y|) + Q_U(x,y) \tag{1.2.5}$$
where Q_U is the unique symmetric function on $U \times U$, harmonic in each variable, which coincides with $\log(|x-y|)$ whenever, $x \in \partial U$ or $y \in \partial U$. Furthermore, if $\Phi : U \to U'$ is a conformal isomorphism and G_U and $G_{U'}$ are the Green's functions on U and U', then, we have (see [Ber15]):
$$G_U\left(\Phi(\cdot), \Phi(\cdot)\right) = G_{U'}\,.$$
We say that the GFF is **conformally invariant** in dimension two. We will not dwell any further on this matter since it is not related to the contents of this thesis but more information can be found in [Aru15] and [Sep17].

Of course, since the initial motivation was to model a random interface via the graph of f, having f be a distribution is hardly satisfying. To deal with this issue, one can regularize f. Here are three possible regularizations:

- **The Discrete Gaussian Free Field** or (DGFF): In this solution, one discretizes space. Take $U \subset \mathbb{R}^n$ an open bounded subset with smooth boundary. For each $N > 0$, let $U_N = NU \cap \mathbb{Z}^n$. In general, the DGFF is well defined on finite regular graphs with different boundary conditions. Here, we will focus on $\mathbb{Z}^n$ with Dirichlet boundary conditions associated to U_N. Hence, in this discussion, for each $N > 0$, DGFF on U_N with Dirichlet boundary conditions will be a random element ϕ_N in the space of functions $\mathbb{R}^{\mathbb{Z}^n}$ that vanish outside of U_N, i.e, a real valued function on the vertices of $\mathbb{Z}^n$ that vanishes outside of U_N. On the graph $\mathbb{Z}^n$ there is a discrete Laplacian Δ, defined as $\Delta u(x) = \sum_{|y-x|=1} u(x) - u(y)$ (here $|x-y|$ is the Euclidean distance between x and y) from which one can define the Hamiltonian $H_N(\phi_N) = \frac{1}{8}\sum_{v \in U_\varepsilon} \phi_N(x)\Delta\phi_N(x)$ by analogy with the continuous Hamiltonaian $H(f) = \int_U |\nabla f|^2 = \int_U f\Delta f$. Since the space of functions on U_N is finite dimensional (recall that U is bounded), the Gibbs measure $e^{-H_N(\phi_N)}d\phi_N$ is well defined. This solution is well known, and has the advantage that it retains many of the algebraic properties of the GFF. It is indeed a regularization since, by a Riemman sum argument, one can show that the measure $\sum_{v \in U_N} \phi_N(v)\delta_{v/N}$ converges weakly to f in distribution when $N \to +\infty$ (up to a multiplicative constant).
- **The Smoothed Gaussian Free Field** or (SGFF): As its name suggests, this solution amounts to taking a sequence of smooth (or at least somewhat regular) kernels $I_\varepsilon(x,y)$ such that the operator $I_\varepsilon u \mapsto \int_M I_\varepsilon(\cdot,y)u(y)|dV_g|(y)$ converges to the identity as $\varepsilon \to 0$ and setting $f_\varepsilon = I_\varepsilon f$. This solution, while less common, has

been used in the case of circle averages of the GFF (see [Wer14]) or in the case of a smooth convolution kernel to study Liouville Quantum Gravity (see for instance [Ber17]). It has the advantage that its is naturally coupled with the GFF and is defined over the same base space. However, it loses most algebraic properties of the GFF.

- **The Cut-off Gaussian Free Field** or (CGFF): The idea here is to take the series representation of the GFF defined above and to truncate it so as to obtain a smooth function. That is, we approximate the GFF by a sequence $(f_L)_{L>0}$ of smooth fields defined as

$$f_L = \sum_{0<\lambda_k\leq L} \frac{\xi_k}{\sqrt{\lambda_k}}\psi_k \,.$$

  While mentioned in a couple of surveys ([Sch07] and [Zel13]), to our knowledge there are no results concerning this random field prior to this thesis. It is a particular case of the fractional Gaussian fields introduced in Subsection 1.2.2 for $s = 1$.

Here is not the place to present a full panorama of the results concerning the Gaussian Free Field. We will just mention three different research topics where a lot of progress has been made in the past ten years:

- Starting with Schramm and Sheffield's papers [SS09], [SS13], on the two dimensional GFF and DGFF, people have been studying level sets of the Gaussian Free Field. Schramm and Sheffield considered the field on simply connected proper subdomains of $\mathbb{C}$ with boundary conditions in which they cut the boundary into two arcs where the field must take values $+\lambda$ and $-\lambda$ respectively. This forces the field to have an interface separating these two level sets. In [SS09], they prove that, for an adequate choice of λ, the discrete level lines converge in law to a certain explicit process called SLE(4). In [SS13] they give an indirect definition of the continuum level line (which is not obvious since the field is a distribution) that has the law of SLE(4) by construction. These papers have lead to a systematic study of level lines of the (continuum) GFF by Aru, Sepúlveda, Werner, Holden, Lupu, Miller and many others.

- Let f be the Gaussian Free Field with Dirichlet boundary conditions on a simply connected bounded domain $D \subset \mathbb{C}$ with smooth boundary and take $\gamma > 0$. Let dx^2 be the Euclidean metric on D. Liouville Quantum Gravity (or LQG) on Q is, formally, the square Q equipped with the metric tensor $e^{\gamma f}dx^2$. The associated area measure can be defined by a renormalization procedure (see for instance [RV10] and [DS11]). It is linked both to the Brownian map and to the KPZ formula (see [Gar13] and the references therein for more details). In recent years, a lot of effort has been put in better understanding the links between the GFF and Liouville Quantum Gravity (see for instance [DMS14], [DKV16] and [KRV17]).

- There is also an independent genealogy of works, starting from [BDG01] and [Dav06] among others, culminating in works like [BDZ16] and [BL18], that aim to describe the behavior of the maximum of the DGFF on a discrete box and the process of the points where it takes its extremal values, as well as the probability that it does not vanish on a box (see Subsection 1.2.9 for more details). These results are quite robust and seem, at their core, to be related to log-correlated Gaussian fields (as suggested for instance by [DRZ17]).

Let us consider $B_N = [-N, N]^2 \cap \mathbb{Z}^2$ a discrete box seen as a graph. Let ϕ_N be the DGFF on B_N. Then, in [BDG01], Bolthausen, Deuschel and Giacomin, prove the following theorem:

Theorem 1.2.9. *Let $\Omega \subset]0,1[^2$ an compact subset with smooth boundary. For each $N \in \mathbb{N}$, $N \geq 1$ set $\Omega_N := \Omega \cap \frac{1}{N}\mathbb{Z}^2$. Then, as $N \to +\infty$,*

$$\mathbb{P}\left[\forall x \in \Omega_N,\ \phi_N(x) \geq 0\right] = \exp\left(-\frac{8}{\pi}\mathrm{cap}_{[0,1]^2}(\Omega)(1+o(1))\ln\left(N\right)^2\right)$$

where

$$\mathrm{cap}_{[0,1]^2}(\Omega) = \inf\left\{\frac{1}{2}\|\nabla u\|_{L^2}^2 : u \in C_c^\infty([0,1]),\ u|_\Omega \geq 1\right\}.$$

This theorem is later used by the authors to study the field ϕ_N conditioned on staying positive on a given domain Ω. In order to prove Theorm 1.2.9, they prove the following estimate on the supremum of the DGFF:

Theorem 1.2.10. *Let $\Omega \subset]0,1[^2$ be a compact subset. Let $(\phi_N)_N$ be the DGFF on $U =]0,1[^2$.*

- *For each $\eta > 0$, there exists $\delta = \delta(\eta) > 0$ such that for all large enough values of N,*

$$\mathbb{P}\left[\max_{B_N}\phi_N \geq \sqrt{\frac{8}{\pi}}\ln(N) + \eta\ln(N)\right] \leq N^{-\delta}.$$

- *For each $\eta > 0$ there exists $c = c(\eta, \Omega) > 0$ such that*

$$\mathbb{P}\left[\max_{\Omega_N}\phi_N \leq \sqrt{\frac{8}{\pi}}\ln(N) - \eta\ln(N)\right] \leq \exp\left(-c\left(\ln(N)\right)^2\right).$$

The variance of $\phi_N(x)$ for a fixed x is of order $\ln(N)$ and the supremum of N^2 independent Gaussians with unit variance is of order $\sqrt{\ln(N)}$ so the upper bound $\ln(N) = \sqrt{\ln(N)} \times \sqrt{\ln(N)}$ is not surprising and indeed follows from a union bound. The lower bound is trickier because the field has non-trivial correlations at large distances. The proof uses the spatial Markov property of the DGFF. Theorem 1.2.10 has been followed in the past eighteen years by a sequence of steps leading up to a description of the law of the maximum coupled with the set at which it is attained (see [BDZ16] and [BL18]).

1.2.5 Computing the covariance function for cut-off fractional Gaussian fields †

The results presented here correspond to those of Chapter 2 which corresponds to [Riv18b].

As explained in Section 1.1, to handle a Gaussian field, one should start by looking at its covariance function. In this section, we describe the asymptotic behavior of the covariance function of cut-off fractional Gaussian fields when the cut-off threshold goes to infinity. The guiding example is that of the CGFF on a closed manifold (see Subsection 1.2.4), which converges to the GFF. Since the GFF behaves qualitatively differently in dimension two and since this dimension is the first one for which the Green's function is not continuous, it should also appear in the following calculations. Actually in general, this 'critical dimension' coincides with the order of the operator. More precisely, let (M,μ), A, $(\psi_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$ be as in Theorem 1.2.1. Take $(\xi_k)_{k\geq 0}$ a sequence of independent centered Gaussians with unit variance and for each $s\in\mathbb{R}$, $L>0$,

$$f_{s,L}=\sum_{0<\lambda_k\leq L}\frac{\xi_k}{\lambda_k^{s/2}}\psi_k\,.$$

Then, the covariance function K_L^s for $f_{s,L}$ has the following asymptotic:

Theorem 1.2.11 (Theorems 1.1 and 1.2 of [Riv18b] or Theorems 2.1.1 and 2.1.2)**.** *Write both K_L^s and σ_A, the principal symbol of A, in measure preserving local coordinates.*

1. *Assume that $s<n/m$. Fix a compact subset Ω of the local chart and $R\in]0,+\infty[$. For any $w\in\Omega$ and $x,y\in\mathbb{R}^n$ such that $|x|,|y|\leq R$, for $L>0$ large enough, let $K_{w,L}^s(x,y)=L^{s-n/m}K_L^s\left(w+L^{-1/m}x,w+L^{-1/m}y\right)$. Then, for each $\alpha,\beta\in\mathbb{N}^n$,*

$$\left(\partial^\alpha\otimes\partial^\beta\right)K_{w,L}^s(x,y)=\frac{1}{(2\pi)^n}\int_{\sigma_A(w,\xi)\leq 1}e^{i\langle x-y,\xi\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{\sigma_A(w,\xi)^s}d\xi+O\left(L^{-1/m}\ln(L)^\eta\right)$$

   where $\eta=1$ if $s=(n+|\alpha|+|\beta|-1)/m$ and 0 otherwise, where $|\alpha|=\alpha_1+\cdots+\alpha_n$, and where $\xi^\alpha=\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$. The estimate is uniform in $w\in\Omega$ and $x,y\in\mathbb{R}^n$ such that $|x|,|y|\leq R$. Moreover, for each $\varepsilon>0$, uniformly for $x,y\in\Omega$ such that $|x-y|\geq\varepsilon$, and for each $L>0$,

$$\left(\partial^\alpha\otimes\partial^\beta\right)K_L^s(x,y)=O\left(L^{(n+|\alpha|+|\beta|-s-1)/m}\ln(L)^\eta\right)\,.$$

2. *Assume now that $s=n/m$. Then the same result holds as long as $(\alpha,\beta)\neq 0$. Moreover, there exists a smooth positive valued function g_A on $M\times M$ such that, uniformly in $(x,y)\in M\times M$,*

$$K_L^s(x,y)=g_A(x,y)\left[\ln\left(L^{1/m}\right)-\ln_+\left(L^{1/m}|x-y|\right)\right]+O(1)\,.$$

Here, $\ln_+(t) := \max\{\ln(t), 0\}$. *The function* g_A *is defined as follows. For each* $x \in \Omega$, *let* $S_x^* = \sigma_A(x,\cdot)^{-1}(1)$ *and let* $d\omega$ *the area measure on* S_x^*. *Then,*

$$\frac{1}{(2\pi)^n} \times \frac{1}{2}\left(\int_{S_x^*} \frac{m d\omega}{|\partial_\omega \sigma_A(x,\omega)|} + \int_{S_y^*} \frac{m d\omega}{|\partial_\omega \sigma_A(y,\omega)|}\right).$$

Finally, there exists a bounded symmetric function $Q : \Omega \times \Omega \to \mathbb{R}$ *such that for any* $\kappa \geq 1$, $L \geq 1$ *and* $x, y \in \Omega$ *such that* $|x-y| \geq \kappa L^{-1/m}$,

$$K_L^s(x,y) = -g_A(x,y)\ln(|x-y|) + Q(x,y) + O\left(\kappa^{-1/k}\right)$$

uniformly in $x, y \in \Omega$, $\kappa \geq 1$ *and* $L \geq 1$, *where* $k = 1$ *for* $n = 1$ *and* $k = m$ *for* $n \geq 2$.

In particular, more generally, the behavior changes when $s = n/m$. When $s < n/m$, the result is analogous to the case $s = 0$ described in Theorem 1.2.4, that is, the kernel converges in C^∞ at spatial scale $L^{-1/m}$ when rescaled by $L^{s-n/m}$. In this case, the high eigenvalues seem to predominate, which would explain the L^{-s} factor in the growth order of K_L^s. On the other hand, for $s = n/m$, the kernel exhibits macroscopic correlations with a logarithmic singularity near the diagonal, smoothed at scale $L^{-1/m}$. For example, in the case of the Laplacian, on a compact surface, we have

- for $s < -1$, in measure preserving local charts, as $L \to +\infty$,

$$\lim_{L\to+\infty} L^{s-1} K_L^s\left(w + L^{-1/2}x, w + L^{-1/2}y\right) = \frac{1}{(2\pi)^n}\int_{|\xi|^2\leq 1} e^{i\langle x-y,\xi\rangle}|\xi|^{-s} d\xi$$

  and the convergence is locally uniform in w and is in the C^∞ sense. Notice that since $s < -1$, the integrand is L^1 at 0 so it is well defined.

- for $s = -1$, the covariance function has a logarithmic singularity around the diagonal, smoothed at scale $L^{-1/2}$. In other words, if f_L is the CGFF in dimension 2, for each $x, y \in \Omega$,

$$\mathbb{E}\left[f_L(x) f_L(y)\right] = K_L^s(x,y) = \frac{1}{2\pi}\ln\left[\left(L^{1/2}\right) - \ln_+\left(L^{1/2}|x-y|\right)\right] + O(1).$$

  This is not surprising since in this case, K_L^s converges in distribution to the Green function on M, which has a logarithmic singularity on the diagonal exactly in dimension $n = 2$.

Proof sketch for Theorem 1.2.11:
The proof of Theorem 1.2.11 is based on the following observation. If one considers $K_L(x,y)$ as a distribution in L, then

$$\partial_L K_L(x,y) = \sum_k \lambda_k^{-s}\psi_k(x)\psi_k(y)\delta_{L=\lambda_k} = L^{-s}\sum_k \psi_k(x)\psi_k(y)\delta_{L=\lambda_k} = L^{-s}\partial_L K_L^0(x,y).$$

But K_L^0 is just the spectral function E_L which was computed by Hörmander in [Hör68]. The ideas will be clearer if we forget about the derivatives and focus instead on estimating K_L^s itself. As a first try, one might plug Hörmander's asymptotic in the identity $\partial_L K_L^s = L^{-s}\partial_L E_L$ and one would get the right polynomial order of growth, but this would be cheating since the asymptotic with respect to L and the derivative with respect to L need not commute, and this brash approximation would thus miss the logarithmic term in the case $s = n/m$. To make this rigorous, K_L^s must be expressed as a function of E_L without the derivative, for instance by integrating by parts along the L variable:

$$K_L^s = L^{-s}E_L + s\int_1^L \lambda^{-s-1}E_\lambda d\lambda + O(1)\,. \tag{1.2.6}$$

The order of growth of E_λ as described in [Hör68] (see also Theorem 1.2.4) is of $\lambda^{n/m}$ near the diagonal. In particular, on the diagonal $E_L(x,x) = C_A L^{n/m} + O\left(L^{n/m-1}\right)$ where $C_A > 0$ is a positive constant. Thus,

$$K_L^s(x,x) = C_A\left(L^{n/m-s} + s\int_0^L \lambda^{n/m-s-1}d\lambda\right) + O\left(L^{n/m-s-1}\right)\,.$$

When $s < n/m$, we have $K_L^s(x,x) \asymp L^{n/m-s}$ whereas when $s = n/m$, the integral grows logarithmically so $K_L^s(x,x) \asymp \ln(L)$, just as announced in Theorem 1.2.11. To prove the case $s < n/m$ of Theorem 1.2.11, we need to extend this calculation to a $L^{-1/m}$-neighborhood of the diagonal. Theorem 1.2.4 works up to distance $L^{-1/m}$ so for the case $s < n/m$, the above calculation works well, with the added (technical) difficulty of integrating over converging functions. For the case $s = n/m$, more work is needed since the statement is macroscopic in scale. First of all, recall that Hörmander's theorem from [Hör68] is stated up to macroscopic scale, though the description is not canonical and not entirely explicit. For simplicity, we will assume that we have the following expression for E_L up to macroscopic scale:

$$E_L(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma_A(x,\xi)\leq L} e^{i\langle x-y,\xi\rangle}d\xi\,. \tag{1.2.7}$$

This is a good toy model since σ_A is m-homogenous so applying the change of variables $\zeta = L^{-1/m}\xi$ and assuming that $y = L^{-1/m}\tau$, we obtain

$$E_L(x, x+L^{-1/m}\tau) = \frac{1}{(2\pi)^n}\int_{\sigma_A(x,\zeta)\leq 1} e^{i\langle\tau,\zeta\rangle}d\zeta \times L^{n/m}$$

just as in Theorem 1.2.4. Let us see what happens if one replaces E_L in (1.2.6) by the expression (1.2.7). The $L^{-s}E_L$ is $O(1)$ since $s = n/m$. On the other hand, the integral term looks like

$$\int_1^L s\lambda^{-s-1}\int_{0\leq\sigma_A(x,\xi)\leq\lambda} e^{i\langle x-y,\xi\rangle}d\xi d\lambda\,.$$

In particular, since σ_A is homogeneous in ξ and given the shape of the integration domain, it seems natural at this point to introduce the change of variables $\xi = r\zeta$ where $r > 0$

and $\sigma_A(x,\zeta) = 1$. This change of variables reduces the problem to the study of integrals of the form:

$$\int_{S_w^*} e^{ir\langle \tau,\zeta\rangle} d\zeta$$

where τ is a unit vector and $r \gg 1$. The order of decay of this integral depends on the singularities of the function $\xi \mapsto \langle \tau, \xi\rangle$ restricted to $S_w^* = \{\zeta \in T_x^*M \ : \ \sigma_A(x,\zeta) = 1\}$. If $\sigma(x,\xi) = |\xi|^m$ then the stationary phase formula[9] applies. The general case is trickier, but fortunately we can appeal to the rich theory of oscillatory integrals (see [AGV12]).

1.2.6 Counting connected components of nodal sets of fractional Gaussian fields †

The results presented here correspond to those of Chapter 3 which corresponds to [Riv18a].

In this subsection, we will explain how to apply Theorem 1.2.11 to the problem of counting the connected components of the zero set of the fields $f_{s,L}$ for a fixed $s \leq n/m$ in the limit $L \to +\infty$. This problem, motivated by Courant's nodal domain theorem (Theorem 1.2.2), was solved for random spherical harmonics in [NS09] in the sense that they gave a leading term asymptotic for the expectation of nodal the number of nodal domains and proved its exponential concentration in probability around this leading term. After this paper various generalizations were provided in [GW11], [Nic15], [GW14], [GW16b], [GW16a], [GW17], [NS16] and [SW16]. In this section, we will be taking a closer look at [NS16]. Indeed, the assumptions made in [NS16] were axiomatic and covered in particular, the case of band limited random functions, $f_{0,L}$. It follows easily from Theorem 1.2.11 that their theorem also applies to $f_{s,L}$ for $s < n/m$. However, for $s = n/m$, other ideas are necessary. We will start by explaining the context [NS16] work in, state their result and explain how it applies to the fields we are studying. The results in Chapter 3 only concern the case where M is a closed Riemannian manifold and A is the associated Laplacian so we will do so here as well, though is likely that the results apply in to the general case.

Heuristics:
Before we state the precise results and explain the proof strategy, we will guess what the answer should be using a heuristic argument. Let Z_L be the number of connected components of $f_{s,L}^{-1}(0)$. The first part of Theorem 1.2.11 says that for $s < n/2$, the field converges to a local non-trivial limit at scale $L^{-1/2}$. Therefore, Z_L could have a connected components inside a small ball of radius $L^{-1/2}$ but not too many. Since we can fit about $L^{n/2}\mathrm{Vol}(M)$ such balls in M, the number of connected components of Z_L should be of order $L^{n/2}\mathrm{Vol}(M)$.
On the other hand, the second point of Theorem 1.2.11 says that for $s = n/2$, $f_{s,L}$ varies at scale $L^{-1/2}$ but has values that fluctuate logarithmically. These variations create regions with a low density in connected components in Z_L. More precisely, for $f_{s,L}$ to

[9] See for instance Section 1.2 of [Dui96].

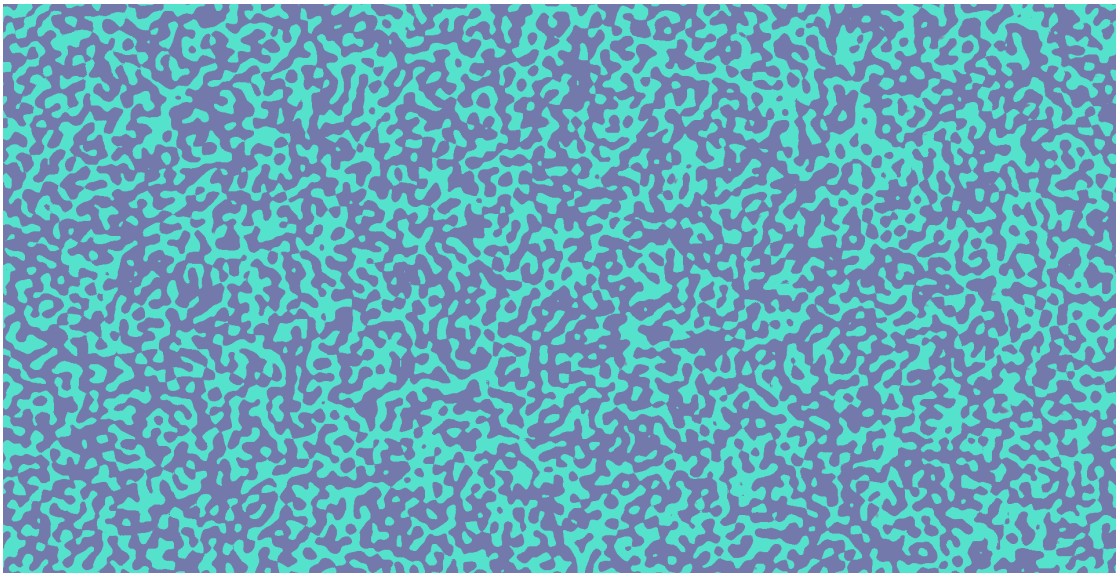


Figure 1.4: Here is a detail of the plot of the sign of a band-limited random function on the torus. The local variation scale $L^{-1/2}$ is quite apparent in the rate of oscillation of the interface and it seems to intersect any disk of radius $100 \times L^{-1/2}$.

have a chance to vanish in a small ball $B(x, L^{-1/2})$, first $f_{s,L}(x)$ must be $O(1)$. Since it is a Gaussian of logarithmic variance, the probability of having a connected component inside a small ball should be of order $1/\sqrt{\ln(L)}$. Thus, the total number of connected components of Z_L should be of order $L^{n/2}/\sqrt{\ln(L)}$.

The case $s < n/2$:
To begin with, in the case of stationary fields on $\mathbb{R}^n$, Theorem 1.2.8 (Theorem 1 of [NS16]) says that, under some mild assumptions on the spectral measure ρ, the number of connected components of the zero set of the Gaussian field inside a ball $B(0, R)$ concentrates around aR^n as $R \to +\infty$, for some constant $a \in [0, +\infty[$. Moreover, they provide a concrete criterion to prove that $a > 0$. This is condition $(\rho 4)$ of their Theorem 1 (which is Theorem 1.2.8 in the present document). As explained in Subsection 1.2.3, to prove this result, they first discard large connected components by controlling the number of intersections of the zero set with large balls, which can be measured via an integral formula, and then see the total number of small components as an average over connected components inside smaller balls. This averaging perspective allows them to apply an ergodic theorem to reach their conclusion. But of course, the fields $f_{0,L}$ are not defined on $\mathbb{R}^n$. However, Theorem 1.2.4 shows that, asymptotically, at a microscopic scale, band-limited random functions behave like a stationary Gaussian field. More precisely, consider measure preserving coordinates on a compact subset $\Omega \subset M$. For each $w \in \Omega$ and $L > 0$ let $g_{w,L} = L^{-1/4} f_{0,L}\left(w + L^{-1/2}\cdot\right)$. Then, according to Theorem

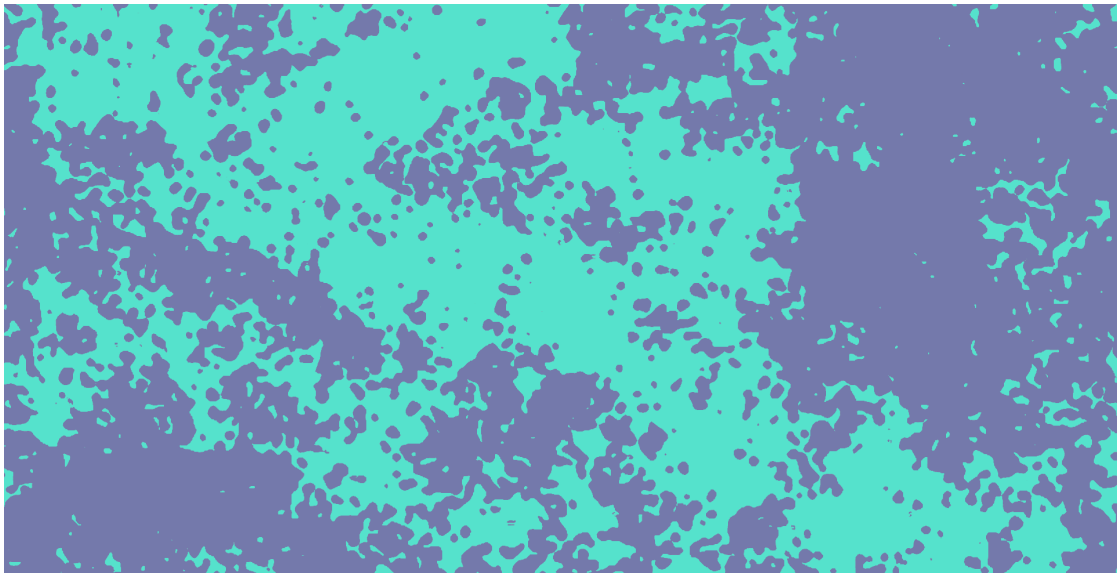


Figure 1.5: Here is part of the plot of the sign of the toral CGFF. The local variation scale $L^{-1/2}$ is still present. However, the macroscopic correlations create almost monochromatic islands with few nodal components.

1.2.4, as $L \to +\infty$,

$$\mathbb{E}\left[g_{w,L}(x)g_{w,L}(y)\right] \to \kappa_0(x-y)$$

uniformly in w for the topology of uniform convergence of derivatives of any order on compact subsets with respect to x and y, and where κ_0 is the Fourier transform of the uniform measure on the unit ball. In other words, $g_{w,L}$ converges uniformly in w in law to a stationary field on $\mathbb{R}^n$. The condition $(\rho 4)$ follows from Majer's criterion, given Appendix C2 of [NS16]. This is the property that Nazarov and Sodin use to extend their results to families of Gaussian fields on manifolds in Theorem 3 of [NS16]. But this property is shared by the fields $f_{s,L}$ for $s < n/2$. Indeed, Theorem 1.2.11 shows that, if we set $g_{w,s,L} = L^{-1/4} f_{s,L}\left(w + L^{-1/2}\cdot\right)$, then,

$$\mathbb{E}\left[g_{w,s,L}(x)g_{w,s,L}(y)\right] \to \kappa_s(x-y)$$

where κ_s is the Fourier transform of the measure $|\xi|^{-2s}\mathbb{1}_{[|\xi|^2\leq 1]}d\xi$. Again, condition $(\rho 4)$ is satisfied by Majer's criterion mentioned above and the other conditions of [NS16]'s Theorem 3 follow directly from the estimates of Theorem 1.2.11. In summary, Theorem 1.2.11 contains enough information to apply the general Theorem 3 of [NS16] and obtain:

Theorem 1.2.12 (Theorem 1.1 of [Riv18b] or Theorem 3.1.1)**.** *Let (M, g) be a Riemannian manifold of dimension $n > 0$, fix $s < n/2$ and let $(f_{s,L})_{L>0}$ be the cut-off fractional Gaussian field associated to the Laplacian with parameter s. For each $L > 0$, let N_L be the number of connected components of $f_{s,L}^{-1}(0)$. Then, there exists a constant $\nu_{n,s} > 0$*

depending only on the dimension n and on the parameter s such that $L^{-n/2}N_L$ converges in L^1 to $\nu_{n,s}$. In other words,

$$\mathbb{E}\left[\left|\frac{N_L}{\sqrt{L}} - \nu_{n,s}\mathrm{Vol}_g(M)\right|\right] \xrightarrow[L\to+\infty]{} 0\,.$$

The case $s = n/2$:
In this case, the field $f_{s,L}$ behaves quite differently. Indeed, while for $s < n/2$, the field had non-trivial stationary local limits at scale $L^{-n/2}$ and decorrelated at macroscopic scale, in the present case, $s = n/2$, the field becomes asymptotically trivial at scale $L^{-n/2}$ and retains some correlations at the macroscopic scale. Therefore, Nazarov and Sodin's theorem has no hope of working. On the other hand, one can try to adapt the Kac-Rice approach and the barrier method approach presented in Subsection 1.2.3. Recall that under a simple non-degeneracy assumption, the zero set is known to be smooth as stated in Lemma 1.1.7. Using Theorem 1.2.11, it is easy to check that $f_{s,L}$ satisfies the non-degeneracy assumptions needed to apply the Kac-Rice critical point upper bound (see (1.2.1)):

$$\mathbb{E}\left[N_L\right] \leq \int_M \mathbb{E}\left[|\det(H_x f_{s,L})| \mid d_x f_{s,L} = 0\right] \gamma_{d_x f_{s,L}}(0)|dV_g|(x)\,.$$

Applying Lemma 1.1.5, the integrand can be expressed in terms of derivatives of the covariance of $f_{s,L}$ at (x,x) which can then be estimated using Theorem 1.2.11. The resulting estimate is of order $L^{n/2}$. The logarithmic factor does not appear! In order to be more precise, one must apply a variation of this argument introduced by Gayet and Welschinger in [GW17]. Instead of looking at critical points of $f_{s,L}$ one can fix a function $p \in C^\infty(M)$ with isolated critical points (p could for instance be a Morse function) and look at its restriction to Z_L. Since Z_L is a smooth submanifold of M, each of its connected components $\mathcal{C}$ is a smooth closed boundary so $p|_{\mathcal{C}}$ has a critical point. Therefore, N_L is no greater than $C_L(p)$, the number of critical points of $p_{|Z_L}$. While this is a slightly more complicated quantity than the number of critical points of $f_{s,L}$, it also has an integral expression as a function of p and the covariance. Using this formula, one can conclude that there exists $C = C(M) < +\infty$ such that

$$\mathbb{E}\left[N_L\right] \leq C\frac{L^{n/2}}{\sqrt{\ln(L)}} \tag{1.2.8}$$

which is the upper bound heuristically predicted. The reason why this method works better than just counting critical points is that according to Theorem 1.2.11, the derivatives of the field vary at scale $L^{-1/2}$ so, by just counting critical points, one will not detect the logarithmic factor that only appears in the values of the field. On the other hand, here we are really restricting p to the zero level set of $f_{s,L}$ which is why we see the logarithmic term appear. A powerful method to show lower bounds on the number of connected components of Z_L is the barrier method presented in Subsection 1.2.3. Recall that in this method, the goal is to fix a domain, say a small ball $B(x,r)$ in M, and

to prove lower bounds for the probability that Z_L has a connected component inside $B(x,r)$. In the barrier method, this is typically done by decomposing the field into a "nice term" and an indepenent fluctuation (see (1.2.2)). Here, that argument is too crude to work as is. Indeed, as explained in the heuristic argument, for field to vanish on a ball of size $L^{-1/2}$ around a point x, with good enough probability then, $f_{s,L}(x)$ must be bounded in L, which is quite rare since its variance blows up. In order to counter this effect, we condition on $f_{s,L}(x)$ for $f_{s,L}(x) \in [-1,0[$. Since $f_{s,L}$ is Gaussian, its conditional law is explicit and still Gaussian. We want to study the probability that it stays positive on the sphere $\partial B(x, L^{-1/2})$. However, given a Gaussian field, it is not clear in general whether it can stay positive on a given set. To prove this, we use the FKG inequality. More precisely, to prove the lower bound, instead of $B(x, L^{-1/2})$ consider $B(x, L^{-1/2}\rho)$ for some $\rho > 0$. If $\rho > 0$ is large enough, the conditioned field is positively correlated on $\partial B(x, L^{-1/2}\rho)$ and essentially constant on $L^{-1/2}$ sized hulls on this sphere. Thus, it stays positive on each hull. If $B_1, \dots, B_N$ is a covering of the sphere by such hulls, in fact, for a good choice of parameters, for each $j \in \{1, \dots, N\}$, on the event $f_{s,L}(x) \in]0,-1]$,

$$\mathbb{P}[f_{s,L|B_j} > 0 \mid f_{s,L}(x)] \geq 1/3\,.$$

We would like to estimate the probability that $f_{s,L}$ stays positive on all the balls B_j at once. To do so, we use the FKG inequality (see Pitt, [Pit82b] or Theorem 1.3.7), according to which, the positive correlation of $f_{s,L|\partial B(x,L^{-1/2}\rho)}$ implies that, on the event $f_{s,L}(x) \in]0,-1]$,

$$\mathbb{P}[f_{s,L|\partial B_L(x,L^{-1/2}\rho)} \mid f_{s,L}(0)] \geq \prod_{j=1}^{N} \mathbb{P}[f_{s,L|B_j} > 0 \mid f_{s,L}(x)] \geq 3^{-N} > 0\,.$$

Taking expectations over $f_{s,L}(x)$ shows that Z_L has a connected component inside $B(x, L^{-1/2}\rho)$ with probability $\geq a\,(\ln(L))^{-1/2}$ for some $a > 0$. Summing over such small balls proves that

$$\mathbb{E}[N_L] \geq c\frac{L^{n/2}}{\sqrt{\ln(L)}}$$

for some $c = c(M) > 0$. To sum up, by reasoning as described above, one can prove:

Theorem 1.2.13 (Theorem 1.2 of [Riv18a] or Theorem 3.1.2)**.** *Let (M,g) be a closed Riemannian manifold of dimension $n > 0$, let $(f_{s,L})$ be the cut-off fractional Gaussian field assoticated to (M,g) with parameter $s = n/2$. Let N_L be the number of connected components of $f_{s,L}^{-1}(0)$. Then, there exist two constants $0 < c < C < +\infty$ that depend only on n such that for each $L > 1$ large enough,*

$$c\frac{L^{n/2}}{\sqrt{\ln(L)}} \leq \mathbb{E}\left[N_L\right] \leq C\frac{L^{n/2}}{\sqrt{\ln(L)}}\,.$$

1.2.7 Concentration of the maximum and hole probability for the CGFF †

Here we present results from Chapter 4 which corresponds to [Riv17].

For each $L > 0$, let f_L be the cut-off GFF on a closed surface Σ, introduced in Subsection 1.2.4. By Theorem 1.2.11, in measure preserving local charts, the covariance G_L of f_L satisfies

$$G_L(x,y) = \frac{1}{2\pi}\left[\ln\left(\sqrt{L}\right) - \ln_+\left(\sqrt{L}|x-y|\right)\right] + O(1)$$
$$\text{and for all } \alpha,\beta \in \mathbb{N}^2,\ (\alpha,\beta) \neq (0,0),\ \left(\partial^\alpha \otimes \partial^\beta\right) G_L(x,y) = O\left(L\right) .$$

In particular, the CGFF has logarithmic correlations and has a natural variation scale of $1/\sqrt{L}$. We will therefore express this estimate and the following results in terms of $\sqrt{L}$ instead of L. From these covariance estimates, one can prove the following analog of Theorem 1.2.10.

Theorem 1.2.14 (Theorem 4 of [Riv17] or Theorem 4.1.4)**.** *Let (Σ, g) be a closed surface with a Riemannian metric g. Let $(f_L)_{L\geq 0}$ be the CGFF on (Σ, g). Let $\Omega \subset \Sigma$ be a compact subset.*

- *For each $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ such that for all large enough values of L,*

$$\mathbb{P}\left[\max_{\Sigma} |f_L| \geq \sqrt{\frac{2}{\pi}}\ln\left(\sqrt{L}\right) + \eta \ln\left(\sqrt{L}\right)\right] \leq \sqrt{L}^{-\delta}$$

- *For each $\eta > 0$ there exists $c = c(\eta, \Omega) > 0$ such that*

$$\mathbb{P}\left[\max_{\Omega} |f_L| \leq \sqrt{\frac{2}{\pi}}\ln\left(\sqrt{L}\right) - \eta \ln\left(\sqrt{L}\right)\right] \leq \exp\left(-c\ln\left(\sqrt{L}\right)^2\right) .$$

Intuition behind Theorem 1.2.14:
In Theorem 1.2.10, the upper bound follows by a union bound over the all the sites of the space of definition. The only real property that is used is that the values of the field at each site have Gaussian tails. To adapt this argument, one can cover Σ with $\frac{1}{\sqrt{L}}$-balls and, using a Sobolev inequality with the derivative estimates mentioned above, prove that for each such ball B_L, $\mathbb{E}\left[\sup_{B_L} |f_L|\right] = O\left(\sqrt{\ln(\sqrt{L})}\right)$. By the Borell Tsirelson-Ibragimov-Sudakov inequality (see Theorem 2.1.1 of [AT07]), the centered maximum of the field on B_L has Gaussian tails. The result then follows as before by union bound. The lower bound in Theorem 1.2.10 is trickier since it uses the Markov property, which is not available for the CGFF. Instead, one can restrict the field to a well chosen small lattice and use Slepian's lemma (Corollary 2.2 of [AW09]) to show that the supremum

discretized CGFF stochastically dominates the supremum of the DGFF for $N \simeq \sqrt{L}$. This approach is inspired by the proof of Lemma 2.8 of [DRZ17].

Theorem 1.2.14 is leads to the proof of a version of Theorem 1.2.9 for the CGFF.

Theorem 1.2.15 (Theorem 1 of [Riv17] or Theorem 4.1.1)**.** *Let (Σ, g) be a closed surface with a Riemannian metric g. Let $(f_L)_{L\geq 0}$ be the CGFF on (Σ, g). Let $\Omega \subset \Sigma$ be a compact subset with smooth boundary. Then, as $L \to +\infty$,*

$$\mathbb{P}\left[f_L \geq 0 \text{ on } \Omega\right] = \exp\left(-\frac{2}{\pi}\mathrm{cap}_\Sigma(\Omega)(1+o(1))\ln\left(\sqrt{L}\right)^2\right)$$

where

$$\mathrm{cap}_\Sigma(\Omega) = \inf\left\{\frac{1}{2}\|\nabla u\|_{L^2}^2 \ : \ u \in C_0^\infty(\Sigma), u|_\Omega \geq 1\right\}.$$

Proof sketch for Theorem 1.2.15:
While the proof of Theorem 1.2.14 follows that of Theorem 1.2.10 quite closely, the proof of Theorem 1.2.15 is only loosely inspired by that of Theorem 1.2.9. It is split into two parts, one for the upper bound and one for the lower bound. For the lower bound, it is enough to find an explicit event that implies that $f_L \geq 0$ and that is easy to estimate. Let us consider for instance the following event. First, let u be a function in the support of the law of f_L such that $u \geq 1$ on Ω and u has zero mean. This is possible for $L > 0$ large enough. Then, one can decompose f_L as an independent sum

$$f_L = \xi\frac{u}{\|\nabla u\|_{L^2}^2} + \tilde{f}_L$$

where $\xi \sim \mathcal{N}(0,1)$ and $\tilde{f}_L$ is a smooth Gaussian field as suggested in Remark 1.1.2 and Subsection 1.2.3 (more precisely, $\tilde{f}_L$ is the Gaussian field associated to the orthogonal of u in the Hilbert space associated to f_L.). Then, as long as $\xi > \|\nabla u\|_{L^2}^2 \sup_\Sigma \tilde{f}_L$, f_L must stay positive on Ω. It is easy to see that the supremum of $\tilde{f}_L$ is dominated by that of f_L so one can use Theorem 1.2.14 to control its behavior and obtain the lower bound. To prove the upper bound for the case of the DGFF, the authors of [BDG01] partition the square into small boxes and rely on the Markov property to deduce that the extremes of the field in separate boxes are essentially independent. We will first explain how to obtain this independence without the Markov property. Recall that f_L is defined as a sum with random independent coefficients:

$$f_L = \sum_{0<\lambda_k\leq L}\frac{\xi_k}{\sqrt{\lambda_k}}\psi_k\,.$$

This sum can be split into two independent parts

$$f_L = \sum_{0<\lambda_k\leq L^\delta}\frac{\xi_k}{\sqrt{\lambda_k}}\psi_k + \sum_{L^\delta<\lambda_k\leq L}\frac{\xi_k}{\sqrt{\lambda_k}}\psi_k$$

for some small parameter $\delta > 0$. Then, the first part is just f_{L^δ} so its extremes will be of size $O\left(\delta \ln\left(\sqrt{L}\right)\right)$ and it will vary at scale $L^{-\delta/2}$. On the other hand, one can show that the second part, $f_L - f_{L^\delta}$, decorrelates at macroscopic distances, and that, seen at scale $L^{-\delta/2}$, it behaves essentially like a CGFF with parameter $L^{1-\delta}$. The idea is then to show that the first sum is negligible and to use the independence in the second sum. We call A_L the event that $f_{L|\Omega} > 0$ and we consider the following dichotomy, depending on parameters $\delta > 0$, $\eta > 0$ and $K \in]0, +\infty[$ that we will fix later:

1. The field f_{L^δ} takes values below $\left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln\left(\sqrt{L}\right)$ on a subset of Ω of size at least $\frac{K}{L^{\delta/2}}$.

2. The subset $\mathcal{E}_L \subset \Omega$ of points where f_{L^δ} takes values above $\left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln\left(\sqrt{L}\right)$ is large enough: $\mathrm{Vol}\left(\Omega \setminus \mathcal{E}_L\right) \leq \frac{K}{L^{\delta/2}}$.

In the first situation, one can imagine that there are roughly K boxes of radius $L^{-\delta/2}$ on which f_{L^δ} takes values below $\left(\sqrt{\frac{2}{\pi}} - \eta\right) \ln\left(\sqrt{L}\right)$ (at this scale, f_{L^δ} typically has fluctuations of size $O(1)$). On each of these boxes, $f_L - f_{L^\delta}$ behaves like a CGFF of parameter $L^{1-\delta}$ so by the lower bound in Theorem 1.2.14, there exists $a > 0$ (independent of δ, η, K and L) so that it will take values below $-\sqrt{\frac{2}{\pi}}(1-2\delta) \ln\left(\sqrt{L}\right)$ except on an event of probability $\exp\left(-a \ln\left(\sqrt{L}\right)^2\right)$. Here, if we take $\delta < \eta/2$, the fact that $f_L - f_{L^\delta}$ takes values smaller than $-\sqrt{\frac{2}{\pi}}(1-2\delta) \ln\left(\sqrt{L}\right)$ on this box implies that f_L takes negative values on this box so $f_L \notin A_L$. Using the fact that $f_L - f_{L^\delta}$ decorrelates at macroscopic distances, we can treat each of these rare events as independent. Thus, on the first alternative, $f_L \notin A_L$ except on an event of probability at most $\exp\left(-aK \ln\left(\sqrt{L}\right)^2\right)$. We then take $K \gg 1$ so that $\mathbb{P}[A_L] \gg \exp\left(-aK \ln\left(\sqrt{L}\right)^2\right)$. Once we have chosen δ depending on η and K to reach this conclusion, we can discard this alternative and move on to the second one. In this case, the (Gaussian) random variable $X_L := \int_\Omega u(x) f_{L^\delta}(x) |dV_g|(x)$ will be greater than $\sqrt{\frac{2}{\pi}}(1-\eta)\left(\int_\Omega u\right) \ln\left(\sqrt{L}\right) - 2K \|f_{L^\delta} u\|_\infty L^{-\delta/2}$. An estimate similar to the upper bound in Theorem 1.2.14 allows us ensure that $\|f_{L^\delta} u\|_\infty L^{-\delta/2} = o(1)$ (as $L \to +\infty$) outside of a negligible event so it can be ignored. Now the variance of X_L converges to $\sigma_\Omega(u) := \int_\Omega \int_\Omega G(x,y) u(x) u(y) |dV_g|(x) |dV_g(y)|$ so by standard Gaussian tail estimates, as $L \to +\infty$,

$$\ln\left(\mathbb{P}\left[X_L \geq \sqrt{\frac{2}{\pi}}(1-\eta)\left(\int_\Omega u\right) \ln\left(\sqrt{L}\right)\right]\right) \sim -\frac{1}{2}\sqrt{\frac{2}{\pi}}(1-\eta) \frac{\int_\Omega u^2}{\sigma_\Omega(u)} \ln\left(\sqrt{L}\right)^2 .$$

It is an exercise in spectral theory to prove that

$$\sup\left\{\frac{\int_\Omega u^2}{\sigma_\Omega(u)} \ : \ u \in C^\infty(\Sigma),\ u|_\Omega > 0\right\} = \mathrm{cap}_\Sigma(\Omega)$$

which yields the upper bound announced in Theorem 1.2.15.

1.2.8 Estimating the Bogomolny-Schmit constant for monochromatic random waves †

The result presented here is that of Chapter 5, which corresponds to [IR18]. It is the result of a collaboration with Maxime Ingremeau.

The number of nodal domains of Laplace eigenfunctions is bounded from above by Courant's nodal domain theorem (Theorem 1.2.2 presented above). However, no deterministic lower bounds are known in full generality. In [Zel16], following the works referenced therein, Zelditch provides logarithmic lower bounds for the number of nodal domains on certain negatively curved surfaces. The proof relies on a result by Hezari and Rivière [HR16] that uses a quantum ergodicity type argument. This suggests that on manifolds whose geodesic flow is "chaotic" in some sense, one should have lower bounds for the nodal set of Laplace eigenfunctions. As explained in Subsection 1.2.1, monochromatic random waves (defined in Subsection 1.2.2) are of special interest for this question because of Berry's conjecture. In particular, whenever Berry's conjecture holds, information on the nodal set of monochromatic random waves provides information of high-frequency eigenfunctions of the Laplacian when the geodesic flow is chaotic. In [Ing17], Ingremeau gives a precise interpretation for Berry's conjecture that would imply lower bounds for the number of nodal domains in quantum chaotic eigenfunctions. This interpretation was inspired by results by Bourgain [Bou14] as well as Buckley and Wigman, [BW16b]. Bourgain proves this version of Berry's conjecture holds for a density one sequence of toral eigenfunctions (although the flow is not chaotic in that setting). As an application, Bourgain provides a lower bound for the number of nodal domains of the eigenfunctions on this sequence (see see Theorem 2 of [Bou14]). More precisely, he applies Theorem 1.2.8 (from [NS16]) to the monochromatic random wave model. According to this theorem, there exists a constant $\nu_{BS} \in]0, +\infty[$ such that if N_R is the number of nodal domains of a monochromatic random wave that are contained in the disk of radius $R > 0$ centered at 0, then, almost surely,

$$N_L \sim \frac{\nu_{BS}}{4\pi} \times \pi R^2 \, .$$

The "BS" stands for Bogomolny and Schmit, who introduced this constant in [BS02]. However, the proof of this result is not constructive so it gives no information about the value of ν_{BS}. Knowing Bourgain's result and Ingremeau's interpretation of Berry's conjecture, it would be interesting to estimate the constant ν_{BS}. In [BS02], Bogomolny and Schmit predicted that $\nu_{BS} \approx 0.0624$ using an analogy with bond percolation. Numerical experiments later showed that $\nu_{BS} \approx 0.0589$ (see [Nas11], [Kon12] and [BK13]), correcting the predicted value by a few digits. As for rigorous estimates, first, there is Pleijel's result optimizing Courant's theorem for bounded open subsets of $\mathbb{R}^2$ (see [Ple56]). It yields the upper bound $\nu_{BS} \leq \frac{4}{j_0^2} \approx 0.692$ (where j_0 is the first zero of the zeroth Bessel function of the first kind). More recently [Nic15], Nicolaescu proves that

$\nu_{BS} \leq \frac{1}{2\sqrt{3}} \approx 0.288$ by counting the expected number of critical points of random spherical harmonics with a Kac-Rice formula (Theorem 2.2 of [Nic15] or (1.2.1)), analogous to Lemma 1.1.8. A possible variation of this result could be, as in [GW17], to fix a Morse function $p : S^2 \to \mathbb{R}$ and count the number of local maxima of $p_{|f^{-1}(0)}$ where f is a random spherical harmonic of high degree. In any case, Nicolaescu's bound is already only off by a factor of 4, which is not so bad. In comparison, before [IR18], the best rigorous lower bound was of order 10^{-319}. It was due to Nastasescu, who followed Nazarov and Sodin's argument in [NS09] and retraced the constants used at every step. In [IR18], we improved this result and proved that:

Theorem 1.2.16 ([IR18] or Theorem 5.1.1)**.**

$$\nu_{BS} \geq 1.39 \times 10^{-4}\,.$$

Proof sketch for Theorem 1.2.16:
The proof goes roughly as follows. First, we estimate the probability that a monochromatic random wave f does not vanish on a circle of radius $r > 0$. We do this by first conditioning on the value of f at the center of the circle and then applying the Kac-Rice formula (see Lemma 1.1.8) to f on the circle itself. Tuning the parameters gives a lower bound on this probability that is not too small. Finally, if f does not vanish on the circle of radius r centered at x and if r is between the first two zeros of the zeroth Bessel function of the first kind, then it is easy to see (using some basic properties of Laplace eigenfunctions on the plane) that the nodal domain containing x must be included in the disk bounded by said circle. Estimating the expected size of the set of points x with this property then leads to a lower bound for ν_{BS}.

1.2.9 Further directions

To conclude Section 1.2, here are a few open questions relating to what has been discussed so far.

- Since [BDG01], the maximum of the DGFF and the set of points where it takes extremal values have been studied in great detail (see [Dav06], [BDZ11], [BZ12], [Din13], [BDZ16], [BL16b], [BL16a], [BL14] and [BL18]). It has been studied in the context of the maximum of log-correlated Gaussian fields in general (see for instance [Aco14], [Mad15] and [DRZ17]). The CGFF has the same scaling limit as the DGFF and is also log-correlated by Theorem 1.2.11 so it seems quite likely that most of these results could be adapted to it. Theorem 1.2.14 is the first step in this program. The main obstacle is the absence of the spatial Markov property of the DGFF, though it could perhaps be replaced by the decomposition described in Subsection 1.2.7.

- Though the previous question seems promising, it is quite likely that the answer will be the same as for the DGFF. Something that has not been studied at all for the DGFF (since there is no clear analog for it) is the component counting

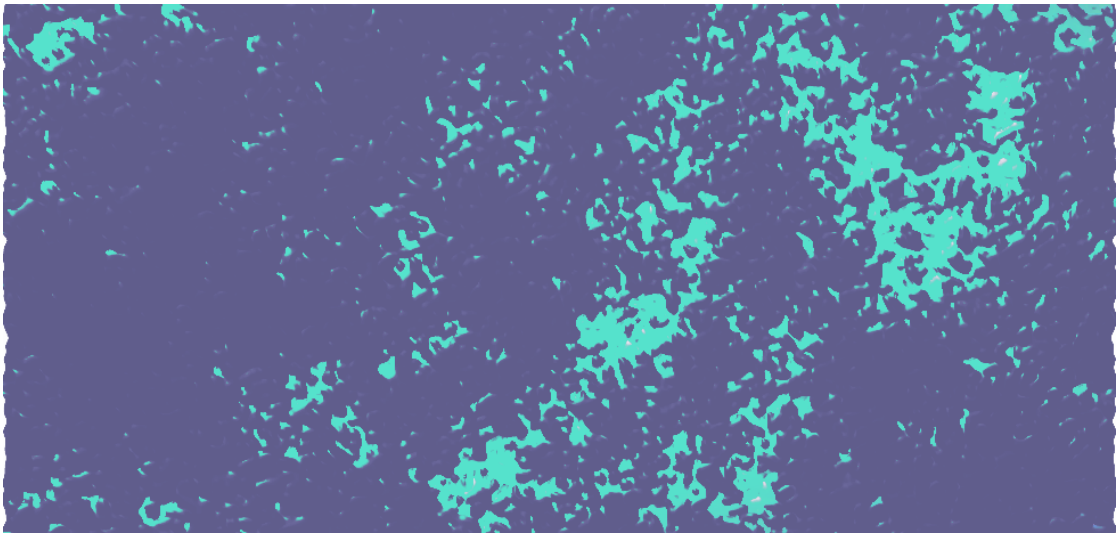


Figure 1.6: Here the values above 0.30 times the maximum of the CGFF are highlighted in turquoise. According to [Dav06] (who studies the DGFF), the Hausdorff dimension of the set $\{f_L \geq \eta \ln\left(\sqrt{L}\right)\}$ should in some sense vary with η.

process for the CGFF. Roughly speaking, it should be a process that associates to as subset A of the domain of the CGFF f_L the number $N_L(A) = "\langle \eta_L, A \rangle"$ of connected components of the zero set of f_L contained inside A. This is not really a measure but is could be approximated by a measure supported on the set $\{f_L = 0\}$ giving mass one to each connected component (as was done in [NS16]). Unlike the case of stationary Gaussian fields for which, as was shown in [NS16], the component counting process equidistributes, the macroscopic fluctuations of the field could cause variations in the density of connected components (see Figure 1.5) so the scaling limit could be non-trivial.

- In Subsection 1.2.6 we restricted our attention to the parameters $s < n/2$, where the field had non-trivial stationary local limits, and $s = n/2$, where the field was log-correlated. However, for $s > n/2$ there are still interesting questions to look at. For instance, if $s < n/2 + 1$, the field converges in C^0 but not in C^1 so its zero set will probably be fractal but not equidistributed. Indeed, away from the zero set of the limit, the field f_L will, for large enough values of L, have the same sign as this limit. This will prevent the nodal set from being equidistributed. Moreover, since the limit is not C^1, its zero set will probably be rough, which will make the (smooth) nodal set of f_L very long and curvy, which will likely produce lots of small nodal components. This means that these processes will have non-trivial scaling limits. Though part of them will probably depend on the geometry (like the overall distribution of the zero set), the fluctuations could still be universal.

- Hole probabilities and large deviation results for the size of the zero set have been studied in many contexts since [BDG01] (see [ST05], [NS09], [SZZ08], [Nis11], [GW11], [Nis12], [BNPS18], [FFN17] and [FFN17]). It could be interesting to see if similar results could be obtained for the cut-off fractional Gaussian fields introduced in Subsection 1.2.2. For the CGFF, Theorem 1.2.15 adapts the result of [BDG01]. In this case, the rate function for large deviations contains a capacity intimately related to the geometry of the field. Perhaps something similar happens for other models as well.

- This problem dates back to [Sch07] (problem 24). The CGFF is an example of a smooth field that converges in distribution to the DGFF. In [SS09], Schramm and Sheffield considered a DGFF on a simply connected domain with boundary conditions $+\lambda$ and $-\lambda$ on the two halves of the boundary (for $\lambda > 0$). They showed that for a good choice of λ the zero-level line separating the $+\lambda$ and $-\lambda$ parts converges in law to SLE(4) as the mesh size goes to 0. One could ask the same question of a smooth approximation of the GFF with similar boundary conditions. More formally, consider for instance f_L the CGFF on a simply connected bounded domain $D \subset \mathbb{C}$ with smooth boundary ∂D and two marked points $a, b \in \partial D$. Let γ_+ and γ_- be the two arcs of ∂D joining a and b and let h be the unique harmonic function on D with boundary conditions $+\lambda$ on γ_+ and $-\lambda$ on γ_- (where $\lambda > 0$ must be chosen with care). Then, $f_L + h$ is positive on γ_+ and negative on γ_- and smooth inside D so there exists a curve γ_L on which $f_L + h$ vanishes, and that goes from a to b. Then, as $L \to +\infty$, γ_L should converge in law to SLE(4).

- It could be interesting to see if one could prove results about deterministic functions using results concerning random functions. Berry's conjecture (see [Ber77]) is an example of such a possible result (see Subsection 1.2.1). Along these lines, in [BW16b], Buckley and Wigman prove lower bounds for the number of nodal components for a density one subsequence of toral eigenfunctions. While proving Berry's conjecture itself seems very difficult, maybe even a weaker result, such as a lower bound on the entropy of the local measures of a fixed eigenfunction could give new information about the nodal set of eigenfunctions on chaotic manifolds.

1.3 Percolation of Gaussian fields

This section concerns results from Chapters 6 and 7 which are the fruit of a collaboration with Hugo Vanneuville (see [RV17a] and [RV17b]). The overarching principle of this project is to adapt Bernoulli percolation, which is traditionally defined on a lattice (or more generally on a graph) to the context of Gaussian fields. We will therefore start with some generalities on Bernoulli percolation in Subsections 1.3.1, 1.3.2 and 1.3.3. This is followed by a discussion of decorrelation inequalities for Gaussian vectors in Subsection 1.3.5. In Subsections 1.3.6 and 1.3.7 we present the results from Chapters 6 and 7. Finally, Subsection 1.3.8 contains a list of questions we think would be interesting to look at in the future.

1.3.1 Bernoulli percolation

Bernoulli percolation is a sprawling topic with classical textbooks such as [Gri99] and [BR06b]. It can be defined in many different settings. In this short account, planar Bernoulli percolation will simply be a random i.i.d. coloring of the edges of $\mathbb{Z}^2$ in black or white. More precisely, let E be the set of edges of the graph $\mathbb{Z}^2$. For each $p \in [0,1]$, Bernoulli percolation is a random element of $\Omega = \{-1,+1\}^E$ chosen with the probability measure $\mathbb{P}_p = \mathrm{Ber}(p)^{\otimes E}$ where $\mathrm{Ber}(p) = p\delta_1 + (1-p)\delta_{-1}$ is the Bernoulli law of parameter p (hence the name). Elements $\omega = (\omega_e)_{e\in E}$ of Ω are called **percolation configurations**. An edge $e \in E$ is **open** in ω or **black** if $\omega_e = +1$ and **closed** or **white** otherwise. Research in Bernoulli percolation consists in the study of the large scale connectivity properties of a random percolation configuration chosen with law $\mathbb{P}_p$. Possibly the most classical theorem in Bernoulli percolation is Kesten's theorem:

Theorem 1.3.1 (Kesten's theorem, [Kes80])**.** *Consider planar Bernoulli percolation on $\mathbb{Z}^2$ with parameter $p \in [0,1]$. If $p > 1/2$, then, a.s., there exists exactly one infinite connected component made of black edges. On the other hand, if $p \leq 1/2$, then a.s., there exists no such unbounded component.*

Kesten's theorem comes with a companion result that follows from one of its proofs, which is

Theorem 1.3.2 (Exponential decay, [Kes80])**.** *Consider planar Bernoulli percolation on $\mathbb{Z}^2$ with parameter $p > 1/2$. Then, for each $\rho > 0$, there exists $c = c(p,\rho) > 0$ such that for each $R > 0$, the probability that there exists a continuous black path in $[0,\rho R] \times [0,R]$ joining $\{0\} \times [0,R]$ and $\{\rho R\} \times [0,R]$ is at least $1 - e^{-cR}$.*

Such continuous paths are called **left to right black crossings**, **lengthwise black crossings** or just **black crossings** of the rectangle $[0,\rho R] \times [0,R]$ when there is no possible ambiguity.

The goal of Chapter 7 will be to adapt Theorems 1.3.1 and 1.3.2 to the case of Gaussian fields. In the following subsection we outline a proof of these results using differential inequalities. We will need three facts that are considered elementary in Bernoulli percolation:

- The Fortuyn-Kasteleyn-Ginibre or **FKG** inequality (see [Gri99] or [BR06b]): By comparing open edges one can define a partial order on configurations. That is, for any two configurations ω and $\tilde{\omega}$ we write $\omega \leq \tilde{\omega}$ if any open edge in ω is also open in $\tilde{\omega}$. Then, one can study events that are **increasing** for this order. That is A is increasing if $\omega \leq \tilde{\omega}$ and $\omega \in A$ together imply $\omega \tilde{\in} A$. In other words, A is stable by the operation of opening edges in a configuration. The FKG inequality says that increasing events are positively correlated. In other words, if A and B are increasing events,
$$\mathbb{P}_p[A \cap B] \geq \mathbb{P}_p[A]\mathbb{P}_p[B]\,. \tag{1.3.1}$$

The main examples of increasing events we will encounter are the existence of black crossings of rectangles (as in Theorem 1.3.2). These events are increasing because if there is a black crossing of a given rectangle in ω, turning some white edges black in ω cannot affect this crossing.

- Let $\theta(p)$ be the probability that there exists a black path connecting 0 to infinity in Bernoulli percolation with parameter p. Then, we claim that the function $\theta(p)$ is non-decreasing. Indeed, one can couple Bernoulli percolation configurations of all parameters as the super-level sets of a family $(U_e)_{e\in E}$ of independent uniform random variables in $[0,1]$. In other words, for each $p \in [0,1]$ the family $(\mathbb{1}_{[U_e \geq 1-p]})_{e\in E}$ has law $\mathbb{P}_p$. On the other hand, any realization of $(U_e)_{e\in E}$ makes the corresponding coupling increasing: if $p < p'$ are two possible parameters and e is open at parameter p, it must be open at parameter p'. More generally, if A is an increasing event, $\mathbb{P}_p[A]$ is a non-decreasing function of p.

- The graph $\mathbb{Z}^2$ is **self-dual**. This means that its dual graph, $\left(\mathbb{Z}^2\right)^*$ is just $\mathbb{Z}^2$ shifted by the vector $(1/\sqrt{2}, 1/\sqrt{2})$. The edges of $\mathbb{Z}^2$ and edges of its dual are naturally in bijection: each edge e intersects exactly one dual edge e^* and vice versa. Given a percolation configuration ω on $\mathbb{Z}^2$, one can define a **dual percolation configuration** ω^* on $\left(\mathbb{Z}^2\right)^*$ by $\omega^*_{e^*} = -\omega_e$. If ω is chosen with parameter p, ω^* is also a random Bernoulli percolation configuration with parameter $1-p$. With this in mind, one can already predict that the parameter $p = 1/2$ will play a special role.

We will also need a crucial result that describes the situation in the case $p = 1/2$.

Theorem 1.3.3 (The Russo, Seymour-Welsh theorem, or **RSW theorem**, see Lemma 4 of Chapter 3 of [BR06b] or Theorem 11.70 and Equation (11.72) of [Gri99])**.** *Consider planar Bernoulli percolation with parameter $p = 1/2$. For each $\rho > 0$ there exists $c = c(\rho) \in]0, 1/2]$ such that for each $R > 0$, the probability that there exists a continuous left to right black crossing of $[0, \rho R] \times [0, R]$ is bounded between c and $1 - c$.*

Using the FKG inequality (1.3.1) and the symmetries of the lattice $\mathbb{Z}^2$ it is easy to deduce the following corollary from Theorem 1.3.3.

Corollary 1.3.4. *There exist $\delta > 0$ and $C < +\infty$ such that for each $R > 0$, the probability that there exists a black path connecting 0 to a vertex x such that $|x| \geq R$ in planar Bernoulli percolation with parameter $p = 1/2$ is at most $CR^{-\delta}$.*

In particular $\theta(p) = 0$ for each $p \leq 1/2$, which implies that a.s. there is no infinite connected component for the same parameters.

This leaves the case $p > 1/2$ which is the real content of Kesten's theorem.

1.3.2 Interlude on boolean functions, influences and the KKL theorem

Bernoulli percolation concerns events measurable with respect to a finite or countable number of independent Bernoulli random variables. In particular, finitely supported events, are just functions $f : \{-1,1\}^\Lambda \to \{0,1\}$ for some finite set Λ, where $\{-1,1\}^\Lambda$ is seen as a measured space with the product probability measure that is Bernoulli along each factor with the same parameter $p \in [0,1]$. This remark may seem trivial but it places the topic in a wider context: that of Boolean functions. The proof of Theorem 1.3.1 presented in Subsection 1.3.3 hinges on a couple of ideas from the analysis of Boolean functions. To place these ideas in their proper context, a short discussion of Boolean functions is in order. The main reference will be [GS14].

Fix $N \in \mathbb{N}$ and let $\mathcal{H}_N = \{-1,1\}^N$. For each $p \in [0,1]$, denote by $\mathbb{E}_p$ and $\mathbb{P}_p$ the probability and expectation of a random element on $\mathcal{H}_N$ whose coordinates are independent Bernoulli random variables of parameter p. A **boolean function** is just a function $f : \mathcal{H}_N \to \{0,1\}$. The hypercube $\mathcal{H}_N$ is also equipped with the partial order defined by comparing each coordinate. That is, $\omega^1 \leq \omega^2$ when for each $i \in \{1,\dots,N\}$, $\omega^1_i \leq \omega^2_i$. A boolean function f is called **increasing**[10] if $\omega^1 \leq \omega^2$ implies $f(\omega^1) \leq f(\omega^2)$. One can couple all the laws $(\mathbb{P}_p)_{p\in[0,1]}$ together by modelling each coordinate as an independent copy of $\mathbb{1}_{[U\geq 1-p]}$ where U is uniform in $[0,1]$. This coupling shows that for any increasing function f, the map $p \mapsto \mathbb{E}_p[f]$ is non-decreasing. Theorem 1.3.2 is a phase transition result. It implies that, if $f : \mathcal{H}_N \to \{0,1\}$ is a crossing event for a $2R \times R$ rectangle (with $N \asymp R^2$ sites), then the function

$$p \mapsto \mathbb{E}_p[f]$$

looks more and more like a step function as $N \to +\infty$. As we shall see in Section 1.3.3, this is an illustration of the following general principle from boolean analysis.

If $f : \mathcal{H}_N \to \{0,1\}$ is an increasing function that does not depend disproportionately on any coordinate, then the function $p \mapsto \mathbb{E}_p[f]$ will look more and more like a step function as $N \to +\infty$.

This principle will be cristallized in Theorem 1.3.5 below. We first need to introduce the terms used in its statement. Given f a boolean function and $i \in \{1,\dots,n\}$, the i-th partial derivative $\partial_i f$ of f is defined as follows: Let $\omega \in \mathcal{H}_N$. Let ω^i_+ be ω with the ith coordinate is replaced by $+1$ and let ω^i_- be defined similarly with the ith coordinate replaced by -1. Then,

$$\partial_i f(\omega) := f(\omega^i_+) - f(\omega^i_-) \in \{-1,0,+1\}\,.$$

[10] Given the definition, one might think that non-decreasing would be a more accurate name. However, since boolean functions are bound to be very much non-injective, none of them are actually increasing. With this in mind, it seems sensible to choose the shorter term without the double negative.

This measures the variation of f as one switches the coordinate i from -1 to 1. The partial derivatives taken together define the gradient of f, $\nabla f : \mathcal{H}_N \to \{-1, 0, +1\}^N$. For $q \in [1, +\infty[$, and $p \in [0, 1]$, the L^q norm of a function f will just be

$$\|f\|_q = \mathbb{E}_p \left[|f|^q\right]^{1/q} = \mathbb{P}_p[f = 1]^{1/q} .$$

Also, the q norm of the gradient of f will be

$$\|\nabla f\|_q = \left(\sum_{i=1}^{N} \mathbb{E}_p[|\partial_i f|^q] \right)^{1/q} .$$

As with f, for each $i \in \{1, \ldots, N\}$, $\mathbb{E}_p[|\partial_i f|^q]$ is just the probability that $f(\omega_+^i)$ is different from $f(\omega_-^i)$. This quantity is called the **influence** of i on f and denote it by $I_i^p(f)$. By definition,

$$I_i^p(f) = \mathbb{E}_p \left[|\partial_i f|\right] \tag{1.3.2}$$

and $\|\nabla f\|_q^q = \sum_{i=1}^N I_i^p(f)$.

Following a qualitative result by Russo [Rus82] and a more quantitative version by Kahn, Kalai and Linial [KKL88], in [BKK+92], Bourgain, Kahn, Kalai and Katznelson proved the following result:

Theorem 1.3.5 ([BKK+92])**.** *Fix $N \in \mathbb{N}$ and $p \in [0, 1]$. There exists an absolute constant $c > 0$ such that for any boolean function f on $\mathcal{H}_N$,*

$$\sum_{i=1}^{N} I_i^p(f) \geq c \mathbb{P}_p[f = 1] \mathbb{P}_p[f = 0] \log \left(\frac{1}{\max_{1 \leq i \leq N} I_i^p(f)} \right) .$$

In particular, c is independent of both N and p.

Now, if f does not depend disproportionately on any coordinate, and if this this remains true for parameters close to p, then the influences must be $O(1/N)$ on most of this interval. But if they are indeed of order $O(1/N)$, then Theorem 1.3.5 implies

$$\mathbb{P}_p[f = 1] \mathbb{P}_p[f = 0] = O\left(\frac{1}{\log(N)} \right) \ll 1 .$$

This means that $\mathbb{E}_p[f]$ must be close to either zero or one. But f is increasing so, if $\mathbb{E}_p[f]$ is close to 0 or 1 for most $p \in [0, 1]$, the map $p \mapsto \mathbb{E}_p[f]$ must look like a step function.

1.3.3 Proof of Theorem 1.3.1, the case $p > 1/2$

In this subsection, we provide a proof of Theorem 1.3.1, due to Kesten.

Consider the assembly of homothetic rectangles of Figure 1.7. Assuming the result of Theorem 1.3.2, because of the symmetries of the $\mathbb{Z}^2$ lattice and by the Borel-Cantelli

Lemma, it is almost certain that all but a finite number of these rectangles admit lengthwise black crossings. But these crossings must therefore intersect and form an infinite black path. To prove that it is unique, we consider a second construction, also made up of homothetic rectangles, in Figure 1.8. In this construction, rectangles can be partitioned in groups of four. Moreover, simultaneous crossings of the four rectangles in a given group implies the existence of a black loop inside the annulus formed by these rectangles, separating 0 from infinity. As before, Theorem 1.3.2 implies that in this construction, all but a finite number of rectangles are crossed. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two unbounded black clusters, they must intersect all of the large enough annuli formed by the rectangles. In particular, they will intersect a common black loop, which will connect them together. Thus $\mathcal{C}_1 = \mathcal{C}_2$, so we have shown that the unbounded black cluster is a.s. unique. This completes the proof of Theorem 1.3.1. **The rest of this section will therefore be devoted to the proof of Theorem 1.3.2**.

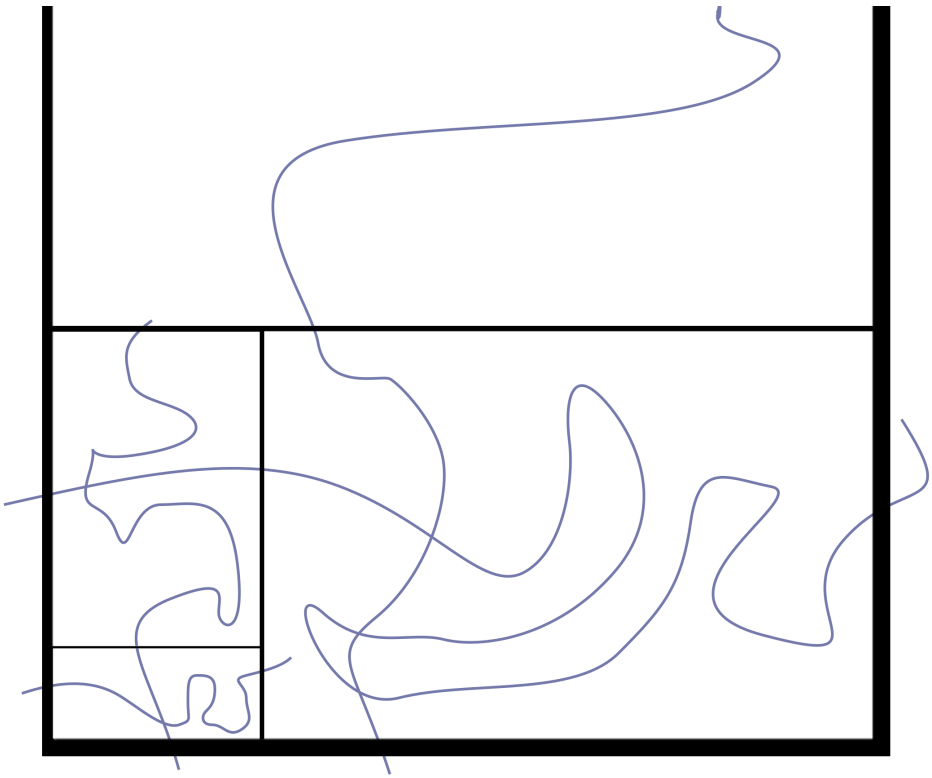


Figure 1.7: A detail from a sequence of homothetic two by one rectangles. Each rectangle is exactly large enough that one can fit the previous largest one perpendicularly inside it. Crossings of all these rectangles are shown in blue.

Fix $p \in]1/2, 1]$ and for $k \in \mathbb{N}$, let a_k be the probability that a rectangle $[0, 2^{k+1}] \times [0, 2^k]$

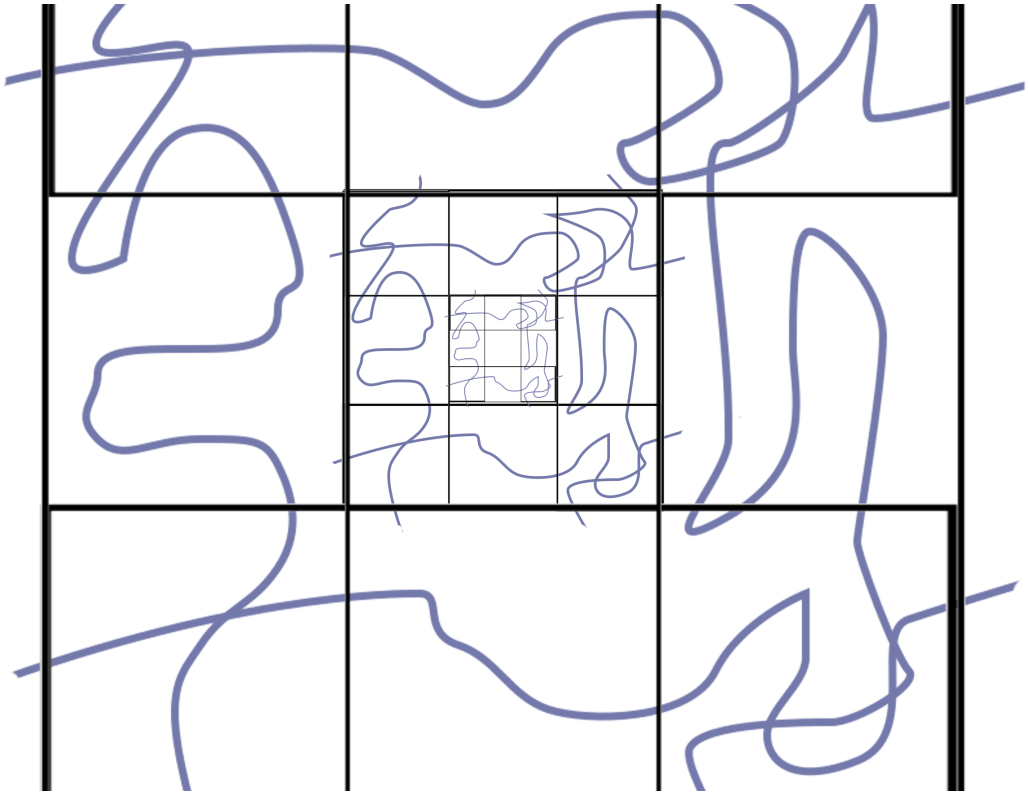


Figure 1.8: A detail from a sequence of homothetic two by one rectangles. Here, the rectangles are assembled in groups of four forming a kind of frame. This frame is then rescaled an infinite number of times. Crossings of all these rectangles are shown in blue.

does not admit a lengthwise black crossing. The absence of a lengthwise black crossing of this rectangle implies the existence of a white crossing from $[0, 2^{k+1}] \times \{0\}$ to $[0, 2^{k+1}] \times \{2^k\}$. Consider the following assembly of rectangles, shown in Figure 1.9:

- The left rectangles are $[-2^k, 0] \times [0, 2^{k-1}]$, $[-2^k, 0] \times [-2^{k-1}, 0]$, $[-2^{k-1}, 0] \times [0, 2^k]$, $[-2^{k-1}, 0] \times [-2^{k-1}, 2^{k-1}]$ and $[-2^{k-1}, 0] \times [-2^k, 0]$.
- The right rectangles are five rectangles symmetric to the five left rectangles with respect to the reflexion along the $\{0\} \times \mathbb{R}$ axis.
- The large rectangle is the rectangle $[-2^{k-1}, 2^{k-1}] \times [-2^k, 2^k]$.

The large rectangle is isometric to the rectangle $[0, 2^{k+1}] \times [0, 2^k]$ while the left and right rectangles are isometric to $[0, 2^k] \times [0, 2^{k-1}]$. Let us call a widthwise crossing of any (non-square) rectangle a continuous path inside this rectangle, joining the two longest sides among its four sides. With this terminology, in the assemblage just described, any path that induces a widthwise crossing of the large rectangle must also induce a

widthwise crossing of a left rectangle as well as a widthwise crossing of a right rectangle (see Figure 1.9). In particular, since the left rectangles are independent from the right rectangles, by union bound, for each $k \in \mathbb{N}$,

$$a_{k+1} \leq 25 a_k^2 \,. \tag{1.3.3}$$

This induction formula is very powerful since if for some $k_0 \in \mathbb{N}$, $a_{k_0} < 1/25$, then there exists $c > 0$ such that $a_k \leq \exp\left(-c2^k\right)$ which yields the conclusion of Theorem 1.3.2, at least for the subsequence $R = 2^k$ and $\rho = 2$. Using the FKG inequality, it is easy to construct yet another gluing scheme that extends this result to any other value $\rho > 0$. Moreover, it turns out that the sequence $(2^k)_k$ has sufficient density in $]0, +\infty[$ to fill the gaps and recover the result for all R. We will leave aside this part of the proof and focus instead on finding k_0 such that $a_{k_0} < 1/25$.

Let $\mathrm{Cross}(R)$ be the event that there is a lengthwise black crossing of the rectangle $[0, 2R] \times [0, R]$. In order to find k_0 such that $a_{k_0} < 1/25$, we will prove the following, stronger estimate. For each $p > 1/2$,

$$\mathbb{P}_p[\mathrm{Cross}(R)] \xrightarrow[R \to +\infty]{} 1 \,. \tag{1.3.4}$$

Since there are a finite number of sites involved in the event $\mathrm{Cross}(R)$, the probability of this event depends polynomially in p. In particular, it is differentiable in p. Equation (1.3.4) will follow from the following estimate:

Claim 1.3.6. *Let $F_R(p) = \mathbb{P}_p[\mathrm{Cross}(R)]$. Then, there exists a positive sequence $(M_R)_{R>0}$ such that:*

- *F_R satisfies the following differential inequality:*

$$\forall p \geq 1/2, \;\; F_R'(p) \geq F_R(p)(1 - F_R(p)) M_R \tag{1.3.5}$$

- *The sequence $(M_R)_R$ diverges to $+\infty$ when $R \to +\infty$.*

Indeed, since $\mathrm{Cross}(R)$ is an increasing event, F_R is non-decreasing so, by Theorem 1.3.3, there exists $c > 0$ such that $F_R(p) \geq F_R(1/2) \geq c > 0$. Next, by (1.3.5) for any $p \geq 1/2$:

$$\log\left(1 - F_R(p)\right) - \log\left(1 - F_R(1/2)\right) \leq \int_{1/2}^{p} \frac{d}{dq} \log\left(1 - F_R(q)\right) dq \leq -c M_R (p - 1/2) \,.$$

Since $M_R \xrightarrow[R \to +\infty]{} +\infty$, this implies that for any $p > 1/2$, $F_R(p) \to 1$ as $R \to +\infty$.

To establish Claim 1.3.6 we will use the ideas of Subsection 1.3.2. Recall that, given a function $f : \{-1, 1\}^\Lambda \to \{0, 1\}$, and an edge $e \in \Lambda$ the **influence** of e on f is the probability $I_e(f)$ that the state of ω_e matters in deciding the value of f. More precisely, given $\omega \in \{-1, +1\}^\Lambda$ a percolation configuration, we say that e is **pivotal** for f in ω if changing the only value of ω_e changes the value of f. Then,

$$I_e^p(f) := \mathbb{P}_p[e \text{ is pivotal for } f] \,.$$

In Subsection 1.3.2, we also expressed influences in terms of partial derivatives of f: $I_e^p(f) = \mathbb{E}_p[|\partial_e f|]$. For any event $A \subset \Omega$ depending only on a finite set of edges Λ, $\mathbb{1}_A$ can be seen as a function $\{-1,1\}^\Lambda \to \{0,1\}$. In this case, the notation $I_e^p(A)$ is often used in place of $I_e^p(\mathbb{1}_A)$. By a simple chain rule calculation, one can check that for any Boolean function f,

$$\frac{d}{dp}\mathbb{E}_p[f] = \sum_{e\in\Lambda} \mathbb{E}_p[\partial_e f]\,.$$

If f is increasing, then the partial derivatives are all non-negative so they can be replaced by their absolute value:

$$\frac{d}{dp}\mathbb{E}_p[f] = \sum_{e\in\Lambda} I_e^p(f)\,. \tag{1.3.6}$$

This is known as **Russo's formula** (see Section 2.4 of [Gri99]). But by Theorem 1.3.5 there exists an absolute constant $c > 0$ such that for any finite set of edges Λ and any $f : \{-1,+1\}^\Lambda \to \{0,1\}$,

$$\sum_{e\in\Lambda} I_e^p(f) \geq c\mathbb{P}_p[f(\omega)=1]\mathbb{P}_p[f(\omega)=0]\log\left(\frac{1}{\max_{e\in\Lambda} I_e^p(f)}\right)\,. \tag{1.3.7}$$

Taking $f = \mathbb{1}_{[\mathrm{Cross}(R)]}$, and combining Equations (1.3.6) and (1.3.6) we get

$$F_R'(p) \geq cF_R(p)(1-F_R(p))\log\left(\frac{1}{\max_{e\in\Lambda_R} I_e^p(\mathrm{Cross}(R))}\right)$$

where Λ_R is the set of edges intersecting the rectangle $[0,2R]\times[0,R]$. To establish the inequality announced in Claim 1.3.6, we just need to show that $\max_{e\in\Lambda_R} I_e^p(\mathrm{Cross}(R))$ tends to 0 when $R \to +\infty$ uniformly in $p \geq 1/2$. Recall that for each edge e, $I_e^p(\mathrm{Cross}(R))$ is the probability that changing the value of ω_e changes the answer to the question of whether or not $\omega \in \mathrm{Cross}(R)$. For this to be the case, there must be a white path connecting the two longest sides of the rectangle, that is interrupted when $\omega_e = 1$ and that is completed when $\omega_e = -1$. In particular, one of the extremities x of ω_e must be connected by a white path at distance at least $R/2$, as shown in Figure **??**.
By stationarity and duality, the fact that x is connected by a white path to a vertex at distance at least $R/2$ has the same as the probability that 0 is connected by a black path to a vertex at distance at least $R/2$, for Bernoulli percolation with parameter $q = 1-p$. But for all $p \geq 1/2$, $q \leq 1/2$. In particular, if $\mathrm{Arm}(R)$ is the event that 0 is to connected to a vertex y with $|y| \geq R$, then, for each edge e, each $p \geq 1/2$ and each $R > 0$,

$$I_e^p(\mathrm{Cross}(R)) \leq \mathbb{P}_{1-p}[\mathrm{Arm}(R)] \leq \mathbb{P}_{1/2}[\mathrm{Arm}(R)]\,. \tag{1.3.8}$$

The second inequality is true because $\mathrm{Arm}(R)$ is an increasing event. Let $M_R = -\log\big(\mathbb{P}_{1/2}[\mathrm{Arm}(R)]\big)$. Then, for each $R > 0$ and each $p \in [1/2,1]$,

$$F_R'(p) \geq cF_R(p)(1-F_R(p))M_R\,.$$

But by Corollary 1.3.4, M_R blows up as $R \to +\infty$ which proves Claim 1.3.6 and concludes the proof of Theorem 1.3.1.

1.3.4 What is percolation for Gaussian fields and why is it significant?

Let f be a continuous Gaussian field on $\mathbb{R}^2$. For each $p \in \mathbb{R}$, let $\mathcal{D}_p = \{x \in \mathbb{R}^2 \ : \ f(x) \geq -p\}$. This defines a random coloring of $\mathbb{R}^2$ where the set $\mathcal{D}_p$ is **black** while its complement is **white**.

One can ask whether there is an unbounded black connected component, or maybe an unbounded connected component of $\mathcal{N}_p = \{x \in \mathbb{R}^2 \ : \ f(x) = -p\}$ the level set of f of height p, that would correspond to percolation interfaces between black and white regions. To make the analogy with percolation more plausible, we impose once and for all some conditions on f:

- The field f should be stationary (See 1.1.3). This was used almost everywhere in the proof of Kesten's theorem and without this restriction, there is no control on how f could behave at infinity.
- The field f should be symmetric by rotation by some angle in $]0, \pi[$ and by reflexion through the horizontal axis. This is useful to rotate and flip rectangles in gluing schemes.
- The field f should be a.s. C^1. This helps avoid wild behavior at a local scale and replaces the discreteness of the space $\mathbb{Z}^2$.

There is a fourth condition which is crucial for any percolation techniques to apply but might not be necessary for the results to be true, which is positive correlation. Indeed, in the proof of Kesten's theorem, the gluing constructions all relied on the FKG inequality. In the case of Gaussian fields, this inequality applies exactly when the covariance is non-negative:

Theorem 1.3.7 (Pitt, [Pit82b]). *Let $X = (X_1, \dots, X_n)$ be a Gaussian vector with covariance $\Sigma = (\Sigma_{ij})_{1 \leq i,j \leq n}$. A set $A \subset \mathbb{R}^n$ is said to be increasing if for any $x \in A$ and $y \in \mathbb{R}^n$ such that $x_1 \leq y_1, \dots, x_n \leq y_n$, we have $y \in A$. Then, the following assertions are equivalent:*

- *For any $i, j \in \{1, \dots, n\}$, $\Sigma_{ij} \geq 0$.*
- *For any increasing Borel subsets $A, B \subset \mathbb{R}^n$,*

$$\mathbb{P}[X \in A \cap B] \geq \mathbb{P}[X \in A]\mathbb{P}[X \in B] \, .$$

By standard differential geometry techniques, continuous crossing events may be approximated by discrete crossing events which depend only on a finite number of sites so the FKG inequality applies to them as well.

Finally, the independence of distinct sites in Bernoulli percolation is usually replaced by a decay of the covariance function, which we will discuss in detail in the next section.

If f satisfies the three basic assumptions and if its covariance function is non-negative and decays fast enough, at large scales, it should behave like Bernoulli percolation. This means that Theorems 1.3.3, 1.3.1 and 1.3.2, but also hopefully finer results such as arm exponents and the scaling limit for percolation interfaces (see Werner's lecture notes [Wer09]) should hold for the random coloring $\mathcal{D}_p$. Percolation questions for Gaussian fields have been studied since the eighties (see [MS83a], [MS83b], [MS86], [Ale96] and [Gar04]). Alexander obtained the following result in the nineties:

Theorem 1.3.8 (Theorem 2.2 of [Ale96]). *Assume that for each $x \in \mathbb{R}^2$, $\kappa(x) \geq 0$, that f is a.s. C^1, and that it is ergodic with respect to translations. Assume that for each $p \in \mathbb{R}$, f has a.s. no critical points at level p. Then, a.s. all the connected components of $\mathcal{D}_0$ are bounded.*

though the methods used were very soft and general. There was no hope of obtaining such precise quantitative results as Theorem 1.3.2, say. The first concrete step in this direction is an analog of Theorem 1.3.3 due to Beffara and Gayet, with a later improvement by Beliaev and Muirhead [BM18]:

Theorem 1.3.9 (Theorem 4.9, [BG17a] (see also Theorem 1.7 [BM18])). *Let f be a stationary Gaussian field on $\mathbb{R}^2$, invariant by $\frac{\pi}{2}$ rotations and reflection along the horizontal axis, whose covariance function κ is non-negative, C^6 at the origin and non-degenerate in the sense that the matrix $(\partial_i \partial_j \kappa(0))_{1 \leq i,j \leq 2}$ is non-degenerate. Assume finally that for all $x \in \mathbb{R}^2$, $\kappa(x) \leq C|x|^{-\alpha}$ for some $\alpha > 16$ and some $C < +\infty$ independent of x. Consider the random coloring of the plane by $\mathcal{D}_p$ defined above. Then, there exists $c = c(\kappa) \in]0, 1[$ such that for each $R > 0$, the probability that there exists a black path in $[0, 2R] \times [0, R]$ joining $\{0\} \times [0, R]$ and $\{2R\} \times [0, R]$ is at least c and at most $1 - c$.*

We have stated the theorem with the assumptions of [BM18] since, leaving regularity considerations aside, their assumptions are weaker than in [BG17a], where it was assumed that $\alpha > 144 + 128 \log_{3/4}(3/2)$. This theorem opened the way for other possible bridges from Gaussian fields into the realm of Bernoulli percolation. Results in this direction could be interesting for percolation theory since many results that are conjectured to be universal, are proven only for certain lattice models. In the framework of universality phenomena, it seems relevant to aks which results hold for Gaussian field percolation. There is also the Bogomolny-Schmit conjecture. In [BS02], the authors suggested the following:

Conjecture 1.3.10 (Bogomolny-Schmit, [BS02]). *The nodal lines of a planar random monochromatic wave in $\mathbb{R}^2$ behave like planar Bernoulli percolation.*

This conjecture is surprising and seems very challenging because the covariance κ of planar random monochromatic waves is the zeroth Bessel function of the first kind

$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\cos(\theta)|x|}d\theta$. In particular, it takes negative values and decays very slowly: at speed $|x|^{-1/2}$. Therefore, the techniques introduced by Beffara and Gayet do not apply to it. The significance of the conjecture comes from Berry's conjecture (see [Ber77]), according to which, random monchromatic waves should be linked to eigenfunctions of manifolds with chaotic geodesic flow. So far, there have been no results comparable to Theorem 1.3.9 with weak enough assumptions to hold for random monochromatic waves, and even if there were, gluing constructions are impossible without FKG. To our knowledge, the only result holding without the FKG inequality is [**?**], which holds for perturbations of fields that satisfy FKG.

1.3.5 Decorrelation inequalities for Gaussian fields

The induction formula (1.3.3) on crossing probabilities in the proof of Kesten's theorem (Theorem 1.3.1) used the fact that the restrictions of ω to disjoint sets of edges are independent. This is obviously false for Gaussian fields, but one can hope for approximate independence in some sense if the covariance K decays at large distance. This question has some subtleties since, for instance, if f is an a.s. analytic Gaussian field on $\mathbb{R}$, then the law of f restricted to $[0,1]$ determines f on all of $\mathbb{R}$. In particular, f on $[0,1]$ and f on $[t,t+1]$ cannot be asymptotically independent for $t \to +\infty$. In this subsection we present a few ways to express asymptotic independence for Gaussian fields.

Let B_1 and B_2 be two balls in $\mathbb{R}^2$ with diameter R and at distance R. In what sense are $f_{|B_1}$ and $f_{|B_2}$ asymptotically independent when $R \to +\infty$?

- A first solution is to consider $\varepsilon > 0$ and to replace B_i by $B_i^\varepsilon = B_i \cap (\varepsilon\mathbb{Z})^2$ for $i \in \{1,2\}$. The field f should then be replaced by $f^\varepsilon = f_{|(\varepsilon\mathbb{Z})^2}$. If one is studying events that are well approximated by events measurable with respect to f^ε, then one can focus on showing that the vectors $(f^\varepsilon(x))_{x\in B_1^\varepsilon}$ and $(f^\varepsilon(x))_{x\in B_2^\varepsilon}$ are asymptotically independent as $R \to +\infty$. In [Pit82a], Piterbarg proved a result about decorrelations of signs of Gaussian vectors. Unkowning of this, in [BG17a], Beffara and Gayet proved a result of a similar nature that was later optimized by Beliaev and Muirhead in [BM18]. Still, the best bound turns out to be Piterbarg's. When applied to the Gaussian vectors $(f^\varepsilon(x))_{x\in B_1^\varepsilon}$ and $(f^\varepsilon(x))_{x\in B_2^\varepsilon}$, his result implies that if A_1 (resp. A_2) is measurable with respect to the signs of $(f^\varepsilon(x))_{x\in B_1^\varepsilon}$ (resp. $(f^\varepsilon(x))_{x\in B_2^\varepsilon}$), then

  $$|\mathbb{P}[A_1 \cap A_2] - \mathbb{P}[A_1]\mathbb{P}[A_2]| \leq CR^4\varepsilon^{-4}\eta(R)$$

  where $\eta(R) = \sup_{|x|\geq R}|\kappa(x)|$. In particular, there is a tension between having to take a very fine mesh to have a good approximation of f by f^ε and preserving the independence between A_1 and A_2. In any case, for the error to be negligible, even with a very large $\varepsilon \leq 1$, we would need $\eta(R) \ll R^{-4}$.

- To say something about events in the continuum, one should take $\varepsilon = \varepsilon(R)$ going to 0 when $R \to +\infty$. In [BG17a], Beffara and Gayet proved a comparison result

between the topology of discretized and continuous nodal lines. This was later improved in [BM18] where Beliaev and Muirhead showed that, in some sense, topological events for the nodal set are faithfullly discretized as long as $\varepsilon(R) = O(R^{-1-\delta})$ for some $\delta > 0$. In particular, circuit and crossing events inside the two boxes B_1 and B_2 decorrelate as long as

$$\eta(R) \leq CR^{-8-\delta}$$

for some $\delta > 0$.

- A second solution is to find a coupling of f with two fields f_1 on B_1 and f_2 on B_2 such that f_1 and f_2 are independent and $f_{|B_i}$ is close to f_i in the C^k topology for some $k \in \mathbb{N}$, for $i \in \{1, 2\}$. This was done in a specific case in [NSV07] (see also Theorem 3.2 of [BG17b] for a variation of this idea in the discrete setting). During the elaboration of this manuscript, this method was also implemented to obtain a quasi-independence result in the continuum for monotonic events (see [MV18]). Using this method, in [MV18], the authors show the following estimate. Consider f a smooth Gaussian $\mathbb{R}^n$ with covariance of the form $\kappa = q * q$ where $q \in L^2(\mathbb{R}^n)$ is of class C^1 and satisfies, for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 1$ and for each $x \in \mathbb{R}^n$, $0 \leq \partial^\alpha q(x) \leq C|x|^{-\beta}$ for some $C = C(q) < +\infty$ and some $\beta = \beta(q) > 0$. Then, under some additional non-degeneracy and regularity assumptions on q, the field f satisfies the following quasi-independence estimate. Let Q_1 and Q_2 be two cubes of sidelength $R > 0$ and at mutual distance at least R. Let A_1 and A_2 be two increasing events[11] such that A_1 (resp. A_2) depends only on $f_{|Q_1}$ (resp. $f_{|Q_2}$). Then, for each $\delta > 0$, there exist $C' = C'(q, \delta) < +\infty$ and $R_0 = R_0(q, \delta) < +\infty$ such that, if $R > R_0$, we have

$$|\mathbb{P}[A_1 \cap A_2] - \mathbb{P}[A_1]\mathbb{P}[A_2]| \leq C' R^{2-\beta+\delta}\,.$$

This estimate, while stronger than the one presented in Subsection 1.3.6 (see Theorem 1.3.11), relies crucially the montonicity assumption $q \geq 0$ which is absent. Whether this assumption can be weakened is as of yet unclear and will probably be the subject of further investigation.

The variety of approaches suggests there is room for improvement, especially if one is interested in specific kinds of events. In the following subsection, we present a new decorrelation result, inspired by Piterbarg's estimate.

1.3.6 Decorrelation for crossings of smooth Gaussian fields †

Here we present results from Chapter 6, which were found in collaboration with Hugo Vanneuville (see [RV17a]).

[11] That is, events A such that if $f \in A$ and h is positive valued, then $f + h \in A$.

We will be working in the setting of Subsection 1.3.4 and using the notations f, $\mathcal{N}_p$ and $\mathcal{D}_p$ introduced therein. Let us choose $p \in \mathbb{R}$ and consider the black and white coloring of the plane defined by $\mathcal{D}_p$. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two translates of the box $[0,2] \times [0,1]$ at distance 1 from each other. For each $R > 0$, and for $i \in \{1,2\}$ let $\mathrm{Cross}_i(R)$ be the event that there exists a continuous black path inside $R\mathcal{Q}_i$ connecting the left and right sides of $R\mathcal{Q}_i$. Then, we show that, as long as $\eta(R) := \sup_{|x| \geq R} |\kappa(x)| \leq 1/2$,

$$|\mathbb{P}[\mathrm{Cross}_1(R) \cap \mathrm{Cross}_2(R)] - \mathbb{P}[\mathrm{Cross}_1(R)]\,\mathbb{P}[\mathrm{Cross}_2(R)]| \leq CR^4\eta(R) \tag{1.3.9}$$

where $C < +\infty$ depends only on κ. In particular, the two crossing events are asymptotically independent as $R \to +\infty$ as long as $\eta(R) = o(R^{-4})$. This is therefore a kind of continuous analog of Piterbarg's estimate from Subsection 1.3.5. The result is weaker than that of [MV18] but works in a significantly wider setting since there is no positivity assumption on the covariance square root (see Subsection 1.3.5 for more details). Equation (1.3.9) actually works in a somewhat more general setup, which we now introduce. Consider $k_1, k_2 \in \mathbb{N}$ and let $(\mathcal{E}_i)_{1 \leq i \leq k_1+k_2}$ be a collection of either rectangles of the form $[a,b] \times [c,d]$ for some $a < b$ and $c < d$ or annuli of the form $x + [-a,a]^2 \backslash]-b,b[^2$ for some $a > b$. A **left-right** crossing of a rectangle $[a,b] \times [c,d]$ above (resp. below) level $-p$ is a continuous black (resp. white) path inside this rectangle, connecting $[a,b] \times \{c\}$ and $[a,b] \times \{d\}$. A **circuit** in an annulus $x + [-a,a]^2 \backslash]-b,b[^2$ above (resp. below) level $-p$ is a black (resp. white) jordan curve $\gamma : S^1 \to x + [-a,a]^2 \backslash]-b,b[^2$ that separates $x +]-b,b[^2$ from infinity. Finally, in addition to these kinds of events, our result also applies to **component counting events** at level $-p$ inside the $\mathcal{E}_i$, which are events measurable with respect to the number of connected components of $\mathcal{N}_p$ contained inside $\mathcal{E}_i$. We will use the following notation:

$$\mathcal{K}_1 = \cup_{i=1}^{k_1} \mathcal{E}_i;\ \mathcal{K}_2 = \cup_{j=k_1+1}^{k_1+k_2} \mathcal{E}_j;\ \mathcal{C}_1 = \cup_{i=1}^{k_1} \partial\mathcal{E}_i;\ \mathcal{C}_1 = \cup_{j=1}^{k_1} \partial\mathcal{E}_j\,.$$

We prove the following theorem:

Theorem 1.3.11 (Theorem 1.12 of [RV17a] or Theorem 6.1.12)**.** *Assume that f is stationary with covariance κ of class C^8 and that for each pairwise distinct $x_1, \ldots, x_k \in \mathbb{R}^2$, the vector $(f(x_1), \ldots, f(x_k))$ is non-degenerate. Assume that f is normalized so that $\kappa(0) = 1$. There exist $C = C(\kappa) < +\infty$ and $d = d(\kappa) < +\infty$ such that the following holds. Fix $p \in \mathbb{R}$. Let $\eta = \sup_{x \in \mathcal{K}_1,\ y \in \mathcal{K}_2} |\kappa(x-y)|$. Let A (resp. B) be an event generated by crossing and circuit events above or below level $-p$ as well as component counting events at level $-p$ in the $\mathcal{E}_i$ for $i \in \{1, \ldots, k_1\}$ (resp. $i \in \{k_1+1, \ldots, k_1+k_2\}$). If $\mathcal{K}_1$ and $\mathcal{K}_2$ are at distance at least d, then:*

$$|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]| \leq \frac{C\eta}{\sqrt{1-\eta^2}}(1+|p|)^4 e^{-p^2} \prod_{i=1}^{2} \left(\mathrm{Area}(\mathcal{K}_i) + \mathrm{Length}(\mathcal{C}_i) + k_i\right)\,.$$

The proof of this result relies on a general quasi-independence formula for finite dimensional Gaussian vectors originally established by Piterbarg (in [Pit82a]) and inspired by the Slepian inequality (see [Sle62]). In [RV17a], we formulate Piterbarg's formula in a

new way. To explain this formulation, we will start with the following definition, which should be reminiscent of the definition of pivotal sites introduced in Subsection 1.3.3 for Bernoulli percolation:

Definition 1.3.12. Let $X = (X_i)_{1 \le i \le N}$ be a Gaussian vector. Fix $p \in \mathbb{R}$. Let $A \subset \mathbb{R}^N$ be an event measurable with respect to the signs of $X_i + p$ for $i \in \{1, \dots, N\}$. Let $x \in \mathbb{R}^N$ and $i \in \{1, \dots, N\}$. We say that i is **pivotal** for A at x if, by changing only the i-th coordinate of x, one can find points $y, z \in \mathbb{R}^N$ such that $y \in A$ and $z \notin A$. We denote by $\mathrm{Piv}_i(A)$ the set of $x \in \mathbb{R}^N$ at which i is pivotal for A.

Let $k, l \in \mathbb{N}$ and consider a centered Gaussian vector $(X, Y) \in \mathbb{R}^k \times \mathbb{R}^l$ with covariance $\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$. Let $\tilde{Y}$ be an independent copy of Y. Also, fix $p \in \mathbb{R}$ and consider $A \subset \mathbb{R}^k$ belong to the boolean algebra generated by the half-spaces $\{x_i + p \ge 0\}$ for $i \in \{1, \dots, k\}$ and $B \subset \mathbb{R}^l$ be generated by the halfspaces $\{y_j + p \ge 0\}$ for $j \in \{1, \dots, l\}$. Piterbarg's quasi-independence formula is a comparison inequality between $\mathbb{P}[X \in A \cap Y \in B]$ and $\mathbb{P}[X \in A]\mathbb{P}[Y \in B]$. It follows from an interpolation argument. More precisely, for each $t \in [0, 1]$, let $Z_t = (X_t, Y_t) = (X, \sqrt{t}Y + \sqrt{1-t}\tilde{Y})$. Then, the covariance of Z_t is

$$\begin{pmatrix} \Sigma_{11} & t\Sigma_{12} \\ t\Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

In particular, $\mathbb{P}[Z_1 \in A \times B] = \mathbb{P}[X \in A,\ Y \in B]$ while

$$\mathbb{P}[Z_0 \in A \times B] = \mathbb{P}[X \in A,\ \tilde{Y} \in B] = \mathbb{P}[X \in A]\mathbb{P}[Y \in B]\,.$$

In other words, $\mathbb{P}[Z_t \in A \times B]$ interpolates between the two quantities we want to compare. Let $\eta = \sup_{ij} |\Sigma_{12}(ij)|$. The quasi-independence formula is the following:

Proposition 1.3.13. *There exists $C < +\infty$ such that*

$$\begin{aligned} &|\mathbb{P}[Z_1 \in A \times B] - \mathbb{P}[Z_0 \in A \times B]| \\ &\le \frac{\eta}{2\pi\sqrt{1-\eta^2}} e^{-p^2} \sum_{i=1}^{k} \sum_{j=1}^{l} \int_0^1 \mathbb{P}\left[X_t \in \mathrm{Piv}_i(A),\ Y_t \in \mathrm{Piv}_j(B) \mid X_{t,i} = Y_{t,j} = -p\right] dt\,. \end{aligned}$$

In particular, if $\eta < 1/2$, there exists $C < +\infty$ such that the right hand side is no greater than

$$C\eta \times k \times l \times \sup_{i,j,t} \mathbb{P}\left[X_t \in \mathrm{Piv}_i(A),\ Y_t \in \mathrm{Piv}_j(B) \mid X_{t,i} = Y_{t,j} = -p\right]. \tag{1.3.10}$$

Proof sketch for Theorem 1.3.11:
Let us sketch the proof of Theorem 1.3.11 starting from Proposition 1.3.13. To simplify the presentation, we will assume that $k_1 = k_2 = 1$ that $\mathcal{E}_1$ and $\mathcal{E}_2$ are rectangles of area $\asymp R^2$ for some $R \gg 1$ and that A and B are crossing events. We will focus on the case $p = 0$. Let us discretize both rectangles at scale $\varepsilon > 0$ and obtain $\mathcal{E}_1^\varepsilon$ and $\mathcal{E}_2^\varepsilon$ finite

ε-nets covering the two rectangles. Then, $k, l \asymp (R/\varepsilon)^2$. Let A^ε and B^ε are discrete approximations of the events A and B that depend only on the signs of $X = f_{|\mathcal{E}_1^\varepsilon}$ and $Y = f_{|\mathcal{E}_2^\varepsilon}$ respectively. Then, bounding the probabilities in(1.3.10) by 1, we get

$$|\mathbb{P}[X \in A^\varepsilon,\ Y \in B^\varepsilon] - \mathbb{P}[X \in A^\varepsilon]\mathbb{P}[Y \in B^\varepsilon]| \leq C\eta R^4 \varepsilon^{-4}$$

which is the result by Piterbarg mentioned in Subsection 1.3.5. The key novelty is that by interpreting the pivotal events geometrically, we are able to control the probabilites to cancel the ε^{-4} term. To explain this interpretation, We will fix $i \in \mathcal{E}_1^\varepsilon$ and $j \in \mathcal{E}_2^\varepsilon$ and study

$$\sup_t \mathbb{P}\left[X_t \in \mathrm{Piv}_i(A^\varepsilon),\ Y_t \in \mathrm{Piv}_j(B^\varepsilon) \mid X_{t,i} = Y_{t,j} = 0\right].$$

We will take the liberty of assuming that i (resp. j) lies in the interior of $\mathcal{E}_1^\varepsilon$ (resp. $\mathcal{E}_2^\varepsilon$). For the boundary case the proof is analogous.

Now, A^ε is a discretized crossing event. In particular, it depends only on the topology of the discretized nodal set. Therefore, $x \in \mathrm{Piv}_i(A^\varepsilon)$ implies[12] that the discrete field x has a "discrete saddle point" at i. Since our field X is a discretization of f, we interpret $X_t \in \mathrm{Piv}_i(A)$ as implying that f has an ε-saddle point near i. On the other hand, the probability is taken conditioned on $f(i) = 0$. Given $z \in \mathbb{R}^2$ and $\varepsilon > 0$, the Kac-Rice formula shows that

$$\mathbb{E}[\mathrm{Card}\{w \in D(z,\varepsilon)\ :\ d_w f = 0\} |\ f(z) = 0] \asymp \varepsilon^2 .$$

We can of course play the same game with intersections of pivotal events in two disjoint boxes: these imply ε-saddle points at two fixed points in different boxes. In particular,

$$\sup_t \mathbb{P}\left[X_t \in \mathrm{Piv}_i(A^\varepsilon),\ Y_t \in \mathrm{Piv}_j(B^\varepsilon) \mid X_{t,i} = Y_{t,j} = 0\right] = O(\varepsilon^4) .$$

Since this is a first moment bound, no quasi-independence estimates are necessary. Plugging this estimate into (1.3.10), we get, for each $\varepsilon > 0$,

$$|\mathbb{P}[X \in A^\varepsilon,\ Y \in B^\varepsilon] - \mathbb{P}[X \in A^\varepsilon]\mathbb{P}[Y \in B^\varepsilon]| \leq C \times \eta \times (R/\varepsilon)^4 \times \varepsilon^2 \times \varepsilon^4 = C\eta R^4 .$$

Notice that while the left-hand side depends on ε, the right-hand side does not. Theorem 1.3.11 follows by letting $\varepsilon \to 0$.

We applied Theorem 1.3.11 to two problems: First we use the decorrelation estimate for crossing events to improve the Russo-Seymour-Welsh theorem (Theorem 1.3.9) for Gaussian fields obtained by Gayet and Beffara in [BG17a] (see also [BM18]).

Theorem 1.3.14 (Theorem 1.1 of [RV17a] or Theorem 6.1.1)**.** *The conclusion of Theorem 1.3.9 holds for any $\alpha > 4$.*

[12]When studying crossing events, pivotality will actually imply the existence of an arm connecting the site i to a point at large distance, as in Subsection 1.3.3, but this is very specific to crossing events.

The second application is related to Nazarov and Sodin's result: Theorem 1.2.8 on the concentration of the number of connected components of the nodal set of a smooth stationary Gaussian field. In this theorem, they take a Gaussian field f on $\mathbb{R}^n$ and study the behavior of N_R, the number of connected components of $f^{-1}(0)$ contained $B(0,R)$ in the ball of radius $R>0$ centered at 0. They prove that under certain non-degeneracy and regularity assumptions on the field $R^{-n}N_R$ converges in L^1 and a.s. as $R\to+\infty$ to a positive constant $\nu\in\mathbb{R}$. However, their theorem says nothing concerning the speed of convergence. We apply Theorem 1.3.11 to a (partial) large deviation bound on N_R. We prove the following result:

Theorem 1.3.15 (Theorem 1.4 of [RV17a] or Theorem 6.1.2)**.** *Let f be a Gaussian field on $\mathbb{R}^2$ with covariance κ satisfying the assumptions of Theorem 1.2.8. Assume also that there exists $\alpha>4$ and $C<+\infty$ such that for each $x\in\mathbb{R}^2$, $|\kappa(x)|\leq C|x|^{-\alpha}$. Let $\nu=\lim_{R\to+\infty}R^{-2}\mathbb{E}[N_R]$. Then, for each $\varepsilon>0$ and $\delta\in]0,\alpha-4]$ there exists $C_0=C_0(\varepsilon,\delta,\kappa)<+\infty$ such that for each $R>0$,*

$$\mathbb{P}\left[N_R\leq(\nu-\varepsilon)R^2\right]\leq C_0R^{4-\alpha+\delta}\,.$$

Moreover, if for some $c>0$, for each $x\in\mathbb{R}^2$, $|\kappa(x)|\leq C\exp(-c|x|^2)$, then, for each $\varepsilon>0$, there exist $C_1=C_1(\kappa,\varepsilon)<+\infty$ and $c_1=c_1(\kappa,\varepsilon)>0$ such that for each $R>0$,

$$\mathbb{P}\left[N_R\leq(\nu-\varepsilon)R^2\right]\leq C_0\exp\left(-c_0R\right)\,.$$

Note that this is only a lower concentration result. General upper concentration seems much more difficult to obtain and might be actually much slower than the lower concentration. We highlight the case of the covariance decaying as $\exp(-c_1|x|^2)$ because it corresponds to the Bargmann-Fock field, which we will discuss in Subsection 1.3.7. Upper and lower concentration results do exist in some specific cases. See for instance [NS09] and [GW11].

1.3.7 A sharp threshold result for the Bargmann-Fock percolation †

The results of this Subsection correspond to those presented in Chapter 7. The author worked on these questions with Hugo Vanneuville (see [RV17b]).

In Subsection 1.3.1 we stated Kesten's theorem (Theorem 1.3.1) for Bernoulli percolation. This theorem says that for Bernoulli edge percolation on $\mathbb{Z}^2$, at parameter $p\leq1/2$ there is a.s. no unbounded open cluster, while at parameter $p>1/2$ there is a.s. a unique unbounded open cluster. This theorem was followed by a second one, according to which rectangle crossing probabilities converge to one exponentially fast in the scale parameter, as long as $p>1/2$ (see Theorem 1.3.2). In [RV17b], we aimed to prove an analogue of Theorems 1.3.1 and 1.3.2 for Gaussian field percolation. The prototypical field we had in mind was the Bargmann-Fock field. This is the smooth stationary Gaussian field on $\mathbb{R}^2$ whose spectral measure is proportional to the Gaussian:

$$\rho(\xi)=\frac{1}{2\pi}\exp\left(-\frac{1}{2}|\xi|^2\right)d\xi\,.$$

Thus, the covariance of the field is $\kappa(x) = \exp\left(-\frac{1}{2}|x|^2\right)$. Just as the random monochromatic wave introduced in Subsection 1.2.2 has a natural interpretation as a local limit for random spherical harmonics as well as band-limited functions on chaotic manifolds (see [CH15]), the Bargmann-Fock field has a natural interpretation as the local limit for random real homogeneous polynomials of high degree. We refer the reader to [BG17a] for more details. Moreover, the Bargmann-Fock field has two very useful properties:

- Its covariance κ decays very fast. In particular, by Theorem 1.3.11, crossings of R-scale rectangles at distance $\sqrt{3\log(R)}$, are asymptotically independent as $R \to +\infty$ (though the decay is so fast that one could instead use the previous estimates described in Subsection 1.3.5 and obtain comparable results).
- For each $x \in \mathbb{R}^2$, we have $\kappa(x) \geq 0$ so by Theorem 1.3.7, it satisfies the FKG inequality, at least for increasing events depending on a finite number of sites, or those that can be approximated by increasing events depending on a finite number of sites.

These two properties imply in particular that Theorem 1.3.9 holds from the Bargmann-Fock field. Using this fact as well as the properties themselves, we obtained the two following theorems. The first is an analog of Theorem 1.3.2.

Theorem 1.3.16 (Theorem 1.8 of [RV17b] or Theorem 7.1.8)**.** *Let f be the Bargmann-Fock field, let $p \in \mathbb{R}$ and consider the random coloring of the plane defined by the excursion sets $\mathcal{D}_p$. For each $p > 0$ and $\rho \in]0, +\infty[$ there exists $c = c(p, \rho) > 0$ such that for each $R > 0$ the probability that there is a lenghtwise black crossing of the rectangle $[0, \rho R] \times [0, R]$ is at least $1 - e^{-cR}$.*

Theorem 1.3.1 had a partial analog for Gaussian fields, due to Alexander (see Theorem 1.3.8). We obtained the missing half of this analog for the Bargmann-Fock field:

Theorem 1.3.17 (Theorem 2.2 of [Ale96], Theorem 1.3 of [RV17b] or Theorem 7.1.3)**.** *Let f be the Bargmann-Fock field, let $p \in \mathbb{R}$ and consider the random coloring of the plane defined by the excursion sets $\mathcal{D}_p$. Then, if $p \leq 0$, a.s. there is no unbounded black component. On the other hand, if $p > 0$, a.s. there is an unbounded black component.*

While our approach was fairly general, there was one crucial point where we used more specific properties of the field. As a result, most of the proof works in a very general setting but the final result is stated for the Bargmann-Fock field.

For the proof of Theorems 1.3.16 and 1.3.17, we followed the strategy outlined in Subsection 1.3.3 for the analogous theorems in Bernoulli percolation. At each step, we had to develop analogous tools for Gaussian field percolation. To begin with, since we still have FKG and since the Bargmann-Fock model is rotation invariant (indeed, κ is radial), both rectangle gluing constructions (Figures 1.7 and 1.8) still work and show that Theorem 1.3.16 implies Theorem 1.3.17. Therefore, we may focus on the proof of Theorem 1.3.16.

Proof sketch for Theorem 1.3.16:
In Subsection 1.3.3, the next step was to establish the recursive formula (1.3.3) to bootstrap crossing estimates. In this step of the proof, independence is crucial so Theorem 1.3.11 will help for the adaptation. But for Theorem 1.3.11 to work, in Figure 1.9 we need to separate the left and right halfes of the picture a little bit. This means that the size of the rectangles at each step cannot be exactly 2^k. They have to be somewhat larger, but not too much if we want to get actual exponential decay. We fix $r_0 > 0$ and define the sequence of scales recursively as

$$r_{k+1} = 2r_k + \sqrt{r_k}\,.$$

Here, the fast decay of κ gives us some leeway so the choice $\sqrt{r_k}$ is somewhat arbitrary. In any case, this recursive relation implies first that $r_k \geq 2^k r_0$, which in turn implies that $r_{k+1} \leq (2 + 2^{-k/2} r_0^{-1/2}) r_k$, from which one can prove that $r_k \leq C2^k$ for some $C = C(r_0) < +\infty$. All in all, this definition implies that for each $k \in \mathbb{N}$,

$$2^k r_0 \leq r_k \leq C 2^k r_0\,. \tag{1.3.11}$$

We now consider the following assembly rectangles (see Figure 1.12):

- On the left, there are seven r_k-scale rectangles: $[-r_k - \sqrt{r_k}/2, -\sqrt{r_k}/2] \times [-2r_k, 0]$, $[-r_k - \sqrt{r_k}/2, -\sqrt{r_k}/2] \times [-r_k, r_k]$, $[-r_k - \sqrt{r_k}/2, -\sqrt{r_k}/2] \times [0, 2r_k]$, $[-r_k - \sqrt{r_k}/2, -\sqrt{r_k}/2] \times [1, 3r_k]$, $[-2r_k - \sqrt{r_k}/2, -\sqrt{r_k}/2] \times [-r_k, 0]$, $[-2r_k - \sqrt{r_k}/2, -\sqrt{r_k}/2] \times [0, r_k]$ and $[-2r_k - \sqrt{r_k}/2, -\sqrt{r_k}/2] \times [r_k, 2r_k]$.
- There are seven other rectangles on the right, symmetric to the seven rectangles on the left by the reflection along the $\{0\} \times \mathbb{R}$ axis.
- There is one large rectangle $[-r_{k+1}/2, r_{k+1}/2] \times [-(3/2)r_k, -(3/2)r_k + 2r_{k+1}]$

By Theorem 1.3.11, crossings of rectangles on the left half are independent from crossings of rectangles on the right half up to an error of order $O(e^{-r_k/3})$. Let a_k denote the probability of the absence of a lengthwise crossing of a $r_k \times 2r_k$ rectangle. Reasoning as in Subsection 1.3.3, one establishes the recurrence relation

$$\forall k \in \mathbb{N},\ a_{k+1} \leq 49 a_k^2 + \exp\left(-r_k/3\right)\,. \tag{1.3.12}$$

By considering $b_k = \max\{a_k, \exp\left(-r_k/6\right)\}$ it is easy to deduce from Equations (1.3.11) and (1.3.12) that, if a_0 is small enough, then there will exist $C_0 = C_0(a_0, r_0) < +\infty$ and $c_0 = c_0(a_0, r_0) > 0$ such that for each $k \in \mathbb{N}$,

$$a_k \leq C_0 \exp\left(-c_0 r_k\right)\,.$$

Since the sequence $(r_k)_k$ grows at most geometrically, this relation quickly leads to the conclusion of Theorem 1.3.16. Thus, we have reduced the proof to finding an initial scale $r_0 > 0$ so that the probability of a lengthwise black crossing of a $2r_0 \times r_0$ rectangle is very close to 1. Let $\mathrm{Cross}^p(R)$ be the event that there is a lengthwise black crossing of

the rectangle $[0, 2R] \times [0, R]$. Just as for Bernoulli percolation, it is now enough to prove the following relation:

$$\forall p > 0, \ \limsup_{R \to +\infty} \mathbb{P}\left[\text{Cross}^p(R)\right] = 1 \,. \tag{1.3.13}$$

In order to follow the same strategy as for Bernoulli percolation, we look some kind of analog of Russo's formula (1.3.6). In particular, we need to define an adequate notion of influence. We start by restricting the field to a fine grid $\varepsilon\mathcal{T}$ and considering a discretized version[13] of $\mathcal{D}_p$. This is helpful because discrete crossing events depend only on a finite number of values of the field. We will denote by $X^\varepsilon = (X_1^\varepsilon, \dots, X_N^\varepsilon)$ the field f restricted to the sites of the fine mesh inside the rectangle whose crossings we are studying. Recall that Russo's formula involved influences I_i^q of coordinates i on boolean functions on the discrete hypercube at some level $q \in [0, 1]$ (see (1.3.2)). For any $A \subset \{-1, +1\}^N$ and each coordinate $i \in \{1, \dots, N\}$, the influence $I_i^p(A)$ can be seen as the size of the projection of the boundary of A along the coordinate i. The natural analogy for this interpretation goes as follows. Take $U \subset \mathbb{R}^N$ with piecewise smooth boundary ∂U, and let ν be the outward normal unit vector to U defined on ∂U except for a set of zero $N-1$ dimensional Hausdorff measure. Then, for each $i \in \{1, \dots, N\}$, **the influence of i on U** is

$$J_i(U) := \int_{\partial U} |\nu_i(x)| d\mathcal{H}^{N-1}(x) \,. \tag{1.3.14}$$

Here $d\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. We stress that this is a notion of influence different from the one one the hypercube. Given $A \subset \{-1, +1\}^N$ an increasing subset and $p \in \mathbb{R}$, we denote by A_p^* the event that $(\text{sgn}(X_i^\varepsilon + p))_i \in A$ and set $J_i^p(A) := J_i(A_p^*)$. We prove the following formula, valid for all $p \in \mathbb{R}$:

$$\frac{d}{dp}\mathbb{P}[A_p^*] = \sum_{i=1}^N J_i^p(A) \,. \tag{1.3.15}$$

This formula is actually valid for any non-degenerate Gaussian probability measure on $\mathbb{R}^N$.

Equation (1.3.15) seems like a satisfactory analogy for (1.3.6), but how far does this analogy go? Is there a corresponding Theorem 1.3.5 for Gaussian vectors? In [KMS12], Keller, Mossel and Sen prove the following theorem:

Theorem 1.3.18 ([KMS12])**.** *There exists a constant $c > 0$ such that for any $N \in \mathbb{N}$, the following holds. Let X be a centered Gaussian vector in $\mathbb{R}^N$ with covariance Id_N. Then, for any event $U \subset \mathbb{R}^N$ which is either monotonic or semi-algebraic[14],*

$$\sum_{i=1}^N \frac{J_i(U)}{\sqrt{-\log(J_i(U))}} \geq c \times \mathbb{P}[U] \times \mathbb{P}\left[\neg U\right] \,.$$

[13] More precisely, we work with site percolation on the face centered square lattice where each site is colored in black if and only if if belongs to $\mathcal{D}_p$, see [RV17b] for more details.

[14] This means that it belongs to the Boolean algebra generated by sets of the form $P^{-1}([a, +\infty[)$ for some $a \in \mathbb{R}$ and $P \in \mathbb{R}[T_1, \dots, T_N]$.

In [KMS12], the assumptions on U are weaker and the influences are defined in a different way but we prove that, for monotonic or semi-algebraic sets U, the definition given in [KMS12] coincides with(1.3.14). A more serious problem is that in the present context, the covariance Gaussian vector X^ε is very different from the identity. Indeed, since f is a.s. smooth and X^ε was obtained by discretizing f at scale ε, one can expect two neighboring values X_i^ε and X_j^ε to have covariance $\mathbb{E}[X_i^\varepsilon X_j^\varepsilon] = 1 - O(\varepsilon^2)$. Moreover, since κ never vanishes, neither does its discretization K_ε, which is the covariance of X. To solve this issue, we adapt Theorem 1.3.18 to general centered Gaussian vectors:

Theorem 1.3.19 (Theorem 2.19 of [RV17a]). *There exists a constant $c > 0$ such that for any $N \in \mathbb{N}$, the following holds. Let X be a centered Gaussian vector in $\mathbb{R}^N$ with non-degenerate covariance K. Let $\sqrt{K}$ be the symmetric matrix square root of K. Then, for any event $U \subset \mathbb{R}^N$ which is either monotonic or semi-algebraic,*

$$\sum_{i=1}^{N} J_i(U) \geq c\|\sqrt{K}\|_{\infty,op}^{-1} \times \mathbb{P}[U] \times \mathbb{P}[\neg U] \times \max_i \sqrt{-\log\left(\|\sqrt{K}\|_{\infty,op} J_i(U)\right)}$$

Here $\|\cdot\|_{\infty,op}$ is the operator norm associated to the l^∞ norm on $\mathbb{R}^N$.

The drawback that appears from correlations takes the form of $\|\sqrt{K}\|_{\infty,op}$. More precisely, we want to avoid the possibility that this norm is too big. This is the point where the argument is specific to the Bargmann-Fock model. Let K^ε be the restriction of the covariance $(x, y) \mapsto \kappa(x - y)$ of the Bargmann-Fock field to a fine grid $\varepsilon\mathcal{T}$. Then we prove that

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} = O\left(\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)\right). \tag{1.3.16}$$

We postpone the explanation of this estimate till the end of the present discussion. The final ingredient in the proof of Theorem 1.3.1 is to compare the influences of the crossing event to arm event probabilities (see Equation 1.3.8), which we know decay polynomially. To do this, we have to reinterpret the geometric influences (1.3.14) as pivotal events of some kind. We actually relate our new influences to the pivotal events from Definition 1.3.12. More precisely, we prove that for any $A \subset \{-1, +1\}^N$ and $p \in \mathbb{R}$,

$$J_i^p(A) = \mathbb{P}\left[X^\varepsilon \in \mathrm{Piv}_i(A) \mid X_i^\varepsilon = -p\right] \frac{e^{-\frac{1}{2\,\mathrm{Var}(X_i^\varepsilon)}p^2}}{\sqrt{2\pi\,\mathrm{Var}(X_i^\varepsilon)}}. \tag{1.3.17}$$

Here the conditioning is present because for the coordinate i to influence A, we must have $X_i^\varepsilon = -p$. Let $R > 0$, number the sites of $\varepsilon\mathcal{T}$ contained inside of $[0, 2R] \times [0, R]$ from 1 to N and let $A = \mathrm{Cross}_\varepsilon(R) \subset \{-1, +1\}^N$ be the event that if we color the $+1$ sites in black and the -1 sites in white, there is a black lengthwise crossing of the rectangle. For each $p \in \mathbb{R}$, $\mathrm{Cross}_\varepsilon^p(R) := A_p^*$ is then the discrete analog of $\mathrm{Cross}^p(R)$. Then, Equation

(1.3.15), Theorem 1.3.19, Equation (1.3.16) and Equation (1.3.17) strongly suggest that

$$\frac{d}{dp}\mathbb{P}[\mathrm{Cross}_\varepsilon^p(R)] \geq$$
$$\frac{c\varepsilon}{|\log(\varepsilon)|}\mathbb{P}[\mathrm{Cross}_\varepsilon^p(R)]\mathbb{P}[\neg\mathrm{Cross}_\varepsilon^p(R)]\max_i \sqrt{\left|\log\left(\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)J_i^p(\mathrm{Cross}_\varepsilon(R))\right)\right|}. \quad (1.3.18)$$

The reason why Equation (1.3.18), does not follow directly from the results mentioned above is that the operator norm was computed on the whole grid $\varepsilon\mathcal{T}$ while here we are considering just the portion of the grid contained inside the rectangle. We go from one to the other by a Gaussian approximation argument.

In any case we will use (1.3.18) as follows. The hope is that either the derivative probability or the probability itself of $\mathrm{Cross}_\varepsilon^p(R)$ blows up as $R \to +\infty$ uniformly for $p \geq 0$. The parameter ε will shrink to 0 at a rate well chosen with respect to R that is fast enough for the discrete crossing to faithfully approximate a continuous crossing but slow enough so that the factor $\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)$ does not cause the right-hand side to converge to 0. The factor $\mathbb{P}[\neg\mathrm{Cross}_\varepsilon^p(R)]$ is at least $1/2$ because $p \geq 0$ and we are looking at a lengthwise rectangle crossing. Finally, using Equation (1.3.17), following the proof of Corollary 1.3.4, we show that there exist $C < +\infty$ and $\gamma > 0$ such that for all $p \geq 0$, $R > 0$ and $\varepsilon > 0$:

$$J_i^p(\mathrm{Cross}_\varepsilon(R)) \leq CR^{-\gamma}.$$

This reduces (1.3.18) to

$$\frac{1}{1-\mathbb{P}[\mathrm{Cross}_\varepsilon^p(R)]}\frac{d}{dp}\mathbb{P}[\mathrm{Cross}_\varepsilon^p(R)] \geq \frac{c\varepsilon}{2|\log(\varepsilon)|}\sqrt{|\log(\varepsilon|\log(\varepsilon)|R^{-\gamma})|}.$$

To make either $\mathbb{P}[\mathrm{Cross}_\varepsilon^p(R)]$ or its derivative blow up, we just have to choose $\varepsilon(R)$ large enough for the right-hand side to blow up. For instance, if we could choose $\varepsilon(R) = \log(R)^{1/2-\delta}$ for some $\delta > 0$ we would be done. On the other hand, $\varepsilon(R) = R^{-\delta}$ would make the estimate trivial. Here we face a final difficulty because the discretization schemes presented in [BG17a] and [BM18] need ε to decrease polynomially in R to work. Here, we take advantage of the fact that we are working in the supercritical regime: We compare continuous crossings above level $-p$ to discrete crossings above level $-p/2$. In particular, we show that, for $\varepsilon = \varepsilon(R) = \log(R)^{1/4+\delta}$ for some $\delta > 0$, as $R \to +\infty$,

$$\mathbb{P}[\mathrm{Cross}_p(R)] \geq \mathbb{P}[\mathrm{Cross}_{p/2}^\varepsilon(R)] + o(1).$$

This shows in particular that, for each $p > 0$, there exists $c = c(p) > 0$ such that for each $R \geq 1$,

$$\mathbb{P}[\mathrm{Cross}_p(R)] \geq 1 - \exp(-c\log^{1/7}(R))$$

which implies Equation (1.3.13).

We finish off with a discussion of (1.3.16). First off, notice that K^ε is a symmetric positive definite map $\mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ invariant by translations. That is, there exists an even function $\kappa^\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$ such that $K^\varepsilon(n,m) = \kappa^\varepsilon(n-m)$ for all $n, m \in \mathbb{Z}^2$. To compute $\sqrt{K^\varepsilon}$ it is enough to find a symmetric map $\eta^\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$ solving the equation

$$\eta^\varepsilon * \eta^\varepsilon = \kappa^\varepsilon \,. \tag{1.3.19}$$

To solve this equation, we consider functions $\check{\eta^\varepsilon}$ and $\check{\kappa^\varepsilon}$ on $\varepsilon^{-1}\mathbb{T}^2$ whose Fourier series are η^ε and κ^ε respectively. Then, (1.3.19) becomes

$$\check{\eta^\varepsilon}^2 = \check{\kappa^\varepsilon} \,.$$

To find an amenable expression for the right-hand side, one can use Poisson's inversion formula and the fact that the Gaussian function is invariant by the Fourier transform. Rather than detailing this calculation to extract estimate (1.3.16), let us consider the case where K^ε is an infinite direct product of square matrices of size $\varepsilon^{-2} \times \varepsilon^{-2}$ (assuming that $\varepsilon^{-2} \in \mathbb{N}$) whose coefficients are all equal to 1. This models the fact that on an ε-grid, each point is at distance $O(1)$ to $\asymp \varepsilon^{-2}$ other points and that since the Gaussian decays very fast, sites at larger distances are almost independent from each other. Let M_ε be one such matrix. Then, it is easy to see that $M_\varepsilon^2 = \varepsilon^{-2} M_\varepsilon$ so that $\sqrt{K^\varepsilon} = \varepsilon K^\varepsilon$. Now, each row of K^ε has ε^{-2} terms so that $\|K^\varepsilon\|_{\infty,op} = \varepsilon^{-2}$. Consequently, $\|\sqrt{K^\varepsilon}\|_{\infty,op} = \varepsilon^{-1}$, thus yielding the correct polynomial order of growth in (1.3.16).

1.3.8 Further directions

Theorems 1.3.11 and 1.3.17 leave many questions unanswered and leads to other questions in percolation of Gaussian field. Here we compile a few questions related to the results discussed in Section 1.3.

- Theorem 1.3.11 strongly suggests that we should depart from discretization techniques and work directly in the continuum. Moreover, while the techniques worked only in the planar case, there is no good reason for the result to fail in higher dimension, or even for other more complex topological events. A promising strategy seems to be to work with fields supported in finite-dimensional function spaces $V \subset C^\infty(\mathbb{R}^n)$ and study the boundary ∂A of topological events $A \subset V$.

- Theorem 1.3.17 should hold for smooth fields with sufficient regularity, non-degeneracy and covariance decay as long as the covariance is positive. Indeed, in the very recent [MV18], Muirhead and Vanneuville provide a broad generalization of the result. However, firstly, they impose conditions on the convolution square root, which we believe are not optimal and which are harder to check in practice. Secondly, their proof follows quite a different route, that of randomized algorithms. The general strategy involving functional inequalities such as Theorem [BKK$^+$92] could perhaps, if applied more naturally than with Theorem 1.3.19, cover the remaining cases. Indeed, when we use this last theorem, we forget local correlations of the

field. This occurs first because of the discretization procedure, and second because it is an $L^\infty - L^1$ inequality. The natural setting for Gaussian fields would be to work in the continuum with L^2 inequalities such as Talagrand's inequality (see Corollary 3 of [CEL12]).

- Bernoulli percolation can of course be defined on other graphs than $\mathbb{Z}^2$ or planar lattices. It is well known that percolation in $\mathbb{Z}^3$ exhibits a phase transition at a parameter p_c that is strictly lower than $p_c(\mathbb{Z}^2) = 1/2$. In other words, percolation is easier in dimension three than in dimension two. A natural follow-up question to Theorem 1.3.17 would be to show that if f is the Bargmann-Fock field on $\mathbb{R}^3$ (defined in exactly the same way as for $\mathbb{R}^2$) for $p < 0$ close enough to 0, $\mathcal{D}_p$ has a.s. a unique unbounded connected component.

- In [CN07] using an argument by Schramm and Smirnov, Camia and Newman prove that Bernoulli percolation interfaces on the triangular lattice converge to SLE(6). If one believes that nodal lines of smooth Gaussian fields look like percolation interface, it seems natural to wonder whether they also behave in this way at large scales. This question has been studied numerically in [BDS07] (see also [BG17a]).

- The Bogomolny and Schmit conjecture (Conjecture 1.3.10 from [BS07]) says that the nodal lines of monochromatic random waves f_{MRW} should behave like percolation. So far, no percolation results are general enough to apply to this model. We do not even know whether or not there exists $p \in \mathbb{R}$ for which the super level sets $\{f_{MRW} \geq -p\}$ contain an unbounded connected component. Any information of this kind would be quite significant.

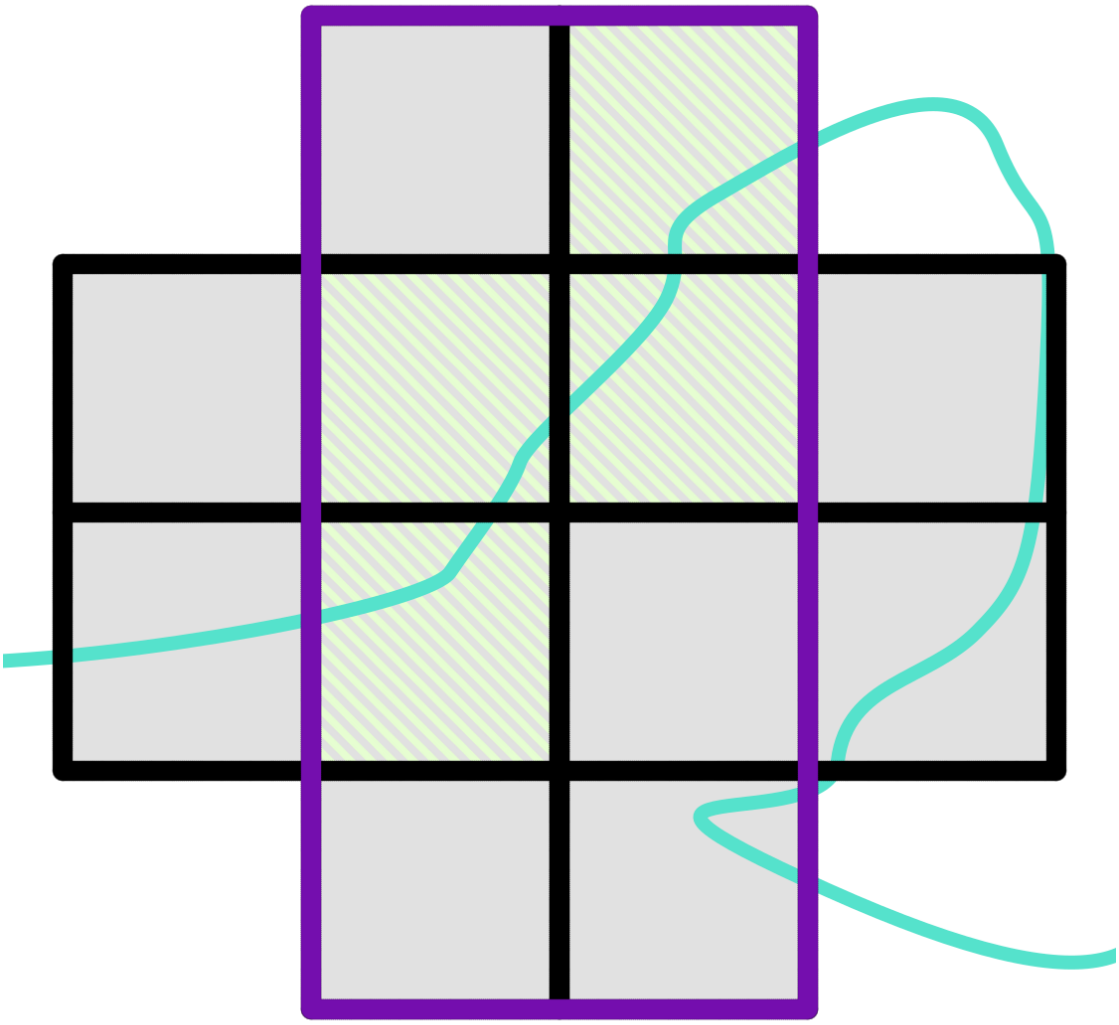


Figure 1.9: In both the left and right half of this assembly of rectangles, there are five small two by one rectangles (three vertical ones ant two horizontal ones). Three vertical ones with two sides on a common line and two horizontal ones. There is also a large vertical rectangle in purple at the center of the picture. The turquoise path, which crosses the large purple rectangle from left to right, induces crossings of two small rectangles, dashed over in light green.

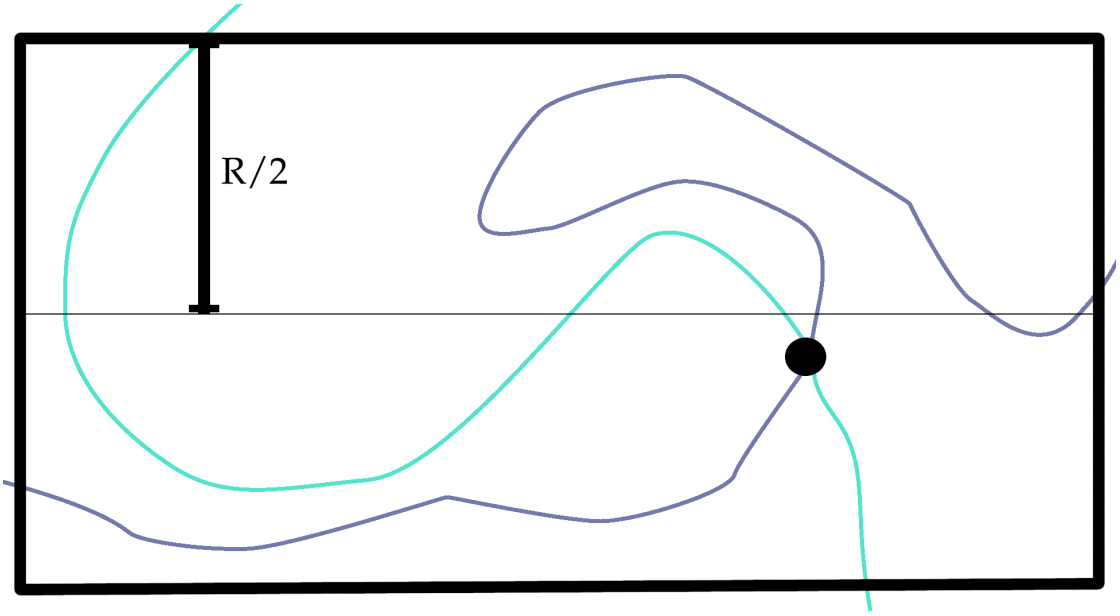


Figure 1.10: Here the black dot inside the rectangle hides a pivotal edge for the crossing event. We assume that it is in the lower half of the rectangle so it is at distance at least $R/2$ of the upper side of its border. The dark blue path represents a lengthwise crossing of the rectangle and the light turquoise path represents a dual crossing. The turquoise path connects the black dot to the top side of the rectangle, which is at distance at least $R/2$ of the black dot.

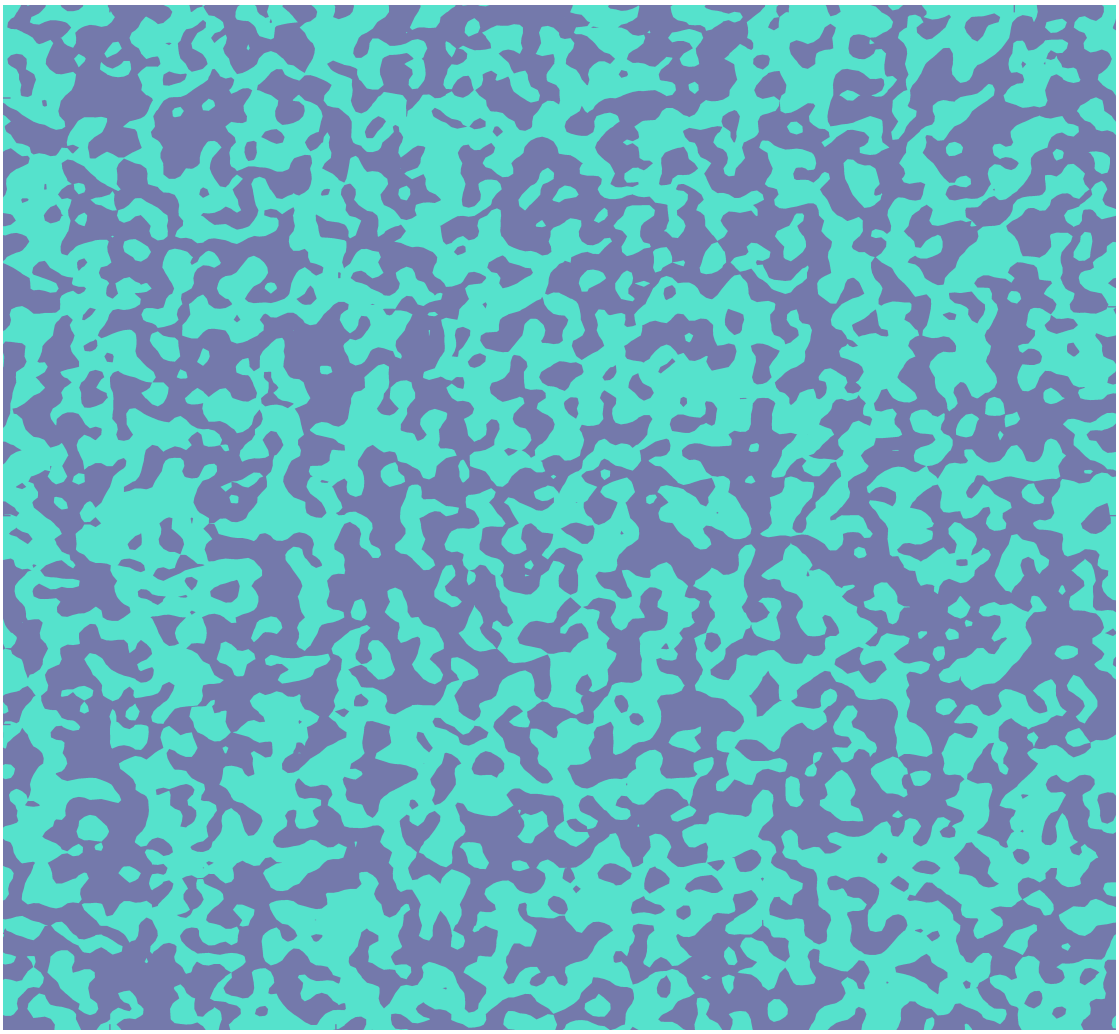


Figure 1.11: Here is an instance of $\mathcal{D}_0$ (in dark blue) where f is the Bargmann-Fock field (defined in Subsection 1.3.7).

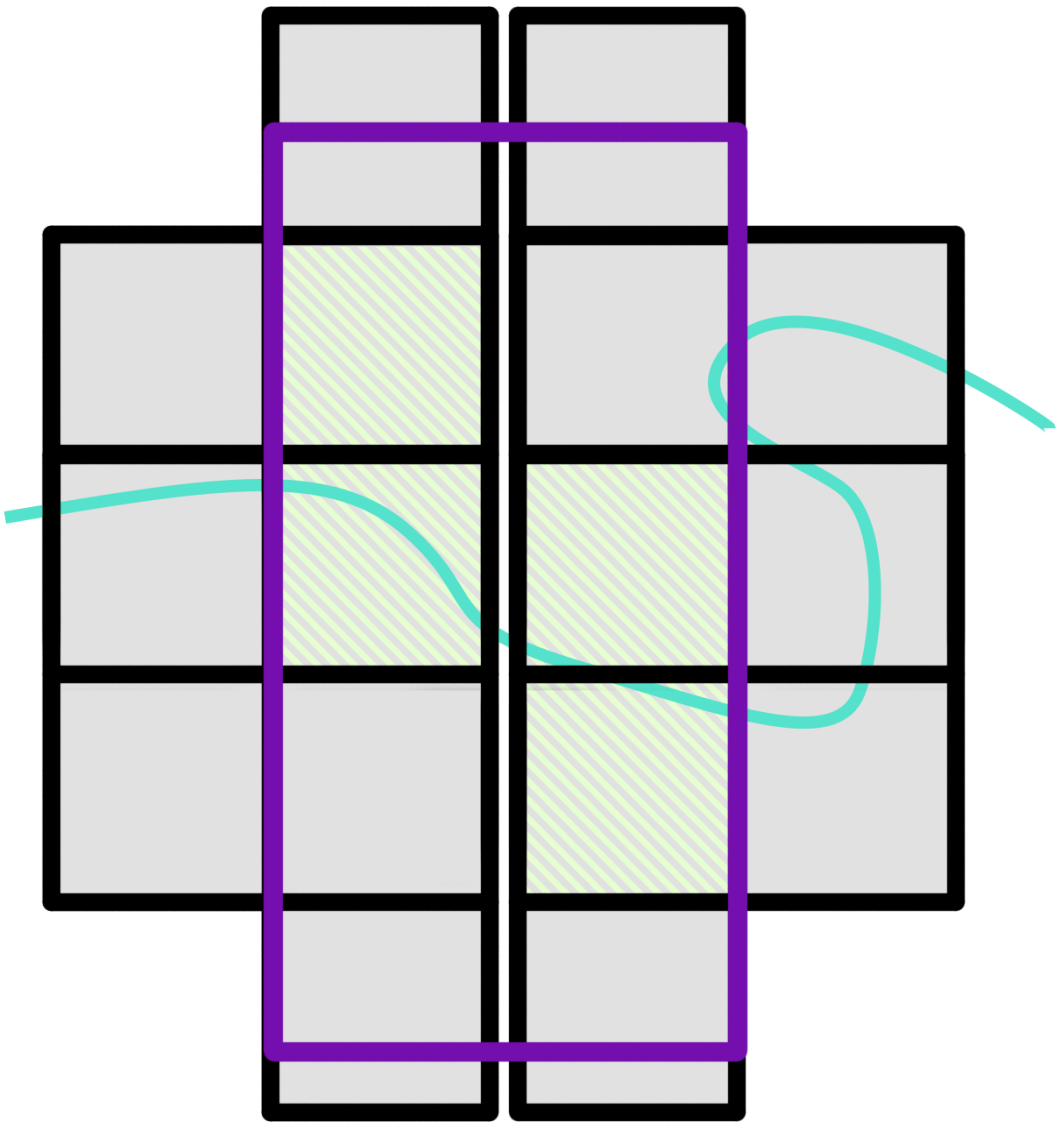


Figure 1.12: Here we superimpose a vertical $r_{k+1} \times 2r_{k+1}$ purple rectangle and fourteen $r_k \times 2r_k$ rectangles. More precisely, each half of the diagram contains eight grey squares, three on the first column starting from the left and five on the second column. Each pair of adjacent squares on the second column forms one of the rectangles. Each pair of squares on the same row also forms a rectangle. Adding to these the seven rectangles symmetric to these on the right half of the picture, we obtain the fourteen small rectangles. The turquoise path, as it crosses the large rectangle from left to right, induces crossings of two smaller rectangles, which are dashed over in green.

Chapter 2

Weighted local Weyl laws for elliptic operators

Abstract

Let A be an elliptic pseudo-differential operator of order m on a closed manifold $\mathcal{X}$ of dimension $n > 0$, formally positive self-ajdoint with respect to some positive smooth density $d\mu_{\mathcal{X}}$. Then, the spectrum of A is made up of a sequence of eigenvalues $(\lambda_k)_{k\geq 1}$ whose corresponding eigenfunctions $(e_k)_{k\geq 1}$ are C^∞ smooth. Fix $s \in \mathbb{R}$ and define

$$K_L^s(x,y) = \sum_{0<\lambda_k\leq L} \lambda_k^{-s} e_k(x)\overline{e_k(y)}\,.$$

We derive asymptotic formulae near the diagonal for the kernels $K_L^s(x,y)$ when $L \to +\infty$ with fixed s. For $s = 0$, K_L^0 is the kernel of the spectral projector studied by Hörmander in [Hör68]. In the present work we build on Hörmander's result to study the kernels K_L^s. If $s < \frac{n}{m}$, K_L^s is of order $L^{-s+n/m}$ and near the diagonal, the rescaled leading term behaves like the Fourier transform of an explicit function of the symbol of A. If $s = \frac{n}{m}$, under some explicit generic condition on the principal symbol of A, which holds if A is a differential operator, the kernel has order $\ln(L)$ and the leading term has a logarithmic divergence smoothed at scale $L^{-1/m}$. Our results also hold for elliptic differential Dirichlet eigenvalue problems.

This chapter is based on [Riv18b].

2.1 Introduction

The purpose of the present work is to compute pointwise asymptotics of the integral kernels of certain operators defined by functional calculus from either elliptic self-adjoint pseudo-differential operators on a closed manifold or an elliptic self-adjoint Dirichlet boundary problem. Stating the results in full generality requires some vocabulary from semi-classical analysis and some additional definitions. For this reason, we start by stating our results in the simpler case of elliptic self-adjoint differential operators on a closed manifold. The general case is presented in Section 2.2. This, of course, leads to some redundancy between different statements which we accept for the sake of accessibility and transparency of the main results.

Let $\mathcal{X}$ be a smooth compact manifold without boundary, of positive dimension $n > 0$ and equipped with a smooth positive density $d\mu_{\mathcal{X}}$. Let A be a positive elliptic differential operator on $\mathcal{X}$ of positive order m. By this we mean that in any local coordinate system $x = (x_1, \dots, x_n)$ on $\mathcal{X}$ defined on $U \subset \mathbb{R}^n$, A, acts on $C_c^\infty(U)$ as

$$\sum_{0 \leq |\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

where $\alpha \in \mathbb{N}^n$ and $a_\alpha \in C^\infty(\mathbb{R}^n)$ and for each $\xi \in \mathbb{R}^n \setminus \{0\}$, we have

$$\sigma_A(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha > 0 \,.$$

The function σ_A is called the **principal symbol** of A in these coordinates. It is well known (and easy to check) that the principal symbol of A read in different coordinates pieces together as a smooth function on the complement of the zero section in $T^*\mathcal{X}$. We assume that A is symmetric with respect to the L^2-scalar product on $(\mathcal{X}, d\mu_{\mathcal{X}})$. Then one can show (see Subsection 2.2.1) that A has a unique self-adjoint extension whose spectrum is made up of a non-decreasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ of positive eigenvalues diverging to $+\infty$ with smooth L^2-normalized eigenfunctions $(e_k)_{k \in \mathbb{N}}$ forming a Hilbert basis for $L^2(\mathcal{X}, d\mu_{\mathcal{X}})$. For each $L \geq 0$, let Π_L be the L^2 orthogonal projector on the space spanned by the eigenfunctions e_k such that $\lambda_k \leq L$. Since this space is finite-dimensional, Π_L has a smooth integral kernel $E_L \in C^\infty(\mathcal{X} \times \mathcal{X})$. More explicitely,

$$\forall (x, y) \in \mathcal{X} \times \mathcal{X},\ E_L(x, y) = \sum_{\lambda_k \leq L} e_k(x) \overline{e_k(y)} \,.$$

In [Hör68], Hörmander studied the behavior of this kernel on a neighborhood of the diagonal as $L \to +\infty$. Integrating E_L over the diagonal he recovered the following estimate, also known as **Weyl's law**:

$$\mathrm{Card}\{k \in \mathbb{N} \mid \lambda_k \leq L\} \sim \frac{1}{(2\pi)^n} \int_{\mathcal{X}} \int_{\sigma_A(x,\xi) \leq 1} \widetilde{d_x \mu_{\mathcal{X}}}(\xi) d\mu_{\mathcal{X}}(x) \times L^{\frac{n}{m}}$$

where $\widetilde{d_x\mu_{\mathcal{X}}}(\xi)$ is the density induced on $T_x^*\mathcal{X}$ by $d\mu_{\mathcal{X}}$. Hörmander's result is stronger than the above estimates in two respects. First because the error term obtained is smaller than the ones known before and is sharp in all generality. Secondly, the result actually provides local information concerning the behavior of the kernel E_L near the diagonal, which is why is sometimes called the **local Weyl law**. We will state this theorem in Subsection 2.2.2 (see Theorem 2.2.2). In recent years, Hörmander's local Weyl law has received a lot of attention because E_L turns out to be the covariance of a certain Gaussian field on $\mathcal{X}$ defined as a random linear combination of eigenfunctions of A (see for instance [Bér85], [NS09], [Zel13], [GW16a], [Let16a], [GW16b], [NR15], [HZZ15], [NS16],[SW16] and [CS16]). In [Riv17] we studied a natural variation of this random linear combination of eigenfunctions in dimension $n = 2$ and observed a very different asymptotic behavior of the covariance function. Following this work, we are interested in studying more general random linear combinations of these eigenfunctions. To this end, it is essential to gather some information about the corresponding covariance function. The purpose of this article is to provide an asymptotic for these kernels similar to the one we have for E_L. For each $s \in \mathbb{R}$ we consider the kernel

$$K_L^s(x,y) = \sum_{\lambda_k \leq L} \lambda_k^{-s} e_k(x)\overline{e_k(y)}\,.$$

These kernels converge in distribution to the integral kernels of A^{-s} as $L \to +\infty$ but diverge on the diagonal for small or negative values of s. The pointwise behavior of the limiting kernel on the diagonal, which is well defined for large values of s, has been studied for instance in [See67] and [Sch86]. In [Sch86], the author proved that, as a function of s, the limit admitted a meromorphic extension to the whole complex plane. We focus instead on a fixed s for which the kernel diverges and study its pointwise divergence near the diagonal. We call these results **weighted local Weyl laws** by analogy with E_L (which is just K_L^0) because of the weights λ_n^{-s} on the terms of the sum defining K_L^s. As we shall see, the kernels K_L^s experience a sudden change in their asymptotic behavior between the phases $s < \frac{n}{m}$ and $s = \frac{n}{m}$. All our results will be local so we take the liberty of omitting with the composition with the chart when writing functions on $\mathcal{X}$ in local coordinates. Our first result provides information when $s < \frac{n}{m}$.

Theorem 2.1.1. *Assume that $s < \frac{n}{m}$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \ldots, x_n)$ for $\mathcal{X}$ centered at x_0 and defined on an open subset $U \subset \mathbb{R}^n$ such that $d\mu_{\mathcal{X}}$ agrees with the Lebesgue measure in these coordinates. Then for each $\alpha, \beta \in \mathbb{N}^n$, there exists $V \subset U$ an open neighborhood of 0 such that, in these coordinates, we have the following estimates.*

1. *Uniformly for $w, x, y \in V$ and $L \geq 1$ we have $w + L^{-1/m}x, w + L^{-1/m}y \in U$ and*

$$L^{s-(n+|\alpha|+|\beta|)/m}\partial_x^\alpha\partial_y^\beta K_L^s\left(w+L^{-1/m}x,w+L^{-1/m}y\right)=$$
$$\frac{1}{(2\pi)^n}\int_{\sigma_A(w,\xi)\leq 1}e^{i\langle\xi,x-y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{\sigma_A(w,\xi)^s}d\xi+O\left(L^{-1/m}\ln(L)^\eta\right)$$

where $\eta=1$ *if* $s=(n+|\alpha|+|\beta|-1)/m$ *and* 0 *otherwise.*

2. *Let* $\varepsilon>0$*. Then, uniformly for* $x,y\in V$ *such that* $|x-y|>\varepsilon$ *and* $L\geq 1$*,*

$$L^{s-(n+|\alpha|+|\beta|)/m}\partial_x^\alpha\partial_y^\beta K_L^s(x,y)=O\left(L^{-1/m}\ln(L)^\eta\right)$$

where $\eta=1$ *if* $s=(n+|\alpha|+|\beta|-1)/m$ *and* 0 *otherwise.*

Here $|\alpha|=\alpha_1+\cdots+\alpha_n$ *and* $\langle\cdot,\cdot\rangle$ *is the Euclidean scalar product.*

Note that the case where $s=0$ and $\alpha=\beta=0$ is Theorem 5.1 of [Hör68] (see Theorem 2.2.2 and the discussion below for more details about this case). We prove Theorem 2.1.1 at the end of Section 2.2. Before stating the second result, we introduce the following notation. Firstly, $d\mu_{\mathcal{X}}$ defines a canonical dual density $\widetilde{d\mu_{\mathcal{X}}}$ on $T^*\mathcal{X}$. Around any $x\in\mathcal{X}$, there exist local coordinates in which $d\mu_{\mathcal{X}}$ corresponds to the Lebesgue measure. Then $\widetilde{d\mu_{\mathcal{X}}}$ is the unique density on $T^*\mathcal{X}$ who in these coordinates corresponds to the Lebesgue measure. For each $x\in\mathcal{X}$, let $S_x^*=\{\xi\in T_x^*\mathcal{X}\,|\,\sigma_A(x,\xi)=1\}$. Since σ_A is m-homogeneous, S_x^* is a smooth compact hypersurface of $T_x^*\mathcal{X}$ strictly star-shaped[1] around the origin and there exists a smooth density $d_x\nu$ on S_x^* such that for each $u\in C_c^\infty(T_x^*\mathcal{X})$,

$$\int_{T_x^*\mathcal{X}}u(\xi)\widetilde{d_x\mu_{\mathcal{X}}}(\xi)=\int_0^{+\infty}\int_{S_x^*}u(t\xi)d_x\nu(\xi)t^{n-1}dt\,. \tag{2.1.1}$$

Our second result deals with the case where $s=\frac{n}{m}$. While Theorem 2.1.1 proves that the rate of growth of K_L^s does not depend on s for $s<n/m$, and that the main term depends continuously on s, the following result shows that this is not true for $s=n/m$. Indeed, while the first point is analogous to the results of Theorem 2.1.1 , the second point is quite different (and requires additional tools).

Theorem 2.1.2. *Assume that* $s=\frac{n}{m}$*. Fix* $x_0\in\mathcal{X}$ *and consider local coordinates* $x=(x_1,\ldots,x_n)$ *for* $\mathcal{X}$ *centered at* x_0 *and defined on an open subset* $U\subset\mathbb{R}^n$ *such that* $d\mu_{\mathcal{X}}$ *agrees with the Lebesgue measure in these coordinates. Then, for each* $\alpha,\beta\in\mathbb{N}^n$*, there exists an open neighborhood* $V\subset U$ *of* 0 *such that the following holds.*

1. • *Assume that* $(\alpha,\beta)\neq(0,0)$*. In these coordinates, uniformly for* $w,x,y\in V$ *and* $L\geq 1$*,* $w+L^{-1/m}x,w+L^{-1/m}y\in U$ *and*

[1] More precisely, for each $\xi\in T_x^*\mathcal{X}\setminus\{0\}$, the ray $\{t\xi\,|\,t>0\}$ intersects S^* exactly at $\sigma_A(x,\xi)^{-1/m}\xi$ and does so transversally.

$$L^{-(|\alpha|+|\beta|)/m}\partial_x^\alpha\partial_y^\beta K_L^s\left(w+L^{-1/m}x,w+L^{-1/m}y\right)=$$
$$\frac{1}{(2\pi)^n}\int_{\sigma_A(w,\xi)\leq 1}e^{i\langle\xi,x-y\rangle}\frac{(i\xi)^\alpha(-i\xi)^\beta}{\sigma_A(w,\xi)^{n/m}}d\xi+O\left(L\right.$$

where $\eta=1$ *if* $1=|\alpha|+|\beta|$ *and* 0 *otherwise.*

- *Assume* $(\alpha,\beta)\neq(0,0)$ *and let* $\varepsilon>0$*. Then, uniformly for* $x,y\in V$ *such that* $|x-y|>\varepsilon$*,*

$$L^{-(|\alpha|+|\beta|)/m}\partial_x^\alpha\partial_y^\beta K_L^s(x,y)=O\left(L^{-1/m}\ln(L)^\eta\right)$$

where $\eta=1$ *if* $1=|\alpha|+|\beta|$ *and* 0 *otherwise.*

2. - *Uniformly for* $x,y\in V$ *and* $L\geq 1$*, in these coordinates,*

$$K_L^s(x,y)=g_A(x,y)\left[\ln\left(L^{1/m}\right)-\ln_+\left(L^{1/m}|x-y|\right)\right]+O(1)$$

where

$$g_A(x,y)=\frac{1}{(2\pi)^n}\times\frac{\nu_x\left(S_x^*\right)+\nu_y\left(S_y^*\right)}{2}$$

and $\ln_+(t)=\ln(t)\vee 0$*.*

- *There exists a symmetric bounded function* $Q:U\times U\to\mathbb{R}$ *such that, uniformly for* $\kappa\geq 1$*,* $L\geq 1$ *and* $x,y\in V$ *such that* $|x-y|\geq\kappa L^{-1/m}$*, in these coordinates,*

$$K_L^s(x,y)=-g_A(x,y)\ln(|x-y|)+Q(x,y)+O\left(\kappa^{-1/k}\right)$$

where, if $n=1$ *then* $k=1$ *and if* $n\geq 2$ *then* $k=m$*.*

Here $|\alpha|=\alpha_1+\cdots+\alpha_n$ *and* $\langle\cdot,\cdot\rangle$ *is the Euclidean scalar product.*

This theorem (especially the second point) generalizes Theorem 3 of [Riv17], which proved the second point in the case where $s=1$, $\mathcal{X}$ was a closed surface (so $n=2$) with a Riemmanian metric and A was the associated Laplacian (so $m=2$). The main challenge in the extension comes from the need to apply a generalized stationary phase formula on the level sets of the symbol. In [Riv17], this is Proposition 23, where the traditional stationary phase formula applies directly. This general setting requires tools from singularity theory that are deployed in Section 2.7. The second point of Theorem 2.1.2 will follow from Theorem 2.2.11 below. As is apparent, in Figure 1, the proof of this result is more complex than that of the others. We prove Theorem 2.1.2 at the end of Section 2.2.

Corollary 2.1.3. *The Schwartz kernel $K \in \mathcal{D}'(\mathcal{X} \times \mathcal{X})$ of $A^{-n/m}$ belongs to $L^1(\mathcal{X} \times \mathcal{X})$. Moreover, for each smooth distance function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ on $\mathcal{X}$ there exists a bounded symmetric function $Q_{A,d} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, smooth on the complement of the diagonal, such that, for any distinct $x, y \in \mathcal{X}$,*

$$K(x,y) = -g_A(x,y) \ln\left(d\left(x,y\right)\right) + Q_{A,d}(x,y)\,.$$

We prove Corollary 2.1.3 at the end of Section 2.2.

2.1.1 An important example: the Laplacian

As explained above, this work is motivated by recent interest in the kernel K_L^0 as the covariance function of a Gaussian field. In further work, we wish to study certain Gaussian fields arising naturally in geometry and statistical mechanics with covariance K_L^s. One such field is the Gaussian Free Field, which is a central object in statistical mechanics today. In Corollaries 2.1.4 and 2.1.5 we detail our main results in this special case.

Let $(\mathcal{X}, g)$ be a closed Riemmanian manifold of dimension $n \geq 2$. Let $\Delta = -div \circ \nabla$ be the Laplace operator on $\mathcal{X}$ and let $|dV_g|$ be the Riemmanian volume density on $\mathcal{X}$. Then, Δ is an elliptic differential operator with principal symbol $\sigma(x,\xi) = g_x^{-1}\left(\xi,\xi\right)$ where g_x^{-1} is the scalar product induced on $T_x^*\mathcal{X}$ by g_x. Moreover, Δ is symmetric with respect to the L^2-scalar product induced by the density $|dV_g|$ on $\mathcal{X}$. Let $(\lambda_k)_{k\in\mathbb{N}}$ be the sequence of eigenvalues of Δ (counted with multiplicity) and arranged in increasing order. Let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert basis of $L^2\left(\mathcal{X}, |dV_g|\right)$ made up of real valued functions, such that for each $k \in \mathbb{N}$, $\Delta e_k = \lambda_k e_k$. For each $L > 0$, each $s > 0$ and each $(x,y) \in \mathcal{X} \times \mathcal{X}$, let

$$K_L^s(x,y) = \sum_{0<\lambda_k \leq L} \lambda_k^{-1} e_k(x) e_k(y)\,.$$

Then, K_L^1 converges in distribution as $L \to +\infty$ to the Green function on $\mathcal{X}$ which is the (generalized) covariance function for the Gaussian Free Field (see for instance [She07]). We have the following results. In the case where $s < n/2$, K_L^s converges at scale $L^{-1/2}$ to a non-trivial function after rescaling by a polynomial factor.

Corollary 2.1.4. *Assume that $s < n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \ldots, x_n)$ for $\mathcal{X}$ centered at x_0 such that $|dV_g|$ agrees with the Lebesgue measure in these coordinates. Then, for each $\alpha, \beta \in \mathbb{N}^n$ there exists $V \subset U$ an open neighborhood of 0 such that, in these coordinates, we have the following estimates.*

1. *Uniformly for $w, x, y \in V$ and $L \geq 1$ we have $w + L^{-1/2}x, w + L^{-1/2}y \in U$ and*

$$\partial_x^\alpha \partial_y^\beta K_L^s\left(w + L^{-1/2}x, w + L^{-1/2}y\right) =$$
$$\frac{1}{(2\pi)^n} \int_{|\xi|_w^2 \leq 1} e^{i\langle \xi, x-y\rangle} \frac{(i\xi)^\alpha (-i\xi)^\beta}{|\xi|_w^{2s}} d\xi L^{(n+|\alpha|+|\beta|-2s)/2} + O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2} \ln\left(L\right)^\eta\right)$$

where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/2$ and 0 otherwise. Here $|\xi|_w^2 = g_w^{-1}(\xi, \xi)$ and $d\xi$ is the Lebesgue measure.

2. *Let $\varepsilon > 0$. Then, uniformly for $x, y \in V$ such that $|x - y| > \varepsilon$ and for $L \geq 1$,*

$$\partial_x^\alpha \partial_y^\beta K_L^s \left(w + L^{-1/2} x, w + L^{-1/2} y \right) = O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2} \ln(L)^\eta \right)$$

where $\eta = 1$ if $s = (n + |\alpha| + |\beta| - 1)/2$ and 0 otherwise.

Proof. This follows directly from Theorem 2.1.1 with $m = 2$, $s < n/2$, $A = \Delta$ and $\sigma_A(w, \xi) = |\xi|_w^2$. □

On the other hand, if $s = n/2$, although the derivatives of K_L^s also have non-trivial local limits at scale $L^{-1/2}$, K_L^s itself converges pointwise to a distribution with a logarithmic singularity on the diagonal. Note that when $s = 1$, the first part of the second point of Corollary 2.1.5 below yields Theorem 3 of [Riv17].

Corollary 2.1.5. *Assume that $s = n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, x_2)$ for $\mathcal{X}$ centered at x_0 defined on an open subset $U \subset \mathbb{R}^2$ such that $|dV_g|$ agrees with the Lebesgue measure in these coordinates. Then, for each $\alpha, \beta \in \mathbb{N}^2$, there exists an open neighborhood $V \subset U$ of 0 such that the following holds.*

1. - *Assume that $(\alpha, \beta) \neq (0, 0)$. In these coordinates, uniformly for $w, x, y \in V$ and $L \geq 1$, we have $w + L^{-1/2} x, w + L^{-1/2} y \in U$ and*

$$\partial_x^\alpha \partial_y^\beta K_L^s \left(w + L^{-1/2} x, w + L^{-1/2} y \right) = \frac{1}{(2\pi)^n} \int_{|\xi|_w^2 \leq 1} e^{i\langle \xi, x - y \rangle} \frac{(i\xi)^\alpha (-i\xi)^\beta}{|\xi|_w^n} d\xi L^{(n+|\alpha|+|\beta|-2s)/2} + O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2} \ln(L)^\eta \right)$$

   where $\eta = 1$ if $1 = |\alpha| + |\beta|$ and 0 otherwise. Here $|\xi|_w^2 = g_w^{-1}(\xi, \xi)$ and $d\xi$ is the Lebesgue measure.

   - *Let $\varepsilon > 0$. Then, uniformly for $x, y \in V$ such that $|x - y| > \varepsilon$ and for $L \geq 1$,*

$$\partial_x^\alpha \partial_y^\beta K_L^s \left(w + L^{-1/2} x, w + L^{-1/2} y \right) = O\left(L^{(n+|\alpha|+|\beta|-2s-1)/2} \ln(L)^\eta \right)$$

   where $\eta = 1$ if $1 = |\alpha| + |\beta|$ and 0 otherwise.

2. - *Uniformly for $x, y \in V$ and $L \geq 1$, in these coordinates,*

$$G_L(x, y) = \frac{|S^{n-1}|}{(2\pi)^n} \left[\ln\left(L^{1/2} \right) - \ln_+\left(L^{1/2} |x - y| \right) \right] + O(1)$$

   where $\ln_+(t) = \ln(t) \vee 0$.

- *There exists a symmetric bounded function $Q : U \times U \to \mathbb{R}$ such that, uniformly for $\kappa \geq 1$, $L \geq 1$ and $x, y \in V$ such that $|x - y| \geq \kappa L^{-1/2}$, in these coordinates,*

$$G_L(x,y) = \frac{\left|S^{n-1}\right|}{(2\pi)^n} \ln(|x-y|) + Q(x,y) + O\left(\kappa^{-1/2}\right) .$$

Proof. This follows directly from Theorem 2.1.2 with $m = 2$, $s = n/2$, $A = \Delta$ and $\sigma_A(w,\xi) = |\xi|_w^2$. □

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2.2 Statement of the main results

In this section, we present the main objects of study and state our results in full generality. In Subsection 2.2.1 we present the general framework of the article. In Subsection 2.2.2 we state Hörmander's local Weyl law. In Subsection 2.2.3 we state the generalizations of the local Weyl law proved in this paper. We finish off by deducing Theorems 2.1.1 and 2.1.2 as well as Corollary 2.1.3.

2.2.1 General setting

In this article, we consider simultaneously two different elliptic eigenvalue problems. Since our arguments hold indifferently for the two cases, we present them in this section using the same notations. The first case is a closed eigenvalue problem. In this case we will follow [Hör68]. In the second case, we consider a Dirichlet eigenvalue problem, for which our main reference will be [Vas84].

1. In this setting we follow [Hör68]. Here $\mathcal{X}$ is a compact manifold without boundary. We consider a classical elliptic pseudo-differential operator A of positive order m acting on $C_c^\infty(\mathcal{X})$. We assume A is symmetric for the L^2-scalar product on $(\mathcal{X}, d\mu_{\mathcal{X}})$. This implies that the principal symbol σ_A of A is real valued and positive homogeneous of order m. Moreover, under these assumptions, A has a unique self-adjoint extension in $L^2(\mathcal{X}, d\mu_{\mathcal{X}})$ whose spectrum forms a discrete non-decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of real numbers diverging to $+\infty$ and whose corresponding eigenfunctions e_k are of class C^∞ (see for instance Section 29.1 of [Hör09]). For each $L > 0$, set

$$\forall x, y \in \overline{\mathcal{X}}, E_L(x,y) = \sum_{\lambda_j \leq L} e_j(x)\overline{e_j(y)}\,.$$

2. In this setting, we follow [Vas84]. Here $\mathcal{X}$ is the interior of a compact manifold $\overline{\mathcal{X}}$ with non-empty boundary $\partial\mathcal{X}$. We assume that $\mathcal{X}$ is equipped with a positive density $d\mu_{\mathcal{X}}$. We consider the Dirichlet eigenvalue problem

$$\begin{aligned} Au &= \lambda u \text{ on } \mathcal{X}\,; \\ \forall j = 1, \ldots, j_0,\, B_j u &= 0 \text{ on } \partial\mathcal{X} \end{aligned}$$

where A is an elliptic differential operator of even order $m \geq 1$ with principal symbol σ_A, B_j are boundary differential operators (see Chapter 2, Section 1.4 of [LM72]) and $\lambda \in \mathbb{C}$. We assume that the problem is elliptic, formally self-adjoint with respect to $d\mu_{\mathcal{X}}$ and semi-bounded from below (see [Vas84] Section 1). As is well known, under these assumptions, the values of λ for which this problem has a non-trivial solution with sufficient regularity form an non-decreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ of real numbers diverging to $+\infty$ and the corresponding eigenfunctions e_k are smooth in $\mathcal{X}$ up to the boundary (the proof goes along the same lines as in the closed case treated in Section 29.1 of [Hör09] and is easily adapted using results from Chapter 20 of [Hör07]). For each $L > 0$, set

$$\forall x, y \in \overline{\mathcal{X}}, E_L(x,y) = \sum_{\lambda_j \leq L} e_j(x)\overline{e_j(y)}\,.$$

2.2.2 Hörmander's local Weyl law

Let us consider the sequence of real numbers $(\lambda_k)_k$ and the sequence of smooth functions $(e_k)_k$ from either of the two settings presented in Subsection 2.2.1. Recall that σ_A is the principal symbol of A, which we assumed to be positive homogeneous of order $m > 0$ in the second variable. We begin by stating Hörmander's local Weyl law, for which we need the following definition[2].

Definition 2.2.1. Given an open subset $U \subset \mathbb{R}^n$, we will say that a function $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ is a **proper phase function** if it satisfies the following conditions.

1. The function ψ is a symbol of order one in its third variable.
2. For each $(x, y, \xi) \in U \times U \times \mathbb{R}^n$, $\langle x - y, \xi\rangle = 0$ implies that $\psi(x, y, \xi) = 0$.
3. For each $x \in U$ and $\xi \in \mathbb{R}^n$, $\partial_x\psi(x, y, \xi)|_{y=x} = \xi$.

[2]This definition is inspired by Definition 2.3 of [Hör68]. However, our notion of proper phase function is more restrictive than Hörmander's notion of phase function.

4. There exists $\psi_\infty \in C^\infty(U \times U \times \mathbb{R}^n \setminus \{0\})$ satisfying all of the above properties and 1-homogeneous in ξ such that

$$t^{-1}\psi(x, y, t\xi) \xrightarrow[t\to+\infty]{} \psi_\infty(x, y, \xi)$$

where the convergence takes place in $S^1(U \times U \times \mathbb{R}^n \setminus \{0\})$.

An important example of proper phase function to have in mind is the phase function $\psi(x, y, \xi) = \langle x - y, \xi\rangle$. Hörmander's local Weyl law may be stated as follows.

Theorem 2.2.2 ([Hör68], Theorem 5.1 for $P = Id$)**.** *Let P be a differential operator of order d acting on $C^\infty(\mathcal{X} \times \mathcal{X})$. Fix a point in $\mathcal{X}$ and consider local coordinates $(x_1, \dots, x_n)$ around it. Suppose further that the density $d\mu_\mathcal{X}$ agrees with the Lebesgue measure in these coordinates. Let σ_A (resp. σ_P) be the principal symbol of A (resp. P) in these coordinates. Then, there exists an open neighborhood U of $0 \in \mathbb{R}^n$, a proper phase function function $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ and a constant $C < +\infty$, such that, in these coordinates, for each $x, y \in U$ and $L > 0$,*

$$\left| PE_L(x, y) - \frac{1}{(2\pi)^n} \int_{\sigma_A(x,\xi)\le L} e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x, y, \xi))d\xi \right| \le C(1 + L)^{(n+d-1)/m}.$$

Moreover, for each neighborhood $W \subset U \times U$ of the diagonal there exists $C > 0$ such that in local coordinates, for each $(x, y) \in (U \times U) \setminus W$ and $L > 0$,

$$\left| PE_L(x, y) \right| \le C(1 + L)^{(n+d-1)/m}.$$

Finally, there exists a symbol $\sigma \in S^1$ such that $\sigma_A^{1/m} - \sigma \in S^0$ and for each $x, y \in U$ and $\xi \in \mathbb{R}^n$,

$$\sigma(x, \partial_x\psi(x, y, \xi)) = \sigma(y, \xi). \tag{2.2.1}$$

Here $\partial_{x,y}\psi$ denotes the partial derivative of ψ with respect to the couple (x, y).

Remark 2.2.3. The asymptotic provided by Theorem 2.2.11 is coordinate dependent since the notion of proper phase function is not invariant.

Remark 2.2.4. Equation (2.2.1) is called the **eikonal equation** and it has a unique solution with the boundary conditions imposed by the admissibility condition (see Section 3 of [Hör68]). The part concerning the eikonal equation is not usually stated as part of the local Weyl law but the function ψ provided by the theorem does satisfy this property and it will be useful in our proofs.

Remark 2.2.5. The case where $P = Id$ and was proved by Hörmander in [Hör68]. The case where $x = y$ and $\mathcal{X}$ is a closed manifold was treated in [SV97] with some restrictions on P. Finally, Gayet and Welschinger extended this result to a general P (see Theorem 2.3 of [GW16a]) on a closed manifold. While in their statement, $x = y$, their proof yields the off-diagonal case with only minor modifications.

Remark 2.2.6. Hörmander manages to lift the compactness assumption using results on the local nature of the spectral projector Π_L. It is not clear that this approach could be applied for a general P.

Remark 2.2.7. Notice that in the boundary problem case (as in setting 2. form Subsection 2.2.1) we only get estimates in the interior of the domain.

Remark 2.2.8. One recent result closely related to this theorem is Canzani and Hanin's asymptotics for the monochromatic spectral projector of the Laplacian under some dynamical assumption on the geodesic flow (see [CH15] and [CH18]).

For the convenience of the reader, in Appendix 2.A we provide a proof of the full result relying on the wave kernel asymptotics provided in [Hör68].

2.2.3 Weighted local Weyl laws

In the present article, we generalize Theorem 2.2.2 in the following way. Consider A and P as in Theorem 2.2.2 and take U and ψ as provided by this theorem.

Theorem 2.2.9. *Fix $z = z_1 + iz_2 \in \mathbb{C}$. Let $f \in C^\infty(\mathbb{R})$ such that $f(t) = t^z$ for t large enough. Let K_L be the Schwartz kernel of $\Pi_L f(A)$. Suppose that $n + d + mz_1 > 0$. For each $x, y \in U$ and $L \geq 1$, let $R_L(x, y)$ equal*

$$L^{-z_1-(n+d)/m}\Big[PK_L(x,y) - \frac{1}{(2\pi)^n}\int_{\sigma_A(0,\xi)\leq 1} e^{i\langle \xi, x-y\rangle L^{1/m}}\sigma_A(0,\xi)^z\sigma_P(0,0,(\xi,-\xi))d\xi\Big].$$

Then, there exists an open neighborhood V of $0 \in U$ such that the following holds.

1. *Uniformly for $L \geq 1$ and $(w, x, y) \in V \times V \times V$, $w + L^{-1/m}x, w + L^{-1/m}y \in U$ and*

$$R_L\left(w + L^{-1/m}x, w + L^{-1/m}y\right) = O\left(L^{-1/m}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise.

2. *Uniformly for $L \geq 1$ and $(x, y) \in V \times V \setminus W$,*

$$PK_L(x,y) = O\left(L^{z_1+(n+d-1)/m}\ln(L)^\eta\right)$$

   where $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise.

We prove Theorem 2.2.9 in Section 2.5. As we will see below, Theorem 2.1.1 and the first assertion of Theorem 2.1.2 are both direct consequences of Theorem 2.2.9. Before stating Theorem 2.2.11 below, we must introduce some more terminology. One key ingredient of the proof will be the decay of certain oscillatory integrals depending on the level sets of σ_A. To observe this behavior we must impose certain condition on σ_A. This is the object of Definition 2.2.10.

Definition 2.2.10. Fix $m \in \mathbb{R}$, $m > 0$ and $k_0 \in \mathbb{N}$, $k_0 \geq 2$. We say that a positive m-homogeneous symbol σ on U is k_0-admissible there exists $k_0 \geq 2$ such that

$$\forall (x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})\ \exists k \in \{2,\dots,k_0\},$$
$$\sigma(x,\xi)^{k-1}\partial_\xi^k \sigma(x,\xi) \neq \frac{m(m-1)\dots(m-k+1)}{m^k}(\partial_\xi \sigma(x,\xi))^{\otimes k}. \quad (2.2.2)$$

This condition is invariant if we see σ as a function on $T^*\mathcal{X}$ because coordinate changes act linearly on the fibers of $T^*\mathcal{X}$. It is stable and generic for k_0 large enough, as explained in Proposition 2.7.8.

Theorem 2.2.11. *We use the same notations as in Theorem 2.2.9. Suppose that $n + d + mz = 0$ and that either $n = 1$ or σ_A is a k_0-admissible symbol for some $k_0 \geq 2$. For each $x, y \in U$ let*

$$Y_P(x,y) = \int_{S_y^*} \sigma_P\left(x, y, \partial_{x,y}(\partial_\xi \psi(x,y,0)\xi)\right) d_y\nu(\xi)\,.$$

Then, there exists $V \subset U$ an open neighborhood of 0 such that the following holds.

1. *Uniformly for $(x,y) \in V \times V$ and $L \geq 1$,*

$$PK_L(x,y) = \frac{1}{(2\pi)^n} Y_P(x,y)\Big[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\Big] + O(1)\,.$$

2. *There exists $Q \in L^\infty(V \times V)$ such that, uniformly for $\kappa \geq 1$, $L \geq 1$ and $(x,y) \in V \times V$ such that $|x-y| \geq \kappa L^{-1/m}$,*

$$PK_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(x,y)\ln(|x-y|) + Q(x,y) + O\left(\kappa^{-1/k_0}\right)\,.$$

Here, if $n = 1$ we set $k_0 = 1$.

We prove Theorem 2.2.11 in Section 2.6. As we will see below, the second point of Theorem 2.1.2 follows directly from this theorem.

Remark 2.2.12. The admissibility condition on the symbol of A may appear to be unfamiliar. However, in practice, it is often satisfied. Here are two important examples of families of admissible symbols:

- If $n \geq 2$ and the level sets S_x^* are strictly convex, $\partial_\xi^2 \sigma_A$ is positive when restricted to their tangent spaces. Therefore, it cannot be a multiple of $(\partial_\xi \sigma)^{\otimes 2}$ so Theorem 2.2.11 applies with $k_0 = 2$.

- If σ_A is a positive homogeneous polynomial of degree $m \in \mathbb{N}$ in ξ, $m \geq 1$, then it is m-admissible. Indeed, otherwise, taking $k = k_0 = m$, we would have, for some $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$, $\sigma_A(x,\xi)^{m-1}\partial_\xi^m \sigma_A(x,\xi) = 0$. But since $\sigma_A(x,\xi) > 0$ we have $\partial_\xi^m \sigma_A(x,\xi) = 0$ which implies that all the coefficients of $\sigma_A(x,\cdot)$ vanish. This contradicts the fact that $\sigma_A(x,\xi) > 0$. In particular, Theorem 2.2.11 applies for all differential operators.

In addition to the two examples of the last remark, we prove the following theorem.

Theorem 2.2.13. *Fix $n \in \mathbb{N}$, $n \geq 2$ and let $U \subset \mathbb{R}^n$ be an open subset. There exists $k_0 = k_0(n) \in \mathbb{N}$ such that for each $m > 0$, the set of k_0-admissible symbols is open and dense in the set of positive m-homogeneous symbols on U for the topology induced by the Whitney topology through the restriction map to the (Euclidean) unit sphere bundle on U.*

Theorem 2.2.13 follows immediately from Proposition 2.7.8, which is proved Subsection 2.7.2. The integer k_0 is explicit (see Proposition 2.7.8).

Finally, though we do not use this in the proof of Theorems 2.1.1 and 2.1.2, we prove the following result, which might be useful in further applications.

Theorem 2.2.14. *We use the same notations as in Theorem 2.2.9. Suppose that $n + d + mz_1 < 0$. Then, there exists and a function $K_\infty \in C^d(U \times U)$ such that the following holds. For each compact subset $\Omega \subset U \times U$, uniformly for $(x, y) \in \Omega$,*

$$PK_L(x,y) = PK_\infty(x,y) + O\left(L^{z_1+(n+d)/m}\right).$$

Remark 2.2.15. In Theorems 2.2.9, 2.2.11 and 2.2.14, the setting provided in Subection 2.2.1 only comes into play through Theorem 2.2.2. Therefore, if one could weaken the hypotheses for this theorem, one would automatically extend Theorems 2.2.9 and 2.2.11 as a corollary. In particular, since Hörmander proves Theorem 2.2.2 for $P = Id$ without any compactness assumption or boundary condition, both of these results remain valid in this case.

Let us check that Theorems 2.2.9 and 2.2.11 imply the results presented in the introduction.

Proof of Theorem 2.1.1. Both results follow from Theorem 2.2.9 applied to the first setting of Subsection 2.2.1 with $z = -s$ by taking $P = \partial_x^\alpha \partial_y^\beta$ in a neighborhood of 0. In this case, the order of P is $d = |\alpha| + |\beta|$ and we have

$$\sigma_P(x,y,\xi) = (i\xi)^\alpha (-i\xi)^\beta$$

for any $\xi \in \mathbb{R}^n$ and $x, y \in V$ close enough to 0. □

Proof of Theorem 2.1.2. Set $z = -s = -n/m$. For the first part, set $P = \partial_x^\alpha \partial_y^\beta$ near 0 and proceed as in the proof of Theorem 2.1.1. Indeed, since $(\alpha, \beta) \neq (0,0)$, we have $n + d + mz_1 = |\alpha| + |\beta| > 0$. For the second part, since by Remark 2.2.12 σ_A is m-admissible, and since $n + d + mz_1 = 0$, we apply Theorem 2.2.11 instead. In our case, $P = Id$ so for each $x, y \in U$, $Y_P(x,y) = \nu_y\left(S_y^*\right)$ so

$$K_L^s(x,y) = \frac{1}{(2\pi)^n}\nu_y\left(S_y^*\right)\left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\right] + O(1)\,.$$

But since $K_L^s(x,y) = \overline{K_L^s(y,x)}$, we may replace $\nu_y\left(S_y^*\right)$ in the above expression by $\frac{\nu_x(S_x^*)+\nu_y\left(S_y^*\right)}{2}$. as announced. □

Proof of Corollary 2.1.3. We use the notations of Theorem 2.2.11. First of all, by definition, as $L \to +\infty$, $K_L^s \to K$ in distribution. Moreover, by Theorem 2.1.2, any point in $\mathcal{X}$ has a neighborhood V such that the sequence $(K_L^s)_{L\geq 1}$ is uniformly bounded on $V \times V$ by a locally integrable function and converge pointwise towards $-g_A(x,y)\ln(|x-y|) + Q(x,y)$ where $Q \in L^\infty(V \times V)$ on the complement of the diagonal in $V \times V$. In particular, they converge in distribution to this function. This implies that when restricted to $C^\infty(V \times V)$,

$$K(x,y) = -g_A(x,y)\ln(|x-y|) + Q(x,y)\,.$$

Now, given any smooth distance d on $\mathcal{X}$, for each x, y distinct,

$$\ln(|x-y|) = \ln(d(x,y)) + \ln\left(\frac{|x-y|}{d(x,y)}\right)$$

and the second term is bounded so, on $V \times V$,

$$K(x,y) = -g_A(x,y)\ln(d(x,y)) + Q_A(x,y)$$

for some $Q_A \in L^\infty(V \times V)$. But K is the integral kernel of A^s which is a self-adjoint pseudo-differential operator (see [See67] or Proposition 29.1.9 of [Hör09]). In particular, it is smooth and symmetric outside the diagonal (see for instance Theorem 18.1.16 of [Hör07]). Hence, Q_A must also be symmetric and smooth outside the diagonal. □

Remark 2.2.16. These proofs work also in Setting 2 of Subsection 2.2.1 instead of that of Setting 1, so all of the results from the introduction hold also in Setting 2.

2.3 Heuristics and proof outline

In this section we provide a heuristic justification for Theorems 2.2.9 and 2.2.11 and an outline of the skeleton of the proof. At the end of this section, we also provide a proof map to highlight the dependencies between intermediate results leading to the proofs of Theorems 2.2.9, 2.2.11 and 2.2.14, see Figure 1.

2.3.1 Heuristics

In order to get a sense of the kind of calculations we will carry out in the rest of the article, let us present a simple example, with few non-rigorous steps in order to shorten the argument. We assume that $\mathcal{X}$ is a closed Riemmanian manifold and that A denotes the associated Laplacian[3]. Then, A is indeed elliptic of order $m = 2$ and self-adjoint with respect to the riemmanian volume density $d\mu$. Moreover the symbol of A is $\sigma_A(x,\xi) = \|\xi\|_x^2$ where $\|\cdot\|_x$ is the norm induced by the riemmanian metric on $T_x^*\mathcal{X}$. Thus, in orthonormal coordinates $S_x^* = S^{n-1}$. Finally, we take $P = Id$. Now, if $s \geq \frac{n}{2}$,

$$K_L^s(x,y) = \int_1^L \lambda^{-s} E'_\lambda(x,y)d\lambda + O(1) = s\int_1^L \lambda^{-s-1} E_\lambda(x,y)d\lambda + O(1)\,. \tag{2.3.1}$$

Here, we artificially cut-off the first eigenvalues since they contribute a constant term to the sum defining K_L^s. By Theorem 2.2.2,

$$E_L(x,y) \simeq \frac{1}{(2\pi)^n}\int_{|\xi|^2\leq L} e^{i\psi(x,y,\xi)}d\xi \tag{2.3.2}$$

where

$$\psi(x,y,\xi) \simeq \langle x-y,\xi\rangle\,. \tag{2.3.3}$$

Replacing E'_λ in Equation (2.3.1) the expression given by Equations (2.3.2) and (2.3.3) we get

$$K_L^s(x,y) \simeq \frac{s}{(2\pi)^{n/2}}\int_1^L \lambda^{-s-1}\int_{\|\xi\|^2\leq\lambda} e^{i\langle x-y,\xi\rangle}d\xi d\lambda\,.$$

At this point, we make the additional assumption that $n \geq 2$. The one dimensional case is similar in spirit but follows a different argument. We then continue with a polar change of coordinates: $\xi = t\omega$.

$$K_L^s(x,y) \simeq \frac{s}{(2\pi)^n}\int_1^L \lambda^{-s-1}\int_0^{\lambda^{1/2}} J_{x,y}(t)t^{n-1}dtd\lambda \tag{2.3.4}$$

where

$$J_{x,y}(t) = \int_{S^{n-1}} e^{it\langle x-y,\omega\rangle}d\omega\,.$$

Let us first assume that $s > \frac{n}{2}$ and take $x - y = L^{-1/2}h$ where $h \in \mathbb{R}^n$ is fixed. Then,

$$\begin{aligned}
K_L^s(x,y) &\simeq \frac{1}{(2\pi)^n}\int_{S^{n-1}}\int_1^L \lambda^{-s}\int_0^{\lambda^{1/2}} e^{iL^{-1/2}t\langle h,\omega\rangle}t^{n-1}dtd\lambda d\omega \\
&\simeq \frac{s}{(2\pi)^n}L^{n/2-s}\int_{S^{n-1}}\int_0^1 \lambda^{-s-1}\int_0^{\lambda^{1/2}} e^{it\langle h,\omega\rangle}t^{n-1}dtd\lambda d\omega \\
&\simeq \frac{1}{(2\pi)^n}\int_{\|\xi\|^2\leq 1}\|\xi\|^{-s}e^{i\langle h,\xi\rangle}d\xi L^{n/2-s}\,.
\end{aligned}$$

[3]Here we use the convention that $\Delta = -\mathrm{div}\circ\nabla$ so that the operator is positive.

This is the conclusion of Theorem 2.1.1.

Assume now that $s = \frac{n}{2}$. Starting off from Equation (2.3.4), we get

$$K_L^s(x,y) \simeq \frac{s}{(2\pi)^n} \int_{|x-y|^2}^{|x-y|^2 L} \lambda^{-s-1} \int_0^{\lambda^{1/2}} J_{x,y}(|x-y|^{-1}t) t^{n-1} dt d\lambda \,.$$

Note that, by the stationary phase method,

$$J_{x,y}(t) = O\left((|x-y|t)^{-1/2}\right) \,. \tag{2.3.5}$$

This crucial observation basically allows us to replace $J_{x,y}(|x-y|^{-1}t)$ with $|S^{n-1}|\mathbb{1}[|x-y|^{-1}t \leq 1]$ and get

$$\begin{aligned} K_L^s(x,y) &\simeq \frac{s|S^{n-1}|}{(2\pi)^n} \int_{|x-y|^2}^{|x-y|^2 L} \lambda^{-s-1} \int_0^{\lambda^{1/2}} \mathbb{1}[|x-y|^{-1}t \leq 1] t^{n-1} dt d\lambda \\ &= \dots \\ &= \frac{|S^{n-1}|}{(2\pi)^n} \int_{|x-y|}^{|x-y|L^{1/2}} \mathbb{1}[u \leq 1] \frac{du}{u} + O(1) \\ &= \frac{|S^{n-1}|}{(2\pi)^n} \ln\left(L^{1/2}\right) - \ln_+\left(L^{1/2}|x-y|\right) + O(1) \,. \end{aligned}$$

This is the essential statement of Theorem 2.1.2.

2.3.2 Proof strategy

There are two main obstacles to carry out the above calculation rigorously in the general case and Sections 2.4 and 2.7 are devoted to dealing with them. The first is to justify Equation (2.3.3). This is the role of Lemmas 2.4.1, 2.4.2 and 2.4.3 that roughly state that ψ behaves like $\langle x-y, \xi \rangle$. The second difficulty is to obtain an analog of Equation (2.3.5) when S^{n-1} is replaced by $S_x^* = \{\xi \,,\, \sigma_A(x,\xi) = 1\}$ for a general symbol σ_A. Indeed, in this case, the standard stationary method need not apply and we must use more general results on oscillatory integrals. This requires the assumption that σ_A be admissible (see Definition 2.2.10). To make this point more precise, let us introduce some notation.

As in the previous section, we fix once and for all a point in $\mathcal{X}$ and consider a local chart centered at this point defined on $U \subset \mathbb{R}^n$ given by Theorem 2.2.2. We also take P with principal symbol σ_P, $W \subset U \times U$ and $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ as in this theorem. The following quantity will be central in our proofs. For any $t > 0$, $x, y \in U$ and $\xi \in \mathbb{R}^n$ let

$$H_P(x,y,\xi,t) = e^{i\psi(x,y,t\xi)} \sigma_P\left(x, y, t^{-1}\partial_{x,y}\psi(x,y,t\xi)\right) \tag{2.3.6}$$

and

$$J_A(x,y,t) = \int_{S_y^*} H_P(x,y,\xi,t) d_y\nu(\xi). \tag{2.3.7}$$

Since σ_P is d-homogeneous in its third variable, H_P satisfies the following Equation. For any $s, t > 0$, $x, y \in U$ and $\xi \in \mathbb{R}^n$,

$$H_P(x, y, s\xi, t) = s^d H_P(x, y, \xi, st). \tag{2.3.8}$$

We will prove the following proposition.

Proposition 2.3.1. *Assume that σ_A is k_0-admissible. Then, there exists $V \subset U$ an open neighborhood of 0 and $C < +\infty$ such that, uniformly for distinct $x, y \in V$ and $t > 0$*

$$|J_A(x, y, t)| \leq C \left(t|x - y|\right)^{-\frac{1}{k_0}} .$$

The proof of Proposition 2.3.1 is divided into two steps. First, we will prove that the admissibility condition on σ_A implies a property governing the decay of certain oscillatory integrals over the level sets of σ_A that we define below in Definition 2.3.2. Next, we prove that this property implies the required behavior of J_A. More precisely, we introduce the following terminology.

Definition 2.3.2. Let $\varepsilon > 0$, $m > 0$, $E \subset \mathbb{R}^p$ a neighborhood of 0 and let $U \subset \mathbb{R}^n$ be an open subset. Let $\sigma \in C^\infty(U \times \mathbb{R}^n \setminus \{0\})$ be homogeneous of degree m in the second variable. For each $x \in U$ let $S^*_x = \{\xi \in \mathbb{R}^n \mid \sigma(x, \xi) = 1\}$ and $d_x\mu$ be the area measure on S^*_x. Let $S^*U = \{(x, \xi) \in U \times \mathbb{R}^n \mid \xi \in S^*_x\}$.

1. Given a compact subset $\Omega \subset U \times (\mathbb{R}^n \setminus \{0\})$ let $X = \{(x, \tau, \xi) \in \Omega \times \mathbb{R}^n \mid \xi \in S^*_x\}$. We call a **deformation of the height function for $\boldsymbol{\sigma}$ over $\boldsymbol{\Omega}$** any family $(f_\eta)_{\eta \in E}$ of continuous, real-valued functions on X, smooth in the third variable ξ, with the following properties:

   - for each $(x, \tau, \xi) \in \Omega \times \mathbb{R}^n$ such that $\xi \in S^*_x$, $f_0(x, \tau, \xi) = \langle \tau, \xi \rangle$
   - for each $\alpha \in \mathbb{N}^n$, the map $\eta \mapsto \partial^\alpha_\xi f_\eta$ is continuous for the topology of uniform convergence on compact sets.

2. We say that σ has **$\boldsymbol{\varepsilon}$-non-degenerate level sets** if, for any compact subset Ω of $U \times (\mathbb{R}^n \setminus \{0\})$ and any deformation of the height function $(f_\eta)_\eta$ for σ over Ω there exists $V \subset \mathbb{R}^p$ a neighborhood of 0 depending only on Ω (and ε) such that for each $\gamma \in C^\infty(S^*U)$ and each continuous family of smooth functions on $(u_\eta)_\eta \in \left(C^\infty\left(\mathbb{R}^n\right)\right)^E$, there exists $C < +\infty$ such that for each each $\eta \in V$, each $(x, \tau) \in \Omega$ and each $\lambda > 0$,

$$\left| \int_{S^*_x} e^{i\lambda f_\eta(x,\tau,\xi)} u_\eta(\xi) \gamma(x, \xi) d_x\mu(\xi) \right| \leq C\lambda^{-\varepsilon} . \tag{2.3.9}$$

   We say that σ has **non-degenerate level sets** if it has ε-non-degenerate level sets for some $\varepsilon > 0$.

3. Let $\varepsilon > 0$. We say that a homogeneous symbol on a manifold has non-degenerate (resp. ε-non-degenerate) level sets if it has this property when written in any local coordinate system.

Proposition 2.3.1 will then be a consequence of the following results. On the one hand, we will prove:

Proposition 2.3.3. *Fix $n, k_0 \in \mathbb{N}$, $n \geq 1$, $k_0 \geq 2$. Let $U \subset \mathbb{R}^n$ be an open subset and let $\sigma \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\})$ be positive and homogeneous of degree $m > 0$ in its second variable. If σ is k_0-admissible, then it has $\frac{1}{k_0}$-non-degenerate level sets.*

The proof of this proposition, which is presented in Subsection 2.7.1, is entirely independent of the rest of the present text and uses different techniques. It is followed by Subsection 2.7.2, in which we prove that the admissibility condition is generic in a suitable sense.

On the other hand, in Subsection 2.4.2, we will prove the following result.

Lemma 2.3.4. *Fix $\varepsilon > 0$. Suppose that the symbol σ_A has ε-non-degenerate level sets (see Definition 2.3.2). Then, there exists $V \subset U$ an open neighborhood of 0 and $C < +\infty$ such that, uniformly for distinct $x, y \in V$ and $t > 0$*

$$|J_A(x, y, t)| \leq C \left(t|x - y|\right)^{-\varepsilon} .$$

This corresponds to Proposition 23 of [Riv17] for $\varepsilon = \frac{1}{2}$ although, in that setting, the non-degeneracy condition was always satisfied.

In the one dimensional case, Proposition 2.3.1 is replaced by Lemma 2.4.5.

After proving all of these results, we carry out the calculation sketched in Subsection 2.3.1 in Sections 2.5 and 2.6 as we will now explain in more detail. We therefore suggest that the reader have Subsection 2.3.1 in mind for what follows. The integration by parts is of course valid in a general setting. This allows us to obtain an expression like Equation (2.3.1) where the map λ^{-s} is replaced by $f(\lambda)$ for some adequate function f. More explicitely, in Section 2.5, we derive the following result. We start by introducing a suitable function $f :]0, +\infty[\mapsto \mathbb{C}$ and studying the asymptotics of the following kernel:

$$K_L^f : (x, y) \mapsto \sum_{\lambda_k \leq L} f(\lambda_k) e_k(x) \overline{e_k}(y) .$$

This is again a smooth function. Since all of our results are local, we fix once and for all a point in $\mathcal{X}$ and consider $x = (x_1, \ldots, x_n)$ the local coordinate system at this point provided by Theorem 2.2.2, defined on an open neighborhood U of 0 in $\mathbb{R}^n$.

Proposition 2.3.5. *Take $f : \mathbb{R} \to \mathbb{C}$ with support in $]0, +\infty[$ differentiable almost everywhere. Then, in local coordinates, uniformly for each $x, y \in U$, for each $L > 0$,*

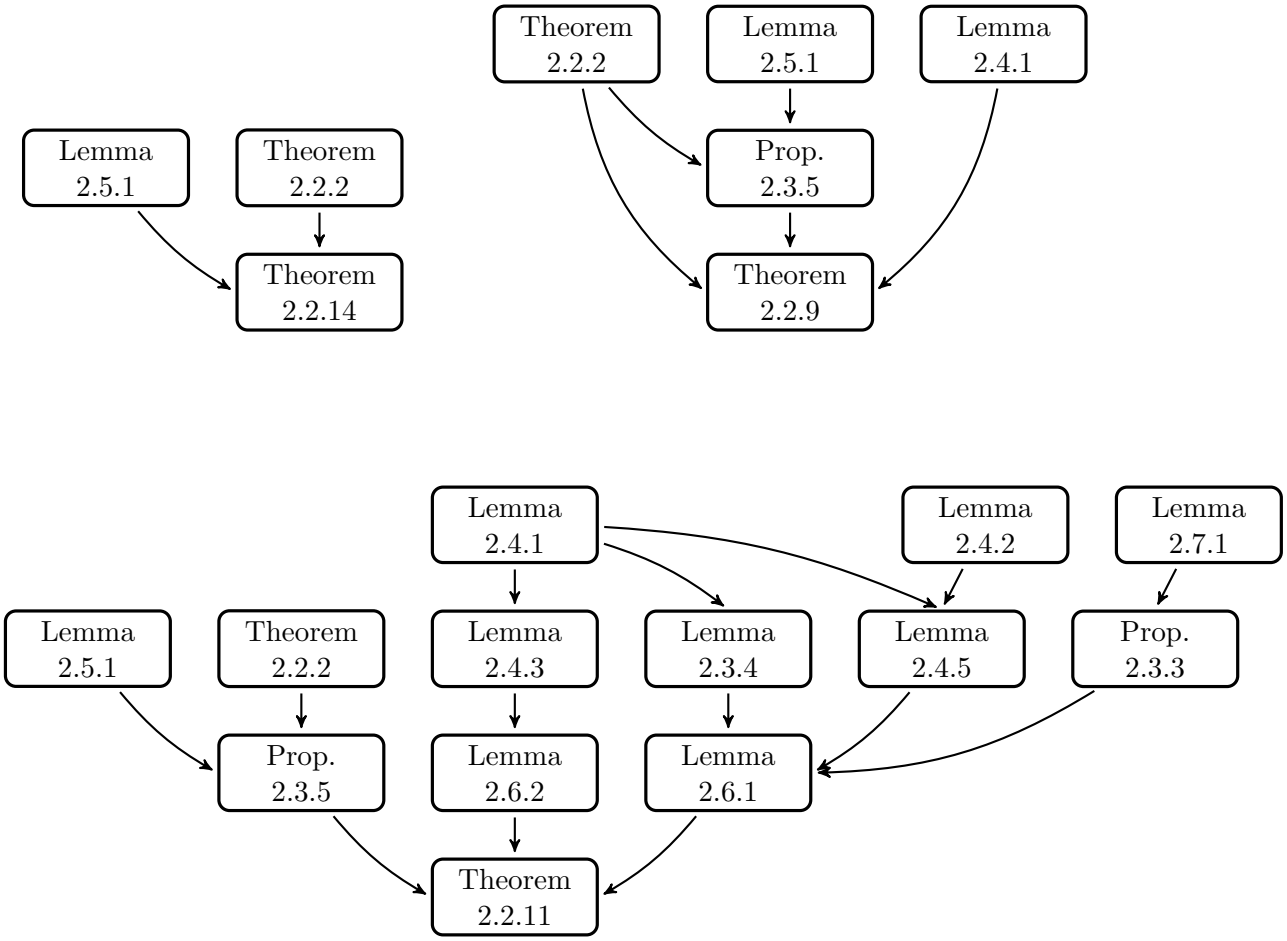
$$PK_L^f(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi) \leq L} e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x, y, \xi)) f(\sigma_A(y, \xi)) d\xi$$
$$+ O\left(f(L) L^{(n+d-1)/m}\right) + O\left(\int_0^L f'(\lambda) \lambda^{(n+d-1)/m} d\lambda\right) .$$

In addition, uniformly for any $(x, y) \in (U \times U) \setminus W$*, for each* $L \geq 1$*,*

$$PK_L^f(x,y) = O\left(f(L)L^{(n+d-1)/m}\right) + O\left(\int_0^L f'(\lambda)\lambda^{(n+d-1)/m}d\lambda\right).$$

Finally, the constants implied by the O*'s do not depend on* f*.*

The condition on the support of f can be achieved by multiplying f by a suitable cut-off function when necessary since the spectrum of A is bounded from below. We prove Proposition 2.3.5 in Section 2.5. Then, we consider the case where f is of the form $f(t) = \chi(t)t^z$ where $z = z_1 + iz_2 \in \mathbb{C}$ and χ is some smooth function with support in $]0, +\infty[$ equal to 1 for t large enough. In Section 2.5, , we prove Theorem 2.2.14 using only a crude estimate from Theorem 2.2.2, and we also deduce Theorem 2.2.9 from Proposition 2.3.5 and results from Section 2.4. Next, in Section 2.6 we prove Theorem 2.2.11 using again Proposition 2.3.5 but also Proposition 2.3.1. We end this section with a diagram detailing the dependencies between different results involved in the proofs of Theorems 2.2.9, 2.2.11 and 2.2.14.



A map of the proofs of Theorems 2.2.9, 2.2.11 and 2.2.14. The result at the origin of each arrow is used in the proof of the result at its target.

2.4 Preliminary results

As before, in this section we fix once and for all a point in $\mathcal{X}$ and consider a local chart centered at this point defined on $U \subset \mathbb{R}^n$ given by Theorem 2.2.2. We also take P with principal symbol σ_P, $W \subset U \times U$ and $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ as in this theorem. The object of this section is to estimate the behavior of the phase ψ near the diagonal and to prove Lemma 2.4.3.

2.4.1 Basic properties of the phase ψ

The phase ψ from Theorem 2.2.2 will frequently appear in the calculations below. We begin by deducing a list of properties of ψ from those given in Definition 2.2.1. We gather these properties in Lemma 2.4.1. It is easy to check that all these properties are satisfied by the function $\psi(x, y, \xi) = \langle x - y, \xi \rangle$. Next, we present an additional lemma, Lemma 2.4.2, for the case $n = 1$. Finally, we use Lemma 2.4.1 to deduce some properties of the function H_P defined in Equation (2.3.6).

Lemma 2.4.1. *Let $U \subset \mathbb{R}^n$ and let $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$ be a proper phase function. For each $t > 0$, let $\psi_t = t^{-1}\psi(\cdot, \cdot, t\cdot)$. Then,*

1. *For each $x, y \in U$ and each $t > 0$, $\psi_t(x, y, 0) = 0$.*
2. *For each $x \in U$, each $t > 0$ and each $\xi \in \mathbb{R}^n$, $\psi_t(x, x, \xi) = 0$.*
3. *For each $x \in U$, each $t > 0$ and each $\xi \in \mathbb{R}^n$, $\partial_{x,y}\psi_t(x, x, \xi) = (\xi, -\xi)$.*
4. *The sequence $(\psi_t)_{t>0}$ converges in $C^\infty(U \times U \times \mathbb{R}^n)$ as $t \to 0$ to the function ψ_0 defined by $\psi_0(x, y, \xi) = \partial_\xi \psi(x, y, 0)\xi$. In other words, for each compact subset $\Omega \subset U$, each $R < +\infty$ and each $\alpha, \beta, \gamma \in \mathbb{R}^n$,*

$$\lim_{t \to 0} \sup_{x,y \in \Omega,\ |\xi| \leq R} \left| \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi) - \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_0(x, y, \xi) \right| = 0.$$

5. *The sequence $(\psi_t)_{t \geq 0}$ is bounded in $C^\infty(U \times U \times \mathbb{R}^n)$.*

Proof of Lemma 2.4.1. Let $t > 0$, $x, y \in U$ and $\xi \in \mathbb{R}^n$. Then, $\langle x - y, 0 \rangle = \langle x - x, t\xi \rangle = 0$ so $\psi_t(x, x, \xi) = \psi_t(x, y, 0) = 0$ by the second point of Definition 2.2.1. This proves the first two points of Lemma 2.4.1. By point 3 of Definition 2.2.1, for each $x \in U$ and $\xi \in \mathbb{R}^n$, $\partial_x \psi(x, x, \xi) = \xi$, so $\partial_x \psi_t(x, x, \xi) = t^{-1}(t\xi) = \xi$. Next, by differentiating the following equality

$$\psi_t(x + sv, x + sv, \xi) = 0$$

with respect to $s \in \mathbb{R}$, at $s = 0$, where $x \in U$, $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, we get

$$\partial_x \psi_t(x, x, \xi) + \partial_y \psi_t(x, x, \xi) = 0.$$

This proves the third point of Lemma 2.4.1.
To prove the fourth point, first, fix $\beta, \gamma \in \mathbb{N}^n$ and let $\Omega \subset U$ be a compact subset and $R < +\infty$. Then, for each $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$ such that $|\xi| \leq R$,

$$\partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi) = t^{-1} \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi).$$

By the first point, of Lemma 2.4.1, $\partial_x^\beta \partial_y^\gamma \psi(x, y, 0) = 0$. We apply Taylor's formula to $t \mapsto \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi)$ uniformly for $t \leq 1$, $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$ such that $|\xi| \leq R$ and get

$$t^{-1} \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi) = 0 + \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi(x, y, 0)\xi) + O(t).$$

In particular, as $t \to 0$, $\partial_x^\beta \partial_y^\gamma \psi_t \to \partial_x^\beta \partial_y^\gamma \psi_0$ uniformly for $x, y \in \Omega$ and $\xi \in \mathbb{R}^n$, $|\xi| \leq R$. Next, fix $\alpha \in \mathbb{N}$ and suppose $|\alpha| \geq 1$. Then, for each $x, y \in K$, $\xi \in \mathbb{R}^n$, $|\xi| \leq R$ and $t > 0$,

$$\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi_t(x, y, \xi) = t^{|\alpha|-1} \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi(x, y, t\xi).$$

If $|\alpha| = 1$, as $t \to 0$ the right hand side converges uniformly to $\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma \psi(x, y, 0) = \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi(x, y, 0)\xi)$. On the other hand, if $|\alpha| > 1$, as $t \to 0$ it converges uniformly to $0 = \partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma (\partial_\xi \psi(x, y, 0)\xi)$. This proves the fourth point of Lemma 2.4.1. Lastly, the family $(\psi_t)_{t>0}$ is obviously continuous into $C^\infty(U \times U \times \mathbb{R}^n)$ for $t > 0$. By the fourth point of Lemma 2.4.1 we may extend it by continuity to $t = 0$. On the other hand, by the fifth point of Definition 2.2.1, it also converges as $t \to \infty$. In particular, the family $(\psi_t)_{t \geq 0}$ is uniformly bounded in $C^\infty(U \times U \times \mathbb{R}^n)$. This proves the fifth point of Lemma 2.4.1. □

We use the following lemma to prove Lemma 2.4.5 below, which is the analog of Proposition 2.3.1 we use in dimension $n = 1$. It is the only place where we use the fact that ψ satisfies the eikonal equation (2.2.1).

Lemma 2.4.2. *Assume that $n = 1$. For each segment $I \subset U$ there exists $c \in]0, +\infty[$ such that for each $x, y \in I$ and $\xi \in \mathbb{R}$, $\frac{1}{c}|x - y| \leq |\partial_\xi \psi(x, y, \xi)| \leq c|x - y|$ and $|\partial_\xi^2 \psi(x, y, \xi)| \leq c|x - y|(1 + |\xi|)^{-1}$.*

Proof of Lemma 2.4.2. Let us fix $I \subset U$ a compact interval. Since the symbol σ_A is m-homogeneous and $dim(\mathcal{X}) = 1$ there exists a positive function $\varrho \in C^\infty(U)$ such that $\sigma_A(x, \xi) = \varrho(x)^m |\xi|^m$ for $\xi \neq 0$ and $x \in U$. By construction of ψ there exist $C_1 < +\infty$ and symbols $\tau \in S^0(U \times \mathbb{R})$ and $\sigma \in S^1(U \times \mathbb{R})$ such that $\sigma(x, \xi) = \varrho(x)|\xi| + \tau(x, \xi)$ for $|\xi| \geq C_1$ and $x \in U$ and such that

$$\forall \xi \in \mathbb{R} \setminus [-C_1, C_1],\ \forall x, y \in U,\ \sigma(x, \partial_x \psi(x, y, \xi)) = \sigma(y, \xi).$$

Since $\tau \in S^0$ and since ϱ, being positive and continuous, is bounded from below on I, there exists $C_2 \in [\max(C_1, 1), +\infty[$ such that for any $x \in I$ and $\xi \in \mathbb{R}$ such that $|\xi| \geq C_2$,

$$\frac{1}{2}\varrho(x)|\xi| \leq \sigma(x, \xi) \leq 2\varrho(x)|\xi|\,;$$
$$C_2^{-1} \leq \mathrm{sign}(\xi)\partial_\xi \sigma(x, \xi) \leq C_2.$$

Let $(\sigma^{-1})(x,\cdot)$ be the inverse of $\sigma(x,\cdot) : [C_2, +\infty[\to [\sigma(x, C_2) + \infty[$. Let us fix $x_0 \in I$. Then, for any $x \in I$,

$$\partial_x \psi(x, x_0, \xi) = (\sigma^{-1})(x, \sigma(x_0, \xi)). \tag{2.4.1}$$

Differentiating this Equation with respect to ξ we obtain the following expression for $\partial_\xi \partial_x \psi$.

$$\partial_\xi \partial_x \psi(x, x_0, \xi) = \partial_\xi(\sigma^{-1})(x, \sigma(x_0, \xi)) \partial_\xi \sigma(x_0, \xi)\,.$$

Now, by definition of σ^{-1}, we have, for $x \in I$ and $\xi \in \mathbb{R}$ such that $\xi \geq C_3 = \max_{y \in I} \sigma(y, C_2)$,

$$\partial_\xi(\sigma^{-1})(x,\xi) = \big(\partial_\xi \sigma(x, \sigma^{-1}(x,\xi))\big)^{-1} = \big(\varrho(x) + \partial_\xi \tau(x, \sigma^{-1}(x,\xi))\big)^{-1}\,,$$

where $\varrho(x)$ is bounded on I from above and below by positive constants and $\partial_\xi \tau(x, \sigma^{-1}(x,\xi))$ is $O(|\sigma^{-1}(x,\xi)|^{-1})$ uniformly for $x \in I$. Since $\sigma^{-1}(x,\xi) \xrightarrow[\xi \to +\infty]{} +\infty$ then there exists $C_4 > 0$ such that for any $x \in I$ and any $\xi \geq C_4 \geq \max(C_3, C_2)$,

$$C_4^{-1} \leq \partial_\xi(\sigma^{-1})(x,\xi) \leq C_4. \tag{2.4.2}$$

Therefore,

$$C_2^{-1} C_4^{-1} \leq \partial_x \partial_\xi \psi(x, x_0, \xi) \leq C_2 C_4\,.$$

Recall that, by the first point of Lemma 2.4.1, $\psi(x, x, \xi) = 0$ for any $x \in U$ and any $\xi \in \mathbb{R}$. Thus, for any $x \in I$, $\xi \geq C_4$,

$$|\partial_\xi \psi(x, x_0, \xi)| = \left| \int_{x_0}^{x} \partial_\xi \partial_x \psi(y, x_0, \xi) dy \right| \in \big[C_5^{-1}|x - x_0|, C_5|x - x_0|\big]$$

where $C_5 = C_2 C_4$ is independent of the choice of x_0. The case where $\xi < 0$ is symmetric and this proves the first identity announced in the lemma. For the second identity, we start by differentiating Equation (2.4.1) with respect to ξ to obtain

$$\partial_\xi^2 \partial_x \psi(x, x_0, \xi) = \partial_\xi^2(\sigma^{-1})(x, \sigma(x_0, \xi))(\partial_\xi \sigma(x_0, \xi))^2 + \partial_\xi(\sigma^{-1})(x, \sigma(x_0, \xi)) \sigma_\xi^2(x_0, \xi)\,. \tag{2.4.3}$$

To deal with the second term of the right hand side, observe that, since σ is a symbol of order one and by Equation (2.4.2), there exists a constant $C_6 < +\infty$ such that for any $x, x_0 \in I$ and any $\xi \in \mathbb{R}$,

$$\big|\partial_\xi(\sigma^{-1})(x, \sigma(x_0, \xi)) \sigma_\xi^2(x_0, \xi)\big| \leq C_6 (1 + |\xi|)^{-1}. \tag{2.4.4}$$

For the first term we proceed as follows. By definition of σ^{-1}, we have, for any $x \in I$ and $\xi \geq C_3$,

$$\partial_\xi^2 \sigma(x, \sigma^{-1}(x,\xi))(\partial_\xi(\sigma^{-1})(x,\xi))^2 + \partial_\xi \sigma(x, \sigma^{-1}(x,\xi)) \partial_\xi^2(\sigma^{-1})(x,\xi) = 0.$$

By Equation (2.4.2), since σ is a symbol of order one and since $\partial_\xi \sigma$ is bounded from below on $[C_2, +\infty[$, there exists $C_7 < +\infty$ such that for each $x, x_0 \in I$ and $\xi \geq C_3$,

$$\big|\partial_\xi^2(\sigma^{-1})(x, \sigma(x_0, \xi))(\partial_\xi \sigma(x_0, \xi))^2\big| \leq C_7 (1 + |\xi|)^{-1}\,. \tag{2.4.5}$$

We use Equations (2.4.4) and (2.4.5) on the right hand side of Equation (2.4.3) and get, for each $x, x_0 \in I$ and $\xi \geq C_3$,

$$\left|\partial_x \partial_\xi^2 \psi(x, x_0, \xi)\right| \leq (C_6 + C_7)(1 + |\xi|)^{-1} .$$

As before, since for all $x \in I$ and $\xi \in \mathbb{R}$, $\psi(x, x, \xi) = 0$, we have

$$\left|\partial_\xi^2 \psi(x, x_0, \xi)\right| \leq \int_{x_0}^{x} \left|\partial_x \partial_\xi^2 \psi(y, x_0, \xi)\right| dy \leq C_8 |x - x_0| (1 + |\xi|)^{-1}$$

where $C_8 = C_6 + C_7$. The case $\xi < 0$ is symmetric. □

From Lemma 2.4.1, we deduce the following properties of the function H_P defined in Equation (2.3.6).

Lemma 2.4.3. *The function H_P satisfies the following properties.*

1. *The function $t \mapsto H_P(\cdot, \cdot, \cdot, t)$ extends continuously to $t = 0$ as a function from $\mathbb{R}_+$ to $C^\infty(U \times U \times \mathbb{R}^n)$ and*

$$H_P(x, y, \xi, 0) = \sigma_P\left(x, y, \partial_{x,y}(\partial_\xi \psi(x, y, 0)\xi)\right) .$$

2. *Uniformly for $t \geq 0$ and x, y in compact subsets of U and $\xi \in \mathbb{R}^n$,*

$$H_P(x, y, \xi, t) - H_P(x, y, \xi, 0) = O\left(t|x - y||\xi|^{d+1}\right) .$$

Note that the assertions are both easy to check for the prototype $H_P(x, y, t) = e^{it\langle x-y,\xi\rangle}\sigma_P(x, y, \xi)$.

Remark 2.4.4. Lemma 2.4.3 implies that the function $t \mapsto J_A(\cdot, \cdot, t)$ extends continuously to $t = 0$ as a function from $\mathbb{R}_+$ to $C^\infty(U \times U)$ and

$$J_A(x, y, 0) = \int_{S_y^*} \sigma_P(x, y, \partial_{x,y}(\partial_\xi \psi(x, y, 0)\xi)) d_y\nu(\xi). \tag{2.4.6}$$

Proof. The first statement follows from the fourth point of Lemma 2.4.1. For the second statement, by Equation (2.3.8), we may therefore restrict our attention to the case where $\xi \in S_y^*$. Next, we observe that by the second point of Lemma 2.4.1, $H_P(y, y, \xi, t) = H_P(y, y, \xi, 0)$. The function H_P is clearly C^1 with respect to its first variable so that $|H_P(x, y, \xi, t) - H_P(x, y, \xi, 0)|$ is no greater than

$$|x - y| \sup_{s \in [0,1]} \left|\partial_x H_P(sx + (1 - s)y, y, \xi, t) - \partial_x H_P(sx + (1 - s)y, y, \xi, 0)\right| .$$

Let us fix $\Omega \subset U$ a compact set. Then by Taylor's inequality, there exists $C_1 < +\infty$ such that for each $x, y \in \Omega$, $\xi \in S_y^*$ and each $t > 0$,

$$|\partial_{x,y}\psi(x, y, t\xi) - \partial_{x,y}\psi(x, y, 0) - \partial_{x,y}(\partial_\xi \psi(x, y, 0)\xi)t| \leq C_1 t^2 .$$

By the first point of Lemma 2.4.1, $\partial_{x,y}\psi(x,y,0)=0$, so that

$$\partial_{x,y}\psi_t(x,y,\xi)=\partial_{x,y}(\partial_\xi\psi(x,y,0)\xi)+O(t) \tag{2.4.7}$$

uniformly in $x,y\in\Omega$ and $\xi\in S_y^*$. On the other hand by the fifth point of Lemma 2.4.1, $(\psi_t)_{t>0}$ is bounded in C^∞. In particular, there exists a constant $C_2<+\infty$ such that for each $t>0$, each $x,y\in K$ and each $\xi\in S_y^*$, $|\psi_t(x,y,\xi)|\leq C_2$. In other words

$$\psi(x,y,t\xi)=O(t) \tag{2.4.8}$$

uniformly in $x,y\in\Omega$ and $\xi\in S_y^*$. Applying estimates (2.4.7) and (2.4.8) to each occurrence of ψ in H_P, we see that uniformly for x,y in compact subsets of U and $\xi\in S_y^*$,

$$\partial_x H_P(x,y,\xi,t)=\partial_x H_P(x,y,\xi,0)+O(t)\,,$$

which completes the proof. □

2.4.2 Proof of Lemma 2.3.4 and its analogue in dimension one

In this subsection, we use the results of the previous subsection to prove Lemma 2.3.4. We will use this lemma in the proof of the multi-dimensional case of Theorem 2.2.11 (see Section 2.6). In the one dimensional case, we will use Lemma 2.4.5 presented below.

Proof of Lemma 2.3.4. To prove this lemma, we interpret J_A as an oscillatory integral whose phase is a deformation of $(\omega,\tau)\mapsto\langle\omega,\tau\rangle$. First, fix $\Omega\subset U$ a compact neighborhood of 0. Let $r_0>0$ be such that $\Omega_0=\{x\in\mathbb{R}^n\mid\exists y\in\Omega,\ |y-x|\leq r_0\}\subset U$. By the fourth point of Lemma 2.4.1 the family $(\psi_t)_{t>0}$ extends by continuity to $t=0$ in C^∞. For each $t\geq 0$, $y\in U$, $0<r\leq r_0$ and $\xi,\tau\in\mathbb{R}^n$ such that $|\tau|\leq 1$, let

$$f_{t,r}(y,\xi,\tau)=r^{-1}\psi_t((y+r\tau),y,\xi).$$

Let $\alpha\in\mathbb{N}^n$. The Taylor expansion of $\partial_\xi^\alpha\psi_t(y+r\tau,y,\xi)$ along r yields, for each $y\in\Omega$, $|\tau|\leq 1$, $0<r\leq r_0$, $t\geq 0$ and $\xi\in S_y^*$,

$$\left|\partial_\xi^\alpha\psi_t(y+r\tau,y,\xi)-\partial_\xi^\alpha\langle\xi,\tau\rangle\right|\leq\frac{1}{2}C_1 r$$

where

$$C_1=\sup\{|\partial_x\partial_\xi\psi_s(w',w,\xi)|\mid w\in\Omega,\ w'\in\Omega_0,\ \xi\in S_w^*,\ s\geq 0\}\,.$$

The constant C_1 is finite by the fifth point of Lemma 2.4.1. In particular,

$$\lim_{r\to 0}f_{t,r}(y,\xi,\tau)=\langle\xi,\tau\rangle$$

smoothly in ξ, uniformly in $t\geq 0$, $y\in\Omega$ and $\tau\in\mathbb{R}^n$ such that $|\tau|\leq 1$. In particular, we have proved first that $f_{t,r}(y,\xi,\tau)\xrightarrow[t,r\to 0]{}\langle\xi,\tau\rangle$ in this same topology, and second that for each $\alpha\in\mathbb{N}^n$, the map $(t,r)\to\partial_\xi^\alpha f_{t,r}$ is continuous at $(t,0)$ for any $t\geq 0$ for the topology

of uniform convergence. Since this map is obviously continuous as long as $r > 0$ we have proved that the family $(f_{t,r})_{t,r}$ is a deformation of the height function in the sense of Definition 2.3.2. Now let $x \in U$ be such that $0 < r := |x - y| \leq r_0$ and let $\tau = \frac{x-y}{|x-y|}$. Then $|\tau| = 1$ and

$$\psi(x, y, t\xi) = t|x - y| f_{t,|x-y|}(y, \xi, \tau).$$

Moreover, by the fifth point of Lemma 2.4.1, the function

$$\xi \mapsto \sigma_P(x, y, \partial_{x,y}\psi_t(x, y, \xi))$$

is bounded in $C^\infty(\mathbb{R}^n)$ uniformly for $x, y \in \Omega$ and $t \geq 1$. Hence, the fact that the function σ_A has ε-non-degenerate level sets (see Definition 2.3.2) implies the existence an open neighborhood $V \subset U$ of 0 and a constant $C > 0$ such that, uniformly for $x, y \in V$ and $t > 0$,

$$\Big| \int_{S_y^*} e^{i\psi(x,y,t\xi)} \sigma_P(x, y, \partial_{x,y}\psi_t(x, y, \xi)) d_y\nu(\xi) \Big| \leq C(t|x - y|)^{-\varepsilon}.$$

Here we took $\omega = t|x - y|$ in Equation (2.3.9). □

In dimension $n = 1$, the symbol will never have non-degenerate level sets (in fact they will be discrete). Instead of Lemma 2.3.4 we will use the following result.

Lemma 2.4.5. *Assume that $n = 1$. For each compact interval $I \subset U$, there exists $C < +\infty$ such that for each $0 < a \leq b$, each $\eta \in \{-1, +1\}$ and each $x, y \in I$*

$$\left| \int_{\eta a}^{\eta b} e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P(x, y, |x - y|\partial_{x,y}\psi(x, y, |x - y|^{-1}\eta)\sigma_A(y, \xi)^{-(d+1)/m} d\eta \right| \leq Ca^{-1}\,. \tag{2.4.9}$$

Proof of Lemma 2.4.5. Let $I \subset U$ be a compact interval. First of all, since σ_A is homogeneous of degree m and $n = 1$, there exists a positive function $\varrho \in C^\infty(U)$ such that $\sigma_A(x, \eta) = \varrho(x)|\eta|^m$. Thus, we may replace $\sigma_A(x, \eta)$ by $|\eta|^m$ in Equation (2.4.9). Observe that for each $t, \lambda > 0$, $x, y \in U$ and $\eta \in \mathbb{R}$,

$$\psi_t(x, y, \lambda\eta) = \lambda\psi_{\lambda t}(x, y, \eta)\,.$$

This Equation, combined with the fifth point of Lemma 2.4.1 implies that there exists $C < +\infty$ such that for each $x, y \in I$, each $t > 0$ and $\eta \in \mathbb{R}$

$$|\partial_{x,y}\psi_t(x, y, \eta)| \leq C|\eta| \text{ and } |\partial_{x,y}\partial_\xi\psi_t(x, y, \eta)| \leq C\,.$$

Since moreover σ_P is homogeneous of degree d in the third variable, we have, uniformly for $x, y \in I$ and for non-zero $\eta \in \mathbb{R} \setminus \{0\}$,

$$\sigma_P(x, y, |x - y|\partial_{x,y}\psi(x, y, |x - y|^{-1}\eta))|\eta|^{-d-1} = \sigma_P(x, y, \partial_{x,y}\psi_{|x-y|^{-1}}(x, y, \eta))|\eta|^{-d-1} = O(|\eta|^{-1})$$

$$\partial_\eta[\sigma_P(x, y, |x - y|\partial_{x,y}\psi(x, y, |x - y|^{-1}\eta))|\eta|^{-d-1}] = \partial_\eta[\sigma_P(x, y, \partial_{x,y}\psi_{|x-y|^{-1}}(x, y, \eta))|\eta|^{-d-1}] = O(|\eta|^{-2}).$$

In addition, again uniformly for $x, y \in I$ and non-zero $\eta \in \mathbb{R} \setminus \{0\}$, by Lemma 2.4.2, $\partial_\eta^2[\psi(x,y,|x-y|^{-1}\eta)] = O(|\eta|^{-1})$ and $\partial_\eta[\psi(x,y,|x-y|^{-1}\eta)]$ is bounded from above and below by a positive constant. Now, setting momentarily $u(\eta) := \psi(x,y,|x-y|^{-1}\eta)$ and $v(\eta) = \sigma_P(x,y,|x-y|\partial_{x,y}\psi(x,y,|x-y|^{-1}\eta))|\eta|^{-d-1}$, we have, for any $a, b > 0$ such that $a \leq b$,

$$\int_a^b e^{iu(\eta)} v(\eta) d\eta = \Big[\frac{1}{i} e^{iu(\eta)} \frac{v(\eta)}{u'(\eta)}\Big]_{\eta=a}^b - \int_a^b \frac{1}{i} e^{iu(\eta)} \left(\frac{v'(\eta)}{u'(\eta)} - \frac{v(\eta)u''(\eta)}{u'(\eta)^2}\right) d\eta.$$

The preceding observations show that, uniformly for $x, y \in I$, $0 < a \leq b$ and $\eta \in [a,b]$, we have $\frac{v(a)}{u'(a)} = O(a^{-1})$, $\frac{v(b)}{u'(b)} = O(b^{-1})$, $\frac{v'(\eta)}{u'(\eta)} = O(\eta^{-2})$ and $\frac{v(\eta)u''(\eta)}{u'(\eta)^2} = O(\eta^{-2})$. Consequently, there exists $C < +\infty$ such that for any $x, y \in \Omega$ and any $0 < a \leq b$,

$$\Big| \int_a^b e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P(x,y,|x-y|\partial_{x,y}\psi(x,y,|x-y|^{-1}\eta))|\eta|^{-d-1} d\eta \Big| \leq Ca^{-1}.$$

The proof for $\int_{-b}^{-a}$ is identical. □

2.5 Proof of Theorem 2.2.14, Proposition 2.3.5 and Theorem 2.2.9

In this section, we prove Theorem 2.2.14, Proposition 2.3.5 and Theorem 2.2.9. We use only Theorem 2.2.2 and Lemma 2.4.1.

Let $f : \mathbb{R} \to \mathbb{R}$ with support in $]0, +\infty[$ differentiable almost everywhere. For each $L \geq 1$, let K_L^f be the integral kernel of $\Pi_L f(A)$. Later in the section, we will be interested in a special case of K_L^f. More precisely, we fix $z = z_1 + iz_2 \in \mathbb{C}$ and set $K_L = K_L^f$ where f is chosen so that $f(t) = t^z$ for $t > 0$ large enough. We begin by linking K_L^f with E_L.

Lemma 2.5.1. *For any $L \in \mathbb{R}$,*

$$K_L^f = f(L)E_L - \int_0^L f'(\lambda) E_\lambda d\lambda.$$

This lemma generalizes Proposition 21 of [Riv17].

Proof. The functions $L \mapsto E_L$ and $L \mapsto K_L^f$ are locally constant and define distributions on $\mathbb{R}$ with values in $C^\infty(\mathcal{X} \times \mathcal{X})$. We denote by $'$ the weak derivative with respect to L of these kernels. For each $u, v \in C^\infty(\mathcal{X})$ we let $u \boxtimes v \in C^\infty(\mathcal{X} \times \mathcal{X})$ be the function $(u \boxtimes v)(x,y) = u(x)v(y)$. For all $L > 0$,

$$E_L = \sum_{\lambda_k \leq L} e_k \boxtimes \overline{e_k};\ K_L^f = \sum_{\lambda_k \leq L} f(\lambda_k) e_k \boxtimes \overline{e_k},$$

so that

$$\left(K_L^f\right)' = \sum_{k\in\mathbb{N}} \delta_{\lambda_k}(L) f(\lambda_k) e_k \boxtimes \overline{e_k} = f(L) \sum_{k\in\mathbb{N}} \delta_{\lambda_k}(L) e_k \boxtimes \overline{e_k} = f(L) E_L'$$

and

$$K_L^f = \int_0^L f(\lambda) E_\lambda' d\lambda.$$

By integration by parts,

$$K_L^f = f(L)E_L - f(0)E_0 - \int_0^L f'(\lambda)E_\lambda d\lambda = f(L)E_L - \int_0^L f'(\lambda)E_\lambda d\lambda$$

since $f(0) = 0$. □

We can now prove both Theorem 2.2.14 and Proposition 2.3.5 using Theorem 2.2.2. We start with Theorem 2.2.14.

Proof of Theorem 2.2.14. Let $L > 0$. Then, by Lemma 2.5.1, we have, for each $t \geq L$,

$$K_{L+t}^f = (L+t)^z E_{L+u} - \int_0^{L+t} f'(\lambda)E_\lambda d\lambda\,.$$

Now, if instead of K_L^f we consider the special case K_L, and if we apply the operator P, then, for all large enough values of $L > 0$ and all $t \geq 0$,

$$PK_{L+t} - PK_L = (L+t)^z PE_{L+t} - L^z PE_L - \int_L^{L+t} z\lambda^{z-1} PE_\lambda d\lambda\,.$$

By Theorem 2.2.2, we have, uniformly for $(x, y) \in U \times U$ and $t \geq 0$,

$$(L+t)^z PE_{L+t}(x,y) = O\left(L^{z_1+(n+d)/m}\right)$$

and

$$\int_L^{L+t} z\lambda^{z-1} E_\lambda d\lambda = O\left(\int_L^{+\infty} \lambda^{-1+z_1+(n+d)/m} d\lambda\right) = O\left(L^{z_1+(n+d)/m}\right)\,.$$

In particular, uniformly for $(x, y) \in U \times U$ and $t \geq 0$,

$$PK_{L+t}(x,y) - PK_L(x,y) = O\left(L^{z_1+(n+d)/m}\right)\,.$$

Since, $z_1 + (n+d)/m < 0$, this last estimate implies that the sequence $(PK_L)_{L>0}$ is a Cauchy sequence in $C^0(U \times U)$. Therefore, it converges uniformly on compact subsets of $U \times U$ to some function $K_\infty^P \in C^0(U \times U)$. Since this is actually true for any differential operator of order at most d (indeed, if $d' \leq d$, we still have $z_1 + (n+d')/m < 0$), all the derivatives of K_L, of order up to d, converge uniformly on compact sets. But this means that the limit K_∞ of $(K_L)_{L>0}$ is actually of class C^d and that the limits of the respective derivatives converge to the derivatives of the limit. In particular, $K_\infty^P = PK_\infty$. □

We now move on to Proposition 2.3.5.

Proof of Proposition 2.3.5. By Theorem 2.2.2, uniformly for $x, y \in U$ and $L \geq 1$,

$$\begin{aligned} PE_L(x,y) &= \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi)\leq L} e^{i\psi(x,y,\xi)} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi)) d\xi + O\left(L^{(n+d-1)/m}\right) \\ &= \frac{1}{(2\pi)^n} \int_0^{L^{1/m}} J_A(x,y,t) t^{n+d-1} dt + O\left(L^{(n+d-1)/m}\right). \end{aligned}$$

In the second equality we used the definition of $d\nu$ (see (2.1.1)) and J_A (see (2.3.7)) as well as the fact that σ_P is d-homogeneous along the fibers. Consequently, uniformly for any $x, y \in U$ and $L \geq 1$,

$$\begin{aligned} &- \int_0^L f'(\lambda) PE_\lambda(x,y) d\lambda = \\ &\quad - \frac{1}{(2\pi)^n} \int_0^L f'(\lambda) \int_0^{\lambda^{1/m}} J_A(x,y,t) t^{n+d-1} dt d\lambda + O\left(\int_{-\infty}^L f'(\lambda) \lambda^{(n+d-1)/m} d\lambda \right). \end{aligned}$$

Integrating by parts along λ the first term in the right hand side, we get

$$-f(L)PE_L(x,y) + \frac{1}{(2\pi)^n} \int_0^L f(\lambda) \frac{1}{m} \lambda^{\frac{1}{m}-1} J_A(x,y,\lambda^{1/m}) \lambda^{(n+d-1)/m} d\lambda + O\left(f(L) L^{(n+d-1)/m}\right).$$

Setting $u = \lambda^{1/m}$ we get

$$\begin{aligned} \int_0^L f(\lambda) \frac{1}{m} \lambda^{\frac{1}{m}-1} J_A(x,y,\lambda^{1/m}) \lambda^{(n+d-1)/m} d\lambda &= \int_0^{L^{1/m}} f(u^m) J_A(x,y,u) u^{n+d-1} du \\ &= \int_{\sigma_A(y,\xi)\leq L} e^{i\psi(x,y,\xi)} f(\sigma_A(y,\xi)) \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi)) d\xi. \end{aligned}$$

By Lemma 2.5.1,

$$PK_L^f = f(L)PE_L - \int_0^L f'(\lambda) PE_\lambda d\lambda.$$

Replacing the integral term by the expression derived above, we see that the $f(L)PE_L$ terms cancel out, leaving the equation from the first result of Proposition 2.3.5. For the case where $(x,y) \in U \times U \setminus W$, we just apply the corresponding estimate from Theorem 2.2.2 and proceed accordingly. □

For the proof of Theorem 2.2.9, we remind teh reader that $K_L = K_L^f$ where $f(t) = t^z$ for $t > 0$ large enough.

Proof of Theorem 2.2.9. Throughout the proof, we let $\eta = 1$ if $n + d + mz = 1$ and 0 otherwise and set $g(L) = L^{(n+d-1)/m+z_1} \ln(L)^\eta$. Let Ω be a compact neighborhood of 0 in U such that for any $w, x \in \Omega$ and $L \geq 1$, $w + L^{-1/m}x$ belongs to U. Firstly, changing f on a compact set affects PK_L by adding a linear combination of smooth functions

(independent of L). Thus, we may assume that $f(t) = t^z \mathbb{1}[t \geq 1]$. By Proposition 2.3.5, uniformly for $w, x, y \in \Omega$ and $L \geq 1$, $PK_L\left(w + L^{-1/m}x, w + L^{-1/m}y\right)$ equals

$$\frac{1}{(2\pi)^n} \int_{1 \leq \sigma_A(w+L^{-1/m}y,\xi) \leq L} \sigma_A(w + L^{-1/m}y, \xi)^z e^{i\psi(w+L^{-1/m}x, w+L^{-1/m}y, \xi)} \tag{2.5.1}$$
$$\times \sigma_P\left(w + L^{-1/m}x, w + L^{-1/m}y, \partial_{x,y}\psi\left(w + L^{-1/m}x, w + L^{-1/m}y, \xi\right)\right) d\xi + O(g(L)) .$$

Indeed, since $n+d+z_1 > 0$, $O(g(L))+O(1) = O(g(L))$. We need to check that replacing each occurrence of $w + L^{-1/m}y$ or $w + L^{-1/m}x$ by w in the integrand will produce an error of order $O(g(L))$. More precisely, we make the following claim.

Claim 2.5.2. *Uniformly for $w, x, y \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $L \geq 1$ such that $1 \leq \sigma_A(w + L^{-1/m}y, \xi) \leq L$, the quantity*

$$\sigma_A(w + L^{-1/m}y, \xi)^z e^{i\psi(w+L^{-1/m}x, w+L^{-1/m}y, \xi)} \tag{2.5.2}$$
$$\times \sigma_P\left(w + L^{-1/m}x, w + L^{-1/m}y, \partial_{x,y}\psi\left(w + L^{-1/m}x, w + L^{-1/m}y, \xi\right)\right)$$

equals

$$e^{iL^{-1/m}\langle \xi, x-y\rangle} \sigma_A(w, \xi)^z \sigma_P(w, w, (\xi, -\xi)) + O\left(|\xi|^{mz_1+d} L^{-1/m}\right) . \tag{2.5.3}$$

Proof. Throughout the proof we fix $w, x, y \in \Omega$, $L \geq 1$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\sigma_A(w + L^{-1/m}y, \xi) \leq L$. Unless otherwise stated, all the O estimates will be uniform with respect to these parameters. First of all, since σ_A is a positive m-homogeneous symbol in its second variable, σ_P is a symbol of order d in its third variable and $\partial_{x,y}\psi$ is a symbol of order 1 in its third variable, applying Taylor's inequality with respect to the L-dependent variables everywhere except the exponential in the quantity (2.5.2) shows that it equals

$$\sigma_A(w, \xi)^z e^{i\psi(w+L^{-1/m}x, w+L^{-1/m}y, \xi)} \sigma_P\left(w, w, \partial_{x,y}\psi\left(w, w, \xi\right)\right) + O\left(|\xi|^{mz_1+d} L^{-1/m}\right) . \tag{2.5.4}$$

Here the $|\xi|^{mz_1}$ appears regardless of the sign of z_1 because σ_A is positive homogeneous. Since ψ is a symbol of order one in ξ and $|\xi| = O\left(L^{1/m}\right)$,

$$\psi(w + L^{-1/m}x, w + L^{-1/m}y, \xi) =$$
$$\psi(w, w, \xi) + \partial_x\psi(w, w, \xi)L^{-1/m}x + \partial_y\psi(w, w, \xi)L^{-1/m}y + O\left(L^{-1/m}\right) .$$

By points two and three of Lemma 2.4.1 we get

$$\psi(w + L^{-1/m}x, w + L^{-1/m}y, \xi) = L^{-1/m}\langle x - y, \xi\rangle + O\left(L^{-1/m}\right) .$$

Using this estimate in the exponential, together with the fact the rest of the integrand is $O\left(|\xi|^{mz_1+d}\right)$ we obtain that the quantity (2.5.4) equals

$$e^{iL^{-1/m}\langle\xi,x-y\rangle}\sigma_A(w,\xi)^z\sigma_P(w,w,(\xi,-\xi))+O\left(|\xi|^{mz_1+d}L^{-1/m}\right)$$

which is exactly (2.5.3). □

By Claim 2.5.2 and Equation (2.5.1) $PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)$ equals

$$PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)= \tag{2.5.5}$$

$$\frac{1}{(2\pi)^n}\int_{1\leq\sigma_A(w+L^{-1/m}y,\xi)\leq L} e^{iL^{-1/m}\langle\xi,x-y\rangle}\sigma_A(w,\xi)^z\sigma_P(w,w,(\xi,-\xi))d\xi \tag{2.5.6}$$

$$+O\left(L^{-1/m}\int_{1\leq\sigma_A(w+L^{-1/m}y,\xi)\leq L}|\xi|^{mz_1+d}d\xi\right).$$

But since $mz_1+d+n>0$ and σ_A is m-homogeneous, the remainder is $O\left(L^{z_1+(n+d-1)/m}\right)=O(g(L))$. For each $y\in\Omega$ and each $L\geq 1$ let $\Delta(y,L)$ be the symmetric difference of the sets $\{\xi\in\mathbb{R}^n \mid 1\leq\sigma_A(w,\xi)\leq L\}$ and $\{\xi\in\mathbb{R}^n \mid 1\leq\sigma_A(w+L^{-1/m}y,\xi)\leq L\}$. Since σ_A is positive m-homogeneous in ξ and smooth in y, there exists $0<C<+\infty$ such that for each $L\geq 1$ and $w\in\Omega$, $Vol(\Delta(w,L))\leq CL^{(n-1)/m}$ and for each $\xi\in\Delta(w,L)$, $C^{-1}L^{1/m}\leq|\xi|\leq CL^{1/m}$. Consequently, in Equation (2.5.5) we can replace the integration domain by $\{\xi\in\mathbb{R}^n \mid 1\leq\sigma_A(w,\xi)\leq L\}$ and produce an error of order $O\left(L^{z_1+(n+d-1)/m}\right)=O(g(L))$ uniformly for $y\in\Omega$ and $L\geq 1$. In other words,

$$PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)=$$
$$\frac{1}{(2\pi)^n}\int_{1\leq\sigma_A(w,\xi)\leq L} e^{iL^{-1/m}\langle\xi,x-y\rangle}\sigma_A(w,\xi)^z\sigma_P(w,w,(\xi,-\xi))d\xi+O(g(L)).$$

Moreover, since $mz_1+d+n>0$ and the integrand scales like $|\xi|^{mz_1+d}$ near 0, adding the region $\sigma_A(w,\xi)\leq 1$ to the integration domain creates a bounded error. Following this by the change of variable $\xi=L^{1/m}\zeta$ shows that uniformly for $w,x,y\in\Omega$ and $L\geq 1$

$$PK_L\left(w+L^{-1/m}x,w+L^{-1/m}y\right)=$$
$$\frac{1}{(2\pi)^n}\int_{\sigma_A(w,\zeta)\leq 1} e^{i\langle\zeta,x-y\rangle}\sigma_A(w,\zeta)^z\sigma_P(w,w,(\zeta,-\zeta))d\zeta L^{z+(n+d)/m}+O(g(L)).$$

This proves the first statement of the theorem for $V=\mathring{\Omega}$. To prove the second statement, observe that by Lemma 2.5.1, uniformly for $L\geq 1$ and $x,y\in\Omega$,

$$PK_L(x,y)=f(L)PE_L-\int_0^L f'(\lambda)PE_\lambda(x,y)d\lambda=L^zPE_L(x,y)-\int_1^L\lambda^{z-1}PE_\lambda(x,y)d\lambda+O(1)$$

Next, fix $W \subset V \times V$ a neighborhood of the diagonal. By Theorem 2.2.2, there exists $C' > 0$ such that for any $(x, y) \in (V \times V) \backslash W$ and any $L \geq 1$, $|PE_L(x, y)| \leq C' L^{(n+d-1)/m}$, which implies

$$|PK_L(x, y)| \leq C' \left(L^{z_1+(n+d-1)/m} + \int_1^L \lambda^{z_1-1+(n+d-1)/m} d\lambda \right) = O(g(L)) .$$

This proves the second statement of Theorem 2.2.9. □

2.6 Proof of Theorem 2.2.11

In this section we prove Theorem 2.2.11. We use the admissibility condition through Proposition 2.3.3. Suppose that $n + d + mz = 0$, so that $z = -\frac{d+n}{m}$. By Proposition 2.3.5, uniformly for $x, y \in U$,

$$PK_L(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi) \leq L} e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x, y, \xi)) f(\sigma_A(y, \xi)) d\xi + O\left(L^{-1/m}\right) .$$

Let $C < +\infty$ be such that $f(t) = t^z$ for $t > C$. Then,

$$PK_L(x, y) = \frac{1}{(2\pi)^n} \int_{C \leq \sigma_A(y,\xi) \leq L} e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x, y, \xi)) \sigma_A(y, \xi)^{-(d+n)/m} d\xi + Q_1(x, y) + O\left(L^{-1/m}\right)$$

where

$$Q_1(x, y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi) \leq C} e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x, y, \xi)) f(\sigma_A(y, \xi)) d\xi .$$

We will split the integral term in the last expression of PK_L as follows. For any $x, y \in U$, let

$$\begin{aligned} I_L(x, y) =& \frac{1}{(2\pi)^n} \int_{C \leq \sigma_A(y,\xi) \leq L} \mathbb{1}[\sigma_A(y, \xi)|x - y|^m \geq 1] \times \\ & e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x, y, \xi)) \sigma_A(y, \xi)^{-(d+n)/m} d\xi \\ II_L(x, y) =& \frac{1}{(2\pi)^n} \int_{C \leq \sigma_A(y,\xi) \leq L} \mathbb{1}[\sigma_A(y, \xi)|x - y|^m < 1] \times \\ & e^{i\psi(x,y,\xi)} \sigma_P(x, y, \partial_{x,y}\psi(x, y, \xi)) \sigma_A(y, \xi)^{-(d+n)/m} d\xi. \end{aligned}$$

Then, uniformly for $x, y \in U$,

$$PK_L(x, y) = I_L(x, y) + II_L(x, y) + Q_1(x, y) + O\left(L^{-1/m}\right) . \tag{2.6.1}$$

Theorem 2.2.11 is an easy consequence of the following two lemmas.

Lemma 2.6.1. *Let $k_0 \in \mathbb{N}$, $k_0 \geq 2$. Suppose that either $n = 1$ or σ_A is $\frac{1}{k_0}$-admissible. There exist an open neighborhood $V \subset U$ of $0 \in \mathbb{R}^n$, a function $Q_2 \in L^\infty(V \times V)$ and a constant $C < +\infty$ such that for any $x, y \in V$ and $L \geq 1$,*

$$\left| I_L(x,y) - Q_2(x,y) \right| \leq C \min\left(L^{-1/k_0 m} |x-y|^{-1/k_0}, 1 \right) .$$

In dimension one, we prove the lemma using Lemma 2.4.5 while in the case of admissible symbols we use Proposition 2.3.1. This proof is the only place where we use these results.

Lemma 2.6.2. *There exist an open neighborhood $V \subset U$ of 0 and a constant $C < +\infty$ such that for all $x, y \in V$ and $L \geq 1$,*

$$\left| II_L(x,y) - \frac{1}{(2\pi)^n} Y_P(x,y) \left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right) \right] \right| \leq C .$$

Moreover $II_L(x,y)$ is independent of L as long as $L \geq 1$ and $L|x-y|^m \geq 1$.

Let us first prove that these lemmas imply Theorem 2.2.11.

Proof of Theorem 2.2.11. Let V be the intersection of the V's appearing in Lemmas 2.6.1 and 2.6.2. Firstly, Lemma 2.6.1 implies that $I_L(x,y)$ is uniformly bounded for $x, y \in V$ and $L \geq 1$. Secondly, Lemma 2.6.2 implies that, uniformly for $x, y \in V$ and $L \geq 1$,

$$II_L(x,y) = \frac{1}{(2\pi)^n} Y_P(x,y) \left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right) \right] + O(1) .$$

Plugging these two estimates in Equation (2.6.1) we get the first point of Theorem 2.2.11. For the second point, we begin by observing that by Lemma 2.6.2, there exists a bounded function $Q_3 \in L^\infty(V \times V)$ such that for each $L \geq |x-y|^{-m}$,

$$II_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(x,y) \ln\left(|x-y|\right) + Q_3(x,y) .$$

Moreover, if $L \geq |x-y|^{-m}$ then $L^{-1/k_0}|x-y|^{-1/k_0 m} \leq 1$ so by Lemma 2.6.1, uniformly for any such x, y and L,

$$I_L(x,y) = Q_2(x,y) + O\left(L^{-1/k_0} |x-y|^{-1/k_0 m} \right) .$$

Applying these two estimates to Equation (2.6.1) we deduce that, uniformly for $x, y \in V$ and $L \geq 1$ such that $|x-y| \geq L^{-1/m}$,

$$PK_L(x,y) = -\frac{1}{(2\pi)^n} Y_P(x,y) \ln\left(|x-y|\right) + Q(x,y) + O\left(L^{-1/k_0} |x-y|^{-1/k_0 m} \right)$$

where $Q = Q_1 + Q_2 + Q_3 \in L^\infty(V \times V)$. This proves the estimate in the second point of Theorem 2.2.11. □

Proof of Lemma 2.6.1. Suppose first that $\mathcal{X}$ has dimension $n = 1$ and fix $\Omega \subset U$ a compact neighborhood of 0. For $x \neq y$, setting $\eta = |x-y|\xi$, the integral $I_L(x,y)$ equals

$$\int_{a(x,y)}^{b(x,y,L)} e^{i\psi(x,y,|x-y|^{-1}\eta)} \sigma_P\left(x, y, |x-y|\partial_{x,y}\psi\left(x, y, |x-y|^{-1}\eta\right)\right) \sigma_A(y,\eta)^{-(d+1)/m} d\eta.$$

where $a(x,y)$ and $b(x,y,L)$ are the positive numbers defined by $\sigma_A(y, a(x,y)) = \max\left(C|x-y|^m, 1\right)$ and $\sigma_A(y, b(x,y,L)) = \max\left(|x-y|^m L, 1\right)$. Since σ_A is elliptic positive homogeneous of degree $m > 0$ there exists $C_1 > 0$ such that for each $x, y \in \Omega$ and $L \geq 1$,

$$b(x,y,L) \geq C_1 \min\left(|x-y|L^{-1/m}\right).$$

By Lemma 2.4.5, $I_L(x,y)$ converges to some limit $Q_2(x,y)$ as $L \to +\infty$ in such a way that the remainder term is $O\left(\min\left(|x-y|^{-1}L^{-1/m}, 1\right)\right)$. The case where $x = y$ follows by continuity and we have proved the lemma in the one-dimensional case with $V = \mathring{\Omega}$.

Suppose now that $n \geq 2$ and σ_A is $\frac{1}{k_0}$-admissible for some integer $k_0 \geq 2$. By Equations (2.1.1) and (2.3.7), for any $L \geq 1$ and $x, y \in U$,

$$\begin{aligned} I_L(x,y) &= \frac{1}{(2\pi)^n} \int_{C^{1/m}}^{L^{1/m}} \mathbb{1}[|x-y|t \geq 1] J_A(x,y,t) \frac{dt}{t} \\ &= \frac{1}{(2\pi)^n} \int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}[s \geq 1] J_A\left(x, y, |x-y|^{-1}s\right) \frac{ds}{s} \end{aligned}$$

By Proposition 2.3.1, there exist an open neighborhood $V \subset U$ of 0 and a constant $C_3 > 0$ such that, uniformly for distinct $x, y \in V$ and $t > 0$, $|J_A(x,y,t)| \leq C_3 \left(|x-y|t\right)^{-1/k_0}$. Therefore, for each $x, y \in V$ and $L > 0$,

$$\begin{aligned} \left|(2\pi)^n I_L(x,y) - \int_{C^{1/m}|x-y|}^{+\infty} \mathbb{1}[s \geq 1] J_A\left(x, y, |x-y|^{-1}s\right) \frac{ds}{s}\right| &\leq C_3 \int_{\max\left(|x-y|L^{\frac{1}{m}}, 1\right)}^{+\infty} s^{-1-1/k_0} ds \\ &= \frac{C_3}{k_0} \min\left(1, L^{-1/k_0 m}|x-y|^{-1/k_0}\right). \end{aligned}$$

By continuity, this stays true for $x = y$. This proves the lemma for σ_A admissible with

$$Q_2(x,y) = \int_{C^{1/m}|x-y|}^{+\infty} \mathbb{1}[s \geq 1] J_A\left(x, y, |x-y|^{-1}s\right) \frac{ds}{s}.$$

□

Proof of Lemma 2.6.2. The proof of the second statement is obvious from the definition of II_L and the expression $\ln\left(L^{\frac{1}{m}}\right) - \ln_+\left(L^{\frac{1}{m}}|x-y|\right)$. We now prove the first statement. For each $y \in U$ and each $0 \leq r_1 \leq r_2$, we set

$$\mathcal{A}_y(r_1, r_2) = \left\{\xi \in \mathbb{R}^n \,|\, r_1 \leq \sigma_A(y,\xi) \leq r_2\right\}.$$

Recall that

$$II_L(x,y) = \frac{1}{(2\pi)^n}\int_{\mathcal{A}_y(C,L)} \mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1] e^{i\psi(x,y,\xi)}\sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))\sigma_A(y,\xi)^{-(n+d)/m} d\xi..$$

By Equation (2.3.6), the integrand equals

$$\mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1] H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, \sigma_A(y,\xi)^{1/m}\right)\sigma_A(y,\xi)^{-n/m}\,.$$

Since σ_A is positive homogeneous of degree m, $\sigma_A(y,\xi)^{-1/m}\xi$ is uniformly bounded for $y \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. By the second point of Lemma 2.4.3, uniformly for $x, y \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, \sigma_A(y,\xi)^{1/m}\right) = H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, 0\right) + O\left(|x-y|\sigma_A(y,\xi)^{1/m}\right)\,.$$

Again by m-homogeneity and positivity, $|x-y|\mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1]\sigma_A(y,\xi)^{(1-n)/m}$ is uniformly integrable in ξ for $x, y \in \Omega$ so

$$II_L(x,y) = \frac{1}{(2\pi)^n}\int_{\mathcal{A}_y(C,L)} \mathbb{1}[\sigma_A(y,\xi)|x-y|^m < 1] H_P\left(x,y,\sigma_A(y,\xi)^{-1/m}\xi, 0\right)\sigma_A(y,\eta)^{-n/m} d\xi + O(1)\,.$$

Fix two distinct points $x, y \in U$. The change of variables $\eta = |x-y|\xi$ in the integral yields

$$\int_{|x-y|\mathcal{A}_y(C,L)} \mathbb{1}[\sigma_A(y,\eta) < 1] H_P(x,y,|x-y|\eta,0)\sigma_A(y,\eta)^{-n/m} d\eta$$

which, by definition of J_A (see Equation (2.3.7)), equals

$$J_A(x,y,0)\int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}[|x-y|s < 1]\frac{ds}{s}\,.$$

Observe that for any $0 < a \le b$,

$$\int_a^b \mathbb{1}[t<1]\frac{dt}{t} = \ln(b) - \ln_+(b) - \ln(a) + \ln_+(a)$$

where $\ln_+(s) = \max(\ln(s), 0)$. In our setting, uniformly for distinct $x, y \in \Omega$,

$$\int_{C^{1/m}}^{L^{1/m}} \mathbb{1}[|x-y|s<1]\frac{ds}{s} = \int_{C^{1/m}|x-y|}^{L^{1/m}|x-y|} \mathbb{1}[t<1]\frac{dt}{t} = \ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right) + O(1)\,.$$

Hence, uniformly for any $(x,y) \in \Omega \times \Omega$ and $L \ge 1$,

$$II_L(x,y) = \frac{1}{(2\pi)^n} J_A(x,y,0)\left[\ln\left(L^{1/m}\right) - \ln_+\left(L^{1/m}|x-y|\right)\right] + O(1).$$

Finally, by Equation (2.4.6) $J_A(x,y,0) = Y_P(x,y)$ so the lemma is proved with $V = \mathring{\Omega}$. □

2.7 Admissible symbols

In this section, we deal with results concerning admissible symbols (see Definition 2.2.10). These results are useful in the proofs of Theorems 2.2.13 and 2.2.11. More precisely, in Subsection 2.7.1 we prove Proposition 2.3.3 which says that admissible symbols have non-degenerate level sets and is used in the proof of Theorem 2.2.11. Then, in Proposition 2.7.8 of Subsection 2.7.2 we prove that admissibility is both stable and generic in a suitable topology. Theorem 2.2.13 follows directly from Proposition 2.7.8.

2.7.1 Proof of Proposition 2.3.3

The object of this subsection is to prove Proposition 2.3.3. To prove this result, we will use partitions of unity and local charts to carry the integral onto $\mathbb{R}^n$ and then apply the following lemma, which we prove later in the section.

Lemma 2.7.1. *Let $n \in \mathbb{N}$, $n \geq 1$. Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 and $(f_\eta)_{\eta \in E}$ be a continuous family of smooth functions on U indexed by $E \subset \mathbb{R}^p$, an open neighborhood of 0. Fix $k \geq 1$ and assume that $d_0^k f_0 \neq 0$. Then, there exist $E' \subset E$ and $U' \subset U$ two open neighborhoods of the origin in $\mathbb{R}^p$ and $\mathbb{R}^n$ respectively, such that for each $u \in C_c^\infty(U')$ there exists $C(u) < +\infty$ such that for each $\lambda > 0$ and each $\eta \in E'$,*

$$\left| \int_{U'} e^{i\lambda f_\eta(x)} u(x) dx \right| \leq C(u) \lambda^{-\frac{1}{k}} .$$

Moreover, $C(u)$ depends continuously on u in the $C_c^\infty(U')$ topology.

We now begin the proof of Proposition 2.3.3.

Proof of Proposition 2.3.3. Take Ω, γ, $(f_\eta)_\eta$ and $(u_\eta)_\eta$ as in Definition 2.3.2. Recall that $d_x\mu$ is the area measure on S_x^*. By using partitions of unity on $\mathbb{R}^n$, we may fix $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and assume that the functions u_η are supported near ξ_0. Let $\xi_1, \dots, \xi_{n-1} \in \mathbb{R}^n$ be such that $(\xi_0, \xi_1, \dots, \xi_{n-1})$ forms a basis for $\mathbb{R}^n$. For any $x \in S^*U$, let

$$\beta_x : (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1} \mapsto \sigma(x, \xi_0 + t_1\xi_1 + \cdots + t_{n-1}\xi_{n-1})^{-\frac{1}{m}} (\xi_0 + t_1\xi_1 + \cdots + t_{n-1}\xi_{n-1}) \in S_x^* .$$

The map β_x defines a local coordinate system at $\sigma(x, \xi_0)^{-\frac{1}{m}} \xi_0 \in S_x^*$. Moreover, the map $x \mapsto \beta_x \in C^\infty(\mathbb{R}^{n-1})$ is continuous. The density $g_x = \frac{\beta_x^*(\gamma(x,\cdot) d_x\mu)}{dt} \in C^\infty(\mathbb{R}^{n-1})$ also depends continously on x in $C^\infty(\mathbb{R}^{n-1})$. Now, for any $\lambda > 0$, $\eta \in E$ and $(x, \tau) \in \Omega$, if u_η is supported close enough to ξ_0,

$$\int_{S_x^*} e^{i\lambda f_\eta(x,\tau,\xi)} u_\eta(\xi) \gamma(x, \xi) d_x\mu(\xi) = \int_{\mathbb{R}^{n-1}} e^{i\lambda f_\eta(x,\tau,\beta_x(t))} u_\eta(\beta_x(t)) g_x(t) dt .$$

We now set $\tilde{E} = U \times \mathbb{R}^n \times E$, for any $\tilde{\eta} = (x, \tau, \eta) \in \tilde{E}$, $\tilde{f}_{\tilde{\eta}} = f_\eta(x, \tau, \beta_x(\cdot)) \in C^\infty(\mathbb{R}^{n-1})$ and $\tilde{u}_{\tilde{\eta}} = u_\eta(\beta_x(\cdot)) g_x \in C^\infty(\mathbb{R}^{n-1})$. By compactness, it is enough to fix $(x_0, \tau_0) \in \Omega$ and prove estimate (2.3.9) for $\tilde{\eta} = (x, \eta, \tau)$ close enough to $\tilde{\eta}_0 = (x_0, 0, \tau_0)$. Also, without

loss of generality, we may assume $x_0 = 0$. Our task is therefore to find $C > 0$ such that for each $\tilde{\eta}$ close enough to $\tilde{\eta}_0$ and each $\lambda > 0$,

$$\left| \int_{\mathbb{R}^{n-1}} e^{i\lambda \tilde{f}_{\tilde{\eta}}(t)} \tilde{u}_{\tilde{\eta}}(t) dt \right| \leq C\lambda^{-\frac{1}{k_0}} .$$

We wish to apply Lemma 2.7.1. The estimate is obvious for $\lambda \leq 1$ while, for $\lambda \geq 1$, replacing k_0 by some smaller integer would improve the estimate. Thus, we need only to check that there exists $k \in \{1, \dots, k_0\}$ such that

$$d_0^k \tilde{f}_{\tilde{\eta}_0} \neq 0 . \tag{2.7.1}$$

Let $g = \tilde{f}_{\tilde{\eta}_0}$. Since $f_0(x, \tau_0, \xi) = \langle \tau_0, \xi \rangle$, we have, for all $t \in \mathbb{R}^{n-1}$,

$$g(t) = (\langle \tau_0, \xi_1 \rangle t_1 + \cdots + \langle \tau_0, \xi_{n-1} \rangle t_{n-1} + \langle \tau_0, \xi_0 \rangle) \sigma(0, \xi_0 + t_1\xi_1 + \cdots + t_{n-1}\xi_{n-1})^{-\frac{1}{m}} .$$

We proceed by contradiction and assume that $d_0^j g = 0$ for each $j \in \{1, \dots, k\}$. To understand how this condition affects σ we use the following claim which we prove at the end.

Claim 2.7.2. *Let $U \subset \mathbb{R}^n$ be an open neighborhood of 0 and $f \in C^\infty(U)$ be positive valued. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{N}$ such that $k \geq 1$. Assume that there exist $b \in \mathbb{R}$ and $\tau \in \mathbb{R}^n$ such that $(\tau, b) \neq (0, 0)$ such that, writing $h : x \in \mathbb{R}^n \mapsto \langle \tau, x \rangle + b \in \mathbb{R}$ we have, for each $j \in \{1, \dots, k\}$,*

$$d_0^j [h f^\alpha] = 0 . \tag{2.7.2}$$

Then,

$$f(0)^{k-1} d_0^k f = (\alpha + 1)(2\alpha + 1) \dots ((k-1)\alpha + 1)(d_0 f)^{\otimes k} . \tag{2.7.3}$$

We wish to use this claim with $\alpha = -\frac{1}{m}$, $h(t) = \langle \tau_0, \xi_1 \rangle t_1 + \cdots + \langle \tau_0, \xi_{n-1} \rangle t_{n-1} + \langle \tau_0, \xi_0 \rangle$ and $f(t) = \sigma(0, \xi_0 + t_1\xi_1 + \cdots + t_{n-1}\xi_{n-1})$. In order to apply it, the only thing to check is that h is not identically 0. But $h = 0$ would imply that $\langle \tau_0, \xi_0 \rangle = \cdots = \langle \tau_0, \xi_{n-1} \rangle = 0$. This cannot happen since $\tau_0 \neq 0$. Hence, by Claim 2.7.2 we have the following equality between (symmetric) k-forms on the hyperplane H spanned by $(\xi_1, \dots, \xi_{n-1})$,

$$\sigma(0, \xi_0)^{k-1} \partial_\xi^k \sigma(0, \xi) = C(m, k) (\partial_\xi \sigma(0, \xi_0))^{\otimes k} \tag{2.7.4}$$

where

$$C(m, k) = \left(-\frac{1}{m} + 1 \right) \cdots \left(-\frac{k-1}{m} + 1 \right) = \frac{m(m-1) \dots (m-k+1)}{m^k} .$$

Next, we make the following claim, which we prove at the end.

Claim 2.7.3. *Let m be a positive real number and let $f \in C^\infty(\mathbb{R}^p \setminus \{0\})$ be a real-valued m-homogeneous function. Then, for each $x \in \mathbb{R}^p \setminus \{0\}$, each hyperplane $H \subset \mathbb{R}^p$ not containing x and each $k_0 \geq 2$,*

$$\forall k \in \{2, \dots, k_0\}, \ f(x)^{k-1} d_x^k f = \frac{m(m-1) \dots (m-k+1)}{m^k} (d_x f)^{\otimes k} \tag{2.7.5}$$

is equivalent to

$$\forall k \in \{2, \dots, k_0\},\ f(x)^{k-1} d_x^k f\big|_H = \frac{m(m-1)\dots(m-k+1)}{m^k} (d_x f)^{\otimes k}\big|_H . \qquad (2.7.6)$$

This claim implies that σ actually satisfies Equation (2.7.4) on the whole of $T^*_\xi \mathbb{R}^n \simeq \mathbb{R}^n$. By the assumption on σ, this Equation cannot be satisfied for all $k \leq k_0$. Hence, $d_0^k g$ cannot vanish for each $k \in \{1, \dots, k_0\}$. In particular, there exists $k \in \{1, \dots, k_0\}$ for which $\tilde{f}_{\tilde{\eta}}$ satisfies Equation (2.7.1). Hence, Lemma 2.7.1 applies for this k and we are done.

Proof of Claim 2.7.2. Let f, τ, b, α, h and k be as in the statement of the claim. Let $g(x) = h(x)^{-\frac{1}{\alpha}}$. First of all, by Equation (2.7.2) with $j = 1$,

$$f(0)\tau = -\alpha b d_0 f$$

In particular, since $(\tau, b) \neq 0$ and $f(0) > 0$, we actually have $b \neq 0$. Thus, the function $g : x \mapsto h(x)^{-\frac{1}{\alpha}}$ is well defined and positive near the origin. Moreover, $hg^\alpha = 1$ so all of its derivatives vanish. Consequently, for each $j \in \{1, \dots, k\}$, $d_0^j(f^\alpha g^{-\alpha}) = 0$ which in turn gives, for each $j \in \{1, \dots, k\}$, $d_0^j(fg^{-1}) = 0$ (here we use the fact that $fg^{-1} = (f^\alpha g^{-\alpha})^{\frac{1}{\alpha}}$ which is well defined near 0). In particular, the Taylor expansions of f and g coincide to the kth order up to a multiplicative constant. By homogeneity of Equation (2.7.3) we may assume that they agree up to order k. But

$$\begin{aligned} d_0^k g &= \prod_{j=0}^{k-1} \left(-\frac{1}{\alpha} - j \right) \times b^{-\frac{1}{\alpha}-k} \tau^{\otimes k} \\ &= \left(b^{-\frac{1}{\alpha}} \right)^{1-k} (\alpha+1)(2\alpha+1)\dots((k-1)\alpha+1) \left(-\alpha b^{\frac{1}{\alpha}+1} \right)^{-k} \tau^{\otimes k} \end{aligned}$$

and $g(0) = b^{-\frac{1}{\alpha}}$ and $d_0 g = \left(-\alpha b^{\frac{1}{\alpha}+1} \right)^{-1} \tau$. Thus,

$$g(0)^{k-1} d_0^k g = (\alpha+1)(2\alpha+1)\dots((k-1)\alpha+1)(d_0 g)^{\otimes k} .$$

Since f agrees with g up to order k, f satisfies Equation (2.7.3). □

Proof of Claim 2.7.3. Equation (2.7.5) implies (2.7.6) by restriction to H. Let us assume (2.7.6) and prove the converse. Since $x \notin H$, $\mathbb{R}x \bigoplus H$ generate $\mathbb{R}^p$. By multilinearity, it is enough to prove (2.7.5) when the k forms are evaluated on families of the form $(x, \dots, x, y_1, \dots, y_h)$ where $y_1, \dots, y_h \in H$ and $h \in \{1, \dots, k\}$. Now, since f is homogeneous, by Euler's Equation, for any $h \leq k$, and for any $y_1, \dots, y_h \in H$,

$$d_x^k f(x, \dots, x, y_1, \dots, y_h) = \underbrace{(m-h)\dots(m-k+1)}_{1 \text{ if } k=h} d_x^h f(y_1, \dots, y_h)$$

$$\text{and } (d_x f)^{\otimes k}(x, \dots, x, y_1, \dots, y_h) = m^{k-h} f^{k-h}(x) (d_x f)^{\otimes h}(y_1, \dots, y_h).$$

Applying (2.7.6) to compare the right hand sides of each line we get Equation (2.7.5). □

□

The proof of Lemma 2.7.1 will combine two theorems from singularity theory and oscillatory integral asymptotics which we state now.

The following theorem is a corollary of the Malgrange preparation theorem presented in [Hör03]. We give a slightly different formulation and add the continuity with respect to smooth perturbations, which actually follows from Hörmander's original proof.

Theorem 2.7.4 ([Hör03], Theorem 7.5.13)**.** *Let $U \subset \mathbb{R}\times\mathbb{R}^n$ (resp. $E \subset \mathbb{R}^p$) be an open neighborhood of $0 \in \mathbb{R}\times\mathbb{R}^n$ (resp. $0 \in \mathbb{R}^p$) and $(f_\eta)_{\eta\in E}$ be a continuous family of smooth functions on U. We denote by (t, x) the elements of U. Let $k \in \mathbb{N}$, $k \geq 2$. Assume that for each $\eta \in E$ and $j \in \{0, \dots, k-1\}$*

$$\partial_t^j f_\eta(0, 0) = 0$$

and that $\partial_t^k f_\eta(0, 0) > 0$. Then, there exist $W \subset \mathbb{R} \times \mathbb{R}^n$ (resp. $V \subset \mathbb{R}^n$) a neighbohood of $0 \in \mathbb{R} \times \mathbb{R}^n$ (resp. $0 \in \mathbb{R}^n$) with $U' \subset \mathbb{R} \times V$ such that for each $\eta \in E$, there exist $\phi_\eta \in C^\infty(W)$ as well as $a_\eta^1, \dots, a_\eta^{k-1} \in C^\infty(V)$, satisfying, for any $\eta \in E$, $(t, x) \in W$,

$$\begin{aligned}
\phi_\eta(0, 0) &= 0\,,\\
\partial_t\phi_\eta(0, 0) &> 0\,,\\
a_\eta^1(0) = \cdots = a_\eta^{k-1}(0) &= 0\,,\\
\text{and } f_\eta(\phi_\eta(t, x), x) &= t^k + \sum_{j=0}^{k-1} a_\eta^j(x)t^j\,.
\end{aligned}$$

Moreover, one can choose these functions such that the maps $\eta \mapsto \phi_\eta$ and $\eta \mapsto a_\eta^j$ are continuous into C^∞.

Proof. First, apply Theorem 7.5.13 of [Hör03] to each f_η and define $\tilde{\phi}_\eta(\cdot, x)$ as the inverse map of $T(\cdot, x)$ for the T corresponding to f_η. That the maps $\tilde{\phi}_\eta$ and $\tilde{a^j}_\eta$ depend continuously on η follows from the proof of the aforementioned result. Indeed, they are built as solutions of ODEs whose initial conditions depend continuously on f in C^∞. Finally, by rescaling the new variable t and thus replacing $\tilde{\phi}_\eta$ (resp. $\tilde{a^j}_\eta$) by ϕ_η (resp. a_η^j) we get rid of the $\frac{1}{k}$ factor appearing in front of T^k in Theorem 7.5.13 of [Hör03]. □

The following theorem is the special case of Theorem 4 of [Col77] (and the remarks 2.3 and 2.4 that follow it) of type A_n singularities.

Theorem 2.7.5 ([Col77], Theorem 4)**.** *Let $k \in \mathbb{N}$, $k \geq 2$. There exist $\delta = \delta(k) > 0$, $V \subset \mathbb{R}^k$ an open neighborhood of 0 such that for all $u \in C_c^\infty(]-\delta, \delta[)$ there exists $C(u) < +\infty$ such that for all $\lambda > 0$ and $(a_0, \dots, a_{k-1}) \in V$,*

$$\left|\int_{\mathbb{R}} e^{i\lambda(t^k + a_{k-1}t^{k-1} + \cdots + a_0)}u(t)dt\right| \leq C(u)\lambda^{-\frac{1}{k}}\,.$$

Moreover $C(u)$ depends continuously on $u \in C_c^\infty(]-\delta, \delta[)$.

Proof. In the terminology of [Col77], the map $(t, a_0, \dots, a_{k-1}) \mapsto t^k + a_{k-1}t^{k-1} + \cdots + a_0$ is the universal unfolding of the singularity type A_{k-1}. In the notations of [Col77] our k corresponds to their n while their k equals 1 in our setting. Moreover, as stated in the table preceding Theorem 4 of [Col77], in the case A_{k-1}, $\varepsilon(\sigma) = \frac{1}{2} - \frac{1}{k}$ so that the integral decays polynomially in λ at order $-\frac{1}{2} + \varepsilon(\sigma) = -\frac{1}{k}$. □

Proof of Lemma 2.7.1. Let $(f_\eta)_\eta$ and $k \geq 1$ be as in the statement of the lemma. To make use of the assumption $d_0^k f_0 \neq 0$ we use the following elementary result in multilinear algebra which we prove at the end.

Claim 2.7.6. *Let ω be a symmetric k-linear form on $\mathbb{R}^n$. Let $q : \mathbb{R}^n \to \mathbb{R}$ be defined as $q(x) = \omega(x, \dots, x)$. Then, $q = 0$ implies $\omega = 0$.*

By Claim 2.7.6 there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $d_0^k f_0(v, v, \dots, v) \neq 0$. Without loss of generality, we may assume that $v = e_n := (0, \dots, 0, 1)$. We write $x = (\tilde{x}, x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Let $u \in C_c^\infty(U)$ be such that $u(\tilde{x}, x_n) \neq 0$ implies that $\|\tilde{x}\|_\infty \leq 1$. Then, for each $\eta \in E$ and $\lambda > 0$,

$$\left| \int_U e^{i\lambda f_\eta(x)} u(x) dx \right| \leq \max_{\tilde{x} \in \mathbb{R}^{n-1}} \left| \int_{\mathbb{R}} e^{i\lambda f_\eta(\tilde{x}, x_n)} u(\tilde{x}, x_n) dx_n \right| .$$

This way by replacing η by $(\eta, \tilde{x})$ and f_η by $f_\eta(\tilde{x}, \cdot)$ we have reduced the problem to the one dimensional case. From now on, we assume that $n = 1$.

For each $\eta \in E$, each $x \in U$, and $q = (q_0, \dots, q_{k-1}) \in \mathbb{R}^k$, let

$$g_\eta(x, q) = f_\eta(x) - f_\eta(0) - f'_\eta(0)x - \cdots - \frac{1}{(k-1)!} f_\eta^{(k-1)}(0) x^{k-1} + q_0 + a_1 x + \cdots + q_{k-1} x^{k-1} .$$

We will first prove the desired bound where we replace f_η by $g_\eta(\cdot, q)$, uniformly for q close enough to 0 and then deduce the result for f_η itself as a phase.

The map $\eta \mapsto g_\eta$ is continuous from E to $C^\infty(U \times \mathbb{R}^k)$. Moreover, for each $\eta \in E$ close enough to 0 we have

$$\begin{aligned} \forall j \in \{0, \dots, k-1\},\ \partial_x^j g_\eta(0, 0) = 0 \\ \partial_x^k g_\eta(0, 0) \neq 0. \end{aligned}$$

Replacing f_η by $-f_\eta$ does not change the estimate since it amounts to complex conjugation of the integrand. With this in mind, we may assume that $\partial_x^k g_\eta(0, 0) > 0$. By Theorem 2.7.4, there exist $W \subset \mathbb{R} \times \mathbb{R}^k$ and continuous families of smooth functions $(a_\eta^1)_\eta \dots, (a_\eta^{k-1})_\eta$, as well as $(\phi_\eta)_\eta$ defined respectively in a neighborhood of $0 \in \mathbb{R}^k$ and a neighborhood of $(0, 0)$ in $U \times \mathbb{R}^k$ such that for each $\eta \in E$ and (x, q) close enough to

0 and $(0,0)$ respectively,

$$\begin{aligned}\phi_\eta(0,0) &= 0\,,\\ \partial_t\phi_\eta(0,0) &> 0\,,\\ a^1_\eta(0) = \cdots = a^{k-1}_\eta(0) &= 0\,,\\ \text{and } g_\eta(\phi_\eta(x,q),q) &= x^k + \sum_0^{k-1} a^j_\eta(q)x^j\,.\end{aligned}$$

Hence, if $u \in C^\infty_c(\mathbb{R})$ is supported close enough to 0, we have, for all $\eta \in E$ close enough to 0 and all $q \in \mathbb{R}^k$,

$$\int_{\mathbb{R}} e^{i\lambda g_\eta(y,q)}u(y)dy = \int_{\mathbb{R}} e^{i\lambda(x^k+a^{k-1}_\eta(q)x^{k-1}+\cdots+a^0_\eta(q))}u(\phi_\eta(x,q))\left(\phi^{-1}_\eta(\cdot,q)\right)'(x)dx\,.$$

By Theorem 2.7.5, there exist $W_1 \subset \mathbb{R}^k \times E$ a neighborhood of 0 such that $\delta > 0$ such that for each $(q,\eta) \in W_1$, for each $v \in C^\infty_c(]-\delta,\delta[)$, there exists $C'(v) < +\infty$ such that for each $\lambda > 0$ and each (q,η) close enough to $(0,0)$,

$$\left|\int_{\mathbb{R}} e^{i\lambda(x^k+a^{k-1}_\eta(q)x^{k-1}+\cdots+a^0_\eta(q))}v(x)dx\right| \leq C'(v)\lambda^{-\frac{1}{k}}\,.$$

Moreover, Theorem 2.7.5 specifies that the map $v \in C^\infty_c(]-\delta,\delta[) \to C'(v) \in \mathbb{R}$ is continuous. By continuity, there exist $\varepsilon > 0$ and $W_2 \subset W_1$ a compact neighborhood of 0 such that for any $(q,\eta) \in W_2$ and any $x \in \mathbb{R}$ with $|x| \geq \delta/2$, $|\phi_\eta(x,q)| \geq \varepsilon$. In particular, the map $(q,\eta,u) \in W_2 \times C^\infty_c(]-\varepsilon,\varepsilon[) \mapsto u(\phi_\eta(\cdot,q))\left(\phi_\eta(\cdot,q)^{-1}\right)' \in C^\infty_c(]-\delta,\delta[)$ is well defined and continuous. Consequently, so is the map

$$\begin{aligned}W_2 \times C^\infty_c(]-\varepsilon,\varepsilon[) &\to \mathbb{R}\\ (q,\eta,u) &\mapsto C_{q,\eta}(u) = C'\left(u(\phi_\eta(\cdot,q))\left(\phi_\eta(\cdot,q)^{-1}\right)'\right)\,.\end{aligned}$$

By compactness, $C(u) = \sup_{(q,\eta)\in W_2} C_{q,\eta}(u)$ is finite and continuous in u. We have proved that for any $(q,\eta) \in W_2$, any $\lambda > 0$ and any $u \in C^\infty_c(]-\varepsilon,\varepsilon[)$,

$$\left|e^{i\lambda g_\eta(y,q)}u(y)dy\right| \leq C(u)\lambda^{-\frac{1}{k}}\,.$$

To obtain the corresponding estimate with f_η instead of $g_\eta(\cdot,q)$, we make the following two observations. First, for each $\eta \in E$, and $x \in U$,

$$g_\eta(x, f_\eta(0), \ldots, f^{(k-1)}_\eta(0)) = f_\eta(x)\,.$$

Second, since $f_0(0) = \cdots = f^{(k-1)}_0(0) = 0$, there exists $E' \subset E$ a neighborhood of 0 such that for each $\eta \in E'$, $(f_\eta(0), \ldots, f^{(k-1)}_\eta(0), \eta) \in W_3$. Thus, for each $\eta \in E'$ each $u \in C^\infty_c(]-\varepsilon,\varepsilon[)$ and each $\lambda > 0$,

$$\left|e^{i\lambda f_\eta(y)}u(y)dy\right| \leq C(u)\lambda^{-\frac{1}{k}}$$

and the proof is over, save for the proof of Claim 2.7.6.

Proof of Claim 2.7.6. Let us prove the following formula.

$$\forall x_1,\dots,x_k\in\mathbb{R}^n,\ \omega(x_1,\dots,x_k)=\frac{1}{2^k}\sum_{\eta\in\{-1,1\}^k}\prod_{i=1}^k\eta_i\times q\left(\sum_{j=1}^k\eta_j x_j\right).$$

For each $x_1,\dots,x_k\in\mathbb{R}^n$ and $p=(p_1,\dots,p_k)\in\mathbb{N}^k$ such that $p_1+\cdots+p_k=k$, we denote by $\omega\left(x_1^{p_1}\dots x_k^{p_k}\right)$ the form ω evaluated in any k-uple with exactly p_j occurrences of x_j $(\forall j\in\{1,\dots,k\})$. Then, for each $x_1,\dots,x_k\in\mathbb{R}^n$,

$$\begin{aligned}\sum_{\eta\in\{-1,1\}^k}\prod_i^k\eta_i\times q\left(\sum_{j=1}^k\eta_j x_j\right)&=\sum_{\eta\in\{-1,1\}^k}\prod_i^k\eta_i\sum_{p_1+\cdots+p_k=k}\binom{k}{p_1,\dots,p_k}\omega\left((\eta_1x_1)^{p_1}\dots(\eta_kx_k)^{p_k}\right)\\&=\sum_{p_1+\cdots+p_k=k}\binom{k}{p_1,\dots,p_k}\omega\left(x_1^{p_1}\dots x_k^{p_k}\right)\sum_{\eta\in\{-1,1\}^k}\prod_i^k\eta_i^{p_i+1}.\end{aligned}$$

Given $j\in\{1,\dots,j\}$ and $(p_1,\dots,p_k)$ such that $p_j=0$, applying the bijection

$$(\eta_1,\dots,\eta_k)\mapsto(\eta_1,\dots,-\eta_j,\dots,\eta_k)$$

shows that $\sum_{\eta\in\{-1,1\}^k}\prod_j\eta_j^{p_j+1}=0$. Thus, the only remaining term is the one corresponding to $p_1=\cdots=p_k=1$ for which the sum of products of the $\varepsilon_j^{p_j+1}$ equals 2^k. Therefore,

$$\sum_{\eta\in\{-1,1\}^k}\prod_i\eta_i q\left(\sum_i\eta_i v_i\right)=2^k\omega(v_1,\dots,v_k)$$

as announced. □

□

2.7.2 Genericity and stability of the non-degeneracy condition

The goal of this subsection is to prove Proposition 2.7.8 below, which says roughly that admissible symbols are stable and generic. To give a precise meaning to this statement, we first need to define a topology on the set of positive homogeneous symbols.

Definition 2.7.7. Fix $n\in\mathbb{N}$, $n\geq 1$ and $m\in]0;+\infty[$. For each $U\subset\mathbb{R}^n$, let $S_h^m(U)\subset C^\infty(U\times(\mathbb{R}^n\setminus\{0\}))$ be the set of smooth functions m-homogeneous in the second variable. We write $S_{h,+}^m(U)$ for the set of positive valued functions in $S_h^m(U)$. The map

$$S_h^m(U)\to C^\infty(U\times S^{n-1}),$$

restricting the second variable to the unit sphere, is a bijection. We endow $S_h^m(U)$ with the topology induced by the Whitney topology on $C^\infty(U\times S^{n-1})$ (see Definition 3.1 of Chapter II of [**?**]).

We have the following proposition.

Proposition 2.7.8. *For all $n \in \mathbb{N}$, $n \geq 2$ we define $k_0 = k_0(n) \in \mathbb{N}$ as follows. We set $k_0(2) = 5$, $k_0(3) = 3$, $k_0(4) = 3$ and $\forall n \geq 5$, $k_0(n) = 2$. Fix $n \geq 2$ and $m > 0$. Let $U \subset \mathbb{R}^n$ be an open subset. Then, the set of $\sigma \in S^m_{h,+}(U)$ such that for each $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ there exists $j \in \{2, \dots, k_0\}$ such that*

$$\sigma^{j-1}(x,\xi)\partial^j_\xi \sigma(x,\xi) \neq \frac{m(m-1)\dots(m-j+1)}{m^j} \left(\partial_\xi \sigma(x,\xi)\right)^{\otimes j} \tag{2.7.7}$$

is open and dense in $S^m_{h,+}(U)$.

To prove this proposition, we will apply Thom's transversality theorem (see Theorem 4.9 of Chapter II of [**?**]) to a well chosen submanifold of the jet bundle of $U \times S^{n-1}$ whose codimension grows with the degree of admissibility we consider. Lemmas 2.7.10, 2.7.12, 2.7.13, 2.7.14 and 2.7.15 below are devoted to the construction of this manifold. The proof of Proposition 2.7.8 is presented only after these are stated and proved. Throughout the rest of the section we fix $n \in \mathbb{N}$, $n \geq 2$, $U \subset \mathbb{R}^n$ an open subset and $m \in \mathbb{R}$, $m > 0$. We start by introducing some notation.

Notation 2.7.9.

1. For each $j, p \in \mathbb{N}$, $p \geq 1$, let Sym^j_p be the space of symmetric j-linear forms over $\mathbb{R}^p$. This is a vector space of dimension $\binom{p+j-1}{j}$. We adopt the convention that $\mathrm{Sym}^0_p = \mathbb{R}$.

2. Let X be a smooth manifold. For each $k \geq 0$ we denote by $\mathcal{J}^k(X)$ the k-th jet space of mappings from X to $\mathbb{R}$, that is, the space $J^k(X, \mathbb{R})$ introduced in Definition 2.1 of Chapter II of [**?**]. For any $p \in \mathbb{N}$ and any open subset $V \subset \mathbb{R}^p$, the space $\mathcal{J}^k(V)$ is canonically isomorphic to $V \times \bigoplus_{j=0}^k \mathrm{Sym}^j_p$. We will denote its elements by (ξ, ω) where $\xi \in V$ and $\omega = (\omega_0, \dots, \omega_k) \in \bigoplus_{j=0}^k \mathrm{Sym}^j_p$.

3. Let X be a smooth manifold and $k \in \mathbb{N}$. For each $f \in C^\infty(X)$, we write $j^k f$ for the section of $\mathcal{J}^k(X)$ whose value at each point is the k-jet of f at this point (see the paragraph below Definition 2.1 of Chapter II of [**?**]).

Since the jet bundle $\mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)$ is quite explicit, we will make most of our contructions inside it and them 'push them down' onto the sphere. In the following lemma, we build the map we need to 'push down' our constructions.

Lemma 2.7.10. *Let $\iota : S^{n-1} \to \mathbb{R}^n$ be the canonical injection. Then, there exists a bundle morphism*

$$\rho : \iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right) \to \mathcal{J}^k\left(S^{n-1}\right)$$

such that the following diagram commutes:

$$\begin{array}{ccc} C^\infty\left(\mathbb{R}^n \setminus \{0\}\right) & \xrightarrow{\iota^*} & C^\infty\left(S^{n-1}\right) \\ {\scriptstyle \iota^*(j^k \cdot)}\downarrow & & \downarrow{\scriptstyle j^k} \\ \iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right) & \xrightarrow{\rho} & \mathcal{J}^k\left(S^{n-1}\right). \end{array}$$

Here the top arrow is the restriction map while the left arrow is the restriction of the k-jet to the sphere.

Proof. We construct ρ by defining its action on each fiber. Let $\xi \in S^{n-1}$ and let (V, ϕ) be a chart $\phi : V \to \mathbb{R}^{n-1}$ of S^{n-1} near ξ. Then, for each $f \in C^\infty(\mathbb{R}^n)$, the k-th order Taylor expansion of $f \circ \phi^{-1}$ at ξ depends only on the k-th order Taylor expansion of f at ξ and the dependence is linear. This defines a linear map $\rho|_\xi : \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})|_\xi \to \mathcal{J}^k\left(S^{n-1}\right)|_\xi$. The corresponding fiberwise map ρ is clearly smooth and defines a morphism of smooth vector bundles. Moreover, by construction, for each $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and each $\xi \in S^{n-1}$, $\rho|_\xi\left(j^k f(\xi)\right) = j^k(f \circ \iota)(\xi)$ so the diagram does indeed commute. □

Notation 2.7.11. For each $k \in \mathbb{N}$, each $\xi \in \mathbb{R}^n$ and each $\omega = (\omega_0, \dots, \omega_k) \in \bigoplus_{j=0}^k \mathrm{Sym}_n^j$ we introduce the following notation. For each $j \in \{0, \dots, k\}$, $\omega_j|_{\xi^\perp}$ is the restriction of ω_j to the orthogonal of ξ in $\mathbb{R}^n$. Moreover, we set $\omega|_{\xi^\perp} = (\omega_0|_{\xi^\perp}, \dots, \omega_k|_{\xi^\perp})$.

In the following lemma, we check that the set of jets of homogeneous maps is a smooth submanifold of $\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ and give an explicit description of it. Moreover, we show that the 'push down' map ρ maps it diffeomorphically on the space $\mathcal{J}^k\left(S^{n-1}\right)$.

Lemma 2.7.12. *Fix $k \in \mathbb{N}$. Let H_m^k be the subset of $\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})$ of jets of m-homogeneous functions. Then,*

1. *The set H_m^k is characterized by the following equations:*
$$H_m^k = \cap_{j=0}^{k-1} \left\{ (\xi, \omega) \in \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \;\middle|\; \omega_{j+1}(\xi, \dots) = (m-j)\omega_j \right\}.$$
2. *The set H_m^k is a submanifold of $\iota^* \mathcal{J}^k(U \times \mathbb{R}^n \setminus \{0\})$ of the same dimension as $\mathcal{J}^k\left(S^{n-1}\right)$.*
3. *The map $\rho|_{H_m^k} : H_m^k \to \mathcal{J}^k\left(U \times S^{n-1}\right)$ is a diffeomorphism.*

Proof. We set
$$\widetilde{H_m^k} = \cap_{j=0}^{k-1} \left\{ (\xi, \omega) \in \iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\}) \;\middle|\; \omega_{j+1}(\xi, \dots) = (m-j)\omega_j \right\}.$$
Firstly, each m-homogeneous $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$, satisfies Euler's equation. That is, for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $d_\xi f(\xi) = m f(\xi)$. Next, notice that if f is m-homogeneous, then, for each $j \in \{1, \dots, k\}$, $\xi \mapsto d_\xi^j f$ is homogeneous of order $m-j$ so that for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $d_\xi\left(d^j f\right)(\xi, \dots) = (m-j) d_\xi^j f$. Therefore, for each $\xi \in \mathbb{R}^n \setminus \{0\}$, $j^k f(\xi) \in \widetilde{H_m^k}$. We have shown that $H_m^k \subset \widetilde{H_m^k}$. Next, notice that for each $f \in C^\infty\left(S^{n-1}\right)$, the m-homogeneous function $\xi \mapsto |\xi|^m f\left(\frac{\xi}{|\xi|}\right)$ restricts back to f on S^{n-1}. Therefore, we have $\mathcal{J}^k\left(S^{n-1}\right) = \rho\left(H_m^k\right) \subset \rho\left(\widetilde{H_m^k}\right) \subset \mathcal{J}^k\left(S^{n-1}\right)$. So we have
$$\rho\left(H_m^k\right) = \rho\left(\widetilde{H_m^k}\right) = \mathcal{J}^k\left(S^{n-1}\right). \tag{2.7.8}$$

Given this equation, in order to prove the lemma, it is enough to prove points 2 and 3 with H_m^k replaced by $\widetilde{H_m^k}$, which we call 2' and 3' respectively. Indeed, point 3' will imply that $\rho|_{\widetilde{H_m^k}}$ is one-to-one so by Equation (2.7.8), we will have $H_m^k = \widetilde{H_m^k}$ which is point 1. Moreover, since we will have already proved points 2 and 3 for $\widetilde{H_m^k}$ we will have them for H_m^k. Let us start by proving 2'. For each $j \in \{0, \dots, k-1\}$ set

$$F_m^j : (\xi, \omega) \mapsto \omega_{j+1}(\xi, \dots) - (m-j)\omega_j$$

so that $\widetilde{H_m^k} = \cap_{j=0}^{k-1} \left(F_m^j\right)^{-1}(0)$. Let us prove that the map

$$F_m = \left(F_m^0, \dots, F_m^{k-1}\right) : \iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right) \to \bigoplus_{j=0}^{k-1} \mathrm{Sym}_n^j$$

is a submersion. Fix $(\xi, \omega) \in \iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)$. Let $(\eta_0, \dots, \eta_{k-1}) \in \bigoplus_{j=0}^{k-1} \mathrm{Sym}_n^j \simeq T_{F_m(\xi,\omega)} \bigoplus_{j=0}^{k-1} \mathrm{Sym}_n^j$. Then, for each $j \in \{0, \dots, k-1\}$,

$$\partial_{\omega_{j+1}} F_m^j(\xi, \omega)\left(|\xi|^{-2}\langle \xi, \cdot \rangle \otimes \eta_j\right) = \eta_j \,.$$

In particular, $d_{(\xi,\omega)}F_m$ is surjective. Therefore $\widetilde{H_m^k}$ is a submanifold of $\iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)$ of codimension

$$codim_{\iota^* \mathcal{J}^k(\mathbb{R}^n \setminus \{0\})}\left(\widetilde{H_m^k}\right) = \sum_{j=0}^{k-1} dim\left(\mathrm{Sym}_n^j\right) = \sum_{j=0}^{k-1} \binom{n+j-1}{j} .$$

Indeed, recall that $dim\left(\mathrm{Sym}_n^j\right) = \binom{n+j-1}{j}$. Using this identity, we also have:

$$dim\left(\iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)\right) = (n-1) + \sum_{j=0}^{k} \binom{n+j-1}{j} ;$$

$$dim\left(\mathcal{J}^k\left(S^{n-1}\right)\right) = (n-1) + \sum_{j=0}^{k} \binom{n+j-2}{j} .$$

Therefore, firstly $dim\left(\widetilde{H_m^k}\right) = (n-1) + \binom{n+k-1}{k}$ and secondly

$$dim\left(\widetilde{H_m^k}\right) - dim\left(\mathcal{J}^k\left(S^{n-1}\right)\right) = \binom{n+k-1}{k} - \sum_{j=0}^{k} \binom{n+j-2}{j} = 0 \,. \qquad (2.7.9)$$

In the last equality we use a well known binomial formula which is easily checked by induction on k. The conclusion here is that $\widetilde{H_m^k}$ has the same dimension as $\mathcal{J}^k\left(S^{n-1}\right)$ so we have proved 2'. To prove 3' observe that ρ is linear on each fiber of $\iota^* \mathcal{J}^k\left(\mathbb{R}^n \setminus \{0\}\right)$ so that its derivative $d\rho$ is constant on each fiber. Moreover, it is equivariant with respect

to the automorphisms of the base space S^{n-1} so its derivative must have the same rank on different fibers. Since ρ is surjective (see Equation (2.7.8)) $d\rho$ must be of maximal rank. This proves that ρ is a local diffeomorphism. But since it is a morphism of vector bundles, it must be a diffeomorphism, which is the claim of 3'. This concludes the proof of the lemma. □

In the following lemma, we build a submanifold of H_m^k that describes the condition of non-admissibility and compute its codimension.

Lemma 2.7.13. *For each $k \in \mathbb{N}$, $k \geq 2$, define*

$$Y_m^k = \cap_{j=2}^k \left\{ (\xi,\omega) \in \iota^* \mathcal{J}^k \left(\mathbb{R}^n \setminus \{0\}\right) \,\middle|\, \omega_0 > 0,\, \omega_0^{j-1}\omega_j|_{\xi^\perp} = \frac{m(m-1)\dots(m-j+1)}{m^j} \left(\omega_1|_{\xi^\perp}\right)^{\otimes j} \right\}.$$

Then, $Y_m^k \cap H_m^k$ is a closed submanifold of H_m^k of codimension $\sum_{j=2}^k \binom{n+j-2}{j}$.

Proof. For each $j \in \{0,\dots,k-1\}$, each $l \in \{2,\dots,k\}$ and each $(\xi,\omega) \in \iota^* \mathcal{J}^k \left(\mathbb{R}^n \setminus \{0\}\right)$, let, as before, $F_m^j(\xi,\omega) = \omega_{j+1}(\xi,\dots) - (m-j)\omega_j \in Sym_n^{j-1}$. Moreover, let $\mathrm{Sym}_n^l|_{\xi^\perp}$ be the set of symmetric l-linear forms acting on the orthogonal of ξ in $\mathbb{R}^n$ and let $G_m^l(\xi,\omega) = \omega_l|_{\xi^\perp} - \frac{m(m-1)\dots(m-l+1)}{m^l}\left(\omega_1|_{\xi^\perp}\right)^{\otimes l} \in \mathrm{Sym}_n^l|_{\xi^\perp}$. Then, $Y_m^k \cap H_m^k$ is the intersection of the zero sets of the functions F_m^j and G_m^l for $j \in \{0,\dots,k-1\}$ and $l \in \{2,\dots,k\}$. In particular, it is closed. Note first that $\partial_{\omega_0} F_m^0 = m \neq 0 \in \mathrm{Hom}\left(\mathrm{Sym}_n^0, \mathrm{Sym}_n^0\right) \simeq \mathbb{R}$. In particular this map is invertible. We will now prove that for each $l \in \{2,\dots,k\}$, the map $(\partial_{\omega_l} F_m^{l-1}, \partial_{\omega_l} G_m^l)$ is of maximal rank on Y_m^k. For any $(\xi,\omega) \in Y_m^k$ and any $l \in \{2,\dots,k\}$, $(\partial_{\omega_l} F_m^{l-1}(\xi,\omega), \partial_{\omega_l} G_m^l(\xi,\omega))$ acts as follows.

$$\begin{aligned} \mathrm{Sym}_n^l &\to \mathrm{Sym}_n^{l-1} \bigoplus \mathrm{Sym}_n^l|_{\xi^\perp} \\ \eta_l &\mapsto (\eta_l(\xi,\dots), \omega_0^{l-1}\eta_l|_{\xi^\perp}). \end{aligned}$$

But this map is invertible. To see this, let $pr_{\xi^\perp}^* : \mathrm{Sym}_n^l|_{\xi^\perp} \to \mathrm{Sym}_n^l$ be the pull-back map by the orthogonal projection onto the orthogonal of ξ. Also, recall that on Y_m^k, we have $\omega_0 > 0$. Then, the inverse of $(\partial_{\omega_l} F_m^{l-1}(\xi,\omega), \partial_{\omega_l} G_m^l(\xi,\omega))$ is

$$\begin{aligned} \mathrm{Sym}_n^{l-1} \bigoplus \mathrm{Sym}_n^l|_{\xi^\perp} &\to \mathrm{Sym}_n^l \\ (\eta_{l-1}, \eta|_\perp) &\mapsto |\xi|^{-2}\langle \xi, \cdot \rangle \otimes \eta_{l-1} + \omega_0^{1-l} pr_{\xi^\perp}^* \eta_\perp . \end{aligned}$$

All in all, we have shown so far that $\partial_{\omega_0} F_m^0$ is surjective and that for each $l \in \{2,\dots,k\}$, $(\partial_{\omega_l} F_m^{l-1}, \partial_{\omega_l} G_m^l)$ is of maximal rank. Therefore, $Y_m^k \cap H_m^k$ is a submanifold of H_m^k of

codimension

$$\begin{aligned} codim_{H_m^k}\left(Y_m^k \cap H_m^k\right) &= codim_{\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})}\left(Y_m^k \cap H_m^k\right) - codim_{\iota^*\mathcal{J}^k(\mathbb{R}^n\setminus\{0\})}\left(H_m^k\right) \\ &= 1 + \sum_{l=2}^{k}\binom{n+l-1}{l} - \sum_{j=0}^{k-1}\binom{n+j-1}{j} \\ &= \binom{n+k-1}{k} - \binom{n+1-1}{1} \\ &= \sum_{j=2}^{k}\binom{n+j-2}{j} \end{aligned}$$

where in the last line we use the same binomial identity as in Equation (2.7.9). □

So far we have neglected the U coordinate in the product $U \times S^{n-1}$. To take this coordinate into account, in the following lemma, we introduce a submersion $pr_2 : \mathcal{J}^k\left(U \times S^{n-1}\right) \to \mathcal{J}^k\left(S^{n-1}\right)$ by which we will pull back the submanifold $\rho\left(Y_m^k\right)$.

Lemma 2.7.14. *Let $k \in \mathbb{N}$. Let $\pi : U \times S^{n-1} \to S^{n-1}$ be the map $(x,\xi) \mapsto \xi$. Also, for each $x \in U$, let $\iota_x : S^{n-1} \to U \times S^{n-1}$ be the map $\xi \mapsto (x,\xi)$. Then, there exists a surjective vector bundle morphism $pr_2 : \mathcal{J}^k\left(U \times S^{n-1}\right) \to \pi^*\mathcal{J}^k\left(S^{n-1}\right)$ such that for each $x \in U$, the following diagram commutes:*

$$\begin{array}{ccccc} C^\infty\left(U \times S^{n-1}\right) & & \xrightarrow{\quad\iota_x^*\quad} & & C^\infty\left(S^{n-1}\right) \\ \downarrow{\scriptstyle j^k} & & & & \downarrow{\scriptstyle j^k} \\ \mathcal{J}^k\left(U \times S^{n-1}\right) & \xrightarrow{pr_2} & \pi^*\mathcal{J}^k\left(S^{n-1}\right) & \xrightarrow{\iota_x^*} & \mathcal{J}^k\left(S^{n-1}\right). \end{array}$$

In particular, pr_2 is a submersion.

Proof. Given $f \in C^\infty\left(U \times S^{n-1}\right)$ and $x \in U$, the k-jet of $f(x,\cdot)$ at $\xi \in S^{n-1}$ depends only on the k-jet of f at (x,ξ). This allows us to define a map $pr_2|_{(x,\xi)} : \mathcal{J}^k\left(U \times S^{n-1}\right)|_{(x,\xi)} \to \pi^*\mathcal{J}^k\left(S^{n-1}\right)_{(x,\xi)}$. This defines a bundle morphism $pr_2 : \mathcal{J}^k\left(U \times S^{n-1}\right) \to \pi^*\mathcal{J}^k()$. The fact that the diagram commutes follows by construction. Finally, since the composition of the top and right arrows : $j^k \circ \iota_x^*$ is onto, so is the composition of the left and bottom arrows. But this implies that the composition of bottom arrows is onto. Since $\pi^*\mathcal{J}^k\left(S^{n-1}\right)$ and $\mathcal{J}^k\left(S^{n-1}\right)$ have the same rank, then pr_2 must also be onto. In particular, it defines a submersion from the manifold $\mathcal{J}^k\left(U \times S^{n-1}\right)$ to the manifold $\pi^*\mathcal{J}^k\left(S^{n-1}\right)$. □

In this last lemma, we check that the previous construction does indeed characterize non-admissibility of a symbol by the intersection of the k-jet with the submanifold constructed in Lemma 2.7.13 and 'pushed down' by ρ.

Lemma 2.7.15. *Let $k \in \mathbb{N}$, $k \geq 2$. Let $\sigma \in S^m_{h,+}(U)$. Then, there exists $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ such that for each $j \in \{2,\dots,k\}$*

$$\sigma^{j-1}(x,\xi)\partial^j_\xi \sigma(x,\xi) = \frac{m(m-1)\dots(m-j+1)}{m^j} \left(\partial_\xi \sigma(x,\xi)\right)^{\otimes j} \tag{2.7.10}$$

if and only if $pr_2 \circ j^k\left(\sigma|_{U\times S^{n-1}}\right)\left(U \times S^{n-1}\right) \cap \rho\left(Y^k_m\right) \neq \emptyset$.

Proof. Firstly, Equation (2.7.10) is homogeneous in ξ so there exists a pair $(x,\xi) \in U\times(\mathbb{R}^n\setminus\{0\})$ satisfying it if and only if there exists such a pair in $U\times S^{n-1}$. Now, since σ is m-homogeneous, for each $x \in U$, $j^k(\sigma(x,\cdot))(S^{n-1}) \subset H^k_m$. Therefore, $(x,\xi) \in U\times S^{n-1}$ satisfy Equation (2.7.10) if and only if $j^k(\sigma(x,\cdot))(\xi) \in Y^k_m \cap H^k_m$ (here we use that the symbols are positive, as well as m-homogeneous). Since, moreover, by Lemma 2.7.13, $\rho|_{H^k_m}$ is bijective, this is equivalent to $\rho \circ j^k(\sigma(x,\cdot))(\xi) \in \rho(Y^k_m)$. But, by Lemmas 2.7.10 and 2.7.14, $\rho \circ j^k(\sigma(x,\cdot)) = j^k(\sigma(x,\cdot)|_{S^{n-1}}) = pr_2 \circ j^k\left(\sigma|_{U\times S^{n-1}}\right)(x,\cdot)$. To conclude, we have proved that for any $(x,\xi) \in U \times S^{n-1}$, (x,ξ) satisfies Equation (2.7.10) if and only if $pr_2 \circ j^k\left(\sigma|_{S^{n-1}}\right)(x,\xi) \in \rho\left(H^k_m\right)$. This concludes the proof of the lemma. □

We are now ready to prove Proposition 2.7.8.

Proof of Proposition 2.7.8. Firstly, by Lemma 2.7.15, Equation (2.7.7) has solutions in $U \times (\mathbb{R}^n \setminus \{0\})$ if and only if $j^k\left(\sigma|_{U\times S^{n-1}}\right)(U \times S^{n-1}) \cap pr_2^{-1}\left(\rho(Y^k_m)\right) \neq \emptyset$. Now, by Lemmas 2.7.12 and 2.7.13, $\rho\left(Y^k_m\right)$ is a closed submanifold of $\mathcal{J}^k\left(S^{n-1}\right)$ of codimension $\sum_{j=2}^k \binom{n+j-2}{j}$. Since moreover, by Lemma 2.7.14, pr_2 is a submersion, $Z^k_m = pr_2^{-1}\left(\rho(Y^k_m)\right)$ has the same codimension in $\mathcal{J}^k\left(U \times S^{n-1}\right)$. At this point, we apply Thom's transversality theorem (Corollary 4.10 of Chapter II of [**?**]). This theorem states that the functions $f \in C^\infty\left(U \times S^{n-1}\right)$ such that $j^k(f)(U \times S^{n-1})$ is transverse to Z^k_m is open and dense. But $j^k(f)\left(U \times S^{n-1}\right)$ has dimension at most $2n-1$ so if k is such that

$$2n - 1 < \sum_{j=2}^{k} \binom{n+j-2}{j} \tag{2.7.11}$$

then such a transverse intersection must be empty. Inequality (2.7.11) is satisfied for instance for $n = 2$ and $k = 5$, for $n \in \{3,4\}$ and $k = 3$ and for $n \geq 5$ and $k = 2$. This ends the proof of the proposition. □

Appendix

2.A Proof of Theorem 2.2.2

In this section, we prove Theorem 2.2.2 by following closely the approach used in [Hör68] and in [GW16a]. As explained above, [GW16a] contains all the essential arguments for Theorem 2.2.2 despite the focus on the case where $x = y$ and $\mathcal{X}$ is closed. In this section we merely wish to confirm this by revisiting the proof. We consider A, σ_A and E_L indifferently as in any of the two settings presented in Subection 2.2.1.

2.A.1 Preliminaries

The following lemma summarizes the results proved in Section 4 of [Hör68] for the closed manifold setting. For the boundary problem, this was proved in Section 3 of [Vas84]. We introduce the following notation. For each $L > 0$, set $\tilde{E}_L = E_{L^m}$.

Lemma 2.A.1. *Firstly, the spectral function $\tilde{E}_L(x, y)$ defines a tempered distribution of the L variable with values in $C^\infty(\mathcal{X} \times \mathcal{X})$. In addition, for each set of local coordinates in which $d\mu_{\mathcal{X}}$ coincides with the Lebesgue measure on $\mathbb{R}^n$, there is an open neighborhood U of $0 \in \mathbb{R}^n$ such that there exist $\varepsilon > 0$, a proper phase function $\psi \in C^\infty(U \times U \times \mathbb{R}^n)$, a symbol $\sigma \in S^1(U, \mathbb{R}^n)$, a function $k \in C^\infty(U \times U \times]-\varepsilon, \varepsilon[)$ and a symbol $q \in S^0(U \times]-\varepsilon, \varepsilon[\times U, \mathbb{R}^n)$, for which*

$$\mathcal{F}_L[\tilde{E}'_L(x, y)](t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} q(x, t, y, \xi) e^{i(\psi(x,y,\xi) - t\sigma(y,\xi))} d\xi + k(x, y, t).$$

Here $\mathcal{F}_L$ (resp. $'$) denotes the Fourier transform (resp. the derivative) with respect to the variable L, in the sense of temperate distributions, and the integral is to be understood in the sense of Fourier integral operators (see Theorem 2.4 of [Hör68]). We have

1. *The function ψ satisfies the Equation*

$$\forall x, y \in U,\ \xi \in \mathbb{R}^n,\ \sigma(x, \partial_x \psi(x, y, \xi)) = \sigma(y, \xi).$$

2. *For each $t \in]-\varepsilon, \varepsilon[$ and $\xi \in \mathbb{R}^n$, the function $q(\cdot, t, \cdot, \xi)$ has compact support in $U \times U$ uniformly in (t, ξ) and $q(x, 0, y, \xi) - 1$ is a symbol of order -1 as long as x, y belong to some open neighborhood U_0 of 0 in U.*

3. $\sigma - \sigma_A^{\frac{1}{m}} \in S^0$.

We will also need the following classical lemma. Here and below, $\mathcal{S}(\mathbb{R})$ will denote the space of Schwartz functions.

Lemma 2.A.2. *For each $\varepsilon > 0$ there is a function $\rho \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{F}(\rho)$ has compact support contained in $]-\varepsilon, \varepsilon[$, $\rho > 0$ and $\mathcal{F}(\rho)(0) = 1$.*

Proof. Choose $f \in \mathcal{S}(\mathbb{R})$ whose Fourier transform has support in $]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. Then it is easy to see that $\rho = f^2 * f^2$ satisfies the required properties. □

Before we proceed, let us fix U, ψ, q, k and ρ as in Lemmas 2.A.1 and 2.A.2 as well as a differential operator P on $\mathcal{X} \times \mathcal{X}$ of order d with principal symbol σ_P. Let $\tilde{E}_{L,P} = P\tilde{E}_L$. In order to estimate this $\tilde{E}_{L,P}$, we will first convolve it with ρ in order to estimate it using Lemma 2.A.1. Then, we will compare $\tilde{E}_{L,P}$ to its convolution with ρ which we denote - somewhat liberally - by

$$\rho * \tilde{E}_{L,P} = \int_{\mathbb{R}} \rho(\lambda) \tilde{E}_{L-\lambda,P} d\lambda \,.$$

The starting point of the following calculations will be the following Equation, which follows from Lemma 2.A.1.

$$\frac{d}{d\lambda}(\rho * e_{\lambda,P}(x,y))|_{\lambda=L} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)(t) P\left(q(x,t,y,\xi) e^{i(\psi(x,t,y,\xi) - t\sigma(y,\xi))}\right)\Big](L) d\xi \tag{2.A.1}$$
$$+ \mathcal{F}_t^{-1}\big[\mathcal{F}(\rho)(t) P k(x,t,y)\big](L).$$

2.A.2 Estimating the convolved kernel

In this section we provide the following expression for $\rho * \tilde{E}_{L,P}$ in the local coordinates chosen in Lemma 2.A.1.

Lemma 2.A.3. *There is an open set $V \subset U$ containing 0 such that, as $L \to \infty$ and uniformly for $(x,y) \in V \times V$,*

$$\rho * \tilde{E}_{L,P}(x,y) = \frac{1}{(2\pi)^n} \int_{\sigma(y,\xi) \leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi)) e^{i\psi(x,y,\xi)} d\xi + O(L^{n+d-1}) \,.$$

In order to do so we use the three lemmas stated below, whose proofs are given at the end of the section. To begin with, we use the information of Lemma 2.A.1 to give a first expression for $\rho * \tilde{E}_{L,P}$.

Lemma 2.A.4. *The quantity*

$$\rho * \tilde{E}_{L,P}(x,y) - \int_{-\infty}^{L} \frac{1}{(2\pi)^n} \int_{T_y^*M} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho) P\left(q(x,t,y,\xi) e^{i(\psi(x,y,t,\xi) - t\sigma(y,\xi))}\right)\Big](\lambda) d\xi d\lambda$$

is bounded uniformly for $(x,y) \in U \times U$.

Here and below $\mathcal{F}$ is the Fourier transform and the occasional subscript indicates the variable on which the transform is taken. Let us now investigate the effect of the differential operator P on the right hand side of this expression. By the Leibniz rule, there is a finite family of symbols $(\sigma_j)_{0\leq j\leq d} \in C^\infty(U\times]-\varepsilon,\varepsilon[\times U,\mathbb{R}^n)^{d+1}$ such that for each j, σ_j is homogeneous of degree j, such that

$$P\Big[q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\Big] = \Big[\sum_{j=0}^{d}\sigma_j(x,t,y,\xi)\Big]e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}$$

and such that

$$\sigma_d(x,t,y,\xi) = q(x,t,y,\xi)\sigma_P(x,y,\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))\,.$$

Now, for each j, let

$$R_j(x,y,L,\xi) = \frac{1}{(2\pi)^{n+1}}\int_{\mathbb{R}}\mathcal{F}(\rho)(t)\sigma_j(x,t,y,\xi)e^{itL}dt$$

and

$$S_j(x,y,L) = \int_{-\infty}^{L}\int_{\mathbb{R}^n} R_j(x,y,\lambda-\sigma(y,\xi),\xi)e^{i\psi(x,y,\xi)}d\xi d\lambda\,.$$

Then,

$$\int_{-\infty}^{L}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\left(q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\right)\Big](\lambda)d\xi d\lambda = \sum_{j=0}^{d}S_j(x,y,L)\,.$$

Each S_j will grow at an order corresponding to the degree of the associated symbol. This is shown in the following lemma.

Lemma 2.A.5. *There is an open set $V\subset U$ containing 0 such that, as $L\to\infty$ and uniformly for $(x,y)\in V\times V$,*

$$S_j(x,y,L) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leq L}\sigma_j(x,0,y,\xi)e^{i\psi(x,y,\xi)}d\xi + O(L^{n+j-1})\,.$$

Similarly since $q(x,0,y,\xi)-1\in S^{-1}(U_0\times U_0,\mathbb{R}^n)$, from a computation analogous to the proof of Lemma 2.A.5 and left to the reader, replacing σ_d by

$$(q(x,0,y,\xi)-1)\sigma_P(x,y,\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))\in S^{d-1}$$

one can remove q from the main term, which results in the following.

Lemma 2.A.6. *There is an open set $V\subset U$ containing 0 such that, as $L\to\infty$ and uniformly for $(x,y)\in V\times V$,*

$$S_d(x,y,L) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leq L}\sigma_P(x,y,\partial_{x,y}(\psi(x,y,\xi)-t\sigma(y,\xi)))e^{i\psi(x,y,\xi)}d\xi + O(L^{n+d-1})\,.$$

The juxtaposition of these results yields Lemma 2.A.3.

Proof of Lemma 2.A.4. Since $k \in C^\infty(U \times U \times]-\varepsilon,\varepsilon[)$ and $\mathcal{F}(\rho)$ is supported in $]-\varepsilon,\varepsilon[$,

$$\mathcal{F}_t^{-1}\big[\mathcal{F}(\rho)(t)Pk(x,t,y)\big](L) \in \mathcal{S}(\mathbb{R})\,.$$

Therefore, by Equation (2.A.1),

$$\rho * \tilde{E}_{L,P}(x,y) - \int_{-\infty}^{L} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}_t^{-1}\Big[\mathcal{F}(\rho)P\left(q(x,t,y,\xi)e^{i(\psi(x,y,\xi)-t\sigma(y,\xi))}\right)\Big](\lambda)d\xi d\lambda$$

is bounded. □

Proof of Lemma 2.A.5. In this proof, all generic constants will be implicitly uniform with respect to $(x,y) \in V \times V$. Let us fix $y \in V$ and define the following three domains of integration.

$$\begin{aligned} D_1 &= \{(\lambda,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda \le L,\ \sigma(y,\xi) \le L\} \\ D_2 &= \{(\lambda,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda \le L,\ \sigma(y,\xi) > L\} \\ D_3 &= \{(\lambda,\xi) \in \mathbb{R} \times \mathbb{R}^n \mid \lambda > L,\ \sigma(y,\xi) \le L\}\,. \end{aligned}$$

Moreover, for $l = 1,2,3$, let $I_l = \int_{D_l} R_j(x,y,\lambda-\sigma(y,\xi),\xi)e^{i\psi(x,y,\xi)}d\xi d\lambda$. We will prove that I_2 and I_3 are $O(L^{n+j-1})$. The following calculation will then yield the desired identity. Here we use Fubini's theorem and the fact that $\mathcal{F}(\rho)(0) = \int_{\mathbb{R}} \rho(\lambda)d\lambda = 1$.

$$\begin{aligned} S_j(x,y,L) &= I_1 + I_2 = I_1 + I_3 + O(L^{n+j-1}) \\ &= \int_{\sigma(y,\xi)\le L} \Big[\int_{\mathbb{R}} R_j(x,y,s,\xi)ds\Big]e^{i\psi(x,y,\xi)}d\xi + O(L^{n+j-1}) \\ &= \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\le L} \sigma_j(x,0,y,\xi)e^{i\psi(x,y,\xi)}d\xi + O(L^{n+j-1}). \end{aligned}$$

First of all, R_j is rapidly decreasing in the third variable and, since σ is elliptic of degree 1, bounded by $\sigma(y,\xi)^j$ with respect to the last variable, ξ. Therefore, for each $N > 0$ there is a constant $C > 0$ such that

$$|R_j(x,y,\lambda,\xi)| \le \frac{C\sigma(y,\xi)^j}{(1+|\lambda|)^N}\,.$$

Since σ is elliptic of order 1, the hypersurface $L^{-1}\{\sigma(y,\xi) = L\} \subset \mathbb{R}^n$ converges smoothly for $L \to \infty$ uniformly in y to $S_y^* = \{\sigma_A(y,\xi) = 1\}$ and the volume of $\{\sigma(x,\xi) = \beta\} \subset \mathbb{R}^n$ is $O(\beta^{n-1})$. Taking $N = 2n+j+1$, we deduce that

$$\begin{aligned} |I_2| &\le C\int_{-\infty}^{L}\int_{\sigma(y,\xi)>L} \frac{\sigma(y,\xi)^j}{(1+|\lambda-\sigma(y,\xi)|)^{2n+j+1}}d\xi d\lambda \le C\int_{-\infty}^{L}\int_{L}^{+\infty} \frac{\beta^{n+j-1}}{(1+|\lambda-\beta|)^{2n+j+1}}d\beta d\lambda \\ &\le C\int_{L}^{+\infty}\int_{-\infty}^{L-\beta} \frac{\beta^{n+j-1}}{(1+|s|)^{2n+j+1}}ds d\beta \le C\int_{L}^{\infty} \frac{\beta^{n+j-1}}{(1+\beta-L)^{2n+j}}d\beta \\ &\le C\int_{0}^{+\infty} \frac{(\gamma+L)^{n+j-1}}{(1+\gamma)^{2n+j}}d\gamma \le CL^{n+j-1}. \end{aligned}$$

Here we applied first the change of variables $s = \lambda - \beta$ and then $\gamma = \beta - L$. The case of I_3 is analogous and by a similar calculation we deduce that I_1 is well defined. □

2.A.3 Comparison of the kernel and its convolution

In this section we set about proving that $\tilde{E}_{L,P}$ is close enough to its convolution with ρ. This is encapsulated in the following lemma.

Lemma 2.A.7. *There is an open set $V \subset U$ containing 0 such that, as $L \to \infty$ and uniformly for $(x, y) \in V \times V$,*

$$\rho * \tilde{E}_{L,P}(x, y) - \tilde{E}_{L,P}(x, y) = O(L^{n+d-1}) .$$

As before, the proofs are relegated to the end of the section. In order to prove Lemma 2.A.7 we first estimate the growth of the R_j as follows.

Lemma 2.A.8. *There is an open set $V \subset U$ containing 0 such that, as $L \to \infty$ and uniformly for $(x, y) \in V \times V$,*

$$\int_{\mathbb{R}^n} R_j(x, y, L - \sigma(y, \xi), \xi) e^{i\psi(x,y,\xi)} d\xi = O(L^{n+j-1}) .$$

This lemma follows from a computation analogous to the bound on I_2 and I_3 given in the proof of Lemma 2.A.5 above and the details are left to the reader. It allows us to prove a second intermediate result from which we obtain Lemma 2.A.7 directly.

Lemma 2.A.9. *There is an open set $V \subset U$ containing 0 such that, as $L \to \infty$ and uniformly for $(x, y) \in V \times V$,*

$$\tilde{E}_{L+1,P}(x, y) - \tilde{E}_{L,P}(x, y) = O(L^{n+d-1}) .$$

Proof of Lemma 2.A.9. We begin with the case where $x = y$ and P is of the form $P_1 \otimes P_1$. For brevity we define

$$u(L) = \tilde{E}_{L,P}(x, x) = \sum_{\lambda_k \leq L} |(P_1 e_k)(x)|^2 .$$

Recall $\rho > 0$ so it stays greater than some constant $a > 0$ on the interval $[-1, 0]$. Moreover u is increasing so by Equation (2.A.1) and Lemma 2.A.8,

$$\begin{aligned} 0 \leq u(L+1) - u(L) &= \int_L^{L+1} u'(\lambda) d\lambda \leq \frac{1}{a} \int_L^{L+1} \rho(L - \lambda) u'(\lambda) d\lambda \\ &\leq \frac{1}{a} \frac{d}{dL} (\rho * u) \leq \frac{1}{a} \sum_{j=0}^{d} \int_{\mathbb{R}^n} R_j(x, y, L - \sigma(y, \xi)) d\xi + O(L^{n+d-1}) = O(L^{n+d-1}). \end{aligned}$$

Now if P is of the form $P_1 \otimes P_2$, and for any x and y, let $X = (P_1 e_k)_{L<\lambda_k \leq L+1}$ and $Y = (P_2 e_k)_{L<\lambda_k \leq L+1}$ be two vectors in some $\mathbb{C}^q$ which we equip with the standard hermitian product "$\star$". Then, $\tilde{E}_{L+1,P}(x,y) - \tilde{E}_{L,P}(x,y) = X \star \overline{Y}$ so

$$\begin{aligned}
|\tilde{E}_{L+1,P}(x,y) - \tilde{E}_{L,P}(x,y)|^2 &\leq |X|^2 |Y|^2 \\
&= |\tilde{E}_{L+1,P_1\otimes P_1}(x,y) - \tilde{E}_{L,P_1\otimes P_1}(x,y)||\tilde{E}_{L+1,P_2\otimes P_2}(x,y) - \tilde{E}_{L,P_2\otimes P_2}(x,y)| \\
&\leq \frac{1}{4}\Big(\tilde{E}_{L+1,P_1\otimes P_1}(x,x) - \tilde{E}_{L,P_1\otimes P_1}(x,x) + \tilde{E}_{L+1,P_1\otimes P_1}(y,y) - \tilde{E}_{L,P_1\otimes P_1}(y,y)\Big) \\
&\times \Big(\tilde{E}_{L+1,P_2\otimes P_2}(x,x) - \tilde{E}_{L,P_2\otimes P_2}(x,x) + \tilde{E}_{L+1,P_2\otimes P_2}(y,y) - \tilde{E}_{L,P_2\otimes P_2}(y,y)\Big) \\
&\leq C L^{2n+2d-2}.
\end{aligned}$$

Here we used first the Cauchy-Schwarz inequality, then the mean value inequality, then on each factor,

$$2|P_1 e_k(x)\overline{P_1 e_k(y)}| \leq |P_1 e_k(x)|^2 + |P_1 e_k(y)|^2$$

and finally the above estimate. In general P is a locally finite sum of operators of the form $P_1 \otimes P_2$. □

Proof of Lemma 2.A.7. First of all, according to Lemma 2.A.9 there is a constant C such that for all $L \geq 0$ and λ,

$$|\tilde{E}_{L+\lambda,P}(x,y) - \tilde{E}_{L,P}(x,y)| \leq C(1+|\lambda|+L)^{n+d-1}(1+|\lambda|).$$

Consequently

$$\begin{aligned}
(\rho * \tilde{E}_{L,P}(x,y) - \tilde{E}_{L,P}(x,y)| &\leq |\int \rho(\lambda)\tilde{E}_{L+\lambda,P}(x,y)d\lambda - \tilde{E}_{L,P}(x,y)| \\
&\leq \int \rho(\lambda)\Big|\tilde{E}_{L+\lambda,P}(x,y) - \tilde{E}_{L,P}(x,y)\Big|d\lambda \\
&\leq C\int \rho(\lambda)(1+|\lambda|+L)^{n+d-1}(1+|\lambda|)d\lambda \\
&\leq C' L^{n+d-1}
\end{aligned}$$

for some $C' > 0$. Here we used that $\rho > 0$, ρ is rapidly decreasing and $\int_{\mathbb{R}} \rho(\lambda)d\lambda = \mathcal{F}(\rho)(0) = 1$. □

2.A.4 Conclusion

Combining Lemmas 2.A.3 and 2.A.7 we obtain the following:

$$\tilde{E}_{L,P}(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma(y,\xi)\leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))e^{i\psi(x,y,\xi)}d\xi + O(L^{n+d-1}).$$

Since $\sigma - \sigma_A^{\frac{1}{m}} \in S^0$, replacing one by the other adds only a $O(L^{n+d-1})$ term. Therefore,

$$\tilde{E}_{L,P}(x,y) = \frac{1}{(2\pi)^n}\int_{\sigma_A(y,\xi)^{1/m}\leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi))e^{i\psi(x,y,\xi)}d\xi + O(L^{n+d-1}).$$

This estimate is valid and uniform for $x, y \in V$. To conclude, notice that $\sigma_A(x,\xi)^{1/m} \leq L$ is equivalent to $\sigma_A(x,\xi) \leq L^m$. Since $\tilde{E}_L = E_{L^m}$, replacing L by $L^{1/m}$ in the last estimate we get

$$E_{L,P}(x,y) = \frac{1}{(2\pi)^n} \int_{\sigma_A(y,\xi)\leq L} \sigma_P(x,y,\partial_{x,y}\psi(x,y,\xi)) e^{i\psi(x,y,\xi)} d\xi + O(L^{(n+d-1)/m})$$

as announced.

Chapter 3

Expected number of nodal components for cut-off fractional Gaussian fields

Abstract

Let $(\mathcal{X}, g)$ be a closed Riemmanian manifold of dimension $n > 0$. Let Δ be the Laplacian on $\mathcal{X}$, with eigenfunctions (resp. eigenvalues) $(e_k)_k$ (resp. $(\lambda_k)_k$). We assume that $(\lambda_k)_k$ is increasing and that the e_k are real-valued and L^2-normalized. Let $(\xi_k)_k$ be a sequence of iid $\mathcal{N}(0,1)$ random variables. For each $L > 0$ and $s \in \mathbb{R}$, set

$$f_L^s = \sum_{0<\lambda_j \leq L} \lambda_j^{-\frac{s}{2}} \xi_j e_j \,.$$

Then, f_L^s is almost surely regular on its zero set. Let N_L be the number of connected components of its zero set. If $s < \frac{n}{2}$, then we deduce from previous results that there exists $\nu = \nu(n,s) > 0$ such that $N_L \sim \nu \mathrm{Vol}_g(\mathcal{X}) L^{n/2}$ in L^1 and almost surely. In particular, $\mathbb{E}[N_L] \asymp L^{n/2}$. On the other hand, we prove that if $s = \frac{n}{2}$ then

$$\mathbb{E}[N_L] \asymp \frac{L^{n/2}}{\sqrt{\ln\left(L^{1/2}\right)}} \,.$$

In the latter case, we also obtain an asymptotic formula for the expected Euler characteristic of the zero set of f_L^s as well as upper bounds for its Betti numbers.

This chapter is based on [Riv18a].

3.1 Introduction

3.1.1 Setting and main results

In this manuscript we study the number of **nodal components** of random linear combinations of eigenfunctions of the Laplacian on a closed Riemmanian manifold, that is, the number of connected components of the zero set of such random functions. The study of such components goes back to [NS09] where the authors consider random eigenfunctions with eigenvalue L of the laplacian on S^2 and prove that the number of components concentrates around cL for some $c \in]0, +\infty[$ as $L \to +\infty$. Later, in [NS16], a similar result[1] was proved regarding the number of components of general Gaussian fields on Riemmanian manifolds. Meanwhile, using different methods, in [GW16a, GW16b], the authors determined the rate of growth of the Betti numbers of the nodal set for a particular model of random linear combination of eigenfunctions on a Riemmanian manifold. Most of the arguments in these two papers were quite general but required inputs from spectral analysis at a few key steps in the proof. More precisely, the authors relied on a result from semi-classical analysis (see Theorem 2.3 of [GW16a] in which the authors extend a result from [Hör68]). Other works in this field are [SW16, BW16a, KW17, CS16]. All of the aforementioned works study parametric families of smooth functions $(f_L)_{L\geq 0}$ on a manifold of dimension n that vary at a natural scale $L^{-1/2}$ and that posess 'local limits'. In the case of [NS16] this is an explicit assumption while in the other cases, it follows from results about spectral asymptotics. As a result, the number of connected components in a fixed compact set is of order $L^{n/2}$. In contrast, the recent [Riv17] introduced a model of random linear combinations of eigenfunctions of the Laplacian on a closed surface that did not have 'local limits'. A natural problem is to determine the rate of growth of the number of nodal domains for this new model. The present work is set in the continuation of the articles mentioned above and provides an answer to this question.

We consider a smooth compact Riemmanian manifold $(\mathcal{X}, g)$ of dimension $n > 0$ with no boundary. Let $|dV_g|$ be Riemmanian density and Δ the Laplacian operator induced by g on $\mathcal{X}$. Let g^{-1} be the metric induced on $T^*\mathcal{X}$ by g. Since $\mathcal{X}$ is closed, the spectrum of Δ is discrete and made up of a sequence of non-negative eigenvalues $(\lambda_k)_{k\geq 0}$ whose corresponding eigenfunctions $(e_k)_{k\in\mathbb{N}}$ we can take to be real valued, smooth and normalized so as to form a Hilbert basis for $L^2(\mathcal{X}, |dVg|)$. Let $(\xi_k)_{k\geq 0}$ be a sequence of independent centered gaussians of unit variance and, for each $s \in \mathbb{R}$ and $L > 0$, set

$$f_L^s = \sum_{0<\lambda_j\leq L} \lambda_j^{-\frac{s}{2}} \xi_j e_j \,. \tag{3.1.1}$$

This formula defines a smooth Gaussian field on $\mathcal{X}$ which we call **cut-off fractional Gaussian field** because of the cut-off $\lambda_j \leq L$ and the fractional power $-\frac{s}{2}$. A simple calculation shows that the covariance function for f_L^s is

[1] Note however that the concentration rate is much weaker than in the former setting [NS09].

$$\mathbb{E}[f_L^s(x)f_L^s(y)] := K_L^s(x,y) = \sum_{0<\lambda_j\leq L} \lambda_j^{-s} e_j(x)e_j(y)\,.$$

The behavior of K_L^s near the diagonal as $L \to +\infty$ was studied in [Riv18b] for $s \leq n/2$. It is well known, at least for $s = 0$, that for L large enough, the nodal set $\{x \in \mathcal{X} \mid f_L^s(x) = 0\}$ is almost surely smooth. We are interested in N_L, the number of connected components of the nodal set. In the case where $s < n/2$, combining results from [Riv18b] and [NS16], we deduce the following result:

Theorem 3.1.1. *Suppose that $s < n/2$. Then, there exists a (deterministic) constant $\nu_{n,s} > 0$ depending only on s and n such that $L^{-n/2}N_L$ converges to $\nu_{n,s}\mathrm{Vol}_g(\mathcal{X})$ in L^1 as $L \to +\infty$.*

On the other hand, in the case where $s = n/2$, the asymptotic behavior of the field is quite different (see Theorem 1.2 of [Riv18b] or Theorem 3.1.6). In particular, it does not have non-trivial local limits. In this case, we prove the following result:

Theorem 3.1.2. *Suppose that $s = n/2$. Then, there exist constants $0 < c < C < +\infty$ where C depends only on n such that for L large enough,*

$$c\mathrm{Vol}_g(\mathcal{X})\frac{L^{n/2}}{\sqrt{\ln\left(L^{1/2}\right)}} \leq \mathbb{E}[N_L] \leq C\mathrm{Vol}_g(\mathcal{X})\frac{L^{n/2}}{\sqrt{\ln\left(L^{1/2}\right)}}\,.$$

Moreover, there exists $\rho = \rho(\mathcal{X}) < +\infty$ such that if $N_L(\rho)$ is the number of connected components with diameter at most $\rho L^{-1/2}$ then

$$\mathbb{E}\left[N_L(\rho)\right] \geq c\mathrm{Vol}_g(\mathcal{X})\frac{L^{n/2}}{\sqrt{\ln\left(L^{1/2}\right)}}\,.$$

Remark 3.1.3. The upper constant is explicit, as explained in Theorem 3.1.5 below.

Remark 3.1.4. In the case where $s = 1$, the field f_L^s is the cut-off Gaussian Free Field introduced in [Riv17]. By construction, this field converges in distribution to the Gaussian Free Field (see [She07]). In this case, Theorem 3.1.2 implies that in dimension $n = 2$, $\mathbb{E}\left[N_L\right] \asymp \frac{L}{\sqrt{\ln(L)}}$ while in dimension $n \geq 3$, $\mathbb{E}\left[N_L\right] \asymp L^{n/2}$.

For the upper bound, we follow the approach of [GW16a]. Indeed, although their main result doesn't apply, their strategy still does. For the lower bound, the most common strategy is to construct a 'barrier' (see for instance Claim 3.2 of [NS09] or Corollary 1.11 of [GW16b]). This amounts to constructing a model function with a nodal component inside a given ball and proving that the field does not deviate too much from this model with positive probability. In this case, the barrier cannot hold with probability bounded from below without contradicting the upper bound. However, by pinning the field near zero at a given point, we manage to construct such a barrier losing only a factor of

$\sqrt{\ln\left(L^{1/2}\right)}$. The strategy of [GW16a] is to count the critical points of a Morse function on Z_L. This provides not only an upper bound on the number of nodal components but also the following result:

Theorem 3.1.5. *For each $L \geq 1$ let $\chi(Z_L)$ be the Euler characteristic of Z_L and for each $i \in \{0, \dots, n-1\}$ let $b_i(Z_L)$ be the i-th Betti number of Z_L. If n is odd, as $L \to +\infty$,*

$$\mathbb{E}[\chi(Z_L)] = \left(A_n \mathrm{Vol}_g(\mathcal{X}) + o(1)\right) \frac{L^{n/2}}{\sqrt{\ln\left(L^{1/2}\right)}}$$

where A_n is defined as follows. If n is even then $A_n = 0$. Otherwise,

$$A_n = \frac{(n-1)!}{2^{3n/2-1}((n-1)/2)!\sqrt{\pi^{n+1} n (n+2)^{n-1}}}\,.$$

Moreover, for each $i \in \{0, \dots, n-1\}$,

$$\limsup_{L \to +\infty} \frac{\sqrt{\ln\left(L^{1/2}\right)}}{L^{n/2}} \mathbb{E}[b_i(Z_L)] \leq A_n^i \mathrm{Vol}_g(\mathcal{X})$$

where A_n^i is defined as follows. Let M be a centered Gaussian vector with values in $\mathrm{Sym}_{n-1}(\mathbb{R})$ with covariance Ξ satisfying for any $i, j, k, l \in \{1, \dots, n-1\}$ such that $i \neq j$ and $(i,j) \neq (k,l)$, $\Xi_{ii,jj} = \Xi_{ij,ij} = 1$, $\Xi_{ii,ii} = 3$ and $\Xi_{ij,kl} = 0$. Then,

$$A_n^i = \frac{\mathbb{E}[|\det(M)| \mathbb{1}[\mathrm{sgn}(M) = i]]}{\sqrt{\pi^{n+1} 2^{2n-1} n (n+2)^{n-1}}}\,.$$

Here sgn *denotes the number of negative eigenvalues of the symmetric matrix M and $b_i(Z_f) = \mathrm{rank}_{\mathbb{Z}} H_i(Z_f; \mathbb{Z})$.*

In particular, for the upper bound in Theorem 3.1.2 one can set $C = A_n^0$. Note that, as far as we know, A_n might vanish for some odd values of n as well as for all even values of n.

3.1.2 Proof strategy

The crucial tool for the proof of Theorems 3.1.1 and 3.1.2 is the following result from [Riv18b]. It provides an estimate for the covariance of K_L^s. Recall that g^{-1} is the metric induced on $T^*\mathcal{X}$ by g. Given $(x, \xi) \in T^*\mathcal{X}$, we write $|\xi|_x^2 := g_x^{-1}(\xi, \xi)$.

Theorem 3.1.6 (Corollaries 1.4 and 1.5 of [Riv18b])**.**

1. *Assume that $s < n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \dots, x_n)$ centered at x such that $|dV_g|$ agrees the Lebesgue measure in these coordinates.*

Then, there exists $U \subset \mathbb{R}^n$ a neighborhood of 0 such that for each $\alpha, \beta \in \mathbb{N}^n$, uniformly for $z \in U$ and $L \geq 1$

$$\lim_{L\to+\infty} L^{s-(n+|\alpha|+|\beta|)/2}\partial_x^\alpha \partial_y^\beta K_L^s(x,y)|_{x=y=z} = \frac{1}{(2\pi)^n}\int_{|\xi|_z^2\leq 1} \frac{(i\xi)^\alpha(-i\xi)^\beta}{|\xi|^{2s}}d\xi$$

where for any $\gamma \in \mathbb{N}^n$ we set $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$.

2. *Assume that $s = n/2$. Fix x_0 and a set of local coordinates centered at x_0 such that the density $|dV_g|$ in these coordinates agreees with the Lebesgue measure. Then, there exists $U \subset \mathbb{R}^n$ a neighborhood of 0 such that uniformly for $x, y \in U$ and $L \geq 1$*

$$K_L^s(x,y) = \frac{|S^{n-1}|}{(2\pi)^n}\left(\ln\left(L^{1/2}\right) - \ln_+\left(L^{1/2}|x-y|\right)\right) + O(1)\,.$$

Here $|S^{n-1}|$ is the area of the Euclidean unit sphere in $\mathbb{R}^n$.

3. *Assume that $s = n/2$. Fix $x_0 \in \mathcal{X}$ and consider local coordinates $x = (x_1, \ldots, x_n)$ centered at x such that $|dV_g|$ agrees the Lebesgue measure in these coordinates. Then, there exists $U \subset \mathbb{R}^n$ a neighborhood of 0 such that for each $\alpha, \beta \in \mathbb{N}^n$ such that $(\alpha, \beta) \neq 0$, uniformly for $z \in U$ and $L \geq 1$,*

$$\lim_{L\to+\infty} L^{-(|\alpha|+|\beta|)/2}\partial_x^\alpha \partial_y^\beta K_L^s(x,y)|_{x=y=z} = \frac{1}{(2\pi)^n}\int_{|\xi|_z^2\leq 1} \frac{(i\xi)^\alpha(-i\xi)^\beta}{|\xi|^{n}}d\xi\,.$$

The original result is somewhat more general and in particular provides an estimate of the error terms in each case but since our results are bounds up to a constant factor, these will not be of use to us.

As we shall see in Section 3.2, this result implies that for $s < n/2$, the family $(f_L^s)_{L\geq 1}$ satisfies the assumptions for the main result of [NS16] which directly implies Theorem 3.1.1 .

We prove the upper and lower bounds Theorem 3.1.2 in two separate sections. For the upper bound, we follow the strategy of [GW16a]. More explicitely, we fix a function $p \in C^\infty(\mathcal{X})$ with at most a countable number of critical points[2]. For each $i \in \{0, \ldots, n-1\}$, each $L \geq 1$ and each Borel subset $B \subset \mathcal{X}$, let $m_i(p, f_L, B)$ be the number of critical points of $p|_{Z_L}$ of index[3] i. In Section 3.4 we prove the following theorem.

Theorem 3.1.7. *Let M be a centered Gaussian vector with covariance Ξ such that for any $i, j, k, l \in \{1, \ldots, n-1\}$ such that $i \neq j$ and $(i,j) \neq (k,l)$, $\Xi_{ii,jj} = \Xi_{ij,ij} = 1$,*

[2]Such functions always exist since for instance Morse functions are dense in $C^\infty(\mathcal{X})$.

[3]Recall that the index of a critical point of a given function is the number of negative eigenvalues of the Hessian of this function at the critical point.

$\Xi_{ii,ii} = 3$ *and* $\Xi_{ij,kl} = 0$. *Then, for* L *large enough,* Z_L *is almost surely smooth,* $p|_{Z_L}$ *is almost surely Morse and for any Borel subset* $B \subset \mathcal{X}$, *as* $L \to +\infty$,

$$\mathbb{E}\left[m_i(p, f_L, B)\right] \sim C_n \mathrm{Vol}_g(B) \mathbb{E}\left[|\det(M)| \mathbb{1}[\mathrm{sgn}(M) = i]\right] \frac{L}{\sqrt{\ln\left(L^{1/2}\right)}} \tag{3.1.2}$$

where

$$C_n = \frac{1}{\sqrt{\pi^{n+1} 2^{2n-1} n (n+2)^{n-1}}}$$

and where $sgn(M)$ *is the number of negative eigenvalues of the matrix* M.

Theorem 3.1.2 as well as Theorem 3.1.5 will then follow from the Morse inequalities. For the lower bound, we prove that given a ball of radius $\asymp L^{-1/2}$, the probability that this ball contains a nodal component is bounded from below by a constant multiple of $\left(\ln\left(L^{1/2}\right)\right)^{-1/2}$. This result actually holds for log-correlated gaussian fields with only Hölder regularity. More precisely, in Section 3.5 we prove the following result.

Theorem 3.1.8. *Fix* $n \in \mathbb{N}$ *and* $U \subset \mathbb{R}^n$ *an open subset containing* $\overline{B(0,1)}$. *Let* $(f_\lambda)_{\lambda \geq 1}$ *be a family of continuous centered Gaussian fields on* U *satisfying the following properties.*

1. *There exists* $a < +\infty$ *for each* $x, y \in U$ *and* $\lambda \geq 1$,
$$|\mathbb{E}\left[f_\lambda(x) f_\lambda(y)\right] - \ln(\lambda) + \ln_+(\lambda |x-y|)| \leq a\,.$$
2. *There exists* $\alpha \in]0, 2]$ *and* $b < +\infty$ *such that for each* $x, y \in U$ *and each* $\lambda \geq 1$ *satisfying* $\lambda |x-y| \leq 1$,
$$\mathbb{E}\left[(f_\lambda(x) - f_\lambda(y))^2\right] \leq b^2 \lambda^\alpha |x-y|^\alpha\,.$$

There exist $\rho = \rho(a, b, \alpha, n) > 0$, $\kappa = \kappa(a, b, \alpha, n) > 0$ *and* $\lambda_0 = \lambda_0(a, b, \alpha, n) \in [1, +\infty[$ *such that the following holds. Let* $\mathcal{C}^\lambda$ *be the event that* $f_\lambda^{-1}(0)$ *has a connected component included in the ball* $B(0, \rho/\lambda)$. *Then, for each* $\lambda \geq \lambda_0$.

$$\mathbb{P}\left[\mathcal{C}^\lambda\right] \geq \kappa \ln(\lambda)^{-1/2}\,.$$

Theorem 3.1.8 plays the same role as the 'barrier lemma' (Claim 3.2) of [NS09] or as Theorem 0.3 of [GW16b]. However, in this setting, we do not (and cannot!) obtain a uniform lower bound on the probability of having a nodal component inside a given small ball. Moreover, the behavior of log-correlated random fields is quite different from that of locally-translation-invariant random fields. Indeed, just as the aforementioned results were used to obtain lower bounds on the expected number (and topology) of connected components of the nodal set, Theorem 3.1.2 will follow by packing $\mathcal{X}$ with disjoint small balls of radius $\asymp \lambda^{-1} = L^{-1/2}$ and adding up the expected nodal components contained in each ball. The proof of Theorem 3.1.8 combines tools (see Lemma 3.5.1) from the theory of smooth Gaussian fields with the FKG inequality (see Lemma 3.5.2) from statistical mechanics.

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3.2 Proof of the main results

The object of this section is to prove Theorems 3.1.1, 3.1.2 and 3.1.5 using the results presented in Subsection 3.1.2 as well as Theorem 3 of [NS16]. We start with the proof of Theorem 3.1.1. As explained above, we simply check that Theorem 3.1.6 implies that the field satisfies the assumptions of Theorem 3 of [NS16].

Proof of Theorem 3.1.1. Fix $2s < n$. By the first point of Theorem 3.1.6, for any $x_0 \in \mathcal{X}$ and any set of local coordinates on a chart $U \subset \mathcal{X}$ centered at x_0 and pushing $|dV_g|$ to the Lebesgue measure, there exists a neighborhood V of 0 in U such that we have

$$\lim_{L\to+\infty} L^{s-n/2} K_L^s\Big(w + L^{-1/2}x, w + L^{-1/2}y\Big) := \frac{1}{(2\pi)^n}\int_{|\xi|_w^2\leq 1} e^{i\langle \xi, x-y\rangle}|\xi|^{-2s}d\xi$$

where the convergence takes place in C^∞ with respect to $(x, y) \in V \times V$, uniformly with respect to $w \in V$. This shows that the kernels K_L^s have, in the terminology of [NS16], translation-invariant limits on V (see Definition 2 of [NS16])[4] and that it satisfies the norm estimates required for the parametric Gaussian ensemble (see Definition 1 of [NS16]) to be locally uniformly controllable (see Definition 4 of [NS16]) on V. The spectral measure at w equals

$$\rho_w(\xi) = |\xi|_w^{-2s}\mathbb{1}\left[|\xi|_w^2 \leq 1\right] d\xi$$

so it has no atoms and for each $i, j \in \{1, \cdots, n\}$, by parity,

$$\begin{aligned}\lim_{L\to+\infty} \partial_{x_i}\partial_{y_j} L^{s-(n+2)/2} K_L^s(x, y)|_{y=x=w} &= -\frac{1}{(2\pi)^n}\int_{|\xi|_w^2\leq 1} \xi_i\xi_j|\xi|_w^{-2s}d\xi \\ &= -\frac{\delta_{ij}}{n(2\pi)^n}\int_{|\xi|_w^2\leq 1} |\xi|_w^{2-2s}d\xi > 0\,.\end{aligned}$$

In particular, the ensemble is locally uniformly non-degenerate (see Definition 3 of [NS16]) and, together with the previous estimate on K_L^s, it is locally uniformly controllable. Since, finally, the spectral measure has no atoms, the Gaussian ensemble $\left(L^{s/2-n/4} f_L\right)_{L>0}$, in local coordinates, defines a tame ensemble (see Definition 5 of [NS16]). Moreover, since such charts exist around each $x_0 \in \mathcal{X}$, by the criterion given in Subsection 1.4.1 of [NS16], the sequence $\left(L^{s/2-n/4} f_L\right)_{L>0}$ forms a tame parametric Gaussian ensemble on $\mathcal{X}$ (see Definition 6 of [NS16]). Thus, by Theorem 3 of [NS16] and the remark 1.5.2 that follows it, there exists a locally finite (and therefore finite since

[4]The parameter L used in [NS16] corresponds to the quantity $L^{1/2}$ of the present work.

$\mathcal{X}$ is compact) Borel measure $\mathfrak{n}_\infty$ on $\mathcal{X}$ such that the sequence $\left(L^{n/2}N_L\right)_L$ converges in L^1 to $\mathfrak{n}_\infty(\mathcal{X}) < +\infty$. Recall that the parameter L in Nazarov and Sodin's theorem corresponds to $L^{1/2}$ of the present work. All that remains is to show that this quantity is positive. Moreover, according to the third item of Theorem 3 of [NS16], the density of $\mathfrak{n}_\infty$ with respect to $|dV_g|$ at a point $x \in \mathcal{X}$ is given by the constant ν given by item (ρ_3) of Theorem [NS16] for the limiting ensemble at the point x. Note that we may always require that the measure-preserving coordinates around x_0 be isometric at x_0. If so, the limiting spectral measure depends only on n and s. Thus, $\nu = \nu_{n,s}$ is constant that depends only on n, s and $\mathfrak{n}_\infty = \nu_{n,s}|dV_g|$. Moreover, since the support of the spectral measure contains 0, it satisfies Pjetro Majer's interior point criterion (see Appendix C.2 of [NS16]) which in turn implies condition (ρ_4) for Theorem 1 of [NS16]. This shows that $\nu_{n,s} > 0$ and so $\mathfrak{n}_\infty(\mathcal{X}) = \nu_{n,s}\mathrm{Vol}_g(\mathcal{X}) > 0$. □

Next we check Theorem 3.1.5.

Proof of Theorem 3.1.5. Fix $p \in C^\infty(\mathcal{X})$. By Theorem 3.1.7, $p|_{Z_L}$ is almost surely a Morse function. By the (weak) Morse inequalities (see Theorem 5.2 of [Mil63]), we have first for each $i \in \{0, \dots, n-1\}$,

$$b_i\left(Z_L\right) \leq m_i(p, f_L, \mathcal{X})$$

where the m_i are as in Theorem 3.1.7. Taking expectations, by Equation (**??**), we obtain the upper bound on $\mathbb{E}\left[b_i(Z_L)\right]$ announced in Theorem 3.1.5. Theorem 5.2 of [Mil63] also implies that

$$\chi(Z_L) = \sum_{i=0}^{n-1} m_i(p, f_L, \mathcal{X})\,.$$

Taking expectations and using Equation (3.1.2) on the right-hand side, we get, as $L \to +\infty$:

$$\mathbb{E}\left[\chi(\mathcal{X})\right] = C_n\mathrm{Vol}_g(\mathcal{X}) \times \sum_{i=0}^{n-1} \mathbb{E}\left[|\det(M)|\mathbb{1}[\mathrm{sgn}(M) = i]\right] \frac{L}{\sqrt{\ln\left(L^{1/2}\right)}} + o\left(\frac{L}{\sqrt{\ln\left(L^{1/2}\right)}}\right) \tag{3.2.1}$$

where $C_n = \frac{1}{\sqrt{\pi^{n+1}2^{2n-1}n(n+2)^{n-1}}}$ and M is as described in the statement of Theorem 3.1.5. Observe now that since M is a symmetric matrix, then

$$\sum_{i=1}^{n-1}(-1)^i|\det(M)|\mathbb{1}[\mathrm{sgn}(M) = i] = \det(M)\,.$$

Therefore,

$$\sum_{i=0}^{n-1} \mathbb{E}\left[|\det(M)|\mathbb{1}[\mathrm{sgn}(M) = i] = \mathbb{E}[\det(M)\right]\,. \tag{3.2.2}$$

The determinant of M is a polynomial in its coefficients in which coefficient appears twice as a factor in a given monomial. Consequently, each monomial involving a non-diagonal coefficient has zero expectation. Hence,

$$\mathbb{E}[\det(M)] = \mathbb{E}\left[\prod_{i=1}^{n-1} M_{ii}\right].$$

Notice that if $X, Y_1, \dots, Y_n - 1$ are i.i.d centered Gaussians with unit variance, then, the Gaussian vector $(X + \sqrt{2}Y_i)_{1\leq i\leq n-1}$ has the same covariance structure, and therefore the same law, as $(M_{ii})_{1\leq i\leq n-1}$. As a consequence,

$$\mathbb{E}[\det(M)] = \mathbb{E}\left[\prod_{i=1}^{n-1} (X + \sqrt{2}Y_i)\right].$$

Developing the product on the right-hand side, the only term with non-zero expectation is X^{n-1} so

$$\mathbb{E}[\det(M)] = \mathbb{E}\left[X^{n-1}\right] = \begin{array}{l} 0 \text{ if } n \text{ is even} \\ \dfrac{(n-1)!}{2^{(n-1)/2}((n-1)/2)!} \text{ if } n \text{ is odd}. \end{array}$$

By Equations (3.2.1) and (3.2.2), we get, as $L \to +\infty$:

$$\mathbb{E}[\chi(\mathcal{X})] A_n \mathrm{Vol}_g(\mathcal{X}) \frac{L}{\sqrt{\ln\left(L^{1/2}\right)}} + o\left(\frac{L}{\sqrt{\ln\left(L^{1/2}\right)}}\right)$$

where A_n is zero for n even and

$$A_n = \frac{(n-1)!}{2^{(n-1)/2}((n-1)/2)!} C_n$$

for n odd. Replacing C_n with its definition we obtain the desired expression for A_n. □

Finally, we check Theorem 3.1.2.

Proof of Theorem 3.1.2. The upper bound follows with $C = A_n^0$ by Theorem 3.1.5. For the lower bound, first, by the second point of Theorem 3.1.6 together with the compactness of $\mathcal{X}$, there exist a constant $a = a(\mathcal{X}) < +\infty$ such that the first assumption of Theorem 3.1.8 is satisfied by $(f_\lambda)_{\lambda\geq 1} = \left(f_{\sqrt{L}}\right)_{L\geq 1}$ in local charts of some atlas. Let us check that the second assumption of Theorem 3.1.8 is also satisfied by this family of fields with $\alpha = 2$. Consider U a local chart given by Theorem 3.1.6. Let $V \subset U$ a convex neighborhood of 0. Fix $x, y \in U$. For each $t \in [0, 1]$ we have $x + t(y - x) \in U$.

For such t, let $u(t) = \mathbb{E}\left[\left(f_{\sqrt{L}}(x) - f_{\sqrt{L}}(x + t(y-x))\right)^2\right]$. Then, u is twice continuously differentiable and we have $u(0) = u'(0) = 0$. Moreover, for each $t \in [0,1]$,

$$\begin{aligned} u''(t) =& 2\mathbb{E}\left[\left(d_{x+t(y-x)} f_{\sqrt{L}}(y-x)\right)^2\right] \\ &+ 2\mathbb{E}\left[\left(f_{\sqrt{L}}(x+t(y-x)) - f_{\sqrt{L}}(x)\right) d^2_{x+t(y-x)} f_{\sqrt{L}}(y-x, y-x)\right] \\ \leq& 2\partial_z \partial_w K_L(z,w)\Big|_{z=w=x+t(y-x)}(y-x, y-x) \\ &+ 2\left(u(t)\partial_z^2 \partial_w^2 K_L(z,w)\Big|_{z=w=x+t(y-x)}(y-x, y-x, y-x, y-x)\right)^{1/2}. \end{aligned}$$

Here we used the definition of K_L as well as the Cauchy-Schwarz inequality. Applying the third point of Theorem 3.1.6 to the derivatives of K_L in the right hand side of the last line of the above computation, we have the following estimate: There exists $C = C(n) < +\infty$ such that for each $L \geq 1$, and for any choice of $x, y \in V$,

$$u''(t) \leq CL|x-y|^2 \left(1 + \sqrt{u(t)}\right).$$

Now, applying Taylor's inequality up to order 2 to u from 0 to any $t \in [0,1]$, we get

$$u(t) \leq \frac{C}{2} L|x-y|^2 \left(1 + \sqrt{\sup_{0 \leq s \leq t} u(s)}\right).$$

Assume now that $\sqrt{L}|x-y| \leq 1$. For each $t \in [0,1]$, let $v(t) = \sup_{0 \leq s \leq t} u(s)$. We have shown that for each $t \in [0,1]$,

$$\frac{v(t)}{1 + \sqrt{v(t)}} \leq \frac{C}{2} L|x-y|^2 \leq \frac{C}{2}.$$

Comparing the first and third term, we get that $v(t)$ is uniformly bounded. Then, with this information, the first inequality shows that there exists $b < +\infty$ depending only on C such that $v(t) \leq bL|x-y|^2$. In particular, for each $L \geq 1$ and each $x, y \in V$ such that $\sqrt{L}|x-y| \leq 1$,

$$\mathbb{E}\left[\left(f_{\sqrt{L}}(x) - f_{\sqrt{L}}(y)\right)^2\right] = u(1) \leq v(1) \leq b^2 L|x-y|^2.$$

Hence, Theorem 3.1.8 does apply. Let $\rho > 0$, $\kappa > 0$ and $\lambda_0 < +\infty$ be as in Theorem 3.1.8. Then, there exists $c' > 0$ independent of $\mathcal{X}$ and $\lambda_1 = \lambda_1(\mathcal{X})$ such that for each $L \geq \lambda_0^2 \vee \lambda_1^2$, we can find $\lfloor c' L^{n/2} \mathrm{Vol}_g(\mathcal{X}) \rfloor$ disjoint balls of radius $\rho L^{-1/2}$ in $\mathcal{X}$. By Theorem 3.1.8, for such L, each ball contains a nodal component with probability at least $\kappa \left(\ln\left(L^{1/2}\right)\right)^{-1/2}$ and

$$\mathbb{E}[N_L] \geq \kappa \lfloor c' L^{n/2} \mathrm{Vol}_g(\mathcal{X}) \rfloor \left(\ln\left(L^{1/2}\right)\right)^{-1/2}.$$

This proves the lower bound with for instance $c = c'\kappa/2$. Finally, since $\mathcal{X}$ is compact, the Euclidean diameter of balls in local charts such as the one used above and the Riemmanian diameter are comparable. Since the connected components constructed using Theorem 3.1.8 have Euclidean diamter less than ρ/λ then we also have the second part of Theorem 3.1.2. □

3.3 Differential geometry of smooth Gaussian fields

The object of this section is to apply classical results from the theory of smooth Gaussian fields to our setting. These results will be used in Section 3.4. Throughout this section $\mathcal{X}$ will be a smooth manifold of dimension $n > 0$ and f be a real-valued Gaussian field on $\mathcal{X}$ with covariance K. We will assume throughout that f is almost surely of class C^2.

Condition 3.3.1. For each $x \in \mathcal{X}$, the Gaussian vector $(f(x), d_x f)$ is non-degenerate.

Condition 3.3.2. Let $p \in C^\infty(\mathcal{X})$. For each $x \in \mathcal{X}$ such that $d_x p \neq 0$ and for any connection ∇^p on $p^{-1}(x)$ defined near x, the Gaussian vector $(f(x), d_x f, (\nabla^p df|_{p^{-1}(0)})_x)$ is non-degenerate.

Remark 3.3.3. Condition 3.3.2 is satisfied for any connection ∇^p as soon as it is satisfied for one particular connection.

Remark 3.3.4. Conditions 3.3.1 and 3.3.2 are diffeomorphism invariant.

Remark 3.3.5. As we shall see in Lemma 3.4.1 below, for any $p \in C^\infty(\mathcal{X})$ the field (f_L^s) satisfies Condition 3.3.2 for $s = n/2$ and large enough values of L.

3.3.1 Manifold versions of classical lemmas for smooth Gaussian fields

In this subsection we state two general lemmas for smooth Gaussian fields. The statements are slightly altered to fit the case of section-valued fields. We simply check that the euclidean case adapts well to this setting. First, let us introduce the following notation.

Definition 3.3.6. Let V, W be two n-dimensional vector spaces each equipped with a non-zero n-form ω_V and ω_W. Let $A : V \to W$ be a linear map. We define the determinant $\det_{\omega_V,\omega_W}(A)$ of A from (V, ω_V) to (W, ω_W) by the following equation

$$A^*\omega_W = \det_{\omega_V,\omega_W}(A)\omega_V \,.$$

Note that changing the sign of ω_V or ω_W only affects the sign of the determinant so we can speak of $|\det_{|\omega_V|,|\omega_W|}(A)|$ even when $|\omega_V|$ and $|\omega_W|$ are given only up to a sign.

The following lemma is an adaptation of the classical Kac-Rice formula to crossings of sections of vector bundles. It is the main tool used in the proof of Theorem 3.1.7. Before we state the lemma, we introduce the following terminology.

Definition 3.3.7. Let $\mathcal{X}$ be a smooth n-dimensional manifold and let $E \to \mathcal{X}$ be a vector bundle on $\mathcal{X}$. For each $x \in \mathcal{X}$, we denote by E_x the fiber of E at x. A Gaussian E-valued field on $\mathcal{X}$ will be a collection of random variables $(F_x)_{x\in E}$ such that for each $x_1, \dots, x_k \in \mathcal{X}$, the random variable $(F_{x_1}, \dots, F_{x_k})$ is a Gaussian vector in $E_{x_1} \times \cdots \times E_{x_k}$.

We have the following result.

Lemma 3.3.8 (Theorem 6.2 of [AW09])**.** *Let $\mathcal{X}$ be a smooth n-dimensional manifold equipped with a smooth positive density $d\mu$ and let $E \to \mathcal{X}$ be a vector bundle on $\mathcal{X}$ of rank n. We equip E with positive smooth density $d\nu$. Let F be a Gaussian E-valued field on $\mathcal{X}$ that is almost surely of class C^1 and such that for each $x \in \mathcal{X}$, F_x is a non-degenerate E_x-valued Gaussian vector. For each $x \in \mathcal{X}$, let γ_{ν,F_x} be the density of F_x with respect to $d\nu$. Then, for any connection ∇^E on E, any Borel subset $B \subset \mathcal{X}$ and any section σ of E defined in a neighborhood of B,*

$$\mathbb{E}\left[\mathrm{Card}\{x \in B \,|\, F_x = \sigma_x\}\right] = \int_B \mathbb{E}\left[\left|\det_{\mu_x,\nu_x}\left(\left(\nabla^E F\right)_x - \left(\nabla^E \sigma\right)_x\right)\right| \,\middle|\, F_x = \sigma_x\right] \gamma_{\nu,F_x}(\sigma_x) d\mu(x)\,.$$

Here, both quantities may be infinite.

Note that $\left(\nabla^E F\right)_x - \left(\nabla^E \sigma\right)_x$ does not depend on ∇^E at the points x where $F_x = \sigma_x$.

Proof of Lemma 3.3.8. Since both sides of the equality are additive in B, we may restrict ourselves to the case where $E \to \mathcal{X}$ is the trivial bundle $U \times \mathbb{R}^n \to U$ over some open subset $U \subset \mathbb{R}^n$. Moreover, by considering $\tilde{F} = F - \sigma$, it is enough to treat the case where $\sigma = 0$. In this case, F is just an $\mathbb{R}^n$ valued Gaussian field on U and the Rice formula (see Theorem 6.2 of [AW09]) applies. Therefore,

$$\mathbb{E}\left[\mathrm{Card}\{x \in B \,|\, F_x = 0\}\right] = \int_B \mathbb{E}\left[\left|\det\left(d_x F\right)\right| \,\middle|\, F_x = 0\right] \gamma_{F_x}(0) dx\,. \tag{3.3.1}$$

Here, first, γ_{F_x} is the density of the measure of $F(x)$ with respect to the Lebesgue measure dx. Second, we endowed $T^*_x U$ with the Lebesgue density and det is the usual determinant, i.e., $|\det| = |\det_{dx,dv}|$. Let $g, h \in C^\infty(U)$ be such that

$$d\mu(x) = g(x)dx;\; d\nu_x(v) = h(x)dv\,.$$

Then, for any $L \in \mathrm{Hom}(T_x U, \mathbb{R}^n)$, $|\det(L)|g(x) = |\det_{\mu_x,\nu_x}(L)|h(x)$ and $\gamma_{F_x} = \gamma_{\nu,F_x} h(x)$. Applying these identities to the right hand side of equation (3.3.1), we get

$$\mathbb{E}\left[\mathrm{Card}\{x \in B \,|\, F_x = 0\}\right] = \int_B \mathbb{E}\left[\left|\det_{\nu_x,\mu_x}\left(d_x F\right)\right| \,\middle|\, F_x = 0\right] \gamma_{\nu_x,F_x}(0) d\mu(x)\,. \tag{3.3.2}$$

Finally, let ∇ be a connection on E. Then, there is exists smooth family $(A_x)_{x\in U}$ of linear maps $A_x : \mathbb{R}^n \to \mathrm{Hom}(T_x U, \mathbb{R}^n)$ such that for any function $f : U \to \mathbb{R}^n$ and any $x \in U$,

$$\nabla f_x = d_x f + (A_x \circ f)(x)\,.$$

In particular,

$$\mathbb{E}\left[\left|\det_{\nu_x,\mu_x}(d_xF)\right| \mid F_x = 0\right] = \mathbb{E}\left[\left|\det_{\mu_x,\nu_x}((\nabla F)_x)\right| \mid F_x = 0\right]$$

and in view of equation (3.3.2) we are done. □

In Subsection 3.3.2 we will use the following lemma to prove basic facts about the regularity of Z_f and functions defined on it.

Lemma 3.3.9 (Lemma 11.2.10 [AT07])**.** *Let $\mathcal{X}$ be a smooth manifold of dimension n equipped with a rank $n+1$ vector bundle $E \to \mathcal{X}$. Let F be a Gaussian field on $\mathcal{X}$ with values in E. Assume that F is almost surely C^1 on $\mathcal{X}$ and that for every $x \in \mathcal{X}$, the random vector $F(x)$ is non-degenerate. Then, almost surely, for all $x \in \mathcal{X}$, $F(x) \neq 0$.*

Proof. Firstly, since $\mathcal{X}$ is paracompact we may assume that $E \to \mathcal{X}$ is the trivial bundle over some open subset $U \subset \mathbb{R}^n$ so that F is an $\mathbb{R}^{n+1}$ valued Gaussian field on U. Since U is covered by a countable union of closed balls, it suffices to check the conclusion of the lemma for T a compact subset of U of Hausdorff dimension n. The fact that F is almost surely C^1 implies that its covariance is continuous. Since for each $x \in T$, $F(x) = (F_1(x), \cdots, F_n(x))$ is non-degenerate and T is compact, its covariance is bounded from below on T (as a quadratic form) so its probability density is bounded on T. Finally, since T is compact and the partial derivatives of F are almost surely continuous, they are almost surely bounded on T. Hence we can apply Lemma 11.2.10 of [AT07] with $u = 0 \in \mathbb{R}^{n+1}$. □

3.3.2 Almost-sure properties of Z_f

In this subsection, we use Lemma 3.3.9 to prove that Z_f is almost surely smooth and that if we restrict an adequate deterministic function to Z_f then it is almost surely Morse. We begin by treating smoothness.

Lemma 3.3.10. *Assume that f satisfies Condition 3.3.1. Then, Z_f is almost surely a C^2 hypersurface of $\mathcal{X}$.*

Proof. The random field $F = (f, df)$ is a Gaussian random field on $\mathcal{X}$ with values in $\mathbb{R} \times T^*\mathcal{X}$. Moreover, it is almost surely C^1 since f is almost surely C^2. In addition, since f satisfies Condition 3.3.1, the components of F have positive variances. Therefore, by Lemma 3.3.9, almost surely, there is no $x \in \mathcal{X}$ such that $F(x) = 0$. Therefore, f is almost surely regular on Z_f. Since f is of class C^2, Z_f is almost surely a smooth hypersurface of $\mathcal{X}$ of class C^2. □

We now briefly recall the definition of the Hessian of a function at its critical point.

Definition 3.3.11. Let $\mathcal{X}$ be a smooth manifold. We equip $T^*\mathcal{X}$ with a connection ∇. Let $f \in C^2(\mathcal{X})$ and $x \in \mathcal{X}$ be such that $d_xf = 0$. Then, $(\nabla df) : T^*_x\mathcal{X} \times T^*_x\mathcal{X} \to \mathbb{R}$ defines a symmetric bilinear form that does not depend on the choice of connection ∇. We call

this map the Hessian of f at x and denote it by $\mathrm{Hess}(f)(x)$. In particular, for any local chart $\psi : U \subset \mathcal{X} \to \psi(U) \subset \mathbb{R}^n$, any $x \in U$ such that $d_x f = 0$ and any $v, w \in T_x\mathcal{X}$,

$$d^2_{\psi(x)}(f \circ \psi^{-1})(v, w) = \mathrm{Hess}(f)(x)((d_x\psi)^{-1}v, (d_x\psi)^{-1}w)) \,. \tag{3.3.3}$$

The following lemma will be useful first to prove the next almost sure result about Z_f and to characterize the signature of the Hessian of a function restricted to Z_f in terms of f near this point.

Lemma 3.3.12 (Lemma A.3 [GW16a])**.** *Let $\mathcal{X}$ be an n-dimensional smooth manifold. Let $f, p \in C^2(\mathcal{X})$ and fix $x_0 \in \mathcal{X}$. Assume that $d_{x_0}f, d_{x_0}p \neq 0$ and that $d_{x_0}p = \lambda d_{x_0}f$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then $L_f = f^{-1}(f(x_0))$ and $L_p = p^{-1}(p(x_0))$ are both smooth in a neighborhood of x_0 and*

$$\mathrm{Hess}(p|_{L_f})(x_0) = -\lambda \mathrm{Hess}(f|_{L_p})(x_0) \,.$$

Here, note that the condition $d_{x_0}p = \lambda d_{x_0}f$ implies that both $f|_{L_p}$ and $p|_{L_f}$ are singular at x_0 so their Hessian at x_0 are well defined bilinear forms on $T_{x_0}L_p$ and $T_{x_0}L_f$ respectively. But these two tangent spaces are naturally isomorphic to the same subspace of $T_{x_0}\mathbb{R}^n$ so it makes sense to compare the two Hessians. Gayet and Welschinger state and show this lemma in a coordinate free language. Here we provide a proof in local coordinates.

Proof of Lemma 3.3.12. Without loss of generality, we may replace $(\mathcal{X}, x_0)$ by $(U, 0)$ where U is an open neighborhood of 0 in $\mathbb{R}^n$. We may also assume $d_0 f = d_0 p = dx_n$, $f(0) = p(0) = 0$. Since $d_0 f = dx_n$, the following map is a local diffeomorphism at 0:

$$x = (x_1, \ldots, x_n) \mapsto (x_1, \cdots, x_{n-1}, f(x)) \,.$$

Moreover, its inverse F satisfies $d_0 F = Id_n$. The map F is a local diffeomorphism at 0, say from $0 \in W \subset \mathbb{R}^n$ to $x_0 \in V \subset U$. Let $h \in C^2(U)$ be such that $d_0 h = dx_n$. Then,

$$\begin{aligned} d^2_0(h \circ F) &= d^2_0 h \circ (d_0 F)^{\otimes 2} + d_0 h \circ \left(d^2_0 F\right) \\ &= d^2_0 h + d^2_0 F_n \,. \end{aligned}$$

Take first $h = f$. Then, the left hand side vanishes since $f \circ F(x) = x_n$ and

$$0 = d^2_0 f + d^2_0 F_n \,.$$

Next, take $h = p$. Then,

$$d^2_0(p \circ F) = d^2_0 p + d^2_0 F_n \,.$$

Hence,

$$d^2_0(p \circ F) = d^2_0 p - d^2_0 f \,. \tag{3.3.4}$$

By symmetry of the initial assumptions, if P is the local inverse at x_0 of the map

$$x = (x_1, \dots, x_n) \mapsto (x_1, \cdots, x_{n-1}, p(x)) .$$

Then,

$$d_0^2(f \circ P) = d_0^2 f - d_0^2 p . \tag{3.3.5}$$

Therefore, if $H = T_0 L_f = T_0 L_p$, we have

$$\mathrm{Hess}(p|_{L_f})(0) = d_0^2(p \circ F)|_H = -d_0^2(f \circ P)|_H = -\mathrm{Hess}(f|_{L_p})(0)$$

where the middle equality follows from (3.3.4) and (3.3.5) while the two others follow from equation (3.3.3). □

Finally, we prove the following result.

Lemma 3.3.13. *Assume that f satisfies Condition 3.3.2. Let $p \in C^\infty(\mathcal{X})$ with an at most countable set of critical points. Let $\mathcal{X}' \subset \mathcal{X}$ be the regular set of p. Then, almost surely, $Z_f \subset \mathcal{X}'$ and $p|_{Z_f}$ is a Morse function.*

Proof. Firstly, for any critical point x of p, by Condition 3.3.2, $f(x)$ is non-degenerate and therefore almost surely non-zero. Since p has at most countably many critical points, almost surely, Z_f stays in $\mathcal{X}'$. By Lemma 3.3.12, for $p|_{Z_f \cap \mathcal{X}'}$, not to be Morse, there must be $(x, v) \in T\mathcal{X}$ such that $f(x) = 0$, $d_x f$ vanishes on the kernel of $d_x p$, $d_x p(v) = 0$ and $\mathrm{Hess}(f|_{p^{-1}(p(x))})(x)(v, \cdot) = 0$. Let us prove that this is almost surely never the case. On the manifold $\mathcal{X}'$, the kernel of dp defines a smooth rank $n-1$ vector bundle $\mathcal{K}$ on $\mathcal{X}$. Let $S(\mathcal{K})$ be the unit sphere bundle of $\mathcal{K}$ for some auxiliary metric on $\mathcal{K}$. Let ∇^p be an auxiliary connection on $T^* p^{-1}(p(x))$. For each $x \in \mathcal{X}'$ and $v \in S_x(\mathcal{K})$, let $F(x, v) = (f(x), d_x f|_{\mathcal{K}_x}, (\nabla^p(df|_{\mathcal{K}}))(x)(v, \cdot))$. Let π denote the projection map $S(\mathcal{K}) \to \mathcal{X}'$. Then, $(F(x, v))_{(x,v) \in S(\mathcal{K})}$ defines a $\pi^* (\mathbb{R} \otimes \mathcal{K}^* \otimes \mathcal{K}^*)$ valued Gaussian field on $S(\mathcal{K})$. But $S(\mathcal{K})$ is a smooth manifold of dimension $2n-2$ while the image vector bundle has dimension $2n-1$. By Condition 3.3.2, Lemma 3.3.9 applies so almost surely, the field F does not vanish and $p|_{Z_f \cap \mathcal{X}'}$ is a Morse function. □

3.4 The upper bound in the critical case

In this subsection we apply the results of Section 3.3 to prove Theorem 3.1.7. First, in Subsection 3.4.1 we use Theorem 3.1.6 to compute the asymptotic covariance of the two-jet of f_L at a given point. Then, in Subsection 3.4.2 we prove Theorem 3.1.7 using this computation on the integral formula provided by Lemma 3.3.8. In this subsection, we consider a closed riemmanian manifold $(\mathcal{X}, g)$ of positive dimension n and consider the family of fields $(f_L^s)_{L \geq 1}$ defined in Section 3.1 for $s = n/2$. Since we have fixed s, we write f_L instead of f_L^s and K_L instead of K_L^s for the rest of the subsection. Recall that $|dV_g|$ is the Riemmanian density induced by g on $\mathcal{X}$.

3.4.1 Covariance computations

The object of this subsection is to prove the following Lemma.

Lemma 3.4.1. *Let $p \in C^\infty(\mathcal{X})$ be a Morse function on $\mathcal{X}$. Let $\mathcal{K}$ be the vector bundle on $\mathcal{X}' = \{x \in \mathcal{X},\, d_x p \neq 0\}$ whose fiber above x is $Ker(d_x p) \subset T_x\mathcal{X}$. Fix $x_0 \in \mathcal{X}$. There exist ∇^p a connection on $\mathcal{K}$ and local coordinates $x = (x_1, \dots, x_n)$ defined on $0 \in U \subset \mathbb{R}^n$ centered at x such that the density $|dV_g|$ agrees with the Lebesgue measure in these coordinates and the following holds. Let U' be the regular set of p in these coordinates. For each $x \in U'$ and $L \geq 1$, define the centered Gaussian vector $(X^L(x), Y^L(x), Z^L(x))$ with values in $\mathbb{R} \times \mathbb{R}^{n-1} \times \mathrm{Sym}_{n-1}(\mathbb{R})$ as follows. Let $X^L(x) = \frac{1}{\sqrt{\ln(L^{1/2})}} f_L(x)$. Next, let $Y^L(x) = (Y_1^L(x), \cdots, Y_{n-1}^L(x))$ be $L^{-1/2} d_x f_L$ seen as a n-uple in the local coordinates. Finally, let $Z^L(x)$ be $L^{-1}\nabla^p(df_L|_{\mathcal{K}})(x)|_{\mathcal{K}_x}$ seen as a symmetric $(n-1)$-matrix in the local coordinates Then, for any $i, j, k, l \in \{1, \dots, n-1\}$ such that $i \neq j$ and $(i, j) \neq (k, l)$, uniformly for $x \in U'$, the covariance matrix of $(X^L(x), Y^L(x), Z^L(x))$ converges as $L \to +\infty$ to the following matrix.*

$$c_n \times \begin{pmatrix} n & 0 & 0 \\ 0 & \frac{1}{2} Id_n & 0 \\ 0 & 0 & \frac{1}{4(n+2)} \Xi \end{pmatrix}$$

where c_n is the (positive) constant defined in Lemma 3.4.2 and for any $i, j, k, l \in \{1, \dots, n-1\}$ such that $i \neq j$ and $(i, j) \neq (k, l)$, $\Xi_{ii,jj} = \Xi_{ij,ij} = 1$, $\Xi_{ii,ii} = 3$ and $\Xi_{ij,kl} = 0$. In particular, for L large enough, the field f_L satisfies Condition 3.3.2 on U'.

Proof of Lemma 3.4.1. We start with ∇_1^p a connection on $\mathcal{K}$. Fix $x_0 \in \mathcal{X}$ and consider a local coordinate patch $\tilde{U}$ at x_0 given by Theorem 3.1.6 that is also isometric at x_0. We have, in this set of local coordinates, $\nabla_1^p(df_L|_{\mathcal{K}})(x)|_{\mathcal{K}_x} = d_x^2 f_L|_{\mathcal{K}_x} + (A_x(d_x f_L|_{\mathcal{K}_x}))|_{\mathcal{K}_x}$ where $A \in \Gamma(U'; T^*U \otimes \mathcal{K}^* \otimes \mathcal{K})$. Let $\chi \in C_c^\infty(\tilde{U})$ be equal to one in a neighborhood U of 0 and let $\nabla^p = \nabla_1^p - \chi A$. Then, ∇^p defines a connection on $\mathcal{K}$. Let $i, j, k, l \in \{1, \dots, n-1\}$ such that $i \neq j$ and $(i, j) \neq (k, l)$. Moreover, with this choice of ∇^p, we have, for each $x \in U'$, $Z_L(x) = L^{-1} d_x^2 f_L|_{K_x}$. Therefore, the joint covariance of $(X^L(x), Y^L(x), Z^L(x))$ is equal to

$$\begin{pmatrix} \frac{1}{\ln(L^{1/2})} K_L(x,x) & \frac{1}{L^{1/2}\sqrt{\ln(L^{1/2})}} d_y K_L(x,x) & \frac{1}{L\sqrt{\ln(L^{1/2})}} d_y^2 K_L(x,x) \\ \frac{1}{L^{1/2}\sqrt{\ln(L^{1/2})}} d_y K_L(x,x) & \frac{1}{L} d_x d_y K_L(x,x) & \frac{1}{L^{3/2}} d_x d_y^2 K_L(x,x) \\ \frac{1}{L\sqrt{\ln(L^{1/2})}} d_x^2 K_L(x,x) & \frac{1}{L^{3/2}} d_x^2 d_y K_L(x,x) & \frac{1}{L^2} d_x^2 d_y^2 K_L(x,x) \end{pmatrix}$$

for any $x \in U$. By applying the estimates of the second and third of Theorem 3.1.6 and since, by parity, the integrals $\int_{|\xi|^2 \leq 1} |\xi|^{-n} (i\xi)^\alpha (-i\xi)^\beta d\xi$ vanish whenever $\alpha + \beta \in \mathbb{N}^n$ has at least one odd component, we get, for any $i, j, k, l \in \{1, \dots, n\}$ such that $i \neq j$ and

$(i,j) \neq (k,l)$, uniformly for $x \in U'$:

$$\begin{aligned}
\lim_{L\to+\infty} \mathbb{E}\left[X^L(x)^2\right] &= \frac{|S^{n-1}|}{(2\pi)^n} \\
\lim_{L\to+\infty} \mathbb{E}\left[X^L(x)Y_i^L(x)\right] &= 0 \\
\lim_{L\to+\infty} \mathbb{E}\left[Y_i^L(x)Z_{kl}^L(x)\right] &= 0 \\
\lim_{L\to+\infty} \mathbb{E}\left[X^L(x)Z_{kl}^L(x)\right] &= 0 \\
\lim_{L\to+\infty} \mathbb{E}\left[Y_i^L(x)^2\right] &= \frac{1}{(2\pi)^n}\int_{|\xi|^2\leq 1} \frac{\xi_i^2}{|\xi|^n} d\xi \\
\lim_{L\to+\infty} \mathbb{E}\left[Y_i^L(x)Y_j^L(x)\right] &= 0 \\
\text{if } k,l \leq n-1,\ \lim_{L\to+\infty} \mathbb{E}\left[Z_{kk}^L(x)Z_{ll}^L(x)\right] = \lim_{L\to+\infty} \mathbb{E}\left[Z_{kl}^L(x)^2\right] &= \frac{1}{(2\pi)^n}\int_{|\xi|^2\leq 1} \frac{\xi_k^2\xi_l^2}{|\xi|^n} d\xi \\
\text{if } i,j,k,l \leq n-1,\ \lim_{L\to+\infty} \mathbb{E}\left[Z_{ij}^L(x)Z_{kl}^L(x)\right] &= 0\,.
\end{aligned}$$

The first statement then follows by the computations carried out in Lemma 3.4.2. To check Condition 3.3.2 we check that the limit law is non-degenerate. Aside from the diagonal coefficients of the symmetric matrix component, all the components are independent with positive variance. As for the diagonal components of the symmetric matrix, their covariance is a positive multiple of $2Id_{n-1}+J_{n-1}$ where J_{n-1} is the $(n-1)\times(n-1)$ matrix whose coefficients are all equal to 1. But J_{n-1} is the covariance of the constant Gaussian vector with unit variance so it is non-negative. Therefore, $2Id_{n-1}+J_{n-1}$ is positive definite. □

The following lemma contains the integral calculations needed for the proof of Lemma 3.4.1 above.

Lemma 3.4.2. *Fix $n \geq 2$. We define the universal constant c_n as follows.*

$$c_n = \frac{2^{(n+1)/2}((n+1)/2)!}{(n+1)!}\sqrt{\frac{2}{\pi}} \text{ if } n \text{ is odd and } \frac{1}{2^{n/2}(n/2)!} \text{ if } n \text{ is even.}$$

Let B^n denote the euclidean unit ball in $\mathbb{R}^n$. Then, for any $i,j \in \{1,\ldots,n\}$ distinct,

$$\begin{aligned}
\frac{|S^{n-1}|}{(2\pi)^n} &= nc_n \\
\frac{1}{(2\pi)^n}\int_{B^n} \frac{\xi_i^2}{|\xi|^n} d\xi &= \frac{1}{2}c_n \\
\frac{1}{(2\pi)^n}\int_{B^n} \frac{\xi_i^2\xi_j^2}{|\xi|^n} d\xi &= \frac{1}{4(n+2)}c_n \\
\frac{1}{(2\pi)^n}\int_{B^n} \frac{\xi_i^4}{|\xi|^n} d\xi &= \frac{3}{4(n+2)}c_n\,.
\end{aligned}$$

Proof of Lemma 3.4.2. First, by a polar change of coordinates we get

$$\int_{B^n} \frac{\xi_i^2}{|\xi|^n} d\xi = \frac{1}{2} \int_{S^{n-1}} \omega_i^2 d\omega$$
$$\int_{B^n} \frac{\xi_i^2 \xi_j^2}{|\xi|^n} d\xi = \frac{1}{4} \int_{S^{n-1}} \omega_i^2 \omega_j^2 d\omega$$
$$\int_{B^n} \frac{\xi_i^4}{|\xi|^n} d\xi = \frac{1}{4} \int_{S^{n-1}} \omega_i^4 d\omega \,.$$

Here $d\omega$ is the surface area for the unit sphere in $\mathbb{R}^n$. Moreover $|S^{n-1}| = \int_{S^{n-1}} d\omega$. To compute the integrals over the sphere, we compare them to moments of Gaussian random variables. Let X be a centered Gaussian vector in $\mathbb{R}^n$ with covariance Id_n. Another polar change of coordinates yields, for $1 \leq i < j \leq n$,

$$1 = \mathbb{E}[1] = \frac{1}{(2\pi)^{n/2}} \int_0^{+\infty} t^{n-1} e^{-t^2/2} dt \int_{S^{n-1}} d\omega$$
$$1 = \mathbb{E}[X_i^2] = \frac{1}{(2\pi)^{n/2}} \int_0^{+\infty} t^{n+1} e^{-t^2/2} dt \int_{S^{n-1}} \omega_i^2 d\omega$$
$$1 = \mathbb{E}[X_i^2 X_j^2] = \frac{1}{(2\pi)^{n/2}} \int_0^{+\infty} t^{n+3} e^{-t^2/2} dt \int_{S^{n-1}} \omega_i^2 \omega_j^2 d\omega$$
$$3 = \mathbb{E}[X_i^4] = \frac{1}{(2\pi)^{n/2}} \int_0^{+\infty} t^{n+3} e^{-t^2/2} dt \int_{S^{n-1}} \omega_i^4 d\omega \,.$$

Now, for each $k \in \mathbb{N}$, let $J_k = \int_0^{+\infty} t^k e^{-t^2/2} dt$. By integration by parts, for each $k \in \mathbb{N}$, we have $J_{k+2} = (k+1) J_k$. From this we deduce the following:

$$J_k = \frac{1}{2^{k/2}} \frac{k!}{(k/2)!} \sqrt{\frac{\pi}{2}} \text{ if } k \text{ is even and } 2^{(k-1)/2}((k-1)/2)! \text{ if } k \text{ is odd.}$$

With this notation,

$$\int_{S^{n-1}} d\omega = \frac{(2\pi)^{n/2}}{J_{n-1}} = \frac{n(2\pi)^{n/2}}{J_{n+1}}$$
$$\int_{S^{n-1}} \omega_i^2 d\omega = \frac{(2\pi)^{n/2}}{J_{n+1}}$$
$$\int_{S^{n-1}} \omega_i^2 \omega_j^2 d\omega = \frac{(2\pi)^{n/2}}{J_{n+3}} = \frac{(2\pi)^{n/2}}{(n+2) J_{n+1}}$$
$$\int_{S^{n-1}} \omega_i^4 d\omega = \frac{3(2\pi)^{n/2}}{J_{n+3}} = \frac{3(2\pi)^{n/2}}{(n+2) J_{n+1}} \,.$$

Replacing these expressions in the original integrals yields the desired result (with $c_n = 1/J_{n+1}$). □

3.4.2 Proof of Theorem 3.1.7

In this subsection we prove Theorem 3.1.7. The proof relies on Lemmas 3.3.8 and 3.4.1. Let $p \in C^\infty(\mathcal{X})$ with an at most countable number of critical points. Let $\mathcal{X}' \subset \mathcal{X}$ be its regular set and let $\mathcal{K}$ be the sub-bundle of $T\mathcal{X}'$ defined by the kernel of dp. For each $i \in \{0, \cdots, n-1\}$, $L \geq 1$ and $B \subset \mathcal{X}$ Borel subset, let $\nu_i(p, f_L, B)$ be the number of critical points of index i of $p|_{Z_f}$ inside B. In addition to previous results will need the following elementary lemma.

Lemma 3.4.3. *Let $(X_t^L)_{t\in T, L\geq 1} = (X_{1,t}^L, X_{2,t}^L)_{t\in T, L\geq 1}$ be a family of centered Gaussian vectors in $\mathbb{R}^n \times \mathbb{R}^m$ with covariances $\Sigma_{t,L}$. Here T is any index set. Assume that, uniformly for $t \in T$, the sequence $(\Sigma_{t,L})_{L\geq 1}$ converges to some covariance matrix Σ corresponding to the Gaussian vector $X = (X_1, X_2)$ such that the vector X^2 is non-degenerate. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a measurable function such that for each $\varepsilon > 0$ there exist $c = c(\varepsilon) < +\infty$ for which $\forall x \in \mathbb{R}^n \times \mathbb{R}^m$, $|f(x)| \leq ce^{\varepsilon|x|^2}$. Then, uniformly for $t \in T$,*

$$\lim_{L\to+\infty} \mathbb{E}\left[f(X_t^L) \,|\, X_{2,t}^L = 0\right] = \mathbb{E}\left[f(X) \,|\, X_2 = 0\right] .$$

Proof of Lemma 3.4.3. Apply the regression formula (see Proposition 1.2 of [AW09]) to (X_t^L) and use the dominated convergence theorem on f times the conditional density with respect to the Lebesgue measure on $\mathbb{R}^n$. The fact that $X_{2,t}$ is uniformly non-degenerate guarantees that for large enough values of the vectors $X_{2,t}^L$ are all non-degenerate and that the conditional inverse covariances of X_t^L are uniformly bounded from below by a positive multiple of Id_n. This and the sub-exponential bound on f guarantees the uniform integrability needed for dominated convergence. □

We are ready to prove Theorem 3.1.7.

Proof of Theorem 3.1.7. By Lemma 3.4.1 and compactness of $\mathcal{X}$, for L large enough, f_L satisfies Condition 3.3.2 so the smoothness of Z_L and the fact that $p|_{Z_L}$ is almost surely a Morse function follows from Lemmas 3.3.10 and 3.3.13 respectively. By Lemma 3.3.13, almost surely $Z_f \subset \mathcal{X}'$ so it is enough to treat the case where $B \subset \mathcal{X}'$. Secondly, the quantities on both sides of equation (3.1.2) are (at least finitely) additive in B so it is enough to prove the result for B inside any local chart of some atlas. Fix $i \in \{0, \cdots, n-1\}$. Let $x_0 \in \mathcal{X}$ and consider ∇^p a connection on $\mathcal{K}$ and $x = (x_1, \cdots, x_n)$ the local coordinates centered at x_0 provided by Lemma 3.4.1, defined on $0 \in U \subset \mathbb{R}^n$. Let U' be the regular set of p in these coordinates. Let $|dx|$ be the Lebesgue measure on U. Let $|dt|$ be the Lebesgue density on the trivial bundle $\mathbb{R} \times U$ on U. On U', the euclidean scalar product restricts to the fibers of $\mathcal{K}$ (resp. $\mathcal{K}^\perp$, $\mathcal{K}^*$) and defines a density $|d\tilde{x}|$ (resp. $\left|dx^\perp\right|$, $|dx^*|$). Let $\nu = |dt| \otimes |dx^*|$. Given $x \in U$, conditionally on the event that $f(x) = 0$ and $d_x f|_{\mathcal{K}} = 0$, let $S_i^L(x)$ be the event that $\mathrm{Hess}(p|_{Z_L})$ has signature i. Fix $B \subset U'$. Then, by Lemma 3.3.8 (which applies for L large enough by Lemma 3.4.1), the quantity $\mathbb{E}\left[m_i(p, f_L, B)\right]$ is the integral over B and against $|dx|$ of the following density:

$$\mathbb{E}\left[\left|\det_{|dx|,\nu}\left(\left(d_x f_L, \nabla^p df_L|_{\mathcal{K}_x}\right)(x)\right)\right| \mathbb{1}\left[S_i^L(x)\right] \;\middle|\; f_L(x)=0,\, d_x f_L|_{\mathcal{K}}=0\right] \gamma_{\nu,(f_L(x),d_x f_L|_{\mathcal{K}_x})}(0)\,. \tag{3.4.1}$$

Note that $|dx| = \left|dx^{\perp}\right| \otimes |d\tilde{x}|$ so conditionally on $d_x f_L|_{\mathcal{K}_x} = 0$,

$$\left|\det_{|dx|,\nu}\left(\left(d_x f_L, \nabla^p df_L|_{\mathcal{K}_x}\right)(x)\right)\right| = \|d_x f_L\|_{eucl}\left|\det_{|d\tilde{x}|,|dx^*|}(\nabla^p(df_L|_{\mathcal{K}})(x)|_{\mathcal{K}_x})\right|\,.$$

For any $x \in U'$ and $L \geq 1$, let $(X^L(x), Y^L(x), Z^L(x))$ be as in Lemma 3.4.1. Let $\tilde{Y}^L(x)$ be the coordinates of $Y^L(x)|_{\mathcal{K}_x}$ in some orthonormal basis of $\mathcal{K}_x^*$ and let $\Sigma^L(x)$ be the covariance of $(X^L(x), \tilde{Y}^L(x))$. Then,

$$\|d_x f_L\|_{eucl}\left|\det_{|d\tilde{x}|,|dx^*|}(\nabla^p(df_L|_{\mathcal{K}})(x)|_{\mathcal{K}_x})\right| = L^{n-1/2}\|Y^L(x)\|_{eucl}\left|\det\left(Z^L(x)\right)\right|$$

and

$$\gamma_{\nu,(f_L(x),d_x f|_{\mathcal{K}_x})}(0) = \frac{1}{(2\pi)^{n/2}L^{(n-1)/2}\sqrt{\ln\left(L^{1/2}\right)}\sqrt{\det(\Sigma^L(x))}}\,.$$

Therefore, by equation (3.4.1), $\mathbb{E}\left[m_i(p, f_L, B)\right]$ is the integral over B and against $|dx|$ of the following density:

$$\frac{\mathbb{E}\left[\|Y^L(x)\|_{eucl}\left|\det\left(Z^L(x)\right)\right| \mathbb{1}\left[S_i^L(x)\right] \;\middle|\; X^L(x)=0,\, Y^L(x)|_{\mathcal{K}_x}=0\right]}{(2\pi)^{n/2}\sqrt{\det(\Sigma^L(x))}}\frac{L^{n/2}}{\sqrt{\ln\left(L^{1/2}\right)}}\,. \tag{3.4.2}$$

By Lemma 3.4.1, uniformly for $x \in U'$,

$$\lim_{L\to+\infty}\det\left(\Sigma^L(x)\right) = n2^{1-n}c_n^n\,.$$

To deal with the expectation, note first that by Lemma 3.3.12, the event $S_i^L(x)$ is exactly the event that, either $Y^L(x)$ is a positive multiple of $d_x p$ and the signature of $Z^L(x)$ is $n-1-i$, or it is a negative multiple and the signature is i. Let $(X^\infty, Y^\infty, Z^\infty)$ be the centered Gaussian vector with values in $\mathbb{R}\times\mathbb{R}^n\times\mathrm{Sym}_{n-1}(\mathbb{R})$ with the following covariance structure. The three components X^∞, Y^∞ and Z^∞ are independent. X^∞ has variance nc_n, Y^∞ has covariance $(c_n/2)I_n$ and the covariance of Z^∞ is $\frac{c_n}{4(n+2)}\Xi$ where c_n is some positive constant and for any $i,j,k,l \in \{1,\dots,n-1\}$ such that $i \neq j$ and $(i,j) \neq (k,l)$, $\Xi_{ii,jj} = \Xi_{ij,ij} = 1$, $\Xi_{ii,ii} = 3$ and $\Xi_{ij,kl} = 0$. By Lemmas 3.4.1 and 3.4.3, we have, uniformly for $x \in U'$,

$$\mathbb{E}\left[\|Y^L(x)\|_{eucl}\left|\det\left(Z^L(x)\right)\right|\mathbb{1}\left[S_i^L(x)\right]\;\middle|\;X^L(x)=0,\,Y^L(x)|_{\mathcal{K}_x}=0\right]$$

$$\xrightarrow[L\to+\infty]{}$$

$$\mathbb{E}\left[\|Y^\infty\|_{eucl}\left|\det\left(Z^\infty\right)\right|\mathbb{1}\left[S_i^\infty\right]\;\middle|\;X^\infty=0,\,Y^\infty|_{\mathcal{K}_x}=0\right]$$

where S_i^∞ is the event that either Z^∞ has signature $n-1-i$ and Y^∞ is a positive multiple of d_xp or that its signature is i and Y^∞ is a negative multiple of d_xp. Since the components of $(X^\infty, X^\infty, Z^\infty)$ are independent, the above limit equals

$$\mathbb{E}\left[\|Y^\infty\|_{eucl}\,\middle|\,Y^\infty|_{\mathcal{K}_x}=0\right]\mathbb{E}\left[\left|\det\left(Z^\infty\right)\right|\mathbb{1}[\mathrm{sgn}\left(Z^\infty\right)=i]\right]\,. \tag{3.4.3}$$

Let $M=\sqrt{4(n+2)/c_n}Z^\infty$ so that M has covariance Ξ. Then, since Y^∞ has covariance $(c_n/2)Id_n$, the quantity (3.4.3) equals

$$(c_n/\pi)^{1/2}(c_n/(4(n+2)))^{(n-1)/2}\mathbb{E}\left[\left|\det\left(M\right)\right|\mathbb{1}[\mathrm{sgn}\left(M\right)=i]\right]\,.$$

Therefore, by equations (3.4.2) and (3.4.3), as $L\to+\infty$,

$$\mathbb{E}\left[m_i(p,f_L,B)\right]\sim C_n\mathrm{Vol}_{eucl}(B)\mathbb{E}\left[\left|\det\left(M\right)\right|\mathbb{1}[\mathrm{sgn}\left(M\right)=i]\right]\frac{L^{n/2}}{\sqrt{\ln\left(L^{1/2}\right)}}$$

where

$$C_n=\frac{1}{\sqrt{\pi^{n+1}2^{2n-1}n(n+2)^{n-1}}}\,.$$

To conclude note that in the coordinates given by Lemma 3.4.1, the density $|dV_g|$ corresponds to the Lebesgue density so $\mathrm{Vol}_{eucl}(B)=\mathrm{Vol}_g(B)$. □

3.5 The lower bound in the critical case

The object of this section is to prove Theorem 3.1.8. The proofs of this section do not rely on any result from the rest of the article. First, in Subsection 3.5.1 we prove two elementary inequalities. Then, we use these to prove Theorem 3.1.8 in Subsection 3.5.2.

3.5.1 Two useful Gaussian inequalities

In this subection, we state two inequalities that follow easily from known results. The first is an upper bound for the concentration of the maximum and combines the Fernique inequality with the Borell-TIS inequality.

Lemma 3.5.1. *Let g be a centered Gaussian field on a bounded subset V of $\mathbb{R}^n$. Assume that there exist $0<\sigma,D<+\infty$ and $\alpha\in]0,2]$ such that for all $x,y\in V$*

$$\mathbb{E}\left[g(x)^2\right]\le\sigma^2$$
$$\mathbb{E}\left[(g(x)-g(y))^2\right]\le D^2|x-y|^\alpha\,.$$

Then, g is almost surely bounded, its supremum M has finite expectation and there exists $C = C(V, \alpha) < +\infty$ such that

$$\mathbb{E}[M] \leq CD\,.$$

Moreover, for each $u > 0$,

$$\mathbb{P}[M \geq CD + u] \leq 2e^{-\frac{1}{2\sigma^2}u}\,.$$

Proof. By Theorem 2.9 of [AW09], it is enough to obtain a uniform bound on the expectation of M_λ. Let $(X(x))_{x\in\overline{V}}$ be an n-dimensional fractional Brownian motion of index α on $\overline{V}$ (see for instance Definition 3.3.1 [CI13]), that is, X is a centered Gaussian field which is almost surely continuous on V and whose covariance is

$$\mathbb{E}[X(x)X(y)] = \frac{1}{2}\left[|x|^\alpha + |y|^\alpha - |x-y|^\alpha\right]\,.$$

Since X is almost surely continuous on and V is bounded, its maximum on $\overline{V}$ is almost surely finite. Since it is a Gaussian field, its maximum has finite expectation (see once more Theorem 2.9 of [AW09]). Let $m = \mathbb{E}[\max_{\overline{V}} X]$. For any $x, y \in V$,

$$\mathbb{E}\left[(DX(x) - DX(y))^2\right] = D^2|x-y|^\alpha \geq \mathbb{E}\left[(g(x) - g(y))^2\right]$$

so that, by the Sudakov-Fernique inequality (see Theorem 2.4 of [AW09]),

$$\mathbb{E}[M_\lambda] \leq \mathbb{E}[\sup_V DX] = Dm < +\infty$$

and we are done. □

The second lemma deals with a certain type of event that we now define. For any set T, we say that an event $A \subset \mathbb{R}^T$ is **increasing** if for any $x \in A$

$$\{y \in \mathbb{R}^T \,|\, \forall t \in T,\, y(t) \geq x(t)\} \subset A\,.$$

The following result is essentially due to Loren Pitt and says that Gaussian vectors with non-negative covariance satisfy the FKG inequality. Loren Pitt stated it for finite dimensional Gaussian vectors but the general case follows easily (see for instance Theorem A.4 of [RV17a]).

Lemma 3.5.2 ([Pit82b])**.** *Let $(X_t)_{t\in T}$ be an a.s. continuous Gaussian random field on a separable topological space T with covariance $\Sigma = (\sigma_{ij})_{ij}$. Assume that for each $i, j \in \{1, \cdots, n\}$, $\sigma_{ij} \geq 0$. Then, for any two increasing events $A, B \subset \mathbb{R}^T$ (measurable with respect to the product σ-algebra),*

$$\mathbb{P}[X \in A \cap B] \geq \mathbb{P}[X \in A]\mathbb{P}[X \in B]\,.$$

3.5.2 Proof of Theorem 3.1.8

In this subsection, we use the inequalities of Subsection 3.5.1 to prove Theorem 3.1.8. Throughout the proof, the constants implied by the O's will be universal constants.

Proof of Theorem 3.1.8. Take $\rho \geq 1$ a parameter to be fixed later and for each $\lambda \geq \rho$, set S_λ the sphere centered at 0 of radius ρ/λ. For each $\lambda \geq \rho$ and each $x, y \in S_\lambda$,

$$\begin{aligned}\mathbb{E}\left[f_\lambda(0)^2\right] &= \ln(\lambda) + O(a)\\ \mathbb{E}\left[f_\lambda(0) f_\lambda(x)\right] &= \ln(\lambda) - \ln(\rho) + O(a)\\ \mathbb{E}\left[f_\lambda(x) f_\lambda(y)\right] &\geq \ln(\lambda) - \ln(\rho) + O(a)\,.\end{aligned}$$

In particular, there exists $\lambda_1(\rho, a) <, +\infty$ such that for $\lambda \geq \lambda_1(a, \rho)$, $f_\lambda(0)$ is non-degenerate. By the regression formula (see Proposition 1.2 of [AW09]), the field $(f_\lambda(x))_{x \in S_\lambda}$ conditioned on $f_\lambda(0)$ is an almost surely Gaussian field with mean

$$\forall x \in S_\lambda,\ \mathbb{E}\left[f_\lambda(x) \,\middle|\, f_\lambda(0)\right] = \left[1 + O\left((\ln(\lambda))^{-1}(\ln(\rho) + a)\right)\right] f_\lambda(0) \tag{3.5.1}$$

and whose covariance at any $x, y \in S_\lambda$ is by definition

$$\mathbb{E}\left[\left(f_\lambda(x) - \mathbb{E}\left[f_\lambda(x) \,\middle|\, f_\lambda(0)\right]\right)\left(f_\lambda(y) - \mathbb{E}\left[f_\lambda(y) \,\middle|\, f_\lambda(0)\right]\right) \,\middle|\, f_\lambda(0)\right]$$

which equals

$$\ln(\lambda) - \ln(\rho) + O(a) - \frac{\left(\ln(\lambda) - \ln(\rho) + O(a)\right)^2}{\ln(\lambda) + O(a)} = \ln(\rho) + O(\ln(\rho)/\ln(\lambda)) + O(a) + O(1)\,. \tag{3.5.2}$$

In particular, there exists $\rho_0 = \rho_0(a) < +\infty$ such that if $\rho \geq \rho_0$, then, for each $\lambda \geq \rho \vee \lambda_1$, the vector $(f_\lambda(x))_{x \in S_\lambda}$ conditioned on $f_\lambda(0)$ is positively correlated. Take $\lambda \geq \lambda_0$. Let $\mathcal{H}^\lambda$ be the event that for all $x \in S_\lambda$, $f_\lambda(x) > 0$. We want to find a lower bound for the probability of $\mathcal{H}^\lambda$ conditioned on $f_\lambda(0)$, on the event that $f_\lambda(0) > -1$. To this end we start by proving the following estimate.

Claim 3.5.3. *There exists $\rho_1 = \rho_1(a, b, \alpha, n) < +\infty$ such that if $\rho \geq \rho_1$, for each $\lambda \geq \rho \vee \lambda_1$ and each $x \in S_\lambda$, on the event $f_\lambda(0) \geq -1$,*

$$\mathbb{P}\left[\forall y \in S_\lambda \cap B(x, 1/\lambda),\ f_\lambda(y) > 0 \,\middle|\, f_\lambda(0)\right] \geq 1/3\,.$$

Proof. Fix $\rho \geq \rho_0$, $\lambda \geq \rho \vee \lambda_1$ and $x \in S_\lambda$. Set $V_\lambda \subset B(0,1)$ be the set of $z \in B(0,1)$ such that $x + z/\lambda \in S_\lambda$. For each $z \in B(0,1)$,

$$\begin{aligned}m_\lambda(z) &= \mathbb{E}\left[f_\lambda(x + z/\lambda) \,|\, f_\lambda(0)\right]\\ h_\lambda(z) &= f_\lambda(x + z/\lambda) - m_\lambda(z)\\ g_\lambda(z) &= h_\lambda(0) - h_\lambda(z)\,.\end{aligned}$$

Then, we have, for each $z \in B(0,1)$,

$$f_\lambda(x + z/\lambda) = h_\lambda(0) - g_\lambda(z) + m_\lambda(z)\,. \tag{3.5.3}$$

We will now show that, conditionally on $f_\lambda(0)$ and on the event $f_\lambda(0) > -1$, with positive probability, the three terms in the right-hand side of (3.5.3) satisfy inequalities that imply that $f_\lambda(x + z\lambda) > 0$ for each $z \in V_\lambda$. Firstly, Equation (3.5.1) shows that on the event $f_\lambda(0) > -1$, $m_\lambda(z)$ is bounded from below uniformly for each λ and each $z \in V_\lambda$. Let $m = m(a, n) > -\infty$ be a uniform lower bound. Next, conditionally on $f_\lambda(0)$, g_λ is centered and for each $z, z' \in B(0, 1)$,

$$\begin{aligned}\mathbb{E}\left[g_\lambda(z)^2 \,|\, f_\lambda(0)\right] &\leq \mathbb{E}\left[g_\lambda(z)^2\right] \leq 2b^2|z|^\alpha \leq 2b^2\\ \mathbb{E}\left[(g_\lambda(z) - g_\lambda(z'))^2 \,|\, f_\lambda(0)\right] &\leq 4\mathbb{E}\left[(f_\lambda(x + z/\lambda) - f_\lambda(x + z'/\lambda))^2\right] \leq 4b^2|z - z'|^\alpha\,.\end{aligned}$$

Here we used that, by the regression formula (Proposition 1.2 of [AW09]), variances do not increase under gaussian conditioning. By Lemma 3.5.1 (with $V = B(0, 1)$) there exist a constant $C = C(n, \alpha) < +\infty$ and a constant $u_0 = u_0(b) \in]0, +\infty[$ such that for each λ,

$$\mathbb{P}\left[\sup_{B(0,1)} g_\lambda > Cb + u_0 \,\middle|\, f_\lambda(0)\right] \leq 1/9\,. \tag{3.5.4}$$

By equation (3.5.2), uniformly in $\lambda \geq \rho$,

$$\mathbb{E}\left[h_\lambda(0)^2 \,|\, f_\lambda(0)\right] = \ln(\rho) + O(a) + O(1)\,.$$

Also, conditionally on $f_\lambda(0)$, $h_\lambda(0)$ is centered. Hence, there exists $\rho_1 = \rho_1(a, m, u_0, b, C) < +\infty$ such that if $\rho \geq \rho_1$,

$$\mathbb{P}\left[h_\lambda(0) > Cb + u_0 - m \,|\, f_\lambda(0)\right] > 4/9\,. \tag{3.5.5}$$

Here m, C and u_0 depend only on a, b, α and n so ρ_1 depends only on a, b, α and n. Assume now that $f_\lambda(0) > -1$. By Equation (3.5.3), since for each $z \in V_\lambda$ we have $m_\lambda(z) \geq m$, the event

$$\{h_\lambda(0) > Cb + u_0 - m\} \cap \neg\left\{\sup_{B(0,1)} g_\lambda > Cb + u_0\right\}$$

implies that $\forall y \in S_\lambda \cap B(x, 1/\lambda)$, $f_\lambda(y) > 0$. Hence, for each $\rho \geq \rho_1$, each $\lambda \geq \rho \vee \lambda_1$ and each $x \in S_\lambda$, we have shown that on the event $f_\lambda(0) > -1$,

$$\begin{aligned}\mathbb{P}\left[\forall y \in S_\lambda \cap B(x, 1/\lambda)\, f_\lambda(y) > 0 \,\middle|\, f_\lambda(0)\right] &\geq \mathbb{P}\left[h_\lambda(0) > Cb + u_0 - m \,|\, f_\lambda(0)\right]\\ &\quad - \mathbb{P}\left[\sup_{B(0,1)} g_\lambda > Cb + u_0 \,\middle|\, f_\lambda(0)\right]\\ &\geq 4/9 - 1/9 \text{ by equations (3.5.4) and (3.5.5)}\\ &\geq 1/3\,.\end{aligned}$$

□

From now on, we assume that $\rho \geq \rho_0 \vee \rho_1$. Cover S_λ with $N = N(\rho)$ balls $(B_i)_{1 \leq i \leq N}$ of radius $1/\lambda$. Conditionally on $f_\lambda(0)$, for each $i \in \{1, \dots, N\}$, the event that the field f_λ stays positive on B_i is increasing. Therefore, by the FKG inequality (Lemma 3.5.2) we have, for each $\lambda \geq \rho \vee \lambda_1$,

$$\mathbb{P}\left[\mathcal{H}^\lambda \,\middle|\, f_\lambda(0)\right] \geq \prod_{i=1}^{N} \mathbb{P}\left[\forall x \in B_i \, f_\lambda(x) > 0 \,|\, f_\lambda(0)\right] .$$

Now, by the previous claim, on the event that $f_\lambda(0) \geq -1$, the right hand side is greater than 3^{-N}. Consequently, for each $\lambda \geq \rho \vee \lambda_1$,

$$\begin{aligned} \mathbb{P}\left[\mathcal{C}^\lambda\right] \geq \mathbb{P}\left[f_\lambda(0) < 0; \, \mathcal{H}^\lambda\right] &\geq \mathbb{E}\left[\mathbb{P}\left[\mathcal{H}^\lambda \,\middle|\, f_\lambda(0)\right] \mathbb{1}[-1 \leq f_\lambda(0) < 0]\right] \\ &\geq 3^{-N} \mathbb{P}\left[-1 \leq f_\lambda(0) < 0\right] . \end{aligned}$$

Since f_λ is centered and $\mathbb{E}[f_\lambda(0)^2] = \ln(\lambda) + O(a)$, there exist $\lambda_0 = \lambda_0(a, \rho) < +\infty$ and $\kappa = \kappa(a, \rho) > 0$ such that for each $\lambda \geq \lambda_0$,

$$\mathbb{P}\left[\mathcal{C}^\lambda\right] \geq \kappa \left(\ln(\lambda)\right)^{-1/2} .$$

Choosing $\rho = \rho_0 \vee \rho_1$, one can assume that λ_0 and κ depend only on a, b, n and α. $\square$

Chapter 4

Hole probability for nodal sets of the cut-off Gaussian free field

Abstract

Let (Σ, g) be a closed connected surface equipped with a riemannian metric. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be the increasing sequence of eigenvalues and the sequence of corresponding L^2-normalized eigenfunctions of the laplacian on Σ. For each $L > 0$, we consider $\phi_L = \sum_{0<\lambda_n\leq L} \frac{\xi_n}{\sqrt{\lambda_n}}\psi_n$ where the ξ_n are i.i.d centered gaussians with variance 1. As $L \to \infty$, ϕ_L converges a.s. to the Gaussian Free Field on Σ in the sense of distributions. We first compute the asymptotic behavior of the covariance function for this family of fields as $L \to \infty$. We then use this result to obtain the asymptotics of the probability that ϕ_L is positive on a given open proper subset with smooth boundary. In doing so, we also prove the concentration of the supremum of ϕ_L around $\frac{1}{\sqrt{2\pi}} \ln L$.

This chapter is based on [Riv17].

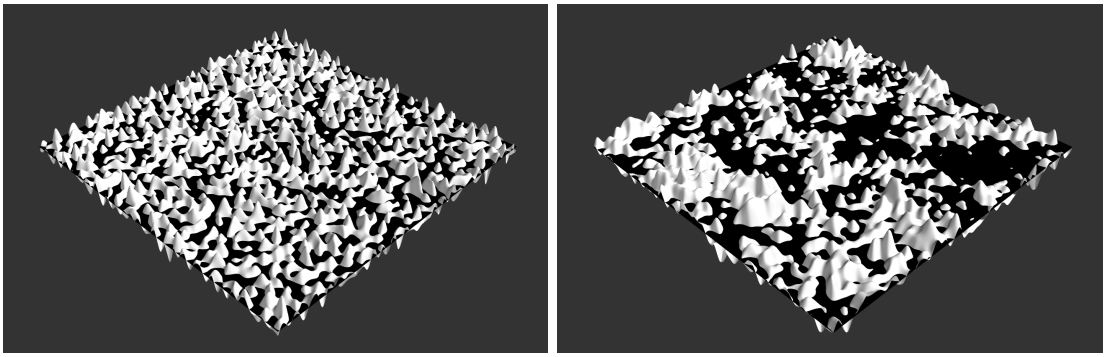
4.1 Introduction

4.1.1 Setting and main results

In recent years, there have been many developments in the study of random linear combinations of eigenfunctions of the laplacian on a closed manifold. In this paper we consider a different model with strong ties to statistical mechanics, mentioned both in [Sch07] (Problem 2.4) and [Zel13] (equation (97)). Let (Σ, g) be a smooth compact connected surface equipped with a riemannian metric. Let $\Delta = d^*d$ be the Laplace operator on Σ associated to g and $|dV_g|$ the volume density defined by g. Note that with our convention, if Σ is the flat torus with coordinates (x_1, x_2), $\Delta = -\partial_{x_1}^2 - \partial_{x_2}^2$. For each $L > 0$, let U_L be the real vector space spanned by the eigenfunctions of Δ whose eigenvalues are positive and smaller than L. Then, U_L is finite dimensional and

$$(u, v) \mapsto \int_\Sigma g(\nabla u, \nabla v)|dV_g|$$

defines a scalar product on each U_L which induces a gaussian probability distribution on U_L. For each $L > 0$, let ϕ_L be random variable chosen with this distribution. Then, each instance of ϕ_L is a smooth function on Σ. In particular, for each $L > 0$, $(\phi_L(x))_{x\in\Sigma}$ defines a gaussian field on Σ. We will see that the field ϕ_L converges $L \to \infty$ almost surely in the sense of distributions to the Gaussian Free Field on Σ, a central object in contemporary statistical mechanics. Following [Zel13], we choose to call ϕ_L **the cut-off Gaussian Free Field** on Σ or COGFF for short. While the definition of the COGFF is formally similar to that of the usual cut-off eigenfunction model (see for instance [Nic15]), it is actually quite different. Indeed, while the cut-off model exhibits a local scale with polynomial correlations, the COGFF has global logarithmic correlations. We will prove that it is actually much closer to the discrete Gaussian Free Field. For this purpose we will combine methods from statistical mechanics and random geometry, thus creating a new interface between the two subjects.



An instance of the cut-off field (resp. the COGFF) on the left (resp. right) on the flat torus with $L = 1000$. The field is colored in white and the black surface is a horizontal square at height zero.

Let $D \subset \Sigma$ be a non-empty proper open subset of Σ with smooth boundary. We ask what the probability is that the the field stays positive on D. The asymptotic behavior of this probability as $L \to \infty$ will be expressed in terms of **the capacity** of D, which we define as the infimum of the quantity $\frac{1}{2}\|\nabla h\|_2^2$ taken over all $h \in C^\infty(\Sigma)$ with zero mean, such that $\forall x \in D$, $h(x) \geq 1$ and which we denote by $cap_\Sigma(D)$ or $cap(D)$ when there is no ambiguity. More precisely, we prove the following theorem.

Theorem 4.1.1. *Let (Σ, g) be a smooth compact connected surface without boundary and let $(\phi_L)_{L>0}$ be the COGFF on (Σ, g). Let D be a non-empty proper open subset of Σ with smooth boundary. Then,*

$$\lim_{L\to\infty} \frac{\ln\left(\mathbb{P}(\forall x \in D, \phi_L(x) > 0)\right)}{\ln^2\left(\sqrt{L}\right)} = -\frac{2}{\pi} cap(D).$$

Along the way, we will also prove that $cap(D) > 0$. The assumption that Σ has no boundary follows the tradition of the study of random sums of eigenfunctions. However, **all of the results presented in this paper stay valid in the case where Σ has a boundary** with minor modifications and provided we change the definitions of the COGFF and capacity accordingly (see section 4.5 for the corresponding statements). This is especially significant for two reasons. First, from Riemann's mapping theorem, two non-empty simply connected proper subsets of $\mathbb{C}$ are conformally equivalent. Second, in this setting $cap(D)$ will be **conformally invariant** (see section 4.5 for more details). To the best of our knowledge, this is the first time a non-trivial conformal invariant emerges from the asymptotics of random sums of eigenfunctions. The event of staying positive on a given set has been studied before in the case of sections of complex line bundles on complex manifolds (see [SZZ08]) and in the case of the discrete Gaussian Free Field (or DGFF) on a box in the square lattice (see [BDG01]). In [SZZ08], Shiffman, Zrebiec and Zelditch actually prove much stronger results relying on large deviation estimates that work because the field they consider has exponential decay in correlations. As will be apparent in the statement of Theorem 4.1.3, this is not the case in our model, which, like [BDG01], has logarithmic correlations. In this article, Bolthausen, Deuschel and Giacomin prove the following result

Theorem 4.1.2. *Let $V_N = \{1, \dots, N\}^2$ be a square box in $\mathbb{Z}^2$. Let ϕ_N be the discrete Gaussian Free Field on V_N with Dirichlet boundary conditions. Let $D \subset [0,1]^2$ be an open subset with smooth boundary and at positive distance of $\partial[0,1]^2$. Let D_N be the set of points $y \in V_N$ such that $\frac{1}{N}y \in D$. Then,*

$$\lim_{N\to\infty} \frac{1}{(\ln N)^2} \ln\Big(\mathbb{P}(\forall x \in D_N, \phi_N(x) \geq 0)\Big) = -\frac{8}{\pi} cap_V(D).$$

Here, $cap_V(D)$ is the infimum of $\frac{1}{2}\|\nabla h\|_2^2$ over all the $h \in C^\infty(V)$ with compact support in $\mathring{V}$ such that $h \geq 1$ in D. In Theorem 4.1.1, the connectedness assumption simplifies the proofs and is not very restrictive since the COGFF is independent between different components. The assumption that D be an open set with smooth boundary allows

us to use classical results concerning the potential $cap(D)$ and is already present in the discrete setting. Finally, since the field we consider has zero mean, it cannot stay positive on $D = \Sigma$ so we assume $D \neq \Sigma$. We keep the square root inside the logarithm in the statement because it emerges from the proof as a more natural scale in this problem. Our approach follows the structure of [BDG01]. However, we consider fields in a continuous setting, and more importantly, unlike the DGFF, the COGFF does not seem to have a Markov property. The relation with [BDG01] as well as the strategies employed to deal with these issues will be explained below. Finally, we need to estimate the covariance function of the field. This step is central in our strategy since it is only through this object that we can manipulate the field. We prove the following theorem, which is significant in its own right.

Theorem 4.1.3. *Let (Σ, g) be a compact riemannian surface without boundary and let $(\phi_L)_{L>0}$ be the COGFF on (Σ, g). For each $L > 0$ and $p, q \in \Sigma$, let*

$$G_L(p,q) = \mathbb{E}[\phi_L(p)\phi_L(q)].$$

Then, there exists $\varepsilon > 0$ such that for each $p, q \in \Sigma$ satisfying $d_g(p,q) \leq \varepsilon$ and for each $L > 0$,

$$G_L(p,q) = \frac{1}{2\pi}\Big(\ln\left(\sqrt{L}\right) - \ln_+\left(\sqrt{L} d_g(p,q)\right)\Big) + \rho_L(p,q)$$

where $\ln_+(a) = \max(\ln(a), 0)$, d_g is the riemannian distance and $\rho_L(p,q)$ is bounded uniformly with respect to p, q and L.

The proof of this theorem relies on Hörmander's estimates for the spectral kernel of an elliptic operator in [Hör68]. The result is reminiscent of the well known analogue for the DGFF (see for instance Lemma 2.2 of [BZ12]). Aside from Theorem 4.1.3, one important step in the proof of Theorem 4.1.1 is to control the supremum of the field on a given domain. We prove the following theorem.

Theorem 4.1.4. *Let (Σ, g) be a smooth compact riemannian surface without boundary and let $(\phi_L)_{L>0}$ be the COGFF on (Σ, g). Let D be a non-empty open subset of Σ. Then for each $\eta > 0$,*

$$\limsup_{L\to\infty} \frac{\ln\Big(\mathbb{P}\Big(\sup_\Sigma \phi_L > \Big(\sqrt{\tfrac{2}{\pi}} + \eta\Big)\ln\left(\sqrt{L}\right)\Big)\Big)}{\ln\left(\sqrt{L}\right)} \leq -2\sqrt{2\pi}\eta + O(\eta^2)$$

and there exists $a > 0$ such that for L large enough,

$$\mathbb{P}\Big(\sup_D \phi_L \leq \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\left(\sqrt{L}\right)\Big) \leq \exp\Big(-a\ln^2\left(\sqrt{L}\right)\Big).$$

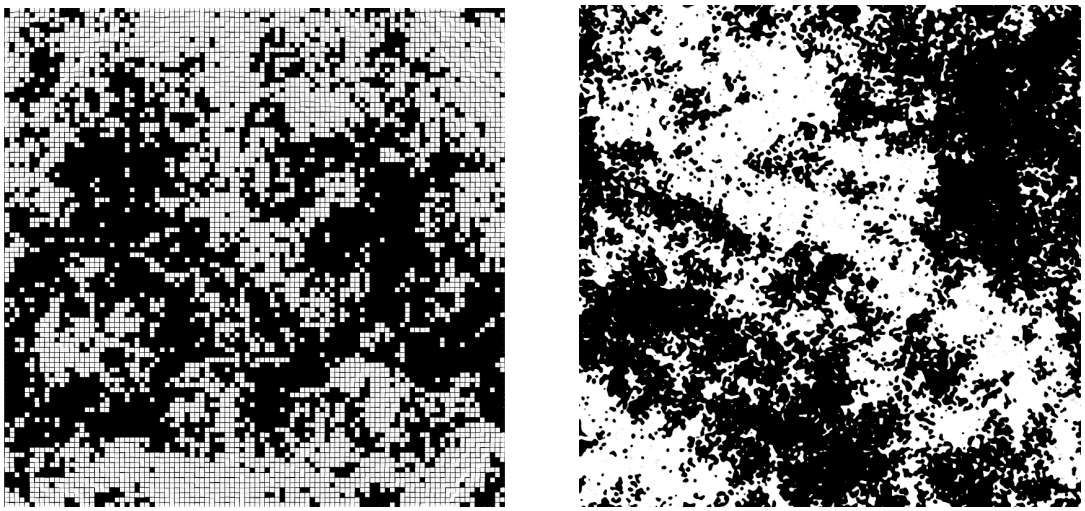
The maxima of random fields on smooth manifolds have been studied, for instance for holomorphic sections of line bundles on Kähler manifolds in [SZ02] and for another eigenfunction model in [BL11]. However, these fields are not log-correlated so the probabilistic

arguments employed are quite different. This theorem is an analogue of Theorem 2 in [BDG01] and the proof relies on it. The supremum of general discrete log-correlated fields has been studied in [DRZ17]. In the case of the DGFF, as well as a large class of continuous log-correlated fields, the law of the supremum has been studied with much higher precision, see for instance [BZ12], [BDZ16], [DRZ17], [BL16a] and [Mad15]. It would be interesting to see if similar results could be obtained for the COGFF.

The paper is organised as follows. In the rest of this section, we give an outline of the proof and introduce some basic notation in order to give a more concrete definition of the COGFF. In section 4.2 we prove Theorem 4.1.4. In section 4.3 we prove Theorem 4.1.1. Section 4.4 is dedicated to the proof of the analytical tools used before, most notably, Theorem 4.1.3. In section 4.5, we cover the case where Σ has a boundary. In the appendix 4.6, we recall some classical results regarding the laplacian and the capacity.

4.1.2 Comparison with the discrete setting

Part of this paper is written in the spirit of [BDG01] which studies the hole probability of the discrete Gaussian Free Field. In this section we outline our proof strategy with [BDG01] in mind and explain the new ideas introduced to deal with this model. We will use the notations introduced in Theorem 4.1.2.



On the left, an instance of the DGFF on the square of side $N = 100$ with periodic boundary conditions. On the right, an instance of the COGFF on the flat torus with $\sqrt{L} = 100$. In both cases, the square is colored white where the field is positive and black where it is negative.

To begin with, the field we consider is defined as a random linear combination of eigenfunctions of the laplacian on a compact surface Σ and some work is required to obtain a tractable expression for the covariance function. This is Theorem 4.1.3 and is proved in section 4.4. In particular we use Hörmander's estimates of **the spectral function of the laplacian** from [Hör68]. The analog in the discrete case is taken for granted in [BDG01]. Since this part uses different techniques than the rest of the paper, we present it at the end. We first relate the kernel G_L to the Schwartz kernel of the orthogonal projector onto U_L in order to apply the aforementioned result by Hörmander. This yields an expression of G_L as an integral involving a kind of generalized Bessel function, which we control thanks to **the stationary phase method**. It emerges from Theorem 4.1.3 that the COGFF (ϕ_L) is a log-correlated field that varies at a scale of $L^{-1/2}$. Consequently, in the analogy with [BDG01], Σ will play the role of the square box of size N where $N \simeq L^{1/2}$. This is why we choose to write $\ln\left(\sqrt{L}\right)$ instead of $\frac{1}{2}\ln L$ in our main results above.

The proof of Theorem 4.1.1 goes as follows. The first step is to estimate the supremum of the field. This is the object of section 4.2 and the result is Theorem 4.1.4. In [BDG01], the bound on the right tail comes from a simple union bound. In our case, since the space is continuous, the field could fluctuate at scales smaller than $L^{-1/2}$. To control these fluctuations, we use **a Sobolev inequality** on smalls disks of this scale (see Lemma 4.2.2). Note that this requires control of the successive derivatives of the field. The result then follows from a union bound applied to a covering of Σ by such disks (see Proposition 4.2.1). To control the left tail of the maximum, Bolthausen, Deuschel and Giacomin use **the Markov property** of the field to construct a tree-like structure and, inspired by branching random walks, use large deviation results to conclude. At this point we cannot follow the original proof because the COGFF does not seem to have a Markov property. Instead, we restrict the COGFF to a discrete box and use a method from the much more recent [DRZ17] in order to apply a gaussian comparison inequality between the restricted field and the DGFF. Thus we recover the bound from the original one (see Proposition 4.2.5).

Once we have Theorem 4.1.4, we can start studying the probability that the COGFF stays positive on a given domain D. This is section 4.3. For the lower bound, [BDG01] uses an entropy inequality and the capacity appears by discrete approximation. In our case (see Proposition 4.3.3) it seemed more natural to apply **the barrier method**, already used in [NS09] and [GW16b]. The idea is to decompose the field into a random multiple of a function h that is greater than one on D and an independent fluctuation. Then, we use the bound on the right tail from Theorem 4.1.4 to control the supremum of the fluctuation. We then vary h to minimize the cost of this procedure and end up with the capacity of D. The lower bound (see Proposition 4.3.4) is more subtle. Indeed, Bolthausen, Deuschel and Giacomin use the Markov property once again to decompose the DGFF (ϕ_N) into two independent gaussian fields. One is "tamer" while the other is "wilder". We call this **the two-scale decomposition** of the COGFF.

Roughly speaking, to stop the wilder field from making ϕ_N negative, the tamer field will need to be close to the expected maximum on a large enough portion of D. This will come at a cost that will be related to the capacity. In our paper the Markov property is once again absent. To construct this decomposition, we split the eigenvalue interval in two and obtain a decomposition of ϕ_L as an independent sum. The tameness of the tamer field will be immediate. On the other hand the wilder field will require a finer analysis. We will first prove an approximate version of the independence afforded by the Markov property of the DGFF in this decomposition. Then, we will apply once again **the gaussian comparison method** from [DRZ17] to conclude.
For the convenience of the reader, we summerize the above discussion in the following table.

Proof step	**Discrete case**	**Continuous case**
Covariance function	$\emptyset$	spectral asymptotics
Right tail for the supremum	union bound	Sobolev inequality + union bound
Left tail of the maximum	Markov property + large deviations	gaussian comparison method
Lower bound for hole probability	entropy inequality	barrier method
Upper bound for hole probability	Markov property + two-scale decomposition	two-scale decomposition + decorrelation estimates + gaussian comparison method

4.1.3 The cut-off Gaussian Free Field

From now on, (Σ, g) will be a smooth, compact, connected surface without boundary, equipped with a riemannian metric. Let $|dV_g|$ be the density and $\Delta = d^*d$ the Laplace operator defined by g. We will denote by $L^2(\Sigma)$ and $H^m(\Sigma)$ for any integer $m \geq 1$ respectively the space of square integrable functions over Σ with respect to the measure $|dV_g|$ and the L^2 Sobolev space of order m with respect to this same measure (see for instance Definition B.1.1 of [**?**]). For any of these spaces, say $E(\Sigma)$, we will denote by $E_0(\Sigma)$ – or E_0 when no ambiguity is possible – the subspace of $E(D)$ consisting of functions of zero mean on Σ. We will denote by $\langle\ ,\ \rangle_2$ the L^2 scalar product on Σ. We will use the same notation in the following case. If X, Y are two vector-fields on Σ, $\langle X, Y\rangle_2 = \int_\Sigma g_p(X_p, Y_p)|dV_g|(p)$. By the Poincaré-Wirtinger inequality (see Theorem 1 of section 5.8.1 of [Eva10]) the bilinear form

$$\langle u, v\rangle_\nabla := \langle \nabla u, \nabla v\rangle_2.$$

defines a scalar product equivalent to the standard one on $H^1_0(\Sigma)$ called the Dirichlet inner product. It is well known (see for instance Theorem 4.43 of [GHL04]) that there exist $(\psi_n)_{n\in\mathbb{N}} \in C^\infty(\Sigma)^{\mathbb{N}}$ and $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \ldots, \lambda_n \xrightarrow[n\to\infty]{} +\infty$, such that $(\psi_n)_{n\in\mathbb{N}}$ is a Hilbert basis for $L^2(\Sigma)$ and such that for each $n \in \mathbb{N}$,

$\Delta\psi_n = \lambda_n\psi_n$. In addition, ψ_0 is constant and, consequently, $\forall n \geq 1$, $\psi_n \in C_0^\infty(\Sigma)$. Stokes' theorem shows that $\left(\frac{1}{\sqrt{\lambda_n}}\psi_n\right)_{n\geq 1}$ is a Hilbert basis of $(H_0^1, \langle\ ,\ \rangle_\nabla)$. For each $L > 0$, let $(U_L, \langle\ ,\ \rangle_\nabla)$ be the subspace of $(H_0^1, \langle\ ,\ \rangle_\nabla)$ spanned by the functions $\frac{1}{\sqrt{\lambda_n}}\psi_n$ such that $0 < \lambda_n \leq L$. Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d real centered gaussian random variables with variance 1. Then, for each $L > 0$ we define the **cut-off Gaussian Free Field**, or COGFF, as

$$\phi_L = \sum_{0<\lambda_n\leq L} \frac{\xi_n}{\sqrt{\lambda_n}}\psi_n. \tag{4.1.1}$$

Hence, for each $L > 0$ and $p, q \in \Sigma$,

$$G_L(p,q) := \mathbb{E}[\phi_L(p)\phi_L(q)] = \sum_{0<\lambda_n\leq L} \frac{1}{\lambda_n}\psi_n(p)\psi_n(q). \tag{4.1.2}$$

Note that defining the COGFF by equation (4.1.1) amounts to saying that it is a random function in U_L with probability density proportional to $e^{-\frac{1}{2}\|\phi\|_\nabla^2}d\phi$ where $d\phi$ is the Lebesgue measure on $(U_L, \langle\ ,\ \rangle_\nabla)$. These definitions imply in particular that ϕ_L converges almost surely to the Gaussian Free Field in the sense of distributions as $L \to \infty$ (see for instance section 2.4 of [She07]).

Acknowledgements: I would like to express my gratitude towards my advisor Damien Gayet for supporting me throughout the course of this project, as well as for his many helpful comments regarding the exposition of these results. I am also grateful to my second advisor Christophe Garban for his encouragements and for sharing his intuitions on the probabilistic aspects of this paper. Finally, I would like to thank Vincent Beffara as well as Luis Alberto Rivera for helping me with the numerical simulations.

4.2 The maximum of the COGFF

The aim of this section is to prove Theorem 4.1.4. The proof is split in two parts, one for the right tail of the maximum and one for the left. More precisely, Theorem 4.1.4 follows immediately from Proposition 4.2.1 and Proposition 4.2.5 below.

4.2.1 Binding the right tail the maximum

The aim of this section is to prove the following proposition.

Proposition 4.2.1. *Let (Σ, g) be a smooth compact surface and let $(\phi_L)_L$ be the COGFF on (Σ, g). Then, for each $\eta > 0$,*

$$\limsup_{L\to\infty} \frac{\ln\Big(\mathbb{P}\Big(\sup_\Sigma \phi_L > \Big(\sqrt{\frac{2}{\pi}} + \eta\Big)\ln\big(\sqrt{L}\big)\Big)\Big)}{\ln\big(\sqrt{L}\big)} \leq -2\sqrt{2\pi}\eta + O(\eta^2).$$

Let us begin by introducing some notation. For each $p \in \Sigma$, $t > 0$ and $L > 0$, let $D_L(p,t)$ be the riemannian disk of radius $\frac{t}{\sqrt{L}}$ around p. For the proof of Proposition 4.2.1, we will need the following two results.

Lemma 4.2.2. *Let (Σ, g) be a smooth compact surface and let $(\phi_L)_L$ be the COGFF on (Σ, g). Then there is a constant $C > 0$ such that for each $p \in \Sigma$ and for $L > 0$ large enough,*

$$\mathbb{E}\Big[\sup_{D_L(p,1)} |\phi_L|\Big] \leq C\sqrt{\ln(L)}$$
$$\mathbb{E}\Big[\sup_{D_L(p,1)} |\nabla\phi_L|\Big] \leq C\sqrt{L}.$$

This lemma is to be compared with Proposition 2.1 of [GW16b]. We postpone its proof till the end of the section. The second result is Theorem 2.1.1 of [AT07] specialised to continuous fields.

Proposition 4.2.3 (Borell-Tsirelson-Ibragimov-Sudakov inequality)**.** *Let T be a separable topological space and $(\phi_t)_{t\in T}$ be a centered gaussian field over T which is almost surely bounded and continuous. Then, $\mathbb{E}[\sup_{t\in T}\phi_t] < \infty$ and for all $u > 0$,*

$$\mathbb{P}\Big(\sup_{t\in T}\phi_t - \mathbb{E}[\sup_{t\in T}\phi_t] > u\Big) \leq \exp\Big(-\frac{u^2}{2\sigma_T^2}\Big)$$

where $\sigma_T^2 = \sup_{t\in T} Var(\phi_t)$.

Let us now prove Proposition 4.2.1 using Lemma 4.2.2 and Proposition 4.2.3.

Proof of Proposition 4.2.1. According to Theorem 4.1.3 for $x = y$, there is a constant C such that for each $p \in \Sigma$,

$$\sup_{q\in D_L(p,1)} Var(\phi_L(q)) = \sup_{q\in D_L(p,1)} G_L(q,q) \leq \frac{1}{2\pi}\ln\big(\sqrt{L}\big) + C.$$

Let $\eta > 0$. We apply Proposition 4.2.3 to $(\phi_L(q))_{q\in D_L(p,1)}$ using Lemma 4.2.2 to deduce that for each $0 < \eta' < \eta$, for L large enough and for all $p \in \Sigma$,

$$\mathbb{P}\Big(\sup_{D_L(p,1)} \phi_L > \Big(\sqrt{\frac{2}{\pi}} + \eta\Big)\ln\big(\sqrt{L}\big)\Big) \leq \exp\Big(-\Big(2 + 2\sqrt{2\pi}\eta' + O(\eta'^2)\Big)\ln\big(\sqrt{L}\big)\Big).$$

Choose some $0 < \eta' < \eta$. Since Σ is compact, there exists another constant which we also denote by C, such that for each L, Σ is covered by CL disks of radius $\frac{1}{\sqrt{L}}$. Thus,

$$\begin{aligned}\mathbb{P}\Big(\sup_{\Sigma}\phi_L > \Big(\sqrt{\frac{2}{\pi}}+\eta\Big)\ln\big(\sqrt{L}\big)\Big) &\leq CL\exp\Big(-\Big(2+2\sqrt{2\pi}\eta'+O(\eta'^2)\Big)\ln\big(\sqrt{L}\big)\Big)\\ &\leq C\big(\sqrt{L}\big)^{-2\sqrt{2\pi}\eta'+O(\eta'^2)}.\end{aligned}$$

Since $-\phi_L$ has the same law as ϕ_L, we have the analogous result for the minimum. Therefore, for each $0 < \eta' < \eta$ there is an L_0 such that for each $L \geq L_0$,

$$\mathbb{P}\Big(\sup_{\Sigma}|\phi_L| > \Big(\sqrt{\frac{2}{\pi}}+\eta\Big)\ln\big(\sqrt{L}\big)\Big) \leq \big(\sqrt{L}\big)^{-2\sqrt{2\pi}\eta'+O(\eta'^2)}.$$

□

To prove Lemma 4.2.2, we use Theorem 4.1.3 and the following proposition. The proof of both of these results is presented in the last section.

Proposition 4.2.4. *Let Q_1 and Q_2 be differential operators on Σ of respective orders d_1 and d_2 and let $d = d_1 + d_2$. Suppose that $d \geq 1$. Then there exists $C > 0$ such that for each $p \in \Sigma$ and $L > 0$,*

$$\Big|(Q_1 \otimes Q_2)G_L(p,p)\Big| \leq C\big(1+L^{d/2}\big).$$

Proof of Lemma 4.2.2. Let $p \in \Sigma$. We apply the Sobolev inequality from paragraph 5.2.4 of [Fed69] with $m = 2$ and $N = 2$. The inequality implies there exist constants $C, L_0 > 0$ such that for all $L \geq L_0$,

$$\begin{aligned}\sup_{q\in D_L(p,1)}|\phi_L(q)| \leq C\Big[&\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}\phi_L(q)^2|dV_g|(q)\Big)^{\frac{1}{2}}\\ &+L^{-\frac{1}{2}}\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}|\nabla\phi_L(q)|^2|dV_g|(q)\Big)^{\frac{1}{2}}\\ &+L^{-1}\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}|\nabla^2\phi_L(q)|^2|dV_g|(q)\Big)^{\frac{1}{2}}\Big].\end{aligned}$$

Here ∇^2 denotes the hessian defined by the metric g. Note that in order to bind the supremum of a function on a two-dimensional space by L^2 Sobolev norms, one must use derivatives up to order at least two. By compactness, C and L_0 may be chosen independent of p. The same inequality holds for expectations and applying Jensen's

inequality to the right hand side we obtain :

$$\mathbb{E}[\sup_{q\in D_L(p,1)}|\phi_L(q)|] \leq C\Big[\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}\mathbb{E}[\phi_L(q)^2]|dV_g|(q)\Big)^{\frac{1}{2}} + L^{-\frac{1}{2}}\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}\mathbb{E}[|\nabla\phi_L(q)|^2|]dV_g|(q)\Big)^{\frac{1}{2}} + L^{-1}\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}\mathbb{E}[|\nabla^2\phi_L(q)|^2|]dV_g|(q)\Big)^{\frac{1}{2}}\Big].$$

In the above inequality, for any tensor T, $|T|$ denotes the norm of T induced by g on the corresponding tensor bundle. Since for any differential operator P over Σ, $\mathbb{E}[P\phi_L(q)^2] = (P\otimes P)G_L(q,q)$, we get

$$\mathbb{E}[\sup_{q\in D_L(p,1)}|\phi_L(q)|] \leq C\Big[\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}G_L(q,q)|dV_g|(q)\Big)^{\frac{1}{2}} + L^{-\frac{1}{2}}\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}|(\nabla\otimes\nabla)G_L(q,q)|\ |dV_g|(q)\Big)^{\frac{1}{2}} + L^{-1}\Big(\frac{1}{Vol(D_L(p,2))}\int_{D_L(p,2)}|(\nabla^2\otimes\nabla^2)G_L(q,q)|\ |dV_g|(q)\Big)^{\frac{1}{2}}\Big]. \tag{4.2.1}$$

Now, from Proposition 4.2.4, there is a constant $C > 0$ such that for all $p \in \Sigma$ and $L > 0$,

$$|G_L(p,p)| \leq C\ln(L); \quad |(\nabla\otimes\nabla)G_L(p,p)| \leq CL; \quad |(\nabla^2\otimes\nabla^2)G_L(p,p)| \leq CL^2.$$

Applying these three inequalities to equation (4.2.1), we deduce that there is a constant $C > 0$ such that for all $p \in \Sigma$ and $L > 0$,

$$\mathbb{E}[\sup_{q\in D_L(p,1)}|\phi_L(q)|] \leq C\sqrt{\ln(L)}.$$

This proves the first statement. The proof carries over to the second statement almost verbatim, using in addition the following estimate from Proposition 4.2.4.

$$|(\nabla^3\otimes\nabla^3)G_L(p,p)| \leq CL^3.$$

□

4.2.2 Binding the left tail of the maximum

In this section we prove the following proposition.

Proposition 4.2.5. *Let (Σ, g) be a smooth compact surface and let $(\phi_L)_L$ be the COGFF on (Σ, g). Let $D \subset \Sigma$ be a non-empty open subset of Σ. Then, for each $\eta > 0$ there is a constant $a > 0$ such that for L large enough,*

$$\mathbb{P}\Big(\sup_D \phi_L \leq \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\big(\sqrt{L}\big)\Big) \leq \exp\Big(-a\ln^2\big(\sqrt{L}\big)\Big).$$

We now introduce some notation. For each $N \in \mathbb{N}_{\geq 1}$, let V_N be the set of points $(x_1, x_2) \in \mathbb{Z}^2$ such that $0 \leq x_1, x_2 \leq N-1$, let V'_N the set of points in V_N at distance at least $N/4$ from the boundary and let x_N be one of the points nearest to its center. For each $x, y \in V_N$, $|x-y|$ will denote the euclidian distance between x and y. For each $t > 0$, $\ln_+ t$ will denote $\max(\ln t, 0)$.

Proposition 4.2.5 will follow from the two following results.

Proposition 4.2.6. *Let $(X_t)_{t>1}$ be a family of random fields such that for all $t > 1$, X_t is defined over the box $V_{\lfloor t \rfloor}$. Suppose there is a constant $C > 0$ such that for each $t > 1$ and $x, y \in V_{\lfloor t \rfloor}$,*

$$\big|\mathbb{E}[X_t(x)X_t(y)] - \ln t + \ln_+ |x-y|\big| \leq C.$$

Then for each $\eta > 0$ there is a constant $a > 0$ depending only on C and η such that for $t > 1$ large enough,

$$\mathbb{P}\Big(\sup_{V_{\lfloor t \rfloor}} X_t \leq (2-\eta)\ln t\Big) \leq \exp\big(-a(\ln t)^2\big).$$

Lemma 4.2.7. *Fix $0 < \delta < \frac{1}{2\sqrt{2}}$ and $L > 0$. Let $\iota : V_{\lfloor \sqrt{L} \rfloor} \to \Sigma$ be an injection of the $\mathbb{Z}^2$-box of side-length $\lfloor \sqrt{L} \rfloor$ into Σ such that for any distinct $x, y \in V_{\lfloor \sqrt{L} \rfloor}$,*

$$\frac{\delta}{2} \leq \frac{\sqrt{L} d_g(\iota(x), \iota(y))}{|x-y|} \leq 2\delta.$$

Then, there is a constant $C(\delta) > 0$ independent of L and ι such that for L large enough and for each $x, y \in V_{\lfloor \sqrt{L} \rfloor}$,

$$\Big|\mathbb{E}[\phi_L(\iota(x))\phi_L(\iota(y))] - \frac{1}{2\pi}\ln\big(\sqrt{L}\big) + \ln_+ |x-y|\Big| \leq C(\delta).$$

Lemma 4.2.7 is just the specialisation of Lemma 4.3.1 from section 4.3 to the case $\alpha = 0$. In the following proof we use Proposition 4.2.6 and Lemma 4.2.7.

Proof of Proposition 4.2.5. Choose some $\delta > 0$ and ι satisfying the properties required to apply Lemma 4.2.7 and such that the image of ι is contained in D for L large enough. Then, by Lemma 4.2.7, the family $(X_t)_t$ defined by, $\forall t > 0$, $X_t := \sqrt{2\pi}\phi_{t^2} \circ \iota$ satisfies the hypotheses of Proposition 4.2.6. In particular, for each $\eta > 0$, there is a constant $a > 0$ such that for all $L > 0$,

$$\begin{aligned}\mathbb{P}(\sup_D \phi_L \leq \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\big(\sqrt{L}\big)\Big) &\leq \mathbb{P}\Big(\sup_{V_{\lfloor \sqrt{L} \rfloor}} X_{\sqrt{L}} \leq (2 - \sqrt{2\pi}\eta)\ln\big(\sqrt{L}\big)\Big) \\ &\leq \exp\Big(-a\ln^2\big(\sqrt{L}\big)\Big).\end{aligned}$$

□

To prove Proposition 4.2.6, we use the two following results. The first is a special case of Theorem 2 (b) of [BDG01].

Theorem 4.2.8. *Let $(\phi_N)_N$ be the DGFF on V_N with the standard normalization. For each $\eta > 0$ there is a constant $c > 0$ such that for each $N \in \mathbb{N}$ large enough,*

$$\mathbb{P}\Big(\sup_{V_N'} \phi_N \leq \Big(\sqrt{\frac{8}{\pi}} - \eta \Big) \ln N \Big) \leq \exp(-c(\ln N)^2).$$

Lemma 4.2.9 (Slepian's Lemma, see Theorem 2.2.1 of [AT07])**.** *Let T be a separable topological space and $(Z(p))_{p\in T}$, $(Y(p))_{p\in T}$ be two continuous centered gaussian fields on T satisfying the following two properties.*

1. *For each $p \in T$, $\mathbb{E}[Z(p)^2] = \mathbb{E}[Y(p)^2]$.*
2. *For each $p, q \in T$, $\mathbb{E}[Z(p)Z(q)] \leq \mathbb{E}[Y(p)Y(q)]$.*

Then, for each $u \in \mathbb{R}$,

$$\mathbb{P}(\sup_{p\in T} Z(p) > u) \geq \mathbb{P}(\sup_{p\in T} Y(p) > u).$$

We now deduce Proposition 4.2.6 from Proposition 4.2.8 and Lemma 4.2.9. The following proof is inspired by that of Lemma 2.8 of [DRZ17].

Proof of Proposition 4.2.6. Choose any $\eta > 0$ and some $j \in \mathbb{N}$ to be fixed later. For each $t > 2^j$, let $N = N(t) = \lfloor 2^{-j} t \rfloor$ and let Z_N be gaussian field defined on V_N by setting, for each $x \in V_N$, $Z_N(x) = X_t(2^j x)$. Then, for distinct $x, y \in V_N$,

$$\begin{aligned} \big| \mathbb{E}[Z_N(x)^2] - \ln N - j \ln 2 \big| &\leq C \\ \big| \mathbb{E}[Z_N(x) Z_N(y)] - \ln N + \ln_+ |x - y| \big| &\leq C. \end{aligned}$$

Let Y_N be the DGFF on V_N multiplied by $\sqrt{\frac{\pi}{2}}$. From Lemma 2.2 of [BZ12] there is a universal constant $C_0 > 0$ such that for any distinct $x, y \in V_N'$,

$$\begin{aligned} \big| \mathbb{E}[Y_N(x)^2] - \ln N \big| &\leq C_0 \\ \big| \mathbb{E}[Y_N(x) Y_N(y)] - \ln N + \ln_+ |x - y| \big| &\leq C_0. \end{aligned}$$

Since $1/4 < \ln 2 < 1$, there is j_0 depending only on C (and the universal constant C_0) such that for each $j \geq j_0$ and for each $x \in V_N'$,

$$j/4 \leq \mathbb{E}[Z_N(x)^2] - \mathbb{E}[Y_N(x)^2] \leq j.$$

Let

$$a_N(x) = \sqrt{j^{-1}(\mathbb{E}[Z_N(x)^2] - \mathbb{E}[Y_N(x)^2])} \in [1/2, 1]$$

and choose ξ a centered gaussian random variable with variance 1 independent from the fields previously introduced. Then, there is $j \geq j_0$ depending only on C such that for each $x, y \in V'_N$ distinct,

$$\mathbb{E}[Z_N(x)^2] = \mathbb{E}[(Y_N(x) + \sqrt{j}\xi a_N(x))^2]$$
$$\mathbb{E}[Z_N(x)Z_N(y)] \leq \mathbb{E}[(Y_N(x) + \sqrt{j}\xi a_N(x))(Y_N(y) + \sqrt{j}\xi a_N(y))].$$

Thus, by Lemma 4.2.9,

$$\begin{aligned}\mathbb{P}\Big(\sup_{V_{\lfloor t \rfloor}} X_t \leq (2-\eta)\ln t\Big) \leq& \mathbb{P}\Big(\sup_{V'_N} Z_N \leq (2-\eta)\ln t\Big)\\ \leq& \mathbb{P}\Big(\sup_{x\in V'_N}\Big[Y_N(x) + \sqrt{j}\xi a_N(x)\Big] \leq (2-\eta)\ln t\Big)\\ \leq& \mathbb{P}\Big(\sup_{x\in V'_N} Y_N(x) \leq (2-(\eta/2))\ln t\Big) +\\ & \mathbb{P}\Big(\xi \geq \frac{\eta}{2\sqrt{j}\sup_{V'_N} a_N}\ln t\Big).\end{aligned}$$

For t large enough, $(2-(\eta/2))\ln t \leq (2-(\eta/3))\ln(N)$. From standard tail estimates for gaussian variables applied to ξ and Theorem 4.2.8 applied to $\sqrt{\frac{2}{\pi}}Y_N$, there is a constant $a > 0$ such that for t large enough,

$$\mathbb{P}\Big(\sup_{V'_{\lfloor t \rfloor}} X_t \leq (2-\eta)\ln t\Big) \leq \exp\big(-a(\ln t)^2\big).$$

Moreover, a depends only on C, η. □

4.3 Hole probabilitiy for the COGFF

The aim of this section is to prove Theorem 4.1.1. From now on, we fix an open proper subset D of Σ with smooth boundary. This assumption implies that there exist functions $f \in C_0^\infty(\Sigma)$ greater or equal to 1 on D. We want to estimate the probability of the event

$$\Omega_L^+ = \Big\{\forall x \in D, \phi_L(x) > 0\Big\}.$$

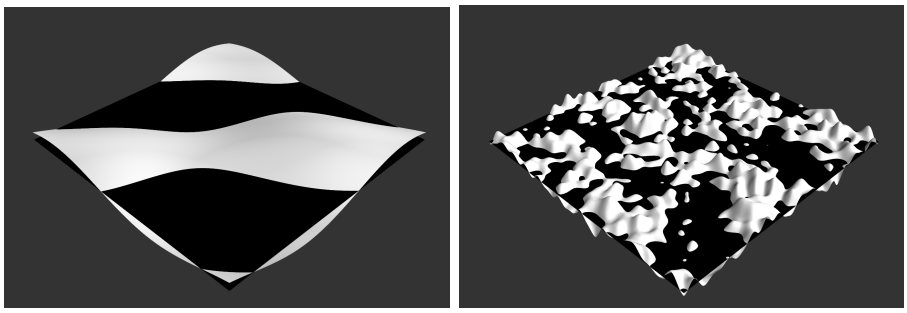
We divide the proof into two parts, one for the lower bound and one for the upper bound. Theorem 4.1.1 will thus follow from immediately from Proposition 4.3.3 and Proposition 4.3.4 below. Note that, if D is non-empty, then Proposition 4.6.2 implies $cap(D) > 0$.

4.3.1 The two-scale decomposition

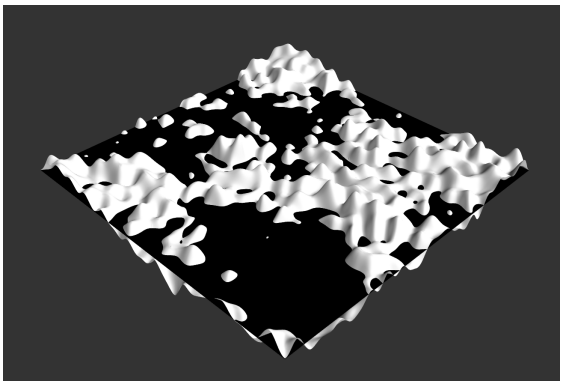
In this section, we introduce the two-scale decomposition of the COGFF used below. We will use notations from sections 4.1.3 and 4.2. For each $\alpha \in]0,1]$ and each $L > 0$, we denote by $\psi_{\alpha,L}$ the field $\psi_{\alpha,L} = \sum_{L^\alpha < \lambda_n \leq L} \frac{\xi_n}{\sqrt{\lambda_n}}\psi_n$. Note that

$$\phi_L = \psi_{\alpha,L} + \phi_{L^\alpha} \tag{4.3.1}$$

and that $\psi_{\alpha,L}$ and ϕ_{L^α} are independent. Moreover, observe that the two point correlation function of $\psi_{\alpha,L}$ is $G_{L^\alpha,L}(p,q) = \sum_{L^\alpha<\lambda_n\leq L} \frac{1}{\lambda_n}\psi_n(p)\psi_n(q) = G_L(p,q) - G_{L^\alpha}(p,q)$. The asymptotics of $G_{L^\alpha,L}$ will follow easily from those of G_L. The field ϕ_{L^α} will vary at scale $L^{-\alpha/2}$ while $\psi_{\alpha,L}$ will vary at scale $L^{-1/2}$ and will decorrelate at large distances. While the first fact is immediate, the other two require some justification.



An instance of the field ϕ_{L^α} on the left and of the field $\psi_{\alpha,L}$ on the right for $\alpha = 0.25$ and $L = 400$ on the flat torus. The fields are colored in white and the black surface is a horizontal square at height zero.



The total field $\phi = \phi_{L^\alpha} + \psi_{\alpha,L}$ obtained from the instances above.

We will prove the following two results.

Lemma 4.3.1. *Fix $0 < \delta < \frac{1}{2\sqrt{2}}$, $\alpha \in [0,1[$ and $L > 0$. Let $\iota : V_{\lfloor L^{(1-\alpha)/2}\rfloor} \to \Sigma$ be an injection of the $\mathbb{Z}^2$-box of side-length $\lfloor L^{(1-\alpha)/2}\rfloor$ into Σ such that for distinct $x, y \in V_{\lfloor L^{(1-\alpha)/2}\rfloor}$,*

$$\frac{\delta}{2} \leq \frac{\sqrt{L} d_g(\iota(x),\iota(y))}{|x-y|} \leq 2\delta.$$

Then, there is a constant $C(\delta) > 0$ independent of α, L and ι such that for L large enough and for each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$\left|\mathbb{E}[\psi_{\alpha,L}(\iota(x))\psi_{\alpha,L}(\iota(y))] - \frac{1-\alpha}{2\pi}\ln\left(\sqrt{L}\right) + \ln_+|x-y|\right| \leq C(\delta).$$

This lemma shows that $\psi_{\alpha,L}$ does indeed vary at scale $L^{-1/2}$ and we will use it to prove that its maximum on a box of sidelength $L^{\alpha/2}$ will be close to $\frac{1-\alpha}{2\pi}\ln\left(\sqrt{L}\right)$. Note that Lemma 4.2.7 is just Lemma 4.3.1 with $\alpha = 0$ as announced above. Before proving Lemma 4.3.1, let us state the second result we will need concerning $\psi_{\alpha,L}$.

Proposition 4.3.2. *Choose $0 < \alpha < 1$. For each $\delta > 0$ and $0 < \beta < \alpha$,*

$$G_{L^\alpha,L}(p,q) \xrightarrow[L\to\infty]{} 0.$$

This proposition shows that the field decorrelates at large distances. The proof is completely analytical so we leave it for section 4.4.

Proof of Lemma 4.3.1. For L large enough, then $V_L := \iota\left(V_{\lfloor L^{(1-\alpha)/2} \rfloor}\right)$ has diameter smaller than the ε in the statement of Theorem 4.1.3. For each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$\begin{aligned}\mathbb{E}[\psi_{\alpha,L}(\iota(x))\psi_{\alpha,L}(\iota(y))] &= G_{L^\alpha,L}(\iota(x),\iota(y)) \\ &= G_L(\iota(x),\iota(y)) - G_{L^\alpha}(\iota(x),\iota(y)).\end{aligned}$$

By Theorem 4.1.3 applied for $L' = L$ or $L' = L^\alpha$, there is a constant $C > 0$ such that for each $L > 0$ large enough, for each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$G_L(\iota(x),\iota(y)) = \frac{1}{2\pi}\Big(\ln\left(\sqrt{L}\right) - \ln_+\left(\sqrt{L}d_g(\iota(x),\iota(y))\right)\Big) + \rho_L^1(x,y).$$

and

$$\begin{aligned}G_{L^\alpha}(\iota(x),\iota(y)) &= \frac{1}{2\pi}\Big(\ln\left(\sqrt{L^\alpha}\right) - \ln_+\left(L^{\alpha/2}d_g(\iota(x),\iota(y))\right)\Big) + \rho_L^2(x,y) \\ &= \frac{1}{2\pi}\Big(\alpha\ln\left(\sqrt{L}\right) - \ln_+\left(L^{\alpha/2}d_g\big(\iota(x),\iota(y)\big)\right)\Big) + \rho_L^2(x,y).\end{aligned}$$

where $|\rho_L^j(x,y)| \leq C$ for $j = 1, 2$. Therefore,

$$\left|\mathbb{E}[\psi_L(\iota(x))\psi_L(\iota(y))] - \frac{1-\alpha}{2\pi}\ln\left(\sqrt{L}\right) + \ln_+\left(\sqrt{L}d_g(\iota(x),\iota(y)) - \ln_+\left(L^{\alpha/2}d_g\big(\iota(x),\iota(y)\big)\right)\right)\right| \leq 2C.$$

For each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$, $|x-y| \leq \sqrt{2}\lfloor L^{(1-\alpha)/2} \rfloor$ so that, since $\delta < \frac{1}{2\sqrt{2}}$,

$$L^{\alpha/2}d_g(\iota(x),\iota(y)) \leq 2\delta L^{(\alpha-1)/2}|x-y| < 1.$$

Therefore, $\ln_+(L^{\alpha/2}|d_g(\iota(x),\iota(y)) = 0$. Now, for each $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$,

$$\frac{1}{2}\delta|x-y| \leq \sqrt{L}d_g(\iota(x),\iota(y)) \leq 2\delta|x-y|$$

so that

$$\ln_+|x-y| - \ln 2 + \ln(\delta) \leq \ln_+\left(\sqrt{L}d_g(\iota(x),\iota(y))\right) \leq \ln_+|x-y| + \ln 2 + \ln(\delta).$$

This concludes the proof of the lemma. □

4.3.2 The lower bound

In this section we use the upper bound in Theorem 4.1.4 to prove the lower bound in Theorem 4.1.1. In other words, we will prove the following.

Proposition 4.3.3. *Let (Σ, g) be a compact smooth surface equipped with a riemannian metric and (ϕ_L) be the COGFF on Σ. Let D be a proper open subset of Σ and Ω_L^+ be the event that $\phi_L(x) > 0$ for each $x \in D$. Then,*

$$\liminf_{L\to\infty} \frac{\ln \mathbb{P}(\Omega_L^+)}{\ln^2\left(\sqrt{L}\right)} \geq -\frac{2}{\pi} cap(D).$$

Our approach in the following proof is inspired by that of Nazarov and Sodin in section 3 of [NS09] and that of Gayet and Welschinger in section 2.2 of [GW16b].

Proof of Proposition 4.3.3. Let us choose $\varepsilon > 0$ and a function $h \in C_0^\infty(\Sigma)$ such that for each $x \in D$, $h(x) \geq 1$. Since $u \mapsto ||\nabla u||_2$ is C^1 continuous, by Lemma 4.6.1, for L large than some L_0, there is a function $f \in U_L$ such that $\forall x \in \Sigma$, $|f(x) - h(x)| \leq \varepsilon$ and such that $\|\nabla f - \nabla h\|_2 \leq \varepsilon$. Now, for L large enough, the random field ϕ_L can be decomposed as the independent sum $\xi \frac{f}{\|\nabla f\|_2} + \tilde{\phi}_L$ where ξ is a real centered gaussian random variable with variance 1 and $\tilde{\phi}_L$ is some gaussian field. Choose $\tilde{\xi}$ another real centered gaussian random variable with variance 1, independent from all the former random variables, and set

$$\phi_L^\pm = \pm\tilde{\xi}\frac{f}{\|\nabla f\|_2} + \tilde{\phi}_L.$$

Then, $\phi_L^\pm$ are random fields with the same law as ϕ_L but independent from ξ. Furthermore,

$$\tilde{\phi}_L = \frac{\phi_L^- + \phi_L^+}{2}.$$

We now introduce a constant $A > 0$ which we will fix later. The field ϕ_L will be positive on D if the following three equations are satisfied.

$$\begin{aligned} \xi &> \|\nabla f\|_2 A \frac{1}{1-\varepsilon} \\ \forall x \in D,\ \phi_L^-(x) &\leq A \\ \forall x \in D,\ \phi_L^+(x) &\leq A. \end{aligned}$$

Therefore, by independence of ξ and $\phi_L^\pm$,

$$\mathbb{P}(\Omega_L^+) \geq \Big(1 - 2\mathbb{P}\Big(\sup_D \phi_L > A\Big)\Big)\mathbb{P}(\xi > \|\nabla f\|_2 A(1-\varepsilon)^{-1}).$$

Choose $\delta > 0$ and $A = \left(\sqrt{\frac{2}{\pi}} + \delta\right)\ln(\sqrt{L})$ for $L \geq L_0$. From Theorem 4.1.4 we have

$$\mathbb{P}\Big(\sup_D \phi_L \geq \left(\sqrt{\frac{2}{\pi}} + \delta\right)\ln\left(\sqrt{L}\right)\Big) \to 0.$$

Moreover, from gaussian tail estimates (see equation (1.2.2) of [AT07]), for L large enough,

$$\mathbb{P}\Big(\xi > (1-\varepsilon)^{-1}||\nabla f||_2\Big(\sqrt{\frac{2}{\pi}}+\delta\Big)\ln\big(\sqrt{L}\big)\Big)$$

is greater than

$$\frac{(1-\varepsilon)\exp\Big(-\frac{1}{2}(1-\varepsilon)^{-2}\|\nabla f\|_2^2\Big(\sqrt{\frac{2}{\pi}}+\delta\Big)^2\ln^2\big(\sqrt{L}\big)\Big)}{2\|\nabla f\|_2\Big(\sqrt{\frac{2}{\pi}}+\delta\Big)\ln\big(\sqrt{L}\big)}.$$

Therefore,

$$\begin{aligned}\liminf_{L\to\infty}\frac{\ln\mathbb{P}(\Omega_L^+)}{\ln^2\big(\sqrt{L}\big)} &\geq -\Big(\sqrt{\frac{2}{\pi}}+\delta\Big)^2\frac{1}{2}(1-\varepsilon)^{-2}\|\nabla f\|_2^2\\ &\geq -\Big(\sqrt{\frac{2}{\pi}}+\delta\Big)^2\frac{1}{2}(1-\varepsilon)^{-2}(\|\nabla h\|_2+\varepsilon)^2.\end{aligned}$$

Taking the infimum over $\delta>0$ and $\varepsilon>0$ we obtain

$$\liminf_{L\to\infty}\frac{\ln\mathbb{P}(\Omega_L^+)}{\ln^2\big(\sqrt{L}\big)} \geq -\frac{2}{\pi}\frac{1}{2}\|\nabla h\|_2^2.$$

By taking the infimum of $\frac{1}{2}\|\nabla h\|_2^2$ over all h, we get

$$\liminf_{L\to\infty}\frac{\ln\mathbb{P}(\Omega_L^+)}{\ln^2\big(\sqrt{L}\big)} \geq -\frac{2}{\pi}cap(D).$$

□

4.3.3 The upper bound

The aim of this section is to prove the following proposition.

Proposition 4.3.4.

$$\limsup_{L\to\infty}\frac{\ln\big(\Omega_L^+\big)}{\ln^2\big(\sqrt{L}\big)} \leq -\frac{2}{\pi}cap(D).$$

Just as in section 3 of [BDG01], we proceed by dichotomy with respect to the following event. Let $K\in\mathbb{N}\setminus\{0\}$, $\eta>0$, $\alpha\in]0,1]$ and

$$A_{K,\eta,\alpha}=\Big\{Vol\Big[x\in D\mid \phi_{L^\alpha}(x)<\Big(\sqrt{\frac{2}{\pi}}-\eta\Big)\ln\big(\sqrt{L}\big)\Big]\leq\frac{K}{L^{\alpha/2}}\Big\}.$$

Proposition 4.3.4 is an immediate consequence of the two following results, to be compared with lemmas 9 and 10 of [BDG01].

Lemma 4.3.5. *Let $\eta > 0$. For any integer $K > 0$ and $\alpha \in]0,1]$,*

$$\limsup_{L\to\infty} \frac{\ln\left(\mathbb{P}(A_{K,\eta,\alpha} \cap \Omega_L^+)\right)}{\ln^2\left(\sqrt{L}\right)} \leq -\Big(\sqrt{\frac{2}{\pi}} - \eta\Big)^2 cap(D).$$

Lemma 4.3.6. *For any $\eta > 0$ and $\kappa > 0$ there exist $\alpha \in]0,1[$ and an integer $K > 0$ such that*

$$\limsup_{L\to\infty} \frac{\ln\left(\mathbb{P}(A^c_{K,\eta,\alpha} \cap \Omega_L^+)\right)}{\ln^2\left(\sqrt{L}\right)} \leq -\kappa.$$

Proof of Lemma 4.3.5. Fix $\eta, K > 0$ and $\alpha \in]0,1]$. Choose $f \in C^\infty(\Sigma)$ positive ond D. For each $L > 0$ let $\mathcal{E}_L$ be the random set and X_L the real random variable defined as follows.

$$\mathcal{E}_L = \Big\{x \in D \mid \phi_{L^\alpha}(x) \geq \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\left(\sqrt{L}\right)\Big\}$$
$$X_L = \int_D f(p)\phi_{L^\alpha}(p)|dV_g|(p).$$

X_L is a centered gaussian variable with variance

$$Var(X_L) = \int_\Sigma\int_\Sigma \mathbf{1}_D(p)f(p)G_{L^\alpha}(p,q)\mathbf{1}_D(q)f(q)|dV_g|(p)|dV_g|(q).$$

Therefore, according to equation (4.6.1),

$$Var(X_L) \xrightarrow[L\to\infty]{} \sigma(\mathbf{1}_D f).$$

Moreover, on $A_{K,\eta,\alpha} \cap \Omega_L^+$, $Vol(D \setminus \mathcal{E}_L) \leq \frac{K}{L^{\alpha/2}}$ and $X_L \geq \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\left(\sqrt{L}\right)\int_{\mathcal{E}_L} f$ so that

$$X_L \geq \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\left(\sqrt{L}\right)\Big(\int_D f - \frac{K||f||_\infty}{L^{\alpha/2}}\Big).$$

Therefore,

$$\mathbb{P}(A_{K,\eta,\alpha} \cap \Omega_L^+) \leq \mathbb{P}\Big(X_L \geq \Big(\int_D f - \frac{K||f||_\infty}{L^{\alpha/2}}\Big)\Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\left(\sqrt{L}\right)\Big).$$

By standard gaussian tail estimates (see again equation (1.2.2) of [AT07]), for L large enough, the right hand side of the above inequality is smaller than

$$\frac{\sqrt{Var(X_L)}\exp\Big(-\frac{\Big(\int_D f - \frac{K||f||_\infty}{L^{\alpha/2}}\Big)^2}{2Var(X_L)}\Big(\sqrt{\frac{2}{\pi}} - \eta\Big)^2\ln^2\left(\sqrt{L}\right)\Big)}{\Big(\int_D f - \frac{K||f||_\infty}{L^{\alpha/2}}\Big)\Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\left(\sqrt{L}\right)}.$$

Thus,

$$\limsup_{L\to\infty} \frac{\ln\big(\mathbb{P}(A_{K,\eta,\alpha}\cap\Omega_L^+)\big)}{\ln^2\big(\sqrt{L}\big)} \le -\Big(\sqrt{\frac{2}{\pi}}-\eta\Big)^2 \frac{\Big(\int_D f\Big)^2}{2\sigma(\mathbf{1}_D f)}.$$

From Proposition 4.6.2, we complete the proof by taking the supremum over f in the last inequality. □

To prove Lemma 4.3.6 we will need the following technical result, which we prove at the end of the section. This result is analogous to Lemma 12 of [BDG01].

Lemma 4.3.7. *Let $\eta > 0$ and $\alpha \in]0,1]$. For each $\delta > 0$, let F_δ be the event defined by*

$$F_\delta = \Big\{ \sup_{d_g(p,q)\le \delta L^{-\alpha/2}} |\phi_{L^\alpha}(p) - \phi_{L^\alpha}(q)| \ge (\eta/2)\ln\big(\sqrt{L}\big)\Big\}.$$

Then, for each $\kappa > 0$ there is $\delta_0 > 0$ such that for each $\delta < \delta_0$,

$$\mathbb{P}\big(F_\delta\big) \le \exp\big(-\kappa \ln^2\big(\sqrt{L}\big)\big).$$

The proof itself goes roughly as follows. On the event $A^c_{K,\eta,\alpha}$ the field ϕ_{L^α} takes low values on an abnormally large set $\mathcal{E}_L^c$. On this set we will consider K small disks that will be so far apart from each other that, from Proposition 4.3.2, the values of the field $\psi_{\alpha,L}$ on different disks will decorrelate. Now Lemma 4.3.1 tells us that for α small enough, on each of these disks, $\psi_{\alpha,L}$ will be likely to spike downwards and make the sum negative. Taking K large enough will yield the desired inequality.

Proof of Lemma 4.3.6. Let us begin by fixing $\eta > 0$ and $\kappa > 0$. We introduce constants $K > 0$ and $\alpha \in]0,1[$ which we will fix later. As in the previous lemma, for every $L > 0$ let

$$\mathcal{E}_L = \Big\{ x \in D \mid \phi_{L^\alpha}(x) \ge \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\big(\sqrt{L}\big)\Big\}$$

By Lemma 4.3.7 there is $\delta_0 > 0$ such that for any $\delta < \delta_0$,

$$\mathbb{P}(\Omega_L^+ \cap A^c_{K,\eta,\alpha} \cap F_\delta) \le \mathbb{P}(F_\delta) \le \exp\Big(-\kappa \ln^2\big(\sqrt{L}\big)\Big).$$

We now provide an upper bound for $\mathbb{P}(\Omega_L^+ \cap A^c_{K,\eta,\alpha} \cap F_\delta^c)$. On the event $A^c_{K,\eta,\alpha}$, there is a constant $c > 0$ such that for L large enough, $D \setminus \mathcal{E}_L$ contains at least K points whose mutual distance and distance to $\mathcal{E}_L$ is at least $2cL^{-\alpha/4}$. When both $A^c_{K,\eta,\alpha}$ and F_δ^c are satisfied, for L large enough, there is a collection $(D_j)_{j\in J}$ of K disks of radius $\delta L^{-\alpha/2}$ on which ϕ_{L^α} is smaller than $\Big(\sqrt{\frac{2}{\pi}} - (\eta/2)\Big)\ln\big(\sqrt{L}\big)$ such that for each $i, j \in J$ distinct,

$$\inf_{p\in D_i,\ q\in D_j} d_g(p,q) \ge cL^{-\alpha/4}.$$

Recall that $\psi_{\alpha,L}$, defined in equation (4.3.1), is independent from ϕ_{L^α}. Le $\tilde{\mathbb{P}}$ be the conditional probability with respect to ϕ_{L^α}. On $A^c_{K,\eta,\alpha} \cap F^c_\delta$ consider the events

$$T^{\pm}_{K,\eta,\alpha} = \Big\{ \phi_{L^\alpha} \in A^c_{K,\eta,\alpha} \cap F^c_\delta \text{ and } \forall j \in J,\ \sup_{D_j} \pm\psi_{\alpha,L} \leq \Big(\sqrt{\frac{2}{\pi}} - (\eta/2) \Big) \ln\big(\sqrt{L}\big) \Big\}.$$

Since, $\psi_{\alpha,L}$ has the same law as $-\psi_{\alpha,L}$, the two events have the same probability. Moreover, since $\phi_L = \phi_{L^\alpha} + \psi_{\alpha,L}$, $\Omega^+_L \cap A^c_{K,\eta,\alpha} \cap F^c_\delta$ clearly implies $T^-_{K,\eta,\alpha}$. We will prove that for an adequate choice of K and α, and for L large enough, on the event $A^c_{K,\eta,\alpha} \cap F^c_\delta$,

$$\tilde{\mathbb{P}}\Big(T^+_{K,\eta,\alpha}\Big) \leq \exp\Big(- \kappa \ln^2\big(\sqrt{L}\big)\Big). \tag{4.3.2}$$

Passing to expectations with respect to ϕ_{L^α}, this will imply that

$$\mathbb{P}(\Omega^+_L \cap A^c_{K,\eta,\alpha} \cap F^c_\delta) \leq \mathbb{P}\Big(T^+_{K,\eta,\alpha}\Big) \leq \exp\Big(- \kappa \ln^2\big(\sqrt{L}\big)\Big).$$

For each $j \in J$, consider $\iota_{j,\alpha,L} : V_{\lfloor L^{(1-\alpha)/2} \rfloor} \to D_j$ such that for any distinct $x, y \in V_{\lfloor L^{(1-\alpha)/2} \rfloor}$

$$\frac{\delta}{8} \leq \frac{\sqrt{L} d_g(\iota_{\alpha,L}(x), \iota_{\alpha,L}(y))}{|x-y|} \leq \frac{\delta}{2}.$$

Here, as in Lemma 4.3.1, V_N is the box of sidelength N in $\mathbb{Z}^2$. Now, for each $j \in J$, define $\psi_{j,\alpha,L}$ a random field on $V_{\lfloor L^{(1-\alpha)/2} \rfloor}$ with the same law as $\psi_{\alpha,L} \circ \iota_{j,\alpha,L}$ such that the collection $(\psi_{j,\alpha,L})_{j\in J}$ is independent overall and of the previously defined fields. According to Lemma 4.3.1 and Proposition 4.2.6, there is a constant $a > 0$ depending only on η and δ such that on the event $A^c_{K,\eta,\alpha} \cap F^c_\delta$,

$$\forall j \in J,\ \tilde{\mathbb{P}}\Big(\sup \psi_{j,\alpha,L} \leq \Big(\sqrt{\frac{2}{\pi}} - (\eta/16) \Big)(1-\alpha) \ln\big(\sqrt{L}\big)\Big) \leq \exp\big(- a \ln^2\big(\sqrt{L}\big)\big). \tag{4.3.3}$$

At this point, we fix $\alpha > 0$ such that $\Big(\sqrt{\frac{2}{\pi}} - (\eta/16) \Big)(1-\alpha) \geq \Big(\sqrt{\frac{2}{\pi}} - (\eta/8) \Big)$. For each $j \in J$, define $V_j = \iota_{j,\alpha,L}(V_{\lfloor L^{(1-\alpha)/2} \rfloor})$. Take $\varepsilon > 0$. By Proposition 4.3.2 with $\beta = \alpha/2$, there is L_0 such that for any $L \geq L_0$, any distinct $i, j \in J$ and any $p \in V_i$, $q \in V_j$,

$$|\mathbb{E}[\psi_{\alpha,L}(p)\psi_{\alpha,L}(q)]| \leq \varepsilon^2. \tag{4.3.4}$$

We now introduce, for each $j \in J$ and $p \in V_j$, a real random variable ξ_p, as well as an additional random variable ξ, all independent from previously introduced variables such that the family $(\xi, (\xi_p)_p)$, is independent and each variable is a centered gaussian with variance 1. We will consider the following events

$$\Lambda_1 = \Big\{\forall j \in J, \sup_{p\in V_j} \psi_{\alpha,L}(p) \leq \Big(\sqrt{\frac{2}{\pi}} - (\eta/2)\Big)\ln\big(\sqrt{L}\big)\Big\}$$
$$\Lambda_2 = \Big\{\exists j \in J,\ p \in V_j \mid \varepsilon\xi_p \geq (\eta/4)\ln\big(\sqrt{L}\big)\Big\}$$
$$\Lambda_3 = \Big\{\forall j \in J, \sup_{p\in V_j} \Big[\psi_{\alpha,L}(p) + \varepsilon\xi_p\Big] \leq \Big(\sqrt{\frac{2}{\pi}} - (\eta/4)\Big)\ln\big(\sqrt{L}\big)\Big\}$$
$$\Lambda_4 = \Big\{\forall j \in J, \sup_{V_{\lfloor L^{(1-\alpha)/2}\rfloor}} \Big[\psi_{j,\alpha,L} + \varepsilon\xi\Big] \leq \Big(\sqrt{\frac{2}{\pi}} - (\eta/4)\Big)\ln\big(\sqrt{L}\big)\Big\}$$
$$\Lambda_5 = \Big\{\forall j \in J, \sup_{V_{\lfloor L^{(1-\alpha)/2}\rfloor}} \psi_{j,\alpha,L} \leq \Big(\sqrt{\frac{2}{\pi}} - (\eta/8)\Big)\ln\big(\sqrt{L}\big)\Big\}$$
$$\Lambda_6 = \Big\{\varepsilon\xi > (\eta/8)\ln\big(\sqrt{L}\big)\Big\}.$$

We have the following inclusions

$$T^+_{K,\eta,\alpha} \subset \Lambda_1 \subset \Lambda_2 \cup \Lambda_3$$
$$\Lambda_4 \subset \Lambda_5 \cup \Lambda_6.$$

Consider the two following gaussian fields defined over $\sqcup_{j\in J} V_j$. First $(\psi_{\alpha,L}(p) + \varepsilon\xi_p)_p$, then $p \mapsto \psi_{j,\alpha,L} \circ \iota^{-1}_{j,\alpha,L}(p) + \varepsilon\xi$ where $j \in J$ is such that $p \in V_j$. From equation (4.3.4), these two fields satisfy the conditions for Lemma 4.2.9 so that

$$\tilde{\mathbb{P}}(\Lambda_3) \leq \tilde{\mathbb{P}}(\Lambda_4).$$

Therefore,

$$\tilde{\mathbb{P}}\Big(T^+_{K,\eta,\alpha}\Big) \leq \tilde{\mathbb{P}}(\Lambda_1) \leq \tilde{\mathbb{P}}(\Lambda_2) + \tilde{\mathbb{P}}(\Lambda_5) + \tilde{\mathbb{P}}(\Lambda_6). \tag{4.3.5}$$

Firstly, by standard estimates on tails of gaussian variables, for L large enough,

$$\tilde{\mathbb{P}}(\Lambda_2) \leq \frac{4K\varepsilon\lfloor L^{(1-\alpha)/2}\rfloor^2}{\eta\ln\big(\sqrt{L}\big)} \exp\Big(-\frac{\eta^2}{32\varepsilon^2}\ln^2\big(\sqrt{L}\big)\Big) \tag{4.3.6}$$
$$\tilde{\mathbb{P}}(\Lambda_6) \leq \frac{8\varepsilon}{\eta\ln\big(\sqrt{L}\big)} \exp\Big(-\frac{\eta^2}{128\varepsilon^2}\ln^2\big(\sqrt{L}\big)\Big)$$

Secondly, by independence of the $\psi_{j,\alpha,L}$ and by equation (4.3.3)

$$\tilde{\mathbb{P}}(\Lambda_5) = \prod_{j\in J} \tilde{\mathbb{P}}\Big(\sup\psi_{j,\alpha,L} \leq \Big(\sqrt{\frac{2}{\pi}} - (\eta/8)\Big)\ln\big(\sqrt{L}\big)\Big).$$
$$\leq \exp\big(-aK\ln^2\big(\sqrt{L}\big)\big).$$

From equations (4.3.5) and (4.3.6), taking K large enough and ε small enough, we obtain inequality 4.3.2 and the lemma is proved. □

We now prove Lemma 4.3.7. The proof is very similar to that of Proposition 4.2.1.

Proof of Lemma 4.3.7. Firstly, note that if $0 < \delta < \delta'$ then $F_\delta \subset F_{\delta'}$. Therefore, for each $C > 0$ it is enough to find $\delta > 0$ such that $\mathbb{P}(F_\delta) \leq e^{-C \ln\left(\sqrt{L}\right)^2}$. Choose $p, q \in \Sigma$ at distance smaller or equal to $\delta L^{-\alpha/2}$. There is a smooth path γ in Σ, parametrized by arclength, such that $\gamma(0) = p$, $\gamma(2\delta L^{-\alpha/2}) = q$ and for each $0 \leq t \leq 2\delta L^{-\alpha/2}$, $|\gamma'(t)| = 1$. Thus,

$$\begin{aligned} |\phi_{L^\alpha}(p) - \phi_{L^\alpha}(q)| &\leq \int_0^{2\delta L^{-\alpha/2}} |(\phi_{L^\alpha} \circ \gamma)'(t)| dt \\ &\leq 2\delta L^{-\alpha/2} \sup_{D_{L^\alpha}(p, 2\delta)} |\nabla \phi_{L^\alpha}|. \end{aligned}$$

We now choose a local trivialisation the tangent bundle of $D_{L^\alpha}(p, 2\delta)$ in which $\nabla \phi_{L^\alpha}$ has coordinates $(\nabla^1 \phi_{L^\alpha}, \nabla^2 \phi_{L^\alpha})$. From Proposition 4.2.4, there is a constant $C > 0$ independent of p such that for any such q and for $j = 1, 2$,

$$Var(\nabla_q^j \phi_{L^\alpha}) \leq C L^\alpha.$$

With this information as well as the second inequality in Lemma 4.2.2, we apply the BTIS inequality (Proposition 4.2.3) to the random fields $(\nabla_q^j \phi_{L^\alpha}(v_q))_{q \in D_{L^\alpha}(p, 2\delta)}$ for $j = 1, 2$ and deduce there is a constant $C > 0$ such that for L large enough,

$$\mathbb{P}\Big(\sup_{D_{L^\alpha}(p, 2\delta)} |\nabla \phi_{L^\alpha}| \geq \frac{\eta L^{\alpha/2}}{4\delta} \ln\left(\sqrt{L}\right) \Big) \leq \exp\Big(- \frac{C\eta^2}{\delta^2} \ln^2\left(\sqrt{L}\right) \Big).$$

By the triangle inequality,

$$\mathbb{P}\Big(\sup_{q_1, q_2 \in D_{L^\alpha}(p, \delta)} |\phi_{L^\alpha}(q_1) - \phi_{L^\alpha}(q_2)| \geq (\eta/2) \ln\left(\sqrt{L}\right) \Big) \leq \exp\Big(- \frac{C\eta^2}{\delta^2} \ln^2\left(\sqrt{L}\right) \Big). \quad (4.3.7)$$

There exists $C > 0$ such that for each $L > 0$, there is a covering of Σ by a collection $(D_j)_{j \in J}$ of at most $CL^\alpha \delta^{-2}$ disks of radius $\delta L^{-\alpha/2}$. For each j, inequality (4.3.7) applies on D_j. Therefore,

$$\begin{aligned} \mathbb{P}\Big(\sup_{d_g(p,q) \leq \delta L^{-\alpha/2}} |\phi_{L^\alpha}(p) - \phi_{L^\alpha}(q)| \geq (\eta/2) \ln\left(\sqrt{L}\right) \Big) &\leq CL^\alpha \exp\Big(- \frac{C\eta^2}{\delta^2} \ln^2\left(\sqrt{L}\right) \Big) \\ &\leq \exp\Big(- \frac{2C\eta^2}{\delta^2} \ln^2\left(\sqrt{L}\right) \Big) \end{aligned}$$

for L large enough. This inequality ends the proof of the lemma. □

4.4 The covariance function

The aim of this section is to prove Theorem 4.1.3, Proposition 4.2.4 and Proposition 4.3.2. For each $L > 0$, let E_L be the Schwartz kernel of the orthogonal projector in $L^2(\Sigma)$ onto U_L. Then,

$$E_L(p,q) = \sum_{0<\lambda_n \leq L} \psi_n(p)\psi_n(q). \tag{4.4.1}$$

The asymptotic behavior of this kernel as $L \to \infty$ has been studied extensively by Hörmander in [Hör68] (see also [Bin04] and [GW16a]). In order to prove Theorem 4.1.3 we will express G_L in terms of E_L and use the aforementioned results to extract an explicit formula for G_L.

4.4.1 Preliminary results

We begin with the following proposition, which establishes the link between G_L and E_L.

Proposition 4.4.1. *There is a function $R \in C^\infty(M \times M, \mathbb{R})$ such that for each $L > 0$,*

$$G_L = \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda + R.$$

Proof. For any $p, q \in \Sigma$, the functions $L \mapsto E_L(p,q)$ and $L \mapsto G_L(p,q)$ define distributions over $]0, +\infty[$. In what follows, ∂_L will mean differentiation in the sense of distributions. First of all, for any $p, q \in \Sigma$

$$\partial_L E_L(p,q) = \sum_{0<\lambda_n} \psi_n(p)\psi_n(q)\delta_{\lambda_n}(L)$$
$$\partial_L G_L(p,q) = \sum_{0<\lambda_n} \frac{1}{\lambda_n}\psi_n(p)\psi_n(q)\delta_{\lambda_n}(L) = \sum_{0<\lambda_n} \frac{1}{L}\psi_n(p)\psi_n(q)\delta_{\lambda_n}(L) = \frac{1}{L}\partial_L E_L(p,q).$$

Consequently $\partial_L\Big(G_L - \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda\Big) = 0$. Therefore

$$R = G_L - \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda$$

is independent of L. The right hand side is a linear combination of functions $(p,q) \mapsto \psi_n(p)\psi_n(q)$ so it belongs to $C^\infty(M \times M)$. □

Now, we use Theorem 5.1 of [Hör68] to obtain an explicit description of the integral term in the equation of the previous proposition. Let us fix $p_0 \in \Sigma$. According to Theorem 5.1 of [Hör68] there is an open neighborhood U of p in Σ with a chart $\phi : U \to \phi(U)$ such that $(d_{p_0}\phi^*)^{-1}$ is an isometry from $T^*_{p_0}\Sigma$ with the metric induced by g to $\mathbb{R}^n$ with the euclidian metric, as well as a real valued function $\theta \in C^\infty(U \times T^*U)$ satisfying the

phase condition (see Definition 2.3 of [Hör68]) and a constant $C > 0$ such that for each $p, q \in U$ and each $L > 0$,

$$\Big| E_L(p,q) - \frac{1}{(2\pi)^2} \int_{|\xi|^2 \leq L} e^{i\theta(p,q,\xi)} d_q\eta(\xi) \Big| \leq C\sqrt{L} \tag{4.4.2}$$

where $d_q\eta$ is the measure associated to the metric induced by g on T^*U. Let $\phi_*\theta \in C^\infty(\phi(U) \times \phi(U) \times \mathbb{R}^2)$ be defined by

$$\forall x, y \in \phi(U),\ w \in \mathbb{R}^2,\ \phi_*\theta(x,y,w) = \theta(\phi^{-1}(x), \phi^{-1}(y), d_{\phi^{-1}(y)}\phi^* w).$$

Here $d_{\phi^{-1}(y)}\phi^*$ is the adjoint of the differential of ϕ at $\phi^{-1}(y)$.

Theorem 5.1 of [Hör68] provides the following information concerning θ.

1. For each $x, y \in \phi(U)$ and $w \in \mathbb{R}^2$ such that $\langle x - y, w\rangle = 0$, $\phi_*\theta(x,y,w) = 0$.
2. For each $y \in \phi(U)$ and $w \in \mathbb{R}^2$, $\partial_x(\phi_*\theta)(x,y,w)|_{x=y} = w$.

In particular, these equations have the following consequence. Choose $y \in \phi(U)$, $t \geq 0$, $v \in S^1$, $\alpha \in \mathbb{N}^2$ a multiindex and $w \in \mathbb{R}^2$. Then, by the Taylor-Young estimate applied to $\partial_w^\alpha \phi_*\theta(y + \lambda v, y, w)$ with respect λ, for λ small enough,

$$\partial_w^\alpha \phi_*\theta(y + \lambda v, y, w) = \lambda \partial_w^\alpha \langle v, w\rangle + O(\lambda^2) \tag{4.4.3}$$

where the $O(\lambda^2)$ is uniform when y and w are restricted to any compact set.

Before we proceed any further, let us introduce some notation. For each $q \in \Sigma$, let S_q be the unit circle in $(T_q^*\Sigma, g_q)$ and $d_q\nu$ the measure induced by the restriction of g_q to S_q. Also, for each $y \in \phi(U)$, let $\tilde{S}_y$ be

$$\tilde{S}_y = \{w \in \mathbb{R}^2 |\ d_{\phi^{-1}(y)}\phi^* w \in S_{\phi^{-1}(y)}\}. \tag{4.4.4}$$

Lemma 4.4.2. *For each $t > 0$ and $p, q \in U$, let*

$$J(p,q,t) = \int_{S_q} e^{i\theta(p,q,t\omega)} d_q\nu(\omega). \tag{4.4.5}$$

Then, there is a constant $C > 0$ such that for each $L > 0$ and $p, q \in U$,

$$\Big| \int_1^L E_\lambda(p,q)\lambda^{-2} d\lambda - \frac{1}{4\pi^2}\Big(\int_1^{\sqrt{L}} J(p,q,t) t^{-1} dt - L^{-1} \int_1^{\sqrt{L}} J(p,q,t) t dt \Big) \Big| \leq C.$$

Proof. First of all, from equation (4.4.2) there is $C > 0$ such that for all p, q and L,

$$\begin{aligned} \Big| \int_1^L E_\lambda(p,q)\lambda^{-2} d\lambda - \frac{1}{4\pi^2} \int_1^L \lambda^{-2} \int_{|\xi|^2 \leq \lambda} e^{i\theta(p,q,\xi)} d_p\eta(\xi) d\lambda \Big| &\leq C \int_1^L \lambda^{-3/2} d\lambda \\ &\leq 2C. \end{aligned} \tag{4.4.6}$$

Now, applying the polar change of coordinates $(t, \omega) \mapsto t\omega = \xi$,

$$\int_{|\xi|^2 \leq \lambda} e^{i\theta(p,q,\xi)} d_p\eta(\xi) = \int_0^{\sqrt{\lambda}} J(p,q,t)tdt.$$

Next, we apply the change of variables $u = \sqrt{\lambda}$.

$$\int_1^L \lambda^{-2} \int_0^{\sqrt{\lambda}} J(p,q,t)tdtd\lambda = \int_1^{\sqrt{L}} \int_0^u J(p,q,t)tdt2u^{-3}du.$$

We split the inner integral in two $\int_0^u = \int_0^1 + \int_1^u$. Note that the integral from 0 to 1 has lost any dependence on u or L. Since $\int_0^\infty u^{-3/2}du < \infty$ that term is bounded. Therefore applying Fubini's theorem,

$$\begin{aligned}
\frac{1}{4\pi^2} \int_1^L \lambda^{-2} \int_0^{\sqrt{\lambda}} J(p,q,t)tdtd\lambda &= \frac{1}{4\pi^2} \int_1^{\sqrt{L}} \int_1^u J(p,q,t)tdt2u^{-3}du \\
&= \frac{1}{4\pi^2} \int_1^{\sqrt{L}} J(p,q,t)t \int_t^{\sqrt{L}} 2u^{-3}dudt \\
&= \frac{1}{4\pi^2} \Big(\int_1^{\sqrt{L}} J(p,q,t)t^{-1}dt - L^{-1} \int_1^{\sqrt{L}} J(p,q,t)tdt \Big).
\end{aligned}$$

Together with equation (4.4.6) this implies that

$$\int_1^L E_\lambda(p,q)\lambda^{-2}d\lambda - \frac{1}{4\pi^2} \Big(\int_1^{\sqrt{L}} J(p,q,t)t^{-1}dt - L^{-1} \int_1^{\sqrt{L}} J(p,q,t)tdt \Big)$$

is bounded as required. □

To proceed any further, we need to control the behavior of $J(p,q,t)$ when $t \to \infty$. We will use the stationary phase method to prove the following proposition.

Proposition 4.4.3. *There exist $V \subset U$ an open neighborhood of p_0 and a constant $C > 0$, such that for all $p, q \in V$ and $t \in [0, +\infty[$,*

$$|J(p,q,t)| \leq \frac{C}{\sqrt{1 + d_g(p,q)t}}. \tag{4.4.7}$$

The function J is an oscillatory integral over the circle. To obtain the bound for large t uniformly with respect to p, q distinct, we should apply the stationary phase method to the phase $w \mapsto \frac{\phi_*\theta(x,y,w)}{|x-y|}$ with parameter $|x-y|t$ where ϕ is the chart appearing just above equation (4.4.2) and $\phi_*\theta$ is defined just below it. In order to obtain these bounds we will first apply the adequate change of variables in order to compactify the space of definition near the diagonal.

Proof of Proposition 4.4.3. Let $K \subset \phi(U)$ be a compact neighborhood of $\phi(p)$. Since K is compact, there exists a constant $\alpha > 0$ such that for each $x \in K$, $\overline{D(x,\alpha)} \subset \phi(U)$. Here $D(x,\alpha)$ is the open disk centered at x and of radius α. Let us define the following sets.

$$A = \{(x,y,w) \mid x,y \in K,\ x \neq y,\ |x-y| < \alpha,\ w \in \tilde{S}_y\}$$
$$B = S^1 \times]0,\alpha[\times \{(y,w) \in K \times \mathbb{R}^2 \mid y \in K,\ w \in \tilde{S}_y\}.$$

where $\tilde{S}_y$ is defined in equation (4.4.4). The following map is a diffeomorphism

$$f : A \to B$$
$$(x,y,w) \mapsto \Big(\frac{x-y}{|x-y|}, |x-y|, y, w\Big).$$

Set

$$\psi : A \to \mathbb{R}$$
$$(x,y,w) \to \frac{\phi_*\theta(x,y,w)}{|x-y|}.$$

Now, applying equation (4.4.3) with $\alpha = 0$ we deduce that for any $(v,\lambda,y,w) \in B$,

$$\psi \circ f^{-1}(v,\lambda,y,w) = \langle v,w\rangle + O(\lambda)$$

where $O(\lambda)$ is uniform with respect to v,y,w. Thus, $\Psi := \psi \circ f^{-1}$ extends by continuity to $\overline{B}$ so that for all $y \in K$, $w \in \tilde{S}_y$ and $v \in S^1$, $\Psi(v,0,y,w) = \langle v,w\rangle$. Equation (4.4.3) also shows that the map

$$\lambda \mapsto \Big((v,y,w) \mapsto \Psi(v,\lambda,y,w)\Big)$$

is continuous from $[0,\alpha]$ to the space of continuous functions on $S^1 \times \{(y,w) \in K \times \mathbb{R}^2 \mid w \in \tilde{S}_y\}$ which are C^∞ with respect to w. Let us momentarily fix v and set $y = \phi(p_0)$. The curve $\tilde{S}_y$ is a circle since $d_{p_0}\phi^{-1}$ is an isometry. Hence, the map $w \mapsto \Psi(v,0,y,w) = \langle v,w\rangle$ defined over $\tilde{S}_y$ is a Morse function with two critical points. Thus, for λ small enough and y close enough to $\phi(p_0)$, $w \mapsto \Psi(v,\lambda,y,w)$ is also a Morse function with two critical points. Moreover, it depends C^4-continuously on v, t, and y. Therefore there exists $\beta > 0$ and a constant $C > 0$ such that for any $x,y \in D(\phi(p_0),\beta)$ distinct we have the following.

- $\psi_{x,y} : w \mapsto \psi(x,y,w)$ is a Morse function with two critical points $z_1(x,y)$ and $z_2(x,y)$.
- For $j = 1,2$, $\big|det(d^2_{z_j(x,y)}\psi_{x,y})\big| > C^{-1}$ (Here $d^2_{z_j(x,y)}\psi_{x,y}$ is the hessian of $\psi_{x,y}$ which is well defined since $z_j(x,y)$ are critical points, see for instance the definition following Lemma 1.6 of [Nic11]).

- $\|\psi_{x,y}\|_{C^4} \leq C$.

Consequently, $u = 1$ and $f = \psi_{x,y}$ satisfy the hypotheses of Theorem 7.7.5 of [**?**] with $k = 1$, from which we deduce that there is $C > 0$ such that for these same x, y and λ,

$$J(\phi^{-1}(x), \phi^{-1}(y), \lambda) \leq \frac{C}{\sqrt{|x-y|\lambda}}.$$

Moreover, for all p, q and t, $|J(p,q,t)| \leq \int_{S_q} 1 d_q\nu = 2\pi$. Let $V = \phi^{-1}(D(\phi(p_0, \beta))$. From the two last inequalities, there is a constant $C > 0$ such that for any $p, q \in V$ and $t \geq 0$,

$$J(p,q,t) \leq \frac{C}{\sqrt{1 + d_g(p,q)t}}.$$

□

4.4.2 Proof of Theorem 4.1.3

We now use the results and notations of the previous section to prove Theorem 4.1.3. We will estimate G_L when p and q are in a neighborhood of p_0 and use the compactness of Σ to make the result global.. Firstly, from Proposition 4.4.1, $G_L = \frac{E_L}{L} + \int_1^L \frac{E_\lambda}{\lambda^2} d\lambda + O(1)$. From equation (4.4.2), $L^{-1}E_L$ is bounded. Now, from Lemma 4.4.2, there is $C > 0$ such that for each $p, q \in U$

$$\Big| \int_1^L E_\lambda(p,q)\lambda^{-2} d\lambda - \frac{1}{4\pi^2}\Big(\int_1^{\sqrt{L}} J(p,q,t)t^{-1}dt - L^{-1}\int_1^{\sqrt{L}} J(p,q,t)tdt\Big)\Big| \leq C.$$

Let $V \subset U$ be as in Proposition 4.4.3. From equation (4.4.7), there is a constant $C > 0$ such that for all $p, q \in V$,

$$\Big| L^{-1}\int_1^{\sqrt{L}} J(p,q,t)tdt\Big| \leq C.$$

Note that, from equation (4.4.3), $\theta(p,q,0) = 0$ so that $J(p,q,0) = 2\pi$. Choose $p, q \in U$ distinct and set $r = d_g(p,q)$. Then, applying the change of variables $a = rt$,

$$\begin{aligned}\int_1^{\sqrt{L}} J(p,q,t)t^{-1}dt &= \int_r^{r\sqrt{L}} J(p,q,r^{-1}a)a^{-1}da \\ &= 2\pi\int_r^{r\sqrt{L}} \frac{\mathbf{1}_{a\leq 1}}{a}da + \int_r^{r\sqrt{L}} \frac{J(p,q,r^{-1}a) - 2\pi\mathbf{1}_{a\leq 1}}{a}da\end{aligned}$$

where $\mathbf{1}_{a\leq 1}$ equals 1 if $a \leq 1$ and 0 otherwise. Recall the notation introduced in equation (4.4.4). Now, from equation (4.4.3), for $0 < a \leq r$ and $x, y \in \phi(V)$, uniformly for $w \in \tilde{S}_y$,

$$\phi_*\theta(x, y, (a/r)w) = \frac{\langle x - y, w\rangle a}{r} + O(|x-y|^2 a/r)$$

so $f : a \mapsto \frac{J(p,q,r^{-1}a)-2\pi \mathbf{1}_{a\leq 1}}{a}$ admits a continuous extension at $a = 0$ equal to

$$\int_{S_q} \frac{i\langle \phi(p) - \phi(q), (d_q\phi^*)^{-1}\omega\rangle}{r} d_q\nu(\omega).$$

Let $W \subset V$ be an open neighboorhood of p_0 on which ϕ is bi-lipschitz and recall that $r = d_g(p, q)$. Then, there exists $C > 0$ independent of L such that for all $p, q \in W$, $|\phi(p) - \phi(q)| \leq Cr$ and such that for all $q \in W$ and $\omega \in S_q$, $\|(d_q\phi^*)^{-1}\omega\|_{eucl} \leq C$. (Here, $\| \|_{eucl}$ denotes the euclidean norm on $\mathbb{R}^2$.) Thus, the aforementioned continuous extension of f is uniformly bounded for $p, q \in W$. Moreover, by equation (4.4.7), f is $O(a^{-3/2})$ uniformly with respect to r when $a \to \infty$. Therefore it is integrable with uniform bounds over p and q. Consequently, for any distinct $p, q \in W$ such that $d_g(p, q) < 1$,

$$\begin{aligned}
\int_1^{\sqrt{L}} \lambda^{-2} E_\lambda d\lambda &= \frac{1}{2\pi} \int_{d_g(p,q)}^{d_g(p,q)\sqrt{L}} \frac{\mathbf{1}_{a\leq 1}}{a} da + O(1) \\
&= \frac{1}{2\pi}\Big(\ln\big(\min(1, \sqrt{L}d_g(p, q))\big) - \ln\big(\min(1, d_g(p, q))\big)\Big) + O(1) \\
&= \frac{1}{2\pi}\Big(\ln\big(\sqrt{L}\big) - \ln_+\big(\sqrt{L}d_g(p, q)\big)\Big) + O(1).
\end{aligned}$$

Here, the bounds implied by the O's are uniform with respect to p and q. For the last equality we use the fact that $\ln_+\big(d_g(p, q)\big)$ is bounded for $p, q \in W$. The case $p = q$ follows by continuity. Moreover, by compactness, one can cover Σ with a finite number of such W's so that there is a constant $\varepsilon > 0$ independent of L for which the constants are uniform with respect to any p, q such that $d_g(p, q) < \varepsilon$.

4.4.3 Proof of Propositions 4.2.4 and 4.3.2

Proof of Proposition 4.2.4. From Theorem 2.3 of [GW16a] with $M = \Sigma$ and $P = \Delta$, there is a constant $C > 0$ such that for all $p \in \Sigma$ and $L \geq 0$,

$$|(Q_1 \otimes Q_2)E_L(p, p)| \leq C\big(1 + L^{1+d/2}\big)$$

Therefore, from Proposition 4.4.1 for all $p \in \Sigma$ and $L > 0$,

$$\begin{aligned}
|(Q_1 \otimes Q_2)G_L(p, p)| &\leq L^{-1}|(Q_1 \otimes Q_2)E_L(p, p)| + \\
&\quad + \int_1^L \lambda^{-2}|(Q_1 \otimes Q_2)E_\lambda(p, p)|d\lambda + |(Q_1 \otimes Q_2)R(p, p)| \\
&\leq C\Big(1 + L^{d/2} + \int_1^L \lambda^{-1+d/2} d\lambda\Big) \\
&\leq C'\big(1 + L^{d/2}\big).
\end{aligned}$$

□

In order to prove Proposition 4.3.2, we will need the following technical result, which we deduce from Theorem 5.1 of [Hör68].

Lemma 4.4.4. *For each $\beta > 0$ and $\delta > 0$, there is $C > 0$ such that for each $L, \lambda > 0$ and $p, q \in \Sigma$ such that $d_g(p,q) \geq \delta L^{-\beta/2}$,*

$$\big|E_\lambda(p,q)\big| \leq CL^{\beta/4}\lambda^{3/4}.$$

Proof. First of all, by Theorem 5.1 of [Hör68], there is $\varepsilon > 0$ and C such that for each $p_0 \in \Sigma$ and $q \in \Sigma$,

1. If $d_g(p_0,q) \geq \varepsilon$ then, for all $\lambda > 0$, $\big|E_\lambda(p_0,q)\big| \leq C\sqrt{\lambda}$ (see equation (5.4) of [Hör68]).

2. There is a chart (ϕ, U) such that $D(p_0,\varepsilon) \subset U$, around p_0 as well as a function θ such that equations (4.4.3) and (4.4.2) hold with the same constant $C > 0$.

From 1. we need only deal with the case where p_0 and q are ε-close. Now, by a polar coordinate change, and with the same notations as in Lemma 4.4.2, for all $\lambda > 0$,

$$\int_{|\xi|^2 \leq \lambda} e^{i\theta(p_0,q,\xi)} d_q\eta(\xi) = \int_0^{\sqrt{\lambda}} J(p_0,q,t)tdt.$$

By equation (4.4.7) there is $C > 0$ independent of p_0 and q such that for all $\lambda > 0$ and all $q \neq p_0$,

$$\Big|\int_0^{\sqrt{\lambda}} J(p_0,q,t)tdt\Big| \leq C\frac{\lambda^{3/4}}{\sqrt{d_g(p_0,q)}}.$$

This concludes the proof. □

Proof of Proposition 4.3.2. From Proposition 4.4.1,

$$G_{L^\alpha,L} = G_L - G_{L^\alpha} = \frac{E_L}{L} - \frac{E_{L^\alpha}}{L^\alpha} + \int_{L^\alpha}^{L} \frac{E_\lambda}{\lambda^2} d\lambda.$$

Now from Lemma 4.4.4 there is a constant $C > 0$ such that for any $L > 0$ and any $p, q \in \Sigma$ such that $d_g(p,q) \geq \delta L^{-\beta/2}$,

$$\begin{aligned}
\big|L^{-1}E_L(p,q)\big| &\leq CL^{(\beta-1)/4} \\
\big|L^{-\alpha}E_{L^\alpha}(p,q)\big| &\leq CL^{(\beta-\alpha)/4} \\
\Big|\int_{L^\alpha}^{L} \frac{E_\lambda}{\lambda^2} d\lambda\Big| &\leq CL^{\beta/4}\int_{L^\alpha}^{L} \lambda^{-5/4} d\lambda \leq 2CL^{(\beta-\alpha)/4}
\end{aligned}$$

and each term tends to 0 since $0 < \beta < \alpha < 1$. □

4.5 The case of a surface with boundary

In this paper, we studied a random zero mean field over a compact surface Σ with $\partial\Sigma = \emptyset$. The bulk of our results remain valid in some sense if Σ has a smooth boundary $\partial\Sigma \neq \emptyset$. The zero mean condition is then replaced by a Dirichlet zero boundary condition. In this section we state definitions and results for this setting and explain the minor modifications needed in each proof. Be advised that **we use the same notations as before to denote slightly different objects**.

4.5.1 Definitions

Let (Σ, g) be a smooth surface with smooth boundary $\partial\Sigma$ equipped with a riemannian metric. We will denote the interior of Σ by $\mathring{\Sigma} = \Sigma \setminus \partial\Sigma$. If $E(\Sigma)$ is a topological vector-space of functions on Σ of which the $C^\infty(\Sigma)$ is a dense subspace, $E_0(\Sigma)$ will denote the closure in $E(\Sigma)$ of smooth functions with compact support in $\mathring{\Sigma}$. We begin by defining the COGFF on Σ. As in the introduction, the bilinear form $\langle u, v\rangle_\nabla := \int_\Sigma g(\nabla u, \nabla v)|dV_g|$ defines a scalar product on $H_0^1(\Sigma)$. From Theorem 4.43 of [GHL04] there exist $(\psi_n)_{n\geq 1} \in C_0^\infty(\Sigma)^{\mathbb{N}}$ and $(\lambda_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$ such that $0 < \lambda_1 \leq \lambda_2 \ldots, \lambda_n \xrightarrow[n\to\infty]{} +\infty$, such that $(\psi_n)_{n\geq 1}$ is a Hilbert basis for $L_0^2(\Sigma)$ and such that for each $n \in \mathbb{N}$, $\Delta\psi_n = \lambda_n\psi_n$. Then $\left(\frac{1}{\sqrt{\lambda_n}}\psi_n\right)_{n\geq 1}$ is a Hilbert basis of $(H_0^1, \langle\ ,\ \rangle_\nabla)$. For each $L > 0$, let $(U_L, \langle\ ,\ \rangle_\nabla)$ be the subspace of $(H_0^1, \langle\ ,\ \rangle_\nabla)$ spanned by the functions $\frac{1}{\sqrt{\lambda_n}}\psi_n$ such that $\lambda_n \leq L$. Let $(\xi_n)_{n\in\mathbb{N}}$ be an i.i.d sequence of centered gaussians with variance one. Then, for each $L > 0$ we define the COGFF on Σ as

$$\phi_L = \sum_{\lambda_n \leq L} \frac{\xi_n}{\sqrt{\lambda_n}}\psi_n. \tag{4.5.1}$$

For each $L > 0$ and $p, q \in \Sigma$, let $G_L(p, q) = \mathbb{E}[\phi_L(p)\phi_L(q)]$.

Let D be an open subset of $\mathring{\Sigma}$ at positive distance from $\partial\Sigma$. Then, we define the capacity of D relative to Σ as the infimum of $\frac{1}{2}\|\nabla h\|^2$ over all $h \in C_0^\infty(\mathring{\Sigma})$ and denote it by $cap_\Sigma(D)$ or $cap(D)$. If D is non-empty and has smooth boundary, we will see that $cap(D) > 0$. Note that in this setting, the capacity is conformally invariant. More precisely, consider (Σ_1, g_1) and (Σ_2, g_2) are two riemannian surfaces with smooth boundary such that there exists a conformal isomorphism, $f : \Sigma_1 \to \Sigma_2$. Let $D_1 \subset \Sigma_1$ and $D_2 = f(D_1)$. We claim that $cap_{\Sigma_1}(D_1) = cap_{\Sigma_2}(D_2)$. Indeed, for each $h \in C_0^\infty(\Sigma_2)$ such that $h \geq 1$ on D_2, we have $h \circ f \in C_0^\infty(\Sigma_1)$ and $h \circ f \geq 1$ on D_1. The same is true for f^{-1} if we exchange 1 and 2 so f defines a bijection between the sets whose infimum define the capacities. Moreover, since f is a conformal map, for any $p \in \Sigma_1$ and $v \in T_p\Sigma_1$, $g_2(d_pfv, d_pfv) = |det(d_pf)|g_1(v, v)$ where $det(d_pf)$ is the determinant of the matrix of d_pf written in orthonormal bases. It follows that from the change of variables formula that for any $h \in C_0^\infty(\Sigma_2)$, $\|h\|_\nabla = \|h \circ f\|_\nabla$. This proves the claim.

4.5.2 Main results

Theorem 4.5.1. *Let (Σ, g) be a smooth compact surface with smooth non-empty boundary and let $(\phi_L)_{L>0}$ be the COGFF on (Σ, g). Let D be a non-empty open subset of $\mathring{\Sigma}$ with smooth boundary and positive distance to $\partial\Sigma$. Then,*

$$\lim_{L\to\infty} \frac{\ln\big(\mathbb{P}(\forall x \in D, \phi_L(x) > 0)\big)}{\ln^2\big(\sqrt{L}\big)} = -\frac{2}{\pi} cap(D).$$

The only change in section 4.3, which contains the heart of the argument, is in the proof of Proposition 4.6.2. Though the statement is the same, Σ has a boundary and the definition of $cap(D)$ has changed. For a proof in this case we refer the reader to Lemma 2.1 of [BD93] where one should simply replace V by Σ and $\langle f, \phi\rangle_V - \frac{1}{2}\langle f, K_V f\rangle_V$ by $\frac{1}{2\sigma(\mathbf{1}_D f)}\Big(\int_D f\Big)^2$.

Theorem 4.5.2. *Let (Σ, g) be a compact riemannian surface with smooth non-empty boundary and let $(\phi_L)_{L>0}$ be the COGFF on (Σ, g). For each $L > 0$ and $p, q \in \Sigma$, let*

$$G_L(p, q) = \mathbb{E}[\phi_L(p)\phi_L(q)].$$

Then, there exists $\varepsilon > 0$ such that for each $p, q \in \mathring{\Sigma}$ satisfying $d_g(p, q) \leq \varepsilon$ and for each $L > 0$,

$$G_L(p, q) = \frac{1}{2\pi}\Big(\ln\big(\sqrt{L}\big) - \ln_+\big(\sqrt{L} d_g(p, q)\big)\Big) + \rho_L(p, q)$$

where, for any compact subset $K \subset \mathring{\Sigma}$, $\rho_L(p, q)$ is uniformly bounded for $L > 0$ and $p, q \in K$.

For the proof of Theorem 4.5.2 we fix a compact subset $K \subset \mathring{\Sigma}$ and proceed as in the original proof but replace the statements "uniform with respect to $p, q \in \Sigma$" by "uniform with respect to $p, q \in K$". This works because Theorem 5.1 of [Hör68] is valid on non-compact manifolds except the bound on the remainder term is only uniform on compact sets.

Theorem 4.5.3. *Let (Σ, g) be a smooth compact riemannian surface with non-empty boundary and let $(\phi_L)_{L>0}$ be the COGFF on (Σ, g). Let D be a non-empty open subset of $\mathring{\Sigma}$ at positive distance of $\partial\Sigma$ and K be a compact subset of $\mathring{\Sigma}$. Then for each $\eta > 0$,*

$$\limsup_{L\to\infty} \frac{\ln\Big(\mathbb{P}\Big(\sup_K \phi_L > \Big(\sqrt{\tfrac{2}{\pi}} + \eta\Big)\ln\big(\sqrt{L}\big)\Big)\Big)}{\ln\big(\sqrt{L}\big)} \leq -2\sqrt{2\pi}\eta + O(\eta^2)$$

and there exists $a > 0$ such that for L large enough,

$$\mathbb{P}\Big(\sup_D \phi_L \leq \Big(\sqrt{\frac{2}{\pi}} - \eta\Big)\ln\big(\sqrt{L}\big)\Big) \leq \exp\Big(-a\ln^2\big(\sqrt{L}\big)\Big).$$

The proof of this theorem is essentially the same as the original. In the statement of Lemma 4.2.2, one should introduce a compact $K \subset \mathring{\Sigma}$. The constant C is then uniform for $p \in K$. In Proposition 4.2.1 the supremum is taken over K and in the proof one should cover K by small disks instead of Σ.

4.6 Appendix

In section we recall some classical results from spectral theory used in the article and give an alternate characterization of the capacity which we use in Lemma 4.3.6.

4.6.1 Classical spectral theory results

In the following discussion we adopt the notations introduced in section 4.1.3. We begin by proving an approximation result for eigenfunctions of the laplacian.

Lemma 4.6.1. *For any integers $h, k \geq 1$, the vector space spanned by the sequence $(\psi_n)_{n\geq 1}$ is dense in $C_0^h(\Sigma)$ and in $H_0^k(\Sigma)$.*

Proof. By the Sobolev inequalities (see for instance, Theorem 5.6 (ii) of [Eva10]), for any h there exists k_0 such that for $k \geq k_0$, $H_0^k \subset C_0^h$ and the inclusion is continuous. On the other hand, if $h \geq k$, $C_0^h \subset H_0^k$ and the inclusion is continuous. Therefore, it suffices to prove that the space spanned by the $(\psi_n)_{n>0}$ is dense in all the H_0^k. By Garding's inequality (see Theorem (1), section 8 of [Tay74]), for any $k \geq 1$, there is $C > 0$ such that the scalar product

$$(u \mid v)_k := \langle u, (C + \Delta^k)v\rangle_2$$

is equivalent to the standard H_0^k scalar product. Let $f \in C_0^\infty$ and suppose that for any $n \geq 1$, $(f \mid \psi_n)_k = 0$. Then

$$0 = (f \mid \psi_n)_k = \langle f, (C + \Delta^k)\psi_n\rangle_2 = (C + \lambda_n^k)\langle f, \psi_n\rangle_2.$$

Since $f \in L_0^2$ and $(\psi_n)_{n\geq 1}$ is dense in L_0^2, $f = 0$. Since C_0^∞ is dense in H_0^k, we conclude that so is $(\psi_n)_{n\geq 1}$. □

Now we discuss the functional calculus of the laplacian. This will be useful in the alternate characterization of the capacity. The operator Δ is a symmetric differential operator of order 2 so it defines a bounded operator from $H^2(\Sigma)$ to $L^2(\Sigma)$ which, according to its spectral decomposition (see section 4.1.3) and by Lemma 4.6.1, defines an isomorphism $\Delta : H_0^2 \to L_0^2$. Let Δ^{-1} be its inverse, which we extend to $L^2(\Sigma)$ by setting $\Delta^{-1}f = 0$ for any constant function f. Since Δ is symmetric, Δ^{-1} is self-adjoint. Moreover, for any $n \geq 1$, $\Delta^{-1}\psi_n = \lambda_n^{-1}\psi_n$. For any $f \in L^2(\Sigma)$, we denote by $\sigma(f)$ the quadratic form $\sigma(f) := \langle f, \Delta^{-1}f\rangle_2$. By Parseval's formula,

$$\sigma(f) = \sum_{n\geq 1} \frac{1}{\lambda_n}\langle \psi_n, f\rangle_2^2. \tag{4.6.1}$$

Finally, since Δ and Δ^{-1} are positive and Δ is a differential operator, then $\Delta : C_0^\infty \to C_0^\infty$ and $\Delta^{-1} : C_0^\infty \to C_0^\infty$ both admit square roots $\Delta^{1/2}$ and $\Delta^{-1/2}$ respectively, which are symmetric pseudo-differential operators of orders 1 and -1 (see [See67]). These

define bounded operators

$$\Delta^{1/2} : H^2 \to H^1$$
$$\Delta^{1/2} : H^1 \to L^2$$
$$\Delta^{-1/2} : L^2 \to L^2.$$

Finally, by construction, these operators restrict to H^k_0 and L^2_0 and for any $f \in H^2_0$, $\Delta^{-1/2}\Delta^{1/2} f = f$.

4.6.2 The capacity

For any subset $D \subset \Sigma$ that is not dense in Σ, there exist functions $h \in C^\infty_0(\Sigma)$ such that $h \geq 1$ on D. We define **the capacity** of D the infimum over all such h of $\frac{1}{2}\|\nabla h\|_2^2$ and denote it by $cap(D)$. This is a variation on the relative capacity used in [BD93] and [BDG01]. Similarly to Lemma 2.1 of [BD93], in the case where D is a non-empty proper open set with smooth boundary we have the following alternative characterization of the capacity.

Proposition 4.6.2. *Let D be a non-empty proper open subset of Σ with smooth boundary. Then,*

$$cap(D) = \sup\Big\{ \frac{1}{2\sigma(\mathbf{1}_D f)} \Big(\int_D f \Big)^2 \;\Big|\; f \in C^\infty(\Sigma)\ f \geq 0 \text{ on } D. \Big\}$$

In the proof of this proposition we will use the maximum principle and existence of solutions to certain elliptic PDEs. For this purpose, recall that we defined the laplacian to be $\Delta = d^* d$.

Proof of Proposition 4.6.2. Let $h \in C^\infty_0(\Sigma)$ such that for each $x \in D$, $h(x) \geq 1$. Let $f \in C^\infty(\Sigma)$, non-negative on D.

$$\begin{aligned}\Big(\int_D f \Big)^2 \leq \Big(\int_D hf \Big)^2 &= \langle \mathbf{1}_D f, h\rangle_2^2 = \langle \mathbf{1}_D f, \Delta^{-1/2}\Delta^{1/2} h\rangle_2^2 = \langle \Delta^{-1/2}\mathbf{1}_D f, \Delta^{1/2} h\rangle_2^2 \\ &\leq \langle \Delta^{-1/2}\mathbf{1}_D f, \Delta^{-1/2}\mathbf{1}_D f\rangle_2 \langle \Delta^{1/2} h, \Delta^{1/2} h\rangle_2 \text{ by Cauchy-Schwarz} \\ &= \langle \mathbf{1}_D f, \Delta^{-1}\mathbf{1}_D f\rangle_2 \langle h, \Delta h\rangle_2 = \sigma(\mathbf{1}_D f)\|\nabla h\|_2^2.\end{aligned}$$

Passing to the infimum and supremum, since D is non-empty, we deduce that

$$\sup\Big\{ \frac{1}{2\sigma(\mathbf{1}_D f)} \Big(\int_D f \Big)^2 \mid f \in C^\infty(\Sigma)\ f \geq 0 \text{ on } D. \Big\} \leq cap(D).$$

We now show that one can find f and h so as to make the above functionals arbitrarily close to each other. The previous Cauchy-Schwarz inequality is close to equality when $\Delta^{-1}\mathbf{1}_D f$ is close to h. Recall that we defined Δ^{-1} to be zero when applied to constant

functions. This means that we must find f so that $\mathbf{1}_D f - \Delta h$ is close to a constant. We now define a would-be minimizer h. Let $\tau > 0$ and h be a continuous function constant equal to 1 on D, smooth on $U = \Sigma \setminus \overline{D}$ such that Δh is equal to $-\tau$ on $\Sigma \setminus \overline{D}$ (see Theorems 5 of section 6.2 and 3 of section 6.3 of [Eva10] for the existence of h). Then h is smooth up to the boundary of U (see Theorem 6 of section 6.3 of [Eva10]). We denote by $\partial_\nu h$ the outward normal of h, defined on ∂U and by $d\sigma$ the volume density induced by g on $\partial U = \partial D$. Since $\Delta h \leq 0$ on U, by the maximum principle, $h < 1$ on U. Therefore, there exists τ such that h has zero mean. Let Δh be the laplacian of h in the sense of distributions, which is well defined since h is continuous. Before approximating h, we must find an explicit expression for Δh. For any $u \in C^\infty(\Sigma)$, by Stokes' theorem,

$$\begin{aligned}
\langle \Delta h, u\rangle_2 =& \langle h, \Delta u\rangle_2 = \int_D h\Delta u |dV_g| + \int_U h\Delta u |dV_g| \\
=& \int_D \langle \nabla h, \nabla u\rangle_g |dV_g| + \int_{\partial D} h(-\partial_\nu u) d\sigma \\
&+ \int_U \langle \nabla h, \nabla u\rangle_g |dV_g| + \int_{\partial U} h\partial_\nu u d\sigma \\
=& \int_U \langle \nabla h, \nabla u\rangle_g |dV_g| = \int_U u\Delta h |dV_g| + \int_{\partial U} u\partial_\nu h d\sigma \\
=& \int_{\partial U} (\partial_\nu h) u d\sigma - \tau \int_U u |dV_g|.
\end{aligned}$$

Therefore,

$$\langle \Delta h + \tau, u\rangle_2 = \tau \int_D u|dV_g| + \int_{\partial(\Sigma\setminus D)} (\partial_\nu h) u d\sigma.$$

Again, by the maximum principle, $\partial_\nu h$ is positive. Let $(\tilde{h}_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth functions with zero mean converging uniformly to h in Σ that coincide with h on $\Sigma \setminus D$. Then, for each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $h_\varepsilon = (1 + \delta(\varepsilon))\tilde{h}_\varepsilon \geq 1$ on D. Let $\tau_\varepsilon = (1 + \delta(\varepsilon))\tau$. Then, in the sense of distributions, $f_\varepsilon = \Delta h_\varepsilon + \tau_\varepsilon$ is non-zero and non-negative on Σ, and vanishes on $\Sigma \setminus D$. Since it is smooth, it satisfies these properties in the classical sense as well. Moreover

$$\begin{aligned}
(\int_D f_\varepsilon)^2 \geq& (\sup_D h_\varepsilon)^{-2} (\int_D f_\varepsilon h_\varepsilon)^2 \\
=& (\sup_D h_\varepsilon)^{-2} \langle f_\varepsilon, h_\varepsilon\rangle_2^2 = (\sup_D h_\varepsilon)^{-2} \langle f_\varepsilon, \Delta^{-1} f_\varepsilon\rangle_{L^2} \langle h_\varepsilon, \Delta h_\varepsilon\rangle_2 \\
=& (\sup_D h_\varepsilon)^{-2} \sigma(f_\varepsilon) ||\nabla h_\varepsilon||_2^2 = (\sup_D h_\varepsilon)^{-2} \sigma(f_\varepsilon \mathbf{1}_D) ||\nabla h_\varepsilon||_2^2.
\end{aligned}$$

Therefore,

$$cap(D) \leq \frac{1}{2} ||\nabla h_\varepsilon||_2^2 \leq \frac{(\sup_D h_\varepsilon)^2}{2\sigma(f_\varepsilon \mathbf{1}_D)} \Big(\int_D f_\varepsilon \Big)^2.$$

Since $\sup_D h_\varepsilon \xrightarrow[\varepsilon\to 0]{} 1$, this concludes the proof of the lemma. □

Chapter 5

A lower bound for Bogomolny and Schmit constant for random monochromatic plane waves

This chapter is based on [IR18], which is joint work with Maxime Ingremeau.

5.1 Introduction

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the stationary, isotropic planar centered almost surely continuous Gaussian field[1] with covariance $\mathbb{E}[f(x)f(y)] = J_0(|x-y|)$ where $J_0(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin(\theta)}d\theta$ is the 0th Bessel function of the first kind. We call this field **the random monochromatic plane wave**. For each $R > 0$, let us denote by $B(0, R)$ the ball of centre 0 and of radius R, and by $N(R, f)$ the number of connected components of $\mathbb{R}^2 \backslash f^{-1}(\{0\})$ included in $B(0, R)$.
In [NS09], [NS16], Nazarov and Sodin showed that the limit

$$\nu_{BS} = 4\pi \times \lim_{R\to\infty} \frac{\mathbb{E}N(R, f)}{\pi R^2} \tag{5.1.1}$$

exists, and is positive. However, their method does not give an explicit value for the constant ν_{BS}, sometimes called the *Bogomolny-Schmit constant* (or also Nazarov-Sodin constant). An equivalent definition for ν_{BS} is that it is the limit of the average number of nodal domains for a random spherical harmonic divided by the degree of this spherical harmonic. The factor 4π can be interpreted as the area of the unit sphere. Bogomolny and Schmit gave in [BS02] a highly heuristical argument, based on percolation, which yielded the value

$$\nu_{BS} \simeq 0.0624.$$

However, numerical simulations carried out by Nastasescu ([Nas11]), Konrad ([Kon12]) and Beliaev and Kereta ([BK13]) showed that $\nu_{BS} \simeq 0.0589$, contradicting Bogomolny and Schmit's prediction by a few percents. Experimental measurements of ν_{BS} were also realised by Kuhl, Höhmann, Stöckmann and Gnutzmann ([**?**]), using Berry's conjecture ([Ber77]) that high frequency eigenmodes in a chaotic cavity should locally behave like the monochromatic random wave f. From the mathematical point of view, though, very little is known regarding the value of ν_{BS}. The best rigorous lower bound so far was the one given in [Nas11], which is of the order of 10^{-319}. The aim of this note is obtain a much better lower bound by elementary means:

Theorem 5.1.1.

$$\nu_{BS} \geq 1.39 \times 10^{-4}.$$

This bound is much smaller than the expected value of ν_{BS}. However, our method does not take into account all nodal domains, but only those which are included in circles of radius 3.8 (the first minimum of the Bessel function J_0). After visual inspection of computer simulations we expect that these are not very common. Aside from this, in our use of Lemma 5.2.2, we ignore the fact that small, isolated nodal domains should be somewhat rare. We hope that our methods can be used to count more general nodal domains, and to obtain sharper lower bounds on ν_{BS}.

Before proving Theorem 5.1.1 let us recall two basic facts about random monochromatic plane waves.

[1] We refer the reader to [**?**] for the definition and properties of random fields.

- The function f almost surely satisfies $\Delta f + f = 0$ where Δ is the Laplace operator. This follows by applying the Laplace operator with respect to x to the expression $\mathbb{E}[f(x)f(y)] = J_0(|x-y|)$ because J_0 is also an eigenfunction of the Laplace operator.

- In polar coordinates, f can be given the simple expression

$$f(r,\theta) = X_0 J_0(r) - \sqrt{2}\sum_{n\geq 1} J_n(r)\left(X_n\cos(n\theta) + Y_n\sin(n\theta)\right). \tag{5.1.2}$$

  where for each $n \in \mathbb{Z}$ we denote by J_n the nth Bessel function of the first kind, that is $J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta - x\sin(\theta)}d\theta$, and where $(X_k)_{k\geq 0}$ and $(Y_k)_{k\geq 1}$ are two families of centred Gaussian random variables with unit variance and independent as a whole. An explanation for this fact can be found for instance in Section 4.2 of [**?**].

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5.2 Proof of Theorem 5.1.1

We now prove Theorem 5.1.1. The idea is to consider points such that f does not vanish on a certain circle around this point. We start by producing a lower bound for the probability that 0 is such a point.

Lemma 5.2.1. *Fix $r > 0$ and write $\Psi : t \mapsto \int_t^{+\infty} e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}$. Let $\mathcal{E}_r$ be the event that the zero set of f does not intersect the circle of radius r. Then, for each $T \in \mathbb{R}$,*

$$\mathbb{P}\left[\mathcal{E}_r\right] \geq 2\Psi(T) - \sqrt{2}r\Psi\left(\frac{T}{\sqrt{1-J_0(r)^2}}\right).$$

Proof. In this proof we use the expression (5.1.2) for f. Fix $r > 0$ and for each $\theta \in [0, 2\pi]$, let $u(\theta) = X_0 J_0(r) - f((r\cos(\theta), r\sin(\theta))$. We first fix $x_0 > 0$ and try to estimate the probability that $u(\theta)$ crosses the level $x_0 J_0(r)$ when θ varies on the unit circle. Also, throughout our calculations, we will use the following two Bessel function identities:

$$J_{-n}(x) = (-1)^n J_n(x); \sum_{n\in\mathbb{Z}} J_n(x)^2 = 1; \sum_{n\in\mathbb{Z}} n^2 J_n(x)^2 = \frac{x^2}{2}. \tag{5.2.1}$$

These identities follow from the following classical formula (cf. [**?**, Chapter 2])

$$\sum_{n\in\mathbb{Z}} t^n J_n(x) = e^{\frac{x}{2}(t-t^{-1})}$$

by setting $t = e^{i\theta}$ and applying Parseval's formula.
Now, observe that, since f is stationary, isotropic and $(f, \nabla f)$ is non-degenerate, u is a stationary Gaussian field on the unit circle and the field (u, u') is non-degenerate so that u crosses the level $x_0 J_0(r)$ almost surely an even number of times. In particular, by Markov's inequality,

$$\mathbb{P}\left[\exists\theta;\, u(\theta) = x_0 J_0(r)\right] \leq \frac{1}{2}\mathbb{E}\left[\mathrm{Card}\{\theta \in [0, 2\pi];\, u(\theta) = x_0 J_0(r)\}\right] .$$

To compute the right-hand side of this inequality, we apply the Kac-Rice formula (see Theorem 6.2 from [AW09]). Define $\alpha(r) > 0$ by $\alpha(r)^2 = \mathrm{Var}(u(\theta)) = 2\sum_{n\geq 1} J_n(r)^2$. Then, by the first and second identity in (5.2.1), $\alpha(r)^2 = 1 - J_0(r)^2$. By the Kac-Rice formula,

$$\mathbb{E}\left[\mathrm{Card}\{\theta \in [0, 2\pi];\, u(\theta) = x_0 J_0(r)\}\right] = \int_0^{2\pi} \mathbb{E}\left[|u'(\theta)| \,|\, u(\theta) = x_0 J_0(r)\right] \frac{e^{-\frac{x_0^2 J_0(r)^2}{2\alpha(r)^2}}}{\alpha(r)\sqrt{2\pi}} d\theta .$$

The fact that u is stationary, implies first that the integrand is independent of θ and second, that $u'(\theta)$ is independent of $u(\theta)$. Observe that $\mathrm{Var}\left(u'(\theta)\right) = 2\sum_{n\geq 1} n^2 J_n(r)^2$. By the first and third identities in (5.2.1), $\mathrm{Var}(u'(\theta)) = \frac{r^2}{2}$. Moreover, if ξ is a centred Gaussian of variance one, $\mathbb{E}\left[|\xi|\right] = \sqrt{\frac{2}{\pi}}$. Therefore,

$$\mathbb{E}\left[\mathrm{Card}\{\theta \in [0, 2\pi];\, u(\theta) = x_0 J_0(r)\}\right] = \frac{\sqrt{2}r}{\alpha(r)} \exp\left(-\frac{x_0^2 J_0(r)^2}{2\alpha(r)^2}\right) .$$

On the other hand, we have

$$\mathbb{P}[\neg\mathcal{E}_r;\, X_0 = x_0] = \mathbb{P}\big[\exists\theta \in [0, 2\pi];\, u(\theta) = x_0 J_0(r)\big] \leq \mathbb{E}\left[\mathrm{Card}\{\theta \in [0, 2\pi];\, u(\theta) = x_0 J_0(r)\}\right]$$

In particular, for each $T \in \mathbb{R}$, the probability that f has a zero on the circle of radius r centred at 0 satisfies

$$\mathbb{P}\left[\mathcal{E}_r\right] \geq \mathbb{E}\left[\left(1 - \frac{r}{\sqrt{2}\alpha(r)} \exp\left(-\frac{X_0^2 J_0(r)^2}{2\alpha(r)^2}\right)\right) \mathbb{1}_{[|X_0|\geq T]}\right] . \tag{5.2.2}$$

Let $\Psi(t) = \mathbb{P}[X_0 \geq t] = \int_t^{+\infty} e^{-t^2/2} \frac{t}{\sqrt{2\pi}}$. Then, the right-hand side is

$$2\Psi(T) - 2\frac{r}{\sqrt{2}\alpha(r)}\mathbb{E}\left[\exp\left(-\frac{X_0^2 J_0(r)^2}{2\alpha(r)^2}\right) \mathbb{1}_{[X_0\geq T]}\right] .$$

But for each $a > 0$, a simple calculation shows that $\mathbb{E}\left[\exp(-aX_0^2)\mathbb{1}_{[X_0\geq T]}\right] = \frac{1}{\sqrt{1+2a}}\Psi\left(\sqrt{1+2a}T\right)$ so

$$\mathbb{P}\left[\mathcal{E}_r\right] \geq 2\Psi(T) - \frac{\sqrt{2}r}{\sqrt{\alpha(r)^2 + J_0(r)^2}}\Psi\left(\sqrt{1 + J_0(r)^2/\alpha(r)^2}T\right)$$

Replacing $\alpha(r)$ by its expression, we get, for any $T > 0$ and any $r > 0$,

$$\mathbb{P}\left[\mathcal{E}_r\right] \geq 2\Psi(T) - \sqrt{2}r\Psi\left(\frac{T}{\sqrt{1 - J_0(r)^2}}\right). \tag{5.2.3}$$

□

Let $G_r = G_r(f) \subset \mathbb{R}^2$ be the (random) set of points $x \in \mathbb{R}^2$ for which f does not vanish on the circle of radius r centred at x. If $x, y \in \mathbb{R}^2$, we will write $x \sim_r y$ if $x, y \in G_r$ and x, y belong to the same connected component of $\mathbb{R}^2 \backslash f^{-1}(\{0\})$. The next step of the proof is to show that the connected components of G_r are not too large.

Lemma 5.2.2. *Let r_1 and r_2 denote the first and second zeros of J_0 and let $r \in]r_1, r_2[$. Then, for each $x \in G_r$, the connected component of x in $\mathbb{R}^2 \backslash f^{-1}(\{0\})$ is included in $B(x, r)$. In particular, if $x, y \in G_r$ are such that $x \sim_r y$, then $|x - y| \leq r$.*

Remark 5.2.3. The result of this lemma may be optimal, deterministically speaking. But, we expect equivalence classes of G_r of diameter close to r to be very rare. We probably lose a large factor in this step. However, this intuition seems difficult to quantify.

Proof of Lemma 5.2.2. In this proof, we use the fact that f satisfies $\Delta f + f = 0$ almost surely. Without loss of generality, we may suppose that $f(x) > 0$. We claim that $f(z) < 0$ for all z such that $|x - z| = r$. We already know that $f(z)$ has constant sign for all z such that $|x - z| = r$. Consider the function

$$g(z) := \frac{1}{2\pi} \int_0^{2\pi} f(R_\theta z) \mathrm{d}\theta,$$

where R_θ is the rotation of centre x and of angle θ. The function g satisfies $(\Delta + 1)g = 0$ (because $f(R_\theta \cdot)$ does for each $\theta \in [0, 2\pi]$), and it is radially symmetric around x. Therefore, we must have

$$g(z) = \lambda J_0(|z - x|)$$

for some $\lambda \neq 0$. Since $f(x) > 0$, we must have $\lambda > 0$, and hence, $f(z) < 0$ for all z such that $|x - z| = r$. Therefore, the connected component of $\mathbb{R}^2 \backslash f^{-1}(\{0\})$ containing x is included in $B(x, r)$ and the Lemma follows. □

We now finish off the proof of Theorem 5.1.1 by estimating the expected size of G_r using Lemma 5.2.1. Let $0 < r_1 < r_2$ be the first two zeros of the Bessel function J_0 and fix $r \in]r_1, r_2[$. Then, by Lemma 5.2.2, for each $x \in G_r$, the equivalence class of x has diameter at most r. By the isodiametric inequality (see paragraph 10 of [**?**]) its area is no greater than $\frac{\pi}{4} r^2$. Also, two different equivalence classes are included in different connected components of $\mathbb{R}^2 \backslash f^{-1}(\{0\})$. Finally, if $R > r$ and $x \in B(0, R - r)$, then the connected component of $\mathbb{R}^2 \setminus f^{-1}(\{0\})$ containing x is included in $B(0, R)$. Thus, for each $R > r$,

$$\mathrm{Vol}\left(G_r \cap B(0, R - r)\right) \leq \sum_c \mathrm{Vol}(c) \leq \frac{\pi}{4} r^2 \times N(R, f)$$

where the sum runs over the equivalence classes of G_r intersecting $B(0, R-r)$. Taking expectations, by stationarity, we get

$$\mathrm{Vol}\left(B(R-r)\right)\mathbb{P}\left[0 \in G_r\right] \leq \frac{\pi}{4} r^2 \times \mathbb{E}\left[N(R, f)\right] .$$

Dividing by $\mathrm{Vol}\left(B(0,R)\right) = \pi R^2$ and letting $R \to +\infty$, we get $\mathbb{P}\left[0 \in G_r\right] \leq \frac{1}{16} r^2 \nu_{BS}$ which yields the following lower bound for the Bogomolny-Schmit constant:

$$\nu_{BS} \geq \frac{16\mathbb{P}[0 \in G_r]}{r^2} .$$

By Lemma 5.2.1, we have, for each $T > 0$ and each $r > r_0$,

$$\nu_{BS} \geq \frac{32}{r^2} \left[\Psi(T) - \frac{r}{\sqrt{2}} \Psi\left(\frac{T}{\sqrt{1 - J_0(r)^2}} \right) \right] .$$

Taking $r = 3.8$ (the first minimum of J_0) and $T = 3.35$ (the smallest T for which in (**??**), the function whose expectation we compute is always positive), we get

$$\frac{32}{r^2} \geq 2.216 ; \ \frac{T}{\sqrt{1 - J_0(r)^2}} \geq 3.659 ; \ \frac{r}{\sqrt{2}} \leq 2.69 .$$

We therefore obtain the announced lower bound

$$\nu_{BS} \geq 1.39 \times 10^{-4} .$$

Chapter 6

Quasi-independence for nodal lines

Abstract

We prove a quasi-independence result for level sets of a planar centered stationary Gaussian field with covariance $(x, y) \mapsto \kappa(x - y)$, with only mild conditions on the regularity of κ. As a first application, we study percolation for nodal lines in the spirit of [BG17a]. In the said article, Beffara and Gayet rely on Tassion's method ([Tas16]) to prove that, under some assumptions on κ, most notably that $\kappa \geq 0$ and $\kappa(x) = O(|x|^{-325})$, the nodal set satisfies a box-crossing property. The decay exponent was then lowered to $16 + \varepsilon$ by Beliaev and Muirhead in [BM18]. In the present work we lower this exponent to $4 + \varepsilon$ thanks to a new approach towards quasi-independence for crossing events. This approach does not rely on quantitative discretization. Our quasi-independence result also applies to events counting nodal components and we obtain a lower concentration result for the density of nodal components around the Nazarov and Sodin constant from [NS16].

This chapter is based on [RV17a], which is joint work with Hugo Vanneuville.

6.1 Introduction

In this article, we prove a quasi-independence result for level lines of planar Gaussian fields and present two applications of this result. First, we use it to revisit and generalize the results by Gayet and Beffara [BG17a] who initiated the study of large scale connectivity properties for nodal lines and nodal domains of planar Gaussian fields. Second, we apply it to the study of the concentration of the number of nodal lines around the Nazarov and Sodin constant (the constant ν of Theorem 1 of [NS16]). Let f be a planar centered Gaussian field. The **covariance function** of f is the function $K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$\forall x, y \in \mathbb{R}^2,\ K(x,y) = \mathbb{E}[f(x)f(y)]\,.$$

We assume that f is normalized so that for each $x \in \mathbb{R}^2$, $K(x,x) = \mathrm{Var}(f(x)) = 1$, that it is non-degenerate (i.e. for any pairwise distinct $x_1, \dots, x_k \in \mathbb{R}^2$, $(f(x_1), \cdots, f(x_k))$ is non-degenerate), and that it is a.s. continuous and stationary. In particular, there exists a strictly positive definite continuous function $\kappa : \mathbb{R}^2 \to [-1,1]$ such that $\kappa(0) = 1$ and, for each $x, y \in \mathbb{R}^2$, $K(x,y) = \kappa(x-y)$. We will also refer to κ as covariance function when there is no possible ambiguity. For each $p \in \mathbb{R}$ we call **level set** of f the random set $\mathcal{N}_p := f^{-1}(-p)$ and **excursion set** of f the random set $\mathcal{D}_p := f^{-1}([-p, +\infty[)$.[1] Let us first state our result regarding planar box-crossing properties.

Box crossing estimates for planar Gaussian fields. In [BG17a], the authors give conditions under which such sets satisfy a box-crossing property at $p = 0$. We say that random sets satisfy a box-crossing property if for any quad (i.e. a topological rectangle with two opposite distinguished sides) $\mathcal{Q}$ there exists a positive constant c such that for any (potentially sufficiently large) scale s, there is a crossing of $s\mathcal{Q}$ between distinguished sides by the random set with probability larger than c. The study of the case $p = 0$ is natural since this is the level at which duality arises, see for instance Remark 6.A.11 in our appendix. The most important conditions asked in [BG17a] were some symmetry conditions, the fact that f is positively correlated (which means that the covariance function κ takes only non-negative values) and a sufficiently fast decay for $\kappa(x)$ as $|x|$ does to $+\infty$, namely $\kappa(x) = O\left(|x|^{-325}\right)$. In [BM18], Beliaev and Muirhead have lowered the exponent 325 to any $\alpha > 16$. In the present paper, we lower this exponent to any $\alpha > 4$, thus obtaining the following result:

Theorem 6.1.1. *Assume that f is a non-degenerate, centered, normalized, continuous, stationary, positively correlated planar Gaussian field that satisfies the symmetry assumption Condition 6.1.8 below. Assume also that κ satisfies the differentiability assumption Condition 6.1.10 below and that $\kappa(x) \leq C|x|^{-\alpha}$ for some $C < +\infty$ and $\alpha > 4$. Let $\mathcal{Q}$ be a quad, i.e. a simply connected bounded open subset of $\mathbb{R}^2$ whose boundary $\partial\mathcal{Q}$ is piecewise smooth boundary with two distinguished disjoint segments on $\partial\mathcal{Q}$. Then, there exists $c = c(\kappa, \mathcal{Q}) > 0$ such that for each $s \in]0, +\infty[$, the probability that there is*

[1] This convention, while it may seem counterintuitive, is convenient because it makes $\mathcal{D}_p$ increasing both in f and in p. See [RV17b].

a continuous path in $\mathcal{D}_0 \cap s\mathcal{Q}$ joining one distinguished side to the other is at least c. Moreover, there exists $s_0 < +\infty$ such that the same result holds for $\mathcal{N}_0$ as long as $s \geq s_0$.

Lowering the exponent α below 4, if at all possible, would require new ideas (see Remark 6.1.13). This result is the analog of the Russo-Seymour-Welsh theorem for planar percolation from [Rus78, SW78], see also Lemma 4 of Chapter 3 of [BR06b], Theorem 11.70 and Equation 11.72 of [Gri99] or Theorem 5.31 of [Gri10]. For more about the links between connectivity properties of nodal lines and domains and percolation, see [MS83a, MS83b, MS86], [Ale96], [BS07], [BG17a], [BM18], [BMW17], [RV17b]. Box-crossing estimates have previously been extended to some other dependent models, see [BR06a, DCHN11, Tas16, ATT16] and also to some non-planar models, see [BS15, NTW17]. It seems also relevant to mention the recent work [BG17b], in which the authors prove that the box-crossing property is stable by perturbations for sufficiently decorrelated discrete Gaussian fields. In particular, they obtain analogs of Theorem 6.1.1 for many discrete Gaussian fields that are not positively associated.

The result analogous to Theorem 6.1.1 in [BG17a] is Theorem 4.9. In [BM18], this is Theorem 1.7. Note that our assumptions about the differentiability and the non-degeneracy of κ are different from those in [BG17a] and [BM18]. Still, we see them essentially as technical conditions, whereas the question of the optimal exponent α seems to be of much more interest.

While our proof differs from the one in [BG17a, BM18] in some key steps, the initial idea is the same, i.e. the use of Tassion's general method to prove box-crossing estimates which goes back to [Tas16]. Let us first be a little more precise about the proof in [BG17a, BM18]. The three main ingredients are: i) a quantitative version of Tassion's method (see Section 2 of [BG17a], ii) a quasi-independence result for finite dimensional Gaussian fields (see Theorem 4.3 of [BG17a] and Proposition C.1 of [BM18]) and iii) a quantitative approximation result (see Theorem 1.5 of [BG17a] and Theorems 1.3 and 1.5 of [BM18]). Steps i) and ii) imply a discrete version of a RSW theorem and Step iii) is then used to deduce a RSW theorem for the continuous model. The most important contribution of [BM18] is an improvement of the approximation result. Another way to prove the box-crossing property is to use prove a quasi-independence in the continuum and then apply Tassion's method (not necessarily in a quantitative way). This strategy was also suggested in [BMW17], where Beliaev, Muirhead and Wigman prove a box-crossing estimate for random Gaussian fields on the sphere and the torus. More precisely, they used analogs of steps ii) and iii) above to prove such a quasi-independence result, see their Proposition 3.4. In the present work, we also prove a quasi-independence result in the continuum (see Theorem 6.1.12) and then apply Tassion's method. However, the way we prove such a quasi-independence result is very different from [BMW17]. In particular, *we do not rely on any quantitative approximation result* and we rather prove a quasi-independence result *uniform in the discretization mesh* (see Proposition 6.3.4). Moreover, our techniques, together with the quantitative adaptation of [Tas16] presented in [BG17a] yield a uniform discrete RSW-estimate without any constraints on the mesh

(see Proposition 6.B.2). This result is quite handy when using discrete techniques to study continuous fields, see for instance [RV17b]. The proof of Theorem 6.1.1 is written in Section 6.4 by relying only on our Sections 6.2 and 6.3 (but not on Subsection 6.3.4) and on [Tas16]. For other works relying on Tassion's method for box crossing estimates, see [ATT16, DCTT16].

Before stating our quasi-independence results, let us state our result regarding the concentration of the number of nodal components of planar Gaussian fields.

A concentration from below around the Nazarov and Sodin constant for the number of nodal components. In [NS09], Nazarov and Sodin prove that, if g is a random spherical harmonic of degree n on the 2-dimensional sphere and if $N_0(n)$ is the number of nodal components (i.e. connected components of the 0-level set) of g, then there exists a constant $c_{NS} \in]0, +\infty[$ such that, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) < +\infty$ and $c = c(\varepsilon) > 0$ such that for every $n \in \mathbb{N}$:

$$\mathbb{P}\left[\left|\frac{N_0(n)}{n^2} - c_{NS}\right| \geq \varepsilon\right] \leq C \exp(-cn)\,. \tag{6.1.1}$$

In other words, the number of nodal components divided by n^2 concentrates exponentially around a constant. In [NS16], the same authors consider a much larger family of fields and obtain the much more general following result but without concentration.

Theorem 6.1.2 (Theorem 1 of [NS16])**.** *Assume that f is a normalized, continuous, stationary planar Gaussian field which satisfies the spectral hypotheses Condition 6.1.11 below. Then, there exists a constant $c_{NS} = c_{NS}(\kappa) \in]0, +\infty[$ such that, if $N_0(s)$ is the number of connected components of the nodal set $\mathcal{N}_0$ contained in the box $[-s/2, s/2]^2$, then $N_0(s)/s^2$ goes to c_{NS} as s goes to $+\infty$ a.s. and in L^1.*

Remark 6.1.3. Their result is actually more general: they obtain a result for families of Gaussian fields on manifolds with translation-invariant local limits (see Subsection 1.2 of [NS16]).

Theorem 6.1.2 and the quasi-independence results of the present paper enable us to obtain a concentration result **from below** of $N_0(s)/s^2$ around c_{NS}:

Theorem 6.1.4. *Assume that f is a normalized, continuous, stationary and non-degenerate planar Gaussian field which satisfies the spectral hypotheses Condition 6.1.11 below and the differentiability assumption Condition 6.1.10 below. With the same notations as Theorem 6.1.2, we have the following:*

1. *if there exists $C < +\infty$ and $c > 0$ such that for every $x \in \mathbb{R}^2$ we have $|\kappa(x)| \leq C \exp(-c|x|^2)$, then for every $\varepsilon > 0$ there exists $C_0 = C_0(\kappa, \varepsilon) < +\infty$ and $c_0 = c_0(\kappa, \varepsilon)$ such that for each $s \in]0, +\infty[$:*

$$\mathbb{P}\left[\frac{N_0(s)}{s^2} \leq c_{NS} - \varepsilon\right] \leq C_0 \exp(-c_0 s)\,;$$

2. *if there exists* $C < +\infty$ *and* $\alpha > 4$ *such that for every* $x \in \mathbb{R}^2$ *we have* $|\kappa(x)| \leq C|x|^{-\alpha}$*, then for every* $\delta > 0$ *and every* $\varepsilon > 0$*, there exists* $C_0 = C_0(\kappa, \alpha, \delta, \varepsilon) < +\infty$ *such that for each* $s \in]0, +\infty[$*:*

$$\mathbb{P}\left[\frac{N_0(s)}{s^2} \leq c_{NS} - \varepsilon\right] \leq C_0 s^{4-\alpha+\delta}\,.$$

An important example of a Gaussian field which satisfies the decorrelation hypothesis of Item 1 above is the **Bargmann-Fock field** which is the analytic Gaussian field : $\mathbb{R}^2 \to \mathbb{R}$ with covariance function $(x, y) \in (\mathbb{R}^2)^2 \mapsto \kappa(x - y) = \exp\left(-\frac{1}{2}|x - y|^2\right)$. In some sense, this field is the local limit of the **Kostlan polynomials** which are random homogeneous polynomials on the sphere which arise naturally from real algebraic geometry, see for instance the introduction of [BG17a] or that of [BMW17]. The analogue of Theorem 6.1.2 is known for these polynomials (see [NS16]), but the concentration inequality (6.1.1) is not known (neither from below nor from above). There are however two relevant results in this direction. The first, Corollary 1.10 of [Let16b], proves that the probability that there are no components in a prescribed region decays polynomially fast. The second, Theorem 1 of [GW11], deals with the other extreme and proves that polynomials of degree $d >> 1$ whose number of nodal components is maximal up to a linear term in d are exponentially rare in d. We hope that the proof of Theorem 6.1.4 can be adapted in order to get the lower concentration part of (6.1.1) with $n = \sqrt{d}$ for Kostlan polynomials of degree $d >> 1$.

Remark 6.1.5. In [NS16], the authors obtain Theorem 6.1.2 in any dimension. We believe that our techniques could be extended to higher dimensions (probably with additional technicalities).

Remark 6.1.6. As explained in the paragraph above about RSW results and as suggested in [BMW17], another way of obtaining quasi-independence results for nodal lines of planar Gaussian fields is to use the quasi-independence results for finite dimensional vectors and the quantitative discretization results, both from [BG17a, BM18]. One could probably deduce Theorem 6.1.4 from either [BG17a] or [BM18], though with slightly different assumptions, and more to the point, with a weaker Item 2 (more precisely, we believe that the exponent in the right hand side would be $16 - \alpha + \delta$ instead.

Before stating our quasi-independence results, we list the conditions on the Gaussian fields under which we work in this article.

Conditions on the planar Gaussian fields. We will assume that Condition 6.1.7 is true in all the present paper. Then, Condition 6.1.8 will be useful to apply classical percolation arguments, Conditions 6.1.9 and 6.1.10 will be useful to obtain quasi-independence results, and finally Conditon 6.1.11 is the assumptions by Nazarov and Sodin to obtain their convergence result.

Condition 6.1.7. The field f is non-degenerate (i.e. for any pairwise distinct $x_1, \dots, x_k \in \mathbb{R}^2$, $(f(x_1), \cdots, f(x_k))$ is non-degenerate), centered, normalized, continuous, and stationary. In particular, there exists a strictly positive definite continuous function $\kappa : \mathbb{R}^2 \to [-1, 1]$ such that $K(x, y) := \mathbb{E}[f(x)f(y)] = \kappa(y - x)$ and $\kappa(0) = 1$.

Condition 6.1.8 (Useful to apply percolation arguments.)**.** The field f is positively correlated, invariant by $\frac{\pi}{2}$-rotation, and reflection through the horizontal axis.

Condition 6.1.9 (Useful to have quasi-independence. Depends on a parameter $\alpha > 0$.)**.** There exists $C < +\infty$ such that for each $x \in \mathbb{R}^2$, $|\kappa(x)| \leq C|x|^{-\alpha}$.

Condition 6.1.10 (Technical conditions to have quasi-independence.)**.** The function κ is C^8 and for each $\beta \in \mathbb{N}^2$ with $\beta_1 + \beta_2 \leq 2$, $\lim_{x\to\infty} \partial^\beta \kappa(x) = 0$.

Condition 6.1.11 (Condition from [NS16])**.** Let ρ be the spectral measure of f which exists by Bochner's theorem (see [NS16]). Then: i) $\int_{\mathbb{R}^2} |\lambda|^4 \rho(d\lambda) < +\infty$, ii) ρ has no atom, iii) ρ is not supported on a linear hyperplane and iv) there exists a compactly supported signed measure μ whose support is included in the support of ρ and a bounded domain $D \subseteq \mathbb{R}^2$ such that $\mathcal{F}(\mu)$ (the Fourier transform of μ) restricted to ∂D is non-positive and there exists $u_0 \in D$ such that $\mathcal{F}(\mu)(u_0) > 0$.

Note that, in the case of the Bargmann-Fock field, the spectral measure is simply a standard Gaussian measure, so this field satisfies Condition 6.1.11 (for the case iv), see Appendix C of [NS16]). Moreover, f is not degenerate since the Fourier transform of a continuous and integrable function : $\mathbb{R}^2 \to \mathbb{R}_+$ which is not 0 is strictly positive definite, see for instance Theorem 3 of Chapter 13 of [CL09] (which is the strictly positive definite version of the easy part of Bochner theorem). Finally, the Bargmann-Fock field satisfies all the conditions above (and for every $\alpha > 0$).

The quasi-independence result. Theorem 6.1.12 below is our quasi-independence result for level lines of planar Gaussian fields. We first need a few more notations. Consider the following setup: let $k_1, k_2 \in \mathbb{Z}_{>0}$ and let $(\mathcal{E}_i)_{1\leq i\leq k_1+k_2}$ be a collections of either rectangles of the from $[a, b] \times [c, d]$ for some $a \leq b$ and $c \leq d$ or annuli of the form $x + [-a, a]^2 \backslash]-b, b[^2$ for some $x \in \mathbb{R}^2$ and $a \geq b$. We say that a rectangle is crossed from left to right above (resp. below) $-p$ if there is a continuous path in $\mathcal{D}_p$ (resp. $\mathcal{D}_p^c$) included in this rectangle that joins its left side to its right side. Of course, an analogous definition holds for top-bottom crossings. Moreover, we say that there is a circuit above (resp. below) $-p$ in an annulus if there is circuit included in $\mathcal{D}_p$ (resp. $\mathcal{D}_p^c$) included in this annulus that separates its inner boundary from its outer boundary. Furthermore, for each $i \in \{1, \cdots, k_1 + k_2\}$, we let $N_p(i)$ denote the number of connected components of the level set $\mathcal{N}_p$ which are included in $\mathcal{E}_i$. We write $\mathcal{K}_1 = \cup_{i=1}^{k_1} \mathcal{E}_i$, $\mathcal{C}_1 = \cup_{i=1}^{k_2} \partial\mathcal{E}_i$, $\mathcal{K}_2 = \cup_{j=k_1+1}^{k_1+k_2} \mathcal{E}_j$, and $\mathcal{C}_2 = \cup_{j=k_1+1}^{k_1+k_2} \partial\mathcal{E}_j$.

Theorem 6.1.12. *Let f be a Gaussian field satisfying Conditions 6.1.7 and 6.1.10 and consider the above setup. There exist $d = d(\kappa) < +\infty$ and $C = C(\kappa) < +\infty$ such*

that we have the following: let $p \in \mathbb{R}$. Let A (resp. B) be an event in the σ-algebra generated by the crossings above $-p$ and below $-p$ of rectangles among the $(\mathcal{E}_i)_{1\leq i\leq k_1}$ (resp. $(\mathcal{E}_j)_{k_1+1\leq j\leq k_1+k_2}$), the circuits above $-p$ and below $-p$ in annuli among the $(\mathcal{E}_i)_{1\leq i\leq k_1}$ (resp. $(\mathcal{E}_j)_{k_1+1\leq j\leq k_1+k_2}$) and the variables $N_p(i)$ for $i \in \{1, \cdots, k_1\}$ (resp. $i \in \{k_1+1, \cdots, k_1+k_2\}$). Let $\eta = \sup_{x\in\mathcal{K}_1, y\in\mathcal{K}_2} |\kappa(x-y)|$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are at distance greater than d, then:

$$|\mathbb{P}[A\cap B] - \mathbb{P}[A]\mathbb{P}[B]| \leq \frac{C\,\eta}{\sqrt{1-\eta^2}}(1+|p|)^4\, e^{-p^2} \prod_{i=1}^{2} (\mathrm{Area}(\mathcal{K}_i) + \mathrm{Length}(\mathcal{C}_i) + k_i) \ .$$

Note that in Theorem 6.1.12 we can consider crossing of rectangles (and similarly circuit in annuli) by **level** lines. Indeed, by Remark 6.A.11, given a rectangle and for each $p \in \mathbb{R}$, a.s. there is a crossing of a rectangle included in $\mathcal{N}_p$ if and only if there is such a crossing above $-p$ and a crossing below $-p$. The proof of Theorem 6.1.12 follows a perturbative technique applied to a discrete approximaion of our model (see Section 6.2). To quantify the perturbation we control certain "pivotal" events using geometric techniques and the Kac-Rice formula (see Section 6.3).

Remark 6.1.13. If the perimeter of each of the rectangles and annuli of Theorem 6.1.12 is at most s, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are at distance more than s and if $\kappa(x) = O\left(|x|^{-\alpha}\right)$ then the right-hand-side of the estimates of Theorem 6.1.12 is:

$$O\left(s^{4-\alpha}\left(1+\frac{k_1+k_2}{s}+\frac{k_1k_2}{s^2}\right)\right) = O\left(k_1k_2s^{4-\alpha}\right) \ ,$$

uniformly in p as $s \to +\infty$ with k_1 and k_2 fixed. Here we see how our condition $\alpha > 4$ from Theorems 6.1.1 and 6.1.4 appears: 4 equals 2 times the dimension. It seems that it would require new ideas to cross this value.

Remark 6.1.14. After the elaboration of this manuscript, the following works were brought to our attention:

- Piterbarg's mixing inequality (see for instance Theorem 1.2 of [Pit96]). This inequality is a more general version of our Proposition 6.2.4 below. We have chosen to keep it in the main body of the proof because we interpret and present it with a different point of view. See also Remark 6.2.5.

- An almost independence result from [NSV07, NSV08, NS10]. In Theorem 3.1 of [NS10] (see also Theorem 3.2 of [NSV07] and Lemma 5 of [NSV08]), the authors derive a quasi-independence result for Gaussian entire functions. The result states roughly that a Gaussian entired function f, when restricted to a disjoint union of compact subsets of $\mathbb{C}$ not too large and far enough from each other, can be realized as a sum of independent copies of itself on each compact subset and a small perturbation. While the result is proved only for Gaussian entire functions, we believe it could apply to general Gaussian fields with sufficient decorrelation and regularity properties. To deduce a result similar to our Theorem 6.1.12 from

Theorem 3.1 of [NS10], one would need to understand how a perturbation of the field affects the events that we consider.

Remark 6.1.15. At least one of the terms Length($\mathcal{C}_i$) and k_i on the right-hand-side of the inequality in Theorem 6.1.12 must be present for the inequality to hold. Indeed, in their absence, we would have a quasi-independence result uniform in the choice (and number) of rectangles involved in the events A and B as long as these rectangles stay within prescribed sets $\mathcal{K}_1$ and $\mathcal{K}_2$. Moreover the excursion set $\mathcal{D}_p$ is measurable with respect to the σ-algebra generated by the crossings of rectangles. Hence, we would have obtained the following result: let $\mathcal{K}_1, \mathcal{K}_2$ be two open subsets of the plane far enough from each other, let $p \in \mathbb{R}$ and let A (resp. B) be an event measurable with respect to the excursion set $\mathcal{D}_p \cap \mathcal{K}_1$ (resp. $\mathcal{D}_p \cap \mathcal{K}_2$). Also, let $\eta = \sup_{x\in\mathcal{K}_1, y\in\mathcal{K}_2} |\kappa(x-y)|$ and assume that $\eta \leq 1/2$, then:

$$|\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]| \leq C' \, \eta \, \mathrm{Area}(\mathcal{K}_1) \, \mathrm{Area}(\mathcal{K}_2) \, . \tag{6.1.2}$$

But this cannot be true in full generality. Indeed, let f be the Bargmann-Fock field[2] described above, that is, the analytic Gaussian field with covariance $K(x,y) = e^{-\frac{1}{2}|x-y|^2}$. Then it is easy to see that f satisfies Conditions 6.1.7 and 6.1.10 so Theorem 6.1.12 applies. For each $s \in]0, +\infty[$, let A_s (resp. B_s) be the event that there is a continuous path in $\mathcal{N}_0$ from $\partial[-s,s]^2$ (resp. $\partial[-3s,3s]^2$) to $\partial[-4s,4s]^2$. But f is analytic and $\mathcal{N}_0$ is a.s. smooth (see Lemma 6.A.9) so A_s is measurable[3] with respect to $\mathcal{D}_0 \cap [-2s,2s]^2$. On the other hand, B_s is measurable with respect to $\mathcal{D}_0 \cap ([-4s,4s]^2 \setminus]-3s,3s[^2)$. But A_s implies B_s. Hence, if Equation (6.1.2) were valid, we would have

$$O\left(s^4 e^{-s^2/2}\right) = |\mathbb{P}\left(A_s \cap B_s\right) - \mathbb{P}\left(A_s\right)\mathbb{P}\left(B_s\right)| = \mathbb{P}\left(A_s\right)\mathbb{P}\left(B_s^c\right) \, .$$

But the Bargmann-Fock field satisfies the hypotheses of Theorem 6.1.1 so both A_s and B_s^c have probability bounded from below as $s \to +\infty$.

Extension of the above results. We believe that Theorem 6.1.12 above can be extended, in at least three directions. First, intead of considering rectangles and square annuli, one could consider quads (i.e. topological rectangles) and more general annuli. It seems that the treatment of the phenomena at the boundary will add new technical difficulties and we believe that, if we considered quads with piecewise smooth boundaries, then we might have obtained the same estimate as in Theorem 6.1.12 but with the following right hand side:

$$\frac{C\,\eta}{\sqrt{1-\eta^2}}(1+|p|)^4 \, e^{-p^2} \prod_{i=1}^{2} \left(\mathrm{Area}(\mathcal{K}_i) + \int_{\mathcal{C}_i} (1 + |\mathbf{k}|(t)) dt + k_i \right) \, ,$$

[2]For more information concerning the Bargmann-Fock field, we refer the reader to [BG17a].

[3]Indeed, a connected component of $\mathcal{N}_0$ is a deterministic function of any segment of this component by unique analytic continuation and by the analytic implicit function theorem.

where dt is the length measure on the boundaries of the quads and $|\mathbf{k}|$ is the curvature (which is a Dirac mass at non-smooth points).

A second extension would be an extension to higher dimensions. We believe that the techniques of the present paper (except when we study the box-crossing property) are not restricted to the planar case. However, it seems that an extension to higher dimensions would add technical difficulties in intermediate lemmas of Section 6.3.

A third extension would be to a larger class of events. It seems to be an interesting question to characterize a class of events for which our methods from Sections 6.2 and 6.3 work.

Proof Sketch. The proof of Theorem 6.1.12 relies on an abstract quasi-independence result for threshold events of Gaussian vectors, namely Proposition 6.2.4. In this proposition, given a Gaussian vector X and two "threshold events" $\{X \in A\}$ and $\{X \in B\}$ measurable with respect to disjoint sets of coordinates (e.g. discrete crossing events of disjoint rectangles), we define a new Gaussian vector Y whose covariance is close to that of X such that $\{Y \in A\}$ and $\{Y \in B\}$ are independent. Next, we create a path $(X_t)_t$ of Gaussian vectors with $X_0 = X$ and $X_1 = Y$ and control the derivative of $\mathbb{P}[X_t \in A \cap B]$ with respect to t via "pivotal" events associated to A and B. The path method we have just sketched is inspired by Slepian's proof of the normal comparison inequality (see Lemma 1 of [Sle62]). The only novelty so far is the interpretation of the quantities which arise as **probabilities of pivotal events**.
Once this core result is established, in Section 6.3, we fix A and B as in[4] the statement of Theorem 6.1.12. Then, we discretize $\mathcal{K}_1 \cup \mathcal{K}_2$ and approximate A and B by some discrete events A^ε, B^ε. We then prove the estimate of Theorem 6.1.12 for A^ε and B^ε with uniform bounds on ε and let ε go to 0. This is the object of Proposition 6.3.4. In order to prove the discrete inequality we first use Proposition 6.2.4 for X equal to f restricted to the discretization, with $U = A^\varepsilon$ and $V = B^\varepsilon$. The right hand side is similar to the right hand side in Proposition 6.3.4. The key is then to find good enough bounds for the probabilities of pivotal events. This is the object of Proposition 6.3.10, at least for crossing events. The general case is dealt with in Subsection 6.3.4. Roughly speaking, if x is an interior point, to be pivotal it must have four neighbors of alternating signs, so there is an ε-approximate saddle point near x, which has probability $O(\varepsilon^2)$. If x is on the boundary (but not a corner), to be pivotal, it must have two neighbors with the same sign separated by a third neighbor with the opposite sign, all three on the same side of a line passing through x. We interpret this as a condition for the tangent of the nodal set at x to belong to an angle of size ε, which has probability $O(\varepsilon)$. The proof of Proposition 6.3.10 is divided in two steps. The first is to show that pivotal events imply the existence of zeros of certain fixed derivatives of f. The arguments are of geometric nature and are presented in Subsection 6.3.2. The second part is to prove

[4] Actually, for simplicity, we begin with the case where A and B are crossing and circuit events. Once the proof is complete, we explain how to deal with the general case in Subsection 6.3.4.

that these events are indeed exceptional using Kac-Rice type arguments. This is done in Subsection 6.3.3.

Outline. In Section 6.2 we recall the key estimate needed to establish Theorem 6.1.12, namely Proposition 6.2.4. We prove Theorem 6.1.12 (the quasi-independence thereorem for nodal lines) in Section 6.3. More precisely, in Subsections 6.3.1, 6.3.2 and 6.3.3 we prove this theorem in the case where A and B are generated by crossing events and then in Subsection 6.3.4 we explain how to take into account the number of level lines components. In Section 6.4, we combine Theorem 6.1.12 (in the case of crossings) with Tassion's method (from [Tas16]) to obtain Theorem 6.1.1. In Section 6.5, we use this theorem (in the case of number of nodal components) to obtain Theorem 6.1.4 (concerning the lower concentration of the number of nodal components). Finally, in Appendix 6.A we recall classical results about Gaussian fields and in Appendix 6.B we prove a discrete box-crossing estimate uniform on the mesh, see Proposition 6.B.2.

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6.2 Quasi-independence for Gaussian vectors

In this section, we reinterpret a classical quasi-independence formula of Gaussian vectors, namely Proposition 6.2.4 below, which is at the heart of the proof of Theorem 6.1.12. We first need to introduce some notation.

Notation 6.2.1. For any subset $U \subseteq \mathbb{R}^n$, write:

$$\mathrm{Piv}_i(U) = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \,:\, \exists y_1, y_2 \in \mathbb{R}, \begin{array}{l} (x_1, \cdots, x_{i-1}, y_1, x_{i+1}, \cdots, x_n) \in U, \\ (x_1, \cdots, x_{i-1}, y_2, x_{i+1}, \cdots, x_n) \notin U \end{array} \right\}.$$

Remark 6.2.2. Note that $\mathrm{Piv}_i(U)$ is a subset of $\mathbb{R}^n$ that does not depend on the i^{th} coordinate. Hence, we will sometimes see $\mathrm{Piv}_i(U)$ as a subset of $\mathbb{R}^{n-1}$ by forgetting the i^{th} coordinate.

Remark 6.2.3. For any $U, V \subseteq \mathbb{R}^n$ and any $i \in \{1, \ldots, n\}$, we have:

$$\mathrm{Piv}_i(U) = \mathrm{Piv}_i(U^c) \text{ and } \mathrm{Piv}_i(U \cap V) \cup \mathrm{Piv}_i(U \cup V) \subseteq \mathrm{Piv}_i(U) \cup \mathrm{Piv}_i(V)\,.$$

Proposition 6.2.4. *Let $k_1, k_2 \in \mathbb{Z}_{>0}$, let X be a non-degenerate centered Gaussian vector of dimension $k_1 + k_2$, and write Σ for the covariance matrix of X. Assume that, for each $i \in \{1, \cdots, k_1 + k_2\}$, $\Sigma_{ii} = 1$. Moreover, let Y be a centered Gaussian*

vector of dimension $k_1 + k_2$ independent of X such that $(Y_i)_{1\leq i\leq k_1}$ has the same law as $(X_i)_{1\leq i\leq k_1}$, $(Y_j)_{k_1+1\leq j\leq k_1+k_2}$ has the same law as $(X_j)_{k_1+1\leq j\leq k_1+k_2}$, and the vectors $(Y_i)_{1\leq i\leq k_1}$ and $(Y_j)_{k_1+1\leq j\leq k_1+k_2}$ are independent. For all $t \in [0,1]$, let $X_t = tX + \sqrt{1-t^2}Y$. Furthermore, let $\overrightarrow{q} \in \mathbb{R}^{k_1+k_2}$ let U (resp. V) belong to the sub-σ-algebra of $\mathcal{B}(\mathbb{R}^{k_1+k_2})$ generated by the sets $\{x_i \geq q_i\}$ for any $i \in \{1, \cdots, k_1\}$ (resp. $i \in \{k_1 + 1, \cdots, k_1 + k_2\}$). Then, we have:

$$\begin{aligned}\big|\mathbb{P}\left[X \in U \cap V\right] - \mathbb{P}\left[X \in U\right]\mathbb{P}\left[X \in V\right]\big| \\ \leq \sum_{\substack{i\in\{1,\cdots,k_1\},\\ j\in\{k_1+1,k_1+k_2\}}} |\Sigma_{ij}| \int_0^1 \mathbb{P}\left[X_t \in \mathrm{Piv}_i(U) \cap \mathrm{Piv}_j(V) \,\middle|\, X_t(i) = q_i,\, X_t(j) = q_j\right] dt \\ \times \frac{1}{2\pi\sqrt{1-\Sigma_{ij}^2}} \exp\left(-\frac{q_i^2+q_j^2}{2}\right).\end{aligned}$$

Remark 6.2.5. Proposition 6.2.4 is a reinterpretation of a classical quasi-independence formula for Gaussian vectors used in quantitative versions of Slepian's Lemma (see [Sle62] and Chapter 1 of [Pit96], especially Theorem 1.1). The proof presented here is very close to that of [Pit96] except that we work in a level of generality more adapted to our purposes and that we introduce the notion of pivotal events, which are central in the proof of Theorem 6.1.12. Later, we use this definition to show that these probabilities are small for discrete approximations of crossing events.

Remark 6.2.6. The proof of Proposition 6.2.4 is an interpolation argument. The path X_t defined in the statement is an interpolation between X and Y. By construction of Y, $\mathbb{P}[Y \in U \cap V] = \mathbb{P}[X \in U]\mathbb{P}[X \in V]$ so the left hand side of the inequality can be written as

$$\int_0^1 \frac{d}{dt}\mathbb{P}[X_t \in U \cap V]dt$$

if you admit that the probability is differentiable. Now the first order of variation of this probability should correspond to how likely the events $X_t \in U$ and $X_t \in V$ are to change when X_t one perturbs one of the $X_{t,i}$ and $X_{t,j}$ jointly, by a bump that depends on the shift in the covariance, which here is Σ_{ij} if $i \leq k_1 < j$ and 0 otherwise. But this is precisely what is expressed in the right hand side of the inequality.

Lemma 6.2.7. *Fix $n \in \mathbb{Z}_{>0}$, $\overrightarrow{q} \in \mathbb{R}^n$ and let U belong to the sub-σ-algebra of $\mathcal{B}(\mathbb{R}^n)$ generated by the sets $\{x_i \geq q_i\}$ for $i \in \{1, \ldots, n\}$. Also, let φ be a function which belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Then, for each $i \in \{1, \ldots, n\}$, there exists a measurable function $\epsilon_i = \epsilon_i(\varphi, U) : \mathbb{R}^{n-1} \to \{-1, 0, 1\}$ such that:*

$$\int_U \frac{\partial\varphi}{\partial x_i}(x)dx = \int_{\mathrm{Piv}_i(U)} \epsilon_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\, \varphi(x_1, \ldots, x_{i-1}, q_i, x_{i+1}, \ldots, x_n) \prod_{j\neq i} dx_j\,.$$

Proof. For each $\tilde{x} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$, let $U_i(\tilde{x})$ be the set of $y \in \mathbb{R}$ such that $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in U$. By Fubini's theorem:

$$\int_U \frac{\partial \varphi}{\partial x_i}(x)\, dx = \int_{\mathbb{R}^{n-1}} \int_{U_i(\tilde{x})} \frac{\partial \varphi}{\partial x_i}(x)\, dx_i\, d\tilde{x}\,.$$

Now, note that, for each $\tilde{x}$, $U_i(\tilde{x})$ equals either $\emptyset$, $\mathbb{R}$, $]-\infty, q_i[$, or $[q_i, +\infty[$. Moreover, if $\tilde{x} \notin \mathrm{Piv}_i(U)$, then $U_i(\tilde{x}) = \mathbb{R}$ or $\emptyset$. Let $\epsilon_i(\tilde{x})$ be 1 if $U_i(\tilde{x}) =]-\infty, q_i[$, -1 if $U_i(\tilde{x}) = [q_i, +\infty[$, and 0 otherwise. By the fundamental theorem of analysis:

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \int_{U_i(\tilde{x})} \frac{\partial \varphi}{\partial x_i}(x)\, dx_i\, d\tilde{x} &= \int_{\mathbb{R}^{n-1}} \epsilon_i(\tilde{x})\, \varphi(x_1, \ldots, x_{i-1}, q_i, x_{i+1}, \ldots, x_n)\, d\tilde{x} \\ &= \int_{\mathrm{Piv}_i(U)} \epsilon_i(\tilde{x})\, \varphi(x_1, \ldots, x_{i-1}, q_i, x_{i+1}, \ldots, x_n)\, d\tilde{x}\,.\end{aligned}$$

Note that Fubini's theorem and the fundamental theorem of analysis can be applied since $\varphi \in \mathcal{S}(\mathbb{R}^n)$. □

Proof of Proposition 6.2.4. Note that we have:

$$\begin{aligned}\mathbb{P}[X \in U \cap V] - \mathbb{P}[X \in U]\,\mathbb{P}[X \in V] &= \mathbb{P}[X \in U \cap V] - \mathbb{P}[Y \in U \cap V] \\ &= \mathbb{P}[X_1 \in U \cap V] - \mathbb{P}[X_0 \in U \cap V]\,.\end{aligned}$$

Hence, it is sufficient to prove that, for each $t \in [0,1]$, we have:

$$\begin{aligned}&\left|\frac{d}{dt}\mathbb{P}[X_t \in U \cap V]\right| \\ &\quad\leq \sum_{\substack{i \in \{1, \cdots, k_1\}, \\ j \in \{k_1+1, k_1+k_2\}}} |\Sigma_{ij}|\, \mathbb{P}\left[X_t \in \mathrm{Piv}_i(U) \cap \mathrm{Piv}_j(V) \,\middle|\, X_t(i) = q_i,\, X_t(j) = q_j\right] \\ &\qquad\qquad \times \frac{1}{2\pi\sqrt{1-\Sigma_{ij}^2}} \exp\left(-\frac{q_i^2 + q_j^2}{2}\right)\,. \quad (6.2.1)\end{aligned}$$

Note that since X and Y are non-degenerate and independent, for every $t \in [0,1]$, X_t is non-degenerate. Moreover, X_t has covariance Σ_t defined as follows: $\Sigma_{t,ij} = \Sigma_{ij}$ if either $1 \leq i, j \leq k_1$ or $k_1 + 1 \leq i, j \leq k_1 + k_2$, and $\Sigma_{t,ij} = t\Sigma_{ij}$ otherwise. Let $\Gamma : S_n^{++}(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}$ be[5] the map that associates to a matrix $\Sigma \in S_n^{++}(\mathbb{R})$ and a point $x \in \mathbb{R}^n$ the Gaussian density at x of a centered gaussian vector of covariance Σ. The function Γ is C^∞ and, for every $1 \leq i < j \leq n$, we have:[6]

$$\frac{\partial \Gamma}{\partial \Sigma_{i,j}} = \frac{\partial^2 \Gamma}{\partial x_i \partial x_j}\,. \quad (6.2.2)$$

[5]Here $S_n^{++}(\mathbb{R})$ is the set of positive definite symmetric matrices of size n, that we see as the corresponding open subset of $\mathbb{R}^{\frac{n(n+1)}{2}} = \{(\Sigma_{i,j})_{1 \leq i \leq j \leq n}\}$.

[6]This is a classical property of Gaussian densities which follows immediately by application of the Fourier transform, see for instance Equation (2.3) of [AW09].

Hence, by using dominated convergence and the chain rule:

$$\begin{aligned}\frac{d}{dt}\mathbb{P}\left[X_t \in U \cap V\right] &= \sum_{1\le i\le j\le k_1+k_2} \frac{d\Sigma_{t,ij}}{dt}\int_{U\cap V}\frac{\partial}{\partial\Sigma_{ij}}\Gamma(\Sigma_t,x)\,dx \\ &= \sum_{\substack{i\in\{1,\cdots,k_1\},\\ j\in\{k_1+1,k_1+k_2\}}} \Sigma_{ij}\int_{U\cap V}\frac{\partial^2}{\partial x_i\partial x_j}\Gamma(\Sigma_t,x)\,dx \text{ by 6.2.2 .} \quad (6.2.3)\end{aligned}$$

Since U depends only on the first k_1 coordinates and V depends only on the k_2 last coordianates, we can apply Lemma 6.2.7 first to $(U, i, \frac{\partial}{\partial x_j}\Gamma(\Sigma_t,\cdot))$ and then to $(V, j, \Gamma(\Sigma_t,\cdot)))$. We obtain that:

$$\begin{aligned}&\left|\int_{U\cap V}\frac{\partial\Gamma}{\partial x_i\partial x_j}(\Sigma_t,x)dx\right| \\ &\le \int_{\mathrm{Piv}_i(U)\cap\mathrm{Piv}_j(V)}\Gamma(\Sigma_t,x_1\ldots,x_{i-1},q_i,x_{i+1},\ldots,x_{j-1},q_j,x_{j+1},\ldots,x_{k_1+k_2})\prod_{\substack{l\in\{1,\cdots,k_1+k_2\},\\ l\notin\{i,j\}}}dx_l \\ &= \mathbb{P}\left[X_t\in\mathrm{Piv}_i(U)\cap\mathrm{Piv}_j(V)\,\middle|\,X_t(i)=q_i,\,X_t(j)=q_j\right]\gamma_t(i,j)\,, \quad (6.2.4)\end{aligned}$$

where $\gamma_t(i,j)$ is the density of $(X_t(i),X_t(j))$ at (q_i,q_j). Note that:

$$\gamma_t(i,j)\le\frac{1}{2\pi\sqrt{1-(t\Sigma_{ij})^2}}\exp\left(-\frac{q_i^2+q_j^2}{2(1-t|\Sigma_{ij}|)}\right)\le\frac{1}{2\pi\sqrt{1-\Sigma_{ij}^2}}\exp\left(-\frac{q_i^2+q_j^2}{2}\right). \quad (6.2.5)$$

Here, in the first inequality, we used the fact that if A is a positive definite symmetric matrix, for any vector X, $\langle X, AX\rangle \ge \min\mathrm{sp}(A)\|X\|^2$. If we combine (6.2.3), (6.2.4) and (6.2.5), we obtain (6.2.1) and we are done. □

6.3 Quasi-independence for planar Gaussian fields: the proof of Theorem 6.1.12

In this section, we prove Theorem 6.1.12. The steps of the proof are the following: we discretize our model, we apply Proposition 6.2.4 to the discrete model, and then we estimate the probability of pivotal events that appear in the proposition. We refer the reader to the introduction for a rough sketch of the proof. Let us now introduce the discretization procedure (by following [BG17a]).

We work with the face-centered square lattice (see Figure 6.1) that we denote by $\mathcal{T}$. We denote by $\mathcal{T}^\varepsilon$ this lattice scaled by a factor ε and we denote by $\mathcal{V}^\varepsilon$ the vertex set of $\mathcal{T}^\varepsilon$. Given a realization of our Gaussian field f, some $p\in\mathbb{R}$ and some $\varepsilon>0$, the signs of the values of $f+p$ on the sites of $\mathcal{T}^\varepsilon$ is a site percolation model on $\mathcal{T}^\varepsilon$. It induces a random coloring of the plane defined as follows: For each $x\in\mathbb{R}^2$, if $x\in\mathcal{V}^\varepsilon$ and $f(x)\ge -p$

or if x belongs to an edge of $\mathcal{T}^\varepsilon$ whose two extremities y_1, y_2 satisfy $f(y_1) \geq -p$ and $f(y_2) \geq -p$, then x is colored black. Otherwise, x is colored white. In other words, we study a **correlated site percolation model** on $\mathcal{T}^\varepsilon$. We also need the following definition.

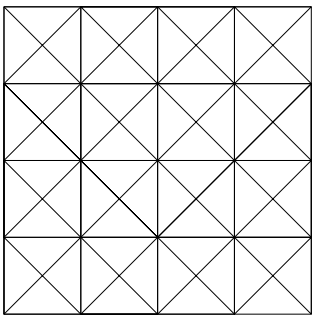


Figure 6.1: The face-centered square lattice (the vertices are the points of $\mathbb{Z}^2$ and the centers of the squares of the $\mathbb{Z}^2$-lattice).

Definition 6.3.1. Given $\varepsilon > 0$, an ε-drawn rectangle is a rectangle of the form $[a, b] \times [c, d]$ where $a \leq b$ and $c \leq d$ are four integer multiples of ε. An integer annulus is an annulus of the form $x + [-a, a]^2 \backslash]-b, b[^2$ where $x \in (\varepsilon\mathbb{Z})^2$ and $a \leq b$ are two positive integer multiples of ε.

The specific choice of the face-centered square lattice is not very important. We will essentially use the following facts: i) $\mathcal{T}$ is a triangulation, so we have nice duality arguments, see Remark 6.3.3 below, ii) $\mathcal{T}$ is translation invariant, iii) any ε-drawn rectangle and any ε-annulus can be drawn by using the edges of $\mathcal{T}$, and iv) $\mathcal{T}$ has nice symmetry properties. Actually, we will use the point iv) only in Section 6.B, but the results of this latter section are not used in the rest of the paper.

We start the proof of Theorem 6.1.12 by showing the result in the case where A and B are generated by crossing and circuit events since the proof is a little less technical in this case. This first part of proof is written in Subsections 6.3.1, 6.3.2 and 6.3.3. Note that this partial result is already sufficient to prove Theorem 6.1.1. We complete the proof of Theorem 6.1.12 by considering also the number of level lines components in Subsection 6.3.4.

6.3.1 The proof of Theorem 6.1.12 in the case of crossing and circuit events

In this subsection, we work only in the case of crossing and circuit events, we state Proposition 6.3.4, a discrete analog of Theorem 6.1.12 with constants *uniform in the mesh* ε, and we deduce Theorem 6.1.12 (in the case of crossing and circuit events) from

Proposition 6.3.4. The proof of Proposition 6.3.4 is written in Subsections 6.3.2 and 6.3.3. Before stating this proposition, we need a definition:

Definition 6.3.2. Let $\varepsilon > 0$, $p \in \mathbb{R}$, and consider the above discrete percolation model. Also, let $\mathcal{E}$ be a rectangle and $\mathcal{A}$ be an annulus. We say that there is a left-right ε-crossing of $\mathcal{E}$ above (resp. below) $-p$ if there is a continuous black (resp. white) path included in $\mathcal{E}$ from the left side of $\mathcal{E}$ to its right side. We define top-bottom ε-crossings similarly. We say that there is an ε-circuit in $\mathcal{A}$ above (resp. below) $-p$ if there is a continuous black (resp. white) path separating the inner boundary of $\mathcal{A}$ from its outer boundary.

Remark 6.3.3. We will use the following duality argument which follows from the fact that $\mathcal{T}$ is a triangulation and that any ε-drawn rectangle and any ε-drawn annulus can be drawn by using edges of $\mathcal{T}^\varepsilon$ (see Definition 6.3.1). Let $\varepsilon > 0$, let $\mathcal{E}$ be an ε-drawn rectangle. Then, there is left-right crossing of $\mathcal{E}$ above level p if and only if there is no top-bottom crossing of $\mathcal{E}$ below level p.

Proposition 6.3.4. *Let f be a Gaussian field satisfying Conditions 6.1.7 and 6.1.10. There exists $d = d(\kappa) < +\infty$ and $C = C(\kappa) < +\infty$ such that we have the following: Let $p \in \mathbb{R}$ and $\varepsilon \in]0,1]$. Also, let $k_1, k_2 \in \mathbb{Z}_{>0}$ and let $(\mathcal{E}_i)_{1 \leq i \leq k_1+k_2}$ be a collections of either ε-drawn rectangles or ε-drawn annuli. Let*

$$\mathcal{K}_1 = \cup_{i=1}^{k_1} \mathcal{E}_i,\ \mathcal{C}_1 = \cup_{i=1}^{k_2} \partial\mathcal{E}_i,\ \mathcal{K}_2 = \cup_{j=k_1+1}^{k_1+k_2} \mathcal{E}_j,\ \mathcal{C}_2 = \cup_{j=k_1+1}^{k_1+k_2} \partial\mathcal{E}_j\,.$$

Let A^ε (resp. B^ε) be an event in the Boolean algebra generated by the left-right and top-bottom ε-crossings above $-p$ and below $-p$ of rectangles among the $(\mathcal{E}_i)_i$ for $1 \leq i \leq k_1$ (resp. $(\mathcal{E}_j)_j$ for $k_1 + 1 \leq j \leq k_1 + k_2$) and the ε-circuits above $-p$ and below $-p$ in annuli among the $(\mathcal{E}_i)_i$ for $1 \leq i \leq k_1$ (resp. $(\mathcal{E}_j)_j$ for $k_1 + 1 \leq j \leq k_1 + k_2$). Let $\eta = \sup_{x \in \mathcal{K}_1, y \in \mathcal{K}_2} |\kappa(x-y)|$. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are at distance greater than d, then:

$$|\mathbb{P}[A^\varepsilon \cap B^\varepsilon] - \mathbb{P}[A^\varepsilon]\mathbb{P}[B^\varepsilon]| \leq \frac{C\,\eta}{\sqrt{1-\eta^2}} (1+|p|)^4\, e^{-p^2} \prod_{i=1}^{2} \left(\mathrm{Area}(\mathcal{K}_i) + \mathrm{Length}(\mathcal{C}_i) + k_i\right)\,.$$

Note that the constant C in Proposition 6.3.4 does not depend on ε. Let us first show how Theorem 6.1.12 follows from Proposition 6.3.4 **in the case where the events A and B are generated by crossing and circuit events**. Also, **here and in all the rest of Section 6.3, we assume that each of the $\mathcal{E}_i$'s are rectangles**. The proof adapts easily to the case where the $\mathcal{E}_i$'s can also be annuli, but would be tedious to spell out.

Proof of Theorem 6.1.12: Part 1 of 2, The case of crossings. We assume that the events A and B are generated by crossing and circuit events. Also, we assume that each $\mathcal{E}_i$ is a rectangle since the proof with annuli is exactly the same. First of all, using Lemma 6.A.9

and reasoning by approximation[7], it is enough to prove the result for rectangles whose sides are integer multiples of some fixed $\eta > 0$. But this is a direct consequence of Proposition 6.3.4 with $\varepsilon_k = \eta/k$, with the same family of rectangles, and by taking the limit as k goes to $+\infty$. Indeed, using Lemma 6.A.9 once more, it is easy to show that, if there is a (left-right, say) crossing of a rectangle above (resp. below) $-p$ in the continuum then a.s. there exists (a random) $\delta > 0$ such that this crossing belongs to a tube of width δ included in $\mathcal{D}_p$ (resp. $\mathcal{D}_p^c$). Hence, such a crossing in the continuum implies the analogous crossing in the discrete as long as $\varepsilon_k < \delta$ and $\mathbb{1}_{A \setminus A^{\varepsilon_k}}$ (resp. $\mathbb{1}_{B \setminus B^{\varepsilon_k}}$) converges a.s. to 0 as $k \to +\infty$. If there is no left-right crossing of a rectangle above (resp. below) $-p$, then (by Remark 6.A.11) a.s. there is a top-bottom crossing below (resp. above) $-p$ of this rectangle so $\mathbb{1}_{A^{\varepsilon_k} \setminus A}$ (resp. $\mathbb{1}_{B^{\varepsilon_k} \setminus B}$) converges a.s. to 0 as $k \to +\infty$. Thus, we have shown Theorem 6.1.12 in the case where A and B are generated by crossing (and circuit) events. □

To prove Proposition 6.3.4, we are going to use Proposition 6.2.4. We first define a Gaussian vector X_t^ε for each $t \in [0,1]$ in the spirit of the Gaussian vector X_t from Proposition 6.2.4. Since we will apply intermediate lemmas to the underlying continuous Gaussian fields, we first define a field f_t for every $t \in [0,1]$ as follows:

Notation 6.3.5. Let f, $(\mathcal{E}_i)_{1 \le i \le k_1+k_2}$, $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{C}_1$ and $\mathcal{C}_2$ be as in Proposition 6.3.4. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be disjoint neighborhoods of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Let g be a continuous Gaussian field indexed[8] by $\mathcal{U}_1 \cup \mathcal{U}_2$ independent of f such that g restricted to either of the $\mathcal{U}_i$'s has the same law as f restricted to $\mathcal{U}_i$ and such that g restricted to $\mathcal{U}_1$ is independent of g restricted to $\mathcal{U}_2$. For each $t \in [0,1]$, let $f_t = tf + \sqrt{1-t^2}g$. Note that (since f is centered and non-degenerate) for each $t \in [0,1]$, f_t is a non-degenerate centered Gaussian field whose covariance function is:

$$\begin{cases} \mathbb{E}\left[f_t(x) f_t(y)\right] = \kappa(x-y) & \text{if } x, y \in \mathcal{U}_1 \text{ or } x, y \in \mathcal{U}_2\,, \\ \mathbb{E}\left[f_t(x) f_t(y)\right] = t\kappa(x-y) & \text{otherwise}\,. \end{cases}$$

Also, for each $i \in \{1,2\}$, let $\mathcal{V}_i^\varepsilon = \mathcal{K}_i \cap \mathcal{V}^\varepsilon$, and let X^ε (resp. X_t^ε) be f (resp. f_t) restricted to $\mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon$.

We need one last notation before beginning the proof:

Notation 6.3.6. Given ε, p, $(\mathcal{E}_i)_{1 \le i \le k_1+k_2}$, A^ε and B^ε as in Proposition 6.3.4, we write $\mathcal{V}_1^\varepsilon$ and $\mathcal{V}_2^\varepsilon$ as in Notation 6.3.5 and we write U^ε and V^ε for the corresponding Borelian subsets of $\mathbb{R}^{\mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon}$ i.e. the elements of the Boolean algebra generated by the sets $\{x_i \ge -p\}$ for any $i \in \mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon$ such that:

$$A^\varepsilon = \{X^\varepsilon \in U^\varepsilon\} \text{ and } B^\varepsilon = \{X^\varepsilon \in V^\varepsilon\}\,.$$

[7] Indeed, Lemma 6.A.9 implies that crossing events for a given rectangle can be approximated by crossing events for approximations of this rectangle. Since A and B are generated by a finite boolean algebra of crossings, they can be obtained by a finite number of intersections and unions of crossings. Approximating each crossing and applying the same operations thus yields an approximation of A and B.

[8] The reason we extend g to open neighborhoods of $\mathcal{K}_1$ and $\mathcal{K}_2$ is largely technical and can be ignored during first reading.

Let us now start the proof of Proposition 6.3.4. By applying Proposition 6.2.4 to X^ε (which is centered, normalized and non-degenerate since f is centered, normalized and non-degenerate), U^ε and V^ε, it is sufficient to prove that there exists $C = C(\kappa) < +\infty$ and $d = d(\kappa) < +\infty$ such that, if $\mathcal{K}_1$ and $\mathcal{K}_2$ are at distance greater than d then for each $t \in [0,1]$ we have:

$$\sum_{\substack{x\in\mathcal{V}_1^\varepsilon,\\ y\in\mathcal{V}_2^\varepsilon}} \mathbb{P}\left[X_t^\varepsilon \in \mathrm{Piv}_x(U^\varepsilon)\cap \mathrm{Piv}_y(V^\varepsilon) \,\middle|\, f_t(x) = f_t(y) = -p\right]$$

$$\leq C\,(1+|p|)^4\, e^{-p^2} \prod_{i=1}^{2} (\mathrm{Area}(\mathcal{K}_i) + \mathrm{Length}(\mathcal{C}_i) + k_i)\ . \quad (6.3.1)$$

To prove (6.3.1), we need to find good enough bounds for the probabilities of pivotal events. This is the purpose of Subsections 6.3.2 and 6.3.3. The proof sketch provided in the introduction can be a useful guide to read the following subsections. Remember also that we have assumed that all of the $\mathcal{E}_i$'s are rectangles.

6.3.2 Pivotal sites imply exceptional geometric events

In this subsection, we fix a point x on the ε-lattice and explain how the fact that x is pivotal for the discretized event U^ε implies the cancellation of certain derivatives of the field. The results are combined in three lemmas that we state together before proving them for future reference. Each proof is independent from the rest.
In the first lemma, we show that, roughly speaking, on the neighbors of a pivotal point x, the field must have alternating signs relative to p.

Lemma 6.3.7. *We use the same notations as in Notation 6.3.6 (remember in particular that $\mathcal{K}_1 = \cup_{i=1}^{k_1}\mathcal{E}_i$ and $\mathcal{C}_1 = \cup_{i=1}^{k_1}\partial\mathcal{E}_i$). Let $x \in \mathcal{V}_1^\varepsilon$, let $\omega^\varepsilon \in \mathrm{Piv}_x(U^\varepsilon) \subseteq \mathbb{R}^{\mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon}$ and call black (resp. white) a vertex $y \in \mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon$ such that $\omega^\varepsilon(y) \geq -p$ (resp. $\omega^\varepsilon(y) < -p$). If the point x belongs to $\mathcal{K}_1 \setminus \mathcal{C}_1$, then it has four neighbors x_1, x_2, x_3, x_4 in anti-clockwise order around x and of alternating color. If the point x belongs to $\mathcal{C}_1$ and is the corner of none of the $\mathcal{E}_i$'s, then x has three neighbors x_1, x_2, x_3 in anti-clockwise order around x belonging to a common half-plane bounded by a line through x and of alternating color.*

In the last two lemmas, we explain how the information obtained in Lemma 6.3.7 implies the cancellation of certain derivatives of the field on fixed segments. The arguments are entirely deterministic.

Lemma 6.3.8. *Consider $\varphi \in C^1(\mathbb{R}^2)$ and $x, x_1, x_2, x_3 \in \mathbb{R}^2$. Assume that any two distinct vectors $x - x_i$ for $i = 1,2,3$ do not point in the same direction and that the x_i are numbered in anti-clockwise order around x. Assume that*

- *We have $\varphi(x) = 0$, $\varphi(x_1), \varphi(x_3) \geq 0$ and $\varphi(x_2) \leq 0$.*
- *There is a closed half plane H such that $x \in \partial H$ and $x_1, x_2, x_3 \in H$.*

Then, there exists $i \in \{1,2,3\}$ such that if $l = [x, x_i]$ has tangent vector v, $\partial_v \varphi$ has a zero on l.

Lemma 6.3.8 essentially states the following: If x is a point on the boundary of the rectangle such that $\varphi(x) = 0$ and such that, as one goes around x along a small half circle inside the rectangle, one encounters alternating color, then, the tangent vector of the nodal line of φ containing x must take some specific values near x. We formalize this by saying that restrictions of φ to certain small segments near x must have critical points.

Lemma 6.3.9. *Consider $\varphi \in C^1(\mathbb{R}^2)$ and $x, x_1, x_2, x_3, x_4 \in \mathbb{R}^2$. Assume that two vectors $x - x_i$ $i = 1,2,3,4$ do not point in the same direction and that the x_i's are numbered in anti-clockwise order around x. Assume also that:*

$$\text{We have } \varphi(x) = 0,\ \varphi(x_1), \varphi(x_3) \geq 0 \text{ and } \varphi(x_2), \varphi(x_4) \leq 0\,.$$

Let d_0 denote the diameter of $\{x, x_1, \cdots, x_4\}$. Then, there exist a finite set $\mathfrak{V}$ of unit vectors and a constant $C_0 < +\infty$ both depending only on the angles between the segments $[x, x_i]$'s such that the following holds: There exist two segments l_1 and l_2 with non-colinear unit tangent vectors $v_1, v_2 \in \mathfrak{V}$, of length at most $C_0\, d_0$ and both passing through at least one of the points $x, x_1, \cdots, x_4$ such that $\partial_{v_1}\varphi$ has a zero on l_1 and $\partial_{v_2}\varphi$ has a zero on l_2.

Lemma 6.3.9 roughly says that if φ changes signs four times when going around x along a small circle, then it must have an approximate saddle point at x. We formalize the notion of approximate saddle point by saying that there are two non-colinear segments of length ε on which the function φ has a vanishing derivative. In the proof we distinguish several cases depending on the relative positions of the x_i's and the gradient of φ at x. This reduces the proof to a planar euclidean geometry problem.

Proof of Lemma 6.3.7. By Remark 6.2.3 we may assume that there exists $i_0 \in \{1, \ldots, k_1\}$ such that U^ε is the Borelian subset of $\mathbb{R}^{\mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon}$ which corresponds to the left-right crossing of $\mathcal{E}_{i_0}$. If $x \notin \mathcal{E}_{i_0}$ then $\mathrm{Piv}_x(U^\varepsilon)$ is empty. If $x \in \mathcal{E}_{i_0} \setminus \partial\mathcal{E}_{i_0}$ and $\omega^\varepsilon \in \mathrm{Piv}_x(U^\varepsilon)$, then there are two paths made of black vertices connecting x to left and right sides of $\mathcal{E}_{i_0}$ and two white paths made of white vertices connecting x to the top and bottom sides of $\mathcal{E}_{i_0}$. These paths are necessarily of alternating color around x, so in particular it has four neighbors of alternating color. This proves the first assertion. Let $x \in \mathcal{C}_1 \cap \mathcal{E}_{i_0}$ such that x is not a corner. If $x \notin \partial\mathcal{E}_{i_0}$ then, as before, x must have four neighbors of alternating color. But then among these, there must be three neighbors belonging to the same half-space bounded by x with the properties required by the second assertion. On the other hand, if $x \in \partial\mathcal{E}_{i_0}$, then there must be a path of one color starting at a neighbor of x and reaching the opposite side of the rectangle and two additional paths of the opposite color connecting neighbors of x to each of the adjacent sides to the one containing x. But then, the three neighbors at which these paths start are in the configuration announced by the second assertion. □

Proof of Lemma 6.3.8. See Figure 6.2 (a) for a snapshot of the proof. If $\nabla\varphi(x) = 0$ then the result is trivial so assume that $\nabla\varphi(x) \neq 0$. Then, this gradient separates the plane into two closed half-spaces H_+ and H_- such that $x \in \partial H_+ = \partial H_-$, $\nabla\varphi(x)$ is orthogonal to this boundary, and $\nabla\varphi(x)$ points toward H_+. We distinguish between two cases: i) There exists $i_0 \in \{1, 3\}$ such that $x_{i_0} \in H_-$. In this case, let $l = [x, x_{i_0}]$ with unit vector v. Then, $\partial_v\varphi(x) \leq 0$, $\varphi(x) = 0$ and $\varphi(x_{i_0}) \geq 0$. Therefore, $\partial_v f$ must vanish somewhere on l. ii) The point x_2 belongs to H_+ (which happens if the case i) does not hold by the existence of the half-plane H and since the x_i's are in anti-clockwise order around x). In this case, the same argument works with $l = [x, x_2]$. □

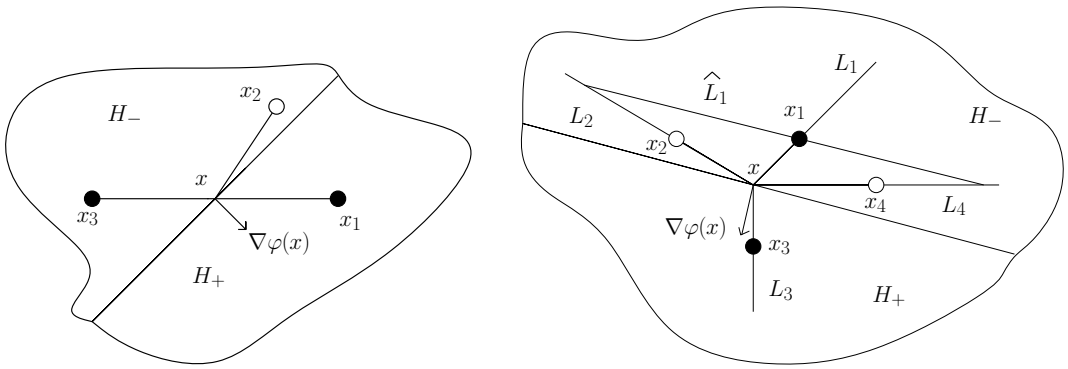


Figure 6.2: (a) The proof of Lemma 6.3.8, more particularly the case i) with $i_0 = 3$. (b) The proof of Lemma 6.3.9, more particularly the case ii).

Proof of Lemma 6.3.9. See Figure 6.2 (b) for an illustration of the proof. For each $i \in \{1, 2, 3, 4\}$, let L_i be the line $[x, x + C_0(x - x_i)]$ for some $C_0 > 0$ to be chosen later. If the anti-clockwise angle θ_i between L_{i-1} and L_{i+1} is less than π (the indices should be read modulo 4), set $\widetilde{L}_i := [x_{i-1}, x_{i+1}]$ and define $\widehat{L}_i$ to be the segment intersecting the bisector of θ_i orthogonally at x_i and whose extremities belong to L_{i-1} and L_{i+1}. We fix C_0 large enough so that whenever θ_i is indeed less than π, L_i is long enough to intersect $\widetilde{L}_i$. We will choose l_1 and l_2 among the L_i's, the $\widehat{L}_i$'s and the $\widetilde{L}_i$'s. The choice will follow by considering several distinct cases. In each case, the critical point will be detected either by finding three consecutive points on the segment on which φ takes alternating signs, or by finding a point on the segment where φ vanishes and has, say, a positive derivative, and proving that φ takes a negative value further along the segment. In both cases, the existence of the critical point follows by Rolle's theorem.

As in the proof of Lemma 6.3.8, note that if $\nabla\varphi(x) = 0$ then the result is trivial so assume that $\nabla\varphi(x) \neq 0$. Then, this gradient separates the plane into two closed half-spaces H_+ and H_- such that $x \in \partial H_+ = \partial H_-$, $\nabla\varphi(x)$ is orthogonal to this boundary, and $\nabla\varphi(x)$ points toward H_+. Note that there are at least two consecutive points among the x_i's in H_- or two consecutive points in H_+, such that they do not both belong to

$\partial H_- = \partial H_+$. Without loss of generality, assume that $x_1, x_2 \in H_-$ and that they do not belong both to ∂H_-. Then, along the segment L_1, φ starts at x with value 0 and a non-positive derivative and $\varphi(x_1) \geq 0$. In particular, its derivative along this segment must vanish. We now distinguish between two cases:

- Assume that there exists $i \in \{2,3,4\}$ with $x_i \in H_-$ such that, first, x_1 and x_i are not both on ∂H_-, and second, $f(x') \geq 0$ for some $x' \in L_i$. Then $\{l_1, l_2\} = \{L_1, L_i\}$ satisfies the required conditions (indeed, with the same argument as for L_1, the derivative of φ vanishes along L_i).

- Otherwise, since $\varphi(x_3) \geq 0$, then on the one hand x_3 necessarily belongs to H_+ (possibly on its common boundary with H_-) and on the other hand φ is necessarily negative on L_2. We distinguish between four subcases: (a) Assume that $x_4 - x$ points in the direction opposite to $x_1 - x$ and that there exists $x' \in L_3$ such that $\varphi(x') \leq 0$. Then L_3 is not colinear to L_1 and $\{l_1, l_2\} = \{L_1, L_3\}$ satisfies the required conditions. (b) Assume that $x_4 - x$ points in the direction opposite to $x_1 - x$ and that there is no $x' \in L_3$ such that $\varphi(x') \leq 0$. Then, the anticlockwise angle θ_3 between by L_2 and L_4 is less than π. Let x' be the intersection of $\tilde{L}_3$ with L_3. Then, $\varphi(x_4) \leq 0$, $\varphi(x_2) \leq 0$ and $\varphi(x') \geq 0$ since $x' \in L_3$ so $\{l_1, l_2\} = \{L_1, \widetilde{L}_3\}$ satisfies the required conditions (in particular the two segments are not colinear). (c) Assume now that $x_4 - x$ does not point in the opposite direction to $x_1 - x$ and that either $x_4 \in H_+$ or $x_4 \notin H_+$ and there is $x' \in L_4$ such that $\varphi(x') \geq 0$ then, as before, one can consider $\{l_1, l_2\} = \{L_1, L_4\}$. (d) Assume finally that $x_4 - x$ does not point in the opposite direction to $x_1 - x$, that $x_4 \notin H_+$ and that there is no $x' \in L_4$ such that $\varphi(x') \geq 0$. Then, the anti-clockwise angle θ_1 between L_4 and L_2 is less than π and one can consider $\{l_1, l_2\} = \{L_1, \widehat{L}_1\}$. Indeed, remember that φ is negative on L_2. Finally, $\widehat{L}_1$ goes through x_1 at which φ is non-negative, and φ is negative at both ends of $\widehat{L}_1$.

This completes the proof. □

6.3.3 End of the proof of Proposition 6.3.4 via Kac-Rice estimates

In this subsection we use results from Subsection 6.3.2 and Kac-Rice estimates to prove Proposition 6.3.4. The only remaining step is the following proposition:

Proposition 6.3.10. *Let f be as in the statement of Proposition 6.3.4. We use Notations 6.3.5 and 6.3.6. There exist $C_1 = C_1(\kappa) < +\infty$, $d_1 = d_1(\kappa) < +\infty$ and $\varepsilon_0 = \varepsilon_0(\kappa) \in]0,1]$ such that, for all $p \in \mathbb{R}$ and $t \in [0,1]$, if $\varepsilon \in]0, \varepsilon_0]$ and if $x \in \mathcal{V}_1^\varepsilon$, $y \in \mathcal{V}_2^\varepsilon$ are such that $|x - y| \geq d_1$ then:*

- *If neither $x \notin \mathcal{C}_1$ nor $y \notin \mathcal{C}_2$ then*

$$\mathbb{P}\left[X_t \in \mathrm{Piv}_x(U^\varepsilon) \cap \mathrm{Piv}_y(V^\varepsilon) \,\middle|\, X_t(x) = X_t(y) = -p\right] \leq C_1(1 + |p|)^4 \varepsilon^4 .$$

- *If among x and y one does not belong to $\mathcal{C}_1 \cup \mathcal{C}_2$ and the other belongs to $\mathcal{C}_1 \cup \mathcal{C}_2$ but is the corner of none of the $\mathcal{E}_i$'s then:*

$$\mathbb{P}\left[X_t \in \mathrm{Piv}_x(U^\varepsilon) \cap \mathrm{Piv}_y(V^\varepsilon) \,\middle|\, X_t(x) = X_t(y) = -p\right] \leq C_1(1+|p|)^3\varepsilon^3\,.$$

- *If x and y both belong to $\mathcal{C}_1 \cup \mathcal{C}_2$ but are the corner of none of the $\mathcal{E}_i$'s or if at least one of them does not belong to $\mathcal{C}_1 \cup \mathcal{C}_2$ then:*

$$\mathbb{P}\left[X_t \in \mathrm{Piv}_x(U^\varepsilon) \cap \mathrm{Piv}_y(V^\varepsilon) \,\middle|\, X_t(x) = X_t(y) = -p\right] \leq C_1(1+|p|)^2\varepsilon^2\,.$$

- *If x or y belongs to $\mathcal{C}_1 \cup \mathcal{C}_2$ but is the corner of none of the $\mathcal{E}_i$'s then:*

$$\mathbb{P}\left[X_t \in \mathrm{Piv}_x(U^\varepsilon) \cap \mathrm{Piv}_y(V^\varepsilon) \,\middle|\, X_t(x) = X_t(y) = -p\right] \leq C_1(1+|p|)\varepsilon\,.$$

Let us first wrap up the proof of Propositon 6.3.4.

Proof of Proposition 6.3.4. Remember that it is enough to prove (6.3.1). First note that if $\varepsilon \in]\varepsilon_0, 1]$ (where ε_0 is as in Propositon 6.3.10) then the result is easily obtained by bounding the probabilities by 1. Now, assume that $\varepsilon \in]0, \varepsilon_0]$. Then, by using Proposition 6.3.10, we obtain that for the $O\left(\varepsilon^{-4}\mathrm{Area}(\mathcal{K}_1)\mathrm{Area}(\mathcal{K}_2)\right)$ couples (x, y) such that $x \in \mathcal{V}_1^\varepsilon \backslash \mathcal{C}_1$ and $y \in \mathcal{V}_2^\varepsilon \backslash \mathcal{C}_1$, the quantitity $\mathbb{P}\left[X_t^\varepsilon \in \mathrm{Piv}_x(U^\varepsilon) \cap \mathrm{Piv}_y(V^\varepsilon) \,\middle|\, f_t(x) = f_t(y) = -p\right]$ is bounded by $C_1(1+|p|)\varepsilon^4$. Consequently, the sum over of all of these couples (x, y) is bounded by $O\left(\varepsilon^{-4}\mathrm{Area}(\mathcal{K}_1)\mathrm{Area}(\mathcal{K}_2)\right)(1+|p|)^4$. We reason similarly by also including the points on the boundary (which corresponds to $O\left(\varepsilon^{-1}\mathrm{Length}(\mathcal{C}_1)\right)$ points $x \in \mathcal{C}_1$ and $O\left(\varepsilon^{-1}\mathrm{Length}(\mathcal{C}_2)\right)$ points $x \in \mathcal{C}_2$) and at the corners (which correspond to $O(k_1)$ points $x \in \mathcal{V}_1^\varepsilon$ and $O(k_2)$ points $y \in \mathcal{V}_2^\varepsilon$). □

We now prove Proposition 6.3.10.

Proof of Proposition 6.3.10. We prove the first item since the proof of the others is the same (possibly by using Lemma 6.3.8 instead if Lemma 6.3.9). Fix $t \in [0, 1]$. Throughout the proof, the bounds will be uniform with respect to t. By combining Lemmas 6.3.7 and 6.3.9, we obtain that there exist a finite set of unit vectors $\mathfrak{V}$ independent of everything else, an absolute constant $C_0 < +\infty$, and a finite set of 4-uples of segments $\mathcal{L} = \mathcal{L}(x, y, \varepsilon)$ such that $\mathrm{Card}\mathcal{L} \leq C_0$ and such that:

- For every $(l_1, l_2, l_1', l_2') \in \mathcal{L}$ we have: The segments l_1, l_2 have non-colinear unit vectors $v_1, v_2 \in \mathfrak{V}$, are of length at most $C_0\varepsilon$, and are at distance at most C_0 from x. Moreover, the same holds for l_1', l_2' near y and with non-colinear unit vectors $v_1', v_2' \in \mathfrak{V}$.

- The probability of the first item of Proposition 6.3.10 is no greater than the sum over all $(l_1, l_2, l_1', l_2') \in \mathcal{L}$ of the expectation of:

$$\mathrm{Card}\{(a_1, a_2, b_1, b_2) \in l_1 \times l_2 \times l_1' \times l_2' \,:\, \forall i, j \in \{1, 2\},\, \partial_{v_i} f_t(a_i) = \partial_{v_j} f_t(b_j) = 0\}\,.$$

To control this expectation, we wish to apply the Kac-Rice formula. In order to do so we introduce the following notation. For each $(a_1, a_2, b_1, b_2) \in l_1 \times l_2 \times l_1' \times l_2'$, let

$$\begin{aligned}
\Phi_t = \Phi_t(a_1, a_2, b_1, b_2) &= (\partial^2_{v_1} f_t(a_1), \partial^2_{v_2} f_t(a_2), \partial^2_{v_1'} f_t(b_1), \partial^2_{v_2'} f_t(b_2)) \,, \\
\Psi_t = \Psi_t(x, y) &= (f_t(x), f_t(y)) \,, \\
\Upsilon_t = \Upsilon_t(a_1, a_2, b_1, b_2) &= (\partial_{v_1} f_t(a_1), \partial_{v_2} f_t(a_2), \partial_{v_1'} f_t(b_1), \partial_{v_2'} f_t(b_2)) \,.
\end{aligned}$$

Since κ satisfies Condition 6.1.10, then the covariance:

$$D_t = D_t(a_1, a_2, b_1, b_2) = \begin{pmatrix} D_t^{11} & D_t^{12} \\ D_t^{21} & D_t^{22} \end{pmatrix}$$

of (Ψ_t, Υ_t) converges as $\varepsilon \to 0$ and $|x - y| \to +\infty$, at a rate depending only on κ, to the following covariance:

$$D_* = \begin{pmatrix} D_*^{11} & D_*^{12} \\ D_*^{21} & D_*^{22} \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & D_*^{22} \end{pmatrix}$$

where:

$$D_*^{22} = \begin{pmatrix} -\partial^2_{v_1}\kappa(0) & -\partial_{v_1}\partial_{v_2}\kappa(0) & 0 & 0 \\ -\partial_{v_1}\partial_{v_2}\kappa(0) & -\partial^2_{v_2}\kappa(0) & 0 & 0 \\ 0 & 0 & -\partial^2_{v_1'}\kappa(0) & -\partial_{v_1'}\partial_{v_2'}\kappa(0) \\ 0 & 0 & -\partial_{v_1'}\partial_{v_2'}\kappa(0) & -\partial^2_{v_2'}\kappa(0) \end{pmatrix} .$$

Here we used Lemma 6.A.1 and Remark 6.A.2. Since v_1 and v_2 (resp. v_1' and v_2') are non-colinear, the vectors $(\partial_{v_1} f(0), \partial_{v_2} f(0))$ and $(\partial_{v_1'} f(0), \partial_{v_2'} f(0))$ are non-degenerate (see Remark 6.A.3) so D_* is non-degenerate. Consequently, there exist $C_2 = C_2(v_1, v_2, v_1', v_2', \kappa) \in]0, +\infty[$, $d_1 = d_1(v_1, v_2, v_1', v_2', \kappa) < +\infty$ and $\varepsilon_0 = \varepsilon_0(v_1, v_2, v_1', v_2', \kappa) \in]0, 1]$ such that, if $\varepsilon \in]0, \varepsilon_0]$ and $|x - y| \geq d_1$ then:

- the matrix D_t^{11} is non-degenerate;
- the matrix $\widetilde{D}_t = D_t^{22} - D_t^{21}(D_t^{11})^{-1} D_t^{12}$ is non-degenerate;
- $\det(\widetilde{D}_t) \geq C_2^{-1}$;
- the coefficients of D_t^{-1} are no greater than C_2.

In addition, κ is of class C^8 so Theorem 6.A.8 applies to the field Υ_t conditionned on $\Psi_t = (-p, -p)$. Since conditioning and differentiation 'commute' (see Remark 6.A.7), we obtain that the aforementioned expectation is no greater than:

$$\int_{l_1 \times l_2 \times l_1' \times l_2'} \frac{\mathbb{E}\left[\prod_{i=1}^4 |(\Phi_t)_i(a_1, a_2, b_1, b_2)| \,\middle|\, \Psi_t(x, y) = (-p, -p) \,,\, \Upsilon_t(a_1, a_2, b_1, b_2) = 0\right]}{(2\pi)^2 \sqrt{\det\left(\widetilde{D}_t(a_1, a_2, b_1, b_2)\right)}} da\, db \,.$$

The denominator is uniformly bounded from below by the previous discussion. We claim that if $\varepsilon \leq \varepsilon_0$ and $|x-y| \geq d_1$, the numerator is $O\left((1+|p|)^4\right)$. To prove this, notice first that D_t is non-degenerate so Lemma 6.A.6 applies. Moreover, the variance of the entries of Φ_t depends only on κ. All that remains is to bound its conditional mean. Firstly, the covariances of the entries of Φ_t and those of (Ψ_t, Υ_t) are bounded[9] by constants depending only on the derivatives up to order three of κ at 0. Moreover, D_t^{-1} has bounded coefficients so the conditional mean of Φ_t is $O(|p|)$. Hence, by Lemma 6.A.6, the numerator is $O\left((1+|p|)^4\right)$. Finally, the integration domain has volume $O(\varepsilon^4)$. □

6.3.4 Completing the proof of Theorem 6.1.12

In this subsection we explain how to complete the proof of Theorem 6.1.12 to take into account events measurable with respect to the number of level lines components inside the rectangles $\mathcal{E}_i$. In particular, this subsection is of no use for the proof of the RSW estimate Theorem 6.1.1. The part of the proof of Theorem 6.1.12 detailed in Subsections 6.3.2 and 6.3.3 hinges on the two following ideas: first, that the crossing events can be approximated by discrete events and second, that the fact that a point x is pivotal for a crossing events implies certain exceptional conditions on its neighbors whose probabilities are easy to control. To complete the proof of of Theorem 6.1.2, we justify that the discretization of the additional events is valid in Lemma 6.3.14 which in turn relies on Lemma 6.3.12. Then, we prove that the additional pivotal events imply the cancellation of certain derivatives in Lemma 6.3.16 and Lemma 6.3.17. The rest of the proof relies on results from Section 6.3.

Remark 6.3.11. Lemmas 6.3.12 and 6.3.14 below could be deduced from Proposition 6.1 of [BM18] and Theorem 1.5 of [BM18] respectively. However, since we do not need to control the rate of convergence when $\varepsilon \to 0$, we do not need a quantitative discretization scheme so instead we present a simpler proof relying only on transversality arguments.

Lemma 6.3.12. *Let $\mathcal{E} \subseteq \mathbb{R}^2$ be a rectangle. Assume that the Gaussian field f satisfies Condition 6.1.7 and that κ is C^6. Fix $p \in \mathbb{R}$. Then, a.s. there exists a (random) constant $\varepsilon_0 > 0$ such that for a.e. $\varepsilon \leq \varepsilon_0$, we have:*

i) $\mathcal{T}^\varepsilon$ and $\mathcal{N}_p$ intersect transversally,

ii) each edge of $\mathcal{T}^\varepsilon$ inside $\mathcal{E}$ has at most two intersection points,

iii) any two distinct intersection points of a common edge e are connected by a smooth path in $\mathcal{N}_p$ inside the union of the two faces adjacent to e,

iv) for each connected component $\mathcal{C}$ of $\mathcal{N}_p$ there exists an edge e of $\mathcal{T}^\varepsilon$ such that $\mathcal{C}$ intersects e exactly once and e has no other intersection with the nodal set,

v) there is no edge of $\mathcal{T}^\varepsilon$ included in the boundary of $\mathcal{E}^\varepsilon$ that is intersected twice by $\mathcal{N}_p$, where $\mathcal{E}^\varepsilon$ is (one of) the largest rectangle whose sides are integer multiples of ε such that $\mathcal{E}^\varepsilon \subseteq \mathcal{E}$.

[9] Indeed, this follows from Lemma 6.A.1 and the fact that for any two L^2 random variables ξ_1 and ξ_2, $|\mathbb{E}[\xi_1\xi_2]| \leq \frac{1}{2}\left(\mathbb{E}\left[\xi_1^2\right] + \mathbb{E}\left[\xi_2^2\right]\right)$.

Proof of Lemma 6.3.12. By Lemma 6.A.9, $\mathcal{N}_p$ is a.s. smooth and intersects $\partial\mathcal{E}$ transversally. Let w be a unit vector tangent to an edge of the lattice. We apply Lemma 6.A.10 to $T = \mathcal{E}$, $g = (f, \partial_w f, \partial_w^2 f)$ and $v = (0,0,0)$ (g has bounded density by Remark 6.A.3 and by stationarity). This shows that the set of points $x \in \mathcal{N}_p$ such that $T_x\mathcal{N}_p$ is tangent to w is a.s. discrete. We then simply apply Lemma 6.A.13 to $\mathcal{C}$ the union of connected components of $\mathcal{N}_p$ intersecting $\mathcal{E}$ (who are a.s. in finite number and a.s. do not intersect 0, possibly modifying them outside of $\mathcal{E}$ to make $\mathcal{C}$ compact). This establishes assertions i), ii) and iii).
To show iv), first take ε smaller than the distance between any two distinct connected components of $\mathcal{N}_p$ intersecting $\mathcal{E}$ so that each edge e can intersect at most one connected component. Assume that $\mathcal{C}$ intersects each edge an even number of times. Then, it must stay in a union of a face and its three adjacent faces. If ε^2 is much smaller than the area of the smallest connected component of $\mathcal{E} \setminus \mathcal{N}_p$ this cannot happen so iv) is satisfied.
In order to show v), use once again Lemma 6.A.9 in order to obtain that $\mathcal{N}_p$ intersects the boundary of $\mathcal{E}$ transversally and only finitely many times. This completes the proof. □

In the arguments below, we will need to discretize level lines of the field. To this end, let us introduce some notations.

Notation 6.3.13. Let $\varepsilon > 0$, $p \in \mathbb{R}$ and $(\mathcal{E}_i)_{1\le i\le k_1+k_2}$, $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $N_p(i)$ be as in Theorem 6.1.12. Let $\mathcal{V}_1^\varepsilon$ and $\mathcal{V}_2^\varepsilon$ as in Notation 6.3.5. Color the plane as explained at the beginning of Section 6.3. Given such a coloring, each face has either zero or two sides whose ends have opposite colors. If a face has two such sides, draw a segment joining the middle of these two sides. This produces a collection of polygonal lines on the plane. We denote by $\mathcal{N}_p^\varepsilon$ the union of these lines. For each $i \in \{1, \dots, k_1 + k_2\}$, let $\mathcal{E}_i^\varepsilon$ be (one of) the largest rectangle whose sides are integer multiples of ε and such that $\mathcal{E}_i^\varepsilon \subseteq \mathcal{E}_i$, let $N_p^\varepsilon(i)$ be the number of connected components of $\mathcal{N}_p^\varepsilon$ contained in $\mathcal{E}_i^\varepsilon$. Let A be an event in the σ-algebra defined by events of the form $\{N_p(i) = m\}$ where $i \in \{1, \dots, k_1\}$ and $m \in \mathbb{N}$. Let A^ε be the same event as A but with the $N_p(i)$'s replaced by the $N_p^\varepsilon(i)$'s. There exists $U^\varepsilon \subseteq \mathbb{R}^{\mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon}$ (resp. $V^\varepsilon \subseteq \mathbb{R}^{\mathcal{V}_1^\varepsilon \cup \mathcal{V}_2^\varepsilon}$) such that $A^\varepsilon = \{X^\varepsilon \in U^\varepsilon\}$. Note that by construction, the events A and B belong to the Boolean algebra generated by events of the form $\{N_p(i) \in S\}$ where $S \subseteq \mathbb{N}$.

Lemma 6.3.14. *Assume that the Gaussian field f satsifies Condition 6.1.7 and that κ is C^6. We use Notation 6.3.13. Then,*

$$\limsup_{\varepsilon\to 0} \mathbb{P}\left[\forall i \in \{1, \dots, k_1 + k_2\},\, N_p(i) = N_p^\varepsilon(i)\right] = 1\,.$$

Proof. We start with the following claim.

Claim 6.3.15. *For each $i \in \{1, \dots, k_1 + k_2\}$ a.s., for Lebesgue-a.e. small enough $\varepsilon > 0$, $N_p(i) = N_p^\varepsilon(i)$.*

Proof. Fix $i \in \{1, \dots, k_1 + k_2\}$. By points i) to iv) of Lemma 6.3.12, a.s., for a.e. $\varepsilon > 0$ small enough, $\mathcal{N}_p$ intersects $\partial\mathcal{E}_i$ and $\mathcal{T}^\varepsilon$ transversally, each edge of $\mathcal{T}^\varepsilon$ included in $\mathcal{E}_i^\varepsilon$ is

crossed at most twice and any two intersection points of the same edge are connected by $\mathcal{N}_p$ inside one of its adjacent faces. Also, each connected component of $\mathcal{N}_p$ must intersect an edge which is crossed exactly once by $\mathcal{N}_p$.
In particular, the following is an equivalent definition of $\mathcal{N}_p^\varepsilon(i)$ for a.e. $\varepsilon > 0$ small enough: i) Let F be a face of the lattice with two sides e, e' that are intersected by $\mathcal{N}_p$ exactly once and consider a path γ included in $F \cap \mathcal{N}_p$ that connects e and e'. Then, replace γ by a straight line as in Figure 6.3 (case 1). ii) Let F be a face of the lattice with two sides e, e' that are intersected by $\mathcal{N}_p$ exactly once, let e'' the third edge adjacent to F and let F' be the other face adjacent to e''. Also, consider a path γ included in $(F \cup F') \cap \mathcal{N}_p$ that connects e and e' and intersects e'' twice. Then, replace γ by a straight line in Figure 6.3 (case 2).

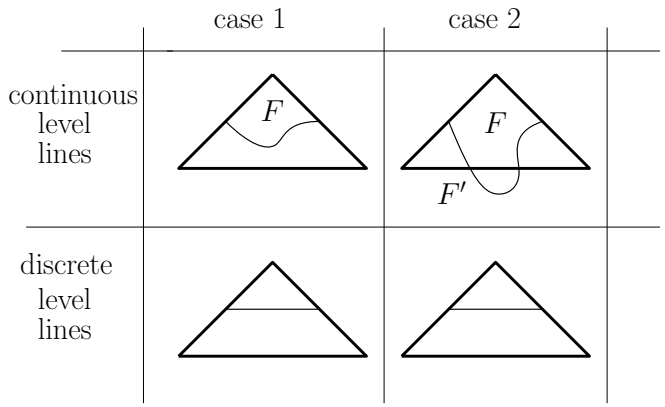


Figure 6.3: An alternative definition of $\mathcal{N}_p^\varepsilon(i)$ when the conclusion of Lemma 6.3.12 holds.

One can see that, doing so, we redefine $\mathcal{N}_p^\varepsilon$ and this alternate definition shows that its connected components are naturally in bijection with those of $\mathcal{N}_p$. Moreover, for all $eps > 0$ small enough, connected components of $\mathcal{N}_p$ included in $\mathcal{E}_i$ are also included in $\mathcal{E}_i^\varepsilon$ so that $N_p(i) \leq N_p^\varepsilon(i)$. On the other hand, if a continuous connected component gives rise to a discrete connected component included in $\mathcal{E}_i^\varepsilon$, it cannot cross edges of $\partial\mathcal{E}_i^\varepsilon$ once. But it cannot cross them twice either by point v) of Lemma 6.3.12. As a result, $N_p^\varepsilon(i) \leq N_p(i)$. □

Let $\Xi(\varepsilon)$ be the event that for all $i \in \{1, \dots, k_1 + k_2\}$, $N_p(i) = N_p^\varepsilon(i)$. Now, by Claim 6.3.15, for each $\delta > 0$ there exists $\tau = \tau(\delta) > 0$ such that, with probability at least $1 - \delta$, for Lebesgue-a.e. $\varepsilon \leq \tau$, $\Xi(\varepsilon)$ is satisfied. Moreover, τ can be chosen so that $\lim_{\delta \to 0} \tau(\delta) = 0$. In particular,

$$\mathbb{E}\left[\int_0^\tau \mathbb{1}_{\Xi(\varepsilon)} d\lambda(\varepsilon)\right] \geq \tau(1 - \delta)\,.$$

By Fubini's theorem, we deduce that

$$\int_0^\tau \mathbb{P}\left[\Xi(\varepsilon)\right] d\lambda(\varepsilon) \geq \tau(1-\delta).$$

In particular, there exists $\varepsilon = \varepsilon(\delta) \in]0, \tau(\delta)]$ such that $\mathbb{P}\left[\Xi(\varepsilon)\right] \geq 1 - 2\delta$. Since this holds for any $\delta > 0$, the proof is complete. □

Lemma 6.3.16. *Use Notation 6.3.13 and, for each $x \in \mathcal{V}_1^\varepsilon$, let $\omega^\varepsilon \in \mathrm{Piv}_x(U^\varepsilon)$. Color the edges $e = (x, y)$ of $\mathcal{T}^\varepsilon$ such that $\omega^\varepsilon(x), \omega^\varepsilon(y) \geq -p$ in black and color the rest of the plane in white. Then:*

1. *if x belongs to $\mathcal{K}_1 \backslash \mathcal{C}_1$ then either the neighbors of x are all of the same color or x has (at least) four neighbors that have alternating color when listed in anti-clockwise order;*

2. *if x belongs to $\mathcal{C}_1$ but is not a corner, then it has three neighbors of alternating color when listed in anti-clockwise order.*

Proof. By Remark 6.2.3, we may assume that $A^\varepsilon = \{N_p^\varepsilon(i) = m\}$ for some $i \in \{1, \ldots, k_1\}$ and $m \in \mathbb{N}$. Fix $\varepsilon > 0$, $x \in \mathcal{V}_1^\varepsilon$ and fix a value of X^ε. If the set of neighbors has exactly one black connected component and one white component, then changing the color of x does not change $N_p^\varepsilon(i)$. Therefore x being pivotal for U^ε implies the two items. □

The following lemma is a trivial application of Rolle's theorem.

Lemma 6.3.17. *Let $\varphi \in C^1(\mathbb{R}^2)$. Fix $x \in \mathbb{R}^2$ and assume that $\varphi(x) = 0$. Then:*

1. *if there exist $x_1, x_2 \in \mathbb{R}^2$ such that for each $i \in \{1, 2\}$, $\varphi(x_i) \leq 0$ and such that $x \in]x_1, x_2[$, then $\varphi|_{[x_1, x_2]}$ has a critical point;*

2. *if there exist $x_1, x_2, x_3, x_4 \in \mathbb{R}^2$ such that for each $i \in \{1, 2, 3, 4\}$, $\varphi(x_i) \leq 0$ and such that $l_1 = [x_1, x_3]$ and $l_2 = [x_2, x_4]$ intersect in their interior at x, then $\varphi|_{l_1}$ and $\varphi|_{l_2}$ have a critical point.*

We now complete the proof of Theorem 6.1.12.

Proof of Theorem 6.1.12: Part 2 of 2 Allowing components as well as crossings. We use Notations 6.3.5 and 6.3.13. According to Lemma 6.3.14

$$\limsup_{\varepsilon \to 0} \mathbb{P}\left[\forall i \in \{1, \ldots, k_1 + k_2\},\, N_p(i) = N_p^\varepsilon(i)\right] = 1\,.$$

We take a subsequence $(\varepsilon_k)_{k \geq 1}$ along which the lim sup is reached. Approximating crossings of the $\mathcal{E}_i$ by discrete crossings of the $\mathcal{E}_i^{\varepsilon_k}$ we get $\lim_{k \to +\infty} \mathbb{P}\left[A^{\varepsilon_k} \triangle A\right] = 0$. Therefore, it is enough to show that for ε small enough

$$\left|\mathbb{P}\left[A^\varepsilon \cap B^\varepsilon\right] - \mathbb{P}\left[A^\varepsilon\right]\mathbb{P}\left[B^\varepsilon\right]\right| \leq \frac{C}{\sqrt{1-\eta^2}}(1+|p|)^4 e^{-p^2} \prod_{i=1}^{2} \left(\mathrm{Area}(\mathcal{K}_i) + \mathrm{Length}(\mathcal{C}_i) + 1\right)$$

for some constant $C = C(\kappa) < +\infty$. Here, unlike in Proposition 6.3.4, A and B are events generated not only by crossing and circuit events but also by the $N_p(i)$'s. Nonetheless the proof is quite similar. Indeed, notice that Proposition 6.3.4 follows from Proposition 6.3.10 which in turn uses only the fact that for two points x, y to be pivotal, certain derivatives of f_t must vanish on certain deterministic segments. This is proved in Lemmas 6.3.7, 6.3.9 and 6.3.8. In our case, first, we combine Lemma 6.3.7 with Lemma 6.3.16 using Remark 6.2.3. Then, we use Lemma 6.3.17 in addition to Lemmas 6.3.9 and 6.3.8. The rest of the proof of Proposition 6.3.4 applies as is. □

6.4 Tassion's RSW theory: the proof of Theorem 6.1.1

In this section, we prove Theorem 6.1.1 by relying on Sections 6.2 and 6.3 (but not on Subsection 6.3.4) and on [Tas16]. Our proof follows [Tas16] so instead of writing the details of each proof, we point out the steps of the original proof that need to be modified to work in our setting. We expect the reader to be familiar with [Tas16] and suggest that this section be read with said work at hand. Note that this simplifies the proof of [BG17a] since we can directly apply Tassion's method in the continuum instead of applying it to different discretizations of the model at each scale. We first prove the following weaker result:

Proposition 6.4.1. *Let f be a Gaussian field satisfying Conditions 6.1.7, 6.1.8, 6.1.10 as well as Condition 6.1.9 for some $\alpha > 4$. Let $\rho > 0$. There exists $c = c(\kappa, \rho) > 0$ such that, for each $s > 0$, the probability that there is a left-right crossing of $[0, \rho s] \times [0, s]$ in $\mathcal{D}_0$ is at least c.*

Throughout the proof, in [Tas16], Tassion uses symmetries of the model such as stationarity (which is satisfied here by Condition 6.1.7), symmetries, and the FKG inequality (which are also valid here by Condition 6.1.8 and Lemma 6.A.12). The final ingredient of the proof is a quasi-independence lemma, which we will state when needed. Otherwise, the proof carries over with only minor changes due to the specificities of the model.

Proof. Step 1: By Remark 6.A.11, the probability that there is a left-right crossing of $[-s, s]^2$ is $1/2$ for any $s \in]0, +\infty[$. In particular, it is uniformly bounded from below by some constant $c_0 > 0$, which is just Equation (1) of [Tas16]. In other words

$$\forall s > 0,\ \mathbb{P}\left[\mathrm{Cross}_0(s, s)\right] \geq c_0\,. \tag{6.4.1}$$

Step 2: Given $s \in]0, +\infty[$ and $\alpha, \beta \in [0, s/2]$ such that $\alpha < \beta$, we define the events $\mathcal{H}_s(\alpha, \beta)$ and $\mathcal{X}_s(\alpha)$ as follows (see Figure 6.1 below): The event $\mathcal{H}_s(\alpha, \beta)$ is satisfied whenever there is a continuous path in $[-s/2, s/2]^2 \cap \mathcal{D}_0$ connecting $\{-s/2\} \times [-s, s]$ to $\{s/2\} \times [\alpha, \beta]$. The event $\mathcal{X}_s(\alpha)$ is the event that there is a path γ_1 in $[-s/2, s/2]^2 \cap \mathcal{D}_0$ connecting $\{-s/2\} \times [-s/2, -\alpha]$ to $\{-s/2\} \times [\alpha, s/2]$, a path γ_2 in $[-s/2, s/2]^2 \cap \mathcal{D}_0$ connecting $\{s/2\} \times [-s/2, -\alpha]$ to $\{s/2\} \times [\alpha, s/2]$ and a path in $[-s/2, s/2]^2 \cap \mathcal{D}_0$ connecting γ_1 to γ_2. As in [Tas16], we define $\phi_s : [0, s/2] \to [-1, 1]$ as

$$\phi_s(\alpha) = \mathbb{P}[\mathcal{H}_s(0, \alpha)] - \mathbb{P}[\mathcal{H}_s(\alpha, s/2)]\,.$$

Then, Lemma 2.1 of [Tas16], says that for each $s \in]0, +\infty[$, there is $\alpha_s \in [0, s/2]$ such that, for some $c_1 > 0$ independent of s,

$$\forall \alpha \in [0, \alpha_s],\ \mathbb{P}\left[\mathcal{X}_s(\alpha)\right] \geq c_1; \forall \alpha \in [\alpha_s, s/2],\ \mathbb{P}\left[\mathcal{H}_s(0, \alpha)\right] \geq c_0/4 + \mathbb{P}\left[\mathcal{H}_s(\alpha, s/2)\right]\ . \quad (6.4.2)$$

To establish this inequality, Tassion uses the fact that ϕ_s is continuous and increasing and defines α_s using the preimage of ϕ_s of a certain value. Here, the continuity of ϕ_s follows easily from the fact the f is a.s. continuous and that for each $x \in \mathbb{R}^2$, $\mathrm{Var}(f(x)) > 0$. Moreover, the fact that ϕ_s is non-decreasing is immediate from its definition and this is sufficient for us since the argument works if one replaces $\min\left\{\phi_s^{-1}(s/4), s/4\right\}$ by $\sup\{\alpha \in]0, s/4[:\ \phi_s(\alpha) \leq c_0/4\}$. The rest of the proof of Lemma 2.1 uses only symmetries, the FKG inequality and Equation 6.4.1 and works as is.

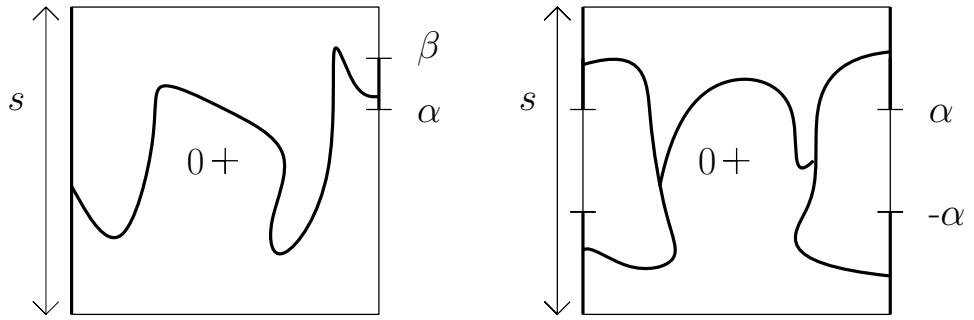


Figure 6.1: The events $\mathcal{H}_s(\alpha, \beta)$ (left-hand-side) and $\mathcal{X}_s(\alpha)$ (right-hand-side). For every $\alpha \in [0, s/2]$, we let $\phi_s(\alpha) = \mathbb{P}[\mathcal{H}_s(0, \alpha)] - \mathbb{P}[\mathcal{H}_s(\alpha, s/2)]$.

Step 3: For each $0 < r < s$, let $\mathrm{Circ}_0(r, s)$ be the event that there is a circuit above level 0 in the annulus $[-s, s]^2 \setminus]-r, r[^2$ separating $[-s, s]^2$ from infinity. In Lemmas 2.2 and 3.1 of [Tas16], Tassion shows that there exist constants $c_2, c_3 \in]0, 1[$ such that

$$\forall s \geq 2,\ \alpha_s \leq 2\alpha_{2s/3} \Rightarrow \mathbb{P}\left[\mathrm{Circ}_0(s, 2s)\right] \geq c_2 \quad (6.4.3)$$

and

$$\forall s \geq 1,\ t \geq 4s,\ \mathbb{P}\left[\mathrm{Circ}_0(s, 2s)\right] \geq c_2 \text{ and } \alpha_t \leq s \Rightarrow \mathbb{P}\left[\mathrm{Circ}_0(t, 2t)\right] \geq c_3\,. \quad (6.4.4)$$

The proof of these two lemmas relies only on the FKG, symmetries and Equations 6.4.2 so it carries over to our setting.

Step 4: This is the step where Tassion uses a quasi-independence lemma. In our case, we will use the following direct consequence of Theorem 6.1.12:

Corollary 6.4.2. *There exists a constant $C_0 = C_0(\kappa) < +\infty$ such that, for every integer N larger than 1, for every $s \in [1, +\infty[$, for every $1 \leq r_1 \leq \cdots \leq r_N < +\infty$ such that $r_2 \geq r_1 + s$, and for every B which belongs to the Boolean algebra generated by the events $\mathrm{Circ}_0(r_i, 2r_i)$, $i = 2, \cdots, N$, we have:*

$$\left|\mathbb{P}\left[\mathrm{Circ}_0(r_1, 2r_1) \cap B\right] - \mathbb{P}\left[\mathrm{Circ}_0(r_1, 2r_1)\right]\mathbb{P}\left[B\right]\right| \leq C_0 N r_N^4\, s^{-\alpha}\,.$$

Using this corollary, we prove an analog of Lemma 3.2 of [Tas16]. Let us first introduce some notation. Given c_0 as in Step 1 and c_2 and c_3 as in Step 3, let $C_1 < +\infty$ be such that $(1-c_3)^{\lfloor C_1/2 \rfloor} < c_0/8$ and let $s_0 < +\infty$ be such that for each $s \geq s_0$, $\frac{C_0}{c_3} \lfloor \log_5(C_1/2) \rfloor (C_1 s)^4 s^{-\alpha} < c_0/8$ (where C_0 is as in Corollary 6.4.2). Then, we prove the following:

Lemma 6.4.3. *Let $s \geq s_0$ such that $\mathbb{P}[\mathrm{Circ}_0(s,2s)] \geq c_2$, then, there exists $s' \in [4s, C_1 s]$ such that $\alpha_{s'} > s$.*

Proof. In the proof of his Lemma 3.2, Tassion uses FKG and the symmetry properties, as well as what we call Equations 6.4.2 and 6.4.3. The only place where he uses a quasi-independence property is where he proves that, if $\mathbb{P}\left[\mathrm{Circ}_0(5^i s, 2 \cdot 5^i s)\right] \geq c_3$ for any $i \in \{0, \cdots, \lfloor \log_5(C_1/2) \rfloor\}$ and if $s \geq s_0$, then:

$$\mathbb{P}[\mathrm{Circ}_0(s, C_1 s)] > 1 - c_0/4\,.$$

In what follows, we prove such a result and we refer to [Tas16] for the rest of the proof. First note that:

$$\mathrm{Circ}_0(s, C_1 s) \subseteq \cup_{i=0}^{\lfloor \log_5(C_1/2) \rfloor} \mathrm{Circ}_0(5^i s, 2 \cdot 5^i s)\,.$$

Now, by Corollary 6.4.2 applied $\lfloor \log_5(C_1/2) \rfloor - 1$ times:

$$\begin{aligned}
&\mathbb{P}\left[\left(\cup_{i=0}^{\lfloor \log_5(C_1/2) \rfloor} \mathrm{Circ}_0(5^i s, 2 \cdot 5^i s)\right)^c\right] \\
&= \mathbb{P}\left[\cap_{i=0}^{\lfloor \log_5(C_1/2) \rfloor} \mathrm{Circ}_0(5^i s, 2 \cdot 5^i s)^c\right] \\
&\leq (1-c_3) \times \mathbb{P}\left[\cap_{i=1}^{\lfloor \log_5(C_1/2) \rfloor} \mathrm{Circ}_0(5^i s, 2 \cdot 5^i s)^c\right] - C_0 \lfloor \log_5(C_1/2) \rfloor (C_1 s)^4 s^{-\alpha} \\
&\leq \cdots \\
&\leq (1-c_3)^{\lfloor \log_5(C_1/2) \rfloor} + C_0 \left(\sum_{j \geq 0} (1-c_3)^j\right) \lfloor \log_5(C_1/2) \rfloor (C_1 s)^4 s^{-\alpha} \\
&\leq (1-c_3)^{\lfloor \log_5(C_1/2) \rfloor} + \frac{C_0}{c_3} \lfloor \log_5(C_1/2) \rfloor (C_1 s)^4 s^{-\alpha} \\
&< c_0/4\,,
\end{aligned}$$

where the last inequality follows from the definition of C_1 and the fact that $s \geq s_0$. □

Step 5: As explained in the proof of Lemma 3.3 of [Tas16] and the final comment that follows it, Proposition 6.4.1 now follows for s large enough from Equations (6.4.2) and (6.4.3) and Lemma 6.4.3 as well as standard gluing constructions that use only the FKG inequality and from symmetries. By the FKG inequality[10] Theorem 6.A.4 applied to events of the form $\{f \geq 1$ on B translated by some vector$\}$, we obtain that, for each $s > 0$, f takes only values larger than or equal to 1 on $[-s, s]^2$ with positive probability. □

[10] More precisely, the events $\{f \geq 1$ on $B\}$ can be approximated by increasing events depending on a finite sets of points, to which one can apply the FKG inequality.

We now prove Theorem 6.1.1.

Proof of Theorem 6.1.1. We prove the result for $\mathcal{N}_0$. This is sufficient since $\mathcal{N}_0 \subseteq \mathcal{D}_0$ and since the result for $\mathcal{D}_0$ for s less than some fixed constant can easily be proved as in the end of Proposition 6.4.1.
Let $\mathcal{Q}$ be a quad and note that there exist $\delta = \delta(\mathcal{Q}) > 0$, $n = n(\mathcal{Q}), m = m(\mathcal{Q}) \in \mathbb{Z}_{>0}$ and two sequences $(\mathcal{E}_i)_{i=1}^n$ and $(\mathcal{E}'_j)_{j=1}^m$ of $2\delta \times \delta$ and $\delta \times 2\delta$ rectangles such that: i) if each $\mathcal{E}_i$ (resp. $\mathcal{E}_j$) is crossed lengthwise then $\mathcal{Q}$ is crossed and ii) $\inf_{x \in \cup_{i=1}^n \mathcal{E}_i,\, y \in \cup_{j=1}^m \mathcal{E}'_j} |x - y| \geq \delta$.
For each $s > 0$, write A_s (resp. B_s) for the event that each $s\mathcal{E}_i$ is crossed (resp. each $s\mathcal{E}'_j$ is dual-crossed) lengthwise. By stationarity, $\frac{\pi}{2}$-rotation invariance and Remark 6.A.11, the crossing events of each of the rectangles above and below 0 are bounded from below by the constant $c = c(\kappa, 2) > 0$ from Proposition 6.4.1. Consequently, by Lemma 6.A.12, for each $s > 0$, $\mathbb{P}[A_s] \geq c^n$ and $\mathbb{P}[B_s] \geq c^m$. But now, by Theorem 6.1.12, there exists $C = C(\kappa) < +\infty$ such that, for each $s > 0$:

$$\mathbb{P}\left[A_s \cap B_s\right] \geq \mathbb{P}\left[A_s\right] \mathbb{P}\left[B_s\right] - C\left(\delta s + 1\right)^{4-\alpha} nm\,.$$

Since, $\alpha > 4$ we have $C\left(\delta s + 1\right)^{4-\alpha} nm \underset{s \to +\infty}{\longrightarrow} 0$ so the left-hand-side is bounded from below by a positive constant for s sufficiently large. But $A_s \cap B_s$ clearly implies the crossing of $s\mathcal{Q}$ by $\mathcal{N}_0$. □

Now that we have established Theorem 6.1.1, we apply it to obtain two results which are well known in Bernoulli percolation. Namely, the polynomial decay of the one-arm event: Proposition 6.4.5, and the absence of unbounded clusters at criticality: Proposition 6.4.6. We are going to use the following notation:

Notation 6.4.4. If $0 < r < s < +\infty$, we write $\mathcal{A}(r, s) = [-s, s]^2 \backslash]-r, r[^2$ and we write $\mathrm{Arm}_0(r, s)$ (resp. $\mathrm{Arm}_0^*(r, s)$) the event that there is a continuous path in $\mathcal{D}_0 \cap \mathcal{A}(r, s)$ (resp. in $\mathcal{A}(r, s) \setminus \mathcal{D}_0$) from the inner boundary of $\mathcal{A}(r, s)$ to its outer boundary.

We start with the following result:

Proposition 6.4.5. *Let f be a Gaussian field that satisfying Conditions 6.1.7, 6.1.8, 6.1.10 as well as Condition 6.1.9 for some $\alpha > 4$. There exists $C = C(\kappa) < +\infty$ and $\eta = \eta(\kappa) > 0$ such that, for each $1 \leq r < s + \infty$:*

$$\mathbb{P}\left[\mathrm{Arm}_0(r, s)\right] = \mathbb{P}\left[\mathrm{Arm}_0^*(r, s)\right] \leq C\left(r/s\right)^{\eta}.$$

Proof. Remark 6.A.11 and the fact that f is centered imply that $\mathbb{P}\left[\mathrm{Arm}_0(r, s)\right] = \mathbb{P}\left[\mathrm{Arm}_0^*(r, s)\right]$. So let us prove the result only for $\mathrm{Arm}_0^*(r, s)$. First fix $h \in [1/2, 1[$ to be determined later. For each $i \in \{0, \cdots, \lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor\}$, let $\mathrm{Circ}_0(i)$ denote the event that there is a circuit at level 0 in the annulus $\mathcal{A}(5^i (rs)^{1-h}, 2 \cdot 5^i (rs)^{1-h})$. Note that:

$$\mathrm{Arm}_0^*(r, s) \subseteq \bigcap_{i=0}^{\lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor} \mathrm{Circ}_0(i)^c\,.$$

Next, note that by Theorem 6.1.1 and by the FKG inequality Lemma 6.A.12, there exists $c = c(\kappa) \in]0,1[$ such that for each $i \in \{0, \cdots, \lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor\}$, $\mathbb{P}[\mathrm{Circ}_0(i)] \geq c$. Next, use the quasi-independence result Theorem 6.1.12 $\lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor$ times to obtain that, for some $C' = C'(\kappa) < +\infty$ we have:

$$
\begin{aligned}
&\mathbb{P}\left[\bigcap_{i=0}^{\lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor} \mathrm{Circ}_0(i)^c\right] \\
&\leq (1-c) \times \mathbb{P}\left[\bigcap_{i=1}^{\lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor} \mathrm{Circ}_0(i)^c\right] - C' \lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor (1+s^4) (rs)^{-\alpha(1-h)} \\
&\leq \cdots \\
&\leq (1-c)^{\lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor} + C' \left(\sum_{j \geq 0} (1-c)^j\right) \lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor (1+s^4) (rs)^{-\alpha(1-h)} \\
&\leq (1-c)^{\lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor} + \frac{C'}{c} \lfloor \log_5(\frac{s^h}{2 \cdot r^{1-h}}) \rfloor (1+s^4) (rs)^{-\alpha(1-h)} .
\end{aligned}
$$

Since $\alpha > 4$, we can find h sufficiently small to obtain what we want. □

From Proposition 6.4.5 we get the following analog of the celebrated theorem by Harris [Har60] (which states that, for Bernoulli percolation on $\mathbb{Z}^2$ with parameter $1/2$, there is no infinite cluster).

Proposition 6.4.6. *With the same hypotheses as Proposition 6.4.5, a.s. every connected component of $\mathcal{D}_0$ is bounded.*

Proof. By a union-bound and translation invariance, it is enough to prove that a.s. there is no unbounded component of $\mathcal{D}_0$ which intersects $[-1,1]^2$, which is the case since by Proposition 6.4.5, $\mathbb{P}[\mathrm{Arm}_0(1,s)]$ goes to 0 as s goes to $+\infty$. □

The natural question arising from this proposition is whether or not this remains true for $\mathcal{D}_p$ with $p > 0$. This is the object of [RV17b] where we prove that, for the Bargmann-Fock field, there is a unique unbounded connected component in $\mathcal{D}_p$ as soon as $p > 0$, thus obtaining the analogue of Kesten's famous theorem [Kes80] (which states that the critical point for Bernoulli percolation on $\mathbb{Z}^2$ is $1/2$).

6.5 Concentration from below of the number of nodal lines: the proof of Theorem 6.1.4

In this section, we prove Theorem 6.1.4 by using Theorem 6.1.2 and our quasi-independence result Theorem 6.1.12. The idea of the proof is the following. Let $\varepsilon > 0$. We first tile

the square $[-s/2, s/2]^2$ with $(r/s)^2$ mesoscopic squares of size r. Then, we use Theorem 6.1.2 and our quasi-independence result Theorem 6.1.12 to prove that the density of $r \times r$ squares containing less than $r^2(c_{NS} - \varepsilon)$ nodal components is asymptotically small. More precisely, we will note that, if the number of such squares is greater than $\delta(s/r)^2$, then there exist $\delta(s/r)^2/8$ such squares that are at distance at least r from each other. By Theorem 6.1.12, this has probability $\mathbb{P}\left[\frac{N_0(r)}{r^2} - c_{NS} \leq -\varepsilon\right]^{\delta(s/r)^2/8}$ up to errors involving terms of the form $\sup_{x\,:\,|x|\geq r} |\kappa(x)|$. The last step is an optimization on the choice of r.
Upper concentration on the other hand seems to require some control of the tail of the density of nodal components. For the moment, it is not even known whether this density is L^2. This type of information seems necessary for the following reason. In Item (1) of Theorem 6.1.4, for instance, we consider exponential concentration of the density of components. To this end we write the number of components as a sum of quasi-independent random variables. But a direct consequence of Cramér's theorem is that, if $X_1, X_2, \cdots$ are i.i.d. L^1 positive random variables such that $\mathbb{E}\left[e^{\theta X_1}\right] = +\infty$ for every $\theta > 0$, then $\left(\frac{X_1+\cdots+X_n}{n}\right)_n$ does not have exponential concentration around its mean. Note finally that to have an upper bound concentration, we need to take care of the mesoscopic components that intersect several $r \times r$ squares. However, these do not add any difficulty. Indeed, by [NS16], if we write $N'(r)$ for the number of nodal components which *intersect* a $r \times r$ box (and are not just included) then Theorem 6.1.2 also holds for $N'(r)$ (with the same constant c_{NS}).

Proof of Theorem 6.1.4. Assume that f is a planar Gaussian field satisfying Conditions 6.1.7, 6.1.10 and 6.1.11. First note that it is sufficient to prove the result for ε sufficiently small and fix $\varepsilon \in]0, c_{NS}/2[$. Let $1 \leq r \leq s$ be such that $s \in r\mathbb{N}$ and tile the square $[-s/2, s/2]^2$ with $(s/r)^2$ $r \times r$ squares $S_1, \cdots, S_{(s/r)^2}$. Throughout the proof, we take the liberty of omitting floor functions. For each $t \in [0, +\infty[$, write $\kappa_t = \sup\{|\kappa(x)| \,:\, |x| \geq t\}$.
By Theorem 6.1.2, for each $h \in]0, 1/2[$, there exist $r_0 = r_0(\varepsilon, h) < +\infty$ such that, if $r \geq r_0$, then:

$$\mathbb{P}\left[\frac{N_0(r)}{r^2} \leq c_{NS} - \varepsilon\right] \leq h\,, \tag{6.5.1}$$

We also assume that r_0 is sufficiently big so that $\kappa_{r_0} \leq 1/2$ and we assume that $r \geq r_0$. For every $i \in \{1, \cdots, (s/r)^2\}$, write N_0^i for the number of connected components of $\mathcal{N}_0$ included in S_i and note that, if $\frac{N_0(s)}{s^2} \leq c_{NS} - 2\varepsilon$, then there exist $(s/r)^2 \frac{\varepsilon}{c_{NS}-\varepsilon}$ squares S_i such that $\frac{N_0^i}{r^2} \leq c_{NS} - \varepsilon$. As a result, if $\eta = \eta(\varepsilon) = \frac{1}{8} \times \frac{\varepsilon}{c_{NS}-\varepsilon}$, there exist $\eta \cdot (s/r)^2$ squares S_i at distance at least r from each other and such that $\frac{N_0^i}{r^2} \leq c_{NS} - \varepsilon$. Let $S_{i_1}, \cdots, S_{i_n}$ be $\eta \cdot (s/r)^2$ pairwise distinct squares among the $(s/r)^2$ squares at distance at least r from each other. In the following, we estimate the probability that for each $j \in \{1, \cdots, \eta \cdot (s/r)^2\}$, $\frac{N_0^{i_j}}{r^2} \leq c_{NS} - \varepsilon$. Recall that $h < 1/2$ and that $0 < \varepsilon < c_{NS}/2$ so $0 < \eta < 1/8$. By Theorem 6.1.12 applied $\eta \cdot (s/r)^2 - 1$ times, by translation invariance

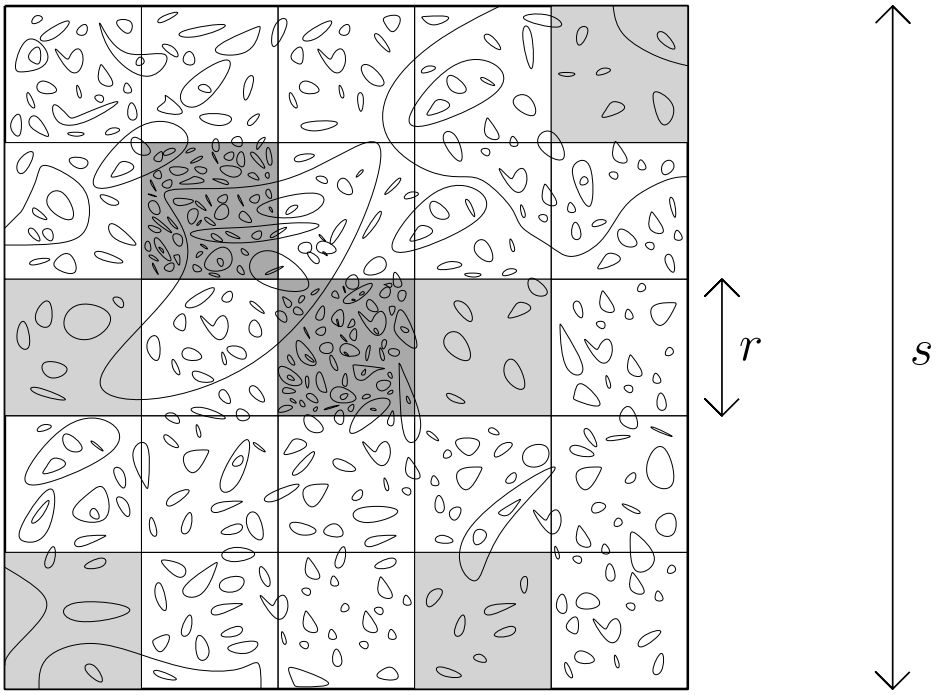


Figure 6.1: The components of $\mathcal{N}_0$ in $[-s/2, s/2]^2$. In light gray: the $r \times r$ squares in which the density of components is smaller than expected. Combining Theorem 6.1.2 with Theorem 6.1.12, we prove that that with high probability there are not too many such squares. In dark gray, the $r \times r$ squares in which the number of components is much greater than expected. Since we do not know whether or not the density of nodal component has an heavy tail, it is very hard to control these exceptional squares.

and by (6.5.1):

$$\begin{aligned}
&\mathbb{P}\left[\forall j \in \{1, \cdots, \eta \cdot (s/r)^2\},\ \frac{N_0^{i_j}}{r^2} \leq c_{NS} - \varepsilon\right] \\
&\leq h \times \mathbb{P}\left[\forall j \in \{2, \cdots, \eta \cdot (s/r)^2\},\ \frac{N_0^{i_j}}{r^2} \leq c_{NS} - \varepsilon\right] + O\left(\kappa_r\, r^2 (s^2 + \eta \cdot (s/r)^2)\right) \\
&\leq h \times \mathbb{P}\left[\forall j \in \{2, \cdots, \eta \cdot (s/r)^2\},\ \frac{N_0^{i_j}}{r^2} \leq c_{NS} - \varepsilon\right] + O\left(\kappa_r r^2 s^2\right) \\
&\leq \cdots \\
&\leq h^{\eta \cdot (s/r)^2} + \left(\sum_{j \geq 0} h^j\right) O\left(\kappa_r\, r^2 s^2\right) = h^{\eta \cdot (s/r)^2} + O\left(\kappa_r\, r^2 s^2\right) \\
&\leq h^{\eta \cdot (s/r)^2} + \frac{1}{1-h} O\left(\kappa_r\, r^2 s^2\right) = h^{\eta \cdot (s/r)^2} + O\left(\kappa_r\, r^2 s^2\right)\ .
\end{aligned}$$

where the constants in the O's depend only on κ. As a result :

$$\begin{aligned}\mathbb{P}\left[\frac{N_0(s)}{s^2} \leq c_{NS} - 2\varepsilon\right] &\leq \binom{(s/r)^2}{\eta\cdot(s/r)^2}\left(h^{\eta\cdot(s/r)^2} + O\left(\kappa_r\, r^2 s^2\right)\right) \\ &\leq (2h^\eta)^{(s/r)^2} + O\left(2^{(s/r)^2}\,\kappa_r\, r^2 s^2\right). \qquad (6.5.2)\end{aligned}$$

Let us first treat the case of Item 1 i.e. assume that there exists $C < +\infty$ and $c > 0$ such that $\kappa_r \leq C\exp(-cr^2)$. Then, the right hand side of (6.5.2) is

$$(2h^\eta)^{(s/r)^2} + O\left(\exp\left((s/r)^2\log(2) - cr^2 + 4\log(s)\right)\right).$$

Taking $h = h(\eta)$ small enough and $r = M\sqrt{s}$ for $M = M(c)$ large enough, this quantity is exponentially small in s so we are done.

Let us now treat the case of Item 2 i.e. assume that there exists $C < +\infty$ and $\alpha > 4$ such that $\kappa_r \leq Cr^{-\alpha}$. Then, the right hand side of (6.5.2) is

$$(2h^\eta)^{(s/r)^2} + O\left(2^{(s/r)^2} s^2 r^{2-\alpha}\right).$$

Fix $\delta > 0$. Choosing $r = s/\sqrt{a\log_2(s)}$ for $a = a(\delta) > 0$ small enough, the second term in the sum is $O\left(s^{4-\alpha+\delta}\right)$. Having chosen a, we choose $h = h(a,\eta)$ such that the first term is also $O\left(s^{4-\alpha+\delta}\right)$. Since this is true for any $\varepsilon \in]0, c_{NS}/2[$ and any $\delta > 0$, we are done. □

Remark 6.5.1. Note that we have used Theorem 6.1.2 only to obtain (6.5.1). Hence, our lower concentration result Theorem 6.1.4 holds if, instead of Condition 6.1.11 (which is the assumption to apply Theorem 6.1.2), we assume that there exists a constant $c_{NS} = c_{NS}(\kappa) \in]0, +\infty[$ such that, for each $\varepsilon > 0$, $\mathbb{P}\left[\frac{N_0(s)}{s^2} \leq c_{NS} - \varepsilon\right]$ goes to 0 as s goes to $+\infty$.

Appendix

6.A Classical tools

In this section we present classical or elementary results about Gaussian vectors and fields.

6.A.1 Classical results for Gaussian vectors and fields

Differentiating Gaussian fields. When one consider derivatives of Gaussian fields, it is important to have the following in mind (see for instance Appendices A.3 and A.9 of [NS16]):

Lemma 6.A.1. *Let f be an a.s. continuous Gaussian field with covariance*[11] *$K \in C^{k+1,k+1}(\mathbb{R}^n \times \mathbb{R}^n)$ and mean $\mu \in C^k(\mathbb{R}^n)$. Then, f is almost surely C^k. Conversely, if a.s. f is C^k, then $K \in C^{k,k}$, $\mu \in C^k$ and for every multi-indices $\beta, \gamma \in \mathbb{N}^2$ such that $\beta_1 + \cdots + \beta_n \leq k$ and $\gamma_1 + \cdots + \gamma_n \leq k$, we have:*

$$\mathrm{Cov}\left(\partial^\beta f(x), \partial^\gamma f(y)\right) = \mathbb{E}\left[(\partial^\beta f(x) - \partial^\beta \mu(x))(\partial^\gamma f(y) - \partial^\gamma \mu(y))\right] = (-1)^{|\gamma|}\partial_x^\beta \partial_y^\gamma K(x,y)\,.$$

Remark 6.A.2. Lemma 6.A.1 has the following consequence: if f satisfies Condition 6.1.7 and is a.s. C^1 then, for each $\beta \in \mathbb{N}^2$ such that $\beta_1 + \beta_2$ is odd, $\partial^\beta \kappa(0) = 0$.

Remark 6.A.3. Another consequence of Lemma 6.A.1 is that if f is a.s. C^1 and satisfies Condition 6.1.7 then for each $x \in \mathbb{R}^2$ and for v, w non-colinear unit vectors, the Gaussian vector $(\partial_v f(x), \partial_w f(x))$ is non-degenerate. Indeed, if this was not the case, then we would obtain the existence of some non-zero vector u such that $\partial_u f$ would a.s. vanish identically, which would contradict the fact that f is non-degenerate. Similarly, if f is a.s. C^2 and satisfies Condition 6.1.7 then for each $x \in \mathbb{R}^2$ and each non-zero vector $w \in \mathbb{R}^2$, $(f(x), \partial_w f(x), \partial_w^2 f(x))$ is non-degenerate. Indeed, $\partial_w f(x)$ is independent of the two other coordinate by Remark 6.A.2 and if $(f(x), \partial_w^2 f(x))$ were degenerate then as above this would contradict the fact that f is non-degenerate.

[11] Here and below, $C^{l,l}$ means that all partial derivatives of K which include at most l differentiations in the first variable and l differentiations in the second variable exist and are continuous.

A FKG inequality for Gaussian vectors. The following result by [Pit82b] says that positively correlated Gaussian vectors satisfy positive association. This is a key result when one wants to use Russo-Seymour-Welsh type techniques. We first need to introduce the following terminology: if I is some set and $A \subseteq \mathbb{R}^I$ then we say that A is increasing if for every $\omega \in A$ and every $\omega' \in \mathbb{R}^I$ such that $\omega'(i) \geq \omega(i)$ for every $i \in I$, we have $\omega' \in A$.

Theorem 6.A.4 ([Pit82b]). *Let $(X_k)_{1 \leq k \leq n}$ be a Gaussian vector such that, for every $k, l \in \{1, \dots, n\}$, $\mathbb{E}[X_k X_l] \geq 0$. Then, For every $A, B \subseteq \mathbb{R}^n$ increasing Borel subsets:*

$$\mathbb{P}[X \in A \cap B] \geq \mathbb{P}[X \in A]\, \mathbb{P}[X \in B]\,.$$

This type of inequality is known as the **Fortuyn-Kasteleyn-Ginibre** (or **FKG**) inequality. Pitt's result easily generalizes to crossing and circuits events by approximation, one just needs to take care that the approximating events are increasing, see Lemma 6.A.12.

Some basic lemmas. The following lemma is useful to bound the expectation of the product of Gaussian variables. The first lemma is known as the regression formula and is quite classical in the field.

Lemma 6.A.5 (Proposition 1.2 of [AW09]). *Let (X, Y) be an $n+m$-dimensional centered Gaussian vector with covariance*

$$\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$$

where A (resp. D) is the covariance of X (resp. Y). Assume Y is non-degenerate. Then, the law of X conditioned on Y is that of a Gaussian vector with covariance $A - BD^{-1}B^t$ and mean $BD^{-1}Y$.

The next lemma is a simple application of the regression formula to the computation of conditional moments of Gaussian vectors.

Lemma 6.A.6. *Let (X, Y) be a centered Gaussian vector in $\mathbb{R}^n \times \mathbb{R}^m$ with covariance*

$$\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}.$$

Assume that D is non-degenerate. Let $\mu \in \mathbb{R}^m$. Then, there exists $C = C(n) < +\infty$ such that

$$\mathbb{E}\left[\prod_{i=1}^{n} |X_i| \,\middle|\, Y = \mu\right] \leq C \max_{i \in \{1,\dots,n\};\, j,k \in \{1,\dots,m\}} \left(\sqrt{A_{ii}} \vee |B_{ik} D^{-1}_{kj} \mu_j|\right)^n$$

Proof. By the regression formula (Lemma 6.A.5), X conditioned on $Y = \mu$ has the law of a Gaussian vector Z with covariance $\widetilde{A} = A - BD^{-1}B^t$ and mean $\widetilde{\mu} = BD^{-1}\mu$. Note that $BD^{-1}B^t$ is symmetric semi-definite. Therefore, its diagonal coefficients must

be non-negative. Therefore, for each $i \in \{1,\dots,n\}$, $\widetilde{A}_{ii} \le A_{ii}$. Moreover, for each $i \in \{1,\dots,n\}$, $|\widetilde{\mu}_i| \le n^2 \max_{j,k\in\{1,\dots,m\}} |B_{ik} D^{-1}_{kj} \mu_j|$. The lemma then follows from the elementary observation that for each $n \ge 1$ there exists $C = C(n) < +\infty$ such that for each Gaussian vector Z with covariance $\widetilde{A}$ and mean $\widetilde{\mu}$,

$$\mathbb{E}\left[\prod_{i=1}^{n} |Z_i|\right] \le C \max_{i\in\{1,\dots,n\}} \left(\sqrt{\widetilde{A}_{ii}} \vee |\widetilde{\mu}_i|\right)^n .$$

□

Remark 6.A.7. From Lemmas 6.A.5 and 6.A.1 we deduce that if f is an a.s. continuous and non-degenerate Gaussian field on $\mathbb{R}^n$ with $C^{k+1,k+1}$ covariance and C^k mean and if $x_1,\dots,x_k \in \mathbb{R}^n$ are such that $(f(x_1),\dots,f(x_k))$ is a non-degenerate Gaussian vector, then, for each $v \in \mathbb{R}^n$ conditionally on $((f(x_1),\dots,f(x_k)) = v$, f is a Gaussian field with $C^{k+1,k+1}$ covariance and C^k mean. Moreover, the covariance (resp. mean) of the derivatives of the conditional field is equal to the covariance (resp. mean) of the derivatives of the field under the same conditioning.

A Kac-Rice formula. The following result is a Kac-Rice type formula, which is for instance a particular case of Theorem 6.2 of [AW09] (together with Proposition 6.5 therein):

Theorem 6.A.8. *Let $\varepsilon \in]0,+\infty[$, let $n \in \mathbb{Z}_{>0}$, and let $\Phi_1,\cdots,\Phi_n$ denote n continuous Gaussian fields $: [0,\varepsilon] \to \mathbb{R}$ that are a.s. C^2 on $]0,\varepsilon[$ and such that, for every $s \in [0,\varepsilon]^n$, $\Phi(s) = (\Phi_1(s_1),\cdots,\Phi_n(s_n))$ is non-degenerate. Then*

$$\mathbb{E}\left[\mathrm{Card}\{s \in [0,\varepsilon]^n \,:\, \Phi(s) = 0\}\right]$$

equals:

$$\int_{]0,\varepsilon[^n} \varphi(s)\, \mathbb{E}\left[\prod_{i=1}^{n} \left|\Phi_i'(s_i)\right| \,\middle|\, \Phi(s) = 0\right] ds\,,$$

where $\varphi(s)$ is the density of $\Phi(s)$ evaluated at 0.

6.A.2 Transversality of the level set and a non-quantitative discretization lemma

In this subsection, we state transversality results which are quite classical in the field and which are very helpful to obtain some continuity results about crossing events. We also prove a non-quantitative discretization lemma useful to justify discrete approximation of certain events.

Lemma 6.A.9. *Assume that f satisfies Condition 6.1.7 and that κ is C^6. Fix $p \in \mathbb{R}$ and fix $(\gamma(t))_{t\in[0,1]}$ a smooth path in the plane. Then:*

1. *A.s.* $f^{-1}([-p,+\infty[) =: \mathcal{D}_p$ *and* $f^{-1}(]-\infty,-p])$ *are two 2-dimensional smooth sub-manifolds of* $\mathbb{R}^2$ *with boundary. Moreover, a.s. their boundaries are equal and are the whole set* $\mathcal{N}_p$*.*

2. *A.s.,* $\mathcal{N}_p$ *intersects* γ *transversally.*

To prove Lemma 6.A.9, we can use the following lemma:

Lemma 6.A.10 (see Lemma 11.2.10 of [AT07])**.** *Let* $n \in \mathbb{N}$*. Let* T *be a compact subset of* $\mathbb{R}^n$ *with Hausdorff dimension* $k \in \mathbb{N}$*. Let* $g = (g_j)_{1\leq j\leq k+1} : \mathbb{R}^n \to \mathbb{R}^{k+1}$ *be a Gaussian field that is a.s.* C^1*. Assume also that* g *has a bounded density on* T*. Then, for each* $v \in \mathbb{R}^{k+1}$*,* $g^{-1}(v) \cap T$ *is a.s. empty.*

Proof of Lemma 6.A.9. First note that the fact that κ is C^6 implies that f is C^2 by Lemma 6.A.1. To prove the first part of the lemma, we fix $R \in]0,+\infty[$ and $p \in \mathbb{R}$ and apply Lemma 6.A.10 to $T = [-R,R]^2$ (of Hausdorff dimension 2) and $g = (f, \partial_1 f, \partial_2 f)$ with $v = (-p,0,0)$. For every x, we have the following: i) by Remark 6.A.2, $f(x)$ is independent of $(\partial_1 f(x), \partial_2 f(x))$ and ii) by Remark 6.A.3, $(\partial_1 f(x), \partial_2 f(x))$ is non-degenerate. As a result, $g(x)$ is non-degenerate. Since g is stationary, this implies that g has bounded density. We obtain that a.s. f vanishes transversally on $\mathcal{N}_p \cap [-R,R]^2$. By taking the intersection of such events for $R = 1, 2, \cdots$ we end the proof of the first statement. For the second part of the statement, we apply Lemma 6.A.10, this time for $T = \{\gamma(t)\}_{t\in[0,1]}$ (of Hausdorff dimension 1) and $g(t) = ((f\circ\gamma)(t), (f\circ\gamma)'(t))$ with $v = (-p,0)$. As before, for every t, $g(t)$ is non-degenerate. By continuity of the covariances, this implies that g restricted to T has bounded density, so Lemma 6.A.10 does apply. □

Remark 6.A.11. The following can easily be deduced from Lemma 6.A.9: Assume that f satisfies Condition 6.1.7 and that κ is C^3. Fix $p \in \mathbb{R}$ and let $\mathcal{Q} \subseteq \mathbb{R}^2$ be a quad (i.e. a region of the plane homeomorphic to a disk, with two distinguished disjoint segments on its boundary). Then a.s. either all or none of the following events hold: (a) there is a continuous path included in $\mathcal{D}_p \cap \mathcal{Q}$ which joins one distinguished side of $\mathcal{Q}$ to the other, (b) there is such a continuous path in $f^{-1}(]-p,+\infty[)$, (c) there is no continuous path included in $f^{-1}(]-\infty,-p]) \cap \mathcal{Q}$ which joins one non-distinguished side of $\mathcal{Q}$ to the other and (d) there is no such path in $f^{-1}(]-\infty,-p[)$. Similarly, if $\mathcal{A}$ is an annulus, then a.s. either all of none of the following events hold: (a) there is a continuous path included in $\mathcal{D}_p \cap \mathcal{A}$ which separates the inner boundary of $\mathcal{A}$ from its outer boundary, (b) there is such a path in $f^{-1}(]-p,+\infty[)$, (c) there is no continuous path in $f^{-1}(]-\infty,-p]) \cap \mathcal{A}$ which joins the inner boundary of $\mathcal{A}$ to its outer boundary and (d) there is no such path in $f^{-1}(]-\infty,-p[)$.
A consequence of these properties and of the fact that f is centered is that, if we assume furthermore that f is invariant by $\frac{\pi}{2}$-rotation, then the probability that there is a left-right crossing at level 0 of the square $[0,s]^2$ is $1/2$ for any $s \in]0,+\infty[$.

The following lemma is a consequence of Lemma 6.A.9 and of Theorem 6.A.4 and is crucial in the proof of box-crossing results.

Lemma 6.A.12 (FKG)**.** *Let f be a Gaussian field on $\mathbb{R}^2$ satisfying Condition 6.1.7 such that κ is C^6. Let $p \in \mathbb{R}$. Assume that for each $x \in \mathbb{R}^2$, $\kappa(x) \geq 0$. Let A, B be obtained by taking as unions and intersections of a finite number of crossings of quads and circuits in annuli above level $-p$. Then,*

$$\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B]\,.$$

Proof. It suffices to approximate the events by increasing events that depend on f restricted a finitely many points and using Theorem 6.A.4. This can easily be done by considering the discrete model introduced in Section 6.3 and by using Lemma 6.A.9 to prove that the discrete crossing events indeed approximate the continuous crossing events (for a similar argument, see the proof of Theorem 6.1.12 in Subsection 6.3.1). □

The following lemma is useful to show that certain discrete approximations of events do converge a.s. to continuous geometric events. In the lemma we refer to the face-centered square lattice defined before (see Figure 6.1). We use this lemma only to study nodal components (see Subsection 6.3.4, but we do not need it in order to study crossing events.

Lemma 6.A.13. *Let $\mathcal{C} \subseteq \mathbb{R}^2 \setminus \{0\}$ be a compact smooth one-dimensional submanifold of $\mathbb{R}^2$ that intersects the axes $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ transversally. Assume that there is a finite number of $x \in \mathcal{C}$ such that $T_x\mathcal{C}$ is colinear to an edge of the face-centered square lattice. Then, for a.e. small enough $\varepsilon > 0$, we have:*

1. *the set $\mathcal{C}$ does not intersect the vertex set and intersects each edge of $\mathcal{T}^\varepsilon$ transversally;*
2. *each edge of $\mathcal{T}^\varepsilon$ is intersected at most twice and any two distinct intersection points of e are connected by a path in $\mathcal{C}$ inside the union of the two faces adjacent to e.*

Proof. By simple application of Sard's theorem, the first property holds for a.e. $\varepsilon > 0$. We now take $\varepsilon > 0$ such that the first property holds and prove that the second property holds for $\varepsilon > 0$ small enough. We begin by defining some constants depending on $\mathcal{C}$ that will determine how small the ε's need to be to satisfy the second property.

- Since there are a finite number of points $x \in \mathcal{C}$ such that $T_x\mathcal{C}$ is colinear to an edge of the lattice, there exists $c_1 > 0$ such that any two such distinct points are at distance greater than $4c_1$.
- The distance between any two distinct components of $\mathcal{C}$ is bounded from below by a constant $c_2 > 0$.
- Each component of $\mathcal{C}$ is the image of some smooth embedding $\gamma : S^1 \to \mathbb{R}^2$ with unit speed such that for each distinct $s, t \in S^1$, $|\gamma(s) - \gamma(t)| \geq \lambda_0 \mathrm{dist}_{S^1}(s,t)$ (here and below, dist_{S^1} denotes the distance function on S^1). Let $\|\mathbf{k}\|_\infty < +\infty$ be the maximum of the curvature $|\mathbf{k}|$ on $\mathcal{C}$ and let $c' > 0$ be such that for any two points x, y on a common edge e and any point z outside of the union of the two faces adjacent to e, the unit vectors v_1 and v_2 pointing in the directions of $z - x$ and $y - z$ satisfy $|v_1 - v_2| \geq c'$. Let $c_3 = c'\lambda_0/\|\mathbf{k}\|_\infty \in]0, +\infty]$.

We take $\varepsilon < \min(c_1, c_2, c_3)$ and prove that the second property holds.
Fix e an edge of $\mathcal{T}^\varepsilon$. Let us prove that any two intersection points on e must be connected by a smooth arc inside the union F of the two faces adjacent to e. If e is intersected at least twice, say at $x, y \in e$, then, x, y are at distance less than c_2 so they must belong to the same component $\mathcal{C}$. This component is parametrized by a smooth embedding $\gamma : S^1 \to \mathbb{R}^2$ with unit speed so there are $s, t \in S^1$, such that $\gamma(s) = x$ and $\gamma(t) = s$. By assumption, $\varepsilon \geq |x - y| \geq \lambda_0 \text{dist}_{S^1}(s, t)$. Assume that x and y are not connected by γ inside the union F of the two faces adjacent to e. Then, there exists $r \in S^1$ belonging to one of the shortest paths between z and t in S^1 such that $\gamma(r) = z \notin F$. We denote by $]s, t[$ the open interval in S^1 containing r, and denote by $]s, r[$ and $]r, t[$ the open sub-intervals with extremities s and r and r and t respectively. Let v_1 and v_2 be the unit tangent vectors pointing in the same directions as $z - x$ and $y - x$ respectively. By construction, $|v_1 - v_2| \geq c'$. By Rolle's theorem, there exist $u_1 \in]s, r[$ and $u_2 \in]r, t[$ such that $\gamma'(u_1) = v_1$ and $\gamma'(u_2) = v_2$. Moreover, by assumption, $\text{dist}_{S^1}(u_1, u_2) \leq \lambda_0^{-1}\varepsilon$. But this means that there exists $u_3 \in]u_1, u_2[$ such that

$$\|\mathbf{k}\|_\infty \geq |\mathbf{k}(\gamma(u_3))| = |\gamma''(u_3)| \geq \lambda_0 c' \varepsilon^{-1} .$$

Consequently, $\varepsilon \geq \lambda_0 c'/\|\mathbf{k}\|_\infty = c_3$ which contradicts our assumption. Therefore, x and y must be connected by a smooth arc.
Now, by Rolle's theorem, for any two distinct intersection points of e connected by a smooth arc inside F, there must be a point x on this connecting arc such that $T_x\mathcal{C}$ is colinear to e. Thus, if e contains three distinct intersection points, then there are two distinct points $x, y \in F$ such that $T_x\mathcal{C}$ and $T_y\mathcal{C}$ are colinear to e. But $x, y \in F$ so they must be at distance at most $4\varepsilon \leq 4c_1$ which contradicts the definition of c_1. Hence, $|\mathcal{C} \cap e| \leq 2$ and we are done. □

6.B A uniform discrete RSW estimate

In this section, we prove a RSW result for the discrete models studied in [BG17a]. As explained in Section 6.1, contrary to [BG17a], we do not use any discrete RSW estimate to deduce the continuous RSW estimate. However, a discrete RSW estimate uniform in the mesh ε can be useful if one wants to apply tools from discrete percolation to our model. The results of this section rely heavily on [BG17a]. We also make a small correction in the arguments made therein. For these reasons, this Appendix should be read as a companion text to [BG17a]. We would like to stress the fact that **the results presented here are not used in the rest of the paper**. We first introduce the following notations:

Notation 6.B.1. Consider the discretized model introduced in the beginning of Section 6.3 and remember Definition 6.3.2. If $\mathcal{Q}$ is a quad, write $\text{Cross}_0^\varepsilon(\mathcal{Q})$ for the event that $\mathcal{Q}$ is ε-crossed at level 0.

We have the following result.

Proposition 6.B.2. *Let f be a Gaussian field satisfying Conditions 6.1.7, 6.1.8, 6.1.10 as well as Condition 6.1.9 for some $\alpha > 4$. For every quad $\mathcal{Q}$, there exist $s_0 = s_0(\kappa, \mathcal{Q}) \in]0, +\infty[$ and $c = c(\kappa, \mathcal{Q}) > 0$ such that for each $s \in [s_0, +\infty[$ and each $\varepsilon \in]0, 1]$ we have:*

$$\mathbb{P}\left[\mathrm{Cross}_0^{\varepsilon}(s\mathcal{Q})\right] \geq c\,.$$

Note that the constant c above does not depend on ε. As in the continuous case, the first result of this kind can be found in [BG17a] by combining Theorem 2.2 of [BG17a] with their Section 4. The novelty here is that the result holds for any $\alpha > 4$ and without any constraint on (s, ε). As in the proof of Theorem 6.1.1, we need a quasi-independence result to prove Proposition 6.B.2. We are going to use Proposition 6.3.4 where the quasi-independence estimate is uniform in ε.

Proof of Proposition 6.B.2. As in Section 6.4, we follow Tassion's strategy from [Tas16]. However, since we need a constant c which is uniform in ε, it is more suitable to follow the quantitative version of Tassion's method presented in Section 2 of [BG17a].
Before going into the proof, let us warn the reader that in Section 6.4 we have used the notations from [Tas16] while in the present appendix we use the notations from [BG17a]. In particular, the notation ϕ_s has two different meanings; we hope that this will not confuse the reader.
We first assume that $\varepsilon^{-1} \in \mathbb{Z}_{>0}$ so that our model is $\mathbb{Z}^2$-periodic. As noted in [BG17a], by a simple duality argument (which works since our lattice is a triangulation), we obtain that the probability that there is a left-right crossing of $[-s/2, s/2]^2$ made of black edges of $\mathcal{T}^{\varepsilon}$ is $1/2$ for any $s \in 2\mathbb{Z}_{>0}$. Hence we have the existence of some $c_0 \in]0, 1[$ such that the probability of this event is at least c_0 for any $s \in 2\mathbb{Z}_{>0}$ as assumed in Condition 3 of Definition 2.1 in [BG17a]. We first prove the following lemma analogous to Lemma 2.7 of [BG17a]. Our way to state this lemma is a little different from [BG17a] since we think that, for the proof of this lemma to be correct, one has to consider variants of the event $\mathcal{H}_s(\cdot, \cdot)$ as we do below. The reason why we need to make such a change is that the models are not continuous, which implies that the function ψ_s (which is defined in the proof) is not continuous, so the proof written in [Tas16] does not work as is. Let us stress that, once one has made this small correction, all the other results of [BG17a] hold without any modification.

Lemma 6.B.3. *For any $s \geq 1$, $-s/2 \leq \alpha \leq \beta \leq s/2$, let $\mathcal{H}_s(\alpha, \beta)$ (resp. $\widetilde{\mathcal{H}}^1(\alpha, \beta)$, $\widetilde{\mathcal{H}}^2(\alpha, \beta)$) be the event that there is a path in $[-s/2, s/2]^2$ from the left side to $\{s/2\} \times [\alpha, \beta]$ (resp. to $\{s/2\} \times]\alpha, \beta]$, to $\{s/2\} \times [\alpha, \beta[$) made of black edges of $\mathcal{T}^{\varepsilon}$. Also, let $\mathcal{X}_s(\alpha)$ be defined exactly as in [Tas16, BG17a] (see for instance Figure 2.2 of [BG17a]). There exists a universal polynomial $Q_1 \in \mathbb{R}[X]$, positive on $]0, 1[$, such that for every $s \in 2\mathbb{Z}_{>0}$, there exists $\alpha_s = \alpha_s(\varepsilon, \kappa) \in [0, s/4]$ satisfying the following properties:*

(P1) $\mathbb{P}\left[\mathcal{X}_s(\alpha_s)\right] \geq Q_1(c_0)\,.$

(P2) *If $\alpha_s < s/4$, then* $\mathbb{P}\left[\mathcal{H}_s(0, \alpha_s)\right] \geq c_0/4 + \mathbb{P}\left[\widetilde{\mathcal{H}}_s^1(\alpha_s, s/2)\right]\,.$

Proof. For every $\alpha \in [0, s/2]$, write:

$$\psi_s(\alpha) = \psi_s(\kappa, \varepsilon, \alpha) = \mathbb{P}\left[\mathcal{H}_s(0, \alpha)\right] - \mathbb{P}\left[\mathcal{H}_s(\alpha, s/2)\right] ,$$

$$\widetilde{\psi}^1_s(\alpha) = \widetilde{\psi}^1_s(\kappa, \varepsilon, \alpha) = \mathbb{P}\left[\mathcal{H}_s(0, \alpha)\right] - \mathbb{P}\left[\widetilde{\mathcal{H}}^1_s(\alpha, s/2)\right] ,$$

and

$$\widetilde{\psi}^2_s(\alpha) = \widetilde{\psi}^2_s(\kappa, \varepsilon, \alpha) = \mathbb{P}\left[\widetilde{\mathcal{H}}^2_s(0, \alpha)\right] - \mathbb{P}\left[\mathcal{H}_s(\alpha, s/2)\right] .$$

Note that:

$$\forall \alpha \in [0, s/2[, \lim_{\substack{\alpha' \to \alpha, \\ \alpha' > \alpha}} \psi_s(\alpha') = \widetilde{\psi}^1_s(\alpha) \, ; \, \forall \alpha \in]0, s/2], \lim_{\substack{\alpha' \to \alpha, \\ \alpha' < \alpha}} \psi_s(\alpha') = \widetilde{\psi}^2_s(\alpha) \, .$$

Now, if $\Psi_s(s/4) > c_0/4$, then let α_s be the infimum over every $\alpha \in [0, s/4]$ such that $\psi_s(\alpha) > c_0/4$; otherwise let $\alpha_s = s/4$. Then, we have $\widetilde{\psi}^2_s(\alpha_s) \leq c_0/4$ and, if $\alpha_s < s/4$, we have $\widetilde{\psi}^1_s(\alpha_s) \geq c_0/4$. Thus, (P2) is satisfied. Concerning (P1), similarly as in Lemma 2.1 of [Tas16] we have:

$$\begin{aligned} c_0 &\leq 2\mathbb{P}\left[\widetilde{\mathcal{H}}^2_s(0, \alpha_s)\right] + 2\mathbb{P}\left[\mathcal{H}_s(\alpha_s, s/2)\right] \\ &\leq 4\mathbb{P}\left[\mathcal{H}_s(\alpha_s, s/2)\right] + 2\widetilde{\psi}^2_s(\alpha_s) \\ &\leq 4\mathbb{P}\left[\mathcal{H}_s(\alpha_s, s/2)\right] + c_0/2 \, . \end{aligned}$$

Finally, $\mathbb{P}\left[\mathcal{H}_s(\alpha_s, s/2)\right] \geq c_0/8$ thus as noted in [Tas16], by a simple construction and by the FKG inequality we obtain that $\mathbb{P}\left[\mathcal{X}(\alpha_s)\right] \geq c_0 \times (c_0/8)^4$. □

Next, Lemmas 2.8 and 2.9 of [BG17a] apply readily. Now, define the universal funciton τ_1 as in (2.5) of [BG17a] and define the following function:

$$\phi_s = \phi_s(\kappa, \varepsilon) = \sup \left| \mathbb{P}\left[A \cap B\right] - \mathbb{P}\left[A\right] \mathbb{P}\left[B\right] \right| ,$$

where the supremum is over any event A of the form $\mathrm{Circ}^{\varepsilon}(\mathcal{A})$ where $\mathcal{A}$ is an ε-drawn annulus centered at 0 and included in $[-s, s]^2$, and any event B which is the intersection of at most $\log(s)$ events of the form $\mathrm{Circ}^{\varepsilon}(\mathcal{A})$ where $\mathcal{A}$ is an ε-drawn annulus centered at 0 and included in $[-s\log(s), s\log(s)]^2 \backslash]-5s, 5s[^2$. Next, write:

$$\hat{s} = \hat{s}(\kappa, \varepsilon) = \max\{s \in \mathbb{Z}_{>0} \, : \, s \geq \exp(\tau_1(c_0)) \text{ and } \phi_s \geq \frac{c_0}{16} Q_3(c_0)\} \, ,$$

where Q_3 is the universal positive function that comes from Lemma 2.9 of [BG17a]. We have the following lemma analogous to Lemma 2.10 of [BG17a], where for any $0 < r < s < +\infty$, $\mathrm{Circ}^{\varepsilon}_0(r, s)$ denotes the event that there is an ε-circuit at level 0 in the annulus $[-r, r]^2 \backslash]-s, s[^2$, and where Q_2 is the universal positive function defined as in Lemma 2.8 of [BG17a].

Lemma 6.B.4. *For any $s \in \mathbb{Z}_{>0}$, $s \geq \hat{s}$, if $\mathbb{P}\left[\mathrm{Circ}^{\varepsilon}_0(s, 2s)\right] \geq Q_2(c_0)$, then there exists $s' \in [4s, \tau_1(c_0)s] \cap \mathbb{Z}_{>0}$ such that $\alpha_{s'} \geq s$.*

Proof. As noted in [BG17a], since the rest of the proof is exactly the same as in [Tas16], it is sufficient to prove that, if $s \geq \hat{s}$, then:

$$\mathbb{P}\left[\bigcap_{i=1}^{\lfloor \log_5(\tau_1)\rfloor} \mathrm{Circ}_0^{\varepsilon}(5^i s, 2\cdot 5^i s)^c\right] < c_0/4\,. \tag{6.B.1}$$

The proof is the same as in [BG17a] since by our definition of $\hat{s}$, if $s \geq \hat{s}$ and if $i_0 \in \{1, \cdots, \lfloor \log_5(\tau_1)\rfloor - 1\}$, we have:

$$\mathbb{P}\left[\bigcap_{i=i_0}^{\lfloor \log_5(\tau_1)\rfloor} \mathrm{Circ}_0^{\varepsilon}(\mathcal{A}_{5^i s, 2\cdot 5^i s}0)^c\right] \leq \mathbb{P}\left[\mathrm{Circ}_0^{\varepsilon}(\mathcal{A}_{5^{i_0} s, 2\cdot 5^{i_0} s})^c\right] \mathbb{P}\left[\bigcap_{i=i_0+1}^{\lfloor \log_5(\tau_1)\rfloor} \mathrm{Circ}_0^{\varepsilon}(\mathcal{A}_{5^i s, 2\cdot 5^i s})^c\right] + \frac{c_0}{16} Q_3\,.$$

Note that here the fact that (P2) in Lemma 6.B.3 is written with $\widetilde{\mathcal{H}}_s^1(\alpha_s, s/2)$ instead of $\mathcal{H}_s(\alpha_s, s/2)$ does not change the proof at all. □

Now, define $\gamma(\nu)$, $t_\nu = t_\nu(\kappa, \varepsilon)$ and $s_\nu = s_\nu(\kappa, \varepsilon)$ as in (2.8), (2.9) and (2.10) of [BG17a] with $\hat{s}$ instead of $s(\Omega)$ i.e.:

$$\begin{aligned}
&\gamma(\nu) = 1 + \log_{4/(3+2\nu)}(3/2 + \nu) > 1\,,\\
&s_\nu = \max(\hat{s}, \lfloor 6/\nu \rfloor + 1)\,,\\
&t_\nu = (3/2+\nu) s_\nu^{\gamma(\nu)} \alpha_{s_\nu}^{1-\gamma(\nu)}\,.
\end{aligned}$$

Then, the proof of Lemma 2.11 of [BG17a] applies readily with our definitions. Finally, as in the proof of Theorem 2.2 of [BG17a], we obtain that for every $\nu \in]0, 1/2[$, there exists a universal positive continuous function P_ν defined on $[1, +\infty[\times]0, 1[$ such that, for every $\rho \geq 1$ and every $s \in \mathbb{Z}_{>0}$ such that $s \geq t_\nu$, the probability that there is a black path in $[0, \rho s] \times [0, s]$ from the left side to the right side is at least $P_\nu(\rho, c_0)$.
At this point, we want to have an upper bound on $t_\nu = t_\nu(\varepsilon)$ independent on ε, i.e. we want to have an upper bound $\hat{s} = \hat{s}(\varepsilon)$ and a lower bound on $\alpha_{s_\nu(\varepsilon)}(\varepsilon)$ that do not depend on ε. To this purpose, first note that the functions Q_2, Q_3 and P_ν are continuous functions of Q_1 and that, as explained in Lemma 4.6 of [BG17a], there exists $a = a(\kappa) > 0$ and $b = b(\kappa) > 0$ such that, if one replace the universal function Q_1 by the function aQ_1 that depends only on κ, then we have $\alpha_s = \alpha_s(\kappa, \varepsilon) \geq b$ for every s. More precisely, we can choose any $a \in]0, 1[$ and $b \in]0, 1/2[$ so that, for every s, the probability that f is positive both in the $4b \times 4b$ box centered at $(-s/2, 0)$ and the $4b \times 4b$ box centered at $(s/2, 0)$ is at least a. Such quantities exist since f is a.s. continuous and thanks to FKG. Secondly, note that, by Proposition 6.3.4, ϕ_s is at most:

$$C\,\log(s)\,(\log(s)s)^2 s^2 s^{-\alpha}\,.$$

for some $C = C(\kappa) < +\infty$. Hence (and since $\alpha > 4$) $\hat{s}$ is less than some finite constant $M = M(\kappa)$ does not depend on ε. Finally, t_ν is less than some finite constant that does not depend on ε, and we have obtained Proposition 6.B.2 for $\varepsilon^{-1} \in \mathbb{Z}_{>0}$ and when the quad is a rectangle $[0, \rho] \times [0, 1]$.
To end the proof, first note one can easily extend the result to any quad by reasoning as in the proof of Theorem 6.1.1. Finally, to extend the result to any $\varepsilon \in]0, 1]$, fix such an ε, let $\lambda \in [1/2, 2]$ such that $(\lambda\varepsilon)^{-1} \in \mathbb{Z}_{>0}$ and define the planar Gaussian field $f_\lambda : x \mapsto f(\lambda x)$ with covariance function $(x, y) \mapsto \kappa_\lambda(x - y)$. For any $\varepsilon' > 0$ and any quad $\mathcal{Q}$, write $\mathrm{Cross}_0^{\varepsilon',\lambda}(\mathcal{Q})$ for the event $\mathrm{Cross}_0^{\varepsilon'}(\mathcal{Q})$ but with f_λ instead of f. Note that we have:

$$\mathrm{Cross}_0^{\varepsilon}(\mathcal{Q}) = \mathrm{Cross}_0^{\lambda\varepsilon,\lambda}(\mathcal{Q}) .$$

Moreover, it is not difficult to see that, since λ belongs to the compact subset of $]0, +\infty[$, $[1/2, 2]$, one can find constant $a = a(\kappa_\lambda)$, $b = b(\kappa_\lambda)$ and $M = M(\kappa_\lambda, c_0)$ as above that are uniform in λ. This ends the proof. □

As in the continuous case, we can deduce that the one-arm event decreases polynomially fast. We first need a notation.

Notation 6.B.5. If $0 < r < s < +\infty$, we write $\mathcal{A}(r, s) = [-s, s]^2 \setminus]-r, r[^2$ and we write $\mathrm{Arm}_0^{\varepsilon}(r, s$ (resp. $\mathrm{Arm}_0^{*,\varepsilon}(r, s)$) for the event that there is an ε-black path rom the inner boundary of $\mathcal{A}(r, s)$ to its outer boundary made of black edges (resp. that lives in the white region of the plane) in the discrete percolation model of mesh ε defined in the beginning of Section 6.3 with $p = 0$.

Proposition 6.B.6. *Assume that f satisfies Conditions 6.1.7, 6.1.8, 6.1.10 as well as Condition 6.1.9 for some $\alpha > 4$. There exists $C = C(\kappa) < +\infty$ and $\eta = \eta(\kappa) > 0$ such that, for each $\varepsilon \in]0, 1]$, for each $s \in [1, +\infty[$ and $r \in [1, s[$:*

$$\mathbb{P}\left[\mathrm{Arm}_0^{\varepsilon}(r, s)\right], \mathbb{P}\left[\mathrm{Arm}_0^{*,\varepsilon}(r, s)\right] \leq C\,(r/s)^{\eta} .$$

Proof. First note that, since f and $-f$ have the same law, we have:[12]

$$\left(\mathbb{P}\left[\mathrm{Arm}_0^{\varepsilon,*}(r + \varepsilon, s - \varepsilon)\right] \leq\right) \mathbb{P}\left[\mathrm{Arm}_0^{\varepsilon}(r, s)\right] \leq \mathbb{P}\left[\mathrm{Arm}_0^{\varepsilon,*}(r, s)\right] .$$

So it is sufficient to prove the result for $\mathrm{Arm}_0^{*,\varepsilon}(r, s)$. The proof is roughly the same as the proof of Proposition 6.4.5 except that we use Propositions 6.B.2 and 6.3.4 instead of Theorem 6.1.1 and Theorem 6.1.12. The only difference is that we have to consider only ε-annuli, but that is not a problem. The constants do not depend on ε since the constants in Propositions 6.B.2 and 6.3.4 do not. □

As in the continuous case, the following is a direct consequence of Proposition 6.B.6:

Proposition 6.B.7. *With the same hypotheses as Proposition 6.B.6, for each $\varepsilon \in]0, 1]$ a.s. there is no unbounded black component in the discrete percolation model of mesh ε defined in the beginning of Section 6.3 with $p = 0$.*

[12] These inequalities are not equalities only because black and white regions of the plane are not totally dual. These would be equalities if we had $\varepsilon^{-1} r \in \mathbb{N}$ and $\varepsilon^{-1} s \in \mathbb{N}$.

Chapter 7

The critical threshold for Bargmann-Fock percolation

Abstract

In this article, we study the excursion sets $\mathcal{D}_p = f^{-1}([-p, +\infty[)$ where f is a natural real-analytic planar Gaussian field called the Bargmann-Fock field. More precisely, f is the centered Gaussian field on $\mathbb{R}^2$ with covariance $(x, y) \mapsto \exp(-\frac{1}{2}|x - y|^2)$. Alexander has proved that, if $p \leq 0$, then a.s. $\mathcal{D}_p$ has no unbounded component. We show that conversely, if $p > 0$, then a.s. $\mathcal{D}_p$ has a unique unbounded component. As a result, the critical level of this percolation model is 0. We also prove exponential decay of crossing probabilities under the critical level. To show these results, we rely on a recent box-crossing estimate by Beffara and Gayet. We also develop several tools including a KKL-type result for biased Gaussian vectors (based on the analogous result for product Gaussian vectors by Keller, Mossel and Sen) and a sprinkling inspired discretization procedure. These intermediate results hold for more general Gaussian fields, for which we prove a discrete version of our main result.

This chapter is based on [RV17b], which is joint work with Hugo Vanneuville.

7.1 Main results

In this article, we study the geometry of excursion sets of a planar centered Gaussian field f. The **covariance function** of f is the function $K : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$\forall x, y \in \mathbb{R}^2,\ K(x,y) = \mathbb{E}[f(x)f(y)]\,.$$

We assume that f is normalized so that for each $x \in \mathbb{R}^2$, $K(x,x) = \mathrm{Var}(f(x)) = 1$, that it is non-degenerate (i.e. for any pairwise distinct $x_1, \dots, x_k \in \mathbb{R}^2$, $(f(x_1), \cdots, f(x_k))$ is non-degenerate), and that it is a.s. continuous and stationary. In particular, there exists a strictly positive definite continuous function $\kappa : \mathbb{R}^2 \to [-1,1]$ such that $\kappa(0) = 1$ and, for each $x, y \in \mathbb{R}^2$, $K(x,y) = \kappa(x-y)$. For our main results (though not for all the intermediate results), we will also assume that f is positively correlated, which means that κ takes only non-negative values. We will also refer to κ as covariance function when there is no possible ambiguity. For each $p \in \mathbb{R}$ we call **level set** of f the random set $\mathcal{N}_p := f^{-1}(-p)$ and **excursion set** of f the random set $\mathcal{D}_p := f^{-1}([-p, +\infty[)$.[1]

These sets have been studied through their connections to percolation theory (see [MS83a], [MS83b], [MS86] [Ale96], [BS07], [BG17a], [BM18], [BMW17], [RV17a]). In this theory, one wishes to determine whether of not there exist unbounded connected components of certain random sets. So far, we know that $\mathcal{D}_0$ has a.s. only bounded components for a very large family of positively correlated Gaussian fields:

Theorem 7.1.1 (Theorem 2.2 of [Ale96])**.** *Assume that for each $x \in \mathbb{R}^2$, $\kappa(x) \geq 0$, that f is a.s. C^1 and ergodic with respect to translations. Assume also that for each $p \in \mathbb{R}$, f has a.s. no critical points at level p. Then, a.s. all the connected components of $\mathcal{D}_0$ are bounded.*

Proof. By [Pit82b], the fact that κ is non-negative implies that f satisfies the FKG inequality so we can apply Theorem 2.2 of [Ale96]. Hence, all its level lines are bounded. By ergodicity, $\mathcal{D}_0$ has either a.s. only bounded connected components or a.s. at least one unbounded connected component. Since f has a.s. no critical points at level 0, a.s., the boundary of $\mathcal{D}_0$ equals $\mathcal{N}_0$ and is a C^1 submanifold of $\mathbb{R}^2$. If $\mathcal{D}_0$ had a.s. an unbounded connected component, then, by symmetry (since f is centered) this would also be the case for $\overline{\mathbb{R}^2 \setminus \mathcal{D}_0}$. But this would imply that $\mathcal{N}_0$ has an unbounded connected component, thus contradicting Theorem 2.2 of [Ale96].- □

More recently, Beffara and Gayet [BG17a] have proved a more quantitative version of Theorem 7.1.1 which holds for a large family of positively correlated stationary Gaussian fields such that $\kappa(x) = O(|x|^{-\alpha})$ for some α sufficiently large. In [RV17a], the authors of the present paper have revisited the results by [BG17a] and weaken the assumptions on α. More precisely, we have the following:

[1] This convention, while it may seem counterintuitive, is convenient because it makes $\mathcal{D}_p$ increasing both in f and in p. See Section 7.2 for more details.

Theorem 7.1.2 ([BG17a] for α sufficiently large,[RV17a]). [2] *Assume that f is a non-degenerate, centered, normalized, continuous, stationary, positively correlated planar Gaussian field that satisfies the symmetry assumption Condition 7.2.2 below. Assume also that κ satisfies the differentiability assumption Condition 7.2.4 below and that $\kappa(x) \leq C|x|^{\alpha}$ for some $C < +\infty$ and $\alpha > 4$. Then, there exist $C' = C'(\kappa) < +\infty$ and $\delta = \delta(\kappa) > 0$ such that for each $r > 0$, the probability that there exists a connected component of $\mathcal{D}_0$ which connects 0 to a point at distance r is at most $C'r^{-\delta}$. In particular, a.s. all the connected components of $\mathcal{D}_0$ are bounded.*

A remaining natural question is whether or not, for $p > 0$, the excursion set $\mathcal{D}_p$ has an unbounded component. Our main result (Theorem 7.1.3 below) provides an answer to this question for a specific, natural choice of f, arising naturally from real algebraic geometry: the Bargmann-Fock model, that we now introduce. **The planar Bargmann-Fock field** is defined as follows. Let $(a_{i,j})_{i,j\in\mathbb{N}}$ be a family of independent centered Gaussian random variables of variance 1. For each $x = (x_1, x_2) \in \mathbb{R}^2$, the Bargmann-Fock field at x is:

$$f(x) = e^{-\frac{1}{2}|x|^2} \sum_{i,j\in\mathbb{N}} a_{i,j} \frac{x_1^i x_2^j}{\sqrt{i!j!}}\,.$$

The sum converges a.s. in $C^\infty(\mathbb{R}^2)$ to a random analytic function. Moreover, for each $x, y \in \mathbb{R}^2$:

$$K(x,y) = \mathbb{E}[f(x)f(y)] = \exp\Big(-\frac{1}{2}|x-y|^2\Big)\,.$$

For a discussion of the relation of the Bargmann-Fock field with algebraic geometry, we refer the reader to the introduction of [BG17a]. Theorem 7.1.2 applies to the Bargmann-Fock model (see Subsection 7.2.2 for the non-degeneracy condition). Hence, if $p \leq 0$ then a.s. all the connected components of $\mathcal{D}_p$ are bounded. In this paper, we prove that on the contrary if $p > 0$ then a.s. $\mathcal{D}_p$ has a unique unbounded component, thus obtaining the following result:

Theorem 7.1.3. *Let f be the planar Bargmann-Fock field. Then, the probability that $\mathcal{D}_p$ has an unbounded connected component is 1 if $p > 0$, and 0 otherwise. Moreover, if it exists, such a component is a.s. unique.*

As a result, the "critical threshold" of this continuum percolation model is $p = 0$. Before saying a few words about the proof of Theorem 7.1.3, let us state the result which is at the heart of the proof of Theorem 7.1.2, both in [BG17a] and in [RV17a]. This result is a **box-crossing estimate**, which is an analog of the classical Russo-Seymour-Welsh theorem for planar percolation. This was proved in [BG17a] for a large family of positively correlated stationary Gaussian fields such that $\kappa(x) = O(|x|^{-325})$. In [BM18], Beliaev and Muirhead have lowered the exponent $\alpha = 325$ to any $\alpha > 16$. In [RV17a],

[2] More precisely, this is Propositions 4.5 and 4.6 of [RV17a]. Moreover, this is Theorem 5.7 of [BG17a] for α sufficiently large and with slightly different assumptions on the differentiability and on the non-degeneracy of κ.

we have obtained that such a result holds with any exponent $\alpha > 4$. More precisely, we have the following:

Theorem 7.1.4 ([BG17a] for $\alpha > 325$, [BM18] for $\alpha > 16$,[RV17a]). [3] *With the same hypotheses as Theorem 7.1.2, for every $\rho \in]0, +\infty[$ there exists $c = c(\rho, \kappa) > 0$ such that for each $R \in]0, +\infty[$, the probability that there is a continuous path in $\mathcal{D}_0 \cap [0, \rho R] \times [0, R]$ joining the left side of $[0, \rho R] \times [0, R]$ to its right side is at least c. Moreover, there exists $R_0 = R_0(\kappa) < +\infty$ such that the same result holds for $\mathcal{N}_0$ as long as $R \geq R_0$.*

In order to prove our main result Theorem 7.1.3, we will use a discrete analog of this box-crossing estimate which goes back to [BG17a] (see Theorem 7.2.13 below). In Section 7.2, we expose the general strategy of the proof of Theorem 7.1.3. This proof can be summed up as follows: i) we discretize our model as was done in [BG17a], ii) we prove that there is a sharp threshold phenomenon at $p = 0$ in the discrete model, iii) we return to the continuum. The results at the heart of our proof of a sharp threshold phenomenon for the discrete model are on the one hand Theorem 7.2.13 (the discrete version of Theorem 7.1.4) and on the other hand a Kahn-Kalai-Linial (KKL)-type estimate for biased Gaussian vectors (see Theorem 7.2.19) that we show by using the analogous estimate for product Gaussian vectors proved by Keller, Mossel and Sen in [KMS12] (the idea to use a KKL theorem to compute the critical point of a percolation model goes back to Bollobás and Riordan [BR06a, BR06d], see Subsection 7.2.1 for more details). To go back to the continuum, we apply a sprinkling argument to a discretization procedure taylor-made for our setting (see Proposition 7.2.22). This step is especially delicate since Theorem 7.2.19 gives no relevant information when the discretization mesh is too fine (see Subsection 7.2.4 for more details).
Most of the intermediate results that we will prove work in a much wider setting, see in particular Proposition 7.3.5 where we explain how, for a large family of Gaussian fields f, the proof of an estimate on the correlation function would imply that Theorem 7.1.3 also holds for f. See also Theorem 7.2.13 which is a discrete analog of Theorem 7.1.3 for more general Gaussian fields.
As in [BG17a], we are inspired by tools from percolation theory. Before going any further, let us make a short detour to present the results of planar percolation we used to guide our research. It will be helpful to have this analogy in mind to appreciate our results.

Planar Bernoulli percolation is a statistical mechanics model defined on a planar lattice, say $\mathbb{Z}^2$, depending on a parameter $p \in [0, 1]$. Consider a family of independent Bernoulli random variables $(\omega_e)_e$ of parameter p indexed by the edges of the graph $\mathbb{Z}^2$. We say that an edge e is black if the corresponding random variable ω_e equals 1 and white otherwise. The analogy with our model becomes apparent when one introduces the following classical coupling of the $(\omega_e)_e$ for various values of p. Consider a family $(U_e)_e$ of

[3] More precisely, this is Theorem 1.1 of [RV17a]. Moreover, this is Theorem 4.9 of [BG17a] (resp. Theorem 1.7 of [BM18]) for $\alpha \geq 325$ (resp. $\alpha > 16$) and with slightly different assumptions on the differentiability and on the non-degeneracy of κ.

independent uniform random variables in $[0,1]$ indexed by the set of edges of $\mathbb{Z}^2$. For each $p \in [0,1]$, let $\omega_e^p = \mathbb{1}_{\{U_e \geq 1-p\}}$. Then, the family $((\omega_e^p)_e)_p$ forms a coupling of Bernoulli percolation with parameters in $[0,1]$. In this coupling, black edges are seen as excursion sets of the random field $(U_e)_e$. Theorem 7.1.4 is the analog of the Russo-Seymour-Welsh (RSW) estimates first proved for planar Bernoulli percolation in [Rus78, SW78] (see also Lemma 4 of Chapter 3 of [BR06b], Theorem 11.70 and Equation 11.72 of [Gri99] or Theorem 5.31 of [Gri10]). We now state the main result of percolation theory on $\mathbb{Z}^2$, a celebrated theorem due to Kesten. Chapter 3 of [BR06b], Chapter 11 of [Gri99] and Chapter 5 of [Gri10] present different approaches to prove this result.

Theorem 7.1.5 (Kesten's Theorem, [Kes80])**.** *Consider planar Bernoulli percolation of parameter $p \in [0,1]$ on $\mathbb{Z}^2$. If $p > 1/2$, then a.s. there exists an unbounded connected component made of black edges. On the other hand, if $p \leq 1/2$ then a.s. there is no unbounded connected component made of black edges.*

The parameter $p_c = 1/2$ is said to be critical for planar Bernoulli percolation on $\mathbb{Z}^2$. It is also known that, if such an unbounded connected component exists, it is a.s. unique. In Theorem 7.1.5, the case where $p = 1/2$ goes back to Harris [Har60] and also follows easily from the RSW estimate. Next, the coupling constructed above yields the result for $p < 1/2$. See Subsection 7.2.1 where we explain which are the main ingredients of a proof of Kesten's theorem that will inspire us.
Kesten's theorem is closely linked with another, more quantitative result:

Theorem 7.1.6 (Exponential decay, [Kes80])**.** *Consider planar Bernoulli percolation with parameter $p > 1/2$ on $\mathbb{Z}^2$. Then, for each $\rho > 0$, there exists a constant $c = c(p,\rho) > 0$ such that for each $R > 0$, the probability that there is a continuous path made of black edges in $[0, \rho R] \times [0, R]$ joining the left side of $[0, \rho R] \times [0, R]$ to its right side is at least $1 - e^{-cR}$.*

The value $p = 1/2$ is significant because with this choice of parameter, the induced percolation model on the dual graph of $\mathbb{Z}^2$ has the same law as the initial one (for more details see Lemma 1 of Chapter 3 of [BR06b] and the preceding discussion, or Chapter 11 of [Gri99]). For this reason, $1/2$ is called the self-dual point. In the case of our planar Gaussian model, self-duality arises at the parameter $p = 0$ (see the duality properties used in [BG17a, RV17a], for instance Remark A.12 of [RV17a]). The results on Bernoulli percolation lead us to the following conjecture:

Conjecture 7.1.7. *For centered, normalized, non-degenerate, sufficiently smooth, stationary, isotropic, and positively correlated random fields on $\mathbb{R}^2$ with sufficient correlation decay, the probability that $\mathcal{D}_p$ has an unbounded connected component is 1 if $p > 0$, and 0 otherwise.*

Of course, our Theorem 7.1.3 is an answer to the above conjecture for a particular model. We also have the following analog analog for Theorem 7.1.6 that we prove in Subsection 7.3.2.

Theorem 7.1.8. *Consider the Bargmann-Fock field and let $p > 0$. Then, for each $\rho > 0$ there exists a constant $c = c(p, \rho) > 0$ such that for each $R > 0$, the probability that there is a continuous path in $\mathcal{D}_p \cap [0, \rho R] \times [0, R]$ joining the left side of $[0, \rho R] \times [0, R]$ to its right side is at least $1 - e^{-cR}$.*

Contrary to the other results of this paper, the fact that the correlation function of the Bargmann-Fock field decreases more than exponentially fast is crucial to prove Theorem 7.1.8.

Organization of the paper. The paper is organized as follows:

- In Section 7.2, we explain the general strategy of the proof of our phase transition result Theorem 7.1.3. In particular, we explain the discretization we use and we state the main intermediate results including the KKL-type inequality for biased Gaussian vectors mentioned above.
- In Section 7.3, we combine all these intermediate results to prove Theorems 7.1.3 and 7.2.13.
- In Sections 7.4 to 7.7, we prove these intermediate results.

Related works. As explained above, the present article is in the continuity of [BG17a] where the authors somewhat initiate the study of a rigorous connection between percolation theory and the behaviour of nodal lines. In [BM18], the authors optimize the results from [BG17a] and the authors of the present paper optimize them further in [RV17a]. See also [BMW17] where the authors prove a box-crossing estimate for Gaussian fields on the sphere or the torus by adapting the strategy of [BG17a]. In [BG17a, BM18, BMW17, RV17a] (while the approaches differ in some key places), the initial idea is the same, namely to use Tassion's general method to prove box-crossing estimates, which goes back to [Tas16]. To apply such a method, we need to have in particular a positive association property and a quasi-independence property. In [RV17a], we have proved such a quasi-independence property for planar Gaussian fields that we will also use in the present paper, see Claim 7.3.7. We will also rely to other results from [RV17a], in particular a discrete RSW estimate. As we will explain in Subsection 7.2.4, we could have rather referred to the slightly weaker analogous results from [BG17a], which would have been sufficient in order to prove our main result Theorem 7.1.3.

The use of a KKL theorem to prove our main result Theorem 7.1.3 shows that our work falls within the approach of recent proofs of phase transition that mix tools from percolation theory and tools from the theory of Boolean functions. See [GS14] for a book about how these theory can combine. Below, we list some related works in this spirit.

- During the elaboration of the present work, Duminil-Copin, Raoufi and Tassion have developed novel techniques based on the theory of randomized algoritheorems and have proved new sharp threshold results, see [DCRT17b, DCRT17a].

This method has proved robust enough to work in a variety of settings: discrete and continuous (the Ising model and Voronoi percolation), with dependent models (such as FK-percolation) and in any dimension. It seems worthwhile to note that the present model resists direct application. At present, we see at least two obstacles: first of all the influences that arise in our setting are not the same as those of [DCRT17b, DCRT17a] (more precisely, the influences studied by Duminil-Copin, Raoufi and Tassion can be expressed as covariances while ours cannot exactly, see Remark **??**). Secondly, right now it is not obvious for us whether or not our measures are monotonic.

- Another related work whose strategy is closer to the present paper is [Rod15], where the author studies similar questions for the d-dimensional discrete (massive) Gaussian free field. Some elements of said work apply to general Gaussian fields. More precisely, following the proof of Proposition 2.2 of [Rod15], one can express the derivative probability with respect to the threshold p as a sum of covariances, which seems promising, especially in view of [DCRT17b, DCRT17a]. However, each covariance is weighted with a sum of coefficients of the inverse covariance matrix of the discretized field and at present we do not know how to deal with these sums. In [Rod15], these coefficients are very simple because we are dealing with the Gaussian free field.
- The idea to use a KKL inequality to compute the critical point of a planar percolation model comes from [BR06d, BR06c, BR06a]. See also[BDC12, DCRT16] where such an inequality is used to study FK percolation. In [BDC12, Rod15, DCRT16], the authors use a KKL inequality for monotonic measures proved by Graham and Grimmett [GG06, GG11]. The same obstacles as in Item 1 above prevented us to use this KKL inequality.

A note on vocabulary. We end the first section by a remark on vocabulary on positive definite matrices and functions.

Remark 7.1.9. In all the paper, we are going to deal with positive definite functions and matrices. The convention seems to be that, on the one hand positive definite matrices are invertible whereas semi-positive definite matrices are not necessarily. On the other hand, a function g is said strictly positive definite if for any $n \in \mathbb{Z}_{>0}$ and any $x_1, \cdots, x_n$, $(g(x_i - x_j))_{1\leq i,j\leq n}$ is a positive definite matrix while g is said positive definite if the matrices $(g(x_i - x_j))_{1\leq i,j\leq n}$ are semi-positive definite. We will follow these conventions and hope this remark will clear up any ambiguities.

Acknowledgments:
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7.2 Proof strategy and intermediate results

In this section, we explain the global strategy of the proof of our main result Theorem 7.1.3. Since the case $p \leq 0$ is already known (see the beginning of Section 7.1), we focus to the case $p > 0$. We first discuss briefly some aspects of the analogous result for Bernoulli percolation: Kesten's theorem. Next, we give an informal explanation of our proof and state the main intermediate results. More precisely, we explain the discretization procedure used in our proof in Subsection 7.2.4. Then, we state the main intermediate results at the discrete level in Subsection 7.2.5, and in Subsection 7.2.6 we explain how to go from the discrete to the continuum.

7.2.1 Some ingredients for the proof of Kesten's theorem

Several proofs of Kesten's theorem (Theorem 7.1.5) are known (see [Gri10, BR06b]). Let us fix a parameter $p \in [0,1]$ and consider Bernoulli percolation on $\mathbb{Z}^2$ with parameter p. As explained before, we are mainly interested in the proof of existence of an unbounded component, i.e. when $p > 1/2$. One possible proof of Kesten's theorem uses the following ingredients, that we will try to adapt to our setting.

- *A box-crossing criterion:* For all $\rho_1, \rho_2 > 0$, let $\mathrm{Cross}_p^{\mathrm{perco}}(\rho_1, \rho_2)$ denote the event that there is a continuous path of black edges that crosses the rectangle $[0, \rho_1] \times [0, \rho_2]$ from left to right, and assume that:

$$\sum_{k \in \mathbb{N}} \mathbb{P}\left[\neg \mathrm{Cross}_p^{\mathrm{perco}}(2^{k+1}, 2^k)\right] < +\infty\,.$$

  Then, a.s. there exists an infinite black component.

- *The RSW (Russo-Seymour-Welsh) theorem* (see Lemma 4 of Chapter 3 of [BR06b], Theorem 11.70 and Equation 11.72 of [Gri99] or Theorem 5.31 of [Gri10]) which implies that, for every $\rho > 0$, there exists a constant $c = c(\rho) > 0$ such that, for every $R > 0$, $\mathbb{P}\left[\mathrm{Cross}_{1/2}^{\mathrm{perco}}(\rho R, R)\right] \geq c(\rho)$.

- *The FKG-Harris inequality* (see [Gri99, BR06b]) that says that increasing events are positively correlated.

- *Russo's differential formula* ([Gri99, BR06b]): Let $n \in \mathbb{N}_+$, $\mathbb{P}_p^n := (p\delta_1 + (1-p)\delta_0)^{\otimes n}$ and $A \subseteq \{0,1\}^n$. For every $i \in \{1, \cdots, n\}$, let $\mathrm{Infl}_i^p(A)$ denote the influence of i on A at the parameter p, which is the probability that changing the value of the coordinate i modifies $\mathbb{1}_A$. If A is increasing, then we have the following differential formula:

$$\frac{d}{dp}\mathbb{P}_p^n[A] = \sum_{i=1}^{n} \mathrm{Infl}_i^p(A)\,.$$

- *A KKL (Kahn-Kalai-Linial) theorem* (see Theorem 4.29 of [Gri10], Theorem 12 of Chapter 2 of [BR06b] or Theorems 1.16 and 3.4 of [GS14]): The sum of influences

can be estimated thanks to the celebrated KKL theorem. Here, we present the version of the KKL theorem that implies that, if all the influences are small, then the sum of the influences is large. A qualitative version of this principle was proved by Russo in [**?**]. The KKL theorem, proved in [KKL88] for $p = 1/2$ and generalized in [BKK$^+$92, **?**] to every p, is a quantitative version of [**?**]. Let $\mathbb{P}_p^n$ be as above. There exists an absolute constant $c > 0$ such that, for every $p \in [0,1]$, every $n \in \mathbb{N}_+$ and every $A \subseteq \{0,1\}^n$, we have:

$$\sum_{i=1}^{n} \mathrm{Infl}_i^p(A) \geq c\, \mathbb{P}_p^n[A] \cdot (1 - \mathbb{P}_p^n[A]) \cdot \log\left(\frac{1}{\max_i \mathrm{Infl}_i^p(A)}\right).$$

The idea to use a KKL theorem to prove Kesten's theorem comes from [BR06d]. These five ingredients can be used as follows to prove that there exists an infinite black component when $p > 1/2$ (for more details, see for instance Section 3.4 of [BR06b], Section 5.8 of [Gri99] or Section 3.4 of [GS14]): the RSW theorem and the FKG-Harris inequality can be used to prove that the influences of the crossing events decrease polynomially fast in R. Thanks to this polynomial decay, Russo's formula, the RSW theorem and the KKL theorem, one can prove that, for every $p > 1/2$, $\mathbb{P}\left[\mathrm{Cross}_p^{\mathrm{perco}}(2R,R)\right] \geq 1 - R^{-a}$ for some $a = a(p) > 0$ (actually, with a bit more work, one can deduce exponential decay, see for instance Section 3.5 of [BR06b]). Thus, the box-crossing criterion is satisfied and we are done.

Our global strategy to prove Theorem 7.1.3 will be based on similar ingredients and on a discretization procedure used in [BG17a, BM18] (together with a sprinkling argument). Some of these ingredients are already known, the others will be proved in our paper. We list them in the remaining subsections of Section 7.2, and in Section 7.3.1 we will explain how we can combine all these ingredients to prove Theorem 7.1.3.

Since most of our intermediate results work in a much wider setting, we first state the conditions on the planar Gaussian field f under which we work.

7.2.2 Conditions on the Gaussian fields

First, we state Condition 7.2.1 **that we will assume during all the work**:

Condition 7.2.1. The field f is non-degenerate (i.e. for any pairwise distinct $x_1, \ldots, x_k \in \mathbb{R}^2$, $(f(x_1), \cdots, f(x_k))$ is non-degenerate), centered, normalized, continuous, and stationary. In particular, there exists a strictly positive definite continuous function $\kappa : \mathbb{R}^2 \to [-1,1]$ such that $K(x,y) := \mathbb{E}[f(x)f(y)] = \kappa(y-x)$ and $\kappa(0) = 1$.

Depending on the intermediate results we prove, we will also need to assume some of the following additional conditions:

Condition 7.2.2 (Useful to apply percolation arguments.)**.** The field f is positively correlated, invariant by $\frac{\pi}{2}$-rotation, and reflection through the horizontal axis.

Condition 7.2.3 (Useful to have quasi-indepence.)**.** Depends on a parameter $\alpha > 0$.] There exists $C < +\infty$ such that for each $x \in \mathbb{R}^2$, $|\kappa(x)| \leq C|x|^{-\alpha}$.

Condition 7.2.4 (Technical conditions to have quasi-independence, see [RV17a].)**.** The function κ is C^8 and for each $\beta \in \mathbb{N}^2$ with $\beta_1 + \beta_2 \leq 2$, $\partial^\beta \kappa(x) \underset{|x|\to+\infty}{\longrightarrow} 0$.

Condition 7.2.5 (Useful to do Fourier calculations on the correlation function. Depends on a parameter $\alpha > 0$.)**.** The Fourier transform of κ takes only positive values. Moreover, κ is C^3 and there exists $C < +\infty$ such that for every $\beta \in \mathbb{N}^2$ with $\beta_1 + \beta_2 \leq 3$, we have:

$$|\partial^\beta \kappa(x)| \leq C|x|^{-\alpha}\,.$$

We will often suppose regularity conditions on f and κ. It will be interesting to have the following in mind (see for instance Appendices A.3 and A.9 of [NS16]):

Lemma 7.2.6. *Assume that f satisfies Condition 7.2.1. Let $k \in \mathbb{N}$. If κ is $C^{2(k+1)}$, then a.s. f is C^k. Conversely, if a.s. f is C^k, then κ is C^{2k} and for every multi-indices $\beta, \gamma \in \mathbb{N}^2$ such that $\beta_1 + \beta_2, \leq k$ and $\gamma_1 + \gamma_2 \leq k$, we have:*

$$\mathrm{Cov}\left(\partial^\beta f(x), \partial^\gamma f(y)\right) = \mathbb{E}\left[\partial^\beta f(x)\partial^\gamma f(y)\right] = (-1)^{|\gamma|}\partial^{\beta+\gamma}\kappa(x-y)\,.$$

It is easy to check that the conditions are all satisfied by the Bargmann-Fock field (see Lemma 7.2.7 for the non-degeneracy condition). In particular, Conditions 7.2.4 and 7.2.5 hold for any $\alpha > 0$. Also, the Bargmann-Fock field is a.s. analytic and its covariance is analytic.

An other example of Gaussian fields in the plane that satisfy the above conditions (with the parameter α which depends on the parameter n) is the field with correlation function:

$$\kappa\,:\, x = (x_1, x_2) \in \mathbb{R}^2 \mapsto \left(\frac{1}{(1+x_1^2)(1+x_2^2)}\right)^n\,, \tag{7.2.1}$$

where $n \in \mathbb{Z}_{>0}$. This is indeed a strictly positive definite function by the following lemma.

Lemma 7.2.7. *The Fourier transform of a continuous and integrable function $\mathbb{R}^2 \to \mathbb{R}_+$ which is not 0 is strictly positive definite. In particular, the Gaussian function $x \mapsto \exp(-\frac{1}{2}|x|^2)$ and the function (7.2.1) (for any $n \in \mathbb{Z}_{>0}$) are strictly positive definite.*

Proof. This is a direct consequence of Theorem 3 of Chapter 13 of [CL09] (which is the strictly positive definite version of the easy part of Bochner theorem). We can apply this to the Bargmann-Fock field and to (7.2.1) since the Fourier transform of a Gaussian function is still a Gaussian function and since the Fourier transform of $x \mapsto \frac{1}{(1+x_1^2)(1+x_2^2)}$ is $\xi \mapsto cst\, \exp(-(|\xi_1| + |\xi_2|))$, hence $x \in \mathbb{R}^2 \mapsto \left(\frac{1}{(1+x_1^2)(1+x_2^2)}\right)^n$ is the Fourier transform of the function $\xi \mapsto cst\, \exp(-(|\xi_1| + |\xi_2|))$ convoluted n times, see Paragraph 1.2.3 of [Rud]. □

If one wants to consider a large family of examples of planar Gaussian fields, one can consider a function $\rho : \mathbb{R}^2 \to \mathbb{R}_+$ sufficiently smooth, that is not 0 and such that ρ and its derivatives decay sufficiently fast. One can note that $\kappa = \rho * \rho$ has the same properties and is strictly positive definite by Lemma 7.2.7 since $\hat{\kappa} = (\hat{\rho})^2$. Moreover, if ρ has sufficiently many symmetries, then the Gaussian field with covariance κ above satisfies Conditions from 7.2.1 to 7.2.5. Now, if W is a two-dimensional white noise, that is, the free field associated to the Hilbert space $L^2(\mathbb{R}^2)$ (see Definition 2.5 of [She07]), then (if ρ is even)

$$f = \rho * W$$

defines a Gaussian field on $\mathbb{R}^2$ with covariance κ.

We now list the main intermediate results of our proof.

7.2.3 A box-crossing criterion

As in the strategy of the proof of Kesten's theorem presented in Subsection 7.2.1, our goal will be to prove a box-crossing criterion. We start by introducing the following notation.

Notation 7.2.8. For each $\rho_1, \rho_2 > 0$ and each $p \in \mathbb{R}$, we write $\mathrm{Cross}_p(\rho_1, \rho_2)$ for the event that there is a continuous path in $\mathcal{D}_p \cap [0, \rho_1] \times [0, \rho_2]$ joining the left side of $[0, \rho_1] \times [0, \rho_2]$ to its right side.

In Subsection 7.4.1, we will prove the following proposition.

Lemma 7.2.9. *Assume that f satisfies Condition 7.2.1 and that κ is invariant by $\frac{\pi}{2}$-rotations. Let $p > 0$ and assume that:*

$$\sum_{k \in \mathbb{N}} \mathbb{P}\left[\neg \mathrm{Cross}_p(2^{k+1}, 2^k)\right] < +\infty\,. \tag{7.2.2}$$

Then, a.s. there exists a unique unbounded component in $\mathcal{D}_p$.

Thus, our goal turns to prove that, if $p > 0$, then $\mathbb{P}\left[\mathrm{Cross}_p(2R, R)\right]$ goes to 1 as R goes to $+\infty$, sufficiently fast so that the above sum is finite. In order to prove such a result, we will show a Russo-type formula and a KKL-type theorem for discrete Gaussian fields. To apply such result, we first need to discretize our model, as it was done in [BG17a].

7.2.4 A discretization procedure and a discrete phase transition theorem for more general fields

We consider the following discrete percolation model: let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be the face-centered square lattice defined in Figure 7.1. Note that the exact choice of lattice is not essential in most of our arguments. We mostly use the fact that it is a periodic triangulation with nice symmetries. However, we do need a little more information to apply Theorem 7.2.13

(see the beginning of Section 3 of [RV17a]) and in Section 7.6 of the present work we use the fact that the sites of $\mathcal{T}$ are the vertices of a rotated and rescaled $\mathbb{Z}^2$-lattice. We will denote by $\mathcal{T}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{E}^\varepsilon)$ this lattice scaled by a factor ε. Given a realization of our Gaussian field f and some $\varepsilon > 0$, we color the plane as follows: For each $x \in \mathbb{R}^2$, if $x \in \mathcal{V}^\varepsilon$ and $f(x) \geq -p$ or if x belongs to an edge of $\mathcal{E}^\varepsilon$ whose two extremities y_1, y_2 satisfy $f(y_1) \geq -p$ and $f(y_2) \geq -p$, then x is colored black. Otherwise, x is colored white. In other words, we study a **correlated site percolation model** on $\mathcal{T}^\varepsilon$.

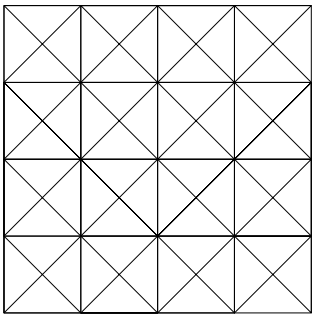


Figure 7.1: The face-centered square lattice (the vertices are the points of $\mathbb{Z}^2$ and the centers of the squares of the $\mathbb{Z}^2$-lattice).

We will use the following notation analogous to Notation **??**.

Notation 7.2.10. Let $\varepsilon > 0$, $p \in \mathbb{R}$, and consider the above discrete percolation model. For every $\rho_1, \rho_2 > 0$, write $\mathrm{Cross}_p^\varepsilon(\rho_1, \rho_2)$ for the event that there is a continuous black path in $[0, \rho_1] \times [0, \rho_2]$ joining the left side of $[0, \rho_1] \times [0, \rho_2]$ to its right side.

As explained in Subsection 7.2.3, we want to estimate the quantities $\mathbb{P}\left[\mathrm{Cross}_p(2R, R)\right]$ for $p > 0$. However, the tools that we develop in our paper are suitable to study the quantities $\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R, R)\right]$. We will thus have to choose ε so that, on the one hand, we can find nice estimates at the discrete level and, on the other hand, the discrete field is a good approximation of the continuous field. As a reading guide for the global strategy, we list below what are the constraints on ε for our intermediate results to hold. See also Figure 7.2. We write them for the Bargmann-Fock field. Actually, most of the intermediate results including Item 2 below hold for more general fields, see the following subsections for more details.

1. We want to establish a lower bound for $\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R, R)\right]$ that goes (sufficiently fast for us) to 1 as R goes to $+\infty$. Our techniques work as long as the number of vertices is not too large (see in particular Subsection **??**). They yield such a bound provided that there exists $\delta > 0$ such that (R, ε) satisfy the condition:

$$\varepsilon \geq \log^{-(1/2-\delta)}(R)\,. \tag{7.2.3}$$

   See Subsubsection **??**.

2. Then, we will have to estimate how much $\mathrm{Cross}_p^\varepsilon(2R,R)$ approximates well $\mathrm{Cross}_p(2R,R)$. At this point, it seems natural to use the quantitative approximation results from [BG17a, BM18]. In particular, as explained briefly in Subsection 7.2.6, it seems that results from [BM18] (based on discretization schemes that generalize the methods in [BG17a]) imply that the event $\mathrm{Cross}_p^\varepsilon(2R,R)$ approximates $\mathrm{Cross}_p(2R,R)$ well if:
$$\varepsilon \leq R^{-(1+\delta)}\,, \tag{7.2.4}$$
for some $\delta > 0$. Unfortunately, this constraint is not compatible with the constraint (7.2.3). For this reason, we will instead use a sprinkling discretization procedure. More precisely, we will see in Proposition 7.2.22 that $\mathrm{Cross}_{p/2}^\varepsilon(2R,R)$ implies $\mathrm{Cross}_p(2R,R)$ with high probability if (R,ε) satisfy the following condition:
$$\varepsilon \geq \log^{-(1/4+\delta)}(R) \tag{7.2.5}$$
for some $\delta > 0$. This time, the constraint combines very well with the constraint (7.2.3) and we will be able to conclude.

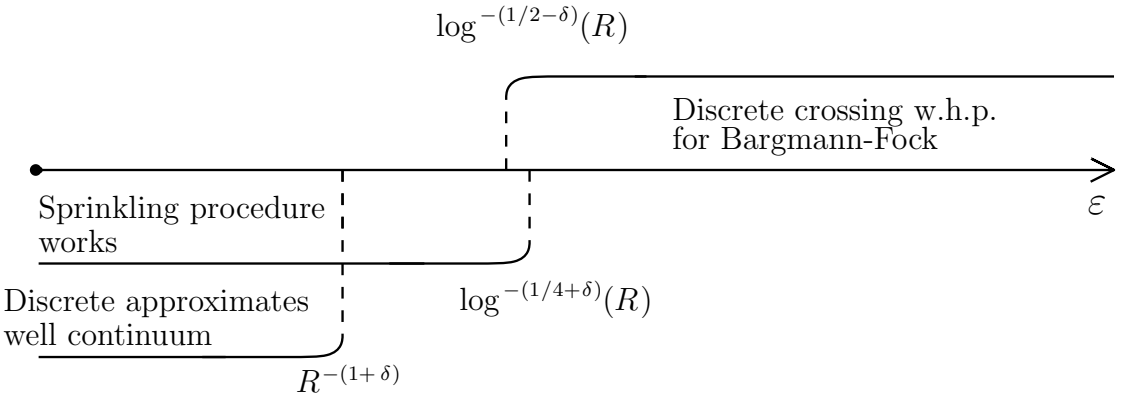


Figure 7.2: The different constraints on (R,ε).

If we work with the Bargmann-Fock field and if we choose for instance $\varepsilon = \varepsilon(R) = \log^{-1/3}(R)$ then, as shown in Figure 7.2, we obtain that $\mathrm{Cross}_p^\varepsilon(2R,R)$ holds with high probability and that the sprinkling discretization procedure works. As explained in Section 7.3.1, we will thus obtain that $\sum_{k\in\mathbb{N}} \mathbb{P}\left[\neg\mathrm{Cross}_p(2^{k+1},2^k)\right] < +\infty$ and conclude thanks to Lemma 7.2.9. See also Figure 7.3.

Note that Item 1 above implies in particular that, if we work with the Bargmann-Fock field and if we fix $\varepsilon \in]0,1]$,[4] then for $p > 0$, $\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]$ goes to 1 (sufficiently fast for us) as R goes to $+\infty$. We will actually obtain such a result for more general

[4] In this paper, we only look at the case $\varepsilon \in]0,1]$ though a lot of results could probably be extended to any $\varepsilon > 0$.

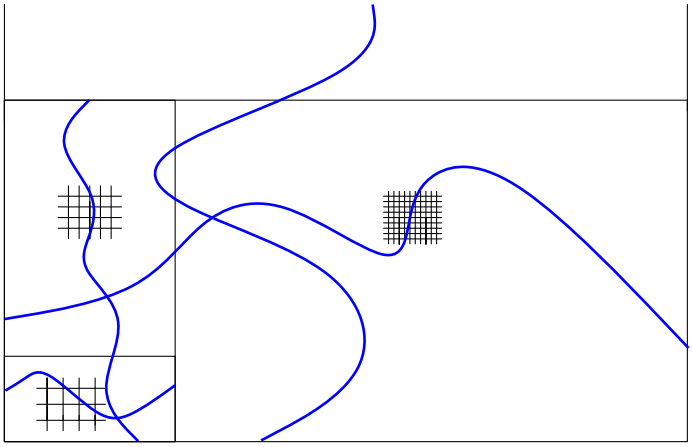


Figure 7.3: A right choice of ε is $\varepsilon = \log^{-1/3}(R)$ with whom we will be able to prove that $\mathbb{P}[\mathrm{Cross}(2R, R)]$ goes sufficiently fast to 1 so that the box-crossing criterion of Lemma 7.2.9 is satisfied. As we will see in the proof of this lemma, this criterion implies that, for a well-chosen of rectangles $(Q_k)_k$ of typical length R_k, a.s. all the Q_k's for k sufficiently large are crossed, which implies that there exists an infinite path. The idea is that, for each k, we study the probability that it is crossed by comparing the continuous event with the discrete event with mesh $\varepsilon = \log^{-1/3}(R_k)$.

fields, which will enable us to prove that a discrete box-crossing criterion analogous to Lemma 7.2.9 is satisfied and deduce the following in Section 7.3.1:

Theorem 7.2.11. *Suppose that f satisfies Conditions 7.2.1 and 7.2.2 and Condition 7.2.5 for some $\alpha > 5$ (thus in particular Condition 7.2.3 is satisfied for some $\alpha > 5$). Then, for each $\varepsilon \in]0, 1]$, the critical threshold of the discrete percolation model on $\mathcal{T}^\varepsilon$ defined in the present subsection is 0. More precisely: the probability that there is an unbounded black connected component is 1 if $p > 0$, and 0 otherwise. Moreover, if it exists, such a component is a.s. unique.*
In particular, this result holds when f is the Bargmann-Fock field or the centered Gaussian field with covariance given by (7.2.1) with $n \geq 3$.

As in the continuous setting, the case $p \leq 0$ of Theorem 7.2.11 goes back to [BG17a], at least for α large enough. In [RV17a], we have optimized this result and obtain the following:

Theorem 7.2.12 ([BG17a] for α sufficiently large,[RV17a])**.** [5] *Assume that f satisfies Conditions 7.2.1, 7.2.2, 7.2.4 as well as Condition 7.2.3 for some $\alpha > 4$. Let $\varepsilon \in]0, 1]$*

[5] More precisely, this is Proposition B.7 of [RV17a]. Moreover, this can be extracted from the proof of Theorem 5.7 of [BG17a] for α sufficiently large and with slightly different assumptions on the differentiability and the non-degeneracy of κ.

and consider the discrete percolation model on $\mathcal{T}^\varepsilon$ defined in the present subsection with parameter $p = 0$. Then, a.s. there is no infinite connected black component.

Again as in the continuous case, this result heavily relies on a RSW estimate, which we state below (see Theorem 7.2.13). This estimate is a uniform lower bound on the crossing probability of a quad scaled by R on a lattice with mesh ε. The proof of such an estimate goes back to [BG17a] (for $\alpha > 16$) and was later optimized to $\alpha > 8$ in [BM18]. The key property in this estimate is that the lower bound of the crossing properties does not depend on the choice of the mesh ε. To obtain such a result, the authors of [BG17a] had to impose some conditons on (R, ε). For instance, if we consider the Bargmann-Fock field, the constraint was $\varepsilon \geq C \exp(-cR^2)$ where $C < +\infty$ and $c > 0$ are fixed. Actually, it seems likely that, one could deduce a discrete RSW estimate with no constraint on (R, ε) by using their quantitative approximation results. In [RV17a], we prove such a discrete RSW estimate with no constraint on (R, ε) and without using quantitative discretization estimates, but rather by using new quasi-independence results. Note that to prove our main result Theorem 7.1.3, it would not have been a problem for us to rather use the result by Beffara and Gayet with constraints on (R, ε) since, as explained above, we will use results from the discrete model with $\varepsilon = \varepsilon(R) = \log^{-1/3}(R)$ - which of course satisfies the condition $\varepsilon \geq C \exp(-cR^2)$. We have the following:

Theorem 7.2.13 ([BG17a] for $\alpha > 16$, [BM18] for $\alpha > 8$,[RV17a])**.** [6] *Assume that f satisfies Conditions 7.2.1, 7.2.2, 7.2.4 as well as Condition 7.2.3 for some $\alpha > 4$. For every $\rho \in]0, +\infty[$, there exists a constant $c = c(\kappa, \rho) > 0$ such that the following holds: let $\varepsilon \in]0, 1]$, and consider the discrete percolation model on $\mathcal{T}^\varepsilon$ defined in the present subsection with parameter $p = 0$. Then, for every $R \in]0, +\infty[$ we have:*

$$\mathbb{P}\left[\mathrm{Cross}_0^\varepsilon(\rho R, R)\right] \geq c\,.$$

What remains to prove in order to show Theorem 7.2.11 is that, if $p > 0$, a.s. there is a unique infinite black component. Exactly as in the continuum, our goal will be to show that a box-crossing criterion is satisfied. The following lemma is proved in Subsection 7.4.1.

Lemma 7.2.14. *Assume that f satisfies Condition 7.2.1 and that κ is invariant by $\frac{\pi}{2}$-rotations. Let $\varepsilon > 0$, let $p \in \mathbb{R}$ and suppose that:*

$$\sum_{k \in \mathbb{N}} \mathbb{P}\left[\neg \mathrm{Cross}_p^\varepsilon(2^{k+1}, 2^k)\right] < +\infty\,. \tag{7.2.6}$$

Then, a.s. there exists a unique unbounded black component in the discrete percolation model defined in the present subsection.

[6] More precisely, this is Proposition B.2 of [RV17a]. Moreover, this is Theorem 2.2 from [BG17a] combined with the results of their Section 4 (resp. this is Appendix C of [BM18]) for $\alpha > 16$ (resp. $\alpha > 8$), with some constraints on (R, ε), and with slightly different assumptions on the differentiability and the non-degeneracy of κ.

7.2.5 Sharp threshold at the discrete level

We list the intermediate results at the discrete level. Among all the following results (and actually among all the intermediate results of the paper) the only result specific to the Bargmann-Fock field is the second result of Proposition 7.2.21. All the others work in a quite general setting.

The FKG inequality for Gaussian vectors

The FKG inequality is a crucial tool to apply percolation arguments. We say that a Borel subset $A \subseteq \mathbb{R}^n$ is **increasing** if for each $x \in A$ and $y \in \mathbb{R}^n$ such that $y_i \geq x_i$ for every $i \in \{1, \cdots, n\}$, we have $y \in A$. We say that A is decreasing if the complement of A is increasing. The FKG inequality for Gaussian vectors was proved by Pitt and can be stated as follows.

Theorem 7.2.15 ([Pit82b])**.** *Let $X = (X_1, \cdots, X_n)$ be a n-dimensional Gaussian vector whose correlation matrix has non-negative entries. Then, for every increasing Borel subsets $A, B \subseteq \mathbb{R}^n$, we have:*

$$\mathbb{P}[X \in A \cap B] \geq \mathbb{P}[X \in A] \cdot \mathbb{P}[X \in B] \, .$$

This result is the reason why we work with positively correlated Gaussian fields. Indeed, the FKG inequality is a crucial ingredient in the proof of RSW-type results. Note however that the very recent [BG17b] proves a box-crossing property without this inequality, albeit only in a discrete setting.

A differential formula

As in the case of Bernoulli percolation, we need to introduce a notion of influence. This notion of influence is inspired by the **geometric influences** studied by Keller, Mossel and Sen in the case of product spaces [KMS12, KMS14]. (For more about the relations between the geometric influences and the influences of Definition (7.2.16) below - which are roughly the same in the case of product measures - see Subsection 7.5.1.)

Definition 7.2.16. Let μ be a finite Borel measure on $\mathbb{R}^n$, let $v \in \mathbb{R}^n$, and let A be a Borel subset of $\mathbb{R}^n$. The influence of v on A under μ is:

$$I_{v,\mu}(A) := \liminf_{r \downarrow 0} \frac{\mu(A + [-r, r]\, v) - \mu(A)}{r} \in [0, +\infty] \, .$$

Write $(e_1, \cdots, e_n)$ for the canonical basis of $\mathbb{R}^n$. We will use the following simplified notations:

$$I_{i,\mu}(A) := I_{e_i,\mu}(A) \, .$$

The events we are interested in are "threshold events" and the measures we are interested in are Gaussian distributions: Let $X = (X_1, \cdots, X_n)$ be a n-dimensional non-degenerate

centered Gaussian vector, write μ_X for the law of X and, for every $\overrightarrow{p} = (p_1, \cdots, p_n) \in \mathbb{R}^n$ and every $i \in \{1, \cdots, n\}$, write:

$$\omega_i^{\overrightarrow{p}} := \mathbb{1}_{\{X_i \geq -p_i\}} .$$

This defines a random variable $\omega^{\overrightarrow{p}}$ with values in $\{0,1\}^n$. If $B \subseteq \{0,1\}^n$, we call the event $\{\omega^{\overrightarrow{p}} \in B\}$ a **threshold event**. For every $i \in \{1, \cdots, n\}$, we let $\mathrm{Piv}_i^{\overrightarrow{p}}(B)$ denote the event that changing the value of the bit i in $\omega^{\overrightarrow{p}}$ modifies $\mathbb{1}_B(\omega^{\overrightarrow{p}})$. In other words,

$$\mathrm{Piv}_i^{\overrightarrow{p}}(B) = \{(\omega_1^{\overrightarrow{p}}, \ldots, \omega_{i-1}^{\overrightarrow{p}}, 0, \omega_{i+1}^{\overrightarrow{p}}, \ldots, \omega_n^{\overrightarrow{p}}) \in B\} \triangle \{(\omega_1^{\overrightarrow{p}}, \ldots, \omega_{i-1}^{\overrightarrow{p}}, 1, \omega_{i+1}^{\overrightarrow{p}}, \ldots, \omega_n^{\overrightarrow{p}}) \in B\} ,$$

where $E \triangle F := (E \setminus F) \cup (F \setminus E)$. Such an event is called a **pivotal event**. We say that a subset $B \subseteq \{0,1\}^n$ is **increasing** if for every $\omega \in B$ and $\omega' \in \{0,1\}^n$, the fact that $\omega'_i \geq \omega_i$ for every i implies that $\omega' \in B$. Moreover, if $p \in \mathbb{R}$, we write $\mathbf{p} = (p, \cdots, p) \in \mathbb{R}^n$ and we use the following notations:

$$\omega^p := \omega^{\mathbf{p}} \text{ and } \mathrm{Piv}_i^p(B) := \mathrm{Piv}_i^{\mathbf{p}}(B) .$$

Proposition 7.2.17. *Assume that X is a n-dimensional non-degenerate centered Gaussian vector and let Σ be its covariance matrix. Let B be an increasing subset of $\{0,1\}^n$. Then:*

$$\frac{\partial \mathbb{P}\left[\omega^{\overrightarrow{p}} \in B\right]}{\partial p_i} = I_{i,\mu_X}(\omega^{\overrightarrow{p}} \in B) = \mathbb{P}\left[\mathrm{Piv}_i^{\overrightarrow{p}}(B) \,\middle|\, X_i = -p_i\right] \frac{1}{\sqrt{2\pi\Sigma_{i,i}}} \exp\left(-\frac{1}{2\Sigma_{i,i}} p_i^2\right) .$$

In particular, if $p \in \mathbb{R}$, then:

$$\frac{d\,\mathbb{P}\left[\omega^p \in B\right]}{dp} = \sum_{i=1}^n I_{i,\mu_X}(\omega^p \in B) = \sum_{i=1}^n \mathbb{P}\left[\mathrm{Piv}_i^p(B) \,\middle|\, X_i = -p\right] \frac{1}{\sqrt{2\pi\Sigma_{i,i}}} \exp\left(-\frac{1}{2\Sigma_{i,i}} p^2\right) .$$

Remark 7.2.18. With the same hypotheses as in Proposition 7.2.17: For $\epsilon \in \{0,1\}$, let $B_\epsilon^i = \{\omega \in \{0,1\}^n : (\omega_1, \cdots, \omega_{i-1}, \epsilon, \omega_{i+1}, \cdots, \omega_n) \in B\}$. Note that we have:

$$\mathbb{P}\left[\mathrm{Piv}_i^{\overrightarrow{p}}(B) \,\middle|\, X_i = -p_i\right] = \mathbb{P}\left[\omega^{\overrightarrow{p}} \in B_1^i \,\middle|\, X_i = -p_i\right] - \mathbb{P}\left[\omega^{\overrightarrow{p}} \in B_0^i \,\middle|\, X_i = -p_i\right] .$$

Hence the form of our influences is close to the covariance between $\mathbb{1}_{\{\omega_i^{\overrightarrow{p}} = 1\}}$ and $\mathbb{1}_{\{\omega^{\overrightarrow{p}} \in B\}}$ but is not exactly a covariance. This is important to have in mind if one wants to try to apply techniques from [DCRT17b, DCRT17a] in future.

Since the proof of Proposition 7.2.17 is rather short, we include this here. The reader essentially interested in the strategy of proof can skip this in a first reading.

Proof of Proposition 7.2.17. Without loss of generality, we assume that $i = n$. For any $C \subseteq \{0,1\}^n$, let $C^{\overrightarrow{p}} \subseteq \mathbb{R}^n$ be the preimage of C by the map $x \in \mathbb{R}^n \mapsto (\mathbb{1}_{x_i \geq -p_i})_{i \in \{1, \cdots, n\}} \in$

$\{0,1\}^n$. Let $e_n = (0,\dots,0,1) \in \mathbb{R}^n$, $h > 0$, let $B \subseteq \{0,1\}^n$ be an increasing event, and let $\tilde{X} = (X_i)_{1\le i\le n-1}$. Then:

$$\mathbb{P}\left[\omega^{\overrightarrow{p}+he_n} \in B\right] = \mathbb{P}\left[X \in B^{\overrightarrow{p}+he_n}\right] = \mathbb{P}\left[X + he_n \in B^{\overrightarrow{p}}\right] = \mathbb{P}\left[(\tilde{X}, X_n + h) \in B^{\overrightarrow{p}}\right],$$

and $\mathbb{P}\left[\omega^{\overrightarrow{p}} \in B\right] = \mathbb{P}\left[(\tilde{X}, X_n) \in B^{\overrightarrow{p}}\right]$.

Also:

$$\begin{aligned}\mathbb{P}\left[(\tilde{X}, X_n + h) \in B^{\overrightarrow{p}}\right] - \mathbb{P}\left[(\tilde{X}, X_n) \in B^{\overrightarrow{p}}\right] &= \mathbb{P}\left[(\tilde{X}, X_n + h) \in B^{\overrightarrow{p}}, (\tilde{X}, X_n) \notin B^{\overrightarrow{p}}\right] \\ &= \mathbb{P}\left[\mathrm{Piv}_n^{\overrightarrow{p}}(B),\ X_n \in [-p_n - h, -p_n[\right].\end{aligned}$$

Since Σ is positive definite, the Gaussian measure has smooth density with respect to the Lebesgue measure. Taking the difference of the two probabilities and letting $h \downarrow 0$, we get:

$$\begin{aligned}&h^{-1}\left(\mathbb{P}\left[\omega^{\overrightarrow{p}+he_n} \in B\right] - \mathbb{P}[\omega^{\overrightarrow{p}} \in B)]\right) \\ &= h^{-1}\mathbb{P}\left[\mathrm{Piv}_n^{\overrightarrow{p}}(B),\ X_n \in [-p_n - h, -p_n[\right] \\ &= \frac{1}{h\sqrt{2\pi\Sigma_{n,n}}} \int_{-p_n-h}^{-p_n} \mathbb{P}\left[\mathrm{Piv}_n^{\overrightarrow{p}}(B) \,\middle|\, X_n = t\right] \exp\left(-\frac{1}{2\Sigma_{n,n}} t^2\right) dt \\ &\xrightarrow[h\downarrow 0]{} \mathbb{P}\left[\mathrm{Piv}_n^{\overrightarrow{p}}(B) \,\middle|\, X_n = -p_n\right] \frac{1}{\sqrt{2\pi\Sigma_{n,n}}} \exp\left(-\frac{1}{2\Sigma_{n,n}} p_n^2\right).\end{aligned}$$

In the last step we use the continuity of the conditional probability of a threshold event with respect to the conditioning value. This is an easy consequence of Proposition 1.2 of [AW09]. The calculation for $h < 0$ is analogous, hence we are done. □

A KKL-KMS (Kahn-Kalai-Linial – Keller-Mossel-Sen) theorem for non-product Gaussian vectors

One of the contributions of this paper is the derivation of a KKL theorem for biased Gaussian vectors, namely Theorem 7.2.19, based on a similar result for product Gaussian vectors by Keller, Mossel and Sen, [KMS12]. (This similar result by [KMS12] is stated in Theorem 7.5.1 of our paper[7].) Actually, with our techniques of Section 7.5, most of the results from [KMS12] could be extended to non-product Gaussian vectors (and to monotonic or semi-algebraic events).

Theorem 7.2.19. *There exists an absolute constant $c > 0$ such that the following holds: Let $n \in \mathbb{Z}_{>0}$, let Σ be a $n \times n$ symmetric positive definite matrix[8] and let $\mu = \mathcal{N}(0,\Sigma)$. Also, let $\sqrt{\Sigma}$ be a symmetric square root of Σ and write $\|\sqrt{\Sigma}\|_{\infty,op}$ for the operator norm*

[7] As we will explain, Theorem 7.5.1 a simple consequence of Item (1) of Theorem 1.5 of [KMS12].

[8] Remember Remark **??**: this means in particular that Σ is non-degenerate.

of $\sqrt{\Sigma}$ for the infinite norm[9] *on $\mathbb{R}^n$. For every A either monotonic or semi-algebraic*[10] *Borel subset of $\mathbb{R}^n$ we have:*

$$\sum_{i=1}^{n} I_{i,\mu}(A) \geq c \, \|\sqrt{\Sigma}\|_{\infty,op}^{-1} \, \mu(A) \, (1 - \mu(A)) \sqrt{\log_+ \left(\frac{1}{\|\sqrt{\Sigma}\|_{\infty,op} \cdot \max\limits_{i \in \{1,\cdots,n\}} I_{i,\mu}(A)} \right)}.$$

This theorem probably holds for a wider class of sets. Since we need it only for monotonic events, we did not try to identify the weakest possible assumptions for the property to hold. In particular, we have not found any examples of Borel sets for which the theorem does not hold.

A way to understand the constant $\|\sqrt{\Sigma}\|_{\infty,op}$ is to see what happens in the extreme cases:

- If Σ is the identity matrix, then $\|\sqrt{\Sigma}\|_{\infty,op} = 1$ and the above result corresponds to the product case from [KMS12].
- If $\Sigma_{i,j} = 1$ for every i, j (which corresponds to the fact that, if $X \sim \Sigma$, then $X_i = X_j$ for every i, j) then $\|\sqrt{\Sigma}\|_{\infty,op} = n$ which is coherent since there is no threshold phenomenon for a single variable.

The proof of Theorem 7.2.19 is organized as follows: In Subsection 7.5.1, we explain how to deduce Theorem 7.2.19 from a sub-linear property of the influences (see Proposition 7.5.3) and from the results of [KMS12]. In Subsection 7.5.2 (respectively in Appendix 7.A), we prove the sub-linearity property for monotonic (respectively semi-algebraic) subsets. The reason why we postpone the proof in the semi-algebraic case to the appendix is that the events that will interest us (namely, the events $\mathrm{Cross}_p^{\varepsilon}(2R, R)$) are both monotonic and semi-algebraic (indeed, any threshold event is semi-algebraic). Hence it is enough for our purpose to prove Theorem 7.2.19 only for A monotonic or A semi-algebraic (and the monotonic case is easier).

Remember that we want to prove that $\mathbb{P}\left[\mathrm{Cross}_p(2R, R)\right]$ is close to 1 and that our first step is to prove that it is the case for $\mathbb{P}\left[\mathrm{Cross}_p^{\varepsilon}(2R, R)\right]$. To do so, we use that for $p = 0$ this probability is bounded from below (by Theorem 7.2.13) and we differentiate the probability with respect to p using Proposition 7.2.17. We then apply Theorem 7.2.19 to the right-hand side so that it is sufficient to prove that the maximum of the corresponding influences is small and that the operator norm for the infinite norm of the correlation matrix of our model is not too large. In the two following subsubsections, we state results in this spirit.

[9] I.e. $\|\sqrt{\Sigma}\|_{\infty,op} = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|\sqrt{\Sigma}(x)|_\infty}{|x|_\infty}$. Equivalently, $\|\sqrt{\Sigma}\|_{\infty,op} = \max\limits_{i \in \{1,\cdots,n\}} \sum_{j=1}^{n} |\sqrt{\Sigma}(i,j)|$.

[10] We say that a set $A \subset \mathbb{R}^n$ is semi-algebraic if it belongs to the Boolean algebra generated by sets of the form $P^{-1}\left(]0, +\infty[\right)$ where $P \in \mathbb{R}[X_1, \ldots, X_n]$.

Polynomial decay of influences

Thanks to the RSW estimate and quasi-independence results, one can obtain that the probability of "discrete arm events" decay polynomially fast, see Subsection 5.3 of [BG17a] and Proposition B.6 of [RV17a]. Such a result together with monotonic arguments imply that we have a polynomial bound on the influences for crossing events. We state this bound here and we prove it in Subsection 7.4.2. We first need some notations. Let $\mathcal{V}_R^\varepsilon$ denote the set of vertices $x \in \mathcal{V}^\varepsilon$ that belong to an edge that intersects $[0, 2R] \times [0, R]$, and let X be our Gaussian field f restricted to $\mathcal{V}_R^\varepsilon$. Thus, X is a finite dimensional Gaussian field. For every $p \in \mathbb{R}$, $x \in \mathcal{V}_R^\varepsilon$ and $B \subseteq \mathbb{R}^{\mathcal{V}_R^\varepsilon}$, we use the notations ω^p and $\mathrm{Piv}_x^p(B)$ from Subsubsection **??**. In particular, for every $x \in \mathcal{V}_R^\varepsilon$ and every $p \in \mathbb{R}$ we write:

$$\omega_x^p = \mathbb{1}_{\{X_x \geq -p\}} = \mathbb{1}_{\{f(x) \geq -p\}} \,.$$

Also, we write $\mathrm{Cross}^\varepsilon(2R, R)$ for the subset of $\{0,1\}^{\mathcal{V}_R^\varepsilon}$ that corresponds to the crossing from left to right of $[0, 2R] \times [0, R]$. Note that $\mathrm{Cross}_p^\varepsilon(2R, R) = \{\omega^p \in \mathrm{Cross}^\varepsilon(2R, R)\}$. We have the following result (see Proposition 7.2.17 for its link with the influences):

Proposition 7.2.20. *Assume that f satisfies Conditions 7.2.1, 7.2.2, 7.2.4 as well as Condition 7.2.3 for some $\alpha > 4$. Then, there exist constants $\eta = \eta(\kappa) > 0$ and $C = C(\kappa) < +\infty$ such that, for every $\varepsilon \in]0, 1]$, every $R \in]0, +\infty[$, every $x \in \mathcal{V}_R^\varepsilon$ and every $p \in [-1, 1]$ we have:*

$$\mathbb{P}\left[\mathrm{Piv}_x^p(\mathrm{Cross}^\varepsilon(2R, R)) \,\middle|\, f(x) = -p\right] \leq C\, R^{-\eta} \,.$$

Note that the constants in Proposition 7.2.20 do not depend on p (as long p belongs to a bounded interval, in our proof we have chosen $[-1, 1]$).

A bound on the infinite norm of the square root matrix

In order to use Theorem 7.2.19, it is important to control the quantity $\|\sqrt{\Sigma}\|_{\infty,op}$. We are interested in the case $\Sigma = K_R^\varepsilon :=$ the correlation matrix K restricted to $\mathcal{V}_R^\varepsilon$ (see Subsubsection **??** the notation $\mathcal{V}_R^\varepsilon$). However, we were not able to estimate $\|\sqrt{K_R^\varepsilon}\|_{\infty,op}$. Instead, we obtained estimates on $\|\sqrt{K^\varepsilon}\|_{\infty,op}$ where K^ε is K restricted to $\mathcal{V}^\varepsilon$, by using Fourier techniques (which work only in a translation invariant setting, hence not for K_R^ε). Here, K^ε is an infinite matrix, so let us be more precise about what we mean by square root of K^ε: To define the product of two infinite matrices, we ask the infinite sums that arise to be absolutely convergent. A square root of K^ε is simply an infinite matrix $\sqrt{K^\varepsilon}$ such that in this sense we have $\sqrt{K^\varepsilon}^2 = K^\varepsilon$. It may be that classical spectral theory arguments imply that such a square root exists. However, we will not need such arguments since we will have to construct quite explicitly $\sqrt{K^\varepsilon}$ in order to estimate its infinite operator norm (where as in the finite dimensional case we let $\|\sqrt{K^\varepsilon}\|_{\infty,op} := \max_i \sum_j |\sqrt{K^\varepsilon}(i,j)|$, one difference being that this sum might be infinite). Let us also note that the matrices $\sqrt{K^\varepsilon}$ that we will construct will be positive definite in the sense that their restriction to finite sets of indices is positive definite. By spectral

theory arguments it may be that only one such matrix exists. However, we do not need such a result either.

Note that, since Theorem 7.2.19 is stated for finite dimensional Gaussian fields, it would have been better to have an estimate on $\|\sqrt{K_R^\varepsilon}\|_{\infty,op}$. See Lemma 7.3.1 for more details on this issue.

We will prove the following in Section 7.6. The reason why our main result Theorem 7.1.3 is only proved for the Bargmann-Fock model is that we were unable to find a general estimate on $\|\sqrt{K^\varepsilon}\|_{\infty,op}$ with explicit dependence on ε as $\varepsilon \downarrow 0$ (this is the only place where we prove a result only for the Bargmann-Fock field). If one could prove a bound by $C\varepsilon^{-2+\delta}$ for some $\delta > 0$ and $C < +\infty$ for another process (which also satisfies some of the conditions of Subsection 7.2.2), the rest of the proof of Theorem 7.1.3 would piece together and yield the same result for this process. See Proposition 7.3.5.

Proposition 7.2.21. *Assume that f satisfies Condition 7.2.1 and Condition 7.2.5 for some $\alpha > 5$. Then, for all $\varepsilon > 0$, there exists a symmetric square root $\sqrt{K^\varepsilon}$ of K^ε such that:*

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} < +\infty\,.$$

Assume further that $\kappa(x) = \exp\left(-\frac{1}{2}|x|^2\right)$ (which is the covariance kernel function of the Bargmann-Fock field). Then, there exist $\varepsilon_0 > 0$ and $C < +\infty$ such that, for all $\varepsilon \in]0, \varepsilon_0]$, there exists a symmetric square root $\sqrt{K^\varepsilon}$ of K^ε such that:

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} \leq C\,\frac{1}{\varepsilon}\,\log\left(\frac{1}{\varepsilon}\right)\,.$$

Combining all the above results

The following Equation (7.2.7) is not used anywhere in the paper though it appears implicitely in Lemma 7.3.1. However, since it sums up the crux of the proof, we state it here as a guide to the reader. Its proof could easily be extracted from the arguments below (see Remark **??**).

As we will explain in Section 7.3.1, by combining the results of Subsubsections **??**, **??**, **??** and **??**, we can obtain the following: Consider the Bargmann-Fock field and let $p \in]0, 1]$. There exists $C < +\infty$ such that, if $\varepsilon \in]0, 1]$ and $R \in]0, +\infty[$, then:

$$\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R, R)\right] \geq 1 - C_1 \exp\left(-p\,c_2\,\frac{\varepsilon}{\log(\frac{1}{\varepsilon})}\sqrt{\log_+\left(C_1 R^{-\eta}\frac{\log(\frac{1}{\varepsilon})}{\varepsilon}\right)}\right)\,, \qquad (7.2.7)$$

where $C_1 < +\infty$ and $c_2 > 0$ are some absolute constants. One can deduce from (7.2.7) that, if $\varepsilon = \varepsilon(R) \geq \log^{-(1/2-\delta)}(R)$, then $\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R, R)\right]$ goes to 1 as R goes to $+\infty$ (sufficiently fast so that $\sum_k \mathbb{P}\left[\mathrm{Cross}_p^{\varepsilon(2^k)}(2^{k+1}, 2^k)\right] < +\infty$ as required, see (7.2.3)).

We will also obtain results analogous to Equation (7.2.7) for more general fields and with $\|\sqrt{K^\varepsilon}\|_{\infty,op}$ instead of $\frac{1}{\varepsilon}\log(\frac{1}{\varepsilon})$, which will enable us to prove (7.2.6) and thus obtain Theorem 7.2.11.

7.2.6 From discrete to continuous

It is helpful to keep in mind in this subsection that, as explained above, in the case of the Bargmann-Fock field, the quantity $\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]$ goes to 1 as $R\to+\infty$ if $\varepsilon=\varepsilon(R)\geq \log^{-(1/2-\delta)}(R)$. In order to obtain an analogous result for the continuum model, we need to measure the extent to which $\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]$ approximates $\mathbb{P}\left[\mathrm{Cross}_p(2R,R)\right]$. To this purpose, it seems natural to use the approximation estimates based on the discretization schemes of [BG17a] which have been generalized in [BM18]. More precisely, we think that the following can easily be extracted from [BM18] (however, we do not write a formal proof since we will not need such a result):[11] Assume that f satisfies Condition 7.2.1 and that κ is C^6. Then, for every $p\in\mathbb{R}$ and $\delta>0$, there exists $C=C(\kappa,p,\delta)<+\infty$ such that for each $\varepsilon\in]0,1]$ and each $R\in\varepsilon\mathbb{Z}_{>0}$ we have:

$$\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\triangle\mathrm{Cross}_p(2R,R)\right]\leq C\varepsilon^{2-\delta}R^2\,. \tag{7.2.8}$$

This would imply in particular that, for every sequence $(\varepsilon(R))_{R>0}$ such that: (i) for each $R>0$, $R\in\varepsilon(R)\mathbb{Z}_{>0}$ and: (ii) $\varepsilon(R)=O(R^{-(1+\delta)})$ for some $\delta>0$, we have:

$$\mathbb{P}\left[\mathrm{Cross}_p^{\varepsilon(R)}(2R,R)\triangle\mathrm{Cross}_p(2R,R)\right]\underset{R\to+\infty}{\longrightarrow}0\,.$$

Unfortunately, the constraint $\varepsilon\leq R^{-(1+\delta)}$ is incompatible with $\varepsilon\geq\log^{-(1/2-\delta)}(R)$. To solve this quandary, we will replace the discretization result with a sprinkling argument.

Let $p>0$. The idea is to compare the probability of a continuum crossing at parameter p to the probability of a discrete crossing at parameter $p/2$. Instead of using the discretization results of [BG17a, BM18], we will use the following result, which we prove in Subsection 7.7. For references about the strategy of sprinkling in the context of Bernoulli percolation, see the notes on Section 2.6 of [Gri99].

Proposition 7.2.22. *Assume that f satisfies Conditions 7.2.1 and that it is a.s. C^2. Let $p>0$. Then, there exist constants $C_1<+\infty$ and $c_2=c_2(\kappa,p)>0$ as well as $\varepsilon_0=\varepsilon_0(\kappa,p)>0$ such that for each $\varepsilon\in]0,\varepsilon_0]$ and $R>1$:*

$$\mathbb{P}\left[\mathrm{Cross}_{p/2}^\varepsilon(2R,R)\setminus\mathrm{Cross}_p(2R,R)\right]\leq C_1R^2\varepsilon^{-2}\exp(-c_2\varepsilon^{-4})\,.$$

In particular, for every sequence $(\varepsilon(R))_{R>0}$ such that $\varepsilon(R)\geq\log^{-(1/4+\delta)}(R)$ for some $\delta>0$, we have:

$$\mathbb{P}\left[\mathrm{Cross}_{p/2}^\varepsilon(2R,R)\setminus\mathrm{Cross}_p(2R,R)\right]\underset{R\to+\infty}{\longrightarrow}0\,.$$

[11] Let us be a little more precise here: one can use for instance Propositions 6.1 and 6.3 of [BM18] and (more classical) results about the regularity of $\mathcal{D}_p$ as for instance Lemma A.10 of [RV17a].

Remark 7.2.23. While this estimate may seem stronger than (7.2.8), we wish to emphasize that it only shows that discretization is legitimate in one direction and also involves a change in threshold from $-p/2$ to $-p$.

If we choose for instance $\varepsilon = \varepsilon(R) = \log^{-1/3}(R)$, then:

$$\log^{-(1/2-\delta)}(R) \ll \varepsilon(R) \ll \log^{-(1/4+\delta)}(R)$$

for any $\delta > 0$ small enough. Hence, with this choice of ε and for the Bargmann-Fock field, on the one hand for any $p > 0$ we have a lower bound on $\mathbb{P}\left[\mathrm{Cross}^{\varepsilon}_{p/2}(2R, R)\right]$ which is close to 1 (see Subsubsection **??**), and on the other hand we can use Proposition 7.2.22 to go back to the continuum and obtain a lower bound on $\mathbb{P}\left[\mathrm{Cross}_p(2R, R)\right]$ which is close to 1. More precisely, we will see in Section 7.3.1 that we can obtain:

$$\mathbb{P}\left[\mathrm{Cross}_p(2R, R)\right] \geq 1 - \exp\left(-c \log^{1/7}(R)\right),$$

for some $c = c(p) > 0$, which implies that the box-crossing criterion of Lemma 7.2.9 is satisfied, and ends the proof of our main result Theorem 7.1.3.

7.3 Proofs of the phase transition theorems and of the exponential decay

7.3.1 Proof of the phase transition theorems

In this subsection, we combine the results stated in Section 7.2 in order to prove Theorem 7.1.3. We also prove Theorem 7.2.11. Most of the proof of the former carries over to that of the latter. The first half of the proof is the following lemma, in which we work with the following function:

$$g^{\varepsilon}_R : p \longmapsto \log\left(\frac{\mathbb{P}\left[\mathrm{Cross}^{\varepsilon}_p(2R, R)\right]}{1 - \mathbb{P}\left[\mathrm{Cross}^{\varepsilon}_p(2R, R)\right]}\right). \tag{7.3.1}$$

Lemma 7.3.1. *There exists an absolute constant $c > 0$ such that the following holds: Assume that f satisfies Condition 7.2.1, and fix $\varepsilon > 0$ and $R > 0$. Moreover, let μ^{ε}_R be the law of f restricted to $\mathcal{V}^{\varepsilon}_R$ (defined in Subsubsection **??**) and let K^{ε} be K restricted to $\mathcal{V}^{\varepsilon}$. Assume that there exists a symmetric square root $\sqrt{K^{\varepsilon}}$ of K^{ε}. Then:*

$$\forall p \in [0,1],\ \frac{d}{dp} g^{\varepsilon}_R(p) \geq c \left\|\sqrt{K^{\varepsilon}}\right\|^{-1}_{\infty,op} \sqrt{\log_+\left(\frac{1}{\left\|\sqrt{K^{\varepsilon}}\right\|_{\infty,op} \cdot \max\limits_{x \in \mathcal{V}^{\varepsilon}_R} I_{x,\mu^{\varepsilon}_R}\left(\mathrm{Cross}^{\varepsilon}_p(2R, R)\right)}\right)}. \tag{7.3.2}$$

Here, square root and infinite norm of an infinite matrix have the same meaning as in Subsubsection **??**.

This lemma is mostly a consequence of Theorem 7.5.1. The only difficulty comes from the fact that, while Theorem 7.2.19 deals with finite dimensional Gaussian fields, the correlation matrix on the right hand side of Equation (7.3.2) is the infinite matrix K^ε (which we need to apply Proposition 7.2.21 later in this section). To overcome such issues, we proceed by approximation and, instead of dealing with a μ_R^ε variable directly, we apply Theorem 7.2.19 to a Gaussian vector defined using $\sqrt{K^\varepsilon}$ (namely, the Gaussian vector $Y^{\varepsilon,\rho}$ below).

Proof of Lemma 7.3.1. Fix $\varepsilon > 0$ and $R > 0$. For each $\rho > 0$, let $H^{\varepsilon,\rho}$ be $\sqrt{K^\varepsilon}$ restricted to $\mathcal{V}^\varepsilon \cap [-\rho, \rho]^2$, let $K^{\varepsilon,\rho} = (H^{\varepsilon,\rho})^2$, and let $Y^{\varepsilon,\rho} \sim \mathcal{N}(0, K^{\varepsilon,\rho}) =: \mu^{\varepsilon,\rho}$. A simple computation shows that since K^ε is non-degenerate,[12] so is its square root. Restriction preserves non-degeneracy so $Y^{\varepsilon,\rho}$ is indeed non-degenerate and we can apply Theorem 7.2.19 to $\mu^{\varepsilon,\rho}$ and the (increasing) event $\mathrm{Cross}_p^\varepsilon(2R, R)$. We obtain that, if ρ is sufficiently large:

$$\sum_{i \in \mathcal{V}_R^\varepsilon} I_{i,\mu^{\varepsilon,\rho}}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) \geq c\|H^{\varepsilon,\rho}\|_{\infty,op}^{-1}\, \mu^{\varepsilon,\rho}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right)\left(1-\mu^{\varepsilon,\rho}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right)\right) \times \sqrt{\log_+\left(\frac{1}{\|H^{\varepsilon,\rho}\|_{\infty,op}\max_{i\in\mathcal{V}_R^\varepsilon} I_{i,\mu^{\varepsilon,\rho}}(\mathrm{Cross}_p^\varepsilon(2R,R))}\right)}. \quad (7.3.3)$$

Here we use the fact that, since $\mathrm{Cross}_p^\varepsilon(2R, R)$ depends only on the sites of $\mathcal{V}_R^\varepsilon$, the influences on this event of all the sites outside of this box vanish. Now, observe that:

$$\frac{d}{dp} g_R^\varepsilon(p) = \frac{d}{dp}\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right] \cdot \frac{1}{\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]\left(1-\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]\right)}.$$

By Proposition 7.2.17,

$$\frac{d}{dp} g_R^\varepsilon(p) = \frac{\exp(-p^2/2)}{\sqrt{2\pi}\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]\left(1-\mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]\right)} \sum_{i\in\mathcal{V}_R^\varepsilon} I_{i,\mu_R^\varepsilon}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right). \quad (7.3.4)$$

Also, by definition of $\|\cdot\|_\infty$, for every ρ, we have:

$$\|H^{\varepsilon,\rho}\|_{\infty,op} \leq \|\sqrt{K^\varepsilon}\|_{\infty,op}. \quad (7.3.5)$$

In view of Equations (7.3.3), (7.3.4) and (7.3.5), we are done as long as we prove that:

1. $\mu^{\varepsilon,\rho}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) \underset{\rho\to+\infty}{\longrightarrow} \mu_R^\varepsilon\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) \left(= \mathbb{P}\left[\mathrm{Cross}_p^\varepsilon(2R,R)\right]\right)$ and:
2. $\forall i \in \mathcal{V}_R^\varepsilon$, $I_{i,\mu^{\varepsilon,\rho}}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) \underset{\rho\to+\infty}{\longrightarrow} I_{i,\mu_R^\varepsilon}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right)$.

[12] Meaning that for any non-zero finitely supported $v \in \mathbb{R}^{\mathcal{V}^\varepsilon}$, $K^\varepsilon v \neq 0$.

To this purpose, we need the following elementary lemma:

Lemma 7.3.2. *Let $(Y_\rho)_{\rho>0}$ be a sequence of non-degenerate Gaussian vectors in $\mathbb{R}^n$ with covariance Σ_ρ and mean m_ρ, assume that Σ_ρ converges to some invertible matrix Σ as ρ goes to $+\infty$, and that m_ρ converges to some $m \in \mathbb{R}^n$ as ρ goes to $+\infty$. Let $Y \sim \mathcal{N}(m, \Sigma)$. Then, for any Borel subset $A \subset \mathbb{R}^n$, any $i \in \{1, \cdots, n\}$ and any $q \in \mathbb{R}$, we have:*

$$\mathbb{P}[Y_\rho \in A] \underset{\rho\to+\infty}{\longrightarrow} \mathbb{P}[Y \in A]\,,$$
$$\mathbb{P}\left[Y_\rho \in A \,\middle|\, Y_\rho(i) = q\right] \underset{\rho\to+\infty}{\longrightarrow} \mathbb{P}\left[Y \in A \,\middle|\, Y(i) = q\right]\,.$$

Proof. For the first statement, just notice that the density function of a non-degenerate Gaussian vector is a continuous function of its covariance and of its mean, and apply dominated convergence. By Proposition 1.2 of [AW09], the law of Y_ρ conditioned on the value of $Y_\rho(i)$ is that of a Gaussian vector whose mean and covariance depend continuously on Σ_ρ and m_ρ. Note also that the law $(Y_\rho(j))_{j\neq i}$ when we condition on $\{Y_\rho(i) = q\}$ (respectively the law $(Y(j))_{j\neq i}$ when we condition on $\{Y(i) = q\}$) is still non-degenerate.[13] We conclude by applying the first statement. □

Item 1 above is a direct consequence of the first part of Lemma 7.3.2. Regarding Item 2, note that, since $\mathrm{Cross}_p^\varepsilon(2R, R)$ is an increasing threshold event, by Proposition 7.2.17, we have the following equalities:

$$I_{i,\mu^{\varepsilon,\rho}}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) = \mathbb{P}\left[Y^{\varepsilon,\rho} \in \mathrm{Piv}_i\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) \,\middle|\, Y^{\varepsilon,\rho}(i) = -p\right] \frac{1}{\sqrt{2\pi K^{\varepsilon,\rho}(i,i)}} e^{-p^2/(2\,K^{\varepsilon,\rho}(i,i))}\,,$$

and:

$$I_{i,\mu_R^\varepsilon}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) = \mathbb{P}\left[Y_R^\varepsilon \in \mathrm{Piv}_i\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) \,\middle|\, Y_R^\varepsilon(i) = -p\right] \frac{1}{\sqrt{2\pi K_{\rho,R}^\varepsilon(i,i)}} e^{-p^2/(2\,K_{\rho,R}^\varepsilon(i,i))}\,,$$

where $Y_R^\varepsilon \sim \mathcal{N}(0, K_R^\varepsilon)$. We conclude by applying the second part of Lemma 7.3.2. □

We are now ready to prove our main result.

Proof of Theorem 7.1.3. We consider the Bargmann-Fock field f. By Theorem 7.1.2, for every $p \leq 0$, a.s. there is no unbounded connected component in $\mathcal{D}_p$. We henceforth consider some parameter $p_0 \in]0, 1]$. By Lemma 7.2.9, it is enough to prove that f satisfies criterion (7.2.2) (note that if we prove that this criterion is satisfied for every

[13] To see this, complete the vector e_i into an orthogonal basis for Σ_ρ (respectively Σ) and express Y_ρ (respectively Y) in this basis.

$p_0 \in]0,1]$ then we obtain the result for every p_0 since the quantities $\mathbb{P}\left[\neg \mathrm{Cross}_{p_0}(2R,R)\right]$ are non-increasing in p_0). To this end we first fix $R_0 < +\infty$ to be determined later, we let $R > R_0$, and we use the discretization procedure introduced in Subsection 7.2.4 with:

$$\varepsilon = \varepsilon(R) = \log^{-1/3}(R)\,.$$

Let $g_R := g_R^{\varepsilon(R)}$ be as in (7.3.1). We are going to apply Lemma 7.3.1. Let us estimate the quantities that appear in this lemma. By Proposition 7.2.21, there exists a constant $C < +\infty$ independent of R such that:

$$\|\sqrt{K^{\varepsilon(R)}}\|_{\infty,op} \leq C\frac{1}{\varepsilon(R)}\log\left(\frac{1}{\varepsilon(R)}\right).$$

Moreover by Propositions 7.2.17 and 7.2.20, if R_0 is large enough, there exists $\eta > 0$ independent of $R > R_0$ and of $p \in [-1,1]$ such that:

$$\forall x \in \mathcal{V}_R^{\varepsilon(R)},\ I_{x,\mu_R^{\varepsilon(R)}}\left(\mathrm{Cross}_p^{\varepsilon(R)}(2R,R)\right) \leq R^{-\eta}\,. \tag{7.3.6}$$

In particular, $\|\sqrt{K^{\varepsilon(R)}}\|_{\infty,op}\cdot \max_{x\in\mathcal{V}_R^{\varepsilon(R)}} I_{x,\mu_R^{\varepsilon(R)}}\left(\mathrm{Cross}_p^{\varepsilon(R)}(2R,R)\right) < 1$ if R_0 is large enough, and by applying Lemma 7.3.1 we obtain:

$$\begin{aligned}
\frac{d}{dp}g_R(p) &\geq \frac{c}{C}\varepsilon(R)\log^{-1}\left(\frac{1}{\varepsilon(R)}\right)\sqrt{\eta\log(R) - \log\left(\frac{1}{C\varepsilon(R)}\right) - \log\left(\log\left(\frac{1}{\varepsilon(R)}\right)\right)} \\
&\geq \frac{c}{3C}\log^{-1/3}(R)\log^{-1}\left(\log(R)\right)\sqrt{\eta\log(R) - \frac{1}{3}\log\left(\frac{\log(R)}{C}\right) - \log(3\log(\log(R)))} \\
&\geq \frac{c}{3C}\sqrt{\eta/2}\log^{1/6}(R)\log^{-1}\left(\log(R)\right) \\
&\geq \frac{c}{C}\sqrt{\eta/2}\log^{1/7}(R)\,.
\end{aligned}$$

By integrating from 0 to $p_0/2$, we get:

$$\begin{aligned}
\log\left(\frac{1}{1-\mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R,R)\right]}\right) &\geq \log\left(\frac{\mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R,R)\right]}{1-\mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R,R)\right]}\right) \\
&\geq \frac{p_0}{2}\frac{c}{C}\sqrt{\eta/2}\log^{1/7}(R) + \log\left(\frac{\mathbb{P}\left[\mathrm{Cross}_{0}^{\varepsilon(R)}(2R,R)\right]}{1-\mathbb{P}\left[\mathrm{Cross}_{0}^{\varepsilon(R)}(2R,R)\right]}\right).
\end{aligned}$$

By Theorem 7.2.13, if R_0 is large enough, there exists $C' < +\infty$ independent of $R > R_0$ such that:

$$\log\left(\frac{\mathbb{P}\left[\mathrm{Cross}_{0}^{\varepsilon(R)}(2R,R)\right]}{1-\mathbb{P}\left[\mathrm{Cross}_{0}^{\varepsilon(R)}(2R,R)\right]}\right) \geq -C'\,.$$

Therefore:

$$\mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R,R)\right] \geq 1-\exp\left(-\frac{c}{2C}\sqrt{\eta/2}\,p_0\log^{1/7}(R)+C'\right). \tag{7.3.7}$$

Now:

$$\mathbb{P}\left[\mathrm{Cross}_{p_0}(2R,R)\right] \geq \mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R,R)\right] - \mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R,R)\setminus\left[\mathrm{Cross}_{p_0}(2R,R)\right]\right],$$

and by Proposition 7.2.22, if R_0 is large enough, there exist constants $c_1 > 0$ and $C_2 = C_2(p_0) < +\infty$ indepedent of $R > R_0$ such that:

$$\begin{aligned}\mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R,R)\setminus\left[\mathrm{Cross}_{p_0}(2R,R)\right]\right] &\leq C_2R^2\varepsilon(R)^{-2}\exp\left(-c_1\varepsilon(R)^{-4}\right)\\ &\leq C_2\exp\left(2\log(R)+(2/3)\log(\log(R))-c_1\log^{4/3}(R)\right)\\ &\leq C_2\exp\left(-\frac{c_1}{2}\log^{4/3}(R)\right).\end{aligned}$$

Combining this estimate with Equation (7.3.7) we get:

$$\mathbb{P}\left[\mathrm{Cross}_{p_0}(2R,R)\right] \geq 1-\exp\left(-\frac{c}{2C}\sqrt{\eta/2}\,p_0\log^{1/7}(R)+C'\right) - C_2\exp\left(-\frac{c_1}{2}\log^{4/3}(R)\right).$$

All in all, for a large enough choice of R_0, there exists $c_3 = c_3(p_0) > 0$ such that for each $R > R_0$,

$$\mathbb{P}\left[\mathrm{Cross}_{p_0}(2R,R)\right] \geq 1-\exp\left(-c_3\log^{1/7}(R)\right).$$

Hence, criterion **??** is satisfied and we are done. □

Remark 7.3.3. Note that this proof also implies that:

$$\forall p>0,\ \lim_{R\to+\infty}\mathrm{Cross}_p(2R,R)=1\,, \tag{7.3.8}$$

which will be useful for the proof of Theorem 7.1.8.

Remark 7.3.4. Note that if we follow the above proof without taking $\varepsilon = \log^{-1/3}(R)$ but by taking any $\varepsilon \in]0,1]$ and R larger than some constant that does not depend on ε, we obtain (7.2.7).

The only result that we have used in the proof of Theorem 7.1.3 and that we have managed to prove only for the Bargmann-Fock field is the $\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)$ estimate on the infinite operator norm of $\sqrt{K^\varepsilon}$. We have the following more general result:

Proposition 7.3.5. *Assume that f satisfies Conditions 7.2.1, 7.2.2, 7.2.4 as well as Condition 7.2.3 for some $\alpha > 4$. Assume also that there exist $\delta > 0$ and $C < +\infty$ such that for every ε sufficiently small K^ε admits a symmetric square root and:*

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} \leq C\frac{1}{\varepsilon^{2-\delta}}\,. \tag{7.3.9}$$

Then the critical threshold for the continuous percolation model induced by f is 0. More precisely, the probability that $\mathcal{D}_p$ has an unbounded connected component is 1 if $p > 0$, and 0 if $p \leq 0$. Moreover, in the case where $p > 0$, if it exists, such a component is a.s. unique.

Remark 7.3.6. A way to prove that (7.3.9) is to use our results of Section 7.6 (where we assume that f satisifies Condition 7.2.1 and Condition 7.2.5 for some $\alpha > 5$): Define the quantity $\Upsilon(\varepsilon)$ as in Lemma 7.6.3. If one proves that there exists $\delta > 0$ and $C_1 < +\infty$ such that, for every ε sufficiently small, we have $\Upsilon(\varepsilon) \leq C_1 \varepsilon^{-1+\delta}$, then one obtains that there exists $C_2 < +\infty$ such that, for every ε sufficiently small:

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} \leq C_2 \frac{1}{\varepsilon^{2-\delta}} \log\left(\frac{1}{\varepsilon}\right) .$$

Proof of Proposition 7.3.5. The proof is almost identical to that of Theorem 7.1.3 so we will not detail elementary computations. First fix some $h > 0$ such that:

$$(-1/4 - h)(2 - \delta) + 1/2 > 0 . \tag{7.3.10}$$

By Theorem 7.1.2, for every $p \leq 0$, a.s. there is no unbounded connected component in $\mathcal{D}_p$. We henceforth consider some parameter $p_0 \in]0, 1]$. Fix $R_0 < +\infty$ to be determined later, and let:

$$\varepsilon = \varepsilon(R) = \log^{-1/4-h}(R) .$$

By Propositions 7.2.17 and 7.2.20 if R_0 is large enough, there exists $\eta > 0$ independent of $R > R_0$ and of $p \in [-1, 1]$ such that:

$$\forall x \in \mathcal{V}_R^{\varepsilon(R)},\ I_{x,\mu_R^{\varepsilon(R)}}\left(\mathrm{Cross}_p^{\varepsilon(R)}(2R, R)\right) \leq R^{-\eta} .$$

Hence, by Lemma 7.3.1, if R_0 is large enough:

$$\frac{d}{dp} g_R(p) \geq \frac{c}{C} \log^{(-1/4-h)(2-\delta)+1/2}(R) \sqrt{\eta/2} \log^{1/2}(R) .$$

By Theorem 7.2.13, if R_0 is large enough, there exists $C' < +\infty$ independent of $R > R_0$ such that:

$$\log\left(\frac{\mathbb{P}\left[\mathrm{Cross}_0^{\varepsilon(R)}(2R, R)\right]}{1 - \mathbb{P}\left[\mathrm{Cross}_0^{\varepsilon(R)}(2R, R)\right]}\right) \geq -C' .$$

Integrating from 0 to $p_0/2$ we get:

$$\mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R)\right] \geq 1 - \exp\left(-\frac{c}{2C} \frac{p_0}{2} \log^{(-1/4-h)(2-\delta)+1/2}(R) \sqrt{\eta/2} + C'\right) .$$

Now, if we apply Proposition 7.2.22, we obtain that, if R_0 is large enough, there exist constants $c_1 = c_1(\kappa) > 0$ and $C_2 = C_2(\kappa, p_0) < +\infty$ independent of $R > R_0$ such that:

$$\begin{aligned}
&\mathbb{P}\left[\mathrm{Cross}_{p_0}(2R, R)\right] \\
&\geq \mathbb{P}\left[\mathrm{Cross}_{p_0/2}^{\varepsilon(R)}(2R, R)\right] - C_2 \exp\left(-c_1 \log^{1+4h}(R)\right) \\
&\geq 1 - \exp\left(-\frac{c}{2C} p_0 \log^{(-1/4-h)(2-\delta)+1/2}(R) \sqrt{\eta/2} + C'\right) - C_2 \exp\left(-c_1 \log^{1+4h}(R)\right) .
\end{aligned}$$

Since (by (7.3.10)) we have $(-1/4 - h)(2 - \delta) + 1/2 > 0$, the criterion of Lemma 7.2.9 is satisfied and we obtain that a.s. $\mathcal{D}_{p_0}$ has a unique unbounded component. □

The following proof is also almost identical to that of Theorem 7.1.3 except that we do not have to use discretization estimates to conclude.

Proof of Theorem 7.2.11. Fix $\varepsilon \in]0,1]$. By Theorem 7.2.12, for every $p \leq 0$, a.s., there is no unbounded black component. From now on, we fix some $p_0 \in]0,1]$. By Lemma 7.2.14, it is enough to prove that f satisfies criterion (7.2.6). To this end we first fix $R_0 > 0$ to be determined later and we let $R > R_0$. By Proposition 7.2.21, there exists $C = C(\kappa, \varepsilon) < +\infty$ such that:

$$\|\sqrt{K^\varepsilon}\|_{\infty,op} \leq C\,.$$

Moreover by Propositions 7.2.17 and 7.2.20 if R_0 is large enough, there exists $\eta > 0$ independent of $R > R_0$ and $p \in [-1,1]$ such that:

$$\forall x \in \mathcal{V}_R^\varepsilon,\ I_{x,\mu_R^\varepsilon}\left(\mathrm{Cross}_p^\varepsilon(2R,R)\right) \leq R^{-\eta}\,.$$

Hence, by Lemma 7.3.1, if R_0 is large enough:

$$\frac{d}{dp} g_R(p) \geq \frac{c}{C}\sqrt{\frac{\eta}{2}\log(R)}\,.$$

By Theorem 7.2.13, if R_0 is large enough, there exists $C' < +\infty$ independent of $R > R_0$ such that:

$$\log\left(\frac{\mathbb{P}\left[\mathrm{Cross}_0^\varepsilon(2R,R)\right]}{1-\mathbb{P}\left[\mathrm{Cross}_0^\varepsilon(2R,R)\right]}\right) \geq -C'\,.$$

Integrating from 0 to p_0 we get:

$$\mathbb{P}\left[\mathrm{Cross}_{p_0}^\varepsilon(2R,R)\right] \geq 1 - \exp\left(-\frac{c}{C}p_0\sqrt{\frac{\eta}{2}\log(R)} + C'\right)\,.$$

Hence, f satisfies criterion **??** and we are done. □

7.3.2 Proof of the exponential decay

In this subsection, we prove Theorem 7.1.8 by following classical arguments of planar percolation that involve a recursion on crossing probabilities. In order to start the induction, we use the following property: According to Equation (7.3.8), if f is the Bargmann-Fock field, then:

$$\forall p > 0,\ \mathbb{P}[\mathrm{Cross}_p(2R,R)] \underset{R\to+\infty}{\longrightarrow} 1\,. \tag{7.3.11}$$

Proof of Theorem 7.1.8. Let f be the Bargmann-Fock field. Let us prove that for every $p > 0$, there exists $c = c(p) > 0$ such that for every $R > 0$ we have:

$$\mathbb{P}\left[\mathrm{Cross}_p(2R,R)\right] \geq 1 - \exp(-c\,R)\,. \tag{7.3.12}$$

Theorem 7.1.8 will follow from (7.3.12). Indeed, by a simple gluing argument, for every $\rho_1 > 1$ and $\rho_2 > 0$, there exists $N = N(\rho_1,\rho_2) \in \mathbb{Z}_{>0}$ such that, for every $R > 0$, there

exist N $\rho_1 R \times R$ rectangles such that if these rectangles are crossed lengthwise then $[0, \rho_2 R] \times [0, R]$ is also crossed lengthwise.

In order to prove (7.3.12), we follow classical ideas in planar percolation theory, see for instance the second version of the proof of Theorem 10 in [BR06b] or the proof of Proposition 3.1 in [ATT16]. To begin with, define the sequence $(r_k)_{k\geq 0}$ as follows. $r_0 = 1$ and for each $k \in \mathbb{N}$,

$$r_{k+1} = 2r_k + \sqrt{r_k}\,.$$

From this definition it easily follows that for each k, $r_k \geq 2^k$, which in turn implies that $r_{k+1} \leq (2 + 2^{-k/2})r_k$. Using this last relation one can easily show that $r_k = O(2^k)$ so that there exists $C < +\infty$ such that for each $k \in \mathbb{N}$,

$$2^k \leq r_k \leq C2^k\,. \tag{7.3.13}$$

Write:

$$a_k = 1 - \mathbb{P}\left[\mathrm{Cross}_p(2r_k, r_k)\right] + \exp(-\frac{r_k}{10})\,.$$

We will show that there exists $k_0 = k_0(p) < +\infty$ such that for any $k \geq k_0$:

$$a_{k+1} \leq 49a_k^2\,. \tag{7.3.14}$$

By Equation (7.3.11), $\lim_{k\to+\infty} a_k = 0$. This observation combined with Equation (7.3.14) yields (7.3.12) for $R = r_k$ with $k \geq k_0$ by an elementary induction argument. But since by (7.3.13), the sequence $(r_k)_{k\geq 0}$, grows geometrically, one then obtains (7.3.12) for any $R > 0$ by elementary gluing arguments.

In order to prove Equation (7.3.14) we first introduce two events, see Figure 7.1. First, the event $\mathrm{FiveCross}_p(k)$ is the event that:

- the $2r_k \times r_k$ rectangles $[(ir_k, (i+2)r_k] \times [0, r_k]$ for $i = 0, 1, 2, 3$ are crossed from left to right by a continuous path in $\mathcal{D}_p$;
- the $r_k \times r_k$ squares $[jr_k, (j+1)r_k] \times [0, r_k]$ for $j = 1, 2, 3$ are crossed from top to bottom by a continuous path in $\mathcal{D}_p$.

Second, the event $\mathrm{FiveCross}'_p(k)$ is the event $\mathrm{FiveCross}_p(k)$ translated by $(0, r_k + \sqrt{r_k})$. Note that $\mathrm{FiveCross}_p(k) \cup \mathrm{FiveCross}'_p(k) \subseteq \mathrm{Cross}_p(5r_k, 2r_k + \sqrt{r_k})$ Moreover, there exists $k_1 < +\infty$ such that for each $k \geq k_1$, $5r_k \geq 2r_{k+1} = 4r_{k+1} + 2\sqrt{r_k}$ so that $\mathrm{Cross}_p(5r_k, 2r_k + \sqrt{r_k}) \subseteq \mathrm{Cross}_p(2r_{k+1}, r_{k+1})$. Hence, for each $k \geq k_1$:

$$\mathbb{P}\left[\mathrm{Cross}_p(2r_{k+1}, r_{k+1})\right] \geq 1 - \mathbb{P}\left[\neg\mathrm{FiveCross}_p(k) \cap \neg\mathrm{FiveCross}'_p(k)\right]\,. \tag{7.3.15}$$

We claim that the events $\mathrm{FiveCross}_p(k)$ and $\mathrm{FiveCross}'_p(k)$ are asymptotically independent. More precisely, the following is a direct consequence of Theorem 1.12 from [RV17a].

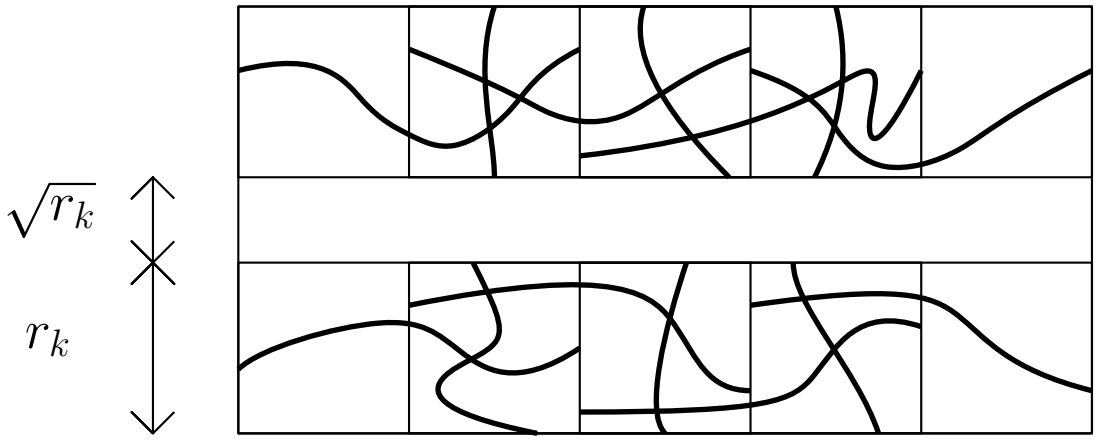


Figure 7.1: The events $\text{FiveCross}_p(k)$ and $\text{FiveCross}'_p(k)$.

Claim 7.3.7. *There exists $k_2 < +\infty$ such that, for every $p \in \mathbb{R}$ and every $k \geq k_2$:*

$$\left|\mathbb{P}\left[\neg\text{FiveCross}_p(k) \cap \neg\text{FiveCross}'_p(k)\right] - \mathbb{P}\left[\neg\text{FiveCross}_p(k)\right]\mathbb{P}\left[\neg\text{FiveCross}'_p(k)\right]\right| \leq O(r_k^4)e^{-\frac{\sqrt{r_k}^2}{2}} \leq e^{-\frac{r_k}{4}}.$$

Let $k \geq \max\{k_1, k_2\}$. Applying Claim 7.3.7 in Equation (7.3.15), we get:

$$\begin{aligned}\mathbb{P}\left[\text{Cross}_p(2r_{k+1}, r_{k+1})\right] &\geq 1 - \mathbb{P}\left[\neg\text{FiveCross}_p(k)\right]\cdot\mathbb{P}\left[\neg\text{FiveCross}'_p(k)\right] - e^{-\frac{r_k}{4}} \\ &= 1 - \mathbb{P}\left[\neg\text{FiveCross}_p(k)\right]^2 - e^{-\frac{r_k}{4}} \text{ (by stationarity).}\end{aligned}$$

By a union bound we have:

$$\mathbb{P}\left[\neg\text{FiveCross}_p(k)\right] \leq 4(1-\mathbb{P}\left[\text{Cross}_p(2r_k, r_k)\right]) + 3(1-\mathbb{P}\left[\text{Cross}_p(r_k, r_k)\right]) \leq 7(1-\mathbb{P}\left[\text{Cross}_p(2r_k, r_k)\right]).$$

Thus:

$$1 - \mathbb{P}\left[\text{Cross}_p(2r_{k+1}, r_{k+1})\right] \leq \left(7(1 - \mathbb{P}\left[\text{Cross}_p(2r_k, r_k)\right])\right)^2 + e^{-\frac{r_k}{4}},$$

and:

$$\begin{aligned}49(a_k)^2 &\geq \left(7(1 - \mathbb{P}\left[\text{Cross}_p(2r_k, r_k)\right])\right)^2 + 49e^{-\frac{2r_k}{10}} \\ &\geq 1 - \mathbb{P}\left[\text{Cross}_p(2r_{k+1}, r_{k+1})\right] + 49e^{-\frac{2r_k}{10}} - e^{-\frac{r_k}{4}} \\ &\geq 1 - \mathbb{P}\left[\text{Cross}_p(2r_{k+1}, r_{k+1})\right] + e^{-\frac{2r_k+\sqrt{r_k}}{10}} \text{ if } k \text{ is sufficiently large} \\ &= a_{k+1}.\end{aligned}$$

This is exactly Equation (7.3.14). □

7.4 Percolation arguments

In this section we prove Lemma 7.2.9 and Proposition 7.2.20. In doing so we also obtain Lemma 7.2.14.

For every $x \in \mathbb{R}^2$ and every $\rho > 0$, we set $B(x, \rho) = x + [-\rho, \rho]^2$. For every $x \in \mathbb{R}^2$ $0 < \rho_1 \leq \rho_2$, we set:

$$\mathrm{Ann}(\rho_1, \rho_2) = [-\rho_2, \rho_2]^2 \backslash]\rho_1, \rho_2[^2 \,,$$

and $\mathrm{Ann}(x, \rho_1, \rho_2) = x + \mathrm{Ann}(\rho_1, \rho_2)$.

7.4.1 Proof of Lemma 7.2.9

In this subsection, we prove Lemma 7.2.9. The proof of Lemma 7.2.9 also works for Lemma 7.2.14 with only a few obvious changes. For the proofs of these results, we never use the fact that our Gaussian field is non-degenerate.

Proof of Lemma 7.2.9. In this proof, we write $\mathrm{Cross}_p(k) = \mathrm{Cross}_p(2^{k+1}, 2^k)$ and we write $\mathrm{Cross}'_p(k)$ for the event that there is a continuous path joining the bottom side of the rectangle $[0, 2^k] \times [0, 2^{k+1}]$ to its top side in $\mathcal{D}_p \cap [0, 2^k] \times [0, 2^{k+1}]$. By translation invariance and $\frac{\pi}{2}$-rotation invariance, $\mathbb{P}\left[\mathrm{Cross}'_p(k)\right] = \mathbb{P}\left[\mathrm{Cross}_p(k)\right]$ for every k. Thus:

$$\sum_{k\in\mathbb{N}} \mathbb{P}\left[\neg\left(\mathrm{Cross}_p(k) \cap \mathrm{Cross}'_p(k)\right)\right] \leq 2 \sum_{k\in\mathbb{N}} \mathbb{P}\left[\neg\mathrm{Cross}_p(k)\right] < +\infty \text{ by assumption.}$$

Together with Borel-Cantelli lemma, it implies that a.s. there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$, $\mathrm{Cross}_p(k) \cap \mathrm{Cross}'_p(k)$ is satisfied. Note that any crossing from left to right of $[0, 2^{k+1}] \times [0, 2^k]$ and any crossing from top to bottom of $[0, 2^{k+1}] \times [0, 2^{k+2}]$ must intersect. Similarly, any crossing from top to bottom of $[0, 2^k] \times [0, 2^{k+1}]$ and any crossing from left to right of $[0, 2^{k+2}] \times [0, 2^{k+1}]$ must intersect. All these intersecting crossings then form an unbounded connected component in $\mathcal{D}_p$.

Let us end the proof by showing that this unbounded component is a.s. unique. The proof follows the same overall structure: First, write $\mathrm{Circ}_p(k)$ for the event that there is a circuit in $\mathcal{D}_p \cap \mathrm{Ann}(2^k, 2^{k+1})$ surrounding the square $[-2^k, 2^k]^2$. Note that, if eight well-chosen $2^{k+1} \times 2^k$ rectangles are crossed lengthwise, two well-chosen $2^k \times 2^k$ squares are crossed from left to right, and two well-chosen $2^k \times 2^k$ squares are crossed from top to bottom, then $\mathrm{Circ}_p(k)$ holds (see Figure 7.1 where we already used such a property). Since the probability that a $2^k \times 2^k$ square is crossed is at least equal to the probability that a $2^{k+1} \times 2^k$ rectangle is crossed lengthwise, we get:

$$\mathbb{P}\left[\neg\mathrm{Circ}_p(k)\right] \leq 12\mathbb{P}\left[\neg\mathrm{Cross}_p(k)\right] \,.$$

Thus, as before,

$$\sum_{k\in\mathbb{N}} \mathbb{P}\left[\neg\mathrm{Circ}_p(k)\right] < +\infty \,,$$

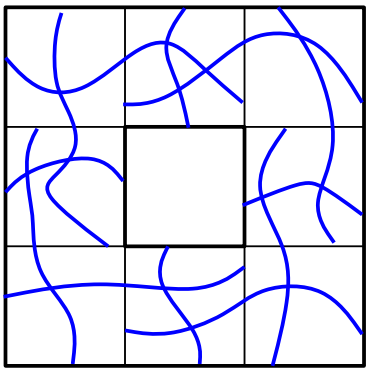


Figure 7.1: If eight $2^{k+1} \times 2^k$ well chosen rectangles and four well chosen $2^k \times 2^k$ squares are crossed, then there is a circuit in $\mathrm{Ann}(2^k, 2^{k+1})$.

and Borel-Cantelli lemma implies that a.s. there exists k_0 such that, for every $k \geq k_0$, $\mathrm{Circ}_p(k)$ holds. Now, note that, for any unbounded connected component, there exists k_1 such that this component crosses the annuli $\mathrm{Ann}(2^k, 2^{k+1})$ for every $k \geq k_1$. Thus, this component contains any circuit of this annulus for every $k \geq k_0 \vee k_1$. In particular, it must be unique. □

7.4.2 Proof of Proposition 7.2.20

The goal of this subsection is to prove Proposition 7.2.20. To this purpose, we use the following result from [RV17a] which is a consequence of the discrete RSW estimate and quasi-independence. Such a result goes back to [BG17a] for $\alpha > 16$. Let $\varepsilon > 0$ and consider the discrete percolation model defined in Subsection 7.2.4. Let $\mathrm{Arm}_p^{\varepsilon}(R)$ (respectively $\mathrm{Arm}_p^{\varepsilon,*}(R)$) be the event that in this model there is a black (respectively white) path in the annulus $\mathrm{Ann}(1, R)$ from the inner boundary of this annulus to its outer boundary. We have the following:

Proposition 7.4.1 ([BG17a] for $\alpha > 16$, [RV17a])**.** [14] *Assume that f satisfies Conditions 7.2.1, 7.2.2, 7.2.4 as well as Condition 7.2.3 for some $\alpha > 4$. Then, there exists $C = C(\kappa) < +\infty$ and $\eta = \eta(\kappa) > 0$ such that, for each $\varepsilon \in]0, 1]$, for each $R \in]0, +\infty[$:*

$$\mathbb{P}\left[\mathrm{Arm}_0^{\varepsilon}(R)\right], \mathbb{P}\left[\mathrm{Arm}_0^{*,\varepsilon}(R)\right] \leq C\, R^{-\eta}\,.$$

To deduce Proposition 7.2.20 from Proposition 7.4.1, we need the following lemma and more particularly its consequence Corollary **??**.

[14] More precisely, this is Proposition B.6 of [RV17a]. Moreover, this can be extracted from the proof of Theorem 5.7 of [BG17a] for $\alpha > 16$ and with slightly different assumptions on the differentiability and the non-degeneracy of κ.

Lemma 7.4.2. *Let $\Sigma = (\Sigma(i,j))_{0\leq i,j\leq n}$ be a $(n+1)\times(n+1)$ symmetric positive definite matrix, let $(m_0, m) = (m_0, m_1, \cdots, m_n) \in \mathbb{R}^{n+1}$ and let $(X_0, X) = (X_0, X_1, \cdots, X_n) \sim \mathcal{N}((m_0, m), \Sigma)$. Assume that $\Sigma(0,i) \geq 0$ for every $i \in \{1, \cdots, n\}$. Then, for every non-decreasing[15] function $\varphi : \mathbb{R}^n \to \mathbb{R}$, the following quantity is non-decreasing in q:*

$$\mathbb{E}\left[\varphi(X) \,\middle|\, X_0 = q\right] .$$

Similarly, if φ is non-increasing, then this quantity is non-increasing in q.

Proof. This is a direct consequence of the following result (see for instance Proposition 1.2 of [AW09]): Under the probability measure $\mathbb{P}\left[\cdot \,\middle|\, X_0 = q\right]$, X is a Gaussian vector whose covariance matrix does not depend on q and whose mean is:

$$m + (q - m_0)v\,,$$

where $v = \left(\frac{\Sigma(0,i)}{\Sigma(0,0)}\right)_{1\leq i\leq n}$ (which has only non-negative entries). □

From this lemma we deduce the following:

Corollary 7.4.3. *Let (X_0, X) be as in Lemma 7.4.2 and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a non-decreasing function. Then, for every $p \in \mathbb{R}$ we have:*

$$\mathbb{E}\left[\varphi(X) \,\middle|\, X_0 = -p\right] \leq \mathbb{E}\left[\varphi(X) \,\middle|\, X_0 \geq -p\right] .$$

Similarly, if φ is non-increasing, then:

$$\mathbb{E}\left[\varphi(X) \,\middle|\, X_0 = -p\right] \leq \mathbb{E}\left[\varphi(X) \,\middle|\, X_0 \leq -p\right] .$$

Proof. Assume that φ is non-decreasing and let γ_0 be the density of X_0, which exists because Σ is positive definite. We have :

$$\begin{aligned}
\mathbb{E}\left[\varphi(X) \,\middle|\, X_0 \geq -p\right] &= \frac{1}{\mathbb{P}\left[X_0 \geq -p\right]} \int_{-p}^{+\infty} \mathbb{E}\left[\varphi(X) \,\middle|\, X_0 = t\right] \gamma_0(t)dt \\
&\geq \frac{1}{\mathbb{P}\left[X_0 \geq -p\right]} \mathbb{E}\left[\varphi(X) \,\middle|\, X_0 = -p\right] \int_{-p}^{+\infty} \gamma_0(t)dt \\
&= \mathbb{E}\left[\varphi(X) \,\middle|\, X_0 = -p\right]
\end{aligned}$$

where the inequality comes from Lemma 7.4.2 applied with $q = -p$. □

We are now ready to prove Proposition 7.2.20. As usual in percolation theory, we are going to control our p-dependent probabilities by probabilities at the self-dual point $p = 0$.

[15] For the partial order $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$ if $x_i \leq y_i$ for every $i \in \{1, \cdots, n\}$.

Proof of Proposition 7.2.20. Fix $R \geq 2$ and $\varepsilon \in]0,1]$. Let us prove the proposition in the case $p \geq 0$. Let $x \in \mathcal{V}_R^\varepsilon$ and let $\mathrm{Arm}_p^{\varepsilon,*}(x, R/2)$ be the event that there is a white path in $\mathrm{Ann}\,(x, 1, R/2)$ from $\partial B(x,1)$ to $\partial B(x, R/2)$. We claim that $\mathrm{Piv}_x^p(\mathrm{Cross}^\varepsilon(2R,R)) \subseteq \mathrm{Arm}_p^{\varepsilon,*}(x, R/2)$. Indeed, $\mathrm{Piv}_x^p(\mathrm{Cross}^\varepsilon(2R,R))$ is the event that there are two white paths from the top and bottom sides of the rectangle $[0,2R] \times [0,R]$ to two neighbors of x and two black paths from the left and right sides of $[0,2R] \times [0,R]$ to two other neighbors of x (with an exception when x does not belong to $[0,2R] \times [0,R]$ but is a neighbor of a point $y \in [0,2R] \times [0,R]$ such that the edge between x and y crosses the left or right side of $[0,2R] \times [0,R]$; in this case $\mathrm{Piv}_x^p(\mathrm{Cross}^\varepsilon(R))$ is the event that there is a white path from top to bottom that goes through y and a black path from y to the oppposite side). In particular, at least one black path reaches a point at distance at least $R/2$ of x.

Since $\mathrm{Arm}_p^{\varepsilon,*}(x, R/2)$ is decreasing and does not depend on $f(x)$, then (by Corollary **??** and with $a := \mathbb{P}\,[Z \leq -1]$ where $Z \sim \mathcal{N}(0,1)$):

$$\begin{aligned}
\mathbb{P}\left[\mathrm{Arm}_p^{\varepsilon,*}(x,R/2) \,\middle|\, f(x) = -p\right] &\leq \mathbb{P}\left[\mathrm{Arm}_p^{\varepsilon,*}(x,R/2) \,\middle|\, f(x) \leq -p\right] \\
&\leq \frac{\mathbb{P}\left[\mathrm{Arm}_p^{\varepsilon,*}(x,R/2)\right]}{\mathbb{P}\left[f(x) \leq -p\right]} \\
&\leq a^{-1}\mathbb{P}\left[\mathrm{Arm}_p^{\varepsilon,*}(x,R/2)\right] \text{ since } p \leq 1 \\
&\leq a^{-1}\mathbb{P}\left[\mathrm{Arm}_0^{\varepsilon,*}(x,R/2)\right] \text{since } p \geq 0\,.
\end{aligned}$$

The result follows from Proposition 7.4.1 (and stationarity). The case $p \leq 0$ is treated similarly (by noting that $\mathrm{Piv}_x^p(\mathrm{Cross}^\varepsilon(2R,R)) \subseteq \mathrm{Arm}_p^\varepsilon(x, R/2)$ and by studying this increasing event exactly as above). □

7.5 A KKL theorem for biased Gaussian vectors: the proof of Theorem 7.2.19

In this section we prove Theorem 7.2.19. The proofs presented here do not rely on any results of the other sections. Recall that we use the following definition for influence. Given a vector $v \in \mathbb{R}^n$ and a Borel probability measure μ on $\mathbb{R}^n$ the influence of v on A under μ is

$$I_{v,\mu}(A) = \liminf_{r \downarrow 0} \frac{\mu\,(A + [-r,r]v) - \mu\,(A)}{r} \in [0, +\infty]\,.$$

7.5.1 Sub-linearity of influences implies Theorem 7.2.19

The aim of this subsection is to prove Theorem 7.2.19. That this theorem holds for product Gaussian measures is a result by Keller, Mossel and Sen, [KMS12] (see Corollary **??** below). In order to extend this result to general Gaussian measures, we use a sub-linearity property for influences. In the present subsection, we state this property

and use it to derive Theorem 7.2.19 from the product case. In Subsection 7.5.2 (respectively in Appendix 7.A), we prove the sub-linearity property for monotonic (respectively semi-algebraic) sets.

A KKL theorem for product Gaussian vectors The authors of [KMS12] introduce and study the notion of *geometric influences* which are defined as follows: Let ν be a probability measure on $\mathbb{R}$ and let A be a Borel subset of $\mathbb{R}^n$. If $i \in \{1, \cdots, n\}$, let:

$$A_i^x := \{y \in \mathbb{R} : (x_1, \cdots, x_{i-1}, y, x_{i+1}, \cdots, x_n) \in A\}.$$

The geometric influence of i on A under the measure $\nu^{\otimes n}$ is:

$$I_{i,\nu}^{\mathcal{G}}(A) := \mathbb{E}_{x \sim \nu^{\otimes n}}\left[\nu^+(A_i^x)\right] \in [0, +\infty],$$

where ν^+ is the lower Minkowski content, defined as follows: for all $B \subseteq \mathbb{R}$ Borel,

$$\nu^+(B) := \liminf_{r \downarrow 0} \frac{\nu(B + [-r, r]) - \nu(B)}{r} \in [0, +\infty].$$

In the case where $\mu = \nu^{\otimes n}$, $I_{i,\nu}^{\mathcal{G}}$ and $I_{i,\mu}$ are closely related. Indeed, firstly, by Fubini's Theorem and Fatou's lemma, for each Borel subset $A \subset \mathbb{R}^n$ and each $i \in \{1, \ldots, n\}$,

$$I_{i,\nu^{\otimes n}}(A) = \liminf_{r \downarrow 0} \mathbb{E}_{x \sim \nu^{\otimes n}}\left[\frac{\nu(A_i^x + [-r, r]) - \nu(A)}{r}\right] \geq I_{i,\nu}^{\mathcal{G}}(A). \tag{7.5.1}$$

While the reverse inequality seems not true in general, we expect it to hold for a wide class of events. In particular, our Lemma 7.5.5 and (2.6) from [KMS12] imply that this is the case for monotonic events. Since it is not useful to us, we do not investigate this matter any further.

We will need the following result, which is a direct consequence of Item (1) of Theorem 1.5 of [KMS12]. Several results of this type can also be found in the more recent [CEL12] (see for instance the paragraph above Corollary 7 therein).

Theorem 7.5.1. *There exists an absolute constant $c > 0$ such that the following holds: Let $\nu = \mathcal{N}(0, 1)$ and let A be a Borel measurable subset of $\mathbb{R}^n$. Then:*

$$\sum_{i=1}^n I_{i,\nu}^{\mathcal{G}}(A) \geq c\, \nu^{\otimes n}(A)\left(1 - \nu^{\otimes n}(A)\right) \sqrt{\log_+\left(\frac{1}{\max_{i \in \{1, \cdots, n\}} I_{i,\nu}^{\mathcal{G}}(A)}\right)}.$$

Corollary 7.5.2. *Let ν and A be as in Theorem 7.5.1. Then:*

$$\sum_{i=1}^n I_{i,\nu^{\otimes n}}(A) \geq c\, \nu^{\otimes n}(A)\left(1 - \nu^{\otimes n}(A)\right) \sqrt{\log_+\left(\frac{1}{\max_{i \in \{1, \cdots, n\}} I_{i,\nu^{\otimes n}}(A)}\right)}.$$

Proof. This is a direct consequence of Theorem 7.5.1 and Equation (7.5.1). □

Now, let us state a sub-linearity property for the influences that we will be proved in Subsection 7.5.2 and Appendix 7.A.

Proposition 7.5.3. *Let Σ be a $n \times n$ symmetric positive definite matrix, let $\mu \sim \mathcal{N}(0, \Sigma)$ and let $(e_1, \cdots, e_n)$ be the canonical basis of $\mathbb{R}^n$. Also, let $v = \sum_{j=1}^n v_i\, e_i \in \mathbb{R}^n$. For every $i \in \{1, \cdots, n\}$ and every A Borel subset of $\mathbb{R}^n$ which is monotonic or semi-algebraic,*[16] *we have:*

$$I_{v,\mu}(A) \leq \sum_{i=1}^n |v_i|\, I_{i,\mu}(A)\,. \tag{7.5.2}$$

We expect the above result to hold for a larger class of Borel subsets A (in particular, we have not found any examples of Borel sets for which this does not hold) and not only for the decomposition on the canonical basis (see in particular Proposition 7.A.1).

Let us now derive Theorem 7.2.19 from Corollary **??** using Proposition7.5.3.

Proof of Theorem 7.2.19. Let $\sqrt{\Sigma}$ be a symmetric square root of Σ and let $\nu = \mathcal{N}(0,1)$. Then, $\mu = \mathcal{N}(0, \Sigma)$ is the pushforward measure of $\nu^{\otimes n}$ by $\sqrt{\Sigma}$. Thanks to Corollary **??** (applied to the event $\sqrt{\Sigma}^{-1}(A)$), it is sufficient to prove the following claim. □

Claim 7.5.4. *Let $\nu = \mathcal{N}(0,1)$. We have:*

$$\max_{j \in \{1,\cdots,n\}} I_{j,\nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq ||\sqrt{\Sigma}||_{\infty,op} \cdot \max_{i \in \{1,\cdots,n\}} I_{i,\mu}(A)\,. \tag{7.5.3}$$

Moreover:

$$\sum_{j=1}^n I_{j,\nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq ||\sqrt{\Sigma}||_{\infty,op} \cdot \sum_{i=1}^n I_{i,\mu}(A)\,. \tag{7.5.4}$$

Proof. For each $j \in \{1, \cdots, n\}$:

$$\begin{aligned} I_{j,\nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) &= \liminf_{r \downarrow 0} \frac{\nu^{\otimes n}\left(\sqrt{\Sigma}^{-1}(A) + [-r,r]\, e_j\right) - \nu^{\otimes n}\left(\sqrt{\Sigma}^{-1}(A)\right)}{r} \\ &= \liminf_{r \downarrow 0} \frac{\mu\left(A + [-r,r]\,\sqrt{\Sigma} \cdot e_j\right) - \mu(A)}{r} \\ &= I_{\sqrt{\Sigma} \cdot e_j, \mu}(A)\,. \end{aligned}$$

By using Proposition 7.5.3, we obtain that:

$$I_{j,\nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq \sum_{i=1}^n |\sqrt{\Sigma}(i,j)| I_{i,\mu}(A)\,. \tag{7.5.5}$$

[16] See Appendix 7.A for the definition of semi-algebraic sets.

Hence:

$$I_{j,\nu^{\otimes n}}(\sqrt{\Sigma}^{-1}(A)) \leq \sum_{i=1}^{n} |\sqrt{\Sigma}(i,j)| \max_{i\in\{1,\cdots,n\}} I_{i,\mu}(A) \leq ||{}^t\sqrt{\Sigma}||_{\infty,op} \max_{i\in\{1,\cdots,n\}} I_{i,\mu}(A)\,.$$

Since $\sqrt{\Sigma}$ is symmetric, we obtain (7.5.3) by taking the supremum over j. Inequality (7.5.5) also implies that:

$$\begin{aligned}\sum_{j=1}^{n} I_{\sqrt{\Sigma}\cdot e_j,\mu}(A) &\leq \sum_{j=1}^{n}\sum_{i=1}^{n} |\sqrt{\Sigma}(i,j)|\, I_{i,\mu}(A)\\ &= \sum_{i=1}^{n} I_{i,\mu}(A) \sum_{j=1}^{n} |\sqrt{\Sigma}(i,j)|\\ &\leq ||\sqrt{\Sigma}||_{\infty,op} \cdot \sum_{i=1}^{n} I_{i,\mu}(A)\,,\end{aligned}$$

which is (7.5.4). □

7.5.2 Sub-linearity of influences for monotonic events

The proof of Proposition 7.5.3 for monotonic events relies on the following lemma.

Lemma 7.5.5. *Let Σ be a $n\times n$ symmetric positive definite matrix and let $\mu = \mathcal{N}(0,\Sigma)$. Moreover, let $v \in \mathbb{R}^n$. For every A monotonic Borel subset of $\mathbb{R}^n$ and every $i \in \{1,\cdots,n\}$, we have:*

$$\exists \lim_{r\downarrow 0} \frac{\mu\left(A + [-r,r]e_i + [-r,r]v\right) - \mu\left(A + [-r,r]v\right)}{r} = I_{i,\mu}(A)\,.$$

We first prove Proposition 7.5.3 using Lemma 7.5.5.

Proof of Proposition 7.5.3 in the monotonic case. Let A be a monotonic Borel subset of $\mathbb{R}^n$ and fix $v \in \mathbb{R}^n$. Let $v^i = \sum_{k=i}^{n} v_i e_i$. We will prove that, for every $i \in \{1,\cdots,n-1\}$:

$$I_{v^i,\mu}(A) \leq |v_i|\, I_{i,\mu}(A) + I_{v^{i+1},\mu}(A)\,. \tag{7.5.6}$$

The result will then follow directly by induction. For any $w_1, w_2 \subseteq \mathbb{R}^n$, we have $[-r,r](w_1+w_2) \subseteq [-r,r]w_1 + [-r,r]w_2$. Hence:

$$\begin{aligned}\mu\left(A + [-r,r]v^i\right) &= \mu\left(A + [-r,r]\left(v_i e_i + v^{i+1}\right)\right)\\ &\leq \mu\left(A + [-|v_i|r, |v_i|r]e_i + [-r,r]v^i\right)\\ &= \mu\left(A + [-|v_i|r, |v_i|r]e_i + [-r,r]v^{i+1}\right)\\ &\quad - \mu\left(A + [-r,r]v^{i+1}\right) + \mu\left(A + [-r,r]v^{i+1}\right)\,.\end{aligned}$$

By Lemma 7.5.5 we have:

$$\frac{\mu\left(A+[-|v_i|r,|v_i|r]\,e_i+[-r,r]\,v^{i+1}\right)-\mu\left(A+[-r,r]v^i\right)}{r}\underset{r\downarrow 0}{\longrightarrow}|v_i|\,I_{i,\mu}(A)\,.$$

Equation (7.5.6) follows. □

Proof of Lemma 7.5.5. We are inspired by the proof of Proposition 1.3 in [KMS12]. We write the proof for A decreasing since the proof for A increasing is identical. Also, we prove the result in the case $i=n$. We write $\tilde{x}$ for the first $(n-1)$ coordinates of any $x\in\mathbb{R}^n$. Let $A_r=A+[-r,r]v$, note that A_r is decreasing, and write for any $\tilde{x}\in\mathbb{R}^{n-1}$:

$$\begin{aligned}s(\tilde{x})&:=\sup\{x_n\in\mathbb{R}\,:\,(\tilde{x},x_n)\in A\}\in[-\infty,+\infty]\,,\\ s_r(\tilde{x})&:=\sup\{x_n\in\mathbb{R}\,:\,(\tilde{x},x_n)\in A_r\}\in[-\infty,+\infty]\,.\end{aligned}$$

Let λ be the density function of μ. Since A and A_r are decreasing, we have:

$$\begin{aligned}\frac{\mu(A+[-r,r]e_n)-\mu(A)}{r}&=\frac{1}{r}\int_{\mathbb{R}^{n-1}}\left(\int_{s(\tilde{x})}^{s(\tilde{x})+r}\lambda(\tilde{x},x_n)dx_n\right)d\tilde{x}\,,\\ \frac{\mu(A_r+[-r,r]e_n)-\mu(A_r)}{r}&=\frac{1}{r}\int_{\mathbb{R}^{n-1}}\left(\int_{s_r(\tilde{x})}^{s_r(\tilde{x})+r}\lambda(\tilde{x},x_n)dx_n\right)d\tilde{x}\,,\end{aligned}\tag{7.5.7}$$

where by convention $\int_{-\infty}^{-\infty+r}=\int_{+\infty}^{+\infty+r}=0$. For each $\tilde{x}\in\mathbb{R}^{n-1}$ let:

$$g_1(\tilde{x})=\sup_{x_n\in\mathbb{R}}\lambda(\tilde{x},x_n);\;\; g_2(\tilde{x})=\sup_{x_n\in\mathbb{R}}\left|\frac{\partial\lambda}{\partial x_n}(\tilde{x},x_n)\right|.$$

Direct computation shows that $g_1,g_2\in L^1(\mathbb{R}^{n-1})$. By the mean value inequality, for each $\tilde{x}\in\mathbb{R}^{n-1}$:

$$\left|\frac{1}{r}\left(\int_{s(\tilde{x})}^{s(\tilde{x})+r}\lambda(\tilde{x},x_n)dx_n\right)-\lambda(\tilde{x},s(\tilde{x}))\right|+\left|\frac{1}{r}\left(\int_{s_r(\tilde{x})}^{s_r(\tilde{x})+r}\lambda(\tilde{x},x_n)dx_n\right)-\lambda(\tilde{x},s_r(\tilde{x}))\right|$$

is no greater than $2rg_2(\tilde{x})$. Combining this with Equation (7.5.7) we get:

$$\begin{aligned}&\left|\frac{\mu(A+[-r,r]e_n)-\mu(A)}{r}-\frac{\mu(A_r+[-r,r]e_n)-\mu(A_r)}{r}\right|\\ &\qquad\qquad\leq\left|\int_{\mathbb{R}^{n-1}}\lambda(\tilde{x},s(\tilde{x}))-\lambda(\tilde{x},s_r(\tilde{x}))\,d\tilde{x}\right|+2r\int_{\mathbb{R}^{n-1}}g_2(\tilde{x})\,d\tilde{x}\,.\end{aligned}$$

Since $g_2\in L^1(\mathbb{R}^{n-1})$, the second integral in the last inequality is finite and independent of r. Moreover:

$$\int_{\mathbb{R}^{n-1}}\lambda(\tilde{x},s_r(\tilde{x}))-\lambda(\tilde{x},s(\tilde{x}))\,d\tilde{x}=\int_{\mathbb{R}^{n-1}}\int_{s(\tilde{x})}^{s_r(\tilde{x})}\frac{\partial\lambda}{\partial x_n}(\tilde{x},x_n)\,dx_nd\tilde{x}\,.$$

Since $\frac{\partial \lambda}{\partial x_n} \in L^1(\mathbb{R}^n)$, by dominated convergence, all that remains is to show that for a.e. $\tilde{x} \in \mathbb{R}^{n-1}$: $s_r(\tilde{x}) \underset{r \downarrow 0}{\longrightarrow} s(\tilde{x})$. Since for each $\tilde{x}$ the sequence $s_r(\tilde{x})$ is decreasing, it converges to some $s_\infty(\tilde{x}) \geq s(\tilde{x})$. Let us prove that, for a.e. $\tilde{x} \in \mathbb{R}^{n-1}$, $s_\infty(\tilde{x}) = s(\tilde{x})$. To do so, first note that, since A is decreasing, we have:

$$0 \leq \mu(A_r) - \mu(A) \leq \mathbb{P}\left[X - \sum_{i=1}^{n} r|v_i|e_i \in A\right] - \mathbb{P}\left[X \in A\right] ,$$

where $X \sim \mathcal{N}(0, \Sigma)$. By dominated convergence, the right hand side tends to 0 when $r \to 0$. Now, note that:

$$\mu(A_r) - \mu(A) = \int_{\mathbb{R}^{n-1}} \int_{s(\tilde{x})}^{s_r(\tilde{x})} \lambda(\tilde{x}, x_n)\, dx_n d\tilde{x} .$$

Hence, by Fatou's lemma:

$$0 = \int_{\mathbb{R}^{n-1}} \int_{s(\tilde{x})}^{s_\infty(\tilde{x})} \lambda(\tilde{x}, x_n)\, dx_n d\tilde{x} .$$

Since the λ takes only positive values, this implies that for a.e. $\tilde{x} \in \mathbb{R}^{n-1}$, $s_\infty(\tilde{x}) = s(\tilde{x})$. □

7.6 An estimate on the infinite operator norm of square root of infinite matrices: the proof of Proposition 7.2.21

The goal of this section is to prove Proposition 7.2.21. We assume that f satisfies Condition 7.2.1 and Condition 7.2.5 for some $\alpha > 5$. In particular, the Fourier transform of κ takes only positive values. Moreover, we assume that κ is C^3 and there exists $C < +\infty$ such that for every $\beta \in \mathbb{N}^2$ such that $\beta_1 + \beta_2 \leq 3$, we have:

$$|\partial^\beta \kappa(x)| \leq C|x|^{-\alpha} , \tag{7.6.1}$$

for some $\alpha > 5$. In this subsection, we never use the fact that our Gaussian field is non-degenerate.

Recall that for each $\varepsilon > 0$, $\mathcal{T}^\varepsilon$ is the lattice $\mathcal{T}$ scaled by a factor ε, that $\mathcal{V}^\varepsilon$ is the set of vertices of $\mathcal{T}^\varepsilon$, and that K^ε is the restriction of K to $\mathcal{V}^\varepsilon$. We begin by observing that the face-centered square lattice $\mathcal{T}$, when rotated by $\frac{\pi}{4}$ and rescaled by $\sqrt{2}$, has the same vertices as $\mathbb{Z}^2$. Since by Condition 7.2.5 our field is invariant by $\frac{\pi}{4}$-rotation, **we will** simply **replace $\mathcal{T}$ by $\mathbb{Z}^2$ throughout the rest of this section**.
Let $\mathbb{T}^2$ be the flat 2D torus corresponding to the circle of length 2π. Throughout this section, we will identify $\lambda\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$ (for some $\lambda > 0$) with functions

on $\lambda\mathbb{T}^2$ and their integrals over the box $[-\lambda\pi, \lambda\pi]^2$ with integrals over $\lambda\mathbb{T}^2$. We will use the following convention for the Fourier transform:

$$\forall \xi \in \mathbb{R}^2,\ \hat{\kappa}(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} e^{-i\langle \xi, x\rangle} \kappa(x) dx\,.$$

Let us begin with a sketch of the construction of the square root $\sqrt{K^\varepsilon}$:

1. First note that if we find some symmetric function $\eta_\varepsilon : \mathbb{Z}^2 \to \mathbb{R}^2$ such that:

$$\forall m \in \mathbb{Z}^2,\ \eta_\varepsilon * \eta_\varepsilon(m) := \sum_{m' \in \varepsilon\mathbb{Z}^2} \eta_\varepsilon(m')\eta_\varepsilon(m - m') = \kappa(\varepsilon m)\,,$$

   then $(\varepsilon m_1, \varepsilon m_2) \in \varepsilon\mathbb{Z}^2 \longmapsto \eta_\varepsilon(m_1 - m_2)$ is a symmetric square root of K^ε.

2. Let κ_ε be κ restricted to $\varepsilon\mathbb{Z}^2$ and let us try to construct η_ε above. The first idea is that, if $\eta_\varepsilon * \eta_\varepsilon = \kappa_\varepsilon$, then the Fourier transform of η_ε should the square root of the Fourier transform of κ_ε. In other words:

$$\mathsf{F}(\eta_\varepsilon) = \sqrt{\mathsf{F}(\kappa_\varepsilon)}\,,$$

   where $\mathsf{F}(\eta_\varepsilon)(\xi) = \sum_{m\in\mathbb{Z}^2} \eta_\varepsilon(m) e^{-i<\xi, m>}$ and similarly for $\mathsf{F}(\kappa_\varepsilon)$.

3. Thus:

$$\begin{aligned} \eta_\varepsilon(m) &= \mathsf{F}^{-1}\left(\sqrt{\mathsf{F}(\kappa_\varepsilon)}\right) \\ &= \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \sqrt{\mathsf{F}(\kappa_\varepsilon)}(\xi)\, d\xi\,. \end{aligned}$$

4. In the expression above, it seems difficult to deal with the term $\mathsf{F}(\kappa_\varepsilon)$. To simplify the expression, we can use the Poisson summation formula and deduce that:

$$\mathsf{F}(\kappa_\varepsilon)(\xi) = 4\pi^2\varepsilon^{-2} \sum_{m\in\mathbb{Z}^2} \hat{\kappa}(\varepsilon^{-1}(2\pi m - \xi))\,.$$

   For the Bargmann-Fock process, $\hat{\kappa}$ is well known since κ is simply the Gaussian function.

There will be two main steps in the proof of Proposition 7.2.21. First, we will make the above construction of η_ε rigorous by considering the four items above in the reverse order. More precisely, we will first set λ_ε as in Lemma 7.6.1 below, then we will apply the Poisson summation formula to prove that $2\pi\varepsilon^{-1}\lambda_\varepsilon = \sqrt{\mathsf{F}(\kappa_\varepsilon)}$. Next, we will define η_ε as in Lemma 7.6.1, we will show that $\mathsf{F}(\eta_\varepsilon) = \sqrt{\mathsf{F}(\kappa_\varepsilon)}$, and we will conclude that η_ε is a convolution square root of κ_ε. All of this will be done in the proof Lemma 7.6.1. Secondly, we will prove estimates on $\sum_{m\in\mathbb{Z}^2} |\eta_\varepsilon(m)|$ when $\kappa(x) = e^{-\frac{1}{2}|x|^2}$. This will be the purpose of Lemma 7.6.2 (and most of the proof of this lemma will be written in a

more general setting than $\kappa(x) = e^{-\frac{1}{2}|x|^2}$).

In the proofs, we will use a subscript ε to denote functions : $\mathbb{Z}^2 \to \mathbb{R}$ or to denote functions defined on $\mathbb{T}^2$ (or equivalently functions $2\pi\mathbb{Z}^2$-periodic). On the other hand, we will use a superscript ε to denote functions : $\varepsilon^{-1}\mathbb{Z}^2 \to \mathbb{R}$ or to denote functions defined on $\varepsilon^{-1}\mathbb{T}^2$ (or equivalently functions $2\pi\varepsilon^{-1}\mathbb{Z}^2$-periodic).

Proposition 7.2.21 is a direct consequence of the following Lemmas 7.6.1 and 7.6.2.

Lemma 7.6.1. *Assume that f satisfies Condition 7.2.1 as well as Condition 7.2.2 for $\alpha > 5$. Fix $\varepsilon > 0$ and let:*

$$\lambda_\varepsilon \,:\, \xi \in \mathbb{R}^2 \mapsto \sqrt{\sum_{m\in\mathbb{Z}^2} \hat{\kappa}(\varepsilon^{-1}(\xi - 2\pi m))}\,.$$

Then, λ_ε is a C^3, positive, even and $2\pi\varepsilon^{-1}\mathbb{Z}^2$-periodic function. Next, define:

$$\eta_\varepsilon \,:\, m \in \mathbb{Z}^2 \mapsto \frac{1}{\varepsilon(2\pi)} \int_{\mathbb{T}^2} e^{i\langle m,\xi\rangle} \lambda_\varepsilon(\xi)\, d\xi\,.$$

Then, $(\varepsilon m_1, \varepsilon m_2) \in \varepsilon\mathbb{Z}^2 \longmapsto \eta_\varepsilon(m_1 - m_2)$ is a symmetric square root of K^ε and we have:

$$\sum_{m\in\mathbb{Z}^2} |\eta_\varepsilon(m)| < +\infty\,.$$

Lemma 7.6.2. *Assume that $\kappa(x) = e^{-\frac{1}{2}|x|^2}$. Then, there exist constants $C_0 < +\infty$ and $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$:*

$$\sum_{m\in\mathbb{Z}^2} |\eta_\varepsilon(m)| \le C_0 \frac{1}{\varepsilon} \log\left(\frac{1}{\varepsilon}\right)\,.$$

Proof of Lemma 7.6.1. First note that (7.6.1) implies that $\hat{\kappa}$ is C^3 and that for every $\beta \in \mathbb{N}^2$ such that $\beta_1 + \beta_2 \le 3$, we have:

$$|\partial^\beta \hat{\kappa}(\xi)| \le C' |\xi|^{-3}\,, \tag{7.6.2}$$

for some $C' = C'(\kappa) < +\infty$. This implies that the series under the square root of the definition of λ_ε converges in C^3-norm towards a C^3 function. Moreover, this series is clearly $2\pi\mathbb{Z}^2$-periodic, even, and positive (since $\hat{\kappa}$ takes only positive values). Thus, λ_ε is well defined and also satisfies these properties.

Let us now prove the second part of the lemma. For each $\varepsilon > 0$ let $\kappa_\varepsilon : \mathbb{Z}^2 \to \mathbb{R}$ be defined as $\kappa_\varepsilon(m) = \kappa(\varepsilon m)$. The discrete Fourier transform of κ_ε is:

$$\mathsf{F}(\kappa_\varepsilon) : \xi \in \mathbb{T}^2 \longmapsto \sum_{m\in\mathbb{Z}^2} \kappa_\varepsilon(m) e^{-i\langle \xi, m\rangle}$$

(since κ and its derivatives of order up to 3 decay polynomially fast with an exponent larger than 5, the above series converges in C^3-norm). Now, since κ and $\hat{\kappa}$ decay polynomially fast with an exponent larger than 2, we can apply the Poisson summation formula (see[17] Theorem 3.1.17 of [Gra09]) which implies that:

$$\forall \xi \in \mathbb{T}^2,\, \mathsf{F}(\kappa_\varepsilon)(\xi) = 4\pi^2\varepsilon^{-2} \sum_{m\in\mathbb{Z}^2} \hat{\kappa}(\varepsilon^{-1}(2\pi m - \xi)) = 4\pi^2\varepsilon^{-2}\lambda_\varepsilon(\xi)^2\,.$$

As a result:

$$\eta_\varepsilon(m) = \frac{1}{4\pi^2}\int_{\mathbb{T}^2} \sqrt{\mathsf{F}(\kappa_\varepsilon)}(\xi) e^{i\langle m,\xi\rangle} d\xi\,.$$

In other words, the $\eta_\varepsilon(m)$'s are the Fourier coefficients of the C^3, positive and $2\pi\mathbb{Z}^2$-periodic function $\sqrt{\mathsf{F}(\kappa_\varepsilon)}$, which implies in particular that $|\eta_\varepsilon(m)| \leq C''|m|^{-3}$ for some $C'' = C''(\kappa,\varepsilon) < +\infty$. As a result:

$$\sum_{m\in\mathbb{Z}^2} |\eta_\varepsilon(m)| < +\infty\,. \tag{7.6.3}$$

Thanks to (7.6.3), we can apply the Fourier inversion formula (see for instance Proposition 3.1.14 of [Gra09]) which implies that:

$$\mathsf{F}(\eta_\varepsilon) = \sqrt{\mathsf{F}(\kappa_\varepsilon)}\,. \tag{7.6.4}$$

Now, let us use the convolution formula (see for instance Paragraph 1.3.3 of [Rud]). Since $\sum_{m\in\mathbb{Z}^2} |\eta_\varepsilon(m)| < +\infty$, we have:

$$\sum_{m\in\mathbb{Z}^2}\sum_{m'\in\mathbb{Z}^2} |\eta_\varepsilon(m')\eta_\varepsilon(m-m')| < +\infty\,, \tag{7.6.5}$$

and:

$$\mathsf{F}(\eta_\varepsilon * \eta_\varepsilon) = \mathsf{F}(\eta_\varepsilon)^2\,, \tag{7.6.6}$$

where $\eta_\varepsilon * \eta_\varepsilon \,:\, m\in\mathbb{Z}^2 \mapsto \sum_{m'\in\mathbb{Z}^2} \eta_\varepsilon(m')\eta_\varepsilon(m-m')$.

We deduce from (7.6.4) and (7.6.6) that $\mathsf{F}(\eta_\varepsilon * \eta_\varepsilon) = \mathsf{F}(\kappa_\varepsilon)$. Since, by the dominated convergence theorem, the Fourier coefficients of $\mathsf{F}(\eta_\varepsilon * \eta_\varepsilon)$ are the $\eta_\varepsilon * \eta_\varepsilon(m)$'s and the Fourier coefficients of $\mathsf{F}(\kappa_\varepsilon)$ are the $\kappa_\varepsilon(m)$'s, we obtain that:

$$\eta_\varepsilon * \eta_\varepsilon = \kappa_\varepsilon\,.$$

This is equivalent to saying that $(\varepsilon m_1, \varepsilon m_2) \in \varepsilon\mathbb{Z}^2 \longmapsto \eta_\varepsilon(m_1 - m_2)$ is a symmetric square root of K^ε. □

The proof of Lemma 7.6.2 is split into two sub-lemmas:

[17] Be aware that the conventions used in [Gra09] are different from ours.

Lemma 7.6.3. *Assume that f satisfies Condition 7.2.1 and Condition 7.2.2 for $\alpha > 5$. In this lemma, we work with the function:*

$$\rho^\varepsilon \,:\, \xi \in \mathbb{R}^2 \mapsto \lambda(\varepsilon\xi) = \sqrt{\sum_{m\in\mathbb{Z}^2} \hat{\kappa}(\xi - 2\pi\varepsilon^{-1}m))}\,.$$

For each $\varepsilon > 0$, let:

$$\Upsilon(\varepsilon) = \max\Big(\int_{\varepsilon^{-1}\mathbb{T}^2} |\rho^\varepsilon(\xi)|d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2} |\Delta\rho^\varepsilon(\xi)|d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2} |\partial_1\Delta\rho^\varepsilon(\xi)| + |\partial_2\Delta\rho^\varepsilon(\xi)|d\xi\Big).$$

Then, for every $\varepsilon > 0$, $\Upsilon(\varepsilon) < +\infty$. Moreover, there exist an absolute constants $C_1 < +\infty$ and a constant $\varepsilon_1 = \varepsilon_1(\kappa) > 0$ such that for every $\varepsilon \in]0, \varepsilon_1]$ we have:

$$\sum_{m\in\mathbb{Z}^2} |\eta_\varepsilon(m)| \le C_1\Upsilon(\varepsilon)\frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right)\,.$$

Lemma 7.6.4. *Assume that $\kappa(x) = e^{-\frac{1}{2}|x|^2}$. Then, there exists $\varepsilon_2 > 0$ such that $\sup_{\varepsilon\in]0,\varepsilon_2]} \Upsilon(\varepsilon) < +\infty$.*

Lemma 7.6.3 follows easily from appropriate integration by parts. The proof of Lemma 7.6.4 is just straightforward elementary computation and uses very crude estimates throughout.

Proof of Lemma 7.6.3. First note that $\Upsilon(\varepsilon) < +\infty$ comes from the fact that (by Lemma 7.6.1) $\lambda_\varepsilon \in C^3(\mathbb{T}^2)$, hence $\rho^\varepsilon \in C^3(\varepsilon^{-1}\mathbb{T}^2)$. Next, note that by an obvious change of variable, we have:

$$\eta_\varepsilon(m) = \frac{\varepsilon}{2\pi}\int_{\varepsilon^{-1}\mathbb{T}^2} e^{i\langle\varepsilon m,\xi\rangle}\rho^\varepsilon(\xi)\,d\xi\,.$$

Hence, $\eta_\varepsilon(0) \le \frac{\varepsilon}{2\pi}\int_{\varepsilon^{-1}\mathbb{T}^2} |\mu_\varepsilon(\xi)|d\xi \le \frac{\varepsilon}{2\pi}\Upsilon(\varepsilon)$. Now, let $m \neq 0$. By integration by parts, we have:

$$\eta_\varepsilon(m) = \frac{1}{2\pi\varepsilon|m|^2}\int_{\varepsilon^{-1}\mathbb{T}^2} \Delta\rho^\varepsilon(\xi)e^{i\langle\varepsilon m,\xi\rangle}d\xi\,, \tag{7.6.7}$$

and:

$$\eta^\varepsilon(m) = \frac{1}{i2\pi\varepsilon^2|m|^2(m_1 \pm m_2)}\int_{\varepsilon^{-1}\mathbb{T}^2} (\partial_1 \pm \partial_2)\Delta\rho^\varepsilon(\xi)e^{i\langle\varepsilon m,\xi\rangle}d\xi\,. \tag{7.6.8}$$

In (7.6.8), since $m \neq 0$, at least one of the two expressions is well defined. Thus, for each $m \in \mathbb{Z}^2$:

$$|\eta_\varepsilon(m)| \le \frac{1}{\varepsilon}\frac{1}{2\pi}\Upsilon(\varepsilon)\min\Big(\varepsilon^2, \frac{1}{|m|^2}, \frac{1}{\varepsilon|m|^2(|m_1| + |m_2|)}\Big)\,.$$

This implies that:

$$\begin{aligned}\sum_{m\in\mathbb{Z}^2}|\eta_\varepsilon(m)| &= |\eta_\varepsilon(0)| + \sum_{m\in\mathbb{Z}^2,\,0<|m|\leq\varepsilon^{-1}}|\eta_\varepsilon(m)| + \sum_{m\in\mathbb{Z}^2,\,|m|>\varepsilon^{-1}}|\eta_\varepsilon(m)| \\ &\leq \frac{1}{\varepsilon}\frac{1}{2\pi}\Upsilon(\varepsilon)\left(\varepsilon + \frac{1}{\varepsilon}\sum_{0<|m|\leq\varepsilon^{-1}}\frac{1}{|m|^2} + \frac{1}{\varepsilon^2}\sum_{|m|>\varepsilon^{-1}}\frac{1}{|m|^2(|m_1|+|m_2|)}\right) \\ &\leq C''\Upsilon(\varepsilon)\left(\varepsilon + \frac{1}{\varepsilon}\log\left(\frac{1}{\varepsilon}\right) + \frac{1}{\varepsilon}\right),\end{aligned}$$

for some $C'' < +\infty$, all this being valid for small enough values of $\varepsilon > 0$. □

Proof of Lemma 7.6.4. Since $\kappa(x) = e^{-\frac{1}{2}|x|^2}$ the Fourier transform of κ is

$$\hat{\kappa} : \xi \in \mathbb{R}^2 \longmapsto \frac{1}{4\pi^2}\int_{\mathbb{R}^2} e^{-i\langle\xi,x\rangle}\kappa(x)dx = \frac{1}{2\pi}e^{-\frac{1}{2}|\xi|^2},$$

so that

$$\rho^\varepsilon(\xi) = \sqrt{\sum_{m\in\mathbb{Z}^2} e^{-\frac{1}{2}|\xi-2\pi\varepsilon^{-1}m|^2}}.$$

Let $P_1 = Id$, $P_2 = \Delta$ and $P_3 = 2\partial_1\Delta$. Then:

$$\Upsilon(\varepsilon) = \max\Big(\int_{\varepsilon^{-1}\mathbb{T}^2}|P_1\rho^\varepsilon(\xi)|d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2}|P_2\rho^\varepsilon(\xi)|d\xi, \int_{\varepsilon^{-1}\mathbb{T}^2}|P_3\rho^\varepsilon(\xi)|d\xi\Big).$$

For the last argument of the max we use the fact that ρ^ε remains unchanged when switching the two coordinates of $\mathbb{R}^2$. We begin by justifying the following claim:

Claim 7.6.5. *There exists $C' < +\infty$ such that for each $\xi \in \varepsilon^{-1}\mathbb{T}^2$ and each $j \in \{1,2,3\}$,*

$$|P_j\rho^\varepsilon(\xi)| \leq C'\left(\sum_{m\in\mathbb{Z}^2} e^{-\frac{1}{13}|\xi-2\pi\varepsilon^{-1}m|^2}\right)^3.$$

Proof. Elementary algebra shows that for each $j \in \{1,2,3\}$, there exists a polynomial function $Q_j : \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ of degree at most j such that for each $\xi \in \varepsilon^{-1}\mathbb{T}^2$, $\rho^\varepsilon(\xi)^5 P_j\rho^\varepsilon(\xi)$ equals:

$$\sum_{m_1,m_2,m_3\in\mathbb{Z}^2} Q_j(\xi-2\pi\varepsilon^{-1}m_1, \xi-2\pi\varepsilon^{-1}m_2, \xi-2\pi\varepsilon^{-1}m_3)\prod_{i=1}^3 e^{-\frac{1}{2}|\xi-2\pi\varepsilon^{-1}m_i|^2}. \quad (7.6.9)$$

By using the very crude bound $\forall m \in \mathbb{Z}^2$, $\rho^\varepsilon(\xi) \geq e^{-\frac{1}{4}|\xi-2\pi\varepsilon^{-1}m|^2}$, we obtain that for each $\xi \in \varepsilon^{-1}\mathbb{T}^2$, $|P_j\rho^\varepsilon(\xi)|$ is at most:

$$\begin{aligned}&\sum_{m_1,m_2,m_3\in\mathbb{Z}} |Q_j(\xi-2\pi\varepsilon^{-1}m_1, \xi-2\pi\varepsilon^{-1}m_2, \xi-2\pi\varepsilon^{-1}m_3)|\prod_{i=1}^3 e^{-\left(\frac{1}{2}-\frac{5}{3}\times\frac{1}{4}\right)|\xi-2\pi\varepsilon^{-1}m_i|^2} \\ &= \sum_{m_1,m_2,m_3\in\mathbb{Z}} |Q_j(\xi-2\pi\varepsilon^{-1}m_1, \xi-2\pi\varepsilon^{-1}m_2, \xi-2\pi\varepsilon^{-1}m_3)|\prod_{i=1}^3 e^{-\frac{1}{12}|\xi-2\pi\varepsilon^{-1}m_i|^2}.\end{aligned}$$

Now, note that there exists a constant $C' < +\infty$ such that, for each $j \in \{1,2,3\}$, for each $m_1, m_2, m_3 \in \mathbb{Z}^2$, and for each $\xi \in \mathbb{R}^2$:

$$\left| Q_j(\xi - 2\pi\varepsilon^{-1}m_1, \xi - 2\pi\varepsilon^{-1}m_2, \xi - 2\pi\varepsilon^{-1}m_3) \prod_{i=1}^{3} e^{-\frac{1}{12}|\xi - 2\pi\varepsilon^{-1}m_i|^2} \right| \leq C' \prod_{i=1}^{3} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m_i|^2},$$

and we are done. □

Let us use the claim to conclude. Let $m \in \mathbb{Z}^2$ be such that $|m| \geq 2$ and $\xi \in [-\varepsilon^{-1}\pi, \varepsilon^{-1}\pi]^2$. Then, $|\xi - 2\pi\varepsilon^{-1}m| \geq \pi\varepsilon^{-1}|m|$. Therefore:

$$\begin{aligned} \sum_{m \in \mathbb{Z}^2,\ |m| \geq 2} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} &\leq \sum_{m \in \mathbb{Z}^2,\ |m| \geq 2} e^{-\frac{\pi}{13}\varepsilon^{-2}|m|^2} \\ &\leq C'' e^{-\frac{4\pi}{13}\varepsilon^{-2}}, \end{aligned}$$

for some $C'' < +\infty$ and if ε is sufficiently small. Moreover, $\sum_{m \in \mathbb{Z}^2,\ |m| \leq 1} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} \leq 5$, hence by expanding the cubed sum in Claim 7.6.5, we obtain that if ε is sufficiently small, then:

$$|P_j \rho^\varepsilon(\xi)| \leq C''' \left(e^{-\frac{4\pi}{13}\varepsilon^{-2}} + \sum_{m \in \mathbb{Z}^2,\ |m| \leq 1} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} \right),$$

for some $C''' < +\infty$. Finally:

$$\begin{aligned} \Upsilon(\varepsilon) &\leq C''' \int_{\varepsilon^{-1}\mathbb{T}^2} \left(e^{-\frac{4\pi}{13}\varepsilon^{-2}} + \sum_{m \in \mathbb{Z}^2,\ |m| \leq 1} e^{-\frac{1}{13}|\xi - 2\pi\varepsilon^{-1}m|^2} \right) d\xi \\ &\leq C'''(2\pi\varepsilon)^2 e^{-\frac{4\pi}{13}\varepsilon^{-2}} + 5C''' \int_{\mathbb{R}^2} e^{-\frac{1}{13}|\xi|^2} d\xi, \end{aligned}$$

which is less than some absolute constant since we consider only ε small. □

7.7 Sprinkling discretization scheme

In this section, we prove Proposition 7.2.22. We do not rely on arguments from other sections. Recall that $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ is the square face-centered lattice and that $\mathcal{T}^\varepsilon = (\mathcal{V}^\varepsilon, \mathcal{E}^\varepsilon)$ denotes $\mathcal{T}$ scaled by ε. Given an edge $e = (x, y)$, we take the liberty of writing "$z \in e$" as a shorthand for "$\exists t \in [0,1]$ such that $z = ty + (1-t)x$". For each $R > 0$ and $\varepsilon > 0$, let $\mathcal{T}_R^\varepsilon = (\mathcal{E}_R^\varepsilon, \mathcal{V}_R^\varepsilon)$ denote the sublattice of $\mathcal{T}^\varepsilon$ generated by the edges e that intersect $[0, 2R] \times [0, R]$.

In this section, we never use the fact that our Gaussian field is non-degenerate. As we shall see at the end of this section, Proposition 7.2.22 is an easy consequence of the following approximation estimate.

Proposition 7.7.1. *Assume that f satisfies Condition 7.2.2 and that it is a.s. C^2. Let $p > 0$. Given $\varepsilon > 0$ and $e = (x, y) \in \mathcal{E}^\varepsilon$, we call Fold$(e)$ the event that there exists $z \in e$ such that $f(x) \geq -\frac{p}{2}$, $f(y) \geq -\frac{p}{2}$, and $f(z) < -p$. There exist constants $c_2 = c_2(\kappa, p) > 0$ and $\varepsilon_0 = \varepsilon_0(\kappa, p) > 0$ such that for each $\varepsilon \in]0, \varepsilon_0]$ we have:*

$$\forall e \in \mathcal{E}^\varepsilon, \, \mathbb{P}\left[\text{Fold}(e)\right] \leq C_1 \exp\left(-c_2 \varepsilon^{-4}\right) .$$

A key ingredient in proving this inequality will be the Borell-Tsirelson-Ibragimov-Sudakov (or BTIS) inequality (see Theorem 2.9 of [AW09]).

Proof of Proposition 7.7.1. Let us fix $e = (x, y) \in \mathcal{E}^\varepsilon$ and consider the vector v defined by $\varepsilon v = y - x$. On the event Fold(e, ε), by Taylor's inequality applied to f between points x, z and y, there exist $w_1, w_2 \in e$ such that $\partial_v f(w_1) > \frac{p|v|}{2\varepsilon}$ and $\partial_v f(w_2) < -\frac{p|v|}{2\varepsilon}$. Applying Taylor's estimate to $\partial_v f$ between w_1 and w_2 we conclude that there exists $w_3 \in e$ such that $|\partial^2_{v,v} f(w_3)| > \frac{p|v|}{\varepsilon^2}$. Hence:

$$\mathbb{P}\left[\text{Fold}(e)\right] \leq \mathbb{P}\left[\sup_{w \in e} |\partial^2_{v,v} f(w)| > \frac{p|v|}{\varepsilon^2}\right] .$$

Let $x_t = \varepsilon t v$ and $g^v_\varepsilon(t) = \partial^2_{v,v} f(x_t)$. By translation invariance of f, we have:

$$\mathbb{P}\left[\sup_{w \in e} |f_{vv}(w)| > \frac{p|v|}{\varepsilon^2}\right] = \mathbb{P}\left[\sup_{t \in [0,1]} |g^v_\varepsilon(t)| > \frac{p|v|}{\varepsilon^2}\right] .$$

The strategy is to apply the BTIS inequality to g^v_ε. Note that g^v_ε is a centered Gaussian field on $[0, 1]$ which is a.s. bounded. Hence, by Theorem 2.9 of [AW09], $\mathbb{E}\left[|\sup_{t \in [0,1]} g^v_\varepsilon(t)|\right] < +\infty$. Note that $\mathbb{E}\left[|\sup_{t \in [0,1]} g^v_\varepsilon(t)|\right]$ is non-decreasing in ε, let $C_4 = \max_{v' \in \Gamma} \mathbb{E}\left[|\sup_{t \in [0,1]} g^{v'}_1(t)|\right]$, and choose $\varepsilon_0 \in]0, 1]$ sufficiently small so that $\min_{v' \in \Gamma} \frac{p|v'|}{\varepsilon_0^2} > 2C_4$. Note that, by translation invariance:

$$\sigma^2 := \sup_{t \in [0,1]} \text{Var}(g^v_\varepsilon(t)) = \text{Var}(g^v_\varepsilon(0)) = \partial^4_{v,v,v,v} \kappa(0) .$$

If $\partial^4_{v,v,v,v} \kappa(0) = 0$, then $g^v_\varepsilon \equiv 0$ a.s. and we are done. Assume now that $\partial^4_{v,v,v,v} \kappa(0) > 0$. Then, by the BTIS inequality (see Theorem 2.9 in [AW09]), for each $\varepsilon \in]0, \varepsilon_0]$:

$$\begin{aligned}
\mathbb{P}\left[\sup_{t \in [0,1]} |g^v_\varepsilon(t)| > \frac{p|v|}{\varepsilon^2}\right] &\leq 2\,\mathbb{P}\left[\sup_{t \in [0,1]} g^v_\varepsilon(t) > \frac{p|v|}{\varepsilon^2}\right] \\
&\leq 2\,\mathbb{P}\left[\left|\sup_{t \in [0,1]} g^v_\varepsilon(t) - \mathbb{E}[\sup_{t \in [0,1]} g^v_\varepsilon(t)]\right| > \frac{p|v|}{2\varepsilon^2}\right] \\
&\leq 4 \exp\left(-\frac{1}{2\sigma^2}\left(\frac{p|v|}{2\varepsilon^2}\right)^2\right) .
\end{aligned}$$

Since v can take only finitely many values, we have obtain what we want. □

Proof of Proposition 7.2.22. By Proposition 7.7.1 (and by a simple union bound), it is enough to prove that $\mathrm{Cross}^{\varepsilon}_{p/2}(2R,R) \cap (\cup_e \mathrm{Fold}(e))^c \subseteq \mathrm{Cross}_p(2R,R)$, where the union is over each $e \in \mathcal{E}^{\varepsilon}_R$. Assume that for every $e \in \mathcal{E}$ $\mathrm{Fold}(e)$ is not satisfied. Then, for each edge $e = (x,y) \in \mathcal{E}^{\varepsilon}_R$ which is colored black in the discrete model of parameter $p/2$, and for each $z \in e$, we have $f(z) \geq -p$. In other words, each black edge is contained in $\mathcal{D}_p$. If in addition $\mathrm{Cross}^{\varepsilon}_{p/2}(2R,R)$ is satisfied, then there exists a crossing of $[0,2R]\times[0,R]$ from left to right made up of black edges. This crossing belongs to $\mathcal{D}_p$ so that $\mathrm{Cross}_p(2R,R)$ is satisfied. □

Appendix

7.A An additional sub-linearity result for influences

In this section, we prove that geometric influences are sub-linear for semi-algebraic sets. The following results are not useful for the proof of the main results of the article. However, they are useful for the proof of Theorem 7.2.19 in the case of semi-algebraic sets. This case is included in the theorem in order to provide a more complete picture of the range of application of Theorem 7.2.19.

Let us first recall their definition. Given $A \subseteq \mathbb{R}^n$, we denote by ∂A its topological boundary. Moreover, for each $k \in \mathbb{N}$, we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. We say that A is semi-algebraic if it belongs to the Boolean algebra generated by sets B of the form $P^{-1}(]0, +\infty[)$ where $P \in \mathbb{R}[X_1, \ldots, X_n]$.

We say that A has piecewise C^1-boundary if there exists $E \subseteq \partial A$ satisfying $\mathcal{H}^{n-1}(E) = 0$ such that for each $x \in \partial A \setminus E$ there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of x and a C^1-diffeomorphism from U to an open subset $V \subseteq \mathbb{R}^n$ sending $A \cap U$ to the intersection of V and the closed upper-half plane.

The goal of this subsection is to prove the following proposition.

Proposition 7.A.1. *Let Σ be a symmetric positive definite $n \times n$ matrix. Let $\mu = \mathcal{N}(0, \Sigma)$ and let λ be density of μ. Let $A \subseteq \mathbb{R}^n$ be a semi-algebraic set. Then, for each $v \in \mathbb{R}^n$:*

$$\exists \lim_{r \downarrow 0} \frac{1}{r} \int_{A + [-r,r]v \setminus A} \lambda(x) dx = I_{v,\mu}(A)\,,$$

and we have the following properties.[18]

$$\begin{aligned} &\forall r \in \mathbb{R},\ v \in \mathbb{R}^n,\ I_{rv,\mu}(A) = |r| I_{v,\mu}(A)\,, \\ &\forall u, v \in \mathbb{R}^n,\ I_{u+v,\mu}(A) \leq I_{u,\mu}(A) + I_{v,\mu}(A)\,. \end{aligned}$$

The semi-algebraic case of Proposition 7.5.3 follows directly from this result. We prove Proposition 7.A.1 at the end of this subsection.

[18] These properties amount to saying that $v \mapsto I_{v,\mu}(A)$ defines a seminorm on $\mathbb{R}^n$.

Remark 7.A.2. As is apparent in its proof, the assumptions made in Proposition 7.A.1 can be substantially weakened. For the sake of simplicity, we did not attempt to prove it in full generality.

Definition 7.A.3. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. For any $\delta > 0$, we denote by $\overline{\varphi}_\delta$ the function defined by $\forall x \in \mathbb{R}^n,\ \overline{\varphi}_\delta(x) = \sup_{|y-x|<\delta} |\varphi(y)|$. A set $A \subseteq \mathbb{R}^n$ will be called **φ-tame** if A has piecewise C^1-boundary and if there exists $\delta > 0$ such that:

$$\int_{\partial A} \overline{\varphi}_\delta(x) d\mathcal{H}^{n-1}(x) < +\infty\,.$$

Here, the integral coincides with the usual surface-area integral $\int \overline{\varphi}_\delta d\sigma_S$ on S the C^1 part of ∂A.

A key step in the proof of Proposition 7.A.1 is the following result.

Proposition 7.A.4. *Fix $\varphi \in C(\mathbb{R}^n, \mathbb{R})$. Let $A \subseteq \mathbb{R}^n$ be a φ-tame set. Let ν be the outward normal vector of A defined $\mathcal{H}^{n-1}$-almost everywhere on ∂A. Then, for each $v \in \mathbb{R}^n$:*

$$J_{v,\varphi}(A) := \lim_{r \downarrow 0} r^{-1} \int_{(A+[-r,r]v)\setminus A} \varphi(x) dx = \int_{\partial A} |v \cdot \nu(x)| \varphi(x) d\mathcal{H}^{n-1}(x)\,.$$

The following result is an immediate consequence of the proposition.

Corollary 7.A.5. *Fix $\varphi \in C(\mathbb{R}^n, \mathbb{R}_+)$. Let $A \subseteq \mathbb{R}^n$ be a φ-tame set. Then, we have the following relations.*

$$\forall \lambda \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\ J_{\lambda v,\varphi}(A) = |\lambda| J_{v,\varphi}(A)\,,$$
$$\forall u, v \in \mathbb{R}^n,\ J_{u+v,\varphi}(A) \leq J_{u,\varphi}(A) + J_{v,\varphi}(A)\,.$$

In order to prove Proposition 7.A.4, we introduce the following notation. Let $\pi_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ (respectively $\pi_2 : \mathbb{R}^n \to \mathbb{R}$) be the projection on the first $n-1$ coordinates (respectively last coordinate). Given $x \in \mathbb{R}^n$, we write $\tilde{x} := \pi_1(x)$. Given $A \subseteq \mathbb{R}^n$ and $\tilde{x} \in \mathbb{R}^{n-1}$ and $r > 0$, we write $A_r(\tilde{x}) = \pi_2(\pi_1^{-1}(\tilde{x}) \cap ((A + [-r, r]e_n) \setminus A)) = \{x_n \in \mathbb{R} : (\tilde{x}, x_n) \in (A + [-r, r]e_n) \setminus A\}$.

The proof of Proposition 7.A.4 will rely on two lemmas which we state along the way and prove at the end of the subsection.

Proof of Proposition 7.A.4. First note that it is enough to prove the result for φ non-negative and for $v \in \S^{n-1}$. Since the statement is invariant by rotations, it is enough to prove the result for $v = e_n := (0, \ldots, 0, 1)$.
Next, we introduce some notation. From this point on, we assume that A has piecewise C^1-boundary. Let $E \subseteq \partial A$ such that $\mathcal{H}^{n-1}(E) = 0$ and $S = \partial A \setminus E$ is a C^1-hypersurface of $\mathbb{R}^n$. Since π_1 is a Lipschitz map, $\mathcal{H}^{n-1}(\pi_1(E)) = 0$. Let $\pi = \pi_1|_S$. This function is of class C^1 so by Sard's lemma the set $\mathrm{Crit}_{val}(\pi)$ of its critical values has measure 0 in

$\mathbb{R}^{n-1}$. For each $x \in S$, write $B_r(x) = \pi_2\left((\{x\} + [-r,r]e_n) \setminus A\right)$. Finally, for any $r > 0$ let:

$$\begin{aligned} F_r : \mathbb{R}^{n-1} &\to [0, +\infty] \\ \tilde{x} &\mapsto r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \,. \end{aligned}$$

For any $r > 0$, by Fubini's theorem,

$$r^{-1} \int_{(A+[-r,r]e_n)\setminus A} \varphi(x) dx = \int_{\mathbb{R}^{n-1}} \Big[r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \Big] d\tilde{x} = \int_{\mathbb{R}^{n-1}} F_r(\tilde{x}) d\tilde{x} \,. \quad (7.A.1)$$

In order to prove Proposition 7.A.4 we will apply the dominated convergence theorem to the sequence $(F_r)_{r>0}$ as $r \downarrow 0$. In order to do so, we begin by finding a uniform bound on the sequence $(F_r)_{r>0}$ by an integrable function. Let us make two observations:

- Each $\tilde{x} \in \mathbb{R}^{n-1} \setminus (\pi_1(E) \cup \mathrm{Crit}_{val}(\pi))$, $\tilde{x}$ is a regular value of π so: for each $x \in \pi^{-1}(\tilde{x})$, $e_n \notin T_x S$ and there exists $r' = r'(x) > 0$ such that for each $r \in]0, r']$:

$$\begin{gathered} B_r(x) =]x_n, x_n + r] \text{ or } [x_n - r, x_n[\qquad (7.A.2) \\ \text{and} \\ \forall y \in \pi^{-1}(\tilde{x}) \setminus \{x\},\ B_r(x) \cap B_r(y) = \emptyset \,. \end{gathered}$$

  (In particular, $\pi^{-1}(\tilde{x})$ is countable and has no accumulation point.)

- For each $\tilde{x} \in \mathbb{R}^{n-1} \setminus (\pi_1(E) \cup \mathrm{Crit}_{val}(\pi))$ and each $r > 0$, since $\varphi \geq 0$ and $A_r(\tilde{x}) = \cup_{x \in \pi^{-1}(\tilde{x})} B_r(x) \subseteq \pi_2\left(\pi^{-1}(\tilde{x}) + [-r, r]e_n\right)$, we have:

$$\begin{aligned} F_r(\tilde{x}) = r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt &\leq \sum_{x \in \pi^{-1}(\tilde{x})} r^{-1} \int_{B_r(x)} \varphi(\tilde{x}, t) dt \qquad (7.A.3) \\ &\leq 2 \sum_{x \in \pi^{-1}(\tilde{x})} \overline{\varphi}_r(x) =: 2G_r(\tilde{x}) \,. \qquad (7.A.4) \end{aligned}$$

By construction, the sequence $(G_r)_{r>0}$ is pointwise non-increasing in r. We wish to prove that G_r is integrable for r small enough. To do so, we use the following lemma, which is a direct application of the coarea formula.

Lemma 7.A.6. *For any non-negative measurable function $\psi : S \to \mathbb{R}$, the function $\tilde{x} \mapsto \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x)$ is measurable on $\mathbb{R}^{n-1} \setminus \mathrm{Crit}_{val}(\pi)$ with values in $[0, +\infty]$ and satisfies:*

$$\int_{\mathbb{R}^{n-1}} \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x) d\tilde{x} = \int_S |\nu(x) \cdot e_n| \, \psi(x) \, d\sigma_S(x) \,,$$

where $d\sigma_S$ is the standard surface-area measure on S.

By Lemma 7.A.6, for each $r > 0$ and almost each $\tilde{x} \in \mathbb{R}^{n-1}$:

$$\int_{\mathbb{R}^{n-1}} G_r(\tilde{x}) d\tilde{x} = \int_S |\nu(x) \cdot e_n| \, \overline{\varphi}_r(x) \, d\mathcal{H}^{n-1}(x)$$

(remember Definition (7.A.3): the integral above coincide with the integral under $d\sigma_S$). Moreover, since A is φ-tame, there exists $r_0 > 0$ such that:

$$\int_S |\nu(x) \cdot e_n| \, \overline{\varphi}_{r_0}(x) d\mathcal{H}^{n-1}(x) \leq \int_S \overline{\varphi}_{r_0}(x) \, |d\mathcal{H}^{n-1}(x)| < +\infty \, .$$

In summary, for each $r \in]0, r_0]$ and almost every $\tilde{x} \in \mathbb{R}^{n-1}$, by (7.A.4) we have:

$$F_r(\tilde{x}) \leq 2G_r(\tilde{x}) \leq 2G_{r_0}(\tilde{x}) \, . \tag{7.A.5}$$

Moreover, G_{r_0} is integrable. This will allow us to apply the dominated convergence theorem once we have established pointwise convergence, which is given by the following lemma:

Lemma 7.A.7. *For a.e. $\tilde{x} \in \mathbb{R}^{n-1}$,*

$$F_r(\tilde{x}) \underset{r \downarrow 0}{\longrightarrow} \sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x) \, .$$

By Lemma 7.A.6, the limit $\sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x)$ is a measurable function of $\tilde{x}$. By Equation (7.A.5), the family $(F_r)_{r>0}$ is pointwise bounded for $r \in]0, r_0]$ by an integrable function, namely G_{r_0}. By dominated convergence,

$$\int_{\mathbb{R}^{n-1}} F_r(\tilde{x}) d\tilde{x} \underset{r \downarrow 0}{\longrightarrow} \int_{\mathbb{R}^{n-1}} \sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x) d\tilde{x} \, .$$

We replace the left-hand side using Equation (7.A.1) and the right-hand side using Lemma 7.A.6 to get:

$$\int_{A + [-r,r]e_n \setminus A} \varphi(x) dx \underset{r \downarrow 0}{\longrightarrow} \int_S |\nu(x) \cdot e_n| \, \varphi(x) \, |dV_S|(x) = \int_{\partial A} |\nu(x) \cdot e_n| \, \varphi(x) \, d\mathcal{H}^{n-1}(x) \, .$$

□

Proof of Lemma 7.A.6. Let $\psi : S \to \mathbb{R}$ be a measurable non-negative function. As a C^1-hypersurface of $\mathbb{R}^n$, S is naturally endowed with a Riemannian metric by restriction of the Euclidean scalar product. By the coarea formula (see Corollary 13.4.6 of [BZ88]) applied to π, the function

$$\begin{aligned} \mathbb{R}^{n-1} &\to \mathbb{R} \\ \tilde{x} &\mapsto \sum_{x \in \pi^{-1}(\tilde{x})} \psi(x) \end{aligned}$$

is measurable, and:

$$\int_{\mathbb{R}^{n-1}} \sum_{x\in\pi^{-1}(\tilde{x})} \psi(x)\, d\tilde{x} = \int_S |J_\pi(x)|\, \psi(x)\, d\mathcal{H}^{n-1}(x)\,,$$

where $J_\pi(x)$ is the determinant of the matrix of $d_x\pi$ between orthonormal bases of T_xS and $\mathbb{R}^{n-1}$. The map $d_x\pi$ is the projection on the first $n-1$ coordinates. Based on this information, we claim that $J_\pi(x) = \pm\nu_n(x) = \pm\nu(x)\cdot e_n$. If $T_xS = \mathbb{R}^{n-1}\times\{0\}$ then $d_x\pi$ acts as the identity so $J_\pi(x) = 1$. On the other hand, if $T_xS \neq \mathbb{R}^{n-1}\times\{0\}$, then the intersection of these two hyperplanes is a linear subspace $H_x \subseteq \mathbb{R}^n$ of dimension $n-2$. Let $u(x)$ (respectively $v(x)$) be a unit vector spanning the orthogonal to H_x in T_xS (respectively $\mathbb{R}^{n-1}\times\{0\}$). Since $d_x\pi$ is an orthogonal projection, it preserves H_x as well as its orthogonal, so $J_\pi(x) = \pm d_x\pi(u(x))\cdot v(x) = \pm u(x)\cdot v(x)$. Note that the vectors $u(x), \nu(x), e_n, v(x)$ all belong to the plane orthogonal to H_x in $\mathbb{R}^n$. Since $v(x)$ is orthogonal to e_n and $u(x)$ is orthogonal to $\nu(x)$, $u(x)\cdot v(x) = \pm\nu(x)\cdot e_n$. Thus, $J_\pi(x) = \pm\nu(x)\cdot e_n$ as announced. Hence,

$$\int_{\mathbb{R}^{n-1}} \sum_{x\in\pi^{-1}(\tilde{x})} \psi(x) d\tilde{x} = \int_S |e_n\cdot\nu(x)|\psi(x) d\mathcal{H}^{n-1}(x).$$

□

Proof of Lemma 7.A.7. First note that, by Equation (7.A.2) (and since φ is continuous), for each $\tilde{x}\in\mathbb{R}^{n-1}\setminus(\pi_1(E)\cup \mathrm{Crit}_{val}(\pi))$:

$$r^{-1}\int_{B_r(x)} \varphi(\tilde{x},t)dt \underset{r\downarrow 0}{\longrightarrow} \varphi(x)\,. \tag{7.A.6}$$

Let us use (7.A.6) and (7.A.3) to prove that if for a.e. $\tilde{x}\in\mathbb{R}^{n-1}\setminus(\pi_1(E)\cup\mathrm{Crit}_{val}(\pi))$, we have:

$$\limsup_{r\downarrow 0} r^{-1}\int_{A_r(\tilde{x})} \varphi(\tilde{x},t)dt \leq \sum_{x\in\pi^{-1}(\tilde{x})} \varphi(x)\,. \tag{7.A.7}$$

To prove the above, on can use the dominated convergence theorem, which can be applied by noting that for every $r\in]0,r_0]$, $r^{-1}\int_{B_r(x)}\varphi(\tilde{x},t)dt \leq \overline{\varphi}_{r_0}(x)$ and that $\sum_{x\in\pi^{-1}(\tilde{x})}\overline{\varphi}_{r_0}(x) = G_{r_0}(\tilde{x})$ is a.e. finite since it is integrable.

Let us now study the liminf. Equation (7.A.2) implies that, for each $\tilde{x}\in\mathbb{R}^{n-1}\setminus\mathrm{Crit}_{val}(\pi)$ and each $y_1,\ldots,y_k\in\pi^{-1}(x)$, there exists $r_1>0$ such that for each $r\in]0,r_1]$, the intervals $B_r(y_j)$ for $j=1,\ldots,k$ have length r and are pairwise disjoint. Hence, for each $r\in]0,r_1]$,

$$r^{-1}\int_{A_r(\tilde{x})}\varphi(\tilde{x},t)dt \geq \sum_{j=1}^{k} r^{-1}\int_{B_r(y_j)}\varphi(\tilde{x},t)dt\,.$$

Taking the liminf followed by the supremum over all such $y_1, \dots, y_k$ we have, by Equation (7.A.6):

$$\liminf_{r \downarrow 0} r^{-1} \int_{A_r(\tilde{x})} \varphi(\tilde{x}, t) dt \geq \sum_{x \in \pi^{-1}(\tilde{x})} \varphi(x) \,.$$

Together with Equation (7.A.7) this concludes the proof of the lemma. □

We are now ready to prove Proposition 7.A.1

Proof of Proposition 7.A.1. Let $P \in \mathbb{R}[X_1, \dots, X_n]$ and let $A = P^{-1}(]0, +\infty[)$. Then A has piecewise C^1-boundary $\tilde{Y} := \partial A = P^{-1}(0)$, which is the image of a projective algebraic hypersurface $Y \subseteq \mathbb{R}P^n$ by a standard chart.[19] Let $g \in C^\infty(\mathbb{R}^n)$ be the density of the Fubini-Study volume measure on $\mathbb{R}P^n$ when read in a canonical chart. Since $\tilde{Y}$ is piecewise C^1, we have:

$$\int_{\tilde{Y}} g(x) d\mathcal{H}^{n-1}(x) = \int_{\tilde{Y}} g(y) d\sigma(y) = \mathrm{Vol}_{\mathbb{R}}(Y) < +\infty \,,$$

where $Vol_{\mathbb{R}}(Y)$ is the volume of Y with respect to the Fubini-Study measure (which is finite: see for instance [Mon12], Corollary A.2) and $d\sigma$ is the usual area measure (see Theorem 3.2.3 of [Fed69]). For each $x \in \mathbb{R}^n$, $g(x) = \frac{4}{(1+|x|^2)^2}$. In particular, for each $\delta > 0$ there exists $C_\delta < +\infty$ such that $\overline{g}_\delta \leq C_\delta g$. This shows that A is g-tame. Moreover, note that there exists a constant $C < +\infty$ such that $0 \leq \lambda \leq Cg$, where λ is the density of our Gaussian measure μ. Thus, A is λ-tame. Since this is valid for any set of the form $P^{-1}(]0, +\infty[)$ and since the set of tame sets is clearly stable by complement, finite union and finite intersection, any semi-algebraic set is λ-tame. We conclude by applying Corollary **??** to $\varphi = \lambda$. □

[19] Indeed, $\tilde{Y}$ is a finite union of irreducible components each of whose singular set is a proper algebraic variety, and thus has (algebraic) dimension at most $n-2$. Moreover the intersection of two such components has also at most dimension $n-2$. By induction, each of these algebraic submanifolds is a finite union of smooth manifolds of dimension $\leq n-2$ and so has Hausdorff dimension $\leq n-2$.

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