Fractional equation of thin films for hydraulic fractures

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Existence et régularité des solutions de deux équations paraboliques, dégénérées et non-locales

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Existence et régularité des solutions de deux équations paraboliques, dégénérées et non-locales

Ces travaux concernent deux équations paraboliques, dégénérées et non-locales. La première équation est une équation de films minces fractionnaire et la deuxième est une équation des milieux poreux fractionnaire. La présentation des problèmes, les résultats existants dans la littérature, ainsi que le résumé de nos résultats font l'objet de l'introduction.

Le deuxième chapitre est consacré à la présentation de la méthode de De Giorgi utilisée pour montrer la régularité Hölder des solutions des équations elliptiques. On présente de plus les résultats utilisant cette approche dans les cas paraboliques local et non-local.

Dans le troisième chapitre, on montre l'existence de solutions faibles d'une équation des films minces fractionnaire. C'est une équation parabolique, dégénérée, non-locale d'ordre $\alpha+2$ où $0<\alpha<2$. C'est une généralisation d'une équation étudiée par Imbert et Mellet en 2011 pour $\alpha=1$. Pour construire les solutions, on passe par un problème régularisé. En utilisant les injections de Sobolev, on passe à la limite pour trouver des solutions faibles. Vu la différence des injections de Sobolev, on distingue deux cas $0<\alpha<1$ et $1\leq\alpha<2$. Dans les deux cas on démontre que la solution est positive si la condition initiale l'est.

Le quatrième chapitre concerne une équation des milieux poreux fractionnaire. On montre la régularité Hölder de solutions faibles positives satisfaisant des estimées d'énergie. D'abord, on montre l'existence de solutions faibles qui satisfont des estimées d'énergie. On distingue deux cas $0<\alpha<1$ et $1\leq\alpha<2$ à cause de problème de divergence. Puis on démontre les lemmes de De Giorgi qui sont des lemmes de réduction de l'oscillation d'en dessus et d'au dessous. Ces deux lemmes ne suffisent pas pour montrer la régularité Hölder. On a besoin d'améliorer le résultat du lemme de réduction de l'oscillation d'en dessus. Donc, on passe par un lemme des valeurs intermédiaires et on montrer un lemme de réduction de l'oscillation d'en dessus amélioré. Enfin, on montre la régularité Hölder des solutions en utilisant la propriété scaling de ces solutions.

Existence and regularity of solutions of two degenerate non-local parabolic equations

In this thesis, we study two degenerate, non-local parabolic equations, a fractional thin film equation and a fractional porous medium equation. The introduction contains a presentation of problems, the previous results in the literature and a brief presentation of our results.

In the second chapter, we present a short overview of the De Giorgi method used to prove Hölder regularity of solutions of elliptic equations. Moreover, we present the results using this approachin the local and non-local parabolic cases.

In the third chapter we prove existence of weak solutions of a fractional thin film equation. It is a non-local degenerate parabolic equation of order $\alpha + 2$ where $0 < \alpha < 2$. It is a generalization of an equation studied by Imbert and Mellet in 2011 for $\alpha = 1$. To construct these solutions, we consider a regularized problem then we pass to the limit using Sobolev embedding theorem, that's why we distinguish two cases $0 < \alpha < 1$ and $1 \leq \alpha < 2$. We also prove that the solution is positive if the initial condition is so.

The fourth chapter is dedicated for a fractional porous medium equation. We prove Hölder regularity of positive weak solutions satisfying energy estimates. First, we prove the existence of weak solutions that satisfy energy estimates. We distiguish two cases $0 < \alpha < 1$ and $1 \leq \alpha < 2$ because of divergence problems. The we prove De Giorgi Lemmas about oscillation reduction from above and from below. This is not suffisant. We need to improve the lemma about oscillation reduction from above. So we pass by an intermediate values lemma and we prove an improved oscillation reduction lemma from above. Finally, we prove Hölder regularity of solutions using the scaling property.

Contents

Chapter 1

Introduction

Dans cette thèse, nous travaillons sur deux problèmes dont chacun concerne une équation parabolique dégénérée non-locale. Dans le premier travail, nous montrons l'existence de solutions faibles d'une famille d'équations paraboliques dégénérées non-locales d'ordre supérieur. Dans le second travail, nous traitons la régularité de solutions faibles d'une équation des milieux poreux fractionnaire.

Cette introduction est composée de deux parties représentant lchaque problème étudié. On y présente les résultats connus dans la littérature, on donne les résultats obtenus durant cette thèse et on ouvre quelques perspectives.

1.1 Equation des films minces fractionnaire

1.1.1 Présentation du problème

Dans cette partie on étudie le problème suivant

$$\begin{cases} \partial_t u + \partial_x (u^n \partial_x I(u)) = 0 & \text{pour } t > 0, \quad x \in \Omega, \\ \partial_x u = 0, u^n \partial_x I(u) = 0 & \text{pour } t > 0, \quad x \in \partial\Omega, \\ u(0,x) = u_0(x) & \text{pour } x \in \Omega, \end{cases} \tag{1.1}$$

où $\Omega = (a,b)$ est un intervalle borné dans $\mathbb{R}$, n est un réel positif et I est un opérateur négatif elliptique non-local d'ordre α défini par $I = -(-\Delta)^{\frac{\alpha}{2}}$ avec des conditions de Neumann aux bords et avec $\alpha \in (0,2)$.

Pour $\alpha = 1$ et $n = 3$ cette équation désigne le modèle physique KGD développé par Geertsma et De Klerk [39] et Khristianovich et Zheltov [81]. Ce modèle représente l'influence de la pression exercée par un fluide visqueux sur une fracture dans un milieu élastique sous la condition de plane strain. Cette équation est obtenue à partir de la conservation de la masse du fluide à l'intérieur de la fracture, la loi de Poiseuille et une loi pour la pression convenable ($p = -I(u)$) [54].

1.1.2 Résultats existants dans les cas $\alpha = 0, \alpha = 2$ et $\alpha = 1$

Pour $\alpha = 0$ on obtient l'équation des milieux poreux en dimension 1 qui est une équation locale

$$\partial_t u - \partial_x(u^n \partial_x u) = 0. \tag{1.2}$$

Nous reviendrons sur cette équation dans la deuxième partie qui a comme sujet une équation des milieux poreux fractionnaire.
Pour $\alpha = 2$ on obtient aussi une équation locale, l'équation des films minces donnée par

$$\partial_t u - \partial_x(u^n \partial^3_{xxx} u) = 0. \tag{1.3}$$

Pour $n = 3$ cette équation représente le mouvement monodimensionnel d'une goutte visqueuse glissante sur une surface solide. L'existence de solutions faibles Hölder continues de (1.3) a été prouvée par Bernis et Friedmann pour $n > 1$ [10]. Pour montrer ce résultat ils utilisent une méthode de régularisation et des estimées d'énergie. De plus ils montrent que la solution est positive si la condition initiale l'est. En utilisant la même approche, Beretta, Bertsch et Dal Passo montrent des réultats analogues pour la même équation mais en considérant d'autres conditions aux bords [7].
D'autre part dans [11, 12] Bertozzi et Pugh montrent aussi des résultats d'existence de solutions faibles qui deviennent fortes après un certain temps fini. De plus ils font des simulations numériques des solutions faibles donnés dans [10]. Notons aussi que Bernis, Peletier et Williams montrent l'existence de source type solutions pour $0 < n < 3$ [8].

L'équation des films minces a été aussi étudiée dans des dimensions supérieures à 1. Elle prend alors la forme suivante:

$$\partial_t u + \nabla.(u^n \nabla \Delta u) = 0 \tag{1.4}$$

Grün [45], Elliott et Garcke [37] montrent l'existence de solutions pour $1 \leq n < 2$ si la condition initiale est positive arbitraire et pour $n \geq 1$ si la condition initiale est strictement positive. De plus Dal Passo, Garcke et Grün [27] étudient l'existence de solutions (pour $\frac{1}{8} < n < 3$ avec une condition initiale positive arbitraire), la non-unicité, la positivité et le comportement asymptotique des solutions de l'équation (1.4).
D'autres propriétés ont été étudiées pour l'équation des films minces comme la vitesse finie de propagation des solutions, le comportement asymptotique des solutions et la propriété de temps d'attente. Bernis [9] a montré en dimension $N = 1$ que pour $0 < n < 3$ les solutions ont une vitesse finie de propagation et que l'interface qui sépare les régions où

u est strictement positive et où u est égale à zéro se déplace avec une vitesse finie. Il utilise des versions locales des estimées d'entropie trouvée dans [7]. Puis Grün [46, 47, 48] montrent que la vitesse finie de propagation est verifiée par les solutions en dimension $N < 4$ pour $2 \leq n < 3$. Dal Passo, Giacomelli et Grün [28] montrent que les solutions présentent un phénomène de temps d'attente c'est à dire il existe un temps positif durant lequel le support de u (localement en espace) ne s'étend pas.

Notons qu'il y a des propriétés communes pour l'équation des films minces et l'équation des milieux poreux. Les deux équations sont paraboliques dégénérées sous la forme de divergence. La vitesse finie de propagation et le phénomène de temps d'attente sont les propriétés les plus communes connues. Dans les deux cas il existe aussi des source type solutions à support compact (sous condition $n > 1$ pour l'équation des milieux poreux [77] et $0 < n < 3$ pour l'équation des films minces [8][26]). La grande différence entre ces deux équations est le manque d'un principe du maximum pour l'équation des films minces [26].

Notre équation (3.1) ressemble à l'équation des films minces en remplaçant le Laplacien par un Laplacien fractionnaire avec des conditions de Neumann aux bords. Donc notre équation est non-locale au contraire de celle des films minces. Cependant le manque d'un principe du maximum nous amènera à utiliser une approche similaire à celle utilisée pour l'équation des films minces.

Pour $\alpha = 1$ cette équation a été d'abord étudiée par Spence et Sharp [72]. Ils travaillent sur les solutions auto-similaires et le comportement asymptotique des solutions. Peirce et al. [69, 70] ont développé des méthodes numériques pour ce modèle. Puis Imbert et Mellet ont montré dans [54], par une technique de régularisation, l'existence de solutions faibles positives de (3.1) pour $n \geq 1$ et une condition initiale positive. De plus ils ont montré un résultat de positivité stricte de solutions sous une condition sur n. Les ingrédients principaux de [54] sont des inégalités d'énergie et l'injection de Sobolev en dimension 1

$$H^{\frac{1}{2}}(\Omega) \hookrightarrow L^p(\Omega) \tag{1.5}$$

pour tout $p < \infty$. Dans un autre article [52] ils ont construit des solutions auto-similaires pour (3.1) avec $n = 3$.

La cas $\alpha \in (-2, 0)$ a été traité dans [16] par Biler, Imbert et Karch qui étudient l'équation suivante

$$u_t = \nabla.\left(u\nabla^{\beta-1}G(u)\right), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N, \tag{1.6}$$

avec $\nabla^{\beta-1} = \nabla(-\Delta)^{\frac{\beta}{2}-1}$ pour $\beta \in (0,2)$ et $G(u) = u^{m-1}$ avec $m > 1$. Pour $\beta = \alpha + 2$, $m = 2$ et $N = 1$ cette équation coïncide avec notre équation (3.1) pour $n = 1$. Ils montrent l'existence de solutions faibles de l'équation sous des conditions sur m suivant la valeur de α. Dans la deuxième partie de la thèse on étudie la régularité Hölder de ces solutions.

1.1.3 Résultats obtenus

Dans la suite de cette section, $\Omega = (a,b)$ désigne un intervalle borné de $\mathbb{R}$. Pour $0 < \alpha \leqslant 1$ des solutions faibles positives de (3.1) existent pour des conditions initiales positives sous des conditions convenables dites conditions d'entropie.

Theorem 1.1 (Existence de solutions pour $0 < \alpha \leqslant 1$)**.** *Soit $n \geqslant 1$ et $\alpha \in (0,1]$. Pour toute condition initiale positive $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$ tel que*

$$\int_\Omega G(u_0)\mathrm{d}x < \infty \tag{1.7}$$

où G est une fonction positive tel que $G''(s) = \frac{1}{s^n}$, il existe une fonction positive

$$u \in L^\infty(0,T; H^{\frac{\alpha}{2}}(\Omega)) \cap L^2(0,T; H_N^{\frac{\alpha}{2}+1}(\Omega))$$

qui satisfait dans $Q = (0,T) \times \Omega$

$$\iint_Q u\partial_t\varphi\mathrm{d}t\mathrm{d}x - \iint_Q nu^{n-1}I(u)\partial_x u\partial_x\varphi\mathrm{d}x\mathrm{d}t - \iint_Q u^n I(u)\partial_{xx}^2\varphi\mathrm{d}x\mathrm{d}t = -\int_\Omega u_0\varphi(0,.)\mathrm{d}x \tag{1.8}$$

pour toute fonction test $\varphi \in \mathcal{D}([0,T) \times \bar{\Omega})$ telle que $\partial_x\varphi = 0$ dans $(0,T) \times \partial\Omega$. De plus u satisfait pour presque tout $t \in (0,T)$

$$\int_\Omega u(t,x)\mathrm{d}x = \int_\Omega u_0(x)\mathrm{d}x, \tag{1.9}$$

$$\int_\Omega G(u(t,x))\mathrm{d}x + \int_0^t |u|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)} \leq \int_\Omega G(u_0)\mathrm{d}x. \tag{1.10}$$

et

$$|u(t,.)|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + 2\int_0^T\int_\Omega g^2\mathrm{d}x \leq |u_0|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} \tag{1.11}$$

où la fonction $g \in L^2(Q)$ définie par $g = \partial_x(u^{\frac{n}{2}}I(u)) - \frac{n}{2}u^{\frac{n-2}{2}}\partial_x uI(u)$ dans $\mathcal{D}'(\Omega)$.

Pour $1 < \alpha < 2$ des solutions faibles positives sont construites pour des conditions initiales positives dans $H^{\frac{\alpha}{2}}(\Omega)$.

Theorem 1.2 (Existence de solutions pour $1 < \alpha < 2$)**.** *Soit $n \geqslant 1$ et $\alpha > 1$. Pour toute condition initiale positive $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$, il existe une fonction positive*

$$u \in C_{t,x}^{\frac{\alpha-1}{2(\alpha+2)},\frac{\alpha-1}{2}}(Q)$$

telle que

$$\partial_x I(u) \in L^2_{loc}(Q_+) \tag{1.12}$$

où $Q_+ = \{u > 0\} \cap Q$, et qui satisfait

$$\iint_Q u\partial_t\varphi \mathrm{d}t\mathrm{d}x + \iint_{Q_+} u^n\partial_x I(u)\partial_x\varphi \mathrm{d}x\mathrm{d}t = -\int_\Omega u_0\varphi(0,.)\mathrm{d}x \tag{1.13}$$

pour tout $\varphi \in \mathcal{D}([0,T)\times\bar{\Omega})$ satisfaisant $\partial_x\varphi = 0$ dans $(0,T)\times\partial\Omega$.

De plus, u est telle que pour presque tout $t \in (0,T)$

$$\int_\Omega u(t,x)\mathrm{d}x = \int_\Omega u_0(x)\mathrm{d}x \tag{1.14}$$

et

$$|u(t,.)|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + 2\iint_{Q_+} u^n(\partial_x I(u))^2\mathrm{d}x \leq |u_0|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}. \tag{1.15}$$

Remark 1.3. *Notons qu'on obtient la formulation faible (3.7) après deux intégrations par parties. Dans le cas $0 < \alpha \leq 1$, le terme $\partial_x I(u)$ est de trop basse régularité donc on a besoin de la seconde intégration par parties pour que tous les termes non-linéaires dans (3.7) aient un sens. Au contraire lorsque $\alpha > 1$, on a*

$$H^{\frac{\alpha}{2}} \subset C^{\frac{\alpha-1}{2}}.$$

En particulier, $u^n \in L^\infty$ et (1.12) garantit que tous les termes de (3.12) ont bien un sens, sans procéder à une seconde intégration par parties.

Finalement un résultat de positivité des solutions est prouvé pour de grandes valeurs de n.

Theorem 1.4 (Solutions strictement positives)**.** *Supposons $0 < \alpha < 2$ et $n > max\{3, 2 + \frac{2}{\alpha+1}\}$. La solution $u(t,.)$ construite dans le Théorème 3.1 est strictement positive dans Ω. De plus, il existe un ensemble $P \subset (0,T)$ de mesure nulle, tel que $u(t,.) \in C^{0,\beta}(\Omega)$ pour tout $t \in (0,T) \setminus P$ et pour tout $\beta < min\{1, \frac{\alpha+1}{2}\}$. Enfin u est une solution de*

$$u_t + \partial_x J = 0 \qquad \text{dans } \mathcal{D}'(\Omega)$$

où

$$J(t,.) = u^n\partial_x I(u) \in L^1(\Omega) \qquad \text{pour tout } t \in P.$$

1.1.4 Etapes de la preuve [74] lorsque $0 < \alpha < 2$

Nos résultats sont des généralisation des résultats obtenus dans [54] pour les deux cas $0 < \alpha < 1$ et $1 < \alpha < 2$ en utilisant la même approche mais en modifiant les résultats de compacité. D'abord on considère le problème régularisé suivant

$$\begin{cases} \partial_t u + \partial_x(f_\epsilon(u)\partial_x I(u)) = 0 & \text{pour } t > 0, \quad x \in \Omega, \\ \partial_x u = 0, f_\epsilon(u)\partial_x I(u) = 0 & \text{pour } t > 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x) & \text{pour } x \in \Omega, \end{cases} \tag{1.16}$$

où $f_\epsilon(s) = s_+^n + \epsilon$, $\epsilon > 0$ et $0 < \alpha < 2$. L'étude d'un problème stationnaire qui permet de montrer l'existence de solutions pour le problème régularisé. Dès que les solutions du problème régularisé sont construites on passe à la limite $\epsilon \to 0$ pour trouver des solutions faibles pour (3.1). En passant à la limite on distingue les deux cas $0 < \alpha \leq 1$ et $1 < \alpha < 2$. Le cas $1 < \alpha < 2$ est le cas le plus simple car dans ce cas on obtient une convergence locale uniforme grâce à l'injection suivante en dimension 1

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow C^{0,\frac{\alpha-1}{2}}(\Omega). \tag{1.17}$$

Cette convergence nous permet de passer à la limite dans le terme nonlinéaire contenant $I(u_\epsilon)$. Ce qui nous permet de construire des solutions faibles positives pour des conditions initiales positives dans $H^{\frac{\alpha}{2}}(\Omega)$. D'abord on construit les solutions pour des conditions initiales positives dans $H^{\frac{\alpha}{2}}(\Omega)$ avec une condition d'entropie, puis on montre que le régularité des solutions en espace implique une régularité en temps comme dans [10] ce qui nous permet de construire des solutions faibles positives sans condition d'entropie sur la condition initiale. On note qu'on utilise une représentation intégrale de l'opérateur I pour pouvoir passer à la limite dans le terme $I(u_\epsilon)$ en utilisant la convergence locale uniforme en temps et en espace.
Le cas $0 < \alpha < 1$ est plus compliqué car on perd la convergence locale uniforme. Dans ce cas on utilise l'injection suivante

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{pour tout } p \leq \frac{2}{1-\alpha}. \tag{1.18}$$

Donc on obtient un résultat de compacité dans $L^p(\Omega)$ seulement lorsque $p \leq \frac{2}{1-\alpha}$. Mais en utilisant le lemme d'Aubin on arrive à un résultat de compacité pour u_ϵ qui nous permet d'affirmer la compacité du terme nonlinéaire contenant $I(u_\epsilon)$ et passer à la limite.

1.1.5 Perspectives

Comme l'équation des films minces et l'équation des milieux poreux vérifient toutes les deux les propriétés de vitesse finie de propagation et de temps d'attente, on s'attend à ce que notre équation en jouisse aussi.

1.2 Equation des milieux poreux fractionnaire

1.2.1 Présentation du problème

Dans cette partie, on étudie le problème suivant

$$u_t = \nabla.\left(u\nabla^{\alpha-1}G(u)\right), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^N, \tag{1.19}$$

où $G(u) = u^{m-1}$, $m \geq 3$ et $\nabla^{\alpha-1}$ est l'opérateur intégro-différentiel d'ordre $\alpha - 1$ $\nabla(-\Delta)^{\frac{\alpha}{2}-1}$, avec $\alpha \in (0,2)$. Cette équation est étudiée avec la condition initiale

$$u(0,x) = u_0(x). \tag{1.20}$$

L'opérateur $\nabla^{\alpha-1}$ est un opérateur nonlocal. Pour une fonction régulière bornée v, on dispose de la représentation intégrale suivante

$$\nabla^{\alpha-1}v(x) = c_\alpha \int (v(y) - v(x)) \frac{y-x}{|y-x|^{N+\alpha}} \mathrm{d}y \tag{1.21}$$

avec une constante convenable $c_\alpha > 0$. De plus, on a $\nabla.\nabla^{\alpha-1} = -(-\Delta)^{\frac{\alpha}{2}}$.

Notons qu'on ne dispose pas d'un principe de comparaison pour cette équation.

1.2.2 Résultats existants

Pour $\alpha = 2$ l'équation (1.19) devient l'équation des milieux poreux avec $m > 1$ et $u \geq 0$

$$\partial_t u = \nabla.(u\nabla u^{m-1}). \tag{1.22}$$

C'est une équation parabolique dégénérée d'ordre 2. Le cas $m = 2$ correspond à l'équation de Boussinesq. Les propriétés de l'équation des milieux poreux (existence de solutions faibles et fortes, unicité des solutions, vitesse finie de propagation, principe du maximum) sont bien connues dans la littérature mathématique, voir [4, 77] pour la théorie concernant cette équation et ses applications. Cette équation décrit différents phénomènes naturels. Ce modèle a été proposé par Boussinesq en 1903 pour étudier l'infiltration des eaux souterraines [17]. En 1950 Zeldovich et al. [80] ont développé une application sur le rayonnement thermique en plasmas. Le modèle le plus connu est la description du flux d'un gaz isentropique dans un milieu poreux modélisé par Leibenzon [62] et Muskat [66] en 1930. Cette équation est dérivée de la conservation de la masse du gaz qui se propage dans un milieu poreux homogène [4, 77]

$$\partial_t u + \nabla.(uv) = 0 \tag{1.23}$$

où $u \geq 0$ représente la densité du gaz et v la vitesse moyenne locale. Puis la loi de Darcy nous permet d'exprimer v en fonction de p la pression

$$v = -\nabla p. \tag{1.24}$$

Enfin une loi sur la pression implique que p est une fonction monotone de u, $p = f(u)$. La fonction f est linéaire quand le flux est isothermique et est un exposant de u quand le flux est adiabatique. Ce qui donne l'équation suivante

$$\partial_t u = \nabla.(u\nabla f(u)). \tag{1.25}$$

Dans notre cas $f(u) = (-\Delta)^{\frac{\alpha}{2}-1}u^{m-1}$ avec $0 < \alpha < 2$. L'opérateur $(-\Delta)^{\frac{\alpha}{2}-1}$ est un Laplacien fractionnaire d'exposant négatif de noyau

$$k(x,y) = c_\alpha |x-y|^{-(N+\alpha-2)}. \tag{1.26}$$

C'est pour cette raison qu'on appelle l'équation (1.19) une équation des milieux poreux fractionnaire.

En dimension 1 et pour $m = 2$ l'équation (1.19) coïncide avec l'équation (3.1) étudiée dans la première partie pour $n = 1$ et $\Omega = \mathbb{R}$. Le Laplacien fractionnaire est au coeur de nombreux modèles récents. Vazquez et al [30, 31] étudient l'équation suivante

$$\partial_t u + (-\Delta)^{\frac{\sigma}{2}}(|u|^{m-1}u) = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \tag{1.27}$$
$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N,$$

avec $0 < \sigma < 2$ et $m > 0$ pour une condition initiale u_0 dans $L^1(\mathbb{R}^N)$. Ils montrent l'existence et l'unicité de solutions faibles sous des conditions sur m. De plus ils étudient le comportement asymptotique des solutions et construisent des solutions auto-similaires. Indépendamment Caffarelli, Chan et Vasseur [18] étudient la régularité Hölder des solutions dans le cas $m = 1$, mais dans un cadre plus général. Ils étudient la famille d'équations non locales paraboliques de la forme

$$u_t(t,x) = \int (u(t,y) - u(t,x))K(t,x,y)\mathrm{d}y \tag{1.28}$$

où k est un noyau symétrique c'est à dire

$$K(t,x,y) = K(t,y,x) \text{ pout tout } x \neq y, \tag{1.29}$$

et pour lequel il existe $0 < s < 2$ et $\Lambda > 0$ tels que

$$\chi_{\{|x-y|\leq 3\}}\frac{1}{\Lambda}\frac{1}{|x-y|^{N+s}} \leq K(t,x,y) \leq \frac{\Lambda}{|x-y|^{N+s}}. \tag{1.30}$$

Pour montrer la régularité Hölder des solutions ils utilisent la méthode de De Giorgi qui sera présentée dans le premier chapitre de cette thèse. Les solutions de l'équation (1.28) changent de signe. On observe cependant que si u est une solution de (1.28) alors $-u$ l'est aussi. Donc il suffit de montrer un lemme de réduction de l'oscillation par au dessus. L'idée est de comparer u à 0. Si u passe un certain temps en dessous de 0 dans un certain cylindre alors u ne peut pas être très proche de 1 dans un cylindre plus petit. En itérant ce lemme pour une famille de solutions de (1.28) on obtient la régularité Hölder des solutions. Notons que la propriété de scaling des solutions est essentielle dans la preuve pour pouvoir itérer le lemme de réduction de l'oscillation.

Caffarelli et Vazquez [21] étudient l'équation

$$\partial_t u = \nabla.(u\nabla(-\Delta)^{-s}u) \quad t > 0, x \in \mathbb{R}^N, \tag{1.31}$$

avec $0 < s < \frac{1}{2}$. L'équation (1.31) coïncide avec notre équation (1.19) dans le cas $\alpha = 2 - 2s$ et $m = 2$. Ils montrent l'existence de solutions faibles pour des conditions initiales bornées avec une décroissance exponentielle à l'infini. Pour construire ces solutions ils ajoutent un terme Laplacien, régularisent le noyau et éliminent la dégénéréscence. De plus comme les solutions de l'équation des milieux poreux, ces solutions ont une vitesse finie de propagation: si la condition initiale $u_0(x)$ est à support compact alors la solution $u(t,x)$ est aussi à support compact pour tout $t > 0$. Dans un autre article [24] ils étudient le comportement asymptotique de ces solutions.
De plus ils montrent avec un autre auteur dans [19] la régularité Hölder de ces solutions pour le cas $s \in (0,1), s \neq 1/2$. Puis ils traitent le cas $s = 1/2$ dans un autre article [22]. Ils utilisent aussi la méthode de De Giorgi pour montrer leur résultat. Ces deux articles ont été une précieuse source d'inspiration nous allons généraliser leur méthode pour établir la régularité des solutions de (1.19). Les ingrédients principaux de la preuve sont les estimées d'énergie pour montrer les premiers lemmes de De Giorgi et les injections de Sobolev. Notons que la propriété de scaling vérifiée par les solutions est aussi importante dans la preuve. L'idée est d'itérer les lemmes de réduction d'oscillation (par au dessus ou par en dessous) pour arriver à la régularité. Notons que par rapport à l'équation (1.28) la différence majeure est qu'ils considèrent des solutions positives. On ne peut donc ni comparer u à 0 ni utiliser $-u$ comme solution de (1.31). Ils comparent plutôt u à $1/2$ et dans ce cas ils ont besoin d'un nouveau lemme de réduction de l'oscillation par en dessous. Plus précisément, si la solution u est presque partout plus grande que $1/2$ dans un certain cylindre alors u est strictement plus grande que 0 dans un cylindre plus petit. Enfin pour arriver à la régularité Hölder ils itèrent alternativement ces deux lemmes.

Notons que dans les deux cas (1.28) et (1.31) la forme bilinéaire

$$B(u,v) = \iint (u(t,x) - u(t,y))K(t,x,y)(v(t,x) - v(t,y))\mathrm{d}x\mathrm{d}y \tag{1.32}$$

joue un rôle crucial dans la preuve des estimées d'énergie. Selon le cas K vérifie les conditions (1.29), (1.30) dans le cas de (1.28) et $K(t,x,y) = |x-y|^{-(N+s)}$ dans le cas de (1.31). Pour arriver aux estimations d'énergie ils utilisent la décomposition suivante de u

$$u = (u-c)_+ - (u-c)_- + c \tag{1.33}$$

où c est une constante positive, $(u-c)_+ = \max(0, u-c)$ et $(u-c)_- = \max(0, c-u)$. Cette décomposition astucieuse est utilisée pour séparer les bons termes des autres termes à estimer.

Récemment Allen, Caffarelli et Vasseur ont étudié une équation sur $\mathbb{R}^N$ avec un potentiel fractionnaire ainsi qu'une dérivée fractionnaire en temps [2]

$$D_t^\alpha u - \nabla.(u\nabla(-\Delta)^{-\sigma}u) = f, \quad u(0,x) = u_0(x),$$

où $0 < \sigma < 1/2$ et les fonctions u_0 et f sont positives. Ils montrent l'existence de solutions faibles positives pour une condition initiale u_0 dans C^2 et une fonction f qui décroît exponentiellement à l'infini. Le résultat principal de leur article est la régularité Hölder de ces solutions. Ils montrent le résultat en utilisant la même approche de De Giorgi.

En ce qui concerne une variante de l'équation des milieux poreux fractionnaire (1.19) Biler, Imbert et Karch [16] étudient le problème de Cauchy suivant

$$u_t = \nabla.\left(|u|\nabla^{\alpha-1}(|u|^{m-2}u)\right), \quad t > 0, \quad x \in \mathbb{R}^N, \tag{1.34}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \tag{1.35}$$

Ils montrent l'existence de solutions faibles bornées pour des conditions initiales intégrables sous des conditions sur m qui dépendent de la valeur de α. De plus, ils prouvent que si la donnée initiale u_0 est positive alors la solution u l'est aussi. Dans ce cas l'équation (4.9) coïncide avec la notre (1.19). Pour construire ces solutions ils passent par un problème régularisé. Dans notre preuve on va aussi utiliser des solutions approchées pour montrer qu'il existe des solutions faibles qui vérifient des estimées d'énergie. Dans un autre article [53] C. Imbert montre que ces solutions ont une vitesse finie de propagation sous des conditions sur m.

1.2.3 Résultats obtenus

On montre la régularité Hölder d'une solution faible de (1.19).

Theorem 1.5 (Régularité Hölder). *Soit $0 < \alpha < 2$ et $m > 3$. Il existe une solution faible nontriviale bornée de (1.19) $u \geq 0$ définie dans $(-1,0) \times \mathbb{R}^N$ tel que u est Hölder continue dans $(-1/2,0) \times \mathbb{R}^N$ et*

$$[u]_{C^{\beta_1,\beta}_{t,x}} \leq C$$

tels que β_1, β et la constante C dépendent seulement de N, m et α.

Remark 1.6. *On obtient une estimée pour des solutions dans $(-T,0) \times \mathbb{R}^N$ par remise à l'échelle.*

1.2.4 Etapes de la preuve

Dans la deuxième partie de cette thèse on montre la régularité Hölder des solutions faibles de (1.19) qui vérifient certaines estimées d'énergie. On suit l'approche utilisée par [18, 19, 22] donc on utilise la méthode de De Giorgi.

Dans un premier temps, on cherche des inégalités d'énergie vérifiées par les solutions pour pouvoir montrer les premiers lemmes de De Giorgi. On considère les solutions approchées u^δ du problème régularisé

$$\partial_t u = \delta\Delta u + \nabla.(u\nabla^{\alpha-1}G(u)). \tag{1.36}$$

On montre les inégalités d'énergie pour les solutions approchées régulières puis on passe à la limite pour arriver aux estimées vérifiées par une solution faible. Pour prouver ces inégalités on distingue les cas $\alpha \in (1,2), \alpha \in (0,1)$ et $\alpha = 1$. Le cas $\alpha \in (1,2)$ est le cas le plus simple car on ne rencontre pas des intégrales divergentes. Dans le cas $\alpha \in (0,1]$ une intégrale utilisée dans les estimations diverge, donc on utilise des transformations géométriques inspirées de [19, 22]. Ces transformations font apparaître un terme qui permet de contrôler le terme divergent.
La nouveauté trouvée en faisant les estimations est qu'on perd la linéarité de u dans la forme bilinéaire (1.32), c'est à dire qu'un terme $G(u)$ remplace u. La décomposition qu'on utilise alors est la suivante, dans l'esprit de (1.33)

$$G(u) = G(c + (u-c)_+) + G(c - (u - c_-)) - G(c). \tag{1.37}$$

Puis on utilise le fait que G est une fonction localement lipschitizienne pour arriver à l'estimation convenable. Une fois que les estimées d'énergie sont vérifiées on démontre les lemmes de réduction d'oscillation qui sont nécessaires pour arriver à la régularité Hölder. On considère une solution positive et majorée par 1. D'abord on montre un lemme de réduction de l'oscillation par au dessus. Si u est presque partout en dessous de $1/2$ dans un certain cylindre (disons $(-4,0) \times B_4$) alors u ne peut être très proche de 1 dans un cylindre plus petit (disons $(-1,0) \times B_1$). Puis on montre un lemme de réduction de l'oscillation par en dessous. Si u est presque partout au dessus de $1/2$ dans un certain cylindre alors u est strictement positive dans un cylindre plus petit. Ces deux lemmes sont appelés les premiers lemmes de De Giorgi. Ces deux lemmes ne sont pas suffisants dans notre cas pour arriver au résultat. Donc on a besoin d'améliorer le lemme de réduction de l'oscillation par au dessus. Pour cela on montre un lemme des valeurs intermédiaires. Ce lemme dit que si u passe un certain temps plus petite qu'une certaine constante c_0 et un certain temps plus grande qu'une autre constante c_2 alors u doit passer du temps entre ces deux constantes. La preuve de ce lemme n'est pas triviale. On a suivi les idées trouvées dans [18] et on a écrit la preuve après des discussions personnelles avec C. Imbert. En considérant une famille convenable d'itérations de la solution et en utilisant le lemme des valeurs intermédiaires pour cette famille on arrive à obtenir un lemme de réduction de l'oscillation par au dessus amélioré. Enfin on arrive à une situation où on a deux alternatives; ou bien u passe un certain temps plus petite que $1/2$ ou bien u passe la plupart du temps plus grande que $1/2$. Dans les deux cas on arrive à réduire l'oscillation de u. En considérant une famille de solutions itérées remises à l'échelle, on arrive enfin à la régularité Hölder de u.

1.2.5 Perspectives

Plusieurs questions sont encore ouvertes à propos notre équation. On souhaite par exemple étudier la régularité de l'interface c'est à dire le support de u, le comportement asymptotique des solutions ou des conditions d'unicité. On peut aussi étudier l'équation des milieux poreux fractionnaire numériquement ou plus généralement étudier ces problèmes nonlocaux dans des domaines bornés. Par ailleurs, Stan, De Teso et Vazquez étudient dans [73] l'équation

$$\partial_t u = \nabla.(u^{m-1}\nabla(-\Delta)^{-s}u) \quad t > 0, \quad x \in \mathbb{R}^N, \tag{1.38}$$

pour $m > 1, 0 < s < 1$ et $u(t,x) \geq 0$. Ils montrent l'existence de solutions faibles pour des conditions initiales positives bornées à support compact. De plus ils étudient la vitesse de propagation des solutions suivant la valeur de m. Une question encore ouverte est la régularité Hölder des solutions de cette équation. L'étude de la régularité de l'équation (1.38) serait un prolongement naturel de notre travail.

Chapter 2

De Giorgi elliptic and parabolic regularity theory: a short overview

2.1 Introduction

In this chapter we present the method introduced by De Giorgi in [29] to prove Hölder regularity of solutions of elliptic equations with rough coefficients.

2.1.1 De Giorgi's legacy: some references

De Giorgi was interested in solving the 19^{th} Hilbert problem about the analytic regularity of solutions to some integral variational problems. Nash introduced different techniques to solve the problem [67]. Then Moser showed the result using a new approach [65]. These methods called De Giorgi-Nash-Moser techniques are by now classical in the regularity theory of partial differential equations. De Giorgi introduced in his paper [29] a class of functions that verify energy estimates and proved that once the function is in this class then it is locally bounded and Hölder continuous.

His ideas was extended to linear parabolic equations with lower order terms and to quasilinear parabolic equations by Ladyzhenskaya and Uralt'seva [60]. They introduced parabolic De Giorgi classes and proved that Hölder estimate holds if $\pm u$ are both in one of this classes. These results are presented in [63]. Then they introduce a more general parabolic De Giorgi class in [59]. Di Benedetto and Trudinger [34] showed that nonnegative function in the elliptic De Giorgi class satisfy Harnack inequality. They use a measure theoretic lemma of Krylov and Safonov [57]. Wang extend their result to the parabolic case in [78] then he establish in [79] Harnack inequality for functions in the general De Giorgi class introduced in [59]. The method of De Giorgi has also been extended to degenerate case like the p-laplacian, first to elliptic case [61] then to parabolic case [32],

[35], [33], [36].

Furthermore, nonlinear nonlocal time-dependent variational problems are studied in [18]. They extend the method of De Giorgi to nonlocal parabolic problems and prove Hölder regularity of solutions for problems with translation invariant kernels. This type of equation has been studied also using Moser's result from [65] in [56] and [38]. Here they prove local regularity results such as a weak Harnack inequality [38]. In [19] the authors also used the De Giorgi method to prove Hölder regularity of solutions of a porous medium equation with nonlocal diffusion effects given by an inverse fractional Laplacian operator. This method was also used to solve regularity issues in fluid mechanics [5], [25], [23], [42], [75]. Recently, the De Giorgi method has been extended to a class of kinetic Fokker-Planck equations [41] and [55]. A Harnack inequality and Hölder regularity are proved for solutions to a general linear equation of Fokker-Planck type whose coefficients are merely measurable and essentially bounded.

2.1.2 The main steps of the method

We present the strategy of De Giorgi presented in [76] and [20] for the elliptic, local and nonlocal parabolic cases. To prove Hölder regularity, the method is to prove a lemma of reduction of the oscillation. In the elliptic case his idea is to compare sign-changing solutions u bounded from above by 1 to zero($u \geq 0$ or $u < 0$). If u is mostly below zero in the ball B_1 then u is far from 1 in a smaller ball $B_{1/2}$. Two main ingredients are used in the proof of this lemma, energy estimate derived from the equation and Sobolev's embedding theorem. Then to pass to a lemma of reduction of the oscillation he proves an isoperimetric inequality which is very crucial in the proof. This inequality has been extended to the local and nonlocal parabolic cases and known by the lemma of intermediate values. Finally using scaling property of the solutions he conclude Hölder regularity. In the parabolic cases, they proceed as in the elliptic case but by considering cylinders instead of balls. We note that proving an energy estimate verified by a solution is very crucial in the proof of Hölder continuity using De Giorgi's approach. Once the energy estimate is proved we can follow the steps of De Giorgi and get the regularity result.

2.1.3 Organization of the chapter

In Section 2 we recall the 19^{th} Hilbert problem. In Section 3 we present De Giorgi's method in the elliptic case. Then we pass to the local parabolic case in Section 4. Finally we present this method in the nonlocal parabolic case in Section 5.

2.2 The 19$^{\text{th}}$ Hilbert problem

The 19$^{\text{th}}$ Hilbert problem consists in showing the regularity of local minimizers of an energy functional of the form

$$\Upsilon(w) = \int_{\Omega} F(\nabla w)dx$$

where Ω is a bounded open set in $\mathbb{R}^N$ and F is a regular function. Note that with local minimizer we mean that

$$\Upsilon(w) \le \Upsilon(w + \varphi)$$

for any φ compactly supported in Ω. Such a minimizer satisfies the Euler-Lagrange equation

$$\nabla.(F'(\nabla w)) = 0, \qquad x \in \Omega. \tag{2.1}$$

De Giorgi showed that if we assume that

$$F \text{ is strictly convex and } \quad \lim_{|p|\to\infty} \frac{F(p)}{|p|^2} = c > 0 \tag{2.2}$$

then any solution of (2.1) is C^∞ in Ω. Note that the assumptions (2.2) on F imply that there exists a constant $\Lambda > 0$ such that

$$\frac{1}{\Lambda} I \le F''(\nabla w) \le \Lambda I, \qquad x \in \Omega,$$

where I is the identity $N \times N$ matrix. If we write the equation in the non-divergence form

$$F''(\nabla w) : D^2 w = 0,$$

then we can use the Calderon-Zygmund theory [40] to say that $\nabla w \in C^\alpha(\Omega)$ implies the $C^{2,\alpha}$ regularity on w in Ω since the equation is linear with C^α coefficients, the argument can be reiterated we arrive to $w \in C^\infty(\Omega)$. So the aim now is to prove that $\nabla w \in C^\alpha(\Omega)$. De Giorgi considered for every $1 \le i \le N$ the derivative with respect to x_i of (2.1) and denoted $u = \partial_i w$, we arrive to

$$\nabla.(F''(\nabla w)\nabla u) = 0.$$

2.3 Elliptic case

We will present in this section the elliptic case which is the original result proved by De Giorgi.

Theorem 2.1 (Hölder regularity). *Let Ω be a bounded open set of $\mathbb{R}^N$ and $\Lambda > 0$. Consider $A(x)$ a measurable matrix valued function defined on Ω such that*

$$\frac{1}{\Lambda} I \leq A(x) \leq \Lambda I, \qquad x \in \Omega. \tag{2.3}$$

Let $u \in H^1(\Omega)$ be a weak solution of

$$-\nabla.(A(x).\nabla u) = 0, \qquad x \in \Omega. \tag{2.4}$$

Then $u \in C^\alpha(\Omega')$ for any $\Omega' \subset\subset \Omega$, with

$$\|u\|_{C^\alpha(\Omega')} \leq C\|u\|_{L^2(\Omega)}.$$

The constant α depends only on Λ and N. The constant C depends on Λ, N, Ω' and Ω.

By weak solution we mean a function $u \in H^1(\Omega)$ which verifies

$$\int A(x).\nabla u \nabla \varphi \mathrm{d}x = 0,$$

for any function $\varphi \in C_c^\infty(\mathbb{R}^N)$. Note that $\Omega' \subset\subset \Omega$ means that Ω' is a relatively compact set of Ω. Let L denote any operator $-\nabla.(A(x)\nabla.)$ where A is an uniformly elliptic matrix that is it verifies (2.3). As we said the proof of this theorem proceeds in two steps. We first pass from L^2 to L^∞ where the main ingredients in this step are the energy estimate derived from the equation and Sobolev's embedding theorem. Then we pass from L^∞ to C^α using the isoperimetric inequality of De Giorgi.

Remark 2.2 (Scaling property). *If $u(x)$ is a solution of $Lu = 0$ so for any $B \in \mathbb{R}$ and $C > 0$ the function*

$$\bar{u}(y) = Bu(x + Cy) + D$$

verifies $\bar{L}\bar{u} = 0$ for an operator $\bar{L}$ which verifies (2.3) for the same value of Λ. This scaling property will be essential in the proof of the regularity. We will prove the result for $\Omega = B_1$ and $\Omega' = B_{1/2}$ then using the scaling property we can conclude the result for general Ω and Ω'.

2.3.1 First lemma of De Giorgi

We will start with the first step, which is the first lemma of De Giorgi. If the solution is bounded in $L^2(B_1)$ then it is bounded in L^∞ in a smaller ball $B_{1/2}$. We write $v_+ = \max(0, v)$.

Lemma 2.3 (From L^2 to L^∞). *There exists a constant $\delta > 0$ depending only on N and Λ such that for any solution $u : \mathbb{R}^N \to \mathbb{R}$ of (2.4) the following implication holds true:*
If

$$\int_{B_1} u_+^2 \mathrm{d}x \leq \delta$$

then we have

$$u_+ \le 1/2 \textit{ in } B_{1/2}.$$

Note that using the scaling property this lemma implies that $u \in L^\infty_{\mathrm{loc}}(\Omega')$. Before giving the proof of this lemma let us describe how it works. It is based on showing a nonlinear estimate of the form

$$U_k \le C^k U_{k-1}^\beta \quad \text{with } \beta > 1, \tag{2.5}$$

where $U_k = \int_{B_{r_k}} (u - c_k)_+^2 \mathrm{d}x$ for some sequence $c_k \to 1/2$ as $k \to \infty$. In this case we can use the following proposition

Proposition 2.4. *Let $(U_k)_k$ be a sequaence that verifies a nonlinear recurrence of the form*

$$U_k \le C^k U_{k-1}^\beta \quad \textit{with } \beta > 1.$$

If U_0 is small enough then U_k converges to zero when k goes to ∞.

The reader can find the proof of this proposition in [76]. Two main ingredients are used to derive the nonlinear estimate (2.5), the Sobolev inequality

$$\|v\|_{L^p(B_1)} \le c\|\nabla v\|_{L^2(B_1)} \qquad \text{with } p = \frac{2N}{N-2} \tag{2.6}$$

and the energy inequality derived from the equation (2.4)

$$\int_{B_1} |\nabla(\xi u_+)|^2 \mathrm{d}x \le c\|\nabla\xi\|^2_{L^\infty(B_1)} \int_{B_1 \cap \operatorname{supp}\xi} u_+^2 \mathrm{d}x \tag{2.7}$$

for any $\xi \in C_0^\infty(B_1)$. The reader can find the proof of this inequality in Lemma 1.3 in [20] and Lemma 7 in [76]. We will now present the proof of Lemma (2.3).

Proof. For $k \in \mathbb{N}$, let us define

$$r_k = \frac{1}{2}\left(1 + \frac{1}{2^k}\right), \quad c_k = \frac{1}{2}\left(1 - \frac{1}{2^k}\right)$$

$$\text{and } \xi_k(x) = \begin{cases} 1 & \text{if } x \in B_{r_k} \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_{r_{k-1}} \\ \text{a } C^2\text{-function} & \text{if } x \in B_{r_{k-1}} \setminus B_{r_k}. \end{cases}$$

such that $|\nabla\xi_k| \le C2^k$. Note that $\chi_{B_{r_k}} \le \xi_k \le \chi_{B_{r_{k-1}}}$ and $(u - c_k)_+ > 0$ implies $(u - c_{k-1})_+ > 2^{-(k+1)}$. Now we define

$$U_k = \int_{B_{r_k}} (u - c_k)_+^2 \mathrm{d}x.$$

We have

$$U_k \leq \int (u - c_k)_+^2 \xi_k^2 \mathrm{d}x.$$

Using Hölder inequality we can write

$$U_k \leq \left(\int ((u - c_k)_+ \xi_k)^{\frac{2N}{N-2}} \mathrm{d}x \right)^{\frac{N-2}{N}} . |\{(u - c_k)_+ \xi_k > 0\}|^{\frac{2}{N}}.$$

From Sobolev inequality (4.17) we arrive to

$$U_k \leq c \left(\int |\nabla((u - c_k)_+ \xi_k)|^2 \mathrm{d}x \right) . |\{(u - c_k)_+ \xi_k > 0\}|^{\frac{2}{N}}.$$

Finally, we use the energy estimate (2.7) to conclude

$$\begin{aligned} U_k &\leq c 2^{2k} \left(\int_{\operatorname{supp} \xi_k} (u - c_k)_+^2 \mathrm{d}x \right) . |\{(u - c_k)_+ \xi_k > 0\}|^{\frac{2}{N}} \\ &\leq c 2^{2k} \left(\int_{B_{r_{k-1}}} (u - c_{k-1})_+^2 \mathrm{d}x \right) . |\{(u - c_{k-1})_+ > 2^{-(k+1)}\} \cap B_{r_{k-1}}|^{\frac{2}{N}} \\ &\leq c 2^{2k + \frac{4(k+1)}{N}} U_{k-1} . (U_{k-1})^{\frac{2}{N}}. \end{aligned}$$

using Chebyshev's inequality. So for $U_0 = \delta$ small enough by Proposition 2.4 U_k converges to zero when k goes to ∞. Note that

$$U_0 = \int_{B_1} u_+^2 \mathrm{d}x$$

and $U_\infty = 0$ implies that $u_+ \leq \frac{1}{2}$ in $B_{1/2}$. □

2.3.2 Second lemma of De Giorgi

In this step one uses measure information to improve an L^∞ bound. It is the key to pass from L^∞ to C^α.

Lemma 2.5 (Lowering the maximum). *Given $\mu > 0$. There exists a constant $\lambda \in (0,1)$ depending only on μ, Λ and N such that for any solution u of (2.4) we have if $u \leq 1$ in B_2 and*

$$|\{u \leq 0\} \cap B_1| \geq \mu$$

then

$$u \leq 1 - \lambda \text{ in } B_{1/2}.$$

This lemma says that if u is a solution of (2.4) smaller than 1 in B_2 and is far from 1 in a subset of B_1 of a nontrivial measure then u is away from 1 in the whole ball $B_{1/2}$. Note that if we have

$$|\{u \leq 0\} \cap B_1| \geq |B_1| - \delta$$

then, as $u \leq 1$ in B_2,

$$\int_{B_1} u_+^2 \mathrm{d}x \leq |\{u > 0\} \cap B_1| \leq \delta$$

and Lemma 2.3 implies

$$u_+ \leq 1/2 \text{ in } B_{1/2}.$$

So we must bridge the gap between knowing that $|\{u \leq 0\} \cap B_1| \geq \mu$ and knowing that $|\{u \leq 0\} \cap B_1| \geq |B_1| - \delta$. The solution is the following De Giorgi isoperimetric inequality.

Lemma 2.6 (De Giorgi isoperimetric inequality). *Consider w such that $\int_{B_1} |\nabla w_+|^2 \mathrm{d}x \leq C_0$. Set*

$$\begin{aligned} A &= \{w \leq 0\} \cap B_1, \\ C &= \{w \geq 1/2\} \cap B_1, \\ D &= \{0 < w < 1/2\} \cap B_1. \end{aligned}$$

Then we have

$$C_0|D| \geq C_1 \left(|A||C|^{1-\frac{1}{n}} \right)^2.$$

In other words we can say that if w is less than zero in a set of nontrivial measure and w is bigger than $1/2$ also in a set of nontrivial measure then the set where w is between 0 and $1/2$ has a nontrivial measure. The proof of this inequality is given in Lemma 1.4 in [20] and Lemma 10 in [76]. We can now pass to the proof of Lemma 2.5.

Proof. of 2.5. For any $k \in \mathbb{N}$ we consider

$$u_k = 2^k \left(u - (1 - 2^{-k}) \right).$$

Note that since $u \leq 1$ in B_2 then for any $k \in \mathbb{N}, u_k \leq 1$ in B_2. So from the energy inequality (2.7) for $\xi \equiv 1$ in B_1 and since $u_k \leq 1$ in B_2 we have

$$\int_{B_1} |\nabla (u_k)_+|^2 \mathrm{d}x \leq C_0.$$

By construction $|\{u_k \leq 0\} \cap B_1|$ is increasing as k increases, thus greater than μ for any k. Hence we can apply Lemma 2.6 on u_k. As long as

$$\int_{B_1} (u_k)_+^2 \mathrm{d}x \geq \delta$$

we get

$$|\{u_{k-1} \geq 1/2\} \cap B_1| = |\{u_k \geq 0\} \cap B_1| \geq \int_{B_1} (u_k)_+^2 \mathrm{d}x \geq \delta.$$

In that case Lemma 2.6 implies that there exists $\gamma > 0$ depending only on N, δ and μ such that

$$|\{0 < u_{k-1} < 1/2\} \cap B_1| \geq \gamma.$$

Then, recursively,

$$\begin{aligned}|\{u_k \leq 0\} \cap B_1| &= |\{u_{k-1} \leq 0\} \cap B_1| + |\{0 < u_{k-1} < 1/2\} \cap B_1| \\ &\geq |\{u_{k-1} \leq 0\} \cap B_1| + \gamma \\ &\geq \mu + k\gamma.\end{aligned}$$

This cannot be true for all k. So for a k_0 we have

$$\int_{B_1} (u_{k_0})_+^2 \mathrm{d}x \leq \delta$$

and Lemma 2.3 implies that $u_{k_0} \leq 1/2$ in $B_{1/2}$ hence

$$u \leq 1 - \lambda \text{ in } B_{1/2}$$

with $\lambda = 2^{-(k_0+1)}$. Note that $k_0 \leq \frac{|B_1|}{\gamma} \leq \frac{C_N}{\mu^2\delta}$ which depends only on N, Λ and μ. □

2.3.3 From L^∞ to C^α

We can now state the lemma of local decrease of the oscillation of the solution.

Lemma 2.7 (Local decrease of the oscillation)**.** *There exists a constant $\theta \in (0,1)$ depending only on Λ and N such for any solution u of (2.4) we have*

$$\operatorname*{osc}_{B_{1/2}} u \leq \theta \operatorname*{osc}_{B_1} u.$$

Proof. As noted after Lemma 2.3, one knows already that $u \in L^\infty(B_1)$. We define the function

$$v(x) = \frac{2}{\operatorname*{osc}_{B_1} u}\left(u(x) - \frac{\sup u + \inf u}{2}\right).$$

We have $-1 \leq v \leq 1$. Assume that $|\{v \leq 0\} \cap B_1| \geq \frac{|B_1|}{2}$, if not we can work with $-v$. Applying Lemma 2.5 with $\mu = \frac{|B_1|}{2}$ we get $\operatorname*{osc}_{B_{1/2}} v \leq 2 - \lambda$ hence $\operatorname*{osc}_{B_{1/2}} u \leq (1-\lambda/2)\operatorname*{osc}_{B_1} u$. □

To prove Hölder regularity we will use the following proposition

Proposition 2.8. *Let u be a function defined in B_1 such that for any $x_0 \in B_{1/2}$ and any $r \in (0,1/2)$ we have*

$$\operatorname*{osc}_{B_r(x_0)} u \leq Cr^\alpha.$$

Then u is α-Hölder continuous in $B_{1/2}(x_0)$.

Finally we can conclude the Hölder regularity of u solution of (2.4) from Lemma 2.7. Here we use the scaling property for the solutions. Consider any $x_0 \in B_{1/2}$. We introduce the rescaled functions

$$u_n(y) = u(x_0 + \frac{y}{2^n}).$$

Note that u_n are solutions of (2.4) with operators L_n that verifies (2.3) for the same value Λ and $\operatorname{osc}_{B_1} u_n = \operatorname{osc}_{B_{1/2}} u_{n-1}$. We apply Lemma 2.7 recursively on u_n. This gives

$$\operatorname*{osc}_{B_{2^{-n}}(x_0)} u \leq C\theta^n.$$

We conclude that u is in C^α with $\alpha = -\frac{\log\theta}{\log 2}$.

2.4 Local parabolic case

In this section, we will describe how the method of De Giorgi can be applied in the local parabolic case. We consider the following equation

$$\partial_t u - \nabla.(A(t,x)\nabla u) = 0, \quad (t,x) \in (0,T) \times \Omega \tag{2.8}$$

where Ω is a bounded open set in $\mathbb{R}^N$ and A is a uniformly elliptic matrix that is it verifies

$$\frac{1}{\Lambda} I \leq A(t,x) \leq \Lambda I, \qquad (t,x) \in (0,T) \times \Omega.$$

Remark 2.9. *A scaling property holds for the set of solutions of (2.8). If u is a solution of (2.8) then the function u_1 defined by $u_1(t,x) = Bu(C^2t, Cx)$ is also solution of (2.8).*

Using De Giorgi techniques, Hölder reguarity of weak solutions of (2.8) was proved.

Theorem 2.10 (Hölder regularity)**.** *Let $u \in L^\infty(0,T;L^2(\Omega))$ be a weak solution of (2.8) in $(0,T) \times \Omega$ such that $\nabla u \in L^2((0,T) \times \Omega)$.*
Then there exists $\alpha > 0$ such that for any $\Omega' \subset\subset \Omega$ and any $0 < s < T$,

$$u \in C^\alpha((s,T) \times \Omega').$$

As in the elliptic case, the proof is split into two steps. We denote $Q_r = (-r, 0) \times B_r$. We will present the result for $\Omega = Q_2$ and $\Omega' = Q_{1/2}$ and using the scaling property and a convering argument we can conclude the result for Ω and Ω'. We start with the first lemma of De Giorgi, passing from L^2 to L^∞. Then we pass to the second step which is passing from L^∞ to C^α.

2.4.1 First lemma of De Giorgi

We will start with the first step, which is the first lemma of De Giorgi. If the nonnegative solution is bounded in $L^2(Q_1)$ then it is bounded in L^∞ in a smaller cylinder $Q_{1/2}$.

Lemma 2.11 (From L^2 to L^∞). *There exists a constant $\delta > 0$ depending only on N and Λ such that for any solution $u : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ of (2.8) the following implication holds true: If*

$$\int_{Q_1} u_+^2 \mathrm{d}x\mathrm{d}t \le \delta$$

then we have

$$u_+ \le 1/2 \text{ in } Q_{1/2}.$$

Proof. **Step 1:** In this we prove an energy estimate verified by the solution u.
For $k \in \mathbb{N}$, let us define

$$T_k = -\frac{1}{2}\left(1 + \frac{1}{2^k}\right), \quad r_k = \frac{1}{2}\left(1 + \frac{1}{2^k}\right), \quad c_k = \frac{1}{2}\left(1 - \frac{1}{2^k}\right)$$

$$\text{and } \xi_k(x) = \begin{cases} 1 & \text{if } x \in B_{r_k} \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_{r_{k-1}} \\ \text{a } C^2\text{function} & \text{if } x \in B_{r_{k-1}} \setminus B_{r_k}. \end{cases}$$

such that $|\nabla \xi_k| \le C 2^k$. To derive the energy estimate, we multiply the equation (2.8) by $(u - c_k)_+ \xi_k^2$ then we integrate in time and space on $(t_1, t_2) \times \mathbb{R}^N$, we get

$$\begin{aligned} \frac{1}{2}\int (u - c_k)_+^2(t_2, x)\xi_k^2(x)\mathrm{d}x - \int_{t_1}^{t_2}\int (u - c_k)_+ \xi_k^2 \nabla.(A\nabla(u - c_k)_+)\mathrm{d}x\mathrm{d}t \\ = \frac{1}{2}\int (u - c_k)_+^2(t_1, x)\xi_k^2(x)\mathrm{d}x. \end{aligned}$$

We estimate now the second term in the following way

$$\begin{aligned} -\int_{t_1}^{t_2}\int (u - c_k)_+ \xi_k^2 \nabla.(A\nabla(u - c_k)_+)\mathrm{d}x\mathrm{d}t &= \int_{t_1}^{t_2}\int A\nabla(u - c_k)_+ . \nabla(u - c_k)_+ \xi_k^2 \mathrm{d}x\mathrm{d}t \\ &+ \int_{t_1}^{t_2}\int A.\nabla(u - c_k)_+ (u - c_k)_+ \nabla(\xi_k^2)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Using the uniform ellipticity of A, see (2.3), we can write

$$\begin{aligned} &\frac{1}{2}\int (u - c_k)_+^2(t_2, x)\xi_k^2(x)\mathrm{d}x + \frac{1}{\Lambda}\int_{t_1}^{t_2}\int |\nabla(u - c_k)_+|^2 \xi_k^2 \mathrm{d}x\mathrm{d}t \\ &\quad \le \frac{1}{2}\int (u - c_k)_+^2(t_1, x)\xi_k^2(x)\mathrm{d}x - 2\int_{t_1}^{t_2}\int A.\nabla(u - c_k)_+ (u - c_k)_+ \xi_k \nabla(\xi_k)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Now we use Young's inequality to write

$$|\int_{t_1}^{t_2}\int A.\nabla(u-c_k)_+(u-c_k)_+\xi_k\nabla(\xi_k)\mathrm{d}x\mathrm{d}t| \le \epsilon\int_{t_1}^{t_2}\int |A\nabla(u-c_k)_+|^2\xi_k^2\mathrm{d}x\mathrm{d}t$$
$$+ c_\epsilon\int_{t_1}^{t_2}\int (u-c_k)_+^2|\nabla(\xi_k)|^2\mathrm{d}x\mathrm{d}t.$$

Using (2.3) twice for ϵ small enough the first term can be absorbed by the LHS. Since $|\nabla(\xi_k)| \lesssim 2^k$ we arrive at

$$\int (u-c_k)_+^2(t_2,x)\xi_k^2(x)\mathrm{d}x + \frac{1}{\Lambda}\int_{t_1}^{t_2}\int |\nabla(u-c_k)_+|^2\xi_k^2\mathrm{d}x\mathrm{d}t \tag{2.9}$$
$$\le \int (u-c_k)_+^2(t_1,x)\xi_k^2(x)\mathrm{d}x + C2^{2k}\int_{t_1}^{t_2}\int_{\operatorname{supp}\xi_k}(u-c_k)_+^2\mathrm{d}x\mathrm{d}t.$$

Step 2: We are going to prove a nonlinear estimate of the form $U_k \le C^k U_{k-1}$..
Now we consider

$$U_k = \sup_{T_k\le t\le 0}\int_{B_{r_k}}(u-c_k)^2\mathrm{d}x + \frac{1}{\Lambda}\int_{T_k}^0\int_{B_{r_k}}|\nabla(u-c_k)_+|^2\mathrm{d}x\mathrm{d}t.$$

From (2.9) taking arbitrary values $T_{k-1} \le t_1 \le T_k \le t_2 \le 0$ we have

$$U_k \le 2\inf_{T_{k-1}\le t_1\le T_k}\left\{\int_{B_{r_{k-1}}}(u-c_k)_+^2(t_1,x)\mathrm{d}x + C2^{2k}\int_{T_{k-1}}^0\int_{B_{r_{k-1}}}(u-c_k)_+^2\mathrm{d}x\mathrm{d}t\right\}.$$

Taking averages in t_1 we arrive at the inequality

$$\inf_{T_{k-1}\le t_1\le T_k}\int_{B_{r_{k-1}}}(u-c_k)_+^2(t_1,x)\mathrm{d}x \le \frac{1}{T_k-T_{k-1}}\int_{T_{k-1}}^{T_k}\int_{B_{r_{k-1}}}(u-c_k)_+^2\mathrm{d}x\mathrm{d}t_1$$
$$\le 2^k\int_{T_{k-1}}^0\int_{B_{r_{k-1}}}(u-c_k)_+^2\mathrm{d}x\mathrm{d}t_1.$$

So we have the same term to estimate. Since $(u-c_k)_+ > 0$ implies that $(u-c_{k-1})_+ > (u-c_k)_+ + 2^{-k+1}$ we can write

$$U_k \le C^k\int_{T_{k-1}}^0\int_{B_{r_{k-1}}}(u-c_{k-1})_+^2\chi_{\{}(u-c_{k-1})_+ > 2^{-k+1}\}\mathrm{d}x\mathrm{d}t.$$

Let $p > 2$ be the exponent corresponding to Sobolev's embedding theorem so that

$$\int_{B_{r_{k-1}}}(u-c_{k-1})_+^p dx \le C\left(\int_{B_{r_{k-1}}}|\nabla(u-c_{k-1})_+|^2\mathrm{d}x\right)^{\frac{p}{2}}.$$

for some constant C. We consider $\theta = \frac{2}{p}$ and $q = 2(1-\theta)+p\theta$. Then, as $2^{(k-1)(q-2)}(u-c_{k-1})_+^{q-2} > 1$ in the next integral,

$$\int_{B_{r_{k-1}}}(u-c_{k-1})_+^2\chi_{(u-c_{k-1})_+>2^{-k+1}}\mathrm{d}x \le C^k\int_{B_{r_{k-1}}}(u-c_{k-1})_+^q\mathrm{d}x$$
$$\le C^k\left(\int_{B_{r_{k-1}}}(u-c_{k-1})_+^2\mathrm{d}x\right)^{1-\theta}\left(\int_{B_{r_{k-1}}}(u-c_{k-1})_+^p\mathrm{d}x\right)^{\theta}$$
$$\le C^k\left(\int_{B_{r_{k-1}}}(u-c_{k-1})_+^2\mathrm{d}x\right)^{1-\theta}\left(\int_{B_{r_{k-1}}}|\nabla(u-c_{k-1})_+|^2\mathrm{d}x\right).$$

Integrating in time along the interval $[T_{k-1}, 0]$ gives us

$$\begin{aligned} U_k &\le C^k \left(\sup_{T_{k-1} \le t \le 0} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^2 \mathrm{d}x \right)^{1-\theta} \left(\int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} |\nabla (u - c_{k-1})_+|^2 \mathrm{d}x \mathrm{d}t \right) \\ &\le C^k U_{k-1}^{1-\theta} U_{k-1}. \end{aligned}$$

Since $1 - \theta > 0$ we conclude that if U_0 is small enough then $U_\infty = 0$. Finally, Note that

$$U_0 = \int_{Q_1} u_+^2 \mathrm{d}x \mathrm{d}t$$

and $U_\infty = 0$ implies that $u_+ \le 1/2$ in $Q_{1/2}$. □

2.4.2 Second lemma of De Giorgi

To pass to the second step i.e. to pass from L^∞ to C^α we need to prove a lemma about lowering the maximum of the solution which leads to a decrease of the oscillation of the solution. We present the result given in [76]. We write $\bar{Q} = (-3/2, -1) \times B_1$.

Lemma 2.12 (Lowering the maximum). *There exists a constant $\lambda \in (0, 1)$ such that for any solution u of (2.8) in Q_2 we have*
if $-1 \le u \le 1$ in Q_2 and

$$|\{u \le 0\} \cap \bar{Q}| \ge \frac{1}{2} |\bar{Q}|$$

then

$$u \le 1 - \lambda \text{ in } Q_{1/2}.$$

This lemma says that if u is below zero for almost every $(t, x) \in (-3/2, -1) \times B_1$ then u is far from 1 in the whole cylinder $Q_{1/2}$. The proof of this lemma needs a lemma similar to the isoperimetric inequality for the parabolic case called lemma on intermediate values.

Lemma 2.13 (Lemma on intermediate values). *There exists a constant $\gamma > 0$ such that for any solution $u \le 1$ of (2.8) we have*
If

$$|\{u \le 0\} \cap \bar{Q}| \ge \frac{1}{2} |\bar{Q}|$$

and

$$|\{u \ge 1/2\} \cap Q_1| \ge \delta$$

then

$$|\{0 < u < 1/2\} \cap (-3/2, 0) \times B_1| \ge \gamma.$$

The proof of this lemma is also given in [76].

Proof. of Lemma 2.12 For any $k \in \mathbb{N}$ we consider

$$u_k = 2^k \left(u - (1 - 2^{-k}) \right).$$

Note that since $u \leq 1$ in Q_2 then for any $k \in \mathbb{N}, u_k \leq 1$ in Q_2. By construction $|\{u_k \leq 0\} \cap B_1|$ is increasing as k increases, thus greater than $\frac{1}{2}|\bar{Q}|$ for any k. Hence we can apply Lemma 2.6 on u_k. As long as

$$\int_{Q_1} (u_k)_+^2 \mathrm{d}x \geq \delta$$

we get

$$|\{u_{k-1} \geq 1/2\} \cap Q_1| = |\{u_k \geq 0\} \cap Q_1| \geq \int_{Q_1} (u_k)_+^2 \mathrm{d}x \geq \delta.$$

So Lemma 2.6 implies that there exists $\gamma > 0$ depending only on N, δ such that

$$|\{0 < u_{k-1} < 1/2\} \cap Q_1| \geq \gamma.$$

Then

$$\begin{aligned}
|\{u_k \leq 0\} \cap Q_1| &= |\{u_{k-1} \leq 0\} \cap Q_1| + |\{0 < u_{k-1} < 1/2\} \cap Q_1| \\
&\geq |\{u_{k-1} \leq 0\} \cap B_1| + \gamma \\
&\geq \frac{1}{2}|\bar{Q}| + k\gamma.
\end{aligned}$$

This cannot be true for all k. So for a k_0 we have

$$\int_{Q_1} (u_{k_0})_+^2 \mathrm{d}x \leq \delta$$

and Lemma 2.11 implies that $u_{k_0} \leq 1/2$ in $Q_{1/2}$ hence

$$u \leq 1 - \lambda \text{ in } Q_{1/2}$$

with $\lambda = 2^{-(k_0+1)}$. □

Following the same steps as in the elliptic case we can prove Theorem 2.10.

2.5 Nonlocal parabolic case

We will present the result of Caffarelli, Chan and Vasseur in [18] where they prove the Hölder regularity of solutions for nonlocal evolution equations of variational type. We note that they follow the De Giorgi idea and techniques to prove their result.

2.5.1 Presentation of the problem

They study the following equation

$$\partial_t u - \int_{\mathbb{R}^N} (u(y) - u(x))K(t,x,y)\mathrm{d}y = 0 \tag{2.10}$$

where K satisfies the following assumptions

$$K(t,x,y) \text{ is symmetric in } x \text{ and } y,$$
$$\chi_{|x-y|\leq 3}\frac{1}{\Lambda|x-y|^{N+\alpha}} \leq K(t,x,y) \leq \frac{\Lambda}{|x-y|^{N+\alpha}}. \tag{2.11}$$

where χ denotes a characteristic function. The aim is to show that the solutions u of (2.10) with initial data in L^2 become instantaneously bounded and Hölder continuous. So the result proved in [18] is the following theorem.

Theorem 2.14 (Hölder regularity for u)**.** *Let $u \in L^2_{\mathrm{loc}}((0,\infty); H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ be a weak solution of (2.10) with initial data $u_0 \in L^2(\mathbb{R}^N)$. Then for every $t_0 > 0$ we have*

$$u \in C^\beta((t_0,\infty)\times\mathbb{R}^N).$$

The constants β and the norm of u depend only on t_0, N, Λ and $\|u_0\|_{L^2(\mathbb{R}^N)}$.

Note that contrary to the parabolic case, β can now depend on t_0. By weak solution we mean a function $u \in L^2_{\mathrm{loc}}((0,\infty); H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ which verifies

$$\int_{\mathbb{R}^N} \partial_t u(t,x).\eta(x)\mathrm{d}x + B(u(t,.),\eta) = 0, \tag{2.12}$$

for any function $\eta \in C_c^\infty(\mathbb{R}^N)$ where

$$B(u,v) = \iint (u(x)-u(y))(v(x)-v(y))K(t,x,y)\mathrm{d}x\mathrm{d}y.$$

The space $H^{\frac{\alpha}{2}}(\mathbb{R}^N))$ is the space defined by

$$H^{\frac{\alpha}{2}}(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N); \nabla^{\frac{\alpha}{2}} u \in L^2(\mathbb{R}^N)\}$$

supplemented with the norm

$$\|u\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N))} = \|u\|^2_{L^2(\mathbb{R}^N)} + C(\alpha,N)\iint \frac{(u(x)-u(y))^2}{|x-y|^{N+\alpha}}\mathrm{d}x\mathrm{d}y.$$

As in the previous cases the proof of this theorem is split into two steps. First we pass from L^2 to L^∞ where the main ingredients of this step are the energy estimate derived from the equation and Sobolev's embedding theorem. Then we pass from L^∞ to C^β where we need a lemma on intermediate values to prove a lemma on local decrease of the oscillation of the solution.

Remark 2.15 (Scaling property)**.** *If u satisfies (2.10) then $\bar{u}(t,x) = u(A^\alpha t, Ax)$ satisfies (2.10) with a kernel that satisfies (2.11). This scaling property will be essential to prove the regularity of solutions.*

2.5.2 First lemma of De Giorgi

As we said the first step is passing from L^2 to L^∞ which is called the first lemma of De Giorgi. The novelty in this case is that they compare the solution u to a function ψ who vanishes close to zero, is locally bounded and verifies

$$\int_{|x|>1} \frac{\psi(x)}{|x|^{N+\alpha}} \mathrm{d}x < \infty. \tag{2.13}$$

So in this way they keep track of the long range behaviour of the solution via the function ψ. We define

$$\psi(x) = (|x|^{\frac{\alpha}{4}} - 1)_+.$$

Lemma 2.16 (First lemma of De Giorgi). *There exists a constant $\delta_0 \in (0,1)$ depending only on N, α and Λ such that for any weak solution $u : [-2,0] \times \mathbb{R}^N \to \mathbb{R}$ of (2.10) the following implication holds true:*
if

$$\int_{-2}^{0} \int_{\mathbb{R}^N} (u(t,x) - \psi(x))_+^2 \, \mathrm{d}x\mathrm{d}t \le \delta_0,$$

then we have

$$u(t,x) \le \frac{1}{2} + \psi(x) \textit{ for } (t,x) \in (-1,0) \times \mathbb{R}^N,$$

hence

$$u(t,x) \le \frac{1}{2} \textit{ for } (t,x) \in (-1,0) \times B_1.$$

Proof. **Step 1:** In this we prove an energy estimate verified by the solution u.
For $0 \le L \le 1$ we consider $\psi_L = L + \psi$ then we take the test function η to be $(u - \psi_L)_+$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int (u - \psi_L)_+^2 \mathrm{d}x + B(u, (u - \psi_L)_+) = 0.$$

Since $u = (u - \psi_L)_+ - (u - \psi_L)_- + \psi_L$ we write

$$B(u, (u-\psi_L)_+) = B((u-\psi_L)_+, (u-\psi_L)_+) - B((u-\psi_L)_-, (u-\psi_L)_+) + B(\psi_L, (u-\psi_L)_+).$$

Note that since $(u - \psi_L)_+ . (u - \psi_L)_- = 0$ and K is symmetric then

$$B((u-\psi_L)_-, (u-\psi_L)_+) = 2 \iint (u-\psi_L)_+(x)(u-\psi_L)_-(y) K(x,y) \mathrm{d}x\mathrm{d}y \ge 0.$$

So we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int (u-\psi_L)_+^2 \mathrm{d}x + B((u-\psi_L)_+, (u-\psi_L)_+) \le -B(\psi_L, (u-\psi_L)_+). \tag{2.14}$$

We will estimate the remainder term $B(\psi_L,(u-\psi_L)_+)$ to arrive to the energy estimate. We write

$$B(\psi_L,(u-\psi_L)_+)=\frac{1}{2}\iint(\psi_L(x)-\psi_L(y))((u-\psi_L)_+(x)-(u-\psi_L)_+(y))K(t,x,y)\mathrm{d}x\mathrm{d}y$$
$$=\frac{1}{2}\iint(\psi(x)-\psi(y))((u-\psi_L)_+(x)-(u-\psi_L)_+(y))K(t,x,y)\mathrm{d}x\mathrm{d}y.$$

We consider the integral in two regions. In the first region we take $|x-y|<1$ so we have

$$|\psi(x)-\psi(y)|\leq|x-y|.$$

We use the fact that

$$|(u-\psi_L)_+(x)-(u-\psi_L)_+(y)|$$
$$\leq\left\{\chi_{\{(u-\psi_L)_+>0\}}(x)+\chi_{\{(u-\psi_L)_+>0\}}(y)\right\}|(u-\psi_L)_+(x)-(u-\psi_L)_+(y)|,$$

and the symmetry in x and y to write

$$|\int\int_{|x-y|<1}(\psi(x)-\psi(y))((u-\psi_L)_+(x)-(u-\psi_L)_+(y))K(t,x,y)\mathrm{d}x\mathrm{d}y|$$
$$\leq 2\int\int_{|x-y|<1}\chi_{\{(u-\psi_L)_+>0\}}(x)|\psi(x)-\psi(y)||(u-\psi_L)_+(x)-(u-\psi_L)_+(y)|K(t,x,y)\mathrm{d}x\mathrm{d}y.$$

Then using Young's inequality we get

$$2\int\int_{|x-y|<1}\chi_{\{(u-\psi_L)_+>0\}}(x)(\psi(x)-\psi(y))((u-\psi_L)_+(x)-(u-\psi_L)_+(y))K(t,x,y)\mathrm{d}x\mathrm{d}y$$
$$\leq\epsilon\int\int_{|x-y|<1}((u-\psi_L)_+(x)-(u-\psi_L)_+(y))^2K(t,x,y)\mathrm{d}x\mathrm{d}y$$
$$+c_\epsilon\int\int_{|x-y|<1}(\psi(x)-\psi(y))^2K(t,x,y)\chi_{\{(u-\psi_L)_+>0\}}(x)\mathrm{d}x\mathrm{d}y$$
$$\leq\epsilon\iint((u-\psi_L)_+(x)-(u-\psi_L)_+(y))^2K(t,x,y)\mathrm{d}x\mathrm{d}y$$
$$+c_\epsilon\int\int_{|x-y|<1}|x-y|^2\frac{\Lambda}{|x-y|^{N+\alpha}}\chi_{\{(u-\psi_L)_+>0\}}(x)\mathrm{d}x\mathrm{d}y$$
$$\leq\epsilon B((u-\psi_L)_+,(u-\psi_L)_+)+C\int\chi_{\{(u-\psi_L)_+>0\}}(x)\mathrm{d}x$$

since $|x-y|^{-(N+\alpha-2)}$ is integrable in this region. For ϵ small enough the first term can be absorbed by the LHS in (2.14). Now we pass to the second region where $|x-y|\geq1$ and we have

$$|\psi(x)-\psi(y)|\leq C|x-y|^{\frac{\alpha}{4}}.$$

By symmetry we have

$$\begin{aligned}
&|\int\int_{|x-y|\geq 1}(\psi(x)-\psi(y))((u-\psi_L)_+(x)-(u-\psi_L)_+(y))K(t,x,y)\mathrm{d}x\mathrm{d}y|\\
&\leq 2\int\int_{|x-y|\geq 1}|\psi(x)-\psi(y)|(u-\psi_L)_+(x)K(t,x,y)\mathrm{d}x\mathrm{d}y\\
&\leq C\int\int_{|x-y|\geq 1}|x-y|^{\frac{3\alpha}{4}}\frac{\Lambda}{|x-y|^{N+\alpha}}(u-\psi_L)_+(x)\mathrm{d}x\mathrm{d}y\\
&\leq C\int(u-\psi_L)_+(x)\mathrm{d}x
\end{aligned}$$

since $|x-y|^{-(N+\frac{3\alpha}{4})}$ is integrable in this region. After these estimations we conclude that for any $t_1<t_2<0$ we have

$$\begin{aligned}
&\int(u-\psi_L)_+^2(t_2,x)\mathrm{d}x+\int_{t_1}^{t_2}B((u-\psi_L)_+,(u-\psi_L)_+)\mathrm{d}t\\
&\quad\leq\int(u-\psi_L)_+^2(t_1,x)\mathrm{d}x+C\int_{t_1}^{t_2}\int\left((u-\psi_L)_+(t,x)+\chi_{\{(u-\psi_L)_+>0\}}(t,x)\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

where C is a constant depending only on N,Λ and α.
In order to employ the Sobolev embedding's theorem we need to compare $B(v,v)$ to $\|v\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)}$.

$$\begin{aligned}
\|v\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)}&=\iint\frac{(v(x)-v(y))^2}{|x-y|^{N+\alpha}}\mathrm{d}x\mathrm{d}y\\
&=\iint_{|x-y|\leq 3}\frac{(v(x)-v(y))^2}{|x-y|^{N+\alpha}}+\iint_{|x-y|>3}\frac{(v(x)-v(y))^2}{|x-y|^{N+\alpha}}\mathrm{d}x\mathrm{d}y\\
&\leq\Lambda B(v,v)+2\iint_{|x-y|\leq 3}\frac{v(x)^2+v(y)^2}{|x-y|^{N+\alpha}}\mathrm{d}x\mathrm{d}y\\
&\leq\Lambda B(v,v)+C\int v(x)^2\mathrm{d}x.
\end{aligned}$$

Hence for $v=(u-\psi_L)_+$ we can write

$$\begin{aligned}
&\int(u-\psi_L)_+^2(t_2,x)\mathrm{d}x+\frac{1}{\Lambda}\int_{t_1}^{t_2}\|(u-\psi_L)_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)}\mathrm{d}t\\
&\quad\leq\int(u-\psi_L)_+^2(t_1,x)\mathrm{d}x+C\int_{t_1}^{t_2}\int\left((u-\psi_L)_++(u-\psi_L)_+^2+\chi_{\{(u-\psi_L)_+>0\}}\right)\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{2.15}$$

Step 2: We are going to prove a nonlinear estimate of the form $U_k\leq C^kU_{k-1}$.
For $k\in\mathbb{N}$ let us define

$$T_k=-1-\frac{1}{2^k},\qquad L_k=\frac{1}{2}(1-\frac{1}{2^k})$$

$$\text{and } U_k=\sup_{t\in[T_k,0]}\int(u-\psi_{L_k})_+^2(t,x)\mathrm{d}x+\int_{T_k}^0\|(u-\psi_{L_K})_+\|^2_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R}^N)}\mathrm{d}t.$$

Using (2.15) for $T_{k-1} \le t_1 \le T_k < t_2 < 0$ we can write

$$U_k \le 2 \inf_{T_{k-1}\le t_1 \le T_k} \int (u-\psi_{L_k})_+^2(t_1,x)\mathrm{d}x + C\int_{T_{k-1}}^0 \int \Big((u-\psi_{L_k})_+ + (u-\psi_{L_k})_+^2 + \chi_{\{(u-\psi_{L_k})_+>0\}}\Big)\,\mathrm{d}x\mathrm{d}t.$$

Taking averages in t_1 we arrive at the inequality

$$\begin{aligned}\inf_{T_{k-1}\le t_1\le T_k}\int (u-\psi_{L_k})_+^2(t_1,x)\mathrm{d}x &\le \frac{1}{T_k - T_{k-1}}\int_{T_{k-1}}^{T_k}\int (u-\psi_{L_k})_+^2(t_1,x)\mathrm{d}x\mathrm{d}t_1\\ &\le 2^k \int_{T_{k-1}}^0 \int (u-\psi_{L_k})_+^2(t_1,x)\mathrm{d}x\mathrm{d}t_1.\end{aligned}$$

Since $(u-\psi_{L_k})_+ > 0$ implies $(u-\psi_{L_{k-1}})_+ > 2^{-(k+1)}$ we can write

$$U_k \le \int_{T_{k-1}}^0 \int \Big((2^k+1)(u-\psi_{L_{k-1}})_+^2 + (u-\psi_{L_{k-1}})_+ + \chi_{\{(u-\psi_{L_{k-1}})_+>2^{-(k+1)}\}}\Big)\,\mathrm{d}x\mathrm{d}t. \tag{2.16}$$

Let $p>2$ be the exponent corresponding to Sobolev's embedding theorem so that

$$\int (u-\psi_{L_{k-1}})_+^p dx \le C(\|(u-\psi_{L_K})_+\|^2_{H^{\frac{\alpha}{2}}(\mathbb{R}^N)})^{\frac{p}{2}}.$$

Now using $(u-\psi_{L_k})_+ \le (u-\psi_{L_{k-1}})_+$ we write

$$\begin{aligned}\int_{T_{k-1}}^0\int (u-\psi_{L_k})_+\mathrm{d}x\mathrm{d}t &\le \int_{T_{k-1}}^0\int (u-\psi_{L_{k-1}})_+\chi_{\{(u-\psi_{L_{k-1}})_+>2^{-(k+1)}\}}\mathrm{d}x\mathrm{d}t\\ &\le (2^{k+1})^{p-1}\int_{T_{k-1}}^0\int (u-\psi_{L_{k-1}})_+^p\mathrm{d}x\mathrm{d}t\\ &\le C^k U_k^{\frac{p}{2}};\\ \int_{T_{k-1}}^0\int \chi_{\{(u-\psi_{L_k})_+>0\}}\mathrm{d}x\mathrm{d}t &\le (2^{k+1})^p\int_{T_{k-1}}^0\int (u-\psi_{L_{k-1}})_+^p\mathrm{d}x\mathrm{d}t\\ &\le C^k U_k^{\frac{p}{2}};\\ \int_{T_{k-1}}^0\int (u-\psi_{L_k})_+^2\mathrm{d}x\mathrm{d}t &\le \int_{T_{k-1}}^0\int (u-\psi_{L_{k-1}})_+^2\chi_{\{(u-\psi_{L_{k-1}})_+>2^{-(k+1)}\}}\mathrm{d}x\mathrm{d}t\\ &\le (2^{k+1})^{p-2}\int_{T_{k-1}}^0\int (u-\psi_{L_{k-1}})_+^p\mathrm{d}x\mathrm{d}t\\ &\le C^k U_k^{\frac{p}{2}}.\end{aligned}$$

The above three inequalities, together with (2.16), give

$$U_k \le C^k U_k^{\frac{p}{2}}.$$

Since $\frac{p}{2} > 1$ we conclude that if U_0 is small enough then $U_\infty = 0$. Finally, Note that

$$U_0 \le \int_{-2}^0\int_{\mathbb{R}^N} (u(t,x)-\psi(x))_+^2\,\mathrm{d}x\mathrm{d}t$$

and $U_\infty = 0$ implies that $u \le \frac{1}{2} + \psi$. □

We will give now a corollary that presents the result in another way: If the solution u is below zero (far from 1) for almost every $(t,x) \in Q_2$ then u cannot be very close to 1 in Q_1.

Corollary 2.17. *There exists a constant $\delta \in (0,1)$ depending only on N, α and Λ such that for any weak solution $u : [-2,0] \times \mathbb{R}^N \to \mathbb{R}$ of (2.10) the following implication holds true If i) u is bounded from above in the following way*

$$u(t,x) \leq 1 + \psi(x) \quad \text{ in } [-2,0] \times \mathbb{R}^N,$$

ii) u is mostly below 0 in $[-2,0] \times B_2$

$$|\{u > 0\} \cap [-2,0] \times B_2| \leq \delta,$$

then we have

$$u \leq \frac{1}{2} \text{ in } [-1,0] \times B_1.$$

The reader can find the proof in Corollary 3.3 in [18].

2.5.3 Second lemma of De Giorgi

It remains to pass from L^∞ to C^β so we need a lemma of local decrease of the oscillation of a weak solution of (2.10). But to do this we need an intermediate lemma as in the previous cases. It is the lemma on intermediate values. In [18] they introduce a new version of the lemma on intermediate values. They do not compare the solution u to constants but to cutoff functions defined in the following way. Let

$$F(x) = \sup(-1, \inf(0, |x|^2 - 9)).$$

Note that F is Lipschitz, compactly supported in B_3 and equal to -1 in B_2. Then for $0 < \lambda < 1/3$ we introduce

$$\Psi_\lambda(x) = \begin{cases} [(|x| - \frac{1}{\lambda^{\frac{4}{\alpha}}})^{\frac{\alpha}{4}} - 1]_+ & \text{if } |x| \geq \frac{1}{\lambda^{\frac{4}{\alpha}}} \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the consecutive cutoffs

$$\begin{aligned} \varphi_0 &= 1 + \Psi_\lambda + F, \\ \varphi_1 &= 1 + \Psi_\lambda + \lambda F, \\ \varphi_2 &= 1 + \Psi_\lambda + \lambda^2 F. \end{aligned}$$

Note that $\varphi_0 \leq \varphi_1 \leq \varphi_2$ and $\varphi_0 \equiv 0$ in B_1.

Lemma 2.18 (Lemma on intermediate values)**.** *Let δ be the constant defined in Corollary 2.17. Then, there exists $\mu > 0, \gamma > 0$ and $\lambda \in (0,1)$, depending only on N, α and Λ, such that for any weak solution $u : [-3,0] \times \mathbb{R}^N \to \mathbb{R}$ of (2.15) satisfying*

$$u \le 1 + \Psi_\lambda \quad \text{in } [-3,0] \times \mathbb{R}^N,$$

if

$$|\{u < \varphi_0\} \cap ([-3,-2] \times B_1)| \ge \mu,$$

and

$$|\{u > \varphi_2\} \cap ([-2,0] \times \mathbb{R}^N)| \ge \delta$$

then

$$|\{\varphi_0 < u < \varphi_2\} \cap ([-3,0] \times \mathbb{R}^N)| \ge \gamma.$$

The proof is given in Lemma 4.1 in [18] (They use the function $(u - \varphi_1)_+$ to get the result). We can now pass to the lemma of local decrease of the oscillation. First for λ given in Lemma 2.18 and for any $\epsilon > 0$, we define

$$\Psi_{\epsilon,\lambda}(x) = \begin{cases} [(|x| - \frac{1}{\lambda^{\frac{4}{\alpha}}})^\epsilon - 1]_+ & \text{if } |x| \ge \frac{1}{\lambda^{\frac{4}{\alpha}}} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.19. *There exists $\epsilon > 0$ and $\lambda^* > 0$ such that for any solution u of (2.10) in $[-3,0] \times \mathbb{R}^N$ such that*

$$-1 - \Psi_{\epsilon,\lambda} \le u \le 1 + \Psi_{\epsilon,\lambda}$$

for λ given in Lemma (2.18), we have

$$\sup_{[-1,0]\times B_1} u - \inf_{[-1,0]\times B_1} u \le 2 - \lambda^*. \tag{2.17}$$

Proof. We may assume that

$$|\{u < \varphi_0\} \cap ([-3,-2] \times B_1)| > \mu.$$

Otherwise this is verified by $-u$ ($\varphi_0 = 0$ on B_1).
Consider $k_0 = \frac{|(-3,0)\times B_3|}{\gamma}$ for γ given in 2.18 and fix ϵ small enough such that

$$\frac{(|x|^\epsilon - 1)_+}{\lambda^{2k_0}} \le (|x|^{\frac{s}{4}} - 1)_+ \quad \text{for all } x,$$

for λ given in 2.18.For $k \le k_0$, we consider the sequence

$$u_{k+1} = \frac{1}{\lambda^2}(u_k - (1 - \lambda^2)), \quad u_0 = u.$$

Then $u_k = \frac{1}{\lambda^{2k}} u + 1 - \frac{1}{\lambda^{2k}}$ and

$$u_k(t,x) \le 1 + \frac{1}{\lambda^{2k}} \Psi_{\epsilon,\lambda}(x) \le 1 + \Psi_\lambda \quad \text{for } k \le k_0.$$

By construction $|\{u_k < \varphi_0\} \cap ([-3,-2] \times B_1)|$ is increasing as k increases so it is greater than μ for any k. Hence as long as

$$|\{u_k > \varphi_2\} \cap ([-2,0] \times \mathbb{R}^N)| \ge \delta,$$

we have, by recursion,

$$|\{u_{k+1} > \varphi_2\}| \le |\{u_{k+1} > \varphi_0\}| - \gamma \le |\{u_k > \varphi_2\}| - \gamma \le |(-3,0) \times B_3| - k\gamma.$$

This cannot be true up to k_0. So there exists $k \le k_0$ such that

$$|\{u_k > \varphi_2\} \cap ((-2,0) \times \mathbb{R}^N)| \le \delta.$$

So we can apply Corollary 2.17 to u_{k+1}, indeed

$$u_{k+1} \le 1 + \Psi_\lambda \le 1 + \psi \quad \text{on } (-3,0) \times \mathbb{R}^N$$

and

$$\begin{aligned} |\{u_{k+1} > 0\} \cap ((-2,0) \times B_2)| &\le |\{u_{k+1} > \varphi_0\} \cap ((-2,0) \times B_2)| \\ &\le |\{u_k > \varphi_2\} \cap ((-2,0) \times \mathbb{R}^N)| \\ &\le \delta. \end{aligned}$$

Hence from Corollary 2.17, we have

$$u_{k+1} \le \frac{1}{2} \quad \text{on } (-1,0) \times B_1.$$

Then $u \le 1 - \frac{\lambda^{2(k+1)}}{2}$ on $(-1,0) \times B_1$. Since $-1 \le u \le 1$ in $(-1,0) \times B_1$ we get (2.17) for $\lambda^* = \frac{\lambda^{2(k+1)}}{2}$. □

Now we can conclude the Hölder regularity of the solution by considering the corresponding rescaled functions using the scaling property.

Proof. of Theorem (2.14). We will prove Hölder regularity of u in $(0,0)$. We consider $A = \frac{1}{1-(\lambda^*/2)}$ for λ^* given in Lemma 2.19 and$K < 1$ such that

$$A \Psi_{\lambda,\epsilon}(Kx) \le \Psi_{\lambda,\epsilon}(x),$$

for λ given in Lemma 2.18 and ϵ given in Lemma 2.19. The coefficient K depends only on λ, λ^* and ϵ. Then we define by induction

$$u_{k+1}(t,x) = A u_k(K^\alpha t, Kx)$$

for $(t,x) \in (-3,0) \times \mathbb{R}^N$. By construction, u_k satisfies the hypothesis of Lemma 2.19 for any k. Hence

$$\underset{(-K^{k\alpha},0)\times B_{K^k}}{\operatorname{osc}} u \leq C(1-\lambda^*)^k.$$

So using Proposition 2.8, u is C^β with $\beta = \frac{\log(1-\lambda^*)}{\log(K^\alpha)}$. □

Chapter 3

Study of a family of higher order nonlocal degenerate parabolic equations: from the porous medium equation to the thin film equation

Abstract

In this chapter, we study a nonlocal degenerate parabolic equation of order $\alpha + 2$ for $\alpha \in (0, 2)$. The equation is a generalization of the one arising in the modeling of hydraulic fractures studied by Imbert and Mellet in 2011. Using the same approach, we prove the existence of solutions for this equation for $0 < \alpha < 2$ and for nonnegative initial data satisfying appropriate assumptions. The main difference is the compactness results due to different Sobolev embeddings. Furthermore, for $\alpha > 1$, we construct a nonnegative solution for nonnegative initial data under weaker assumptions.

3.1 Introduction

In this chapter, we study the following problem

$$\begin{cases} \partial_t u + \partial_x(u^n \partial_x I(u)) = 0 & \text{for } x \in \Omega, \quad t > 0, \\ \partial_x u = 0, u^n \partial_x I(u) = 0 & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded interval in $\mathbb{R}$, n is a positive real number and I is a nonlocal elliptic negative operator of order α defined as the $\alpha/2$ power of the Laplace operator with Neumann boundary conditions $I = -(-\Delta)^{\frac{\alpha}{2}}$ where $\alpha \in (0,2)$; this operator will be defined below by using the spectral decomposition of the Laplacian.

The case $\alpha = 1$ was studied by Imbert and Mellet [54] who proved the existence of nonnegative solutions for nonnegative initial data with appropriate conditions. In this case, when $n = 3$ the equation designs the physical KGD model developed by Geertsma and de Klerk [39] and Khristianovich and Zheltov [81]. It represents the influence of the pressure exerted by a viscous fluid on a fracture in an elastic medium subject only to plane strain. This equation is derived from the conservation of mass for the fluid inside the fracture, the Poiseuille law and an appropriate pressure law (see [54, section 3] and [51] for further details). In [54], weak solutions are constructed by passing to the limit in a regularized problem. The necessary compactness estimates are obtained from appropriate energy estimates.

The equation under consideration

$$u_t + \partial_x(u^n \partial_x I(u)) = 0 \tag{3.2}$$

is a nonlocal degenerate parabolic equation of order $\alpha + 2$.

When $\alpha = 2$, this equation coincides with the thin film equation (TFE for short)

$$u_t + \partial_x(u^n \partial^3_{xxx} u) = 0. \tag{3.3}$$

This is a fourth order nonlinear degenerate parabolic equation originally studied by Bernis and Friedman [10]. This equation arises in many applications like spreading of a liquid film over a solid surface ($n = 3$) and Hele-Shaw flows ($n = 1$) (see [3, 13, 14, 43, 44, 49, 58]). TFE is derived also from a conservation of mass, the Poiseuille law (derived from a lubrication approximation of the Navier-Stokes equations for thin film viscous flows) and various pressure laws. The parameter $n \in (0,3]$ models various boundary conditions at the liquid-solid interface. The case $n > 3$ is mainly of mathematical interest [50]. In [10] weak solutions u are exhibited in a bounded interval under appropriate boundary conditions. In addition, they proved that u is nonnegative if u_0 is also so, and that the support of the solution $u(t,.)$ increases with t if u_0 is nonnegative and $n \geq 4$.

For $\alpha = 0$, the porous medium equation (PME for short) is recovered

$$u_t - \partial_x(u^n \partial_x u) = 0. \tag{3.4}$$

This is a nonlinear degenerate parabolic equation. The simple PME model describes the modeling of the motion of a gas flow through a porous medium [77]. In this case, the PME is derived from mass balance, Darcy's law which describes the dynamics of flows through porous media, and a state equation for the pressure [77]. PME also arises in heat transfer [68] and groundwater flow [71] and was originally proposed by Boussinesq. It took many years to prove that PME is well posed and the famous source type solutions were found by Zel'dovich, Kompanyeets and Barenblatt [77]. The questions of existence, uniqueness, stability, smoothness of solutions together with dynamical properties and asymptotic behavior are well represented in [77] where two main problems are studied. First, the domain space is $\mathbb{R}^d$ and the initial condition u_0 has a compact support so the solution $u(t,x)$ vanishes for all positive times $t > 0$ outside a compact set that changes with time. Secondly, if the initial data has a hole in the support then the solution has a possibly smaller hole for $t > 0$.

Note that TFE can be seen as a fourth order version of the classical PME [50]. Furthermore, both equations are parabolic in divergence form. In both cases, there are compactly supported source type solutions ($n > 1$ for PME [77] and $0 < n < 3$ for TFE [8]) [26]. The most famous common properties are finite speed of propagation and the waiting time phenomenon. Similar properties are expected in our case. Self-similar solutions are constructed in [51] but other properties are still not proved. One striking difference between TFE and PME is the lack of a maximum principle for TFE [26].

The case $\alpha \in (-2, 0)$ corresponds to the fractional porous medium equation studied in [15]. Explicit self-similar solutions are exhibited and, under appropriate conditions, weak solutions are constructed.

In this chapter, we will generalize the result of [54] to the cases $0 < \alpha \leqslant 1$ and $1 < \alpha < 2$. We prove a result of existence with the same approach as that in the case $\alpha = 1$ but by modifying the compactness results. Consequently all cases $\alpha \in [0, 2]$ are now covered.

In the case $\alpha > 1$ we get the local uniform convergence of approximate solutions due to the following embedding in dimension 1

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow C^{0,\frac{\alpha-1}{2}}(\Omega).$$

This convergence allows one to pass to the limit in the nonlinear term and then allows us to construct nonnegative solutions for nonnegative initial data merely in $H^{\frac{\alpha}{2}}(\Omega)$.

In the case $\alpha < 1$ because of the following embedding

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow L^p(\Omega) \text{ for all } p < \frac{2}{1-\alpha},$$

we can get a compactness result in $L^p(\Omega)$ only for $p < \frac{2}{1-\alpha}$ and not for all $p < \infty$ as in the case $\alpha = 1$. Neverthless, we recover a compactness result for the term $I(u)$ which allows us to pass to the limit and conclude.

In both cases, we prove that the solution is strictly positive under a condition on n.

Integral inequalities

Assume that u is a solution of (3.2) then it satisfies the energy inequality

$$-\int_{\mathbb{R}} u(t)I(u(t))dx + 2\int_0^T\int_{\mathbb{R}} u^n \partial_x I(u)^2 dxdt \leqslant -\int_{\mathbb{R}} u_0 I(u_0)dx.$$

Observe that $-\int uI(u)$ is the homogeneous $H^{\frac{\alpha}{2}}$ norm. Let G be a nonnegative function such that $G''(s) = \frac{1}{s^n}$. Then the positive solution satisfies

$$\int_{\mathbb{R}} G(u(t))dx - \int_0^T\int_{\mathbb{R}} \partial_x u \partial_x I(u)dxdt \leqslant \int_{\mathbb{R}} G(u_0)dx.$$

Note that $-\int \partial_x u\partial_x I(u)$ is the homogeneous $H_N^{\frac{\alpha}{2}+1}$ norm (it is in fact a Neumann-Sobolev space, see below). We see that the energy inequality controls the $L^\infty(0,T;H^{\frac{\alpha}{2}}(\mathbb{R}))$ norm of the solution. For the function G mentioned above, we can take

$$G(s) = \int_1^s\int_1^r \frac{1}{t^n}dtdr \tag{3.5}$$

so that G is a nonnegative convex function satisfying $G(1) = G'(1) = 0$, $G(s) = \infty$ for all $s < 0$ and for $s > 0$, we have

$$G(s) = \begin{cases} s\ln s - s + 1 & \text{when } n = 1 \\ -\frac{s^{2-n}}{(2-n)(n-1)} + \frac{s}{n-1} + \frac{1}{2-n} & \text{when } 1 < n < 2 \\ \ln\frac{1}{s} + s - 1 & \text{when } n = 2 \\ \frac{1}{(n-2)(n-1)}\frac{1}{s^{n-2}} + \frac{s}{n-1} - \frac{1}{n-2} & \text{when } n > 2. \end{cases}$$

Main results

In this work, we prove three main results. We first prove the existence of nonnegative weak solutions for the problem with $0 < \alpha \leq 1$ for nonnegative initial data with apropriate conditions. Secondly, for $\alpha > 1$, we construct nonnegative solutions for nonnegative initial data in $H^{\frac{\alpha}{2}}(\Omega)$. Finally, we prove the strict positivity of solutions for large $n's$.

Theorem 3.1 (Existence of solutions for $0 < \alpha \leqslant 1$). *Let $n \geqslant 1$ and $\alpha \in (0,1]$. For any nonnegative initial condition $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$ such that*

$$\int_\Omega G(u_0)dx < \infty \tag{3.6}$$

where G is the nonnegative function (3.5) such that $G''(s) = \frac{1}{s^n}$, there exists a nonnegative function

$$u \in L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega)) \cap L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$$

which satisfies on $Q = (0,T) \times \Omega$

$$\iint_Q u\partial_t\varphi dtdx - \iint_Q nu^{n-1}\partial_x uI(u)\partial_x\varphi dxdt - \iint_Q u^n I(u)\partial^2_{xx}\varphi dxdt = -\int_\Omega u_0\varphi(0,.)dx \quad (3.7)$$

for all $\varphi \in \mathcal{D}([0,T) \times \bar{\Omega})$ satisfying $\partial_x\varphi = 0$ on $(0,T) \times \partial\Omega$.

Furthermore u satisfies for almost every $t \in (0,T)$

$$\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx \quad (3.8)$$

and

$$\|u(t,.)\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + 2\int_0^T \int_\Omega g^2 dxds \leq \|u_0\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} \quad (3.9)$$

where the function $g \in L^2(Q)$ satisfies $g = \partial_x(u^{\frac{n}{2}}I(u)) - \frac{n}{2}u^{\frac{n-2}{2}}\partial_x uI(u)$ in $\mathcal{D}'(\Omega)$, and

$$\int_\Omega G(u(t,x))dx + \int_0^t \|u\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)} ds \leq \int_\Omega G(u_0)dx. \quad (3.10)$$

Remark 3.2. *The weak formulation (3.7) comes after two integrations by parts of the equation (3.2). We recall that the function $G : \mathbb{R}_+ \to \mathbb{R}_+$ is given by (3.5). Note that the space $H_N^s(\Omega)$ is defined via spectral decomposition of $-(-\Delta)^{\frac{s}{2}}$ (see below).*

Theorem 3.3 (Existence of solutions for $1 < \alpha < 2$)**.** *Let $n \geqslant 1$ and $\alpha > 1$. For any nonnegative initial condition $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$, there exists a nonnegative function*

$$u \in C_{t,x}^{\frac{\alpha-1}{2(\alpha+2)},\frac{\alpha-1}{2}}(Q)$$

such that

$$\partial_x I(u) \in L^2_{loc}(Q_+) \quad (3.11)$$

and that satisfies

$$\iint_Q u\partial_t\varphi dtdx + \iint_{Q_+} u^n\partial_x I(u)\partial_x\varphi dxdt = -\int_\Omega u_0\varphi(0,.)dx \quad (3.12)$$

where $Q_+ = \{u > 0\} \cap Q$, for all $\varphi \in \mathcal{D}([0,T) \times \bar{\Omega})$ satisfying $\partial_x\varphi = 0$ on $(0,T) \times \partial\Omega$.

Furthermore, u satisfies for almost every $t \in (0,T)$

$$\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx \quad (3.13)$$

and

$$\|u(t,.)\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + 2\iint_{Q_+} u^n(\partial_x I(u))^2 dxds \leq \|u_0\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} \quad (3.14)$$

Note that in this case $\partial_x I(u) \in L^2_{loc}(Q_+)$ so that we do not need to do an integration by parts to pass to the limit as in (3.7)

Theorem 3.4 (Strictly positive solutions)**.** *Assume $0 < \alpha < 2$ and $n > max\{3, 2 + \frac{2}{\alpha+1}\}$. There exists a set $P \subset (0,T)$ such that $|(0,T) \setminus P| = 0$ and the solution u constructed as in Theorem 3.1 for $0 < \alpha \leq 1$ and as in the first step of the prrof of Theorem 3.3 for $1 < \alpha < 2$ satisfies $u(t,.) \in C^{0,\beta}(\Omega)$ for all $t \in P$ and for all $\beta < min\{1, \frac{\alpha+1}{2}\}$ and $u(t,.)$ is strictly positive in Ω. Furthermore, u is a solution of*

$$u_t + \partial_x J = 0 \qquad \text{in } \mathcal{D}'(\Omega)$$

where

$$J(t,.) = u^n \partial_x I(u) \in L^1(\Omega) \qquad \text{for all } t \in P.$$

Organization of the paper

The paper is organized as follows: in Section 2, we define the nonlocal operator I by using the spectral decomposition of the Laplacian and we write an integral representation for it. Then we prove two important Propositions used in the proofs. In Section 3, we study a regularized problem before proving our Theorems in Section 4.

Notation

In this work, we denote $\Omega = (0,1)$ and $Q = (0,T) \times \Omega$. The space $H^s_N(\Omega)$ is the functional space defined in [54, Section 3.1] by

$$H^s_N(\Omega) = \left\{ u = \sum_{k=0}^{\infty} c_k \varphi_k ; \sum_{k=0}^{\infty} c_k^2 (1 + \lambda_k^s) < +\infty \right\}$$

where $\{\lambda_k, \varphi_k\}_{k \geq 0}$ are the eigenvalues and corresponding eigenvectors of the Laplacian operator in Ω with Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k \text{ in } \Omega, \\ \partial_\nu \varphi_k = 0 \text{ on } \partial\Omega, \\ \int_\Omega \varphi_k^2 dx = 1. \end{cases}$$

with the norm

$$\|u\|^2_{H^s_N(\Omega)} = \sum_{k=0}^{\infty} c_k^2 (1 + \lambda_k^s), \tag{3.15}$$

equivalently to

$$\|u\|^2_{H^s_N(\Omega)} = \left(\int_\Omega u dx \right)^2 + \|u\|^2_{\dot{H}^s_N(\Omega)} \tag{3.16}$$

where the homogeneous norm is given by

$$\|u\|^2_{\dot{H}^s_N(\Omega)} = \sum_{k=0}^{\infty} c_k^2 \lambda_k^s. \tag{3.17}$$

Note that $H^s_N(\Omega) = H^s(\Omega)$ for all $0 \leqslant s < \frac{3}{2}$ (see [1]) with equivalent norms. Indeed,

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|u\|^2_{\dot{H}^s(\Omega)}$$

and since we are in dimension 1 we have for these values of s

$$\|u\|_{\dot{H}^s_N(\Omega)} = \|u\|_{\dot{H}^s(\Omega)}$$

Note also that we have

$$\int_\Omega u dx \leqslant C(\Omega)\|u\|_2 \quad \text{(Hölder inequality)},$$
$$\|u\|_2^2 \leqslant C(\Omega)\|(-\Delta)^{\frac{s}{2}} u\|_2^2 \leqslant c\|u\|^2_{\dot{H}^s_N(\Omega)} \text{(fractional Poincaré's inequality)}.$$

Finally, as usual $s_+ = \max\{0, s\}$.

3.2 Preliminaries

3.2.1 Operator I

Spectral definition. We define the operator I by

$$I : \sum_{k=0}^{\infty} c_k \varphi_k \longrightarrow - \sum_{k=0}^{\infty} c_k \lambda_k^{\frac{\alpha}{2}} \varphi_k \qquad \text{which maps } H^\alpha_N(\Omega) \text{ onto } L^2(\Omega)$$

where $\{\lambda_k, \varphi_k\}_{k\geq 0}$ are the eigenvalues and corresponding eigenvectors of the Laplacian operator in Ω with Neumann boundary conditions on $\partial\Omega$.

Integral representation. The operator I can also be represented as a singular integral operator. We will prove the following.

Proposition 3.5. *Consider a smooth function $u : \Omega \to \mathbb{R}$. Then for all $x \in \Omega$,*

$$I(u)(x) = \int_\Omega (u(y) - u(x))K(x,y)dy$$

where $K(x,y)$ is defined as follows. For all $x, y \in \Omega$

$$K(x,y) = c_\alpha \sum_{k\in\mathbb{Z}} \left(\frac{1}{|\, x - y - 2k\,|^{1+\alpha}} + \frac{1}{|\, x + y - 2k\,|^{1+\alpha}} \right)$$

where c_α is a constant depending only on α.

Proof. Let's replace Ω by $(-1,1)$ and u by its even extension to $(-1,1)$. Then let's extend u periodically to $\mathbb{R}$ and let $\bar{u}$ be this extension. For $x \in \Omega$,

$$\begin{aligned}
I(u)(x) = -(-\Delta)^{\frac{\alpha}{2}}\bar{u}(x) &= c_\alpha \int_{\mathbb{R}} (\bar{u}(y) - \bar{u}(x)) \frac{dy}{\mid y - x \mid^{1+\alpha}} \\
&= c_\alpha \sum_{k \in \mathbb{Z}} \int_{-1+2k}^{1+2k} (\bar{u}(y) - u(x)) \frac{dy}{\mid y - x \mid^{1+\alpha}} \\
&= c_\alpha \int_{-1}^{1} (\bar{u}(y) - u(x)) \left(\sum_{k \in \mathbb{Z}} \frac{1}{\mid y + 2k - x \mid^{1+\alpha}} \right) dy && \text{because } \bar{u} \text{ is 2-periodic} \\
&= c_\alpha \int_{0}^{1} (u(y) - u(x)) \sum_{k \in \mathbb{Z}} \left(\frac{1}{\mid x - y - 2k \mid^{1+\alpha}} + \frac{1}{\mid x + y - 2k \mid^{1+\alpha}} \right) && \text{because } \bar{u} \text{ is even.}
\end{aligned}$$

□

Now we can easily conclude the following Corollary.

Corollary 3.6. *Consider two smooth functions $u, \varphi : \Omega \to \mathbb{R}$. Then*

$$\int_{\Omega} I(u)(x)\varphi(x)dx = \int_{\Omega} u(x)I(\varphi)(x)dx \tag{3.18}$$

3.2.2 Important identities

As [54, Section 3], the semi-norms $\|.\|_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}$, $\|.\|_{\dot{H}^{\alpha}_N(\Omega)}$, $\|.\|_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)}$ and $\|.\|_{\dot{H}_N^{\alpha+1}(\Omega)}$ are related to the operator I by important and very useful equalities.

Proposition 3.7. *1. For all $u \in H^{\frac{\alpha}{2}}(\Omega)$, we have $-\langle I(u), u\rangle = \|u\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}$.*

2. For all $u \in H^{\alpha}_N(\Omega)$, we have $\|u\|^2_{\dot{H}^{\alpha}_N(\Omega)} = \int_\Omega I(u)^2 dx$.

3. For all $u \in H_N^{\frac{\alpha}{2}+1}(\Omega)$, we have $\|u\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)} = -\int_\Omega I(u)_x u_x dx$.

4. For all $u \in H_N^{\alpha+1}(\Omega)$, we have $\|u\|^2_{\dot{H}_N^{\alpha+1}(\Omega)} = \int_\Omega I(u)_x^2 dx$.

Proof. Note that if $u \in H^{\frac{\alpha}{2}}(\Omega)$ then $I(u) \in H^{-\frac{\alpha}{2}}(\Omega)$ and

$$\langle I(u), v\rangle_{H^{-\frac{\alpha}{2}}(\Omega), H^{\frac{\alpha}{2}}(\Omega)} = -\sum_{k=0}^{\infty} c_k \lambda_k^{\frac{\alpha}{2}} d_k$$

where $v = \sum_{k=0}^{\infty} d_k \varphi_k \in H^{\frac{\alpha}{2}}(\Omega)$ and $u = \sum_{k=0}^{\infty} c_k \varphi_k$, so

$$-\int uI(u) = \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{\alpha}{2}} = \|u\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}.$$

The second equality is actually very easy to prove since $I(u) = -\sum_{k=0}^{\infty} c_k \varphi_k^{\frac{\alpha}{2}}$. Indeed,

$$\int_\Omega I(u)^2 dx = \sum_{k=0}^{\infty} c_k^2 \varphi_k^{\alpha} = \|u\|^2_{\dot{H}^{\alpha}_N(\Omega)}.$$

In order to prove the other equalities, we note that $(\partial_x \varphi_k)_k$ form an orthogonal basis of $L^2(\Omega)$. We write

$$u_x = \sum_{k=0}^{\infty} c_k \partial_x \varphi_k \text{ in } L^2(\Omega).$$
$$\text{and } \partial_x I(u) = -\sum_{k=1}^{\infty} c_k \lambda_k^{\frac{\alpha}{2}} \partial_x \varphi_k \text{ in } L^2(\Omega)$$

so

$$\begin{aligned} -\int_\Omega I(u)_x u_x dx &= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{\alpha}{2}} \int_\Omega \partial_x \varphi_k^2 dx = \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{\alpha}{2}} \int_\Omega \varphi_k(-\partial_{xx}\varphi_k) dx \\ &= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{\alpha}{2}} \int_\Omega \lambda_k \varphi_k^2 dx = \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{\alpha}{2}+1} = \|u\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)}. \end{aligned}$$

For the last equality,

$$\int_\Omega I(u)_x^2 dx = \sum_{k=1}^{\infty} c_k^2 \lambda_k^{\alpha} \int_\Omega \partial_x \varphi_k^2 dx = \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\alpha+1} = \|u\|^2_{\dot{H}_N^{\alpha+1}(\Omega)}.$$

□

These propositions are important to derive the properties of the solutions constructed.

3.2.3 The problem $-I(u) = g$

We consider the following problem

$$\begin{cases} \text{For a given } g \in L^2(\Omega), \text{find } u \in H_N^{\alpha}(\Omega) \text{such that} \\ \qquad\qquad -I(u) = g. \end{cases} \tag{3.19}$$

Since $\int_\Omega I(u)dx = 0$ for all $u \in H_N^\alpha(\Omega)$, we must assume that $\int_\Omega g(x)dx = 0$ otherwise (3.19) has no solution.

Proposition 3.8. *For all $g \in L^2(\Omega)$ such that $\int_\Omega g dx = 0$, there exists a unique function $u \in H_N^\alpha(\Omega)$ such that*

$$-I(u) = g \quad \textit{in } L^2(\Omega) \textit{ and } \int_\Omega u dx = 0.$$

Furthermore if $g \in H^1(\Omega)$, then $u \in H_N^{\alpha+1}(\Omega)$.

Proof. Let $g \in L^2(\Omega)$. For $g = \sum_{k=1}^\infty d_k \varphi_k$ with $\sum_{k=1}^\infty d_k^2 < \infty$, we consider

$$u = I^{-1}(g) = \sum_{k=1}^{\infty} \frac{d_k}{\lambda_k^{\frac{\alpha}{2}}} \varphi_k \in H_N^\alpha(\Omega) \text{ and verify } \int_\Omega u dx = 0.$$

Since $(\varphi_k)_k$ form an orthogonal basis of $L^2(\Omega)$, the solution is the unique satisfying $\int_\Omega u dx = 0$. It is clear that every further regularity on g will imply a further regularity on u shifted by an α. □

We thus conclude the following Corollary which will be used to prove the existence of solutions for the stationary problem.

Corollary 3.9. *For all $g \in L^2(\Omega)$, there exists a unique function $v \in H_N^\alpha(\Omega)$ such that*

$$-I(v) + \int_\Omega v dx = g. \tag{3.20}$$

Furthermore if $g \in H^1(\Omega)$, then $u \in H_N^{\alpha+1}(\Omega)$ and the map $g \to u$ is bijective.

Proof. Let $m = \int_\Omega g dx$ and $g' = g - m$. Then $g' \in L^2(\Omega)$(since Ω is bounded) and $\int_\Omega g' dx = 0$. From Proposition 3.8, there exists a function $u \in H_N^\alpha(\Omega)$ such that

$$-I(u) = g' \quad \text{and} \quad \int_\Omega u dx = 0.$$

Let $v = u + m$. Then $\int_\Omega v dx = m$ and

$$-I(v) = -I(u) = g' = g - m = g - \int_\Omega v dx.$$

For the uniqueness, consider two solutions v_1 and v_2 then

$$\int_\Omega v_1 dx = \int_\Omega v_2 dx = \int_\Omega g dx$$

and $w = v_1 - v_2$ satisfies $-I(w) = 0$. Hence, $w = 0$ from the uniqueness given by Proposition 3.8. □

3.3 Regularized problem

We consider the following regularized problem

$$\begin{cases} \partial_t u + \partial_x(f_\epsilon(u)\partial_x I(u)) = 0 & \text{for } x \in \Omega, \quad t > 0, \\ \partial_x u = 0, f_\epsilon(u)\partial_x I(u) = 0 & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{3.21}$$

where $f_\epsilon(s) = s_+^n + \epsilon, \quad \epsilon > 0$ and $0 < \alpha < 2$.

To prove Theorem 3.1 and 3.3, we need to prove the existence of a solution for the regularized problem. Let us introduce the following **stationary problem**

$$\text{For } \tau > 0, g \in H^{\frac{\alpha}{2}}(\Omega), \text{ find } u \in H_N^{\alpha+1}(\Omega) \text{ s.t.} \begin{cases} u + \tau\partial_x(f_\epsilon(u)\partial_x I(u)) = g & \text{in } \Omega, \\ \partial_x u = 0, \partial_x I(u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.22}$$

Once we get a solution for (3.22), we can prove the existence of a solution for (3.21).

3.3.1 Stationary problem

Proposition 3.10. *For all $g \in H^{\frac{\alpha}{2}}(\Omega)$, there exists $u \in H_N^{\alpha+1}(\Omega)$ such that for all $\varphi \in H^1(\Omega)$ we have*

$$\int_\Omega u\varphi dx - \tau \int_\Omega f_\epsilon(u)\partial_x I(u)\partial_x\varphi dx = \int_\Omega g\varphi dx. \tag{3.23}$$

Furthermore, u verifies

$$\int_\Omega u(x)dx = \int_\Omega g(x)dx \tag{3.24}$$

and

$$\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + 2\tau \int_\Omega f_\epsilon(u)\partial_x I(u)^2 dx \leqslant \|g\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}. \tag{3.25}$$

If $\int_\Omega G_\epsilon(g)dx < \infty$ where G_ϵ is a nonnegative function such that $G''_\epsilon(s) = \frac{1}{f_\epsilon(s)}$, then

$$\int_\Omega G_\epsilon(u)dx + \tau\|u\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)} \leqslant \int_\Omega G_\epsilon(g)dx. \tag{3.26}$$

Remark 3.11. *Note that we can consider $G_\epsilon(s) = \int_1^s \int_1^t G''_\epsilon(r)drdt$, so G_ϵ is a non- negative convex function for all $\epsilon > 0$ satisfying $G_\epsilon(1) = G'_\epsilon(1) = 0$.*

Proof. Thanks to Corollary 3.9, we can recover all test functions from $H^1(\Omega)$ by considering

$$\varphi = -I(v) + \int_\Omega vdx$$

for some function $v \in H_N^{\alpha+1}(\Omega)$. So equation (3.23) becomes

$$\begin{aligned} -\int_\Omega uI(v)dx \quad + \left(\int_\Omega udx\right)\left(\int_\Omega vdx\right) + \tau \int_\Omega f_\epsilon(u)\partial_x I(u)\partial_x I(v)dx \\ = -\int_\Omega gI(v)dx + \left(\int_\Omega gdx\right)\left(\int_\Omega vdx\right). \end{aligned} \tag{3.27}$$

Now, we consider the nonlinear operator A defined by

$$A(u)(v) = -\int_\Omega uI(v)dx \quad + \left(\int_\Omega udx\right)\left(\int_\Omega vdx\right) + \tau \int_\Omega f_\epsilon(u)\partial_x I(u)\partial_x I(v)dx \text{ for } u, v \in H_N^{\alpha+1}(\Omega).$$

We prove that this is a continuous, coercive and pseudo-monotone operator. Note that the functional T_g defined by

$$T_g(v) = -\int_\Omega gI(v)dx + \left(\int_\Omega gdx\right)\left(\int_\Omega vdx\right) \text{ for } v \in H_N^{\alpha+1}(\Omega).$$

is a linear form on $H_N^{\alpha+1}(\Omega)$. So our problem reduces to the following

$$\begin{cases} \text{Let } V = H_N^{\alpha+1}(\Omega). \\ A : V \to V^* \text{ coercive, continuous and pseudo-monotone.} \\ T_g \in V^*. \\ \text{Find } u \in H_N^{\alpha+1}(\Omega) \text{ such that } A(u) = T_g \text{ in } V^*. \end{cases} \tag{3.28}$$

The theory of pseudo-monotone operators [64] implies the existence of a solution for (3.28) so there exists $u \in H_N^{\alpha+1}(\Omega)$ such that

$$A(u)(v) = T_g(v) \qquad \text{for all } v \in H_N^{\alpha+1}(\Omega).$$

It remains to prove that A is a continuous, coercive and pseudo-monotone operator on $H_N^{\alpha+1}(\Omega)$. The reader can find the proof in [54, Appendix A] for $V = H_N^2(\Omega)$ but this proof can be easily adapted for our case $V = H_N^{\alpha+1}(\Omega)$.

By using Corollary 3.9 we deduce that u satisfies (3.23) for all $\varphi \in H^1(\Omega)$.
For the properties of u, first by taking $\varphi = 1$ as a test function in (3.23) we obtain mass conservation (3.24). Secondly, take $v = u - \int_\Omega u dx$ in (3.27), by using Proposition 3.7 we have

$$\begin{aligned}
\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + \tau \int_\Omega f_\epsilon(u)\partial_x I(u)^2 &= -\int_\Omega g I(u) dx \\
&\leqslant \|g\|_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} \|u\|_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} \\
&\leqslant \frac{1}{2}\|g\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + \frac{1}{2}\|u\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}
\end{aligned}$$

which (3.25)(Note that the high regularity of g is solely used in this inequality, otherwise $g \in L^2(\Omega)$ is sufficient to prove the existence above). Finally, note that G'_ϵ is smooth with G'_ϵ and G''_ϵ are bounded, and Ω is bounded so we can take $\varphi = G'_\epsilon(u) \in H^1(\Omega)$ as a test function in (3.23),

$$\int_\Omega u G'_\epsilon(u) dx - \tau \int_\Omega f_\epsilon(u)\partial_x I(u)\partial_x u G''_\epsilon(u) dx = \int_\Omega g G'_\epsilon(u) dx.$$

So by using Proposition 3.7 and the fact that $G''_\epsilon(s) = \frac{1}{f_\epsilon(s)}$ we get

$$\tau \|u\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)} = \int_\Omega G'_\epsilon(u)(g-u) dx \leqslant \int_\Omega (G_\epsilon(g) - G_\epsilon(u)) dx$$

because G_ϵ is convex and we deduce (3.26). □

3.3.2 Implicit Euler scheme

We construct a piecewise constant function

$$u^\tau(t,x) = u^k(x) \text{ for } t \in [k\tau, (k+1)\tau), k \in \{0, ..., N-1\}$$

where $\tau = \frac{T}{N}$ and $(u^k)_{k\in\{0,...,N-1\}}$ is such that

$$u^{k+1} + \tau\, \partial_x(f_\epsilon(u^{k+1})\partial_x I(u^{k+1})) = u^k.$$

The existence of the u^k follows from Proposition 3.10 by induction on k with $u^0 = u_0$. We deduce the following

Corollary 3.12. *For any $N > 0$ and $u_0^\epsilon \in H^{\frac{\alpha}{2}}(\Omega)$, there exists a function $u^\tau \in L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ such that*

1. *$t \longrightarrow u^\tau(t,x)$ is constant on $[k\tau,(k+1)\tau), k \in \{0,...,N-1\}$ and $\tau = \frac{T}{N}$.*

2. *$u^\tau = u_0$ on $[0,\tau) \times \Omega$.*

3. *For all $t \in (0,T)$,*
$$\int_\Omega u^\tau(t,x)dx = \int_\Omega u_0(x)dx. \tag{3.29}$$

4. *For all $\varphi \in C_c^1(0,T;H^1(\Omega))$,*
$$\iint_{Q_{\tau,T}} \frac{u^\tau - S_\tau u^\tau}{\tau} \varphi dxdt = \iint_{Q_{\tau,T}} f_\epsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi dxdt \tag{3.30}$$
where $S_\tau u^\tau(t,x) = u^\tau(t-\tau,x)$ and $Q_{\tau,T} = (\tau,T) \times \Omega$.

5. *For all $t \in (0,T)$,*
$$\|u^\tau(t,.)\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + 2\int_0^T \int_\Omega f_\epsilon(u^\tau)\partial_x I(u^\tau)^2 dxdt \leqslant \|u_0\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}. \tag{3.31}$$

6. *If $\int_\Omega G_\epsilon(u_0)dx < \infty$, then for all $t \in (0,T)$*
$$\int_\Omega G_\epsilon(u^\tau(t,x))dx + \int_0^t \|u^\tau(s,.)\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)} ds \leqslant \int_\Omega G_\epsilon(u_0)dx. \tag{3.32}$$

3.3.3 Existence of solution for the regularized problem

Now we are able to prove the existence of a solution for the regularized problem.

Proposition 3.13. *Let $0 < \alpha < 2$. For all $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$ and for all $T > 0$, there exists a function u^ϵ such that*
$$u^\epsilon \in L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega)) \cap L^2(0,T;H_N^{\alpha+1}(\Omega))$$
satisfying
$$\iint_Q u^\epsilon \partial_t\varphi dxdt + \iint_Q f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)\partial_x\varphi dxdt = -\int_\Omega u_0\varphi(0,.)dx \tag{3.33}$$
for all $\varphi \in C^1(0,T;H^1(\Omega))$ with support in $[0,T) \times \bar{\Omega}$.
The function u^ϵ satisfies for almost every $t \in (0,T)$
$$\int_\Omega u^\epsilon(t,x)dx = \int_\Omega u_0(x)dx \tag{3.34}$$
and
$$\|u^\epsilon(t,.)\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)} + 2\int_0^T \int_\Omega f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)^2 dxdt \leqslant \|u_0\|^2_{\dot{H}^{\frac{\alpha}{2}}(\Omega)}. \tag{3.35}$$
Finally, if $\int_\Omega G_\epsilon(u_0)dx < \infty$ then for almost every $t \in (0,T)$,
$$\int_\Omega G_\epsilon(u^\epsilon(t,x))dx + \int_0^t \|u^\epsilon(s,.)\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}(\Omega)} ds \leqslant \int_\Omega G_\epsilon(u_0)dx. \tag{3.36}$$

Proof. We consider the sequence (u^τ) constructed in Corollary 3.12 and let $\tau \to 0$. Bound (3.31) and (3.29) implies that (u^τ) is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ and $(\partial_x I(u^\tau))$ is bounded in $L^2(Q)$.

Case $0 < \alpha \leqslant 1$. Note that

$$\frac{u^\tau - S_\tau u^\tau}{\tau} = \partial_x(f_\epsilon(u^\tau)\partial_x I(u^\tau)).$$

Since $n \geqslant 1$, the function f_ϵ is Lipschitz and so $(f_\epsilon(u^\tau))$ is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ thus by the Sobolev embedding theorem, we deduce that $(f_\epsilon(u^\tau))$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$. We know that $(\partial_x I(u^\tau))$ is bounded in $L^2(0,T;L^2(\Omega))$ so $f_\epsilon(u^\tau)\partial_x I(u^\tau)$ is bounded in $L^2(\tau,T;L^r(\Omega))$ where $\frac{1}{r} = \frac{1}{2} + \frac{1}{p}$. We deduce that

$$\partial_x(f_\epsilon(u^\tau)\partial_x I(u^\tau)) \text{ is bounded in } L^2(\tau,T;W^{-1,r}(\Omega))$$

Since $\alpha \leqslant 1$, we have the following embedding

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow L^p(\Omega) \to W^{-1,l}(\Omega)$$

for all $p < \frac{2}{1-\alpha}$ and for all $l > 2$ (because Ω is bounded and we have a Sobolev space of negative regularity). Aubin's lemma implies that (u^τ) is relatively compact in $C^0(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$. Note that $(\partial_x I(u^\tau))$ is bounded in $L^2(\Omega)$ and (u^τ) is bounded in $L^\infty(0,T;L^1(\Omega))$ (because $1 < \frac{2}{1-\alpha}$). Hence, (u^τ) is bounded in $L^2(0,T;H_N^{\alpha+1}(\Omega))$. Since

$$H_N^{\alpha+1}(\Omega) \hookrightarrow H_N^{\frac{\alpha}{2}+1}(\Omega) \to W^{-1,l}(\Omega),$$

we deduce that (u^τ) is relatively compact in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$. So we can extract a subsequence, also denoted (u^τ), such that when τ tends to zero we have

- $u^\tau \to u^\epsilon \in L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ almost everywhere in Q,
- $u^\tau \to u^\epsilon$ in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$ strongly,
- $\partial_x I(u^\tau) \rightharpoonup \partial_x I(u^\epsilon)$ in $L^2(Q)$ weakly.

Now let us pass to the limit in (3.30). We have

$$\begin{aligned}
\iint_{Q_{\tau,T}} \frac{u^\tau - S_\tau u^\tau}{\tau}\varphi dx dt &= \frac{1}{\tau}\Big[\int_0^T\int_\Omega u^\tau(t,x)\varphi(t,x)dtdx - \int_0^\tau\int_\Omega u^\tau(t,x)\varphi(t,x)dtdx \\
&\quad - \int_0^{T-\tau}\int_\Omega u^\tau(t,x)\varphi(t+\tau,x)dtdx\Big] \\
&= \int_0^T\int_\Omega u^\tau(t,x)\frac{\varphi(t,x)-\varphi(t+\tau,x)}{\tau}dxdt \\
&\quad - \frac{1}{\tau}\int_0^\tau\int_\Omega u^\tau(t,x)\varphi(t,x)dtdx + \frac{1}{\tau}\int_{T-\tau}^T\int_\Omega u^\tau(t,x)\varphi(t+\tau,x)dtdx \\
&\underset{\tau\to 0}{\longrightarrow} -\iint_Q u^\epsilon\partial_t\varphi dxdt - \int_\Omega u^\epsilon(0,x)\varphi(0,x)dx + 0.
\end{aligned}$$

For the nonlinear term, we integrate by parts

$$\iint_Q f_\epsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi = -\iint_Q f_\epsilon(u^\tau)I(u^\tau)\partial^2_{xx}\varphi - \iint_Q n(u^\tau)_+^{n-1}\partial_x u^\tau I(u^\tau)\partial_x\varphi. \quad (3.37)$$

We have

$$u^\tau \to u^\epsilon \text{ in } L^2(0,T;H^s_N(\Omega)) \text{ for all } s < 1+\alpha.$$

So

$$I(u^\tau) \to I(u^\epsilon) \text{ in } L^2(0,T;H^{s'}(\Omega)) \text{ for all } s' < 1$$

and

$$\partial_x u^\tau \to \partial_x u^\epsilon \text{ in } L^2(0,T;H^{s''}(\Omega)) \text{ for all } s'' < \alpha.$$

So we deduce the following convergences

$$I(u^\tau) \to I(u^\epsilon) \text{ in } L^2(0,T;L^q(\Omega)) \text{ for all } q < \infty.$$
$$u^\tau_x \to u^\epsilon_x \text{ in } L^2(0,T;L^p(\Omega)) \text{ for all } p < \frac{2}{1-\alpha}.$$

Furthermore, since $u^\tau \to u^\epsilon$ in $C^0(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$ and f_ϵ is lipschitz then

$$f_\epsilon(u^\tau) \to f_\epsilon(u^\epsilon) \text{ in } C^0(0,T;L^p(\Omega)) \text{ for all } p < \frac{2}{1-\alpha}.$$

For the term $(u^\tau)^{n-1}$, if $n \geqslant 2$ then the function $s \to s^{n-1}$ is lipschitz and

$$(u^\tau)^{n-1} \to (u^\epsilon)^{n-1} \text{ in } C^0(0,T;L^p(\Omega)) \text{ for all } p < \frac{2}{1-\alpha}.$$

If $n < 2$ then $\frac{p}{n-1} \geqslant 1$ and

$$(u^\tau)^{n-1} \to (u^\epsilon)^{n-1} \text{ in } C^0(0,T;L^{\frac{p}{n-1}}(\Omega)) \text{ for all } p < \frac{2}{1-\alpha}.$$

Thus we can pass to the limit in (3.37) and reverse the integration by parts to obtain

$$\begin{aligned}\iint_Q f_\epsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi &\to -\iint_Q f_\epsilon(u^\epsilon)I(u^\epsilon)\partial^2_{xx}\varphi - \iint_Q n(u^\epsilon)_+^{n-1}\partial_x u^\epsilon I(u^\epsilon)\partial_x\varphi \\ &= \iint_Q f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)\partial_x\varphi.\end{aligned}$$

For the properties of u^ϵ, first since $u^\tau \to u^\epsilon$ in $L^\infty(0,T;L^1(\Omega))$ mass conservation equation (3.34) follows from (3.29).

Secondly, we note that (u^τ) is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ so (u^τ) weakly converges to u^ϵ in $H^{\frac{\alpha}{2}}(\Omega)$ and

$$\|u^\epsilon\|_{\dot H^{\frac{\alpha}{2}}(\Omega)} \leq \liminf_{\tau\to 0} \|u^\tau\|_{\dot H^{\frac{\alpha}{2}}(\Omega)}.$$

Note that estimate (3.31) implies that $\sqrt{f_\epsilon(u^\tau)}\partial_x I(u^\tau)$ is bounded in $L^2(0,T;L^2(\Omega))$ thus it weakly converges in $L^2(0,T;L^2(\Omega))$ and the lower semicontinuity permits us to conclude (3.35).

Finally, to derive (3.36) we note that $G_\epsilon(u^\tau) \to G_\epsilon(u^\epsilon)$ almost everywhere and Fatou's lemma implies for almost every $t \in (0,T)$

$$\int_\Omega G_\epsilon(u^\epsilon(t,x))dx \leq \liminf_{\tau\to 0}\int_\Omega G_\epsilon(u^\tau(t,x))dx.$$

Furthermore, (u^τ) is relatively compact in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$ thus

$$\int_0^t \|u^\epsilon(s)\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}} ds = \lim_{\tau\to 0}\int_0^t \|u^\tau(s)\|^2_{\dot{H}_N^{\frac{\alpha}{2}+1}} ds.$$

Hence (3.32) implies (3.36).

Case $1<\alpha<2$. Note that

$$\frac{u^\tau - S_\tau u^\tau}{\tau} = \partial_x(f_\epsilon(u^\tau)\partial_x I(u^\tau)).$$

We have (u^τ) is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ so by the Sobolev embedding theorem, we deduce that (u^τ) is bounded in $L^\infty(0,T;C^{0,\frac{\alpha-1}{2}}(\Omega))$. Thus $(f_\epsilon(u^\tau))$ is bounded in $L^\infty(0,T;L^\infty(\Omega))$. We know that $(\partial_x I(u^\tau))$ is bounded in $L^2(0,T;L^2(\Omega))$ so $(f_\epsilon(u^\tau)\partial_x I(u^\tau))$ is bounded in $L^2(\tau,T;L^2(\Omega))$. We deduce that

$$\partial_x(f_\epsilon(u^\tau)\partial_x I(u^\tau)) \text{ is bounded in } L^2(\tau,T;W^{-1,2}(\Omega)).$$

Since $\alpha>1$ we have the following embedding

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow C^{0,\frac{\alpha-1}{2}}(\Omega) \to W^{-1,2}(\Omega).$$

Aubin's lemma implies that the sequence (u^τ) is relatively compact in $C^0(0,T;C^{0,\frac{\alpha-1}{2}}(\Omega))$. Since $(\partial_x I(u^\tau))$ is bounded in $L^2(\Omega)$ and (u^τ) is bounded in $L^\infty(0,T;L^1(\Omega))$ then, (u^τ) is bounded in $L^2(0,T;H_N^{\alpha+1}(\Omega))$. Using the following embedding

$$H_N^{\alpha+1}(\Omega) \hookrightarrow H_N^{\frac{\alpha}{2}+1}(\Omega) \to W^{-1,2}(\Omega),$$

we deduce that (u^τ) is relatively compact in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$. So for a subsequence we have

- $u^\tau \to u^\epsilon$ locally uniformly,
- $\partial_x I(u^\tau) \rightharpoonup \partial_x I(u^\epsilon)$ in $L^2(Q)$-weakly,
- $u^\tau \to u^\epsilon$ in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$ strongly.

Let us pass to the limit in (3.30). As in the first case

$$\iint_{Q_{\tau,T}} \frac{u^\tau - S_\tau u^\tau}{\tau}\varphi dxdt \underset{\tau\to 0}{\longrightarrow} -\iint_Q u^\epsilon \partial_t\varphi dxdt - \int_\Omega u^\epsilon(0,x)\varphi(0,x)dx.$$

For the nonlinear term, since

$$u^\tau \to u^\epsilon \text{ locally uniformly,}$$

Then

$$f_\epsilon(u^\tau)\partial_x\varphi \to f_\epsilon(u^\epsilon)\partial_x\varphi \text{ in } L^2(0,T;L^2(\Omega)) - \text{strongly.}$$

Furthermore

$$\partial_x I(u^\tau) \rightharpoonup \partial_x I(u^\epsilon) \text{ in } L^2(0,T;L^2(\Omega)) - \text{weakly.}$$

Hence

$$\iint_Q f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi dxdt \to \iint_Q f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)\partial_x\varphi dxdt$$

and the proof is complete.

For the properties of u^ϵ, the proofs of estimates (3.34), (3.36) and (3.35) are the same as in the first case. $\square$

3.4 Proofs of main results

3.4.1 Proof of Theorem 3.1

Consider the sequence (u^ϵ) such that $u^\epsilon \in L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega)) \cap L^2(0,T;H^{\frac{\alpha}{2}+1}(\Omega))$ solution of (3.21). Our goal is to pass to the limit $\epsilon \to 0$.

Note that (3.35) and (3.34) imply that (u^ϵ) is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$. Since f_ϵ is Lipschitz then $f_\epsilon(u^\epsilon)$ is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$. So by using the Sobolev embedding theorem, we deduce that $f_\epsilon(u^\epsilon)$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$.

Furthermore, (3.35) also implies that $f_\epsilon(u^\epsilon)^{\frac{1}{2}}\partial_x I(u^\epsilon)$ is bounded in $L^2(0,T;L^2(\Omega))$. Thus

$$f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon) \text{is bounded in } L^2(0,T;L^r(\Omega)) \text{ where } \frac{1}{r} = \frac{1}{2} + \frac{1}{2p}.$$

Hence

$$\partial_t u^\epsilon = -\partial_x(f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)) \text{ is bounded in } L^2(0,T;W^{-1,r}(\Omega)).$$

Since

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow L^{\frac{2}{1-\alpha}}(\Omega) \to W^{-1,l}(\Omega),$$

Aubin's lemma implies that (u^ϵ) is relatively compact in $C^0(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$. So we can extract a subsequence such that

- $u^\epsilon \to u$ in $C^0(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$.
- $u^\epsilon \to u \in L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$ almost everywhere in Q.

Let us pass to the limit in (3.33). Let $\varphi \in D([0,T)\times\bar{\Omega})$ satisfying $\partial_x\varphi = 0$ on $(0,T)\times\partial\Omega$. Since $u^\epsilon \to u$ in $C^0(0,T;L^1(\Omega))$, we have

$$\iint_Q u^\epsilon \partial_t\varphi dxdt \to \iint_Q u\partial_t\varphi dxdt.$$

Remark that (3.35) implies that

$$\epsilon \iint (\partial_x I(u^\epsilon))^2 \le c.$$

The Cauchy-Schwarz inequality implies

$$\epsilon \iint \partial_x I(u^\epsilon)\partial_x\varphi dxdt \leqslant c(\varphi)\sqrt{\epsilon}(\sqrt{\epsilon}\|\partial_x I(u^\epsilon)\|_2) \to 0.$$

Estimate (3.35) also gives that $(u^\epsilon)_+^{\frac{n}{2}}\partial_x I(u^\epsilon)$ is bounded in $L^2(0,T;L^2(\Omega))$. For the term $(u^\epsilon)^{\frac{n}{2}}$, we consider two cases, if $n \geqslant 2$ then the function $s \to s^{\frac{n}{2}}$ is Lipschitz and $((u^\epsilon)^{\frac{n}{2}})$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$. We deduce that $((u^\epsilon)_+^n\partial_x I(u^\epsilon))$ is bounded in $L^2(0,T;L^m(\Omega))$ where $\frac{1}{m} = \frac{1}{2} + \frac{1}{p}$. If $n < 2$ then $((u^\epsilon)^{\frac{n}{2}})$ is bounded in $L^\infty(0,T;L^{\frac{2p}{n}}(\Omega))$ for all $p < \frac{2}{1-\alpha}$ (in this case $\frac{2p}{n} \geqslant 1$). We deduce that $((u^\epsilon)_+^n\partial_x I(u^\epsilon))$ is bounded in $L^2(0,T;L^m(\Omega))$ where $\frac{1}{m} = \frac{1}{2} + \frac{n}{2p}$, hence

$$h^\epsilon := (u^\epsilon)_+^n\partial_x I(u^\epsilon) \rightharpoonup h \text{ in } L^2(0,T;L^m(\Omega)) \text{ weakly.}$$

Passing to the limit we obtain

$$\iint_Q u\partial_t\varphi dxdt + \iint_Q h\partial_x\varphi dxdt = -\int_\Omega u_0\varphi(0,x)dx.$$

It remains to show that

$$h = u_+^n\partial_x I(u)$$

in the following sense

$$\iint_Q h\varphi dxdt = -\iint_Q nu_+^{n-1}\partial_x uI(u)\varphi dxdt - \iint_Q u_+^n I(u)\partial_x\varphi dxdt \tag{3.38}$$

for all test functions φ such that $\varphi = 0$ on $(0,T)\times\partial\Omega$, that is

$$h = \partial_x(u_+^n I(u)) - nu_+^{n-1}\partial_x uI(u) \text{ in } D'(\Omega).$$

Note that G_ϵ is decreasing with respect to ϵ, so

$$\int_\Omega G_\epsilon(u_0)dx \le \int_\Omega G(u_0)dx \le c.$$

Thus estimate (3.36) implies that (u^ϵ) is bounded in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$. Recall that $(\partial_t u^\epsilon)$ is bounded in $L^2(0,T;W^{-1,l}(\Omega))$. Aubin's lemma implies that

$$(u^\epsilon) \text{ is relatively compact in } L^2(0,T;H_N^s(\Omega)) \text{ for all } s < \frac{\alpha}{2}+1.$$

Hence

$$(\partial_x u^\epsilon) \text{ is relatively compact in } L^2(0,T;H^{s'}(\Omega)) \text{ for all } s' < \frac{\alpha}{2}$$

and

$$(I(u^\epsilon)) \text{ is relatively compact in } L^2(0,T;H_N^{s''}(\Omega)) \text{ for all } s'' < 1-\frac{\alpha}{2}.$$

Thus we can extract a subsequence such that

$$\begin{aligned}
&u^\epsilon \to u \text{ in } C^0(0,T;L^p(\Omega)) \text{ for all } p < \frac{2}{1-\alpha},\\
&I(u^\epsilon) \to I(u) \text{ in } L^2(0,T;L^q(\Omega)) \text{ for all } q < \infty,\\
&\partial_x u^\epsilon \to \partial_x u \text{ in } L^2(0,T;L^p(\Omega)) \text{ for all } p < \frac{2}{1-\alpha}.
\end{aligned}$$

We write

$$\begin{aligned}
\iint_Q h_\epsilon \varphi dx dt &= \iint (u^\epsilon)_+^n \partial_x I(u^\epsilon)\varphi dx dt\\
&= -\iint n(u^\epsilon)_+^{n-1}\partial_x u^\epsilon I(u^\epsilon)\varphi dx dt - \iint (u^\epsilon)_+^n I(u^\epsilon)\partial_x\varphi dx dt.
\end{aligned}$$

Using these convergences and the fact that $I(u^\epsilon)$ converges in $L^2(0,T;L^q(\Omega))$ for all $q<\infty$ we can pass to the limit and obtain (3.38). Note that for the terms $(u^\epsilon)^n$ and $(u^\epsilon)^{n-1}$ we consider two cases $n \geqslant 2$ and $n < 2$ and we proceed as above. In the first case the functions $s \to s^n$ and $s \to s^{n-1}$ are Lipschitz and then $(u^\epsilon)^n \to u^n$ and $(u^\epsilon)^{n-1} \to u^{n-1}$ in $C^0(0,T;L^p(\Omega))$ for all $p < \frac{2}{1-\alpha}$. If $n<2$ then $\frac{p}{n} \geqslant 1$ and $\frac{p}{n-1} \geqslant 1$ and $(u^\epsilon)^n \to u^n$ in $C^0(0,T;L^{\frac{p}{n}}(\Omega))$ and $(u^\epsilon)^{n-1} \to u^{n-1}$ in $C^0(0,T;L^{\frac{p}{n-1}}(\Omega))$ for all $p < \frac{2}{1-\alpha}$.

For the properties of u, passing to the limit in (3.34) implies mass conservation equation (3.13).

Since (u^ϵ) is bounded in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$ then $u^\epsilon \rightharpoonup u$ and

$$\|u\|_{L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))} \le \liminf_{\epsilon\to 0} \|u^\epsilon\|_{L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))}.$$

Note that

$$G_\epsilon(u^\epsilon) \to G(u) \text{ almost everywhere and } G_\epsilon(u_0) \le G(u_0).$$

Then by Fatou's lemma estimate (3.10) follows from (3.36).

Remark that estimate (3.35) implies that $g_\epsilon = (u^\epsilon)_+^{\frac{n}{2}} \partial_x I(u^\epsilon)$ weakly converges in $L^2(Q)$ to a function g and the lower semi-continuity of the norm implies (3.9). It remains to prove that

$$g = \partial_x (u_+^{\frac{n}{2}} I(u)) - \frac{n}{2} u_+^{\frac{n}{2}-1} \partial_x u I(u) \text{ in } D'(\Omega). \tag{3.39}$$

We have

$$\begin{aligned}\iint_Q g_\epsilon \varphi dx dt &= \iint (u^\epsilon)_+^{\frac{n}{2}} \partial_x I(u^\epsilon) \varphi dx dt \\ &= -\iint \frac{n}{2} (u^\epsilon)_+^{\frac{n}{2}-1} \partial_x u^\epsilon I(u^\epsilon) \varphi dx dt - \iint (u^\epsilon)_+^{\frac{n}{2}} I(u^\epsilon) \partial_x \varphi dx dt.\end{aligned}$$

Also, using the convergences above and the fact that $I(u^\epsilon) \to I(u)$ in $L^2(0,T;L^q(\Omega))$ for all $q < \infty$, we can pass to the limit and obtain (3.39). Note also that for the terms $(u^\epsilon)^{\frac{n}{2}-1}$ and $(u^\epsilon)_+^{\frac{n}{2}}$ we consider two cases $n \geqslant 4$ and $n < 4$ and we proceed as above.

It remains to prove that u is a nonnegative function. Note that estimate (3.36) implies that for all $t \in (0,T)$

$$\int_\Omega G_\epsilon(u^\epsilon(t,x)) dx \leq \int_\Omega G_\epsilon(u_0(t,x)) dx.$$

Since

$$\int_\Omega G_\epsilon(u_0(t,x)) dx \leq \int_\Omega G(u_0(t,x)) dx < \infty,$$

we conclude that

$$\limsup_{\epsilon \to 0} \int_\Omega G_\epsilon(u^\epsilon(t,x)) dx < \infty. \tag{3.40}$$

Note that for all $\delta > 0$,

$$\lim_{\epsilon \to 0} G_\epsilon(-\delta) = +\infty.$$

Recall that $u^\epsilon(t,.)$ converges almost everywhere. So for $\eta > 0$, Egorov's theorem implies the existence of a set $A_\eta \subset \Omega$ such that

$$| \Omega \setminus A_\eta | \leq \eta \text{ and } u^\epsilon(t,.) \to u(t,.) \text{ uniformly in } A_\eta.$$

Let $\delta > 0$. We consider

$$C_{\eta,\delta} = A_\eta \cap \{u(t,.) \leq -2\delta\}.$$

For every $\eta, \delta > 0$, there exists $\epsilon_0(\eta,\delta)$ such that if $\epsilon \leq \epsilon_0(\eta,\delta)$ then $u^\epsilon(t,.) \leq -\delta$ in $C_{\eta,\delta}$.

This implies that $C_{\eta,\delta}$ has measure zero. Indeed, if not then for $\epsilon \leq \epsilon_0(\eta,\delta)$ we have

$$G_\epsilon(u^\epsilon(t,x)) \geq G_\epsilon(-\delta) \to +\infty.$$

By Fatou's lemma

$$\liminf_{\epsilon\to 0}\int_{C_{\eta,\delta}} G_\epsilon(u^\epsilon(t,x))dx \geqslant \int_{C_{\eta,\delta}} \liminf_{\epsilon\to 0} G_\epsilon(u^\epsilon(t,x))dx = +\infty$$

which contradicts (3.40).

Hence for all $\delta > 0$ and all $\eta > 0$, we have

$$\mid \{u(t,.) \le -2\delta\} \mid \le \mid C_{\eta,\delta} \mid + \mid \Omega \setminus A_\eta \mid \le \eta.$$

Thus, $\mid \{u(t,.) \le -2\delta\} \mid = 0$ for all $\delta > 0$. We conclude that

$$\{u(t,.) < 0\} = \bigcup_{k\ge 1} \left\{u(t,.) \le \frac{-1}{k}\right\}$$

has measure zero and so $u(t,x) \ge 0$ for almost every $x \in \Omega$ and for all $t > 0$.

3.4.2 Proof of Theorem 3.3

We organize this proof in two steps. In the first step we consider nonnegative $u_0 \in H^{\frac{\alpha}{2}}(\Omega)$ satisfying (3.6) and we prove the existence of solutions as in Theorem 3.1. In the second step we use this information to prove the existence of solutions for nonnegative initial data which belongs to $H^{\frac{\alpha}{2}}(\Omega)$.

First step Consider the sequence (u^ϵ) such that $u^\epsilon \in L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega)) \cap L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$ solution of (3.21). Our goal is to pass to the limit $\epsilon \to 0$. Note that (3.35) implies that (u^ϵ) is bounded in $L^\infty(0,T;H^{\frac{\alpha}{2}}(\Omega))$. So by using the Sobolev embedding theorem, we deduce that (u^ϵ) is bounded in $L^\infty(0,T;C^{0,\frac{\alpha-1}{2}}(\Omega))$. Hence $(f_\epsilon(u^\epsilon))$ is bounded in $L^\infty(0,T;L^\infty(\Omega))$. Furthermore (3.35) gives that $(f_\epsilon(u^\epsilon)^{\frac{1}{2}}\partial_x I(u^\epsilon))$ is bounded in $L^2(0,T;L^2(\Omega))$. We deduce that

$$(f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)) \text{ is bounded in } L^2(0,T;L^2(\Omega)).$$

So

$$\partial_t u^\epsilon = -\partial_x(f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)) \text{ is bounded in } L^2(0,T;W^{-1,2}(\Omega)).$$

Since

$$H^{\frac{\alpha}{2}}(\Omega) \hookrightarrow C^{0,\frac{\alpha-1}{2}}(\Omega) \to W^{-1,2}(\Omega),$$

Aubin's lemma implies that (u^ϵ) is relatively compact in $C^0(0,T;C^{0,\frac{\alpha-1}{2}}(\Omega))$. So we can extract a subsequence such that

$$u^\epsilon \to u \text{ locally uniformly.}$$

Now let us pass to the limit in (3.33). Proceeding as in the case $0 < \alpha \leqslant 1$ we get the same results but it remains to prove the equation on h i.e. (3.38). Since

$$\int_\Omega G_\epsilon(u_0)dx \leq \int_\Omega G(u_0)dx \leq c,$$

estimate (3.36) implies that (u^ϵ) is bounded in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$. Recall that $(\partial_t u^\epsilon)$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$. Aubin's lemma implies that

$$(u^\epsilon) \text{ is relatively compact in } L^2(0,T;H^s(\Omega)) \text{ for all } s < \frac{\alpha}{2}+1.$$

Hence

$$(\partial_x u^\epsilon) \text{ is relatively compact in } L^2(0,T;H^{s'}(\Omega)) \text{ for all } s' < \frac{\alpha}{2}$$

and since $\alpha < 2$ then (u^ϵ) is relatively compact in $L^2(0,T;H^\alpha(\Omega))$ and

$$(I(u^\epsilon)) \text{ is relatively compact in } L^2(0,T;L^2(\Omega)).$$

Thus we can extract a subsequence such that

$$\begin{aligned}
&u^\epsilon \to u \text{ locally uniformly,}\\
&I(u^\epsilon) \to I(u) \text{ in } L^2(0,T;L^2(\Omega)),\\
&\partial_x u^\epsilon \to \partial_x u \text{ in } L^2(0,T;\text{locally uniformly with respect to x}).
\end{aligned}$$

We have

$$\begin{aligned}
\iint_Q h_\epsilon \varphi dxdt &= \iint (u^\epsilon)_+^n \partial_x I(u^\epsilon)\varphi dxdt\\
&= -\iint n(u^\epsilon)_+^{n-1}\partial_x u^\epsilon I(u^\epsilon)\varphi dxdt - \iint (u^\epsilon)_+^n I(u^\epsilon)\partial_x\varphi dxdt.
\end{aligned}$$

Using these convergences we can pass to the limit and obtain (3.38).

For the properties of u, the proofs are the same as in the case $0 < \alpha \leqslant 1$ but we use these convergences above to obtain the equation on g.

We prove also that u is a nonnegative function as in the case $0 < \alpha \leqslant 1$.

Second step Now we consider the case where $u_0 \geqslant 0$ belongs to $H^{\frac{\alpha}{2}}(\Omega)$ without the additional condition (3.6). If we define

$$u_{0\delta}(x) = u_0(x) + \delta$$

and denote u_δ the nonnegative solution u constructed in the first step for the initial data $u_{0\delta}$, which satisfies (3.6), then u_δ satisfies

$$|\, u_\delta \,| \leq A, \quad \iint_Q u_\delta^n \partial_x I(u_\delta)^2 dxdt \leq C, \quad |\, u_\delta(t,x_2) - u_\delta(t,x_1) \,| \leq k \,|\, x_2 - x_1 \,|^{\frac{\alpha-1}{2}}, \qquad (3.41)$$

with constants C, A, K independent of δ and T.

Proposition 3.14. *There exists a constant M independent of δ and T such that*

$$\mid u_\delta(t_2,x) - u_\delta(t_1,x) \mid \leqslant M \mid t_2 - t_1 \mid^{\frac{\alpha-1}{2(\alpha+2)}} \tag{3.42}$$

for all $x \in \Omega$, t_1 and $t_2 \in (0,T)$.

Proof. The proof is given in Appendix A. □

Taking a subsequence

$$u_\delta \to u \quad \text{locally uniformly in } Q,$$

we will prove Theorem 3.3. Let $\varphi \in \mathcal{D}([0,T)\times\bar{\Omega})$ satisfying $\partial_x\varphi = 0$ on $(0,T)\times\partial\Omega$. We have

$$\iint_Q u_\delta \partial_t\varphi dtdx + \iint_Q u_\delta^n \partial_x I(u_\delta)\partial_x\varphi dxdt = -\int_\Omega (u_0+\delta)\varphi(0,.)dx. \tag{3.43}$$

Since $u_\delta \to u$ locally uniformly then

$$\iint_Q u_\delta \partial_t\varphi dtdx \to \iint_Q u\partial_t\varphi dtdx \quad \text{and} \quad \int_\Omega (u_0+\delta)\varphi(0,.)dx \to \int_\Omega u_0\varphi(0,.)dx \quad \text{as } \delta\to 0. \tag{3.44}$$

It remains to pass to the limit in the nonlinear term. We consider

$$h_\delta = u_\delta^n \partial_x I(u_\delta).$$

From (3.41), $((u_\delta)^{\frac{n}{2}}\partial_x I(u_\delta))$ is bounded in $L^2(Q)$ and since (u_δ) is bounded in $L^\infty(Q)$ so (h_δ) is bounded in $L^2(Q)$ and weakly converges to h in $L^2(Q)$. Our aim is to prove that

$$h = \begin{cases} u^n\partial_x I(u) & \text{in } Q_+ := \{u>0\}\cap Q \\ 0 & \text{elsewhere.} \end{cases}$$

For any $\eta > 0$ we have

$$c \geqslant \int_{\{u\geqslant\eta\}\cap Q} u_\delta^n \partial_x I(u_\delta)^2 \geqslant \left(\frac{\eta}{2}\right)^n \int_{\{u\geqslant\eta\}\cap Q} \partial_x I(u_\delta)^2,$$

so $(\partial_x I(u_\delta))$ is bounded in $L^2(\{u\geqslant\eta\}\cap Q)$. Thus for all $k\in\mathbb{N}$,

$$(\partial_x I(u_\delta)) \text{ weakly converges in } L^2(Q_k)$$

where $Q_k := \{u \geqslant \frac{1}{k}\}\cap Q$. So, up to a subsequence,

$$\partial_x I(u_\delta)) \text{ weakly converges to } f \text{ in } L^2_{loc}(Q_+)$$

where $Q_+ = \bigcup_{k\in\mathbb{N}} P_k = \{u>0\}\cap Q$. This implies that

$$\partial_x I(u_\delta) \to f \quad \text{in } \mathcal{D}'(Q_+).$$

It remains to prove that

$$f = \partial_x I(u) \quad \text{in } \mathcal{D}'(Q_+).$$

Since $u_\delta \to u$ locally uniformly in Q then by using Corollary 3.6

$$I(u_\delta) \to I(u) \text{ in } \mathcal{D}'(Q).$$

So, $\partial_x I(u_\delta) \to \partial_x I(u)$ in $\mathcal{D}'(Q)$. Now, let $\varphi \in \mathcal{D}(Q_+)$ we have

$$\langle \partial_x I(u_\delta), \varphi \rangle_{\mathcal{D}'(Q_+)\mathcal{D}(Q_+)} = \langle \partial_x I(u_\delta), \bar{\varphi} \rangle_{\mathcal{D}'(Q)\mathcal{D}(Q)} \underset{\delta \to 0}{\longrightarrow} \langle \partial_x I(u), \bar{\varphi} \rangle_{\mathcal{D}'(Q)\mathcal{D}(Q)}$$

where $\bar{\varphi}$ is the extension by 0 of φ to Q. So

$$f = \partial_x I(u) \text{ in } \mathcal{D}'(Q_+) \text{ and } \partial_x I(u) \in L^2_{loc}(Q_+). \tag{3.45}$$

On the other hand, if δ is sufficiently small, then

$$\Big| \iint_{\{u=0\}\cap Q} u_\delta^n \partial_x I(u_\delta) \partial_x \varphi \Big| \leqslant c\delta^{\frac{n}{2}} \Big(\iint u_\delta^n \partial_x I(u_\delta)^2 \Big)^{\frac{1}{2}} \leqslant C\delta^{\frac{n}{2}} \tag{3.46}$$

Taking $\delta \to 0$ in (3.43) and using (3.44), (3.45) and (3.46) we deduce that (3.12) is satisfied.

For the properties of u, since u_δ satisfies mass conservation and uniformly converges to u then u inherits the same property. Furthermore, remark that estimate (3.41) implies that $g_\delta = u_\delta^{\frac{n}{2}} \partial_x I(u_\delta)$ weakly converges in $L^2(Q)$ to a function g and the lower semi-continuity of the norm implies (3.14). It remains to prove that

$$g = \begin{cases} u^{\frac{n}{2}} \partial_x I(u) & \text{in } Q_+ \\ 0 & \text{elsewhere.} \end{cases} \tag{3.47}$$

We proved that $\partial_x I(u_\delta)$ weakly converges to $\partial_x I(u)$ in $L^2_{loc}(Q_+)$ and u_δ locally converges to u so $g = u^{\frac{n}{2}} \partial_x I(u)$ in $D'(Q_+)$. On the other hand, if δ is sufficiently small, then

$$\Big| \iint_{\{u=0\}\cap Q} u_\delta^{\frac{n}{2}} \partial_x I(u_\delta) \varphi \Big| \leq c\delta^{\frac{n}{2}} \left(\iint \partial_x I(u_\delta)^2 \right)^{\frac{1}{2}}.$$

We deduce that g verifies (3.47).

3.4.3 Proof of Theorem 3.4

Consider the sequence (u^ϵ) such that u^ϵ solution of (3.21) introduced in the proof of Theorem 3.1 and Theorem 3.3. Recall that (28) implies that (u^ϵ) is bounded in $L^2(0,T;H_N^{\frac{\alpha}{2}+1}(\Omega))$.

Case $0 < \alpha < 1$. We recall that $(\partial_t u^\epsilon)$ is bounded in $L^2(0,T;W^{-1,l}(\Omega))$. So Aubin's lemma implies that (u^ϵ) converges in $L^2(0,T;C^{0,\beta}(\Omega))$ for all $\beta < \frac{\alpha+1}{2}$. We can thus find a subsequence, also denoted (u^ϵ), and a set $P \subset (0,T)$ such that $\mid (0,T) \setminus P \mid= 0$ and for all $t \in P$, $u^\epsilon(t,.)$ converges strongly in $C^\beta(\Omega)$.

We note that for all $t \in P$, u is strictly positive. Indeed if there exists $(t_0, x_0) \in P \times \Omega$ such that $u(t_0,x_0) = 0$ then for any $\beta < \frac{\alpha+1}{2}$ there exists a constant c_β such that for all $x \in \Omega$

$$u(t_0,x) \leq c_\beta \mid x - x_0 \mid^\beta .$$

Thus

$$\int G(u(t_0,x))dx \geq \int \frac{1}{(c_\beta \mid x - x_0 \mid^\beta)^{n-2}}dx.$$

Given $n > 4$, we can choose $\beta < \frac{\alpha+1}{2}$ such that $\beta(n-2) > 1$. We deduce

$$\int G(u(x,t_0))dx = \infty$$

which contradicts (3.40).

We deduce that there exists $\delta > 0$ (depending on t) such that for ϵ small enough

$$u^\epsilon(t,.) \geqslant \delta \text{ in } \Omega.$$

Note that

$$\liminf_{\epsilon\to 0} \int_\Omega f_\epsilon(u^\epsilon) \mid \partial_x I(u^\epsilon) \mid^2 dx < \infty \text{ for all } t \in P.$$

Indeed, if we denote

$$A_k = \{t \in P; \liminf_{\epsilon\to 0} \int_\Omega f_\epsilon(u^\epsilon) \mid \partial_x I(u^\epsilon) \mid^2 dx \geqslant k\}$$

then using (3.35) and Fatou's lemma we have

$$\begin{aligned} c &\geqslant \liminf_{\epsilon\to 0} \int_0^T \int_\Omega f_\epsilon(u^\epsilon) \mid \partial_x I(u^\epsilon) \mid^2 dxdt \\ &\geqslant \liminf_{\epsilon\to 0} \int_{A_k} \int_\Omega f_\epsilon(u^\epsilon) \mid \partial_x I(u^\epsilon) \mid^2 dxdt \\ &\geqslant \int_{A_k} \liminf_{\epsilon\to 0} \int_\Omega f_\epsilon(u^\epsilon) \mid \partial_x I(u^\epsilon) \mid^2 dxdt \\ &\geqslant k \mid A_k \mid . \end{aligned}$$

So $\mid A_k \mid\leqslant \frac{c}{k}$ and the set

$$\left\{t \in P; \liminf_{\epsilon\to 0} \int_\Omega f_\epsilon(u^\epsilon) \mid \partial_x I(u^\epsilon) \mid^2 dx = \infty\right\}$$

has measure zero. We deduce that for all $t \in P$

$$\liminf_{\epsilon \to 0} \int_\Omega |\, \partial_x I(u^\epsilon)\,|^2 \, dx < \infty$$

and so for all $t \in P$

$$u^\epsilon(t,.) \rightharpoonup u(t,.) \text{ in } H_N^{\alpha+1}(\Omega) - \text{weakly.}$$

In particular, we can pass to the limit in the flux $J_\epsilon = f_\epsilon(u^\epsilon)\partial_x I(u^\epsilon)$ and write

$$\lim_{\epsilon \to 0} J_\epsilon = J = f(u)\partial_x I(u) \text{ in } L^1(\Omega) \text{ and for almost } t \in (0,T).$$

Finally, since $u \in H_N^{\alpha+1}(\Omega)$, $u_x(t,x) = 0$ for $x \in \partial\Omega$ and almost every $t \in (0,T)$.

Case $1 \leqslant \alpha < 2$. We recall that $(\partial_t u^\epsilon)$ is bounded in $L^2(0,T;W^{-1,2}(\Omega))$. So Aubin's lemma implies that (u^ϵ) converges in $L^2(0,T;C^{0,\beta}(\Omega))$ for all $\beta < 1$. We can thus find a subsequence, also denoted (u^ϵ), and a set $P \subset (0,T)$ such that $|\,(0,T) \setminus P\,| = 0$ and for all $t \in P$, $u^\epsilon(t,.)$ converges strongly in $C^\beta(\Omega)$.

We note that for all $t \in P$, u is strictly positive. Indeed if there exists $(t_0,x_0) \in P \times \Omega$ such that $u(t_0,x_0) = 0$ then for any $\beta < 1$ there exists a constant c_β such that for all $x \in \Omega$

$$u(t_0,x) \leq c_\beta \,|\, x - x_0\,|^\beta \,.$$

Thus

$$\int G(u(x,t_0))dx \geq \int \frac{1}{(c_\beta \,|\, x - x_0\,|^\beta)^{n-2}} dx.$$

Given $n > 3$, we can choose $\beta < 1$ such that $\beta(n-2) > 1$. We deduce

$$\int G(u(x,t_0))dx = \infty$$

which contradicts (3.40).

The rest of the proof is the same as in the first case.

3.5 Proof of Proposition 3.14

Our aim is to prove that if

$$|\, u_\delta(t,x_2) - u_\delta(t,x_1)\,| \leq K \,|\, x_2 - x_1\,|^\gamma \tag{3.48}$$

for all $t \in (0,T), x_1$ and $x_2 \in \Omega$ with constant K independent of δ and T, then there exists a constant M independent of δ and T such that

$$|\, u_\delta(t_2,x_0) - u_\delta(t_1,x_0)\,| \leqslant M \,|\, t_2 - t_1\,|^{\frac{\gamma}{2\gamma+3}} \tag{3.49}$$

for all t_1 and $t_2 \in (0,T), x \in \Omega$. This proof is an adaptation of the proof done by Bernis-Friedman in case $\gamma = \frac{1}{2}$ [10, Lemma 2.1] for a general γ.

We suppose that for all $M > 0$ one can find $x_0 \in \Omega$ and $t_2, t_1 \in (0,T)$ such that

$$| u_\delta(t_2,x_0) - u_\delta(t_1,x_0) | > M \mid t_2 - t_1 \mid^{\frac{\gamma}{2\gamma+3}} . \tag{3.50}$$

We suppose that $u_\delta(t_2,x_0) > u_\delta(t_1,x_0)$ and that $t_2 > t_1$; thus

$$u_\delta(t_2,x_0) - u_\delta(t_1,x_0) > M(t_2-t_1)^\mu, \qquad 0 < t_1 < t_2 < T, \tag{3.51}$$

where $\mu = \frac{\gamma}{2\gamma+3}$. We have

$$\iint u_\delta \partial_t \varphi = -\iint h_\delta \partial_x \varphi \tag{3.52}$$

where $h_\delta = u_\delta^n \partial_x I(u_\delta)$, which is valid for any reasonable testfunction. Consider a testfunction φ of the form

$$\varphi(t,x) = \xi(x)\theta_\rho(t)$$

where ξ and θ_ρ are defined as follows.

$$\xi(x) = \xi_0\left(\frac{x-x_0}{(M/4K)^{\frac{1}{\gamma}}(t_2-t_1)^{\frac{\mu}{\gamma}}}\right)$$

where M is from (3.49) and K is from (3.48), and $\xi_0(x) = \xi_0(-x)$, $\xi_0 \in C_0^\infty(\Omega)$, $\xi_0(x) = 1$ if $0 \leqslant x < \frac{1}{2}$, $\xi_0(x) = 0$ if $x \geqslant 1$ and $\xi_0'(x) \leqslant 0$ if $x \geqslant 0$. Thus

$$\xi(x) = \begin{cases} 0 & \text{if } \mid x - x_0 \mid \geqslant (M/4K)^{\frac{1}{\gamma}} (t_2-t_1)^{\frac{\mu}{\gamma}} \\ 1 & \text{if } \mid x - x_0 \mid \leqslant \frac{1}{2} (M/4K)^{\frac{1}{\gamma}} (t_2-t_1)^{\frac{\mu}{\gamma}}. \end{cases}$$

We take

$$\theta_\rho(t) = \int_{-\infty}^t \theta_\rho'(s)ds \quad \text{where} \quad \theta_\rho'(t) = \begin{cases} \frac{1}{\rho} & \text{if } \mid t-t_2 \mid < \rho \\ \frac{-1}{\rho} & \text{if } \mid t-t_1 \mid < \rho \\ 0 & \text{elsewhere,} \end{cases}$$

and $\rho < \frac{1}{2}(t_2-t_1)$. So, we get

$$\iint u_\delta \xi(x)\theta_\rho'(t) = -\iint h_\delta \xi'(x)\theta_\rho(t).$$

The left-hand side satisfies

$$\iint u_\delta(t,x)\xi(x)\theta_\rho'(t) \to 4\int \xi(x)(u_\delta(t_2,x) - u_\delta(t_1,x))dx \quad \text{ as } \rho \to 0$$

To estimate the last expression, we shall only consider values of x such that

$$\mid x - x_0 \mid \leqslant (M/4K)^{\frac{1}{\gamma}} (t_2-t_1)^{\frac{\mu}{\gamma}}.$$

For such values,

$$\begin{aligned} u_\delta(t_2,x)-u_\delta(t_1,x) &= [u_\delta(t_2,x)-u_\delta(t_2,x_0)]+[u_\delta(t_2,x_0)-u_\delta(t_1,x_0)]+[u_\delta(t_1,x_0)-u_\delta(t_1,x)] \\ &\geqslant -2K \mid x-x_0 \mid^\gamma +M(t_2-t_1)^\mu \\ &\geqslant \frac{M}{2}(t_2-t_1)^\mu. \end{aligned}$$

Hence, by assuming that the set $\{\xi=1\}$ is included in Ω and by a change of variables in x,

$$\int \xi(x)(u_\delta(t_2,x)-u_\delta(t_1,x))dx \geqslant \left(\int \xi_0(x)dx\right)\frac{M}{2}(t_2-t_1)^\mu \frac{M^{\frac{1}{\gamma}}}{(4K)^{\frac{1}{\gamma}}}(t_2-t_1)^{\frac{\mu}{\gamma}}.$$

On the other hand, we have

$$\left|\iint h_\delta \xi'(x)\theta_\rho(t)\right| \leqslant \left(\iint h_\delta^2\right)^{\frac{1}{2}}\left(\iint (\xi'\theta_\rho)^2\right)^{\frac{1}{2}}.$$

But $\xi'(x)=\left((M/4K)^{\frac{1}{\gamma}}(t_2-t_1)^{\frac{\mu}{\gamma}}\right)^{-1}\xi_0'\left(\frac{x-x_0}{(M/4K)^{\frac{1}{\gamma}}(t_2-t_1)^{\frac{\mu}{\gamma}}}\right)$, so since h_δ is uniformly bounded in $L^2(Q)$ we have

$$\left|\iint h_\delta \xi'(x)\theta_\rho(t)\right| \leqslant \frac{C}{\frac{M^{\frac{1}{\gamma}}}{(4K)^{\frac{1}{\gamma}}}(t_2-t_1)^{\frac{\mu}{\gamma}}}\left(\iint h_\delta^2\right)^{\frac{1}{2}}\frac{M^{\frac{1}{2\gamma}}}{(4K)^{\frac{1}{2\gamma}}}(t_2-t_1)^{\frac{\mu}{2\gamma}}(t_2-t_1-2\rho)^{\frac{1}{2}}.$$

Thus by letting $\rho\to 0$ we conclude that

$$M^{1+\frac{1}{\gamma}}(t_2-t_1)^{\mu+\frac{\mu}{\gamma}} \leqslant CM^{-\frac{1}{2\gamma}}(t_2-t_1)^{\frac{\mu}{2\gamma}-\frac{\mu}{\gamma}+\frac{1}{2}},$$

where C is a new constant independent of δ, T and M, thus

$$M \leqslant c^{\frac{2\gamma}{3+2\gamma}}(t_2-t_1)^{-\mu+\frac{\gamma}{2\gamma+3}}.$$

Since $\mu=\frac{\gamma}{2\gamma+3}$, we find that $M\leqslant C^{\frac{2\gamma}{3+2\gamma}}$, and the lemma follows.

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Chapter 4

Regularity of solutions of a fractional porous medium equation

Abstract

This chapter is concerned with a porous medium equation whose pressure law is nonlinear and nonlocal. We prove that the weak solutions constructed by Biler, Imbert and Karch (2015) are locally Hölder continuous in time and space. We adapt the proof of Caffarelli, Soria and Vázquez (2013) who treated the case of a linear pressure law. Classical De Giorgi parabolic regularity techniques are tailored in an appropriate way in order to deal with the equation under study. The two main ingredients are, on the one hand, the derivation of the local energy estimates (in the spirit of parabolic De Giorgi classes) and on the other hand, a so-called "intermediate value lemma", in the spirit of the work of Caffarelli, Chan and Vasseur (2011).

4.1 Introduction

In this work, we study the regularity of nonnegative weak solutions of the following degenerate nonlinear nonlocal evolution equation

$$\partial_t u = \nabla \cdot \left(u \nabla^{\alpha-1} G(u) \right), \quad t > 0, \quad x \in \mathbb{R}^N, \tag{4.1}$$

where $G(u) = u^{m-1}$ with $m \geq 2$. The equation is supplemented with an initial data

$$u(0,x) = u_0(x) \tag{4.2}$$

that we will assume to be both non-negative and integrable on $\mathbb{R}^N$.

For $\alpha \in (0,2)$, the symbol $\nabla^{\alpha-1}$ denotes the integro-differential operator $\nabla(-\Delta)^{\frac{\alpha}{2}-1}$. It is a nonlocal operator of order $\alpha - 1$. For a smooth and bounded function v, it has the following singular integral representation

$$\nabla^{\alpha-1}v(x) = c_{\alpha,N}\int_{\mathbb{R}^N}(v(y)-v(x))\frac{y-x}{|y-x|^{N+\alpha}}dy \tag{4.3}$$

with a suitable constant $c_{\alpha,N}$. Moreover, we have $\nabla\cdot\nabla^{\alpha-1} = -(-\Delta)^{\frac{\alpha}{2}}$.

Our main result is the Hölder regularity of weak solutions of (4.1). For short, let us write $Q_T = (0,T)\times\mathbb{R}^N$.

Definition 4.1 (Weak solutions). *A function $u : Q_T \to \mathbb{R}$ is a weak solution of (4.1)-(4.2) if $u \in L^1(Q_T)$, $\nabla^{\alpha-1}(|u|^{m-2}u) \in L^1_{\mathrm{loc}}(Q_T)$, $|u|\nabla^{\alpha-1}(|u|^{m-2}u) \in L^1_{\mathrm{loc}}(Q_T)$ and*

$$\iint u\partial_t\varphi \mathrm{d}t\mathrm{d}x - \iint |u|\nabla^{\alpha-1}(|u|^{m-2}u).\nabla\varphi \mathrm{d}t\mathrm{d}x = -\int u_0(x)\varphi(0,x)\mathrm{d}x \tag{4.4}$$

for all test functions $\varphi \in C^\infty(Q_T)\cap C^1(\bar{Q}_T)$ with compact support in the space variable x and that vanish near $t = T$.

Theorem 4.2 (Hölder regularity). *Let us assume that $\alpha \in (1,2)$ and $m \geq 2$. For any initial data*

$$u_0 \in L^1 \cap L^\infty(\mathbb{R}^N;\mathbb{R}_+),$$

weak solutions u of (4.1)-(4.2) are Hölder continuous at strictly positive times. More precisely, there is $\beta \in (0,1)$ depending only on N, m and α such that for all $T_0, T_1 > 0$ with $T_0 < T_1$,

$$[u]_{C^\beta([T_0,T_1]\times\mathbb{R}^N)} \leq C\|u\|_{L^\infty([T_0/2,T_1]\times\mathbb{R}^N)} \tag{4.5}$$

where C only depends on N, m and α and T_0.

Remark 4.3. *Weak solutions have been constructed in [16] under the assumptions of Theorem 4.2 and even also for some range of values of $m < 2$. For a precise statement, see Theorem 4.6 below. Our proof can probably be adapted to those small values of m, but it requires some modifications and additional work.*

Remark 4.4. *The (linear) case $m = 2$ was treated in [19, 22] not only for $\alpha \in (1,2)$ but for any $\alpha \in (0,2)$. We believe that the case $\alpha \in (0,1]$ with a general m can probably be treated by adapting the subsequent proof with the ideas from [19, 22], but it still requires additional work.*

Remark 4.5. *One could also study the regularity of unsigned weak solutions of the equation (4.9) below, which is the unsigned version of the equation (4.1). Since the subsequent proof is local, it is probably possible to extend our result in this direction.*

Review of the literature. Let us briefly recall how the porous medium equation is derived from the law of conservation of mass, for a gas propagating in a homogeneous porous medium [4, 77]:

$$\partial_t u + \nabla\cdot(uv) = 0.$$

In this equation, $u \geq 0$ denotes the density of the gas and $v \in \mathbb{R}^N$ is the locally averaged velocity. Darcy's law states that $v = -\nabla p$ where p denotes the pressure. Finally, the pressure law implies that p is a monotone operator of u i.e. $p = f(u)$. This leads us to the following equation

$$\partial_t u = \nabla\cdot(u\nabla f(u)). \tag{4.6}$$

The case $p = u$ is the simplest pressure law and leads to the Boussinesq's equation [6, 17]:

$$\partial_t u = c\Delta(u^2). \tag{4.7}$$

L. Caffarelli and J. L. Vázquez [21] studied the following equation

$$\partial_t u = \nabla\cdot\left(u\nabla(-\Delta)^{-s}u\right), \quad t > 0, \quad x \in \mathbb{R}^N. \tag{4.8}$$

This equation was proposed by [21] to add long-distance effects in the physical model (for further details, see the motivations therein). They study this problem with nonnegative initial data that are integrable and decay at infinity. For $s = \frac{2-\alpha}{2} \in (0,1)$ and $m = 2$, our equation (4.1) coincides with (4.8).

The existence of mass-preserving nonnegative weak solutions satisfying energy estimates has been proved in [21]. Such solutions have a finite propagation speed. Their asymptotic behavior as $t \to \infty$ has been studied in [24]. Moreover, in [19] and [22], the boundedness and the Hölder regularity of nonnegative solutions has been obtained for $s \in (0,1)$.

The proof of the Hölder regularity in the range $s \in (0,1/2)$ is based on De Giorgi-type oscillation lemmas and on the scaling property (see (4.16) below) of the equation. For a general review of the De Giorgi method for classical elliptic and parabolic equations, we refer for instance to [29], [63], [76] and [20]. The regularity result in the case $s \in (1/2,1)$, which corresponds to $\alpha \in (0,1)$ for us, is more difficult due to convection effects that appear and make some integrals diverge. The method proposed in [19] consists in a geometrical transformation that absorbs the uncontrolled growth of one of the integrals that appear in the iterated energy estimates.

The most delicate situation, which is the case $s = 1/2$, has been treated in [22]. The authors performed an iteration analysis that combines consecutive applications of scaling and geometrical transformations.

A similar De Giorgi method is also used in [18] to prove the Hölder regularity for nonlinear nonlocal time-dependent variational equations. In this case however, $-u$ satisfies the same equation as u which slightly simplifies the proof.

In [15] and [16], P. Biler, C. Imbert and G. Karch consider a problem similar to (4.1)-(4.2), but with u unsigned. They prove, under some conditions on m (see (4.13) below), the existence of bounded and mass-preserving weak solutions for the Cauchy problem

$$\partial_t u = \nabla \cdot \left(|u| \nabla^{\alpha-1}(|u|^{m-2}u) \right), \quad t > 0, \quad x \in \mathbb{R}^N, \tag{4.9}$$

with initial condition

$$u(0,x) = u_0(x) \tag{4.10}$$

where u_0 is an integrable but not necessarily positive function on $\mathbb{R}^N$. Moreover, they show that the solution u is nonnegative if the initial condition u_0 is so, in which case the solution is a solution of our problem (4.1)-(4.2). In the sequel of this paper, this existence result is our starting point. The finite speed of propagation for these nonnegative weak solutions has been proved in [53] and holds under the same conditions on m.

A variant of the porous medium equation with both a fractional potential pressure and a fractional time derivative has been studied in [2]

$$D_t^\alpha u - \nabla \cdot (u \nabla (-\Delta)^{-\sigma} u) = f.$$

where D_t^α is a Caputo-type time derivative. The authors study both the existence and the Hölder regularity of the solutions, using the De Giorgi method as in [19].

Organization of the chapter and general ideas. This chapter is organized as follows:

- In Section 2 we recall briefly how the existence Theorem 4.6 was established.
- Section 3 is devoted to the local energy estimates satisfied by a bounded weak solution. We first derive general energy estimates (Theorem 4.11) and then we localize them and we improve them by estimating in a more precise way the "dissipation" terms (Proposition 4.28).
- In Section 4 we prove the first lemmas of De Giorgi. The idea is that a direct application of the energy estimate along a sequence of macroscopic space-time balls leads to a point-wise upper-bound, provided that the measure of the set where u is small is *sufficiently large*. Similarly, one can get a point-wise lower-bound from knowing that u is large enough on a large set.

  In Section 5, we move on to the lemma on intermediate values. It roughly claims that if both the sets where u is small and where u is large are substantial measure-wise, then, thanks to the "good extra term" of the local energy balance, u also has to spend a substantial space-time in-between. In naive words, we quantify the cost of oscillations.

In Section 6, this idea allows us to subtly improve the first lemma of De Giorgi: the point-wise upper bound can be ascertained provided only that the measure of the set where u is small is *not too small*. The proof comes naturally by contradiction: if the upper bound could not be improved, then too much energy would be lost in the oscillations induced between the maximal point and the low values set. Section 6 seems to be a subtle refinement of Section 4, but it suffices to prove Theorem 4.2.

- In Section 7, one follows a “zoom-in and enlarge” sequence of solutions, along which the oscillation is controlled either from above by the refined first De Giorgi lemma of Section 6 or from below by the crude one of Section 4. The improvement of Section 6 was needed to have a clean alternative at this point. This scheme leads directly to the Hölder regularity of the solution u.

4.2 Preliminaries

Notations. In this work, we denote by B_r the ball of $\mathbb{R}^N$ of radius $r > 0$ and of center 0. For any measurable function v we define its positive and negative part by:

$$v_+ = \max(0, v) \quad \text{and} \quad v_- = \max(0, -v). \tag{4.11}$$

We will often use the following notation and identities:

$$a \vee b = \max\{a, b\} = a + (b - a)_+ \quad \text{and} \quad a \wedge b = \min\{a, b\} = a - (b - a)_-.$$

The fractional Laplacian has the following singular integral expression

$$(-\Delta)^{\alpha/2} v(x) = -\int_{\mathbb{R}^N} (v(y) - v(x)) \frac{c^0_{\alpha,N}}{|x-y|^{N+\alpha}} dy \tag{4.12}$$

where $c^0_{\alpha,N}$ is a constant only depending on α and N.

Finally, let us point out that we will usually specify the domain of each integral, except for double space integrals where $\iint f(x, y)\mathrm{d}x\mathrm{d}y$ will denote an integral over $\mathbb{R}^N \times \mathbb{R}^N$, unless stated otherwise.

Weak solutions. The existence of positive weak solutions for our Cauchy problem at hand (4.1)-(4.2) was proved in [16].

Theorem 4.6 (Existence of weak solutions, from [16, Theorem 2.6])**.** *Let $\alpha \in (0, 2)$ and*

$$m > \max\left\{1 + \frac{1-\alpha}{N}; 3 - \frac{2}{\alpha}\right\}. \tag{4.13}$$

For any $u_0 \in L^1(\mathbb{R}^N; \mathbb{R}_+)$, the Cauchy problem (4.1)-(4.2) admits a weak solution u on $(0, +\infty) \times \mathbb{R}^N$. Moreover,

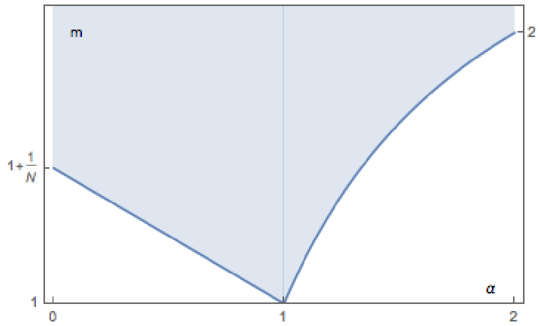
$$\int_{\mathbb{R}^N} u(t, x)\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x \tag{4.14}$$

and for each $p \in [1, \infty]$ and $t > 0$

$$\|u(t)\|_p \leq \min\left\{C_{N,\alpha,m}\|u_0\|_1^{\frac{N(m-1)/p+\alpha}{N(m-1)+\alpha}} t^{-\frac{N}{N(m-1)+\alpha}\left(1-\frac{1}{p}\right)}; \|u_0\|_p\right\}. \tag{4.15}$$

The constant $C_{N,\alpha,m}$ is independent of p, t and u_0.

The admissible pairs of (α, m) in Theorem 4.6 are illustrated on the following drawing.



However, in the rest of this paper, we restrict ourselves to the case $m \geq 2$ and $\alpha \in (1, 2)$.

Scaling invariance of the equation. The solutions of (4.1) have the following scaling property.

Lemma 4.7. *If u satisfies (4.1) then*

$$u_{A,B,C}(t, x) = Au(Bt, Cx) \tag{4.16}$$

also satisfies (4.1) provided that $B = A^{m-1}C^{\alpha}$.

Remark 4.8. *In (4.16), both parameters A and C can take arbitrary values. It is therefore possible to rescale the physical space independently from a change of amplitude of the solution. This double-scaling property plays a key role in the final argument of the proof of Theorem 4.2 (see §4.7).*

A characterization of the Hölder continuity. To prove the Hölder regularity we will use the following lemma which is part of the folklore:

Lemma 4.9. *Let u be a function defined in $(-1, 0) \times B_1$ such that for any $(t_0, x_0) \in (-1/2, 0) \times B_{1/2}$ and any $r \in (0, 1/2)$ we have*

$$\operatorname*{osc}_{(t_0-r,t_0)\times B_r(x_0)} u \leq Cr^{\beta}.$$

Then u is β-Hölder continuous in $(-1/2, 0) \times B_{1/2}$.

Sobolev embedding. The following local Sobolev's embedding theorem will be useful:

$$H^{\frac{\alpha}{2}}(B_r) \subset L^p(B_r)$$

for $p = \frac{2N}{N-\alpha} > 2$ and any $r > 0$. More precisely, there is a constant C independent of r such that:

$$\left(\int_{B_r} u^p \mathrm{d}x\right)^{\frac{2}{p}} \leq C \iint_{B_r \times B_r} (u(y) - u(x))^2 \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}. \tag{4.17}$$

4.3 Energy estimates

In this section, we derive the necessary energy estimates to follow De Giorgi's original path towards the Hölder continuity of the solutions. As we will ultimately use Lemma 4.9 on a dyadic rescaled sequence of solutions, we cannot take for granted the value of the L^∞ bound of the weak-solution. Instead, we have to prove the energy estimates for weak solutions that are potentially allowed to grow as a mild power-law at infinity.

Definition 4.10. *For any $\epsilon > 0$, let us define*

$$\Psi_\epsilon(x) = (|x|^\epsilon - 2)_+. \tag{4.18}$$

Theorem 4.11 (Energy estimates)**.** *Let us assume that $\alpha \in (1,2)$ and that $m \geq 2$. Then there are absolute constants $\epsilon_0 >$ and $C > 0$ (depending only on N, α, m) such that for any weak solution u of (4.1) in $(-2,0] \times \mathbb{R}^N$ satisfying for some $\epsilon \in (0, \epsilon_0)$ that*

$$\forall t \in (-2,0], \quad \forall x \in \mathbb{R}^N, \qquad 0 \leq u(t,x) \leq 1 + \Psi_\epsilon(x) \tag{4.19}$$

and for any smooth truncation functions $\varphi_\pm : \mathbb{R}^N \to [0, +\infty)$ such that

- *$1/4 < \varphi_+ \leq 1 + \Psi_\epsilon$ on $\mathbb{R}^N$ with $\varphi_+ = 1 + \Psi_\epsilon$ outside $B_{2^{1/\epsilon}}$ and*

$$|\nabla\varphi_+/\varphi_+| + |\nabla\varphi_+| + |\nabla\varphi_+/\varphi_+|^2 + |\nabla\varphi_+|^2 \leq C_{\varphi_+},$$

- *$0 < \varphi_- \leq 1$ on B_2 but $\varphi_- \equiv 0$ outside B_2 with*

$$|\nabla\varphi_-/\varphi_-| \leq C_{\varphi_-}\varphi_-^{-1/m_0}$$

on $\bar{B}_2$ for some $m_0 \geq 2$,

the two following energy estimates hold true for any $-2 < t_1 < t_2 < 0$:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} (u(t_2,x) - \varphi_\pm(x))_\pm^2 \, \varphi_\pm^{-1}(x)\mathrm{d}x \\
&\quad + \frac{1}{4}\int_{t_1}^{t_2} \iint \Big((u(t,y) - \varphi_\pm(y))_\pm - (u(t,x) - \varphi_\pm(x))_\pm\Big)^2 \mathcal{D}_G(u(t,x), u(t,y)) \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\mathrm{d}t \\
&\quad + \frac{1}{4}\int_{t_1}^{t_2} \iint (u(t,x) - \varphi_\pm(x))_+ (u(t,y) - \varphi_\pm(y))_- \mathcal{D}_G(u(t,x), u(t,y)) \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\mathrm{d}t
\end{aligned} \tag{4.20}$$

$$\leq \int_{\mathbb{R}^N} (u(t_1,x) - \varphi_\pm(x))_\pm^2 \, \varphi_\pm^{-1}(x)\mathrm{d}x + CC_{\varphi_\pm} \left|\{(u - \varphi_\pm)_\pm > 0\} \cap (t_1,t_2) \times \mathbb{R}^N\right|$$

where $\mathcal{D}_G$ is defined for $a, b \in \mathbb{R}$ by $\mathcal{D}_G(a,b) = \frac{G(a)-G(b)}{a-b}$.

Remark 4.12. *The $\pm$ notation means that the inequality (4.20) stands true if all the symbols $\pm$ are either simultaneously replaced by $+$ or by $-$. Hybrid choices are not allowed.*

Remark 4.13. *The functions $\varphi_\pm$ serve a truncation purpose which should get plain as the proof unfolds. For example, $(u-\varphi_+)_+ \equiv 0$ outside $B_{2^{1/\epsilon}}$ and similarly $(u-\varphi_-)_- \equiv 0$ outside B_2 which in particular takes the ambiguity out of the first integral as φ_-^{-1} does not have to be computed ouside B_2.*

Remark 4.14. *Obviously, each term of (4.20) is nonnegative. The third term in (4.20) that mixes a positive and a negative part is called the "good extra term" in [18]. It will play a crucial role in the proof of the Lemma on intermediate values (see Section 5). By themselves, the other non-negative terms of (4.20) would be sufficient to prove the first lemmas of De Giorgi (see Section 4).*

Remark 4.15. *In order to prove the energy estimates we will introduce an alternate energy functional:*

$$\mathcal{E}_\pm(t) = \int_{\mathbb{R}^N} H\left(1 \pm \frac{(u-\varphi_\pm)_\pm(x)}{\varphi(x)_\pm}\right)\varphi_\pm(x)\mathrm{d}x \tag{4.21}$$

where H is an appropriate convex function. The functional $\mathcal{E}_+$ is well-defined since φ_+ does not vanish. Note also that $\frac{(u-\varphi_+)_+}{\varphi_+} \in [0, (\inf_{B_{2^{1/\epsilon}}} \varphi_+)^{-1}] \subset [0,4]$. As far as $\mathcal{E}_-$ is concerned, we remark that $1 - \frac{(u-\varphi_-)_-}{\varphi_-} = 1 \wedge \frac{u}{\varphi_-} \in [0,1]$. In particular, the spurious fraction simply boils down to $H(1)$ when $x \notin B_2$. Moreover, only the values of $H(1+r)$ for $r \in [-1,4]$ are relevant for (4.21).

The proof of Theorem 4.11 is structured as follows. First we explain why it is enough to consider the alternate energy functional (4.21). Then we estimate the error terms for the energy $\int (u-\varphi_+)_+^2 \varphi_+^{-1}$. Finally, we deal with the case of $\int (u-\varphi_-)_-^2 \varphi_-^{-1}$.

4.3.1 An alternate energy functional

We consider the convex function $H : [0,+\infty) \to [0,+\infty)$ such that

$$H''(r) = r^{-1} \quad \text{and} \quad H(1) = H'(1) = 0.$$

The function H is given by the formula $H(r) = r\ln r - r + 1$. Following [19], we consider the energy functional (4.21). As $\frac{1}{4}r^2 \le H(1+r) \le r^2$ for $r \in [-1,4]$, the proof of Theorem 4.11 is reduced to proving that

$$\begin{aligned} \mathcal{E}_\pm(t_2) + \int_{t_1}^{t_2} \left(\mathcal{B}_G((u(t)-\varphi_\pm)_\pm, (u(t)-\varphi_\pm)_\pm) - \mathcal{B}_G((u(t)-\varphi_\pm)_+, (u(t)-\varphi_\pm)_-)\right) \mathrm{d}t \\ \le \mathcal{E}_\pm(t_1) + C\left|\{(u-\varphi_\pm)_\pm > 0\} \cap (t_1,t_2) \times \mathbb{R}^N\right| \end{aligned} \tag{4.22}$$

where the bilinear form $\mathcal{B}_G$ is defined as follows

$$\mathcal{B}_G(v,w) = \iint (v(y)-v(x))(w(y)-w(x))\mathcal{D}_G(u(x),u(y))\frac{c^0_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}. \tag{4.23}$$

Let us recall that $\mathcal{D}_G(a,b) = \frac{G(a)-G(b)}{a-b}$.

Le us compute first the time derivative of the alternate energy functional:

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_\pm(t) &= \int_{(u-\varphi_\pm)_\pm>0} H'\left(1\pm\frac{(u-\varphi_\pm)_\pm}{\varphi_\pm}\right)\partial_t u\mathrm{d}x\\ &= \int H'\left(1\pm\frac{(u-\varphi_\pm)_\pm}{\varphi_\pm}\right)\partial_t u\mathrm{d}x\\ &= -\int H''\left(1\pm\frac{(u-\varphi_\pm)_\pm}{\varphi_\pm}\right)u\nabla^{\alpha-1}G(u)\cdot\nabla\left(\frac{\pm(u-\varphi_\pm)_\pm}{\varphi_\pm}\right).\end{aligned}$$

This formal computation can be made rigorous thanks to the regularity of some approximate solutions as it was done in [16]. We now remark that, on the set $\{\pm(u-\varphi_\pm)>0\}$, we have the following remarkable identity:

$$H''\left(1\pm\frac{(u-\varphi_\pm)_\pm}{\varphi_\pm}\right)u = \varphi_\pm.$$

This implies that

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_\pm(t) &= -\int\nabla^{\alpha-1}G(u)\cdot\nabla(\pm(u-\varphi_\pm)_\pm)\mathrm{d}x + \int(\pm(u-\varphi_\pm)_\pm)\nabla^{\alpha-1}G(u)\cdot\frac{\nabla\varphi_\pm}{\varphi_\pm}\mathrm{d}x\\ &= -\mathcal{B}_G(\pm(u-\varphi_\pm)_\pm,u) + \mathcal{Q}_G(\pm(u-\varphi_\pm)_\pm,u)\end{aligned} \tag{4.24}$$

with

$$\mathcal{Q}_G(v,w) = \iint v(x)(w(y)-w(x))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_\pm(x)}{\varphi_\pm(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}. \tag{4.25}$$

Up to now, the second variable $w=u$ of $\mathcal{B}_G(\cdot,u)$ or $\mathcal{Q}_G(\cdot,u)$ could have been simplified with the denominator of the kernel $\mathcal{D}_G(u(x),u(y))$. We are now going to split that second variable. More precisely, using the fact that $u = \pm(u-\varphi_\pm)_\pm \mp (u-\varphi_\pm)_\mp + \varphi_\pm$, we get

$$\mathcal{B}_G(\pm(u-\varphi_\pm)_\pm,u) = \mathcal{B}_G((u-\varphi_\pm)_\pm,(u-\varphi_\pm)_\pm) - \mathcal{B}_G((u-\varphi_\pm)_+,(u-\varphi_\pm)_-) + \mathcal{B}_G(\pm(u-\varphi_\pm)_\pm,\varphi_\pm).$$

Combining this with (4.24) yields

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_\pm(t) + \mathcal{B}_G((u-\varphi_\pm)_\pm,(u-\varphi_\pm)_\pm) - \mathcal{B}_G((u-\varphi_\pm)_+,(u-\varphi_\pm)_-)&\\ = -\mathcal{B}_G(\pm(u-\varphi_\pm)_\pm,\varphi_\pm) + \mathcal{Q}_G(\pm(u-\varphi_\pm)_\pm,u).&\end{aligned} \tag{4.26}$$

We remark that the two terms $\mathcal{B}_G((u-\varphi_\pm)_\pm,(u-\varphi_\pm)_\pm)$ and $-\mathcal{B}_G((u-\varphi_\pm)_+,(u-\varphi_\pm)_-)$ are non-negative. Following [18, 19], the first one is referred to as the “coercive term” while the second one is referred to as the “good extra term”. The rest of the proof of Theorem 4.11 consists in controlling the terms in the right hand side of (4.26) by those two non-negative terms plus $C|\{(u-\varphi_\pm)_\pm>0\}\cap(t_1,t_2)\times\mathbb{R}^N|$.

Proof of Theorem 4.11. As far as $\mathcal{E}_+$ is concerned, one has to combine (4.26) with the subsequent Lemmas 4.17, 4.18, 4.19, 4.20, 4.21, 4.22 and 4.23. As far as $\mathcal{E}_-$ is concerned, one has to combine (4.26) with the subsequent Lemmas 4.17, 4.25 and 4.26. Given the range of admissible parameters (α, m) in Theorem 4.11, the critical value for ϵ_0 is

$$\epsilon_0 = (\alpha - 1)/(m - 1).$$

The proof will be complete once the lemmas mentioned above are established. □

In what follows it will be convenient to write

$$u_\varphi^\pm = (u - \varphi_\pm)_\pm.$$

An inequality $A \leq cB$ that involves a universal constant c depending on N, α, m and $C_{\varphi_\pm}$ will be denoted by $A \lesssim B$.

We will use repeatedly that (4.19) implies that

$$\mathcal{D}_G(u(x), u(y)) \leq \sup_{z \in [u(x), u(y)]} |G'(z)| \lesssim (1 \vee |x| \vee |y|)^{\epsilon(m-2)} \tag{4.27}$$

Here in (4.27) we critically used the fact that $m \geq 2$ and $1 + \Psi_\epsilon(x) \leq (1 \vee |x|)^\epsilon$. Another crucial observation that follows from $m \geq 2$ is that G is convex ; one has therefore

$$\mathcal{D}_G(a, b) \geq \mathcal{D}_G(a', b) \geq \mathcal{D}_G(a', b') \tag{4.28}$$

as soon as $0 \leq a' \leq a$ and $0 < b' \leq b$.

Lemma 4.16. *For $m \geq 2$, if at least one of the values $u(x)$ or $u(y)$ is larger than $c > 0$ then*

$$\mathcal{D}_G(u(t, x), u(t, y)) \geq c^{m-2}. \tag{4.29}$$

Proof. If $u(x) \geq c$ and $u(y) \geq c$ then $\mathcal{D}_G(u(x), u(y)) = G'(z)$ for some $z \in [u(x), u(y)]$ with an increasing function $G'(z) = (m-1)z^{m-2}$. One has therefore

$$\mathcal{D}_G(u(x), u(y)) \geq (m-1)c^{m-2} \geq c^{m-2}$$

in that case. On the other hand, if $u(x) \geq c \geq u(y) \geq 0$ then by convexity of G, the inequality (4.28) implies

$$\mathcal{D}_G(u(x), u(y)) \geq \mathcal{D}_G(0, c) = c^{m-2}.$$

The case $u(y) \geq c \geq u(x) \geq 0$ is similar and the lemma follows. □

4.3.2 Common estimate for $\mathcal{E}_+$ and $\mathcal{E}_-$

Controlling $\mathcal{Q}_G$ will require a different approach for $\mathcal{E}_+$ and for $\mathcal{E}_-$. Dealing with the first term on the right-hand side of (4.26) is much easier.

Lemma 4.17. *For $\epsilon < \frac{\alpha}{m}$, we have:*

$$-\mathcal{B}_G(\pm(u-\varphi_\pm)_\pm,\varphi_\pm) \leq \frac{1}{2}\mathcal{B}_G(u_\varphi^\pm,u_\varphi^\pm) + C_{\varphi_\pm}|\{(u-\varphi_\pm)_\pm > 0\}|$$

where $C_{\varphi_\pm} \gtrsim 1 + \|\nabla\varphi_\pm\|_\infty^2$.

Proof. Keeping track of the support of $u_\varphi^\pm$ we write

$$\begin{aligned}
-\mathcal{B}_G(\pm(u-\varphi_\pm)_\pm,\varphi_\pm) = &\mp\iint (u_\varphi^\pm(y)-u_\varphi^\pm(x))(\varphi_\pm(y)-\varphi_\pm(x))\mathcal{D}_G(u(x),u(y))\frac{c_{\alpha,N}^0\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
\leq &\frac{1}{2}\mathcal{B}_G(u_\varphi^\pm,u_\varphi^\pm)\\
&+\frac{1}{2}\iint(\varphi_\pm(y)-\varphi_\pm(x))^2(\mathbb{I}_{u_\varphi^\pm(x)>0}+\mathbb{I}_{u_\varphi^\pm(y)>0})\mathcal{D}_G(u(x),u(y))\frac{c_{\alpha,N}^0\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}.
\end{aligned}$$

Thanks to (4.27) we estimate the second term of the right hand side as follows

$$\begin{aligned}
&\iint(\varphi_\pm(y)-\varphi_\pm(x))^2(\mathbb{I}_{u_\varphi^\pm(x)>0}+\mathbb{I}_{u_\varphi^\pm(y)>0})\mathcal{D}_G(u(x),u(y))\frac{c_{\alpha,N}^0\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\quad\lesssim \int_{u_\varphi^\pm>0}\left\{\int(\varphi_\pm(y)-\varphi_\pm(x))^2(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{\mathrm{d}y}{|y-x|^{N+\alpha}}\right\}\mathrm{d}x.
\end{aligned}$$

Since $\{u_\varphi^\pm > 0\}$ is contained in $B_{2\frac{1}{\epsilon}}$, we have

$$\begin{aligned}
&\int(\varphi_\pm(y)-\varphi_\pm(x))^2(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\qquad\lesssim \|\nabla\varphi_\pm\|_{L^\infty}^2\int_{|y-x|\leq 1}\frac{\mathrm{d}y}{|y-x|^{N+\alpha-2}}+\|\varphi_\pm\|_{L^\infty(B_{2^{1/\epsilon}})}^2\int_{|y-x|\geq 1}|y-x|^{\epsilon m}\frac{\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\qquad\lesssim 1
\end{aligned}$$

provided $\epsilon < \alpha/m$. This yields the desired estimate. □

4.3.3 Estimates for $\mathcal{E}_+$

We first estimate $\mathcal{Q}_G((u-\varphi_+)_+,u)$. In order to do so, we split it as follows (see [19])

$$\mathcal{Q}_G((u-\varphi_+)_+,u) = Q_{\text{int}}^{+,+} + Q_{\text{int}}^{+,-} + Q_{\text{int}}^{+,0} + Q_{\text{out}}^{+,+} + Q_{\text{out}}^{+,-} + Q_{\text{out}}^{+,0}$$

with

$$\begin{cases} Q^{+,+}_{\text{int}} &= Q^{\text{int}}_G((u-\varphi_+)_+,(u-\varphi_+)_+) \\ Q^{+,-}_{\text{int}} &= Q^{\text{int}}_G((u-\varphi_+)_+,-(u-\varphi_+)_-) \\ Q^{+,0}_{\text{int}} &= Q^{\text{int}}_G((u-\varphi_+)_+,\varphi_+) \\ Q^{+,+}_{\text{out}} &= Q^{\text{out}}_G((u-\varphi_+)_+,(u-\varphi_+)_+) \\ Q^{+,-}_{\text{out}} &= Q^{\text{out}}_G((u-\varphi_+)_+,-(u-\varphi_+)_-) \\ Q^{+,0}_{\text{out}} &= Q^{\text{out}}_G((u-\varphi_+)_+,\varphi_+). \end{cases}$$

where Q^{int}_G and Q^{out}_G are defined by

$$Q^{\text{int}}_G(v,w) = \iint_{|x-y|\le\eta} v(x)(w(y)-w(x))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_+(x)}{\varphi_+(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}},$$
$$Q^{\text{out}}_G(v,w) = \iint_{|x-y|\ge\eta} v(x)(w(y)-w(x))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_+(x)}{\varphi_+(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}$$

for some small parameter $\eta\in(0,1)$ to be fixed later (see Lemma 4.19 below). We estimate successively the six terms appearing in this decomposition. Note that we only need upper estimates as negative terms can be discarded from the right-hand side of (4.26).

Lemma 4.18. *For $\alpha<2\le m$, one has*

$$Q^{+,+}_{\text{int}} \le \frac{1}{4}\mathcal{B}_G(u^+_\varphi,u^+_\varphi) + C_{\varphi_+}|\{(u-\varphi_+)>0\}|$$

where $C_{\varphi_+}\gtrsim\|\nabla\varphi_+/\varphi_+\|^2_\infty$.

Proof. We first write a Cauchy-Schwarz type inequality and (4.27):

$$\begin{aligned} Q^{+,+}_{\text{int}} &= Q^{\text{int}}_G(u^+_\varphi,u^+_\varphi) \\ &= \iint_{|x-y|\le\eta} u^+_\varphi(x)(u^+_\varphi(y)-u^+_\varphi(x))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_+(x)}{\varphi_+(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}} \\ &\le \frac{1}{4}\mathcal{B}_G(u^+_\varphi,u^+_\varphi) + \iint_{|x-y|\le\eta}(u^+_\varphi)^2(x)(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{|\nabla\varphi_+(x)|^2}{\varphi_+^2(x)}\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha-2}}. \end{aligned}$$

Since φ_+ is Lipschitz continuous and $u^+_\varphi(x)\le u(x)\le(1\vee|x|)^\epsilon$, we have for $m\ge2$:

$$\iint_{|x-y|\le\eta}(u^+_\varphi)^2(x)(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{|\nabla\varphi_+(x)|^2}{\varphi_+^2(x)}\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha-2}} \lesssim \eta^{2-\alpha}|\{(u-\varphi_+)>0\}|.$$

In this integral, the variable x is confined into $B_{2^{1/\epsilon}}$ and the y variable is controlled by the following fact:

$$\sup_{x\in B_{2^{1/\epsilon}}}\int_{|x-y|\le\eta}(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{c_{\alpha,N}\mathrm{d}y}{|y-x|^{N+\alpha-2}}\lesssim\eta^{2-\alpha} \tag{4.30}$$

since $(1\vee|x|\vee|y|)^{\epsilon(m-2)}\lesssim1$ if $|y-x|\le\eta<1$ and $\alpha<2$. $\square$

Lemma 4.19. *For η such that $\eta \le \frac{c^0_{\alpha,N}}{2c_{\alpha,N}C_{\varphi_+}}$, we have:*

$$Q^{+,-}_{\text{int}} \le \frac{1}{2}\mathcal{B}_G((u-\varphi_+)_+, -(u-\varphi_+)_-).$$

Let us recall that the constants of the singular integrals are defined by (4.3) and (4.12).

Proof. The term $Q^{+,-}_{\text{int}}$ is easy to handle. We simply write

$$\begin{aligned}
Q^{+,-}_{\text{int}} &= -Q^{\text{int}}_G((u-\varphi_+)_+, (u-\varphi_+)-) \\
&= -\iint_{|x-y|\le\eta} (u-\varphi_+)_+(x)(u-\varphi_+)_-(y)\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_+(x)}{\varphi_+(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}} \\
&\le \|\nabla\varphi_+/\varphi_+\|_\infty \eta \iint_{|x-y|\le\eta} (u-\varphi_+)_+(x)(u-\varphi_+)_-(y)\mathcal{D}_G(u(x),u(y))\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}} \\
&\le \frac{1}{2}\mathcal{B}_G((u-\varphi_+)_+, -(u-\varphi_+)_-)
\end{aligned}$$

provided η is chosen small enough to ensure that $\eta c_{\alpha,N}\|\nabla\varphi_+/\varphi_+\|_\infty \le \frac{1}{2}c^0_{\alpha,N}$. □

Lemma 4.20. *For $\alpha < 2 \le m$, one has*

$$Q^{+,0}_{\text{int}} \le C_{\varphi_+}|\{(u-\varphi_+) > 0\}|$$

where $C_{\varphi_+} \gtrsim \|\nabla\varphi_+\|_\infty\|\nabla\varphi_+/\varphi_+\|_\infty$.

Proof. The proof is similar to that of Lemma 4.18, but this time the regularity of φ_+ provides the local integrability, instead of using Cauchy-Schwarz:

$$\begin{aligned}
Q^{+,0}_{\text{int}} &= \iint_{|x-y|\le\eta} u^+_\varphi(x)(\varphi_+(y)-\varphi_+(x))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_+(x)}{\varphi_+(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}} \\
&\le \|\nabla\varphi_+\|_\infty\|\nabla\varphi_+/\varphi_+\|_\infty \int_{B_{2^{1/\epsilon}}} u^+_\varphi(x)\left\{\int_{|x-y|\le\eta}(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{c_{\alpha,N}\mathrm{d}y}{|y-x|^{N+\alpha-2}}\right\}\mathrm{d}x \\
&\lesssim |\{(u-\varphi_+) > 0\}|.
\end{aligned}$$

We used again (4.27), the fact that the variable x is confined into $B_{2^{1/\epsilon}}$ and $\alpha < 2 \le m$. □

Lemma 4.21. *For $\alpha > 1$ and $\epsilon < (\alpha-1)/(m-1)$, we have:*

$$Q^{+,+}_{\text{out}} \le C_{\varphi_+}|\{(u-\varphi_+) > 0\}|$$

with $C_{\varphi_+} \gtrsim \|\nabla\varphi_+/\varphi_+\|_\infty$.

Proof. We use (4.27), the boundedness of $\nabla\varphi_+/\varphi_+$ and $u^+_\varphi(y) \le (1\vee|y|)^\epsilon$ in order to get

$$\begin{aligned}
Q^{+,+}_{\text{out}} &\lesssim \iint_{|x-y|\ge\eta} u^+_\varphi(x)(u^+_\varphi(y)+u^+_\varphi(x))(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha-1}} \\
&\lesssim \int u^+_\varphi(x)\left\{\int_{|y-x|\ge\eta}(1\vee|y|)^{\epsilon(m-1)}\frac{c_{\alpha,N}\mathrm{d}y}{|y-x|^{N+\alpha-1}}\right\}\mathrm{d}x \\
&\quad + \int u^+_\varphi(x)^2\left\{\int_{|y-x|\ge\eta}(1\vee|y|)^{\epsilon(m-2)}\frac{c_{\alpha,N}\mathrm{d}y}{|y-x|^{N+\alpha-1}}\right\}\mathrm{d}x.
\end{aligned}$$

We use here in an essential way that $\alpha > 1$ and $\epsilon < (\alpha-1)/(m-1)$ in order to get that the two terms in curly parentheses are $\lesssim 1$. Note that as $m \geq 2$, one has $\alpha-1 > \epsilon(m-1) > \epsilon(m-2) \geq 0$. This yields the desired estimate. $\square$

Lemma 4.22. *For $\alpha > 1$ and $\epsilon < (\alpha-1)/(m-1)$, we have:*

$$Q_{\text{out}}^{+,-} \leq C_{\varphi_+}|\{(u-\varphi_+) > 0\}|$$

with $C_{\varphi_+} \gtrsim \|\nabla\varphi_+/\varphi_+\|_\infty$.

Proof. We use (4.27), the boundedness of $\nabla\varphi_+/\varphi_+$ and $(u-\varphi_+)_-(y) \leq \varphi_+(y) \leq (1 \vee |y|)^\epsilon$ in order to get

$$\begin{aligned}
Q_{\text{out}}^{+,-} &\lesssim \iint_{|x-y|\geq\eta} (u-\varphi_+)_+(x)(u-\varphi_+)_-(y)(1 \vee |x| \vee |y|)^{\epsilon(m-2)} \frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha-1}} \\
&\lesssim \int (u-\varphi_+)_+(x) \left\{ \int_{|x-y|\geq\eta} (1 \vee |y|)^{\epsilon(m-1)} \frac{c_{\alpha,N}\mathrm{d}y}{|y-x|^{N+\alpha-1}} \right\} \mathrm{d}x \\
&\lesssim \int (u-\varphi_+)_+(x)\mathrm{d}x \\
&\lesssim C|\{(u-\varphi_+) > 0\}|.
\end{aligned}$$

The integral in the curly braces converges because $\epsilon(m-1) < \alpha-1$. This yields the desired estimate. $\square$

Lemma 4.23. *For $\alpha > 1$ and $\epsilon < (\alpha-1)/(m-1)$, we have:*

$$Q_{\text{out}}^{+,0} \leq C_{\varphi_+}|\{(u-\varphi_+) > 0\}|$$

with $C_{\varphi_+} \gtrsim \|\nabla\varphi_+/\varphi_+\|_\infty$.

Proof. We offer here a slightly simpler proof than in [19]. Again, let us observe that x is confined in $B_{2^{1/\epsilon}}$ in the following integral so one can use $|\varphi_+(y) - \varphi_+(x)| \lesssim (1+|x|+|y|)^\epsilon$ and (4.27):

$$\begin{aligned}
Q_{\text{out}}^{+,0} &= \iint_{|x-y|\geq\eta} u_\varphi^+(x)(\varphi_+(y)-\varphi_+(x))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_+(x)}{\varphi_+(x)} \cdot (y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}} \\
&\lesssim \iint_{|x-y|\geq\eta} u_\varphi^+(x)(1 \vee |y|)^{\epsilon(m-1)} \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha-1}} \\
&\lesssim \int u_\varphi^+(x) \left\{ \int_{|y-x|\geq\eta} (1 \vee |y|)^{\epsilon(m-1)} \frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha-1}} \right\} \mathrm{d}x \\
&\lesssim |\{u_\varphi^+ > 0\}|.
\end{aligned}$$

The integral in the curly braces converges because $\epsilon(m-1) < \alpha-1$. $\square$

Remark 4.24. *Note that up to now, as $\alpha < 2 \leq m$, the most stringent condition on ϵ is*

$$0 < \epsilon < \min\left\{\frac{\alpha}{m}, \frac{\alpha-1}{m-1}\right\} = \frac{\alpha-1}{m-1} = \epsilon_0.$$

4.3.4 Estimates for $\mathcal{E}_-$

In order to estimate $\mathcal{Q}_G(-(u-\varphi_-)_-, u)$, we split it again as follows (see [19]), but we group the terms differently:

$$\mathcal{Q}_G(-u_\varphi^-, u) = \mathcal{Q}_G(u_\varphi^-, u_\varphi^-) + \mathcal{Q}_G(-u_\varphi^-, (u-\varphi_-)_+ + \varphi_-).$$

Let us point out that the previous sub-split depending on the size of $|x-y|$ will still be necessary for each term, but the cut-off value η will be different between the proof of Lemma 4.25 and that of Lemma 4.26.

Lemma 4.25. *For $\alpha - 1 > \epsilon(m-2) \geq 0$, we have*

$$\mathcal{Q}_G(u_\varphi^-, u_\varphi^-) \leq \frac{1}{4}\mathcal{B}_G(u_\varphi^-, u_\varphi^-) + CC_{\varphi_-}|\{u_\varphi^- > 0\}|.$$

Proof. We first write

$$\mathcal{Q}_G(u_\varphi^-, u_\varphi^-) = Q_{\mathrm{int}} + Q_{\mathrm{out}}$$

with

$$Q_{\mathrm{int}} = \int_{|x-y|\leq\eta} u_\varphi^-(x)(u_\varphi^-(x) - u_\varphi^-(y))\mathcal{D}_G(u(x), u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}$$
$$Q_{\mathrm{out}} = \int_{|x-y|\geq\eta} u_\varphi^-(x)(u_\varphi^-(x) - u_\varphi^-(y))\mathcal{D}_G(u(x), u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}.$$

for some $\eta > 0$ of arbitrary value.

We argue as in the proof of Lemma 4.18 by writing first, thanks to (4.27) and the properties of φ_- that:

$$\begin{aligned} Q_{\mathrm{int}} &\leq \frac{1}{4}\mathcal{B}_G(u_\varphi^-, u_\varphi^-) + C\iint_{|x-y|\leq\eta}(u_\varphi^-)^2(x)(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{|\nabla\varphi_-(x)|^2}{\varphi_-^2(x)}\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha-2}} \\ &\leq \frac{1}{4}\mathcal{B}_G(u_\varphi^-, u_\varphi^-) + CC_{\varphi_-}\int(u_\varphi^-)^2(x)\varphi_-^{-2/m_0}(x)\mathrm{d}x. \end{aligned}$$

Using $u_\varphi^- \leq \varphi_-$ yields the desired estimate for this term since $m_0 \geq 1$. Note that the integral in $\mathrm{d}y$ did converge because of the assumption $\alpha < 2$.

For the outer part we use (4.27), $\nabla\varphi_-/\varphi_- \leq C_{\varphi_-}\varphi_-^{-1/m_0}$ and $u_\varphi^-(y) \leq \varphi_-(y) \leq 1$ in order to get

$$\begin{aligned} Q_{\mathrm{out}} &\lesssim \int u_\varphi^-(x)\varphi_-^{-1/m_0}(x)\left\{\int_{|x-y|\geq\eta}(1\vee|x|\vee|y|)^{\epsilon(m-2)}\frac{c_{\alpha,N}\mathrm{d}y}{|y-x|^{N+\alpha-1}}\right\}\mathrm{d}x \\ &\lesssim C_{\varphi_-}\int u_\varphi^-(x)\varphi_-^{-1/m_0}(x)\mathrm{d}x \leq C_{\varphi_-}\int_{u_\varphi^->0}\varphi_-^{1-1/m_0}(x)\mathrm{d}x \\ &\lesssim C_{\varphi_-}|\{u_\varphi^- > 0\}|. \end{aligned}$$

We use here in an essential way that $\alpha - 1 > \epsilon(m-2) \geq 0$ to ensure the convergence of the $\mathrm{d}y$ integral in the curly braces. This yields the desired estimate.

$\square$

Lemma 4.26. *For $\epsilon < \frac{\alpha-1}{m-1}$, we have*

$$\mathcal{Q}_G((u-\varphi_-)_+ + \varphi_-, u) \le -\frac{1}{2}\mathcal{B}_G((u-\varphi_-)_+, (u-\varphi_-)_-) + CC_{\varphi_-}|\{u_\varphi^- > 0\}|.$$

Proof. We first write

$$\mathcal{Q}_G((u-\varphi_-)_+ + \varphi_-, u) = Q_{\mathrm{int}} + Q_{\mathrm{out}}$$

with $(u-\varphi_-)_+ + \varphi_- = u \vee \varphi_-$ and

$$\begin{aligned}
Q_{\mathrm{int}} &= \int_{|x-y|\le\eta} u_\varphi^-(x)(u\vee\varphi_-(x) - u\vee\varphi_-(y))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
Q_{\mathrm{out}} &= -\int_{|x-y|\ge\eta} u_\varphi^-(x)u\vee\varphi_-(y)\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}
\end{aligned}$$

for some parameter $\eta > 0$ to be chosen subsequently. Let us point out that we removed the term $u\vee\varphi_-(x)$ for $|x-y|\ge\eta$ since it is away from the singularity and that the kernel is anti-symmetric, which makes the corresponding $\mathrm{d}y$ integral vanish.

Let us observe that u_φ^- is supported in B_2. Choosing η large enough we can ensure that for $|x-y|\ge\eta$ one has $u\vee\varphi_-(y) = u(y)$ and consequently

$$\begin{aligned}
Q_{\mathrm{out}} &= -\int_{|x-y|\ge\eta} u_\varphi^-(x)u(y)\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\lesssim C_{\varphi_-}\int u_\varphi^-(x)\varphi_-^{-1/m_0}(x)\left\{\int_{|y-x|\ge\eta}(1\vee|y|)^{\epsilon(m-1)}\frac{\mathrm{d}y}{|y-x|^{N+\alpha-1}}\right\}\mathrm{d}x\\
&\lesssim C_{\varphi_-}\int u_\varphi^-(x)\varphi_-^{-1/m_0}(x)\mathrm{d}x\\
&\lesssim C_{\varphi_-}|\{u_\varphi^- > 0\}|
\end{aligned}$$

since $m_0 \ge 1$ and $u_\varphi^- \le \varphi_-$.

As far as Q_{int} is concerned, we revert to $u\vee\varphi_- = (u-\varphi_-)_+ + \varphi_-$ and split it as $Q_{\mathrm{int}} = Q_{\mathrm{int}}^{-,+} + Q_{\mathrm{int}}^{-,0}$ with

$$\begin{aligned}
Q_{\mathrm{int}}^{-,+} &= \int_{|x-y|\le\eta} u_\varphi^-(x)((u-\varphi_-)_+(x) - (u-\varphi_-)_+(y))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
Q_{\mathrm{int}}^{-,0} &= \int_{|x-y|\le\eta} u_\varphi^-(x)(\varphi_-(x) - \varphi_-(y))\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}.
\end{aligned}$$

We split the integral further, depending on the size of the unsigned factors:

$$\begin{aligned}
Q_{\mathrm{int}}^{-,+} &= -\int_{|x-y|\le\eta}(u-\varphi_-)_-(x)(u-\varphi_-)_+(y)\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&= -\int_{\substack{\varphi_-^{-1}(x)|\nabla\varphi_-(x)||y-x|\le 1/2\\ |x-y|\le\eta}}(u-\varphi_-)_-(x)(u-\varphi_-)_+(y)\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\quad-\int_{\substack{\varphi_-^{-1}(x)|\nabla\varphi_-(x)||x-y|\ge 1/2\\ |x-y|\le\eta}}(u-\varphi_-)_-(x)(u-\varphi_-)_+(y)\mathcal{D}_G(u(x),u(y))\frac{\nabla\varphi_-(x)}{\varphi_-(x)}\cdot(y-x)\frac{c_{\alpha,N}\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}.
\end{aligned}$$

We then use the good extra term as follows

$$
\begin{aligned}
Q_{\text{int}}^{-,+} \leq &-\frac{1}{2}\mathcal{B}_G((u-\varphi_-)_+,(u-\varphi_-)_-) \\
&+ CC_{\varphi_-}\int (u-\varphi_-)_-(x)\varphi_-^{-1/m_0}(x)\left\{\iint_{\varphi_-^{1/m_0}(x)/(2C_{\varphi_-})\leq|x-y|\leq\eta}\frac{\mathrm{d}y}{|y-x|^{N+\alpha-1}}\right\}\mathrm{d}x \\
\leq &-\frac{1}{2}\mathcal{B}_G((u-\varphi_-)_+,(u-\varphi_-)_-) + CC_{\varphi_-}\int (u-\varphi_-)_-(x)\varphi_-^{-\alpha/m_0}(x)\mathrm{d}x.
\end{aligned}
$$

As $m_0 \geq 2 > \alpha$, the last integral is related to the measure of the set $\{u_\varphi^- > 0\}$ in the following way:

$$
\int_{u_\varphi^->0}(u-\varphi_-)_-(x)\varphi_-^{-\alpha/m_0}(x)\mathrm{d}x \leq \int_{u_\varphi^->0}\varphi_-^{1-\alpha/m_0}(x)\mathrm{d}x \leq C_{\varphi_-}|\{u_\varphi^- > 0\}|.
$$

Similarly, using (4.27), the fact that $\operatorname{supp} u_\varphi^- \in B_2$ and the properties of φ_-, we get for the last term:

$$
\begin{aligned}
Q_{\text{int}}^{-,0} &\lesssim C_{\varphi_-}\int u_\varphi^-(x)\varphi_-^{-1/m_0}(x)\left\{\iint_{|x-y|\leq\eta}(1\vee|y|)^{\epsilon(m-2)}\frac{\mathrm{d}y}{|y-x|^{N+\alpha-2}}\right\}\mathrm{d}x \\
&\lesssim C_{\varphi_-}\int u_\varphi^-(x)\varphi_-^{-1/m_0}(x)\mathrm{d}x \lesssim C_{\varphi_-}|\{u_\varphi^- > 0\}|.
\end{aligned}
$$

This yields the desired estimate. □

Remark 4.27. *Note that for this second half of the proof, as $\alpha < 2 \leq m$, the most stringent condition on ϵ is*

$$
0 < \epsilon < \min\left\{\frac{\alpha}{m}, \frac{\alpha-1}{m-1}, \frac{\alpha-1}{m-2}\right\} = \frac{\alpha-1}{m-1} = \epsilon_0
$$

which is the same critical value as before.

4.3.5 Local energy estimates

Theorem 4.11 provides a global estimate with an embedded cutoff function $\varphi_\pm$. In the sequel, we will need a localized version with the integrals computed on balls.

Proposition 4.28 (Local energy estimates)**.** *Let us assume that $\alpha \in (1,2)$ and $m \geq 2$. We take $\epsilon_0 > 0$ given by Theorem 4.11. There then exists $C > 0$ (only depending on N, α, m) such that for any weak solution u of (4.1) in $(-2,0] \times \mathbb{R}^N$ satisfying (4.19) for some $\epsilon \in (0,\epsilon_0)$, the two following local energy estimates hold true.*

- *For any $r < R$ in $(0, 2^{1/\epsilon})$ and $c > 1/4$ and with $-2 < t_1 < t_2 < 0$, one has:*

$$
\begin{aligned}
\int_{B_r}(u(t_2,x)-c)_+^2\mathrm{d}x &+ \int_{t_1}^{t_2}\left(\int_{B_r}(u-c)_+^p(x)\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}t \\
&+ \int_{t_1}^{t_2}\iint_{B_r\times B_r}(u(t,x)-c)_+(G(c)-G(u(y)))_+\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\alpha}}\mathrm{d}t \\
&\lesssim \int_{B_R}(u(t_1,x)-c)_+^2\mathrm{d}x + C(R-r)^{-2}\left|\{u>c\}\cap(t_1,t_2)\times B_R\right|.
\end{aligned}
\tag{4.31}
$$

- *For any cut-off function φ_- such that $\varphi_- \equiv 0$ outside B_2 and $\varphi_- \equiv c > 0$ in B_r with $|\nabla\varphi_-/\varphi_-| \le C_{\varphi_-}\varphi_-^{-1/m_0}$ for some $m_0 \ge 2$, we have*

$$\int_{\mathbb{R}^N} (u(t_2,x)-\varphi_-(x))_-^2\,\varphi_-^{-1}(x)\mathrm{d}x + \int_{t_1}^{t_2}\left(\int_{B_r}(u(t,x)-c)_-^{\frac{pm}{2}}\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}t$$
$$\lesssim \int_{\mathbb{R}^N}(u(t_1,x)-\varphi_-)_\pm^2\,\varphi_-^{-1}(x)\mathrm{d}x + C_{\varphi_-}\left|\{(u-\varphi_-)_- > 0\}\cap(t_1,t_2)\times\mathbb{R}^N\right|. \tag{4.32}$$

Remark 4.29. *In this proposition, $p = 2N/(N-\alpha)$ is given by the Sobolev embedding (4.17).*

Remark 4.30. *The lower bound $c > 1/4$ in (4.31) is a direct inheritance from the restriction $\varphi_+ > 1/4$ in Theorem 4.11, which in turn was constrained by the range on which the L^2 norm is equivalent to the alternate energy functional (4.21).*

Remark 4.31. *Note that the L^p or $L^{pm/2}$ control of $(u-c)_\pm$ in (4.31)-(4.32) are produced by the coercive term in (4.20). The good extra term appears as the third term on the left-hand side of (4.31). The good-extra term in (4.31), can be replaced for $0 < \tilde{c} < c$ by*

$$(G(c)-G(\tilde{c}))\int_{t_1}^{t_2}\left(\iint_{B_r\times B_r}(u-c)_+(x)\mathbb{I}_{\{u(y)\le\tilde{c}\}}\mathrm{d}x\mathrm{d}y\right)\mathrm{d}t \tag{4.33}$$
$$\lesssim \int_{B_R}(u-c)_+^2(t_1,x)\mathrm{d}x + C(R-r)^{-2}\left|\{u>c\}\cap(t_1,t_2)\times B_R\right|.$$

Proof. We first prove (4.31). We follow [19] by applying the energy estimates from Theorem 4.11 with the cut-off function

$$\varphi_+(x) = 1 + \Psi_\epsilon(x) - (1-c)\xi(x)$$

where ξ is a smooth characteristic function such that $\xi \equiv 1$ on B_r and $\operatorname{supp}\xi \subset B_R$. Remark that this cut-off function satisfies the assumptions of Theorem 4.11 with $C_{\varphi_+} \simeq (R-r)^{-2}$. Moreover, $\varphi_+(x) \equiv c$ for $x \in B_r$. One can apply (4.29) to bound $\mathcal{D}_G$ from below on the complementary set of $\{(x,y)\,;u(x)\vee u(y)\le c\}$ on which the following integrand vanishes anyway. One thus gets, thanks to the Sobolev embedding (4.17), that:

$$\iint((u-c)^+(y)-(u-c)^+(x))^2\mathcal{D}_G(u(t,x),u(t,y))\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}$$
$$\gtrsim c^{m-2}\iint_{B_r\times B_r}((u-c)_+(y)-(u-c)_+(x))^2\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}$$
$$\gtrsim c^{m-2}\left(\int_{B_r}(u-c)_+^p(x)\mathrm{d}x\right)^{\frac{2}{p}}.$$

As far as the good extra term is concerned, we use the convexity inequality (4.28) to assert that:

$$\begin{cases} \text{for } u(x)\ge c, & \mathcal{D}_G(u(x),u(y)) \ge \mathcal{D}_G(c,u(y)), \\ \text{for } u(x)\ge c>\tilde{c}\ge u(y), & (u(y)-c)_-\mathcal{D}_G(c,u(y)) = G(c)-G(u(y)) \ge G(c)-G(\tilde{c}) \end{cases}$$

and in particular

$$\begin{aligned}
&\iint (u(t,x)-\varphi_+(x))_+(u(t,y)-\varphi_+(y))_-\mathcal{D}_G(u(t,x),u(t,y))\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\quad\gtrsim \iint_{B_r\times B_r}(u(t,x)-c)_+\left(G(c)-G(u(y))\right)_+\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\alpha}}\\
&\qquad\gtrsim (G(c)-G(\tilde c))\iint_{B_r\times B_r}(u(t,x)-c)_+\,\mathbb{I}_{\{u(y)\le\tilde c\}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

For the last estimate, we discarded the denominator because $|x-y|^{-N-\alpha}\gtrsim 1$ if $x,y\in B_r$. Applying (4.20) from Theorem 4.11 then yields the first desired estimate (4.31). In particular, (4.33) holds too.

We now turn to the proof of (4.32). Because m can be different from 2, the dissipation term $\mathcal{B}_G(u_\varphi^-,u_\varphi^-)$ appearing in (4.20) is treated in a slightly different way than in [19]. Let us recall that $u_\varphi^-=(u-\varphi_-)_-=u\wedge\varphi_--\varphi_-$ and write

$$\begin{aligned}
\mathcal{B}_G(u_\varphi^-,u_\varphi^-)&\ge\iint_{B_r\times B_r}((u(y)-c)_--(u(x)-c)_-)^2\mathcal{D}_G(u(x),u(y))\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\ge\iint_{B_r\times B_r}(u(y)\wedge c-u(x)\wedge c)^2\mathcal{D}_G(u(x),u(y))\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}.
\end{aligned}$$

Using the convexity inequality (4.28), one gets:

$$\begin{aligned}
\mathcal{B}_G(u_\varphi^-,u_\varphi^-)&\ge\iint_{B_r\times B_r}(u(y)\wedge c-u(x)\wedge c)^2\mathcal{D}_G(u(x)\wedge c,u(y)\wedge c)\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\ge\iint_{B_r\times B_r}\Big(u(y)\wedge c-u(x)\wedge c\Big)\Big(G(u(x)\wedge c)-G(u(y)\wedge c)\Big)\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}\\
&\gtrsim\iint_{B_r\times B_r}\Big((u(y)\wedge c)^{m/2}-(u(x)\wedge c)^{m/2}\Big)^2\frac{\mathrm{d}x\mathrm{d}y}{|y-x|^{N+\alpha}}.
\end{aligned}$$

For the last inequality, we used a well-known identity (4.68) that we recall in the appendix of this paper. Applying the Sobolev embedding (4.17), we finally get

$$\mathcal{B}_G(u_\varphi^-,u_\varphi^-)\gtrsim\left(\int_{B_r}(u\wedge c)^{\frac{pm}{2}}(x)\mathrm{d}x\right)^{\frac{2}{p}}.$$

In particular, Theorem 4.11 implies that for all $-2<t_1<t_2<0$,

$$\begin{aligned}
&\int_{\mathbb{R}^N}(u(t_2,x)-\varphi_-(x))_-^2\,\varphi_-^{-1}(x)\mathrm{d}x+\int_{t_1}^{t_2}\left(\int_{B_r}(u\wedge c)^{\frac{pm}{2}}(x)\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}t\\
&\quad\lesssim\int_{\mathbb{R}^N}(u(t_1)-\varphi_-)_\pm^2\,\varphi_-^{-1}\mathrm{d}x+C_{\varphi_-}\left|\{(u-\varphi_-)_->0\}\cap(t_1,t_2)\times\mathbb{R}^N\right|.
\end{aligned}$$

Using next that $(u-c)_-^m=(c-u\wedge c)^m\lesssim c^m+(u\wedge c)^m$ we can play around with the Lebesgue norm:

$$\begin{aligned}
\left(\int_{B_r}(u-c)_-^{\frac{pm}{2}}(x)\mathrm{d}x\right)^{\frac{2}{p}}&\lesssim\left\|c^m\mathbb{I}_{u(x)<c}+(u(x)\wedge c)^m\right\|_{L^{2/p}(B_r)}\\
&\le c^m|\{u<\varphi_-\}\cap B_r|^{2/p}+\left(\int_{B_r}(u\wedge c)^{\frac{pm}{2}}(x)\mathrm{d}x\right)^{\frac{2}{p}}.
\end{aligned}$$

We thus get the desired estimate (4.32). □

4.4 First lemmas of De Giorgi

This section is devoted to the first lemmas of De Giorgi. These lemmas are concerned with reducing the oscillation of the solution provided u spends "most" of the space-time $Q = (-2,0]\times B_2$ either on the upper side or on the lower side of the a-priori range $[0, \sup_Q u]$. Depending wether the maximum is lowered or the infimum is increased, we get two lemmas.

Let us define some common notations that will be used in both proofs of Lemmas 4.36 and 4.32. For $k \in \mathbb{N}$, let us define $T_k = -1 - \frac{1}{2^k}$ and $r_k = 1 + \frac{1}{2^k}$. One thus has an increasing sequence of times

$$-2 = T_0 < T_1 < T_2 < \ldots < T_k < \ldots < T_\infty = -1$$

and a decreasing sequence of balls:

$$B_2 = B_{r_0} \supset B_{r_1} \supset B_{r_2} \supset \ldots \supset B_{r_k} \supset \ldots \supset B_{r_\infty} = B_1.$$

The idea is to apply recursively the local energy estimates from Proposition 4.28 with well chosen cutoff values. The sequence of nested estimates then provides, for some $c > 0$, that

$$\int_{(-1,0]\times B_1} (u-c)_\pm^2 \mathrm{d}x\mathrm{d}t = 0$$

which either means, depending on each respective case, that $\sup\limits_{(-1,0]\times B_1} u < c$ or $\inf\limits_{(-1,0]\times B_1} u > c$.

4.4.1 Lowering the maximum

Lemma 4.32 (Lowering the maximum). *Let $\alpha \in (1,2)$. For any $\overline{\mu} \in (0,1)$, there exists $\overline{\delta} \in (0,1)$ such that for any function u that satisfies the three assumptions:*

1. *u is locally bounded from above in the following way*

$$\forall (t,x) \in (-2,0]\times B_2, \qquad u(t,x) \le 1, \tag{4.34}$$

2. *the upper local energy-inequality (4.31) is satisfied,*

3. *u is "mostly" low-valued in the sense that*

$$\left|\left\{u \le \tfrac{1+\overline{\mu}}{2}\right\} \cap (-2,0]\times B_2\right| \ge (1-\overline{\delta}) \cdot |(-2,0]\times B_2|, \tag{4.35}$$

then $u(t,x) \le \frac{3+\overline{\mu}}{4}$ in $(-1,0]\times B_1$.

Remark 4.33. *Thanks to Proposition 4.28, weak-solutions of (4.1) that satisfy the mild growth assumption (4.19) will automatically satisfy the first two assumptions of Lemma 4.32. It is interesting to point-out that the PDE is not directly responsible for Lemma 4.32 and that only the local energy inequality matters. We do not require u to be nonnegative ; only (4.34) is necessary. Moreover, the "good extra term" in (4.31) is not required either.*

Remark 4.34. *The admissible values for $\overline{\delta}$ form an interval $(0,\overline{\delta}_*)$ where $\overline{\delta}_*$ is an increasing function of $\overline{\mu}$.*

Remark 4.35. *We will only use Lemma 4.32 in the proof of its improved version, Lemma 4.44. We will need to use a high threshold value for $\overline{\mu}$ (i.e. very close to 1).*

Proof. Let us use the common definition for T_k and r_k from the beginning of §4.4. We now define an increasing sequence (the fact that it is increasing is crucial)

$$c_k = \frac{3+\overline{\mu}}{4} - \frac{1-\overline{\mu}}{4}\frac{1}{2^k} \in \left[\frac{1+\overline{\mu}}{2}, \frac{3+\overline{\mu}}{4}\right]$$

and consider the quantity

$$U_k = \sup_{t\in[T_k,0]} \int_{B_{r_k}} (u-c_k)_+^2(t,x)\mathrm{d}x. \tag{4.36}$$

To study the asymptotic behavior of the sequence $(U_k)_{k\in\mathbb{N}}$, we establish a recurrence inequality. We apply the local upper energy estimate (4.31) with $r = r_k$ and $R = r_{k-1}$ so that $(R-r)^{-2} = 4^k$. Note that $c_k \geq c_0 > 1/4$. For all $T_{k-1} \leq t_1 \leq T_k < t_2 < 0$, we get:

$$\int_{B_{r_k}} (u-c_k)_+^2(t_2,x)\mathrm{d}x + \int_{t_1}^{t_2} \left(\int_{B_{r_k}} (u-c_k)_+^p(x)\mathrm{d}x\right)^{\frac{2}{p}} \mathrm{d}t$$
$$\lesssim \int_{B_{r_{k-1}}} (u-c_k)_+^2(t_1,x)\mathrm{d}x + 4^k \left|\{u>c_k\}\cap(t_1,t_2)\times B_{r_{k-1}}\right|.$$

In particular, U_k satisfies (chose a time t_1 that realizes the following infimum and t_2 that realizes U_k):

$$U_k \leq \inf_{t\in[T_{k-1},T_k]} \int_{B_{r_{k-1}}} (u-c_k)_+^2(t,x)\mathrm{d}x + 4^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{u(t,x)>c_k\}}\mathrm{d}x\mathrm{d}t.$$

We remark that, by positivity of the integral:

$$\inf_{t_1\in[T_{k-1},T_k]} \int_{B_{r_{k-1}}} (u-c_k)_+^2(x,t_1)\mathrm{d}x \leq \frac{1}{T_k - T_{k-1}} \int_{T_{k-1}}^{T_k} \int_{B_{r_{k-1}}} (u-c_k)_+^2(x,t_1)\mathrm{d}x\mathrm{d}t_1$$
$$\leq 2^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} (u-c_k)_+^2(x,t_1)\mathrm{d}x\mathrm{d}t_1$$

and as $(u-c_k)_+ \leq u(x) \leq 1$ on $B_{r_{k-1}} \subset B_2$, it is bounded by the characteristic function:

$$\inf_{t_1\in[T_{k-1},T_k]} \int_{B_{r_{k-1}}} (u-c_k)_+^2(x,t_1)\mathrm{d}x \leq 2^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{u(t,x)>c_k\}}\mathrm{d}x\mathrm{d}t.$$

Let us point out that this is the only point in the proof where the local boundedness assumption (4.34) will be used. We thus got so far that

$$U_k \lesssim 4^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{u(t,x)>c_k\}} \mathrm{d}x\mathrm{d}t. \tag{4.37}$$

Moreover, as the sequence c_k is increasing, we note that

$$(u - c_k)_+ > 0 \qquad \Longrightarrow \qquad (u - c_{k-1})_+ > c_k - c_{k-1} \geq 2^{-k} \left(\frac{1 - \overline{\mu}}{4} \right) > 0,$$

which transforms (4.37) into

$$U_k \lesssim 4^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\left\{(u-c_{k-1})_+ > 2^{-k}\left(\frac{1-\overline{\mu}}{4}\right)\right\}} \mathrm{d}x\mathrm{d}t. \tag{4.38}$$

Now we take $\theta = \frac{2}{p}$ and $q = 2(1-\theta) + p\theta$. Then, using the Markov and Hölder inequalities, we get

$$\begin{aligned}
&\int_{B_{r_{k-1}}} \mathbb{I}_{\left\{(u-c_{k-1})_+ > 2^{-k}\left(\frac{1-\overline{\mu}}{4}\right)\right\}} \mathrm{d}x \\
&\qquad \leq \frac{4^q 2^{qk}}{(1-\overline{\mu})^q} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^q \mathrm{d}x \\
&\qquad \leq \frac{4^q 2^{qk}}{(1-\overline{\mu})^q} \left(\int_{B_{r_{k-1}}} (u - c_{k-1})_+^2 \mathrm{d}x \right)^{1-\theta} \left(\int_{B_{r_{k-1}}} (u - c_{k-1})_+^p \mathrm{d}x \right)^{\theta}.
\end{aligned}$$

Integrating in time t along the interval $[T_{k-1}, 0]$, we get:

$$U_k \leq C^k \left(\sup_{t \in [T_{k-1},0]} \int_{B_{r_{k-1}}} (u - c_{k-1})_+^2(t) \mathrm{d}x \right)^{1-\theta} \left(\int_{T_{k-1}}^0 \left(\int_{B_{r_{k-1}}} ((u - c_{k-1})_+^p \mathrm{d}x \right)^{\frac{2}{p}} \mathrm{d}t \right) \tag{4.39}$$

To control the last factor, we apply (4.31) one last time, but on the time interval $t_1 = T_{k-1}$ and $t_2 = 0$ and with the radii $r = r_{k-1}$ and $R = r_{k-2}$; we get:

$$\int_{T_{k-1}}^0 \left(\int_{B_{r_{k-1}}} ((u - c_{k-1})_+^p \mathrm{d}x \right)^{\frac{2}{p}} \mathrm{d}t \lesssim U_{k-1} + C^k \int_{T_{k-1}}^0 \int_{B_{r_{k-2}}} \mathbb{I}_{\{u(t,x)>c_{k-1}\}} \mathrm{d}x\mathrm{d}t. \tag{4.40}$$

The measure term in (4.40) cannot be removed, but it is harmless. Indeed, let us define

$$M_k = \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{u(t,x)>c_k\}} \mathrm{d}x\mathrm{d}t. \tag{4.41}$$

So far, thanks to (4.37)-(4.40), we have established that for any $k \geq 1$:

$$\begin{cases} U_k \leq C^k M_k, \\ M_k \leq C^k U_{k-1}^{1-\theta} (U_{k-1} + M_{k-1}). \end{cases} \tag{4.42}$$

Therefore, we have $M_k \leq \widetilde{C}^k M_{k-1}^{\sigma}$ with $\sigma = 2 - \theta = 2 - \frac{2}{p} > 1$.

Solving the recurrence equation, we get constants $\bar{C} > 1$ and $C' = \widetilde{C}^{-\sigma(\sigma-1)^{-2}} > 0$ such that

$$0 \le M_k \le \bar{C}^k (C' M_1)^{\sigma^k} \underset{k\to\infty}{\longrightarrow} 0$$

provided $M_1 < 1/C'$ is small enough. In turn, this estimate also implies that $U_k \to 0$.

Using the last assumption (4.35) and the fact that $c_1 > c_0 = \frac{1+\overline{\mu}}{2}$, we get the final control

$$M_1 = \int_{-2}^{0} \int_{B_2} \mathbb{I}_{\{u(t,x)>c_1\}} \mathrm{d}x\mathrm{d}t \le |\{u > (1+\overline{\mu})/2\} \cap (-2,0) \times B_2| < \overline{\delta} \cdot |(-2,0) \times B_2| \tag{4.43}$$

which can be made arbitrary small for a proper choice of $\overline{\delta}$. Adjusting the value of $\overline{\delta}$ properly in (4.43), we get that $U_\infty = 0$ which in turn implies that $u \le \overline{\mu}$ in $(-1,0) \times B_1$. This achieves the proof of this first De Giorgi lemma about lowering the maximum. □

4.4.2 Increasing the infimum

Lemma 4.36 (Increasing the infimum). *Let $\alpha \in (1,2)$ and $m \ge 2$. For any $\underline{\mu} \in (0,1)$, there exists $\underline{\delta} > 0$ such that for any function u that satisfies the three assumptions:*

1. *$u \ge 0$ on $\mathbb{R}^N$,*
2. *the lower local energy-inequality (4.32) is satisfied (with the chosen value for m),*
3. *u is "mostly" high-valued in the sense that*

$$\left| \left\{ u \ge \tfrac{1+2\underline{\mu}}{3} \right\} \cap (-2,0] \times B_2 \right| \ge (1-\underline{\delta}) \cdot |(-2,0] \times B_2| , \tag{4.44}$$

then $u(t,x) \ge \underline{\mu}$ in $(-1,0] \times B_1$.

Remark 4.37. *Again, thanks to Proposition 4.28, weak-solutions of (4.1) that satisfy the mild growth assumption (4.19) will automatically satisfy the first two assumptions of Lemma 4.36. In lemma 4.36, we do not require u to be bounded from above, nor to have a mild growth at infinity. The non-negativity assumption is sufficient. Again, no "good extra term" is required in (4.32) either.*

Remark 4.38. *The admissible values for $\underline{\delta}$ form an interval $(0, \underline{\delta}^*)$ where $\underline{\delta}^*$ is an increasing function of $\underline{\mu}$.*

Remark 4.39. *We have chosen to state (4.44) such that the cut-off value $\frac{1+2\underline{\mu}}{3} \in (1/3, 1)$. Subsequently, we will only use Lemma 4.36 in the final proof of the main theorem, where we intend to use it with $\frac{1+2\underline{\mu}}{3} \ge \frac{1}{2}$ i.e. for $\underline{\mu} \ge 1/4$.*

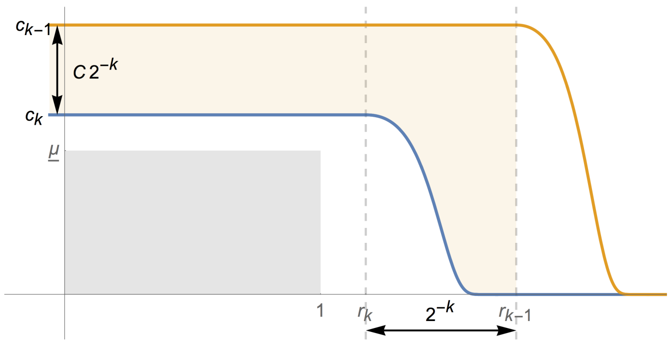
Proof. We use the common definition for T_k and r_k from the beginning of §4.4. To apply (4.32), the key is to chose the sequence of cut-off functions $\varphi_- = \varphi_k$ wisely. Following [19], we define a decreasing sequence

$$c_k = \underline{\mu} + \frac{1-\underline{\mu}}{3} \frac{1}{2^k} \in \left[\underline{\mu}, \frac{1+2\underline{\mu}}{3} \right]$$

and will chose $\varphi_k \to \underline{\mu}\mathbb{I}_{B_1}$ as $k \to +\infty$ while ensuring, for all $k \geq 0$, that:

$$\begin{cases} \varphi_k \equiv c_k \text{ in } B_{r_k}, \\ \varphi_k \equiv 0 \text{ outside } B_{r_{k-1}} \end{cases} \qquad \text{and} \qquad |\nabla \varphi_k / \varphi_k| \leq C^k \varphi_k^{-1/m_0}.$$

For $k = 0$, we set $r_{-1} = 3$ (note that $3 \leq 2^{1/\epsilon_0}$ with ϵ_0 from Theorem 4.11, if $\epsilon_0 \leq \frac{\log 2}{\log 3}$) so that $\varphi_0 \equiv \frac{1+2\underline{\mu}}{3}$ on B_2 with compact support in B_3. The critical properties of φ_k are visible on the graph below.



Similarly as to what we did in the proof of Lemma 4.32, let us define:

$$V_k = \sup_{t \in [T_k, 0]} \int_{\mathbb{R}^N} (u(t,x) - \varphi_k(x))_-^2 \, \varphi_k^{-1}(x) \mathrm{d}x. \tag{4.45}$$

We apply the assumption (4.32) between a starting time $t_1 \in [T_{k-1}, T_k]$ such that

$$\int_{\mathbb{R}^N} (u(t_1) - \varphi_k)_-^2 \, \varphi_k^{-1} = \inf_{t \in [T_{k-1}, T_k]} \int_{\mathbb{R}^N} (u(t) - \varphi_k)_-^2 \, \varphi_k^{-1}$$

and a final time $t_2 \in [T_k, 0]$ that realizes V_k. As $u \geq 0$, the function $(u - \varphi_k)_-$ is supported in $\operatorname{supp} \varphi_k \subset B_{r_{k-1}}$ and as $\varphi_k \leq 1$, we also have $(u - \varphi_k)_-^2 \varphi_k^{-1} \leq \varphi_k \leq 1$ (note that it is the only point in the proof where we use the first assumption). In particular, we get as in the proof of Lemma 4.32:

$$\begin{aligned} \inf_{t \in [T_{k-1}, T_k]} \int_{\mathbb{R}^N} (u(t) - \varphi_k)_-^2 \, \varphi_k^{-1} &\leq \frac{1}{T_k - T_{k-1}} \int_{T_{k-1}}^{T_k} \int_{\mathbb{R}^N} (u(t) - \varphi_k)_-^2 \, \varphi_k^{-1} \mathrm{d}x \mathrm{d}t \\ &\leq 2^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{u(t,x) < \varphi_k\}} \mathrm{d}x \mathrm{d}t. \end{aligned}$$

The measure of $\{u < \varphi_k\} \cap (t_1, t_2) \times \mathbb{R}^N$ is also obviously bound by the same right-hand side. Thus, for this pair of times, assumption (4.32) implies:

$$V_k \lesssim 2^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{u(t,x) < \varphi_k\}} \mathrm{d}x \mathrm{d}t. \tag{4.46}$$

As the sequence φ_k is decreasing both in amplitude and support in a coordinated way, we get:

$$\forall x \in B_{r_{k-1}}, \qquad \varphi_k(x) \leq \varphi_{k-1}(x) - \left(\frac{1 - \underline{\mu}}{3}\right) 2^{-k}$$

and in particular

$$u(t,x) < \varphi_k \qquad \Longrightarrow \qquad (u(t,x) - \varphi_{k-1})_- > \left(\frac{1-\underline{\mu}}{3}\right) 2^{-k}.$$

We are thus allowed to rewrite (4.46) into

$$V_k \lesssim C^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{(u-\varphi_{k-1})_- > (1-\underline{\mu})2^{-k}/3\}} \mathrm{d}x\mathrm{d}t. \tag{4.47}$$

Now we take $\theta = \frac{2}{p}$ and $q = 2(1-\theta) + \theta(pm/2)$ and apply the Markov inequality to (4.47), then the Hölder inequality in the space variable and subsequently integrate in time ; we get

$$\begin{aligned} V_k &\le C^k \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} (u-\varphi_{k-1})_-^q \mathrm{d}x\mathrm{d}t \\ &\le C^k \left(\sup_{t\in[T_{k-1},0]} \int_{B_{r_{k-1}}} (u-\varphi_{k-1})_-^2 \varphi_{k-1}^{-1} \mathrm{d}x \right)^{1-\theta} \left(\int_{T_{k-1}}^0 \left(\int_{B_{r_{k-1}}} (u-\varphi_{k-1})_-^{\frac{pm}{2}} \mathrm{d}x \right)^{\frac{2}{p}} \mathrm{d}t \right) \end{aligned}$$

Note that $\varphi_{k-1}^{-1} \equiv c_{k-1}^{-1} \ge 1$ on $B_{r_{k-1}}$ so we can add it freely at the end of the computation. Finally, let us apply (4.32) one more time, but between $t_1 = T_{k-1}$ and $t_2 = 0$ and with the truncation φ_{k-1}. We then get:

$$\int_{T_{k-1}}^0 \left(\int_{B_{r_{k-1}}} (u-c_{k-1})_-^{\frac{pm}{2}} \mathrm{d}x \right)^{\frac{2}{p}} \mathrm{d}t \lesssim V_{k-1} + C^k \int_{T_{k-1}}^0 \int_{B_{r_{k-2}}} \mathbb{I}_{\{u(t,x)<\varphi_{k-1}\}} \mathrm{d}x\mathrm{d}t. \tag{4.48}$$

Roughly speaking, if we discard the measure term, the flavor of this recurrence equation is $V_k \le C^k V_{k-1}^{1-\theta} V_{k-1}$. However, as there is no hope to control $\mathbb{I}_{\{u(t,x)<\varphi_{k-1}\}}$ by $(u-\varphi_{k-1})_-$, we have to consider the recurrence equation as a system. For this purpose, let us define

$$N_k = \int_{T_{k-1}}^0 \int_{B_{r_{k-1}}} \mathbb{I}_{\{u(t,x)<\varphi_k\}} \mathrm{d}x\mathrm{d}t. \tag{4.49}$$

What we have proved so far with (4.46)-(4.48) is the existence of a universal constant C such that:

$$\begin{cases} V_k \le C^k N_k \\ N_k \le C^k V_{k-1}^{1-\theta} (V_{k-1} + N_{k-1}). \end{cases} \tag{4.50}$$

From this system, we can infer that $N_k \le \widetilde{C}^k N_{k-1}^{2-\theta}$. Provided that N_1 is small enough, we will then get, as in the proof of Lemma 4.32, that $N_k \to 0$ super-exponentially fast as $k \to \infty$ (namely $N_k \le \bar{C}^k (C' N_1)^{\sigma^k}$ with $\sigma = 2-\theta > 1$ so $\sigma^k \gg k$, and $C'N_1 < 1$) and therefore $V_k \to 0$ too.

Let us check that N_1 is indeed small enough. As $\varphi_1 \le \varphi_0 = \frac{1+2\underline{\mu}}{3}$ on B_2, our assumption (4.44) allows us to write

$$N_1 = \int_{-2}^0 \int_{B_2} \mathbb{I}_{\{u<\varphi_1\}} \mathrm{d}x\mathrm{d}t \le |\{u < (1+2\underline{\mu})/3\} \cap (-2,0)\times B_2| < \underline{\delta} \cdot |(-2,0)\times B_2|. \tag{4.51}$$

so it can be made arbitrary small for a proper choice of $\underline{\delta}$. This achieves the proof of the De Giorgi lemma about increasing the infimum. $\square$

4.5 Lemma on intermediate values

To prove the Hölder regularity of the weak solution, we need to improve Lemma 4.32 by showing that a uniform reduction of the maximum in a smaller ball can be obtained not only if u is below $1/2$ for most of the space-time domain $(-2,0]\times B_2$ but that it is also true under the milder assumption that it happens for only a few events $(t,x)\in(-2,0]\times B_2$.

Remark 4.40. *This type of result on intermediate values is sometimes called a "second De Giorgi lemma" (e.g. in [18]) in reference to the historical papers of E. De Giorgi on elliptic PDEs. As we have already established two De Giorgi lemmas of the first kind, it would probably be more proper to call it "a De Giorgi lemma of the second kind".*

Lemma 4.41 (Intermediate values, or De Giorgi Lemma of the Second Kind)**.** *Let $\alpha\in(1,2)$. For any $\rho\in(0,\frac{1}{8})$ and $\delta\in(0,\frac{1}{2})$, there exist $\lambda\in(0,\frac{1}{2})$ and $\gamma\in(0,1)$ such that for any function u that satisfies the following assumptions:*

1. $u(t,x)\le 1$ *in* $(-2,0]\times B_2$,
2. *the upper local energy-inequality (4.31) is satisfied,*
3. *u takes "some" early low values in the sense that*
$$\left|\left\{u<\tfrac{1}{2}\right\}\cap(-2,-1]\times B_1\right|\ge\rho|B_1|,$$
4. *u takes "enough" late high values in the sense that*
$$\left|\left\{u>1-\lambda^2/2\right\}\cap(-1,0]\times B_1\right|\ge\delta|B_1|,$$

then
$$\left|\left\{1/2\le u\le 1-\lambda^2/2\right\}\cap\left((-2,0]\times B_1\right)\right|\ge\gamma\,|(-2,0]\times B_1|. \tag{4.52}$$

Remark 4.42. *In view of the proof, a formula can be given for γ as a function of ρ,δ and constants only depending on N,m and α (see C_+,C_1,C_2,C_D in the proof below). The admissible values for λ form an interval of the form$(0,C\rho\delta^2)$ defined precisely by (4.62).*

Remark 4.43. *Subsequently, we will use this result with some $\delta=\overline{\delta}$ given by Lemma 4.32.*

Proof. We will follow closely the proofs given in Section 4 of [18] and in Section 9 of [19]. As pointed out in [19], the key point is to collect a super-linear control of the good extra term

$$\int_{-2}^{0}\iint_{B_1\times B_1}\left(u(t,x)-(1-\tfrac{\lambda}{2})\right)_+\left(G(1-\tfrac{\lambda}{2})-G(u(y))\right)_+\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+\alpha}}\mathrm{d}t\lesssim\lambda^{1+\varepsilon}$$

for $\lambda<1/2$ and with some $\varepsilon>1$. In what follows (as in [18], [19]), we will have $\varepsilon+1=2$. Once this goal has been achieved, then the subsequent steps are a straightforward adaptation of the end of Section 4 of [18]. For the convenience of the reader, we will sketch how the end of the argument goes.

We define for $\lambda < 1/2$,

$$c_0 = \frac{1}{2}, \quad c_1 = 1 - \frac{\lambda}{2}, \quad c_2 = 1 - \frac{\lambda^2}{2}.$$

We fix $\rho \in (0, 1/8)$ and we consider

$$\mathcal{E}_+(t) = \int_{B_1} (u - c_1)_+^2(t, x)\mathrm{d}x,$$
$$\mathfrak{G}(t) = \iint_{B_1 \times B_1} (u - c_1)_+(t, x)(G(c_1) - G(u(y)))_+\mathrm{d}x\mathrm{d}y.$$

The proof proceeds in several steps. During the proof, we will use freely that on B_2

$$(u - c_1)_+ \le \frac{\lambda}{2} \cdot \mathbb{I}_{\{u(x) > c_1\}}. \tag{4.53}$$

Step 1: Using the energy estimate we first prove in this step that

$$\mathcal{E}_+'(t) \le C_+\lambda^2 \quad \text{and} \quad \int_{T_1}^{T_2} \mathfrak{G}(t)\mathrm{d}t \le C_+\lambda^2. \tag{4.54}$$

for all $-2 \le T_1 < T_2 < 0$. For any $c_0 \in (0, c_1)$, we can express our assumption (4.31) about the local energy estimate using its alternate form (4.33) and get that:

$$\frac{d}{dt}\int_{B_1} (u - c_1)_+^2(t_2, x)\mathrm{d}x + (G(c_1) - G(c_0))\left(\int_{B_1 \times B_1} (u - c_1)_+(x)\mathbb{I}_{\{u(y) < c_0\}}\mathrm{d}x\mathrm{d}y\right)$$
$$\lesssim C\left|\{u > c_1\} \cap (t_1, t_2) \times B_{3/2}\right| \lesssim C\lambda^2.$$

For the last inequality, we followed [18]-[19] and used (4.53).

Step 2: We construct a set of "early times" for which the energy is "small". More precisely, in order to do this, we consider

$$\Sigma_0 = \{t \in (-2, 0) : \ |\{u(t, .) < c_0\} \cap B_1| \ge \rho|B_1|/4\}.$$

and we prove next that

$$|\Sigma_0 \cap (-2, -1)| \ge \frac{\rho}{2}, \tag{4.55}$$
$$\int_{\Sigma_0} \mathcal{E}_+(t)\mathrm{d}t \le C_1\frac{\lambda^3}{\rho}. \tag{4.56}$$

As far as (4.55) is concerned, we remark that the assumptions of the lemma implies

$$\begin{aligned}
\rho|B_1| &\le \int_{-2}^{-1} |\{u(t) < c_0\} \cap B_1|\mathrm{d}t \\
&\le \int_{\Sigma_0 \cap (-2,-1)} |\{u(t) < c_0\} \cap B_1|\mathrm{d}t + \int_{(-2,-1)\setminus\Sigma_0} \frac{\rho}{4}|B_1|\mathrm{d}t \\
&\le |B_1||\Sigma_0 \cap (-2, -1)| + \frac{\rho}{2}|B_1|.
\end{aligned}$$

In order to get (4.56), we first remark that (4.54) yields

$$C\lambda^2 \geq (G(c_1) - G(c_0)) \int_{\Sigma_0} \int_{B_1} \int_{\{u(t,y)<c_0\}\cap B_1} (u - c_1)_+(t,x)\mathrm{d}x\mathrm{d}y\mathrm{d}t$$

Now we use $G(c_1) - G(c_0) = \left(1 - \frac{1}{2}\lambda\right)^{m-1} - \left(\frac{1}{2}\right)^{m-1} \geq \left(\frac{5}{6}\right)^{m-1} - \left(\frac{1}{2}\right)^{m-1} \gtrsim 1$ and get

$$\begin{aligned} C\lambda^2 &\gtrsim \rho \int_{\Sigma_0} \int_{B_1} (u - c_1)_+(t,x)\mathrm{d}x\mathrm{d}t \\ &\gtrsim \frac{\rho}{\lambda} \int_{\Sigma_0} \int_{B_1} (u - c_1)_+^2(t,x)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

using (4.53) again.

Step 3: We now consider the following set of "early times" for which the energy is small and we prove that it has a positive measure,

$$\tilde{\Sigma_0} = \{t \in \Sigma_0 \cap (-2,-1) : \mathcal{E}_+(t) \leq \frac{C_2}{2}\delta\lambda^2\}.$$

for C_2 to be chosen later and we prove that

$$|\tilde{\Sigma_0}| \geq \frac{\rho}{4} \tag{4.57}$$

for λ small enough. Let F denote $\Sigma_0 \setminus \tilde{\Sigma_0}$. Using (4.56) we can write

$$|F| = \int_F \mathrm{d}t \leq \frac{2}{C_2\lambda^2\delta} \int_F \mathcal{E}_+(t)\mathrm{d}t \leq \frac{2C_1\lambda}{C_2\delta\rho} \leq \frac{\rho}{2}$$

as soon as

$$\lambda \leq \frac{C_2\delta\rho^2}{4C_1}$$

and we get (4.57) from (4.55).

Step 4: We next construct a set of "late times" for which the energy is "large". Precisely, we consider

$$\Sigma_2 = \{t \in (-2,0); |\{u(t) > c_2\} \cap B_1| \geq \frac{\delta}{4}|B_1|\}$$

and we prove that

$$|\Sigma_2 \cap (-1,0)| \geq \frac{\delta}{2} \tag{4.58}$$

$$\forall t \in \Sigma_2, \qquad \mathcal{E}_+(t) \geq C_2\delta\lambda^2 \tag{4.59}$$

for $C_2 = |B_1|/64$. Estimate (4.58) is obtained as above from the assumption of the lemma. As far as (4.59) is concerned, we write for all $t \in \Sigma_2$ that

$$\begin{aligned} \mathcal{E}_+(t) &= \int_{B_1} (u - c_1)_+^2(t,x)\mathrm{d}x \\ &\geq (c_2 - c_1)^2 |B_1 \cap \{\{u(t,x) > c_2\}| \\ &\geq \frac{\lambda^2(1-\lambda)^2\delta|B_1|}{16} \\ &\geq \frac{|B_1|}{64}\delta\lambda^2 \end{aligned}$$

for $\lambda \le \frac{1}{2}$.

Step 5: In this step, we prove that the energy $\mathcal{E}_+$ takes intermediate values between $C_2\delta\lambda^2/2$ and $C_2\delta\lambda^2$ "often enough". Precisely, we consder

$$D = \{t \in (-2,0); \frac{C_2}{2}\lambda^2\delta \le \mathcal{E}_+(t) \le C_2\lambda^2\delta\}$$

and we prove that

$$|D| \ge \delta C_D \tag{4.60}$$

$$|D \setminus \Sigma_0| \ge \frac{|D|}{2}. \tag{4.61}$$

with $C_D = |C_2|/(2C_+)$. We start with (4.60) by picking a time $T_0 \in \tilde{\Sigma}_0$ where $\tilde{\Sigma}_0$ (it has positive measure thanks to (4.57)) and $T_2 \in \Sigma_2 \cap (-1,0)$ (it has positive measure thanks to (4.58)). Consider the truncature function

$$\mathcal{T}(r) = \max(\min(r, C_2\delta\lambda^2), C_2\delta\lambda^2/2).$$

Remark that $\mathcal{T}'(r) = \mathbb{I}_{\{C_2\delta\lambda^2/2 \le r \le C_2\delta\lambda^2\}}$. Then

$$\begin{aligned}\frac{C_2}{2}\delta\lambda^2 &\le \mathcal{T}\circ E(T_2) - \mathcal{T}\circ E(T_0)\\ &\le \int_{T_0}^{T_2} E'(t)\mathcal{T}'(E(t))\mathrm{d}t\\ &\le C_+\lambda^2|D|\end{aligned}$$

where we used (4.31).

As far as (4.61) is concerned, we use the definition of σ_2, (4.56) and (4.61) in order to get

$$|D\cap\Sigma_0|\frac{C_2\lambda^2\delta}{2} \le \int_{D\cap\Sigma_0} E(t)\mathrm{d}t \le C_1\frac{\lambda^3}{\rho} \le \frac{C_D\delta}{2}\times\frac{C_2\lambda^2\delta}{2} \le \frac{|D|}{2}\frac{C_2\lambda^2\delta}{2},$$

as soon as

$$0 < \lambda \le \frac{C_2C_D\rho\delta^2}{4C_1}. \tag{4.62}$$

Step 6: We will pick up an intermediate set in $D \setminus \tilde{\Sigma}_0$ with nontrivial measure. Precisely, for $t \in (-2,0)\setminus(\tilde{\Sigma}_0 \cup \Sigma_2)$, we have (recall $\delta \le \frac{1}{2}$ and $\rho \le \frac{1}{2}$)

$$\begin{aligned}|B_1| &= |\{u(t) < c_0\}\cap B_1| + |\{u(t) > c_2\}\cap B_1| + |\{c_0 < u(t) < c_2\}\cap B_1|\\ &\le \frac{\rho}{2}|B_1| + \frac{\delta}{2}|B_1| + |\{c_0 < u(t) < c_2\}\cap B_1|\\ &\le \frac{1}{2}|B_1| + |\{c_0 < u(t) < c_2\}\cap B_1|.\end{aligned}$$

Hence for all $t \in (-2,0) \setminus (\tilde{\Sigma_0} \cup \Sigma_2)$ we have

$$|\{c_0 < u(t) < c_2\} \cap B_1| \geq \frac{1}{2}|B_1|.$$

Moreover $D \setminus \tilde{\Sigma_0} \subset (-2,0) \setminus (\tilde{\Sigma_0} \cup \Sigma_2)$. So we conclude

$$\begin{aligned}
\int_{-2}^{0} |\{c_0 < u(t) < c_2\} \cap B_1| \mathrm{d}t &\geq \int_{(-2,0)\setminus(\tilde{\Sigma_0}\cup\Sigma_2)} |\{c_0 < u(t) < c_2\} \cap B_1| \mathrm{d}t \\
&\geq \frac{1}{2}|B_1||(-2,0) \setminus (\tilde{\Sigma_0} \cup \Sigma_2)| \\
&\geq \frac{1}{2}|B_1||D \setminus \tilde{\Sigma_0}| \\
&\geq \frac{1}{2}|B_1|.\frac{|D|}{2} \\
&\geq \frac{C_D \delta}{4}|B_1|.
\end{aligned}$$

Hence the Lemma is proved with $\gamma = \frac{C_D\delta}{4}$. □

4.6 Lowering the maximum improved

We are now in a position to prove the improved oscillation reduction result from above. We follow the argument given in Section 10 of [19]. The key will be a proper rescaling of the solution.

Lemma 4.44 (Lowering the maximum improved)**.** *Let $\alpha \in (1,2)$. We take ϵ_0 from Theorem 4.11. For any $\mu \in (0,1/2]$ and $\rho \in (0,1)$, there exists $\mu^* \in (0,1)$ such that for any function u that satisfies the following assumptions:*

1. *u satisfies*
$$u(t,x) \leq 1 \qquad \textit{in } (-2,0] \times B_2$$
2. *the upper local energy-inequality (4.31) is satisfied,*
3. *u takes "some" early low values in the sense that*
$$|\{u < \mu\} \cap (-2,-1] \times B_2| \geq \rho|B_2|,$$

then $u(t,x) \leq \mu^*$ in $(-1/2,0] \times B_{1/2}$. Note that the value of μ^* depends only on the dimension N, on γ, λ from Lemma 4.41, on ρ, μ and ϵ_0.

Remark 4.45. *Note that the major difference with the first De Giorgi Lemma 4.32 is that the value of ρ is now arbitrary, while before it was fixed to $\rho = 1 - \overline{\delta}$. Also note that now, as we apply the Intermediate Values Lemma 4.41, the full length of (4.31) is required, i.e. the "good extra term" plays a crucial role. Lastly, their is a time-gap (from $t = -1$ to $t = -1/2$) between the third assumption and the conclusion.*

Proof. The key of the proof consists in applying Lemma 4.41 to a sequence of functions until all the space-time available for the intermediary values is spend. From then on, we will know that u is mostly low-valued on the "late" times, i.e. on $(-1,0] \times B_1$. The first De Giorgi Lemma 4.32 will then be applied with a high threshold and will reduce the maximum, but only on "late" times compared to its domain of application. This step is thus responsible for a small but necessary time-gap between the assumptions and the conclusion and we can only improve the maximum on $(-1/2,0] \times B_{1/2}$. The first steps consists in checking the assumptions of Lemma 4.41 on a sequence of "pushed down and rescaled" versions of u.

Choice of constants. First, we take the values of $\lambda < 1/2$ and γ given in Lemma 4.41. Next, we consider

$$j_0 = \left\lceil \frac{|(-2,0] \times B_1|}{\gamma} \right\rceil.$$

Finally, we take the value $\overline{\delta}$ given by Lemma 4.32 when applied to $\overline{\mu} = 1 - \lambda^{2j_0+2}$.

Claim 1. Our first claim is that the functions defined for $1 \le j \le j_0$ by

$$u_j(t,x) = \frac{u(t,x) - (1-\lambda^2)(1+\lambda^2+\lambda^4+\ldots+\lambda^{2j-2})}{\lambda^{2j}} = \frac{u(t,x)-(1-\lambda^{2j})}{\lambda^{2j}}$$

satisfy the local energy estimates (4.31) with uniform constants. Let us observe that as $j \to \infty$, one has $\lambda^{2j} u_j(t,x) \to u(t,x) - 1 \le 0$ on $B_{2^{1/\epsilon_0}}$ so that u_j may take some negatives values. Equivalently, the sequence is defined iteratively by

$$u_{j+1}(t,x) = \frac{1}{\lambda^2}\left(u_j(t,x) - (1-\lambda^2)\right),$$

starting from $u_1(t,x) = \lambda^{-2}(u(t,x)-(1-\lambda^2))$.

For any $c_j > 0$, let us repeatedly apply our assumption (4.31) to the function u, with the cutoff constant

$$c = \lambda^{2j} c_j + (1-\lambda^{2j}) > 1/4,$$

radii $0 < r < R < 2^{1/\epsilon_0}$ and start and stop times $-2 < t_1 < t_2 < 0$. Using (4.33) to express the good extra term, we get:

$$\begin{aligned}
&\int_{B_r} (u-(1-\lambda^{2j})-\lambda^{2j}c_j)_+^2(t_2,x)\mathrm{d}x + \int_{t_1}^{t_2}\left(\int_{B_r}(u-(1-\lambda^{2j})-\lambda^{2j}c_j)_+^p(x)\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2}\left(\iint_{B_r\times B_r}\left(u(x)-(1-\lambda^{2j})-\lambda^{2j}c_j\right)_+\left(G((1-\lambda^{2j})+\lambda^{2j}c_j)-G(u(y))\right)_+\mathrm{d}x\mathrm{d}y\right)\mathrm{d}t \\
&\quad \lesssim \int_{B_R}(u(t_1,x)-(1-\lambda^{2j})-\lambda^{2j}c_j)_+^2\mathrm{d}x + C(R-r)^{-2}\left|\{u-(1-\lambda^{2j}) > \lambda^{2j}c_j\} \cap (t_1,t_2)\times B_R\right|.
\end{aligned}$$

We deduce from the previous inequality that $u_j - c_j = (u-c)/\lambda^{2j}$ satisfies the following local energy estimate

$$\int_{B_r} (u_j(t_2,x)-c_j)_+^2 \mathrm{d}x + \int_{t_1}^{t_2} \left(\int_{B_r} (u_j(t,x)-c_j)_+^p \mathrm{d}x \right)^{\frac{2}{p}} \mathrm{d}t$$
$$+ \int_{t_1}^{t_2} \left(\iint_{B_r \times B_r} (u_j-c_j)_+(x)(G((1-\lambda^{2j})+\lambda^{2j}c_j) - G(u(y)))\mathrm{d}x\mathrm{d}y \right) \mathrm{d}t$$
$$\lesssim \int_{B_R} (u_j(t_1,x)-c_j)_+^2 \mathrm{d}x + C(R-r)^{-2}\lambda^{-4j} \left|\{u_j > c_j\} \cap (t_1,t_2) \times B_R\right|$$

As $\lambda^{-1} > 1$, we have $\lambda^{-4j} \le \lambda^{-4j_0}$ if $j \le j_0$. We conclude that, as long as $1 \le j \le j_0$, all the functions u_j satisfies the local energy estimates (4.31) with uniform constants. Moreover, $c_j > 0$ can be arbitrary.

Claim 2. We also claim that the early low-values assumption of Lemma 4.41 does hold for u_j. Indeed, as $\lambda < 1$ then for any $\mu < 1$, the inequality $u_j(t,x) < \mu$ implies $u_{j+1}(t,x) < \mu$ hence

$$|\{u_j < 1/2\} \cap (-2,-1] \times B_1| \ge |\{u_j < \mu\} \cap (-2,-1] \times B_1| \ge |\{u < \mu\} \cap (-2,-1] \times B_1| \ge \rho |B_1|.$$

As $\mu \le 1/2$, the early low-values assumption of Lemma 4.41 is satisfied.

Main Step. Let us now reason by contradiction. We assume that for any $j \in [1, j_0]$ one has

$$|\{u_j > 1 - \lambda^2/2\} \cap (-1,0] \times B_1| \ge \bar{\delta} |B_1|.$$

Then the Lemma 4.41 on intermediate values can be applied to u_j and implies that

$$|\{1/2 \le u_j \le 1 - \lambda^2/2\} \cap (-2,0] \times B_1| \ge \gamma |(-2,0] \times B_1|.$$

Translating this for the function u, we get

$$\left| \left\{ 1 - \frac{\lambda^{2j}}{2} \le u \le 1 - \frac{\lambda^{2j}}{2} - \frac{\lambda^{2j+1}}{2} \right\} \cap (-2,0] \times B_1 \right| \ge \gamma \, |(-2,0] \times B_1|.$$

This implies in particular

$$\left| \left\{ 1 - \frac{\lambda^{2j}}{2} \le u \le 1 - \frac{\lambda^{2j+2}}{2} \right\} \cap (-2,0] \times B_1 \right| \ge \gamma \, |(-2,0] \times B_1|.$$

But these intermediate level sets are disjoint and of positive measure so there can be only at most $j_0 - 1$ of them in the space-time ball $(-2,0] \times B_1$. The original assumption is false.

In particular, there exists $j_1 \le j_0$ such that

$$|\{u_{j_1} > 1 - \lambda^2/2\} \cap (-1,0] \times B_1| < \bar{\delta} |B_1|.$$

As $\overline{\mu} = 1 - \lambda^{2j_0+2}$, this translates back to u as

$$|\{u > \tfrac{1+\overline{\mu}}{2}\} \cap (-1,0] \times B_1| \le |\{u > 1 - \lambda^{2j_1+2}/2\} \cap (-1,0] \times B_1| < \overline{\delta}|B_1|. \qquad (4.63)$$

We want to apply the first De Giorgi Lemma 4.32 to u with $\overline{\mu} = 1 - \lambda^{2j_0+2}$. However, (4.63) only states that u is "mostly low valued" at late times while Lemma 4.32 requires u to be "mostly low valued" for all times.

We thus consider $\tilde{u}(t,x) = u(t/2^\alpha, x/2)$ which satisfies

$$|\{\tilde{u} > \tfrac{1+\overline{\mu}}{2}\} \cap (-2,0] \times B_2| < \overline{\delta}|B_1|.$$

because $2^\alpha > 2$ (note that we use here again that $\alpha \ge 1$). Applying Lemma 4.32 to $\tilde{u}$, we get:

$$\tilde{u}(t,x) \le \frac{3+\overline{\mu}}{4} = \mu^* \quad \text{ on } (-1,0] \times B_1$$

with $\mu^* = 1 - \lambda^{2(j_0+1)}/4$. Hence $u(t,x) \le \mu^*$ on $(-1/2, 0] \times B_{1/2}$. □

4.7 Proof of the main theorem

In this section, we use alternatively the Lemma of De Giorgi on increasing the infimum (Lemma 4.36) and the improved lemma about lowering the maximum (Lemma 4.44) in order to prove Theorem 4.2.

Proof of Theorem 4.2. We now consider a weak solution of (4.1)-(4.2) associated with an initial data

$$u_0 \in L^1 \cap L^\infty(\mathbb{R}^N; \mathbb{R}_+).$$

We know from [16, Theorem 2.6] which we recalled here as Theorem 4.6, that this solution is globally bounded in $[0,+\infty)\times\mathbb{R}^N$, by a constant M which depends on $\|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)}$. To prove Theorem 4.2, we want to study its Hölder regularity on some interval $[T_0, T_1]$ with $0 < T_0 < T_1$.

We can translate the time interval and study the equation in $(-2,0] \times \mathbb{R}^N$. It is then sufficient to prove that it is Hölder continuous at the point $(t_0, x_0) = (0,0)$. Using the scale invariance, we can assume without loss of generality that $M = 1$ (by choosing $A = 1/M$ and $C = M^{\frac{m-1}{\alpha}}$ in (4.16)). In particular,

$$0 \le u(t,x) \le 1 + \Psi_{\epsilon_0}(x) \quad \text{ for } (t,x) \in (-2,0] \times \mathbb{R}^N. \qquad (4.64)$$

where Ψ_{ϵ_0} is the mildly-growing function defined by (4.18).

In order to apply Lemma 4.9, we are going to prove that the oscillation of u around the point $(t_0, x_0) = (0,0)$ decays algebraically on $(-r, 0] \times B_r$ as $r \to 0$. More precisely, we will show subsequently that if the solution u satisfies (4.64), then $\operatorname*{osc}_{(-1,0]\times B_1} u \le \omega_0 < 1$. Thanks to (4.16) we can then construct a sequence of rescaled solutions

$$v_{n+1}(t,x) = \frac{v_n(t/\tau_n, x/\kappa_n)}{(\tau_n/\kappa_n^\alpha)^{1/(m-1)}} \qquad \text{with} \qquad v_1(t,x) = u(t,x)$$

and scaling parameters $\tau_n, \kappa_n \ge 2$ such that $\tau_n/\kappa_n^\alpha \le 1$. One can adjust the parameters such that all the v_n satisfy (4.64). Note that the values of the pair (τ_n, κ_n) can alternate between a few universal choices from one iteration to the next, but overall, it has no detrimental effect.

Iterating this construct gives a dyadic formulation of the assumption of Lemma 4.9, which can then ultimately be applied to u and provides the desired Hölder regularity.

Let us now explain the fine details of the process that reduces the oscillation of v_n on $(-1,0] \times B_1$. We consider an increasing sequence of thresholds

$$\mu_n = 1 - \frac{1}{2^n}$$

We take δ_n to be the value of $\underline{\delta}$ associated with $\underline{\mu} = \mu_n$ by Lemma 4.36. We will successively distinguish two mutually exclusive cases.

- The first possibility is that

$$|\{v_n \ge (1+2\mu_n)/3\} \cap (-2,-1] \times B_2| \ge (1-\delta_n)|(-2,0) \times B_2|. \tag{4.65}$$

In particular, one has

$$|\{v_n \ge (1+2\mu_n)/3\} \cap (-2,0] \times B_2| \ge (1-\delta_n)|(-2,0) \times B_2|.$$

In this case, we can apply Lemma 4.36 with $\underline{\mu} = \mu_n$ and get that u satisfies

$$v_n(t,x) \ge \mu_n \text{ in } (-1,0] \times B_1.$$

The oscillation of v_n has thus decreased from 1 on $(-2,0] \times B_2$ to $\operatorname*{osc}_{(-1,0]\times B_1} v_n \le 1 - \mu_n = 2^{-n}$.

For the subsequent rescaling, we take

$$v_{n+1}(t,x) = v_n(t/\tau, x/\kappa) \qquad \text{with} \qquad \kappa = 2 \quad \text{and} \quad \tau = \kappa^\alpha \ge 2$$

which, according to (4.16), is also a solution of (4.1). Moreover, it satisfies

$$0 \le (1-2^{-n})\,\mathbb{I}_{B_2}(x) \le v_{n+1}(t,x) \le 1 + \Psi_{\epsilon_0}(x) \qquad \text{on} \qquad (-2,0] \times \mathbb{R}^N \tag{4.66}$$

so in particular (4.64) holds again for v_{n+1}.

- In case (4.65) fails, the alternative reads

$$|\{v_n \ge (1+2\mu_n)/3\} \cap (-2,-1] \times B_2| < (1-\delta_n)|(-2,0] \times B_2| \tag{4.67}$$

which implies

$$|\{v_n < (1+2\mu_n)/3\} \cap (-2,-1] \times B_2| \geq 2\delta_n |B_2|.$$

In this case, we can apply Lemma 4.44 with $\mu = (1+2\mu_n)/3$ and $\rho = 2\delta_n$ and get that

$$v_n(t,x) \leq \mu^* \text{ in } (-1/2, 0] \times B_{1/2}.$$

The oscillation of v_n has thus decreased to $\operatorname*{osc}_{(-1/2,0]\times B_{1/2}} v_n \leq \mu^*$.

We then consider the function

$$v_{n+1}(t,x) = \frac{v_n(t/\tau, x/\kappa)}{\mu^*}$$

with $\tau = (\mu^*)^{m-1}\kappa^\alpha$. Note that $\tau/\kappa^\alpha < 1$. Thanks to (4.16), we know that v_{n+1} is still a weak solution of (4.1) and that

$$\begin{cases} v \leq 1 & \text{in } (-\tau/2, 0] \times B_{\kappa/2}, \\ v \leq \frac{(|x/\kappa|^{\epsilon_0} - 2)_+ + 1}{\mu^*} & \text{in } (-\tau, 0] \times \mathbb{R}^N. \end{cases}$$

It is not difficult to check that for $\kappa \geq 2(1 + (\mu^*)^{-1})^{1/\epsilon} > 4$, then

$$\frac{(|x/\kappa|^{\epsilon_0} - 2)_+ + 1}{\mu^*} \leq 1 + \Psi_{\epsilon_0}(x) \qquad \text{ouside } B_{\kappa/2}.$$

By also choosing $\kappa \geq (4(\mu^*)^{-(m-1)})^{1/\alpha}$, we get $\tau \geq 4$ and therefore

$$0 \leq v_{n+1}(t,x) \leq 1 + \Psi_{\epsilon_0}(x) \qquad \text{on} \qquad (-2, 0] \times \mathbb{R}^N.$$

This concludes the treatment of the alternative case (4.67).

Conclusion. In both cases (4.65)-(4.67), we have reduced the oscillation of u by at least a universal factor

$$\omega_0 = \frac{1}{2} \wedge \mu^* < 1$$

and proposed a universal rescaling process that brings us back to the initial situation (4.64).

As explained in the introduction of this proof, the oscillation then decays algebraically when zooming-in and this fact achieves the proof of the main theorem. □

4.8 Useful inequalities

Lemma 4.46. *The following inequalities are valid for any $m > 1$*

$$\forall a,b \geq 0, \qquad (a-b)(a^{m-1} - b^{m-1}) \geq \tfrac{4(m-1)}{m^2} \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \tag{4.68}$$

$$\forall a,b \geq 0, \qquad (a^2-b^2)(a^{m-1} - b^{m-1}) \geq \tfrac{8(m-1)}{(m+1)^2} \left(a^{\frac{m+1}{2}} - b^{\frac{m+1}{2}}\right)^2 \tag{4.69}$$

and

$$\forall a,b \geq c > 0, \qquad (a-b)^2 \leq c^{-(m-1)} (a^{\frac{m+1}{2}} - b^{\frac{m+1}{2}})^2. \tag{4.70}$$

Remark 4.47. *The following proof also shows that converse inequalities to (4.68)-(4.69) are also true:*

$$(a-b)(a^{m-1}-b^{m-1}) \le \left(a^{\frac{m}{2}}-b^{\frac{m}{2}}\right)^2 \tag{4.71}$$
$$(a^2-b^2)(a^{m-1}-b^{m-1}) \le \left(a^{\frac{m+1}{2}}-b^{\frac{m+1}{2}}\right)^2 \tag{4.72}$$

for any $a,b \ge 0$.

Proof. When $a=0$, all inequalities are obvious, at least once we observe that

$$m^2-4(m-1)=(m-2)^2 \ge 0 \qquad \text{and} \qquad (m+1)^2-8(m-1)=(m-3)^2 \ge 0.$$

Again, they are also true when $a=b$. We can thus assume that $a \ne 0$ and $a \ne b$ and consider $\theta = b/a \in \mathbb{R}_+$ but with $\theta \ne 1$. We claim that the functions

$$f(\theta)=\frac{(1-\theta^{\frac{m}{2}})^2}{(1-\theta)(1-\theta^{m-1})}, \qquad g(\theta)=\frac{(1-\theta^{\frac{m+1}{2}})^2}{(1-\theta^2)(1-\theta^{m-1})} \qquad \text{and} \qquad h(\theta)=\frac{1-\theta}{1-\theta^{\frac{m+1}{2}}}$$

are continuous through $\theta=1$ and satisfy $\|f\|_{L^\infty(\mathbb{R}_+)} \le \frac{m^2}{4(m-1)}$, $\|g\|_{L^\infty(\mathbb{R}_+)} \le \frac{(m+1)^2}{8(m-1)}$ and $\|h\|_{L^\infty(\mathbb{R}_+)} \le 1$. The inequalities (4.68)-(4.69) then follows from $(1-\theta)(1-\theta^{m-1})>0$ and $(1-\theta^2)(1-\theta^{m-1})>0$ while (4.70) comes from

$$(a-b)^2=a^2g(\theta)^2(1-\theta^{\frac{m+1}{2}})^2 \le a^{-(m-1)}(a^{\frac{m+1}{2}}-b^{\frac{m+1}{2}})^2\|g\|_{L^\infty}^2$$

and the final restriction $a \ge c$.

To back up our claim, let us briefly study the functions f, g and h. The continuity around $\theta=1$ comes from a simple Taylor expansion:

$$\begin{aligned}
f(\theta) &= \frac{m^2}{4(m-1)}-\frac{m^2(m-2)^2}{192(m-1)}(\theta-1)^2+O[(\theta-1)^3]\\
g(\theta) &= \frac{(m+1)^2}{8(m-1)}-\frac{(m+1)^2(m-3)^2}{384(m-1)}(\theta-1)^2+O[(\theta-1)^3]\\
h(\theta) &= \frac{2}{m+1}-\frac{m-1}{2(m+1)}(\theta-1)+O[(\theta-1)^2].
\end{aligned}$$

Moreover, one can check that $f(\theta)$ and $g(\theta)$ reach a global maximum when $\theta=1$ while $h(\theta)$ is maximal at $\theta=0$, i.e.:

$$\forall \theta>0, \qquad \begin{cases} 1 \le f(\theta) \le f(1), \\ 1 \le g(\theta) \le g(1) \\ 0 \le h(\theta) \le h(0) \end{cases}$$

The lower values follow from the limits at 0 and $+\infty$, once we know the variations of f,g,h.

For example, for any $\theta > 0$, one has

$$h'(\theta) = -\frac{(m-1)\theta^{\frac{m+1}{2}} - (m+1)\theta^{\frac{m-1}{2}} + 2}{2\theta\left(1-\theta^{\frac{m+1}{2}}\right)^2} \leq 0$$

because $(m-1)\theta^{\frac{m+1}{2}} - (m+1)\theta^{\frac{m-1}{2}} + 2$ is a positive functions that vanishes only for $\theta = 1$. Indeed, we can rewrite it as a balance of two signed terms

$$(m-1)\theta^{\frac{m+1}{2}} - (m+1)\theta^{\frac{m-1}{2}} + 2 = \theta^{\frac{m-1}{2}}(m-1)(\theta-1) + 2(1-\theta^{\frac{m-1}{2}})$$

whose derivative is

$$\frac{d}{d\theta}\left[\theta^{\frac{m-1}{2}}(m-1)(\theta-1) + 2(1-\theta^{\frac{m-1}{2}})\right] = \theta^{\frac{m-3}{2}}(m-1)(\theta-1)\left(\theta+\frac{m+1}{2}\right)$$

and has therefore the same sign as $\theta - 1$.

Similarly, one has $f'(\theta) = \frac{(1-\theta^{m/2})(1-\theta^{(m-2)/2})\mathfrak{F}(\theta)}{(1-\theta)^2(1-\theta^{m-1})^2}$ with $\mathfrak{F}(\theta) = 1-\theta^{m-1} + (m-1)\theta^{m/2}(1-\theta^{-1})$ which (based on a quick study of $\mathcal{F}'$) has the sign of $(m-1)(2-m)(\theta-1)$. Therefore $f'(\theta)$ has the same sign as

$$(m-1)(2-m)(\theta-1)(1-\theta^{m/2})(1-\theta^{(m-2)/2})$$

which, for $m > 1$ is positive on $(0,1)$ and negative on $(1,+\infty)$.

In the same spirit, one gets $g'(\theta) = \frac{(1-\theta^{\frac{m+1}{2}})\mathfrak{G}(\theta)}{\theta^2(1-\theta^2)^2(1-\theta^{m-1})^2}$ for some auxiliary function $\mathfrak{G}(\theta) \geq 0$ and thus $g'(\theta)$ has the same sign as $1-\theta^{\frac{m+1}{2}}$. □

Bibliography

[1] M. S. Agranovich and B. A. Amosov. On Fourier series in eigenfunctions of elliptic boundary value problems. *Georgian Math. J.*, 10(3):401–410, 2003.

[2] Mark Allen, Luis Caffarelli, and Alexis Vasseur. Porous medium flow with both a fractional potential pressure and fractional time derivative. *Chin. Ann. Math. Ser. B*, 38(1):45–82, 2017.

[3] Robert Almgren. Singularity formation in Hele-Shaw bubbles. *Phys. Fluids*, 8(2):344–352, 1996.

[4] D. G. Aronson. The porous medium equation. In *Nonlinear diffusion problems (Montecatini Terme, 1985)*, volume 1224 of *Lecture Notes in Math.*, pages 1–46. Springer, Berlin, 1986.

[5] Eric Baer and Alexis Vasseur. A bound from below on the temperature for the Navier-Stokes-Fourier system. *SIAM J. Math. Anal.*, 45(4):2046–2063, 2013.

[6] J. Bear. *Dynamics of fluids in porous media*, volume 27. Dover, New York, 1972.

[7] Elena Beretta, Michiel Bertsch, and Roberta Dal Passo. Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. *Arch. Rational Mech. Anal.*, 129(2):175–200, 1995.

[8] F. Bernis, L. A. Peletier, and S. M. Williams. Source type solutions of a fourth order nonlinear degenerate parabolic equation. *Nonlinear Anal.*, 18(3):217–234, 1992.

[9] Francisco Bernis. Finite speed of propagation and continuity of the interface for thin viscous flows. *Adv. Differential Equations*, 1(3):337–368, 1996.

[10] Francisco Bernis and Avner Friedman. Higher order nonlinear degenerate parabolic equations. *J. Differential Equations*, 83(1):179–206, 1990.

[11] A. L. Bertozzi and M. Pugh. The lubrication approximation for thin viscous films: the moving contact line with a “porous media” cut-off of van der Waals interactions. *Nonlinearity*, 7(6):1535–1564, 1994.

[12] Aa La Bertozzi and Ma Pugh. The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Comm. Pure Appl. Math.*, 49(2):85–123, 1996.

[13] Andrea L. Bertozzi. The mathematics of moving contact lines in thin liquid films. *Notices Amer. Math. Soc.*, 45(6):689–697, 1998.

[14] Andrea L. Bertozzi, Michael P. Brenner, Todd F. Dupont, and Leo P. Kadanoff. Singularities and similarities in interface flows. In *Trends and perspectives in applied mathematics*, volume 100 of *Appl. Math. Sci.*, pages 155–208. Springer, New York, (1994).

[15] Piotr Biler, Cyril Imbert, and Grzegorz Karch. Barenblatt profiles for a nonlocal porous medium equation. *C. R. Math. Acad. Sci. Paris*, 349(11-12):641–645, 2011.

[16] Piotr Biler, Cyril Imbert, and Grzegorz Karch. The nonlocal porous medium equation: Barenblatt profiles and other weak solutions. *Arch. Ration. Mech. Anal.*, 215(2):497–529, 2015.

[17] J Boussinesq. Recherches théoriques sur l'écoulement des nappes d'eau infiltrées dans le sol et sur le débit des sources. *Journal de mathématiques pures et appliquées*, 10:5–78, 1904.

[18] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, 2011.

[19] Luis Caffarelli, Fernando Soria, and Juan Luis Vázquez. Regularity of solutions of the fractional porous medium flow. *J. Eur. Math. Soc. (JEMS)*, 15(5):1701–1746, 2013.

[20] Luis Caffarelli and Alexis Vasseur. The De Giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics. *Discrete Contin. Dyn. Syst. Ser. S*, 3(3):409–427, 2010.

[21] Luis Caffarelli and Juan Luis Vazquez. Nonlinear porous medium flow with fractional potential pressure. *Arch. Ration. Mech. Anal.*, 202(2):537–565, 2011.

[22] Luis Caffarelli and Juan Luis Vázquez. Regularity of solutions of the fractional porous medium flow with exponent 1/2. *Algebra i Analiz [St. Petersburg Mathematical Journal]*, 27(3), 2015.

[23] Luis A. Caffarelli and Alexis F. Vasseur. The De Giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics. *Discrete Contin. Dyn. Syst. Ser. S*, 3(3):409–427, 2010.

[24] Luis A. Caffarelli and Juan Luis Vázquez. Asymptotic behaviour of a porous medium equation with fractional diffusion. *Discrete Contin. Dyn. Syst.*, 29(4):1393–1404, 2011.

[25] M. Cristina Caputo and Alexis Vasseur. Global regularity of solutions to systems of reaction-diffusion with sub-quadratic growth in any dimension. *Comm. Partial Differential Equations*, 34(10-12):1228–1250, 2009.

[26] J. A. Carrillo and G. Toscani. Long-time asymptotics for strong solutions of the thin film equation. *Comm. Math. Phys.*, 225(3):551–571, 2002.

[27] Roberta Dal Passo, Harald Garcke, and Günther Grün. On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions. *SIAM J. Math. Anal.*, 29(2):321–342, 1998.

[28] Roberta Dal Passo, Lorenzo Giacomelli, and Günther Grün. Waiting time phenomena for degenerate parabolic equations—a unifying approach. In *Geometric analysis and nonlinear partial differential equations*, pages 637–648. Springer, Berlin, 2003.

[29] Ennio De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[30] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A fractional porous medium equation. *Advances in Mathematics*, 226(2):1378 – 1409, 2011.

[31] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A general fractional porous medium equation. *Communications on Pure and Applied Mathematics*, 65(9):1242–1284, 2012.

[32] E. DiBenedetto. On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 13(3):487–535, 1986.

[33] E. DiBenedetto, U. Gianazza, and V. Vespri. Harnack type estimates and Hölder continuity for non-negative solutions to certain sub-critically singular parabolic partial differential equations. *Manuscripta Math.*, 131(1-2):231–245, 2010.

[34] E. DiBenedetto and Neil S. Trudinger. Harnack inequalities for quasiminima of variational integrals. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(4):295–308, 1984.

[35] Emmanuele DiBenedetto, Ugo Gianazza, and Vincenzo Vespri. Forward, backward and elliptic Harnack inequalities for non-negative solutions to certain singular parabolic partial differential equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 9(2):385–422, 2010.

[36] Emmanuele DiBenedetto, Ugo Gianazza, and Vincenzo Vespri. *Harnack's inequality for degenerate and singular parabolic equations*. Springer Monographs in Mathematics. Springer, New York, 2012.

[37] Charles M. Elliott and Harald Garcke. On the Cahn-Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.*, 27(2):404–423, 1996.

[38] Matthieu Felsinger and Moritz Kassmann. Local regularity for parabolic nonlocal operators. *Comm. Partial Differential Equations*, 38(9):1539–1573, 2013.

[39] J. Geertsma and F. de Klerk. A rapid method of predicting width and extent of hydraulically induced fractures. *J. Pet. Technol.*, 21:1571–81, 1969.

[40] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[41] François Golse, Cyril Imbert, Clément Mouhot, and Alexis Vasseur. Harnack inequality for kinetic fokker-planck equations with rough coefficients and application to the landau equation. *arXiv preprint arXiv:1607.08068*, 2017.

[42] Thierry Goudon and Alexis Vasseur. Regularity analysis for systems of reaction-diffusion equations. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(1):117–142, 2010.

[43] H. P. Greenspan. On the motion of a small viscous droplet that wets a surface. *J. Fluid Mech.*, 1978.

[44] H. P. Greenspan and B. M. McCay. On the wetting of a surface by a very viscous fluid. *Stud. Appl. Math.*, 1981.

[45] G. Grün. Degenerate parabolic differential equations of fourth order and a plasticity model with non-local hardening. *Z. Anal. Anwendungen*, 14(3):541–574, 1995.

[46] G. Grün. On Bernis' interpolation inequalities in multiple space dimensions. *Z. Anal. Anwendungen*, 20(4):987–998, 2001.

[47] Günther Grün. Droplet spreading under weak slippage: the optimal asymptotic propagation rate in the multi-dimensional case. *Interfaces Free Bound.*, 4(3):309–323, 2002.

[48] Günther Grün. Droplet spreading under weak slippage: a basic result on finite speed of propagation. *SIAM J. Math. Anal.*, 34(4):992–1006, 2003.

[49] L. M. Hocking. Sliding and spreading of thin two-dimensional drops. *Quart. J. Mech. Appl. Math.*, 1981.

[50] Josephus Hulshof. Some aspects of the thin film equation. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 291–301. Birkhäuser, Basel, (2001).

[51] C. Imbert and A. Mellet. Self-similar solutions for a fractional thin film equation governing hydraulic fractures. *Preprint, arxiv:1403.7491.*

[52] C. Imbert and A. Mellet. Self-similar solutions for a fractional thin film equation governing hydraulic fractures. *Communications in Mathematical Physics*, 340(3):1187–1229, Dec 2015.

[53] Cyril Imbert. Finite speed of propagation for a non-local porous medium equation. *Colloq. Math.*, 143(2):149–157, 2016.

[54] Cyril Imbert and Antoine Mellet. Existence of solutions for a higher order non-local equation appearing in crack dynamics. *Nonlinearity*, 24(12):3487–3514, 2011.

[55] Cyril Imbert and Luis Silvestre. Weak harnack inequality for the boltzmann equation without cut-off. *arXiv preprint arXiv:1608.07571*, 2016.

[56] Moritz Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differential Equations*, 34(1):1–21, 2009.

[57] N. V. Krylov and M. V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):161–175, 239, 1980.

[58] A. A. Lacey. The motion with slip of a thin viscous droplet over a solid surface. *Stud. Appl. Math.*, 67(3):217–230, 1982.

[59] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tceva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[60] Olga A. Ladyzhenskaya and Nina N. Ural'tseva. *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.

[61] Olga A. Ladyzhenskaya and Nina N. Ural'tseva. *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.

[62] L.S. Leibenzon. The motion of a gas in a porous medium. *Complete Works, Acad. Sciences URSS, Moscow*, 2, 1930.

[63] Gary M Lieberman. *Second order parabolic differential equations.* World scientific, 1996.

[64] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod; Gauthier-Villars, Paris, (1969).

[65] Jürgen Moser. A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.*, 13:457–468, 1960.

[66] M. Muskat. The flow of homogeneous fluids through porous media. *McGraw-Hill, New York*, 1937.

[67] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.*, 80:931–954, 1958.

[68] John Ockendon, Sam Howison, Andrew Lacey, and Alexander Movchan. *Applied partial differential equations.* Oxford University Press, Oxford, revised edition, (2003).

[69] A. P. Peirce and E. Siebrits. A dual mesh multigrid preconditioner for the efficient solution of hydraulically driven fracture problems. *Internat. J. Numer. Methods Engrg.*, 63(13):1797–1823, 2005.

[70] Anthony Peirce and Emmanuel Detournay. An Eulerian moving front algorithm with weak-form tip asymptotics for modeling hydraulically driven fractures. *Comm. Numer. Methods Engrg.*, 25(2):185–200, 2009.

[71] P. Ya. Polubarinova-Kochina. *Theory of ground water movement.* Translated from the Russian by J. M. Roger De Wiest. Princeton University Press, Princeton, N.J., (1962).

[72] D. A. Spence and P. Sharp. Self-similar solutions for elastohydrodynamic cavity flow. *Proc. Roy. Soc. London Ser. A*, 400(1819):289–313, 1985.

[73] Diana Stan, Félix del Teso, and Juan Luis Vázquez. Finite and infinite speed of propagation for porous medium equations with nonlocal pressure. *J. Differential Equations*, 260(2):1154–1199, 2016.

[74] Rana Tarhini. Study of a family of higher order nonlocal degenerate parabolic equations: from the porous medium equation to the thin film equation. *J. Differential Equations*, 259(11):5782–5812, 2015.

[75] Alexis Vasseur. Higher derivatives estimate for the 3D Navier-Stokes equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(5):1189–1204, 2010.

[76] Alexis F. Vasseur. The De Giorgi method for elliptic and parabolic equations and some applications. In *Lectures on the analysis of nonlinear partial differential equations. Part 4*, volume 4 of *Morningside Lect. Math.*, pages 195–222. Int. Press, Somerville, MA, 2016.

[77] Juan Luis Vázquez. *The porous medium equation*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. Mathematical theory.

[78] Guang Lie Wang. Harnack inequalities for functions in De Giorgi parabolic class. In *Partial differential equations (Tianjin, 1986)*, volume 1306 of *Lecture Notes in Math.*, pages 182–201. Springer, Berlin, 1988.

[79] Guang Lie Wang and Zhaubo Liu. Harnack inequalities for functions in the general De Giorgi parabolic class. *Chinese Ann. Math. Ser. B*, 10(2):154–168, 1989. A Chinese summary appears in Chinese Ann. Math. Ser. A **10** (1989), no. 3, 387.

[80] Ya.B. Zel'dovich and Yu.P. Raizer. Physics of shock waves and high-temperature hydrodynamic phenomena. *Academic Press, New York*, II, 1966.

[81] Y.P. Zheltov and S.A. Khristianovich. On hydraulic fracturing of an oil-bearing stratum. *Izv. Akad. Nauk. SSSR. Otdel Tekhn. Nauk*, 5:3–41, 1955.