Contributions to tensor models, Hurwitz numbers and Macdonald-Koornwinder polynomials

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JURY

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Introduction

Three related subjects: tensor models, Hurwitz numbers and Macdonald-Koornwinder polynomials are studied in this thesis. I would like to make some general remarks about these subjects and the structure of my thesis. More details will be given right below in each dedicated section of this introduction and in the corresponding chapters in the main text.

Tensor models are generalizations of matrix models as an approach to quantum gravity in arbitrary dimensions (matrix models give a $2D$ version). Matrix models (and more generally, random matrices) have been playing an important role in mathematics and physics, particularly in integrable systems, quantum field theories, string theory and statistical mechanics. Tensor models are comparatively young, and their significance has been continuously rising due to their rich properties. In this thesis, I study a specific model called the quartic melonic tensor model. The specialty of this model is that it can be transformed into a matrix model, which is very interesting by itself. With the help of well-established tools, I make an analysis of this equivalent matrix model and compute the first two leading orders of its $1/N$ expansion.

Hurwitz numbers count the number of weighted ramified coverings of Riemann surfaces. They can also be interpreted as the number of ways of factorizing permutations in the symmetric groups. They are important in enumerative geometry and combinatorics. On one hand, there has been a sustained effort to obtain explicit formulas for them. My main contribution lies in this direction. I found an explicit formula for a class of numbers called one-part double Hurwitz numbers with completed 3-cycles. On the other hand, Hurwitz numbers have been shown to have connection with a handful of other significant objects such as matrix models, integrable equations and moduli spaces of complex curves. The best understood case is single (or simple) Hurwitz numbers. As applications of my explicit formula, I show that the one-part double Hurwitz numbers with completed 3-cycles satisfy some similar properties as in the case of simple Hurwitz numbers. This suggests that there should be a deeper (algebraic/geometric) structure behind these new numbers.

From their very definition, Hurwitz numbers are linked to symmetric polynomials which are fundamental objects in algebraic combinatorics and representation theory. The algebra of symmetric polynomials, as a vector space, has a particularly important basis formed by the Schur polynomials. One of the most important advances in the theory of symmetric functions is the introduction of a two parameter deformation of Schur polynomials by Macdonald in 1987. He also realized that his polynomials can be defined as multivariate orthogonal polynomials associated to the root system of type A and thus naturally generalized to every root system. For the case of non-reduced root system BC_n, the definition was made previously by Koornwinder, and thus the polynomials are named after him. They are very general in the sense that all the Macdonald polynomials associated to the classical (i.e., non-exceptional) root systems are specializations of the Koornwinder polynomials. One of the most important problems in symmetric function theory is to decompose a symmetric polynomial in the Macdonald basis (it is an established convention that when one talks about Macdonald polynomials without further specifications, one means those associated to the root system of type A). The obtained decomposition (in particular, if the coefficients are explicit and reasonably compact) are called a Littlewood identity. In this thesis, I study many recent Littlewood identities proved by Rains and Warnaar by combining many old and new results about Macdonald and Koornwinder polynomials. My own contributions include a proof of an extension of one of their identities and partial progress towards generalization of one another.

My thesis is organized as follows.

The first chapter studies the quartic melonic tensor model. In the first part, I review some basic tools of matrix models. The tools are the $1/N$ expansion, the saddle point method and the Schwinger-Dyson equations. Then a general introduction about tensor models is given. The quartic melonic tensor model is then introduced and shown to be equivalent to a multi-matrix model. Using the mentioned tools, I make an analysis of this matrix model. The main results are stated and proved in Sect.1.5 and Sect.1.6.

The second chapter collects some well-known material about symmetric functions that will be used in the next two chapters. It also serves the purpose of fixing many notations. In particular, the Schur polynomials, and their famous two-parameter deformation, the Macdonald polynomials are discussed.

The subject of the third chapter is Hurwitz numbers. First, I recall some familiar facts about the symmetric groups. The general Hurwitz numbers are defined in a combinatorial (and algebraic) way, and shown to count ramified coverings of Riemann surfaces. Then, I review some connections between (certain classes of) Hurwitz numbers and integrable equations, moduli spaces of curves, matrix models and the Chekhov-Eynard-Orantin topological recursion. Subsequently, in Sec.3.8, I define the double Hurwitz numbers with completed cycles. In fact, the special case of one-part double Hurwitz numbers with completed 3-cycles will be studied in detail. In the last two sections, I present my own contributions: an explicit formula for these particular numbers and some interesting implications.

In the fourth and last chapter, I study Littlewood identities. First, I define the Koornwinder polynomials and recall some of their essential properties. Second, I define the virtual Koornwinder integrals, which are the main technical tools. The known evaluations of these integrals are stated. Third, I review the new bounded Littlewood identities discovered by Rains and Warnaar. Finally, in the last section, I present a full proof of an extension of one of these identities and some partial progresses towards another case.

The appendices contain some complementary materials on technical results used in this thesis.

0.1 Tensor models

The first part of this thesis is about tensor models, which were introduced in physics as an approach to the fundamental problem of quantum gravity in arbitrary dimensions [4, 80]. Since the emphasis in the main text is their mathematical aspects, let us in this introduction discuss their physics. As it is well known, the two pillars of modern physics are quantum mechanics and Einstein's general theory of relativity . While quantum mechanics governs the physical universe at small scales, general relativity governs it at large scales. Both of them have been extensively confirmed by experiments. However, they seem to contradict each other. Indeed, no one has been able to combine them into a single consistent framework (while the combination of quantum mechanics and Einstein's special theory of relativity was a tremendous success). This is the problem of quantum gravity.

Quantum mechanics carries the message that the physical world is inherently random. The word "inherently" is crucial: the phenomena do not just *appear* random to us because we lack information, they *are* random by nature. General relativity carries another message, that the physical world is geometric: the stress-energy tensor equals a combination of the Ricci tensor and the metric tensor. It then becomes natural to approach quantum gravity via "random geometry". Not only the particles are behaving randomly, the space-time itself can be random. Besides, not only matter is discrete, but space-time could also not be smooth as it apparently appears. In other words, one wants a theory of a quantum particle in a quantum space-time, if one is ready to take the language that far.

Among many problems, one is that it is already difficult to define what random geometry means. There may be not a canonical answer to that question. Indeed, at the moment, the researchers pursue many different definitions. Tensor models give an approach. A good account of the state of the art can be found, for example, in [41, 79]. The fundamental idea is discretization of geometric "spaces" (I want to avoid the mathematical word "manifolds" because physicists also consider geometric objects which are not manifolds in the mathematical sense). For example, in two dimensions, one can ask if one glues randomly the triangles along the edges, what type of surfaces one will obtain at the end? The same question is posed for higher dimensions.

Mathematically, a tensor model is an integral over the space of tensors. Ultimately it is just a multiple (real or complex) integral. The main question concerning such integrals is their behavior in the limit $N \to \infty$, where N is the dimension of the underlying vector spaces (suppose that all the spaces involved are of the same dimension). More specifically, the model considered in this thesis is called *quartic melonic* model. All the definitions and notations will be explained in the main text. Let us here resume the main results.

The quartic melonic tensor model is defined by the following measure (Def.1.4.6):

$$d\nu_T := \frac{1}{Z_{T,0}} \exp\left[\frac{N^{D-1}}{2}\left(\overline{T} \cdot T\right)\right] d\mu_T,$$

In particular, we want to study the partition function

$$\begin{aligned} Z_T &= \frac{1}{Z_{T,0}} \int d\mu_T \exp\left[-N^{D-1}\left(\frac{1}{2}(\overline{T}\cdot T) + \frac{\lambda}{4}\sum_{c=1}^{D} (\overline{T}\cdot_{\hat{c}} T)\cdot_c (\overline{T}\cdot_{\hat{c}} T) \right)\right] \\ &=: \left\langle \exp\left[\frac{N^{D-1}}{2}\cdot\frac{\lambda}{4}\sum_{c=1}^{D} (\overline{T}\cdot_{\hat{c}} T)\cdot_c (\overline{T}\cdot_{\hat{c}} T) \right]\right\rangle_{\nu_T} . \end{aligned}$$

Here, $\overline{T}\cdot T$ and $\overline{T}\cdot_{\hat{c}} T$ are certain sums over repeated indices of the tensors. Introducing the intermediate Hermitian matrix fields, we can write Z_T as a multi-matrix integral. Technically, the transformation is via the Gaussian integral formula. Matrix integrals are special cases of tensor integrals, and much more well studied. It turns out that the obtained matrix model is very interesting on its own. The observables of the two models are related explicitly via the Hermite polynomials in Thm.1.4.2.

Then the main result concerns the matrix model (Thm.1.5.1):

Theorem 0.1.1. *The eigenvalue resolvent $W_c(x)$ of a matrix of any color $c \in [\![1, D]\!]$ expands, up to next-to-leading order, as:*

$$W_c(x) = \frac{1}{x-\alpha} + \frac{1}{\sqrt{N^{D-2}}}(1-\alpha^2)\left(x \pm \sqrt{x^2 - \frac{1}{(1-\alpha^2)}} \right) + o\left(N^{\frac{D-2}{2}} \right). \quad (1)$$

This is equivalent to the following statements concerning the "generalized" eigenvalues (the meaning of the word "generalized" will be clear in the main text). In the limit $N \to \infty$, the eigenvalues of each matrix $M^{(c)}$ collapse to a point $\alpha \in \mathbb{C}$ in the leading order, and are distributed with respect to the Wigner semi-circle law in the next-to-leading order. However, as it will be explicitly mentioned and explained, my collaborators and I have not succeeded in proving that the saddle point method used to derive this result is rigorous. However, the final result is correct as an equivalent formulation of Thm.1.5.1 is also made. It is Thm.1.6.1, whose rigorous proof via the Schwinger-Dyson equations is given:

Theorem 0.1.2. *In the $N \to \infty$ limit, the matrix resolvent $\tilde{W}_c(z)$ satisfies*

$$\tilde{W}_c(z)^2 = (1-\alpha^2) z \tilde{W}_c(z) - (1-\alpha^2). \quad (2)$$

One immediate corollary is the dominant (leading) order of Z_T (Cor. 1.5.2):

Corollary 0.1.3. *The partition function Z_T is given by:*

$$Z_T = c_{N,D,\lambda} \exp\left[N^D \left(-\frac{D\alpha^2}{2} - \log\left(1 + iD\alpha\sqrt{\lambda/2} \right) \right) + o\left(N^D \right) \right], \quad (3)$$

where $c_{N,D,\lambda}$ is a certain constant.

In fact, the next-to-leading order behavior of the eigenvalues also give us the next order inside the exponential of Z_T. However, the expression is too cumbersome to be recorded.

0.2 Hurwitz numbers

The objects of the second part of the thesis are Hurwitz numbers, introduced by Hurwitz in 1891 [46, 47]. They count automorphism classes of branched covers of Riemann surfaces. Equivalently, Hurwitz numbers count the number of ways a permutation can be factorized into others under certain constraints. This in turn implies formulas for them which contain the irreducible characters of the symmetric groups. In this thesis, I will start with the algebraic definition and show that the geometric interpretation holds.

It turns out the Hurwitz numbers are connected with a host of domains of contemporary mathematics. In the main text, I will discuss their connection with matrix models [13], integrable systems [72], moduli spaces of complex curves [29] and the topological recursion of Chekhov-Eynard-Orantin [32].

The direction that I pursue is to find explicit formulas. My main contribution is an explicit formula for one-part Hurwitz numbers with completed 3-cycles (Thm.3.9.1) (the notations will be explained in the main text):

Theorem 0.2.1. *Given $g \geq 0$, $d > 0$, let β be a partition of odd length of d and s be an integer such that $2s = 2g - 1 + l(\beta)$. Then we have:*

$$H^{g,(2)}_{(d),\beta} = \frac{s!d^{s-1}}{2^s} \sum_{h=0}^{g} \frac{(2s-2h)!}{h!(s-h)!12^h} d^{2h} \left[z^{2(g-h)}\right] \prod_{i\geq 1} \left(\frac{\sinh(iz/2)}{iz/2}\right)^{c_i} \tag{4}$$

$$= \frac{s!d^{s-1}}{2^{s+2g}} \sum_{h=0}^{g} \frac{(2s-2h)!}{h!(s-h)!3^h} d^{2h} \sum_{\lambda \vdash (g-h)} \frac{\xi_{2\lambda} S_{2\lambda}}{|Aut\lambda|}. \tag{5}$$

From this theorem, I also prove some corollaries. The first is the strong polynomiality of one-part Hurwitz numbers with completed 3-cycles (Cor.3.10.1).

Corollary 0.2.2. *$H^{g,(2)}_{(d),\beta}$, for fixed g, is a polynomial of the parts of β and satisfies the strong polynomiality property, i.e. it is polynomial in $\beta_1, \beta_2, \ldots$ with highest and lowest degrees respectively $3g + \frac{l(\beta)-3}{2}$ and $g + \frac{l(\beta)-3}{2}$.*

Secondly, I define and compute some "combinatorial Hodge integrals" to give some clues and supports to the conjecture that an ELSV-type formula should exist for one-part Hurwitz numbers with completed 3-cycles. I find that these combintorial Hodge integrals satisfy a property similar to the λ_g theorem of Faber and Pandharipande [35]. It is the content of Thm.3.10.2:

Theorem 0.2.3. *For $b_1 + \ldots + b_n = g + \frac{n-1}{2}$, the lowest combinatorial Hodge integral is given by:*

$$\langle\langle \tau_{b_1} \ldots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g = \binom{g + \frac{n-1}{2}}{b_1, \ldots, b_n} \mathbf{C}_{g,n}, \tag{6}$$

with

$$\mathbf{C}_{g,n} = \frac{(2g+n-1)!\,(2^{2g-1}-1)}{(2g)!\,\left(g + \frac{n-1}{2}\right)! 2^{3g+\frac{n-3}{2}}} |B_{2g}|. \tag{7}$$

Furthermore, I prove that the lowest degree combinatorial Hodge integrals satisfy the string and dilaton equations (Thm.3.10.3):

Theorem 0.2.4. ***String equation:*** *For $g \geq 0$, $n \geq 1$, n odd, $b_1, \ldots, b_n \geq 0$, $b_1 + \ldots + b_n = g + \frac{n+1}{2}$:*

$$\langle\langle \tau_0^2 \tau_{b_1} \ldots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g = (2g+n) \sum_{i=1}^{n} \langle\langle \tau_{b_1} \ldots \tau_{b_{i-1}} \tau_{b_i - 1} \tau_{b_{i+1}} \ldots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g. \qquad (8)$$

Dilaton equation: *For $g \geq 0$, $n \geq 1$, n odd, $b_1, \ldots, b_n \geq 0$, $b_1 + \ldots + b_n = g + \frac{n-1}{2}$ (minus here is not a misprint):*

$$\langle\langle \tau_0 \tau_1 \tau_{b_1} \ldots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g = (2g+n) \left(g + \frac{n+1}{2} \right) \langle\langle \tau_{b_1} \ldots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g. \qquad (9)$$

Here we assume that $\langle\langle . \rangle\rangle$=0 if there is some $\tau_{<0}$ inside.

Finally, I deduce an explicit formula for the top degree combinatorial Hodge integrals (Thm.3.10.4).

Theorem 0.2.5. *For $b_1, \ldots, b_n \geq 0$, $b_1 + \ldots + b_n = 3g + \frac{n-1}{2}$, we have:*

$$\langle\langle \tau_{b_1} \ldots \tau_{b_n} \rangle\rangle_g$$
$$= \frac{1}{2^{3g + \frac{n-1}{2}}} \sum_{h=0}^{g} \frac{(2s-2h)!}{h!(s-h)!3^h} \sum_{\lambda \vdash (g-h)} \sum_{\vec{a} \in D_{2g-2h}(\vec{b})} \frac{\xi_{2\lambda} R_{2\lambda, P_{\vec{a}}}}{|Aut \lambda|} \binom{g + \frac{n-1}{2} + 2h}{b_1 - a_1, \ldots, b_n - a_n}. \qquad (10)$$

0.3 Koornwinder polynomials and Littlewood identities

As we will see, Hurwitz numbers are by definition intimately connected to symmetric polynomials. The third and final object of my study is symmetric orthogonal polynomials associated to the (classical) root systems. Specifically, I study the most general family for the classical root systems, the Koornwinder BC_n symmetric polynomials.

Symmetric polynomials have a long history and remain among the main objects studied in algebraic combinatorics and representation theory. It was realized very early that certain symmetric combinations of the zeros of a polynomial are given neatly by its coefficients. However, the most interesting symmetric polynomials were discovered later. They are called Schur polynomials (although first appeared in Jacobi's works [49]). The main reason for the particular importance of Schur polynomials is that they occur naturally in the representation of the symmetric groups, general linear groups, and unitary groups. For example, they are the characters of finite-dimensional irreducible representations of the general linear groups (this significant fact is due to Schur).

Schur polynomials satisfy a lot of remarkable algebraic and combinatorial properties, some of which will be described in Chapter 2. There are many generalizations (deformations by one or many parameters) of Schur polynomials. The most important may be the two parameter deformation defined by Macdonald in 1987 [64]. The Macdonald polynomials also generalize many famous polynomials such as those of Hall-Littlewood [62] and Jack [48].

Macdonald observed further that his polynomials can be defined as orthogonal polynomials with respect to a certain density attached to root systems, the original case corresponding to the root system of type A. So Macdonald generalized his construction to all root systems [63] (he also defined non-symmetric versions). For the non-reduced root system BC_n, the definition was first given by Koornwinder [58], thus the polynomials in this case are named BC_n Koornwinder polynomials. They are very general as all the Macdonald polynomials associated to the non-exceptional root systems are specialization of the Koornwinder polynomials.

Beside being important objects of study by themselves, these polynomials are connected to many other significant topics such as double affine Hecke algebra [24], discrete integrable systems [11, 20], Hurwitz numbers [43], elliptic hypergeometric series and identities [77].

In this thesis, I am interested in Littlewood identities. A Littlewood identity is a decomposition of a symmetric polynomial in the basis of Macdonald polynomials $P_\lambda(;q,t)$ [1], in particular if the coefficients of this decomposition are reasonably explicit and compact. The origin of my work is the paper [75] of Rains and Warnaar. They use new properties of Koornwinder polynomials to prove several new bounded Littlewood identities. As it will be explained in the main text, their identities are indexed by rectangular partitions or half-partitions of maximal length. They conjecture some identities indexed by partitions or half-partitions [2] of different shapes, in particular, near-rectangular ones. I present the proof of one of their conjectured formulas, and partial progress towards another.

For example, Rains and Warnaar proved the following decomposition of a (C_n, B_n) polynomial [3] indexed by rectangular partitions of maximal length (times a simple factor) into Macdonald polynomials of type A (all the notations will be explained in the main text):

Theorem 0.3.1. *For $\mathbf{x} = (x_1, \ldots, x_n)$, m a nonnegative integer and a a complex number,*

$$\sum_\lambda a^{odd(\lambda)} b^{oa}_{\lambda;m}(q,t) P_\lambda(\mathbf{x};q,t) = \left(\prod_{j=1}^n x_j^m(1+ax_j)\right) P^{(C_n,B_n)}_{m^n}(\mathbf{x};q,t,qt), \tag{11}$$

where $odd(\lambda)$ is the number of odd parts of λ and

$$b^{oa}_{\lambda;m}(q,t) := b^{oa}_\lambda(q,t) \prod_{\substack{s\in\lambda \\ a'(s)\ odd}} \frac{1-q^{2m-a'(s)+1}t^{l'(s)}}{1-q^{2m-a'(s)}t^{l'(s)+1}}.$$

Indeed, the polynomial in the right hand side is indexed by the maximal rectangular partition m^n (if λ is a partition whose length exceeds n then $P^{(C_n,B_n)}_\lambda(\mathbf{x};q,t,qt) = 0$). As one can see, the coefficients are uniformly given by a reasonably compact expression.

My main result is the following extension to the case in which the polynomial is indexed by near-rectangular partitions (Thm.4.4.5)

1. Without further specifications, Macdonald polynomials mean those attached to the root system of type A.

2. A half-partition is a non-increasing finite sequence $(a_1, a_2, \ldots, a_n)$ such that $a_i \in \{\frac{1}{2}, \frac{3}{2}, \ldots\}$ for $1 \le i \le n$.

3. As it will be explained, this polynomial is a specialization of an approriate Koornwinder polynomial.

Theorem 0.3.2. *For positive integers m, n and r an integer such that $0 \le r \le n$, one has*

$$\sum_{\substack{\lambda \\ odd(\lambda)=r}} b^{oa}_{\lambda;m,r}(q,t) P_\lambda(x;q,t) = (x_1 \dots x_n)^m P^{(C_n,B_n)}_{m^{n-r}(m-1)^r}(x;q,t,qt), \tag{12}$$

where

$$b^{oa}_{\lambda;m,r}(q,t) = b^{oa}_\lambda(q,t) \prod_{\substack{s \in \lambda/1^r \\ a'_\lambda(s) even}} \frac{1 - q^{2m - a'_\lambda(s)} t^{l'_\lambda(s)}}{1 - q^{2m - a'_\lambda(s) - 1} t^{l'_\lambda(s)+1}}.$$

The previous theorem is the $r = 0$ case of this theorem. I also obtain some partial results for the case (B_n, B_n) which will be presented in Subsec.4.4.3.

As applications, Rains and Warnaar show that their Littlewood identities, together with the Weyl-Kac formula for the character of integrable highest-weight modules of affine Lie algebras [53] and hypergeometric identities associated to root systems, imply new combinatorial character formulas. Under specialization of variables, these character formulas become new Rogers-Ramanujan-type identities associated to affine Lie algebras. They also conjecture many generalizations of their Littlewood identities. Studying these questions is a future project of mine.

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On a more personal note, I dedicate this thesis to my beloved parents, who among many things, never failed to assure that I have everything I need for my education.

I feel blessed to have shared my life with a wonderful woman. Duyen, without you by my side during all ups and downs, I could have messed it up badly!

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Contents

1

Tensor models in mathematical physics

First, I will recall the necessary concepts and tools for matrix models. Then, I will introduce the general framework of colored tensor models and define the quartic melonic model that will be studied in detail. Using Gaussian integral formula, this model is transformed to a multi-matrix model. With the help of well-known tools, I compute the first two orders of the $1/N$ expansion of this matrix model. This in turn implies the corresponding expansion of the tensor model. My new results are contained in Sec.1.5 and Sec.1.6.

1.1 Introduction to matrix models

Before studying tensor models, we review its origin, matrix models for two reasons. First, tensor models are direct generalizations of matrix models. Second, matrix models have been more developed in the sense of that there are more tools and results. Indeed, in this thesis, I translate a tensor model into a matrix model, and use tools for the latter to obtain information about both of them.

To start with, matrix models are the name used by physicists for matrix integrals. A matrix integral is simply an integral over a set of matrices. In the language of probability, studying matrix integrals also means studying (particular classes of) random matrices. Random matrices were first considered in statistics by Wishart [89](the reader is invited to consult the classic book by Mehta, one of the main contributors to the field [66]). Later, physicists realized that matrix integrals could be used to investigate the two dimensional case of quantum gravity (see for instance the survey [27] or the book [65]). Pursuing this idea, tensor models were introduced to study quantum gravity in higher dimensions.

One of the most fascinating aspects of matrix integrals is that they are intimately related to a whole range of important concepts of contemporary mathematics such as integrable hierarchies, enumerative geometry, orthogonal polynomials, intersection theory of moduli spaces of complex curves, $2D$ topological field theories and Frobenius manifolds. Due to this universality, they might be named "special functions for 21st century". Without trying to be exhaustive, I just want to direct the reader's attention to a sample of recent research [3, 14, 19, 31]. In another direction, random matrices are now a very active topic in probability and applications. They pose challenging analytic and probabilistic questions, and find a wide variety of applications in statistics, wireless communication, statistical physics, and many other disciplines; see for instance [1, 5, 92].

For this thesis, the most important tool is the $1/N$ expansion which was first observed by 't Hooft [85] (N is the size of the matrices). It was a combinatorial observation based on a fundamental concept in theoretical physics, namely Feynman diagrams. In subsequent developments, Feynman diagrams have invaded a significant part of mathematics (even some domains which do not have direct connection with physics such as number theory). This expansion enables elegant solutions to some difficult combinatorial and geometrical problems (for instance [7], [59, Chapter 3]). Computing the coefficients of the $1/N$ expansion is a problem which has led to important and surprising advances in many other questions across mathematical physics, combinatorics and algebraic geometry.

The hope is that the same richness will be discovered in tensor models. However, at the moment, despite many impressive advances, the corresponding analytic, algebraic and combinatorial tools are still considerably less developed in the tensor models framework.

Finally, although not technically used in this thesis, I want to mention the two following issues of matrix models whose corresponding versions for tensor models have been the long term goal for this work.

The first is the widely open problem of studying the double (or more) scaling limits of the models. This limit means letting the size of matrices tend to infinity and some other parameters tend to certain critical values in keeping certain constraints between them. In physics, the double scaling limit might provide a road to the non-perturbative definition of string theory and thus has attracted activities from this area. In fact, physicists have derived many results about this limit for various models using physical intuition and arguments. Mathematicians have succeeded in proving many predictions of physicists, but many physical results are still waiting for mathematical proofs.

The second is the Chekhov-Eynard-Orantin (CEO for short) topological recursion [33, 34]. Initially, this recursion of topological nature was found to solve certain classes of matrix models. One now knows a handful of mathematical objects, some quite distant to matrices at first look, which are governed by the same recurrence. The recurrence is the same, only the initial data differ case by case! Our hope is that by solving tensor models, we may someday discover some similar universal recurrences. Indeed, after this work was done, Dartois et Bonzom [8] showed that the melonic quartic tensor model considered in this thesis satisfies the blobbed topological recursion, a new extension of the CEO recursion introduced by Borot [17].

1.2 A review of tools via specific models

In this chapter, given a (probability) measure $d\mu$ and a function f, we denote the expectation of f by

$$\langle f \rangle_\mu := \int f d\mu.$$

In physics language, such expectations are often called *correlation functions* or *correlators*. I will often follow the physics terminology.

1.2.1 A prototypical matrix integral: Hermitian one-matrix integral with polynomial potential

There is a natural measure on the space of $N \times N$ Hermitian matrices $\mathcal{H}_N$ which is invariant by conjugation:

$$dM = \prod_{j=1}^{N} dM_{jj} \prod_{1 \le j < k \le N} d(\Re M_{jk}) d(\Im M_{jk}).$$

It is easily seen that a $N \times N$ Hermitian matrix M is uniquely parametrized by N^2 real parameters

$$M_{ii}(i = 1, \dots, N), \Re M_{jk}, \Im M_{jk}(1 \le j < k \le N).$$

Let $d\mu_H$ be the following Gaussian measure on $\mathcal{H}_N$

$$d\mu_H := \frac{1}{Z_0} e^{-\frac{N}{2}\mathrm{Tr}\left(M^2\right)} dM,$$

where Z_0 is the constant such that $d\mu_H$ is a probability measure, i.e.

$$Z_0 = \int_{\mathcal{H}_N} dM \exp\left[-\frac{N}{2}\mathrm{Tr}\left(M^2\right)\right]. \tag{1.1}$$

As it is well known, Gaussian integrals are the cornerstones of (perturbative) quantum field theories. This is the reason for which this and other classes of integrals (measures) were introduced in mathematical physics. In the case at hand, we want to study the following integral:

$$Z[\underline{t}, N] = \frac{1}{Z_0} \int_{\mathcal{H}_N} dM \exp\left[-N\left(\frac{1}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}\left[V_{\underline{t}}(M)\right]\right)\right] =: \left\langle e^{-N\mathrm{Tr}\left[V_{\underline{t}}(M)\right]}\right\rangle_{\mu_H}, \tag{1.2}$$

where $\underline{t} = (t_1, t_2, \dots, t_n)$ for a fixed $n \in \mathbb{N}$ is a set of parameters and

$$V_{\underline{t}}(M) := \sum_{k=1}^{n} t_k M^k.$$

The parameters are chosen such that the integral converges. For example, we can assume that n is even and $t_n > 0$. If the reader wants, she/he can assume that $n = 4$ and only t_4 does not vanish. The function $Z[\underline{t}, N]$ is often called a *partition function*.

There are two approaches to study $Z[\underline{t}, N]$. The first is the analytic approach in which the first main question is about the global dependence on $\underline{t}$, particularly in the limit $N \to \infty$. Analytic questions are receiving much attention; however, in this work, I follow the other route. I consider the integral as a formal integral. What this means is that I essentially only try to compute the derivatives of the integral with respect to $\underline{t}$ at $\underline{t} = 0$. More generally, a formal identity of the form

$$F(z_1, \dots, z_m) \overset{\text{formal}}{=} \sum_{k_1,\dots,k_m=0}^{\infty} F_{k_1,\dots,k_m} z^{k_1} \dots z^{k_m}/k_1! \dots k_m!$$

means simply

$$\frac{\partial^{k_1+\dots+k_m} F}{\partial z_1^{k_1} \dots \partial z_m^{k_m}}(0) = F_{k_1,\dots,k_m}.$$

1.2.2 $1/N$ expansion

The most important result (although it has only been proved for a limited numbers of models) that we will need is the $1/N$ *expansion*. It goes as follows. In the formal interpretation, we have

$$Z[\underline{t}, N] \overset{\text{formal}}{=} \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{1}{k_1! \dots k_n!} (-N)^{k_1+\dots+k_n} t_1^{k_1} \dots t_n^{k_n} \left\langle \left[\mathrm{Tr}\left(M^1\right)\right]^{k_1} \dots \left[\mathrm{Tr}\left(M^n\right)\right]^{k_n} \right\rangle_{\mu_H}. \tag{1.3}$$

Since $Z[\underline{t}, N] = 1 + O(\underline{t})$, we can define the free energy $F[\underline{t}, N] = \ln Z[\underline{t}, N]$, again as a formal series in $\underline{t}$.

A surprising thing happens when $F[\underline{t}, N]$ is considered as a series in N. Regrouping the powers of N order by order, we get

$$F[\underline{t}, N] \overset{\text{regrouping powers of } N}{=} \sum_{g \geq 0} N^{2-2g} F_g(\underline{t}) \tag{1.4}$$

It is nontrivial that only the even powers N^{2-2g} do not vanish. Such an expansion is called a $1/N$ expansion. In this particular case, it is also aptly called a *topological* expansion, with g playing the role of the genus of surfaces as explained below.

Notice that here, we have made two consecutive non-rigorous steps. First we exchanged the sum and the integral in (1.3), and second we regrouped the powers of N in (1.4). The interesting fact is that in this case, the final result is correct. That is the content of the following two theorems of Ercolani and McLaughlin [30]. The first theorem justifies the asymptotic expansion.

Theorem 1.2.1. *[30] There is an open region $U \subset \mathbb{R}^n$ containing 0 , and $N_0 > 0$, so that for $\underline{t} \in U$ and $N > N_0$, the $N \to \infty$ asymptotic expansion*

$$F[\underline{t}, N] = N^2 F_0(\underline{t}) + F_1(\underline{t}) + N^{-2} F_2(\underline{t}) + \dots \tag{1.5}$$

holds true. The meaning of the expansion is that if terms up to order N^{-2k} are kept, the error term is bounded by CN^{-2k-2}, where the constant C is independent of $\underline{t}$ for all $\underline{t} \in U$. For each j, the function $F_j(\underline{t})$ is an analytic function of the vector $\underline{t}$ in a neighborhood of 0. Moreover, the asymptotic expansion of the derivatives of $F[\underline{t}, N]$ can be calculated via term-by-term differentiation of the above series.

The second theorem gives the following interpretation of $F_g(\underline{t})$:

Theorem 1.2.2. *[30] For $g \geq 0$, we have*

$$F_g(\underline{t}) = \sum_{k_i \geq 1} \frac{1}{k_1! \dots k_n!} (-t_1)^{k_1} \dots (-t_n)^{k_n} m_g(k_1, \dots, k_n), \tag{1.6}$$

where $m_g(k_3, \dots, k_n)$ is the number of maps of genus g with k_j j-valent vertices for $j = 1, \dots, n$.

Here, a map of genus g means a map on a surface of genus g.

Definition 1.2.1. A *map* $M = (G, S)$ on a surface S is a graph G embedded in the surface S such that if we cut S along the edges of G, all the pieces we get are homeomorphic to the open disc $D = \{z \in \mathbb{C}, |z| < 1\}$.

The same phenomenon happens for many other matrix integrals. The proofs often involve very technical analytic details. I do not try to survey the strongest results in this direction; the interested reader can consult for instance the paper [15] in which strong results are obtained via a delicate study of the Schwinger-Dyson equations. These equations, introduced below, are also important in my study. I will return to analytic issues when we transform matrix integrals into ordinary multiple integrals for the eigenvalues. What I am interested in is how to compute explicitly the expansion given its existence.

In general, the $1/N$ expansion of matrix integrals often counts interesting combinatorial and geometrical objects. In fact, many objects which seem totally unrelated to matrices have been shown to be counted by (often ingenious) matrix integrals. The computation of this expansion has led to many important results, one of which is the topological recursion of Chekhov, Eynard and Orantin mentioned above. To be historically correct, Eynard and Orantin put the solution of the Schwinger-Dyson equations in a universal form but the solution had been obtained in important cases (but written in forms which obscured its universal structure) in the works of Ambjorn, Chekhov, Jurkiewicz, Kristjansen and Makeenko.

Let us discuss how we can compute $Z[\underline{t}, N]$ (in the formal sense). To make the discussion simple, yet retain the most essential points, suppose that $n = 4$ and only t_4 does not vanish. Denote the partition function by $Z[t_4, N]$ for short. We have

$$Z[t_4, N] \overset{\text{formal}}{=} \frac{1}{Z_0} \sum_{k=0}^{\infty} \left(\frac{-Nt_4}{4} \right)^k \int_{\mathcal{H}_N} dM \exp\left[-\frac{N}{2} \mathrm{Tr}\left(M^2\right) \right] \left(\mathrm{Tr}(M^4) \right)^k . \tag{1.7}$$

We can identify $H_N \cong \mathbb{R}^{N^2}$, where $M \mapsto (x_1, \dots, x_{N^2})$ such that $x_1, \dots, x_N$ are M_{ii}; $x_{N+1}, \dots, x_{N^2}$ are $\Re M_{jk}, \Im M_{jk}$ (the order is not important here).

For $M \in H_N$, one observes the following fact

$$\mathrm{Tr}(M^2) = \sum_{i,j=1}^{N} M_{ij}M_{ji} = \sum_{i=1}^{N} M_{ii}^2 + 2 \sum_{1\leq j<k\leq N} ((\Re M_{jk})^2 + (\Im M_{jk})^2) = (Bx, x),$$

where $B = \text{diagonal}(1, \dots, 1, 2, \dots, 2)$, where there are N entries equal to 1 and $N^2 - N$ entries equal to 2. Thus the matrix integral is a Gaussian integral. Applying Eq.(B.1) in the Appendix, one obtains

$$Z_0 = \left(\frac{2\pi}{N}\right)^{N^2/2} 2^{-N(N-1)/2}.$$

We also have

Lemma 1.2.3. *For arbitrary indices i, j, k, l, we have*

$$\langle M_{ij} M_{kl} \rangle = \frac{1}{N} \delta_{il} \delta_{jk}. \tag{1.8}$$

Now we can use the Wick formula B.0.1 to compute any correlation functions of interest. For example

$$\begin{aligned}
\langle \mathrm{Tr}(M^4) \rangle &= \sum_{i,j,k,l} \langle M_{ij} M_{jk} M_{kl} M_{li} \rangle \\
&= \sum_{i,j,k,l} (\langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle + \langle M_{ij} M_{kl} \rangle \langle M_{jk} M_{li} \rangle + \langle M_{ij} M_{li} \rangle \langle M_{jk} M_{kl} \rangle) \\
&= 2N + \frac{1}{N}.
\end{aligned}$$

The same approach can be used to compute more complicated correlation functions of the form $\langle (\mathrm{Tr}\,(M^{2m}))^n \rangle$, and eventually $Z[t_4, N]$. However, this method cannot be carried out very far in practice due to huge complexities as the power of $1/N$ grows. A more efficient method is to transform the matrix integral into a multiple integral of eigenvalues as follows.

Writing $M = U^{-1}DU$ where U is unitary and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ and integrating over the unitary group, one finds that $Z[\underline{t}, N]$ can be expressed in the eigenvalue variables as (the proportional constant can be evaluated explicitly)

$$Z[\underline{t}, N] \propto \int \prod_{i=1}^{N} d\lambda_i \Delta(\{\lambda_j\})^2 \exp\left[-N\left(\frac{1}{2}\sum_{j=1}^{N} \lambda_j^2 + \sum_{j=1}^{N} V_{\underline{t}}(\lambda_j)\right)\right], \tag{1.9}$$

where Δ is the Vandermonde determinant:

$$\Delta(x_1, \dots, x_n) := \prod_{1\leq i<j\leq n} (x_i - x_j).$$

Then, one can use orthogonal polynomials to compute the integral on the right hand side. This approach is very powerful for studying asymptotic behavior of this integral, see for instance [25, 26].

1.2.3 Saddle Point Method

If we cannot compute all F_g and Z_g, we may set for ourselves the more modest goal of computing the first few orders. Due to Eq.(1.9), one needs to study a multiple integral. One basic tool to study the asymptotic of multiple integrals is the classical saddle point method which is recalled in App.C.

The integral on the right hand side of Eq.(1.9) is nearly an integral of Laplace type. The difference is that the number of variables is N grows to infinity (by the way, the parameter α in Thm.C.0.1 equals N in this case). The Laplace's theorem does not work in this situation. However, it can be used to get a quick understanding. Furthermore, the results it gives are in many cases the right ones (this is true for the integral we are considering and for the integral issued from the quartic melonic tensor model). Once one gets the results, one may try to find a proof by other methods.

Let us apply the saddle point method for the integral on the right hand side of Eq.(1.9). The integrand can be rewritten as $\exp\left(-N^2 S(\{\lambda_k\})\right)$ where

$$S(\{\lambda_k\}) = \frac{1}{N}\left(\frac{1}{2}\sum_j \lambda_j^2 + \sum_{j=1}^N V_{\underline{t}}(\lambda_j)\right) - \log\left(\Delta(\{\lambda_j\})^2\right). \tag{1.10}$$

In this case, the saddle point approximation gives the correct asymptotics. Thus one has to look for the extremum of $S(\{\lambda_k\})$. The equations for that are

$$0 = \frac{1}{N}\lambda_\nu + \frac{1}{N}V'_{\underline{t}}(\lambda_\nu) - \frac{1}{N^2}\sum_{i\neq\nu}\frac{1}{\lambda_\nu - \lambda_i}, \tag{1.11}$$

for $\nu = 1, \ldots, N$. These equations can be solved in the $N \to \infty$ limit by introducing the resolvent

$$W(x) := \frac{1}{N}\sum_{i=1}^N \frac{1}{x - \lambda_i}.$$

We have

$$W(x)^2 = \frac{1}{N^2}\sum_{\substack{1\le k,j\le N\\ k\neq j}}\left(\frac{1}{x-\lambda_k} - \frac{1}{x-\lambda_j}\right)\frac{1}{\lambda_k - \lambda_j} - \frac{1}{N}W'(x).$$

The first term of the RHS can be computed from the saddle point equations, giving

$$W(x)^2 = \frac{2}{N}\sum_{k=1}^N \frac{\lambda_k + V'_{\underline{t}}(\lambda_k)}{x - \lambda_k} - \frac{1}{N}W'(x).$$

Therefore

$$W(x)^2 = 2(x + V'_{\underline{t}}(x))W(x) - \frac{1}{N}W'(x) - 2 + \sum_{k=1}^N \frac{V'_{\underline{t}}(\lambda_k) - V'_{\underline{t}}(x)}{x - \lambda_k}. \tag{1.12}$$

The last term is a polynomial because $V'_{\underline{t}}$ is a polynomial. At the leading order, the second term of the right hand side of Eq(1.12) can be discarded and the equation becomes algebraic and solvable. We shall follow this strategy to compute the next-to-leading order distribution of the matrix formulation of the quartic tensor model.

1.2.4 Schwinger-Dyson Constraints (Equations)

Schwinger-Dyson constraints (or equations) are relations between correlation functions coming from integration by parts. Simple idea as it may seem, these equations are very powerful. In the subsequent section, I will use them to study the quartic melonic tensor model. In a larger context, they are the starting point for the discovery of the Chekhov-Eynard-Orantin topological recursion.

They are often derived from the fact the integration of a total derivative is zero (one chooses the measures which satisfy this condition). Consider the following probability measure on $\mathcal{H}_N$ (the Schwinger-Dyson equations are more cumbersome when written with respect to the Gaussian measure)

$$d\mu_{H,\underline{t}} = \frac{1}{Z_{0,\underline{t}}} \exp\left[-N\left(\frac{1}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}V_{\underline{t}}(M)\right)\right] dM.$$

To keep the expressions compact, in this subsection, denote $\langle f\rangle_{H,\underline{t}}$ simply by $\langle f\rangle$. With proper assumptions about $V_{\underline{t}}$, we have

$$0 = \frac{1}{Z_{0,\underline{t}}}\int_{\mathcal{H}_N} dM \frac{\partial}{\partial M_{ij}}\left[\left(M^{k+1}\right)_{ij}\exp\left(-N\left(\frac{1}{2}\mathrm{Tr}(M^2) + \mathrm{Tr}V_{\underline{t}}(M)\right)\right)\right].$$

A simple manipulation of this equation gives

$$0 = \frac{1}{N}\sum_{i=0}^{k}\left\langle \mathrm{Tr}(M^i)\mathrm{Tr}(M^{k-i})\right\rangle - \left\langle \mathrm{Tr}(M^{k+2})\right\rangle - \left\langle \mathrm{Tr}\left(M^{k+1}V'_{\underline{t}}(M)\right)\right\rangle. \tag{1.13}$$

Define the *connected resolvents* by

$$W_n(x_1,\ldots,x_n) := \left\langle \mathrm{Tr}\frac{1}{x_1 - M}\ldots\mathrm{Tr}\frac{1}{x_n - M}\right\rangle_c,$$

Remark. For functions $f_1(M),\ldots,f_n(M)$, the number $\langle f_1(M)\ldots f_n(M)\rangle_c$ is called a *joint cumulant* or a *connected correlator*. It is defined as follows

$$\sum \frac{t_1^{k_1}\ldots t_n^{k_n}}{k_1!\ldots k_n!}\langle f_1^{k_1}\ldots f_n^{k_n}\rangle_c := \log\langle e^{t_1 f_1+\cdots+t_n f_n}\rangle.$$

Thus one has for example $\langle f\rangle_c = \langle f\rangle$ and $\langle fg\rangle_c = \langle fg\rangle - \langle f\rangle\langle g\rangle$.

Multiplying both sides of the equality (1.13) for each value of k by x^{-k-2} and summing over k, we get

$$0 = W(x,x) + W(x)^2 - \left(x + V'_{\underline{t}}(x)\right)W(x) - P(x), \tag{1.14}$$

where $P(x)$ is a polynomial. This is the first (lowest order) Schwinger-Dyson equation. We can repeat the same trick by observing that

$$0 = \sum_{ij}\int_{\mathcal{H}_N} dM\partial_{M_{ij}}\left[\left(M^{n_1}\right)_{ij}\mathrm{Tr}M^{n_2}\ldots\mathrm{Tr}M^{n_k}\exp\left[-N\left(\frac{1}{2}\mathrm{Tr}\left(M^2\right) + \mathrm{Tr}V_{\underline{t}}(M)\right)\right]\right].$$

for every $n_1, \ldots, n_k$. The manipulations are not too difficult, yielding the higher Schwinger-Dyson equations

$$\begin{aligned}&W_{n+2}(x,x,I)+\sum_{J\subset I}W_{1+|J|}(x,J)W_{1+|I-J|}(x,I-J)+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{W_n(x,I-x_i)-W_n(I)}{x-x_i}\\&=N\left[\left(x+V'_{\underline{t}}(x)\right)W_{n+1}(x,I)-P_n(x;I)\right],\end{aligned}\tag{1.15}$$

where $I=(x_1,\ldots,x_n)$ and P_n is a polynomial.

At first sight, it seems impossible to solve these equations. However, under the hypothesis that all the resolvents have $1/N$ expansion, it was shown that the coefficients of these expansions can be computed by a simpler recursion called the topological recursion. This is the important result of Chekhov, Eynard and Orantin [23, 33]. In fact, this recursion works for more general matrix models; however one should know that in many cases, the existence of the $1/N$ expansion has not been proven. Even more surprising and important is the fact that this recursion is universal in some sense. It has been shown to govern many combinatorial and geometric objects, several of which are not obviously related to matrix models. Universality means different problems require different initial conditions, but the recursion is basically always the same. Indeed, one motivation for our work in tensor models is to see whether some new universal recursions can be discovered.

1.3 Introduction to tensor models in physics and combinatorics

In this and subsequent sections, I describe the general framework of tensor models and study in details the quartic melonic model. My contributions, obtained with Dartois and Eynard, are presented in Sec.1.5 and Sec.1.6. The results are published in [71].

Tensor models are generalization of matrix models. They were first studied in [4, 80] in order to give a description of quantum gravity in dimension $D > 2$ as a field theory *of* space-time (and not *on* space-time). Indeed, the field theory thus obtained generates Feynman graphs that may have an interpretation as a D-dimensional space (but this space is not a manifold in general).

Like in other quantum field theories, each of these graphs comes with a quantum amplitude computed by the Feynman rules. Unfortunately, unlike the case of matrices, these amplitudes turn out to be very difficult to handle analytically because of the lack of tools and theoretical understanding of this new tensor world. In addition, it is a well known fact that the geometry of three and higher dimensional spaces is considerably more involved than the $2D$ geometry. This is the source of difficulties when one tries to give a combinatorial and geometrical description of the theory.

To avoid these difficulties, *colored* tensor models were introduced by Gurau in 2011 [39]. Many seemingly unsolvable difficulties of the early tensor models can be overcome in this setting. The most important issue solved by colored tensor models is the lack of $1/N$ expansion. Gurau [40] showed that colored tensor models possess $1/N$ expansion and gave a combinatorial description of all the orders of this expansion. In contrast with matrix models, this expansion is not topological. Indeed the parameter governing this expansion, called the *degree*, is not a topological invariant of the space corresponding to the Feynman graph. This is expected since the topology of three dimensional spaces is not simple. While the geometric interpretation of this parameter is still unclear, it can be computed quite easily from the combinatorial description of the Feynman graphs.

Another important advantage of these new tensor models is that they enable a non-ambiguous description of the observables [1] of the models [9]. This allows leading as well as next-to-leading orders computations. In fact, in our work, we recompute these orders using matrix model techniques.

The tensor model which I will study is among the simplest ones. It is called the *quartic melonic* model. It can be written equivalently as a multi-matrix model. The saddle point equations and Schwinger-Dyson equations can then be used to study it.

The plan of our study of tensor models is as follows:

— Sec.1.4 introduces the general setting of colored tensor models. Then the quartic melonic tensor model is defined and its matrix formulation is derived. We end this section by establishing Thm.1.4.2, which gives a simple relation between the observables of the tensor model and the associated matrix model.

— Sec.1.5 is devoted to saddle point computations of the eigenvalue distribution at leading order and next-to-leading order (NLO).

— Sec.1.6 describes the Schwinger-Dyson equations, which are more suited for rigorous computations. The results of Sec.1.5 are proved from these equations.

1.4 General framework of tensor models

In this section, I introduce the general framework of tensor models so that the reader can appreciate their mathematical and physical significance. For more details, the reader can consult for example the survey [79] by Rivasseau, one of the main contributors to this field. In this work, my collaborators and I transform the tensor model of focus into a matrix model and entirely use matrix tools to analyze it.

1.4.1 Tensor invariants and generic 1-tensor models

We construct tensor models in a similar way to the construction of matrix models. In almost every case, the integrand of a matrix model is constructed from the $GL(N)$ invariants of the matrices.

Consider a rank D tensor T and its complex conjugate $\overline{T}$, i.e., a tensor with complex conjugated entries, once a particular choice of basis has been made. Thus T belongs to the space $V_1 \otimes \cdots \otimes V_D$ endowed with a Hermitian product and $\overline{T}$ belongs to the canonical dual $V_1^* \otimes \cdots \otimes V_D^*$. Here, $V_1, \ldots, V_D$ are complex vector spaces. Fix a basis of $V_1 \otimes \cdots \otimes V_D$. Denote by $T_{i_1 \cdots i_D}$ and $\overline{T}_{i_1 \cdots i_D}$ the components of T and $\overline{T}$ in this and the dual basis. Require for simplicity that $\dim V_j = N$ for all j.

1. An observable can be simply understood as an expectation value.

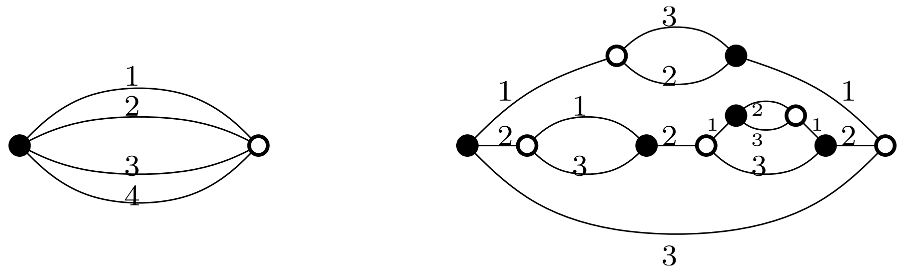

Figure 1.1 – On the left the 4-colored fundamental melon. On the right, an example of a 3-colored melonic graph.

The tensor model should be invariant under the action of $GL(V_1) \times \cdots \times GL(V_D)$. Let $(g_1, \ldots, g_D) \in GL(V_1) \times \cdots \times GL(V_D)$, denote by $R(g_i)$ the matrix representation of g_i on V_i. The action of $(g_1, \ldots, g_D)$ on the tensor T transforms it to:

$$T'_{j_1 \cdots j_D} = R(g_1)_{j_1 i_1} \cdots R(g_D)_{j_D i_D} T_{i_1 \cdots i_D}. \tag{1.16}$$

In the dual vector space, the action of $(g_1, \ldots, g_D)$ on the tensor $\overline{T}$ is

$$\overline{T}'_{j_1 \cdots j_D} = R(g_1)^{-1}_{j_1 i_1} \cdots R(g_D)^{-1}_{j_D i_D} \overline{T}_{i_1 \cdots i_D}. \tag{1.17}$$

Here, and in the following, the repeated indices are summed. Thus all the polynomial invariants are obtained by contracting the j^{th} index of a T with the j^{th} index of a $\overline{T}$, in which case all the R and R^{-1} matrices cancel out. One obtains the tensor invariants as $T\overline{T}T\overline{T} \cdots T\overline{T}$ with a contraction pattern between them that respects the position. Such invariants are often called *trace invariants*.

The index contraction patterns can be graphically represented by D-edge-colored bipartite graphs (hence the name *colored* tensor models).

Definition 1.4.1. A *D-edge-colored bipartite graph* is a graph with, say v black vertices (standing for T) and v white vertices (standing for $\overline{T}$) such that only vertices of different colors are connected by edges and exactly D edges of D different colors are attached to each vertex.

The color of an edge indicates the position of the index being contracted. We draw some examples of such graphs for $D = 3$ in Fig.1.1. The colors are indicated as numbers along the edges.

Thus, the first graph corresponds to the invariant $T_{i_1 \ldots i_4} \overline{T}_{i_1 \ldots i_4}$ (remember that the repeated indices are summed), while the second graph corresponds to the following invariant (see Fig.1.2, the example is intentionally complicated to illustrate that the graphical representation is quite efficient to encode cumbersome algebraic expressions)

$$T_{i_1 i_2 i_3} \overline{T}_{j_1 i_2 j_3} T_{j_1 k_2 j_3} \overline{T}_{i_1 l_2 l_3} T_{m_1 l_2 l_3} \overline{T}_{n_1 k_2 n_3} T_{n_1 p_2 p_3} \overline{T}_{q_1 p_2 p_3} T_{q_1 q_2 n_3} \overline{T}_{m_1 q_2 i_3}.$$

Note that until now, D-edge-colored bipartite graphs only serve the purpose of being alternative encoding of tensor invariants which will appear in the definition of tensor

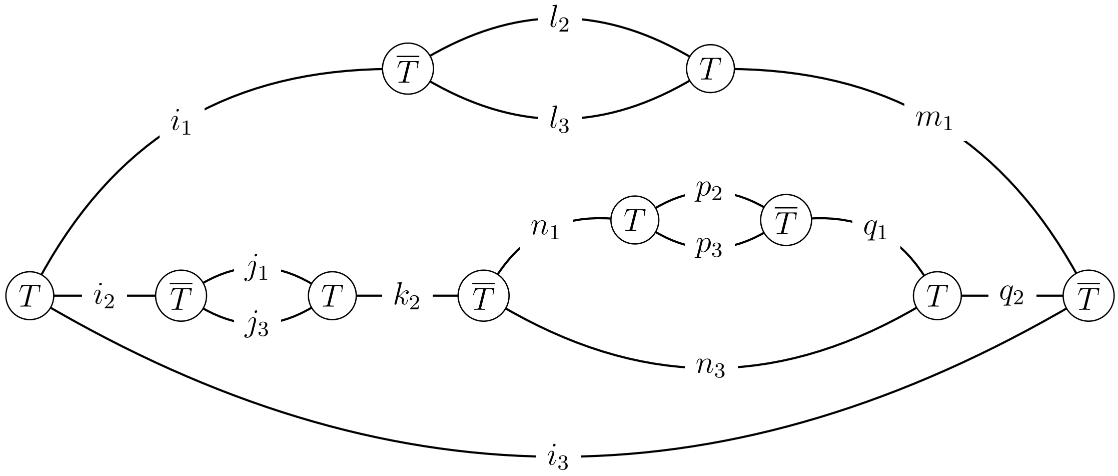

Figure 1.2 – The second graph of Fig.1.1 with explicit indices.

models. The interesting and important fact is that the Feynman diagrams which encode the $1/N$ expansion of colored tensor models are also D-edge-colored bipartite graphs. Thus, the combinatorics of colored tensor models is the combinatorics of these graphs. The two most important notions concerning these graphs are the *jackets* and the *degree*. Since combinatorics is not the tool that I will use, I will only record here the formal definitions without trying to clarify many associated conceptual and technical issues. The definitions look quite abstract and unmotivated without deeper insights. The interested reader is invited to consult the original paper [40].

Definition 1.4.2. Let τ be cyclic permutation of D colors. A colored *jacket* $\mathcal{J}(\tau)$ is an edge-colored ribbon graph associated to a D-colored graph $\mathcal{G}$ with as 1-skeleton the graph $\mathcal{G}$ and with faces made of graph cycles of colors $(\tau^q(1), \tau^{q+1}(1))$, modulo the orientation of the cycle (i.e. τ^{-1} leads to the same jacket).

As such there are $\frac{(D-1)!}{2}$ jackets for a D-edge-colored graph. Each jacket $\mathcal{J}$ leads to a cellular decomposition of a surface and thus comes with a genus $g_{\mathcal{J}}$.

Definition 1.4.3. The *degree* $\omega(\mathcal{G})$ of a colored graph $\mathcal{G}$ is defined as

$$\omega(\mathcal{G}) := \sum_{\mathcal{J}(\mathcal{G})} g_{\mathcal{J}}, \tag{1.18}$$

where the sum runs over all jackets associated to the graph.

Remark. For $D = 2$ (matrix case), the degree reduces to the genus of the only jacket associated to the graph.

Now we can give a definition of tensor models. Denote by $d\mu_T$ the following measure

$$d\mu_T = \prod_{i_1,\dots,i_D=1}^{N} d\Re(T_{i_1,\dots,i_D}) d\Im(T_{i_1,\dots,i_D}).$$

The domain of integration is $\mathbb{R}^{2ND}$ unless explicitly mentioned otherwise. Tensor models are integrals of the following type

Definition 1.4.4. A D-dimensional tensor model is defined by the partition function:

$$Z[N,\{t_{\mathcal{B}}\}] = \int d\mu_T \exp\Big(-N^{D-1}\sum_{\mathcal{B}} N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})} t_{\mathcal{B}}\mathcal{B}(T,\overline{T})\Big), \tag{1.19}$$

where $\mathcal{B}$ runs over certain D-colored graphs indexing the invariants. The $t_{\mathcal{B}}$ are the coupling constants; the one corresponding to the only invariant of order 2 is often fixed to $1/2$. The number $\mathcal{B}(T,\overline{T})$ is the invariant of T and $\overline{T}$ indexed by the graph $\mathcal{B}$.

Remark. In this thesis, studying a model given by a partition function means firstly studying the derivatives of the partition functions with respect to the coupling constants at 0. As one can see immediately, it is equivalent to the problem of studying moments of a certain measure (not always a probability measure).

Definition 1.4.5. A D-colored graph $\mathcal{G}$ is said to be *melonic* if and only if $\omega(\mathcal{G}) = 0$.

In exact analogy with the matrix models case, the integral (1.19) can be interpreted as a sum over certain Feynman diagrams. These Feynman diagrams are discretisation of D-dimensional "pseudo-manifolds" [79]. It is for this reason that tensor models were introduced to study random geometries and quantum gravity in arbitrary dimensions. The interested reader is invited to consult the literature for more details. I just want to make the following comment. The Feynman diagrams for colored tensor models are also D-edge-colored bipartite graphs. [2] It is proved in [40] that the $1/N$ expansion is ordered by the degree of the Feynman graphs: the smaller the degree of a Feynman graph, the more dominant it is. In other words, the degree plays the same role for tensor models as the genus plays for matrix models.

1.4.2 Quartic melonic tensor models and intermediate field representation.

The specific model that I am going to study is the *quartic melonic* tensor model in D dimension. The adjective "melonic" indicates that we choose as interaction terms the simplest ones, i.e., represented by melonic D-colored graphs such as those in Fig. 1.1.

In order to write the model, let us introduce some notations. Denote by $\mathcal{C} = \{1,\dots,D\}$ the set of colors. We write $\overline{T}\cdot T$ for the contraction of all the indices of $\overline{T}$ with all the indices of T. We also introduce the following partial scalar product between $\overline{T}$ and T. For $\mathcal{Y} \subset \mathcal{C}$, we denote by $\overline{T}\cdot_{\mathcal{Y}} T$ the contraction of indices of $\overline{T}$ and T which belong to $\mathcal{Y}$. Moreover we denote $\hat{\mathcal{Y}} = \mathcal{C} - \mathcal{Y}$. If $\mathcal{Y}$ contains one element, we denote it by its element for simplicity.

2. Do not confuse Feynman graphs and graphs indexing the invariants in the definition of the models although both of them are D-edge-colored bipartite graphs!

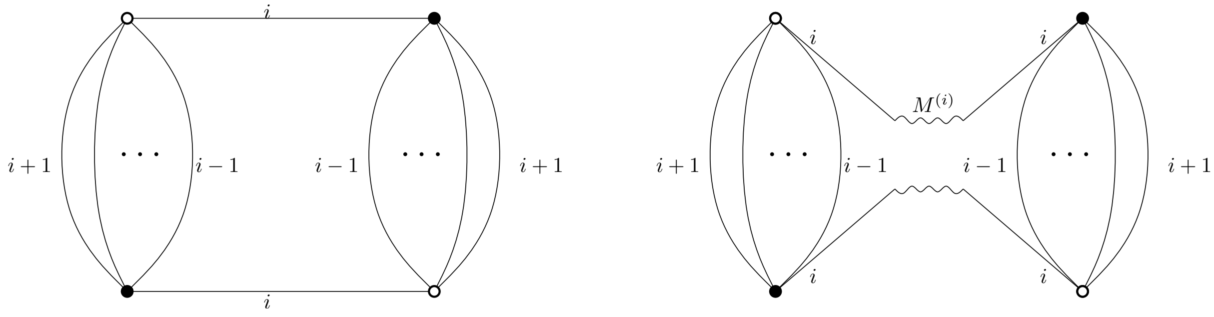

Figure 1.3 – On the left: one of the interaction term. On the right: its splitting with the intermediate matrix field $M^{(i)}$ of color i (the image is just suggestive and does not contain any "hidden" meaning).

Definition 1.4.6. Let λ be a positive real number. The quartic melonic tensor measure is defined as follows

$$d\nu_T := \frac{1}{Z_{T,0}} \exp\left[-N^{D-1}\left(\frac{1}{2}(\overline{T}\cdot T) + \frac{\lambda}{4}\sum_{c=1}^{D}(\overline{T}\cdot_{\hat{c}} T)\cdot_c(\overline{T}\cdot_{\hat{c}} T)\right)\right] d\mu_T, \tag{1.20}$$

where

$$Z_{T,0} = \int d\mu_T \exp\left(-\frac{N^{D-1}}{2}\overline{T}\cdot T\right). \tag{1.21}$$

The quartic interaction term (the term following λ) is represented graphically in Fig.1.3. For any reasonable function $f(T)$, denote the corresponding correlator by $\langle f\rangle_T$. Of particular interest is the partition function of the quartic melonic model

$$\boxed{\begin{aligned} Z_T &= Z_T(N,\lambda) := \langle 1\rangle_T \\ &= \frac{1}{Z_{T,0}}\int d\mu_T \exp\left[-N^{D-1}\left(\frac{1}{2}(\overline{T}\cdot T) + \frac{\lambda}{4}\sum_{c=1}^{D}(\overline{T}\cdot_{\hat{c}} T)\cdot_c(\overline{T}\cdot_{\hat{c}} T)\right)\right], \quad (1.22)\end{aligned}}$$

where λ is a positive real number (the coupling constant).

Now we will write Z_T in terms of a matrix integral. For each color $c\in\mathcal{C}$, we introduce an *intermediate Hermitian matrix field* $M^{(c)}$ to split the interaction terms $(\overline{T}\cdot_{\hat{c}}T)\cdot_c(\overline{T}\cdot_{\hat{c}}T)$. This is pictured on the right hand side of Fig. 1.3 (note that the image is just suggestive and does not contain any "deeper" meaning). This allows us to construct a matrix model that is equivalent to the tensor model under consideration; equivalent in the sense that all correlation functions of the two models match.

Technically, we use the following case of the Gaussian integral formula:

$$\begin{aligned} &\exp\Big(-N^{D-1}\frac{\lambda}{4}(\overline{T}\cdot_{\hat{c}} T)\cdot_c(\overline{T}\cdot_{\hat{c}} T)\Big) \\ =&\tilde{C}_{N,D,\lambda}\int_{\mathcal{H}_N} dM^{(c)}\exp\left[-\frac{N^{D-1}}{2}\mathrm{Tr}\left(\left(M^{(c)}\right)^2\right)-i\sqrt{\frac{\lambda}{2}}N^{D-1}\mathrm{Tr}\left(\left(\overline{T}\cdot_{\hat{c}} T\right)M^{(c)}\right)\right]. \end{aligned} \tag{1.23}$$

where $\tilde{C}_{N,D,\lambda}$ is a constant, i is the imaginary unit, and $\int_{\mathcal{H}_N} dM^{(c)}$ is the integral over Hermitian matrices as before.

Rewriting the tensor model using this representation of the interaction term, we get the mixed form of the partition function:

$$\boxed{\begin{aligned} Z_T=&\frac{\tilde{C}_{N,D,\lambda}}{Z_{T,0}}\int d\mu_T\int_{(\mathcal{H}_N)^D}\prod_{c=1}^{D}dM^{(c)} \\ &\exp\left[-\frac{N^{D-1}}{2}\overline{T}\Big(\mathbb{1}^{\otimes D}+i\sqrt{\frac{\lambda}{2}}\sum_{c=1}^{D}\mathcal{M}_c\Big)T\right]\exp\left[-\frac{1}{2}\sum_{c=1}^{D}\mathrm{Tr}(\mathcal{M}_c^2)\right], \end{aligned}} \tag{1.24}$$

where we have introduced the notation $\mathcal{M}_c = \mathbb{1}^{\otimes(c-1)}\otimes M^{(c)}\otimes\mathbb{1}^{\otimes(D-c)}$ for any $c=1,\dots,D$. The symbol $\otimes$ denotes the *Kronecker product*. For our purpose, we will only need the following property of this product.

Lemma 1.4.1. *Suppose that A and B are two square matrices of size m and n respectively. Let $\lambda_1,\dots,\lambda_m$ be the eigenvalues of A and $\mu_1,\dots,\mu_m$ be those of B. Then the eigenvalues of $A\times B$ are*

$$\lambda_i\mu_j,\quad i=1,\dots,m; j=1,\dots n.$$

It follows that

$$\mathrm{Tr}(A\otimes B)=\mathrm{Tr}A\cdot\mathrm{Tr}B \quad \textit{and} \ \det(A\otimes B)=(\det A)^m(\det B)^n.$$

Integrating out the T and $\overline{T}$'s (using again the Gaussian integral formula), we obtain the matrix form of Z_T:

$$\boxed{\begin{aligned} Z_T=C_{N,D,\lambda}\int_{(\mathcal{H}_N)^D}\prod_{c=1}^{D}dM^{(c)}\det{}^{-1}\Big(\mathbb{1}^{\otimes D}+i\sqrt{\lambda/2}\sum_{c=1}^{D}\mathcal{M}_c\Big)& \\ \times\exp\Bigg(-\frac{1}{2}\sum_{c=1}^{D}\mathrm{Tr}(\mathcal{M}_c^2)\Bigg),& \end{aligned}} \tag{1.25}$$

where $C_{N,D,\lambda}$ is a constant. This is the *intermediate field representation* of the T^4 melonic tensor model.

Definition 1.4.7. The corresponding matrix model measure is defined to be

$$d\nu_M := C_{N,D,\lambda} \prod_{c=1}^{D} dM^{(c)} \det{}^{-1}\Big(\mathbb{1}^{\otimes D} + i\sqrt{\lambda/2}\sum_{c=1}^{D} \mathcal{M}_c\Big) \exp\Big(-\frac{1}{2}\sum_{c=1}^{D} \mathrm{Tr}(\mathcal{M}_c^2)\Big). \tag{1.26}$$

For a reasonable function $f\left(M^{(1)}, \dots, M^{(D)}\right)$, denote by $\langle f\rangle_M$ the corresponding matrix correlator. The relation between the tensor and matrix correlators is given in the following

Theorem 1.4.2. *We have:*

$$\langle \mathrm{Tr}\,(\Theta_c)^p\rangle_T = \Big(\frac{2i\sqrt{2}}{\sqrt{\lambda}}\Big)^p \left\langle \mathrm{Tr} H_p\left(M^{(c)}\right)\right\rangle_M, \tag{1.27}$$

and

$$\left\langle \mathrm{Tr}\left(M^{(c)}\right)^p \right\rangle_M = \left\langle \mathrm{Tr} H_p\left(\frac{\sqrt{\lambda}}{2i\sqrt{2}}\Theta_c\right)\right\rangle_T, \tag{1.28}$$

where $\Theta_c = (\bar{T}\cdot_{\hat{c}} T)$ is a matrix, and H_p is the Hermite polynomial of order p.

Proof. Consider the mixed matrix-tensor representation 1.24. One can write $\langle \mathrm{Tr}(\Theta_c^p)\rangle$ as:

$$\begin{aligned}
&\left(\frac{N^{D-1}\sqrt{\lambda/2}}{2i}\right)^p \langle \mathrm{Tr}(\Theta_c^p)\rangle \\
=&\frac{\tilde{C}_{N,D,\lambda}}{Z_0}\int d\mu_T \int_{(H_N)^D} \prod_{d=1}^{D} dM^{(d)} \exp\Big(-\frac{1}{2}\sum_{d=1}^{D}\mathrm{Tr}(\mathcal{M}_d^2)\Big)\times \\
&\left(\frac{\partial^p}{\partial M^{(c)}_{a_1a_2}\partial M^{(c)}_{a_2a_3}\cdots\partial M^{(c)}_{a_pa_1}} \exp\Big(-\frac{N^{D-1}}{2}\bar{T}\Big(\mathbb{1}^{\otimes D} + i\sqrt{\frac{\lambda}{2}}\sum_{d=1}^{D}\mathcal{M}_d\Big)T\Big)\right)
\end{aligned}$$

with the convention that repeated indices are summed. Via integration by parts, one gets

$$\begin{aligned}
&\left(\frac{-iN^{D-1}\sqrt{\lambda/2}}{2}\right)^p \langle \mathrm{Tr}(\Theta_c^p)\rangle \\
=&(-1)^p\frac{\tilde{C}_{N,D,\lambda}}{Z_0}\int d\mu_T \int_{(H_N)^D} \prod_{d=1}^{D} dM^{(d)} \exp\Big(-\frac{N^{D-1}}{2}\bar{T}\Big(\mathbb{1}^{\otimes D} + i\sqrt{\lambda/2}\sum_{d=1}^{D}\mathcal{M}_d\Big)T\Big)\times \\
&\left(\frac{\partial^p}{\partial M^{(c)}_{a_1a_2}\partial M^{(c)}_{a_2a_3}\cdots\partial M^{(c)}_{a_pa_1}} \exp\Big(-\frac{1}{2}\sum_{d=1}^{D}\mathrm{Tr}(\mathcal{M}_d^2)\Big)\right).
\end{aligned}$$

Recall the definition of Hermite polynomials $H_p(x) = (-1)^p \exp(\frac{x^2}{2}) \frac{d^p}{dx^p} \exp(-\frac{x^2}{2})$. This leads to:

$$\left(\frac{-iN^{D-1}\sqrt{\lambda/2}}{2}\right)^p \langle \mathrm{Tr}(\Theta_c^p) \rangle_T = N^{p(D-1)} \langle H_p(M^{(c)}) \rangle_M$$

which simplifies to the first claimed equation. For the second equation it suffices to use the Weierstrass transform. It is defined as the linear operator sending a monomial of degree n to the corresponding Hermite polynomial H_n. Explicitly we have:

$$H_n(x) = e^{-\frac{1}{4}\frac{d^2}{dx^2}} x^n, \forall x \in \mathbb{R}.$$

Inverting the operator and using the property of Hermite polynomials $\frac{d}{dx} H_n(x) = n H_{n-1}(x)$ we get:

$$x^n = \sum_{k=0}^{[n/2]} \frac{1}{4^k} \frac{n!}{(n-2k)!k!} H_{n-2k}(x).$$

This can be used to obtain:

$$\left\langle \mathrm{Tr}\left(M^{(c)}\right)^n \right\rangle_M = \sum_{k=0}^{[n/2]} \frac{1}{4^k} \frac{n!}{(n-2k)!k!} \left(\frac{\sqrt{\lambda}}{2i\sqrt{2}}\right)^{n-2k} \left\langle \mathrm{Tr}\left(\Theta_c^{n-2k}\right) \right\rangle_T,$$

hence

$$\left\langle \mathrm{Tr}\left(M^{(c)}\right)^n \right\rangle_M = \left\langle \mathrm{Tr}\left(H_n\left(\frac{\sqrt{\lambda}}{2i\sqrt{2}}\Theta_c\right)\right) \right\rangle_T.$$

□

So this Theorem helps us to transit easily between the two models. This is the equivalence between the two models that I have mentioned.

1.5 Saddle Point Equation of the Matrix Model

Although we could not prove that the saddle point method is rigorous for our problem, it can give us a quick glimpse of the final results. In our case, it actually gives the correct answer, as we will prove the results by the rigorous Schwinger-Dyson equations in the next section. Without the results obtained by the saddle point method, it is difficult to guess how to solve the Schwinger-Dyson equations. In the next two subsections, I am going to carry out the saddle point computations for the leading and next-to-leading orders of the eigenvalues of the matrices $M^{(c)}$. In order to state the main result, let us prepare some notations.

For $1 \leq c \leq D$, denote the eigenvalues of $M^{(c)}$ by $\lambda_j^{(c)}$, $j = 1, \ldots, N$. First we write the matrix model (1.25) in terms of eigenvalues.

$$Z = c_{N,D,\lambda} \int_{\mathbb{R}^{ND}} \prod_{c=1}^{D} \prod_{j=1}^{N} d\lambda_j^{(c)} \exp\Big(-\frac{N^{D-1}}{2} \sum_{c=1}^{D} \sum_{j=1}^{N} \left(\lambda_j^{(c)}\right)^2 \Big)$$
$$\prod_{\{j_c=1\}_{c=1\cdots D}}^{N} \frac{1}{1 + i\sqrt{\lambda/2} \sum_{c=1}^{D} \lambda_{j_c}^{(c)}} \prod_{c=1}^{D} \Delta(\{\lambda_j^{(c)}\}_{j=1\cdots N})^2, \tag{1.29}$$

where $c_{N,D,\lambda}$ is a constant (which can be computed explicitly, though we do not need that). This can be rewritten as:

$$Z = c_{N,D,\lambda} \int_{U} \prod_{c=1}^{D} \prod_{j=1}^{N} d\lambda_j^{(c)} \exp\left(-N^D S\left(\{\lambda_j^{(c)}\}_{j=1\cdots N}^{c=1\cdots D} \right) \right), \tag{1.30}$$

where

$$S\left(\{\lambda_j^{(c)}\}_{j=1\cdots N}^{c=1\cdots D} \right) = -\frac{1}{2N} \sum_{c=1}^{D} \sum_{j=1}^{N} \left(\lambda_j^{(c)}\right)^2 + \frac{1}{N^D} \log\left[\prod_{c=1}^{D} \Delta(\{\lambda_j^{(c)}\}_{j=1\cdots N})^2 \right]$$
$$+ \frac{1}{N^D} \log\left[\prod_{\{j_c=1\}_{c=1\cdots D}}^{N} \frac{1}{1 + i\sqrt{\lambda/2} \sum_{c=1}^{D} \lambda_{j_c}^{(c)}} \right]. \tag{1.31}$$

In the integral (1.30), we have changed the domain of integration $\mathbb{R}^{ND}$ to a complex domain U which contains the extremum of S. This is possible thank to the Cauchy theorem. The $\lambda_j^{(c)}$ are then not the eigenvalues of $M^{(c)}$ anymore. We will however call them "generalized" eigenvalues.

Remark. With this change of the domain of integration, the physical meaning of each term of S is well known. As usual, we have the Coulomb potential coming from the Vandermonde determinant which separates the "generalized" eigenvalues away from each other. The tensor product interaction between the different matrices leads to an interaction term that pushes all the "generalized" eigenvalues towards $i\sqrt{\frac{2}{\lambda}}$. Finally the Gaussian term attracts all the "generalized" eigenvalues to zero.

Denote by $\{\lambda_j^{(c)\star}\}_{j=1\cdots N}^{c=1\cdots D}$ the critical point of $S\left(\{\lambda_j^{(c)}\}_{j=1\cdots N}^{c=1\cdots D} \right)$ (under the hypothesis that S has a unique critical point).

Definition 1.5.1. For each color c, the corresponding eigenvalue *resolvent* W_c is defined as

$$W_c(x) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{x - \lambda_k^{(c)\star}},$$

Then the main result is

Theorem 1.5.1. *The eigenvalue resolvent $W_c(x)$ of a matrix of any color $c \in [\![1, D]\!]$ expands, up to next-to-leading order, as:*

$$W_c(x) = \frac{1}{x-\alpha} + \frac{1}{\sqrt{N^{D-2}}}(1-\alpha^2)\left(x \pm \sqrt{x^2 - \frac{1}{(1-\alpha^2)}}\right) + o\left(N^{\frac{D-2}{2}}\right), \quad (1.32)$$

where

$$\alpha = \frac{-1+\sqrt{1+2D\lambda}}{2iD\sqrt{\lambda/2}}$$

In particular, W_c does not depend on c up to the first two orders

The two next subsections are spent to derive this result.

1.5.1 Leading Order (LO) $1/N$ Computation

To keep the writing reasonably neat, since now, denote $\lambda_j^{(c)\star}$ simply by $\lambda_j^{(c)}$ (we will not need to use $\lambda_j^{(c)}$ with the meaning of being integration variables, so it is hoped that no confusion will arise). The saddle point equations are given by $\frac{\partial S}{\partial \lambda_k^{(c)}} = 0$ for $(k, c) \in [\![1, N]\!] \times [\![1, D]\!]$. Thus we obtain the following equations:

$$\begin{aligned} 0 =& \frac{\partial S}{\partial \lambda_k^{(c)}} \qquad (1.33) \\ =& -\frac{\lambda_k^{(c)}}{N} + \frac{1}{N^D}\sum_{\substack{1\le l\le N\\ l\neq k}} \frac{1}{\lambda_k^{(c)} - \lambda_l^{(c)}} - \frac{i\sqrt{\lambda/2}}{N^D}\sum_{\{j_b=1\}_{b\neq c}}^{N} \frac{1}{1+i\sqrt{\lambda/2}(\lambda_k^{(c)} + \sum_{\substack{1\le b\le D\\ b\neq c}} \lambda_{j_b}^{(b)})} \end{aligned}$$

Suppose that the eigenvalues can be expanded in the powers of N in the large N limit. The expansion of the eigenvalues in the powers of N coming from the tensor model scaling can be very different from that of matrix models. Since we do not know how to solve these equations exactly, we make some hypotheses. First we see that the equations are symmetric under the permutations of the color index c. This indicates that the saddle point might obey $\lambda_k^{(c)} = \lambda_k^{(d)}$ for any $c, d = 1 \cdots D$. So we postulate this property. With this in mind the equations rewrite:

$$0 = \frac{\lambda_k^{(c)}}{N} - \frac{2}{N^D}\sum_{\substack{1\le l\le N\\ l\neq k}} \frac{1}{\lambda_k^{(c)} - \lambda_l^{(c)}} + \frac{i\sqrt{\lambda/2}}{N^D}\sum_{\{j_r=1\}_{r=1,\cdots,D-1}}^{N} \frac{1}{1+i\sqrt{\lambda/2}(\lambda_k^{(c)} + \sum_{r=1}^{D-1}\lambda_{j_r}^{(c)})}. \quad (1.34)$$

Assume the following Ansatz concerning the expansion of the eigenvalues $\lambda_k^{(c)}$ in $1/N$:

$$\lambda_k^{(c)} = \lambda_{k,0}^{(c)} + \frac{\lambda_{k,1}^{(c)}}{\sqrt{N^{(D-2)}}} + \frac{\lambda_{k,2}^{(c)}}{N^{(D-2)}} + \cdots .$$

In fact, this Ansatz will give us reasonable results, although I do not claim that this is the only possible expansion (but I have tried many). Under this assumption, the first and third terms of the right hand side are leading whereas the second term is a sub-leading $O(\frac{1}{N^{D/2}})$ term, by a simple counting argument.

First we compute $\lambda_{k,0}^{(c)} = \beta$ (which do not depend either on k or c because the formulation of the matrix model in terms of eigenvalues is totally symmetric with respect to their exchange). As just discussed, we can neglect the second term in Eq. (1.34) and obtain:

$$\beta_{\pm} = \frac{-1 \pm \sqrt{1+2D\lambda}}{2iD\sqrt{\lambda/2}}. \tag{1.35}$$

We have to choose the $'+'$ root in order to get a finite limit when $\lambda \to 0$. This is α in Thm.1.5.1. Note that since we changed the domain of integration, the fact that α is complex is not contradictory, i.e. $\lambda_k^{(c)}$ are no longer interpreted as eigenvalues of Hermitian matrices $M^{(c)}$. In resume, the term $\frac{1}{x-\alpha}$ in $W_c(x)$ is explained.

The immediate consequence is

Corollary 1.5.2. *The partition function Z_T is given by:*

$$Z_T = c_{N,D,\lambda} \exp\left[N^D\left(-\frac{D\alpha^2}{2} - \log\left(1 + iD\alpha\sqrt{\lambda/2}\right)\right) + o\left(N^D\right)\right] \tag{1.36}$$

We also get the 2-point function of the tensor model.

Corollary 1.5.3. *The 2-point function $G_2(\lambda) = \frac{1}{N}\langle \overline{T} \cdot T\rangle_T$ is given in the $N \to \infty$ limit by:*

$$\lim_{N\to\infty} G_2(\lambda) = \lim_{N\to\infty} \frac{1}{N}\langle \mathrm{Tr}\Theta_c\rangle_T = \frac{2i\sqrt{2}}{\sqrt{\lambda}}\alpha = \frac{2}{D\lambda}\left(-1 + \sqrt{1+2D\lambda}\right). \tag{1.37}$$

Proof. Recall the relation in Thm.1.4.2

$$\langle \mathrm{Tr}(\Theta_c^p)\rangle_T = \left(\frac{2i\sqrt{2}}{\sqrt{\lambda}}\right)^p \langle \mathrm{Tr}(H_p(M^{(c)}))\rangle_M.$$

In the $N \to \infty$ limit, we can compute $\langle \mathrm{Tr}(M^{(c)})\rangle_M$ at the saddle point approximation as $\sum_j \lambda_j^c = N\alpha$. Since $H_1(x) = x$, we get

$$\langle \mathrm{Tr}(\Theta_c)\rangle_T = \frac{2i\sqrt{2}}{\sqrt{\lambda}} N\alpha.$$

□

Remark. It is also feasible to compute the leading order of all the $\mathrm{Tr}(\Theta_c^p)$.

1.5.2 Next-to-Leading Order (NLO) Computation.

In this section, we want to compute $\lambda_{k,1}^{(c)}$. In particular we see that it has interesting "statistical distribution" properties. Inserting the expansion $\lambda_k^{(c)} = \alpha + \frac{\lambda_{k,1}^{(c)}}{\sqrt{N^{D-2}}} + O(\frac{1}{N^{D-2}})$ in Eq. 1.34, we get, for every $k = 1, \ldots, N$,

$$0 = \frac{\alpha}{N} + \frac{\lambda_{k,1}^{(c)}}{\sqrt{N^D}} - \frac{2}{\sqrt{N^{D+2}}} \sum_{\substack{1 \le l \le N \\ l \neq k}} \frac{1}{\lambda_{k,1}^{(c)} - \lambda_{l,1}^{(c)}} \tag{1.38}$$
$$+ \frac{i\sqrt{\lambda/2}}{N^D} \sum_{\{j_r=1\}_{r=1,\cdots,D-1}}^{N} \frac{i\sqrt{\lambda/2}}{1 + i\sqrt{\lambda/2}D\alpha + \frac{i\sqrt{\lambda/2}}{\sqrt{N^{D-2}}}\left(\lambda_{k,1}^{(c)} + \sum_{r=1}^{D-1} \lambda_{j_r,1}^{(c)} + O(1/N^{D-2})\right)}.$$

Keeping the dominant terms, we obtain

$$0 = \lambda_{k,1}^{(c)}(1-\alpha^2) - \frac{2}{N} \sum_{\substack{1 \le l \le N \\ l \neq k}} \frac{1}{\lambda_{k,1}^{(c)} - \lambda_{l,1}^{(c)}} - \frac{\alpha^2(D-1)}{N^{D-1}} \sum_{\{j_r=1\}_{r=1,\cdots,D-1}}^{N} \lambda_{j_r,1}^{(c)}. \tag{1.39}$$

By antisymmetry of the Vandermonde factor, summing over k gives

$$0 = (1-D)\alpha^2 \sum_{k=1}^{N} \lambda_{k,1}^{(c)} \Rightarrow \sum_{k=1}^{N} \lambda_{k,1}^{(c)} = 0.$$

Thus the third term of the right hand side of the Eq. (1.39) vanishes. We therefore obtain:

$$(1-\alpha^2)\lambda_{k,1}^{(c)} - \frac{2}{N} \sum_{\substack{1 \le l \le N \\ l \neq k}} \frac{1}{\lambda_{k,1}^{(c)} - \lambda_{l,1}^{(c)}} = 0, \tag{1.40}$$

which is the well known equation determining the critical point of the action for the GUE ensemble and describing the large N limit of the *Wigner's semi-circle law*. In order to solve it, we introduce the (colored) resolvent for the NLO eigenvalues

$$V_c(x) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{x - \lambda_{k,1}^{(c)}}.$$

As it is done in Subsec.1.2.3, Eq.(1.40) becomes

$$V_c(x)^2 = (1-\alpha^2)(xV_c(x) - 1) - \frac{1}{N} V_c'(x)$$

In the $N \to \infty$ limit, the second term of the right hand side is subleading and can be discarded. Hence the last equation becomes quadratic:

$$V_c(x)^2 = (1-\alpha^2)(xV_c(x) - 1), \tag{1.41}$$

and has the following solutions

$$V_{c,\pm}(x) = (1-\alpha^2)\Big(x \pm \sqrt{x^2 - \frac{1}{(1-\alpha^2)}}\Big). \tag{1.42}$$

The correct answer is the one with the minus sign because of the behavior of $V_c(x)$ at the limit $x \to \infty$, and we denote it simply by V_c. This explains the second term in $W_c(x)$.

Remark. One notices that the NLO term for the 2-point function vanishes in this context. Indeed the resolvent is the generating function of the traces of the matrix and the term in front of $1/x^2$ is vanishing in the expansion of $W_c(x)$.

1.6 Schwinger-Dyson Equations.

In this section, we use the Schwinger-Dyson equations to give a rigorous proof of an equivalent result to Thm.1.5.1. In order to state the result, we need some new notations.

As suggested by the above study, we will consider the loop equations in terms of new renormalized variables $\tilde{M}^{(c)}$ defined by

$$M^{(c)} = \alpha \mathbb{1} + \frac{\tilde{M}^{(c)}}{\sqrt{N^{D-2}}}.$$

In fact the previous study showed that in the $N \to \infty$ limit, all the "eigenvalues" collapse to a point α and the next to leading order term follows a distribution which is more regular for a matrix model. The partition function Z_T can be written in the following form:

$$\begin{aligned} Z_T &= C_{N,D,\lambda} \int_{(H_N)^D} \prod_{c=1}^{D} dM^{(c)} \det{}^{-1}\Big(\mathbb{1}^{\otimes D} + i\sqrt{\lambda/2}\sum_{c=1}^{D} \mathcal{M}_c\Big) \exp\Bigg(-\frac{1}{2}\sum_{c=1}^{D} \mathrm{Tr}(\mathcal{M}_c^2)\Bigg) \\ &= C_{N,D,\lambda} \int_{(H_N)^D} \prod_{c=1}^{D} dM^{(c)} \exp\Bigg(-\frac{1}{2}\sum_{c=1}^{D} \mathrm{Tr}(\mathcal{M}_c^2) - \mathrm{Tr}\log\Big(\mathbb{1}^{\otimes D} + i\sqrt{\lambda/2}\sum_{c=1}^{D} \mathcal{M}_c\Big)\Bigg). \end{aligned} \tag{1.43}$$

In the new matrix variables, it becomes

$$\begin{aligned} Z_T = C_{N,D,\lambda} \frac{\exp\big(-\frac{N^D}{2}\alpha^2\big)}{N^{D-2}} \int_{(H_N)^D} \prod_{c=1}^{D} d\tilde{M}^{(c)} \exp\Big[-\frac{N}{2}\sum_{c=1}^{D} \mathrm{Tr}\tilde{M}_c^2 - \alpha N^{\frac{D}{2}} \sum_{c=1}^{D} \mathrm{Tr}\tilde{M}_c \\ - \mathrm{Tr}\log\big((1+i\sqrt{\lambda/2}\alpha)\mathbb{1}^{\otimes D} + i\sqrt{\frac{\lambda}{2N^{D-2}}}\sum_{c=1}^{D} \tilde{\mathcal{M}}_c\big)\Big], \end{aligned} \tag{1.44}$$

with the obvious extension of the previous notation: $\tilde{\mathcal{M}}_c = \mathbb{1}^{\otimes(c-1)} \otimes \tilde{M}_c \otimes \mathbb{1}^{\otimes(D-c)}$. For simplicity, and because no confusion can arise, let us denote the correlators according to the $\tilde{M}$ measure simply by $\langle\cdot\rangle$.

Definition 1.6.1. For each color c, define the matrix resolvent

$$\tilde{W}_c(z) = \frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{z - \tilde{M}^{(c)}}\right)\right\rangle.$$

Then we have the following equivalent to Thm.1.5.1

Theorem 1.6.1. *In the $N \to \infty$ limit,*

$$\tilde{W}_c(z)^2 = (1-\alpha^2) z \tilde{W}_c(z) - (1-\alpha^2). \tag{1.45}$$

*In other words, $\tilde{W}_c$ satisfies the same Eq.(**??**) as V_c.*

Proof. First, we construct the Schwinger-Dyson equations in terms of the $\tilde{M}$'s matrices. For $c = 1, \ldots, D$ and for every positive integer k, it follows from

$$\begin{aligned}0 = \sum_{i,j=1}^{N} \int_{(H_N)^D} \prod_{d=1}^{D} d\tilde{M}^{(d)} \frac{\partial}{\partial \tilde{M}^{(c)}_{ij}} \Big[(\tilde{M}^{(c)})^k_{ij} \exp\Big\{ -\frac{N}{2}\sum_{d=1}^{D} \mathrm{Tr}\tilde{M}_d^2 \\ - \alpha N^{\frac{D}{2}} \sum_{d=1}^{D} \mathrm{Tr}\tilde{M}_d - \mathrm{Tr}\log\left(\mathbb{1}^{\otimes D} - \frac{\alpha}{N^{(D-2)/2}} \sum_{d=1}^{D} \tilde{\mathcal{M}}_d\right)\Big\}\Big]\end{aligned}$$

that

$$\begin{aligned}0 = &\left\langle \sum_{n=0}^{k-1} \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^n \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k-1-n}\right\rangle \\ &- N\left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k+1}\right\rangle - \alpha N^{\frac{D}{2}}\left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k}\right\rangle \\ &+ \left\langle \sum_{p\geq 0}\Big(\frac{\alpha}{N^{(D-2)/2}}\Big)^{p+1} \sum_{\substack{\{q_i\}_{i=1\cdots D} \\ \sum_i q_i = p}} \binom{p}{q_1, \cdots, q_D} \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{q_c+k} \prod_{i\neq c} \mathrm{Tr}\left(\tilde{M}^{(i)}\right)^{q_i}\right\rangle.\end{aligned}$$

The third term cancels with the $p = 0$ term of the last sum and gives us the Schwinger-Dyson equations which are constraints between correlators

$$\begin{aligned}0 = &\left\langle \sum_{n=0}^{k-1} \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^n \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k-1-n}\right\rangle - N\left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k+1}\right\rangle \\ &+ \left\langle \sum_{p\geq 1}\Big(\frac{\alpha}{N^{(D-2)/2}}\Big)^{p+1} \sum_{\substack{\{q_i\}_{i=1\cdots D} \\ \sum_i q_i = p}} \binom{p}{q_1, \cdots, q_D} \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{q_c+k} \prod_{i\neq c} \mathrm{Tr}\left(\tilde{M}^{(i)}\right)^{q_i}\right\rangle.\end{aligned} \tag{1.46}$$

In the sum of the Eq.(1.46), the only leading term in the $N \to \infty$ limit is the $p = 1$ term. In this limit, the relevant equations read:

$$\begin{aligned}0 = &\left\langle \sum_{n=0}^{k-1} \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^n \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k-1-n}\right\rangle - N\left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k+1}\right\rangle \\ &+\alpha^2 N\left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k+1}\right\rangle + \alpha^2 \sum_{j\neq c}\left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k} \mathrm{Tr}\tilde{M}^{(j)}\right\rangle.\end{aligned} \tag{1.47}$$

In the limit $N \to \infty$, the correlators *factorize*. It means that for $l \neq m$

$$\left\langle \mathrm{Tr}\left(\tilde{M}^{(l)}\right)^s \mathrm{Tr}\left(\tilde{M}^{(m)}\right)^t \right\rangle = \left\langle \mathrm{Tr}\left(\tilde{M}^{(l)}\right)^s \right\rangle \left\langle \mathrm{Tr}\left(\tilde{M}^{(m)}\right)^t \right\rangle + O\left(N^{-(D-2)}\right). \quad (1.48)$$

This factorization property can be proved by looking at the Feynman graphs of the model 1.44. Let us compute the contribution of a Feynman graph G. Let E, F, V be respectively the number of edges, faces and vertices of G. It can be easily seen that the edges contribute as N^{-E}, and the faces contribute as N^F.

The contribution of the vertices is more involved. Expanding the potential we notice that the linear term of the expansion vanishes with the term $\alpha N^{\frac{D}{2}} \sum_c \mathrm{Tr}\tilde{M}_c$. The remaining term of the expansion can be represented as vertices of Feynman graphs that are themselves made of k fat vertices of different colors $c \in \mathcal{S} \subseteq [\![1, D]\!]$, $|\mathcal{S}| = k$ for $1 \leq k \leq D$. Each fat vertex of color c is of valence $p_c \geq 2$. Each of this vertex comes with a factor $N^{\frac{2-D}{2}\sum p_c + (D-k)}$ (where $D \geq 3$).

Since we are interested in the $N \to \infty$ limit we focus on graphs that are made out of "leading" vertices. These are the ones for which $k = 1$ and $p := p_c = 2$ for a given c. The factor coming with these vertices is N, it is the usual scaling for matrix models. One can extend the argument for $p \geq 2$ and find the scaling for such graphs G with E edges, F faces and V vertices is

$$N^{F-E+\sum_{v\in G}[(2-D)+(p_v-2)\frac{2-D}{2}+(D-1)]} = N^{\chi(G)-(D-2)(E-V)},$$

with $\chi(G)$ the Euler characteristic of G. The leading graphs are thus the ones for which $(E - V)$ vanishes and χ is maximum. Finally this scaling favors at leading order disconnected contributions maximizing $\chi(G)$. Thus the observables factorize as above.

The Eq.(1.47) becomes thus

$$\begin{aligned} 0 = &\left\langle \sum_{n=0}^{k-1} \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^n \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k-1-n} \right\rangle - N \left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k+1} \right\rangle \\ &+ \alpha^2 N \left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k+1} \right\rangle + \alpha^2 \sum_{j\neq c} \left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^k \right\rangle \left\langle \mathrm{Tr}\tilde{M}^{(j)} \right\rangle. \end{aligned} \quad (1.49)$$

Because of the symmetry, we have $\langle \mathrm{Tr}(\tilde{M}^{(c)}) \rangle = 0$ for every c. Therefore Eq.(1.49) becomes

$$0 = \sum_{n=0}^{k-1} \left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^n \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k-1-n} \right\rangle - N\left(1-\alpha^2\right) \left\langle \mathrm{Tr}\left(\tilde{M}^{(c)}\right)^{k+1} \right\rangle. \quad (1.50)$$

Summing Eq.1.50 over k weighted with z^k and applying the same manipulation tricks as in Subsec.1.5.2, we obtain the desired claim. Thus we have completed the proof of the results obtained by saddle point computations.

$\square$

1.7 Conclusion and perspective

In this chapter, I have presented the computation of the leading and next-to-leading orders of the quartic melonic tensor model via techniques from matrix models. The natural next step is to compute all the subsequent orders, and/or to show that they satisfy some structural properties. Indeed, Bonzom and Dartois [8] have developed further this tensor-matrix interplay initiated in this work and proved that the all-order correlation functions of the quartic melonic tensor model satisfy the blobbed topological recursion defined by Borot [12]. Are more general tensor models also linked to the topological recursion? This is a question for the future.

2

Symmetric functions

This chapter serves the purpose of recalling well-known facts and fixing notations about partitions and symmetric functions. Most of the material is taken from the classic book of Macdonald [64]. The reader can go directly to the next two chapters for new results.

2.1 Partitions

A *(half-)partition* λ is a sequence of non-increasing positive (half-)integers containing only finitely many non-zero terms. To simplify the writing, we will only discuss partitions if the analogues for half-partitions are obvious.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition. The *length* of λ, denoted by $l(\lambda)$, is the number of non-zero terms. The *weight* of λ, denoted by $|\lambda|$, is the sum of its part. We say λ is a partition of $|\lambda|$ and denote this by $\lambda \vdash |\lambda|$. We identify two sequences which differ only by a string of zeros at the end. Denote the set of partitions of n by $\mathcal{P}_n$, and let $\mathcal{P} = \cup_{n\geq 0}\mathcal{P}_n$.

Another convenient way to write the partition λ is $\lambda = (1^{m_1}2^{m_2}\dots)$ which says that exactly m_i parts are equal to i. A partition can be visually represented by a Young *diagram* (also called Ferrer diagram by some authors) (for example Fig.2.1). We identify a partition and its diagram. The coordinate (i,j) of the boxes follows the same convention as with matrices.

The *conjugate* of λ is the partition λ' whose the diagram is obtained by reflection in the main diagonal. For example, if $\lambda = (4,3,2)$ then $\lambda' = (3,3,2,1)$.

If λ and μ are partitions, we write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all $i \geq 1$, i.e. the diagram λ contains the diagram μ. The set-theoretic difference $\theta = \lambda - \mu$, more often denoted by λ/μ, is called a *skew diagram*.

A skew diagram θ is called a *horizontal m-strip* (resp. a *vertical m-strip*) if $|\theta| = m$ and $\theta_i' \leq 1$ (resp. $\theta_i \leq 1$) for all $i \geq 1$. In other words, a horizontal (resp. vertical) strip has at most one square in each column (resp. row).

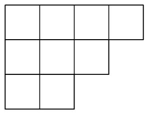

Figure 2.1 – Young diagram $\lambda = (4, 3, 2)$

1	3	4	7
2	5	9	
6	8		

Figure 2.2 – A standard tableau of shape $(4, 3, 2)$.

For a diagram λ, a *tableau* of shape λ is obtained by filling the squares of λ with symbols taken from some alphabet. In this thesis, the alphabet is always taken to be the set of integers from 1 to $|\lambda|$. A *semi-standard tableau* of shape λ is a tableau of shape λ such that the entries increase weakly along each row and strictly along each column. A standard tableau is a semi-standard tableau such that each number $1, 2, \dots, |\lambda|$ appears exactly once (for example Fig.2.2).

Further definitions on partitions/diagrams are listed below. Let λ and μ be partitions/diagrams.

1. We define $\mu \leq \lambda$ if $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$ for all $i \geq 1$. If $\mu \leq \lambda$ and $\mu \neq \lambda$, we write $\mu < \lambda$. This gives a partial order, called the *natural* or *dominance* order, on the set of partitions. One total order which extends this partial order is the *reverse lexicographic ordering*: $\lambda >_{RL} \mu$ if the first non-vanishing difference $\lambda_i - \mu_i$ is strictly positive.

2. The sum of two partitions λ and μ is defined as $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$.

3. The number of even/odd parts of λ is denoted by even(λ)/odd(λ).

4. If $\lambda \subseteq m^n$ then $m^n - \lambda := (m - \lambda_n, \dots, m - \lambda_1)$.

5. The arm-length, arm-colength, leg-length and leg-colength of the square $s = (i, j) \in \lambda$ are defined as follows (the subscript λ can be added if ambiguity needs to be avoided):

$$
\begin{aligned}
a(s) &:= a_\lambda(s) = \lambda_i - j, & a'(s) &:= a'_\lambda(s) = j - 1 \\
l(s) &:= l_\lambda(s) = \lambda'_j - i, & l'(s) &:= l'_\lambda(s) = i - 1.
\end{aligned}
$$

Thus $a(s)$ (resp. $a'(s)$) is the number of squares on the row i which are on the right (resp. left) hand side of s. Similarly, $l(s)$ (resp. $l'(s)$) is the number of squares on the column j which are below (resp. above) s. The hook length of s is $h(s) = a(s) + l(s) + 1$. The statistic $n(\lambda)$ is given by

$$
n(\lambda) := \sum_{s \in \lambda} l'(s) = \sum_{i \geq 1} (i-1)\lambda_i = \sum_{i \geq 1} \binom{\lambda'_i}{2}.
$$

6. For $x, q, t \in \mathbb{C}$, define

$$\begin{aligned} C^+_\lambda(x;q,t) &:= \prod_{(i,j)\in\lambda} (1 - q^{\lambda_i+j-1} t^{2-\lambda'_j-i} x) \\ &= \prod_{1\le i\le l(\lambda)=l} \frac{(q^{\lambda_i} t^{2-l-i} x;q)_\infty}{(q^{2\lambda_i} t^{2-2i} x;q)_\infty} \prod_{1\le i<j\le l} \frac{(q^{\lambda_i+\lambda_j} t^{3-i-j} x;q)_\infty}{(q^{\lambda_i+\lambda_j} t^{2-i-j} x;q)_\infty}, \qquad (2.1) \\ C^-_\lambda(x;q,t) &:= \prod_{(i,j)\in\lambda} (1 - q^{\lambda_i-j} t^{\lambda'_j-i} x) \\ &= \prod_{1\le i\le l} \frac{(x;q)_\infty}{(q^{\lambda_i} t^{l-i} x;q)_\infty} \prod_{1\le i<j\le l} \frac{(q^{\lambda_i-\lambda_j} t^{j-i} x;q)_\infty}{(q^{\lambda_i-\lambda_j} t^{j-i-1} x;q)_\infty}, \qquad (2.2) \\ C^0_\lambda(x;q,t) &:= \prod_{(i,j)\in\lambda} (1 - q^{j-1} t^{1-i} x) \\ &= \prod_{1\le i\le l} (t^{1-i} x;q)_{\lambda_i}. \qquad (2.3) \end{aligned}$$

In particular, for $\lambda = (r)$, we have

$$\begin{aligned} C^0_{(r)}(x;q,t) = C^-_{(r)}(x;q,t) &= \prod_{j=0}^{r-1} (1 - q^j x) = (x;q)_r, \\ C^+_{(r)}(x;q,t) &= \prod_{j=0}^{r-1} (1 - q^{r+j} x) = (q^r x;q)_r, \end{aligned}$$

As usual, the Pochhammer symbols are defined as

$$\begin{aligned} (a;q)_n &:= (1-a)(1-aq)\dots(1-aq^{n-1}), \\ (a;q)_\infty &:= (1-a)(1-aq)(1-aq^2)\dots, \\ (a_1,a_2,\dots,a_m;q)_n &:= (a_1;q)_n \dots (a_m;q)_n, \\ (a_1,a_2,\dots,a_m;q)_\infty &:= (a_1;q)_\infty \dots (a_m;q)_\infty. \end{aligned}$$

7. For $s \in \lambda$, let

$$b_\lambda(s;q,t) := \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}, \qquad (2.4)$$

and

$$b_\lambda(q,t) := \prod_{s\in\lambda} b_\lambda(s;q,t). \qquad (2.5)$$

2.2 The algebra of symmetric functions

Let F be a field of characteristic 0. Consider the algebra $F[x_1,\dots,x_n]$ of polynomials in n independent variables $x_1,\dots,x_n$ with coefficients in F. The symmetric group S_n

acts on $F[x_1, \dots, x_n]$ by permuting the variables. A *symmetric* polynomial is one that is invariant under this action. The symmetric polynomials form a subalgebra, denoted by

$$\Lambda_n := F[x_1, \dots, x_n]^{S_n}.$$

It is a graded ring:

$$\Lambda_n := \oplus_{k \geq 0} \Lambda_n^k,$$

where Λ_n^k contains the homogeneous symmetric polynomials of degree k and the zero polynomial.

We also often want to work with symmetric functions [1] in infinitely many variables. Define the algebra of symmetric functions as the projective limit

$$\Lambda := \varprojlim \Lambda_n,$$

where the projective limit is taken in the category of filtered algebras with respect to the homomorphism which sends the last variable to 0. Concretely, an element of this algebra is a sequence $f = \{f^{(d)}\}_{d \geq 1}$, $f^{(d)} \in \Lambda_d$ such that the polynomials $f^{(d)}$ are of uniformly bounded degree and stable under the restriction, i.e $f^{(d+1)}|_{x_{d+1}=0} = f^{(d)}$. However, if confusion is absent, we will use the same notation for polynomials and their projective limit version.

The algebra Λ has the following bases which are all indexed by partitions.

1. **Monomial symmetric functions**
   For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, let

   $$x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

   For a partition λ of length no more than n, the *monomial* symmetric polynomial m_λ is defined as

   $$m_\lambda(x_1, \dots, x_n) := \sum x^\alpha,$$

   where the sum is over all distinct permutation α of λ. We shall use the same notation m_λ for the infinite variable version. The polynomials $\{m_\lambda, l(\lambda) \leq n\}$ form an F-basis of Λ_n. The symmetric functions $\{m_\lambda, \lambda \in \mathcal{P}\}$ form a F-basis of Λ.

2. **Elementary symmetric functions**
   For each integer $r \geq 1$, the rth *elementary* symmetric function e_r is

   $$e_r := \sum_{i_1 < i_2 < i_r} x_{i_1} \dots x_{i_r} = m_{(1^r)}.$$

   Also, let $e_0 := 1$. For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$, define

   $$e_\lambda := e_{\lambda_1} e_{\lambda_2} \dots.$$

   The functions $\{e_\lambda, \lambda \in \mathcal{P}\}$ form a F-basis of Λ. One also has

   $$\Lambda = F[e_1, e_2, \dots]$$

   and the e_r are algebraically independent over F.

1. The word "function" is just conventional. They are in fact (formal) power series.

3. **Complete symmetric functions**
   For each integer $r \geq 0$, the rth *complete* symmetric function h_r is defined by
   $$h_r := \sum_{\lambda \vdash r} m_\lambda.$$
   As before, let $h_\lambda := h_{\lambda_1} h_{\lambda_2} \dots$. Then the $\{h_\lambda, \lambda \in \mathcal{P}\}$ form a F-basis of Λ. One also has
   $$\Lambda = F[h_1, h_2, \dots]$$
   and the h_r are algebraically independent over F as in the case of elementary symmetric functions.
4. **Power sums**
   For each $r \geq 1$, the rth *power sum* p_r is
   $$p_r := \sum x_i^r = m_{(r)}.$$
   As before define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots$. The $\{p_\lambda, \lambda \in \mathcal{P}\}$ form a F-basis of Λ. One also has
   $$\Lambda = F[p_1, p_2, \dots]$$
   and the p_r are algebraically independent over F.
5. **Schur functions**
   Let δ be the partition $(n-1, n-2, \dots, 0)$. For a partition λ of length at most n, define the following homogeneous anti-symmetric polynomials of degree $|\lambda| + n(n-1)/2$ in $x_1, \dots, x_n$
   $$a_{\lambda+\delta} := \det\left(x_j^{\lambda_i+n-i}\right)_{i,j=1,\dots,n}.$$
   In particular
   $$a_\delta = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$
   is the Vandermonde determinant that we have seen in the precedent chapter. Since $a_{\lambda+\delta}$ is divisible by each of the differences $x_i - x_j (1 \leq i < j \leq n)$, $a_{\lambda+\delta}$ is divisible by a_δ. The Schur function s_λ is thus well defined by
   $$s_\lambda(x_1, \dots, x_n) := a_{\lambda+\delta}/a_\delta.$$
   It is obvious then that $s_\lambda \in \Lambda_n$. In fact, the Schur polynomials $s_\lambda(x_1, \dots, x_n)$ where $l(\lambda) \leq n$ form an F-basis of Λ_n. Their infinite-variable version, also denoted by s_λ for convenience, form a basis of Λ.

   Proposition 2.2.1. *We have*
   $$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}$$
   for every n such that $l(\lambda) \leq n$, and
   $$s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \leq i,j \leq m}$$
   for every m such that $l(\lambda') = \lambda_1 \leq m$.

2.3 Properties of the Schur functions

The Schur functions form perhaps the most important basis of Λ. In fact, they have a fundamental significance in the representation theory of Lie groups, i.e., they are the irreducible characters of the general linear group GL_n. However, we shall not discuss this story; instead, we shall discuss its properties related to the symmetric groups and orthogonal polynomials of several variables.

Proposition 2.3.1. *(Cauchy). The following identity holds*

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\lambda} s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}),$$

where the sum is over all partitions.

Define a symmetric scalar product on Λ by declaring that

$$\langle p_\lambda, p_\mu\rangle := \delta_{\lambda\mu}z_\lambda.$$

Then the Schur functions form an orthonormal basis for this pairing. This orthonormality is equivalent to the Cauchy identity.

The Schur functions are intimately related to the combinatorics of Young diagrams and tableaux. This connection makes them one of the most important objects in algebraic combinatorics. Let us discuss some classic results in that direction.

For a tableau T filled with positive integers, write x^T for $\prod_{s\in T} x_s$. Let λ and μ be partitions, define the skew Schur functions $s_{\lambda/\mu}$ by requiring that the relation

$$\langle s_{\lambda/\mu}, s_\nu\rangle = \langle s_\lambda, s_\mu s_\nu\rangle$$

holds for all the partitions ν. In particular, $s_{\lambda/\emptyset} = s_\lambda$. We have

$$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j})_{1\le i,j\le n}$$

for every $n \ge l(\lambda)$, and

$$s_{\lambda/\mu} = \det(e_{\lambda'_i-\mu'_j-i+j})_{1\le i,j\le m}$$

for every $m \ge l(\lambda')$. It follows from either of these identities that $s_{\lambda/\mu} = 0$ unless $\lambda \supset \mu$. The following result establishes a very nice relation between Schur functions and tableaux

Proposition 2.3.2. *Let $\lambda \supset \mu$ be diagrams, we have*

$$s_{\lambda/\mu} = \sum_T x^T,$$

where the sum is over all semi-standard tableaux T of shape $\lambda - \mu$.

We have the Pieri's rule for multiplying s_λ with h_r and e_r:

Proposition 2.3.3. *For μ a partition and $r \geq 1$,*

$$s_\mu h_r = \sum_{\lambda - \mu \text{ vertical } r\text{-strip}} s_\lambda,$$
$$s_\mu e_r = \sum_{\lambda - \mu \text{ horizontal } r\text{-strip}} s_\lambda.$$

Finally, the Schur functions satisfy various Littlewood identities. Studying Littlewood identities for more general classes of symmetric functions constitutes the main objective of Chapter 4. Here are some examples from [64, p.76-79]

$$\sum_\lambda s_\lambda = \prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_i x_j)^{-1},$$
$$\sum_{\lambda \text{ even}} s_\lambda = \prod_i (1-x_i^2)^{-1} \prod_{i<j} (1-x_i x_j)^{-1},$$
$$\sum_{\lambda' \text{ even}} s_\lambda = \prod_{i<j} (1-x_i x_j)^{-1},$$
$$\sum_\lambda (-1)^{n(\lambda)} s_\lambda = \prod_i (1-x_i)^{-1} \prod_{i<j} (1+x_i x_j)^{-1}.$$

Among many applications, these identities can be used to find explicit generating functions of plane partitions [64, p.80]. In the next section, the q,t-deformation of these identities will be stated and proved.

2.4 Macdonald polynomials

The Macdonald polynomials are two-parameter deformation of the Schur functions which preserve many essential properties.[2] Let q,t be complex parameters, and $\mathbb{F} = \mathbb{Q}(q,t)$. To keep the notations simple, denote again by Λ_n be the sub-algebra of symmetric polynomials in $\mathbb{F}[x_1,\ldots,x_n]$. All the bases above are also bases of this new Λ_n. In the following, an element of $\mathbb{F}[x_1,\ldots,x_n]$ is denoted by $f(;q,t)$ if the dependence on q,t needs to be emphasized.

Definition 2.4.1. For any partitions λ and μ of length less than or equal to n, define the q,t-Hall scalar product on Λ_n

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{n} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}, \tag{2.6}$$

where $z_\lambda := \prod_{i\geq 1} m_i(\lambda)! i^{m_i(\lambda)}$.

2. When one says Macdonald polynomials without any other specifications, it is implicitly supposed that one is talking about those attached to the root systems of type A. As it will be explained soon, there are Macdonald polynomials attached to other root systems.

Theorem 2.4.1. *There is a unique family of symmetric polynomials $P_\lambda(x_1, \ldots, x_n; q, t)$ indexed by partitions of length less than or equal to n and depending on two parameters q, t such that the following two conditions hold*

1. *Triangularity:* $P_\lambda(; q, t) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu$, *and*
2. *Orthogonality:* $\langle P_\lambda(; q, t), P_\mu(; q, t)\rangle_{q,t} = 0$ *if* $\lambda \neq \mu$.

Definition 2.4.2. The polynomials P_λ in the previous theorem are called the Macdonald polynomials. If $l(\lambda) > n$, we set $P_\lambda(x_1, \ldots, x_n; q, t) = 0$.

When $q = t$ we recover the Schur polynomials, i.e.,

$$P_\lambda(\mathbf{x}; t, t) = s_\lambda(\mathbf{x}).$$

Other specializations of q, t give well known families of polynomials such as those of Hall-Littlewood and of Jack. Currently, there are many families of multivariate symmetric and/or orthogonal polynomials related to the Macdonald polynomials which play a crucial role in various domains such as integrable probability, algebraic combinatorics and asymptotic representation theory (see for instance [10]).

The skew Macdonald polynomials $P_{\lambda/\mu}(; q, t)$ are defined by requiring that

$$\langle P_{\lambda/\mu}(; q, t), P_\nu(; q, t)\rangle_{q,t} = \langle P_\lambda(; q, t), P_\mu(; q, t) P_\nu(; q, t)\rangle_{q,t} \tag{2.7}$$

for every partition ν. One can prove that $P_{\lambda/\mu}(; q, t) = 0$ unless $\mu \subset \lambda$.

The Macdonald polynomials are also orthogonal with respect to another scalar product associated to type A root system. This property has been generalized to define (symmetric and non symmetric) multivariate orthogonal polynomials associated to all root systems. Such a generalization to the non-reduced root system BC_n is described in Chap.4. In that case, the polynomials are called the Koornwinder BC_n symmetric polynomials. In fact, all the Macdonald polynomials associated to classical (i.e., non-exceptional) root systems are specialization of the Koornwinder polynomials.

Definition 2.4.3. Let $|q|, |t| < 1$, define the Macdonald density

$$\Delta(\mathbf{x}; q, t) := \prod_{1\le i<j\le n} \frac{(x_i/x_j, x_j/x_i; q)_\infty}{(tx_i/x_j, tx_j/x_i; q)_\infty}.$$

Definition 2.4.4. Let $|q|, |t| < 1$, define the following scalar product on $\mathbb{F}[x_1, \ldots, x_n]$:

$$\begin{aligned}\langle f, g\rangle'_{q,t} :=& \frac{1}{n!}\int_{\mathbb{T}^n} f(\mathbf{x}) g(\mathbf{x}^{-1}) \Delta(\mathbf{x}; q, t) d\mathbb{T} \\ =& \frac{1}{n!}[1] f(\mathbf{x}) g(\mathbf{x}^{-1}) \Delta(\mathbf{x}; q, t),\end{aligned} \tag{2.8}$$

where $\mathbb{T}^n$ is the torus $\{|x_1| = \ldots = |x_n| = 1\}$, $[1]f$ means the constant term of f, and

$$d\mathbb{T} = \frac{dx_1}{2\pi i x_1} \cdots \frac{dx_n}{2\pi i x_n}.$$

We have

Proposition 2.4.2. *For any partitions λ and μ,*

$$\langle P_\lambda(;q,t), P_\mu(;q,t)\rangle'_{q,t} = 0 \quad \text{if } \lambda \neq \mu.$$

The polynomials $P_\lambda(q,t)$ satisfy many remarkable properties. I only list here the results that will be needed later. First, we can explicitly compute the quadratic norm for $\langle\cdot\rangle_{q,t}$

Proposition 2.4.3. *We have*

$$\langle P_\lambda(;q,t), P_\mu(;q,t)\rangle_{q,t} = \frac{1}{b_\lambda(q,t)}\delta_{\lambda\mu}, \tag{2.9}$$

where $b_\lambda(q,t)$ is introduced in Eq.(2.5).

This equation is equivalent to the Cauchy-Macdonald identity:

Proposition 2.4.4. *Let m, n be positive integers. Then*

$$\sum_{\lambda\subset m^n}(-1)^{|\lambda|}P_\lambda(x_1,\dots,x_n;q,t)P_{\lambda'}(y_1,\dots,y_m;t,q) = \prod_{i=1}^n\prod_{j=1}^m(1-x_iy_j). \tag{2.10}$$

Second, we have the g- and e-Pieri rules:

Proposition 2.4.5. *Let μ be a partition and r a positive integer.*

1. *g-Pieri rule: Let $g_r(\mathbf{x};q,t)$ be the symmetric polynomials defined by the generating function*

$$\prod_{i=1}^n \frac{(tux_i;q)_\infty}{(ux_i;q)_\infty} =: \sum_{k\geq 0} g_k(\mathbf{x};q,t)u^k.$$

   Then we have

$$P_\mu g_r = \sum_\lambda \varphi_{\lambda/\mu}P_\lambda, \tag{2.11}$$

2. *e-Pieri rule: We have*

$$P_\mu e_r = \sum_\lambda \psi'_{\lambda/\mu}P_\lambda. \tag{2.12}$$

In (2.11) *(resp.* (2.12)*), the sum is over partitions λ such that λ/μ is a horizontal (resp. vertical) r-strip. The coefficients are given by*

$$\varphi_{\lambda/\mu} = \prod_{s\in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)},$$

$$\psi'_{\lambda/\mu} = \prod_{s\in C_{\lambda/\mu}-R_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}.$$

Here, for λ and μ such that $\lambda \supset \mu$, $C_{\lambda/\mu}$ (resp. $R_{\lambda/\mu}$) is the union of the columns (resp. rows) that intersect $\lambda - \mu$.

The Cauchy-Macdonald identity gives the *branching* formula for Macdonald polynomials

Proposition 2.4.6. *Let $x_1, \dots, x_n, z_1, \dots, z_m$ be variables. Then*

$$P_\lambda(x_1, \dots, x_n, z_1, \dots, z_m; q, t) = \sum_{\mu \subset \lambda} P_{\lambda/\mu}(x_1, \dots, x_n; q, t) P_\mu(z_1, \dots, z_m; q, t). \quad (2.13)$$

In particular, in combination with the Pieri rules, we have

Corollary 2.4.7. *Let $n \in \mathbb{N}$. Then*

$$P_\lambda(x_1, \dots, x_n, y; q, t) = \sum_{\substack{\mu \\ \lambda/\mu \text{ is a horizontal strip}}} \psi'_{\lambda'/\mu'}(t, q) y^{|\lambda/\mu|} P_\mu(x_1, \dots, x_n; q, t) \quad (2.14)$$

The Macdonald polynomials satisfy the following Littlewood identities, which can be viewed either as evaluation of their generating functions or decomposition of symmetric functions into the Macdonald basis. The following theorem is due to Macdonald [64, p.349], and written in the following form by Warnaar [88].

Theorem 2.4.8. *Let $n \in \mathbb{N}$ and $a \in \mathbb{C}$. We have*

$$\sum_\lambda a^{odd(\lambda)} b_\lambda^{oa}(q, t) P_\lambda(\mathbf{x}; q, t) = \prod_{i=1}^n \frac{(1 + a x_i)(q t x_i^2; q^2)_\infty}{(x_i^2; q^2)_\infty} \prod_{1 \le i < j \le n} \frac{(t x_i x_j; q)_\infty}{(x_i x_j; q)_\infty}, \quad (2.15)$$

$$\sum_\lambda a^{odd(\lambda')} b_\lambda^{el}(q, t) P_\lambda(\mathbf{x}; q, t) = \prod_{i=1}^n \frac{(a t x_i, q)_\infty}{(a x_i; q)_\infty} \prod_{1 \le i < j \le n} \frac{(t x_i x_j; q)_\infty}{(x_i x_j; q)_\infty}. \quad (2.16)$$

Here, we define (oa (resp. el) stands for odd arm (resp. even leg))

$$b_\lambda^{oa}(q, t) := \prod_{\substack{s \in \lambda \\ a(s) \text{ odd}}} b_\lambda(s; q, t) \quad \text{and} \quad b_\lambda^{el}(q, t) := \prod_{\substack{s \in \lambda \\ l(s) \text{ even}}} b_\lambda(s; q, t).$$

Proof. The two identities are in fact dual to each other via the involution $\omega_{q,t}$

$$\omega_{q,t} P_\lambda(; q, t) := b_{\lambda'}(t, q) P_{\lambda'}(; t, q).$$

The reader is invited to consult Macdonald's book for more information concerning this duality. Given this, we thus only need to prove the second identity Eq.(2.16). First, let us prove the case $a = 0$, i.e.,

$$\sum_{\lambda' \text{ even}} b_\lambda^{el}(q, t) P_\lambda(x_1, \dots, x_n; q, t) = \prod_{1 \le i < j \le n} \frac{(t x_i x_j; q)_\infty}{(x_i x_j; q)_\infty} \quad (2.17)$$

Denote the left hand side of (2.17) by $A(x_1, \dots, x_n; q, t)$. The case $n = 1$ trivially holds because both sides of (2.17) equal 0. So by induction, we need to prove that

$$A(x_1, \dots, x_n, y; q, t) = A(x_1, \dots, x_n; q, t) \prod_{i=1}^n \frac{(t x_i y; q)_\infty}{(x_i y; q)_\infty}.$$

We have

$$
\begin{aligned}
&A(x_1,\ldots,x_n;q,t)\prod_{i=1}^{n}\frac{(tx_iy;q)_\infty}{(x_iy;q)_\infty}\\
=&\sum_{r\geq 0} g_r(x_1,\ldots,x_n;q,t)y^r \sum_{\nu'\text{ even}} b_\nu^{el}(q,t)P_\nu(x_1,\ldots,x_n;q,t)\\
=&\sum_{r\geq 0}\sum_{\nu'\text{ even}} b_\nu^{el}(q,t)y^r g_r(x_1,\ldots,x_n;q,t)P_\nu(x_1,\ldots,x_n;q,t)\\
=&\sum_{\nu'\text{ even}}\sum_{\substack{\mu\\ \mu/\nu\text{ horizontal strip}}} b_\nu^{el}(q,t)y^{|\mu/\nu|}\varphi_{\mu/\nu}(q,t)P_\mu(x_1,\ldots,x_n;q,t)\quad\text{(Eq.(2.11))}\\
=&\sum_{\mu}\sum_{\substack{\nu'\text{ even}\\ \mu/\nu\text{ horizontal strip}}} b_\nu^{el}(q,t)\varphi_{\mu/\nu}(q,t)y^{|\mu/\nu|}P_\mu(x_1,\ldots,x_n;q,t)
\end{aligned}
$$

For a given partition μ, there exists a unique partition ν satisfying both conditions in the inner sum, i.e., $\nu_i' = 2\lfloor\mu_i'/2\rfloor$. There exists also a unique partition λ such that λ' is even and λ/μ is a horizontal strip, i.e. $\lambda_i' = 2\lceil\mu_i'/2\rfloor$. It is easy to show that for λ,μ,ν related in this way, one has

$$
b_\lambda^{\mathrm{el}}(q,t)\psi_{\lambda'/\mu'}(t,q) = b_\nu^{\mathrm{el}}(q,t)\varphi_{\mu/\nu}(q,t).
$$

Given this equality, one can write

$$
\begin{aligned}
&A(x_1,\ldots,x_n;q,t)\prod_{i=1}^{n}\frac{(tx_iy;q)_\infty}{(x_iy;q)_\infty}\\
=&\sum_{\mu}\sum_{\substack{\lambda'\text{ even}\\ \lambda/\mu\text{ horizontal strip}}} b_\lambda^{\mathrm{el}}(q,t)\psi_{\lambda'/\mu'}(t,q)y^{|\mu/\nu|}P_\mu(x_1,\ldots,x_n;q,t)\\
=&\sum_{\lambda'\text{ even}}\sum_{\substack{\mu\\ \lambda/\mu\text{ horizontal strip}}} b_\lambda^{\mathrm{el}}(q,t)\psi_{\lambda'/\mu'}(t,q)y^{|\mu/\nu|}P_\mu(x_1,\ldots,x_n;q,t)\\
=&\sum_{\lambda'\text{ even}} b_\lambda^{\mathrm{el}}(q,t)P_\lambda(x_1,\ldots,x_n,y;q,t)\quad\text{(Eq.(2.14))}\\
=&A(x_1,\ldots,x_n,y;q,t).
\end{aligned}
$$

Thus Eq.(2.17) is proven. For the case a arbitrary, we just need to expand the product

$$
\prod_{i=1}^{n}\frac{(atx_i,q)_\infty}{(ax_i;q)_\infty} = \sum_{r\geq 0} a^r g_r(x_1,\ldots,x_n;q,t),
$$

and use the g-Pieri rule (2.11) again. The details are analogous to the above manipulations. □

Another identity was conjectured by Kawanaka in [54] and proven in [60]. The proof is via a new identity of theta functions.

Theorem 2.4.9. *[54, 60] Let $n \in \mathbb{N}$, we have*

$$\sum_{\lambda} b_{\lambda}^{-}(q,t) P_{\lambda}(\mathbf{x}; q^2, t^2) = \prod_{i=1}^{n} \frac{(-tx_i, q)_\infty}{(x_i; q)_\infty} \prod_{1 \le i < j \le n} \frac{(t^2 x_i x_j; q^2)_\infty}{(x_i x_j; q^2)_\infty}, \tag{2.18}$$

where

$$b_{\lambda}^{-}(q,t) := \prod_{s \in \lambda} \frac{1 + q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}.$$

In [75], Rains and Warnaar prove many generalizations of these identities. They also prove that their identities imply new combinatorial character identities for affine Lie algebras, which in turn, imply new Rogers-Ramanujan-type identities associated to affine Lie algebras. These Littlewood identities of Rains and Warnaar are the object of study in Chap.4. In particular, I have been able to prove one of their conjectured identity and make partial progress on another.

3

Hurwitz numbers

The symmetric groups are very classical objects. Indeed, they are the genesis of modern group theory. After a quite long dormant time, they have become again, since the eighties of the twentieth century, a central object of study, particularly in algebraic combinatorics, (ordinary and modular) representation theory and probability. Hurwitz numbers play an important role in the renaissance of this interest. These numbers are beautifully connected with other parts of mathematics such as matrix models, integrable systems and enumerative algebraic geometry. After some general discussions about the whole context of Hurwitz numbers, my own contributions are described in the sections 3.9 and 3.10. The main results are an explicit formula for one-part double Hurwitz numbers with completed 3-cycles and its implications.

3.1 Irreducible characters of the symmetric groups

I will briefly recall the representation theory of the symmetric groups (for a detailed account, see for instance [22]). The main objects that we need are actually (complex) irreducible characters, not irreducible representations themselves. The reader can skip this section and the next one if the material is familiar to her/him. We always assume that the ground field is $\mathbb{C}$.

Let S_d be the symmetric group, i.e., the group of all permutations of $\{1, 2, \ldots, d\}$. It is well-known that the irreducible representations of a finite group G are indexed exactly by its conjugacy classes. In the case of S_d, a conjugacy class consists of permutations with identical cycle factoring structure. Thus, the irreducible representations of S_d are indexed by the partitions of d.

For λ and μ partitions of d, denote by χ^λ_μ the evaluation of the irreducible character χ^λ at the conjugacy class μ. It can be shown that χ^λ_μ is an integer (à priori, it is a complex number). In particular, the dimension of the irrep λ is given by

$$\dim \lambda = \chi^{\lambda}_{(1^d)}.$$

The dimension can be computed by the following nice combinatorial rule

Proposition 3.1.1. *For $\lambda \vdash d$, $\dim \lambda$ is the number of standard tableaux of shape λ.*

The dimension of λ can also be computed by the famous hook-length formula of Frame, Robinson, and Thrall [36]:

$$\dim \lambda = \frac{d!}{\prod_{s\in\lambda} h(s)}, \tag{3.1}$$

where for each box $s \in \lambda$, $h(s)$ is its hook length. Later, it will be convenient to have the following normalized characters:

$$\hat{\chi}^{\lambda}_{\mu} := \frac{\chi^{\lambda}_{\mu}}{\dim \lambda},$$

and

$$f_\mu(\lambda) := |\text{cyc}(\mu)| \frac{\chi^{\lambda}_{\mu}}{\text{dim}\lambda} = |\text{cyc}(\mu)| \hat{\chi}^{\lambda}_{\mu}, \tag{3.2}$$

where cyc(μ) is the set of permutations whose cycle structure is μ. The irreducible characters satisfy the following orthogonality relations

Proposition 3.1.2.

$$\frac{1}{d!} \sum_{\nu \vdash d} |cyc(\nu)| \chi^{\lambda}_{\nu} \chi^{\mu}_{\nu} = \delta_{\lambda\mu}, \tag{3.3}$$

and

$$\frac{|cyc(\mu)|}{d!} \sum_{\lambda \vdash d} \chi^{\lambda}_{\mu} \chi^{\lambda}_{\nu} = \delta_{\mu\nu}. \tag{3.4}$$

The irreducible characters can be calculated by the following formula of Frobenius, although it is not the most efficient way.

Proposition 3.1.3. *(Frobenius) For $\mu \vdash n$, we have*

$$p_\mu = \sum_{\lambda \vdash n} \chi^{\lambda}_{\mu} s_\lambda, \tag{3.5}$$

where p and s are the power sums and the Schur polynomials respectively.

Reciprocally, by the orthogonality of irreducible characters, Frobenius's formula is equivalent to

$$s_\lambda = \sum_{\mu \vdash n} z_\mu^{-1} \chi^{\lambda}_{\mu} p_\mu. \tag{3.6}$$

Although the irreducible characters χ^λ are classical, they remain largely mysterious, mostly because of their great combinatorial complexity. I list below some open questions that have attracted my attention. It might be helpful for the readers of this thesis.

1. The Kronecker coefficients $g(\lambda, \mu, \nu)$, for λ, μ, ν partitions of n, are defined as the multiplicity the irreducible representation ν in the representation $\lambda \otimes \mu$. By definition, they are non-negative integers. They are the coefficients in the following decomposition
$$\chi^\lambda \chi^\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) \chi^\nu.$$
It is not obvious from the definition but it can be shown that $g(\lambda, \mu, \nu)$ is symmetric in all three arguments. Some experts have called them the most challenging, deep and mysterious objects in Algebraic Combinatorics [74]. The outstanding open question is to find a combinatorial or geometrical description for these coefficients. Their cousin, the Littlewood-Richardson coefficients have indeed beautiful combinatorial and geometrical interpretation. In particular, the problem of positivity, i.e., deciding whether $g(\lambda, \mu, \nu) > 0$ is still unsolved (while the same question for the Littlewood-Richardson coefficients has been answered).
A concrete version of the positivity problem is the Saxl conjecture which states that
$$g(\rho_k, \rho_k, \lambda) > 0$$
for all $\lambda \vdash k(k+1)/2$ and $\rho_k = (k, k-1, \dots, 1)$.
2. A reciprocal view of $\hat{\chi}^\lambda_\mu$ was put forward by Kerov and Vershik [87] to study their asymptotics (the normalised version turns out be more suitable than the original one for this purpose). Rather than considering $\hat{\chi}^\lambda_\mu$ as function of μ, these authors consider them as function of λ. Fixing the "shape" of μ, they ask questions about the asymptotics of $\hat{\chi}^\lambda_\mu$ when $n = |\lambda|$ tends to infinite. Of course, one does not expect to get a uniform answer for the limits, i.e., certainly the way we fix μ and let λ grow influences the final limit.
To fix the idea, let us state the celebrated Kerov-Vershik asymptotic formula, although it might not be the strongest one available currently. A good exposition of this and relevant results is given in the book [22]. Let h be a positive integer and $\rho = (\rho_1, \dots, \rho_h)$ be a partition such that $\rho_h \geq 2$. Let $\mu = \left(\rho, 1^{n-|\rho|}\right)$ and $\sigma = (\rho_1 - 1, \dots, \rho_h - 1)$.

Proposition 3.1.4. *We have the following asymptotic formula*
$$\hat{\chi}^\lambda_\mu = \frac{\rho_1 \cdots \rho_h}{[n]_{|\rho|}} p_\sigma[Cont(\lambda)] + O\left(\frac{1}{|\lambda|}\right),$$
where the constant in $O\left(\frac{1}{|\lambda|}\right)$ *depends only on* ρ. *Here* p_σ *is the power sum,* $Cont(\lambda) = \{\{i - j, (i,j) \in \lambda\}\}$ *is the multi-set called the content of* λ, *and*
$$[n]_k := n(n-1)\dots(n-k+1)$$
for $n, k \in \mathbb{Z}$.

Similar asymptotic problems for generalizations (deformations) of the characters of the symmetric groups have also been studied (the reader can start with the short survey article [84]).

3. A vaguer question concerns explicit formulas for the characters χ^λ_μ. In fact, there is only one explicit formula which was discovered ten years ago by Lassalle [22, 61]. Unfortunately, the formula is extremely complicated (as expected!).

3.2 Group algebra of the symmetric groups

Let S_d be the symmetric group. Rather than following the convention for composing functions, we multiply the elements of S_d from left to right. As usual, for a ring $\mathcal{R}$, define the $\mathcal{R}$-group algebra $\mathcal{R}S_d$ as the set of formal linear combinations of elements of S_d with coefficients in $\mathcal{R}$. The addition and multiplication on $\mathcal{R}S_d$ are naturally defined. In this thesis, I will only use $\mathcal{R} = \mathbb{Q}$. Let $\mathbb{Q}S = \bigoplus_{d=0}^{\infty} \mathbb{Q}S_d$. Let $\mathbf{Z}_d$ be the center of $\mathbb{Q}S_d$ and $\mathbf{Z} = \bigoplus_{d=0}^{\infty} \mathbf{Z}_d$. An element $a \in \mathbf{Z}_d$ is also called a *central* element.

Definition 3.2.1. For a partition μ of d, let $C_\mu := \sum_{g \in \text{cyc}(\mu)} g$.

The following result is fundamental.

Proposition 3.2.1. *The elements $\{C_\mu, \mu \vdash d\}$ form a basis for the algebra $\mathbf{Z}_d$.*

Proof. We observe that an element $A = \sum_{g \in S_d} a_g g$ of $\mathbb{Q}S_d$ is central if and only if $a_g = a_{wgw^{-1}}$ for every $w, g \in S_d$. Since the conjugate action of S_d on itself, i.e. $w.g := wgw^{-1}$ is transitive, every element C_μ is central and every central element is a linear combination of the C_μ. Furthermore, it is obvious from the definition that the elements C_μ are linearly independent. Thus they make a basis of $\mathbf{Z}_d$.

□

The main problem concerning us is to study the structure constants of the commutative algebra $\mathbf{Z}_d$ (and $\mathbf{Z}$) with respect to the basis C_μ, and related ones. These constants can be interpreted as counting factorization of a permutation into others with certain conditions. They (with different normalisations) are commonly called (connected or disconnected) **Hurwitz numbers** (the precise definition will be given in the next section).

The algebra $\mathbf{Z}_d$ has another basis consisting of elements F_λ, $\lambda \vdash d$ defined as follows

$$F_\lambda = \frac{\dim(\lambda)}{d!} \sum_{\mu \vdash d} \chi^\lambda_\mu C_\mu. \tag{3.7}$$

Equivalently, due to the orthogonality of the irreducible characters, one has

$$C_\mu = |\text{cyc}(\mu)| \sum_{\lambda \vdash d} \hat{\chi}^\lambda_\mu F_\lambda.$$

Proposition 3.2.2. *The central elements F_λ form a basis of orthogonal idempotents, i.e., $F_\lambda F_\mu = \delta_{\lambda\mu} F_\lambda$.*

Proof. Let λ and μ be two partitions of d. Let V^λ be the irreducible module corresponding to λ. Each central element acts on V^λ as a multiplication by a scalar due to the Schur lemma. Thus the matrix representation of C_μ on V^λ is $bI_{\dim\lambda}$, where b is a complex number and I is the identity matrix. We have

$$b = \frac{1}{\dim\lambda}\mathrm{Tr}(C_\mu) = \frac{1}{\dim\lambda}\sum_{g\in \mathrm{cyc}(\mu)} \mathrm{Tr}(g) = f_\mu(\lambda).$$

This equation, together with the definition of F_λ and the first orthogonality of the irreducible characters, implies that that F_λ acts identically on V^λ, and trivially on V^ν for $\nu \neq \lambda$. This is equivalent to the claim. $\square$

For each $a \in \mathbf{Z}_d$, let $g_a(\lambda)$ be the coefficients in the decomposition

$$a = \sum_{\lambda\vdash d} g_a(\lambda)F_\lambda,$$

i.e., $aF_\lambda = g_a(\lambda)F_\lambda$. The number $g_a(\lambda)$ is the constant by which the element a acts in the irreducible representation λ as multiplication. We have the following proposition which will be used to connect the combinatorics of the symmetric groups with integrable equations:

Proposition 3.2.3. *For any element*

$$a = \sum_{\mu\vdash d} a_\mu \frac{C_\mu}{|cyc(\mu)|} \in \mathbf{Z}_d$$

the following equation holds

$$\sum_{\mu\vdash d} a_\mu p_{\mu_1}p_{\mu_2}\cdots = \sum_{\lambda\vdash d} g_a(\lambda)\dim(\lambda)s_\lambda(p),$$

where p and s are respectively the power sums and the Schur functions.

Proof. By definition,

$$g_a(\lambda) = \sum_\mu a_\mu \hat{\chi}^\lambda_\mu.$$

Thus, we have

$$\sum_\lambda g_a(\lambda)\dim(\lambda)s_\lambda = \sum_\lambda\sum_\mu a_\mu\chi^\lambda_\mu s_\lambda(p) = \sum_\mu a_\mu p_\mu$$

according to the Frobenius character formula Eq.3.5. $\square$

We will often need to compute $g_a(\lambda)$ for an arbitrary central element a and irreducible representation λ. The tool is the *Jucys-Murphy elements* [52, 68].

Definition 3.2.2. The Jucys-Murphy elements of the group algebra $\mathbf{Z}_d$ are the following sums of transpositions: $X_1 := 0$ and

$$X_k := (1,k) + \cdots + (k-1,k) \quad \text{for } k = 2,\ldots,d. \tag{3.8}$$

These elements are not central, however we have the following results

Proposition 3.2.4. *1. The elements $X_1,\ldots,X_d$ commute pair-wise and thus generate a commutative subalgebra of $\mathbb{Q}S_d$.*

2. *An element of $\mathbb{Q}S_d$ is central if and only if it can be written as a symmetric polynomial of the elements $X_1,\ldots,X_d$.*
3. *Let $a = P(X_1,\ldots,X_d) \in \mathbf{Z}_d$, where P is a symmetric polynomial. Then*

$$P(X_1,\ldots,X_d)F_\lambda = P\left(\mathit{Cont}\left(\lambda\right)\right)F_\lambda,$$

where $\mathit{Cont}(\lambda)$ is the content of the Young diagram λ. In other words,

$$g_a(\lambda) = P\left(\mathit{Cont}\left(\lambda\right)\right).$$

Proof. 1. Let $2 \le k < l \le d$. The difference between X_kX_l and X_lX_k can only happen at the multiplications $(jk)(jl)$ versus $(jl)(jk)$ and $(jk)(kl)$ versus $(kl)(jk)$ for $j < k$. However one has

$$(jk)(jl) + (jk)(kl) = (jkl) + (jlk) = (kl)(jk) + (jl)(jk).$$

Thus $X_kX_l = X_lX_k$.

2. First to prove that symmetric polynomials in the JM elements are central, let us prove a stronger statement. For $1 \le s \le d$, we have

$$e_s(X_1,X_2,\ldots,X_d) := \sum_{1\le i_1<\cdots<i_s\le d} X_{i_1}\ldots X_{i_s} = \sum_{\substack{\mu\vdash d\\ l(\mu)=d-s}} C_\mu.$$

This follows from the fact that for a permutation of cycle structure μ, the minimal number of transpositions in its factorization as product of transposition is $k = |\mu| - l(\mu)$; and there is a unique minimal factorization $(a_1b_1)\ldots(a_kb_k)$ such that $b_1 < \cdots < b_k$ (we always write a transposition (ab) with convention $a < b$). Since the elementary symmetric polynomials e_s generate the algebra of symmetric polynomials, it follows that every symmetric polynomial of the JM elements is central.

Now we prove that every central element is a symmetric polynomial of the JM elements. It suffices to construct a set of linearly independent symmetric polynomials in $X_2,\ldots,X_d$ indexed by the partitions of d. For $\mu = (\mu_1,\ldots,\mu_l) \vdash d$ with $\mu_l > 0$, let

$$X_\mu = \sum X_{i_1}^{\mu_1-1}\ldots X_{i_l}^{\mu_l-1},$$

where the sum is over all distinct monomials with $i_1,\ldots,i_l$ distinct indices in $\{2,3,\ldots,d\}$. Then it can be verified that X_μ's are linearly independent. However, the verification is long. The reader is invited to consult [22] for details.

3. To prove this equation, we need to invoke some representation theoretic results. The irreducible $\mathbb{Q}S_d$ module V^λ has a $\mathbb{Q}$-basis $\{v_T\}$ indexed by the set of standard Young tableaux T of shape λ, called the *Young's orthogonal basis*. It is an important fact that these vectors are eigenvectors of the JM elements

$$X_k v_T = c_T(k) v_T, \tag{3.9}$$

where $c_T(k)$ is the content of the box numbered k of T (remember that for a standard tableau T, there exists a unique box of T which is numbered k). Thus for any polynomial $f(X_1, \dots, X_d)$, we have

$$f(X_1, \dots, X_d) v_T = f\left(c_T(1), \dots, c_T(d)\right) v_T. \tag{3.10}$$

In particular, if P is a symmetric polynomial (such that $P(X_1, \dots, X_d)$ is central) then we can write

$$P(X_1, \dots, X_d) F_\lambda = P(\text{Cont}\,(\lambda)) F_\lambda.$$

□

3.3 Combinatorial definition of Hurwitz numbers

Let $\mu^1, \dots, \mu^k$ be partitions of d. Define the number $N\left(d; \mu^1, \dots, \mu^k\right)$ by

$$N\left(d; \mu^1, \dots, \mu^k\right) := \left[C_{(1^d)}\right] C_{\mu^1} \dots C_{\mu^k} = \left[\frac{C_{\mu^1}}{|\text{cyc}(\mu^1)|}\right] C_{\mu^2} \dots C_{\mu^k},$$

i.e., the coefficient of $C_{(1^d)}$ in the linear decomposition of $C_{\mu^1} \dots C_{\mu^k}$. With varying normalization, these numbers are called Hurwitz numbers.

From the definition of C_μ, one immediately notices that

$$\begin{aligned} N\left(d; \mu^1, \dots, \mu^k\right) &= \#\{(w_1, \dots, w_k) \in \text{cyc}(\mu^1) \times \dots \text{cyc}(\mu^k) \mid w_1 \dots w_k = 1\} \\ &= \#\{(w_2, \dots, w_k) \in \text{cyc}(\mu^2) \times \dots \text{cyc}(\mu^k) \mid w_2 \dots w_k \in \text{cyc}(\mu^1)\}. \end{aligned}$$

In general, a problem of counting permutation factorization is called a Hurwitz enumeration problem. One has the following "explicit" formula due to Frobenius and Burnside:

Theorem 3.3.1. *With the same notation as above,*

$$N\left(d; \mu^1, \dots, \mu^k\right) = \frac{|cyc(\mu^1)| \dots |cyc(\mu^k)|}{d!} \sum_{\lambda \vdash n} \frac{\chi^\lambda_{\mu^1} \cdots \chi^\lambda_{\mu^k}}{\dim(\lambda)^{k-2}}. \tag{3.11}$$

Proof. The formula follows immediately from the formula expressing C_μ in terms of F_λ and the idempotency of F_λ. □

This formula looks compact and is a useful theoretical tool, but of course not practical in most cases due to the lack of simple formulas for the irreducible characters. Besides, there are just too many summands (the number of partitions of n grows as $\exp(\pi\sqrt{2n/3})/4n\sqrt{3})$. However, in certain cases, most of the terms vanish and the formula simplifies a lot. I will exploit this observation to calculate explicitly the one-part double Hurwitz numbers with completed 3-cycles.

A surprising property of Hurwitz numbers is that while the dependency of each summand on μ^i is virtually incomprehensible, the final sum can be very nice. For example, a proper normalization makes the sum polynomial (or piece-wise polynomials) in the parts of μ^i in some cases.

We now give a geometric definition (in fact it was the original definition given by Hurwitz [46]) and prove its equivalency with the combinatorial one.

3.4 Hurwitz numbers count ramified coverings of the 2-sphere

Let $\mathrm{Cov}_d(\mu^1, \dots, \mu^k)$ be the set of isomorphism classes of weighted degree d (both connected and disconnected) coverings of $\mathbb{P}^1$ ramified over k fixed points of $\mathbb{P}^1$ with ramification profiles given by $\mu^1, \dots, \mu^k$. The weight of a covering is defined to be the inverse of the order of its finite automorphism group. [1] The disconnected Hurwitz number $H(d; \mu^1, \dots, \mu^k)$ is defined to be the weighted sum over $\mathrm{Cov}_d(\mu^1, \dots, \mu^k)$, i.e.,

$$H(d; \mu^1, \dots, \mu^k) := \sum_{f \in \mathrm{Cov}_d(\mu^1, \dots, \mu^k)} \frac{1}{|\mathrm{Aut}(f)|} \tag{3.12}$$

If we restrict ourselves to connected coverings, we have connected Hurwitz numbers, denoted by $H^\star(d; \mu^1, \dots, \mu^k)$. The connection between connected and disconnected numbers follows the general inclusion-exclusion principle of enumerative combinatorics. That is, the generating function of disconnected numbers is the exponential of a well chosen generating function of connected ones.

The Riemann-Hurwitz formula gives us the Euler-Poincaré characteristic of the cover:

$$2 - 2g = l(\mu^1) + \cdots + l(\mu^k) - (k-2)d. \tag{3.13}$$

Thus one might prefer keeping track of g rather than d. We will denote $H_g(\mu^1, \dots, \mu^k) = H(d; \mu^1, \dots, \mu^k)$ in such occasions (and always suppose that the Riemann-Hurwitz formula holds).

Theorem 3.4.1. *We have the following equality between geometric and combinatorial Hurwitz numbers:*

$$H(d; \mu^1, \dots, \mu^k) = \frac{1}{d!} N\left(d; \mu^1, \dots, \mu^k\right).$$

1. Let $f, g : X \to Y$ be two coverings of Riemann surfaces. They are called isomorphic if there is a biholomorphic mapping $\sigma : X \to X$ such that $f = \sigma \circ g$. An automorphism of f is a biholomorphic mapping $\tau : X \to X$ such that $f = \tau \circ f$. Two isomorphic coverings have isomorphic automorphism groups, Eq.3.12 is thus well defined.

Proof. For each tuple of permutations $(w_1, \ldots, w_k) \in \text{cyc}(\mu^1)\times \ldots \text{cyc}(\mu^k)$, the Riemann existence theorem assures that there exists a covering of the sphere with branching at the points $P_1, \ldots, P_k \in \mathbb{C}P^1$ specified by w_i's precisely when $w_1 \ldots w_k = 1$. For each tuple $(w_1, \ldots, w_k)$ satisfying this property, the corresponding covering is unique up to isomorphism. It follows that one has a well-defined map

$$\psi : \{(w_1, \ldots, w_k) \in \text{cyc}(\mu^1) \times \ldots \text{cyc}(\mu^k) \mid w_1 \ldots w_k = 1\} \to \text{Cov}_d(\mu^1, \ldots, \mu^k).$$

Since for each $f \in \text{Cov}_d(\mu^1, \ldots, \mu^k)$, $|\psi^{-1}(f)| = d!/|\text{Aut}(f)|$, it follows that

$$H(d; \mu^1, \ldots, \mu^k) = \frac{1}{d!}\#\{(w_1, \ldots, w_k) \in \text{cyc}(\mu^1) \times \ldots \text{cyc}(\mu^k) \mid w_1 \ldots w_k = 1\}.$$

□

Remark. — The cover corresponding to $w_1, \ldots w_k$ is connected if and only if theses permutations generate a transitive subgroup of S_d.

— The counting of covering of surfaces of higher genus is also linked to the counting of permutation factorization. Let $\text{Cov}_d^g(\mu^1, \ldots, \mu^k)$ be the isomorphism classes of coverings of degree d of a surface S of genus g ramified at exactly k points with profiles $\mu^1, \ldots, \mu^k \vdash d$. Then

$$\begin{aligned}
&\sum_{f\in \text{Cov}_d^g(\mu^1,\ldots,\mu^k)} \frac{1}{|\text{Aut}(f)|} \\
=&\frac{1}{d!}\#\{(a_1, \ldots, a_g, b_1, \ldots, b_g, w_1, \ldots, w_k) \in S_d^{2g} \times \text{cyc}(\mu^1) \times \ldots \text{cyc}(\mu^k) \mid \\
&\qquad [a_1, b_1] \ldots [a_g, b_g]w_1 \ldots w_k = 1\} \\
=&(d!)^{2g-2}|\text{cyc}(\mu_1)| \ldots |\text{cyc}(\mu_k)| \sum_\lambda \frac{\chi^\lambda_{\mu^1} \cdots \chi^\lambda_{\mu^k}}{(\dim \lambda)^{k+2g-2}}.
\end{aligned} \tag{3.14}$$

There are still at least two other ways to view and compute Hurwitz numbers. The first one is about *maps* on surfaces (see Def.1.2.1). For a ramified covering $\pi : X \to \mathbb{P}^1$, we can associate two maps on X. Let $y_1, \ldots, y_k \in \mathbb{P}^1$ be the ramification points and an arbitrary point $y \in \mathbb{P}^1 - \{y_1, \ldots, y_k\}$. Let P be a polygon whose vertices are $y_1, \ldots, y_k$. Then $\pi^{-1}(P)$ is a map on X. Alternatively, let $St(y, y_1, \ldots, y_k)$ a "star" connecting y with $y_1, \ldots, y_k$. Then $\pi^{-1}(St)$ is a also map on X. With careful definitions, we can interpret Hurwitz numbers as numbers of maps satisfying certain conditions. See [59] for a gentle introduction, and [28, 81] for newer results.

Another way to interpret Hurwitz numbers is via *tropical graphs* [21]. The advantage of the graph-theoretic approaches is that one can sometimes obtain elementary proofs via explicit bijections. Furthermore, these graphs and bijections are interesting subjects of study by themselves.

3.5 Hurwitz numbers and integrable hierarchies

The first connection between Hurwitz numbers and integrable equations was made explicit by Okounkov [72], although the general connection between enumeration problems and integrable equations had been known before. More specifically, Okounkov proved the following theorem about the double Hurwitz numbers. In our notation, they are $H\left(d;\mu,\nu,\underbrace{(21^{d-2}),\dots,(21^{d-2})}_{k}\right)$.

Theorem 3.5.1. *Let $q, u, p_1, p_2, \dots, p'_1, p'_2, \dots$ be variables. The generating function*

$$H(\mathbf{p},\mathbf{p}',u,q) := \sum_{d,k,\mu,\nu} q^d u^k p_\mu p'_\nu H\left(d;\mu,\nu,\underbrace{(21^{d-2}),\dots,(21^{d-2})}_{k}\right)/k!$$

is a tau function of the Toda lattice hiearchy (where the "time" variables are $p_1, p_2, \dots, p'_1, p'_2, \dots$). The sum runs over all $d, k \in \mathbb{Z}_+$, and μ, ν partitions of d such that

$$g = 1 + \frac{k - l(\mu) - l(\nu)}{2} \in \mathbb{Z}_+.$$

As usual $p_\mu := p_{\mu_1} p_{\mu_2} \dots$.[2]

A particular case of this result is that the generating function of single (or simple) Hurwitz numbers is a tau function of the Kadomtsev-Petviashvili (KP) hierarchy (which is a reduction of the Toda lattice hierarchy). To simplify the exposition, I will restrict myself to this special case. The Toda lattice hierarchy is in fact defined in the same framework, with slightly more complicated details. Further developments concerning the connection between Hurwitz numbers and tau functions can be found for example in the works of Alexandrov, Guay-Paquet, Harnad, Natanzon, Orlov among others [2,38,43,44,69]. The basics of KP integrable hierarchy are recalled in App.A.

Theorem 3.5.2. *Let $u, p_1, p_2, \dots$ be variables. The generating function*

$$H(u;p_1,p_2,\dots) := \sum_{k=0}^{\infty}\sum_{d=0}^{\infty}\sum_{\mu\vdash d} H\left(d;\mu,\underbrace{(21^{d-2}),\dots,(21^{d-2})}_{k}\right) p_{\mu_1}p_{\mu_2}\cdots\frac{u^k}{k!}$$

is a KP τ-function.

Proof. Denote for short the central element $C_{(21^{d-2})}$ by C_2. Applying Prop. 3.2.3 for the element $a = C_2^k$, we obtain

$$\begin{aligned} H(u;p_1,p_2,\dots) &= \sum_{d\geq 0}\sum_{\mu\vdash d}\frac{1}{d!}p_{\mu_1}p_{\mu_2}\cdots\left[\frac{C_\mu}{|\mathrm{cyc}(\mu)|}\right]e^{uC_2} \\ &= \sum_{\mu} e^{g_2(\mu)u}\frac{\dim(\mu)}{|\mu|!}s_\lambda(p) \end{aligned}$$

2. Do not confuse p_μ for power sums, although there is eventually a reason to use this notation. In this theorem, just consider p_i as variables.

where $g_2(\mu)$ is a constant such that $C_2F_\mu = g_2(\mu)F_\mu$. Since $C_2 = X_2 + \cdots + X_d$, Prop. 3.2.4 implies that

$$g_2(\mu) = \sum_{s\in\mu} c(s) = \frac{1}{2}\sum_{i=1}^{l(\mu)}\left[\left(\mu_i - i + \frac{1}{2}\right)^2 - \left(-i + \frac{1}{2}\right)^2\right].$$

Thus the generating function $H(u; p_1, p_2, \dots)$ belongs to the Orlov-Shcherbin family described in Thm.A.0.1 with $y_i = e^{ui}$. □

Remark. The function g_2 is a shifted power sum that will be introduced in a subsequent section. We also have

$$g_2(\lambda) = \frac{d(d-1)}{2}\hat{\chi}^\lambda_{(2^1 1^{d-2})}.$$

3.6 Connection with moduli spaces of curves

The celebrated Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula connects single Hurwitz numbers with intersection numbers on the moduli space of stable curves

Theorem 3.6.1. *[29] For any partition β of d whose length is n, we have*

$$H(g,\beta) := H\left(d; \beta, \underbrace{(21^{d-2}), \dots, (21^{d-2})}_{k}\right) = C(g,\beta)\int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \ldots + (-1)^g\lambda_g}{(1-\beta_1\psi_1)\ldots(1-\beta_n\psi_n)}, \tag{3.15}$$

where

$$C(g,\beta) = k!\prod_{i=1}^{n}\frac{\beta_i^{\beta_i}}{\beta_i!}.$$

Here, the parameters are connected by the Riemann-Hurwitz formula

$$k = 2g - 2 + d + n.$$

The space $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of genus g with n marked points. On this space, λ_i is a certain cohomology class of dimension i, and ψ_i is a certain cohomology class of dimension 1.

The integral on the right hand side of Eq.(3.15) are certain intersection numbers on this moduli space. The precise definitions are not needed for us. The reader is invited to consult the book [59] for a excellent exposition of this important result.

In other words, $P^g(\beta) := H(g,\beta)/C(g,\beta)$ is polynomial in $\beta_1, \dots, \beta_n$ and the (linear) Hodge integrals are given by:

$$\langle\tau_{b_1}\ldots\tau_{b_n}\lambda_k\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{b_1}\ldots\psi_n^{b_n}\lambda_k = (-1)^k\left[\beta_1^{b_1}\ldots\beta_n^{b_n}\right]P^g(\beta). \tag{3.16}$$

Another ELSV type formula has been found for the so-called orbifold Hurwitz numbers, i.e, double Hurwitz numbers $H\left(d;\alpha,\beta,\underbrace{(21^{d-2}),\dots,(21^{d-2})}_{k}\right)$ with $\alpha=(a^m)$ by Johnson, Pandharipande and Tseng [51]. And more recently, some special cases of the orbifold version of Zvonkine's r-ELSV formula are proved by Borot and collaborators in [16], which are based on the anterior works of Shadrin and collaborators (cited in the preprint). The r-spin Hurwitz numbers studied in this preprint are the single version of Hurwitz numbers with completed cycles, whose double version will be the object of my study in later sections. It is an important and challenging problem to find other ELSV type formulas.

To widen the perspective and connect with the first chapter, it is worth mentioning that the intersection numbers can be calculated via a matrix integral. This is a classic result of Kontsevich [57]. Again, an excellent exposition is available in the book [59]. For the sake of completeness, let us state Kontsevich's theorem. Let Λ be a diagonal $N\times N$ matrix with positive entries $\Lambda_1,\dots,\Lambda_N$ on the diagonal. Consider the *Konsevich model* given by the following integral

$$K := \log\left[\frac{1}{C_{\Lambda,N}}\int_{\mathcal{H}_N} e^{\frac{i}{6}\mathrm{Tr}M^3-\frac{1}{2}\mathrm{Tr}M^2\Lambda}dM\right], \tag{3.17}$$

where

$$C_{\Lambda,N} := \int_{\mathcal{H}_N} e^{-\frac{1}{2}\mathrm{Tr}M^2\Lambda}dM. \tag{3.18}$$

It is obvious that K is a symmetric function of $\Lambda_1,\dots,\Lambda_N$. We will regard it as a function in a different set of variables. Let $t_0,t_1,\dots$ be the following variables

$$t_j := -(2j-1)!!\mathrm{Tr}\left(\Lambda^{-2j-1}\right).$$

Kontsevich proved the following

Theorem 3.6.2. *The function K is a formal power series in the variables $t_0,t_1,\dots$ with rational coefficients and a τ- function of the KdV hierarchy (which is a reduction of the KP hierarchy). Furthermore*

$$\left[\frac{t_0^{b_0}\dots t_s^{b_s}}{b_0!\dots b_s!}\right]K(t_0,t_1,\dots) = \langle\tau_0^{b_0}\dots\tau_s^{b_s}\rangle. \tag{3.19}$$

3.7 Hurwitz numbers, matrix models, and the topological recursion

Single Hurwitz numbers can be computed from a matrix integral. Let $u,t,p_1,p_2,\dots$ be variables, consider the following generating function for the disconnected single Hurwitz numbers $H(g,\mu)$:

$$Z(\mathbf{p},u,t) := \sum_{g=0}^{\infty}\sum_{\mu} H(g,\mu)p_\mu u^{2g-2}\frac{t^{|\mu|}}{(2g-2+|\mu|+l(\mu))!}. \tag{3.20}$$

The generating function for the connected numbers is

$$F(\mathbf{p}, u, t) := \sum_{g=0}^{\infty} \sum_{\mu} H^{\star}(g, \mu) p_{\mu} u^{2g-2} \frac{t^{|\mu|}}{(2g-2+|\mu|+l(\mu))!} = \log Z(\mathbf{p}, u, t). \quad (3.21)$$

Borot, Eynard, Mulase and Safnuk [13] proved the following theorem

Theorem 3.7.1. *[13] Let N be a positive integer and $V : \mathbb{C} \to \mathbb{C}$ be the function*

$$V(x) = -\frac{x^2}{2} + u(N - 1/2)x + (\log(u/t) + i\pi)\, x - u \log\left(\Gamma(-x/u)\right) + C_t,$$

with

$$C_t = -\frac{1}{3}u^2 \left(N^2 - \frac{3}{2}N + 2 \right) + \frac{1}{2}u(N-1)\log\left(u/t\right).$$

Then we have the following equality

$$Z(\mathbf{p}, u, t) = \frac{u^{-N^2}}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \int_{\mathcal{H}_N} e^{-\frac{1}{u}\mathrm{Tr}(V(M) - M\mathbf{R})} dM,$$

where $\mathbf{v} = (v_1, \ldots, v_N)$ is a tuple of N parameters such that

$$p_k = u \sum_{i=1}^{N} v_i^k,$$

and, $\mathbf{R} := diag(\log v_1, \ldots, \log v_N)$.

Finally, Hurwitz numbers have been shown to satisfy the topological recursion by Eynard, Mulase and Safnuk [32]. The proof is based on the cut-and-joint equations. To state this property, we need to use yet another generating function. Let $t, x_1, x_2, \ldots$ be variables. For each $n \in \mathbb{N}$, define

$$H^{(g)}(x_1, \ldots, x_n) := \sum_{l(\mu)=n} t^{|\mu|} \frac{\prod_{i=1}^{n} \mu_i M_{\mu}(x_1, \ldots, x_n)}{(2g-2+|\mu|+n)!} H(q, \mu),$$

where $M_{\mu}(\mathbf{x}) := \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{\sigma(i)}^{\mu_i}$ are the un-normalized symmetric monomials.

Theorem 3.7.2. *Let $\mathcal{E} = (\mathcal{L}, x, y)$ be the following Lambert spectral curve $\mathcal{L} = \{(x, y) \in \mathbb{C}^2 | ye^{-y} = te^{x}\}$, $x(z) = -z + \log(z/t)$ and $y(z) = z$. Denote by $\omega_n^{(g)}(z_1, \ldots, z_n)$ the symmetric differential forms defined by the topological recursion as in Def.D.0.4 with the initial data being the Lambert spectral curve. Then the following equations hold true*

$$H^{(g)}(v_1, \ldots, v_n) = \frac{\omega_n^{(g)}(z_1, \ldots, z_n)}{dx(z_1) \ldots dx(z_n)},$$

where the variables v_i are such that $v_i = e^{x(z_i)}$.

3.8 Double Hurwitz numbers with completed cycles

In this and the remaining sections, I describe my own results concerning double Hurwitz numbers with completed cycles [70]. First, I follow closely the exposition in [82] to give an algebraic definition of these numbers.

3.8.1 Shifted symmetric functions

Let $\mathbb{Q}[x_1, \ldots, x_d]$ be the algebra of d-variable polynomials over $\mathbb{Q}$. The shifted action of the symmetric group S_d on this algebra is defined by:

$$\sigma(f(x_1 - 1, \ldots, x_d - d)) := f(x_{\sigma(1)} - \sigma(1), \ldots, x_{\sigma(d)} - \sigma(d)) \tag{3.22}$$

for $\sigma \in S_d$ and for any polynomial written in the variables $x_i - i$.

Example 3.8.1. $\sigma(x_1^2 x_2) = (x_{\sigma(1)} - \sigma(1) + 1)^2 (x_{\sigma(2)} - \sigma(2) + 2)$.

Denote by $\mathbb{Q}[x_1, \ldots, x_d]^\star$ the sub-algebra of polynomials which are invariant under this action. It is isomorphic with the usual algebra of symmetric polynomials. Define the algebra of shifted symmetric functions as the projective limit

$$\Lambda^\star := \lim_{\longleftarrow} \mathbb{Q}[x_1, \ldots, x_d]^\star,$$

where the projective limit is taken in the category of filtered algebras with respect to the homomorphism which sends the last variable to 0. Concretely, an element of this algebra is a sequence $f = \{f^{(d)}\}_{d \geq 1}$, $f^{(d)} \in \mathbb{Q}[x_1, \ldots, x_d]^\star$ such that the polynomials $f^{(d)}$ are of uniformly bounded degree and stable under the restriction, i.e $f^{(d+1)}|_{x_{d+1}=0} = f^{(d)}$.

3.8.2 Two bases of the algebra of shifted symmetric functions

Definition 3.8.1. For any positive integer k, define the corresponding shifted symmetric power sum:

$$p_k(x_1, x_2, \ldots) := \sum_{i=1}^{\infty} \left(\left(x_i - i + \frac{1}{2} \right)^k - \left(-i + \frac{1}{2} \right)^k \right). \tag{3.23}$$

In the following, we are only interested in evaluating these functions on partitions. That is, for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots.)$, we define $p_k(\lambda) := p_k(\lambda_1, \lambda_2, \ldots)$. As usual in symmetric function theory, for any partition μ, define $p_\mu = p_{\mu_1} p_{\mu_2} \ldots$

The functions $\{p_\mu, \mu \in \mathcal{P}\}$ form a basis of $\Lambda^\star$. Another important basis is given in the following proposition of Kerov and Olshanski [56].

Proposition 3.8.1. *The functions $\{f_\mu, \mu \in \mathcal{P}\}$ given in Eq.*(3.2) *are shifted symmetric functions, and form a basis of $\Lambda^\star$.*

3.8.3 Completed cycles

Consider the following linear isomorphism

$$\begin{aligned} \phi : \mathbf{Z} &\to \Lambda^\star \\ C_\mu &\mapsto f_\mu. \end{aligned} \tag{3.24}$$

Definition 3.8.2. For any partition μ, the *completed* μ-conjugacy class $\overline{C}_\mu$ is defined as

$$\overline{C}_\mu := \phi^{-1}(p_\mu)/\prod_{i\geq 1}\mu_i!.$$

Of special interest are the *completed* r*-cycles* $\overline{(r)} := \overline{C}_{(r)}, r \in \mathbb{N}$. Here, (r) is the 1-part partition of r.

Some first completed cycles are:

$$\begin{aligned} 0!\overline{(1)} &= (1) \\ 1!\overline{(2)} &= (2) \\ 2!\overline{(3)} &= (3) + (1,1) + \frac{1}{12}(1) \\ 3!\overline{(4)} &= (4) + 2(2,1) + \frac{5}{4}(2) \end{aligned}$$

3.8.4 Double Hurwitz numbers with completed cycles

Let α and β be two partitions of a positive integer d, whose lengths are m and n respectively. Let g, r and s be three non-negative integers such that $rs = 2g - 2 + m + n$.

Definition 3.8.3. *Disconnected double Hurwitz numbers with completed* $(r+1)$*-cycles* are defined by the following formula

$$H^{g,(r)}_{\alpha,\beta} := \frac{1}{\prod_{i\geq 1}\alpha_i \prod_{j\geq 1}\beta_j}\sum_{\lambda\vdash d}\chi^\lambda_\alpha\left(\frac{p_{r+1}(\lambda)}{(r+1)!}\right)^s \chi^\lambda_\beta. \tag{3.25}$$

We often omit the superscript (r) if it is fixed in advance. Since the completed 2-cycle is equal to the ordinary 2-cycle, the double Hurwitz numbers $H^{g,(1)}_{\alpha,\beta}$ are just the ordinary double Hurwitz numbers.

We are mostly interested in the dependence of $H^{g,(r)}_{\alpha,\beta}$ on (the parts of) α and β, given fixed $g, r, l(\alpha)$ and $l(\beta)$. The numbers obtained in the case $m = 1$, i.e. $\alpha = (d)$ are called one-part double numbers. In this case, the sum is simplified a lot, and we can get an explicit and compact formula for $r = 2$.

The double Hurwitz numbers with completed cycles are the answers for the following factorization counting problem. We just need a simple adaptation of what Shadrin, Spitz and Zvonkine did for simple Hurwitz numbers with completed cycles in [83, Section 2.2]. Define a (g, r, α, β)*-factorization* $\mathbf{fac}(g, r, \alpha, \beta)$ as a factorization in S_d of the following form:

$$h_1 \dots h_s g_1 g_2 = 1, \tag{3.26}$$

where $rs = 2g - 2 + l(\alpha) + l(\beta)$, $g \in \mathbb{Z}_+$, $g_1 \in \text{cyc}(\alpha)$, $g_2 \in \text{cyc}(\beta)$, and each $h_i \in S_d$ appears in $\overline{(r+1)}$ with a coefficient $c_i \neq 0$. The weight of this factorization is defined as

$$w(\mathbf{fac}) := \prod_{i=1}^{s} c_i.$$

Proposition 3.8.2. *We have the following equality:*

$$\sum_{\textit{\textbf{fac}} \in \{(g,r,\alpha,\beta)-\textit{factorizations}\}} w(\textit{\textbf{fac}}) = \frac{d!}{|Aut(\alpha)||Aut(\beta)|} H^{g,(r)}_{\alpha,\beta}. \tag{3.27}$$

Proof. Since $\{C_\lambda | \lambda \vdash d\}$ form a basis of $Z\mathbb{Q}S_d$, we can write:

$$\overline{(r+1)}^s C_\alpha C_\beta = \sum_{\lambda \vdash d} a_\lambda C_\lambda. \tag{3.28}$$

By definition,

$$\sum_{\mathbf{fac} \in \{(g,r,\alpha,\beta)-\textit{factorizations}\}} w(\mathbf{fac}) = \left[C_{(1^d)}\right] \underbrace{\overline{(r+1)} \dots \overline{(r+1)}}_{s} C_\alpha C_\beta,$$

where the right hand side means the coefficient of $C_{(1^d)} = \text{Id} \equiv 1$ in the product following it. Consider the left regular representation of $\mathbb{Q}S_d$, i.e. the action of $\mathbb{Q}S_d$ on itself by multiplication on the left. A main theorem of the representation theory of the symmetric groups [59, Thm.A.1.5] gives us the decomposition of this representation into irreducible ones:

$$\mathbb{Q}S_d = \bigoplus_{\lambda \vdash d} \dim(\lambda) V_\lambda.$$

Here, $\dim(\lambda)$ is the dimension of the irreducible representation λ of S_d (as we defined in the section 3.8.2) and $\dim V_\lambda = \dim(\lambda)$. The action of an element $B \in Z\mathbb{Q}S_d$ in V_λ is multiplication by a number $L_\lambda(B)$, i.e. the matrix $L(B)$ representing B is diagonal:

$$L(B) = \text{diag}_{\lambda \vdash d} \left(\underbrace{L_\lambda(B)}_{\dim(\lambda)^2 \text{times}} \right).$$

In particular, we can compute:

$$L_\lambda(C_\alpha) = f_\alpha(\lambda) = |\text{cyc}(\alpha)| \frac{\chi^\lambda_\alpha}{\dim(\lambda)},$$

$$L_\lambda(\overline{(r+1)}) = \frac{1}{(r+1)!} p_{r+1}(\lambda) = \frac{1}{(r+1)!} \sum_{i=1}^{l(\lambda)} \left(\left(\lambda_i - i + \frac{1}{2}\right)^{r+1} - \left(-i + \frac{1}{2}\right)^{r+1} \right).$$

Now let us take the trace of the action in the left regular representation of the two sides of the equation (3.28). The right hand side gives $d! a_{(1^d)}$ since

$$\text{Tr} L(g) = \begin{cases} d! & \text{if } g = 1, \\ 0 & \text{otherwise,} \end{cases}$$

while the left hand side gives

$$\mathrm{Tr}\left(L(\overline{(r+1)})^s L(C_\alpha)L(C_\beta)\right) = \sum_{\lambda\vdash d} \dim(\lambda)^2 L_\lambda(C_\alpha)L_\lambda(C_\beta)L_\lambda(\overline{(r+1)})^s.$$

Finally, we get:

$$\begin{aligned}
[C_{(1^d)}]\underbrace{\overline{(r+1)}\dots\overline{(r+1)}}_{s}C_\alpha C_\beta &= \frac{1}{d!}\sum_{\lambda\vdash d}\dim(\lambda)^2 L_\lambda(C_\alpha)L_\lambda(C_\beta)L_\lambda(\overline{(r+1)})^s \\
&= \frac{|\mathrm{Per}(\alpha)||\mathrm{Per}(\beta)|}{d!}\sum_{\lambda\vdash d}\chi^\lambda_\alpha\chi^\lambda_\beta\left(\frac{p_{r+1}(\lambda)}{(r+1)!}\right)^s \\
&= \frac{d!}{|\mathrm{Aut}(\alpha)||\mathrm{Aut}(\beta)|}H^{g,(r)}_{\alpha,\beta}.
\end{aligned}$$

In the last line, we used:

$$|\mathrm{cyc}(\alpha)| = \frac{d!}{|\mathrm{Aut}(\alpha)|\prod \alpha_i}.$$

□

3.9 One-part numbers with completed 3-cycles

This section contains my main contribution, an explicit formula for one-part double Hurwitz numbers with completed 3-cycles, i.e. the case $r = 2$. Let β be a partition of d of odd length n (the constraint $rs = 2g - 2 + l(\alpha) + l(\beta)$ forces so). Let $s = g + \frac{n-1}{2}$. We write β in three ways, each of which is convenient in each specific context.

$$(\beta_1, \beta_2, \dots) = (1^{n_1}2^{n_2}\dots) = (\ell^{n_\ell}\dots q^{n_q})\,. \tag{3.29}$$

Here, ℓ and q are the smallest and greatest numbers appearing in β. If a number i does not appear in β, we have $n_i = 0$. For later use in the proofs, introduce also $c_i = n_i$ for $i \geq 2$ and $c_1 = n_1 - 1$. We have $\sum_i c_i = n - 1$ and $\sum ic_i = d - 1$.

The main theorem that I obtained is

Theorem 3.9.1. *[70] Given $g \geq 0$, $d > 0$, let β be a partition of odd length of d and s be an integer such that $2s = 2g - 1 + l(\beta)$. Then we have:*

$$H^{g,(2)}_{(d),\beta} = \frac{s!d^{s-1}}{2^s}\sum_{h=0}^{g}\frac{(2s-2h)!}{h!(s-h)!12^h}d^{2h}\left[z^{2(g-h)}\right]\prod_{i\geq 1}\left(\frac{\sinh(iz/2)}{iz/2}\right)^{c_i} \tag{3.30}$$

$$= \frac{s!d^{s-1}}{2^{s+2g}}\sum_{h=0}^{g}\frac{(2s-2h)!}{h!(s-h)!3^h}d^{2h}\sum_{\lambda\vdash(g-h)}\frac{\xi_{2\lambda}S_{2\lambda}}{|Aut\lambda|}. \tag{3.31}$$

There is strong similarity with the case of ordinary one-part double Hurwitz numbers obtained by Goulden, Jackson and Vakil [37, Thm.3.1]. Their formula is

Theorem 3.9.2. *[37, Thm.3.1] For $g \geq 0$, $\beta \vdash d$ and $s = 2g - 1 + l(\beta)$,*

$$\begin{aligned} H^{g,(1)}_{(d),\beta} &= s! d^{s-1} \left[z^{2g}\right] \prod_{i\geq 1} \left(\frac{\sinh(iz/2)}{iz/2}\right)^{c_i} \\ &= \frac{s! d^{s-1}}{2^{2g}} \sum_{\lambda \vdash g} \frac{\xi_{2\lambda} S_{2\lambda}}{|Aut\lambda|}. \end{aligned}$$

For the proof of Thm.3.9.1, we need the following preparatory results. First, one computes that:

$$p_3\left((d-k, 1^k)\right) = \left(d - k - \frac{1}{2}\right)^3 - \left(-k - \frac{1}{2}\right)^3 = 3d\left(\left(k - \frac{d-1}{2}\right)^2 + \frac{d^2}{12}\right). \tag{3.32}$$

Remark. The fact that $p_3\left((d-k, 1^k)\right)$ has the form $a(k+b)^2 + c$, where a, b, c do not depend on k turns out to be crucial for my method. Unfortunately, $p_{r+1}((d-k, 1^k))$ for $r \geq 3$ do not have the form $a(k+b)^r + c$, so I could not obtain a compact formula by the same strategy.

Lemma 3.9.3. *We have the following irreducible character evaluation:*

$$\begin{aligned} \chi_\beta^{(d-k,1^k)} &= (-1)^k \left[z^k\right] (1 + z + \ldots + z^{\ell-1})(1 - z^\ell)^{n_\ell - 1} \prod_{i \geq \ell+1} (1 - z^i)^{n_i} \\ &= (-1)^k \left[z^k\right] \prod_{i\geq 1} (1 - z^i)^{c_i} \\ &= (-1)^k \sum_{h=0}^{\ell-1} \sum_{j_\ell=0}^{n_\ell - 1} \sum_{j_{\ell+1}=0}^{n_{\ell+1}} \cdots \sum_{j_q=0}^{n_q} (-1)^{\sum_{i\geq l} j_i} \binom{n_\ell - 1}{j_\ell} \binom{n_{\ell+1}}{j_{\ell+1}} \cdots \binom{n_q}{j_q} \delta_{k, h + \sum_{i=\ell}^q i j_i}. \end{aligned} \tag{3.33}$$

Here, $\delta_{x,y} := 1$ if $x = y$, and 0 otherwise. This lemma is well known, and can be derived from the Murnaghan-Nakayama rule (see, for instance, [37, p.59]).

For $j \geq 1$, let $\xi_{2j} = [x^{2j}] \log(\sinh x/x)$ and

$$S_{2j} = \sum_{k\geq 1} k^{2j} c_k = -1 + \sum_{k \geq 1} k^{2j} n_k = -1 + \sum_{k\geq 1} {\beta_k}^{2j},$$

i.e. S_{2j} is a power sum for the partition, shifted by 1. For a partition λ , let $\xi_\lambda = \xi_{\lambda_1} \xi_{\lambda_2} \ldots$ and $S_\lambda = S_{\lambda_1} S_{\lambda_2} \ldots$ and $2\lambda = (2\lambda_1, 2\lambda_2, \ldots)$.

Lemma 3.9.4. *The following formula holds true:*

$$\left[z^{2k}\right] \prod_{i\geq 1} \left(\frac{\sinh(iz/2)}{iz/2}\right)^{c_i} = 2^{-2k} \sum_{\lambda \vdash k} \frac{\xi_{2\lambda} S_{2\lambda}}{|Aut\lambda|}. \tag{3.34}$$

Proof. We have

$$\prod_{i\geq 1}\left(\frac{\sinh(ix)}{ix}\right)^{c_i} = \exp\left(\sum_{i\geq 1} c_i \sum_{j\geq 1} \xi_{2j} i^{2j} x^{2j}\right) = \exp\left(\sum_{j\geq 1} \xi_{2j} S_{2j} x^{2j}\right) = \sum_{\lambda} \frac{\xi_{2\lambda} S_{2\lambda}}{|\mathrm{Aut}\lambda|} x^{2|\lambda|}.$$

The proof is finished upon setting $x = z/2$. □

Lemma 3.9.5. *Let $S_p(k,x) := \sum_{h=0}^{k-1}(h+x)^p$. Then*

$$S_p(k,x) = \left[\frac{z^p}{p!}\right] e^{xz}\left(1 + e^z + \ldots + e^{(k-1)z}\right). \tag{3.35}$$

Proof. Indeed,

$$\sum_{p=0}^{\infty} S_p(k,x)\frac{z^p}{p!} = \sum_{h=0}^{k-1} e^{z(h+x)} = e^{zx}\left(1 + e^z + \ldots + e^{(k-1)z}\right).$$

□

Let us now prove Thm.3.9.1.

Proof. By definition, we have

$$H^{g,(2)}_{(d),\beta} = \frac{1}{d\prod \beta_j} \sum_{\lambda\vdash d} \chi^{\lambda}_{(d)} \left(\frac{p_3(\lambda)}{6}\right)^s \chi^{\lambda}_{\beta}.$$

It is well known that $\chi^{\lambda}_{(d)} = 0$ except for $\lambda = (d-k, 1^k), k = 0, \ldots, d-1$, in which case it is equal to $(-1)^k$. So

$$\begin{aligned} H^{g,(2)}_{(d),\beta} &= \frac{d^{s-1}}{2^s\prod \beta_j} \sum_{k=0}^{d-1} \left(\left(k - \frac{d-1}{2}\right)^2 + \frac{d^2}{12}\right)^s (-1)^k \chi^{(d-k,1^k)}_{\beta} \\ &= \frac{d^{s-1}}{2^s\prod \beta_j} \left[\frac{t^s}{s!}\right] \sum_{k=0}^{d-1} \exp\left\{t\left(\left(k - \frac{d-1}{2}\right)^2 + \frac{d^2}{12}\right)\right\} (-1)^k \chi^{(d-k,1^k)}_{\beta} \\ &= \frac{s!\,d^{s-1}}{2^s\prod \beta_j} [t^s] \exp\left(\frac{td^2}{12}\right) \sum_{k=0}^{d-1} \exp\left\{t\left(k - \frac{d-1}{2}\right)^2\right\} (-1)^k \chi^{(d-k,1^k)}_{\beta}. \end{aligned}$$

We first treat the sum separately:

$$\begin{aligned} A &= \sum_{k=0}^{d-1} \exp\left\{t\left(k - \frac{d-1}{2}\right)^2\right\} (-1)^k \chi^{(d-k,1^k)}_{\beta} \\ &= \sum_{h=0}^{\ell-1} \sum_{j_\ell=0}^{n_\ell-1} \sum_{j_{\ell+1}=0}^{n_{\ell+1}} \cdots \sum_{j_q=0}^{n_q} \exp\left\{t\left(h + \sum_{i\geq \ell} i j_i - \frac{d-1}{2}\right)^2\right\} \\ &\qquad \times (-1)^{\sum_{i\geq l} j_i} \binom{n_\ell - 1}{j_\ell}\binom{n_{\ell+1}}{j_{\ell+1}} \cdots \binom{n_q}{j_q} \quad \text{(Lem.3.9.3)}. \end{aligned}$$

. Now expand the exponential and sum over h first :

$$\begin{aligned}
A &= \sum_{j_\ell=0}^{n_l-1}\sum_{j_{\ell+1}=0}^{n_{\ell+1}}\dots\sum_{j_q=0}^{n_q}(-1)^{\sum_{i\geq\ell}j_i}\binom{n_\ell-1}{j_\ell}\binom{n_{\ell+1}}{j_{\ell+1}}\dots\binom{n_q}{j_q}\sum_{p=0}^{\infty}\frac{t^p}{p!}\sum_{h=0}^{\ell-1}\left(h+\sum_{i\geq\ell}ij_i-\frac{d-1}{2}\right)^{2p}\\
&= \sum_{p=0}^{\infty}\frac{(2p)!t^p}{p!}\left[z^{2p}\right]\left(1+e^z+\ldots+e^{(\ell-1)z}\right)e^{-\frac{(d-1)z}{2}}\times\\
&\times\sum_{j_\ell=0}^{n_\ell-1}\sum_{j_{\ell+1}=0}^{n_{\ell+1}}\dots\sum_{j_q=0}^{n_q}e^{z\sum_{i\geq\ell}ij_i}(-1)^{\sum_{i\geq\ell}j_i}\binom{n_\ell-1}{j_l}\binom{n_{\ell+1}}{j_{\ell+1}}\dots\binom{n_q}{j_q}\quad\text{(Lem.3.9.5)}.\\
&= \sum_{p=0}^{\infty}\frac{(2p)!t^p}{p!}\left[z^{2p}\right]e^{-\frac{(d-1)z}{2}}\left(1+e^z+\ldots+e^{(\ell-1)z}\right)\left(1-e^{\ell z}\right)^{n_\ell-1}\prod_{i\geq\ell+1}\left(1-e^{iz}\right)^{n_i}\\
&= \sum_{p=0}^{\infty}\frac{(2p)!t^p}{p!}\left[z^{2p}\right]e^{-\frac{(d-1)z}{2}}\prod_{i\geq1}\left(1-e^{iz}\right)^{c_i}.
\end{aligned}$$

Finally we get:

$$\begin{aligned}
H^{g,(2)}_{(d),\beta} &= \frac{s!d^{s-1}}{2^s\prod\beta_j}\left[t^s\right]\exp\left(\frac{td^2}{12}\right)\sum_{p=0}^{\infty}\frac{(2p)!t^p}{p!}\left[z^{2p}\right]e^{-\frac{(d-1)z}{2}}\prod_{i\geq1}\left(1-e^{iz}\right)^{c_i}\\
&= \frac{s!d^{s-1}}{2^s}\left[t^s\right]\exp\left(\frac{td^2}{12}\right)\sum_{p=0}^{\infty}\frac{(2p)!t^p}{p!}\left[z^{2p-n+1}\right]\prod_{i\geq1}\left(\frac{\sinh(iz/2)}{iz/2}\right)^{c_i}\\
&= \frac{s!d^{s-1}}{2^s}\sum_{h=0}^{s}\frac{(2s-2h)!}{h!(s-h)!12^h}d^{2h}\left[z^{2s-2h-n+1}\right]\prod_{i\geq1}\left(\frac{\sinh(iz/2)}{iz/2}\right)^{c_i}.
\end{aligned}$$

To pass from the first line to the second, we write $1-e^{iz}=-2e^{iz/2}\sinh(iz/2)$ and use $\sum_i c_i=n-1$, $\sum_i ic_i=d-1$ and $\prod_i i^{c_i}=\prod\beta_j$. There is also the factor $(-1)^{\sum c_i}=(-1)^{n-1}=1$ since n is odd.

Note that $2s=2g-1+n$, so we are taking the coefficient of $z^{2(g-h)}$. Because the lowest degree of the series in z is 0, the summing index h actually runs from 0 to g. Finally, we get the first claimed equality:

$$H^{g,(2)}_{(d),\beta}=\frac{s!d^{s-1}}{2^s}\sum_{h=0}^{g}\frac{(2s-2h)!}{h!(s-h)!12^h}d^{2h}\left[z^{2(g-h)}\right]\prod_{i\geq1}\left(\frac{\sinh(iz/2)}{iz/2}\right)^{c_i}.$$

The second equation in the theorem follows from Lem.3.9.4,. □

3.10 Some corollaries

As it was done in [37], I will prove some fairly important implications of Thm.3.9.1.

3.10.1 Strong Polynomiality

Our formula gives immediately the strong polynomialty of 1-part double Hurwitz numbers with completed 3-cycles. In fact, double Hurwitz numbers with completed cycles of any size satisfy the strong piecewise polynomiality, i.e. they are piecewise polynomial with the highest and lowest orders respectively $(r+1)s+1-m-n$ and $(r+1)s+1-m-n-2g$. This is proved in [82]. For one-part numbers, piecewise polynomiality becomes polynomiality. Our formula can be viewed as an illustration of this fact through an explicitly computable case.

Corollary 3.10.1. *$H^{g,(2)}_{(d),\beta}$, for fixed g and n, is a polynomial of the parts of β and satisfies the strong polynomiality property, i.e. it is polynomial in $\beta_1, \beta_2, \ldots$ with highest and lowest degrees respectively $3g + \frac{n-3}{2}$ and $g + \frac{n-3}{2}$.*

The polynomial is divisible by d^{s-1}, but unlike the case of ordinary double Hurwitz numbers, $2^s H^{g,(2)}_{(d),\beta}/s!d^{s-1}$ depends on the number of parts of β for $g \geq 1$. See the comment after [37, Corollary 3.2].

3.10.2 Connection with intersection theory on moduli spaces of curves and "the λ_g theorem"

In [82], the authors conjecture that for every $r \geq 1$, there exist moduli spaces $X^{(r)}_{g,n}$ of complex dimension $2g(r+1)+n-1$ such that we have the following ELSV formula:

$$H^{g,(r)}_{(d),\beta} = \frac{s!}{d} \int_{X_{g,n}} \frac{1 - \Lambda_2 + \Lambda_4 - \ldots + (-1)^g \Lambda_{2g}}{(1-\beta_1\Psi_1)\ldots(1-\beta_n\Psi_n)}, \tag{3.36}$$

where we fix the degrees of the rational cohomology classes $\Lambda_{2k} \in H^{4rk}\left(X^{(r)}_{g,n}\right)$ and $\Psi_i \in H^{2r}\left(X^{(r)}_{g,n}\right)$.

A similar conjecture was previously made by Goulden, Jackson and Vakil [37] for ordinary double Hurwitz numbers, i.e. the case $r = 1$. To support their conjecture, they made a thorough combinatorial study and found many similarity between "combinatorial Hodge integrals" and the "real" ones such as those defined by Eq.(3.16) .

Following them, let us define the *combinatorial Hodge integrals* for $b_1, \ldots, b_n \geq 0$ and $0 \leq k \leq g$:

$$\begin{aligned} \langle\langle \tau_{b_1} \ldots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g &:= (-1)^k \left[\beta_1^{b_1} \ldots \beta_n^{b_n}\right] \left(d \frac{H^{g,(2)}_{(d),\beta}}{s!} \right) \\ &= (-1)^k \left[\beta_1^{b_1} \ldots \beta_n^{b_n}\right] \left(d \frac{H^{g,(2)}_{(d),\beta}}{\left(g + \frac{n-1}{2}\right)!} \right). \end{aligned} \tag{3.37}$$

We will not keep the superscript (2) to save space. This "intersection" number vanishes unless $b_1 + \ldots + b_n + 2k = 3g + \frac{n-1}{2}$. The order of $\langle\langle \tau_{b_1} \ldots \tau_{b_n} \Lambda_{2k} \rangle\rangle_g$ is defined to be $b_1 + \ldots + b_n$.

We are going to evaluate the lowest order terms, i.e. the terms with $k = g$. In [35], Faber and Pandharipande proved the so-called λ_g conjecture (which is also a consequence of the unresolved Virasoro conjecture):

$$\langle \tau_{b_1} \dots \tau_{b_n} \lambda_g \rangle_g = \mathbf{c}_g \binom{2g-3+n}{b_1, \dots, b_n}. \tag{3.38}$$

Recall the left hand side notation in Eq.(3.16). By computing $\langle \tau^{2g-2} \lambda_g \rangle_g$, they found

$$\mathbf{c}_g = \frac{2^{2g-1}-1}{2^{2g-1}(2g)!} |B_{2g}|,$$

where B_{2g} is a Bernoulli number ($B_0 = 1, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, \dots$). In analogy with Eq.(3.38), the following combinatorial version is proved in [37, Prop.3.12]:

$$\langle\langle \tau_{b_1} \dots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g^{r=1} = \mathbf{c}_g \binom{2g-3+n}{b_1, \dots, b_n}. \tag{3.39}$$

Their symbol $\langle\langle . \rangle\rangle_g^{r=1}$ is defined in a similar way as in Eq (3.37), with a different normalisation; for precise details, see [37, Eq.25]. It is quite remarkable that the same constant $\mathbf{c}_g$ appears in both cases.

Here, thank to Thm.3.9.1, we can also easily evaluate $\langle\langle \tau_{b_1} \dots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g$.

Theorem 3.10.2. *For $b_1 + \dots + b_n = g + \frac{n-1}{2}$, the lowest combinatorial Hodge integral is given by:*

$$\langle\langle \tau_{b_1} \dots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g = \binom{g + \frac{n-1}{2}}{b_1, \dots, b_n} \mathbf{C}_{g,n}, \tag{3.40}$$

with

$$\mathbf{C}_{g,n} = \frac{(2g+n-1)!\,(2^{2g-1}-1)}{(2g)!\,\big(g+\frac{n-1}{2}\big)!\,2^{3g+\frac{n-3}{2}}} |B_{2g}|. \tag{3.41}$$

Proof. To compute $\langle\langle \tau_{b_1} \dots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g$, we have to do two steps. The first consists of extracting the lowest term in the polynomial $dH^{g,(2)}_{(d),\beta}/s!$, i.e. the $h = 0$ term in the sum (3.30), and extracting the constant term of $S_{2\lambda}$, which is $(-1)^{l(\lambda)}$, in the sum over all partitions λ of g. The result is:

$$\frac{(2s)!d^s}{s!2^{s+2g}} \sum_{\lambda \vdash g} \frac{(-1)^{l(\lambda)} \xi_{2\lambda}}{|\mathrm{Aut}\lambda|} = \frac{(2g+n-1)!d^{g+\frac{n-1}{2}}}{\big(g+\frac{n-1}{2}\big)!2^{3g+\frac{n-1}{2}}} \sum_{\lambda \vdash g} \frac{(-1)^{l(\lambda)} \xi_{2\lambda}}{|\mathrm{Aut}\lambda|}.$$

Then the second step is computing the coefficient of $\beta_1^{b_1} \dots \beta_n^{b_n}$ of this expression. The final result is

$$\langle\langle \tau_{b_1} \dots \tau_{b_n} \Lambda_{2g} \rangle\rangle_g = (-1)^g \binom{g + \frac{n-1}{2}}{b_1, \dots, b_n} \frac{(2g+n-1)!}{\big(g+\frac{n-1}{2}\big)!2^{3g+\frac{n-1}{2}}} \sum_{\lambda \vdash g} \frac{(-1)^{l(\lambda)} \xi_{2\lambda}}{|\mathrm{Aut}\lambda|}. \tag{3.42}$$

On the other hand, using (3.30), we have:

$$\begin{aligned}\langle\langle\tau_g\Lambda_{2g}\rangle\rangle_g &= (-1)^g\,[d^g]\,\frac{dH_{(d),(d)}}{g!}\\ &= (-1)^g\,[d^g]\,\frac{d^g(2g)!}{2^g g!}\left[z^{2g}\right]\frac{z/2}{\sinh z/2}\frac{\sinh dz/2}{dz/2}\\ &= \frac{(2g)!}{g!2^g}(-1)^g\left[z^{2g}\right]\frac{z/2}{\sinh z/2}\\ &= \frac{2^{2g-1}-1}{g!2^{3g-1}}|B_{2g}|. \end{aligned}\tag{3.43}$$

Comparing (3.42) and (3.43), we obtain the explicit evaluation for the sum $\sum_{\lambda\vdash g}\frac{(-1)^{l(\lambda)}\xi_{2\lambda}}{|\mathrm{Aut}\lambda|}$ and get the desired claim. $\square$

We observe a strong similarity with the results quoted above. The main difference is the dependence on n of the factor $\mathbf{C}_{g,n}$. A geometric explanation would be of great interest.

3.10.3 Dilaton and string equations

Goulden, Jackson and Vakil proved that their combinatorial Hodge integrals for ordinary double Hurwitz numbers satisfy the string and dilation equations [37, Prop.3.10]. Here I prove that the lowest terms satisfy the (modified) string and dilaton equations for every genus g. The situation for higher terms is not clear to me.

Theorem 3.10.3. ***String equation:*** *For $g\geq 0$, $n\geq 1$, n odd, $b_1,\dots,b_n\geq 0$, $b_1+\dots+b_n=g+\frac{n+1}{2}$:*

$$\langle\langle\tau_0^2\tau_{b_1}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g=(2g+n)\sum_{i=1}^{n}\langle\langle\tau_{b_1}\dots\tau_{b_{i-1}}\tau_{b_i-1}\tau_{b_{i+1}}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g. \tag{3.44}$$

Dilaton equation: *For $g\geq 0$, $n\geq 1$, n odd, $b_1,\dots,b_n\geq 0$, $b_1+\dots+b_n=g+\frac{n-1}{2}$ (minus here is not a misprint):*

$$\langle\langle\tau_0\tau_1\tau_{b_1}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g=(2g+n)\left(g+\frac{n+1}{2}\right)\langle\langle\tau_{b_1}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g. \tag{3.45}$$

Here we assume that $\langle\langle.\rangle\rangle$=0 if there is some $\tau_{<0}$ inside the brackets.

Proof. For the string equation:

$$\begin{aligned}\langle\langle\tau_0^2\tau_{b_1}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g &= \binom{g+\frac{n+1}{2}}{b_1,\dots,b_n}\mathbf{C}_{g,n+1}\\ &= \frac{\left(g+\frac{n-1}{2}\right)!(b_1+\dots+b_n)}{b_1!\dots b_n!}\mathbf{C}_{g,n}\frac{(2g+n+1)(2g+n)}{2\left(g+\frac{n+1}{2}\right)}\\ &= (2g+n)\sum_{i=1}^{n}\langle\langle\tau_{b_1}\dots\tau_{b_{i-1}}\tau_{b_i-1}\tau_{b_{i+1}}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g.\end{aligned}$$

For the dilaton equation:

$$\begin{aligned}\langle\langle\tau_0\tau_1\tau_{b_1}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g &= \binom{g+\frac{n+1}{2}}{0,1,b_1,\dots,b_{2s+1}}\mathbf{C}_{g,n+1}\\ &= \binom{g+\frac{n-1}{2}}{b_1,\dots,b_n}\mathbf{C}_{g,n}(2g+n)\left(g+\frac{n+1}{2}\right)\\ &= (2g+n)\left(g+\frac{n+1}{2}\right)\langle\langle\tau_{b_1}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g.\end{aligned}$$

□

Finally, let us make the following remark concerning the Virasoro constraints. Consider the following generating function

$$F := \sum_{n\geq 1}\frac{1}{n!}\sum_{b_1,\dots,b_n\geq 0}\langle\langle\tau_{b_1}\dots\tau_{b_n}\Lambda_{2g}\rangle\rangle_g\frac{t_{b_1}\dots t_{b_n}}{(2g+n-2)!!} \tag{3.46}$$

Then the string and dilaton equations can be written as follows:

$$\left(-\frac{\partial^2}{\partial t_0^2}+\sum_{i=0}^{\infty}t_{i+1}\frac{\partial}{\partial t_i}\right)F := L_{-1}F = 0 \tag{3.47}$$

$$\left(-\frac{\partial^2}{\partial t_0\partial t_1}+1+\sum_{i=0}^{\infty}it_i\frac{\partial}{\partial t_i}\right)F := L_0F = 0 \tag{3.48}$$

It is easy to check that $[L_0, L_{-1}] = L_{-1}$. They look like two lowest Virasoro constraints. It would be interesting to investigate if we have higher Virasoro-like constraints as well. And of course, it would be of great interest to investigate string and dilaton equations for higher order integrals, i.e. for $\langle\tau_{b_1}\dots\tau_{b_n}\Lambda_{2k}\rangle$ with $k < g$.

3.10.4 Explicit formulae for top degree terms

Finally, let us show how to compute the top degree terms

$$\langle\langle\tau_{b_1}\dots\tau_{b_n}\rangle\rangle_g := \langle\langle\tau_{b_1}\dots\tau_{b_n}\Lambda_0\rangle\rangle_g.$$

They are sometimes called *Witten terms* because of the celebrated Witten's conjecture [90]. In order to state the result, we need some notations. For any partition λ, we have the expansion of p_λ in the monomial symmetric functions m_μ:

$$p_\lambda = \sum_{\mu\vdash|\lambda|}R_{\lambda\mu}m_\mu. \tag{3.49}$$

From the definition of p_λ and m_μ, one can see that $R_{\lambda\mu}$ is equal to the number of ordered partitions [3] $\pi = \big(A_1,\dots,A_{l(\mu)}\big)$ of the set $\{1,\dots,l(\lambda)\}$ such that for $1\leq j\leq l(\mu)$:

$$\mu_j = \sum_{i\in A_j}\lambda_i.$$

3. Do not confuse between a partition of a number and a partition of a set. A partition of a set S is a set of pairwise disjoint subsets $\{S_1, S_2, \dots\}$ of S such that $S = S_1\cup S_2\cup\dots$.

For $2j \leq b_1 + \ldots + b_n$, denote

$$D_{2j}(\vec{b}) := \{(a_1, \ldots, a_n), a_i \text{ even}, a_i \leq b_i, a_1 + \ldots + a_n = 2j\}.$$

For a vector $\vec{a}$, denote $P_{\vec{a}}$ the associated partition, i.e. the rearrangement of the components of $\vec{a}$ in non-decreasing order.

Theorem 3.10.4. *For $b_1, \ldots, b_n \geq 0$, $b_1 + \ldots + b_n = 3g + \frac{n-1}{2}$, we have:*

$$\begin{aligned}&\langle\langle \tau_{b_1} \ldots \tau_{b_n} \rangle\rangle_g \\ =&\frac{1}{2^{3g+\frac{n-1}{2}}} \sum_{h=0}^{g} \frac{(2s-2h)!}{h!(s-h)!3^h} \sum_{\lambda \vdash (g-h)} \sum_{\vec{a} \in D_{2g-2h}(\vec{b})} \frac{\xi_{2\lambda} R_{2\lambda, P_{\vec{a}}}}{|Aut\lambda|} \binom{g + \frac{n-1}{2} + 2h}{b_1 - a_1, \ldots, b_n - a_n}.\end{aligned} \tag{3.50}$$

Proof. To compute $\langle\langle \tau_{b_1} \ldots \tau_{b_n} \rangle\rangle_g$, we have to do two steps. First, we need to extract the highest degree term in $dH^g_{(d),\beta}/s!$. The result is:

$$\frac{d^s}{2^{s+2g}} \sum_{h=0}^{g} \frac{(2s-2h)!}{h!(s-h)!3^h} d^{2h} \sum_{\lambda \vdash (g-h)} \frac{\xi_{2\lambda} p_{2\lambda}}{|\mathrm{Aut}\lambda|}, \tag{3.51}$$

where $p_{2\lambda} = p_{2\lambda}(\beta_1, \beta_2, \ldots)$ is the power sum. Then we compute the coefficient of $\beta_1^{b_1} \ldots \beta_n^{b_n}$ of this expression to obtain $\langle\langle \tau_{b_1} \ldots \tau_{b_n} \rangle\rangle_g$.

Using the obvious fact that $[x_1^{a_1} x_2^{a_2} \ldots] \, m_\mu(x) = 1$ if $\mu = P_{\vec{a}}$ and 0 otherwise, we obtain the desired result. □

In particular, for $g = 1$, we have

Corollary 3.10.5. *For $n \geq 1$, $b_1, \ldots, b_n \geq 0$ and $b_1 + \ldots + b_n = \frac{n+5}{2}$:*

$$\langle\langle \tau_{b_1} \ldots \tau_{b_n} \rangle\rangle_1 = \frac{(n+1)!}{3 \times 2^{\frac{n+7}{2}}} \left[\frac{1}{n} \binom{\frac{n+5}{2}}{b_1, \ldots, b_n} + \sum_{i=1}^{n} \binom{\frac{n+1}{2}}{b_1, \ldots, b_{i-1}, b_i - 2, b_{i+1}, \ldots, b_n} \right], \tag{3.52}$$

where $\binom{\frac{n+1}{2}}{b_1, \ldots, b_{i-1}, b_i-2, b_{i+1}, \ldots, b_n} = 0$ if $b_i - 2 < 0$.

One can compare this formula with the following Hodge integrals over $\overline{\mathcal{M}}_{1,n}$ which can be found, for instance, in [59, Prop.4.6.11]:

Proposition 3.10.6. *For $d_1 + \ldots d_n = n$, we have:*

$$\langle \tau_{d_1} \ldots \tau_{d_n} \rangle_1 := \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} = \frac{1}{24} \binom{n}{d_1 \ldots d_n} \left(1 - \sum_{i=2}^{n} \frac{(i-2)!(n-i)!}{n!} e_i(d_1, \ldots, d_n) \right), \tag{3.53}$$

where e_i is the i-th elementary symmetric function:

$$e_i(d_1, \ldots, d_n) = \sum_{j_1 < \ldots < j_i} d_{j_1} \ldots d_{j_i}$$

4

Koornwinder polynomials and Littlewood identities

The subject of this chapter is bounded Littlewood identities. First, I define and prove some basic properties of Koornwinder polynomials. Then the virtual Koornwinder integrals are introduced. Their known evaluations are the main technical tool. After that, I discuss several new bounded Littlewood identities proved by Rains and Warnaar in [75]. My main results, presented in the last section, concern the extension of two of these identities. More specifically, I give a full proof of a conjectured formula stated in their paper, and make partial progress towards another (see Subsec.4.4.2 and Subsec.4.4.3).

4.1 Koornwinder polynomials

The polynomials which concern us are the BC_n symmetric Koornwinder polynomials, which were introduced by Koornwinder in [58]. There are many equivalent ways to define them. One of those is via a q-difference equation as follows.

Let $q, t, \mathbf{t} = (t_0, t_1, t_2, t_3)$ be complex parameters and $\mathbb{K} = \mathbb{Q}(q, t, \mathbf{t})$. Let n be a positive integer and $\mathbf{x} = (x_1, \dots, x_n)$ be indeterminates. We denote $\mathbf{x}^{-1} := (x_1^{-1}, \dots, x_n^{-1})$, $\overline{\mathbf{x}} = (x_1, \dots, x_{n-1}, x_n^{-1})$.

Let $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ be the hyper-octahedral group, which is the Weyl group of the finite root system of type B_n (and C_n). The group W can be viewed as the group of permutations of $\{1, \overline{1}, \dots, n, \overline{n}\}$ such that $\overline{w(i)} = w(\overline{i})$ for any $w \in W$ and $i = 1, \dots, n$ (with the convention that $\overline{\overline{i}} = i$). This group acts on $\mathbf{x}$ by permutation and inversion. More specifically, each $w \in W$ is a permutation of $2n$ symbols $\{\mathbf{x}, \mathbf{x}^{-1}\}$ which satisfies $w(x_j^{-1}) = w(x_j)^{-1}$. As usual, it acts on $\mathbb{K}[\mathbf{x}^{\pm 1}]$ by transposition, i.e.,

$$(wf)(x_1, \dots, x_n) := f(wx_1, \dots, wx_n).$$

Denote by $\Lambda^{BC_n} = \mathbb{K}\left[\mathbf{x}^{\pm 1}\right]^W$ the subalgebra of Laurent polynomials which are invariant under W. It has the following basis indexed by the partitions of length less than or equal to n:

$$m_\lambda = \sum_{w \in W} \mathbf{x}^{w\lambda} = \sum_{w \in W} (wx_1)^{\lambda_1} \dots (wx_n)^{\lambda_n}. \tag{4.1}$$

The polynomials m_λ are called monomial BC_n symmetric functions.

Definition 4.1.1. Define the *Koornwinder difference operator*

$$\begin{aligned} D_{\mathbf{x}}^{(n)} &= D_{\mathbf{x}}^{(n)}(q, t; t_0, \dots, t_3) \\ &:= \sum_{j=1}^{n} \left[a_j(x_1, \dots, x_n)\left(T_{x_j,q} - 1\right) + a_j(x_1^{-1}, \dots, x_n^{-1})\left(T_{x_j,q^{-1}} - 1\right)\right], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} a_j(x_1, \dots, x_n) = & \frac{(1 - t_0 x_j)(1 - t_1 x_j)(1 - t_2 x_j)(1 - t_3 x_j)}{(1 - x_j^2)(1 - q x_j^2)} \\ & \times \prod_{\substack{1 \le k \le n \\ k \ne j}} \frac{(1 - t x_j x_k)(1 - t x_j x_k^{-1})}{(1 - x_j x_k)(1 - x_j x_k^{-1})}, \end{aligned} \tag{4.3}$$

and $(T_{x_j,q} f)(x_1, \dots, x_n) = f(x_1, \dots, q x_j, \dots, x_n)$ for any function f of n variables.

For each partition λ of length less than or equal to n, define the *Koornwinder eigenvalue*

$$E_\lambda^{(n)} = E_\lambda^{(n)}(q, t; t_0, \dots, t_3) = \sum_{j=1}^{n} \left[t_0 t_1 t_2 t_3 q^{-1} t^{2n-j-1}\left(q^{\lambda_j} - 1\right) + t^{j-1}\left(q^{-\lambda_j} - 1\right)\right], \tag{4.4}$$

It is not obvious from the expression of $D_{\mathbf{x}}^{(n)}$ that the image of a BC_n symmetric Laurent polynomial is a Laurent polynomial since à priori, the images are rational functions. In fact, one can prove that

Proposition 4.1.1. *$D_{\mathbf{x}}^{(n)}$ is an endomorphism of the vector space Λ^{BC_n}.*

Proof. Let $f \in \Lambda^{BC_n}$. Since $D_{\mathbf{x}}^{(n)}$ is BC_n symmetric, $g := D_{\mathbf{x}}^{(n)} f$ is BC_n symmetric. One thus only needs to verify that g is a Laurent polynomial, that is the denominators in $D_{\mathbf{x}}^{(n)}$ are finally all canceled. From the expression of $D_{\mathbf{x}}^{(n)}$, what we have to prove is that for every j, the function g, considered as a complex function of x_j, has vanishing residue at the following points

$$x_j = \pm 1, \pm q^{1/2}, \pm q^{-1/2}, x_k^{\pm 1} \quad (\forall k \ne j).$$

We compute the residue at $x_j = 1, x_k$ to illustrate; the remaining points are completely analogous. For $x_j = 1$,

$$\begin{aligned}\operatorname*{Res}_{x_j=1} g = &- \frac{(1-t_0)(1-t_1)(1-t_2)(1-t_3)}{2(1-q)} \prod_{\substack{1\le k\le n\\ k\ne j}} \frac{(1-tx_k)(1-tx_k^{-1})}{(1-x_k)(1-x_k^{-1})}\\ &\times (f(x_1,\dots,q,\dots,x_n) - f(x_1,\dots,1,\dots,x_n))\\ &+ \frac{(1-t_0)(1-t_1)(1-t_2)(1-t_3)}{2(1-q)} \prod_{\substack{1\le k\le n\\ k\ne j}} \frac{(1-tx_k)(1-tx_k^{-1})}{(1-x_k)(1-x_k^{-1})}\\ &\times \big(f(x_1,\dots,q^{-1},\dots,x_n) - f(x_1,\dots,1,\dots,x_n)\big)\\ =&0,\end{aligned}$$

since

$$f(x_1,\dots,q,\dots,x_n) = f(x_1,\dots,q^{-1},\dots,x_n).$$

Now for $x_j = x_k$ (without loss of generality, suppose that $j < k$), we first prove that

$$\operatorname*{Res}_{x_j=x_k} \left[a_j(\mathbf{x})\,(T_{x_i,q}-1) + a_k(\mathbf{x})\,(T_{x_k,q}-1)\right] f = 0.$$

Indeed, this residue is equal to

$$\begin{aligned}&\frac{(1-t_0x_k)(1-t_1x_k)(1-t_2x_k)(1-t_3x_k)}{(1-x_k^2)(1-qx_k^2)} \prod_{\substack{1\le l\le n\\ l\ne j,k}} \frac{(1-tx_kx_l)(1-tx_kx_l^{-1})}{(1-x_kx_l)(1-x_kx_l^{-1})}\\ &\times \frac{x_k(1-tx_k^2)(1-t)}{1-x_k^2}\left[-f(x_1,\dots,qx_k,\dots,x_k,\dots,x_n) + f(x_1,\dots,x_k,\dots,qx_k,\dots,x_n)\right]\\ &= 0,\end{aligned}$$

due to the symmetry of f. By the same calculation, we also have

$$\operatorname*{Res}_{x_j=x_k} \left[a_j(\mathbf{x}^{-1})\,(T_{x_i,q^{-1}}-1) + a_k(\mathbf{x}^{-1})\,(T_{x_k,q^{-1}}-1)\right] f = 0.$$

Thus $\operatorname*{Res}_{x_j=x_k} g = 0$. □

Remark. A more conceptual proof based on the language of root systems is given in the original paper of Koornwinder [58].

The operator $D_{\mathbf{x}}^{(n)}$ has the following property which is essential for the construction of the Koornwinder polynomials.

Proposition 4.1.2. *The operator $D_{\mathbf{x}}^{(n)}$ is upper-triangular in the basis $\{m_\lambda\}$, more specifically*

$$D_{\mathbf{x}}^{(n)} m_\lambda = \sum_{\mu\le\lambda} c_{\lambda\mu}(q,t;t_0,t_1,t_2,t_3) m_\mu.$$

Its eigenvalues (diagonal entries) are $E_\lambda^{(n)}$. The eigenvalues are generically (i.e., for generic values of parameters) pairwise distinct.

Proof. The partial order on partitions defines a natural partial order on monomials. One simply defines $x^\lambda \geq x^\mu$ if and only if $\lambda \geq \mu$. First, we show that only partitions μ such that $\mu \leq \lambda$ enter into the decomposition. It is equivalent to showing that every monomial $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}$ in $D_{\mathbf{x}}^{(n)} m_\lambda$ such that $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$ satisfies $x^\mu \leq x^\lambda$. This follows immediately from the observation that the factors in front of the scaling operators $T_{x_i,\cdot}$ of $D_{\mathbf{x}}^{(n)}$ all have the form $\frac{P(x)}{Q(x)}$ where for every j, the degree of x_j in P is equal to its degree in Q.

Now we show that the diagonal entries are $E_\lambda^{(n)}$. The coefficient of m_λ in $D_{\mathbf{x}}^{(n)} m_\lambda$ is equal to the coefficient of x^λ in $D_{\mathbf{x}}^{(n)} x^\lambda$. For $1 \leq j \leq n$, we have

$$\begin{aligned}
&a_j(x_1, \dots, x_n) \left(T_{x_j,q} - 1\right) x_1^{\lambda_1} \dots x_n^{\lambda_n} \\
=&x_1^{\lambda_1} \dots x_n^{\lambda_n} \left(q^{\lambda_j} - 1\right) \frac{(1-t_0x_j)(1-t_1x_j)(1-t_2x_j)(1-t_3x_j)}{(1-x_j^2)(1-qx_j^2)} \prod_{\substack{1\leq k\leq n \\ k\neq j}} \frac{1-tx_jx_k}{1-x_jx_k} \cdot \frac{x_k - tx_j}{x_k - x_j} \\
=&t_0t_1t_2t_3q^{-1}t^{n-1} \left(q^{\lambda_j} - 1\right) x_1^{\lambda_1} \dots x_n^{\lambda_n} \prod_{\substack{1\leq k\leq n \\ k\neq j}} \frac{x_k - tx_j}{x_k - x_j} + \text{ lower terms} \\
=&t_0t_1t_2t_3q^{-1}t^{n-1} \left(q^{\lambda_j} - 1\right) x_1^{\lambda_1} \dots x_n^{\lambda_n} \\
&\times \left(1 + \frac{(1-t)x_j}{x_1 - x_j}\right) \dots \left(1 + \frac{(1-t)x_j}{x_{j-1} - x_j}\right) \left(t - \frac{(1-t)x_{j+1}}{x_j - x_{j+1}}\right) \dots \left(t - \frac{(1-t)x_n}{x_j - x_n}\right) \\
&+ \text{ lower terms} \\
=&t_0t_1t_2t_3q^{-1}t^{2n-j-1} \left(q^{\lambda_j} - 1\right) x_1^{\lambda_1} \dots x_n^{\lambda_n} + \text{ lower terms.}
\end{aligned}$$

In the above equations, "lower terms" are understood in the sense of the mentioned partial order on monomials. The last equation is just the natural consequence of this partial order that for $i < j$, x_i is privileged in calculating the higher term. Similarly

$$\begin{aligned}
&a_j(x_1^{-1}, \dots, x_n^{-1}) \left(T_{x_j,q^{-1}} - 1\right) x_1^{\lambda_1} \dots x_n^{\lambda_n} \\
=&(q^{-\lambda_j} - 1)x^\lambda \frac{(x_j - t_0)(x_j - t_1)(x_j - t_2)(x_j - t_3)}{(x_j^2 - 1)(x_j^2 - q)} \prod_{\substack{1\leq k\leq n \\ k\neq j}} \frac{(x_jx_k - t)(x_j - tx_k)}{(x_jx_k - 1)(x_j - x_k)} \\
=&(q^{-\lambda_j} - 1)x^\lambda \prod_{\substack{1\leq k\leq n \\ k\neq j}} \frac{x_j - tx_k}{x_j - x_k} + \text{ lower terms} \\
=&(q^{-\lambda_j} - 1)x^\lambda \\
&\times \left(t - \frac{(1-t)x_j}{x_1 - x_j}\right) \dots \left(t - \frac{(1-t)x_j}{x_{j-1} - x_j}\right) \left(1 + \frac{(1-t)x_{j+1}}{x_j - x_{j+1}}\right) \dots \left(1 + \frac{(1-t)x_n}{x_j - x_n}\right) \\
&+ \text{ lower terms} \\
=&(q^{-\lambda_j} - 1)t^{j-1}x^\lambda + \text{ lower terms.}
\end{aligned}$$

Summing over j the two results, we get the desired claim about $E_\lambda^{(n)}$. □

The main consequence is that this operator is generically diagonalisable with 1-dimensional eigenspaces. It leads us to the following

Definition 4.1.2. For each partition λ of length less than or equal to n, the BC_n *Koornwinder polynomial* $K_\lambda(\mathbf{x}) = K_\lambda(x_1, \ldots, x_n; q, t; t_0, \ldots, t_3)$ is the unique solution of the q-difference equation

$$\left[D_{\mathbf{x}}^{(n)} - E_\lambda^{(n)}\right] K_\lambda(\mathbf{x}) = 0 \tag{4.5}$$

such that

$$K_\lambda(\mathbf{x}) = m_\lambda(\mathbf{x}) + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu(\mathbf{x}) \in \mathbb{K}[\mathbf{x}^{\pm 1}]^W. \tag{4.6}$$

If $l(\lambda) > n$, we set $K_\lambda = 0$.

In particular, $\{K_\lambda\}$ form a basis of $\mathbb{K}[\mathbf{x}^{\pm 1}]^W$. It is clear from the definition that K_λ is invariant under permutations of t_0, t_1, t_2, t_3. In the following, in order to make the expressions look neat, we shall often omit the parameters whenever no confusion can arise. Without explicit writing, we always assume that the parameters $q, t; t_0, t_1, t_2, t_3$ appear in that order.

To the best of my knowledge, there is no known algorithm to compute the Koornwinder polynomials. Even for small partitions λ, the expression of K_λ can be very complicated. There is actually an explicit formula for polynomials corresponding to one-part partitions $\lambda = (r)$ found by Hoshino, Noumi and Shiraishi [45] (see Thm.4.4.2).

The Koornwinder polynomials satisfy many remarkable properties. First, they are orthogonal polynomials which generalize the celebrated 5-parameter univariate orthogonal polynomials of Askey and Wilson [6, 58]. To state this orthogonality property, let us define the *Koornwinder density*

$$\Delta_K = \Delta_K(\mathbf{x}; q, t; t_0, \ldots, t_3) := \prod_{j=1}^{n} \frac{\left(x_j^{\pm 2}; q\right)_\infty}{\prod_{r=0}^{3}\left(t_r x_j^{\pm 1}; q\right)_\infty} \prod_{1\le j<k\le n} \frac{\left(x_j^{\pm 1} x_k^{\pm 1}; q\right)_\infty}{\left(t x_j^{\pm 1} x_k^{\pm 1}; q\right)_\infty}, \tag{4.7}$$

where in the last product, all four combinations of signs are allowed. For complex parameters q, t, t_0, t_1, t_2, t_3 of modulus strictly less than 1, one can define the following scalar product on $\mathbb{K}\left[\mathbf{x}^{\pm 1}\right]$:

$$\begin{aligned}\langle f, g\rangle :=& \frac{1}{|W|}\int_{\mathbb{T}^n} f(\mathbf{x}) g(\mathbf{x}^{-1}) \Delta_K(\mathbf{x}) d\mathbb{T} \\ =& \frac{1}{2^n n!}[1] f(\mathbf{x}) g(\mathbf{x}^{-1}) \Delta_K(\mathbf{x}),\end{aligned} \tag{4.8}$$

where $\mathbb{T}$ and $d\mathbb{T}$ are respectively the n-dimensional unit torus and the natural measure on it defined in Def.2.4.4. Now we can state an equivalent definition of K_λ.

Proposition 4.1.3. *The K_λ are the unique family of polynomials in $\mathbb{K}[\mathbf{x}^{\pm 1}]^W$ such that:*

1. $K_\lambda(\mathbf{x}) = m_\lambda(\mathbf{x}) + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu(\mathbf{x})$*, and*
2. $\langle K_\lambda, K_\mu\rangle = 0$ *if* $\lambda \neq \mu$.

Proof. It suffices to prove that $D_{\mathbf{x}}^{(n)}$ is symmetric with respect to the scalar product, i.e.,

$$\langle D_{\mathbf{x}}^{(n)} f, g\rangle = \langle f, D_{\mathbf{x}}^{(n)} g\rangle,$$

for every $f, g \in \Lambda^{BC_n}$. Let

$$\Delta_K^+(\mathbf{x}) = \prod_{j=1}^{n} \frac{(x_j^2; q)_\infty}{\prod_{r=0}^{3} (t_r x_j; q)_\infty} \prod_{1\le j<k\le n} \frac{(x_j x_k^{\pm 1}; q)_\infty}{(t x_j x_k^{\pm 1}; q)_\infty}$$

so that $\Delta_K = \Delta_K^+(\mathbf{x})\Delta_K^+(\mathbf{x}^{-1})$. We have

$$\Delta_K^+(\mathbf{x})^{-1} T_{x_1,q} \Delta_K^+(\mathbf{x}) = \frac{\prod_{r=0}^{3}(1-t_r x_j)}{(1-x_j^2)(1-qx_j^2)} \prod_{k=2}^{n} \frac{(1-tx_1x_k)(1-tx_1/x_k)}{(1-x_1x_k)(1-x_1/x_k)} = a_1(\mathbf{x}).$$

Thus

$$\left(a_1(\mathbf{x}) T_{x_1,q} f(\mathbf{x})\right) \Delta_K^+(\mathbf{x}) = T_{x_1,q}\left(f(\mathbf{x})\Delta_K^+(\mathbf{x})\right).$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{T}^n} \left(a_1(\mathbf{x}) T_{x_1,q} f(\mathbf{x})\right) \Delta_K^+(\mathbf{x}) g(\mathbf{x}^{-1}) \Delta_K^+(\mathbf{x}^{-1}) d\mathbb{T} \\
&= \int_{\mathbb{T}^n} T_{x_1,q}\left(f(\mathbf{x})\Delta_K^+(\mathbf{x})\right) g(\mathbf{x}^{-1}) \Delta_K^+(\mathbf{x}^{-1}) d\mathbb{T} \\
&= \int_{\mathbb{T}^n} f(\mathbf{x})\Delta_K^+(\mathbf{x}) T_{x_1,q^{-1}}\left(g(\mathbf{x}^{-1})\Delta_K^+(\mathbf{x}^{-1})\right) d\mathbb{T} \\
&= \int_{\mathbb{T}^n} f(\mathbf{x}) \left(a_1(\mathbf{x}^{-1}) T_{x_1,q^{-1}} g(\mathbf{x}^{-1})\right) \Delta_K^+(\mathbf{x}) \Delta_K^+(\mathbf{x}^{-1}) d\mathbb{T}.
\end{aligned}$$

By interchanging x_1 and $x_i^{\pm 1}$, and summing over all of the obtained equations, we indeed get the claimed symmetry of $D_{\mathbf{x}}^{(n)}$. $\square$

A key property of the Koornwinder polynomials that will be used to prove the bounded Littlewood identities is the following BC_n Cauchy identity of Mimachi:

Proposition 4.1.4. *[67]For $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$, we have*

$$\begin{aligned}
&\sum_{\lambda \subset m^n} K_{m^n - \lambda}(\mathbf{x}; q, t; t_0, \dots, t_3) K_{\lambda'}(\mathbf{y}; t, q; t_0, \dots, t_3) \\
&= \prod_{j=1}^{n}\prod_{k=1}^{m} \left(x_j + x_j^{-1} - y_k - y_k^{-1}\right) = (x_1 \dots x_n)^{-m} \prod_{j=1}^{n}\prod_{k=1}^{m} (1 - x_j y_k^{\pm 1})
\end{aligned} \tag{4.9}$$

The proof of this theorem is long and involved, the interested reader is invited to consult the original paper [67]. Note that q and t are exchanged in the second polynomial in the sum. We will also need the explicit quadratic norm evaluation:

Proposition 4.1.5. *[76] The quadratic norm of Koornwinder polynomials with respect to the Koornwinder density is*

$$N_\lambda^{(n)}(q,t;t_0,t_1,t_2,t_3) := \frac{\langle K_\lambda(;q,t;t_0,t_1,t_2,t_3), K_\lambda(;q,t;t_0,t_1,t_2,t_3)\rangle}{\langle 1,1\rangle}$$
$$= \frac{C_\lambda^-(q;q,t)C_\lambda^+(t^{2n-3}t_0t_1t_2t_3;q,t)C_\lambda^0(t^n,t^{n-2}t_0t_1t_2t_3;q,t)\prod_{0\le i<j\le 3}C_\lambda^0(t^{n-1}t_it_j;q,t)}{C_\lambda^-(t;q,t)C_\lambda^+(t^{2n-2}t_0t_1t_2t_3/q;q,t)C_{2\lambda^2}^0(t^{2n-2}t_0t_1t_2t_3;q,t)},$$

where all the notations are defined in Sec.2.1.

It is known that all other Macdonald polynomials associated to classical root systems can be obtained from the BC_n Koornwinder polynomials. More specifically, we have

Proposition 4.1.6. *[75,86] The A_{n-1} Macdonald polynomial $P_\lambda(q,t)$ is the highest order term of K_λ (note that $P_\lambda(q,t)$ is homogeneous). The other Macdonald polynomials can be obtained by specialising the parameters as follows.*

1. *For λ a partition of length at most n,*

$$P_\lambda^{(C_n,B_n)}(x;q,t,t_2) = K_\lambda(x;q,t;\pm q^{1/2},\pm t_2^{1/2}). \tag{4.10}$$

2. *For λ a partition or a half-partition of length at most n,*

$$P_\lambda^{(B_n,B_n)}(x;q,t,t_2) = K_\lambda(x;q,t;t_2,q^{1/2}), \tag{4.11}$$
$$P_\lambda^{(B_n,C_n)}(x;q,t;t_2) = K_\lambda(x;q,t;t_2,t_2q^{1/2}), \tag{4.12}$$

where

$$K_\lambda(x;q,t;t_2,t_3)$$
$$= \begin{cases} K_\lambda(x;q,t;-1,-q^{1/2},t_2,t_3) & \lambda \text{ is a partition,} \\ K_{\lambda-(1/2)^n}(x;q,t;-q,-q^{1/2},t_2,t_3)\prod_{i=1}^n(x_i^{1/2}+x_i^{-1/2}) & \lambda \text{ is a half-partition.} \end{cases}$$

3. *Let λ be a (generalized) partition or a half-partition of length at most n, where the part λ_n can be negative but satisfies $-\lambda_{n-1}\le\lambda_n\le\lambda_{n-1}$. Denote $\bar\lambda := (\lambda_1,\ldots,\lambda_{n-1},-\lambda_n)$. Then, we have*

$$P_\lambda^{(D_n,D_n)}(\mathbf{x};q,t) = P_\lambda^{(B_n,B_n)}(\mathbf{x};q,t,1), \tag{4.13}$$

if $\lambda_n = 0$, and

$$P_\lambda^{(D_n,D_n)}(\mathbf{x};q,t) + P_{\bar\lambda}^{(D_n,D_n)}(\mathbf{x};q,t) = P_\lambda^{(B_n,B_n)}(\mathbf{x};q,t,1), \tag{4.14}$$
$$P_\lambda^{(D_n,D_n)}(\mathbf{x};q,t) - P_{\bar\lambda}^{(D_n,D_n)}(\mathbf{x};q,t) = P_{\lambda-(1/2)^n}^{(B_n,C_n)}(\mathbf{x};q,t,q^{1/2})\prod_{i=1}^n\left(x_i^{-1/2}-x_i^{1/2}\right), \tag{4.15}$$

if $\lambda_n \neq 0$.

I will not define what $P_\lambda^{(R,S)}$ for a pair of admissible root systems (R,S) are. The general construction is completely analogous to that of Koornwinder polynomials (by triangularity with respect to the monomial symmetric functions and orthogonality with respect to a well chosen density attached to the root system R). We can of course take the above formulas as the definition for $(R,S) = (B_n, B_n), (B_n, C_n), (C_n, B_n)$, and (D_n, D_n). The interested reader should consult the book [63] for a uniform treatment of all root systems (including exceptional ones).

4.2 Virtual Koornwinder integrals

The *virtual Koornwinder integrals* are the main technical tools used by Rains and Warnaar to prove new bounded Littlewood identities [75,76]. They are defined as follows. For $f \in \Lambda^{BC_n}$, define

$$I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) := [K_0(; q, t; t_0, t_1, t_2, t_3)]f. \tag{4.16}$$

Note that although $K_0 = 1$, $I_K^{(n)}$ is not the constant term of f. Similarly, for $f \in \Lambda$, define

$$I_K(f; q, t, T; t_0, t_1, t_2, t_3) := \left[\tilde{K}_0(; q, t, T; t_0, t_1, t_2, t_3)\right] f, \tag{4.17}$$

where $\tilde{K}$ is the infinite variable version of a lifting of K (see [76, Sec.7] for details). If the specialization of parameters hits the poles of of $\tilde{K}_\lambda$, i.e., if

$$t_0 t_1 t_2 t_3 = q^{2-\lambda_i - j} t^{i+\lambda'_j} T^{-2}, \quad (i,j) \in \lambda,$$

then T must be specialized before other parameters. The relationship between the two virtual integrals is:

$$I_K^{(n)}(f; q, t; t_0, t_1, t_2, t_3) = I_K(f; q, t, t^n; t_0, t_1, t_2, t_3). \tag{4.18}$$

Remind that if no confusion arises, we will omit writing the parameters. A corollary of the orthogonality property 4.1.3 is that:

Corollary 4.2.1. *For $q, t, t_0, \ldots, t_3$ of magnitude < 1, and $f \in \Lambda^{BC_n}$,*

$$I_K^{(n)}(f) = Z^{-1} \int_{\mathbb{T}^n} f(\mathbf{x}) \Delta_K(\mathbf{x}) d\mathbb{T} = \langle f, 1 \rangle / \langle 1, 1 \rangle, \tag{4.19}$$

where

$$Z = \langle 1, 1 \rangle = \int_{\mathbb{T}^n} \Delta_K(\mathbf{x}) d\mathbb{T} = \prod_{j=1}^{n} \frac{(t, t_0 t_1 t_2 t_3 t^{n+i-2}; q)_\infty}{(q, t^i; q)_\infty \prod_{0 \le r < s \le 3} (t_r t_s t^{i-1}; q)_\infty}. \tag{4.20}$$

The parameters are understood in a consistent way.

The evaluation of $\langle 1, 1 \rangle$ is known as Gustafson's integral [42]. In general, it is hard to compute explicitly the virtual Koornwinder integrals. The known evaluations are as follows.

Theorem 4.2.2. *(Rains and Vazirani [75, p.30], [78, Thm.4.1]) For every partition μ, we have*

$$I_K(P_\mu(;q,t);q,t,T;\pm t^{1/2},\pm(qt)^{1/2}) = \chi(\mu' even)\frac{C^0_\nu(T^2;q,t^2)}{C^0_\nu(qT^2/t;q,t^2)}\frac{C^-_\nu(qt;q,t^2)}{C^-_\nu(t^2;q,t^2)} \quad (4.21)$$

where $\nu := (\mu'/2)' = (\mu_1,\mu_3,\dots)$ and $\chi(\mu' even) = 0$ if μ' is not even, i.e., at least one part of μ' is odd, and equals 1 if μ' is even. Recall the definition of $C^{0,-}_\lambda$ in Sec.2.1.

The proof of this theorem is based on the affine Hecke algebras. The next proposition is due to Rains [75, p.30]. It is a consequence of the recent works of Rains on multivariate elliptic hypergeometric functions and identities.

Theorem 4.2.3. *For every partition μ,*

$$\begin{aligned} &I_K(P(;q,t);q,t,T;-1,-q^{1/2},-t^{1/2},-(qt)^{1/2}) \\ &= (-1)^{|\mu|}\frac{C^0_\mu(T;q^{1/2},t^{1/2})}{C^0_\mu(-q^{1/2}T/t^{1/2};q^{1/2},t^{1/2})}\frac{C^-_\mu(-q^{1/2};q^{1/2},t^{1/2})}{C^-_\mu(t^{1/2};q^{1/2},t^{1/2})} \end{aligned} \quad (4.22)$$

Two variants for $I^{(n)}_K$ are available [75, Thm.3.4, Thm.3.5].

Theorem 4.2.4. *For μ a partition of length at most $2n$, let*

$$\tilde\mu = (\mu_1-\mu_{2n},\dots,\mu_{2n-1}-\mu_{2n}).$$

Then we have:

$$\begin{aligned} &I^{(n)}_K\left(P_\mu(x_1^\pm,\dots,x_n^\pm;q,t);q,t;\pm1,\pm t^{1/2}\right) \\ &= (-1)^{\mu_{2n}} I^{(n-1)}_K\left(P_\mu(x_1^\pm,\dots,x_{n-1}^\pm,\pm1;q,t);q,t;\pm t,\pm t^{1/2}\right) \\ &= \begin{cases} A^{(2n)}_{\mu/2}(q,t) & \text{if } \tilde\mu \text{ is even} \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (4.23)$$

where $\mu/2 := (\mu_1/2,\mu_2/2,\dots)$.

Theorem 4.2.5. *For ν a partition of length at most $2n+1$, let*

$$\tilde\nu := (\nu_1-\nu_{2n+1},\dots,\nu_{2n}-\nu_{2n+1}).$$

Then we have

$$\begin{aligned} &I^{(n)}_K\left(P_\nu(x_1^\pm,\dots,x_n^\pm,1;q,t);q,t;-1,t,\pm t^{1/2}\right) \\ &= (-1)^{\nu_{2n+1}} I^{(n)}_K\left(P_\nu(x_1^\pm,\dots,x_n^\pm,-1;q,t);q,t;1,-t,\pm t^{1/2}\right) \\ &= \begin{cases} A^{(2n+1)}_{\nu/2}(q,t) & \text{if } \tilde\nu \text{ is even} \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (4.24)$$

4.3 Bounded Littlewood identities

One fundamental problem in the theory of symmetric functions is calculating the transition matrix between two bases (one of them is often the Schur or Macdonald basis). Of course, due to the great complexity of involved objects, it is hopeless to compute the full transition matrix. However, we are only interested in those matrix coefficients which have a nice and reasonably compact expression. Let us formalize the question

Problem: Let R be a symmetric polynomial, calculate the coefficients of the expansion of R in the Macdonald basis P_λ, i.e. calculate c_λ such that

$$R = \sum_\lambda c_\lambda P_\lambda.$$

Such an equation, particularly when R and c_λ are all reasonably simple, is called a Littlewood identity. If the sum runs over a finite number of partitions, it is further called a bounded Littlewood identity.

Lemma 4.3.1. *Let* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, \ldots, y_m)$*. Then*

$$\begin{aligned}&(-1)^{|\mu|}[P_\lambda(\mathbf{x}; q, t)](x_1 \ldots x_n)^m K_{m^n-\mu}(\mathbf{x}; q, t; t_0, t_1, t_2, t_3)\\ =&(-1)^{|\lambda|}[K_{\mu'}(\mathbf{y}; t, q; t_0, t_1, t_2, t_3)]P_{\lambda'}(\mathbf{y}^\pm; t, q)\end{aligned} \tag{4.25}$$

Proof. Indeed, combining the Macdonald-Cauchy identity 2.4.4 and the Mamichi-Cauchy identity 4.1.4, we obtain

$$\begin{aligned}&\sum_{\lambda \subset m^n} (-1)^{|\lambda|}(x_1 \ldots x_n)^m K_{m^n-\lambda}(\mathbf{x}; q, t; t_0, t_1, t_2, t_3)K_{\lambda'}(\mathbf{y}; t, q; t_0, t_1, t_2, t_3)\\ =&\sum_{\lambda \subset (2m)^n} (-1)^{|\lambda|}P_\lambda(\mathbf{x}; q, t)P_{\lambda'}(\mathbf{y}^\pm; t, q).\end{aligned}$$

Equating the coefficients of $P_\lambda(\mathbf{x}; q, t)K_{\mu'}(\mathbf{y}; t, q; t_0, t_1, t_2, t_3)$, one finds the desired equation. $\square$

Thus one can interchangeably do the computation in the bases P_λ or K_λ. Taking $\mu = 0$ in Eq.(4.25), one obtains

Corollary 4.3.2. *For any partition* λ*,*

$$\begin{aligned}&[P_\lambda(\mathbf{x}; q, t)](x_1 \ldots x_n)^m K_{m^n}(\mathbf{x}; q, t; t_0, t_1, t_2, t_3)\\ =&(-1)^{|\lambda|}[K_0(\mathbf{y}; t, q; t_0, t_1, t_2, t_3)]P_{\lambda'}(\mathbf{y}^\pm; t, q)\\ =&(-1)^{|\lambda|}I_K^{(m)}(P_{\lambda'}(t, q); t, q; t_0, t_1, t_2, t_3).\end{aligned} \tag{4.26}$$

Rains and Warnaar [75] proved various bounded Littlewood identities which express the decomposition of rectangular-shaped Macdonald polynomials [1] associated to B, C, D root systems into those associated to A system. The implications of these identities include new combinatorial formulas for highest weight characters of affine Lie algebras, new identities of Rogers-Ramanujan type associated with affine Lie algebras, and new Kaneko-Macdonald-type hypergeometric identities.

Their Littlewood identities are stated in the following theorems. The reader is invited to look at the discussions in [75, Ch.4] to know about the earlier identities of which these identities are generalization.

The first bounded Littlewood identity of Rains and Warnaar is for $P_{m^n}^{(C_n,B_n)}(\mathbf{x}; q, t, qt)$. In the next section, I will prove an extension to the case of near-rectangular polynomials $P_{m^{n-r}(m-1)^r}^{(C_n,B_n)}(x; q, t, qt)$; it is Thm.4.4.5. It was conjectured by Rains and Warnaar in their paper.

Theorem 4.3.3. *For* $\mathbf{x} = (x_1, \ldots, x_n)$*,* m *a nonnegative integer and* a *a complex number,*

$$\sum_{\lambda} a^{odd(\lambda)} b^{oa}_{\lambda;m}(q,t) P_\lambda(\mathbf{x}; q, t) = \left(\prod_{j=1}^{n} x_j^m (1 + a x_j) \right) P_{m^n}^{(C_n,B_n)}(\mathbf{x}; q, t, qt), \qquad (4.27)$$

where

$$b^{oa}_{\lambda;m}(q,t) := b^{oa}_\lambda(q,t) \prod_{\substack{s\in\lambda \\ a'(s)\ odd}} \frac{1 - q^{2m-a'(s)+1} t^{l'(s)}}{1 - q^{2m-a'(s)} t^{l'(s)+1}}.$$

Note that $b^{\mathrm{oa}}_{\lambda;m}(q,t) = 0$ if $\lambda_1 > 2m + 1$, thus the sum is over partitions lying inside the rectangular $((2m + 1)^n)$. This explains the adjective "bounded". Also, for λ an even partition

$$b^{\mathrm{oa}}_{\lambda;m}(q,t) = b^{\mathrm{oa}}_\lambda(q,t) \prod_{\substack{s\in\lambda \\ a'(s)\ \text{even}}} \frac{1 - q^{2m-a'(s)} t^{l'(s)}}{1 - q^{2m-a'(s)-1} t^{l'(s)+1}}. \qquad (4.28)$$

Proof. First, let us prove the case $a = 0$, in which Eq.(4.27) is equivalent to

$$(-1)^{|\lambda|} I_K^{(m)}(P_{\lambda'}(t,q); t, q; \pm q^{1/2}, \pm (qt)^{1/2}) = \begin{cases} b^{\mathrm{oa}}_{\lambda;m}(q,t) & \text{if } \lambda \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Following Thm.4.2.2 with $(T, \mu) = (t^m, \lambda')$, we have $(-1)^{|\lambda|} I_K^{(m)}(P_{\lambda'}(t,q); t, q; \pm q^{1/2}, \pm (qt)^{1/2})$ vanishes unless λ is even in which case it is equal to

$$\frac{C^0_{(\lambda/2)'}(q^{2m}; t, q^2)}{C^0_{(\lambda/2)'}(q^{2m-1}t; t, q^2)} \cdot \frac{C^-_{(\lambda/2)'}(qt; t, q^2)}{C^-_{(\lambda/2)'}(q^2; t, q^2)}.$$

1. It means the polynomials indexed by rectangular Young diagrams.

Now suppose that λ is even, we show that this expression equals $b^{\mathrm{oa}}_{\lambda;m}(q,t)$. From the definition, one has, for every partition ν

$$\begin{aligned}C^0_\nu(a;q,t) &= (-a)^{|\nu|}q^{n(\nu)}t^{-n(\nu')}C^0_\nu(a^{-1};t,q),\\ C^-_{\nu'}(a;q,t) &= C^-_\nu(a;t,q).\end{aligned}$$

Thus

$$\begin{aligned}&(-1)^{|\lambda|}I^{(m)}_K(P_{\lambda'}(t,q);t,q;\pm q^{1/2},\pm(qt)^{1/2})\\ &= \left(\frac{q}{t}\right)^{|\lambda|/2}\frac{C^0_{\lambda/2}(q^{-2m};q^2,t)}{C^0_{\lambda/2}(q^{1-2m}/t;q^2,t)}\cdot\frac{C^-_{\lambda/2}(qt;q^2,t)}{C^-_{\lambda/2}(q^2;q^2,t)}\\ &= \prod_{s\in\lambda/2}\frac{1-q^{2m-2a'(s)}t^{l'(s)}}{1-q^{2m-2a'(s)-1}t^{l'(s)+1}}\frac{1-q^{2a(s)+1}t^{l(s)+1}}{1-q^{2a(s)+2}t^{l(s)}}\\ &= \prod_{\substack{s\in\lambda\\ a(s)\text{ odd}}}\frac{1-q^{2m-a'(s)}t^{l'(s)}}{1-q^{2m-a'(s)-1}t^{l'(s)+1}}\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}.\end{aligned}$$

To save space, I did not put subscripts under the arm-length etc. It is understood that when the product is over the boxes of a partition, the lengths are calculated with respect to this partition. Thus in the last line, the lengths are calculated with respect to λ, while in the line before it, they are calculated with respect to $\lambda/2$. The reader is advised to draw a small example of an even partition λ to see the last equality.

Because λ is even, odd arm-lengths correspond to even arm-colengths. Therefore, the last product is indeed equal to $b^{\mathrm{oa}}_{\lambda;m}(q,t)$ in the form given by Eq.(4.28). Thus the case $a=0$ of Eq.(4.27) is proved.

Now, for the case a arbitrary, note that

$$\prod_{i=1}^n(1+ax_i) = \sum_{r=0}^n a^r e_r(x_1,\dots,x_n).$$

Therefore, by the e-Pieri rule Eq.(2.12), it suffices to show that

$$a^{\mathrm{odd}(\lambda)}b^{\mathrm{oa}}_{\lambda;m}(q,t) = \sum_{\substack{\mu\text{ even}\\ \lambda/\mu\text{ vertical strip}}} a^{|\lambda/\mu|}\psi'_{\lambda/\mu}(q,t)b^{\mathrm{oa}}_{\mu;m}(q,t)$$

Given λ, there is a unique μ satisfying the two conditions in the sum, i.e., $\mu_i = 2\lfloor\lambda_i/2\rfloor$. Note that then $|\lambda/\mu| = \mathrm{odd}(\lambda)$. Thus, we are led to showing that

$$b^{\mathrm{oa}}_{\lambda;m}(q,t) = \psi'_{\lambda/\mu}(q,t)b^{\mathrm{oa}}_{\mu;m}(q,t).$$

The m-dependent parts on both sides agree:

$$\prod_{\substack{s\in\lambda\\ a'_\lambda(s)\text{ even}}}\frac{1-q^{2m-a'_\lambda(s)}t^{l'_\lambda(s)}}{1-q^{2m-a'_\lambda(s)-1}t^{l'\lambda(s)+1}} = \prod_{\substack{s\in\mu\\ a'_\mu(s)\text{ even}}}\frac{1-q^{2m-a'_\mu(s)}t^{l'_\mu(s)}}{1-q^{2m-a'_\mu(s)-1}t^{l'\mu(s)+1}}.$$

So it remains to show that

$$b^{\mathrm{oa}}_{\lambda}(q,t) = \psi'_{\lambda/\mu}(q,t) b^{\mathrm{oa}}_{\mu}(q,t).$$

This follows easily from the definition of these quantities. □

The second identity of Rains and Warnaar concerns $P^{(B_n,B_n)}_{(\frac{m}{2})^n}$. In Subsec.4.4.3, I obtain some partial progresses towards the generalization to the near-rectangular case. For m a positive integer and λ a partition, let (ol stands for odd leg)

$$b^{\mathrm{ol}}_{\lambda;m}(q,t) := \prod_{\substack{s\in\lambda \\ l(s)\text{ odd}}} \frac{1-q^{a(s)}t^{l(s)}}{1-q^{a(s)+1}t^{l(s)-1}} \prod_{\substack{s\in\lambda \\ l'(s)\text{ odd}}} \frac{1-q^{m-a'(s)}t^{l'(s)-1}}{1-q^{m-a'(s)-1}t^{l'(s)}}.$$

We note that $b^{\mathrm{ol}}_{\lambda;m}(q,t) = 0$ if $\lambda_2 > m$. Also note that unlike the case of $b^{\mathrm{oa}}_{\lambda;m}$, the m-independent part of $b^{\mathrm{ol}}_{\lambda;m}$ is slightly different from the conventional $b^{\mathrm{ol}}_{\lambda}$.

Theorem 4.3.4. *For $\mathbf{x} = (x_1, \dots, x_n)$ and m a nonnegative integer,*

$$\sum b^{ol}_{\lambda;m}(q,t) P_\lambda(\mathbf{x};q,t) = (x_1 \dots x_n)^{\frac{m}{2}} P^{(B_n,B_n)}_{(\frac{m}{2})^n}(\mathbf{x};q,t,1) \tag{4.29}$$

where the sum is over partitions $\lambda \subset m^n$ such that $m_i(\lambda)$ [2] is even for $1 \le i \le m-1$.

The identity (4.29) can be dissected into two following identities:

Theorem 4.3.5. *For $\mathbf{x} = (x_1, \dots, x_n)$ and m a nonnegative integer,*

$$\sum_{\lambda' \text{ even}} b^{ol}_{\lambda;m}(q,t) P_\lambda(\mathbf{x};q,t) = (x_1 \dots x_n)^{\frac{m}{2}} P^{(D_n,D_n)}_{(\frac{m}{2})^n}(\mathbf{x};q,t), \tag{4.30}$$

$$\sum_{\substack{\lambda' \text{ odd} \\ \lambda_1 = m}} b^{ol}_{\lambda;m}(q,t) P_\lambda(\mathbf{x};q,t) = (x_1 \dots x_n)^{\frac{m}{2}} P^{(D_n,D_n)}_{(\frac{m}{2})^n}(\bar{\mathbf{x}};q,t). \tag{4.31}$$

For m a positive integer and λ a partition, let

$$b^{\mathrm{el}}_{\lambda;m}(q,t) := b^{\mathrm{el}}_{\lambda}(q,t) \prod_{\substack{s\in\lambda \\ l'(s)\text{ even}}} \frac{1-q^{m-a'(s)}t^{l'(s)}}{1-q^{m-a'(s)-1}t^{l'(s)+1}}.$$

Note that $b^{\mathrm{el}}_{\lambda;m}(q,t)$ vanishes unless $\lambda_1 \le m$.

Theorem 4.3.6. *For $\mathbf{x} = (x_1, \dots, x_n)$ and m a nonnegative integer,*

$$\sum_{\lambda} b^{el}_{\lambda;m}(q,t) P_\lambda(\mathbf{x};q,t) = (x_1 \dots x_n)^{\frac{m}{2}} P^{(B_n,B_n)}_{(\frac{m}{2})^n}(\mathbf{x};q,t,t), \tag{4.32}$$

Furthermore, they proved the following generalization of the Kaneko identity (2.18)

2. Recall that $m_i(\lambda)$ is the multiplicity of i in λ.

Theorem 4.3.7. *For* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* m *a nonnegative integer,*

$$\sum_{\lambda} b^{-}_{\lambda;m}(q,t) P_\lambda(\mathbf{x}; q^2, t^2) = (x_1 \ldots x_n)^{\frac{m}{2}} P^{(B_n, C_n)}_{\left(\frac{m}{2}\right)^n}(\mathbf{x}; q^2, t^2, -t), \tag{4.33}$$

where

$$b^{-}_{\lambda;m}(q,t) := b^{-}_{\lambda}(q,t) \prod_{s \in \lambda} \frac{1 - q^{m - a'(s)} t^{l'(s)}}{1 - q^{m - a'(s) - 1} t^{l'(s)+1}}.$$

They also proved two identities for the Hall-Littlewood polynomials whose q-analogues are not known.

Theorem 4.3.8. *For* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* m *a nonnegative integer,*

$$\sum_{\lambda_1 \leq 2m} h^{(2m)}_{\lambda}(t_2, t_3; t) P_\lambda(\mathbf{x}; t) = (x_1 \ldots x_n)^m P^{(BC_n)}_{m^n}(\mathbf{x}; t, t_2, t_3) \tag{4.34}$$

Theorem 4.3.9. *For* $\mathbf{x} = (x_1, \ldots, x_n)$ *and* m *a nonnegative integer,*

$$\sum_{\lambda_1 \leq m} h^{(m)}_{\lambda}(t_2; t) P_\lambda(\mathbf{x}; t) = (x_1 \ldots x_n)^{\frac{m}{2}} P^{(B_n)}_{\left(\frac{m}{2}\right)^n}(\mathbf{x}; t, t_2) \tag{4.35}$$

Here $h^{(m)}_{\lambda}(a, b; q)$ is a certain generalisation of the Rogers-Szego polynomials; the reader is invited to consult [75] for details.

4.4 Near-rectangular bounded Littlewood identities

The bounded Littlewood identities in the previous section are decomposition of (R, S) Macdonald polynomials indexed by rectangular partitions or half-partitions of maximal length (which is the number of variables). A natural next step is to consider near-rectangular (half-)partitions of maximal length, i.e., (half-)partitions $m^n - 1^r = (m^{n-r}(m-1)^r)$. In fact, Rains and Warnaar propose this question at the end of their paper. I would like to thank Warnaar for suggesting to me that this question is doable.

Problem: Compute the decomposition number $[P_\lambda]\,(x_1 \ldots x_n)^m P^{(R,S)}_{m^n - 1^r}$.

4.4.1 Preparatory results

First we transform this question into a computation of virtual Koornwinder integrals. Let $\mu = (1^r)$ in Lem.4.25, we obtain

$$\begin{aligned}&[P_\lambda(\mathbf{x}; q, t)](x_1 \ldots x_n)^m K_{m^n - 1^r}(\mathbf{x}; q, t; t_0, t_1, t_2, t_3)\\ &= (-1)^{|\lambda| + r} [K_{(r)}(\mathbf{y}; t, q; t_0, t_1, t_2, t_3)] P_{\lambda'}(\mathbf{y}^{\pm}; t, q),\end{aligned} \tag{4.36}$$

for $\lambda \subset (2m)^n$.

Due to orthogonality, we have the following formula for a BC_n symmetric polynomial F (the parameters are understood in a consistent way):

$$[K_\mu]F = \frac{\langle K_\mu, F\rangle}{\langle K_\mu, K_\mu\rangle} = \frac{\langle 1, K_\mu F\rangle}{\langle K_\mu, K_\mu\rangle} = \frac{I_K^{(n)}(K_\mu F)}{N_\mu}.$$

Thus, we get

$$\boxed{\begin{aligned} &[P_\lambda(\mathbf{x}; q, t)](x_1 \dots x_n)^m K_{m^n-1^r}(\mathbf{x}; q, t; t_0, t_1, t_2, t_3) \\ =&(-1)^{|\lambda|+r} \frac{I_K^{(m)}\left(K_{(r)}(\mathbf{y}; t, q; t_0, t_1, t_2, t_3) P_{\lambda'}(\mathbf{y}^{\pm}; t, q)\right)}{N_{(r)}^{(n)}(t, q; t_0, t_1, t_2, t_3)}. \end{aligned}} \tag{4.37}$$

Our strategy is that if $K_{(r)}$ is simple enough, we can expand $K_{(r)}P_{\lambda'}$ in the Macdonald basis and use the known evaluations of virtual Koornwinder integrals to proceed.

Remark. We must be careful in specializing the parameters. The rule of thumb is that we should specialize only after having simplified every repeated factor. Sometimes, the order in which we specialize the parameters does matter. We shall explicitly mention that in necessary cases.

For example, we make the following remark concerning the specialization $t = q^{-1}$. We shall see in a moment that it is more convenient to work with the following normalized version of one-row Koornwinder polynomials

$$f_r(\mathbf{y}; t, q; t_0, t_1, t_2, t_3) := \frac{(q;t)_r}{(t;t)_r} K_{(r)}(\mathbf{y}; t, q; t_0, t_1, t_2, t_3).$$

We also notice that $N_{(r)}^{(n)}(t, q; t_0, t_1, t_2, t_3)$ always has the following factorization

$$N_{(r)}^{(n)}(t, q; t_0, t_1, t_2, t_3) = \frac{(t;t)_r}{(q;t)_r} \tilde{N}_{(r)}^{(n)}(t, q; t_0, t_1, t_2, t_3),$$

where $\tilde{N}_{(r)}^{(n)}(t, q; t_0, t_1, t_2, t_3)$ is finite at $t = q^{-1}$. Of course, $(t^{-1};t)_r = 0$ for $r \geq 1$ by definition, thus $N_{(r)}(t, t^{-1}; t_0, t_1, t_2, t_3)$ is not defined for $r \geq 1$. But with this new normalization, we have

$$\boxed{\begin{aligned} &[P_\lambda(\mathbf{x}; q, t)](x_1 \dots x_n)^m K_{m^n-1^r}(\mathbf{x}; q, t; t_0, t_1, t_2, t_3) \\ =&(-1)^{|\lambda|+r} \frac{I_K^{(m)}\left(f_r(\mathbf{y}; t, q; t_0, t_1, t_2, t_3) P_{\lambda'}(\mathbf{y}^{\pm}; t, q)\right)}{\tilde{N}_{(r)}^{(n)}(t, q; t_0, t_1, t_2, t_3)}. \end{aligned}} \tag{4.38}$$

Now, there is no problem in specializing $t = q^{-1}$, for all r.

The cases we are going to consider only require the following special evaluations of the quadratic norm:

Lemma 4.4.1. *1. For* $(t_0, t_1, t_2, t_3) = (\pm t^{1/2}, \pm(qt)^{1/2})$*, we have*

$$\tilde{N}^{(n)}_{(r)}(q, t; \pm t^{1/2}, \pm(qt)^{1/2}) = \frac{(t^{2n}; q)_r}{(qt^{2n-1}; q)_r}. \tag{4.39}$$

2. For $(t_0, t_1, t_2, t_3) = (\pm 1, \pm t^{1/2})$*, we have*

$$\tilde{N}^{(n)}_{(r)}(q, t; \pm 1, \pm t^{1/2}) = \frac{(q^r t^{2n-2}; q)_r (t^{2n}, t^{2n-1}; q^2)_r}{(q^{r-1} t^{2n-1}; q)_r (qt^{2n-1}, qt^{2n-2}; q^2)_r}. \tag{4.40}$$

Proof. We just need to apply the general formula. For the first identity, we get

$$\begin{aligned}
&\tilde{N}^{(n)}_{(r)}(; q, t; (\pm t^{1/2}, \pm(qt)^{1/2}) \\
=&\frac{(q^{r+1} t^{2n-1}, \pm t^n, \pm qt^n, \pm q^{1/2} t^n, \pm q^{1/2} t^n; q)_r}{(q^r t^{2n}; q)_r (qt^{2n}, qt^{2n-1}; q)_{2r}} \\
=&\frac{(t^{2n}; q)_r}{(qt^{2n-1}; q)_r}.
\end{aligned}$$

Notice that there is a huge cancellation in the above product which is the result of straightforward computation. For the second identity,

$$\begin{aligned}
&\tilde{N}^{(n)}_{(r)}(q, t; \pm 1, \pm t^{1/2}) \\
=&\frac{(q^r t^{2n-2}, \pm t^n, \pm t^{n-1}, \pm t^{n-1/2}, \pm t^{n-1/2}; q)_r}{(q^{r-1} t^{2n-1}; q)_r (t^{2n-1}, t^{2n-2}; q)_{2r}} \\
=&\frac{(q^r t^{2n-2}; q)_r (t^{2n}, t^{2n-1}; q^2)_r}{(q^{r-1} t^{2n-1}; q)_r (qt^{2n-1}, qt^{2n-2}; q^2)_r}.
\end{aligned}$$

Two nice particular cases are $q = t$ and $q = t^{-1}$:

$$\begin{aligned}
\tilde{N}^{(n)}_{(r)}(t, t; \pm 1, \pm t^{1/2}) &= 1 \\
\tilde{N}^{(n)}_{(r)}(t^{-1}, t; \pm 1, \pm t^{1/2}) &= \frac{(1 - t^{2n})(1 - t^{2n-1})}{(1 - t^{2n-r})(1 - t^{2n-r-1})}.
\end{aligned}$$

□

Definition 4.4.1. Define the symmetric polynomial $g_r(\mathbf{x}; q, t)$ by

$$\prod_{i=1}^{n} \frac{(tux_i; q)_\infty}{(ux_i; q)_\infty} = \sum_{r \geq 0} g_r(\mathbf{x}; q, t) u^r,$$

and the symmetric Laurent polynomial $G_r(\mathbf{x}; q, t) := g_r(\mathbf{x}^{\pm}; q, t)$.

Hoshino, Noumi and Shiraishi [45] proved the following formulas for one-row Koornwinder polynomials:

Theorem 4.4.2. *[45] Let* $f_r(\mathbf{x};q,t;t_0,t_1,t_2,t_3) := \frac{(t;q)_r}{(q;q)_r}K_{(r)}(\mathbf{x};q,t;t_0,t_1,t_2,t_3)$*. Then we have - we denote the parameters by* a,b,c,d *to make the formula look nicer:*

$$\begin{aligned} f_r(\mathbf{x};q,t;a,-a,c,-c) = &\sum_{\substack{k,l\geq 0\\ 2k+2l\leq r}} G_{r-2k-2l}(\mathbf{x};q,t) \\ &\times \frac{(qa^2/c^2;q^2)_k\,(q^{3-r}t^{1-n}/c^2;q^2)_k\,(q^{2-2r}t^{2-2n}/c^4;q^2)_k}{(q^2;q^2)_k\,(q^{1-r}t^{1-n}/c^2;q^2)_k\,(q^{3-2r}t^{2-2n}/a^2c^2;q^2)_k}\left(\frac{t^2}{a^2}\right)^k \\ &\times \frac{(c^2/qt;q)_l\,(q^{-r}t^{-n};q)_{2k+l}}{(q;q)_l\,(q^{2-r}t^{1-n}/c^2;q)_{2k+l}}\cdot\frac{1-q^{-r+2k+2l}t^{-n}}{1-q^{-r}t^{-n}}\left(\frac{t^2}{c^2}\right)^l, \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} &f_r(\mathbf{x};q,t;a,b,c,d) \\ &= \sum_{\substack{i,j\geq 0\\ i+j\leq r}} f_{r-i-j}(\mathbf{x};q,t;\pm a,\pm c)\frac{(-b/a;q)_i(q^{1-r}t^{1-n}/cd;q)_i}{(q;q)_i(-q^{1-r}t^{1-n}/ac;q)_i} \\ &\times \frac{(q^{1-r}t^{-n};q)_{i+j}(-q^{1-r}t^{1-n}/ac;q)_{i+j}(q^{1-2r}t^{2-2n}/a^2c^2;q)_{i+j}}{(q^{2-2r}t^{2-2n}/abcd;q)_{i+j}(q^{1/2-r}t^{1-n}/ac;q)_{i+j}(-q^{1/2-r}t^{1-n}/ac;q)_{i+j}}(t/b)^i \\ &\times \frac{(-d/c;q)_j(q^{1-r}t^{1-n}/ab;q)_j}{(q;q)_j(-q^{1-r}t^{1-n}/ac;q)_j}(t/d)^j \end{aligned} \tag{4.42}$$

Fortunately, for special parameters which we need, the formula is significantly simplified.

Lemma 4.4.3. *We have*

$$f_r(\mathbf{x};q,t;\pm t^{1/2};\pm(qt)^{1/2}) = G_r(\mathbf{x};q,t). \tag{4.43}$$

Proof. According to the formula (4.41), we have

$$\begin{aligned} &f_r(\mathbf{x};q,t;\pm t^{1/2};\pm(qt)^{1/2}) \\ &= \sum_{\substack{k,l\geq 0\\ 2k+2l\leq r}} G_{r-2k-2l}(\mathbf{x};q,t)\frac{(1;q^2)_k(q^{2-r}t^{-n};q^2)_k(q^{-2r}t^{-2n};q^2)_k}{(q^2;q^2)_k(q^{-r}t^{-n};q^2)_k(q^{2-2r}t^{-2n};q^2)_k}t^k \\ &\quad\times\frac{(1;q)_l(q^{-r}t^{-n};q)_{2k+l}}{(q;q)_l(q^{1-r}t^{-n};q)_{2k+l}}\frac{1-q^{-r+2k+2l}t^{-n}}{1-q^{-r}t^{-n}}\left(\frac{t}{q}\right)^l. \end{aligned}$$

Because $(1;q)_n = 0$ for $n>0$ by definition, and $(1;q)_0 = 1$ by convention, only the term with $k=l=0$ survives. Thus we get the claimed equality. □

Lemma 4.4.4. *We have the following identity*

$$\begin{aligned} &f_r\left(\mathbf{y};q,t;\pm 1,\pm t^{1/2}\right) = G_r(\mathbf{y};q,t) + \\ &\sum_{0\leq k\leq\left\lfloor\frac{r}{2}\right\rfloor-1} G_{r-2-2k}(\mathbf{y};q,t)\frac{(q/t;q^2)_k(q^{2-2r}t^{-2n};q^2)_k}{(q^2;q^2)_{k+1}(q^{3-2r}t^{1-2n};q^2)_{k+1}}\cdot\frac{t^{2k+1}(qt-1)(1-q^{4-2r+4k}t^{-2n})}{q}. \end{aligned}$$

Proof. According to the HNS formula, we have

$$\begin{aligned}
&f_r\left(\mathbf{y};q,t;\pm 1,\pm t^{1/2}\right)\\
&= \sum_{\substack{k,l\geq 0\\ 2k+2l\leq r}} G_{r-2k-2l}(\mathbf{y};q,t)\frac{(q/t;q^2)_k(q^{3-r}t^{-n};q^2)_k(q^{2-2r}t^{-2n};q^2)_k}{(q^2;q^2)_k(q^{1-r}t^{-n};q^2)_k(q^{3-2r}t^{1-2n};q^2)_k}t^{2k}\\
&\quad\times\frac{(1/q;q)_l(q^{-r}t^{-n};q)_{2k+l}}{(q;q)_l(q^{2-r}t^{-n};q)_{2k+l}}\frac{1-q^{-r+2k+2l}t^{-n}}{1-q^{-r}t^{-n}}t^l
\end{aligned}$$

Because of the factor $(1/q;q)_l$, l can only be 0 or 1. For these two values of l, the following simplification takes place:

$$\frac{(q^{3-r}t^{-n};q^2)_k}{(q^{1-r}t^{-n};q^2)_k}\cdot\frac{(q^{-r}t^{-n};q)_{2k+l}}{(q^{2-r}t^{-n};q)_{2k+l}}\cdot\frac{1-q^{-r+2k+2l}t^{-n}}{1-q^{-r}t^{-n}}=1,$$

for all k. Thus

$$\begin{aligned}
&f_r\left(\mathbf{y};q,t;\pm 1,\pm t^{1/2}\right)\\
&= \sum_{0\leq k\leq\lfloor\frac{r}{2}\rfloor} G_{r-2k}(\mathbf{y};q,t)\frac{(q/t;q^2)_k(q^{2-2r}t^{-2n};q^2)_k}{(q^2;q^2)_k(q^{3-2r}t^{1-2n};q^2)_k}t^{2k}\\
&\quad-\sum_{0\leq k\leq\lfloor\frac{r}{2}\rfloor-1} G_{r-2-2k}(\mathbf{y};q,t)\frac{(q/t;q^2)_k(q^{2-2r}t^{-2n};q^2)_k}{(q^2;q^2)_k(q^{3-2r}t^{1-2n};q^2)_k}\cdot\frac{t^{2k+1}}{q}.
\end{aligned}$$

Finally, we can gather the two terms of each G_i together and get:

$$\begin{aligned}
&f_r\left(\mathbf{y};q,t;\pm 1,\pm t^{1/2}\right)=G_r(\mathbf{y};q,t)-\\
&\sum_{0\leq k\leq\lfloor\frac{r}{2}\rfloor-1} G_{r-2-2k}(\mathbf{y};q,t)\frac{(q/t;q^2)_k(q^{2-2r}t^{-2n};q^2)_k}{(q^2;q^2)_{k+1}(q^{3-2r}t^{1-2n};q^2)_{k+1}}\cdot\frac{t^{2k+1}(1-qt)(1-q^{4-2r+4k}t^{-2n})}{q}.
\end{aligned}$$

□

Two particularly nice cases are $t=q$ and $t=q^{-1}$:

$$\begin{aligned}
f_r\left(\mathbf{y};t,t;\pm 1,\pm t^{1/2}\right)&=G_r(\mathbf{y};t,t)-G_{r-2}(\mathbf{y};t,t),\\
f_r\left(\mathbf{y};t^{-1},t;\pm 1,\pm t^{1/2}\right)&=G_r(\mathbf{y};t^{-1},t).
\end{aligned}$$

4.4.2 Near-rectangular (C_n,B_n) bounded Littlewood identities

Having set up all the necessary tools, we can now give a proof for the following conjectured formula of Rains and Warnaar, which is a generalization of Thm.4.3.3 (which corresponds to the case $r=0$).

Theorem 4.4.5. *For positive integers m, n and r an integer such that $0\leq r\leq n$, one has*

$$\sum_{\substack{\lambda\\ odd(\lambda)=r}} b^{oa}_{\lambda;m,r}(q,t)P_\lambda(\mathbf{x};q,t)=(x_1\dots x_n)^m P^{(C_n,B_n)}_{m^{n-r}(m-1)^r}(\mathbf{x};q,t,qt),\qquad(4.44)$$

where

$$b^{oa}_{\lambda;m,r}(q,t) = b^{oa}_{\lambda}(q,t) \prod_{\substack{s\in\lambda/1^r \\ a'_\lambda(s) even}} \frac{1-q^{2m-a'_\lambda(s)}t^{l'_\lambda(s)}}{1-q^{2m-a'_\lambda(s)-1}t^{l'_\lambda(s)+1}}.$$

Proof. Following Eqs.(4.38),(4.43), (4.39), Eq.(4.44) is equivalent to

$$\begin{aligned}
&(-1)^{|\lambda|+r}\frac{(tq^{2m-1};t)_r}{(q^{2m};t)_r} I_K^{(m)}\left(g_r(\mathbf{y}^{\pm};t,q)P_{\lambda'}(\mathbf{y}^{\pm 1};t,q);t,q;\pm q^{1/2},\pm(qt)^{1/2}\right)\\
&= \begin{cases} b^{\mathrm{oa}}_{\lambda;m,r}(q,t) & \text{if } \mathrm{odd}(\lambda)=r,\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

for $\lambda \subset (2m)^n$. Using the g-Pieri rule

$$g_r P_{\lambda'} = \sum_{\substack{\mu \\ \mu/\lambda \text{ vertical r-strip}}} \varphi_{\mu'/\lambda'} P_{\mu'},$$

we get

$$\begin{aligned}
&I_K^{(m)}\left(g_r(\mathbf{y}^{\pm};t,q)P_{\lambda'}(\mathbf{y}^{\pm 1};t,q);t,q;\pm q^{1/2},\pm(qt)^{1/2}\right)\\
&= \sum_{\substack{\mu \\ \mu/\lambda \text{ vertical r-strip}}} \varphi_{\mu'/\lambda'}(t,q) I_K^{(m)}\left(P_{\mu'}(\mathbf{y}^{\pm 1};t,q);t,q;\pm q^{1/2},\pm(qt)^{1/2}\right)\\
&= \sum_{\substack{\mu,\ \mu \text{ even} \\ \mu/\lambda \text{ vertical r-strip}}} \varphi_{\mu'/\lambda'}(t,q) \frac{C^0_\nu(q^{2m};t,q^2)}{C^0_\nu(q^{2m-1}t;t,q^2)} \frac{C^-_\nu(qt;t,q^2)}{C^-_\nu(q^2;t,q^2)},
\end{aligned}$$

where $\nu = (\mu/2)' = (\mu'_1, \mu'_3, \dots)$. Note that the condition μ is even is equivalent to $\mu'_{2i-1} = \mu'_{2i}$ for $i = 1, 2 \dots$.

Given λ, the two conditions in the sum uniquely fix μ. More specifically, $\mu_i = 2\lceil \lambda_i/2 \rceil$. In particular, $|\mu/\lambda|$ is the number of odd rows of λ. Thus for the sum to be non vanishing, the number of odd rows of λ must be r. For such λ we have

$$\begin{aligned}
&I_K^{(m)}\left(g_r(\mathbf{y}^{\pm};t,q)P_{\lambda'}(\mathbf{y}^{\pm 1};q,t);t,q;\pm q^{1/2},\pm(qt)^{1/2}\right)\\
&= \varphi_{\mu'/\lambda'}(t,q) \frac{C^0_\nu(q^{2m};t,q^2)}{C^0_\nu(q^{2m-1}t;t,q^2)} \frac{C^-_\nu(qt;t,q^2)}{C^-_\nu(q^2;t,q^2)}.
\end{aligned}$$

In the proof of Thm.4.3.3, we have shown that

$$\frac{C^0_\nu(q^{2m};t,q^2)}{C^0_\nu(q^{2m-1}t;t,q^2)} \frac{C^-_\nu(qt;t,q^2)}{C^-_\nu(q^2;t,q^2)} = b^{\mathrm{oa}}_{\mu;m}(q,t).$$

Therefore we are left with proving that

$$\frac{(tq^{2m-1};t)_r}{(q^{2m};t)_r} \varphi_{\mu'/\lambda'}(t,q) b^{\mathrm{oa}}_{\mu;m}(q,t) = b^{\mathrm{oa}}_{\lambda;m,r}(q,t). \tag{4.45}$$

First we prove that

$$\varphi_{\mu'/\lambda'}(t,q)b^{\mathrm{oa}}_{\mu}(q,t) = b^{\mathrm{oa}}_{\lambda}(q,t). \tag{4.46}$$

Indeed, we have

$$\varphi_{\mu'/\lambda'}(t,q) = \varphi'_{\mu/\lambda}(q,t) = \prod_{s\in R_{\mu/\lambda}} \frac{b_\lambda(s;q,t)}{b_\mu(s;q,t)},$$

where $R_{\mu/\lambda}$ is the set of rows of μ which intersect μ/λ. Consider such a row, and denote its boxes from left to right by $1,2,\dots,2k$. Note that this row of λ contains boxes $1,\dots,2k-1$. Then the contribution of this row to $\varphi'_{\mu/\lambda}(q,t)$ is (we will omit the parameters (q,t) for conciseness)

$$\frac{b_\lambda(1)\dots b_\lambda(2k-1)}{b_\mu(1)\dots b_\mu(2k)}.$$

We see that $b_\lambda(2i-1) = b_\mu(2i)$ for $i=1,\dots,k$. Furthermore $b_\lambda(2)b_\lambda(4)\dots$ and $b_\mu(1)b_\mu(3)\dots$ are exactly the contribution of this row to b^{oa}_λ and b^{oa}_μ respectively. Since this is true for every row in $R_{\mu/\lambda}$, and since the contributions of remaining rows to b^{oa}_λ and b^{oa}_μ are equal, (4.46) is true.

It remains to prove that the factors containing m and r match. Indeed, we have

$$\prod_{\substack{s\in\mu\\ a'_\mu(s)\text{ odd}}} \frac{1-q^{2m-a'_\mu(s)+1}t^{l'_\mu(s)}}{1-q^{2m-a'_\mu(s)}t^{l'_\mu(s)+1}} = \prod_{\substack{s\in\lambda\\ a'_\lambda(s)\text{ even}}} \frac{1-q^{2m-a'_\lambda(s)}t^{l'_\lambda(s)}}{1-q^{2m-a'_\lambda(s)-1}t^{l'_\lambda(s)+1}}.$$

Multiplying this with $\frac{(tq^{2m-1};t)_r}{(q^{2m};t)_r}$ indeed gives the m,r-dependent part of $b^{\mathrm{oa}}_{\lambda;m,r}(q,t)$, which is

$$\prod_{\substack{s\in\lambda/1^r\\ a'_\lambda(s)\text{ even}}} \frac{1-q^{2m-a'_\lambda(s)}t^{l'_\lambda(s)}}{1-q^{2m-a'_\lambda(s)-1}t^{l'_\lambda(s)+1}}.$$

The proof is therefore completed. □

4.4.3 Near-rectangular (B_n, B_n) Littlewood identity

Now we try to find the near-rectangular generalization of Thm.4.3.4. In this case, we have to deal with the computation of

$$\begin{aligned}&[f_r(y;t,q;\pm1,\pm q^{1/2})]P_{\lambda'}(y^{\pm1};t,q)\\ =&\frac{I^{(m)}_K\left(f_r\left(y^{\pm1};t,q;\pm1,\pm q^{1/2}\right)P_{\lambda'}\left(y^{\pm1};t,q\right);t,q;\pm1,\pm q^{1/2}\right)}{\tilde N^{(m)}_{(r)}\left(t,q;\pm1,\pm q^{1/2}\right)}\end{aligned}$$

Following Lem.4.4.4, the first step is to compute $I^{(m)}_K\left(g_k\left(\mathbf{y}^{\pm1};t,q\right)P_{\lambda'}\left(\mathbf{y}^{\pm1};t,q\right);t,q;\pm1,\pm q^{1/2}\right)$. Although I cannot find a complete answer for this, I can characterize the condition under which this quantity does not vanish and compute it for a special case.

Theorem 4.4.6. *We have*

$$I_K^{(m)}\left(g_k\left(\mathbf{y}^{\pm 1};t,q\right)P_{\lambda'}\left(\mathbf{y}^{\pm 1};t,q\right);t,q;\pm 1,\pm q^{1/2}\right)=0$$

unless $|\lambda|+k$ is even and one of the two following sets of conditions holds (they are not exclusive, however):

1. *$\lambda_1-\lambda_2\leq 1$, $\lambda_3-\lambda_4\leq 1$,... and the number of pairs $(\lambda_{2i-1},\lambda_{2i}), i\geq 1$ whose difference is 1, denoted by d_e, satisfies $d_e\leq k$. The subscript e is for "even"; the reason will be explained in the proof.*
2. *$\lambda_2-\lambda_3\leq 1$, $\lambda_4-\lambda_5\leq 1$,... and the number of pairs $(\lambda_{2i},\lambda_{2i+1}), i\geq 1$ whose difference is 1, denoted by d_o, satisfies $d_o\leq k$. The subscript o is for "odd".*

Proof. Using the g-Pieri rule and Thm.4.2.4, we have

$$\begin{aligned}
&I_K^{(m)}\left(g_k\left(y^{\pm 1};t,q\right)P_{\lambda'}\left(y^{\pm 1};t,q\right);t,q;\pm 1,\pm q^{1/2}\right)\\
&=\sum_{\substack{\mu,\mu_1\leq 2m\\ \mu/\lambda\text{ vertical k-strip}}}\varphi_{\mu'/\lambda'}(t,q)I_K^{m}\left(P_{\mu'}\left(y^{\pm 1};t,q\right);t,q;\pm 1,\pm q^{1/2}\right)\\
&=\sum_{\substack{\mu,\mu_1\leq 2m\\ \mu/\lambda\text{ vertical k-strip}\\ \tilde{\mu'}\text{ even}}}\varphi_{\mu'/\lambda'}(t,q)A^{(2m)}_{\mu'/2}(t,q),
\end{aligned}\tag{4.47}$$

where

$$\tilde{\mu'}:=\left(\mu'_1-\mu'_{2m},\ldots,\mu'_{2m-1}-\mu'_{2m},0\right),$$

and for ν a partition or half-partition of length at most n,

$$A_\nu^{(n)}(q,t):=\prod_{1\leq i<j\leq n}\frac{\left(qt^{j-i-1},t^{j-i+1};q^2\right)_{\nu_i-\nu_j}}{\left(qt^{j-i},t^{j-i};q^2\right)_{\nu_i-\nu_j}}.$$

If $\tilde{\mu'}$ is even then $|\mu|=|\lambda|+k$ is even. So for the sum (4.47) to be non zero, λ and k must satisfy this condition.

Now, $\tilde{\mu'}$ is even if and only if either μ' is even or μ' is odd. Note that μ' is even if and only if $\mu_1=\mu_2$, $\mu_3=\mu_4,\ldots$. And μ' is odd if and only if $\mu_2=\mu_3$, $\mu_4=\mu_5,\ldots$.

Given λ, let us analyze how a partition μ satisfying three conditions in the sum of Eq.(4.47) is obtained. We examine separately two cases.

1. Case 1: μ' is even.

   Because μ'/λ' is a vertical strip, we must have $\lambda_1-\lambda_2\leq 1$, $\lambda_3-\lambda_4\leq 1,\ldots$. We must also have $d_e\leq k$. Then, the condition that $|\lambda|+k$ is even implies that $k-d_e$ is even. We see that μ is obtained by adding a box to each row λ_{2i} if $\lambda_{2i}=\lambda_{2i-1}-1$. The remaining $k-d_e$ boxes are added arbitrarily (but making μ/λ a vertical strip) to pairs $\lambda_{2i}=\lambda_{2i-1}$ (including pairs of empty rows).

   For later use, we observe one case in which μ is uniquely determined: $d_e=k$. In that case, $\mu'/2=\lceil\lambda'/2\rceil$.

2. Case 2: μ' is odd, i.e. $\mu'_1, \dots, \mu'_{2m}$ are odd.
   In particular $\mu'_{2m} \geq 1$, thus $\mu_1 = 2m$. Therefore $\lambda_1 = 2m - 1$ or $2m$. And again, because μ'/λ' is a vertical strip, we must have $\lambda_2 - \lambda_3 \leq 1$, $\lambda_4 - \lambda_5 \leq 1, \dots$. We also have $d_o \leq k$ and $\lambda_1 + k - d_o$ is even. If $\lambda_1 = 2m - 1$, we add a box to it. The way we add remaining boxes to the other rows of λ is similar to the first case above.
   For later use, we observe two cases in which μ is uniquely determined: $\lambda_1 = 2m$ and $d_o = k$; or $\lambda_1 = 2m - 1$ and $d_o = k - 1$. In both cases, we have $\mu'/2 = \lfloor \lambda'/2 \rfloor + (1/2)^{2m}$.

In conclusion, the claim is proved. □

Remark. Note that a partition λ can give rise to both even and odd μ' (there may be many even and many odd). This is reason for which I can not figure out how to simplify Eq.(4.47). For example, consider $m = 1$ and $k = 1$. For $\lambda = (2, 1)$, we can get either $\mu = (2^2)$ or $\mu = (2, 1^2)$.

However, there is one case in which the sum in Eq.(4.47) is simplified. It is the cases where only even μ' arises, and is uniquely determined. For example this condition is satisfied if $d_e = k$.

Proposition 4.4.7. *Suppose that $d_e = k$. Then*

$$I_K^{(m)}\left(g_k\left(y^{\pm 1}; t, q\right) P_{\lambda'}\left(y^{\pm 1}; t, q\right); t, q; \pm 1, \pm q^{1/2}\right) = \varphi'_{\mu/\lambda}(q, t) A^{(2m)}_{\mu'/2}(t, q),$$

where $\mu' = 2\lceil \lambda'/2 \rceil$.

We want to transform the result to an expression without reference to μ. At the moment, I can only do that for the following partitions:

Theorem 4.4.8. 1. *Let $a_1 > a_2 > \cdots > a_r > 0$ be integers. Consider the partition $\lambda = (a_1, a_1 - 1, a_2, a_2 - 1, \dots, a_r, a_r - 1)$. Then there is a unique partition μ such that μ' is even and μ/λ is a r-vertical strip, that is $\mu = (a_1, a_1, a_2, a_2, \dots, a_r, a_r)$. The consequence is that*

$$\begin{aligned} &I_K^{(m)}\left(g_k\left(y^{\pm 1}; t, q\right) P_{\lambda'}\left(y^{\pm 1}; t, q\right); t, q; \pm 1, \pm q^{1/2}\right) \\ =&b_\lambda^{ocl}(q, t) \prod_{\substack{s \in \lambda \\ l'(s) even}} \frac{1 - q^{2m - a'_\lambda(s)} t^{l'_\lambda(s)}}{1 - q^{2m - a'_\lambda(s) - 1} t^{l'_\lambda(s)} + 1}, \end{aligned} \tag{4.48}$$

where "ocl" stands for odd co-leg:

$$b_\lambda^{ocl}(q, t) := \prod_{\substack{s \in \lambda \\ l'(s)\ odd}} b_\lambda(s; q, t).$$

2. *Let $2m > a_1 > a_2 > \cdots > a_r > 0$ be integers. For the partition $\lambda = (2m, a_1, a_1 - 1, \dots, a_r, a_r - 1)$, there is a unique partition μ such that μ' is odd and μ/λ is a r-vertical strip, that is $\mu = (2m, a_1, a_1, \dots, a_r, a_r)$. The consequence is that*

$$\begin{aligned} &I_K^{(m)}\left(g_k\left(y^{\pm 1}; t, q\right) P_{\lambda'}\left(y^{\pm 1}; t, q\right); t, q; \pm 1, \pm q^{1/2}\right) \\ =&b_\lambda^{pecl}(q, t) \prod_{\substack{s \in \lambda \\ l'(s) odd}} \frac{1 - q^{2m - a'_\lambda(s)} t^{l'_\lambda(s) - 1}}{1 - q^{2m - a'_\lambda(s) - 1} t^{l'_\lambda(s)}}, \end{aligned} \tag{4.49}$$

where "pecl" stands for positive even co-leg, i.e., 3rd, 5th,... rows:

$$b_\lambda^{pecl}(q,t) := \prod_{\substack{s\in\lambda \\ l'(s)=3,5,7,\ldots}} b_\lambda(s;q,t).$$

Proof. I will only give a proof for the first case. The second case is just a minor variation. Denote $\nu = (\mu'/2)'$. By definition of φ', we have in this case

$$\varphi'_{\mu/\lambda}(q,t) = \prod_{\substack{s\in\mu \\ l'_\mu(s) \text{ odd}}} \frac{b_\lambda(s;q,t)}{b_\mu(s;q,t)}$$

We have

$$A_{\nu'}^{(2m)}(t,q) = \prod_{s\in\nu} \frac{1-q^{2m-a'_\nu(s)}t^{2l'_\nu(s)}}{1-q^{2m-a'_\nu(s)-1}t^{2l'_\nu(s)+1}} \prod_{s\in\nu} \frac{1-q^{a_\nu(s)}t^{2l_\nu(s)+1}}{1-q^{a_\nu(s)+1}t^{2l_\nu(s)}}.$$

Under the 1-to-1 correspondence which associates to each box $(i,j)\in\nu$ the box $(2i,j)\in\mu$, the second product of $A_{\nu'}^{(2m)}(t,q)$ can be seen to be equal to

$$\prod_{\substack{s\in\mu \\ l'_\mu(s) \text{ odd}}} b_\mu(s;q,t).$$

Furthermore, under the 1-to-1 correspondence which associates to each box $(i,j)\in\nu$ the box $(2i-1,j)\in\lambda$, the first product in $A_{\nu'}^{(2m)}(t,q)$ can be written as

$$\prod_{\substack{s\in\lambda \\ l'(s)\text{even}}} \frac{1-q^{2m-a'_\lambda(s)}t^{l'_\lambda(s)}}{1-q^{2m-a'_\lambda(s)-1}t^{l'_\lambda(s)}+1}.$$

By combining these observations, one obtains the claimed identity (4.48). □

A

KP tau functions

I follow closely the exposition of Kazarian and Lando in [55]. Let V be the infinite dimensional vector space of semi-infinite Laurent series in one variable z. Elements of V are series of the form

$$c_{-k}z^{-k} + c_{-k+1}z^{-k+1} + \dots .$$

The vector space V has the standard basis consisting of the monomials z^k, $k \in \mathbb{Z}$.

Definition A.0.1. The *semi-infinite wedge space* $\Lambda^{\frac{\infty}{2}}V$ (of charge 0) is the span of vectors of the form

$$v_\mu = z^{m_1} \wedge z^{m_2} \wedge \dots ,$$

where μ is a partition and $m_i = \mu_i - i$. Note that the condition that μ is a partition is equivalent to $m_i \geq -i$ for all $i \geq 1$, $m_1 > m_2 > \dots$, and $m_i = -i$ for all sufficiently large i.

Define a scalar product on $\Lambda^{\frac{\infty}{2}}V$ by declaring that the basis $\{v_\mu\}$ is orthonormal.

A special role is played by the *vacuum vector* which corresponds to the empty partition

$$v_\emptyset := z^{-1} \wedge z^{-2} \wedge \dots .$$

A more picturesque description for $\Lambda^{\frac{\infty}{2}}V$ is available via the Maya diagrams and Dirac's sea. I refer the reader to [50], where the link between the semi-infinite wedge and double Hurwitz numbers is also discussed.

Let Λ be the algebra of symmetric functions. We consider the Schur functions s_λ as functions, more precisely (formal) power series, of power sums p_k (rather than the more "primitive" variables on which p_k depend). The p_k will be used as variables for the integrable PDEs.

We have the following isomorphism, called the *boson-fermion correspondence*, between $\Lambda_0^{\frac{\infty}{2}}V$ and Λ. For $i \in \mathbb{Z}$, denote by $\widehat{z^i}$ the shift operator acting on V

$$\begin{aligned}\widehat{z^i} : V &\to V \\ z^m &\mapsto z^{m+i}.\end{aligned}$$

for $m \in \mathbb{Z}$. This action extends to the space $\Lambda^{\frac{\infty}{2}}V$ by the Leibniz rule. [1]

For a vector $v \in \Lambda^{\frac{\infty}{2}}V$, denote

$$\langle v \rangle := \langle v, v_\emptyset \rangle .$$

Definition A.0.2. The boson-fermion correspondence is the following isomorphism:

$$BS : \Lambda_0^{\frac{\infty}{2}}V \to \Lambda \tag{A.2}$$

$$v \mapsto \left\langle \exp\left(\sum_{i=1}^{\infty} \frac{p_i}{i} \widehat{z^{-i}} \right) v \right\rangle . \tag{A.3}$$

For $i \in \mathbb{Z}_+$, this correspondence takes the operator $\widehat{z^{-i}}$ to the operator $i\frac{\partial}{\partial p_i}$, and $\widehat{z^i}$ to the multiplication by p_i. In fact, this correspondence is simply the isomorphism which sends v_λ to s_λ for every partition λ.

Now let us introduce directly the famous τ-functions of the KP hierarchy without defining the equations for which they solve.

Definition A.0.3. Let $\beta_1(z), \beta_2(z), \dots$ be Laurent series in the variable z such that for all i large enough,

$$\beta_i(z) = z^{-i} + c_{i1}z^{-i+1} + c_{i2}z^{-i+2} + \dots .$$

The τ-function (corresponding to $\{\beta_i\}$) is defined as

$$\tau(p_1, p_2, \dots) := BS\left(\beta_1 \wedge \beta_2 \wedge \dots\right). \tag{A.4}$$

In other words, expand $\beta_1 \wedge \beta_2 \wedge \dots$ in the basis $\{v_\lambda\}$:

$$\beta_1 \wedge \beta_2 \wedge \dots = \sum_\lambda c_\lambda v_\lambda .$$

Then,

$$\tau(p_1, p_2, \dots) = \sum_\lambda c_\lambda s_\lambda(p_1, p_2, \dots).$$

1. For any operator g acting on V, define its (linear) action on $\Lambda^{\frac{\infty}{2}}V$ in the following way:

$$g\left(z^{i_1} \wedge z^{i_2} \wedge \dots\right) := g\left(z^{i_1}\right) \wedge z^{i_2} \wedge \dots + z^{i_1} \wedge g\left(z^{i_2}\right) \wedge \dots . \tag{A.1}$$

An important family of τ-functions of the KP hierarchy which are closely related to the combinatorics of the symmetric groups is the Orlov-Shcherbin family [73]. For a set of variables y_i, $i \in \mathbb{Z}$, and a partition μ, let

$$y_\mu := \prod_{s\in\mu} y_{c(s)},$$

where $c(s)$ is the content of the box s. Recall that if s is at row i and column j, $c(s) := j - i$.

Theorem A.0.1. *The generating function*

$$\sum_\mu y_\mu \frac{\dim(\mu)}{|\mu|!} s_\mu(p_1, p_2, \dots),$$

where the sum is over all partitions, is a τ-function for the KP hierarchy.

Proof. It can be checked directly that this generating function is the τ-function corresponding to the following β_i:

$$\beta_i := \frac{1}{u_{-i}} \sum_{j=0}^{\infty} u_{j-i} \frac{z^{j-i}}{j!},$$

where $u_0 = 1$ and

$$u_i = \begin{cases} \prod_{j=1}^{i} y_j & i > 0, \\ 1/\prod_{j=i+1}^{0} y_j & i < 0. \end{cases}$$

$\square$

B

Gaussian integrals

Let A be a positive definite symmetric $n \times n$ matrix, and $(,)$ be the usual scalar product in $\mathbb{R}^n$, then the following famous identity holds

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(Ax,x)} dx_1 \dots dx_n = (2\pi)^{n/2} (\det A)^{-1/2} . \tag{B.1}$$

The integral on the left hand side is called a *Gaussian integral*. We will also need the following variant (by a simple shifting of variables). Let $b \in \mathbb{R}^n$, then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(Ax,x)+(b,x)} dx_1 \dots dx_n = e^{\frac{1}{2}(b,A^{-1}b)} (2\pi)^{n/2} (\det A)^{-1/2} . \tag{B.2}$$

The measure $d\mu = (2\pi)^{-n/2} (\det A)^{1/2} e^{-\frac{1}{2}(Ax,x)} dx$ is called the *Gaussian measure*. The matrix $C = A^{-1}$ is called the *covariance matrix*, and we have [1]

$$\langle x_1 \rangle = 0, \langle x_i x_j \rangle = C_{ij}.$$

The pleasant feature of the Gaussian measure is that we can compute $\langle f \rangle$ for any polynomial f in the following simple way. First, by symmetry, notice that if $f(x)$ is a monomial of odd degree then $\langle f \rangle = 0$. Second, we can reduce the computation for arbitrary polynomial to that of degree 2 by the Wick formula:

Theorem B.0.1. *Let $f_1, \dots, f_{2k}$ be linear functions of $x_1, \dots, x_n$. Then*

$$\langle f_1 \dots f_{2k} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \dots \langle f_{p_k} f_{q_k} \rangle,$$

where the sum is taken over all permutations $p_1 q_1 \dots p_k q_k$ of $\{1, 2, \dots, 2k\}$ such that $p_1 < p_2 \dots < p_k$, $p_1 < q_1, \dots p_k < q_k$.

1. Please do not confuse with the notation of the previous appendix. Here, $\langle\rangle$ denotes the expectation value with respect to the measure $d\mu$.

For more details, the reader can consult for example the book [59].

C

Saddle point method

A comprehensive account can be found in [91]. I only quote a theorem which is relevant to this thesis. The saddle point method concerns Laplace-type integrals, which are of the following form

$$J(\alpha) = \int_D g(x)e^{-\alpha f(x)}dx, \tag{C.1}$$

where D is a possibly unbounded domain in $\mathbb{R}^n$ and α is a large positive parameter. Assume that both f and g are infinitely differentiable in D. Furthermore, assume the following conditions

— The integral $J(\alpha)$ converges absolutely for all $\alpha > \alpha_0$.

— For every $\epsilon > 0$, $\rho(\epsilon) > 0$ where

$$\rho(\epsilon) = \inf\{f(x) - f(x_0) : x \in D \quad \text{and } |x - x_0| \geq \epsilon\}.$$

— The Hessian matrix

$$H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{x=x_0}$$

is positive definite.

The second condition implies that the function f has a minimum at a unique point, say x_0. Under these assumptions, we have

Proposition C.0.1. ***Laplace's asymptotic expansion:***

If x_0 is an interior point of D, then the integral $J(\alpha)$ has an asymptotic expansion of the form

$$J(\alpha) \sim e^{-\alpha f(x_0)} \sum_{k=0}^{\infty} c_k \alpha^{-n/2-k}, \quad \textit{as } \alpha \to \infty,$$

where the c_k are constants. In particular,

$$c_0 = (2\pi)^{n/2} \, (\det H)^{-1/2} \, g(x_0).$$

D

All-order expansion and the topological recursion

One of the motivations behind our work on tensor models is to see whether the Chekhov-Eynard-Orantin topological recursion, or a modified version of it, works in this context as well. In fact, after the work presented in this thesis, Dartois and Bonzom [8] have succeeded in showing that the quartic melonic tensor model satisfies an extension named the blobbed topological recursion proposed by Borot [12]. For the sake of completeness, in this appendix, I define this topological recursion in following [34]. The reader is invited to look at this review article for much more information.

First, we recall some facts about Riemann surfaces.

Definition D.0.1. A *spectral curve* $\mathcal{E}$ is a triple $(\mathcal{L}, x, y)$ where $\mathcal{L}$ is a compact Riemann surface and x, y are two analytic functions on some open domains of $\mathcal{L}$.

Definition D.0.2. A spectral curve $(\mathcal{L}, x, y)$ is called *regular* if two conditions hold

- the differential form dx has a finite number of zeros, all of which are simple,
- the differential form dy does not vanish at the zeros of dx.

From now on, let $(\mathcal{L}, x, y)$ be a regular spectral curve. Denote by $a_1, \dots, a_m \in \mathcal{L}$ the zeros of dx. They are also called branch points. Their simplicity condition implies that for any z close to a_i, there is exactly one point $\overline{z} \neq z$ in the neighborhood of a_i such that

$$x(\overline{z}) = x(z).$$

The point $\overline{z}$ is called the *conjugate* of z. A compact Riemann surface $\mathcal{L}$ of genus $\overline{g} \geq 1$ [1] can be equipped with a symplectic basis (not unique, but we will fix an arbitrary choice)

1. The letter g without bar will be reserved for the "genus" of the topological recursion; the two things are unrelated

of $2\overline{g}$ non-contractible cycles such that

$$A_i \cap B_j = \delta_{ij}, \quad A_i \cap A_j = 0, \quad B_i \cap B_j = 0.$$

The surface $\mathcal{L}$ has a $\overline{g}$-dimensional vector space of holomorphic differential forms. There exists a unique basis $du_1, \ldots, du_{\overline{g}}$ such that

$$\oint_{A_i} du_j = \delta_{ij}.$$

The corresponding Riemann matrix of periods τ is defined to be

$$\tau_{i,j} = \oint_{B_i} du_j.$$

This matrix is symmetric and its imaginary part is positive definite.

Proposition D.0.1. *On $\mathcal{L} \times \mathcal{L}$, there is a unique bilinear differential $B(z_1, z_2)$ having one double pole at $z_1 = z_2$ and no other pole, and such that,*

$$B(z_1, z_2) \sim_{z_1 \to z_2} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \textit{regular}, \quad \textit{and}\ \forall i = 1, \ldots, \overline{g}, \quad \oint_{A_i} B(z_1, z_2) = 0.$$

The bidifferential B is called the Bergman kernel.

Definition D.0.3. For any $z_0 \in \mathcal{L}$ and any z close to a branch point, define the recursion kernel

$$K(z_0, z) := -\frac{1}{2} \frac{\int_{z'=\overline{z}}^{z} B(z_0, z')}{(y(z) - y(\overline{z}))dx(z)},$$

where the path in the integral is taken in a small neighborhood of the concerned branch point.

Now we have all the necessary notions to define the topological recursion. We are going to define recursively a sequence of symmetric meromorphic n-forms $\omega_n^{(g)}(z_1, \ldots, z_n)$ on $\mathcal{L}^{\otimes n}$ with $n = 1, 2, \ldots$ and $g = 0, 1, 2, \ldots$.

Definition D.0.4. Let $(\mathcal{L}, x, y)$ be a regular spectral curve. Denote $J = \{z_1, \ldots, z_n\}$. Define

$$\omega_1^{(0)}(z) := -y(z)dx(z), \quad \omega_2^{(0)}(z_1, z_2) := B(z_1, z_2), \tag{D.1}$$

and for $2g - 2 + n \geq 0$,

$$\omega_{n+1}^{(g)}(z_0, J) := \sum \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[\omega_{n+2}^{(g-1)}(z, \overline{z}, J) + \sum_{h=0}^{g} \sideset{}{'}\sum_{I \subset J} \omega_{1+|I|}^{(h)}(z, I) \omega_{1+n-|I|}^{(g-h)}(\overline{z}, J - I) \right], \tag{D.2}$$

where the leftmost sum is over all the branch points (zeros of dx). The apostrophe over the rightmost sum means that we exclude the terms with $(h, I) = (0, \emptyset)$ or (g, J).

Notice that Eq.(D.2) is a recurrence in $2g-2+n$ (of course this quantity is understood to be associated to $\omega_n^{(g)}$)).

The $\omega_n^{(g)}$ satisfy many remarkable properties, one of which is that they are symmetric. This is not obvious from the definition because, in Eq.(D.2), the first variable is treated in a completely different way from others. The interested reader is invited to consult [34] for more details.

List of Figures

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UNIVERSITE BRETAGNE LOIRE

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Thèse de Doctorat

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Contributions aux modèles de tenseurs, nombres de Hurwitz et polynômes de Macdonald-Koornwinder

Contributions to tensor models, Hurwitz numbers and Macdonald-Koornwinder polynomials

Résumé

Dans cette thèse, j'étudie trois sujets reliés : les modèles de tenseurs, les nombres de Hurwitz et les polynômes de Macdonald-Koornwinder. Les modèles de tenseurs géneralisent les modèles de matrices en tant qu'une approache à la gravité quantique en dimension arbitraire (les modèles de matrices donnent une version bidimensionnelle). J'étudie un modèle particulier qui s'appelle le modèle quartique mélonique. Sa spécialité est qu'il s'écrit en termes d'un modèle de matrices qui est lui-même aussi intéressant. En utilisant les outils bien établis, je calcule les deux premiers ordres de leur $1/N$ expansion. Parmi plusieurs intépretations, les nombres de Hurwitz comptent le nombre de revêtements ramifiés de surfaces de Riemann. Ils sont connectés avec de nombreux sujets en mathématiques contemporaines telles que les modèles de matrices, les équations intégrables et les espaces de modules. Ma contribution principale est une formule explicite pour les nombres doubles avec 3-cycles complétées d'une part. Cette formule me permet de prouver plusieurs propriétés intéressantes de ces nombres. Le dernier sujet de mon étude est les polynôme de Macdonald et Koornwinder, plus précisément les identités de Littlewood. Ces polynômes forment les bases importantes de l'algèbre des polynômes symmétriques. Un des problèmes intrinsèques dans la théorie des fonctions symmétriques est la décomposation d'un polynôme symmétrique dans la base de Macdonald. La décompostion obtenue (notamment si les coefficients sont raisonablement explicits et compacts) est nommée une identité de Littlewood. Dans cette thèse, j'étudie les identités démontrées récemment par Rains et Warnaar. Mes contributions incluent une preuve d'une extension d'une telle identité et quelques progrès partiels vers la généralisation d'une autre.

Mots clés

modèles de tenseurs et matrices, nombres de Hurwitz, fonctions symmétriques, polynômes de Macdonald-Koornwinder, identités de Littlewood.

Abstract

In this thesis, I study three related subjects: tensor models, Hurwitz numbers and Macdonald-Koornwinder polynomials. Tensor models are generalizations of matrix models as an approach to quantum gravity in arbitrary dimensions (matrix models give a $2D$ version). I study a specific model called the quartic melonic tensor model. Its specialty is that it can be transformed into a multi-matrix model which is very interesting by itself. With the help of well-established tools, I am able to compute the first two leading orders of their $1/N$ expansion. Among many interpretations, Hurwitz numbers count the number of weighted ramified coverings of Riemann surfaces. They are connected to many subjects of contemporary mathematics such as matrix models, integrable equations and moduli spaces of complex curves. My main contribution is an explicit formula for one-part double Hurwitz numbers with completed 3-cycles. This explicit formula also allows me to prove many interesting properties of these numbers. The final subject of my study is Macdonald-Koornwinder polynomials, in particular their Littlewood identities. These polynomials form important bases of the algebra of symmetric polynomials. One of the most important problems in symmetric function theory is to decompose a symmetric polynomial into the Macdonald basis. The obtained decomposition (in particular, if the coefficients are explicit and reasonably compact) is called a Littlewood identity. In this thesis, I study many recent Littlewood identities of Rains and Warnaar. My own contributions include a proof of an extension of one of their identities and partial progress towards generalization of one another.

Key Words

tensor models, matrix models, Hurwitz numbers, symmetric functions, Macdonald-Koornwinder polynomials, Littlewood identities.