

Optimal stopping problem in mean field games

Charles Bertucci

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Contributions à la théorie des jeux à champ moyen

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Contributions à la théorie des jeux à champ moyen

Charles Bertucci
sous la direction de Pierre-Louis Lions

11 décembre 2018

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Introduction

Introduction générale

L'objet principal de cette thèse est l'étude de modèles nouveaux de jeux faisant intervenir une infinité de joueurs. Les modèles que nous allons étudier s'inscrivent dans la théorie récente des jeux à champ moyen, en abrégé MFG pour l'anglais *Mean Field Games*. Les éléments caractéristiques d'un jeu à champ moyen sont : le nombre infini de joueurs, l'indiscernabilité de ceux-ci et les interactions entre les joueurs, qui sont supposées être de type "champ moyen". Plus précisément, les interactions entre joueurs ont uniquement lieu aux travers de couplages qui dépendent de mesures induites par les autres joueurs, comme la mesure induite par leur état ou par leur contrôles. Du point de vue du jeu sous-jacent, ces hypothèses sont cruciales. L'indiscernabilité des joueurs et la nature des interactions nous amènent alors à considérer la fonction valeur d'un joueur générique qui dépend, en général, de mesures associées à la "foule" des autres joueurs. Ces mesures évoluent suivant les comportements optimaux des joueurs.

La théorie des jeux à champ moyen a permis de modéliser de nombreuses situations faisant intervenir un grand nombre de joueurs. On peut par exemple approcher sous certaines hypothèses les équilibres de Nash d'un jeu avec un grand nombre de joueurs par ceux du jeu à champ moyen correspondant. Les jeux à champ moyen sont par ailleurs particulièrement adaptés pour modéliser des situations économiques où de nombreux agents font face à un problème de maximisation d'utilité qui dépend des autres joueurs au travers de grandeurs macro-économiques. Il convient aussi de préciser que la théorie des jeux à champ moyen est évidemment concernée par des jeux comportant des aspects stochastiques. On peut distinguer deux grandes classes de structures stochastiques dans les jeux à champ moyen. La première comprend les situations sans aléa ou celles où celui-ci est indépendant d'un joueur à l'autre. La seconde classe est composée des jeux à champ moyen où les aléas des différents joueurs sont corrélés. L'indépendance des bruits permet une grande simplification mathématique des problèmes de la première classe. On sépare donc souvent l'analyse de ces deux classes de problèmes, même si on peut

bien sûr combiner les deux structures.

La prochaine partie de cette introduction porte sur quelques notations utilisées dans ce chapitre. On présente ensuite quelques enjeux et résultats majeurs de la théorie des jeux à champ moyen ainsi que le contexte dans lequel s'inscrit ce travail. Le reste de cette introduction porte sur les principaux résultats de cette thèse.

Notations

On introduit ici quelques notations et conventions utilisées dans le reste de cette introduction.

— Le tore unitaire d dimensionnel est noté $\mathbb{T}^d$

— $\mathcal{P}(A)$ désigne l'ensemble des mesures de probabilités sur un ensemble A.

— La dérivée directionnelle d'une fonction $U : \mathcal{P}(A) \to \mathbb{R}$ est notée (lorsqu'elle existe)

$$\frac{\delta U}{\delta m}.$$

— La dérivée intrinsèque d'une fonction $U : \mathcal{P}(A) \to \mathbb{R}$ est notée (lorsqu'elle existe)

$$D_m U.$$

— On note

$$||f(\cdot,\cdot,\cdot)||_{(n+\alpha,n+\alpha,n+\alpha)} = \sum_{|l|,|l'|,|l''|\leq n} ||D^{(l,l',l'')}f||_\infty + \sum_{|l|=|l'|=|l''|=n} \sup_{(x,y,z)\neq(x',y',z')} \frac{D^{(l,l',l'')}f(x,y,z) - D^{(l,l',l'')}f(x',y',z')}{|x-x'|^\alpha + |y-y'|^\alpha + |z-z'|^\alpha}.$$

— Pour une fonction $U : \mathcal{P}(A) \to \mathbb{R}$, on note

$$Lip_n(U) = \sup_{m_1\neq m_2} \mathbf{d}_1^{-1}(m_1,m_2)\Big|\Big|\frac{\delta U}{\delta m}(m_1,\cdot) - \frac{\delta U}{\delta m}(m_2,\cdot)\Big|\Big|_{(n+\alpha)}$$

où $\alpha \in (0,1)$ et $\mathbf{d}_1(m_1,m_2)$ désigne la distance de Monge-Kantorovich entre m_1 et m_2.

— Dans un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$, on note

$$X \sim m$$

si m est la loi de probabilité de la variable aléatoire X.

— Une application $f : E \to E'$ d'un espace vectoriel E dans son dual E' est dite monotone si

$$\Big(f(x) - f(y)\Big)(x - y) \geq 0.$$

— Une application $f : E \to E'$ d'un espace vectoriel métrique $(E, d(\cdot, \cdot))$ dans son dual topologique E' est dite α monotone si

$$\Big(f(x) - f(y)\Big)(x - y) \geq \alpha d(x, y).$$

— On utilise les notations standards $W^{k,p}$ pour les espaces de Sobolev, avec la convention classique $W^{k,2} = H^k$.
— Pour $T > 0$, on utilise également la notation $W^{n,m,p}((0,T), \mathbb{T}^d)$ pour l'ensemble des fonctions qui ont n dérivées faible en temps et m dérivées faibles en espace dans L^p.

0.1 Quelques résultats majeurs de la théorie des jeux à champ moyen

La théorie des jeux à champ moyen a donné lieu à de nombreux modèles qu'il n'est pas possible de tous mentionner ici. Nous allons dans cette partie nous intéresser particulièrement à deux types de modèles.

Le premier est un modèle en temps continu dans un espace d'états continu qui sera $\mathbb{T}^d$. Chaque joueur contrôle sa trajectoire qui évolue suivant une équation différentielle stochastique et a des coûts qui dépendent de la mesure associée à l'état des autres joueurs. Dans ce cadre, en général, c'est à dire en présence d'un bruit commun, une équation typique qui décrit les équilibres de jeu à champ moyen est :

$$\begin{cases} -\partial_t U - (\nu + \beta)\Delta_x U + H(x, \nabla_x U, m) - (\nu + \beta) \int_{\mathbb{R}^d} div_y(D_m U) dm(y) + \\ + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, \nabla_x U, m) dm(y) - 2\beta \int_{\mathbb{R}^d} div_x(D_m U) dm(y) \\ \quad - \beta \int_{\mathbb{R}^{2d}} Tr[D^2_{mm} U] dm \otimes dm = 0 \text{ dans } (0,T) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d); \\ U(T, x, m) = g(x, m) \text{ dans } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d). \end{cases} \tag{1}$$

Cette équation aux dérivées partielles est appelée "*master equation*". C'est l'équation satisfaite par la fonction valeur U d'un joueur générique. Cette fonction U dépend du temps t, de l'état du joueur x et de la mesure des états des autres joueurs m. L'interprétation de cette *master equation* est que la trajectoire $(X_t)_{t \geq 0}$

d'un joueur générique est donnée par :

$$\begin{cases} dX_t = \alpha_t dt + \sqrt{2\nu} dW_t + \sqrt{2\beta} dB_t; \\ X_0 \sim m_0; \end{cases} \tag{2}$$

où $(W_t)_{t\geq 0}$ et $(B_t)_{t\geq 0}$ sont deux mouvement browniens d dimensionnels indépendants sur un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$. Le contrôle du joueur est ici le processus $(\alpha_t)_{t\geq 0}$ à valeur dans $\mathbb{R}^d$. Les joueurs sont initialement distribués suivant la mesure de probabilité m_0 et ν et β sont deux paramètres positifs. Le mouvement brownien $(W_t)_{t\geq 0}$ est le bruit propre au joueur et $(B_t)_{t\geq 0}$ est le bruit commun qui affecte tous les joueurs de la même façon. Plus précisément, deux joueurs différents ont des trajectoires $(X_t)_{t\geq 0}$ et $(Y_t)_{t\geq 0}$ qui sont solutions de (2) avec deux réalisations indépendantes du mouvement brownien $(W_t)_{t\geq 0}$ mais font face à la même réalisation de $(B_t)_{t\geq 0}$. La fonction U est alors la fonction valeur du problème suivant :

$$\inf_{(\alpha_t)_{t\geq 0}} \mathbb{E}\left[\int_0^T L(X_t, \alpha_t, m_t)dt + g(X_T, m_T)\right]; \tag{3}$$

où $(m_t)_{t\geq 0}$ est le processus qui décrit l'évolution de la mesure des états des joueurs. On se restreint ici à des contrôles en boucle fermée et la fonction de coût L est telle que

$$H(x, p, m) = L^*(x, p, m) := \inf_{\alpha\in\mathbb{R}^d}\{L(x, \alpha, m) - \alpha \cdot p\}.$$

Une remarque fondamentale dans la théorie des jeux à champ moyen est qu'en l'absence d'un bruit commun (c'est à dire $\beta = 0$), étant donnée la distribution initiale des joueurs $m_0 \in \mathcal{P}(\mathbb{T}^d)$, la *master equation* se réduit au système suivant :

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, \nabla u, m) = 0 \text{ dans } (0, T) \times \mathbb{T}^d; \\ \partial_t m - \nu\Delta m - div(D_p H(x, \nabla u, m)m) = 0 \text{ dans } (0, T) \times \mathbb{T}^d; \\ m(0) = m_0; u(T) = g(m_T) \text{ dans } \mathbb{T}^d. \end{cases} \tag{4}$$

Ce système, composé d'une équation d'Hamilton-Jacobi-Bellmann (HJB) rétrograde et d'une équation de Fokker-Planck (FP), décrit les équilibres de Nash du jeu à champ moyen. En effet si la fonction valeur u des joueurs satisfait l'équation de HJB (dans un sens classique) de (4), alors les joueurs utilisent le contrôle $-D_p H(x, \nabla u(t, x), m)$ lorsqu'ils se trouvent en (t, x). La mesure des joueurs m satisfait donc bien l'équation de FP dans (4). On remarque que le système (4) est un système de caractéristiques pour la *master equation* (1) dans le cas $\beta = 0$. En effet la relation suivante est vérifiée. :

$$U(t, x, m(t)) = u(t, x) \text{ dans } (0, T) \times \mathbb{T}^d; \tag{5}$$

où U est une solution classique de (1) et (u, m) une solution du système (4).

Le système (4) et ses variantes constituent le jeu à champ moyen le plus largement étudié. Il a été introduit et analysé en détails par Jean-Michel Lasry et Pierre-Louis Lions dans [54] ainsi que par P.-L. Lions dans son cours [57]. Ce système a également été indépendamment introduit dans un contexte plus particuliers par M. Huang, P. Caines et R. Malhamé dans [48]. La *master equation* (1) a elle été étudiée en détails par P.-L. Lions dans [57] et par Pierre Cardaliaguet, François Delarue, J.-M. Lasry et P.-L. Lions dans [27].

Le second modèle de jeux à champ moyen que l'on présente ici est un modèle à espace d'états discret en temps continu. On note d le nombre d'états dans ce modèle. La *master equation* typique pour ce genre de modèle, en l'absence de bruit commun, est donnée par :

$$\begin{cases} -\partial_t U + (F(x, U) \cdot \nabla_x) U = G(x, U) \text{ dans } (0, T) \times \mathbb{R}^d; \\ U(T) = U_0 \text{ dans } \mathbb{R}^d. \end{cases} \tag{6}$$

L'inconnu $U = (U^i)_{1 \leq i \leq d}$ de cette équation s'interprète comme la valeur du jeu pour un joueur générique . C'est à dire qu'à l'instant t et dans l'état i, lorsque l'histogramme (non normalisé) $x \in \mathbb{R}^d$ décrit la quantité de joueurs dans chaque état, la valeur du MFG pour un joueur est $U^i(t, x)$. Le système hyperbolique non conservatif du premier ordre (6) est l'analogue en dimension fini de (1). De la même façon, le système caractéristique analogue à (4) associé à (6) est :

$$\begin{cases} -\dot{V}(t) = G(y(t), V(t)) \text{ pour } 0 \leq t \leq T \text{ (i) ;} \\ \dot{y}(t) = -F(y(t), V(t)) \text{ pour } 0 \leq t \leq T \text{ (ii) ;} \\ y(0) = y_0; V(T) = U_0(y(T)). \end{cases} \tag{7}$$

Si (V, y) est solution de (7) et U est solution de (6) alors on a, au moins formellement, la relation suivante :

$$U(t, y(t)) = V(t) \text{ pour } 0 \leq t \leq T.$$

L'interprétation de ce jeu à champ moyen à d états est que les joueurs contrôlent leurs probabilités de transition au sein des états. Les transitions agrégées des joueurs (modélisées par F) induisent une évolution de la densité des joueurs. Cette évolution est modélisée par (ii)-(7). La valeur du jeu évolue elle suivant (i)-(7) où G est interprété comme un coût.

Le système (6) a été introduit J.-M. Lasry et P.-L. Lions et étudié par P.-L. Lions dans [57].

Nous présentons maintenant quelques résultats majeurs concernant ces deux modèles MFG. On commence par rappeler le résultat d'existence de J.-M. Lasry et P.-L. Lions [54, 57] sur le système (4) :

Théorème 0.1.1. *On suppose que $\nu > 0$ et que le hamiltonien H est de la forme*

$$H(x,p,m)(t,x) = \tilde{H}(x,p) - f[m](t,x);$$

où $\tilde{H}$ est régulier sur $\mathbb{T}^d \times \mathbb{R}^d$ et vérifie pour une constante $C \geq 0$:

$$\left|D_p\tilde{H}\right| + \left|\nabla_x\tilde{H}\right| \leq C(1+|p|), \forall x,p \in \mathbb{T}^d \times \mathbb{R}^d.$$

On suppose de plus que f et g sont bornés de $C^{k,\alpha}$ dans $C^{k+1,\alpha}$ ($\forall k \geq 0, \alpha \in (0,1)$). On suppose enfin que f (respectivement g) est continu du sous ensemble X de $\mathcal{C}([0,T], L^1(\mathbb{T}^d))$ (respectivement $L^1(\mathbb{T}^d)$) des fonctions $m \geq 0$, $\int_{\mathbb{T}^d} m = 1$ dans $L^\infty((0,T), W^{1,\infty}(\mathbb{T}^d))$ (respectivement $W^{1,\infty}(\mathbb{T}^d)$). Alors il existe au moins une solution régulière de (4).

Remarque 0.1.1. *Il existe de nombreuses autres hypothèses différentes sur la structure du hamiltonien qui permettent d'obtenir l'existence de solutions pour le système (4).*

Nous présentons maintenant un résultat d'unicité majeur dans la théorie des MFG. Ce résultat de J.-M. Lasry et P.-L. Lions a été présenté dans [54, 57].

Théorème 0.1.2. *Supposons que g est monotone. Supposons également que*

— *ou bien H est local en m et vérifie*

$$\begin{pmatrix} zD^2_{pp}H(x,p,z) & \frac{z}{2}D^2_{pz}H(x,p,z) \\ \frac{z}{2}D^2_{pz}H(x,p,z) & -D_zH(x,p,z) \end{pmatrix} \geq 0$$

au sens des matrices symétriques pour tout $(x,p,z) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}_+$

— *ou bien H est de la forme*

$$H(x,p,m) = \tilde{H}(x,p) - f[m]$$

avec possiblement une dépendance non-locale en m, avec $\tilde{H}$ convexe en p et f strictement monotone en m

— *ou bien H est de la forme*

$$H(x,p,m) = \tilde{H}(x,p) - f[m]$$

avec possiblement une dépendance non-locale en m, avec f monotone en m et $\tilde{H}$ convexe en p qui vérifie pour tout $x \in \mathbb{T}^d$

$$\tilde{H}(x,p) - \tilde{H}(x,q) - D_p\tilde{H}(x,q)\cdot(p-q) = 0 \Rightarrow p = q.$$

Alors il existe au plus une solution au système (4).

Ce résultat d'unicité est fondamental dans l'étude des jeux à champ moyen. En effet comme nous l'avons expliqué plus haut, le système (4) joue le rôle de caractéristiques pour la *master equation* (1) dans le cas $\beta = 0$. L'unicité pour ce système est donc directement liée au caractère bien posé de la *master equation*. En particuliers, en l'absence d'unicité pour le système MFG, on s'attend à avoir des chocs dans les solutions de la *master equation*. Les résultats obtenus jusqu'à présent sur l'existence et l'unicité de solutions pour la *master equation* utilisent tous des hypothèses de monotonie. On peut classer ces résultats dans deux catégories différentes, suivant qu'ils utilisent les caractéristiques ou non. Le résultat le plus général obtenu utilisant les caractéristiques est le suivant et il s'intéresse notamment au cas $\nu > 0$. Il témoigne de l'importance de la monotonie dans les jeux à champ moyen. On trouve ce résultat dans le travail de P. Cardaliaguet, F. Delarue, J.-M. Lasry et P.-L. Lions [27].

Théorème 0.1.3. *On suppose que $\nu > 0$ et que H est de la forme :*

$$H(x, p, m) = \tilde{H}(x, p) - f[m].$$

On suppose également que $\tilde{H}, f$ et g sont globalement lipschitziennes et vérifient :

$$\forall (x, p) \in \mathbb{T}^d \times \mathbb{R}^d \text{ pour une constante } C > 0\,;$$

$$C^{-1}\frac{Id}{1+|p|} \leq D^2_{pp}\tilde{H}(x, p) \leq CId;$$

$$\sup_{m\in\mathcal{P}(\mathbb{T}^d)} \left(||f(\cdot, m)||_{3+\alpha} + \left|\left|\frac{\delta f(\cdot, m, \cdot)}{\delta m}\right|\right|_{(3+\alpha,3+\alpha)} \right) + \sup_{m\in\mathcal{P}(\mathbb{T}^d)} \left|\left|\frac{\delta^2 f(\cdot, m, \cdot, \cdot)}{\delta m^2}\right|\right|_{(3+\alpha,3+\alpha,3+\alpha)}$$
$$+ Lip_3(\frac{\delta^2 f}{\delta m^2}) + \sup_{m\in\mathcal{P}(\mathbb{T}^d)} \left(||g(\cdot, m)||_{4+\alpha} + \left|\left|\frac{\delta g(\cdot, m, \cdot)}{\delta m}\right|\right|_{(4+\alpha,4+\alpha)} \right)$$
$$+ \sup_{m\in\mathcal{P}(\mathbb{T}^d)} \left|\left|\frac{\delta^2 g(\cdot, m, \cdot, \cdot)}{\delta m^2}\right|\right|_{(4+\alpha,4+\alpha,4+\alpha)} + Lip_4(\frac{\delta^2 g}{\delta m^2}) < \infty$$

pour un certain $\alpha \in (0, 1)$. On suppose de plus que f et g sont monotones. Alors il existe une unique solution de classe C^2 de la master equation (1).

Ce résultat repose sur une étude détaillée du système caractéristique (4) ainsi que de son équivalent stochastique dans le cas $\beta > 0$. Il est important de remarquer que les hypothèses de régularité demandées sur f et g dans ce résultat sont très fortes, même si elles peuvent être légèrement affaiblies dans le cas $\beta = 0$.

La seconde catégorie de résultats concernant la *master equation* est composée des résultats qui reposent uniquement sur des arguments de monotonie. Ces

résultats de J.-M. Lasry et P.-L. Lions sont présentés dans [57]. Ces résultats s'intéressent à (6) et à (1) dans le cas $\nu = 0$. Ce dernier cas est traité par l'approche dite hilbertienne que l'on ne présente pas ici par soucis de concision. Dans le cas espace d'états discret, on a alors le résultat suivant :

Théorème 0.1.4. *Soit U une solution régulière de (6). On suppose que F, G et U_0 sont globalement lipschitziennes. On suppose également que U_0 et (G, F) sont monotones de respectivement $\mathbb{R}^d$ dans $\mathbb{R}^d$ et $\mathbb{R}^{2d}$ dans $\mathbb{R}^{2d}$. S'il existe $\alpha > 0$ tel que soit F est α monotone, soit U_0 et G sont α monotones, alors U est lipschitzienne en espace, localement uniformément en temps, avec une constante de Lipschitz qui ne dépend que de α et des constantes de Lipschitz de F, G et U_0.*

Bien que cela ne soit pas le point de vue adopté dans cette thèse, il convient de mentionner qu'il existe une théorie probabiliste des jeux à champ moyen. On renvoie notamment au livre de F. Delarue et R. Carmona [30] pour plus d'informations sur cette vision des jeux à champ moyen. Les points de vue équations aux dérivées partielles d'un côté et probabilités de l'autre conduisent évidemment à des résultats très proches les uns des autres dans la plupart des cas.

0.2 Organisation du manuscrit

Cette thèse est composée de résultats dans différentes directions concernant les jeux à champ moyen. Elle est divisée en cinq chapitres qui sont indépendants les uns des autres pour la majorité de leur contenu. Chacun des chapitres un à cinq est directement issu d'un article de recherche dont les références sont données en début de chapitre. Une part importante de ce manuscrit est dédiée à l'étude de jeux à champ moyen où les actions des joueurs sont de natures différentes que celles que nous venons de présenter. Le premier chapitre de ce manuscrit est consacré à l'étude de l'analogue du système (4) dans un cas où les joueurs peuvent sortir du jeu à champ moyen. Le temps d'arrêt des joueurs fait parti de leurs contrôles et ils cherchent donc à optimiser ce temps. On montre notamment dans ce premier chapitre que la structure d'un jeu à champ moyen avec arrêt optimal est moins régulière que celle de (4). Cela est notamment dû au fait que l'ensemble des stratégies des joueurs n'est pas convexe si l'on considère uniquement des solutions du système MFG correspondant à des équilibres de Nash en stratégie pure. Dans ce premier chapitre on étend alors la notion de solution du système MFG de façon à ce que ces nouvelles solutions modélisent des équilibres de Nash en stratégies mixtes. On montre l'existence de telles solutions sous certaines hypothèses de continuité des couplages. On montre également l'unicité de ces solutions sous des hypothèses

de monotonie analogues à celles du théorème 0.1.2.

Le second chapitre de cette thèse est consacré à une extension des résultats du premier chapitre dans le cas où le problème d'optimisation des joueurs est de type contrôle impulsionnel. C'est à dire que les joueurs peuvent sauter dans l'espace d'états et qu'ils contrôlent ces sauts. L'équation de Hamilton-Jacobi-Bellmann vérifiée par la valeur du jeu pour un joueur générique est alors une inéquation quasi-variationnelle. Tout comme dans le chapitre précédent on montre que l'on peut caractériser la densité des joueurs par une formulation variationnelle. On montre alors l'existence de solutions au système analogue à (4) dans le cas contrôle impulsionnel. On prouve aussi l'unicité de ces solutions sous des hypothèses de monotonie. À l'instar du chapitre précédent, les solutions que l'on considère dans ce deuxième chapitre modélisent des équilibres de Nash en stratégies mixtes pour le jeu à champ moyen sous-jacent.

Le troisième chapitre de ce travail porte sur quelques digressions autour des jeux à champ moyen. On étudie d'abord l'unicité des solutions de systèmes de type (4) lorsque le couplage entre les joueurs a également lieu au travers de la mesure de leur contrôles. On montre notamment que sous certaines hypothèses, il y a unicité des solutions pour un tel système. On s'intéresse également à la limite des modèles de type MFG lorsque le paramètre de préférence intertemporelle des joueurs tend vers l'infini. Le modèle MFG converge alors vers un modèle d'évolution pure de type *agent-based model*. Enfin on montre dans ce troisième chapitre un résultat de régularité analogue au théorème 0.1.4 sur une *master equation* de type (6) avec des termes qui modélisent la présence de bruit commun. La preuve de ce résultat repose uniquement sur des arguments de monotonie et non sur l'étude des caractéristiques stochastiques de cette équation. Ce chapitre est issu d'un travail réalisé en collaboration avec J.-M. Lasry et P.-L. Lions.

Le quatrième chapitre de ce manuscrit est consacré à l'étude d'un algorithme qui produit une suite convergeant vers la solution d'un système d'inégalités variationnelles. L'algorithme étudié est une extension de l'algorithme classique d'Uzawa pour la recherche de point selle. On montre ensuite que l'on peut caractériser les solutions de systèmes MFG comme (4) avec un système d'inégalités variationnelles. La méthode précédemment mentionnée s'applique alors et permet d'approcher la solution du système MFG sous certaines hypothèses de monotonie.

Le cinquième et dernier chapitre de cette thèse est l'étude d'un modèle de communications sans-fil entre un grand nombre d'appareils et une antenne. Ce chapitre porte sur la résolution numérique du système MFG qui modélise ce problème ob-

jets connectés-antenne. Ce travail a été réalisé en collaboration avec S. Vassilaras, G. Paschos, M. Debbah, J.-M. Lasry et P.-L. Lions.

La suite de cette introduction est dédiée à la présentation des principaux résultats des chapitres suivants.

0.3 Problèmes d'arrêt optimal et de contrôle impulsionnel dans les jeux à champ moyen

Dans cette partie on présente les systèmes analogues à (4) qui modélisent des jeux à champ moyen d'arrêt optimal ou de contrôle impulsionnel, en l'absence de bruit commun. L'enjeu majeur de cette partie est de caractériser la densité des joueurs dans ce type de situation. On donne également des résultats d'unicité analogue au théorème 0.1.2 sous des hypothèses de monotonie adaptées. Les résultats présentés dans cette partie sont issus des articles [11] et [12].

Plusieurs travaux ont été réalisés sur des problèmes de type arrêt optimal ou contrôle impulsionnel dans les MFG. R. Carmona, F. Delarue et D. Lacker ont obtenu un résultat d'existence dans un cadre probabiliste pour un problème d'arrêt optimal dans [32]. D. Gomes et S. Patrizi ont étudié certains systèmes MFG qui modélisent des situations d'arrêt optimal ou d' *optimal switching* dans [42, 43]. Enfin M. Nutz a résolu un cas particuliers de problème d'arrêt optimal dans [61].

0.3.1 Présentation formelle du problème d'arrêt optimal

Le problème MFG d'arrêt optimal que l'on va étudier ici est le suivant. On travaille sur $\mathbb{T}^d$ et on se fixe un temps final $T > 0$. On suppose que la trajectoire d'un joueur générique est donnée par

$$\begin{cases} dX_t = \alpha_t dt + \sqrt{2\nu} dW_t; \\ X_0 \sim m_0; \end{cases} \tag{8}$$

où $m_0 \in \mathcal{M}(\mathbb{T}^d)$ est une mesure positive à densité $m_0 \in L^2(\mathbb{T}^d)$ et $(W_t)_{t\geq 0}$ est un mouvement brownien d dimensionnel sur $\mathbb{T}^d$ pour un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$. Le mouvement brownien $(W_t)_{t\geq 0}$ est associé à un bruit individuel et $\nu > 0$ est une constante. Le joueur contrôle toujours le processus $(\alpha_t)_{t\geq 0}$ mais il contrôle désormais aussi un temps de sortie. Étant donnée l'évolution de la densité m des joueurs, le problème d'optimisation du joueur générique est donné par

$$\inf_{\alpha,\tau} \mathbb{E}\left[\int_0^\tau L(X_t, \alpha_t) + f(m(t))dt + \psi(X_\tau, m(\tau))\right]; \tag{9}$$

où l'infimum est pris sur les contrôles en boucle fermée $(\alpha_t)_{t\geq 0}$ et sur les temps d'arrêt $\tau \leq T$ du processus $(X_t)_{t\geq 0}$. La fonction ψ est le coût de sortie des joueurs et L le coût courant. On note $H(x,p) = L^*(x,p)$ la conjuguée de Fenchel de L par rapport à sa seconde variable. La fonction valeur u du joueur générique satisfait donc (sous des hypothèses sur H et ψ) le problème d'obstacle :

$$\begin{cases} \max(-\partial_t u - \nu\Delta u + H(x,\nabla u) - f(m), u - \psi(m)) = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ u(T) = \psi(m)(T) \text{ dans } \mathbb{T}^d. \end{cases} \tag{10}$$

La question est alors de savoir comment, étant données prescrites les stratégies des joueurs, on peut déterminer l'évolution de leur densité m. Si on connaît la solution u de (10), on sait qu'il est optimal de sortir du jeu lorsque la trajectoire $(X_t)_{t\geq 0}$ atteint l'ensemble $\{u = \psi(m)\}$. Si cet ensemble est fermé, une équation naturelle pour modéliser la densité m des joueurs est :

$$\begin{cases} \partial_t m - \nu\Delta m - div(D_p H(x,\nabla u)m) = 0 \text{ dans } \{u < \psi(m)\}; \\ m = 0 \text{ dans } \{u = \psi(m)\}; m = m_0 \text{ dans } \mathbb{T}^d. \end{cases} \tag{11}$$

Cependant, il n'existe pas en général de couple (u,m) solution de (10)-(11), peu importe la régularité du hamiltonien H, de l'obstacle ψ ou de f. Cela est notamment dû à la discontinuité de (11) par rapport à u. On peut par exemple considérer la suite $(u_n)_{n\geq 0}$ définie par

$$u_n = \psi - \frac{1}{n+1}$$

pour observer la discontinuité en question. On peut noter que l'existence d'un couple (u,m) qui satisfait (10)-(11) est associée à l'existence d'un équilibre de Nash symétrqiue en stratégie pure pour le MFG sous jacent. Or, dans un jeu d'arrêt optimal, si on limite les joueurs à des stratégies pures, leur ensemble de stratégies possibles n'est pas convexe. Cela justifie formellement la non existence de solutions à (10)-(11). Pour palier cette non existence de solution, on va considérer une inégalité variationnelle pour m qui va étendre l'équation (11). On part de la remarque suivante : étant donnée une fonction λ de $(0,T)\times\mathbb{T}^d$ dans $\mathbb{R}_+ \cup \{+\infty\}$, on voudrait résoudre l'équation aux dérivées partielles suivante :

$$\begin{cases} \partial_t m - \nu\Delta m - div(D_p H(x,\nabla u)m) + \lambda m = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0 \text{ dans } \mathbb{T}^d. \end{cases} \tag{12}$$

On s'attend à avoir $m \geq 0$ et on remarque alors que pour toute fonction v régulière négative à support compact dans $(0,T)\times\mathbb{T}^d$ on obtient

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t v - \nu\Delta v + D_p H(x,\nabla u)\cdot\nabla v + \lambda v)m = 0;$$

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t v - \nu \Delta v + D_p H(x, \nabla u) \cdot \nabla v) m \geq 0;$$

avec une égalité si $\lambda v \equiv 0$.

Il faut noter que si λ est un taux de sorti qui correspond à un comportement optimal pour (9), alors si u est solution de (10) on a $(u - \psi(m))\lambda \equiv 0$. En effet formellement il est strictement optimal de rester dans le jeu dans l'ensemble $\{u < \psi(m)\}$. On attend donc ici naturellement que la densité m vérifie *a minima* :

$$\begin{cases} m \geq 0; \\ \forall v \in W^{1,2,2}((0,T), \mathbb{T}^d), v \leq \psi(m), v(T) = \psi(m)(T) : \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t (v-u) - \nu \Delta (v-u) + D_p H(x, \nabla u) \cdot \nabla (v-u)) m - \\ \qquad - \int_{\mathbb{T}^d} (v-u)(0) m_0 \geq 0. \end{cases} \tag{13}$$

On a ici supposé que $u \in W^{1,2,2}((0,T), \mathbb{T}^d)$ est solution de (10) dans L^2. Comme on l'a vu plus haut, (13) est formellement vérifiée par toute solution de (12) dès que λ vérifie $\lambda(u - \psi(m)) \equiv 0$. Il faut donc introduire un critère de sélection parmi toutes ces fonctions m. Par exemple le critère choisit dans (11) est de dire que m est nul dans $\{u = \psi(m)\}$. Cela correspond à dire que tous les joueurs sortent du jeu lorsque cela est optimal. On introduit ici le critère suivant :

$$m = 0 \text{ dans } \{-\partial_t u - \nu \Delta u + H(x, \nabla u) < f(m)\}$$

que l'on réécrit

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u + H(x, \nabla u) - f(m)) m = 0. \tag{14}$$

On rappelle que l'ensemble $\{-\partial_t u - \nu \Delta u + H(x, \nabla u) < f(m)\}$ correspond (s'il est bien défini) à l'ensemble où il est strictement sous optimal de rester dans le jeu. L'ensemble $\{u = \psi(m)\} \cap \{-\partial_t u - \nu \Delta u + H(x, \nabla u) = f(m)\}$ correspond lui à l'ensemble où il est à la fois optimal de rester et de partir du jeu.

On appelle donc solution mixte du jeu un couple $(u, m) \in W^{1,2,2}((0,T), \mathbb{T}^d) \times L^2((0,T), H^1(\mathbb{T}^d))$ tel que :

$$\begin{cases} \max(-\partial_t u - \nu \Delta u + H(x, \nabla u) - f(m), u - \psi(m)) = 0 \text{ dans } (0,T) \times \mathbb{T}^d; \\ u(T) = \psi(m)(T) \text{ dans } \mathbb{T}^d; \\ m \geq 0; \\ \forall v \in W^{1,2,2}((0,T), \mathbb{T}^d), v \leq \psi(m), v(T) = \psi(m)(T) : \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t (v-u) - \nu \Delta (v-u) + D_p H(x, \nabla u) \cdot \nabla (v-u)) m - \\ \qquad - \int_{\mathbb{T}^d} (v-u)(0) m_0 \geq 0; \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u + H(x, \nabla u) - f(m)) m = 0. \end{cases}$$

La terminologie solution mixte fait ici référence au fait que de telles solutions s'interprètent comme des équilibres de Nash en stratégies mixtes pour le jeu à champ moyen sous-jacent. En effet, formellement, l'inégalité variationnelle satisfaite par m implique que m vérifie une équation de type (12) pour un certain taux de départ λ. La contrainte (14) implique elle que λ est nul dans $\{u < \psi(m)\}$ et infini dans $\{-\partial_t u - \nu\Delta u + H(x, \nabla u) < f(m)\}$. Sur l'ensemble restant on a seulement $0 \leq \lambda \leq +\infty$. Cela s'interprète comme le fait que les joueurs choisissent de sortir de façon aléatoire avec un taux de départ donné par λ. Ils jouent donc en stratégies mixtes.

Pour terminer cette présentation formelle du problème, on précise que l'inégalité variationnelle (13) satisfaite par m avec la contrainte (14) est très proche de la notion de dérivée directionnelle d'une inéquation variationnelle introduite par F. Mignot dans [60]. Cela rappelle le fait que dans de nombreux cas de systèmes MFG, l'équation satisfaite par la densité des joueurs est l'équation adjointe du linéarisé de l'équation satisfaite par la fonction valeur du joueur générique.

0.3.2 Principaux résultats sur le cas arrêt optimal

On rappelle que l'on s'intéresse ici aux couples $(u, m) \in W^{1,2,2}((0,T), \mathbb{T}^d) \times L^2((0,T), H^1(\mathbb{T}^d))$ tels que

$$\begin{cases} \max(-\partial_t u - \nu\Delta u + H(x, \nabla u) - f(m), u - \psi(m)) = 0 \text{ dans } (0,T) \times \mathbb{T}^d; \\ u(T) = \psi(m)(T) \text{ dans } \mathbb{T}^d; \\ m \geq 0; \\ \forall v \in W^{1,2,2}((0,T), \mathbb{T}^d), v \leq \psi(m), v(T) = \psi(m)(T) : \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t(v-u) - \nu\Delta(v-u) + D_p H(x, \nabla u) \cdot \nabla(v-u))m - \\ \qquad - \int_{\mathbb{T}^d} (v-u)(0) m_0 \geq 0; \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u + H(x, \nabla u) - f(m))m = 0. \end{cases} \tag{15}$$

On établit dans le premier chapitre de cette thèse le résultat suivant :

Théorème 0.3.1. *On fait les hypothèses suivantes :*
- *f est continue de $L^2((0,T) \times \mathbb{T}^d)$ dans lui même pour la topologie faible.*
- *ψ est continue de $L^2((0,T) \times \mathbb{T}^d)$ dans $W^{1,2,2}((0,T), \mathbb{T}^d)$ pour la topologie faible.*
- *H est régulier, globalement lipschitzien et convexe en sa seconde variable. Il est uniformément borné par dessous.*

Alors il existe un couple $(u,m) \in W^{1,2,2}((0,T),\mathbb{T}^d)\times L^2((0,T),H^1(\mathbb{T}^d))$ *qui satisfait (15).*

On donne maintenant quelques éléments de preuve de ce résultat, la preuve complète étant détaillée plus loin dans cette thèse. On fixe un terme de pénalisation $\epsilon > 0$ et on s'intéresse au système suivant :

$$\begin{cases} -\partial_t u_\epsilon - \nu\Delta u_\epsilon + H(x,\nabla u_\epsilon) + \frac{1}{\epsilon}(u_\epsilon - \psi(m_\epsilon))^+ = f(m_\epsilon) \text{ dans } (0,T)\times\mathbb{T}^d; \\ u_\epsilon = \psi(m_\epsilon)(T) \text{ dans } \mathbb{T}^d; \\ \partial_t m_\epsilon - \nu\Delta m_\epsilon - div(D_pH(x,\nabla u_\epsilon)m_\epsilon) + \frac{\alpha_\epsilon}{\epsilon}\mathbb{1}_{\{u_\epsilon \geq \psi(m_\epsilon)\}} = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ m_\epsilon(0) = m_0 \text{ dans } \mathbb{T}^d; \end{cases} \tag{16}$$

avec $\alpha_\epsilon \in L^\infty((0,T)\times\mathbb{T}^d)$ qui vérifie

$$\begin{cases} 0 \leq \alpha_\epsilon \leq 1 \text{ dans } (0,T)\times\mathbb{T}^d; \\ \alpha_\epsilon = 1 \text{ dans } \{u_\epsilon \neq \psi(m_\epsilon)\}. \end{cases} \tag{17}$$

L'existence, à $\epsilon > 0$ fixé, d'un triplet $(u_\epsilon, m_\epsilon, \alpha_\epsilon)$ vérifiant (16)-(17) est donnée par un théorème de point fixe de Kakutani. Il est important de noter que le terme α_ϵ est l'introduction, au niveau pénalisé, du fait que l'on cherche des solutions du système MFG qui modélisent des équilibres de Nash en stratégies mixtes. Formellement, $\frac{\alpha_\epsilon}{\epsilon}$ tend, lorsque ϵ tend vers 0, vers le taux de sortie λ des joueurs.
On déduit ensuite de :

$$m_\epsilon \geq 0; \partial_t m_\epsilon - \nu\Delta m_\epsilon - div(D_pH(x,\nabla u_\epsilon)m_\epsilon) \leq 0$$

de la compacité pour la suite $(u_\epsilon, m_\epsilon)_{\epsilon>0}$. On déduit alors des hypothèses sur f, ψ et H que, à extraction d'une sous suite près, la suite (u_ϵ, m_ϵ) a une limite qui vérifie

$$\begin{cases} \max(-\partial_t u - \nu\Delta u + H(x,\nabla u) - f(m), u - \psi(m)) = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ u(T) = \psi(m)(T) \text{ dans } \mathbb{T}^d; \\ m \geq 0; \\ \forall v \in W^{1,2,2}((0,T),\mathbb{T}^d), v \leq \psi(m), v(T) = \psi(m)(T) : \\ \int_0^T \int_{\mathbb{T}^d} (-\partial_t(v-u) - \nu\Delta(v-u) + D_pH(x,\nabla u)\cdot\nabla(v-u))m - \\ \qquad - \int_{\mathbb{T}^d}(v-u)(0)m_0 \geq 0. \end{cases}$$

En observant certaines bornes sur $(\sqrt{\frac{\alpha_\epsilon}{\epsilon}}\mathbb{1}_{\{u_\epsilon\geq\psi(m_\epsilon)\}}m_\epsilon)_{\epsilon>0}$, on déduit enfin que le couple limite (u,m) vérifie :

$$\int_0^T \int_{\mathbb{T}^d} (f(m) + \partial_t u + \nu\Delta u - H(x,\nabla u))m = 0.$$

On précise que le précédent théorème n'apporte aucune information sur la régularité en temps de m.

On énonce maintenant un résultat d'unicité sur les couples (u, m) vérifiant (15) :

Théorème 0.3.2. *On fait l'hypothèse que f est strictement monotone de L^2 dans lui même. On suppose également que*
— *soit $H = 0$ et $\partial_t\psi(\cdot) + \nu\Delta\psi(\cdot)$ est monotone de L^2 dans lui même.*
— *soit $\psi = 0$ (et H est convexe en sa seconde variable).*
Alors il existe au plus un couple $(u, m) \in W^{1,2,2}((0,T), \mathbb{T}^d) \times L^2((0,T), H^1(\mathbb{T}^d))$ qui vérifie (15).

Démonstration. On présente ici la preuve de ce résultat dans le cas $\psi = 0$ et H convexe. On renvoie au premier chapitre de cette thèse pour la preuve complète de ce résultat.
Soient (u_1, m_1) et (u_2, m_2) deux solutions de (15). Comme u_1 et u_2 vérifient deux problèmes de l'obstacle, on déduit, à l'aide des relations

$$\int_0^T \int_{\mathbb{T}^d} (f(m_i) + \partial_t u_i + \nu\Delta u_i - H(x, \nabla u_i))m_i = 0$$

que l'on a bien

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} &(-\partial_t(u_1 - u_2) - \nu\Delta(u_1 - u_2) + H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) \\ &\geq \int_0^T \int_{\mathbb{T}^d} (f(m_1) - f(m_2))(m_1 - m_2).\end{aligned}$$

Par ailleurs, comme le hamiltonien H est convexe, on obtient que

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} &(-\partial_t(u_1 - u_2) - \nu\Delta(u_1 - u_2) + H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) \\ &\leq \int_0^T \int_{\mathbb{T}^d} (-\partial_t(u_1 - u_2) - \nu\Delta(u_1 - u_2) + D_p H(x, \nabla u_1) \cdot \nabla(u_1 - u_2))m_1 \\ &\quad + \int_0^T \int_{\mathbb{T}^d} (-\partial_t(u_2 - u_1) - \nu\Delta(u_2 - u_1) + D_p H(x, \nabla u_2) \cdot \nabla(u_2 - u_1))m_2.\end{aligned}$$

En utilisant les deux inégalités variationnelles satisfaites par m_1 et m_2, on déduit finalement que

$$\int_0^T \int_{\mathbb{T}^d} (f(m_1) - f(m_2))(m_1 - m_2) \leq 0.$$

On conclut donc par la stricte monotonie de f que $m_1 = m_2$. □

0.3.3 Présentation formelle du problème de contrôle impulsionnel

Le problème MFG de contrôle impulsionnel auquel on s'intéresse est le suivant. On suppose que la trajectoire d'un joueur générique est donnée par :

$$\begin{cases} dX_t = \sqrt{2\nu} dW_t, \forall i, \forall t \in (\tau_i, \tau_{i+1}); \\ X_{\tau_i^+} = X_{\tau_i^-} + \xi_i; \\ X_0 \sim m_0; \end{cases} \tag{18}$$

où $m_0 \in \mathcal{P}(\mathbb{T}^d)$ est la mesure initiale de la répartition des joueurs, de densité $m_0 \in L^2(\mathbb{T}^d)$. Le processus $(W_t)_{t\geq 0}$ est un mouvement brownien d dimensionnel sur $\mathbb{T}^d$ sous un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$. Le contrôle du joueur est ici la suite de couples $(\tau_i, \xi_i)_{i\geq 0}$. Cette suite doit être progressivement mesurable par rapport à la tribu générée par le processus $(X_t)_{t\geq 0}$. La suite $(\tau_i)_{i\geq 0}$ est une suite de temps d'arrêt pour ce processus et $(\xi_i)_{i\geq 0}$ est une suite à valeurs dans un ensemble fini $K \subset \mathbb{T}^d$. Les joueurs peuvent donc décider de "sauter" dans l'espace d'états ξ plus loin, pour tout ξ dans K. Étant donnée l'évolution de la densité m des joueurs entre 0 et $T > 0$, le joueur générique fait face au problème d'optimisation suivant :

$$\inf_{(\tau_i,\xi_i)_{i\geq 0}} \mathbb{E}\left[\int_0^T f(X_s, m(s))ds + \sum_{i=0}^{\infty} \mathbb{1}_{\{\tau_i \leq T\}} k(X_{\tau_i}, \xi_i)\right] \tag{19}$$

où la trajectoire $(X_t)_{t\geq 0}$ vérifie (18) et l' *infimum* est pris sur les suites $(\tau_i, \xi_i)_{i\geq 0}$ vérifiant les propriétés énoncées dans le paragraphe précédent. Sous certaines hypothèses sur les fonctions de coûts f et k, la fonction valeur u associée à (19) vérifie dans L^2 :

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f(m), u - Mu) = 0 \text{ dans } (0,T) \times \mathbb{T}^d; \\ u(T) = 0 \text{ dans } \mathbb{T}^d; \end{cases} \tag{20}$$

où Mu est défini pour tout u par

$$Mu(t,x) = \inf_{\xi \in K} \{u(t, x+\xi) + k(x, \xi)\}. \tag{21}$$

En général on associe à l'équation (20) l'inéquation quasi-variationnelle (IQV) suivante :

$$\begin{cases} u \leq Mu, u(T) = 0; \\ \forall v \in L^2((0,T), H^1(\mathbb{T}^d)), v \leq Mu : \\ \int_0^T (-\partial_t u - \nu\Delta u - f(m), v - u)_{H^{-1}, H^1} \geq 0. \end{cases} \tag{22}$$

Il reste donc maintenant à trouver une caractérisation de la densité m des joueurs. Au vu des résultats que l'on a exposés dans le cas arrêt optimal, on s'attend à

n'avoir des résultats d'existence que pour des couples (u, m) associés à des équilibres de Nash en stratégies mixtes pour le MFG sous-jacent. On cherche donc une caractérisation variationnelle de la densité m des joueurs, dont les trajectoires évoluent suivant une équation différentielle stochastique de type (18). Pour simplifier les notations et introduire la formulation variationnelle que l'on va utiliser, on se place brièvement dans le cas $K = \{\xi\}$, c'est à dire qu'il n'existe qu'un seul saut possible ξ. Si les joueurs utilisent le saut ξ avec un certain taux $\lambda \in L^\infty((0,T) \times \mathbb{T}^d), \lambda \geq 0$, la densité m des joueurs vérifie :

$$\begin{cases} \partial_t m - \nu \Delta m + \lambda m - (\lambda m)(t, x - \xi) = 0 \text{ dans } (0,T) \times \mathbb{T}^d; \\ m(0) = m_0 \text{ dans } \mathbb{T}^d. \end{cases} \tag{23}$$

Le dernier terme de la première ligne de cette équation traduit l'arrivée des joueurs qui utilisent le contrôle ξ lorsqu'ils sont en $(t, x - \xi)$. On remarque que si on multiplie la première ligne de (23) par une fonction v régulière telle que $v \leq Mv$ et que l'on intègre, on obtient :

$$\int_0^T \int_{\mathbb{T}^d} (\partial_t m - \nu \Delta m) v = - \int_0^T \int_{\mathbb{T}^d} (\lambda m)(t,x) \Big(v(t,x) - v(t, x + \xi) \Big) dx dt.$$

Comme $v \leq Mv$ on en déduit que

$$\int_0^T \int_{\mathbb{T}^d} (\partial_t m - \nu \Delta m) v \geq - \int_0^T \int_{\mathbb{T}^d} (\lambda m) k,$$

avec égalité si v vérifie $\lambda(v - Mv) \equiv 0$. Ainsi on attend de la densité m qu'elle vérifie :

$$\begin{aligned} &\forall v \in L^2((0,T), H^1(\mathbb{T}^d)) \cap H^1((0,T), H^{-1}(\mathbb{T}^d)), v(T) = 0, v \leq Mv : \\ &\int_0^T (-\partial_t (v - u) - \nu \Delta (v - u), m)_{H^{-1}, H^1} - \int_{\mathbb{T}^d} (v - u)(0) m_0 \geq 0, \end{aligned} \tag{24}$$

où u est solution de (20). Comme dans le cas arrêt optimal, cette seule inégalité variationnelle ne suffit pas à caractériser m. On introduit donc le même critère que dans le cas arrêt optimal afin d'obtenir des couples (u, m) qui modélisent des équilibres de Nash en stratégies mixtes. Ce critère est donc :

$$\int_0^T \int_{\mathbb{T}^d} (f(m) + \partial_t u + \nu \Delta u) m = 0. \tag{25}$$

On appelle donc solution mixte du problème un couple (u, m) vérifiant (20)-(24)-(25).

0.3.4 Principaux résultats sur le cas contrôle impulsionnel

On présente ici les quelques résultats majeurs de cette thèse concernant les couples (u, m) vérifiant :

$$\begin{cases} max(-\partial_t u - \nu\Delta u - f(m), u - Mu) = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ u(T) = 0 \text{ dans } \mathbb{T}^d; \\ \forall v \in L^2((0,T), H^1(\mathbb{T}^d)) \cap H^1((0,T), H^{-1}(\mathbb{T}^d)), v(T) = 0, v \leq Mv : \\ \int_0^T (-\partial_t(v-u) - \nu\Delta(v-u), m)_{H^{-1},H^1} - \int_{\mathbb{T}^d}(v-u)(0)m_0 \geq 0; \\ \int_0^T \int_{\mathbb{T}^d}(f(m) + \partial_t u + \nu\Delta u)m = 0. \end{cases} \tag{26}$$

Les résultats de cette partie se trouvent, ainsi que leur preuves détaillées, dans le second chapitre de ce manuscrit. On commence par donner un résultat d'existence :

Théorème 0.3.3. *On fait les hypothèses suivantes :*
- *f est continue de $L^2((0,T), H^1(\mathbb{T}^d))$ (équipé de la topologie faible) vers $L^2((0,T), H^{-1}(\mathbb{T}^d))$.*
- *f est uniformément bornée par dessous sur $\{m \in L^2((0,T), H^1(\mathbb{T}^d)), m \geq 0\}$.*
- *$f\left(L^2((0,T), H^1(\mathbb{T}^d))\right)$ est un borné de $L^p((0,T)\times\mathbb{T}^d)$ avec $p > d$.*
- *Il existe $k_0 > 0$ tel que $k(x,\xi) \geq k_0 \forall x \in \mathbb{T}^d, \forall \xi \in K$.*
- *La fonction $x \to \inf_{\xi\in K} k(x,\xi)$ appartient à $W^{2,\infty}(\mathbb{T}^d)$.*

Alors il existe $(u,m) \in W^{1,2,2}((0,T),\mathbb{T}^d) \times L^2((0,T), H^1(\mathbb{T}^d))$ vérifiant (26).

La preuve de ce résultat repose, comme dans le cas arrêt optimal, sur l'étude d'un système pénalisé. On ne précise pas ici ce système. On insiste cependant sur le critère de compacité qui nous permet de passer à la limite. On utilise le fait que si $m \in L^2((0,T), H^1(\mathbb{T}^d)) \cap H^1((0,T), H^{-1}(\mathbb{T}^f))$ vérifie une équation du type de (23), pour un certain λ vérifiant $\lambda(u^* - Mu^*) \equiv 0$ pour u^* régulier donné, alors

$$\begin{aligned} ||m||^2_{L^2(H^1)} \leq - \inf_{v,\lambda(v-Mv)\leq 0} \left\{ \int_0^T (\partial_t m - \nu\Delta m, v)_{H^{-1}\times H^1} \right\} \\ + C(1 + ||m||_{L^2(H^1)})||m_0||_{L^2}, \end{aligned} \tag{27}$$

où C ne dépend que de k (qui apparaît à travers M) et de $||u^*||_{L^\infty}$. On établit cette estimée dans le chapitre 2 en utilisant une fonction test v bien choisie dans l'*infimum*. Formellement, pour construire cette fonction test v, on résout le problème de contrôle impulsionnel

$$\begin{cases} \max(-\partial_t v - \nu\Delta v + \Delta m, v - Mv) = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ v(T) = 0 \text{ dans } \mathbb{T}^d. \end{cases} \tag{28}$$

Ensuite, si v est suffisamment régulière, on peut écrire

$$\begin{aligned}\inf_{w,\lambda(w-Mw)\leq 0}\left\{\int_0^T(\partial_t m-\nu\Delta m,w)_{H^{-1}\times H^1}\right\}&\leq\int_0^T(\partial_t m-\nu\Delta m,v)_{H^{-1}\times H^1}\\&=\int_0^T(-\partial_t v-\nu\Delta v,m)_{H^{-1}\times H^1}\\&\quad-\int_{\mathbb{T}^d}m_0v(0)\\&\leq-\int_0^T\int_{\mathbb{T}^d}|\nabla m|^2-\int_{\mathbb{T}^d}m_0v(0).\end{aligned}$$

D'où on déduit l'estimée si on peut majorer d'une façon convenable $||v||_{L^\infty(L^2)}$. Dans le cas général, le problème (28) n'admet pas de solution et donc en particuliers pas de solutions régulières. Dans le chapitre 2, on introduit une version affaiblie de (28). On montre que sa solution v, bien que pas assez régulière pour effectuer l'intégration par parties faite plus haut, suffit à obtenir cette estimée.

Encore une fois, ce résultat n'apporte aucune information sur la régularité en temps de m.

On présente maintenant un résultat d'unicité concernant les couples (u,m) vérifiant (26).

Théorème 0.3.4. *Si f est strictement monotone de $L^2((0,T)\times\mathbb{T}^d)$ dans lui même, alors il existe au plus un couple $(u,m)\in W^{1,2,2}((0,T),\mathbb{T}^d)\times L^2((0,T),H^1(\mathbb{T}^d))$ vérifiant (26).*

La preuve de ce résultat suit exactement celle du théorème 0.3.2 en prenant $H=0$.
Il est important de remarquer que cette preuve d'unicité, ainsi que l'estimée (27) montrent un lien fort, de type adjoint du linéarisé, entre l'IQV (22) et l'inégalité variationnelle (24) avec la condition (25).

0.4 Deux remarques sur les MFG

Bien que nous ne prenons pas le temps de détailler les deux sujets que nous allons présenter ici, nous les mentionnons car ils sont traités dans le troisième chapitre de ce manuscrit. Le premier problème s'intéresse à la convergence lorsque

λ tend vers l'infini de la solution du système MFG :

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, \nabla u, m) + \lambda u = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ \partial_t m - \nu\Delta m - div(D_p H(x,\nabla u, m)m) = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0; u(T) = g(m_T) \text{ dans } \mathbb{T}^d. \end{cases} \tag{29}$$

On montre alors que pour ce système (ainsi que pour d'autres problèmes) la limite formelle de (29) est une équation d'évolution pour la densité m des joueurs.

Le second sujet abordé dans le troisième chapitre, et que nous ne développerons pas ici, est la question de l'unicité des solutions de système de type :

$$\begin{cases} -\partial_t u - \nu\Delta u + H(\nabla u) - G[\nabla u, m]\cdot\nabla u = f(m) \text{ dans } (0,T)\times\mathbb{T}^d; \\ \partial_t m - \nu\Delta m - div\Big((D_p H(x,\nabla u, m) - G[\nabla u, m])m\Big) = 0 \text{ dans } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0; u(T) = g(m_T) \text{ dans } \mathbb{T}^d; \end{cases} \tag{30}$$

où "$G[.]$" signifie une dépendance fonctionnelle. Sous certaines hypothèses sur la fonction G, le système (30) modélise un MFG dans lequel les trajectoires des joueurs sont impactées par la mesure des contrôles des autres joueurs. On donne donc un résultat d'unicité sur (30) dans le chapitre 3 qui est issu de l'article [14] rédigé en collaboration avec J.-M. Lasry et P.-L. Lions.

0.5 Master equation en espace d'états fini avec du bruit commun

On s'intéresse dans cette partie à des systèmes non conservatifs hyperboliques de la forme :

$$\begin{cases} \partial_t U + (F(x,U)\cdot\nabla)U = G(x,U) \text{ dans } \mathbb{R}_+\times\mathbb{R}^d; \\ U_{|t=0} = U_0 \text{ dans } \mathbb{R}^d. \end{cases} \tag{31}$$

On renvoie à la seconde partie de ce chapitre introductif pour l'interprétation en termes de MFG de ce système. Le système (31) a été introduit par J.-M. Lasry et P.-L. Lions et largement étudié par P.-L. Lions dans son cours [57]. Nous présentons ici des résultats similaires à ceux de P.-L. Lions pour des systèmes de type (31) auxquels on a ajouté des termes modélisant du bruit commun. Ces résultats se trouvent dans l'article [14]. On se concentre ici sur des bruits communs de la forme suivante : étant donné un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$, un processus de Poisson de paramètre $\lambda > 0$ produit une séquence aléatoire de temps auxquels l'histogramme

x de la répartition des joueurs est instantanément changé en $T(x)$, où T est une application de $\mathbb{R}^d$ dans lui même. Le système qui modélise ce type de bruit est

$$\begin{cases} \partial_t U + (F(x,U)\cdot\nabla)U + \lambda\Big(U - (DT(Tx))^*(U(t,Tx))\Big) = G(x,U) \text{ dans } \mathbb{R}_+ \times \mathbb{R}^d; \\ U_{|t=0} = U_0 \text{ dans } \mathbb{R}^d. \end{cases} \tag{32}$$

Le terme

$$\lambda\Big(U - (DT(Tx))^*(U(t,Tx))\Big)$$

modélise les anticipations que les joueurs font sur le bruit. Les deux résultats que l'on va énoncer ici sont de natures différentes. Le premier est un résultat de propagation de la monotonie pour l'application U. C'est à dire que sous certaines hypothèses on aura :

$$\begin{aligned} &\forall t \geq 0, \forall x, y \in \mathbb{R}^d : \\ &< U(t,x) - U(t,y), x - y > \geq 0. \end{aligned}$$

Le second résultat est un résultat concernant la propagation de régularité pour U. Ils font tous les deux partie du troisième chapitre de cette thèse qui a été réalisé en collaboration avec J.-M. Lasry et P.-L. Lions. Le premier résultat est le suivant :

Théorème 0.5.1. *On suppose que T est affine. Si U_0 est monotone de $\mathbb{R}^d$ dans lui même et (G,F) est monotone de $\mathbb{R}^{2d}$ dans lui même, alors toute solution régulière U de (32) est monotone pour tout temps.*

Démonstration. On définit $V = U$ et on introduit W défini par :

$$W(t,x,y) = < U(t,x) - V(t,y), x - y > . \tag{33}$$

On note $T = e + S$ avec S linéaire et on remarque que W vérifie l'équation suivante :

$$\begin{aligned} \partial_t W + F(x,U)\cdot\nabla_x W + F(y,V)\cdot\nabla_y W + \lambda\Big(W - < S^*U(t,Tx) - S^*V(t,Ty), x - y > \Big) \\ = < G(x,U) - G(y,V), x - y > + < F(x,U) - F(y,V), U - V > \text{ dans } \mathbb{R}_+ \times \mathbb{R}^d. \end{aligned}$$

On déduit alors en utilisant la monotonie de (G,F) que

$$\partial_t W + F(x,U)\cdot\nabla_x W + F(y,V)\cdot\nabla_y W + \lambda\Big(W - W(t,Tx,Ty)\Big) \geq 0$$

Un résultat de type principe du maximum fort (que l'on peut trouver dans l'appendice du troisième chapitre) nous permet alors d'établir que comme $W(0) \geq 0$ (car U_0 est monotone), on a bien

$$W(t,x,y) \geq 0, \forall t, x, y.$$

□

La preuve de ce résultat implique également que si on prend deux solutions régulières U et V de (32), alors la fonction W définie par (33) est positive pour tout temps. On en déduit donc qu'il existe une seule solution régulière (lipschitzienne en espace) de (32) sous les hypothèses du théorème 0.5.1. On observe donc encore une fois que la monotonie des coûts pour le MFG est fortement liée au caractère bien posé du problème.
Le résultat de régularité que l'on présente maintenant est le suivant :

Théorème 0.5.2. *On fait les hypothèses suivantes :*
— *T est affine.*
— *U_0, F et G sont lipschitziennes.*
— *U_0 et (G, F) sont monotones.*
— *Il existe $\alpha > 0$ tel que soit F est α monotone, soit U_0 et G sont α monotones.*
Alors si U est solution de (32), U est lipschitzienne en espace, localement uniformément en temps.

La preuve de ce résultat étant fastidieuse, on se contente simplement d'en énoncer les étapes clefs. On renvoie au troisième chapitre de cette thèse pour sa preuve détaillée.
L'étape principale de cette preuve consiste à écrire l'EDP vérifiée par la fonction Z_β définie par

$$Z_\beta(t, x, \xi) =< \xi, \nabla_x W(t, x, \xi) > -\beta(t)|\nabla_x W(t, x, \xi)|^2$$

où W est définie ici par

$$W(t, x, \xi) =< U(t, x), \xi > .$$

En utilisant les hypothèses de monotonie et de régularité, on peut ensuite montrer qu'il existe une fonction β strictement positive pour tout temps, pour laquelle $Z_\beta \geq 0$ pour tout temps, ce qui entraîne une estimée sur le gradient en espace de U.

Comme on l'a mentionné dans la deuxième section de ce chapitre, il est important de noter que la preuve de ce résultat ne fait pas intervenir de système caractéristique pour (32).

0.6 Itérations d'Uzawa et structure variationnelle des MFG

Dans le cas dit potentiel, défini plus loin, les solutions de systèmes MFG du même type que (4) peuvent s'interpréter comme des points selle de certains lagragiens. On s'intéresse donc dans cette partie à la construction de suites convergeant

vers la solution d'un système d'inégalités variationnelles, système qui rappelle les conditions satisfaites par un point selle. Par inégalités variationnelles on entend ici une formulation du type :

Trouver $x \in A$ tel que pour tout $y \in B$:

$$F(x, y) \geq 0.$$

Cette formulation est donc en particuliers plus générale que la notion usuelle d'inéquation variationnelle étudiée notamment par J.-L. Lions et G. Stampacchia qui correspond au cas $A = B$.

On présente ici pourquoi l'algorithme d'Uzawa nous permet de construire une suite qui converge vers la solution de certains systèmes d'inégalités variationnelles, même lorsqu'il n'existe pas de lagrangien pour lequel ce type de systèmes caractérise un point selle. Ensuite, on montre que les systèmes MFG (4), (15) et (26), pour certaines formes de hamiltonien, peuvent être interprétés comme des systèmes d'inégalités variationnelles et que ceux-ci tombent dans le champ d'application de la remarque précédente. Les résultats de cette partie sont détaillés, accompagnés de simulations numériques, dans le quatrième chapitre de cette thèse, qui est composé de l'article [13]. On note également que depuis les premières méthodes numériques pour résoudre (4) présentées par Yves Achdou et Italo Capuzzo-Dolcetta dans [2], de nombreuses méthodes se sont développées. On renvoie ici le lecteur à [17] et [7] pour des techniques relativement proches de celle que nous présentons ici.

On se donne un lagrangien L défini par

$$L(x, y) = F(x) + < a(x), b(y) >, \forall x \in K_1, y \in K_2$$

où K_1 est une partie convexe fermée d'un espace de Hilbert $(H_1, (\cdot, \cdot))$ et K_2 une partie convexe fermée d'un espace de Hilbert $\Big(H_2, ((\cdot, \cdot))\Big)$. On note $(H_3, < \cdot, \cdot >)$ un troisième espace de Hilbert et $a : H_1 \to H_3$ et $b : H_2 \to H_3$ sont deux applications continues. On suppose que $F : H_1 \to \mathbb{R}$ est une application C^1. Un point selle du lagrangien L est un couple (x, y) vérifiant

$$\inf_{x' \in K_1} \sup_{y' \in K_2} L(x', y') = \sup_{y' \in K_2} \inf_{x' \in K_1} L(x', y') = L(x, y) \tag{34}$$

On note $\mathbf{P}_A$ la projection sur $A \subset H_3$ dans H_3. On rappelle ici le résultat classique de convergence de l'algorithme d'Uzawa :

Théorème 0.6.1. *On suppose que la différentielle f de F est une application α monotone de H_1 dans lui même pour un certain $\alpha > 0$. On suppose également que $\tilde{K_2} := b(K_2)$ est un convexe fermé de H_3 et que a est une application C*

lipschitzienne pour $C > 0$. Alors pour $\delta < \frac{2\alpha}{C^2}$ la suite $(x_n, y_n)_{n\geq 0}$ définie par

$$\begin{cases} y_0 \in K_2 \\ x_n = \underset{x\in K_1}{argmin}\ \{L(x, y_n)\} \\ y_{n+1} \in b^{-1}\Big(\{\mathbf{P}_{\tilde{K_2}}(b(y_n) + \delta a(x))\}\Big) \end{cases} \tag{35}$$

est telle que $(x_n)_{n\geq 0}$ converge vers x, où (x, y) est l'unique couple vérifiant (34).

Démonstration. Vu les hypothèses, on sait que le couple (x, y) caractérisé par (34) satisfait :

$$\begin{cases} (f(x), x' - x) + < a(x') - a(x), b(y) > \geq 0, \forall x' \in K_1 \\ < a(x), b(y') - b(y) > \leq 0, \forall y' \in K_2. \end{cases}$$

Par construction, (x_n, y_n) vérifie pour tout $n \geq 0$

$$(f(x_n), x' - x_n) + < a(x') - a(x_n), b(y_n) > \geq 0, \forall x' \in K_1.$$

On déduit donc que

$$\begin{aligned} < a(x_n) - a(x), b(y) - b(y_n) > &\geq (f(x) - f(x_n), x - x_n) \\ &\geq \alpha ||x - x_n||^2_{H^1}. \end{aligned}$$

On remarque ensuite que, comme $\mathbf{P}_{\tilde{K_2}}$ est contractante :

$$\begin{aligned} ||b(y_{n+1}) - b(y)||^2_{H^3} \leq & ||b(y_n) - b(y) + \delta(a(x_n) - a(x))||^2_{H^3} \\ \leq & ||b(y_n) - b(y)||^2_{H^3} + 2\delta < b(y_n) - b(y), a(x_n) - a(x) > \\ & + \delta^2 ||a(x_n) - a(x)||^2_{H^3}. \end{aligned}$$

D'où on déduit donc que

$$||b(y_{n+1}) - b(y)||^2_{H^3} \leq ||b(y_n) - b(y)||^2_{H^3} - 2\delta ||x_n - x||^2_{H^1} + \delta^2 ||a(x_n) - a(x)||^2_{H^3}.$$

Or comme $\delta < \frac{2\alpha}{C^2}$, on en déduit qu'il existe $\beta > 0$ tel que

$$\beta ||x_n - x||^2_{H^1} \leq ||b(y_n) - b(y)||^2_{H^3} - ||b(y_{n+1}) - b(y)||^2_{H^3}.$$

Ce qui implique directement le résultat annoncé. □

La remarque que nous faisons maintenant porte sur les solutions (x, y) du système

$$\begin{cases} (f(x), x' - x) + < a(x') - a(x), b(y) > \geq 0, \forall x' \in K_1 \\ < a(x), b(y') - b(y) > \leq 0, \forall y' \in K_2. \end{cases} \tag{36}$$

Pour approcher les solutions de ce système, on considère les itérations :

$$\begin{cases} y_0 \in K_2; \\ x_n \text{ est défini par } (f(x_n), x' - x_n) + < a(x') - a(x_n), b(y_n) > \geq 0; \\ y_{n+1} \in b^{-1}\Big(\{\mathbf{P}_{\tilde{K_2}}(b(y_n) + \delta a(x))\}\Big). \end{cases} \tag{37}$$

En particuliers on ne fait plus l'hypothèse ici que f est la différentielle d'une application F. On a alors le résultat suivant :

Théorème 0.6.2. *On suppose que f est une application α monotone de H_1 dans lui même pour un certain $\alpha > 0$. Alors sous les mêmes hypothèses que le théorème 0.6.1, les itérations (37) définissent bien une suite $(x_n, y_n)_{n\geq 0}$ telle que $(x_n)_{n\geq 0}$ converge vers x, où (x, y) est l'unique couple vérifiant (36).*

La preuve de ce résultat est essentiellement la même que celle du théorème précédent. Nous ne la détaillons donc pas ici et renvoyons au quatrième chapitre pour sa preuve détaillée.

Deux remarques découlent des résultats précédents. La première est que l'on peut construire des approximations des systèmes MFG (4), (15) et (26), dans le cas potentiel, à l'aide des itérations d'Uzawa (35) (sous l'hypothèse de α monotonie). Le cas dit potentiel est celui où il existe une fonction $\mathcal{F}$, que nous prenons de $L^2(\mathbb{T}^d)$ dans $\mathbb{R}$ pour fixer les idées, telle que pour tout $m, m' \in L^2(\mathbb{T}^d)$:

$$\lim_{t\to 0} \frac{\mathcal{F}(m + tm') - \mathcal{F}(m)}{t} = \int_{\mathbb{T}^d} f(m)(m' - m).$$

où f est la fonction apparaissant dans (15), (26) et dans (4) dans le cas $H(x, p, m) = \tilde{H}(x, p) - f(m)$. Il est bien connu que dans ce cas potentiel, la solution du système MFG peut s'interpréter comme le point selle d'un lagrangien. Nous renvoyons à [54] ainsi qu'aux chapitres 1 et 2 pour la présentation de ces interprétations.

La deuxième remarque est que même en dehors du cas dit potentiel, si f est α monotone on peut toujours approcher des solutions de systèmes MFG par des itérations de type (37). On se place ici dans le cas stationnaire pour simplifier les notations. Alors si $(u, m) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ est solution de

$$\begin{cases} \max(-\Delta u + u - f(m), u - M(u)) = 0 \text{ dans } \mathbb{T}^d; \\ \forall v \in H^1(\mathbb{T}^d), v \leq M(v) : \\ \int_{\mathbb{T}^d}(-\Delta(v - u) + (v - u))m - \int_{\mathbb{T}^d} \rho(v - u) \geq 0; \\ \int_{\mathbb{T}^d}(f(m) + \Delta u - u)m = 0; m \geq 0; u \leq M(u); \end{cases}$$

on observe que (u, m) est également l'unique solution de

$$\begin{cases} \forall m' \in L^2(\mathbb{T}^d), m' \geq 0 : \\ \int_{\mathbb{T}^d}(f(m) + \Delta u - u)(m' - m) \geq 0; \\ \forall v \in H^1(\mathbb{T}^d), v \leq M(v) : \\ \int_{\mathbb{T}^d}(-\Delta(v-u) + (v-u))m - \int_{\mathbb{T}^d} \rho(v-u) \geq 0; \\ m \geq 0; u \leq M(u). \end{cases}$$

Remarquons qu'en prenant $M(u)$ défini soit comme l'opérateur de saut du cas contrôle impulsionnel, soit comme une constante, on retrouve la formulation des solutions MFG des cas arrêt optimal et contrôle impulsionnel.

On remarque alors que si f est α monotone, on peut appliquer directement le théorème 0.6.2 en prenant $H_1 = H_3 = L^2(\mathbb{T}^d)$, $H_2 = H^2(\mathbb{T}^d)$, $K_1 = \{m \in H_1, m \geq 0\}$ et $K_2 = \{u \in H^2, u \leq M(u)\}$. On équipe dans ce cas $L^2(\mathbb{T}^d)$ de son produit scalaire usuel et $H^2(\mathbb{T}^d)$ du produit scalaire $< \cdot, \cdot >$ défini par l'opérateur auto-adjoint $L = -\Delta + Id$, c'est à dire

$$< u, v >= \int_{\mathbb{T}^d} (Lu)v.$$

Le cas du système MFG (4) étant plus complexe à cause du hamiltonien, on modifie les itérations (37) en

$$\begin{cases} u_0 \in H^2(\mathbb{T}^d); \\ m_n \text{ est défini par } \forall m' \in L^2(\mathbb{T}^d), m' \geq 0 : \\ \int_{\mathbb{T}^d}(f(m_n) + \Delta u_n - u_n - H(x, \nabla u_n))(m' - m_n) \geq 0; \\ u_{n+1} \text{ est solution de } : \\ -\Delta u_{n+1} + u_{n+1} + H(x, \nabla u_{n+1}) = -\Delta u_n + u_n + H(x, \nabla u_n) \\ \qquad\qquad + \delta_n(m_n - (L_{u_n})^{*-1}\rho) \text{ dans } \mathbb{T}^d; \end{cases} \tag{38}$$

où $(\delta_n)_{n\geq 0}$ est une suite de réels strictement positifs et pour tout $v \in H^1(\mathbb{T}^d)$, L_v est l'opérateur défini par

$$L_v u = -\Delta u + u + D_p H(x, \nabla v) \cdot \nabla u.$$

On montre alors dans le quatrième chapitre le résultat suivant :

Théorème 0.6.3. *S'il existe un couple $(u, m) \in H^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ alors si*
- *f est α monotone.*
- *Le hamiltonien H est convexe.*

— *H est globalement lipschitz.*
alors il existe une suite de réels $(\delta_n)_{n\geq 0}$ telle que la suite (u_n, m_n) définie par (38) soit bien définie et $(m_n)_{n\geq 0}$ converge vers m dans $L^2(\mathbb{T}^d)$.

On illustre également ces différentes itérations d'Uzawa par des simulations numériques présentées dans le chapitre 4.

0.7 Un cas concret de modèle MFG

On présente dans cette partie un problème concret de MFG étudié dans le cinquième chapitre de cette thèse. Ce travail a été réalisé en collaboration avec S. Vassilaras, G. Paschos, M. Debbah, J.-M. Lasry et P.-L. Lions et fait l'object de l'article [15]. Ce modèle a pour objet d'étudier les interactions qui existent entre de nombreux objets connectés qui communiquent tous avec une seule antenne. Ce type de modèle est actuellement l'objet de nombreux travaux de recherche, on renvoie à l'article [59] de F. Mériaux, S. Lasaulce, and H. Tembine pour une présentation plus détaillée de ce problème. L'espace d'états des objets connectés (agents) est de dimension trois. Les trajectoires des agents sont données par :

$$\begin{cases} dE_t = -p_t dt; \\ dB_t = -f(\frac{h_t p_t}{\Gamma})dt \\ dh_t = \sqrt{2\nu} dW_t; \end{cases} \tag{39}$$

où le processus $(p_t)_{t\geq 0}$ est le contrôle de l'agent, $(W_t)_{t\geq 0}$ est un mouvement brownien réfléchi sur l'intervalle $[(2\nu)^{-\frac{1}{2}} h_{min}, (2\nu)^{-\frac{1}{2}} h_{max}]$ pour un espace probabilisé $(\Omega, \mathcal{A}, \mathbb{P})$, f est une fonction croissante donnée, $\nu > 0$ une constante et Γ un terme de couplage que l'on précisera plus tard. La variable d'état E représente l'énergie encore disponible pour un agent, B la quantité d'information qu'un agent a encore à transmettre et h la qualité de sa liaison avec l'antenne. Un agent peut donc transmettre de l'information en dépensant de l'énergie, mais son taux de transmission d'information dépend du paramètre de couplage Γ et de la qualité h de sa relation avec l'antenne.

Le problème d'un agent est alors

$$\sup_{0\leq p\leq p_{max}} \mathbb{E}\left[-\int_0^\tau e^{-\lambda}\theta dt + e^{-\tau}\psi(E_\tau, B_\tau) \right], \tag{40}$$

où $(E_s, B_s, h_s)_{s\geq 0}$ satisfait (39). Le temps d'arrêt τ est défini comme le temps de sortie de l'ensemble $\{E > 0\} \cup \{B > 0\}$ pour le processus $(E_s, B_s)_{s\geq 0}$. Le gain de sortie ψ est une fonction donnée et $p_{max}, \lambda, \theta > 0$ sont des paramètres du modèle. Les agents cherchent donc dans ce jeu à maximiser un gain de sortie qui croît

avec l'énergie finale restante E_τ et décroit avec la quantité d'information restant à transmettre B_τ (on choisit ψ de telle sorte). Par ailleurs on pénalise le temps passé dans le jeu pour diminuer les interférences entre appareils. On s'intéresse ici à des équilibres stationnaires de ce MFG et on cherche donc des couples (u, m) solutions de

$$\begin{cases} -\nu\partial_{hh}u + \lambda u + p^*\partial_E u + f(\frac{hp^*}{\Gamma})\partial_B u = -\theta \text{ dans } \mathbb{R}^2_+ \times [h_{min}, h_{max}]; \\ -\nu\partial_{hh}m - \partial_E(p^*m) - \partial_B\left(f(\frac{hp^*}{\Gamma})m\right) = m_s \text{ dans } \mathbb{R}^2_+ \times [h_{min}, h_{max}]; \\ u = \psi \text{ sur } \{E > 0\} \cup \{B > 0\}; \\ m = 0 \text{ sur } \{E > 0\} \cup \{B > 0\}; \\ \partial_h u = \partial_h m = 0 \text{ sur } \{h = h_{min}\} \cup \{h = h_{max}\} \end{cases} \tag{41}$$

où m_s est le taux d'entrée des agents dans le MFG et p^* est défini par

$$p^*(E, B, h) = \underset{0 \le p \le p_{max}}{argmax} \left\{ -p\partial_E u - f(\frac{hp}{\Gamma})\partial_B u \right\}. \tag{42}$$

Le terme de couplage Γ est lui défini par

$$\Gamma = \sigma^2 + c \int\int\int_{\mathbb{R}^2_+ \times [h_{min}, h_{max}]} m(E, B, h) h p^*(E, B, h) dE dB dh; \tag{43}$$

où $\sigma, c > 0$ sont des constantes du modèle. Dans le cinquième chapitre de ce manuscrit, on présente des simulations numériques portant sur des solutions de (41)-(42)-(43).

0.8 Perspectives et futurs développements

Bien sûr, de nombreuses questions sont toujours sans réponse dans la théorie des jeux à champ moyen. Certainement la plus importante de toutes concerne une meilleure compréhension de la *master equation*. Il semble en effet fondamental de comprendre par exemple si les hypothèses du théorème 0.1.3 peuvent être affaiblies, et si oui comment. Il est fortement probable qu'il faille pour cela définir une notion plus faible de solution de la *master equation*. Il est également crucial de trouver des méthodes numériques qui pourraient permettre de résoudre la *master equation*. En effet dans de nombreux cas pratiques, on ne peut pas ramener le problème à un système MFG de dimension fini, or à l'heure actuelle c'est dans ce genre de cas que se concentrent la plupart des méthodes numériques.

Concernant plus particulièrement les problèmes abordés dans ce manuscrit, on peut noter qu'il reste encore à écrire une *master equation* dans le cas arrêt optimal ou contrôle impulsionnel. Il reste également à prouver un résultat analogue au

théorème 0.1.3 dans ces cas. La difficulté majeure pour établir ce genre de résultats réside dans l'absence de régularité de l'évolution en temps de la densité des joueurs. Ce manque de régularité se traduit, au niveau de la *master equation*, par une perte de différentiabilité de la solution. Il faut donc a priori trouver une notion de solution plus faible pour pouvoir établir un résultat analogue au théorème 0.1.3.

Par ailleurs, il est fondamental de continuer à développer de nouvelles applications des MFG. Que ce soit pour des modèles de décentralisation de prises de décisions comme dans le chapitre 5 ou pour des modèles de jeux plus standards, la théorie des MFG a de nombreux aspects très généraux qui permettent d'aborder des problèmes diverses dans un cadre théorique solide. C'est pourquoi il est fort probable que de nombreux problèmes puissent être résolus grâce aux MFG. Inversement le développement de nouveaux modèles permettra certainement d'avoir une meilleure compréhension de la théorie des MFG ainsi que le développement de nouvelles méthodes numériques.

Chapitre 1

Arrêt optimal dans les jeux à champ moyen, une approche par problème de l'obstacle

Ce chapitre s'intéresse au problème d'arrêt optimal dans un contexte de jeu à champ moyen. On introduit la notion de solution mixte pour résoudre le système d'équations aux dérivées partielles qui modélise ce genre de problème. Cette notion de solution traduit le fait que les équilibres de Nash du jeu à champ moyen sont en stratégies mixtes. L'existence et l'unicité de telles solutions sont prouvées sous des hypothèses générales pour le problème stationnaire et celui dépendant du temps.

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1.1 Introduction

1.1.1 General introduction

This paper is the first of a series devoted to the systematic study of mean field games (MFG) of optimal stopping or impulse controls. We solve here a system of forward-backward obstacle problems which models MFG of optimal stopping without common noise. The case of the "Master equation" will be considered in a subsequent work.

In the past decade, MFG have been broadly studied since their introduction by Lasry and Lions in their series of paper [52, 53, 54] and it has been shown they have lots of applications [45, 46]. From the well posedness of the problem [21, 57] to the difficult question of the master equation [27, 57] through numerical questions [2, 7] and developments like long time average [26] or learning in MFG [23], the original model has been the source of a huge number of mathematical questions. A very powerful probabilistic point of vue has also been developed [29, 49]. We refer to [57] for a large study of the problem. We briefly recall the model proposed by Lasry and Lions when there is no common noise. A continuum of indiscernable players, which is characterized by a measure m, faces an optimal control problem (which is stochastic here but can also be deterministic) whose value function is denoted by u. The players are initially described by the measure m_0 and are only interested in the distribution of the other players. Under general assumptions on the regularity and geometry of the cost functions (involved in both the running and terminal costs, respectively f and g), the value function u satisfies a Hamilton-Jacobi-Bellman equation given the evolution of the measure of the player. On the other hand, given the value function and then the behavior of the players, the measure m satisfies a transport equation. Nash equilibria of the game are then given by solutions of the MFG system :

$$\begin{cases} -\partial_t u - \Delta u + H(x, \nabla u) = f(x, m) \text{ in } \mathbb{R}^d \times (0, T); \\ \partial_t m - \Delta m - div(D_p(H(x, \nabla u))m) = 0 \text{ in } \mathbb{R}^d \times (0, T); \\ u(T) = g(T, m(T)), m(0) = m_0. \end{cases} \tag{1.1}$$

Our purpose is to study the analogue of (1.1) for an optimal stopping problem. In this setting, players do not affect their velocity anymore but choose a time after which they definitively leave the game. We present our results in three different situations : a stationary case, a time dependent case with a general exit cost and a last case in which players can both leave the game and affect their velocity like in the case we just described. We are going to introduce the notion of mixed solutions for MFG, which describes Nash equilibria for MFG in mixed strategies. As we shall see, this notion is both analytically natural and the good notion of

solution for MFG with optimal stopping. Recently some results have been obtained concerning the problem of optimal stopping in MFG. In [61], the author solves a game of optimal stopping in a MFG and in [32], the authors prove an interesting and general result of existence in a probabilistic approach of the problem. We may also mention [42], in which the authors studied a MFG type system for an obstacle problem, which is the natural Hamilton-Jacobi-Bellman equation for an optimal stopping problem. Finally let us mention [20] in which the authors study a MFG in which the players are absorbed once they reach the boundary of the domain, meaning that the leaving time is not a control but determined by the model. Note that MFG with optimal stopping are very natural, from at least two perspectives. First, optimal stopping problems are interesting in themselves like for example resistance games or the modeling of american options. Then, it appears natural to allow the players to leave the MFG, because it is very restrictive to impose that all the players have to stay until the end like it is done in the classical MFG model. Indeed, it is more realistic to allow the players to leave the game by paying an exit cost.

1.1.2 The model

We present here the typical framework which is beneath the idea of optimal stopping in MFG. We refer to the next parts for more precise statements as we only want to give an intuition on why the model we are going to study is general and natural. We assume that there is an infinite number of players and we associate to each player i a diffusion which satisfies :

$$\begin{cases} dX_t^i = \sqrt{2} dW_t^i; \\ X_0^i = x_i \text{ in } \Omega; \end{cases}$$

where Ω is a bounded, smooth open set of $\mathbb{R}^d$ with $d \geq 1$. The player i has to make the choice of a stopping time τ which has to be measurable for the σ-algebra generated by the d-dimensional brownian motion $(W_t^i)_{t\geq 0}$. We assume that all the brownian motions $((W_t^i)_{t\geq 0})_i$ are independent. The cost is then defined by

$$\int_0^\tau f(X_s^i, m(s))ds + \psi(X_\tau^i, m(\tau));$$

where $m(t)$ is the measure which characterizes the distribution of the player in Ω at time t. The players minimize the expectation of this cost. We work with a finite horizon T and assume that if the diffusion (X_t^i) reaches the boundary $\partial\Omega$ of Ω at time t^*, then the player i exits the game paying the cost $\int_0^{t^*} f(X_s^i, m(s))ds$. Given the evolution of the measure $m(t)$, the value function of the problem is a solution

of the obstacle problem :

$$\begin{cases} \max(-\partial_t u(t,x) - \Delta u(t,x) - f(x,m(t)), u(t,x) - \psi(x,m(t))) = 0; \\ u(T,x) = \psi(x,m(T)) \\ \forall t \leq T, u(t,x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

See for example [6] for more details on Hamilton-Jacobi-Bellman equations. On the other hand, given the value function u and the fact that it is optimal to leave the game at (t,x) if $u(t,x) = \psi(t,x)$, m satisfies

$$\begin{cases} \partial_t m - \Delta m = 0 \text{ on } \{(t,x) \in (0,T) \times \Omega / u(t,x) < \psi(x,m(t))\}; \\ m(0) = m_0; \\ m = 0 \text{ elsewhere;} \end{cases} \tag{1.3}$$

because the players move freely in the set $\{u < \psi(m)\}$ since it is optimal to stay in the game. In this set, the players evolve only through the diffusion to which they are associated (they use no control). On the other hand, because it is optimal to leave the game in the set $\{u = \psi\}$, we want m to be 0 in this set, as all the players are leaving. We do not make precise here the sense in which systems (1.2) and (1.3) have to be taken as it will be the subject of later discussions. In the same way Nash equilibria in the classical MFG are given by solutions of the system (1.1), we expect Nash equilibria of our problem to be the solutions of

$$(OSMFG) \begin{cases} \max(-\partial_t u(t,x) - \Delta u(t,x) - f(x,m(t)), u(t,x) - \psi(x,m(t))) = 0; \\ \forall (t,x) \in (0,T) \times \Omega; \\ \partial_t m - \Delta m = 0 \text{ on } \{(t,x) \in]0,T[\times \Omega / u(t,x) < \psi(x,m(t))\}; \\ m = 0 \text{ elsewhere;} \\ u(T,x) = \psi(x,m(T)) \text{ on } \Omega; \\ \forall t \leq T, u(t,x) = 0 \text{ on } \partial\Omega. \end{cases}$$

For pedagogical reasons, we first solve a stationary setting in which the time variable has disappeared and the leaving cost is 0. In this setting, we are forced to introduce a source of players. This term can be interpreted as some players (who are still identical to all the others) entering the game uniformly in time. If we do not add this term, then the only equilibrium is $m = 0$ because, almost surely, the trajectories of the players touch the boundary of Ω and thus the associated players leave the game. We denote it by ρ. This term is completely arbitrary and do not play a strong role in the qualitative approach we present. We also add a first order term in the equations in "u" and "m". The first one stands for a preference for the present over the future while the second one stands for a "natural" death rate of

players. The "simpler" version of $(OSMFG)$ is then

$$(SMFG)\begin{cases}\max(-\Delta u + u - f(x,m), u) = 0 \text{ in } \Omega;\\ -\Delta m + m = \rho \text{ on } \{u < 0\};\\ m = 0 \text{ on } \{u = 0\};\\ u = m = 0 \text{ on } \partial\Omega.\end{cases}$$

This system is the subject of the first part of this article and we extend our results to the case of $(OSMFG)$ in the second part. In the last part we present the case where both optimal stopping and continuous control are possible. This last setting leads to the following system :

$$(MFG)\begin{cases}\max(-\partial_t u - \Delta u + H(x,\nabla u) - f(m), u) = 0 \text{ in } (0,T)\times\Omega;\\ \partial_t m - \Delta m - div(mD_pH(x,\nabla u)) = 0 \text{ in } \{u < 0\};\\ m = 0 \text{ in } \{u = 0\};\\ u = m = 0 \text{ on } \partial\Omega;\\ m(0) = m_0 \text{ and } u(T) = 0 \text{ in } \Omega;\end{cases}$$

where H is the Fenchel conjugate of the part of the running cost which depends on the control. More details on this problem are given in the last part.

Please note that every time we consider the case $\psi(m) = 0$, we are in fact considering the case where ψ does not depend on m (it could depend on x), as we can change the cost f with the addition of $\partial_t\psi + \Delta\psi$ to pass from a problem in which there is an exit cost to the case in which this cost is 0.

1.1.3 Assumptions

We present here the assumptions which hold for the rest of this discussion as well as some notations

- Ω is an open bounded subset of $\mathbb{R}^d$ with a smooth (say C^2) boundary ;
- $m_0 \in L^2(\Omega)$ and $m_0 \geq 0$;
- f is a contiuous application from $L^2(\Omega)$ to $L^2(\Omega)$ which maps $L^\infty(\Omega)$ onto itself ;
- ψ is a continuous application from $L^2((0,T), L^2(\Omega))$ to $L^2((0,T), H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1((0,T), L^2(\Omega))$ and ψ is valued in the set of continuous functions ;
- $\rho \in C^\infty(\Omega)$ with compact support in Ω and $\rho \geq 0$.

1.2 The stationary problem

1.2.1 Preliminary results

In this section, we present useful results regarding the (stationary) problems (1.2) and (1.3). We address here the question of regularity of those problems in order to present the results of regularity for the MFG problem later on.

The obstacle problem

This problem is classical (see for instance [18]). For any f of $L^2(\Omega)$, a solution u of the obstacle problem is such that

$$\max(-\Delta u + u - f, u) = 0 \text{ in } \Omega,\ u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Existence and uniqueness of solutions of this problem can be found in [18, 55]. Also, the mapping from $L^2(\Omega)$ to $H^2(\Omega) \cap H_0^1(\Omega)$ which associates to each f the solution of the previous problem is continuous with respect to the canonical norms of those sets. Moreover, the sequence $(u_\epsilon)_{\epsilon>0}$ of $H^2(\Omega) \cap H_0^1(\Omega)$ converges to u, solution of the obstacle problem with source f, for the $H^1(\Omega)$ norm, if u_ϵ is defined for all ϵ as the unique solution of

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon + \frac{1}{\epsilon}(u_\epsilon)^+ = f \text{ in } \Omega; \\ u_\epsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

Moreover, if $f \in L^\infty(\Omega)$, then $u \in \mathcal{C}^{1,\alpha}(\Omega)$ for any $0 < \alpha < 1$.

The equation in m

We are here interested in the regularity of the solution m of

$$\begin{cases} -\Delta m + m = \rho \text{ in } \omega; \\ m = 0 \text{ on } \partial\omega; \end{cases}$$

where ω is any open subset included in Ω. We extend m by 0 on $\Omega \setminus \omega$. Obviously, there exists a unique solution of this problem in $H_0^1(\omega)$ and we cannot expect further regularity without any assumptions on the regularity of ω. ($H_0^1(\omega)$ is here defined as the closure of the C^∞ functions with compact support in ω for the $H^1(\Omega)$ norm.) Then, if $(\omega_n)_{n\geq 0}$ is a sequence of open subsets of Ω which converges towards ω, open set of Ω, with respect to the Hausdorff distance then the sequence of

associated solutions $(m_n)_{n\geq 0}$ converges towards m for the norm of $H^1(\Omega)$. Finally, if we define m_ϵ as the unique solution of

$$\begin{cases} -\Delta m_\epsilon + m_\epsilon + \frac{1}{\epsilon}\mathbb{1}_{\omega^c} m_\epsilon = \rho \text{ in } \Omega; \\ m_\epsilon = 0 \text{ on } \partial\Omega; \end{cases}$$

then $(m_\epsilon)_{\epsilon>0}$ converges to m for the norm of $L^2(\Omega)$. We also recall the following result, which is very simple and that we shall need later.

Lemma 1.2.1. *Let $m \in H_0^1(\Omega)$ be such that*
— *there exists an open set $\omega \subset \Omega$ such that $-\Delta m + m = \rho$ on ω ;*
— *$-\Delta m + m \leq \rho$ in the sense of distributions on Ω ;*
and denote by $u \in H^2(\Omega) \cap H_0^1(\Omega)$ a negative function, then

$$\int_\Omega (-\Delta u + u)m \geq \int_\Omega u\rho;$$

and we have an equality if the support of u is included in ω.

Proof. This result is simply the translation in a variational form of the assumption $-\Delta m + m \leq \rho$.

□

1.2.2 First properties of the system

We now turn to the presentation of some results which give a better idea of the situation we are trying to model. Mainly we prove uniqueness and non existence of solutions for $(SMFG)$. Those results highlight that we have to adopt a new definition of solutions in order to find Nash equilibria of the MFG. We recall the notion of monotone applications on functions' spaces which is, as usual, crucial in the study of MFG. An application T from $L^2(\Omega)$ into itself is said to be monotone if

$$\forall m_1, m_2 \in L^2(\Omega), \int_\Omega (T(m_1) - T(m_2))(m_1 - m_2) \geq 0.$$

It is strictly monotone if the inequality is strict as soon as $m_1 \neq m_2$. T is said to be anti-monotone if $-T$ is monotone.

We first prove a uniqueness result which depends on the monotonicity of the cost f.

Theorem 1.2.1. *If f is strictly monotone, then there exists at most one solution of $(SMFG)$.*

Proof. We shall present here an analogue of the proof of uniqueness of Lasry and Lions in [54]. We denote by (u_1, m_1) and (u_2, m_2) two solutions of $(SMFG)$ and note $u = u_1 - u_2$, $m = m_1 - m_2$ and the continuation sets $\Omega_1 = \{u_1 < 0\}$ and $\Omega_2 = \{u_2 < 0\}$. We then compute :

$$\begin{aligned}\int_{\Omega_1 \cup \Omega_2} (-\Delta u + u)m = &\int_{\Omega_1 \cap \Omega_2} (f(m_1) - f(m_2))m \\ &+ \int_{\Omega_1 \setminus \Omega_2} f(m_1)m_1 + \int_{\Omega_2 \setminus \Omega_1} f(m_2)m_2.\end{aligned} \tag{1.4}$$

We here use the fact that on Ω_i^c, $-\Delta u_i = 0$ which holds because the obstacle problem

$$\max(-\Delta u_i + u_i - f(m_i), u_i) = 0 \text{ in } \Omega$$

holds true in $L^2(\Omega)$.

Let us remark that on Ω_2^c, $m_2 = 0$ and $f(m_2) \geq 0$ because (u_2, m_2) is a solution of $(SMFG)$. Thus, it is in particular true on $\Omega_1 \setminus \Omega_2$. Hence the following inequality is true.

$$\int_{\Omega_1 \setminus \Omega_2} f(m_1)m_1 \geq \int_{\Omega_1 \setminus \Omega_2} (f(m_1) - f(m_2))(m_1 - m_2).$$

With the same argument on the third term of the right hand side of (1.4), we obtain

$$\int_{\Omega_1 \cup \Omega_2} (-\Delta u + u)m \geq \int_{\Omega_1 \cup \Omega_2} (f(m_1) - f(m_2))m \geq 0.$$

We now evaluate the sign of this term in a different way. First, let us remark that because of the fact that (u_1, m_1) and (u_2, m_2) are solutions of $(SMFG)$, we know using lemma 1.2.1 that

$$\int_\Omega (-\Delta u_1 + u_1)m_1 + \int_\Omega (-\Delta u_2 + u_2)m_2 = \int_\Omega (u_1 + u_2)\rho.$$

This equality gives

$$\begin{aligned}\int_\Omega (-\Delta u + u)m = &- \int_\Omega (-\Delta u_1 + u_1)m_2 + \int_\Omega u_1\rho \\ &- \int_\Omega (-\Delta u_2 + u_2)m_1 + \int_\Omega u_2\rho.\end{aligned}$$

Using once again lemma 1.2.1 we deduce

$$\int_\Omega (-\Delta u + u)m \leq 0.$$

And finally we obtain

$$\int_\Omega (-\Delta u + u)m = \int_{\Omega_1 \cup \Omega_2} (f(m_1) - f(m_2))m = 0.$$

So by strict monotonicity, $m_1 = m_2$ and the result is proved.

□

We observe that the monotonicity of f is important for the question of uniqueness.We now show it is in general necessary.

Proposition 1.2.1. *Uniqueness does not hold in general for* $(SMFG)$.

Proof. We denote by $m^* \in H_0^1(\Omega)$ the unique solution of

$$-\Delta m^* + m^* = \rho \text{ in } \Omega.$$

We note $E(m) = \int_\Omega |x| m(x) dx$. We define $f(m)$ by the following equation for $m \in L^2(\Omega)$:

$$\forall x \in \Omega, f(x, m) = -2\frac{E(m)}{E(m^*)} + 1.$$

We observe that $f(m^*) = -1$ and $f(0) = 1$. It is clear that (0,0) is a solution of $(SMFG)$. Now if we denote by u^* the solution of the obstacle problem with cost $f(m^*)$, it is clear by the maximum principle that the contact zone with the obstacle $\{u = 0\}$ is empty and thus that (u^*, m^*) is also a solution of $(SMFG)$. □

We take advantage of this collection of remarks around this new system to present the fact that uniqueness of solutions does not hold in general, even if f is strictly monotone, if the obstacle ψ depends on m. However, we are not interested in giving too much detail about which assumption has to be made on ψ for uniqueness, as it is a subject of the next part.

Proposition 1.2.2. *For all strictly monotone* f*, there exists an obstacle* ψ *such that there is no uniqueness of solutions.*

Proof. Once again we denote by $m^* \in H_0^1(\Omega)$ the unique solution of

$$-\Delta m^* + m^* = \rho \text{ in } \Omega;$$

which is strictly positive by the strong maximum principle. We define $u^* \in H_0^1(\Omega) \cap H^2(\Omega)$ by the unique solution of

$$-\Delta u^* + u^* = f(m^*) \text{ in } \Omega;$$

and $u_* \in H_0^1(\Omega) \cap H^2(\Omega)$ by the unique solution of

$$-\Delta u_* + u_* = f(0).$$

Now define ψ by the following :

$$\psi(m) = (u^*)\frac{m}{m^*} + \frac{m^* - m}{m^*}(u_*).$$

It is then easy to verify that both (u^*, m^*) and $(u_*, 0)$ are solutions of $(SMFG)$. □

Remark 1.2.1. *We can notice that here ψ is monotone. We shall see later that in order to have uniqueness of solutions, we have to make an assumption which is somehow related to the anti-monotonicity of ψ.*

We now present an example of non existence of solutions for $(SMFG)$. It is very general in its construction and the reader could easily understand how it could be adapted for different models.

Proposition 1.2.3. *There exists f such that there is no solution for the system $(SMFG)$.*

Proof. We define by m^* the unique solution of the equation :

$$\begin{cases} -\Delta m + m = \rho \text{ in } \Omega; \\ m = 0 \text{ on } \partial\Omega. \end{cases}$$

We then choose a non-positive smooth function (C^2) denoted by u^* which vanishes only on $\{x_0\}$ and on the boundary of the domain, where $x_0 \in \Omega$. We then define f on $L^2(\Omega)$ by

$$f(x, m) = -\Delta u^*(x) + u^*(x) + m(x) - m^*(x), \forall x \in \Omega.$$

By construction

$$\begin{cases} -\Delta u^* + u^* = f(m^*), \forall x \in \Omega; \\ u \leq 0. \end{cases}$$

Hence, u^* satisfies the obstacle problem with source $f(m^*)$. By the strong maximum principle, m^* is strictly positive on Ω, so m^* does not satisfy $m^* = 0$ on $\{u^* = 0\}$. Thus the couple (u^*, m^*) is not a solution of $(SMFG)$. Suppose that there exists a solution (u, m) of $(OSMFG)$, then necessarily $m < m^*$ (i.e. $m \leq m^*$ and $m \neq m^*$). Then recalling the strict monotonicity of f and the strong maximum

principle we obtain that $u < u^*$ everywhere on the domain. So the contact zone $\{u = 0\}$ is empty. This contradicts the fact that $m < m^*$, thus we have proven that there is no solution to $(SMFG)$ in this case.

□

Remark 1.2.2. *One can object to this conclusion that imposing that m vanishes (in some sense) on $\{u^* = 0\}$ is very restrictive, regarding the problem we are trying to model, because this set has a 0 Lebesgue measure. One can think that we should not have any constraint in this case. The interested reader could easily note that if u^* is such that it vanishes only on a small ball around x_0, then we could also have proven non existence in this case.*

We have now proven that a solution of the system may not exist and that it may not be unique if f is not monotone. This observation leads us to consider a relaxed notion of solutions of $(SMFG)$.

1.2.3 Towards the good notion of solutions

We now present the notion of mixed solutions of $(SMFG)$ for which we can establish better results of existence than for the notion of solution we used in the previous section. We give two different approaches for $(SMFG)$, each of them leads to this notion of mixed solution. The first one is an optimal control interpretation of $(SMFG)$. The second one is a penalized version of the problem.

The optimal control interpretation

We here assume that the dependence of f in m is local, meaning that

$$\forall m \in L^2(\Omega), x \in \Omega, f(x, m) = f(x, m(x)).$$

Thus, there exists $\mathcal{F}$ such that :

$$\forall (x, p) \in \Omega \times \mathbb{R}, \frac{\partial \mathcal{F}}{\partial p}(x, p) = f(x, p).$$

We assume that $\mathcal{F}(x, \cdot) \in C^1(\mathbb{R})$ and that $\mathcal{F}$ is continuous as seen as a functional from $H_0^1(\Omega)$ into $L^1(\Omega)$ (which is of course compatible with the fact that $f \in C^0(L^2(\Omega))$).

We now introduce an optimal control problem using $\mathcal{F}$ which leads to $(SMFG)$. We start by recalling the optimal control interpretation for the classical MFG system (1.1). This approach was presented in [54] and has been well studied since then (see [7, 22, 28] for example). Each time the authors of those articles link

the system (1.1) with both an optimal control of a Fokker-Planck equation and an optimal control of a Hamilton-Jacobi-Bellman equation. Here we only present (formally) the link with the Fokker-Planck equation. For each control α, we define m as the solution of the Fokker-Planck equation :

$$\begin{cases} \partial_t m - \Delta m - div(\alpha m) = 0 \text{ in }]0,T[\times\Omega; \\ m(0) = m_0 \text{ in } \Omega; \end{cases}$$

and we introduce the following problem

$$\inf_{\alpha} \Big\{ \int_0^T \int_\Omega \mathcal{F}(m) + L(x,\alpha)m \Big\}.$$

Where L is the Fenchel conjugate of the Hamiltonian H with respect to the p variable. Then it has been proved ([22] for example) that the infimum is reached when $\alpha(t,x) = -D_p H(x, \nabla u(t,x))$, where (u,m) is the solution of (1.1). Our aim is now to prove a similar result for $(SMFG)$. We start by a remark : in the optimal control problem for the players, the term $-D_p H(x, \nabla u(t,x))$ is an optimal choice to minimize their cost (it is their best answer), if (u,m) is a solution of (1.1). Thus, the problem of controlling the Fokker-Planck equation is interpreted as prescribing a behavior for the players (it is the α, the control) and then minimizing a quantity depending on the distribution of the players (m) induced by the prescribed behavior (α). For the optimal stopping case, we can describe a behavior as an exit set in Ω, on which the players leave the game. Thus, we can search for an optimal control problem where the control α is now a set and m (the associated distribution of players) satisfies the equation associated with this behavior :

$$\begin{cases} -\Delta m + m = \rho \text{ in } \alpha; \\ m = 0 \text{ on } \partial\alpha. \end{cases}$$

We then minimize the following

$$\inf_{\alpha} \Big\{ \int_\Omega \mathcal{F}(m) \Big\};$$

where the infimum is taken over the open sets of Ω. As it is well known minimization problems over a collection of open sets are difficult in general. This is why, we shall work on a relaxed version of the problem. We make two remarks : for all α, m is positive by the maximum principle and the following inequality is satisfied in the sense of distributions

$$-\Delta m + m \leq \rho.$$

So a possible relaxation for the previous problem is

$$\inf_{m \in \mathcal{H}} \Big\{ \int_\Omega \mathcal{F}(m) \Big\}; \tag{1.5}$$

where $\mathcal{H} = \{m \in H_0^1(\Omega), m \geq 0, -\Delta m + m \leq \rho\}$ in which the second inequality holds in the sense of distributions.

Theorem 1.2.2. *Assume $\mathcal{F}$ is strictly convex, then the previous relaxed problem admits a unique solution. Moreover, any minimizer m of the problem satisfies the Euler-Lagrange optimal conditions :*

$$\forall m' \in \mathcal{H}, \int_\Omega f(m)(m' - m) \geq 0.$$

Proof. The functional $\mathcal{F}$ is weakly sequentially lower semi continuous (ws lsc) for the topology of $H_0^1(\Omega)$ as it is both continuous and convex. Take a minimizing sequence $(m_n)_{n\in\mathbb{N}}$, by definition,

$$-\Delta m_n + m_n \leq \rho;$$

hence testing this relation against m_n itself,

$$\int_\Omega |\nabla m_n|^2 + \int_\Omega m_n^2 \leq \int_\Omega \rho m_n \leq ||\rho||_{L^2(\Omega)}||m_n||_{L^2(\Omega)};$$

which implies a bound in $H_0^1(\Omega)$. So there is a subsequence of $(m_n)_{n\in\mathbb{N}}$ which converges toward a limit m in $H_0^1(\Omega)$ for the weak topology. It is easy to check that the limit still belongs to $\mathcal{H}$. Then because $\mathcal{F}$ is ws lsc, we get that m is a minimizer of the problem. And it is obviously unique if $\mathcal{F}$ is strictly convex. Because of the regularity of $\mathcal{F}$, the verification of the Euler-Lagrange inequality is trivial. □

From now on we will denote by m the unique minimizer of this problem and u the solution of

$$\begin{cases} \max(-\Delta u + u - f(m), u) = 0 \text{ in } \Omega; \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Let us remark that $\mathcal{H}$ is bounded in $L^\infty(\Omega)$, hence $f(m) \in L^\infty(\Omega)$ and thus u is continuous. Our goal is to show that the couple (u, m) is a solution of $(SMFG)$ in a certain sense. More precisely we obtain the following :

Theorem 1.2.3. (u, m) *satisfies :*

$$\begin{cases} \max(-\Delta u + u - f(m), u) = 0 \ \textit{in } \Omega; \\ -\Delta m + m = \rho \ \textit{on } \{u < 0\}; \\ m = 0 \ \textit{on } \partial\Omega; \\ \int_\Omega f(m)m = \int_\Omega u\rho. \end{cases}$$

Proof. For all m' in $\mathcal{H}$, we compute

$$\begin{aligned}\int_\Omega f(m)(m'-m) &= \int_{\{u<0\}} f(m)(m'-m) + \int_{\{u=0\}} f(m)(m'-m)\\ &= \int_{\{u<0\}} (-\Delta u + u)(m'-m) + \int_{\{u=0\}} f(m)(m'-m)\\ &= \int_\Omega (-\Delta u + u)(m'-m) + \int_{\{u=0\}} f(m)(m'-m).\end{aligned}$$

Where the last equality holds true because $u \in H^2(\Omega)$. Take the sequence $(m_1^\epsilon)_{\epsilon>0}$ defined as solutions of

$$\begin{cases} -\Delta m_1^\epsilon + m_1^\epsilon + \frac{1}{\epsilon}\mathbb{1}_{\{u=0\}}(m_1^\epsilon - m)^+ = \rho \text{ in } \Omega;\\ m_1^\epsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

For all $\epsilon > 0$, $m_1^\epsilon \in H^2(\Omega) \cap H_0^1(\Omega)$. Next, observe that in the sense of distributions

$$-\Delta m_1^\epsilon + m_1^\epsilon \leq \rho;$$

$$m_1^\epsilon \geq 0 \text{ in } \Omega.$$

Hence, extracting a subsequence if necessary, $(m_1^\epsilon)_\epsilon$ converges weakly to a limit m_1 in $\mathcal{H}$ such that $m_1 \leq m$ on $\{u = 0\}$ and

$$\int_\Omega (-\Delta u + u) m_1 = \int_\Omega u\rho.$$

Thus we can write

$$\int_\Omega f(m)(m_1 - m) = \int_\Omega u\rho + \int_\Omega (-\Delta u + u)(-m) + \int_{\{u=0\}} f(m)(m_1 - m).$$

Now, note that because u satisfies an obstacle problem

$$f(m) \geq 0 = -\Delta u + u \text{ on } \{u = 0\}.$$

Using this remark and the fact that $m_1 \leq m$ on $\{u = 0\}$ we obtain

$$\int_{\{u=0\}} f(m)(m_1 - m) \leq 0.$$

Using lemma 1.2.1 we deduce that

$$\int_\Omega f(m)(m_1 - m) \leq 0;$$

and the inequality is strict if

$$-\Delta m + m \neq \rho \text{ on } \{u < 0\}.$$

Thus, because of the Euler-Lagrange conditions of optimality it follows that

$$-\Delta m + m = \rho \text{ on } \{u < 0\}.$$

Now we define a sequence $(m_2^\epsilon)_{\epsilon>0}$ as the solutions of

$$\begin{cases} -\Delta m_2^\epsilon + m_2^\epsilon + \frac{1}{\epsilon}\mathbb{1}_{\{u=0\}} m_2 = \rho \text{ in } \Omega; \\ m_2^\epsilon = 0 \text{ on } \partial\Omega. \end{cases}$$

Once again, extracting a subsequence if necessary, $(m_2^\epsilon)_\epsilon$ converges weakly to a limit $m_2 \in \mathcal{H}$ and we deduce

$$\int_\Omega f(m)(m_2 - m) = \int_{\{u=0\}} f(m)(m_2 - m) = -\int_\Omega f(m)m;$$

because both m and m_2 satisfy

$$-\Delta m' + m' = \rho \text{ in } \{u < 0\}.$$

Remark that because of the Euler-Lagrange conditions, we obtain that

$$\int_{\{u=0\}} f(m)m \leq 0.$$

But because of the obstacle problem satisfied by u, this quantity is also positive. Hence

$$\int_{\{u=0\}} f(m)m = 0. \tag{1.6}$$

Now we only have to remark that because (1.6) holds true,

$$\begin{aligned} \int_\Omega f(m)m &= \int_{\{u<0\}} f(m)m \\ &= \int_{\{u<0\}} (-\Delta u + u)m \\ &= \int_\Omega u\rho. \end{aligned}$$

□

We now introduce the following definition :

Definition 1.2.1. *A pair $(u, m) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ is a mixed solution of $(SMFG)$ if :*

— $\max(-\Delta u + u - f(m), u) = 0$ *in* Ω *;*

— $-\Delta m + m = \rho$ *on* $\{u < 0\}$ *in* $\mathcal{D}'(\Omega)$ *;*

— $-\Delta m + m \leq \rho$ *in* $\mathcal{D}'(\Omega)$ *;*
— $\int_{\{u=0\}} f(m)m = 0$ *;*
— $m \geq 0$.

Let us remark that the third and fifth point of the definition imply together that $m \in L^\infty(\Omega)$ and thus that u is continuous and that $\{u < 0\}$ is open. Obviously if m is a minimizer of (1.5) and u is the solution of the obstacle problem with source $f(m)$, then (u, m) is a mixed solution of $(SMFG)$. We present here in which extent it is natural to adopt this definition. This discussion is quite formal and we begin by taking a mixed solution (u, m) of $(SMFG)$. Let us remark that formally there exits a potential V such that

$$-\Delta m + m + Vm = \rho \text{ in } \Omega;$$

with the convention that V equals ∞ when m equals 0. Because $-\Delta m + m \leq \rho$, $V \geq 0$. The classical interpretation for such a term in mathematical modeling is usually a death rate. It is more appropriate to talk about a leaving rate here as we are interested in an optimal stopping game. We can now describe a Nash equilibrium of the MFG using this potential V : in the zone $\{V = 0\}$ it is optimal for the players to stay in the game ; in the zone $\{V = \infty\}$ it is optimal for the players to leave the game ; the zone $\{0 < V < \infty\}$ describes an indifference region where it is both optimal to leave and to stay and where players do leave with a non constant leaving rate given by V. We can remark we can still have a Nash equilibrium even if the players behave differently in the same situation, as soon as the way with which they choose between the options is random and that the law of this choice is the same for all the players. A Nash equilibrium of the MFG is then obtained when the leaving rate V of the players in this region is such that $0 = -\Delta u + u = f(m)$. Because the first equality comes from the fact that it is optimal to leave, and the second one from the fact that it is optimal to stay.
It justifies our choice of the terminology mixed solution, as it corresponds to the players playing in mixed strategies. Remark that here a strategy for the players is not just an exit set $\{u = 0\}$ but also a leaving rate V on this set which characterizes the probability they are playing as a strategy.

From a game theory point of vue it is very common to have existence of equilibria in mixed strategies but not in pure strategies. But from a MFG point of vue, it is quite a surprise that Nash equilibria for the game cannot be found in pure strategies for the optimal stopping problem. We recall that in the classical MFG setting, under the monotonicity assumptions on the costs f and g, there exists a unique solution for the system (1.1), which is interpreted as a Nash equilibrium in pure strategies.

The notion of mixed solutions of MFG we just presented is highly related to the notion of weak solutions which is introduced in [32]. In [32], the measure of stopping time is a weak solution, when it satisfies a fixed point property in a weak sense. It can be seen as letting the players play in mixed strategies. The authors of this article oppose their notion of weak solution to a notion of strong solution, which is more or less a classical notion of solution for $(SMFG)$. They prove their existence under a monotonicity assumption on the cost. We present the same result with a partial differential equations point of vue. We begin with a lemma.

Lemma 1.2.2. *Suppose that f is anti-monotone and define $T : H_0^1(\Omega) \to H_0^1(\Omega)$ by the fact that $T(m)$ is the only solution of*

$$-\Delta T(m) + T(m) = \rho \text{ in } \{u < 0\};$$

where u is the only solution of

$$\max(-\Delta u + u - f(m), u) = 0 \text{ in } \Omega;$$

Then $T(m_1) \leq T(m_2)$ if $m_1 \leq m_2$.

Proof. Take $m_1 \leq m_2$ and define u_1 and u_2 by

$$\max(-\Delta u_i + u_i - f(m_i), u_i) = 0 \text{ in } \Omega;$$

for $i = 1, 2$. Since $u_1 \geq u_2$ (f is anti-monotone) we deduce that $\{u_1 < 0\} \subset \{u_2 < 0\}$. Thus m defined by $m = T(m_2) - T(m_1)$ satisfies

$$-\Delta m + m = 0 \text{ on } \{u_1 < 0\}.$$

Let us note that $T(m_2)$ is positive on $\{u_1 < 0\} \subset \{u_2 < 0\}$ by the maximum principle. Since $T(m_1)$ vanishes everywhere on $\partial\{u_1 < 0\}$ we conclude by the maximum principle that $T(m_1) \leq T(m_2)$. □

We can now prove the following :

Theorem 1.2.4. *Suppose that f is anti-monotone, then there exists a smallest "classical" solution (u^*, m^*) of $(SMFG)$ such that all equations are satisfied in $L^2(\Omega)$. It is the smallest solution in the following sense : if (u, m) is a solution of $(SMFG)$, then $m^* \leq m$.*

Proof. We set $m_0 = 0$. For all $n \in \mathbb{N}$, we define :
— $m_{n+1} = T(m_n)$;
— u_n the solution of the obstacle problem with source $f(m_n)$.

By the maximum principle, $m_1 \geq 0 = m_0$. Using the previous lemma we get by induction that $(m_n)_{n\in\mathbb{N}}$ is an increasing sequence, while $(u_n)_{n\in\mathbb{N}}$ is a decreasing one. Recalling the estimates on the obstacle problem [18] , the sequence $(u_n)_{n\in\mathbb{N}}$ converges pointwise to a limit we call $u^* \in H^2(\Omega) \cap H_0^1(\Omega)$. Since $(u_n)_{n\in\mathbb{N}}$ is decreasing, the sequence of open sets $\{u_n < 0\}$ converges to $\{u^* < 0\}$ for the Hausdorff distance. Then recalling the result we gave in the previous section, we deduce that $(m_n)_{n\in\mathbb{N}}$ converges to m^* which satisfies

$$-\Delta m^* + m^* = \rho \text{ in } \{u^* < 0\};$$

and is equal to 0 elsewhere. Thus, the couple (u^*, m^*) is a solution of the system $(SMFG)$. It is then easy to prove that it is the smallest solution of the system. Indeed, if we take another solution (u, m) of the system then obviously $m \geq m_0 = 0$. Since m is a fixed point for the application T, using the previous lemma, for all $n \in \mathbb{N}$, $m \geq m_n$, which proves the last point of the theorem by passing to the limit. □

The penalized system

We now present a penalized version of $(SMFG)$ which leads to the existence of mixed solutions. A natural penalization for $(SMFG)$ is the coupling of the penalized version of both the obstacle problem and the Laplace's equation on the domain $\{u < 0\}$. This leads to the following penalized system of partial differential equations :

$$\begin{cases} -\Delta u + u + \frac{1}{\epsilon}(u)^+ = f(m) \text{ in } \Omega; \\ -\Delta m + m + \frac{1}{\epsilon}\mathbb{1}_{\{u\geq 0\}} m = \rho \text{ in } \Omega; \\ u = m = 0 \text{ on } \partial\Omega. \end{cases}$$

The lack of continuity of this system prevents us from proving existence of solutions. Indeed the equation in m has no continuity with respect to u and this problem cannot be overcome. The reason why is basically the same as the reason why there is no existence of classical solutions for $(SMFG)$. The proof of non-existence of solutions for $(SMFG)$ can be easily adapted to prove its counterpart for this penalized system so we shall not present it here once again. We add a new unknown α to introduce the leaving rate V in the penalized system. This approach makes the problem more convex and allows us to prove existence of solutions. We say that (u, m, α) is a solution of the penalized system if

$$\begin{cases} -\Delta u + u + \frac{1}{\epsilon}(u)^+ = f(m) \text{ in } \Omega; \\ -\Delta m + m + \frac{1}{\epsilon}\alpha\mathbb{1}_{\{u\geq 0\}} m = \rho \text{ in } \Omega; \\ u = m = 0 \text{ on } \partial\Omega; \\ 0 \leq \alpha \leq 1; u \neq 0 \Rightarrow \alpha = 1. \end{cases} \tag{1.7}$$

We are now able to prove the following :

Theorem 1.2.5. *For all $\epsilon > 0$ there exists a solution $(u_\epsilon, m_\epsilon, \alpha_\epsilon) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times L^\infty(\Omega)$ of (1.7).*

Proof. We define first the application $\mathcal{F}_1$ from $L^2(\Omega)$ into $H_0^1(\Omega) \cap H^2(\Omega)$ by : $\mathcal{F}_1(m)$ is the only solution of the obstacle problem with source $f(m)$. We also define the correspondance $\mathcal{F}_2$ from $H_0^1(\Omega) \cap H^2(\Omega)$ into $L^2(\Omega)$ by :

$$\mathcal{F}_2(u) = \{m \in H_0^1(\Omega) \cap H^2(\Omega), \exists \alpha \in \mathcal{D}(u), (u, m, \alpha) \text{ solve } (1.3^*)\};$$

where $\mathcal{D}(u) = \{\alpha \in L^\infty(\Omega), 0 \leq \alpha \leq 1, u \neq 0 \Rightarrow \alpha = 1\}$ and (1.3^*) is the following

$$(1.3^*) \begin{cases} -\Delta m + m + \frac{1}{\epsilon}\alpha \mathbb{1}_{\{u \geq 0\}} m = \rho; \\ m = 0 \text{ in } \partial\Omega. \end{cases}$$

Thus finding a solution (u, m, α) of the penalized system is equivalent to finding $m \in \mathcal{F}_2(\mathcal{F}_1(m))$. Recalling the results of section 2 the application $\mathcal{F}_1$ is well defined and continuous. As we are going to apply Kakutani's fixed point theorem, we just have to verify that the correspondance $\mathcal{F}_2$ is upper semicontinuous (i.e. that for all open set $\mathcal{O} \subset H_0^1(\Omega)$, $\{u \in H_0^1(\Omega) \cap H^2(\Omega), \mathcal{F}_2(u) \subset \mathcal{O}\}$ is open) and takes values in the set of convex closed subsets of $H_0^1(\Omega)$. As the last point is trivial we focus on the upper semicontinuity. We take an open set $\mathcal{O} \subset H_0^1(\Omega)$ and $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $\mathcal{F}_2(u) \subset \mathcal{O}$.

We are now going to find a small enough $\delta > 0$ such that for any $v \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $||u - v||_{H^2(\Omega)} \leq \delta$, $\mathcal{F}_2(v) \subset \mathcal{O}$. First we remark that

$$dist(\mathcal{F}_2(u), \mathcal{O}^c) > 0.$$

Indeed, $\mathcal{F}_2(u)$ is a compact subset of $\mathcal{O}$ because it is a bounded subset of $H_0^1(\Omega) \cap H^2(\Omega)$. We call $a = dist(\mathcal{F}_2(u), \mathcal{O}^c) > 0$. We now prove that for every $m' \in \mathcal{F}_2(v)$, there exists $m \in \mathcal{F}_2(u)$ such that $||m - m'||_{L^2} \leq \frac{a}{2}$, given that δ is small enough. Hence, $\mathcal{F}_2(v) \subset \mathcal{O}$ shall hold.

Take $m' \in \mathcal{F}_2(v)$, there exists α', taking values between 0 and 1, such that

$$\begin{cases} -\Delta m' + m' + \frac{1}{\epsilon}\alpha' \mathbb{1}_{\{v \geq 0\}} m' = \rho \text{ in } \Omega; \\ m' = 0 \text{ on } \partial\Omega. \end{cases}$$

We divide Ω into three zones and define α by :

— on $\{v \geq 0\} \cap \{u = 0\}$, $\alpha = \alpha'$;

— on $\{v < 0\} \cap \{u = 0\}$, $\alpha = 0$;
— on the rest of Ω, $\alpha = 1$.
Then we define m as the solution of

$$\begin{cases} -\Delta m + m + \frac{1}{\epsilon}\alpha \mathbb{1}_{\{v \geq 0\}} m = \rho; \\ m = 0 \text{ on } \partial\Omega. \end{cases}$$

and we set $\mu = m' - m$. μ solves

$$\begin{cases} -\Delta\mu + \mu + \dfrac{1}{\epsilon}\alpha' \mathbb{1}_{(\{v \geq 0\} \cap \{u \geq 0\})}\mu = -\dfrac{1}{\epsilon}(\mathbb{1}_{(\{v>0\}\cap\{u<0\})}m' - \mathbb{1}_{(\{v>0\}\cap\{u<0\})}m \\ \qquad\qquad + \mathbb{1}_{(\{v=0\}\cap\{u<0\})}m' + \mathbb{1}_{(\{v=0\}\cap\{u>0\})}(\alpha' m' - m)) \\ \qquad\qquad \text{in } \Omega; \\ \mu = 0 \text{ on } \partial\Omega. \end{cases}$$

All the terms of the right hand side involve characteristic functions of subsets where u and v do not have strictly the same sign. We claim that we can always choose δ sufficiently small to make the second term has small as we want for the norm of $L^2(\Omega)$, independently of m and m'. Hence taking δ small enough, we obtain that $||m' - m||_{L^2} \leq \frac{a}{2}$. Thus we can apply Kakutani's fixed point theorem and find a solution of the penalized system.

□

Let us remark that we could have used a smoother version of the penalized system to prove existence of solutions at a penalized level, but it would have been less clear to show how this sequence of penalized solutions converges to a mixed solution of the problem, which is the result we now present.

Theorem 1.2.6. *There exists at least one mixed solution of* $(SMFG)$.

Proof. We begin by introducing the penalized system, using the previous result, we obtain that for all $\epsilon > 0$, there exists a solution $(u_\epsilon, m_\epsilon, \alpha_\epsilon)$ of the system (1.7). We now prove uniform estimates and show that the limit is indeed a mixed solution of $(SMFG)$. First using m_ϵ as a test function in the equation satisfied by m_ϵ itself we get the bound

$$||m_\epsilon||_{H_0^1(\Omega)} \leq C;$$

where C only depends on Ω and ρ. Hence $(f(m_\epsilon))_{\epsilon>0}$ is uniformly bounded in $L^2(\Omega)$. So we get a uniform bound in $H^2(\Omega)$ for $(u_\epsilon)_{\epsilon>0}$. Thus we can find a limit (u, m) in $(H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega)$ for a subsequence of $((u_\epsilon, m_\epsilon))_{\epsilon>0}$, for the weak topology of $H^2(\Omega) \times H_0^1(\Omega)$. Because of the regularity of f it is clear that u solves the obstacle problem with cost $f(m)$.

Since $-\Delta m_\epsilon + m_\epsilon \leq \rho$ for all $\epsilon > 0$, we deduce that $-\Delta m + m \leq \rho$. Hence, $m \in L^\infty(\Omega)$ which implies that u is continuous. Taking a smooth function ϕ with support in $\{u < 0\}$, we have for all ϵ :

$$\int_\Omega (-\Delta\phi + \phi) m_\epsilon = -\int_\Omega \mathbb{1}_{\{u_\epsilon \geq 0\} \cap \{u < 0\}} \frac{\alpha_\epsilon}{\epsilon} m_\epsilon \phi + \rho\phi.$$

Let us note that in view of classical results on variational inequalities [9], extracting a subsequence if necessary, $(u_\epsilon)_{\epsilon>0}$ converges uniformly toward u. Moreover because ϕ has support in $\{u < 0\}$, there exists some $\delta > 0$ such that ϕ is supported in $\{u \leq -\delta\}$. Thus for ϵ sufficiently small

$$\mathbb{1}_{\{u_\epsilon \geq 0\} \cap \{u < 0\}} \phi = 0.$$

Hence passing to the limit in a subsequence which converges to (u, m), we deduce

$$\int_\Omega (-\Delta\phi + \phi) m = \int_\Omega \rho\phi.$$

Thus,

$$-\Delta m + m = \rho \text{ in } \{u < 0\}.$$

Now we are going to test the variational formulation of the penalized equation in u_ϵ on m_ϵ and vice versa. Substracting the two equalities yields

$$\int_\Omega (-\Delta u_\epsilon + u_\epsilon + \frac{1}{\epsilon}(u_\epsilon)^+ - f(m_\epsilon)) m_\epsilon - \int_\Omega (-\Delta m_\epsilon + m_\epsilon + \frac{\alpha_\epsilon}{\epsilon} \mathbb{1}_{\{u_\epsilon \geq 0\}} m_\epsilon - \rho) u_\epsilon = 0.$$

Hence,

$$\int_\Omega (\frac{1}{\epsilon}(u_\epsilon)^+ - f(m_\epsilon)) m_\epsilon - \frac{\alpha_\epsilon}{\epsilon} \mathbb{1}_{\{u_\epsilon \geq\}} m_\epsilon u_\epsilon = -\int_\Omega \rho u_\epsilon.$$

From which we deduce

$$\int_\Omega f(m_\epsilon) m_\epsilon = \int_\Omega u_\epsilon \rho.$$

Thanks to the regularity of f we can pass to the limit in the previous equation and we get

$$\int_\Omega f(m) m = \int_\Omega u\rho.$$

Let us remark that we can write

$$\begin{aligned}
\int_{\{u<0\}} f(m) m &= \int_{\{u<0\}} (-\Delta u + u) m \\
&= \int_{\{u<0\}} (-\Delta m + m) u \\
&= \int_{\{u<0\}} \rho u \\
&= \int_\Omega \rho u.
\end{aligned}$$

which completes the proof of the fact that (u, m) is a mixed solution of $(SMFG)$. □

The following result also holds :

Theorem 1.2.7. *If f is strictly monotone, then there exists at most one solution of $(SMFG)$.*

The proof of this result is essentially the same as the one we present for the uniqueness of classical solutions so we do not re-write it here.

1.3 The time dependent problem

We study in this part the solution of the time dependent MFG obstacle problem with a smooth obstacle ψ which depends on m : $(OSMFG)$. We recall that from the assumption we made on ψ at the end of the introduction, the weak derivative of ψ in time and its space laplacian are both in L^2. We here define the notion of mixed solutions and present results of existence, and uniqueness which holds under a monotonicity assumption on the costs.

Definition 1.3.1. *A pair $(u, m) \in L^2((0,T), H_0^1(\Omega) \cap H^2(\Omega)) \times L^2((0,T), H_0^1(\Omega))$ is a mixed solution of $(OSMFG)$ if*

— $\max(-\partial_t u - \Delta u - f(m), u - \psi(m)) = 0$ *in* $\mathcal{D}'((0,T) \times \Omega)$;
— $\partial_t m - \Delta m \leq 0$ *in* $\mathcal{D}'((0,T) \times \Omega)$;
— $m \geq 0$;
— $u = \psi(m)(T)$ *at* $t = T$;
— *for any smooth v such that $v \leq 0$ and $v(T) = 0$:*

$$\int_0^T \int_\Omega (-\partial_t v - \Delta v)m - \int_\Omega v(0)m_0 \geq 0;$$

with an equality if $v = u - \psi(m)$;
— $\partial_t m - \Delta m = 0$ *in* $\mathcal{D}'(\{u < \psi(m)\})$;
— $\int_{\{u=\psi(m)\}} (f(m) + (\partial_t + \Delta)\psi(m))m = 0$;

Let us note that the fifth point is a variational formulation of the boundary condition one has to impose on m at time $t = 0$. This formulation implies that $m(0) \leq m_0$ with possibly not an equality, as well as the second point of this definition. This inequality at $t = 0$ comes from the fact that a shock can occur at this time, with lots of players leaving instantaneously the game. From a game theory point of view, this property for m is quite reasonable and the uniqueness of such mixed solutions justified it. As in the stationary case, the second and third

point of this definition imply that $m \in L^\infty$ and thus, because of the assumption on f, u is continuous and the set $\{u < \psi(m)\}$ is open. This definition is the adaptation of the definition 1.2.1 for the case of time dependent problems with an obstacle which depends on m. Note that the condition

$$\int_{\{u=\psi(m)\}} (f(m) + (\partial_t + \Delta)\psi(m))m = 0$$

is the analogue of

$$\int_{\{u=0\}} f(m)m = 0$$

in the stationary case. Indeed this condition is interpreted as letting possible the fact for m to be strictly positive in the contact region $\{u = \psi(m)\}$, when u satisfies the Hamilton-Jacobi-Bellman equation. Note that we apply the derivatives on $\psi(m)$ which we see as an element of $L^2((0,T), H^2(\Omega)) \cap H^1((0,T), L^2(\Omega))$.

1.3.1 Preliminary results on the time dependent obstacle problem

For any $f_1 \in L^2((0,T), L^2(\Omega))$ and an obstacle $\psi_1 \in L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1((0,T), L^2(\Omega))$ there exists a unique $u \in L^2((0,T), H_0^1(\Omega) \cap H^2(\Omega))$ which solves

$$\max(-\partial_t u - \Delta u - f_1, u - \psi_1) = 0;$$

in the sense of distributions in $(0,T) \times \Omega$ with the terminal condition $u(T) = \psi_1(T)$. This result is the exact analogue of the one we gave in the stationary case. Details about this problem can be found in [9]. Such a u is called the solution of the time dependent obstacle problem with source f_1, cost ψ_1 and terminal condition $\psi_1(T)$. We recall that the mapping which associates to each pair (f_1, ψ_1) a solution u of the time dependent obstacle problem is continuous from $L^2((0,T), L^2(\Omega)) \times \Big(H^1((0,T), L^2(\Omega)) \cap L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega))\Big)$ to $L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega))$. More importantly the sequence of solutions of the penalized system

$$\begin{cases} -\partial_t u_\epsilon - \Delta u_\epsilon + \frac{1}{\epsilon}(u_\epsilon - \psi_1)^+ = f_1 \text{ in } (0,T) \times \Omega; \\ u_\epsilon(T) = \psi_1(T); \\ \forall 0 \leq t \leq T : u_\epsilon(t) = 0 \text{ on } \partial\Omega; \end{cases}$$

converges towards the solution of the obstacle problem in $L^2((0,T), H_0^1(\Omega))$. Moreover, if $f_1 \in L^\infty((0,T) \times \Omega)$, then u is continuous if ψ_1 is.

1.3.2 Existence of mixed solutions

We now turn to the proof of existence of mixed solutions for $(OSMFG)$. As in the stationary case, this result follows from the use of a penalized version of the problem. This is why we introduce the following system for all $\epsilon > 0$:

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{\epsilon}(u - \psi(m))^+ = f(m) \text{ in} (0,T) \times \Omega; \\ \partial_t m - \Delta m + \frac{1}{\epsilon}\alpha \mathbb{1}_{\{u \geq \psi(m)\}} m = 0 \text{ in } (0,T) \times \Omega; \\ u = m = 0 \text{ on } \partial\Omega; \\ u(T) = \psi(m)(T) \text{ in } \Omega; \\ m(0) = m_0 \text{ in } \Omega; \\ 0 \leq \alpha \leq 1; u \neq \psi(m) \Rightarrow \alpha = 1. \end{cases} \tag{1.8}$$

There exist solutions for this system and the proof of this statement is step by step the same as the one we did in the stationary case, so we do not present it here. The interested reader shall easily be able to adapt all the elliptic arguments into parabolic ones. We prove the following :

Theorem 1.3.1. *Assume f is continuous from L^2 to itself and ψ is continuous from $L^2((0,T), L^2(\Omega))$ to $L^2((0,T), H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1((0,T), L^2(\Omega))$, then there exists a mixed solution of $(OSMFG)$.*

Proof. We take $(u_\epsilon, m_\epsilon, \alpha_\epsilon)_{\epsilon>0}$ a sequence of solutions of the penalized system. Using the same arguments as in the proof of this statement for the stationary case, we obtain uniform bounds on $(u_\epsilon, m_\epsilon)_{\epsilon>0}$ and we find a limit (u, m) such that

— $(u, m) \in L^2((0,T), H_0^1(\Omega) \cap H^2(\Omega)) \cap H^1((0,T), L^2(\Omega)) \times L^2((0,T), H_0^1(\Omega))$;
— u solves the obstacle problem with terminal condition $\psi(m)(T)$, source $f(m)$ and obstacle $\psi(m)$;
— $\partial_t m - \Delta m \leq 0$ in $\mathcal{D}'((0,T) \times \Omega)$;
— $m \in L^\infty$ and thus u is continuous ;
— $\partial_t m - \Delta m = 0$ in $\{u < \psi(m)\}$ in $\mathcal{D}'(\Omega)$;
— for any smooth v such that $v \leq 0$ and $v(T) = 0$:

$$\int_0^T \int_\Omega (-\partial_t v - \Delta v) m - \int_\Omega m_0 v(0) \geq 0.$$

We will here show why $\int_{\{u=\psi(m)\}} (f(m) + (\partial_t + \Delta)\psi(m)) m = 0$. Using the equations satisfied by u_ϵ and m_ϵ, and multiplying the equation in u_ϵ by m_ϵ and the one in m_ϵ by u_ϵ we can write :

$$\begin{aligned} 0 = &\int_0^T \int_\Omega (-\partial_t u_\epsilon - \Delta u_\epsilon + \frac{1}{\epsilon}(u_\epsilon - \psi(m_\epsilon))^+ - f(m_\epsilon)) m_\epsilon \\ &- \int_0^T \int_\Omega (\partial_t m_\epsilon - \Delta m_\epsilon + \frac{\alpha_\epsilon}{\epsilon} \mathbb{1}_{\{u_\epsilon \geq \psi(m_\epsilon)\}} m_\epsilon) u_\epsilon. \end{aligned}$$

Hence,

$$\int_\Omega \psi(m_\epsilon)(T)m_\epsilon(T) - u_\epsilon(0)m_0 = \int_0^T \int_\Omega (\frac{1}{\epsilon}(u_\epsilon - \psi(m_\epsilon))^+ - f(m_\epsilon))m_\epsilon \\ - \int_0^T \int_\Omega \frac{\alpha_\epsilon}{\epsilon} \mathbb{1}_{\{u_\epsilon \geq \psi(m_\epsilon)\}} m_\epsilon u_\epsilon.$$

We now use the equation verified by m_ϵ to interpret the term $\int_\Omega \psi(m_\epsilon(T))m_\epsilon(T)$

$$\int_0^T \int_\Omega (\frac{1}{\epsilon}(u_\epsilon - \psi(m_\epsilon))^+ - f(m_\epsilon))m_\epsilon - \frac{\alpha_\epsilon}{\epsilon} \mathbb{1}_{\{u_\epsilon \geq \psi(m_\epsilon)\}} m_\epsilon u_\epsilon = \int_\Omega -u_\epsilon(0)m_0 \\ + \int_\Omega \psi(m)(0)m_0 + \int_0^T \int_\Omega (\partial_t \psi(m_\epsilon) + \Delta\psi(m_\epsilon))m_\epsilon \\ - \int_0^T \int_\Omega (\frac{\alpha_\epsilon}{\epsilon} \mathbb{1}_{\{u_\epsilon \geq \psi(m_\epsilon)\}} \psi(m_\epsilon))m_\epsilon.$$

Using the fact that $(u_\epsilon - \psi(m_\epsilon))^+ = \alpha_\epsilon \mathbb{1}_{\{u_\epsilon \geq \psi(m_\epsilon)\}}(u_\epsilon - \psi(m_\epsilon))$, we deduce

$$\int_0^T \int_\Omega (f(m_\epsilon) + \partial_t \psi(m_\epsilon) + \Delta\psi(m_\epsilon))m_\epsilon = \int_\Omega (u_\epsilon(0) - \psi(m)(0))m_0.$$

Passing to the limit we obtain

$$\int_0^T \int_\Omega (f(m) + \partial_t \psi(m) + \Delta\psi(m))m = \int_\Omega (u(0) - \psi(m)(0))m_0.$$

Let us note that we are able to pass to the limit in the right hand side because $(u_\epsilon)_{\epsilon>0}$ is uniformly continuous in time with value in L^2. This is fact is a consequence of the L^∞ bound on $(f(m_\epsilon))_{\epsilon>0}$ and of classical results concerning the obstacle problem. Now, because $\partial_t m - \Delta m = 0$ on $\{u < \psi(m)\}$, we derive finally

$$\int_0^T \int_\Omega (f(m) + \partial_t u + \Delta u)m = 0;$$

$$\int_{\{u=\psi(m)\}} (f(m) + \partial_t \psi(m) + \Delta\psi(m))m = 0;$$

by using that

$$\int_0^T \int_\Omega (\partial_t(u - \psi(m)) + \Delta(u - \psi(m)))m = -\int_\Omega (u(0) - \psi(m)(0))m_0$$

which we obtain by passing to the limit at a penalized level on this equation. This completes the proof of the fact that (u, m) is a mixed solution of $(OSMFG)$. $\square$

1.3.3 Uniqueness of mixed solutions

We now turn to the question of uniqueness of such mixed solutions. We recall that regarding the counter example presented in the first part, we cannot expect uniqueness to hold in general and some assumptions must be made on the two costs f and ψ. The previous section strongly suggests to make an assumption on the term $f(m) + (\partial_t + \Delta)\psi(m)$, as it is involved in most of the calculations. One can look as this term using the following remark. The formal obstacle problem

$$\max(-\partial_t u - \Delta u - f, u - \psi) = 0;$$

can be equivalently reformulated as

$$\max(-\partial_t v - \Delta v - \tilde{f}, v) = 0;$$

where $\tilde{f} = f + \partial_t \psi + \Delta\psi$ and $v = u - \psi$, given that ψ is smooth enough. Hence it is natural to adapt the result of uniqueness in the stationary case by making an assumption on the "new cost" $\tilde{f}$.

Theorem 1.3.2. *Assume that $f+(\partial_t+\Delta)\psi$ is strictly monotone in $L^2((0,T), H_0^1(\Omega))$, then there is a unique solution of the penalized system and a unique mixed solution for $(OSMFG)$.*

Proof. This proof relies on the same arguments as the one we gave for classical solutions of $(SMFG)$. We only present here the proof of uniqueness for mixed solutions, since the proof of uniqueness for the penalized system is the same as this one except for the fact that $(u - \psi)^+ = \mathbb{1}_{\{u \geq \psi\}}(u - \psi)$ plays the role of the integral relation satisfied by the mixed solutions.
We denote by (u_1, m_1) and (u_2, m_2) two mixed solutions, $v_i = u_i - \psi(m_i)$, $\Omega_i = \{v_i < 0\}$ for $i = 1, 2$ and $v = v_1 - v_2$ and $m = m_1 - m_2$ and by $\tilde{f} = f + \partial_t\psi + \Delta\psi$. We can compute

$$\begin{aligned}\int_{\Omega_1 \cup \Omega_2} (-\partial_t v - \Delta v)m = &\int_{\Omega_1 \cap \Omega_2} (\tilde{f}(m_1) - \tilde{f}(m_2))m \\ &- \int_{\Omega_1^c \cap \Omega_2} \tilde{f}(m_2)m + \int_{\Omega_2^c \cap \Omega_1} \tilde{f}(m_1)m.\end{aligned}$$

From which we deduce

$$\begin{aligned}\int_{\Omega_1 \cup \Omega_2} (-\partial_t v - \Delta v)m = &\int_{\Omega_1 \cup \Omega_2} (\tilde{f}(m_1) - \tilde{f}(m_2))m \\ &- \int_{\Omega_1^c \cap \Omega_2} \tilde{f}(m_1)m + \int_{\Omega_2^c \cap \Omega_1} \tilde{f}(m_2)m.\end{aligned}$$

But we know that (u_1, m_1) and (u_2, m_2) are mixed solutions so

$$\int_{\Omega_1^c} \tilde{f}(m_1)m_1 = \int_{\Omega_2^c} \tilde{f}(m_2)m_2 = 0.$$

Hence using this relation in the previous equality, we get

$$\int_{\Omega_1\cup\Omega_2}(-\partial_t v-\Delta v)m=\int_{\Omega_1\cup\Omega_2}(\tilde f(m_1)-\tilde f(m_2))m \\ +\int_{\Omega_1^c\cap\Omega_2}\tilde f(m_1)m_2+\int_{\Omega_2^c\cap\Omega_1}\tilde f(m_2)m_1.$$

Now recalling the argument that because of the obstacles problem $\tilde f(m_i)\geq 0$ on Ω_i^c and the monotony of $\tilde f$ we obtain, as in the stationary case,

$$\int_{\Omega_1\cup\Omega_2}(-\partial_t v-\Delta v)m\geq\int_{\Omega_1\cup\Omega_2}(\tilde f(m_1)-\tilde f(m_2))(m_1-m_2).$$

Next, let us remark that

$$\int_0^T\int_\Omega(-\partial_t v-\Delta v)m=\int_0^T\int_\Omega(-\partial_t v-\Delta v)m_1-\int_\Omega v(0)m_0; \\ -\Big(\int_0^T\int_\Omega(-\partial_t v-\Delta v)m_2-\int_\Omega v(0)m_0\Big).$$

Using the fifth point of the definition of mixed solutions, we obtain for $i=1,2$ and $j\neq i$

$$\int_0^T\int_\Omega(-\partial_t v_i-\Delta v_i)m_i-\int_\Omega v_i(0)m_0=0;$$

$$\int_0^T\int_\Omega(-\partial_t v_i-\Delta v_i)m_j-\int_\Omega v_i(0)m_0\geq 0.$$

Then, we deduce successively

$$\int_{\Omega_1\cup\Omega_2}(-\partial_t v-\Delta v)m\leq 0;$$

$$\int_{\Omega_1\cup\Omega_2}(-\partial_t v-\Delta v)m=0.$$

We can now state as in the stationary case that

$$\int_{\Omega_1\cup\Omega_2}(\tilde f(m_1)-\tilde f(m_2))m=0.$$

So by strict monotonicity of $\tilde f$, $m_1=m_2$ and there exists a unique solution of the system.

□

1.3.4 The optimal control interpretation

We present here the analogue of the optimal control approach for this more difficult problem. Even if this approach is more restrictive than the approach we just presented, we believe it is useful.

Let $\mathcal{H} := \{m \in L^2((0,T), H_0^1(\Omega)), s.t., m \geq 0, m(0) \leq m_0, \partial_t m - \Delta m \leq 0\}$ where the last inequality is taken in the sense of distributions. Suppose there exist C^1 potentials $\mathcal{F}$ and Ψ such that

$$\frac{\partial \mathcal{F}}{\partial p}(x, m(t,x)) = f(x, m(t,x));$$

and

$$\frac{\partial \Psi}{\partial p}(x, m(t,x)) = (\partial_t + \Delta)\psi(t, m).$$

Where p stands for the second variable. Then we can prove :

Theorem 1.3.3. *If $\mathcal{F} + \Psi$ is strictly convex and a continuous functional from $L^2((0,T), H_0^1(\Omega))$ into $L^1((0,T) \times \Omega)$, then there exists a unique minimizer m of*

$$\inf_{m' \in \mathcal{H}} \int_0^T \int_\Omega \mathcal{F}(m') + \Psi(m');$$

and (u, m) is the only mixed solutions of $(OSMFG)$ if u is set to be the only solution of the time dependent obstacle problem with source $f(m)$, obstacle $\psi(m)$ and terminal cost $\psi(m(T))$.

This theorem relies on the same arguments than the ones we used in the first part. It is natural having in mind the definition of mixed solutions. Indeed in the stationary case the relation satisfied by m in the mixed zone was that $f(m) = 0$, meaning that the derivative of the function we want to minimize is 0 when evaluated in m in this zone. Here the relation which is satisfied in the mixed zone is $f(m) + \partial_t \psi(m) + \Delta\psi(m) = 0$, so it is natural to choose a functional such that its derivative with respect to m gives the relation we are looking for. This explanation is of course a heuristic one, but the proof is step by step the same as the one in the stationary case, so we do not present it here.

1.3.5 Remarks on the assumptions on ψ

Assumption on the monotonicity

We made some strong assumptions on ψ in the previous section in order to find uniqueness of solutions. Indeed we assumed that $(\partial_t + \Delta)\psi$ seen as a functional

from $L^2((0,T),H_0^1(\Omega))$ to $L^2((0,T),H^2(\Omega)\cap H_0^1(\Omega))\cap H^1((0,T),L^2(\Omega))$ is strictly monotone. It is not obvious that non-trivial obstacles satisfy such a condition. This is the question we want to address here.

We begin with a statement about the model this problem is concerned with. We recall that uniqueness in [54] is obtained under the assumption that f is strictly monotone, meaning that for m' bigger than m, $f(m')$ is bigger than $f(m)$ and the players pay a higher cost. This assumption tends to force the players to spread. It is then natural to think that under such kind of assumptions, uniqueness may hold. We proved that if f is strictly monotone, then if $\partial_t\psi+\Delta\psi$ is monotone, then uniqueness holds. Remark that the operator $\partial_t+\Delta$ (which is defined only when boundary conditions are imposed, but this is not the purpose of this formal discussion) is decreasing in the following sense if $\partial_t f_1+\Delta f_1$ is bigger that $\partial_t f_2+\Delta f_2$ then f_2 is bigger than f_1. This means that making the assumption that $\partial_t\psi+\Delta\psi$ is monotone implies that if m_1 is bigger than m_2, then $\psi(m_2)$ is bigger than $\psi(m_1)$. Note this can be thought as an incitation to leave the game when the players are more. This can be interpreted as in [21, 54] as a tendency for the players not to accumulate. From this point of vue the assumption we make in order to have uniqueness is natural. Moreover, it is easy to show that under the assumption that $\partial_t\psi+\Delta\psi$ is monotone, ψ is anti-monotone.

We are going to show here that there are non-trivial ψ such that for all m_1 and m_2 in $L^2((0,T),H_0^1(\Omega))$ the following inequality holds true

$$\int_0^T\int_\Omega(\partial_t(\psi(m_1)-\psi(m_2))+\Delta(\psi(m_1)-\psi(m_2)))(m_1-m_2)\geq 0. \tag{1.9}$$

Now let us remark that any ψ which does not depend on m satisfies such an inequality. This makes this assumption coherent with the results of the first part. We here denote by g a monotone operator from L^2 to itself and we define ψ by : given m in $L^2((0,T),H_0^1(\Omega))$, $\psi(m)$ is the only solution of

$$\begin{cases}\partial_t\psi(m)+\Delta\psi(m)=g(m) \text{ in } (0,T)\times\Omega;\\ \psi(m)(T)=0 \text{ in } \Omega;\\ \forall 0\leq t\leq T, \psi(m)(t)=0 \text{ on } \partial\Omega.\end{cases}$$

Clearly the application ψ defined with this equation is regular enough to be taken as an obstacle in $(OSMFG)$ and satisfy (1.9).

Assumption on the existence of a primitive

We make here precise the sense in which we can understand the assumption we made in the optimal control approach. We assume there exists a Ψ such that the

derivative of Ψ with respect to m is $\partial_t\psi + \Delta\psi$. Of course derivatives in the space of measure are now well understood, see for instance section 2 in [27], however we are not interested in presenting this formalism here and we want to introduce a primitive in m in a local sense, just like we did for the cost f. That is why we want $\partial_t\psi + \Delta\psi$ to have a local dependence in m, meaning that

$$(\partial_t\psi(m) + \Delta\psi(m))(t,x) = g(t,x,m(t,x)).$$

In which g is a smooth function in the second variable. The functional Ψ is then simply the primitive of g with respect to its second argument. Note that there are ψ such that such a g exists, just define for example $\psi(m)$ implicitly as the only solution of

$$\begin{cases} \partial_t\psi(m) + \Delta\psi(m) = g(m) \text{ in } (0,T)\times\Omega; \\ \psi(m) = 0 \text{ on } \Big(\{T\}\times\Omega\Big) \cup \Big((0,T)\times\partial\Omega\Big). \end{cases}$$

Hence there exist non trivial ψ such that we can find a potential Ψ for which

$$\frac{\partial\Psi}{\partial p}(t,x,m(t,x)) = (\partial_t + \Delta)(\psi(m))(t,x).$$

1.4 The case of optimal stopping with continuous control

In this last part, we extend the case of optimal stopping to a case in which both the classical optimal control and optimal stopping can occur, in a MFG setting. We work in the case in which ψ is equal to 0 because some technical difficulties arise for general obstacles and we do not want to enter in those details. Mainly we are interested with the existence and the uniqueness of mixed solutions of the following forward-backward system :

$$(MFG)\begin{cases} \max(-\partial_t u - \Delta u + H(x,\nabla u) - f(m), u) = 0 \text{ in } (0,T)\times\Omega; \\ \partial_t m - \Delta m - div(mD_pH(x,\nabla u)) = 0 \text{ in } \{u<0\}; \\ m = 0 \text{ in } \{u=0\}; \\ m(0) = m_0 \text{ and } u(T) = 0 \text{ in } \Omega. \end{cases}$$

We call mixed solution of this problem a couple $(u,m) \in L^2((0,T),H^2(\Omega)\cap H_0^1(\Omega))\times L^2((0,T),H_0^1(\Omega))$ such that

— $\max(-\partial_t u - \Delta u + H(x,\nabla u) - f(m), u) = 0$ in $\mathcal{D}'((0,T)\times\Omega)$;
— $\partial_t m - \Delta m - div(mD_pH(x,\nabla u)) = 0$ in $\mathcal{D}'(\{u<0\})$;
— $\partial_t m - \Delta m - div(mD_pH(x,\nabla u)) \leq 0$ in $\mathcal{D}'((0,T)\times\Omega)$;

— $\int_{\{u=0\}}(f(m) - H(x,0))m = 0$;
— $u(T) = 0$ in Ω ;
— for any smooth v such that $v \leq 0$ and $v(T) = 0$:

$$\int_0^T \int_\Omega (-\partial_t v - \Delta v + D_p H(x, \nabla u) \cdot \nabla v)m \geq \int_\Omega v(0)m_0$$

with an equality if $v = u$.

We keep on with the assumptions we made on the regularity of f and we precise here assumptions on the hamiltonian H. As we do not want to enter into details about the more general assumption on the growth of the hamiltonian, we here assume the following :

— The hamiltonian H is lipschitz in both variables and convex in the second variable, moreover, it is bounded from below by a constant.

This regularity assumption together with the previous definition implies that if (u, m) is a mixed solution, then $m \in L^\infty$ and thus that u is continuous. Hence, $\{u < 0\}$ is open. We also present the optimal control interpretation of this problem.

1.4.1 Existence of mixed solutions

Once again we prove that there exist mixed solutions for (MFG) using a penalized version of it. Since the proof is again a very easy adaptation of the proof of existence for $(SMFG)$, we do not detail some arguments. We prove the following result

Theorem 1.4.1. *There exists at least one mixed solution of* (MFG).

Proof. As we did in the previous two parts we are going to introduce the following penalized version of (MFG)

$$\begin{cases} -\partial_t u - \Delta u + H(x, \nabla u) + \frac{1}{\epsilon}(u)^+ = f(m) \text{ in } (0,T) \times \Omega; \\ \partial_t m - \Delta m - div(mD_p H(x, \nabla u)) + \frac{\alpha}{\epsilon}\mathbb{1}_{\{u \geq 0\}} m = 0 \text{ in } (0,T) \times \Omega; \\ m(0) = m_0 \text{ and } u(T) = 0 \text{ in } \Omega; \\ m = u = 0 \text{ on } \partial\Omega; \\ u \neq 0 \Rightarrow \alpha = 1. \end{cases}$$

The proof of existence of solutions for such a system is the same as the one we gave in the case of $(SMFG)$, so we do not present it here. We have a sequence $((u_\epsilon, m_\epsilon))_{\epsilon>0}$ of elements of $(L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega)))^2$ of solutions of the penalized system. Using a priori estimates on the equations ([5]) we deduce that, up to a subsequence, $((u_\epsilon, m_\epsilon))_{\epsilon>0}$ converges to (u, m), strongly in $L^2((0,T), H_0^1(\Omega))$ for

$(u_\epsilon)_{\epsilon>0}$ and weakly in $L^2((0,T),H^1_0(\Omega))$ for $(m_\epsilon)_{\epsilon>0}$ with $(u,m)\in L^2((0,T),H^2(\Omega)\cap H^1_0(\Omega))\times L^2((0,T),H^1_0(\Omega))$. Because of the regularity of f and H, u solves

$$\max(-\partial_t u-\Delta u+H(x,\nabla u)-f(m),u)=0 \text{ in } (0,T)\times\Omega;$$

and we obtain for m that

$$\begin{cases}\partial_t m-\Delta m-div(mD_pH(x,\nabla u))\le 0 \text{ in } (0,T)\times\Omega;\\ \partial_t m-\Delta m-div(mD_pH(x,\nabla u))=0 \text{ in } \{u<0\}.\end{cases}$$

We now check that $\int_{\{u=0\}}(f(m)-H(x,0))m=0$ to prove that (u,m) is indeed a mixed solution of (MFG). As in the previous parts we multiply the equation in u by m and we integrate, and vice-versa for the equation in m (at the penalized level). This leads to

$$\begin{aligned}0=&\int_0^T\int_\Omega(-\partial_t u_\epsilon-\Delta u_\epsilon+H(x,\nabla u_\epsilon)+\frac{1}{\epsilon}(u_\epsilon)^+-f(m_\epsilon))m_\epsilon\\ &-\int_0^T\int_\Omega(\partial_t m_\epsilon-\Delta m_\epsilon-div(m_\epsilon D_pH(x,\nabla u_\epsilon))+\frac{\alpha}{\epsilon}\mathbb{1}_{\{u_\epsilon\ge 0\}}m_\epsilon)(u_\epsilon).\end{aligned}$$

By integration by parts in time, we obtain, using the fact that $(u)^+=\mathbb{1}_{\{u\ge 0\}}u$

$$\begin{aligned}0=&\int_\Omega(u_\epsilon(0))m_0+\int_0^T\int_\Omega(H(x,\nabla u_\epsilon)-f(m_\epsilon))m_\epsilon\\ &+\int_0^T\int_\Omega div(m_\epsilon D_pH(x,\nabla u_\epsilon))(u_\epsilon).\end{aligned}$$

because of the convergence (up to a subsequence) of $((u_\epsilon,m_\epsilon))$ we can pass to the limit in this equality and we get

$$\begin{aligned}0=&\int_\Omega(u(0))m_0+\int_0^T\int_\Omega(H(x,\nabla u)-f(m))m;\\ &+\int_0^T\int_\Omega div(mD_pH(x,\nabla u))(u).\end{aligned}$$

Then using that

$$\int_\Omega u(0)m_0+\int_0^T\int_\Omega div(mD_pH(x,\nabla u))u=\int_0^T\int_\Omega(-\partial_t u-\Delta u)m;$$

we deduce that

$$0=\int_0^T\int_\Omega(-\partial_t u-\Delta u+H(x,\nabla u)-f(m))m;$$

which leads to

$$\int_{\{u=0\}}(f(m)-H(x,0))m=0.$$

The variational formulation of the initial condition on m (which is the last point of the definition of mixed solutions) follows as in the previous part. □

1.4.2 Uniqueness of mixed solutions

We here show that adding a convex hamiltonian do not alter the property of uniqueness of the problem.

Theorem 1.4.2. *If f is strictly monotone, then there exists a unique mixed solution of (MFG).*

Proof. The proof here is once again the adaptation of the one we gave in the first part. We take (u_1, m_1) and (u_2, m_2) two mixed solutions of (MFG) and we are going to show that

$$\int_0^T \int_\Omega ((-\partial_t - \Delta)(u_1 - u_2) + H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) = 0.$$

First we note the sets $\Omega_i = \{u_i < 0\}$ for $i = 1, 2$ and we compute

$$\begin{aligned}
&\int_0^T \int_\Omega ((-\partial_t - \Delta)(u_1 - u_2) + H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) = \\
&\int_{\Omega_1 \cap \Omega_2} (f(m_1) - f(m_2))(m_1 - m_2) + \int_{\Omega_1 \cap \Omega_2^c} (f(m_1) - H(x, 0))(m_1 - m_2) \\
&+ \int_{\Omega_2 \cap \Omega_1^c} (f(m_2) - H(x, 0))(m_2 - m_1).
\end{aligned}$$

Once again, using the same arguments we used in the previous part we deduce

$$\begin{aligned}
&\int_0^T \int_\Omega ((-\partial_t - \Delta)(u_1 - u_2) + H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) \geq \\
&\int_0^T \int_\Omega (f(m_1) - f(m_2))(m_1 - m_2).
\end{aligned}$$

Now we obtain using the last point of the definition of mixed solutions, that

$$\begin{aligned}
\int_0^T \int_\Omega ((-\partial_t - \Delta)(u_1 - u_2))(m_1 - m_2) \leq &\int_0^T \int_\Omega (D_p H(x, \nabla u_1) \cdot \nabla(u_2 - u_1)) m_1 \\
&+ \int_0^T \int_\Omega (D_p H(x, \nabla u_2) \cdot \nabla(u_1 - u_2)) m_2.
\end{aligned}$$

Hence, using the convexity of H we get that

$$\int_0^T \int_\Omega ((-\partial_t - \Delta)(u_1 - u_2))(m_1 - m_2) \leq \int_0^T \int_\Omega (H(x, \nabla u_2) - H(x, \nabla u_1))(m_1 - m_2).$$

So we have proven

$$\int_0^T \int_\Omega ((-\partial_t - \Delta)(u_1 - u_2) + H(x, \nabla u_1) - H(x, \nabla u_2))(m_1 - m_2) = 0;$$

and we conclude as we did before. □

1.4.3 The optimal control interpretation

We end this part with the presentation of the optimal control interpretation of (MFG). We are not interested in giving results or proofs in this part as it is only the mix of what we can found in [22, 28] and what we did in the optimal control interpretation of $(SMFG)$. We just want to show to the reader to which optimization problem (MFG) is linked. Moreover we do not want to add some technical difficulties here. We suppose that there exists $\mathcal{F} \in C^1(\Omega \times \mathbb{R})$ such that

$$\forall (x,p) \in \Omega \times \mathbb{R}, \frac{\partial \mathcal{F}}{\partial p}(x,p) = f(x,p).$$

The following optimization problem is then naturally linked to (MFG)

$$\inf_{(\alpha,m)\in\mathcal{H}} \Big\{ \int_0^T \int_\Omega \mathcal{F}(m) - H(x,0)m + L(x,\alpha)m \Big\};$$

where $\mathcal{H}$ is the set of (α, m) in $L^2((0,T), H^1(\Omega)) \times L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega))$, such that $m(0) \leq m_0$ and

$$\partial_t m - \Delta m - div(\alpha m) \leq 0$$

hold sin the sense of distributions. L is here the Fenchel conjugate of H with respect to the second variable. Such a problem is convex and under strict convexity of $\mathcal{F}$, we have existence of a unique minimizer.

Chapitre 2

Équations de Fokker-Planck pour des particules sautantes et jeux à champ moyen de contrôle impulsionnel

Ce chapitre s'intéresse à la description de la densité de particules évoluant suivant des trajectoires optimales associées à un problème de contrôle impulsionnel. On commence par fixer une politique de sauts optimale et expliquer comment on peut caractériser la densité de telles particules. On s'intéresse ensuite au cas avec couplage où le problème de contrôle impulsionnel sous-jacent dépend de la densité des particules que l'on cherche à modéliser : les jeux à champ moyen de contrôle impulsionnel. Dans les deux cas, on donne une caractérisation variationnelle de la densité des particules sautantes.

Ce chapitre a été soumis pour une publication dans les Annales de l'Institut Henri Poincaré C : Analyse Non Linéaire.

2.1 Introduction

2.1.1 General introduction

This paper is the second of a series devoted to the systematic study of mean field games (MFG for short) with optimal stopping or impulse control. In [11] we developed an obstacle problem approach to solve a forward-backward system which models MFG with optimal stopping (without common noise). We here develop the same point of view for MFG with impulse control. As the definition of a Fokker-Planck equation associated with a density of players playing an impulse control problem is a difficult question in itself, it is the subject of the first part of this article. This part is independent from the MFG theory. The case of the master equation (i.e. when there is a common noise) will be treated in a subsequent work.

MFG model situations in which a continuum of indistinguishable players are playing a differential game. The evolution of the density of players is induced by the optimal choices the players make. In many situations, the costs involved in the game depend only on the density of players. Denoting by u the value function of a generic player and by m the density of players, a classical forward-backward MFG system during the time interval $(0, T)$ is

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, \nabla u) = f(m); \\ \partial_t m - \nu\Delta m - div(D_p H(x, \nabla u)m) = 0; \\ m(0) = m_0; u(T) = g(m(T)). \end{cases}$$

where $H(x, p)$ is the hamiltonian of a continuous optimal control problem, m_0 is the initial condition for the density of players, g and f are respectively the terminal and running costs and $\nu > 0$ characterizes the intensity of the individual noises. A solution (u, m) of this system corresponds to a Nash equilibrium for the game with an infinite number of players. This system, as well as MFG, have been introduced in [52, 53, 54] by J.-M. Lasry and P.-L. Lions. In these papers they proved general conditions under which the existence and uniqueness hold for this problem. Such games have been introduced independently in [48]. We also refer to [21, 57] for more results on this system. MFG models have a wide range of applications, see [3, 45, 46] for examples. Many interesting questions have also been raised around this system, we can cite for example long time average [25], learning [23], the difficult problem of the convergence of the system of N players as N goes to infinity and the presence of a common noise [27]. Numerical methods are also being developed, let us cite [2, 17] for instance. Let us also mention that a powerful probabilistic point of view on MFG has been developed, we refer to [29, 30, 49] for more details on this point of view. In this paper, we generalize the results of the existence and uniqueness of the previous system to the case

in which the players face an impulse control problem. Concerning closely related works, several optimal control problems, related to the impulse control problem, have been studied in a MFG setting. Optimal stopping or obstacle problems have been studied in [11, 32, 42, 61], singular controls in [40, 47] and optimal switching in [43].

2.1.2 Impulse control problems

Impulse control problems have been studied since the 70s. We refer to the work of A. Bensoussan and J.-L. Lions (see [10]) for a complete presentation of the problem. The terminology impulse control refers to an optimization problem in which the state is driven by a stochastic ordinary differential equation with jumps :

$$\begin{cases} \forall t \in (\tau_i, \tau_{i+1}), dX_t = \sqrt{2\nu} dW_t; \\ X_{\tau_i^+} = X_{\tau_i^-} + \xi_i; \end{cases} \tag{2.1}$$

where $(W_t)_{t\geq 0}$ is a brownian motion under a standard probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The jump ξ_i occurs at time τ_i and is controlled by the player. The jumps are characterized by the random sequence of stopping times $(\tau_i)_{i\geq 0}$ and the random sequence of jumps $(\xi_i)_{i\geq 0}$. Those two sequences are the controls and are adapted to the brownian motion $(W_t)_{t\geq 0}$ in the sense that the sequence $(\tau_i)_{i\geq 0}$ is indeed a sequence of stopping times for this brownian motion and that (ξ_i, τ_i) is measurable with respect to the σ-algebra generated by $(W_t)_{0\leq t\leq \tau_i}$. We assume that $(\xi_i)_{i\geq 0}$ is valued in a set $K \subset \mathbb{T}^d$. Denoting by f the running cost and by $k(x, \xi)$ the cost paid to use the jump ξ while on the position x, we define the value function u by

$$u(t,x) := \inf_{(\tau_i)_i, (\xi_i)_i} \mathbb{E}[\int_t^T f(s, X_s) ds + \sum_{i=1}^{\#(\tau_j)_j} k(X_{\tau_i^-}, \xi_i)];$$

where the infimum is taken over all sequences $((\tau_i)_i, (\xi_i)_i)$ which are adapted to the brownian motion in the sense prescribed above and which satisfy the fact that $(\tau_i)_i$ is an increasing sequence and that $(\xi_i)_i$ is valued in K. The trajectory $(X_s)_{s\geq t}$ is given by (2.1) with the initial condition $X_t = x$. Under some assumptions on the costs(which are detailed in the appendix), the value function u satisfies, in the sense of quasi variational inequality (QVI) (which is also detailed in the appendix), the Hamilton-Jacobi-Bellman equation :

$$\begin{cases} \max(-\partial_t u - \nu \Delta u - f, u(t,x) - \inf_{\xi \in K}\{k(x,\xi) + u(t, x+\xi)\}) = 0.; \\ u(T) = 0. \end{cases}$$

2.1.3 The density of players

In a MFG context, the main question we are addressing is how will evolve an initial density of players, if those players are facing the same impulse control problem. Intuitively, the density of players m has to satisfy (formally) at least some requirements :

— $\partial_t m - \nu \Delta m = 0$ where it is optimal for the players to wait and not to jump and where no player is arriving.
— $m = 0$ where it is strictly suboptimal not to jump.
— The flux of arriving players at x is equal to the sum over ξ of the flows of players which choose to use the jump ξ at $x - \xi$.

Let us note that, at least formally, we talk about parts of the space on which it is optimal to jump (i.e. to use a control to make the process $(X_t)_t$ jumps) because all the players being indistinguishable, if it is optimal for one player to jump, then it is optimal for all the players to jump.

The problem of finding a density m which satisfies the above requirements is not classical, mostly because there is no particular assumption on how the players use their controls. We focus on the problem of modeling the evolution of a density of jumping players in the first part of this article. We build a dual characterization of the solution of the Fokker-Planck "equation". We fix a function $V(t, x, \xi)$ which describes whether or not the players use the jump ξ at the position x and time t. Then we construct a density of players m which satisfies the required properties and thus solves a Fokker-Planck equation of jumping particles. The characterization of this density relies on the fact that we can interpret such Fokker-Planck equations as dual equations of QVI. The construction of such a solution uses a penalized version of the problem in which we can write rigorously the PDE satisfied by the density m. We then find a priori estimates which allow us to pass to the limit in this penalized version of the problem, while obtaining a characterization of the limit density. We also give results concerning the stationary case.

In the second part of this article, we present results on the uniqueness and existence for the impulse control problem in MFG. We also recall that in view of the results of [11] we expect the solutions of the MFG system to be mixed solutions, meaning that optimal strategies are random in general as the Nash equilibria of the MFG can be mixed equilibria. We end this second part by giving results on the stationary case and on the optimal control interpretation of such MFG.

We present in two appendixes new results concerning QVI and the heat equation in a time dependent domain, which we need in this article and which are somehow independent of it.

2.2 Fokker-Planck equation of jumping particles

In this part we present the variational formulation of the Fokker-Planck equation satisfied by the density of jumping particles. We work in the d dimensional torus (denoted by $\mathbb{T}^d$) in a time dependent setting, except for the last section which is concerned with the stationary case. The positive real number T is the final time and $m_0 \in L^2(\mathbb{T}^d)$ is the initial density at time $t = 0$. The aim of this part is to construct a suitable notion to characterize densities of jumping particles. By opposition to Fokker-Planck equations of jumping processes, we do not want to model populations of particles which are driven by Poisson processes or other jump processes of the sort. Namely if a unique jump ξ is possible, we are interested in building solutions for $\epsilon > 0$ of

$$\begin{cases} \partial_t m_\epsilon - \nu \Delta m_\epsilon + \frac{1}{\epsilon} \mathbb{1}_A m_\epsilon - \frac{1}{\epsilon} (\mathbb{1}_A m_\epsilon)(t, x - \xi) = 0; \\ m(0) = m_0; \end{cases} \tag{2.2}$$

and passing to the limit $\epsilon \to 0$. The interpretation of (2.2) is that the particles jump ξ further if they are in the set A with a probability given by an exponential law of parameter ϵ^{-1}. The interpretation of the limit m of solutions of (2.2) is that it describes particles evolving only along brownian trajectories in A^c and which make a jump of ξ once they reach A. If A is a smooth closed set such that the reaching time of ∂A is well defined, then the trajectory $(X_s)_{s \geq 0}$ of a generic particle is defined by

$$\begin{cases} dX_s = \sqrt{2\nu} dW_s, \forall i, \forall s \in (t_i, t_{i+1}); \\ X_{t_i^+} = X_{t_i^-} + n(X_{t_i^-})\xi; \end{cases}$$

where $n(x)$ is the smallest integer p such that $x + p\xi \notin A$ and where t_{i+1} is the stopping time defined by the reaching time of A by the process $(X_s)_{s \geq t_i}$. We recall that this interpretation is given in the case in which a unique jump is possible, a similar interpretation also exists in the case of a finite number of possible jumps.

Finding solutions of the penalized equation (2.2) does not require new techniques and is not a difficult question in itself. The majority of this part is concerned with building, uniform in ϵ, a priori estimates (lemma 2.2.2) to pass to the limit $\epsilon \to 0$. Even though we use these uniform a priori estimates to prove proposition 2.2.1 and theorem 2.2.1, this estimate is crucial only to prove the existence of a limit as ϵ goes to 0 (theorem 2.2.2).

As explained in the introduction, we shall characterize in this part the solution of a Fokker-Planck equation (the limit density) of jumping particles with dual properties and not with a PDE. The main duality idea of this part is that a Fokker-Planck

equation of jumping particle is somehow the dual or adjoint equation of a QVI, which by the way describes how it is optimal to jump and thus how dynamics of jumping particles evolve. Thus QVI are crucial for the study of the density of jumping particles. For the sake of clarity, the results needed on QVI are given in the appendix. We define here the notion of a smooth cost of jumps k. A function k is said to be a smooth cost of jumps if it satisfies :

$$\begin{cases} \forall \xi \in K, k(\cdot, \xi) \in H^2(\mathbb{T}^d); \\ k^* : x \to \inf_{\xi \in K} k(x, \xi) \in W^{2,\infty}(\mathbb{T}^d); \\ \exists k_0 > 0 \text{ such that } k \geq k_0; \end{cases} \tag{2.3}$$

where K is a finite subset of $\mathbb{T}^d$. The interpretation of $k(x, \xi)$ is that it is the cost paid by the player (or the energy used by a particle) to instantaneously go from x to $x + \xi$. We also define the operator M which plays an important role in the study of QVI by :

$$M(k, u)(t, x) = \inf_{\xi \in K} \{k(x, \xi) + u(t, x + \xi)\}. \tag{2.4}$$

When there is no ambiguity on k, we shall write only $Mu = M(k, u)$.

We begin the study of Fokker-Planck equations of jumping particles with the simpler case of a unique possible jump before addressing the case of a finite number of possible jumps. In each of these situations, we begin by constructing a penalized version of the problem and we then pass to the limit in the resulting penalized equation.

2.2.1 The case of a unique possible jump

We work here in the case in which a single jump ξ is possible. We also assume that there is a measurable set A on which the particles jump. The first part of this section is devoted to the study of a penalized equation. We then prove the existence and uniqueness of the limit density under a certain assumption on the set A. The study of the penalized equation is quite simple however we warn the reader that the estimate of lemma 2.2.2 that we use at a penalized level is crucial to study the limit case. An important feature of our model is that some assumption has to be made on the set on which the particles jump. This assumption shows in some sense the limit of this model. It can be formulated in the following way.

Hypothesis 2.2.1. *The set A is such that there exists $k \in L^\infty$ satisfying* (2.3)*, and $u \in L^2((0,T), H^2(\mathbb{T}^d)) \cap H^1((0,T), L^2(\mathbb{T}^d)) \cap L^\infty((0,T) \times \mathbb{T}^d)$ such that*

$$\begin{cases} u(t, x) = k(x, \xi) + u(t, x + \xi) \text{ for } (t, x) \in A; \\ u(T) = 0. \end{cases}$$

Formally, this assumption restricts the situations which we are able to model to a case in which the particles do not jump an infinite number of time in a finite time interval. Let us also note that if we interpret u as the value function of some impulse control problem, then we are assuming that it is optimal to use its control on the set A. For instance if $A = \mathbb{T}^d$ or if $A = ([x_1, x_2] + vect(\xi))/\mathbb{Z}^d$ for some $x_1, x_2 \in \mathbb{T}^d$ (i.e. A is a strip of "direction" ξ), then the hypothesis does not hold. Indeed in those cases, it is not possible to exit the set A by making jumps of ξ. On the other hand, if $A \subset B(x_0, \alpha)$ for $x_0 \in \mathbb{T}^d, \alpha < |\xi|/2$, then A satisfies the hypothesis (the construction of a function u satisfying the requirements is in this case not trivial but fairly simple and we do not detail it here).

A penalized version of the problem

In order to understand how the density of jumping particles behave we first introduce a smoother version of the problem. We here assume that the particles do not simply jump when they are in A but that they have a given uniform probability of jumping in this set. This method allows us to work with a PDE. We naturally work with the equation :

$$\begin{cases} \partial_t m_\epsilon(t,x) - \nu\Delta m_\epsilon(t,x) + \frac{1}{\epsilon}\mathbb{1}_A m_\epsilon(t,x) - \frac{1}{\epsilon}(\mathbb{1}_A m_\epsilon)(t, x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0,x) = m_0(x) \text{ in } \mathbb{T}^d; \end{cases}$$

where $\epsilon > 0$ is a real number which describes the probability of jumping in A. The term $\frac{1}{\epsilon}\mathbb{1}_A m(t,x)$ stands for the leaving rate of particles which jump in A. The term $-\frac{1}{\epsilon}(\mathbb{1}_A m)(t, x-\xi)$ stands for the arriving rate of particles which jump at $(t, x-\xi)$ and thus which arrive at (t,x). As ϵ goes to 0, the probability of jumping becomes more and more important. Thus finding the limit as ϵ goes to 0 gives the desired density of particles.

For the rest of this paper we may not always write the "(t,x)" in order to lighten the notations. So by default, if the variable considered is not written, it is (t,x).

We begin by showing how we can find estimates on such a penalized equation. Let us introduce the set H defined by

$$H := \{v \in L^2((0,T), H^1(\mathbb{T}^d)), \partial_t v \in L^2((0,T), H^{-1}(\mathbb{T}^d))\}. \tag{2.5}$$

The following lemma will be useful to establish a priori estimates on m.

Lemma 2.2.1. *Let $m \in L^2((0,T), H^2(\mathbb{T}^d))$ be a solution of (2.2) and k satisfying (2.3). If $m \geq 0$, $u \in H$ and $u \leq Mu$ almost everywhere on A, then*

$$\int_0^T \int_{\mathbb{T}^d} (\partial_t m - \nu\Delta m)u + \frac{1}{\epsilon}\int_A km \geq 0.$$

Proof. We multiply (2.2) by u and we integrate, we then obtain after a change of variable :

$$\int_0^T \int_{\mathbb{T}^d} (\partial_t m - \nu \Delta m) u + \frac{1}{\epsilon} \int_A mu = \frac{1}{\epsilon} \int_A m(t,x) u(t, x+\xi) dt dx.$$

Using the fact that $u \leq Mu$ on A we deduce the desired result. □

The previous result suggests to work with the set $\mathcal{H}(k)$ for some k where $\mathcal{H}(k)$ is defined by :

$$\mathcal{H}(k) := \{m \in L^2((0,T), H^1(\mathbb{T}^d)), D(k,m) > -\infty\};$$

where $D(k,m)$ is defined by

$$D(k,m) := \inf \left\{ \int_0^T (-\partial_t u - \nu \Delta u, m)_{H^{-1} \times H^1} - \int_{\mathbb{T}^d} u(0) m_0 | u \in \mathcal{AD}(k) \right\};$$

where $\mathcal{AD}(k)$ is defined by :

$$\mathcal{AD}(k) := \{u \in H, u \leq M(k,u) \text{ on } A, u(T) = 0\}.$$

We recall that M is defined in (2.4). When no confusion is possible for k, as we do for M, we shall only write $D(m)$ and $\mathcal{H}$.

The set $\mathcal{H}$ has to be interpreted as the set of admissible solutions of the Fokker-Planck equation. Indeed if m is a density of jumping particles, then when particles leave (or jump) we should have exactly the same arriving particles ξ further. We recall that $\partial_t m - \nu \Delta m \leq 0$ is interpreted as particles leaving and $\partial_t m - \nu \Delta m \geq 0$ as particles arriving. Thus it is natural to measure the variation of $\partial_t m - \nu \Delta m$ with functions u such that $u(x) - u(x+\xi) \leq k$. This "test" quantifies the fact that some negativity for $\partial_t m - \nu \Delta m$ has to be compensated by some positivity of this quantity ξ further. Moreover, as jumps can only occur on A we restrict ourselves to the case in which those conditions are satisfied only on A. We now prove the following lemma, which states that in some sense, the quantity $D(m)$ is of interest to bound m in some functional space (this lemma is crucial to study the limit $\epsilon \to 0$) :

Lemma 2.2.2. *Let k be such that it satisfies* (2.3) *and hypothesis* 1 *with a given $u \in H$. For any $m \in H \cap \mathcal{H}(k)$, $m \geq 0$, there exists $C(k) > 0$ depending only on k and $||u||_{L^\infty}$ such that*

$$||m||^2_{L^2((0,T),H^1)} \leq -D(k,m) + C(k)(1 + ||m||_{L^2((0,T),H^1)})||m_0||_{L^2}.$$

Remark 2.2.1. *The assumption $m \in H$ is crucial as it allows us to deal with the problem of the time regularity. Moreover, the constant $C(k)$ does not depend on A.*

Proof. Because hypothesis 1 is satisfied, we are able to apply proposition 2.A.2 (appendix) and we deduce that there exists $\tilde{u} \in L^2((0,T), H^1(\mathbb{T}^d))$ such that :

$$\begin{cases} \tilde{u} \leq M(k,\tilde{u}) \text{ on } A; \\ \forall v \in H, v \leq M(k,\tilde{u}) \text{ on A}; \\ -\int_0^T \int_{\mathbb{T}^d} \partial_t v(v-\tilde{u}) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla\tilde{u} \cdot \nabla(v-\tilde{u}) + \frac{1}{2}\int_{\mathbb{T}^d} |v(T)|^2 \geq \int_0^T \int_{\mathbb{T}^d} (\Delta m)(v-\tilde{u}). \end{cases}$$

Because we made the assumption $m \in H$, we can remark that by integration by parts in time :

$$D(k,m) = \inf\{\int_0^T (\partial_t m - \nu\Delta m, v)_{H^{-1}\times H^1} | v \in H, v \leq M(k,v) \text{ on } A, v(T) = 0\}.$$

We obtain by density that

$$D(k,m) = \inf\{\int_0^T (\partial_t m - \nu\Delta m, v)_{H^{-1}\times H^1} | v \in L^2((0,T), H^1(\mathbb{T}^d)), v \leq M(k,v) \text{ on } A\}.$$

Thus we deduce

$$D(k,m) \leq \int_0^T (\partial_t m - \nu\Delta m, \tilde{u})_{H^{-1}\times H^1}. \tag{2.6}$$

One would like to write :

$$\begin{aligned} \int_0^T (\partial_t m - \nu\Delta m, \tilde{u})_{H^{-1}\times H^1} &= \int_0^T (-\partial_t\tilde{u} - \nu\Delta\tilde{u}, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} \tilde{u}(0)m_0; \\ &\leq \int_0^T (\Delta m, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} \tilde{u}(0)m_0; \end{aligned}$$

but since $\tilde{u} \notin H$, this does not make sense. However, because of the weak variational inequality satisfied by $\tilde{u}$, we can deduce that

$$\int_0^T (\partial_t m - \nu\Delta m, \tilde{u})_{H^{-1}\times H^1} \leq \int_0^T (\Delta m, m)_{H^{-1}\times H^1} + ||m_0||_{L^2}||\tilde{u}||_{L^\infty(L^2)}.$$

Hence we obtain that

$$D(k,m) \leq -\int_0^T \int_{\mathbb{T}^d} |\nabla m|^2 + ||m_0||_{L^2}||\tilde{u}||_{L^\infty(L^2)}.$$

Recalling the estimate of proposition 2.A.2 (appendix), the result is proved. □

This lemma suggests to find a priori estimates for solutions of (2.2) by looking at the quantity $D(m)$. However, this estimate requires the positivity of m. In order to use this estimate to exhibit solutions of (2.2), we prove a maximum principle for this equation. This proof is very general and can be applied to more general equations. See [57] for an example of the use of this proof for systems of conservation laws for instance.

Proposition 2.2.1. *Let $\lambda \in L^\infty((0,T)\times\mathbb{T}^d)$, $\lambda \geq 0$, $m_0 \in L^2(\mathbb{T}^d)$, $m_0 \geq 0$, and $m \in L^2((0,T)\times\mathbb{T}^d)$ be a solution of*

$$\begin{cases} \partial_t m(t,x) - \nu\Delta m(t,x) + \lambda(t,x)m(t,x) - \lambda(t,x-\xi)m(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases} \tag{2.7}$$

Then, $m \geq 0$.

Proof. We assume in a first time that λ and m_0 are smooth functions and that $m_0 > 0$. Then by classical parabolic estimates, m is also smooth (C^1 in time and C^2 in space). If there exists $(t_0,x_0) \in (0,T)\times\mathbb{T}^d$ such that $m(t_0,x_0) < 0$, then there exists $\delta > 0$ such that $m(t_0,x_0) + \delta t_0 < 0$. We define μ by :

$$\mu(t,x) = m(t,x) + \delta t, \forall (t,x) \in (0,T)\times\mathbb{T}^d.$$

For any $x \in \mathbb{T}^d$, $\mu(0,x) > 0$ and $\mu(t_0,x_0) < 0$, thus there exists $(t_1,x_1) \in (0,T)\times\mathbb{T}^d$ such that

$$\begin{cases} \mu(t) \geq 0, \forall t \leq t_1; \\ \mu(t_1,x_1) = 0; \\ \partial_t\mu(t_1,x_1) \leq 0; \\ \Delta\mu(t_1,x_1) \geq 0. \end{cases}$$

Let us remark that

$$\begin{aligned} \partial_t\mu &= \partial_t m + \delta; \\ &= \delta + \nu\Delta m - \lambda m + (\lambda m)(\cdot - \xi). \end{aligned}$$

Evaluating this last expression at (t_1,x_1) we obtain that $\partial_t\mu(t_1,x_1) > 0$ which is impossible. So we have proven that if λ and m_0 are smooth, then $m \geq 0$. Because of the uniqueness of solutions of (2.7) (which will be independently proved in the theorem 2.2.1) this result extends to non smooth λ and m_0 with only $m_0 \geq 0$. □

Next, we establish the main result of this section : the existence and uniqueness of a solution of (2.2).

Theorem 2.2.1. *For any $m_0 \in L^2(\mathbb{T}^d)$, $m_0 \geq 0$, there exists a unique $m \in H$ such that*

$$\begin{cases} \partial_t m - \nu\Delta m + \frac{1}{\epsilon}\mathbb{1}_A(t,x)m(t,x) - \frac{1}{\epsilon}\mathbb{1}_A(t,x-\xi)m(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases}$$

where the first line has to be taken in the sense of distributions. Moreover, $m \geq 0$.

Proof. We define $\lambda \in L^\infty$ by

$$\lambda = \frac{1}{\epsilon}\mathbb{1}_A$$

We then define the application $\mathcal{F}$ from $L^2((0,T)\times\mathbb{T}^d)$ to itself by : $\mathcal{F}(m)$ is the unique solution in H of

$$\begin{cases} \partial_t \mathcal{F}(m) - \nu\Delta\mathcal{F}(m) + \lambda(t,x)m(t,x) - \lambda(t,x-\xi)m(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases}$$

By standard parabolic estimates, $\mathcal{F}$ is continuous and compact. Let us take $\mu \in [0,1]$ and $m \in L^2((0,T)\times\mathbb{T}^d)$ such that $m = \mu\mathcal{F}(m)$, m satisfies

$$\begin{cases} \partial_t m - \nu\Delta m + \mu\lambda(t,x)m(t,x) - \mu\lambda(t,x-\xi)m(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases}$$

By proposition 2.2.1, $m \geq 0$. For any $v \in H$ such that $v \leq Mv$ on A, we obtain using lemma 2.2.1 :

$$\int_0^T \int_{\mathbb{T}^d} (\partial_t m - \nu\Delta m)v \geq -\mu \int_0^T \int_{\mathbb{T}^d} \lambda m.$$

Hence by the lemma 2.2.2 we deduce that there exists $C > 0$ independent of μ such that :

$$\begin{aligned} ||m||^2_{L^2((0,T),H^1(\mathbb{T}^d))} &\leq C||m||_{L^2((0,T),H^1(\mathbb{T}^d))}||m_0||_{L^2} + \mu\int_0^T\int_{\mathbb{T}^d}\lambda m; \\ &\leq C||m||_{L^2((0,T),H^1(\mathbb{T}^d))}||m_0||_{L^2} + \mu||\lambda||_{L^2}||m||_{L^2}. \end{aligned}$$

From which we deduce that the set

$$\{m \in L^2, \exists \mu \in [0,1], m = \mu\mathcal{F}(m)\}$$

is bounded in L^2. Applying Schaefer's fixed point theorem, we obtain that there exists a solution of (2.2).

For any m_1 and m_2 solutions of (2.2), we denote by $\delta m = m_1 - m_2$ the difference of these solutions. The function δm satisfies

$$\begin{cases} \partial_t \delta m - \nu\Delta\delta m + \lambda(t,x)\delta m(t,x) - \lambda(t,x-\xi)\delta m(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ \delta m(0) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Multiplying this equation by δm and integrating in space, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}(\delta m)^2 + \nu\int_{\mathbb{T}^d}|\nabla\delta m|^2 + \int_{\mathbb{T}^d}\lambda(\delta m)^2 = \int_{\mathbb{T}^d}(\lambda\delta m)(t,x)(\delta m)(t,x+\xi)dx;$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}(\delta m)^2 \leq ||\lambda||_{L^\infty}||\delta m(t)||^2_{L^2(\mathbb{T}^d)}.$$

Finally, we deduce, using Gronwall's lemma, that $\delta m = 0$ and thus that there exists a unique solution of (2.2).

□

Existence of a limit density

We show here how we can pass to the limit in the equation (2.2) and hence obtain a characterization of the density of jumping particles. For the rest of this section, we fix k given in hypothesis 1. We now describe the behavior of the solutions of (2.2) as ϵ goes to 0.

Theorem 2.2.2. *Assume hypothesis* 1 *holds, then there exists* $m \in L^2((0,T), H^1(\mathbb{T}^d))$ *such that, extracting a subsequence if necessary,* $(m_\epsilon)_\epsilon$ *converges weakly in* $L^2((0,T), H^1(\mathbb{T}^d))$ *toward* m *which satisfies*

$$\begin{cases} D(m) > -\infty; \\ m = 0 \text{ a.e. in } A. \end{cases}$$

Moreover, for all $v \in H$ *such that* $v \leq Mv$ *on* A *and* $v(T) = 0$,

$$\begin{aligned} &\int_0^T (-\partial_t v - \nu\Delta v, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v(0)m_0 \\ &\geq \int_0^T (-\partial_t u - \nu\Delta u, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} u(0)m_0 = D(m); \end{aligned}$$

for any $u \in H$ *which satisfies* $u = Mu$ *on* A, $u(T) = 0$.

Remark 2.2.2. *Let us note that the trace condition* $m(0) = m_0$ *is not satisfied, we send the reader to the paragraph following the proof for the interpretation of this fact.*

Proof. We define m_ϵ for all $\epsilon > 0$ as the unique solution of

$$\begin{cases} \partial_t m_\epsilon - \nu\Delta m_\epsilon + \frac{1}{\epsilon}\mathbb{1}_A(t,x)m_\epsilon(t,x) - \frac{1}{\epsilon}\mathbb{1}_A(t,x-\xi)m_\epsilon(t,x-\xi) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ m_\epsilon(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases}$$

For all $\epsilon > 0$, we can observe that

$$-\infty < D(m_\epsilon) = \int_0^T (-\partial_t u - \nu\Delta u, m_\epsilon)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} u(0)m_0;$$

where $u \in L^\infty \cap L^2((0,T), H^1(\mathbb{T}^d))$ is given by hypothesis 1. We known from lemma 2.2.2 that

$$||m_\epsilon||^2_{L^2(H^1)} \leq -D(m_\epsilon) + C(k)(1 + ||m_\epsilon||_{L^2(H^1)})||m_0||_{L^2}.$$

Thus there exists $C(k,u)$ depending on k and u such that

$$||m_\epsilon||^2_{L^2(H^1)} \leq C(k,u)(1 + ||m_\epsilon||_{L^2(H^1)})(1 + ||m_0||_{L^2}).$$

Hence we deduce that $(m_\epsilon)_{\epsilon>0}$ is a bounded sequence of $L^2((0,T), H^1(\mathbb{T}^d))$ and as a consequence, that $(D(m_\epsilon))_{\epsilon>0}$ is also a bounded sequence. Thus, extracting a subsequence if necessary, $(m_\epsilon)_{\epsilon>0}$ converges weakly toward a limit $m \in L^2((0,T), H^1(\mathbb{T}^d))$. Now take any $v \in H$ such that $v \leq Mv$ on A and any $u \in H$ such that $u = Mu$ on A and $u(T) = v(T) = 0$. For all $\epsilon > 0$:

$$\int_0^T (\partial_t m_\epsilon, u)_{H^{-1}\times H^1} + \nu \int_0^T \int_{\mathbb{T}^d} \nabla m_\epsilon \nabla u \leq \int_0^T (\partial_t m_\epsilon, v)_{H^{-1}\times H^1} + \nu \int_0^T \int_{\mathbb{T}^d} \nabla m_\epsilon \nabla v.$$

Thus we deduce that

$$\begin{aligned} &\int_0^T (-\partial_t u - \nu\Delta u, m_\epsilon)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} u(0)m_0 \\ \leq &\int_0^T (-\partial_t v - \nu\Delta v, m_\epsilon)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v(0)m_0. \end{aligned}$$

Passing to the limit ϵ goes to 0 we deduce :

$$\begin{aligned} &\int_0^T (-\partial_t v - \nu\Delta v, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v(0)m_0 \\ \geq &\int_0^T (-\partial_t u - \nu\Delta u, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} u(0)m_0. \end{aligned}$$

In particular,

$$D(m) > -\infty.$$

Let us note that for all $\epsilon > 0$

$$D(m_\epsilon) = -\frac{1}{\epsilon}\int_A k m_\epsilon \leq -\frac{1}{\epsilon}\int_A k_0 m_\epsilon.$$

Thus, $m = 0$ almost everywhere on A because $(D(m_\epsilon))_\epsilon$ is bounded. □

Interpretation of the limit density

From a variational point of view, the properties of the limit density m given in this theorem are characterizing what we expect for such a density. Indeed, as we

mentioned earlier, $D(m) > -\infty$ stands for the fact that m is an admissible density for describing jumping particles. The condition $m = 0$ on A stands for the fact that m is a density of particles which are actually jumping on A because otherwise there will be particles on A. The condition :

$$\begin{cases} \forall u \in H, u = Mu \text{ on } A, u(T) = 0 : \\ D(m) = \int_0^T (-\partial_t u - \nu \Delta u, m)_{H^{-1} \times H^1} - \int_{\mathbb{T}^d} u(0) m_0; \end{cases} \tag{2.8}$$

stands formally from the fact those particles are not jumping elsewhere than on A. Indeed at a penalized level we know that

$$D(m_\epsilon) = -\frac{1}{\epsilon} \int_A k m_\epsilon.$$

Thus the quantity $D(m)$ is closely related to the set A on which the particles actually jump. It appears that the quantity $D(m)$ measures the aggregate cost "paid" by all the particles which jump. Because $m = 0$ on A, we know that the particles jump at least on A. Formally (2.8) states that the particles do not jump elsewhere than on A ; because the total "cost" is minimum for particles which jump from at least A. To understand why we state that (2.8) stands for the fact that the particles do not jump elsewhere than on A, let us look at an example : we take B a measurable subset of $\mathbb{T}^d$ which satisfies hypothesis 1 and such that $A \subset B$. We define μ_ϵ by :

$$\begin{cases} \partial_t \mu_\epsilon - \nu \Delta \mu_\epsilon + \frac{1}{\epsilon} \mathbb{1}_B \mu_\epsilon - \frac{1}{\epsilon} (\mathbb{1}_B \mu_\epsilon)(t, x - \xi) = 0; \\ \mu(0) = m_0. \end{cases}$$

Letting ϵ go to 0, recalling the previous theorem, μ satisfies $D(\mu) > -\infty$ and $\mu = 0$ on A because $\mu = 0$ on B. What is differentiating μ from m is that, for $v \in H, v \leq Mv, v = Mv$ on A, we do not necessary have an equality in :

$$D(\mu) \leq \int_0^T (\partial_t \mu, v)_{H^{-1} \times H^1} + \nu \int_0^T \int_{\mathbb{T}^d} \nabla \mu \nabla v;$$

if $v \neq Mv$ on B.

Finally, let us note that the initial condition $m(0) = m_0$ may not be satisfied. This is a consequence of the fact that if A is not negligible near $\{t = 0\}$, then particles are jumping instantaneously. We cannot expect in such a case for the initial condition to be satisfied. However the variational relation satisfied by m is sufficient to "remember" that the density starts from m_0. In general, there is no time regularity for the solution m. This discussion leads to the following definition :

Definition 2.2.1. *For any positive $m_0 \in L^2(\mathbb{T}^d)$, A measurable subset of $\mathbb{T}^d$, $m \in L^2((0,T), H^1(\mathbb{T}^d))$ is called a solution of the Fokker-Planck equation of particles jumping on A if there exists k satisfying (2.3) such that*

— *$m = 0$ on A ;*

—

$$\forall v, u \in H, v \leq M(k,v) \text{ on } A, u = M(k,u) \text{ on } A, v(T) = 0, u(T) = 0 :$$
$$\int_0^T (-\partial_t(u-v) - \nu\Delta(u-v), m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d}(u-v)(0)m_0 \leq 0.$$

Even though we just explain in which extent this definition is legitimate, the following section on the uniqueness, even though the assumptions on A will be a bit more restrictive, justifies this choice of definition.

Uniqueness of the limit density

We now discuss the uniqueness of such solutions. We state that the uniqueness holds under certain assumptions on the set A. Let us note that it is classical to have some assumptions on the domain in the study of parabolic PDE in time dependent domain, see [19, 41]. We make here the following assumption :

Hypothesis 2.2.2. *The set A is either a closed set with Lipschitz boundary such that $(\{T\} \times \mathbb{T}^d) \cap A$ is an open set with Lipschitz boundary, or it is non-decreasing in time (for the inclusion).*

Our main argument is that the uniqueness of solutions of the Fokker-Planck equation can be deduced from an existence result for an "adjoint" equation. Under hypothesis 2.2.2, we have the following result :

Lemma 2.2.3. *Assume hypothesis 2.2.1 and 2.2.2 hold and take (k,u) given by hypothesis 2.2.1. There exists $\epsilon > 0$ such that for any $f \in L^\infty((0,T)\times\mathbb{T}^d)$ $||f||_{L^\infty} \leq \epsilon$, there exists $v \in H$ such that, for all $\mu \in L^2((0,T), H^1(\mathbb{T}^d))$, $\mu = 0$ on A :*

$$\begin{cases} \int_0^T (-\partial_t v - \nu\Delta v, \mu)_{H^{-1}\times H^1} = \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u + f, \mu)_{H^{-1}\times H^1}; \\ v = M(k,v) \text{ on } A; \\ v(T) = u(T). \end{cases}$$

Proof. See appendix B, theorem 2.B.1. □

We are now able to prove the following result :

Theorem 2.2.3. *Assume hypothesis 2.2.1 and 2.2.2 hold, for any non negative $m_0 \in L^2(\mathbb{T}^d)$ there exists at most one $m \in L^2((0,T), H^1(\mathbb{T}^d))$ solution of the problem in the sense of definition 2.2.1.*

Proof. We denote by m_1 and m_2 two solutions. The idea of the proof consists in constructing for $i = 1, 2$, $v_i \in H$ as in lemma 2.2.3 with respective data $f = f_1$ and $f = f_2$, where $||f_1||_{L^\infty}, ||f_2||_{L^\infty} \leq \epsilon$. We then evaluate

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1}.$$

This proof is the adaptation of the uniqueness proof we give for the MFG problem at the end of this paper, which is itself an adaptation of the proof of uniqueness of J.-M. Lasry and P.-L. Lions [54]. Because m_1 and m_2 are solutions, the following holds :

$$D(m_j) = \int_0^T (-\partial_t(v_i) - \nu\Delta(v_i), m_j)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v_i(0)m_0.$$

Hence,

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1} = 0. \tag{2.9}$$

On the other hand, using lemma 2.2.3, we derive that

$$\int_0^T (-\partial_t(v_1 - v_2) - \nu\Delta(v_1 - v_2), m_1 - m_2)_{H^{-1}\times H^1} = \int_0^T \int_{\mathbb{T}^d} (m_1 - m_2)(f_1 - f_2).$$

Recalling (2.9), we obtain that

$$\int_0^T \int_{\mathbb{T}^d} (m_1 - m_2)(f_1 - f_2) = 0.$$

Thus we deduce that $m_1 = m_2$ because $m_1 - m_2$ is orthogonal to the ball of center 0 and radius 2ϵ of $L^\infty((0, T) \times \mathbb{T}^d)$. □

A remark on the hypothesis for uniqueness

Even though we are only able to prove the uniqueness of solutions of the Fokker-Planck equation under hypothesis 2.2.2, we conjecture that uniqueness is a more general property for this equation. Indeed we believe that an approach similar to the one we present in the second part of Appendix B (theorem 2.B.2) can be adapted to this situation. The technical difficulty which we have not been able to overcome for the moment is to prove some time regularity for m when the set A is decreasing.

However, we hope that the range of applications of hypothesis 2.2.2 is large enough to convince the reader that the notion of solution of the Fokker-Planck equation we present is the correct one.

2.2.2 The case of a finite number of possible jumps

We now address a more general model as we look at situations in which different jumps can occur. As we shall see, all the results of the case of a single jump are adaptable to the case of a finite number of jumps. However there are in this section more notations and we advise not to read this section before the previous one. We denote by $K \subset \mathbb{T}^d$ the finite set of possible jumps. In this setting a single set A is no longer sufficient to describe all the jumps. We introduce V which describes the jumps by : $V(\xi, t, x)$ is the proportion of particles which use the jump ξ at (t, x). We assume the following :

$$\begin{cases} V \in L^\infty(K, (0,T), \mathbb{T}^d); \\ V \geq 0; \\ \sum_{\xi \in K} V(\xi, t, x) \leq 1. \end{cases} \tag{2.10}$$

We also define the following sets :

$$\forall \xi \in K, A_\xi := \{V(\xi, \cdot, \cdot) > 0\};$$

$$A = \cup_{\xi \in K} A_\xi.$$

As in the case of a single jump, an assumption on the sets on which the particles jump is still needed. We make here the following assumption :

Hypothesis 2.2.3. *There exists k satisfying* (2.3) *and $u \in H \cap L^\infty$ such that*

$$\begin{cases} u \leq M(k, u) \text{ in } (0,T) \times \mathbb{T}^d; \\ u(T) = 0; \\ \forall \xi \in K, V(\xi, t, x)(k(x, \xi) + u(t, x + \xi) - u(t, x)) = 0 \text{ a.e. in } (0,T) \times \mathbb{T}^d. \end{cases} \tag{2.11}$$

We recall that the set H is defined by

$$H := \{v \in L^2((0,T), H^1(\mathbb{T}^d)), \partial_t v \in L^2((0,T), H^{-1}(\mathbb{T}^d))\}.$$

This hypothesis is slightly more sophisticated than hypothesis 1. This is due to the fact that multiple jumps being possible, we have to be more precise. We still assume that the sets on which the particles are jumping are given as a result of an impulse control problem but we precise for which ξ the minimum is reached.

The penalized equation

We introduce first a penalized version of the problem. We recall that this penalization models situations in which the particles have a certain probability to jump in the prescribed sets, and that the limit as ϵ goes to 0 corresponds to the probability of jumping going to 1. At this penalized level, we expect the density of particles m to satisfy :

$$\begin{cases} \partial_t m_\epsilon - \nu\Delta m_\epsilon + \frac{1}{\epsilon} m_\epsilon(t,x)\Big(\sum_{\xi\in K} V(\xi,t,x)\Big) - \frac{1}{\epsilon}\sum_{\xi\in K} m_\epsilon(t,x-\xi)V(\xi,t,x-\xi) = 0 \\ \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases} \tag{2.12}$$

We define the set $\mathcal{AD}(k)$ by :

$$\mathcal{AD}(k) := \{u \in H, u(T) = 0, \forall \xi \in K, V(t,x,\xi)(u(t,x) - k(x,\xi) - u(t,x+\xi)) \leq 0 \text{ on } A\}.$$

This set represents the set of admissible solutions of an impulse control problem in which one can only use the jump $\xi \in K$ at (t,x) if $V(t,x,\xi) > 0$. Let us assume that m is a smooth solution of the previous PDE. Then, as in the case of a single jump, for any $u \in \mathcal{AD}(k)$, after a simple change of variable we obtain that :

$$\begin{aligned} \int_0^T (\partial_t m - \nu\Delta m, u)_{H^{-1}\times H^1} &= -\frac{1}{\epsilon}\int_0^T \int_{\mathbb{T}^d} m(t,x)\left(\sum_{\xi\in K} V(\xi,t,x)\right) u(t,x)dtdx + \\ &\quad + \frac{1}{\epsilon}\int_0^T \int_{\mathbb{T}^d} m(t,x)\left(\sum_{\xi\in K} V(\xi,t,x)u(t,x+\xi)\right) dtdx; \\ &\geq -\frac{1}{\epsilon}\int_0^T \int_{\mathbb{T}^d} m(t,x)(\sum_{\xi\in K} V(\xi,t,x)k(t,x,\xi))dtdx; \end{aligned}$$

This leads us to define the meaningful quantity :

$$D(k,m) := \inf\{\int_0^T (-\partial_t u - \nu\Delta u, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} m_0 u(0) | u \in \mathcal{AD}(k)\}; \tag{2.13}$$

We introduce the set $\mathcal{H}(k) := \{m \in L^2((0,T), H^1(\mathbb{T}^d)), D(k,m) > -\infty\}$. Let us remark that for any u satisfying (2.11) :

$$D(k,m_\epsilon) = \int_0^T (\partial_t m_\epsilon - \nu\Delta m_\epsilon, u)_{H^{-1}\times H^1}.$$

The proofs of the following lemmata are the exact analogous of the proof we did earlier in the case of a single jump so we do not present them here.

Lemma 2.2.4. *Assume hypothesis 2.2.3 holds for (k, u). Then there exists $C > 0$ (independent on m and depending on k and $||u||_{L^\infty}$) such that for any $m \in \mathcal{H}(k) \cap H$, with $m \geq 0$:*

$$||m||^2_{L^2((0,T),H^1)} \leq -D(k,m) + C(1 + ||m||_{L^2((0,T),H^1)})||m_0||_{L^2}.$$

Lemma 2.2.5. *Let $m_0 \in L^2(\mathbb{T}^d)$, $m_0 \geq 0$, and $m \in L^2((0,T), H^1(\mathbb{T}^d))$ be a solution of (2.12). Then, $m \geq 0$.*

Furthermore, as in the case of a single jump, we can prove the following result.

Theorem 2.2.4. *Assume hypothesis 2.2.3 holds, then for all $m_0 \in L^2(\mathbb{T}^d)$ there exists a unique solution $m \in H$ of the penalized equation (2.12).*

The limit density

We now address the passage to the limit in (2.12) using the previous result. As in the case of a single jump, the existence follows from lemma 2.2.4 and we prove uniqueness under a more restrictive assumption in order to avoid technical difficulties.

Theorem 2.2.5. *If there exists k satisfying (2.3) such that hypothesis 2.2.3 holds, then there exists $m \in L^2((0,T), H^1(\mathbb{T}^d))$ such that*

$$\begin{cases} D(k,m) > -\infty; \\ m = 0 \text{ in } A; \end{cases}$$

and, for all $v \in \mathcal{AD}(k)$:

$$\begin{aligned} \int_0^T (-\partial_t v - \nu\Delta v, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v(0)m_0 \geq & \int_0^T (-\partial_t u - \nu\Delta u, m)_{H^{-1}\times H^1} \\ & - \int_{\mathbb{T}^d} u(0)m_0 \\ = & D(k,m); \end{aligned}$$

where $u \in H$ satisfies (2.11).

We recall that A is defined by

$$A = \cup_{\xi\in K}\{V(\xi,\cdot,\cdot) > 0\}.$$

We do not present the proof of this result as it is the same as the one we presented in the case of a unique jump. Like we did in the case of a single jump, we give the following definition :

Definition 2.2.2. *For any $m_0 \in L^2(\mathbb{T}^d)$, if there exists k satisfying (2.3) and V measurable function satisfying* (2.10)*, $m \in L^2((0,T), H^1(\mathbb{T}^d))$ is called a solution of the Fokker-Planck equation of particles jumping with jumps described by V if*

— *$m = 0$ on A ;*

—

$$\begin{cases} \forall v \in H, v \leq M(k,v), \forall u \in H \text{ satisfying (2.11)}: \\ \int_0^T (-\partial_t v - \nu\Delta v, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} v(0)m_0 \geq \int_0^T (-\partial_t u - \nu\Delta u, m)_{H^{-1}\times H^1} \\ \qquad\qquad - \int_{\mathbb{T}^d} u(0)m_0 \\ \qquad\qquad = D(k,m). \end{cases}$$

We now turn to the question of the uniqueness of such solutions. As in the case of a single jump, uniqueness is a consequence of an existence result for an adjoint equation. Similarly as we proceeded in the particular case of a single jump, we are going to make some assumption on A. Moreover here we add an hypothesis on the function u given by hypothesis 2.2.3. We state the following hypothesis :

Hypothesis 2.2.4. *The set A is either a closed set with Lipschitz boundary such that $(\{T\} \times \mathbb{T}^d) \cap A$ is an open set with Lipschitz boundary, or it is non-decreasing in time (for the inclusion). Moreover, there exists (k,u) satisfying hypothesis 2.2.3 such that :*

$$\begin{cases} \forall \xi \in K, x \in A_\xi \Rightarrow (k(x,\xi) + u(t,x+\xi) - u(t,x)) = 0; \\ \forall \xi \in K, x \in A_\xi \Rightarrow \forall \xi' \neq \xi, u(t,x) < k(x,\xi') + u(t,x+\xi'). \end{cases} \tag{2.14}$$

We are now able to state the uniqueness of solutions of Fokker-Planck equation of jumping particles in the case of a finite number of possible jumps for a set A verifying hypothesis 2.2.4.

Theorem 2.2.6. *Under hypothesis 2.2.4, for any positive $m_0 \in L^2(\mathbb{T}^d)$, there exists at most one $m \in L^2((0,T), H^1(\mathbb{T}^d))$ solution of the Fokker-Planck equation of particles jumping with jumps described by V.*

This result is proved following the same argument as the one for a unique possible jump.

2.2.3 The stationary case

We now turn to the question of the stationary setting. Most of the arguments of the proofs of the results below follow the ones from the time dependent setting. Thus, we only details the arguments which differ form the one in the time

dependent case.

We assume that there exists V such that :

$$\begin{cases} V \in L^\infty(K, \mathbb{T}^d); \\ V \geq 0; \\ \sum_{\xi \in K} V(\xi, x) \leq 1. \end{cases}$$

We denote by $K \subset \mathbb{T}^d$ the finite set of possible jumps. We assume that the following assumption is satisfied :

Hypothesis 2.2.5. *There exists $u \in H^2(\mathbb{T}^d) \cap L^\infty$ sand $k \in L^\infty(K \times \mathbb{T}^d)$ such that*

$$\begin{cases} k \geq k_0 > 0; \\ \forall \xi \in K, k(\xi, \cdot) \in H^2(\mathbb{T}^d); \\ \forall \xi \in K, V(\xi, x)\,(k(x, \xi) + u(x + \xi) - u(x)) = 0 \text{ a.e. in } \mathbb{T}^d; \end{cases}$$

where $M(k, u)$ is defined by :

$$M(k, u)(x) = \inf_{\xi \in K} k(\xi, x) + u(x + \xi).$$

We study here a stationary Fokker-Planck equation in which there is a fixed leaving rate of players $\delta > 0$ and a constant entry of players $\rho \in L^\infty(\mathbb{T}^d)$, $\rho \geq 0$. Namely, at a penalized level, we are interested in :

$$-\nu \Delta m_\epsilon + \delta m_\epsilon + \frac{1}{\epsilon} m_\epsilon(x) \Big(\sum_{\xi \in K} V(\xi, x) \Big) - \frac{1}{\epsilon} \sum_{\xi \in K} m_\epsilon(x - \xi) V(\xi, x - \xi) = \rho \text{ in } \mathbb{T}^d. \tag{2.15}$$

This section is organized as follow : we first show the existence and uniqueness of solutions of the penalized equation (2.15). We then show the existence of a limit as ϵ goes to 0 which satisfies the Fokker-Planck equation in a weak sense. We then prove the uniqueness of such limits.

The penalized equation

We begin this section by showing a general uniqueness result for equations of the type of (2.15).

Proposition 2.2.2. *Let $(\lambda_1, .., \lambda_n) \in L^\infty(\mathbb{T}^d)^n$. Then, for any $\rho \in L^2(\mathbb{T}^d)$, there exists at most one solution $m \in H^2(\mathbb{T}^d)$ of :*

$$-\nu \Delta m + \delta m + m(x) \Big(\sum_{i=1}^n \lambda_i(x) \Big) - \sum_{i=1}^n m(x - \xi) \lambda_i(x - \xi) = \rho \text{ in } \mathbb{T}^d. \tag{2.16}$$

Proof. Let us assume that there exist two such solutions m_1 and m_2. Then if we note $\mu = m_1 - m_2$, it satisfies :

$$-\nu\Delta\mu + \delta\mu + \mu(x)\Big(\sum_{i=1}^{n}\lambda_i(x)\Big) - \sum_{i=1}^{n}\mu(x-\xi)\lambda_i(x-\xi) = 0 \text{ in } \mathbb{T}^d.$$

Let us assume $\mu \neq 0$, then the operator $\mathcal{T}$ has a spectral radius $r \geq 1$, where $\mathcal{T}$ is defined from $L^1(\mathbb{T}^d)$ into itself as follows : for any $m \in L^1(\mathbb{T}^d)$, $\mathcal{T}(m)$ is the unique solution of

$$-\nu\Delta\mathcal{T}(m) + \delta\mathcal{T}(m) + \mathcal{T}(m)(x)\Big(\sum_{i=1}^{n}\lambda_i(x)\Big) = \sum_{i=1}^{n} m(x-\xi)\lambda_i(x-\xi) \text{ in } \mathbb{T}^d.$$

The operator $\mathcal{T}$ is compact and

$$\mathcal{T}(\{m \in L^1(\mathbb{T}^d), m \geq 0\}) \subset \{m \in L^1(\mathbb{T}^d), m \geq 0\}.$$

Thus by the Krein-Rutman theorem, there exists $r \geq 1$, $w \in \{m \in L^1(\mathbb{T}^d), m \geq 0\}, w \neq 0$ such that :

$$-\nu\Delta w + \delta w + w\Big(\sum_{i=1}^{n}\lambda_i(x)\Big) = \frac{1}{r}\sum_{i=1}^{n} w(x-\xi)\lambda_i(x-\xi) \text{ in } \mathbb{T}^d.$$

Integrating in space, we obtain that

$$\int_{\mathbb{T}^d}\delta w + \int_{\mathbb{T}^d}\left(1 - \frac{1}{r}\right)(\sum_{i=1}^{n}\lambda_i)w = 0.$$

The two terms of the left hand side are positive so we deduce that $w = 0$, which is a contradiction. Thus $\mu = 0$ and there exists a unique solution of this PDE. □

We now show that there exists a solution of (2.15).

Proposition 2.2.3. *For any $\epsilon > 0$, there exists a unique solution $m \in L^1(\mathbb{T}^d)$ of (2.15). This solution m is positive.*

Proof. We define the application $\mathcal{T}$ from $L^1(\mathbb{T}^d)$ into itself by : for any $m \in L^1(\mathbb{T}^d)$, $\mathcal{T}(m)$ is the unique solution of

$$-\nu\Delta\mathcal{T}(m)+\delta\mathcal{T}(m)+\epsilon^{-1}\mathcal{T}(m)(x)\Big(\sum_{\xi\in K}V(\xi,x)\Big) = \epsilon^{-1}\sum_{\xi\in K} m(x-\xi)V(\xi,x-\xi)+\rho \text{ in } \mathbb{T}^d.$$

Let us observe that if $m \geq 0$, then by the maximum principle $\mathcal{T}(m) \geq 0$. Moreover, if $m \in \Omega$ defined by :

$$\Omega := \{m \in L^1(\mathbb{T}^d), m \geq 0, \int_{\mathbb{T}^d} m(x)\left(\sum_{\xi\in K} V(\xi,x)\right)dx \leq \delta^{-1}\int_{\mathbb{T}^d}\rho\};$$

then $\mathcal{T}(m) \in \Omega$. Indeed, integrating in space the equation which defines $\mathcal{T}(m)$, we obtain that

$$\delta \int_{\mathbb{T}^d} \mathcal{T}(m) + \frac{1}{\epsilon} \int_{\mathbb{T}^d} \mathcal{T}(m) \left(\sum_{\xi \in K} V(\xi) \right) = \int_{\mathbb{T}^d} \rho + \frac{1}{\epsilon} \int_{\mathbb{T}^d} m \left(\sum_{\xi \in K} V(\xi) \right).$$

Thus, Ω is stable by $\mathcal{T}$. Applying Schauder's fixed point theorem, we deduce that $\mathcal{T}$ has a fixed point. The uniqueness is given by the previous proposition. □

Remark 2.2.3. *This result does not depend on the sets on which the particles jump, i.e. it does not depend on the function V except for the fact that $V \geq 0$ and $\sum_{\xi \in K} V(\xi) \leq 1$.*

Existence and uniqueness of the solution of the stationary Fokker-Planck equaiton

We now turn to the existence of a limit density as ϵ goes to 0. We define the following set :

$$\mathcal{AD}(k) := \{v \in H^1(\mathbb{T}^d), \forall \xi \in K, V(t,x,\xi)(v(x) - k(x,\xi) - v(x+\xi)) \leq 0 \text{ on } A\}.$$

A crucial result to pass to the limit is the following :

Lemma 2.2.6. *Assume hypothesis 2.2.5 holds true. Let $m \in H^1(\mathbb{T}^d)$ be such that :*

$$\begin{cases} m \geq 0, \\ \delta \int_{\mathbb{T}^d} m = \int_{\mathbb{T}^d} \rho. \end{cases}$$

Then for any k satisfying (2.3) :

$$||m||^2_{H^1(\mathbb{T}^d)} \leq C \left(-D(k,m) + ||m||_{H^1} ||\rho||_{L^2} \right);$$

where $D(k,m)$ is defined by :

$$D(k,m) = \{\nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla v + \delta \int_{\mathbb{T}^d} vm - \int_{\mathbb{T}^d} \rho v | v \in \mathcal{AD}(k)\}.$$

The following result holds true :

Theorem 2.2.7. *Assume hypothesis 2.2.5 holds true. Then, there exists a unique $m \in H^1(\mathbb{T}^d)$ such that :*

— $\forall \xi \in K : V(\xi, x) m(x) = 0$ almost everywhere in $\mathbb{T}^d$.

— *For any* $u \in H^1(\mathbb{T}^d)$ *which satisfies*

$$\forall \xi \in K, V(\xi, x)\,(k(x,\xi) + u(x+\xi) - u(x)) = 0 \text{ a.e. in } \mathbb{T}^d;$$

the following holds :

$$\forall v \in \mathcal{AD}(k) :$$
$$\nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla (v-u) + \delta \int_{\mathbb{T}^d} m(v-u) \geq \int_{\mathbb{T}^d} \rho(v-u).$$

Proof. The proof of both existence and uniqueness are the exact analogous of the ones in the time dependent case, see proposition 2.A.4 in the appendix for the analogous in the stationary setting of proposition 2.A.2. □

Remark 2.2.4. *Let us note that the question of the uniqueness of solutions is a lot more simpler in the stationary case. Indeed the time regularity is no longer a problem and the existence of solutions for the "adjoint" problem is then easily proved following the argument of the time-dependent case.*

2.2.4 A remark on the generality of this method

Before using this notion of solution of Fokker-Planck equation in a MFG of impulse control, we precise some straightforward generalizations of the results of the previous part. First working on the torus $\mathbb{T}^d$ does not play any role but simplifying the notations and fixing a framework. Thus those results generalize to more complex domain and boundary conditions. Secondly, the cost of jumps k can be allowed to depend on the time variable. If this dependence is smooth, this does not change our results. Also more general jumps can be modeled with this kind of method. For example one can think of impulse control in which any jump is possible ; or a problem of optimal stopping time type, except that instead of leaving, "stopping" the trajectories restarts it at the origin (or at any given point). In this second problem, it is not the jumps but the destination which belongs to a finite set. Both of these optimization problems have value functions which solves QVI under some assumptions. In these two examples continuity of the value function is crucial, hence appropriate assumptions have to be made on the regularity of solutions and a solution of the Fokker-Planck equation may not be more regular than a measure. The important point is that with the QVI comes a notion of "admissible" solutions for the QVI (in this article being admissible is satisfying $u \leq Mu$). With this notion comes the notion of admissible density of particles which is, in this article, the fact that $D(m) > -\infty$. Then a priori estimates on the solutions of the Fokker-Planck equation are available and we can continue developing such

solutions.

Finally, let us note that optimal switching problems can be formulated in terms of QVI, see [10]. Thus we can model a density of particles, whose trajectories are given by optimal trajectories of an optimal switching problem, using the same technique as the one we have just presented in this section.

2.3 Mean field games of impulse control through quasi-variational inequalities

We present in this part an application of the notion of solutions of a Fokker-Planck equation of jumping particles. We study a MFG of impulse control where the density of players is naturally a solution of this kind of equation. We work here in the case of a finite number of possible jumps. We denote by K the set of jumps. We denote by k satisfying (2.3) the cost of the different jumps depending on the position. For any $v \in L^2((0,T), H^1(\mathbb{T}^d))$, we define Mv by

$$Mv(t,x) = \inf_{\xi \in K} \{k(x,\xi) + v(t, x+\xi)\}.$$

We denote by f the running cost of the problem. The function f depends on space, time and on the repartition of the player (i.e. $f = f(t,x,m)$). We make the following assumptions on f :

— f is continuous from $L^2((0,T), H^1(\mathbb{T}^d))$ endowed with its weak topology to $L^2((0,T), H^{-1}(\mathbb{T}^d))$.
— f is uniformly bounded from below by a constant $-C$ (where $C > 0$) on the positive elements of $L^2((0,T), H^1(\mathbb{T}^d))$
— f maps $L^2((0,T), H^1(\mathbb{T}^d))$ to a bounded set of $L^p((0,T) \times \mathbb{T}^d)$ with $p > d$.

Remark 2.3.1. *The last assumption on f can be replaced by any assumption which yields a uniform bound by above for the solution of the heat equation with source $f(m)$.*

As in the previous part, we work on the d dimensional torus to simplify the notations but all the following results are adaptable to more complex situations. We once again use the notation

$$H = \{v \in L^2((0,T), H^1(\mathbb{T}^d)), \partial_t v \in L^2((0,T), H^{-1}(\mathbb{T}^d))\}.$$

The problem we are interested in, is finding (u,m) such that :

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f(m), u - Mu) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ u(T) = 0 \text{ in } \mathbb{T}^d; \\ D(m) > -\infty; \\ \forall v \in H, v \leq Mv, v(T) = 0 : \\ \int_0^T (-\partial_t(v-u) - \nu\Delta(v-u), m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d}(v-u)(0)m_0 \geq 0; \\ \int_0^T \int_{\mathbb{T}^d}(-\partial_t u - \nu\Delta u - f(m))m = 0; \end{cases} \tag{2.17}$$

where $D(m)$ is defined by

$$D(m) := \inf_{v\in H_{ad}} \int_0^T (-\partial_t v - \nu\Delta v, m)_{H^{-1}\times H^1} - \int_{\mathbb{T}^d} m_0 v(0); \tag{2.18}$$

where $H_{ad} := \{v \in H, v \leq Mv, v(T) = 0\}$. The function u denotes the value function of the impulse control problem for a generic player of the MFG and m is the density of players.The first two lines have to be taken in the sense that u is the solution of the associated QVI. Thus that it formally solves the impulse control problem for the generic player in which the running cost is $f(m)$ and k is the cost for the jumps. We refer to the appendix for some details on QVI and to [10] for a complete study of the problem. In view of the previous part, m is a solution of a Fokker -Planck equation which models the density of (jumping) players of the game. Let us note that because there is no constraint such that $m = 0$ on $\{u = Mu\}$, m is not necessary the solution of a limit problem of the previous part with some V (describing the jumps) well chosen depending on u. Indeed, in view of [11], we do not expect the existence of solutions if we impose such strong conditions which are assimilated with Nash equilibria in pure strategies for the MFG. That is why we do not impose the condition $m = 0$ on $\{u = Mu\}$ but the integral relation of the last line. It is the formulation of the fact that (u, m) is a mixed solution of the MFG, i.e. that this system characterizes Nash equilibria in mixed strategies. We recall the interpretation of such a relation.

Formally, a natural requirement for the solution of such a MFG shall be to impose that $m = 0$ on $\{u = Mu\}$ which is the set where it is optimal to use an impulse control. The integral formulation in this system requires that $m = 0$ on $\{-\partial_t u - \Delta u < f(m)\}$ which is the set where it is strictly optimal to use an impulse control. The difference here is that on $\{u = Mu\}$ one can still have $-\partial_t u - \Delta u = f(m)$ and thus that it is both optimal to stay and to use a control. We do not impose that m vanishes in such a situation. Such a relaxation makes the problem more convex and allows us to prove an existence result while still conserving a uniqueness property.

The methodology to work on (2.17) is the following : we first introduce a penalized version of this problem and then we show how we can pass to the limit to obtain the existence of solutions of (2.17). Later on we prove a result of uniqueness for such solutions.

2.3.1 The penalized problem

We introduce here the penalized problem :

$$\begin{cases} \max(-\partial_t u - \nu\Delta u - f(m), u - Mu) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ u(T) = 0 \text{ in } \mathbb{T}^d; \\ \partial_t m - \nu\Delta m = \displaystyle\sum_{\xi\in K} V(t,x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_{\{u=Mu\}}(t,x-\xi)m(t,x-\xi) \\ \qquad\qquad - \dfrac{1}{\epsilon}(\displaystyle\sum_{\xi\in K} V)\mathbb{1}_{\{u=Mu\}}m \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d; \\ \sum_{\xi\in K} V \geq 1 \text{ on } \{-\partial_t u - \nu\Delta u < f(m)\}; \\ \forall \xi\in K, u(t,x) \neq k(t,x,\xi) + u(t,x+\xi) \Rightarrow V(t,x,\xi) = 0; \\ \forall \xi\in K, \forall (t,x)\in(0,T)\times\mathbb{T}^d, 0 \leq V(t,x,\xi) \leq 1. \end{cases} \tag{2.19}$$

Let us remark that we do not impose that the sum over $\xi \in K$ of the functions V is bounded by 1. Up to a change of the constant ϵ, this does not play any role in the following. The important part is that this sum is bounded from below by a non negative constant on the set $\{-\partial_t u - \nu\Delta u < f(m)\}$. This means that players are actually jumping when it is strictly suboptimal to stay where they are.

Recalling the previous part, it is natural to introduce first such a penalized system, and then pass to the limit ϵ goes to 0. Indeed the equation satisfied by m cannot be easily written in terms of a partial differential equation whereas it can at a penalized level. The potential V gives at each point (t,x) the jump used by the players at this point. If $V(t,x,\xi) \neq 0$ then some players use the jump ξ at (t,x). The way V is defined adds convexity to the problem and makes possible the existence of a solution ; it is the translation at a penalized level that we are looking for Nash equilibria of the MFG in mixed strategies. The technique here is inspired from [11] where it is shown that such a system, for variational inequalities instead of quasi variational inequalities, leads to a solution of the MFG system for optimal stopping.

Theorem 2.3.1. *There exists a solution* (u, m, V) *of* (2.19) *such that* $(u, m, V) \in (L^2((0,T), H^2(\mathbb{T}^d)) \cap H^1((0,T), L^2(\mathbb{T}^d))) \times L^2((0,T), H^1(\mathbb{T}^d)) \times \mathbb{T}^d) \times L^\infty(K \times (0,T) \times \mathbb{T}^d)$ *of* (2.19) *where that* u *solves the associated QVI and the equation in* m *is satisfied in a weak sense.*

Proof. We define an application $\mathcal{F}_1$ from $L^2((0,T), H^1(\mathbb{T}^d))$ (endowed with its weak topology) into itself (endowed with its strong topology) by : for any $m \in L^2((0,T), H^1(\mathbb{T}^d))$, $\mathcal{F}_1(m)$ is the unique solution of the QVI with costs $f(m)$ and k. $\mathcal{F}_1$ is well defined and continuous from well known results on QVI (see appendix). Then, we define the set-valued function $\mathcal{F}_2$ from $L^2((0,T) \times \mathbb{T}^d)$ to $L^2((0,T), H^1(\mathbb{T}^d))$ (endowed with its weak topology) by : for any $\mu \in L^2$, (we define $u = \mathcal{F}_1(\mu)$) :

$$\mathcal{F}_2(\mu) := \{m \in L^2((0,T), H^1(\mathbb{T}^d)), \exists V \in L^\infty, (m, V) \text{ solves } (2.19)(u)\};$$

where $(2.19)(u)$ is

$$(2.19)(u) \begin{cases} \partial_t m - \nu \Delta m - \sum_{\xi \in K} V(t, x-\xi, \xi) \frac{1}{\epsilon} \mathbb{1}_{\{u = Mu\}}(t, x-\xi) m(t, x-\xi) + \\ \qquad + \frac{1}{\epsilon}(\sum_{\xi \in K} V) \mathbb{1}_{\{u=Mu\}} m = 0 \text{ in } (0,T) \times \mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d; \\ \sum_{\xi \in K} V \geq 1 \text{ on } \{-\partial_t u - \nu \Delta u < f(\mu)\}; \\ \forall \xi \in K, u(t,x) \neq k(x,\xi) + u(t, x+\xi) \Rightarrow V(t,x,\xi) = 0; \\ 0 \leq V(t,x,\xi) \leq 1. \end{cases}$$

Let us note that for any $\mu \in L^2$, $\mathcal{F}_2(\mu) \neq \emptyset$ because of the results of the first part. As we want to apply Kakutani's fixed point theorem on $\mathcal{F}_2$, we need to prove that $\mathcal{F}_2$ is upper semicontinuous and that it is valued in the set of convex and closed subsets of $L^2((0,T), H^1(\mathbb{T}^d))$. The last point comes easily from the linearity of the equation in m. We focus in this proof on the upper semicontinuity. We recall that a set valued function F from A to B is upper semicontinuous if for any open set $\mathcal{O} \subset B$, $\{x \in A, F(x) \subset \mathcal{O}\}$ is open in A. Let us take an open set $\mathcal{O} \subset L^2$ and $m \in L^2((0,T), H^1(\mathbb{T}^d))$ such that $\mathcal{F}(m) \subset \mathcal{O}$. In view of the previous part (namely lemma 2.2.4), we know that that $\mathcal{F}_2(m)$ is a compact subset of $L^2((0,T), H^1(\mathbb{T}^d))$ endowed with its weak topology, hence

$$dist(\mathcal{F}_2(m), \mathcal{O}^c) > 0.$$

Let us take $\delta > 0$ and $m' \in L^2((0,T), H^1(\mathbb{T}^d))$ such that $||m - m'||_{L^2(H^1)} \leq \delta$. We now prove that if δ is small enough, then $\mathcal{F}_2(m') \subset \mathcal{O}$, and thus that $\mathcal{F}_2$ is upper

semi continuous. First, we define u_1 and u_2 by

$$u_1 = \mathcal{F}_1(m); u_2 = \mathcal{F}_1(m').$$

For any $\mu_2 \in \mathcal{F}_2(m')$, there exists V_2 such that (μ_2, V_2) solves

$$\begin{cases} \partial_t \mu_2 - \nu\Delta\mu_2 - \sum_{\xi\in K} V_2(t, x-\xi, \xi)\frac{1}{\epsilon}\mathbb{1}_{\{u_2 = Mu_2\}}(t, x-\xi)\mu_2(t, x-\xi) + \\ \qquad + \frac{1}{\epsilon}(\sum_{\xi\in K} V)\mathbb{1}_{\{u_2=Mu_2\}}\mu_2 = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ \mu_2(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases}$$

Now we define V_1 by

— $\forall(t, x, \xi)$ such that $u_i(t, x) = k(x, \xi) + u_i(t, x, x + \xi)$, for $i \in \{1; 2\}$, then

$$V_1(t, x, \xi) = V_2(t, x, \xi);$$

— $\forall(t, x, \xi)$ such that $u_1(t, x) = k(x, \xi)+u_1(t, x+\xi)$ and $-\partial_t u_1-\nu\Delta u_1 = f(m)$:

$$V_1(t, x, \xi) = V_2(t, x, \xi)$$

— $\forall(t, x, \xi)$ such that $u_1(t, x) = k(x, \xi)+u_1(t, x+\xi)$ and $-\partial_t u_1-\nu\Delta u_1 < f(m)$

$$V_1(t, x, \xi) = 1$$

— and $V_1 = 0$ elsewhere.

Let us define μ_1 as the unique solution of

$$\begin{cases} \partial_t \mu_1 - \nu\Delta\mu_1 - \sum_{\xi\in K} V_1(t, x-\xi, \xi)\frac{1}{\epsilon}\mathbb{1}_{\{u_1 = Mu_1\}}(t, x-\xi)\mu_1(t, x-\xi) + \\ \qquad + \frac{1}{\epsilon}(\sum_{\xi\in K} V)\mathbb{1}_{\{u_1=Mu_1\}}\mu_1 = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ \mu_1(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases}$$

By construction, $\mu_1 \in \mathcal{F}_2(m)$. We then define $\mu := \mu_1-\mu_2$. Once again by construc-

tion, $\mu(0) = 0$ and μ satisfies

$$\begin{aligned}
&\partial_t \mu - \nu\Delta\mu + \frac{1}{\epsilon}\mathbb{1}_A(\sum_{\xi\in K} V_2)\mu - \sum_{\xi\in K} V_2(t,x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_A(t,x-\xi)\mu(t,x-\xi)\\
&= \frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}\cap\{u_1<Mu_1\}}(\sum_{\xi\in K} V_2)\mu_2\\
&- \sum_{\xi\in K} V_2(t,x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}\cap\{u_1<Mu_1\}}(t,x-\xi)\mu_2(t,x-\xi)\\
&+ \frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}\cap\{u_1=Mu_1\}\cap A^c}(\sum_{\xi\in K} V_2)\mu_2\\
&- \sum_{\xi\in K} V_2(t,x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}\cap\{u_1=Mu_1\}\cap A^c}(t,x-\xi)\mu_2(t,x-\xi)\\
&- \frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}\cap\{u_1=Mu_1\}\cap A^c}(\sum_{\xi\in K} V_1)\mu_1\\
&+ \sum_{\xi\in K} V_1(t,x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}\cap\{u_1=Mu_1\}\cap A^c}(t,x-\xi)\mu_1(t,x-\xi)\\
&- \frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}^c\cap\{P(u_2)=f(m')\}\cap\{P(u_1)<f(m)\}\cap A^c}(\sum_{\xi\in K} V_1)\mu_1\\
&+ \sum_{\xi\in K} V_1(t,x-\xi,\xi)\frac{1}{\epsilon}\mathbb{1}_{\{u_2=Mu_2\}^c\cap\{P(u_2)=f(m')\}\cap\{P(u_1)<f(m)\}\cap A^c}(t,x-\xi)\mu_1(t,x-\xi).
\end{aligned}$$

Where we have used for A the set of coincidence defined by :

$$\Big\{(t,x)/\forall\xi, V_2(t,x,\xi)\mathbb{1}_{\{u_2=Mu_2\}}(t,x) = V_1(t,x,\xi)\mathbb{1}_{\{u_1=Mu_1\}}(t,x)\Big\}.$$

The operator P is here defined by being the parabolic operator :

$$P(u) := -\partial_t u - \nu\Delta u.$$

Let us remark that all the terms in the right hand side of this expression are the ones which involve different coefficients (i.e. different V) in front of μ_1 and μ_2. We can also note that all the terms of the right hand side are multiplied by a characteristic function of a subsets whose Lebesgue measure goes to 0 as $||m'-m||_{L^2(H^1)}$ goes to 0. Indeed, because $\mathcal{F}_1$ is continuous,

$$||u_2 - u_1||_{L^2((0,T),H^1(\mathbb{T}^d))} \to_{\delta\to 0} 0.$$

Hence, taking δ sufficiently small, we obtain that the Lebesgue measure of the following sets are as small as we want :

— for all $\xi_1 \neq \xi_2 \in K$:

$$\{u_1(t,x) = k(x,\xi_1) + u_1(t,x+\xi_1)\} \cap \{u_1(t,x) < k(x,\xi_2) + u_1(t,x+\xi_2)\} \cap$$
$$\cap \{u_2(t,x) = k(x,\xi_2) + u_2(t,x+\xi_2)\} \cap \{u_2(t,x) < k(x,\xi_1) + u_2(t,x+\xi_1)\}$$

— $\{P(u_1) < f(m)\} \cap \{P(u_2) = f(m')\}$
— $\{u_1 < Mu_1\} \cap \{u_2 = Mu_2\}$.

Thus because μ_1 and μ_2 are bounded in $L^2((0,T)\times\mathbb{T}^d)$ independently of δ (lemma 2.2.4), we deduce that taking δ sufficiently small, the right hand side of the previous equation is as small as necessary in $L^2((0,T)\times\mathbb{T}^d)$. Thus, we fix $\eta > 0$ and we choose δ such that the right hand side is smaller than η in the $L^2((0,T)\times\mathbb{T}^d)$ norm. Multiplying by μ and integrating over $\mathbb{T}^d$ the equation μ satisfies, we deduce that

$$\frac{1}{2}\frac{d}{dt}||\mu(t)||^2_{L^2(\mathbb{T}^d)} \leq \frac{1}{\epsilon}\#(K)||V||_{L^\infty}||\mu||^2_{L^2(\mathbb{T}^d)} + \eta||\mu||_{L^2(\mathbb{T}^d)};$$

where $\#(K)$ stands for the cardinal of the set K. From this inequality, it follows that

$$\frac{d}{dt}||\mu(t)||_{L^2(\mathbb{T}^d)} \leq \frac{1}{\epsilon}\#(K)||V||_{L^\infty}||\mu||_{L^2(\mathbb{T}^d)} + \eta.$$

Hence, we conclude with Gronwall's lemma that taking η small enough, μ is as small as necessary in $L^\infty((0,T),L^2(\mathbb{T}^d))$ (we recall that $\mu(0)=0$). Because of the partial differential equations satisfied by μ, it follows that taking δ small enough :

$$||\mu||_{L^2((0,T),H^1(\mathbb{T}^d))} < \frac{dist(\mathcal{F}_2(m),\mathcal{O}^c)}{2};$$

which proves that $\mu_2 \in \mathcal{O}$ and thus that $\mathcal{F}_2(m') \subset \mathcal{O}$. Hence $\mathcal{F}_2$ is upper semi continuous and we deduce from Kakutani's fixed point theorem the existence of a solution. □

2.3.2 Existence of solutions of the MFG system

In this section we discuss the existence of solutions of the MFG system (2.17). The proof of this result consists in passing to the limit in the penalized system. Let us remark that in the first part of this article, we use either hypothesis 2.2.1 or 2.2.3 to pass to the limit $\epsilon \to 0$ in a Fokker-Planck equation of jumping particles. Here such an assumption is no more required, as the jumps the players are using, are by definition optimal for a certain optimization problem. Obviously this problem is the optimization problem the players have to solve.

Theorem 2.3.2. *There exists a solution* $(u,m) \in L^2((0,T),H^2(\mathbb{T}^d)) \cap H^1((0,T),L^2(\mathbb{T}^d)) \times L^2((0,T),H^1(\mathbb{T}^d))$ *of* (2.17).

Proof. For $\epsilon > 0$ we denote by $(u_\epsilon, m_\epsilon, V_\epsilon)$ a solution of the penalized system (2.19). We first show some compactness for the sequence $(m_\epsilon)_{\epsilon>0}$. Let us remark that

$$D(m_\epsilon) = \int_0^T \int_{\mathbb{T}^d} (-\partial_t u_\epsilon - \nu \Delta u_\epsilon) m_\epsilon - \int_{\mathbb{T}^d} u_\epsilon(0) m_0.$$

Furthermore, because of lemma 2.2.4,

$$-D(m_\epsilon) + C_\epsilon ||m_0||_{L^2} ||m_\epsilon||_{L^2((0,T),H^1)} \geq ||m_\epsilon||^2_{L^2((0,T),H^1)};$$

where C_ϵ only depends on $||u_\epsilon||_{L^\infty}$ (and on k which is fixed here). We then deduce from the assumptions we made on f, that there exists C independent of ϵ such that :

$$\begin{aligned} ||m_\epsilon||^2_{L^2((0,T),H^1)} \leq & \int_0^T \int_{\mathbb{T}^d} (\partial_t u_\epsilon + \nu \Delta u_\epsilon) m_\epsilon + \int_{\mathbb{T}^d} u_\epsilon(0) m_0 \\ & + C ||m_0||_{L^2} ||m_\epsilon||_{L^2((0,T),H^1)}; \end{aligned}$$

Thus we deduce from the assumptions we made on f that $(m_\epsilon)_{\epsilon>0}$ is a bounded sequence of $L^2((0,T), H^1(\mathbb{T}^d))$. So there exists $m \in L^2((0,T), H^1(\mathbb{T}^d))$ such that, extracting a subsequence if necessary, $(m_\epsilon)_\epsilon$ weakly converges toward m in $L^2((0,T), H^1(\mathbb{T}^d))$. Because f is continuous for the weak topology, we deduce from lemma 2.A.1 (see appendix), that $(u_\epsilon)_\epsilon$ converges toward u solution of the quasi variational inequality associated to

$$\begin{cases} \max(-\partial_t u - \nu \Delta u - f(m), u - Mu) = 0 \text{ in } (0,T) \times \mathbb{T}^d; \\ u(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Moreover, by passing to the limit in

$$\begin{aligned} &\forall v \in H, v \leq Mv, v(T) = 0 : \\ &\int_0^T (-\partial_t v - \nu \Delta v, m_\epsilon)_{H^{-1} \times H^1} - \int_{\mathbb{T}^d} v(0) m_0 \geq \int_0^T (-\partial_t u_\epsilon - \nu \Delta u_\epsilon, m_\epsilon)_{H^{-1} \times H^1} \\ &\qquad - \int_{\mathbb{T}^d} u_\epsilon(0) m_0; \\ &\qquad = D(m_\epsilon); \end{aligned} \tag{2.20}$$

we obtain

$$\begin{cases} D(m) > -\infty; \\ \forall v, v \leq Mv, v(T) = 0 : \\ \int_0^T (-\partial_t v - \nu \Delta v, m)_{H^{-1} \times H^1} - \int_{\mathbb{T}^d} v(0) m_0 \geq \int_0^T (-\partial_t u - \nu \Delta u, m)_{H^{-1} \times H^1} \\ \qquad - \int_{\mathbb{T}^d} u(0) m_0 \\ \qquad = D(m). \end{cases}$$

Let us note that we can pass to the limit in the right hand side of (2.20) because the uniform bounds on f yields some uniform Hölder estimates in time for u_ϵ. Thus, we can easily deduce that extracting a subsequence if necessary :

$$u_\epsilon(0) \xrightarrow[\epsilon\to 0]{L^2} u(0);$$

$$\int_0^T \int_{\mathbb{T}^d} m_\epsilon \partial_t u_\epsilon \xrightarrow[\epsilon\to 0]{} \int_0^T \int_{\mathbb{T}^d} (\partial_t u, m)_{H^{-1}\times H^1}.$$

Now let us remark that for all $\epsilon > 0$,

$$\begin{aligned}
-D(m_\epsilon) &= \frac{1}{\epsilon}\int_0^T \int_{\mathbb{T}^d} (\sum_{\xi\in K} V_\epsilon(t,x,\xi)k(x,\xi)) m_\epsilon(t,x)dxdt;\\
&\geq \frac{1}{\epsilon}\int_0^T \int_{\mathbb{T}^d} (\sum_{\xi\in K} V_\epsilon(t,x,\xi)) k_0 m_\epsilon(t,x)\mathbb{1}_{\{-\partial_t u_\epsilon - \nu\Delta u_\epsilon < f(m_\epsilon)\}}(t,x)dxdt;\\
&\geq \frac{k_0}{\epsilon}\int_0^T \int_{\mathbb{T}^d} m_\epsilon \mathbb{1}_{\{-\partial_t u_\epsilon - \nu\Delta u_\epsilon < f(m_\epsilon)\}},
\end{aligned}$$

where we have used that

$$\sum_{\xi\in K} V_\epsilon \geq 1 \text{ on } \{-\partial_t u_\epsilon - \nu\Delta u_\epsilon < f(m_\epsilon)\}.$$

Since $(D(m_\epsilon))_{\epsilon>0}$ is a bounded sequence (c.f. (2.20)), we deduce that

$$\Big(\frac{k_0}{\epsilon}\int_0^T \int_{\mathbb{T}^d} m_\epsilon \mathbb{1}_{\{-\partial_t u_\epsilon - \nu\Delta u_\epsilon < f(m_\epsilon)\}}\Big)_{\epsilon>0}$$

is also a bounded sequence and thus that :

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u - f(m))m = 0.$$

This ends the proof of the fact that (u, m) is a solution of (2.17).

□

2.3.3 Uniqueness of solutions of the MFG system

We now turn to the question of the uniqueness of solutions of (2.17). As it is the case in MFG of continuous control [54], uniqueness does not hold in general. However it does under an assumption on the monotonicity of the costs of the MFG (i.e. the coupling) with respect to the density of players. In our model the density of players appears only in the running cost f and thus only an assumption on f is required for uniqueness to hold. We recall that f is said to be strictly monotone if :

$$\int_0^T \int_{\mathbb{T}^d} (f(m_1) - f(m_2))(m_1 - m_2) > 0 \text{ if } m_1 \neq m_2.$$

Theorem 2.3.3. *Assume that f is strictly monotone, then there exists at most one solution of* (2.17).

Proof. The proof of this statement is directly inspired from the original proof of uniqueness in MFG of continuous control [54]. Let us take (u_1, m_1) and (u_2, m_2) two solutions of (2.17). We denote by u and m the differences $u_1 - u_2$ and $m_1 - m_2$. Let us observe that because of the optimality of u_1 in $D(m_1)$ and similarly for u_2 in $D(m_2)$ we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u)m = \int_0^T \int_{\mathbb{T}^d} (-\partial_t(u_1 - u_2) - \nu\Delta(u_1 - u_2))m_1 \\ + \int_0^T \int_{\mathbb{T}^d} (-\partial_t(u_2 - u_1) - \nu\Delta(u_2 - u_1))m_2; \\ \leq 0.\end{aligned}$$

On the other hand, because $m_2 \geq 0$, and

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t u_1 - \nu\Delta u_1 - f(m_1))m_1 = 0;$$

we deduce that

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} (-\partial_t u_1 - \nu\Delta u_1)m = \int_0^T \int_{\mathbb{T}^d} f(m_1)m_1 + m_2(\partial_t u_1 + \nu\Delta u_1); \\ \geq \int_0^T \int_{\mathbb{T}^d} f(m_1)(m_1 - m_2).\end{aligned}$$

Where we have used the fact that $-\partial_t u_1 - \nu\Delta u_1 \leq f(m_1)$. Obviously we have the analogous relation for u_2. Putting the pieces together we finally obtain

$$\int_0^T \int_{\mathbb{T}^d} (f(m_1) - f(m_2))(m_1 - m_2) \leq \int_0^T \int_{\mathbb{T}^d} (-\partial_t u - \nu\Delta u)m \leq 0.$$

Using the strict monotonicity of f, we have just proven that $m_1 = m_2$ and thus that there exists at most one solution of (2.17).

□

2.3.4 The stationary setting

In this section, we present a stationary setting for a MFG of impulse control. We denote by k the cost of jumps and we assume that it satisfies (2.3). We denote by f the running cost for the players. We assume that

— f is continuous from $H^1(\mathbb{T}^d)$ endowed with its the weak topology to $H^{-1}(\mathbb{T}^d)$.

— f is uniformly bounded from below by a constant $-C$ (where $C > 0$) on the positive elements of $H^1(\mathbb{T}^d)$.

— f maps $H^1(\mathbb{T}^d)$ into a bounded subset of $L^d(\mathbb{T}^d)$.
We denote by $\delta > 0$ the death rate of the players and by $\lambda > 0$ their intertemporal preference rate. We denote by $\rho \in L^2(\mathbb{T}^d)$, $\rho \geq 0$ the entry rate of the players. The jump operator M is defined by :

$$Mu(x) = \inf_{\xi \in K} k(x, \xi) + u(x + \xi).$$

We are interested in the following MFG system :

$$\begin{cases} \max(-\nu\Delta u + \lambda u - f(m), u - Mu) = 0 \text{ in } \mathbb{T}^d; \\ \forall v \in H^1(\mathbb{T}^d), v \leq Mv : \\ \nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla(v - u) + \delta \int_{\mathbb{T}^d} m(v - u) \geq \int_{\mathbb{T}^d} \rho(v - u); \\ \int_{\mathbb{T}^d} (-\nu\Delta u + \lambda u - f(m))m = 0. \end{cases} \tag{2.21}$$

The following result holds true :

Theorem 2.3.4. *There exists a solution $(u, m) \in H^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$ of (2.21). It is unique under the assumption that f is strictly monotone.*

We do not detail the proof of this result as its argument follows step by step the one of the time dependent case, namely by passing to the limit in the following penalized system :

$$\begin{cases} \max(-\nu\Delta u + \lambda u - f(m), u - Mu) = 0 \text{ in } \mathbb{T}^d; \\ -\nu\Delta m + \delta m = \rho + \sum_{\xi \in K} V(x - \xi, \xi) \frac{1}{\epsilon} \mathbb{1}_{\{u = Mu\}}(x - \xi) m(x - \xi) \\ \qquad\qquad - \frac{1}{\epsilon} \mathbb{1}_{\{u = Mu\}} m \text{ in } \mathbb{T}^d; \\ \sum_{\xi \in K} V \geq 1 \text{ on } \{-\nu\Delta u + \lambda u < f(\mu)\}; \\ \forall \xi \in K, u(x) \neq k(x, \xi) + u(x + \xi) \Rightarrow V(x, \xi) = 0; \\ \forall \xi \in K, \forall (x) \in (0, T) \times \mathbb{T}^d, 0 \leq V(x, \xi) \leq 1; \end{cases}$$

using lemma (2.2.6).

2.3.5 The optimal control interpretation

In this section, we present an optimal control interpretation of a MFG of impulse control. We do not make use of this interpretation to prove the existence of a solution of the MFG. We just show that a certain optimization problem has a solution, and that the solution of the MFG, for which we have proven the existence in the previous part, is the solution of this optimization problem. To make

this section simpler we work only in the stationary setting. The case of the time dependent problem is mentioned at the end of this section. Let us note that the optimal control interpretation of MFG has been introduced in [54] and that it can be used to show the existence of solutions for certain MFG system, see for instance [28].

We denote by $\mathcal{F}$ a strictly convex function from $H^1(\mathbb{T}^d)$ to $\mathbb{R}$, bounded from below. We assume that there exists f, satisfying the requirements of the previous section, such that for any $m, m' \in H^1(\mathbb{T}^d)$:

$$\lim_{\theta \to 0} \frac{\mathcal{F}((1-\theta)m + \theta m') - \mathcal{F}(m)}{\theta} = \int_{\mathbb{T}^d} f(m)(m' - m).$$

We define the following application from $H^1(\mathbb{T}^d)$ to $\mathbb{R}_- \cup \{-\infty\}$:

$$D(m) = \inf\{\nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla v + \delta \int_{\mathbb{T}^d} mv - \int_{\mathbb{T}^d} \rho v | v \in H^1(\mathbb{T}^d), v \leq Mv\}.$$

The optimal control interpretation of the MFG of impulse control leads to the following optimization problem :

$$\inf_{m \in \mathcal{H}} \mathcal{F}(m) - D(m) \tag{2.22}$$

where $\mathcal{H} = \{m \in H^1(\mathbb{T}^d), m \geq 0\}$. We now establish the following result :

Proposition 2.3.1. *Under the previous assumptions on $\mathcal{F}$, the problem (2.22) admits a unique minimizer $m^* \in \mathcal{H}$.*

Proof. We first prove that the function $D(\cdot)$ is concave and upper semi continuous on $\mathcal{H}$. Assume that the sequence $(m_n)_{n \geq 0}$ converges weakly toward m in $\mathcal{H}$. Let us remark that because $\mathcal{F}$ is bounded from below : $D(m) > -\infty$. Moreover for any $v \in H^1(\mathbb{T}^d), v \leq Mv$ we obtain

$$\nu \int_{\mathbb{T}^d} \nabla(m_n - m) \cdot \nabla v + \delta \int_{\mathbb{T}^d} (m_n - m)v \to_{n \to \infty} 0.$$

Thus taking a sequence $(v_p)_{p \geq 0}$ in $H^1(\mathbb{T}^d)$ such that

$$\nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla v_p + \delta \int_{\mathbb{T}^d} m v_p - \int_{\mathbb{T}^d} \rho v_p \leq D(m) + \frac{1}{p+1};$$

we deduce that

$$\begin{aligned} D(m_n) &\leq \nu \int_{\mathbb{T}^d} \nabla m_n \cdot \nabla v_p + \delta \int_{\mathbb{T}^d} m_n v_p - \int_{\mathbb{T}^d} \rho v_p; \\ &\to_{n \to \infty} \nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla v_p + \delta \int_{\mathbb{T}^d} m v_p - \int_{\mathbb{T}^d} \rho v_p; \\ &\leq D(m) + \frac{1}{p}. \end{aligned}$$

Thus $D(\cdot)$ is upper semi continuous, it is obviously concave. Now let $(m_n)_{n\geq 0}$ be a minimizing sequence of (2.22). If we denote by μ the solution of

$$-\nu\Delta\mu + \delta\mu = \rho;$$

then we observe that $\mu \in \mathcal{H}$ and $D(\mu) = 0$. Thus we deduce that

$$0 \leq -D(m_n) \leq \mathcal{F}(\mu) - \inf \mathcal{F}.$$

Recalling lemma 2.2.6, $(m_n)_{n\geq 0}$ is thus a bounded sequence of $H^1(\mathbb{T}^d)$. Thus it converges weakly to $m^* \in H^1(\mathbb{T}^d)$. Because $\mathcal{F}$ is weakly sequentially lower semi continuous (it is continuous and convex) and $D(\cdot)$ is weakly sequentially upper semi continuous, we deduce that m^* is a minimizer of (2.22). This minimizer is unique because $\mathcal{F}$ is strictly convex and D is concave. □

Now let us remark that the solution (u, m) of the MFG system (2.21) (with $\lambda = \delta$) given by theorem 2.3.4 satisfies the following system of variational inequalities :

$$\begin{cases} \forall \mu \in H^1(\mathbb{T}^d), \mu \geq 0 : \\ \int_{\mathbb{T}^d}(-\nu\Delta u + \delta u - f(m))(\mu - m) \leq 0; \\ \forall v \in H^1(\mathbb{T}^d), v \leq Mv : \\ \nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla(v-u) + \delta \int_{\mathbb{T}^d} m(v-u) \geq \int_{\mathbb{T}^d} \rho(v-u). \end{cases} \tag{2.23}$$

This system of variational inequalities is the characterization of a saddle point of (2.22). From this observation we deduce the following :

Theorem 2.3.5. *The unique minimizer of (2.22) is the density of players m of the MFG of impulse control. This is if (u, m) is the solution of (2.21) given by theorem 2.3.4, then m is the unique minimizer of (2.22).*

Proof. We denote by (u, m) the unique solution of (2.21). For any $m' \in \mathcal{H}$, $0 < \theta < 1$, using the second variational inequality of (2.23) we deduce that :

$$\begin{aligned} &\mathcal{F}\left((1-\theta)m + \theta m'\right) - D\left((1-\theta)m + \theta m'\right) - \mathcal{F}(m) + D(m) \\ &= \mathcal{F}\left((1-\theta)m + \theta m'\right) - D\left((1-\theta)m + \theta m'\right) - \mathcal{F}(m) \\ &\qquad + \nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla u + \delta \int_{\mathbb{T}^d} mu - \int_{\mathbb{T}^d} \rho u. \end{aligned}$$

Using the definition of $D(\cdot)$, we now obtain that :

$$\begin{aligned} &\mathcal{F}\left((1-\theta)m + \theta m'\right) - D\left((1-\theta)m + \theta m'\right) - \mathcal{F}(m) + D(m) \\ &\geq \mathcal{F}\left((1-\theta)m + \theta m'\right) - \mathcal{F}(m) - \nu \int_{\mathbb{T}^d} \nabla\left((1-\theta)m + \theta m'\right) \cdot \nabla u \\ &\qquad -\delta \int_{\mathbb{T}^d} ((1-\theta)m + \theta m')u + \int_{\mathbb{T}^d} \rho u + \nu \int_{\mathbb{T}^d} \nabla m \cdot \nabla u + \delta \int_{\mathbb{T}^d} mu - \int_{\mathbb{T}^d} \rho u; \\ &= \mathcal{F}\left((1-\theta)m + \theta m'\right) - \mathcal{F}(m) - \theta \int_{\mathbb{T}^d} (-\nu\Delta u + \delta u)(m' - m). \end{aligned}$$

Hence, using the first variational inequality of (2.23), we obtain that :

$$\lim_{\theta \to 0^+} \frac{\mathcal{F}\left((1-\theta)m + \theta m'\right) - D\left((1-\theta)m + \theta m'\right) - \mathcal{F}(m) + D(m)}{\theta} \geq 0.$$

We deduce from the previous line that m is a local minimum of the functional $\mathcal{F}(\cdot) - D(\cdot)$, because this functional is strictly convex, m is the unique minimizer of (2.22). □

The time dependent case

We only indicate the optimal control interpretation in the time dependent case. We do not give any proofs are precise statements as they are the exact analogous of the one we just establish in the stationary setting. In a time dependent setting with time horizon $T > 0$ and initial condition $m_0 \in L^2(\mathbb{T}^d), m_0 \geq 0$, let us assume that there exists $\mathcal{F}$ such that for any $m, m' \in L^2((0,T), H^1(\mathbb{T}^d))$:

$$\lim_{\theta \to 0} \frac{\mathcal{F}((1-\theta)m + \theta m') - \mathcal{F}(m)}{\theta} = \int_0^T \int_{\mathbb{T}^d} f(m)(m' - m).$$

Then the optimal control problem associated with (2.17) is :

$$\inf_{m \in \mathcal{H}} \mathcal{F}(m) - D(m).$$

where $\mathcal{H} := \{m \in L^2((0,T), H^1(\mathbb{T}^d)), m \geq 0\}$ and $D(m)$ is defined by (2.18).

Appendix

2.A Results on the impulse control problem

The problem of impulse control is classical, we refer to the book of A. Bensoussan and J.-L. Lions [10] for a more complete presentation of the problem. The first part of this appendix is dedicated to time dependent quasi-variational inequalities(QVI), the second one to stationary QVI. We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

2.A.1 The time dependent setting

In this time dependent setting, we fix a final time T. The problem of impulse control consists in minimizing the following expectation :

$$\inf_{(\tau_i)_i, (\xi_i)_i} \mathbb{E}[\int_0^T f(s, X_s) ds + \sum_{i=1}^{\#(\tau_j)_j} k(\tau_i, X_{\tau_i^-}, \xi_i)]; \tag{2.24}$$

where the infimum is taken over the (finite and infinite) sequences $(\tau_i)_i$ of times such that $0 \leq \tau_i < \tau_{i+1}$ and over the sequences $(\xi_i)_i$ valued in the finite set K. The (random) sequences $(\tau_i)_i$ and $(\xi_i)_i$ are measurable with respect to the σ-algebra generated by the process $(X_s)_{s\geq 0}$, which is defined below in (2.25). The function $f \in L^2((0,T), \mathbb{T}^d)$ denotes the running cost and $k \in L^\infty((0,T) \times \mathbb{T}^d \times K)$ denotes the cost of the jumps (i.e. $k(t,x,\xi)$ is the cost paid to use the jump ξ at the time t and the position x). In (2.24), $(X_s)_s$ is the process given by

$$\begin{cases} \forall s \in (\tau_i, \tau_{i+1}), dX_s = \sqrt{2\nu} dW_s; \\ X_{\tau_i^+} = X_{\tau_i^-} + \xi_i; \\ X_0 = x \in \mathbb{T}^d; \end{cases} \tag{2.25}$$

where $(W_s)_s$ is a standard brownian motion under $(\Omega, \mathcal{A}, \mathbb{P})$. The problem of impulse control then consists in choosing the optimal jumps (defined by a time and an element of K) to impose on the state $(X_s)_{s\geq 0}$ in order to minimize (2.24). We define $M(k,u)$ by

$$M(k,u)(t,x) = \inf_{\xi \in K} \{u(t, x+\xi) + k(t,x,\xi)\}.$$

We shall note Mu instead of $M(k,u)$ when there is no ambiguity on k. Several assumptions can be made on the regularity of k as well as on its dependence on the variable of the problem. We here assume that the following holds in order to work with solutions of the problem which are smooth enough :

$$\begin{cases} \forall \xi \in K, k(\cdot, \xi) \in H^2(\mathbb{T}^d); \\ k^* : x \to \inf_{\xi \in K} k(x,\xi) \in W^{2,\infty}(\mathbb{T}^d); \\ \exists k_0 > 0 \text{ such that } k \geq k_0. \end{cases} \tag{2.26}$$

We define H by :

$$H = \{v \in L^2((0,T), H^1(\mathbb{T}^d)), \partial_t v \in L^2((0,T), H^{-1}(\mathbb{T}^d))\}.$$

In the same way value functions of optimal stopping problems can be solutions of obstacle problems [9], we expect the value function of this impulse control problem to be a solution of

$$\begin{cases} \max(u - Mu, -\partial_t u - \nu \Delta u - f) = 0 \text{ in } (0,T) \times \mathbb{T}^d; \\ u(T) = 0 \text{ in } \mathbb{T}^d. \end{cases} \tag{2.27}$$

However, just as variational inequalities are the most natural object to represent solutions of obstacle problems, quasi-variational inequalities (QVI) are a natural

object associated to (2.27). The QVI for this impulse control problem, which we denote by $QVI(f,k)$, is :

$$QVI(f,k)\begin{cases} u \leq Mu \text{ a.e. in } (0,T)\times\mathbb{T}^d; \\ \forall v \in L^2((0,T),H^1(\mathbb{T}^d)), v \leq Mu, v(T)=0; \\ -\int_0^T\int_{\mathbb{T}^d}\partial_t u(v-u)+\nu\int_0^T\int_{\mathbb{T}^d}\nabla u\cdot\nabla(v-u)\geq\int_0^T\int_{\mathbb{T}^d}f(v-u); \\ u(T)=0 \text{ in } \mathbb{T}^d. \end{cases} \tag{2.28}$$

The function u is here the solution/unknown of $QVI(f,k)$. Finding a solution of $QVI(f,k)$ is not possible for any $f \in L^2$ in any dimension. This is a consequence of the fact that if f is not bounded from below, then we cannot expect in general u to be bounded from below. Indeed in such a case, it is unclear in which sense the condition $u \leq Mu$ has to be understood. Usually a solution of (2.28) is build as the limit of the sequence $(u_n)_{n\in\mathbb{N}}$ defined by :

$$\begin{cases} \max(-\partial_t u_{n+1}-\nu\Delta u_{n+1}-f, u_{n+1}-Mu_n)=0 \text{ in } (0,T)\times\mathbb{T}^d; \\ u_{n+1}(T)=0 \text{ in } \mathbb{T}^d; \end{cases} \tag{2.29}$$

with the convention $u_{-1}=+\infty$. The obstacle problem (2.29) is understood in the sense of variational inequalities. For all $n\in\mathbb{N}$, $u_n \in L^2((0,T),H^2(\mathbb{T}^d))\cap H^1((0,T),L^2(\mathbb{T}^d))$ because (2.26) holds. Moreover, $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence because M is monotone in the sense that it is order preserving. If one can find $v \in L^2((0,T),H^1(\mathbb{T}^d))\cap H^1((0,T),H^{-1}(\mathbb{T}^d))$ such that $v\leq u_n$ for all $n\in\mathbb{N}$ then we deduce that :

$$-\int_0^T\int_{\mathbb{T}^d}\partial_t u_n(v-u_n)+\nu\int_0^T\int_{\mathbb{T}^d}\nabla u_n\cdot\nabla(v-u_n)\geq\int_0^T\int_{\mathbb{T}^d}f(v-u_n).$$

Rearranging this inequality we deduce that :

$$||u_n(t)||_{L^2}^2+\nu\int_0^t\int_{\mathbb{T}^d}|\nabla u_n|^2 \leq \int_{\mathbb{T}^d}u_n(0)v(0)+\int_0^t\int_{\mathbb{T}^d}(\partial_t v-\nu\Delta v+f,u_n)_{H^{-1}\times H^1} - \int_0^t\int_{\mathbb{T}^d}fv \tag{2.30}$$

Thus we obtain estimates on the sequence $(u_n)_{n\in\mathbb{N}}$ from the existence of a uniform lower bound v. Let us note that if $f\geq 0$, then $u_n\geq 0$ for all $n\in\mathbb{N}$ so we can choose $v=0$ in (2.30). More generally if f is bounded from below by a constant $-C$ then for all $n\in\mathbb{N}$ we deduce that $u_n(t,x)\geq -Ct$ for all $(t,x)\in(0,T)\times\mathbb{T}^d$ and we can choose $v=-Ct$ in (2.30). Moreover, if f is bounded from below by a constant $-C$, because k satisfies (2.26), the following estimate is classical :

$$||\partial_t u||_{L^2}+||u||_{L^2(H^2)}\leq C_1(1+||f||_{L^2}) \tag{2.31}$$

where C_1 depends only on C and on k. We present a result of stability concerning solutions of regular QVI. This result does not seem to be new but we detail the proof for the sake of completeness.

Proposition 2.A.1. *Let us take any sequence $(f_n)_n$ and a constant $C > 0$, such that for all $n \in \mathbb{N}$, $f_n \in L^2((0,T) \times \mathbb{T}^d)$ and $f_n \geq -C$. We also assume that $k \in L^\infty(\mathbb{T}^d \times K)$ satisfies (2.26). If $(f_n)_n$ is bounded in $L^2((0,T)\times\mathbb{T}^d)$ and converges toward $f \in L^2((0,T) \times \mathbb{T}^d)$ in $L^2((0,T), H^{-1}(\mathbb{T}^d))$ with $f \geq -C$, then the sequence $(u_n)_n$ of solutions of $QVI(f_n, k)$ converges toward the solution u of $QVI(f, k)$ in $L^2((0,T), H^1(\mathbb{T}^d))$.*

Proof. The sequence $(||f_n||_{L^2})_n$ is bounded. Hence $(u_n)_n$ is a bounded sequence of $L^2((0,T), H^2(\mathbb{T}^d)) \cap H^1((0,T), L^2(\mathbb{T}^d))$. Extracting a subsequence if necessary, it converges to a limit $u^* \in L^2((0,T), H^2(\mathbb{T}^d))$ for the $L^2((0,T), H^1(\mathbb{T}^d))$ norm. The limit u^* satisfies $u^* \leq Mu^*$ almost everywhere. Let us take $v \in L^2((0,T), H^1(\mathbb{T}^d))$ such that $v \leq Mu^*$. Obviously the following holds :

$$v_n := v - Mu^* + Mu_n \leq Mu_n.$$

Thus because of $QVI(f_n, k)$, we obtain

$$-\int_0^T \int_{\mathbb{T}^d} \partial_t u_n (v_n - u_n) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u_n \cdot \nabla (v_n - u_n) \geq \int_0^T \int_{\mathbb{T}^d} f_n (v_n - u_n).$$

Re arranging this inequality leads to

$$\begin{aligned}\int_0^T \int_{\mathbb{T}^d} f_n (v - u_n) \leq &- \int_0^T \int_{\mathbb{T}^d} \partial_t u_n (v - u_n) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u_n \cdot \nabla (v - u_n); \\ &- \int_0^T \int_{\mathbb{T}^d} (\partial_t u_n + \nu \Delta u_n - f_n)(Mu_n - Mu^*).\end{aligned}$$

Let us remark that $(||Mu_n - Mu^*||_{L^2})_n$ converges to 0 as n goes to infinity. Thus, because $(\partial_t u_n + \Delta u_n - f_n)_n$ is bounded in L^2, passing to the limit in the previous equation we obtain

$$-\int_0^T \int_{\mathbb{T}^d} \partial_t u^* (v - u^*) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u^* \cdot \nabla (v - u^*) \geq \int_0^T \int_{\mathbb{T}^d} f (v - u^*).$$

We conclude by the uniqueness of solutions of QVI [50], that $u^* = u$, the unique solution of this QVI. □

We now present a result on weaker QVI. To pass to the limit $\epsilon \to 0$ in (2.2), we need an estimate for right hand sides f which are only in $L^2((0,T), H^{-1}(\mathbb{T}^d))$. The following lemma gives such an estimate for a QVI in which we do not impose

the constraint $u \leq M(k,u)$ on the whole space but only on the part which is of interest in (2.2), for a given cost function k. The new constraint we impose is that

$$\forall \xi \in K : \mathbb{1}_{\{V(\xi)>0\}}(u(t,x) - k(x,\xi) - u(t,x+\xi)) \leq 0.$$

We note $\mathcal{K}(k,u)$ the convex closed set :

$$\mathcal{K}(k,u) := \{v \in L^2((0,T), H^1(\mathbb{T}^d)), \forall \xi \in K : \mathbb{1}_{\{V(\xi)>0\}}(v(t,x) - k(x,\xi) - u(t,x+\xi)) \leq 0\}.$$

Proposition 2.A.2. *Assume that there exists V satisfying (2.10) for which hypothesis 2.2.3 (in part 1) holds. We note k and w the couple given by hypothesis 2.2.3. Then for any $f \in L^2((0,T), H^{-1}(\mathbb{T}^d)) \cap \mathcal{M}_b((0,T) \times \mathbb{T}^d)$ there exists $u \in L^2((0,T), H^1(\mathbb{T}^d))$ such that :*

$$\begin{cases} u \in \mathcal{K}(k,u); \\ \forall v \in H, v \in \mathcal{K}(k,u); \\ -\int_0^T \int_{\mathbb{T}^d} \partial_t v(v-u) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u \cdot \nabla(v-u) + \frac{1}{2} \int_{\mathbb{T}^d} |v(T)|^2 \geq \int_0^T \int_{\mathbb{T}^d} f(v-u); \end{cases} \tag{2.32}$$

Moreover we have the estimate

$$||u||_{L^\infty(L^2)} + ||u||_{L^2(H^1)} \leq C(1 + ||f||_{L^2(H^{-1})});$$

where C only depends on K and $\frac{||w||_\infty}{\inf k}$.

The idea of the proof is that the QVI (2.32) is associated to a formal impulse control problem in which one can only use the impulse control ξ on $\{V(\xi) > 0\}$. Because hypothesis 2.2.3 is satisfied, the QVI is somehow well defined and thus we can solve it for unbounded cost functions f.

Proof. Denoting k and w the functions given by hypothesis 2.2.3, there exists $n^* \in \mathbb{N}$ such that :

$$\forall (t,x) s.t. \exists \xi, V(\xi,t,x) > 0, \nexists (\xi_1, ..., \xi_p) \in K^p, p \geq n^*, \forall k \leq (p-1), V(\xi_{k+1}, t, x + \sum_{i=1}^{k} \xi_i) > 0.$$

This fact is a direct consequence of $w \in L^\infty$ and is obtained by evaluating $w(t, x + \sum_{i=1}^k \xi_i)$. Moreover, we have :

$$n^* \leq \frac{2||w||_{L^\infty}}{\inf_{x,\xi} k(x,\xi)}$$

We now define $\tilde{f}$ by

$$\tilde{f} = \min_{p \leq n^*} \min_{(\xi_1,..,\xi_p)} f(\cdot, \cdot + \sum_{i=1}^{p} \xi_i);$$

We understand in the previous equation the *min* as the minimum of a finite number measures. The function $\tilde{f} \in L^2((0,T), H^{-1}(\mathbb{T}^d))$ is well defined because $f \in L^2((0,T), H^{-1}(\mathbb{T}^d)) \cap \mathcal{M}_b((0,T)\times\mathbb{T}^d)$. It represents the best running cost one can face by jumping at the same time a maximum of n^* times. We define $\tilde{u}$ by :

$$\begin{cases} -\partial_t \tilde{u} - \nu\Delta\tilde{u} = \tilde{f} \text{ in } (0,T)\times\mathbb{T}^d; \\ \tilde{u}(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

As already mentioned above, an existence result for QVI usually comes from the existence of a lower bound for an approximating sequence. The function $\tilde{u}$ plays the role of a lower bound for the sequence $(u_n)_{n\in\mathbb{N}}$ that we now define. We denote by $u_0 \in H$ the unique solution of

$$\begin{cases} -\partial_t u_0 - \nu\Delta u_0 = f \text{ in } (0,T)\times\mathbb{T}^d; \\ u_0(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

We then define for all $n \in \mathbb{N}$, $u_n \in L^2((0,T), H^1(\mathbb{T}^d)) : u_{n+1} \in L^2((0,T), H^1(\mathbb{T}^d))$ as the solution of the weak variational inequality (we refer to [10] for a presentation of weak variational inequalities) :

$$\begin{cases} u_{n+1} \in \mathcal{K}(k, u_n); \\ \forall v \in H, v \in \mathcal{K}(k, u_n); \\ -\displaystyle\int_0^T \int_{\mathbb{T}^d} \partial_t v(v - u_{n+1}) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u_{n+1} \cdot \nabla(v - u_{n+1}) + \\ \qquad + \dfrac{1}{2}\displaystyle\int_{\mathbb{T}^d} |v(T)|^2 \geq \int_0^T \int_{\mathbb{T}^d} f(v - u_{n+1}). \end{cases} \tag{2.33}$$

Straightforwardly, we deduce iteratively that for every $n \in \mathbb{N}$, $(u_n)_{n\in\mathbb{N}}$ is well defined, $u_{n+1} \leq u_n$, $\tilde{u} \in \mathcal{K}(k, u_n)$. The last point is a direct consequence of the definition of $\mathcal{K}(k,\cdot)$ and $\tilde{u}$. Evaluating the second line of (2.33) with $v = \tilde{u}$, we deduce :

$$\nu \int_0^T \int_{\mathbb{T}^d} |\nabla u_n|^2 \leq -\int_0^T (-\partial_t \tilde{u} - f, \tilde{u} - u_n)_{H^{-1}\times H^1} + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u_n \cdot \nabla \tilde{u}. \tag{2.34}$$

Thus, $(u_n)_{n\in\mathbb{T}^d}$ is a bounded sequence of $L^2((0,T), H^1(\mathbb{T}^d))$. Because, it is also a decreasing sequence, it converges weakly in $L^2((0,T), H^1(\mathbb{T}^d))$ to a limit $u \in L^2((0,T), H^1(\mathbb{T}^d))$. It follows that

$$\begin{cases} u \in \mathcal{K}(k, u); \\ \forall v \in H, v \in \mathcal{K}(k, u); \\ -\int_0^T \int_{\mathbb{T}^d} \partial_t v(v-u) + \nu \int_0^T \int_{\mathbb{T}^d} \nabla u \cdot \nabla(v-u) + \frac{1}{2}\int_{\mathbb{T}^d} |v(T)|^2 \geq \int_0^T \int_{\mathbb{T}^d} f(v-u). \end{cases}$$

Moreover, because

$$\tilde{u} \leq u \leq u_0,$$

and $\tilde{u}, u_0 \in L^\infty((0,T), L^2(\mathbb{T}^d))$, we obtain that $u \in L^\infty((0,T), L^2(\mathbb{T}^d))$. Finally, let us remark that

$$||\tilde{u}||_{L^\infty(L^2)} + ||\tilde{u}||_{L^2(H^1)} \leq C(1 + ||\tilde{f}||_{L^2(H^{-1})}),$$

where C does not depend on f. Moreover, by construction of $\tilde{f}$,

$$||\tilde{f}||_{L^2(H^{-1})} \leq \tilde{C}||f||_{L^2(H^{-1})},$$

where $\tilde{C}$ depends only on K, and n^*. Hence, u satisfies :

$$||u||_{L^\infty(L^2)} + ||u||_{L^2(H^1)} \leq C(1 + ||f||_{L^2(H^{-1})}),$$

where C depends only on K and $\frac{||w||_\infty}{\inf k}$ (and on ν and d). □

2.A.2 The stationary setting

In this section we give the analogue of the results of the previous section in a stationary setting. The two results in question are proved by following exactly the same argument as in the previous part. Thus we do not detail the proofs of those results. We still fix a finite set $K \subset \mathbb{T}^d$ and we define $M(k,u)$ by

$$M(k,u)(x) = \inf_{\xi \in K} k(x,\xi) + u(x+\xi).$$

We fix a parameter $\lambda > 0$ which describes the intertemporal preference rate in the following impulse control problem :

$$\inf_{(\tau_i)_i,(\xi_i)_i} \mathbb{E}[\int_0^\infty e^{-\lambda s} f(X_s) ds + \sum_{i=1}^{\#(\tau_j)_j} e^{-\lambda \tau_i} k(X_{\tau_i^-}, \xi_i)]; \tag{2.35}$$

where the trajectories are given by :

$$\begin{cases} \forall s \in (\tau_i, \tau_{i+1}), dX_s = \sqrt{2\nu} dW_s; \\ X_{\tau_i^+} = X_{\tau_i^-} + \xi_i; \\ X_0 = x \in \mathbb{T}^d; \end{cases}$$

where $(W_s)_{s \geq 0}$ is a standard brownian motion under $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\xi_i)_i$ and $(\tau_i)_i$ are the controls. In this problem f is the running cost and k the cost of jumps. We made for the rest of this section the assumption that k satisfies (2.26). If the

running cost $f \in L^2(\mathbb{T}^d)$ and $f \geq -C$ for some positive constant C, then the value function u of (2.35) is the unique solution in $H^1(\mathbb{T}^d)$ of the following QVI :

$$SQVI(f,k)\begin{cases} u \leq Mu \text{ a.e. in } \mathbb{T}^d; \\ \forall v \in H^1(\mathbb{T}^d), v \leq M(k,u); \\ \nu \int_{\mathbb{T}^d} \nabla u \cdot \nabla(v-u) + \int_{\mathbb{T}^d} \lambda u(v-u) \geq \int_{\mathbb{T}^d} f(v-u). \end{cases} \tag{2.36}$$

Moreover, u is in fact in $H^2(\mathbb{T}^d)$ and satisfies in L^2 :

$$\max(-\nu\Delta u + \lambda u - f, u - Mu) = 0 \text{ in } \mathbb{T}^d.$$

We have the following result :

Proposition 2.A.3. *Let us take any sequence $(f_n)_n$ and a constant $C > 0$, such that for all $n \in \mathbb{N}$, $f_n \in L^2(\mathbb{T}^d)$ and $f_n \geq -C$. We also assume that $k \in L^\infty(\mathbb{T}^d \times K)$ satisfies (2.26). If $(f_n)_n$ converges toward $f \in L^2(\mathbb{T}^d)$ in L^2 with $f \geq -C$, then the sequence $(u_n)_n$ of solutions of $SQVI(f_n,k)$ converges toward the solution u of $SQVI(f,k)$ in $H^1(\mathbb{T}^d)$.*

We introduce the following notation : We note $\mathcal{K}(k,u)$ the convex closed set :

$$\mathcal{K}(k,u) := \{v \in H^1(\mathbb{T}^d), \forall \xi \in K : \mathbb{1}_{\{V(\xi)>0\}}(v(x) - k(x,\xi) - u(x+\xi)) \leq 0\}.$$

Proposition 2.A.4. *Assume that V satisfies hypothesis 2.2.5 (in part 1). We denote by k and w the couple given by hypothesis 2.2.5. Then for any $f \in H^{-1}(\mathbb{T}^d) \cap \mathcal{M}_b(\mathbb{T}^d)$ there exists $u \in H^1(\mathbb{T}^d)$ such that :*

$$\begin{cases} u \in \mathcal{K}(k,u); \\ \forall v \in \mathcal{K}(k,u); \\ \nu \int_{\mathbb{T}^d} \nabla u \cdot \nabla(v-u) + \int_{\mathbb{T}^d} \lambda u(v-u) \geq \int_{\mathbb{T}^d} f(v-u); \end{cases} \tag{2.37}$$

Moreover we have the estimate

$$||u||_{L^2} + ||u||_{H^1} \leq C(1 + ||f||_{H^{-1}});$$

where C only depends on K, λ, ν and $\frac{||w||_\infty}{\inf k}$.

2.B Some results on parabolic PDE in time dependent domains

In this appendix, we are interested in two topics. The first one is the existence of solutions of parabolic PDE in a time dependent domain with non linear boundary

conditions of Dirichlet type. The second one is the uniqueness of solutions of a similar problem with Dirichlet boundary conditions. The first results allow us to build functions needed in the proof of the uniqueness of solutions of the Fokker-Planck equations in the first part. The second one is a result which generalizes existing results on the well-posedness of a parabolic PDE in a time dependent domain since we allow the domain to evolve more generally than in the existing literature.

In this appendix, we focus on the heat operator

$$\partial_t u - \nu \Delta u$$

where $\nu > 0$, and the domain B on which is posed the PDE is such that $B \subset [0,T] \times \mathbb{T}^d$, where $T > 0$ is the final time. We denote by $A = [0,T] \times \mathbb{T}^d \setminus B$.

Given a function $\tilde{u} \in L^2(\mathbb{T}^d)$ and a running cost $f \in L^\infty((0,T) \times \mathbb{T}^d)$, the standard Dirichlet problem for a parabolic PDE is :

$$\begin{cases} \partial_t u - \nu \Delta u = f \text{ in } B; \\ u = \tilde{u} \text{ on } A \cup \{0\} \times \mathbb{T}^d. \end{cases} \tag{2.38}$$

We denote by $H := L^2((0,T), H^1(\mathbb{T}^d)) \cap H^1((0,T), H^{-1}(\mathbb{T}^d))$. Here we say that $u \in H$ is a strong solution of (2.38) if for any $v \in L^2((0,T), H^1(\mathbb{T}^d))$ such that $v = 0$ almost everywhere on A :

$$\begin{cases} \int_0^T \int_{\mathbb{T}^d} (\partial_t u - \nu \Delta u - f, v)_{H^{-1} \times H^1} = 0; \\ u(0) = \tilde{u} \text{ a. e. on } A \cup \{0\} \times \mathbb{T}^d. \end{cases}$$

We say that $u \in L^2((0,T), H^1(\mathbb{T}^d))$ is a weak solution of (2.38) if $u = \tilde{u}$ almost everywhere on A and for any $v \in H$, such that $v = 0$ almost everywhere on A and $v(T) = 0$:

$$\int_0^T \int_{\mathbb{T}^d} (-\partial_t v - \nu \Delta v - f, u)_{H^{-1} \times H^1} - \int_{\mathbb{T}^d} \tilde{u}(0) v(0) = 0.$$

Regularity of solutions of (2.38) usually comes from two types of assumptions on B. The first class of assumptions consists in assuming that the set B has a smooth boundary, namely that it evolves smoothly in time, see [19]. The second class of assumptions is concerned with monotonicity assumption on the set B see [41]. In this last paper, the authors assume that B is non decreasing in time (for the inclusion), that is A is non-increasing in time (for the inclusion).

2.B.1 Parabolic PDE in a time dependent domain with non linear boundary conditions

Through this section, we assume that there exists $V, k, \tilde{u}$ such that

$$\begin{cases} V \in L^\infty(K \times (0,T) \times \mathbb{T}^d); \\ V \geq 0; \\ \sum_{\xi \in K} V(\xi,t,x) \leq 1; \end{cases}$$

the function k satisfies (2.3) and $u^* \in H$ is such that

$$\begin{cases} u^* \leq M(k,u^*) \text{ in } (0,T) \times \mathbb{T}^d; \\ u(T) = 0; \\ \forall \xi \in K, V(\xi,t,x)(k(x,\xi) + u^*(t,x+\xi) - u^*(t,x)) = 0 \text{ a.e. in } (0,T) \times \mathbb{T}^d; \\ \forall \xi \in K, x \in \tilde{A}_\xi \Rightarrow (k(x,\xi) + u^*(t,x+\xi) - u^*(t,x)) = 0; \\ \forall \xi \in K, x \in \tilde{A}_\xi \Rightarrow \forall \xi' \neq \xi, u^*(t,x) < k(x,\xi') + u^*(t,x+\xi'). \end{cases} \tag{2.39}$$

where $\tilde{A}_\xi$ is defined by

$$\tilde{A}_\xi := \{V(\xi,\cdot,\cdot) > 0\}.$$

We are here interested in the existence of solutions of the following problem

$$\begin{cases} \partial_t u - \nu \Delta u = f \text{ in } B; \\ \forall \xi \in K, V(\xi,t,x)(k(x,\xi) + u(t,x+\xi) - u(t,x)) = 0 \text{ a.e. in } (0,T) \times \mathbb{T}^d; \\ u(0) = u^*(0). \end{cases} \tag{2.40}$$

The problem (2.40) is almost of the type of (2.38), but here the function which gives the boundary conditions is itself a function of u. The set A is in this case given by

$$A = \cup_{\xi \in K} \{V(\xi,t,x) > 0\}.$$

We define B by

$$B = [0,T] \times \mathbb{T}^d \setminus A.$$

The existence of u^* allows us to state the following result :

Theorem 2.B.1. *Let us assume that (2.39) holds true. If either B is an open set with Lipschitz boundary such that $B \cap (\{0\} \times \mathbb{T}^d)$ is an open set with Lipschitz boundary, or B is non-decreasing in time (for the inclusion), then there exists $\epsilon > 0$ such that for any $f \in L^\infty((0,T) \times \mathbb{T}^d)$ such that $||f||_{L^\infty} \leq \epsilon$ there exists a strong solution u of :*

$$\begin{cases} \partial_t u - \nu \Delta u = \partial_t u^* - \nu \Delta u^* + f \text{ in } B; \\ \forall \xi \in K, V(\xi,t,x)(k(x,\xi) + u(t,x+\xi) - u(t,x)) = 0 \text{ a.e. in } (0,T) \times \mathbb{T}^d; \\ u(0) = u^*(0). \end{cases}$$

Remark 2.B.1. *This result is restricted to small f. This is a consequence of the choice of M we made. There are more regular M for which such results can be establish for large f. For instance if $K = \{\xi\}$ is a singleton, then the previous result holds true for any $f \in L^\infty((0,T) \times \mathbb{T}^d)$.*

Proof. We fix $f \in L^\infty((0,T) \times \mathbb{T}^d)$ and we define $\tilde{f} := \partial_t u^* - \nu \Delta u^* + f$. Under any of the two assumptions for B, the following operator is well defined :

$$\begin{aligned} \mathcal{T} : & H \to H \\ & u \to \mathcal{T}(u) \end{aligned}$$

where $\mathcal{T}(u)$ is the unique solution of

$$\begin{cases} \partial_t \mathcal{T}(u) - \nu \Delta \mathcal{T}(u) = \tilde{f} \text{ in } B; \\ \mathcal{T}(u) = Mu \text{ a.e. in } A; \\ u(0) = u^*(0); \end{cases}$$

where Mu is defined by

$$Mu(t,x) = \inf_{\xi \in K} k(t,x) + u(t, x+\xi).$$

For the case of a Lipschitz boundary we refer to [19] and to [41] for the case in which B is increasing. Moreover the map $\mathcal{T}$ is monotone (order preserving). Next we claim using classical results on QVI (see [10]) that there exists $v_1, \tilde{v}_2 \in H$, respectively solutions of the two following QVI :

$$\begin{cases} v_1 \leq M v_1; \\ \forall w \in H, w \leq M v_1 : \\ \int_0^T \int_{\mathbb{T}^d} (\partial_t v_1 - \nu \Delta v_1 - \tilde{f}, w - v_1)_{H^{-1} \times H^1} \geq 0; \\ v_1(0) = u^*(0). \end{cases}$$

$$\begin{cases} \tilde{v}_2 \leq M \tilde{v}_2; \\ \forall w \in H, w \leq M \tilde{v}_2 : \\ \int_0^T \int_{\mathbb{T}^d} (\partial_t \tilde{v}_2 - \nu \Delta \tilde{v}_2 + \tilde{f} - 2(\partial_t u^* - \nu \Delta u^*), w - \tilde{v}_2)_{H^{-1} \times H^1} \geq 0; \\ \tilde{v}_2(0) = u^*(0). \end{cases}$$

We define $v_2 = 2u^* - \tilde{v_2}$. Because $\mathcal{T}$ is monotone, the set $\mathcal{J} := \{v \in H, v_1 \leq v \leq v_2\}$ is invariant by $\mathcal{T}$. Moreover this set is non-empty because $v_1 \leq v_2$. Thus $\mathcal{T}$ has a fixed point $u \in \mathcal{J}$. Now let us remark that as $||f||_{L^\infty}$ goes to 0, v_1 and v_2 converges to u^*. Thus because of the assumption satisfied by u^*, namely the last line of (2.39), if $||f||_{L^\infty}$ is small enough, then because $u = Mu$ on A, u satisfies

$$\forall \xi \in K, V(\xi, t, x)(k(x, \xi) + u(t, x+\xi) - u(t,x)) = 0 \text{ a.e. in } (0,T) \times \mathbb{T}^d.$$

□

2.B.2 Parabolic PDE in general time dependent domains

In this section, we present a result for the heat equation on a time dependent domain with Dirichlet boundary conditions. We detail why under a rather general assumption on the evolution of the domain, we obtain a unique solution of the problem. We are interested in the following PDE :

$$\begin{cases} \partial_t u - \nu \Delta u = f \text{ in } B; \\ u = \phi \text{ in } A; \\ u(0) = \phi(0) \text{ in } \mathbb{T}^d. \end{cases} \tag{2.41}$$

In this PDE, $B \subset (0,T) \times \mathbb{T}^d$ and $A = \Big((0,T) \times \mathbb{T}^d\Big) \setminus B$. We assume that :

$$\phi \in H^1((0,T), H^1(\mathbb{T}^d)).$$

Available results of uniqueness concern situations in which the evolution of B in time is either monotone, see [41], or smooth, see [19]. Here we make the following assumption :

Hypothesis 2.B.1. *The set B is such that $\mu = \partial_t \mathbb{1}_B$ is a measure whose positive and negative parts are strictly separated, i.e. there exists an open set $\mathcal{O}$ with C^1 boundary such that $supp(\mu^-) \subset \mathcal{O}$ and $supp(\mu^+) \cap \mathcal{O} = \emptyset$.*

We plan our study as follows : proposition 2.B.1 states the existence of weak solution of this problem ; lemma 2.B.1 states a local regularity result needed for the uniqueness and theorem 2.B.2 is our main result of uniqueness.

Proposition 2.B.1. *There exists a weak solution $u \in L^2((0,T), H^1(\mathbb{T}^d)) \cap L^\infty((0,T), L^2(\mathbb{T}^d))$ of (2.41).*

Proof. For any $\epsilon > 0$, there exists a unique $u_\epsilon \in H$ solution of

$$\begin{cases} \partial_t u_\epsilon - \nu \Delta u_\epsilon + \frac{1}{\epsilon} \mathbb{1}_A (u_\epsilon - \phi) = f \text{ in } (0,T) \times \mathbb{T}^d; \\ u_\epsilon(0) = \phi(0). \end{cases}$$

Multiplying this equation by $(u_\epsilon - \phi)$ and integrating in space we deduce that :

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathbb{T}^d} u_\epsilon^2(t) + \nu \int_{\mathbb{T}^d} |\nabla u_\epsilon|^2(t) + \frac{1}{\epsilon} \int_{A(t)} (u_\epsilon - \phi)^2(t) = -\frac{1}{2} \int_{\mathbb{T}^d} \phi(0)^2 \\ + \int_{\mathbb{T}^d} f(u_\epsilon - \phi)(t) + \nu \int_{\mathbb{T}^d} \nabla \phi \cdot \nabla u_\epsilon(t) + \int_{\mathbb{T}^d} \partial_t u_\epsilon \phi(t). \end{aligned}$$

Integrating between 0 and t, we obtain :

$$\frac{1}{2}\int_{\mathbb{T}^d} u_\epsilon^2(t) + \nu\int_0^t\int_{\mathbb{T}^d}|\nabla u_\epsilon|^2 + \frac{1}{\epsilon}\int_{\cup_{s\leq t}A(s)}(u_\epsilon-\phi)^2$$
$$\leq \int_0^t\int_{\mathbb{T}^d} f(u_\epsilon-\phi)(t) + \nu\int_0^t\int_{\mathbb{T}^d}\nabla\phi\cdot\nabla u_\epsilon - \int_0^t\int_{\mathbb{T}^d} u_\epsilon\partial_t\phi + \int_{\mathbb{T}^d}\phi(t)u_\epsilon(t).$$

Thus, the sequence $(u_\epsilon)_\epsilon$ is bounded in $L^2((0,T),H^1(\mathbb{T}^d))\cap L^\infty((0,T),L^2(\mathbb{T}^d))$, hence it has a weak limit u which satisfies all the required properties. □

We now pass to the proof of the lemma on local regularity.

Lemma 2.B.1. *Let us define u_ϵ as in the proof of proposition 2.B.1. Assume that $u_\epsilon\in H$ is such that on an open set $(t_1,t_2)\times\Omega\subset(0,T)\times\mathbb{T}^d$:*

$$\partial_t u_\epsilon - \nu\Delta u_\epsilon + \frac{1}{\epsilon}\lambda(u_\epsilon-\phi) = f;$$

where $\lambda\in L^\infty((0,T)\times\mathbb{T}^d)$ is such that $0\leq\lambda\leq 1$ and $\partial_t\lambda\leq 0$.

Then, for every open set Ω' such that $\overline{\Omega'}\subset\Omega$, there exists C independent of ϵ such that

$$||\partial_t u_\epsilon||^2_{L^2((t_1,t_2)\times\Omega)}\leq C$$

Proof. We denote by ζ a positive C^∞ function with compact support in $(t_1,t_2]\times\Omega$ such that $\zeta=1$ on Ω'. We multiply the equation satisfied by u_ϵ by $\zeta\partial_t(u_\epsilon-\phi)$ and we integrate. We obtain :

$$\int_\Omega\zeta\partial_t u_\epsilon^2 + \nu\frac{d}{dt}\frac{1}{2}\int_\Omega\zeta|\nabla u_\epsilon|^2 + \frac{d}{dt}\frac{1}{2\epsilon}\int_\Omega\lambda\zeta(u_\epsilon-\phi)^2 = \int_\Omega f\zeta\partial_t(u_\epsilon-\phi) + \int_\Omega\zeta\partial_t u_\epsilon\partial_t\phi$$
$$-\nu\int_\Omega\zeta\Delta u_\epsilon\partial_t\phi + \frac{1}{2\epsilon}\int_\Omega\lambda(u_\epsilon-\phi)^2\partial_t\zeta + \frac{1}{2\epsilon}\int_\Omega\zeta(u_\epsilon-\phi)^2\partial_t\lambda + \nu\frac{1}{2}\int_\Omega|\nabla u_\epsilon|^2\partial_t\zeta.$$

Integrating between t_1 and s such that $t_1\leq s\leq t_2$, we deduce that (because $\partial_t\lambda\leq 0$ and $\zeta(t_1)=0$) :

$$\int_{t_1}^s\int_{\Omega'}\partial_t u_\epsilon^2 + \nu\frac{1}{2}\int_{\Omega'}|\nabla u_\epsilon|^2(s) + \frac{1}{2\epsilon}\int_{\Omega'}\lambda(u_\epsilon-\phi)^2(s) \leq \int_{t_1}^s\int_\Omega f\zeta\partial_t(u_\epsilon-\phi) +$$
$$+\int_{t_1}^s\int_\Omega\zeta\partial_t u_\epsilon\partial_t\phi - \nu\int_{t_1}^s\int_\Omega\zeta\Delta u_\epsilon\partial_t\phi +$$
$$+\frac{1}{2\epsilon}\int_{t_1}^s\int_\Omega\lambda(u_\epsilon-\phi)^2\partial_t\zeta + \nu\frac{1}{2}\int_{t_1}^s\int_\Omega|\nabla u_\epsilon|^2\partial_t\zeta.$$

Thus the required estimate follows. □

We can now pass to the main result of this section.

Theorem 2.B.2. *Assume hypothesis 2.B.1 holds, then there exists a unique weak solution $u \in L^2((0,T), H^1(\mathbb{T}^d))$ of (2.41) such that for any set $\Omega \subset \mathcal{O}$ (where $\mathcal{O}$ is given by hypothesis 2.B.1), $\partial_t u \in L^2(\Omega)$.*

Proof. A weak solution u of (2.41) exists by proposition 2.B.1. By lemma 2.B.1, it has the required time regularity. Let us assume that there exists two such weak solutions u_1 and u_2. We then define for $i = 1, 2$ $v_i \in L^2((0,T), H^1(\mathbb{T}^d))$, a weak solution of respectively

$$\begin{cases} -\partial_t v_i - \nu \Delta v_i = u_i \text{ in } B; \\ v_i = 0 \text{ in } A; \\ v_i(T) = 0 \text{ in } \mathbb{T}^d. \end{cases}$$

Using lemma 2.B.1 (in the other time direction), we can assume that there exists an open set $\Omega_1 \subset (0,T) \times \mathbb{T}^d$ with smooth boundary such that

$$\begin{cases} u_1, u_2 \in H^1(\Omega_1); \\ v_1, v_2 \in H^1(\Omega_2); \end{cases}$$

where $\Omega_2 = \overset{\circ}{\Omega_1^c}$. The weak formulations satisfied by u_1, u_2, v_1 and v_2 can now be written. Namely, for any $w \in L^2((0,T), H^1(\mathbb{T}^d))$ such that $w \in H^1(\Omega^1)$ and $w = 0$ almost everywhere on A, the following holds for $i = 1, 2$:

$$\int_{\Omega_1} v_i \partial_t w - \int_{\Omega_2} w \partial_t v_i + \nu \int_0^T \int_{\mathbb{T}^d} \nabla w \cdot \nabla v_i = \int_0^T \int_{\mathbb{T}^d} u_i w + \int_{\partial \Omega_1} w v_i \eta_t^1;$$

where $\eta^1 = (\eta_t^1, \eta_x^1)$ is the exterior unit normal vector to Ω_1. Conversely, for any $w \in L^2((0,T), H^1(\mathbb{T}^d))$ such that $w \in H^1(\Omega^2)$ and $w = 0$ almost everywhere on A, the following holds for $i = 1, 2$

$$-\int_{\Omega_2} u_i \partial_t w + \int_{\Omega_1} w \partial_t u_i + \nu \int_0^T \int_{\mathbb{T}^d} \nabla w \cdot \nabla u_i = \int_0^T \int_{\mathbb{T}^d} f w + \int_{\partial \Omega_2} w u_i \eta_t^2;$$

where $\eta^2 = (\eta_t^2, \eta_x^2)$ is the exterior unit normal vector to Ω_2.Thus we deduce, using this relation for $w = u_1 - u_2$ on v_1 and v_2 ; and for $w = v_1 - v_2$ on u_1 and u_2, that

$$\int_0^T \int_{\mathbb{T}^d} (u_1 - u_2)^2 = 0.$$

Thus there exists a unique weak solution which has the time regularity required. □

Chapitre 3

Quelques remarques sur les jeux à champ moyen

On étudie dans ce chapitre trois aspects des jeux à champ moyen. Le premier est le cas où les dynamiques des joueurs dépendent des stratégies des autres joueurs. Le second cas concerne la modélisation de bruit dans des modèles d'espace d'états discret et la formulation de la *master equation* dans cette situation. Enfin, on montre que les jeux à champ moyen se réduisent à des modèles du type *agent based models* lorsque le taux de préférence intertemporel des joueurs tend vers l'infini, c'est à dire lorsque l'anticipation des joueurs tend vers 0.

Ce chapitre, réalisé en collaboration avec J.-M. Lasry et P.-L. Lions, a été accepté pour publication dans Communications in Partial Differential Equations.

Introduction

This paper presents some results on three topics in the theory of Mean Field Games (MFG). It is divided in three parts, independent of each other for the most part. We present first some results on the uniqueness of solutions for MFG in the case of coupled dynamics. Then, we introduce MFG models with noise in the discrete state space case. We give some conditions for the problem to be well-posed. Finally we discuss the limit of MFG models as the intertemporal preference rate of the players goes to infinity. Those three subjects are described in details later on in each section. We briefly recall now a few well known facts on MFG.

MFG have been introduced by the last two authors in [52, 53, 54] and the field has known a tremendous development in the past ten years. MFG models address differential games involving an infinite number of indistinguishable players. In the absence of a common noise, to find a Nash equilibrium of the game reduces to solve a system of forward backward partial differential equations (PDE). This system consists in a backward Hamilton-Jacobi-Bellman equation, which is solved by the value function of the differential game a generic player is facing, and a forward Fokker-Planck equation which models the evolution of the population. Namely in the case of a game set on the d dimensional torus $\mathbb{T}^d$ with a time horizon T, the MFG reduces to systems of the following kind

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, \nabla_x u, m) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ \partial_t m - \nu\Delta m - div(D_p H(x, \nabla_x u, m)m) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ u(T) = G(x,m); m(0) = m_0 \text{ in } \mathbb{T}^d; \end{cases} \tag{3.1}$$

where H is the Hamiltonian of the optimal control problem for the players, G the terminal cost of this problem and m_0 the initial distribution of players. The function u represents here the value function for a generic player and m is the density of players. Uniqueness (under suitable monotonicity conditions) and existence have been proved for this system in [54, 57]. Numerical methods are being developed, let us cite [2, 17] for examples of this growing literature. Let us also mention the questions of long time average [25, 26] or learning [23]. In some very particular cases (the so-called "potential case") solutions of the MFG system can be obtain from a PDE optimal control problem [8]. This approach is particularly successful to find weak solutions of (3.1) [28]. This particular class of MFG is also somewhat similar to the optimal transport theory. Moreover a major axis of development of MFG is the probabilistic interpretation of such differential games. We refer to [29, 49] for examples of such a point of view and to [30] for a complete presentation of this probabilistic approach. Let us also recall that MFG are not limited to continuous control problem and that other setting have been studied

like optimal stopping [11, 32, 61] or impulse control [12] for instance. Moreover we insist that in the presence of a common noise, the reduction of the MFG to the system (3.1) collapses and the study of the master equation is crucial for the understanding of the MFG. We refer the reader to [27, 57] for a detailed study of the master equation and to [25, 36] for examples of applications.

3.1 Uniqueness in strongly coupled Mean Field Games

3.1.1 Formulation of the problem

In this section, we are interested in MFG in the case when the coupling between the players depend on their strategies. We refer to [57] for details on this question. In the case without common noise, the study of this strong coupling is the study of MFG system in which the Hamiltonian of the problem now depends on the measure associated to the distribution of the strategies of the players. We work here on the d dimensional torus $\mathbb{T}^d$ and the system we are interested in is :

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, \nabla u, m, \mu) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ \partial_t m - \nu\Delta m - div(D_p H(x, \nabla u, m, \mu)m) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ m(0) = m_0; u(T) = \phi(m) \text{ in } \mathbb{T}^d; \end{cases} \tag{3.2}$$

where $H : \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ denotes the Hamiltonian associated with the optimal control problem the players are facing. It is assumed to be smooth and $D_p H$ stands for the derivative with respect to its second variable. We denote by u the value function of the players, m the measure which describes their distribution in the state space $\mathbb{T}^d$ and μ the measure associated to the controls of the players. Let us remark that because μ is the measure of the optimal controls used by the players, it satisfies the following relation :

$$\mu = (-D_p H(x, \nabla u, m, \mu))_{\#} m; \tag{3.3}$$

where $G_{\#}m$ stands for the image measure of m by the measurable application G. Existence of solutions of such a system was proven for particular Hamiltonians in [44] ; we refer to [24] for a more general result of existence for (3.2) and to [30, 31] for the study of such a coupling in a probabilistic setup. In general, uniqueness of such solutions is not known.

Before presenting the main result of this part, we make some preliminaries observations. More precisely, we state that the quantity

$$\int_{\mathbb{T}^d} \nabla u m \tag{3.4}$$

is conserved in time under the assumption that the Hamiltonian H is of the form : $H(\nabla u, m)(x) = \tilde{H}(\nabla u(x)) - f(m(x))$. This result is quite simple but we believe it is worth mentioning as it does not seem to appear in the literature. And it shows that there may be properties enjoyed by the solutions which may play a role in uniqueness properties. Moreover, it will be useful for the study of the example we give at the end of this part.

Lemma 3.1.1. *Assume that H does not depend on the space variable x and that it is of the form $H(\nabla u, m)(x) = \tilde{H}(\nabla u(x)) - f(m(x))$, then for any solution (u, m) of (3.1) the quantity*

$$\int_{\mathbb{T}^d} \nabla u m \tag{3.5}$$

does not depend on time.

Proof. We only prove the statement in the case of a smooth solution (u, m) as it easily extends to weaker notions of solutions. Multiplying by $\partial_{x_i} m$ the equation satisfied by u and integrating the PDE satisfied by u on $(0, t) \times \mathbb{T}^d$, we obtain

$$\begin{aligned}
0 &= \int_0^t \int_{\mathbb{T}^d} (-\partial_t u - \nu \Delta u + \tilde{H}(\nabla u) - f(m)) \partial_{x_i} m; \\
&= \int_0^t \int_{\mathbb{T}^d} (-\partial_t m + \nu \Delta m) \partial_{x_i} u + \tilde{H}(\nabla u) \partial_{x_i} m - \int_{\mathbb{T}^d} f(m) \partial_{x_i} m \\
&\qquad + \int_{\mathbb{T}^d} (u(0) \partial_{x_i} m_0 - u(t) \partial_{x_i} m(t)); \\
&= \int_0^t \int_{\mathbb{T}^d} -div(D_p \tilde{H}(\nabla u) m) \partial_{x_i} u + \tilde{H}(\nabla u) \partial_{x_i} m + \int_{\mathbb{T}^d} (u(0) \partial_{x_i} m_0 - u(t) \partial_{x_i} m(t)).
\end{aligned}$$

We then deduce by integrating by parts all the terms of the last line :

$$\begin{aligned}
0 &= \int_0^t \int_{\mathbb{T}^d} m D_p \tilde{H}(\nabla u) \cdot \partial_{x_i} \nabla u - D_p \tilde{H}(\nabla u) \cdot m \partial_{x_i} \nabla u - \int_{\mathbb{T}^d} (m_0 \partial_{x_i} u(0) - m(t) \partial_{x_i} u(t)); \\
&= - \int_{\mathbb{T}^d} (m_0 \partial_{x_i} u(0) - m(t) \partial_{x_i} u(t)).
\end{aligned}$$

□

Remark 3.1.1. *We thus see that the invariance by translation of the Hamiltonian implies that (3.5) is independent of time. Such a quantity is closely related to the average control of the players. Hence such a lemma leads us to think that one can obtain information on the average control using assumptions on the space dependence of the Hamiltonian.*

As we are led to distinguish local and global dependence of the Hamiltonian in ∇u, we introduce the following notation :

— We note $G(f)$ when the dependence of the function G in the function f is local, i.e. $G(f)(x) = G(f(x))$.
— We note $G(f(\cdot))$ for a general dependence of the function G in the function f.

We shall only use this notation in the case when there might be an ambiguity.

3.1.2 A uniqueness result

We present here a uniqueness result for the system :

$$\begin{cases} -\partial_t u - \nu\Delta u + H(\nabla u) - G(\nabla u(\cdot), m(\cdot)) \cdot \nabla u = f(u,m) \text{ in } (0,T) \times \mathbb{T}^d; \\ u(T) = \phi(m(T)) \text{ in } \mathbb{T}^d; \\ \partial_t m - \nu\Delta m - div(m(D_p H(\nabla u) - G(\nabla u(\cdot), m(\cdot)))) = g(u,m) \text{ in } (0,T) \times \mathbb{T}^d; \\ m(0) = m_0 \text{ in } \mathbb{T}^d. \end{cases} \tag{3.6}$$

Such a system models a MFG in which the players are "pushed" by the mean field control of the other players. We recall the usual conditions under which we expect uniqueness to hold for solutions of (3.6) when $G = 0$:

— The hamiltonian H is convex.
— The terminal cost ϕ is monotone.
— There exist α and β such that $(g, f) + (\alpha m, \beta u)$ is monotone in (u, m).
— Either H is strictly convex, $(g, f) + (\alpha m, \beta u)$ is strictly monotone in (u, m), ϕ is strictly monotone or (g, f) is local in (u, m).

We say that the hypothesis $(H1)$ is satisfied if those four requirements are satisfied. We also recall the notion of monotonicity for the sake of completeness. To define this notion properly one usually needs to make precise the functions spaces on which f and g are defined and take their values. To fix ideas, we recall the notion of monotonicity in the case of maps of L^2 : a map F from $L^2(\Omega)$ into itself is said to be (resp. strictly) monotone if for any $u_1, u_2 \in L^2(\Omega)$:

$$\int_\Omega (F(u_1) - F(u_2))(u_1 - u_2) \geq 0 \text{ (resp } > 0 \text{ if } u_1 \neq u_2 \text{)}$$

We define by $\mathcal{T}$ the group of transformation :

$$\mathcal{T} := \{(v, m) \to (v \circ \tau_x, m \circ \tau_x) | \tau_x \text{ is the translation by } x \in \mathbb{T}^d\}$$

We are now able to state the following result :

Theorem 3.1.1. *Let us assume that G and ϕ are invariant by translations, i.e. that they do not depend on the space variable x and that G is invariant under the group of transformation $\mathcal{T}$. Then under the usual condition that $(H1)$ is satisfied, there exists at most one solution (u, m) of (3.6).*

Proof. We denote by (u_1, m_1) and (u_2, m_2) two solutions of the system (3.6). For $i \in \{1, 2\}$ we define $(\tilde{u}_i, \tilde{m}_i)$ by

$$(\tilde{u}_i, \tilde{m}_i)(t, x) = (u, m)(t, x + \int_0^t G(\nabla u_i(s), m_i(s))ds).$$

Because G and ϕ are invariant by translations, we deduce that for $i \in \{1, 2\}$, $(\tilde{u}_i, \tilde{m}_i)$ is a solution of the MFG system :

$$\begin{cases} -\partial_t \tilde{u}_i - \nu \Delta \tilde{u}_i + H(\nabla \tilde{u}_i) = f(\tilde{u}_i, \tilde{m}_i) \text{ in } (0, T) \times \mathbb{T}^d; \\ \partial_t \tilde{m}_i - \nu \Delta \tilde{m}_i - div(D_p H(\nabla \tilde{u}_i)\tilde{m}_i) = g(\tilde{u}_i, \tilde{m}_i) \text{ in } (0, T) \times \mathbb{T}^d; \\ \tilde{m}_i(0) = m_0; \tilde{u}_i(T) = \phi(\tilde{m}_i(T)) \text{ in } \mathbb{T}^d. \end{cases}$$

Let us note that the assumption that H does not depend explicitly on the space variable is crucial. Because the assumption (H1) holds, there is a unique solution of this system and $(\tilde{u}_1, \tilde{m}_1) = (\tilde{u}_2, \tilde{m}_2)$. Now let us remark that because G is invariant by the transformation which sends (u, m) on $(\tilde{u}, \tilde{m})$ we obtain that

$$G(\nabla u_1, m_1) = G(\nabla u_2, m_2) = G(\nabla \tilde{u}, \tilde{m}).$$

Thus, $(u_1, m_1) = (u_2, m_2)$ and there is a unique solution of (3.6). □

3.1.3 A simple example

The previous proof relies strongly on the form of the Hamiltonian and on the invariance by translations of the problem. We now discuss to what extent those assumptions are needed. First, it is very natural to allow the measure of controls μ in (3.2) to appear in the Hamiltonian only through terms like $G(\nabla u, m)$ because we are interested in a mean field interaction with the control of the other players. Thus terms of the form

$$G(\nabla u(\cdot), m(\cdot)) = \int_{\mathbb{T}^d} g(\nabla u(y))m(y)dy$$

seem to be quite general for applications. See the example studied in [24] for example. Moreover such a term satisfies the assumption of theorem 3.1.1. However it is true that the assumption that this term is linear in the "p" variable for the Hamiltonian is rather strong (i.e. that G appears in the Hamiltonian only through $G(\nabla u, m) \cdot \nabla u$).

Concerning the question of the invariance by translation, the following example shows that this assumption can be extended in some cases. But it also shows that there is a complex interaction between the uniqueness and the type of dependence

of the Hamiltonian in the control of the other players. Let us introduce the following system which satisfies all the assumptions of the theorem 3.1.1 except for the fact that the terminal cost ϕ now depends on the space variable.

$$\begin{cases} -\partial_t u - \nu\Delta u + \frac{1}{2}|\nabla u|^2 - \lambda(\int \nabla u m)\cdot\nabla u = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ \partial_t m - \nu\Delta m + div((\nabla u - \lambda(\int \nabla u m))m) = 0 \text{ in } (0,T)\times\mathbb{T}^d; \\ u(T) = \phi(x); m(0) = m_0(x) \text{ in } \mathbb{T}^d. \end{cases} \tag{3.7}$$

Even though this section is not particularly concerned with the question of existence, let us mention that a solution of such a system exists as soon as ϕ is a Lipschitz function. The following lemma gives a general property satisfied by the solutions of (3.7). Proposition 3.1.1 details a precise example of a function ϕ which yields uniqueness even though it is not constant :

Lemma 3.1.2. *Let us denote by (u,m) a solution of (3.7), we define $A = \int_{\mathbb{T}^d} \nabla u m$ and we denote by u_0 the solution of the HJB equation*

$$\begin{cases} -\partial_t u_0 - \nu\Delta u_0 + \frac{1}{2}|\nabla u_0|^2 = 0; \\ u_0(T) = \phi. \end{cases}$$

Then the following holds

$$A = \int_{\mathbb{T}^d} \nabla u_0(x + \lambda A T, 0) m_0(x) dx. \tag{3.8}$$

Proof. Let us remark that the proof of lemma 3.1.1 applies also for (3.7), hence (u,m) satisfies :

$$\frac{d}{dt}\int_{\mathbb{T}^d} \nabla u m = 0.$$

Thus A is well defined. Now let us remark that a simple change of variable yields :

$$u(t,x) = u_0(x + \lambda A(T-t)).$$

Hence, evaluating $\int_{\mathbb{T}^d} \nabla u m$ at $t = 0$ we obtain that

$$A = \int_{\mathbb{T}^d} \nabla u_0(x + \lambda A T, 0) m_0(x) dx.$$

□

Each solution A of this equation produces a solution for the system (3.7). Hence if ϕ and m_0 are such that the previous equation admits a unique solution then we have proved uniqueness. Such a situation is possible if we obtain a precise bound on ∇u_0 for instance. We give now an example of such a situation :

Proposition 3.1.1. *If $D^2\phi(x) \leq c\, Id$ holds for some $c > 0$ in the sense of distributions, then uniqueness in (3.7) holds if*

$$\lambda < \frac{1+cT}{cT}.$$

Proof. Under those assumptions, in view of classical semi-concavity estimates for HJB equations, the Hessian matrix D^2u_0 of u_0 also satisfies in the sense of distributions for any $0 \leq t \leq T$:

$$D^2u_0(t) \leq \frac{c}{1+c(T-t)} Id.$$

Hence the equation (3.8) satisfied by A admits a unique fixed point if :

$$\lambda < \frac{1+cT}{cT};$$

because then $A - \int_{\mathbb{T}^d} \nabla u_0(x+\lambda AT, 0) m_0(x)dx$ is a strictly increasing function of A. □

Remark 3.1.2. *It is clear from (3.8) that without an assumption on ϕ, uniqueness may not hold. Moreover such a proof can be adapted to a case in which we add a dependence $f(m)$ in the HJB equation.*

Remark 3.1.3. *Let us note that the condition in this proposition can be interpreted as the bigger c is (i.e. the more the system depends on the space variable x), the more the effect of the average strategy (the parameter λ) needs to be small. Also, if $\lambda < 1$, then the condition is satisfied independently of c, thus the condition $\lambda < 1$ yields uniqueness even in the case $c = +\infty$, that is for general ϕ.*

3.2 Common noise in discrete state space MFG

The addition of a common noise in the MFG setting remains one of the most important questions in the MFG theory. Even though [27, 57] provide a detailed study of the so-called master equation, several problems remain open. We can cite for example whether or not there exist weaker regularity assumptions than the one needed in [27], or the possibility to give an extended sense of solutions which yields existence results in the non-monotone case. We present here a discrete state space master equation which models the presence of a common noise in the MFG. We first recall some facts on the master equation without common noise before explaining what we mean by common noise in the discrete state space. We then prove the well-posedness of such an equation as well as some limit equations which can be obtained from our model.

3.2.1 The master equation for a discrete state space

In a discrete state space, without a common noise, the infinite dimensional master equation reduces to a first order non conservative hyperbolic system of the following form (see [57] for instance) :

$$\begin{cases} \partial_t U(t,x) + (F(x,U)\cdot\nabla)U = G(x,U) \text{ in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}. \end{cases} \tag{3.9}$$

Let us recall that in such a system, U is a map from $(0,t_f)\times\mathbb{R}^d$ to $\mathbb{R}^d$, F and G are applications from $\mathbb{R}^{2d}$ to $\mathbb{R}^d$. The time t_f is the final time (fixed in $(0,\infty)$) and the initial condition is U_0. We also recall that the system has to be understood as follows : for all $1\leq i\leq d$, the ith component U_i of U satisfies :

$$\begin{cases} \partial_t U_i(t,x) + F(x,U)\cdot\nabla U_i = G_i(x,U) \text{ in } (0,t_f)\times\mathbb{R}^{\mathbb{d}} \\ U_i(0,x) = (U_0)_i(x) \text{ in } \mathbb{R}^{\mathbb{d}}. \end{cases}$$

The MFG interpretation of (3.9) is that x represents the number of players in the d possible states, meaning that x_1 is the number of players in the first state, x_2 the number of players in the second and so on. The interpretation of the map U is that the ith component $U_i(t,x)$ of $U(t,x)$ is the value of the optimal control problem for a generic player in the state i, with the repartition of other players being x at the time t. Let us note that contrary to the classical MFG system (3.1) or to the master equation presented in [27], the equation is written here forward in time, instead of backward (i.e. we reversed time with $t\to T-t$).

We also recall that there exists a system of characteristics for (3.9). It is defined for any $x_0\in\mathbb{R}^d$ by

$$\begin{cases} \frac{d}{ds}V(s) = G(Y(s),V(s)); \\ \frac{d}{ds}Y(s) = F(Y(s),V(s)); \\ Y(0) = x_0; V(0) = U_0(x_0). \end{cases} \tag{3.10}$$

Using the previous characteristics, we can define a solution U of (3.9) by :

$$U(t,Y(t)) = V(t);$$

One can show ([57]) that it is possible to define a solution of (3.9) using the characteristics under a monotonicity assumptions. The system (3.10) is the analogous of the usual MFG system (3.1) in a discrete state space. Thus, many results on (3.9) can be proven using this system of characteristics (3.10). However, addressing the regularity and the well-posedness of (3.9) without using (3.10) leads to techniques

and proofs which are adaptable to the addition of a common noise, whereas the use of the system (3.10) collapses in the presence of a common noise.

As the notion of monotonicity is crucial in all the results we present in this section, we define here what we mean by monotone. A map V from $\mathbb{R}^d$ into itself is said to be monotone if

$$\forall x, y \in \mathbb{R}^d, < V(x) - V(y), x - y > \geq 0.$$

The map V is said to be α monotone if

$$\forall x, y \in \mathbb{R}^d, < V(x) - V(y), x - y > \geq \alpha |x - y|^2.$$

Here, $< x, y >= x \cdot y$ denotes the usual scalar product between x and y.

3.2.2 The master equation in the presence of noise

We propose here a master equation in finite dimension with additional terms which model jumps which can occur in the population. Those jumps can be random and thus can be assimilated to noise in the MFG. The jumps are described by an application T. We give two examples of such maps T to fix ideas, and explain how the jumps have to be interpreted, depending on the nature of the noises.

— If the map T is defined by :

$$T(x) = (x_2, x_1, x_3, ..., x_d);$$

then the map T models jumps which only concerns players in the first or second state. The players in the first state are sent to the second one and vice versa. If all the players face this jump at the same time, then all the players in the first state and the ones in the second one are exchanged. If only a proportion $0 \leq \theta \leq 1$ of the players in each states jump simultaneously, then the state of the population after the jump is given by $\theta T(x) + (1 - \theta)x$, if x is the state of the population before.

— If the map T is defined by :

$$T(x) = (\frac{x_1}{2} + \delta, x_2 + \frac{x_1}{2}, x_3, .., x_d);$$

where $\delta > 0$ describes an entry rate of players. Then, if a player is in the first state, it has a probability one half to jump to the second state. This jump is naturally associated with a structure of noise, independently of the randomness that may exist concerning the players this jump affects or the time at which it occurs. If a proportion $0 \leq \theta \leq 1$ of the players in all states are simultaneously affected by this jump, then a fraction $\frac{\theta}{2}$ of the players in the first state jump in the second one and an amount of players corresponding to $\theta\delta$ is added to the game in the first state.

The noise structures modeled by this kind of terms are numerous. Indeed with this second example, we show that there can be randomness in the fact that a player will jump or not, just as there can be randomness on the state in which it is going to jump. There can also be randomness on the times at which those jumps occur. The correlations between those times can be quite general as we shall now see through some examples of master equations.

We now make precise how jumps are taken into account in the master equation, depending on the correlations of the jumps. We give three examples of master equation which models jumps occurring at deterministic times, or when the times of the jumps are random but common to every players and finally when they are random and independent identically distributed (iid) for the players. In those examples, T is a differentiable map from $\mathbb{R}^d$ to itself.

If there is only one jump characterized by T which occurs at the deterministic time $t_1 \in (0, t_f)$, then the master equation is :

$$\begin{cases} \partial_t U(t,x) + (F(x,U)\cdot\nabla)U = G(x,U) \text{ in } (t_1,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(t_1^+,x) = (DT(Tx))^* U(t_1^-,T(x)) \text{ in } \mathbb{R}^d; \\ \partial_t U(t,x) + (F(x,U)\cdot\nabla)U = G(x,U) \text{ in } (0,t_1)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}; \end{cases} \tag{3.11}$$

where $t_f > 0$ is the time horizon and $(DT)^*$ stands for the adjoint of the differential of the map T.

If at random times given by an exponential law of parameter λ, all the players jump according to T, then the master equation is :

$$\begin{cases} \partial_t U + (F(x,U)\cdot\nabla)U + \lambda(U - (DT(T(x)))^* U(t,T(x))) = G(x,U) \text{ in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}; \end{cases} \tag{3.12}$$

where λ is the parameter of the Poisson process which describes the jumps. In (3.12), the term

$$\lambda(U - (DT(Tx))^* U(t,Tx))$$

is understood as the players anticipating the noise. The proper derivation of such a term is technical and we do not enter in such calculations here. We refer to [33, 34] for a proper derivation of this term in a different setting, when T is a linear map.

If all the players jump according to T at times given by iid Poisson processes

of parameter λ, then the master equation is given by :

$$\begin{cases} G(x,U) = \partial_t U(t,x) + (F(x,U)\cdot\nabla)U \\ \qquad\qquad + \lambda(Id - (DT(x))^*)U(t,x) + \lambda((Id - T(x))x\cdot\nabla)U \\ \qquad\qquad \text{in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}. \end{cases} \tag{3.13}$$

In this equation, the drift term $\lambda((Id - T(x))x\cdot\nabla)U$ is interpreted as the effect of the flow of players which are jumping. Indeed, because the jumps are independent and there is an infinity of players, there is always a constant proportion of players which are jumping. This phenomenon induces a dynamic in the population which is taken into account in this drift term.

Remark 3.2.1. *Let us remark that a master equation can be written in a similar way if some of the above three phenomena happen together in the MFG.*

3.2.3 Propagation of monotonicity and regularity

In this section, we shall show that under some assumptions on the form of T , the solutions of (3.11), (3.12) and (3.13) have some monotonicity properties. We then show how we can establish regularity properties from the propagation of the monotonicity.

Proposition 3.2.1. *Assume that T is affine. Let U be a classical solution of (3.11), (3.12) or (3.13). If (G,F) (seen as a map from $\mathbb{R}^{2d}$ into itself) is monotone and U_0 is monotone, then U is monotone for all time.*

Proof. The proof of this result follows the arguments in [57]. We write the affine map as $T = S + e$ where $e \in \mathbb{R}^d$ and S is a linear map.

The case of (3.11) :
Let U be a classical solution of (3.11), we denote by V another classical solution of (3.11) and we define W by

$$W(t,x,y) = < U(t,x) - V(t,y), x - y > .$$

Let us note that W satisfies the following equation in $((0,t_1)\cup(t_1,t_f))\times\mathbb{R}^{\mathbb{d}}$:

$$\begin{aligned} \partial_t W + F(x,U)\cdot\nabla_x W + F(y,V)\cdot\nabla_y W = &< G(x,U) - G(y,V), x - y > \\ &+ < F(x,U) - F(y,V), U(t,x) - V(t,y) >; \end{aligned}$$

Let us now take $V = U$. Because U_0 is monotone, we deduce that $W(0,x,y) \geq 0$ for all $x, y \in \mathbb{R}^d$. We also deduce from the monotonicity of (G,F) that the right

hand side of the previous equation is positive everywhere. Thus we deduce form the maximum principle (lemma 3.A.1 in the appendix) that U is monotone (W is positive) in $(0, t_1)$. Now let us remark that the relation satisfied by U at t_1 yields for W :

$$W(t_1^+, x, y) = W(t_1^-, Tx, Ty).$$

Thus W is positive at t_1^+ and thus W is positive for all time $t \in (0, t_f)$, by repeating the same argument.

The case of (3.12) :
Let U be a classical solution of (3.12), we denote by V another classical solution of (3.12) and we define again W by

$$W(t, x, y) =< U(t, x) - V(t, y), x - y > .$$

Let us note that W satisfies the following equation :

$$\begin{aligned}\partial_t W + & F(x, U) \cdot \nabla_x W + F(y, V) \cdot \nabla_y W + \lambda W - \lambda < S^*U(t, Tx) - S^*V(t, Ty), x - y > \\ & =< G(x, U) - G(y, V), x - y > + < F(x, U) - F(y, V), U(t, x) - V(t, y) >;\end{aligned}$$

which can be rewritten in :

$$\begin{aligned}\partial_t W + & F(x, U) \cdot \nabla_x W + F(y, V) \cdot \nabla_y W + \lambda W - \lambda W(t, Tx, Ty) \\ & =< G(x, U) - G(y, V), x - y > + < F(x, U) - F(y, V), U(t, x) - V(t, y) > .\end{aligned}$$

By the maximum principle (lemma 3.A.1 in the appendix), we deduce that U is monotone for all time. This is a straightforward consequence of the monotonicity of (G, F) and U_0, taking $V = U$ in W.

The case of (3.13) :
Let U be a classical solution of (3.13), we denote by V another classical solution of (3.13) and we define once again W by

$$W(t, x, y) =< U(t, x) - V(t, y), x - y > .$$

The application W satisfies the following equation :

$$\begin{aligned}\partial_t W + & F(x, U) \cdot \nabla_x W + F(y, V) \cdot \nabla_y W + \lambda (S - Id)x \cdot \nabla_x W \\ & + \lambda (S - Id)y \cdot \nabla_y W + \lambda < U - V, (Id - S)(x - y) > \\ & =< G(x, U) - G(y, V), x - y > + < F(x, U) - F(y, V), U(t, x) - V(t, y) > \\ & + \lambda < U - V, (Id - S)(x - y) >;\end{aligned}$$

Thus W satisfies :

$$\begin{aligned}&\partial_t W + F(x,U)\cdot\nabla_x W + F(y,V)\cdot\nabla_y W + \lambda(S-Id)x\cdot\nabla_x W\\&\quad +\lambda(S-Id)y\cdot\nabla_y W\\&\quad =< G(x,U)-G(y,V), x-y > + < F(x,U)-F(y,V), U(t,x)-V(t,y) >;\end{aligned}$$

Once again, by the maximum principle (lemma 3.A.1 in the appendix), we deduce that U is monotone for all time by taking $V = U$ and using the monotonicity of U_0 and (G, F). □

Remark 3.2.2. *Let us note that in particular, under the assumptions of proposition 3.2.1, there exists at most one classical solution of (3.11), (3.12) and (3.13). Indeed uniqueness follows readily from the positivity of W (see [57]).*

As we can see in the proof of the previous statement, the propagation of the monotonicity of U is really a "strong" property of (3.11), (3.12) and (3.13), in the sense that it does not depend on λ, t_1 or on T. The different terms in each system only reinforce the propagation of monotonicity.

We now prove that under stronger assumptions on U_0 and (G, F), we can deduce from the propagation of monotonicity, some regularity for the system (3.12). Similar results can be obtain for (3.11) and (3.13) by similar arguments.

Theorem 3.2.1. *Assume that the map T is affine and that U_0, F and G are Lipschitz continuous. Let U be a solution of (3.12). If both U_0 and (G, F) are monotone and either U_0 and G are α monotone or F is α monotone (for some $\alpha > 0$), then U is Lipschitz in space, uniformly on $(0, t_f)$, for all $t_f > 0$.*

Proof. We here only prove the result in the case in which both U_0 and (G, F) are α monotone. The proof of the complete statement is given in the appendix. This proof also follows the argument in [57]. Once again we note the affine map $T = S + e$ with S linear and $e \in \mathbb{R}^d$ constant. We define W and Z_β by

$$W(t,x,\xi) =< U(t,x), \xi >;$$

$$Z_\beta(t,x,\xi) =< \xi, \nabla_x W(t,x,\xi) > -\beta|\nabla_x W(t,x,\xi)|^2.$$

We now write the PDE satisfied by Z_β :

$$
\begin{aligned}
&\partial_t Z_\beta + < F(x,\nabla_\xi W), \nabla_x Z_\beta > + < D_p F(x,\nabla_\xi W)\nabla_\xi Z_\beta, \nabla_x W > - \\
&\quad - < D_p G(x,\nabla_\xi W)\nabla_\xi Z_\beta, \xi > + \lambda(Z_\beta - Z_\beta(t,Tx,T\xi - e)) \\
&= < \nabla_x G(x,\nabla_x W)\xi, \xi > - < D_p G(x,\nabla_\xi W)\nabla_x W, \xi > - \\
&\quad - < \nabla_x F(x,\nabla_\xi W)\nabla_x W, \xi > + < D_p F(x,\nabla_\xi W)\nabla_x W, \nabla_x W > \\
&\quad - 2\beta < \nabla_x G(x,\nabla_x W)\xi, \nabla_x W > + 2\beta < \nabla_x F(\nabla_\xi W)\nabla_x W, \nabla_x W > \\
&\quad + \beta\lambda\Big(|\nabla_x W|^2 - 2 < \nabla_x W(t,Tx,T\xi - e), S\nabla_x W > + |\nabla_x W(t,Tx,T\xi - e)|^2\Big).
\end{aligned}
\tag{3.14}
$$

For the sake of clarity, we do not detail the calculation which leads to the previous equation, which easily follows from the chain rule. Let us remark that the following holds :

$$
\begin{aligned}
|\nabla_x W|^2 - |S\nabla_x W|^2 \leq & |\nabla_x W|^2 - 2 < \nabla_x W(t,Tx,T\xi - e), S\nabla_x W > \\
& + |\nabla_x W(t,Tx,T\xi - e)|^2.
\end{aligned}
$$

Because (G,F) and U_0 are α monotone, we deduce that :

$$
\begin{aligned}
&\partial_t Z_\beta + < F(x,\nabla_\xi W), \nabla_x Z_\beta > + < D_p F(x,\nabla_\xi W)\nabla_\xi Z_\beta, \nabla_x W > - \\
&\quad - < D_p G(x,\nabla_\xi W)\nabla_\xi Z_\beta, \xi > + \lambda(Z_\beta - Z_\beta(t,Tx,T\xi - e)) \\
&\geq \alpha(|\xi|^2 + |\nabla_x W|^2) + \beta\lambda(|\nabla_x W|^2 - |S\nabla_x W|^2) \\
&\quad - 2\beta < \nabla_x G(x,\nabla_x W)\xi, \nabla_x W > + 2\beta < \nabla_x F(\nabla_\xi W)\nabla_x W, \nabla_x W >; \\
&\geq \alpha(|\xi|^2 + |\nabla_x W|^2) + \beta\lambda(1 - ||S||^2)|\nabla_x W|^2 \\
&\quad - 2\beta(||\nabla_x G|| \cdot |\xi| \cdot |\nabla_x W| + ||\nabla_x F|| \cdot |\nabla_x W|^2).
\end{aligned}
$$

Hence, if β satisfies the inequality (3.15), then Z_β is positive for all time (by lemma 3.A.1 in appendix).

$$
\beta \leq \min\left(\frac{\alpha}{\lambda(||S||^2 - 1)^+ + ||\nabla_x G|| + 2||\nabla_x F||}, ||\nabla_x U_0||\right). \tag{3.15}
$$

From the fact that Z_β is positive for all time, we deduce that :

$$
\forall t, x, \xi : |\nabla_x W(t,x,\xi)| \leq \beta^{-1}|\xi|.
$$

Hence U is β^{-1} Lipschitz, uniformly in time.

□

Remark 3.2.3. *Let us notice that the constant β involved in the proof only depends on α, t_f, $\lambda(||DT||^2 - 1)^+$ and on the Lipschitz constants of U_0, F and G. Thus if the application S is non-expansive, then the constant β can be chosen independently of λ and T, thus we can expect that the sequence of solutions $(U_\lambda)_{\lambda>0}$ has a limit as λ goes to infinity, extracting a subsequence if necessary.*

This result emphasizes the fact that the monotonicity of the system "implies" the regularity of the solution. Indeed the Lipschitz regularity is enough to prove higher order regularity see [57] for instance. Finally we end this discussion on the well-posedness of this system of conservation laws with this straightforward application :

Theorem 3.2.2. *Let $\mathcal{X}$ be a compact set of affine functions and μ a measure on $\mathcal{X}$. Then, under the assumptions of theorem 3.2.1, there exists a unique smooth monotone solution U of :*

$$\begin{cases} \partial_t U + (F(x,U)\cdot\nabla)U + \lambda \int_{\mathcal{X}} (U - (DT)^* U(t,Tx)) d\mu(T) = G(x,U) \text{ in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}. \end{cases}$$

3.2.4 Asymptotic differential operators

In this section, we present how we can derive, from a common noise term, higher order terms in (3.12)(first order and second order terms) which conserve the monotonicity property, and thus the well-posedness of this equation. In the previous section we showed that adding the term (T is assumed to be linear in all the rest of this section) :

$$\lambda(U - T^*U(T\cdot))$$

does not alter the propagation of monotonicity. Formally, if we set $\lambda = \epsilon^{-1}$ and $T = Id + \epsilon S$, then

$$\epsilon^{-1}(U - T^*U(T\cdot)) \underset{\epsilon\to 0}{\longrightarrow} -(Sx\cdot\nabla)U(x) - S^*U(x).$$

Also if we define $\lambda = \epsilon^{-2}$; $T_+ = Id + \epsilon S$ and $T_- = Id - \epsilon S$, then once again formally :

$$\epsilon^{-2}(U - T_+^*U(T_+\cdot) + U - T_-^*U(T_-\cdot)) \underset{\epsilon\to 0}{\longrightarrow} -2S^*(S\cdot\nabla)U(x) - (Sx\cdot\nabla^2\cdot Sx)U(x).$$

Thus the propagation of monotonicity shall hold for these two equations :

$$\begin{cases} \partial_t U + (F(x,U)\cdot\nabla)U - (Sx\cdot\nabla)U(x) - S^*U(x) = G(x,U) \text{ in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}; \end{cases} \tag{3.16}$$

$$\begin{cases} \partial_t U + (F(x,U)\cdot\nabla)U - 2S^*(Sx\cdot\nabla)U(x) - 2(Sx\cdot\nabla^2\cdot Sx)U = G(x,U) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}. \end{cases} \tag{3.17}$$

To avoid going into technical difficulties we do not address here the question of the regularity of (3.16) and (3.17). We restrict ourselves to the proof of the fact that the monotonicity indeed propagates, as it is expected. The following result presents this fact.

Theorem 3.2.3. *Assume that U_0 and (G, F) are monotone and let U be a solution of either* (3.16) *or* (3.17)*, then U is monotone for all time.*

Proof. We begin with the case of (3.16). We consider U and V two solutions of (3.16). As we did in the proof of the proposition 3.2.1, we define W by

$$W(t,x,y) =< U(t,x) - V(t,y), x - y > .$$

Let us remark that W satisfies :

$$\begin{aligned}\partial_t W + & F(x,U)\cdot\nabla_x W + F(y,V)\cdot\nabla_y W - < (\nabla_x W - \nabla_y W), Sx - Sy > \\ & =< G(x,U) - G(y,V), x - y > + < F(x,U) - F(y,V), U(t,x) - V(t,y) > .\end{aligned}$$

Hence, concluding as we did in proposition 3.2.1, if W is positive at the initial time, it stays positive for all time and thus the propagation of monotonicity is proved. If U and V are solutions of (3.17), then W satisfies :

$$\begin{aligned}\partial_t W + & F(x,U)\cdot\nabla_x W + F(y,V)\cdot\nabla_y W - (Sx\cdot\nabla_{xx}\cdot Sx)W \\ & - (Sy\cdot\nabla_{yy}\cdot Sy)W - 2(Sx\cdot\nabla_{yx}\cdot Sy)W \\ & =< G(x,U) - G(y,V), x - y > + < F(x,U) - F(y,V), U(t,x) - V(t,y) > .\end{aligned}$$

Once again, we can conclude with lemma 3.A.1 (in appendix) that if W is positive at the initial time it remains positive for all time. □

3.3 From MFG to agent based models

In this last part, we investigate to which extent MFG models can approach agent based models (ABM). The terminology ABM is often used in Economics and other sciences to refer to a model in which macroscopic phenomena are captured by aggregating individual actions. The main difference with MFG models is that in the ABM, the agents do not anticipate on the behavior of the other agents, hence they are not entirely rational but obey a given "mechanical" rule. For this reason, ABM are purely forward models. The main idea of this section is that one can measure the weight of anticipation for a given agent in a MFG model through the inter temporal preference parameter, thus taking this parameter to infinity should allow to recover an agent-based model from a MFG model. Indeed the players will no longer anticipate as they have no interest in the future. In the following MFG

system the inter temporal preference parameter is always denoted by λ (which has nothing to do with the parameter λ in the part on the common noise) :

$$\begin{cases} -\partial_t u - \nu\Delta u + H(t,x,\nabla u, m) + \lambda u = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ \partial_t m - \nu\Delta m - div(D_p H(t,x,\nabla u, m)m) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ u(T) = \phi(m(T)); m(0) = m_0 \text{ in } \mathbb{R}^d. \end{cases} \tag{3.18}$$

In this system, $H(x,p,m)$ still denotes the Hamiltonian of the optimal control problem faced by the players. As λ increases, the players are more and more interested in the present over the future. The main idea of this section is that letting λ go to infinity, one expects that u goes to 0 and that (3.18) reduces to

$$\begin{cases} \partial_t m - \nu\Delta m - div(D_p H(t,x,0,m)m) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ m(0) = m_0. \end{cases} \tag{3.19}$$

In the first section we work in a particular case and show how we can establish such a convergence. In the second section, we indicate some extensions of this model without giving proofs. Because this remark on MFG is quite general, we prefer to explain the approach on a simple example rather than trying to obtain the most general results.

3.3.1 An example of convergence results

We here give some assumptions we make on the Hamiltonian for this section :
- H is Lipschitz in x, t and p, uniformly in m.
- $H : m \to ((t,x,p) \to H(t,x,p,m))$ is continuous from $\mathcal{C}((0,T),\mathcal{P})$ to the set of Lipschitz functions.

The following result captures the idea that MFG models approach ABM.

Theorem 3.3.1. *Let $m_0 \in L^2(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} m_0(x)|x|^2 dx < \infty$. For any $\lambda > 0$, we denote by $(u_\lambda, m_\lambda) \in C^{1,2,\alpha} \times \mathcal{C}((0,T),\mathcal{P})$ a solution of (3.18) (with $0 < \alpha < 1$). This sequence is bounded and any limit point m_* of the sequence $(m_\lambda)_{\lambda>0}$ is a weak solution of :*

$$\begin{cases} \partial_t m - \nu\Delta m - div(D_p H(t,x,0,m)m) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ m(0) = m_0 \text{ in } \mathbb{R}^d. \end{cases}$$

Proof. From classical estimates on Hamilton-Jacobi-Bellman equations and on Fokker-Planck equations, we know that solutions of (3.18) exist for all $\lambda > 0$, as a consequence of Schauder's fixed point theorem. For any sequence $(u_\lambda, m_\lambda)_\lambda$

of solutions of (3.18), $(u_\lambda)_{\lambda>0}$ is bounded in $L^2(H^2)$. Indeed, if we multiply the equation satisfied by u_λ by $-\Delta u_\lambda$ and we integrate, we obtain

$$-\int_t^T \int_{\mathbb{R}^d} \frac{1}{2}\frac{d}{dt}|\nabla u_\lambda|^2 + \nu \int_t^T \int_{\mathbb{R}^d} (\Delta u_\lambda)^2 + \lambda \int_t^T \int_{\mathbb{R}^d} |\nabla u_\lambda|^2 = \int_t^T \int_{\mathbb{R}^d} \Delta u_\lambda H(s,x,\nabla u_\lambda, m_\lambda) dx ds; \tag{3.20}$$

we then deduce

$$\nu \int_t^T \int_{\mathbb{R}^d} (\Delta u_\lambda)^2 + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_\lambda|^2(t) \leq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_\lambda|^2(T) + \int_t^T \int_{\mathbb{R}^d} \Delta u_\lambda H(s,x,\nabla u_\lambda, m_\lambda) dx ds.$$

We obtain from the growth assumption on H that $(\Delta u_\lambda)_\lambda$ is a bounded sequence of $L^2((0,T)\times\mathbb{R}^d)$. Thus, $(u_\lambda)_\lambda$ converges to 0 in $L^2((0,T), H^1(\mathbb{R}^d))$. Because $(D_p H(\cdot,\cdot,\nabla u_\lambda, m_\lambda))_\lambda$ is a bounded sequence in L^∞, $(m_\lambda)_\lambda$ is a bounded sequence of $\mathcal{C}((0,T),\mathcal{P})$ where we have equipped $\mathcal{P}$ with the Monge-Kantorovich distance. In fact, for any $\lambda > 0$, m_λ is 1/2-Hölder continuous with a modulus of continuity which depends only on T and the bound on $(D_p H(\cdot,\cdot,\nabla_\lambda, m_\lambda))_\lambda$. The sequence $(m_\lambda)_{\lambda>0}$ has also a bounded second order moment, uniformly in $t \leq T$ and λ. Thus $(m_\lambda)_{\lambda>0}$ is compact in $\mathcal{C}((0,T),\mathcal{P})$. By passing to the limit in the weak formulation of the equation satisfied by m_λ, we deduce that any limit point of this sequence is a weak solution of the Fokker-Planck equation (3.19). □

We now show that an ABM can be approximated by a MFG system. The result we present here is that, if it is regular enough, there is always at least one solution of the agent based model which is the limit of a MFG. Indeed let us take an ABM defined by a smooth vector field B such that :

$$\exists C > 0, \forall t \in (0,T), \forall x \in \mathbb{R}^d, \forall m \in \mathcal{C}((0,T),\mathcal{P}), ||B(t,x,m)||_{L^\infty} \leq C.$$

The corresponding so called agent-based model is then :

$$\begin{cases} \partial_t m - \nu\Delta m - div(B(t,x,m)m) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ m(0) = m_0. \end{cases} \tag{3.21}$$

Let us take any $\delta > 0$ and define by H the following hamiltonian :

$$H(t,x,p,m) = B(t,x,m)\cdot p + \delta|p|^2.$$

Proposition 3.3.1. *For any sequence of solutions $(u_\lambda, m_\lambda)_\lambda$ of :*

$$\begin{cases} -\partial_t u - \nu\Delta u + H(t,x,\nabla u, m) + \lambda u = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ \partial_t m - \nu\Delta m - div(D_p H(t,x,\nabla u, m)m) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ u(T) = 0; m(0) = m_0 \text{ in } \mathbb{T}^d; \end{cases} \tag{3.22}$$

any accumulating point of the bounded sequence $(m_\lambda)_\lambda$ is a solution of the ABM (3.21).

Proof. In view of theorem 3.3.1, the sequence of solutions of the MFG system (3.22) is compact and its accumulating points are solutions of the agent-based model(3.21). □

3.3.2 More general models

In this section we give examples of how one can apply the previous remark to more general models. The first one is a MFG, where a part of the running cost has a proportion of the total cost, which varies with the parameter λ. The second one is a higher order approximation of the limit model. We conclude with cases in which a common noise can occur.

Relative running cost

We here present the limit as λ goes to infinity of the following MFG system :

$$\begin{cases} -\partial_t u - \nu\Delta u + H(x, \nabla u, m) + \lambda(u - \psi(m)) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ \partial_t m - \nu\Delta m - div(mD_pH(x, \nabla u, m)) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ u(T) = \phi(m(T)); m(0) = m_0 \text{ in } \mathbb{R}^d; \end{cases} \tag{3.23}$$

where ψ is a smooth function of m. The interpretation of such a system, in particular of such an HJB equation, is that $\psi(m)$ stands for a cost which has an increasing importance for the players as λ goes to infinity. This cost should be interpreted as a cost the players anticipate with the same weight, whatever the length of the game is. The ABM limit of (3.23) is the following :

$$\begin{cases} \partial_t m - \nu\Delta m - div(mD_pH(x, \nabla_x(\psi(m)), m)) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ m(0) = m_0 \text{ in } \mathbb{R}^d. \end{cases} \tag{3.24}$$

Let us remark that from the results on uniqueness in MFG ([54, 57]), the usual conditions under which (3.23) has a unique solution, implies in particular that ψ is monotone in m and H is convex in p. On the other hand, the higher order term in the first line of (3.24) is (in the case of a local ψ to simplify notation) :

$$-\nu\Delta m - m\psi'(m)tr(D_{pp}H(x, \nabla_x(\psi(m)), m)\cdot\nabla^2_{xx}m);$$

where tr stands for the trace operator. Hence, if the Hamiltonian H is convex in p and ψ is monotone in m, we deduce that (3.24) is a non-linear parabolic equation. This remark makes an obvious link between the well-posedness of the two models and the monotonicity of ψ.

Higher order approximation

We present here a higher order approximation of the Fokker-Planck equation in (3.18) in the limit λ goes to infinity. Let us observe that, at least formally, we have the following in $(0,T)\times\mathbb{T}^d$:

$$u(t,x) \underset{\lambda\to+\infty}{\approx} \frac{-1}{\lambda}H(x,0,m) + O(\lambda^{-2}).$$

Thus if we neglect terms of order λ^{-2}, the resulting Fokker-Planck equation is :

$$\begin{cases} \partial_t m - \nu\Delta m - div(mD_pH(x,\nabla_x(\frac{-1}{\lambda}H(x,0,m)),m)) = 0 \text{ in } (0,T)\times\mathbb{R}^d; \\ m(0) = m_0 \text{ in } \mathbb{R}^d. \end{cases} \tag{3.25}$$

As we did for the relative running cost, we show that under the usual condition of uniqueness for (3.18), the equation (3.25) is parabolic. The higher order term in m in (3.25) is, when H is local in m :

$$-\nu\Delta m + \frac{1}{\lambda}mD_zH(x,0,m)tr(D_{pp}H(x,\nabla_x(\frac{-1}{\lambda}H(x,0,m)),m)\cdot\nabla^2_{xx}m);$$

where tr still stands for the trace operator and D_zH for the derivative of H with respect to its third variable. We recall here the usual conditions under which (3.18) has a unique solution ([57]) : for any $(x,p,z)\in\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}_+$:

$$\begin{pmatrix} zD_{pp}H(x,p,z) & \frac{z}{2}D_{pz}H(x,p,z) \\ \frac{z}{2}D_{pz}H(x,p,z) & -D_zH(x,p,z) \end{pmatrix} \geq 0.$$

Thus, as in the previous case, the uniqueness of solutions of (3.18) implies the well-posedness of (3.25). This strongly suggests that the previous assumption on the derivative of H is close from being necessary for the uniqueness in (3.18).

The case of common noise : discrete state space

We now explain why, formally, the addition of a common noise should not alter the property that, in the limit λ goes to ∞, a MFG converges toward an ABM. We present first a discrete state space case (cf section 3.2), and next a continuous state space case with the notations of [27].

For the discrete state space case, we work with a common noise described by an intensity of jump $\beta > 0$ and a linear map T. The map T is assumed to be linear for the sake of clarity of the derivation of the equation (3.28), but this result apply

to more general maps T. We introduce the solution U of

$$\begin{cases} \partial_t U(t,x) + (F(x,U)\cdot\nabla)U + \lambda U + \beta(U - T^*U(t,Tx)) = G(x,U) \text{ in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(t_f,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}. \end{cases} \tag{3.26}$$

We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $(\tau_i)_{i\in\mathbb{N}}$ a sequence of independent random variables of exponential law of parameter β. We define the sequence $(t_i)_{i\in\mathbb{N}}$ by

$$t_i = \sum_{k=0}^{i} \tau_k.$$

We define for any $t > 0$, $x_0 \in \mathbb{R}^d$ the couple $(V_s, Y_s)_{0\le s\le t}$ by the following :

$$\begin{cases} d(e^{\lambda s}V_s) = G(Y_s, U(s,Y_s))ds; \forall s \in (t_i, t_{i+1}); \\ dY_s = F(Y_s, U(s,Y_s))ds; \forall s \in (t_i, t_{i+1}); \\ Y_{t_i^+} = TY_{t_i^-}; V_{t_i^+} = T^*V_{t_i^-} \\ Y_t = x_0; V_0 = U_0(Y(0)). \end{cases} \tag{3.27}$$

This couple (V_s, Y_s) represents the characteristic associated to (3.26) in the sense that if we defined $\tilde{U}$ by

$$\tilde{U}(t,x_0) = \mathbb{E}[V_t]$$

with $(V_s, Y_s)_{s\ge 0}$ being defined by (3.27), then a formal calculation yields that $\tilde{U}$ solves :

$$\begin{cases} \partial_t \tilde{U}(t,x) + (F(x,U)\cdot\nabla)\tilde{U} + \lambda\tilde{U} + \beta(\tilde{U} - T^*\tilde{U}(t,Tx)) = G(x,U) \text{ in } (0,t_f)\times\mathbb{R}^{\mathbb{d}}; \\ U(0,x) = U_0(x) \text{ in } \mathbb{R}^{\mathbb{d}}. \end{cases} \tag{3.28}$$

Thus formally $\tilde{U} = U$. Hence, the process $(Y_s)_{s\ge 0}$ should represent a realization of the evolution of a population of player facing the MFG modeled by (3.26) and starting from the configuration x_0. Formally, in the limit λ goes to infinity, we obtain that that U goes to 0 uniformly in x, and thus the evolution of the process $(Y_s)_{s\ge 0}$ is given by :

$$\begin{cases} dY_s = F(Y_s, 0)ds; \forall s \in (t_i, t_{i+1}); \\ Y_0 = x_0; Y_{t_i^+} = TY_{t_i^-}; \end{cases} \tag{3.29}$$

where the sequence $(t_i)_{i\in\mathbb{N}}$ has the same law as in (3.27). Hence, the "model" (3.29) is the limit ABM one should obtain in the limit λ goes to infinity for (3.26).

The case of common noise : the infinite dimensional case

Assume now that the MFG is in infinite dimension and that the master equation is given by :

$$\begin{cases} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) - (1+\beta)\int_{\mathbb{R}^d} div_y[D_m U]dm(y) \\ + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U)dm(y) - 2\beta \int_{\mathbb{R}^d} div_x[D_m U]dm(y) \\ -\beta \int_{\mathbb{R}^{2d}} Tr[D^2_{mm} U]dm \otimes dm + \lambda U = F(x,m) \text{ in } [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d); \\ U(T,x,m) = G(x,m) \text{ in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d). \end{cases} \tag{3.30}$$

In this setting H is the hamiltonian of the optimal control problem faced by the players, F the running cost and G the terminal cost. We refer to [27] for a precise description of the MFG this master equation models and for definitions of the derivatives in the space of measures. We just recall here that β is positive parameter which describes the intensity of the common noise ; that F and G satisfies some strong regularity assumptions and that H is smooth, globally Lipschitz and coercive in its second argument. We denote by $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space. In the context modeled by (3.30), the forward-backward system (3.1) takes the form :

$$\begin{cases} d_t u_t = \{-(1+\beta)\Delta u_t + H(x, D_x u_t) + \lambda u_t - F(x, m_t) - \sqrt{2\beta} div(v_t)\}dt + v_t \cdot \sqrt{2\beta} dW_t \\ \text{ in } (0,T) \times \mathbb{R}^d; \\ d_t m_t = \{(1+\beta)\Delta m_t + div(m_t D_p H(x, D_x u_t))\}dt - div(m_t \sqrt{2\beta} W_t) \text{ in } (0,T) \times \mathbb{R}^d; \\ m_0 = m_0; u_T = G(x, m_T) \text{ in } \mathbb{R}^d. \end{cases} \tag{3.31}$$

The process $(W_t)_{0 \le t \le T}$ is a standard d dimensional bronwnian motion under the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The unknown are the three random functions $(u_t, m_t, v_t)_{0 \le t \le T}$ and $(v_t)_{0 \le t \le T}$ is a term which forces u_t to be measurable with respect to the filtration generated by $(W_t)_{0 \le t}$. Formally as λ goes to infinity, the solution U of (3.30) goes to 0 and thus (3.31) reduces to :

$$\begin{cases} d_t m_t = \{(1+\beta)\Delta m_t + div(m_t D_p H(x, 0))\}dt - div(m_t \sqrt{2\beta} W_t) \text{ in } (0,T) \times \mathbb{R}^d; \\ m_0 = m_0 \text{ in } \mathbb{R}^d. \end{cases} \tag{3.32}$$

This nonlinear stochastic Fokker-Planck equation is the limit ABM.

Appendix

3.A A maximum principle result

We present a maximum principle result which is quite general and that we use several times in this paper. Although this result is not new (see [57] for example), we recall it and give a simple proof for the sake of completeness.

Lemma 3.A.1. *Let u be a smooth function from $(0,T)\times\mathbb{R}^d$ to $\mathbb{R}$ which satisfies :*

$$\begin{cases}\partial_t u - A(t,x,\nabla_x^2 u) + B(t,x,\nabla_x u) + \alpha(t,x)(u - u(t,f(x))) \geq 0 \text{ in } (0,T)\times\mathbb{R}^d;\\ u(0)\geq 0 \text{ in } \mathbb{R}^d;\end{cases} \tag{3.33}$$

where A is a uniform elliptic term which is linear in its third argument, B is linear with respect to its third argument, $\alpha\in L^\infty((0,T)\times\mathbb{R}^d)$, $\alpha\geq 0$ and f maps $\mathbb{R}^d$ into itself. Then $u\geq 0$ on $(0,T)\times\mathbb{R}^d$.

Proof. Let us assume that there exists $(t_0,x_0)\in(0,T)\times\mathbb{R}^d$ such that $u(t_0,x_0)<0$. Thus, for any $\lambda\geq 0$, there exists $\delta>0$ such that

$$e^{-\lambda t_0}u(t_0,x_0) + \delta t_0 < 0. \tag{3.34}$$

We define v by

$$v(t,x) = e^{-\lambda t}u(t,x) + \delta(t+1).$$

This function satisfies

$$\begin{cases}\partial_t v - A(t,x,\nabla_x^2 v) + e^{-\lambda t}B(t,x,\nabla_x u) + \alpha(t,x)(v - v(t,f(x))) - \lambda v - \lambda\delta(t+1)\geq\delta\\ \quad\text{in } (0,T)\times\mathbb{R}^d;\\ v(0)>0 \text{ in } \mathbb{R}^d;\\ v(t_0,x_0)<0.\end{cases} \tag{3.35}$$

Thus, there exists (t_1,x_1) such that :

$$\begin{cases}v(t_1,x_1)=0\\ \partial_t v(t_1,x_1)\leq 0\\ \nabla_x^2 v(t_1,x_1)\geq 0\\ \nabla_x v(t_1,x_1)=0\\ v(t,x)\geq 0, \forall t\leq t_1, \forall x.\end{cases}$$

Because $\nabla_x v(t_1,x_1)=0$ implies that $\nabla_x u(t_1,x_1)=0$, evaluating the PDE satisfied by u, we deduce that

$$-\delta(1+\lambda(t_1+1))\geq 0.$$

Thus we obtain the desired contradiction. □

3.B Proof of Theorem 3.2.1 under general assumptions

The proof of Theorem 3.2.1 follows the argument from [57]. We show first the case U_0 and G are α monotone, and then the case F is α monotone.

3.B.1 U_0 and G are α monotone

As we did above, we define W and Z_β by

$$\begin{aligned} W(t,x,\xi) &=< U(t,x), \xi >; \\ Z_\beta(t,x,\xi) &=< \xi, \nabla_x W(t,x,\xi) > -\beta(t)|\nabla_x W(t,x,\xi)|^2; \end{aligned} \tag{3.36}$$

except that now β can be a function of the time. The PDE satisfied by Z_β is :

$$\begin{aligned} &\partial_t Z_\beta + < F(x,\nabla_\xi W), \nabla_x Z_\beta > + < D_p F(x,\nabla_\xi W)\nabla_\xi Z_\beta, \nabla_x W > - \\ &\quad - < D_p G(x,\nabla_\xi W)\nabla_\xi Z_\beta, \xi > +\lambda(Z_\beta - Z_\beta(t,Tx,T\xi - e)) \\ &= < \nabla_x G(x,\nabla_x W)\xi, \xi > - < D_p G(x,\nabla_\xi W)\nabla_x W, \xi > - \\ &\quad - < \nabla_x F(x,\nabla_\xi W)\nabla_x W, \xi > + < D_p F(x,\nabla_\xi W)\nabla_x W, \nabla_x W > \\ &\quad - 2\beta < \nabla_x G(x,\nabla_x W)\xi, \nabla_x W > +2\beta < \nabla_x F(\nabla_\xi W)\nabla_x W, \nabla_x W > \\ &\quad + \beta\lambda\left(|\nabla_x W|^2 - 2 < \nabla_x W(t,Tx,T\xi - e), S\nabla_x W > +|\nabla_x W(t,Tx,T\xi - e)|^2\right) \\ &\quad - \frac{d}{dt}\beta|\nabla_x W|^2. \end{aligned} \tag{3.37}$$

This equation reduces to (using this time only the α monotonicity of G) :

$$\begin{aligned} &\partial_t Z_\beta + < F(x,\nabla_\xi W), \nabla_x Z_\beta > + < D_p F(x,\nabla_\xi W)\nabla_\xi Z_\beta, \nabla_x W > - \\ &\quad - < D_p G(x,\nabla_\xi W)\nabla_\xi Z_\beta, \xi > +\lambda(Z_\beta - Z_\beta(t,Tx,T\xi - e)) \\ &\geq \alpha|\xi|^2 - 2\beta < \nabla_x G(x,\nabla_x W)\xi, \nabla_x W > +2\beta < \nabla_x F(\nabla_\xi W)\nabla_x W, \nabla_x W > \\ &\quad + \beta\lambda(|\nabla_x W|^2 - |S\nabla_x W|^2) - \frac{d}{dt}\beta|\nabla_x W|^2. \end{aligned} \tag{3.38}$$

We then deduce that :

$$\begin{aligned} &\partial_t Z_\beta + < F(x,\nabla_\xi W), \nabla_x Z_\beta > + < D_p F(x,\nabla_\xi W)\nabla_\xi Z_\beta, \nabla_x W > - \\ &\quad - < D_p G(x,\nabla_\xi W)\nabla_\xi Z_\beta, \xi > +\lambda(Z_\beta - Z_\beta(t,Tx,T\xi - e)) \\ &\geq \alpha|\xi|^2 - \beta||\nabla_x G|| \cdot |\xi|^2 - \beta||\nabla_x G|| \cdot |\nabla_x W|^2 - 2\beta||\nabla_x F(\nabla_\xi W)|| \cdot |\nabla_x W|^2 \\ &\quad + \beta\lambda(|\nabla_x W|^2 - |S\nabla_x W|^2) - \frac{d}{dt}\beta|\nabla_x W|^2. \end{aligned} \tag{3.39}$$

Thus if we define β by :

$$\beta(t) = \alpha e^{-(2||\nabla_x F||+||\nabla_x G||+(||S||^2-1)^+\lambda)t}; \tag{3.40}$$

then Z_β is positive for all time because of lemma 3.A.1. We then conclude the proof as we did in the case in which U_0, G and F are α monotone.

3.B.2 F is α monotone

We still define W by (3.36) and we define $Z_{\beta,\gamma}$ by :

$$Z_\beta(t,x,\xi) =< \xi, \nabla_x W(t,x,\xi) > -\beta(t)|\nabla_x W(t,x,\xi)|^2 + \gamma(t)|\xi|^2.$$

Here, both β and γ are function of the time we shall define later. The PDE satisfied by $Z_{\beta,\gamma}$ is :

$$\begin{aligned}
&\partial_t Z_{\beta,\gamma} + < F(x,\nabla_\xi W), \nabla_x Z_{\beta,\gamma} > + < D_p F(x,\nabla_\xi W)\nabla_\xi Z_{\beta,\gamma}, \nabla_x W > - \\
&\quad - < D_p G(x,\nabla_\xi W)\nabla_\xi Z_{\beta,\gamma}, \xi > + \lambda(Z_{\beta,\gamma} - Z_{\beta,\gamma}(t,Tx,T\xi - e)) \\
&= < \nabla_x G(x,\nabla_x W)\xi, \xi > - < D_p G(x,\nabla_\xi W)\nabla_x W, \xi > - \\
&\quad - < \nabla_x F(x,\nabla_\xi W)\nabla_x W, \xi > + < D_p F(x,\nabla_\xi W)\nabla_x W, \nabla_x W > \\
&\quad - 2\beta < \nabla_x G(x,\nabla_x W)\xi, \nabla_x W > + 2\beta < \nabla_x F(\nabla_\xi W)\nabla_x W, \nabla_x W > \\
&\quad + \beta\lambda\left(|\nabla_x W|^2 - 2 < \nabla_x W(t,Tx,T\xi - e), S\nabla_x W > + |\nabla_x W(t,Tx,T\xi - e)|^2\right) \\
&\quad - \frac{d}{dt}\beta|\nabla_x W|^2 + 2\gamma(< D_p F(x,\nabla_\xi W)\xi, \nabla_x W > - < D_p G(x,\nabla_\xi W)\xi, \xi >) \\
&\quad + \frac{d}{dt}\gamma|\xi|^2 + \lambda\gamma(|\xi|^2 - |T\xi - e|^2).
\end{aligned} \tag{3.41}$$

Thus, we deduce the inequality :

$$\begin{aligned}
&\partial_t Z_{\beta,\gamma} + < F(x,\nabla_\xi W), \nabla_x Z_{\beta,\gamma} > + < D_p F(x,\nabla_\xi W)\nabla_\xi Z_{\beta,\gamma}, \nabla_x W > - \\
&\quad - < D_p G(x,\nabla_\xi W)\nabla_\xi Z_{\beta,\gamma}, \xi > + \lambda(Z_{\beta,\gamma} - Z_{\beta,\gamma}(t,Tx,T\xi - e)) \\
&\geq \alpha|\nabla_x W|^2 - \beta||\nabla_x G|| \cdot |\xi|^2 - \beta||\nabla_x G|| \cdot |\nabla_x W|^2 - 2\beta||\nabla_x F(\nabla_\xi W)|| \cdot |\nabla_x W|^2 \\
&\quad + \beta\lambda(|\nabla_x W|^2 - |S\nabla_x W|^2) - \frac{d}{dt}\beta|\nabla_x W|^2 - \gamma(||D_p F|| - 2||D_p G||)|\xi|^2 \\
&\quad - \gamma||D_p F|| \cdot |\nabla_x W|^2 + \frac{d}{dt}\gamma|\xi|^2 + \lambda\gamma(|\xi|^2 - |T\xi - e|^2).
\end{aligned} \tag{3.42}$$

Hence, if β and γ satisfies

$$\begin{cases}
\alpha - \beta[||\nabla_x G|| + 2||\nabla_x F|| - \lambda(1 - ||S||^2)] - \frac{d}{dt}\beta - \gamma||D_p F|| \geq 0; \\
\frac{d}{dt}\gamma + \gamma[\lambda(1 - ||S||^2) - ||D_p F|| - 2||D_p G||] - \beta||\nabla_x G|| \geq 0;
\end{cases} \tag{3.43}$$

then the right hand side of (3.42) is positive for all time $t \in (0, t_f)$. Moreover let us remark that, at the initial time, because U_0 is monotone :

$$Z_{\beta,\gamma} \geq (\gamma(0) - \beta(0)||D_x U_0||^2)|\xi|^2.$$

Thus if we define β and γ by

$$\begin{cases} \frac{d}{dt}\beta = \alpha - \beta[||\nabla_x G|| + 2||\nabla_x F|| + \lambda(1 - ||S||^2)^-] - \gamma||D_p F||; \\ \frac{d}{dt}\gamma = \gamma[-\lambda(1 - ||S||^2)^- + ||D_p F|| + 2||D_p G||] + \beta||\nabla_x G||; \\ \beta(0) > 0; \gamma(0) = \beta(0)||D_x U_0||^2; \end{cases} \tag{3.44}$$

then $Z_{\beta,\gamma}$ is positive at $t = 0$ and the right hand side of (3.42) is positive for all time. Thus, if $\beta(t) > 0$ for all $t \in [0, t_F]$, then by lemma 3.A.1 we conclude that $Z_{\beta,\gamma}$ is positive for all time and the theorem follows immediately. Now let us remark that one can always choose $\beta(0)$ small enough so that for all time $t \in [0, t_f]$:

$$\begin{cases} 0 \leq \beta(t) \leq \alpha t + \beta(0); \\ \gamma(t) \leq \beta(0)||D_x U_0||^2 e^{(\lambda(1-||S||^2)^- + ||D_p F|| + 2||D_p G||)t} + \beta(0)||D_x U_0||^2 \alpha \frac{t^2}{2} + \beta(0)||\nabla_x G||t. \end{cases}$$

Thus the theorem is proved.

Chapitre 4

Une remarque sur l'algorithme d'Uzawa et une application aux systèmes de jeux à champ moyen

On présente dans ce chapitre une extension de l'algorithme d'Uzawa que nous appliquons à la construction de suites approchant des solutions de systèmes MFG. On prouve que les itérations d'Uzawa peuvent être utilisées dans des situations plus générales que celles relevant du cas classique. On présente ensuite quelques résultats numériques issus de ces itérations sur des systèmes de MFG d'arrêt optimal, de contrôle impulsionnel et de contrôle continu classique.

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Introduction

This paper is concerned with the study of an extension of Uzawa's algorithm. We show that the standard Uzawa's algorithm can be used to find solutions of systems similar to the ones characterizing saddle points of lagrangians, even though there is not a proper langrangian associated with this system. The second part of this paper is concerned with the application of this remark to build approximating sequences of solutions of Mean Field Games (MFG) systems.

Uzawa's algorithm was introduced to solve minimization problems with constraints. The main idea of this algorithm is to use a projected gradient descent on the dual problem. Because of its simplicity and efficiency, Uzawa's algorithm is often used in practical problems. We recall that the output of this algorithm is a sequence which converges toward the solution of the primal minimization problem. In the first part of this paper we prove that we can use the same algorithm to find solutions of a wider class of systems than the ones which characterize saddle points of lagrangians.

Next, we take full advantage of this remark to build approximating sequences for several MFG systems. MFG have been introduced by J.-M. Lasry and P.-L. Lions in [54] and independently and in a particular case by M. Huang, P. Caines and R. Malhamme in [48]. The theory of MFG is concerned with Nash equilibria of differential games with infinitely mainly indiscernable players, who interacts only through mean field type terms. We refer to [57] for a detailed presentation of MFG and to [30] for a complete presentation of the probabilistic theory of MFG. In general, the study of a MFG requires to solve the so-called master equation, see [27, 57], but in the case when there is no common noise, the problem reduces to a system of Partial Differential Equations (PDE). It is well known that in the so-called potential case, MFG systems can be interpreted as the optimality conditions for an optimal control problem of a PDE, see [21, 54] for instance. Thus Uzawa's algorithm is a natural method we can apply to such optimal control problems. We show that, under monotonicity assumptions, we can apply an Uzawa's like algorithm to approximate solutions of MFG systems even in the non-potential case. In this paper we shall apply this algorithm to MFG systems of continuous control (i.e. as in [54]), optimal stopping (see [11]) and impulse control (see [12]).

The last part of this paper presents the results of the implementation of Uzawa's iterations to the discretized problems of MFG of optimal stopping, impulse control and continuous control.

Bibliographical comments

We here give some details on the bibliographical context in which this article takes place. Concerning the literature regarding Uzawa's algorithm, there exist plenty of results on this well known algorithm. Although, using this algorithm to find solutions of systems of inequalities seems to be used only in the case of linear system, as in [16, 38] for instance.

Concerning the MFG literature, the first numerical methods for MFG systems have been developed by Y. Achdou and I. Capuzzo-Dolcetta in [2]. Several other methods have been studied and some of them involved the optimal control interpretation in the potential case. Such methods are somehow similar to the one we present here because they are also the implementation of a search for saddle points. We refer to [7, 17] for example. The main novelties of our work is to consider the non potential case and that we consider the cases of optimal stopping and impulse control. Furthermore, we mention the papers [4, 39] of R. Ferreira, D. Gomes and al. in which the first order MFG system of continuous control is interpreted as a system of variational inequalities and solve numerically. The interpretation in terms of variational inequalities of the MFG system is central in the rest of this paper.

4.1 A remark on Uzawa's algorithm

4.1.1 Presentation of the standard algorithm

We present here the classical result of convergence of Uzawa's algorithm. Although we are going to present this algorithm from the point of view of the search of a saddle point, let us recall the well-known fact that given a convex minimization problem, Uzawa's algorithm is only the projected gradient ascent method applied on the dual problem. Let us take a lagrangian L defined by :

$$L(x,y) = F(x) + < a(x), b(y) >, \forall x \in K_1, \forall y \in K_2; \tag{4.1}$$

where K_1 is a closed convex subset of the Hilbert space $(H_1, (\cdot,\cdot))$ and K_2 is a closed convex subset of the Hilbert space $\Big(H_2, ((\cdot,\cdot))\Big)$. We denote by $(H_3, <\cdot,\cdot>)$ a third Hilbert space. The applications $a : H_1 \to H_3$ and $b : H_2 \to H_3$ are such that for all $y \in K_2$, $x \to < a(x), b(y) >$ is a convex application and $\tilde{K}_2 := b(K_2)$ is closed and convex. Moreover $F : H_1 \to \mathbb{R}$ is a convex function. The lagrangian L is associated to the following minimization problem :

$$\inf_{x\in K_1} \{F(x) + \sup_{y\in K_2} < a(x), b(y) >\}.$$

We recall that a saddle point of L is a couple $(x, y) \in K_1 \times K_2$ such that

$$\inf_{x' \in K_1} \sup_{y' \in K_2} L(x', y') = \sup_{y' \in K_2} \inf_{x' \in K_1} L(x', y') = L(x, y). \tag{4.2}$$

We fix a real number $\delta > 0$ and we denote by $\mathbf{P}_A$ the orthogonal projection on the set A in H_3. Uzawa's algorithm (with step δ) consists in building the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ as follows :

$$\begin{cases} y_0 \in K_2; \\ x_n = \underset{x \in K_1}{\operatorname{arginf}} \{F(x) + < a(x), b(y_n) >\}; \\ y_{n+1} \in b^{-1}\Big(\{\mathbf{P}_{\tilde{K_2}}(b(y_n) + \delta a(x_n))\}\Big); \end{cases} \tag{4.3}$$

where we recall that $\tilde{K_2} = b(K_2)$. Before presenting a convergence result for those iterations, we introduce the following definition. An application f from the Hilbert space $(H, < \cdot, \cdot >)$ into itself is said to be α monotone if for any $x, y \in H$,

$$< f(x) - f(y), x - y > \geq \alpha < x - y, x - y > .$$

An application 0 monotone is simply called monotone and an application f is said to be strictly monotone if for any $x, y \in H$ such that $x \neq y$ the following holds

$$< f(x) - f(y), x - y > > 0.$$

A classical convergence result concerning the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ is the following :

Theorem 4.1.1. *Let us assume that :*
- *The application F is differentiable with differential f which is α monotone.*
- *The application a is C lipschitz for some constant $C > 0$.*

Then if $\delta < \frac{2\alpha}{C^2}$, for any $y_0 \in K_2$, the sequence $(x_n, y_n)_{n \in \mathbb{N}}$ defined by (4.3) is well defined and $(x_n)_{n \in \mathbb{N}}$ converges toward x^* in H_1, where (x^*, y^*) is the unique saddle point of L.

We briefly recall here the proof of this result. *Proof.* Given the assumptions we made, we know that there exists a unique couple $(x^*, y^*) \in H_1 \times H_2$ verifying (4.2). It satisfies

$$\begin{cases} (f(x^*), x' - x^*) + < a(x') - a(x^*), b(y^*) > \geq 0, \forall x' \in K_1; \\ < a(x^*), b(y') - b(y^*) > \leq 0, \forall y' \in K_2. \end{cases}$$

Moreover by construction, $(x_n)_{n \in \mathbb{N}}$ satisfies for all $n \in \mathbb{N}$:

$$(f(x_n), x' - x_n) + < a(x') - a(x_n), b(y_n) > \geq 0, \forall x' \in K_1.$$

Thus we deduce that

$$\begin{aligned} < a(x_n) - a(x^*), b(y^*) - b(y_n) > &\geq (f(x_n) - f(x^*), x_n - x^*) \\ &\geq \alpha ||x_n - x^*||_{H_1}. \end{aligned}$$

Because $\mathbf{P}_{\tilde{K_2}}$ is a contraction, we obtain that

$$\begin{aligned} ||b(y_{n+1}) - b(y^*)||^2_{H^3} \leq & ||b(y_n) - b(y^*) + \delta(a(x_n) - a(x^*))||^2_{H^3} \\ \leq & ||b(y_n) - b(y^*)||^2_{H^3} + 2\delta < b(y_n) - b(y^*), a(x_n) - a(x^*) > \\ & + \delta^2 ||a(x_n) - a(x^*)||^2_{H^3}. \end{aligned}$$

We then deduce that

$$||b(y_{n+1}) - b(y)||^2_{H^3} \leq ||b(y_n) - b(y)||^2_{H^3} - 2\delta\alpha ||x_n - x||^2_{H^1} + \delta^2 ||a(x_n) - a(x)||^2_{H^3}.$$

Finally, because $\delta < \frac{2\alpha}{C^2}$ we obtain that there exists $0 < \beta < 1$ such that :

$$\beta ||x_n - x||^2_{H^1} \leq ||b(y_n) - b(y)||^2_{H^3} - ||b(y_{n+1}) - b(y)||^2_{H^3},$$

which concludes the proof of the result. □

Remark 4.1.1. *The use of the application b and of the Hilbert space H_2 is somewhat artificial. We only use this formalism because it is closer to the set up needed for the applications of the next section. Moreover let us note that we do not state any convergence for the sequence $(y_n)_{n\geq 0}$.*

4.1.2 A generalization of Uzawa's algorithm

We now remark that instead of using (4.3) to define a sequence $(x_n, y_n)_{n\in\mathbb{N}}$, we can use the following :

$$\begin{cases} y_0 \in K_2; \\ x_n \text{ is defined by } (f(x_n), x' - x_n) + < a(x') - a(x_n), b(y_n) > \geq 0, \forall x' \in K_1; \\ y_{n+1} \in b^{-1}\Big(\{\mathbf{P}_{\tilde{K_2}}(b(y_n) + \delta a(x_n))\}\Big). \end{cases} \tag{4.4}$$

Let us note that if F is a convex differentiable function, then (4.3) and (4.4) are equivalent, but the second one is more general in the sense that it allows us to build the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ even in the case in which there is no function F for which f is the differential. Under the assumptions of theorem 4.1.1, the sequence

$(x_n)_{n\in\mathbb{N}}$ converges toward x^* where (x^*, y^*) is the saddle point of L. Here we are interested in approximating the couples $(x_*, y_*) \in H_1 \times H_2$ solutions of

$$\begin{cases} (f(x_*), x' - x_*) + < a(x') - a(x_*), b(y_*) > \geq 0, \forall x' \in K_1; \\ < a(x_*), b(y') - b(y_*) > \leq 0, \forall y' \in K_2; \\ x \in K_1; y \in K_2. \end{cases} \tag{4.5}$$

We establish the following result :

Theorem 4.1.2. *Let us take $f : H_1 \to H_1$. We assume that :*
- *The application f is α monotone and locally lipschitz.*
- *The application a is C lipschitz for some constant $C > 0$ and has a differential Da which is locally lipschitz.*
- *There exists a couple (x_*, y_*) satisfying (4.5).*

Then, for any $y_0 \in K_2$, if $\delta < \frac{2\alpha}{C^2}$, (4.4) defines indeed a sequence $(x_n, y_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ converges toward x_* in H_1.

Proof. First let us remark that for any $y_0 \in K_2$, the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ is well defined. Indeed the second line of (4.4) defines a unique element $x_n \in H_1$ for any $n \in \mathbb{N}$. This comes from the fact that for any $y \in K_2$, $\epsilon > 0$, we can define the sequence $(\xi_p)_{p\in\mathbb{N}}$ by :

$$\Big\{\xi_0 \in H_1; \xi_{p+1} = \mathbf{P}_{K_1}\Big(\xi_p - \epsilon\Big(f(\xi_p) + (Da(\xi_p))^*(b(y))\Big)\Big);$$

where $\mathbf{P}_{K_1}$ stands for the orthogonal projection (for H_1) onto K_1. If ϵ is small enough, the sequence $(\xi_p)_{p\in\mathbb{N}}$ is a Cauchy sequence. Indeed the following holds :

$$\xi_{p+1} - \xi_p = \mathbf{P}_{K_1}\Big(\xi_p - \xi_{p-1} - \epsilon\Big(f(\xi_p) - f(\xi_{p-1}) + (Da(\xi_p) - Da(\xi_{p-1}))^*(b(y))\Big)\Big).$$

Thus we deduce that

$$\begin{aligned} ||\xi_{p+1} - \xi_p||^2_{H_1} \leq & ||\xi_p - \xi_{p-1}||^2_{H_1} + \epsilon^2 ||f(\xi_p) - f(\xi_{p-1}) + (Da(\xi_p) - Da(\xi_{p-1}))^*(b(y))||^2_{H_1} \\ & - 2\epsilon\Big(f(\xi_p) - f(\xi_{p-1}) + (Da(\xi_p) - Da(\xi_{p-1}))^*(b(y)), \xi_p - \xi_{p-1}\Big) \\ \leq & ||\xi_p - \xi_{p-1}||^2_{H_1} + \epsilon^2 ||f(\xi_p) - f(\xi_{p-1}) + (Da(\xi_p) - Da(\xi_{p-1}))^*(b(y))||^2_{H_1} \\ & - 2\epsilon\alpha ||\xi_p - \xi_{p-1}||^2_{H_1} \end{aligned}$$

where we have used the convexity of $x \to < a(x), b(y) >$ for all y and the α monotonicity of f. Using the regularity of f and Da we deduce that there exists ϵ small enough so that for some $0 < \beta < 1$:

$$||\xi_{p+1} - \xi_p||^2_{H_1} \leq \beta ||\xi_p - \xi_{p-1}||^2_{H_1}.$$

Thus $(\xi_p)_{p\in\mathbb{N}}$ is indeed a Cauchy sequence and its limit ξ_* satisfies

$$(f(\xi_*), x' - \xi_*) + < a(x') - a(\xi_*), b(y) > \geq 0, \forall x' \in K_1.$$

Such an element is unique because of the α monotonicity of f.

The rest of the proof follows the same argument as in the proof of theorem 4.1.1. □

Remark 4.1.2. *The existence of a couple (x_*, y_*) satisfying (4.5) can be obtained directly under some assumptions on the applications a and b via a Kakutani's type fixed point theorem.*

4.2 Application of Uzawa's iterations to mean field games

We now present how we can use the previous results to approximate some MFG systems. We shall apply this remark on Uzawa's algorithm to three different MFG systems. The first one is a system modeling a MFG of optimal stopping as in [11]. The second one is a MFG system modeling an impulse control problem, following [12] and we finish with the classical case of continuous control as in [54]. To simplify notations, we present the following results in a stationary setting in which the state space is the d dimensional torus $\mathbb{T}^d$. Other boundary conditions such as Neumann or Dirichlet boundary conditions in more general domain can be treated by similar approaches.

4.2.1 The case of optimal stopping

We are here interested in approximating the solution of the following system of unknown (u, m) :

$$\begin{cases} \forall v \in H^1(\mathbb{T}^d), v \leq 0 : \\ (f(m) + \nu\Delta u - \lambda u, v - u)_{H^{-1}\times H^1} \leq 0; \\ (-\nu\Delta m + \lambda m - \rho, v - u)_{H^{-1}\times H^1} \geq 0; \\ \int_{\mathbb{T}^d}(f(m) + \nu\Delta u - \lambda u)m = 0; \\ u \leq 0; m \geq 0; \end{cases} \tag{4.6}$$

where f is a continuous application from $L^2(\mathbb{T}^d)$ into itself, $\nu, \lambda > 0$ are two parameters of the model and $\rho \in H^{-1}(\mathbb{T}^d)$ is the entry rate of the players. The

exit cost of the MFG is here 0. The first variational inequality of this system arises from the obstacle problem satisfied by the value function u of a generic player. The second variational inequality and the integral relation arise from the "Fokker-Planck equation" satisfied by the density of players m. Let us remark that we have here abused the name variational inequality as we only refer to a variational formulation which is an inequality and not to the famous concept introduced in [56] by Lions and Stampacchia. This system models Nash equilibria in mixed strategies of a MFG of optimal stopping, we refer to [11] for more details on this system. From [11] we know that there exists a unique solution $(u, m) \in H^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$ of (4.6) under the assumption that f is strictly monotone, i. e. that it satisfies for all $m, m' \in L^2(\mathbb{T}^d)$:

$$\int_{\mathbb{T}^d} (f(m) - f(m'))(m - m') > 0 \text{ if } m \neq m'.$$

Let us remark that for any $m' \in H^1(\mathbb{T}^d), m' \geq 0$, the following holds

$$u - m' \leq 0.$$

Thus (u, m) also satisfies

$$\begin{cases} \forall \mu \in L^2(\mathbb{T}^d), \mu \geq 0 : \\ \int_{\mathbb{T}^d} (f(m) + \nu \Delta u - \lambda u)(\mu - m) \geq 0; \\ \forall v \in H^1(\mathbb{T}^d), v \leq 0 : \\ (-\nu \Delta m + \lambda m - \rho, v - u)_{H^{-1} \times H^1} \geq 0; \\ u \leq 0; m \geq 0. \end{cases} \tag{4.7}$$

In the case when f is strictly monotone, there is a unique $m \in H^1(\mathbb{T}^d)$ such that there exists $u \in H^2(\mathbb{T}^d)$ such that (u, m) is a solution of (4.7). The system (4.7) falls under the scope of application of the previous section. Thus we define for $\delta > 0$ the following Uzawa's iterations :

$$\begin{cases} u_0 \in H^2(\mathbb{T}^d), u_0 \leq 0. \\ m_n \in L^2(\mathbb{T}^d) \text{ defined by :} \\ m_n \geq 0; \forall \mu \in L^2(\mathbb{T}^d), \mu \geq 0 : \int_{\mathbb{T}^d} (f(m_n) + \nu \Delta u_n - \lambda u_n)(\mu - m_n) \geq 0. \\ u_{n+1} \in H^2(\mathbb{T}^d) \text{ defined by : } L u_{n+1} = \mathbf{P}_{\tilde{K}}\Big(L u_n - \delta(m_n - L^{-1}\rho)\Big). \end{cases} \tag{4.8}$$

where L is the linear operator

$$L = -\nu \Delta + \lambda Id,$$

the closed convex set $\tilde{K}$ is defined by

$$\tilde{K} := \{g \in L^2(\mathbb{T}^d), L^{-1} g \leq 0\}$$

and $\mathbf{P}_A$ stands for the orthogonal projection in $L^2(\mathbb{T}^d)$ onto the set A. Let us note that from classical results on variational inequalities (see [56] for instance), $(u_n)_{n\geq 0}$ is a well defined sequence of $H^2(\mathbb{T}^d)$ because for all $n \in \mathbb{N}$, $m_n - L^{-1}\rho \in L^2(\mathbb{T}^d)$. Recalling the results of the previous section, $(m_n)_{n\geq 0}$ is well defined under some monotonicity assumptions on f. We have the following result :

Theorem 4.2.1. *Assume that f is α monotone from $L^2(\mathbb{T}^d)$ into itself for some $\alpha > 0$ and that $\delta < 2\alpha$, then for any $u_0 \in H^2(\mathbb{T}^d)$, the sequence $(m_n)_{n\geq 0}$ defined by (4.8) converges toward m in $L^2(\mathbb{T}^d)$, where (u, m) is the unique solution of (4.6).*

Proof. This result is a direct application of theorem 4.1.2. □

Let us remark that the projection involved in (4.8) is similar to the resolution of a bi-laplacian obstacle problem. Indeed, given u_n, m_n, we are looking for u_{n+1} such that :

$$\forall v \in H^2(\mathbb{T}^d), v \leq 0 :$$
$$\int_{\mathbb{T}^d} (Lu_{n+1} - Lv)(Lu_{n+1} + \delta(m_n - L^{-1}\rho) - Lu_n) \leq 0.$$

Let us also make a remark on the potential case. The potential case is the case when there exists $\mathcal{F} : L^2(\mathbb{T}^d) \to \mathbb{R}$ such that for every $m, m' \in L^2(\mathbb{T}^d)$:

$$\lim_{t\to 0} \frac{\mathcal{F}(m + tm') - \mathcal{F}(m)}{t} = \int_{\mathbb{T}^d} f(m)(m' - m). \tag{4.9}$$

In such a situation, if f is strictly monotone, following the result of [11], the unique solution (u, m) of (4.6) is also the saddle point of the lagrangian $\mathcal{L}$ defined on $\{\mu \in L^2(\mathbb{T}^d), \mu \geq 0\} \times \{v \in H^2(\mathbb{T}^d), v \leq 0\}$ by

$$\mathcal{L}(\mu, v) = \mathcal{F}(\mu) + \int_{\mathbb{T}^d} (-\nu\Delta v + \lambda v)m - \int_{\mathbb{T}^d} v\rho.$$

The iterations (4.8) are then the result of the classical Uzawa's algorithm on $\mathcal{L}$.

Let us also mention that any coupling f which satisfies $f - \alpha Id$ is monotone is α monotone (in fact this is equivalent to being α monotone).

4.2.2 The case of impulse control

In this section we are interested in building approximations of solutions of the following system :

$$\begin{cases} \forall v \in H^1(\mathbb{T}^d), v \leq Mv : \\ (f(m) + \nu\Delta u - \lambda u, v - u)_{H^{-1}\times H^1} \leq 0; \\ (-\nu\Delta m + \lambda m - \rho, v - u)_{H^{-1}\times H^1} \geq 0; \\ \int_{\mathbb{T}^d} (f(m) + \nu\Delta u - \lambda u)m = 0; \\ u \leq Mu; m \geq 0; \end{cases} \tag{4.10}$$

where f is a continuous application from $L^2(\mathbb{T}^d)$ into itself, bounded uniformly from below on the positive elements of $L^2(\mathbb{T}^d)$, $\nu, \lambda > 0$ are two parameters of the model, $\rho \in H^{-1}(\mathbb{T}^d)$ is the entry rate of players and M is the operator defined by

$$Mv(x) := \inf_{\xi \in J}\{k(x,\xi) + v(x+\xi)\}$$

where J is a finite set of $\mathbb{T}^d$ and k is a smooth non-negative function.

The system (4.10) models Nash equilibria of MFG of impulse control in which the players face the running cost $f(m)$ and have to pay $k(x,\xi)$ if they are in x to make a jump of ξ. The density of players is m and u represents the value function of a generic player. We refer to [12] for more details on this problem and for the following result. If f is strictly monotone and k satisfies

$$\begin{cases} x \to \inf_{\xi \in J}\{k(x,\xi)\} \in W^{2,\infty}; \\ \exists k_0 > 0, \forall x \in \mathbb{T}^d, \xi \in J : k(x,\xi) \geq k_0 \end{cases}$$

then there exists a unique solution $(u,m) \in H^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$ of (4.10), moreover, this couple (u,m) satisfies :

$$\begin{cases} \forall \mu \in L^2(\mathbb{T}^d), \mu \geq 0 : \\ \int_{\mathbb{T}^d}(f(m) + \nu\Delta u - \lambda u)(\mu - m) \geq 0; \\ \forall v \in H^1(\mathbb{T}^d), v \leq Mv : \\ (-\nu\Delta m + \lambda m - \rho, v - u)_{H^{-1}\times H^1} \geq 0; \\ u \leq Mu; m \geq 0. \end{cases} \tag{4.11}$$

Thus we define, as in the case of optimal stopping, the following Uzawa's iterations for $\delta > 0$:

$$\begin{cases} u_0 \in H^2(\mathbb{T}^d), u_0 \leq Mu_0. \\ m_n \in L^2(\mathbb{T}^d) \text{ defined by :} \\ m_n \geq 0; \forall \mu \in L^2(\mathbb{T}^d), \mu \geq 0 : \int_{\mathbb{T}^d}(f(m_n) + \nu\Delta u_n - \lambda u_n)(\mu - m_n) \geq 0. \\ u_{n+1} \in H^2(\mathbb{T}^d) \text{ defined by : } Lu_{n+1} = \mathbf{P}_{K'}\Big(Lu_n - \delta(m_n - L^{-1}\rho)\Big); \end{cases} \tag{4.12}$$

where L still denotes the linear operator

$$L = -\nu\Delta + \lambda Id,$$

the closed convex set K' is defined by

$$K' := \{g \in L^2(\mathbb{T}^d), L^{-1}g \leq M(L^{-1}g)\}$$

and $\mathbf{P}_A$ stands for the orthogonal projection onto A in $L^2(\mathbb{T}^d)$. We have the following result of convergence :

Theorem 4.2.2. *Assume that f is α monotone from $L^2(\mathbb{T}^d)$ into itself for some $\alpha > 0$ and that $\delta < 2\alpha$, then for any $u_0 \in H^2(\mathbb{T}^d)$, the sequence $(m_n)_{n\geq 0}$ defined by (4.8) converges toward m in $L^2(\mathbb{T}^d)$, where (u, m) is the unique solution of (4.6).*

Proof. This result is once again a direct application of theorem 4.1.2. □

Let us remark that although the Hamilton-Jacobi-Bellmann equation in (4.10) is a quasi-variational inequality, the equation we have to solve at each iteration in (4.12) to update the lagrange multiplier u_n is a variational inequality, which is in principle easier to solve than a quasi-variational inequality.

Moreover, in the potential case, i. e. when there exists $\mathcal{F}$ satisfying (4.9), if f is strictly monotone, the solution (u, m) of (4.10) is the saddle point of $\mathcal{L}$ defined on $\{m \in L^2(\mathbb{T}^d), m \geq 0\} \times \{u \in H^2(\mathbb{T}^d), u \leq Mu\}$ by :

$$\mathcal{L}(\mu, v) = \mathcal{F}(\mu) + \int_{\mathbb{T}^d} (-\nu\Delta v + \lambda v)m + \int_{\mathbb{T}^d} v\rho.$$

This results can be found in [12]. The iterations (4.12) are then the ones from the use of the classical Uzawa's algorithm on $\mathcal{L}$.

4.2.3 The case of continuous control

We end this list of applications of Uzawa's iterations with the construction of approximating sequences for the following MFG system :

$$\begin{cases} -\nu\Delta u + \lambda u + H(x, \nabla u) = f(m) \text{ in } \mathbb{T}^d; \\ -\nu\Delta m + \lambda m - \operatorname{div}(D_p H(x, \nabla u)m) = \rho \text{ in } \mathbb{T}^d; \end{cases} \tag{4.13}$$

where f is the running cost of the players and the hamiltonian $H(x, p)$ is assumed to be convex in its second variable and uniformly lispchitz. We refer the reader to [54, 57] for a full presentation and results on the system (4.13). If $(u, m) \in H^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ is a solution of (4.13) (with m being a weak solution of the Fokker-Planck equation), then it is also a solution of :

$$\begin{cases} \forall \mu \in L^2(\mathbb{T}^d), \mu \geq 0 : \\ \int_{\mathbb{T}^d} (f(m) + \nu\Delta u - \lambda u - H(x, \nabla u))(\mu - m) \geq 0; \\ \forall v \in H^2(\mathbb{T}^d) : \\ \int_{\mathbb{T}^d} (-\nu\Delta(v - u) + \lambda(v - u) + D_p H(x, \nabla u) \cdot \nabla(v - u))m - \int_{\mathbb{T}^d} \rho(v - u) \geq 0; \\ m \geq 0. \end{cases} \tag{4.14}$$

Under the assumption that f is strictly monotone, there exists at most one m such that $(u,m) \in H^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)$ is a solution of (4.14). Although this system does not allow a direct application of theorem 4.1.2, the convexity of the hamiltonian allows us to prove a result of convergence for Uzawa's like iterations. Given a sequence of non-negative real numbers $(\delta_n)_{n\geq 0}$, we define Uzawa's iterations in this case by :

$$\begin{cases} u_0 \in H^2(\mathbb{T}^d); \\ m_n \text{ is defined by } \forall \mu \in L^2(\mathbb{T}^d), \mu \geq 0 : \\ \int_{\mathbb{T}^d}(f(m_n) + \nu\Delta u_n - \lambda u_n - H(x,\nabla u_n))(\mu - m_n) \geq 0; \\ u_{n+1} \text{ is defined by } : \\ -\nu\Delta u_{n+1} + \lambda u_{n+1} + H(x,\nabla u_{n+1}) = -\nu\Delta u_n + \lambda u_n + H(x,\nabla u_n) \\ \qquad\qquad - \delta_n\Big(m_n - L_{u_n}^{*-1}(\rho)\Big); \end{cases} \tag{4.15}$$

where for all $v \in H^1(\mathbb{T}^d)$, L_v is the operator defined by :

$$L_v w = -\nu\Delta w + \lambda w - D_p H(x,\nabla v)\cdot\nabla w.$$

We now establish the following result :

Theorem 4.2.3. *Assume that there exists a solution $(u,m) \in H^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)$ of (4.13) and that f is α monotone. Then there exists a sequence of non-negative real number $(\delta_n)_{n\geq 0}$ such that the iterations $(u_n,m_n)_{n\geq 0}$ defined by (4.15) are such that $(m_n)_{n\geq 0}$ converges toward m in $L^2(\mathbb{T}^d)$.*

Proof. We denote by $(u,m) \in H^2(\mathbb{T}^d)\times L^2(\mathbb{T}^d)$ the unique solution of (4.14). We take a sequence $(\delta_n)_{n\geq 0}$, $\delta_n > 0$ for all $n \geq 0$ and we consider the iterations $(u_n,m_n)_{n\geq 0}$ given by (4.15) for a fixed $u_0 \in H^2(\mathbb{T}^d)$. We introduce the notation

$$HJB(v) := -\nu\Delta v + \lambda v + H(x,\nabla v).$$

Let us remark that for all $n \geq 0$:

$$HJB(u_{n+1}) - HJB(u) = HJB(u_n) - HJB(u) - \delta_n(m_n - L_{u_n}^{*-1}\rho).$$

Thus we obtain that :

$$\begin{aligned} ||HJB(u_{n+1}) - HJB(u)||_{L^2}^2 =& ||HJB(u_n) - HJB(u)||_{L^2}^2 + \delta_n^2||m_n - L_{u_n}^{*-1}\rho||_{L^2}^2 \\ & - 2\delta_n\int_{\mathbb{T}^d}(HJB(u_n) - HJB(u))(m_n - L_{u_n}^{*-1}\rho). \end{aligned} \tag{4.16}$$

We now make some calculations around the third term of the right hand side of the previous equality. We compute :

$$\begin{aligned} \int_{\mathbb{T}^d}(HJB(u_n) - HJB(u))(m_n - L_{u_n}^{*-1}\rho) =& \int_{\mathbb{T}^d}(HJB(u_n) - HJB(u))(m_n - m) \\ & + \int_{\mathbb{T}^d}\Big(HJB(u_n) - HJB(u)\Big)(m - L_{u_n}^{*-1}\rho). \end{aligned}$$

Because m is the solution of

$$L_u^* m = \rho,$$

we deduce from the convexity of H :

$$\int_{\mathbb{T}^d} \Big(HJB(u_n) - HJB(u)\Big)m \geq \int_{\mathbb{T}^d} \rho(u_n - u).$$

On the other hand :

$$\begin{aligned}\int_{\mathbb{T}^d} \Big(HJB(u) - HJB(u_n)\Big)L_{u_n}^{*-1}\rho = &\int_{\mathbb{T}^d} L_{u_n}(u-u_n)(L_{u_n}^{*-1}\rho) \\ &- \int_{\mathbb{T}^d} D_p H(x,\nabla u_n)\cdot\nabla(u-u_n)(L_{u_n}^{*-1}\rho) \\ &+ \int_{\mathbb{T}^d} \Big(H(x,\nabla u) - H(x,\nabla u_n)\Big)(L_{u_n}^{*-1}\rho)\end{aligned}$$

By the maximum principle, $L_{u_n}^{*-1}\rho \geq 0$, thus we deduce from the convexity of the hamiltonian that :

$$\int_{\mathbb{T}^d} \Big(HJB(u) - HJB(u_n)\Big)L_{u_n}^{*-1}\rho \geq \int_{\mathbb{T}^d} \rho(u - u_n).$$

This inequality, together with the previous one implies that :

$$\int_{\mathbb{T}^d} (HJB(u_n) - HJB(u))(m_n - L_{u_n}^{-1}\rho) \geq \int_{\mathbb{T}^d} (HJB(u_n) - HJB(u))(m_n - m).$$

Using the equation satisfied by u and the definition of m_n, we obtain that :

$$\int_{\mathbb{T}^d} (HJB(u_n) - HJB(u))(m_n - m) \geq \int_{\mathbb{T}^d} \Big(f(m_n) - f(m)\Big)(m_n - m).$$

The α convexity of f yields finally :

$$\int_{\mathbb{T}^d} (HJB(u_n) - HJB(u))(m_n - L_{u_n}^{-1}\rho) \geq \alpha ||m_n - m||_{L^2}^2.$$

Using this inequality in (4.16) we obtain that :

$$\begin{aligned}\delta_n\Big(2\alpha||m_n - m||_{L^2}^2 - \delta_n||m_n - L_{u_n}^{*-1}\rho||_{L^2}^2\Big) \leq &||HJB(u_n) - HJB(u)||_{L^2}^2 \\ &- ||HJB(u_{n+1}) - HJB(u)||_{L^2}^2.\end{aligned}$$

We assume in a first time that for all $n \in \mathbb{N}$:

$$\begin{cases} ||m_n - m||_{L^2} > 0; \\ ||m_n - L_{u_n}^{*-1}\rho||_{L^2} > 0. \end{cases} \tag{4.17}$$

Then we define for all $n \in \mathbb{N}$

$$\delta_n = \frac{\alpha||m_n - m||_{L^2}^2}{||m_n - L_{u_n}^{*-1}\rho||_{L^2}^2}.$$

Let us observe that in this situation the sequence $(||HJB(u_n) - HJB(u)||_{L^2})_{n\geq 0}$ is decreasing and thus it has a limit and $(u_n)_{n\geq 0}$ is a bounded sequence of $H^2(\mathbb{T}^d)$. We also remark that we deduce from the convergence of $(||HJB(u_n) - HJB(u)||_{L^2})_{n\geq 0}$ that

$$\frac{\alpha||m_n - m||_{L^2}^2}{||m_n - L_{u_n}^{*-1}\rho||_{L^2}} \underset{n\to 0}{\longrightarrow} 0.$$

Because $(||HJB(u_n)||_{L^2})_{n\geq 0}$ is bounded and f is α monotone, we deduce that $(m_n)_{n\geq 0}$ is bounded in L^2 and thus that :

$$||m_n - m||_{L^2} \underset{n\to 0}{\longrightarrow} 0.$$

To complete the proof of the theorem, let us remark that if (4.17) is not satisfied for $n^* \in \mathbb{N}$, then $m_{n^*} = m$ and the convergence is also proved. □

Remark 4.2.1. *Let us remark that because of the α monotonicity of f, there are obvious estimates in $L^2(\mathbb{T}^d)$ for $(m_n)_{n\geq 0}$, thus the sequence $(\delta_n)_{n\geq 0}$ can be chosen to be an explicit constant.*

In the potential case, when there exists $\mathcal{F}$ satisfying (4.9), and when f is strictly monotone, the solution (u, m) of (4.13) is also the saddle point of the lagrangian $\mathcal{L}$ defined on $\{\mu \in L^2(\mathbb{T}^d), \mu \geq 0\} \times H^2(\mathbb{T}^d)$ by :

$$\mathcal{L}(\mu, v) = \mathcal{F}(\mu) + \int_{\mathbb{T}^d} (\nu\Delta v - \lambda v - H(x, \nabla v))m - \int_{\mathbb{T}^d} \rho v.$$

Even though the optimal control interpretation presented in [54] is not exactly written in this form, it can be easily checked that the formulations are equivalent, at least formally. The iterations (4.15) are in this case the result of Uzawa's algorithm on $\mathcal{L}$, in the sense that they are formally the result of a gradient ascent method on the dual problem :

$$\sup_{v\in H^2(\mathbb{T}^d)} \inf_{\mu\geq 0, \in L^2(\mathbb{T}^d)} \mathcal{L}(\mu, v).$$

Remark 4.2.2. *In the three cases mentioned above (optimal stopping, impulse control and continuous control), the sequence $(u_n)_{n\geq 0}$ defined by the Uzawa's iterations is always a bounded sequence of $H^2(\mathbb{T}^d)$. Therefore, up to a subsequence, $(u_n)_{n\geq 0}$ converges in $H^1(\mathbb{T}^d)$ toward $u \in H^2(\mathbb{T}^d)$. It can be shown that under some additional assumptions, this function u is in fact such that (u, m) is the solution of the MFG system and thus the whole sequence $(u_n)_{n\geq 0}$ converges toward u.*

4.2.4 Other possible applications of Uzawa's iterations

We give here some immediate applications of Uzawa's iterations. First let us note that the operator $-\nu\Delta + \lambda Id$ involved in the three problems above can be replaced by more general elliptic linear operators. Let us also mention that this method is also valid in more general domains than $\mathbb{T}^d$. This method can also be applied in time dependent situations.

Another important remark is that Uzawa's iterations can also be applied in the optimal control of PDE governed by inequalities, such that

$$\inf_{m, A(m) \leq 0} F(m),$$

where A is a partial differential operator. Such a class of problem is of some importance. For instance we refer to [11] for a heuristic argument on why (4.6) can be interpreted as the optimality conditions for the relaxation of an optimal shape problem. The relaxation is then of the form just mentioned above.

4.3 Numerical framework and numerical results

We present here the discrete versions of the three problems mentioned in the previous section (optimal stopping, impulse control and continuous control MFG systems). We also present numerical results of the implementation of Uzawa's iteration for those problems.

4.3.1 Notations and presentation of the problem

We give here the notations we are going to use to present the discretized problem we are interested in. We fix a non-negative integer d and we define $h > 0$ by $h = d^{-1}$. We work here on a grid $G_d = \{(i, j), 1 \leq i, j \leq d\}$ which we interpret as a discretization of the 2 dimensional torus. Let $f_d : \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ be a continuous application. We fix $\xi \in G_d$ and $k_0 > 0$ a real number. We then define for all $v \in \mathbb{R}^{d^2}$ Mv by :

$$(Mv)_{i,j} = k_0 + v_{(i,j)+\xi}.$$

We denote by $g : (p_1, p_2, p_3, p_4) \to g(p_1, p_2, p_3, p_4)$ a discretization of the hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ defined by $H(p) = \sqrt{1 + |p|^2}$. Thus g is such that for $p_1, p_2 \in \mathbb{R}$:

$$g(p_1, p_1, p_2, p_2) = \sqrt{1 + (p_1)^2 + (p_2)^2};$$

and g is non decreasing with respect to p_1 and p_3 and non decreasing with respect to p_2 and p_4. We denote by $\nabla_p g$ the gradient of g. We also define the vector of

derivatives $D_h v$ of a vector $v \in \mathbb{R}^{d^2}$ by :

$$(D_h v)_{i,j} = \Big(\frac{v_{i+1,j} - v_{i,j}}{h}, \frac{v_{i,j} - v_{i-1,j}}{h}, \frac{v_{i,j+1} - v_{i,j}}{h}, \frac{v_{i,j} - v_{i,j-1}}{h}\Big).$$

For $\nu, \lambda > 0$, we also define the discrete operator $A : \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ by :

$$(Av)_{i,j} = \nu \frac{4v_{i,j} - v_{i+1,j} - v_{i-1,j} - v_{i,j+1} - v_{i,j-1}}{h} + \lambda v_{i,j}, \forall (i,j) \in G_d;$$

where we use periodic boundary condition on G_d.
For $v \in \mathbb{R}^p$ for some $p \geq 0$, we use the notation $v \geq 0$ when for all $1 \leq i \leq p$, $v_i \geq 0$. We take an element $\rho_d \in \mathbb{R}^d$, $\rho_d \geq 0$.
For the rest of this section, $\mathbb{R}^{d^2}$ is endowed with the scalar product :

$$< x, y >= \sum_{1 \leq i,j \leq d} h^2 x_{i,j} y_{i,j}$$

In this section, we present the results of the implementation of Uzawa's iterations to approximate the solutions of the following three problems (each time the unknown is the couple (u, m)) :

$$\begin{cases} < f_d(m) - Au, v - u > \leq 0, \forall v \in \mathbb{R}^{d^2}, v \leq 0; \\ < Au - Av, m > + < u - v, \rho_d > \leq 0; \forall v \in \mathbb{R}^{d^2}, v \leq 0; \\ < Au - f_d(m), m >= 0; \\ u \leq 0; m \geq 0. \end{cases} \tag{4.18}$$

$$\begin{cases} < f_d(m) - Au, v - u > \leq 0, \forall v \in \mathbb{R}^{d^2}, v \leq Mv; \\ < Au - Av, m > + < u - v, \rho_d > \leq 0; \forall v \in \mathbb{R}^{d^2}, v \leq Mv; \\ < Au - f_d(m), m >= 0; \\ u \leq Mu; m \geq 0. \end{cases} \tag{4.19}$$

$$\begin{cases} Au + g(Du) = f_d(m); \\ < Av + \nabla_p g(D_h u) \cdot D_h(v), m > - < v, \rho_d >= 0; \forall v \in \mathbb{R}^{d^2}; \\ m \geq 0. \end{cases} \tag{4.20}$$

Those problems are the discretized version of respectively (4.6), (4.10) and (4.13).

4.3.2 A remark on the convergence of the discretized problems toward the continuous ones

Although the convergence of (4.18), (4.19) and (4.20) toward their continuous version is not the objective of this article, we explain here briefly why such a

convergence is expected. We give some results on the case of (4.18). We refer to [2] for results on (4.20).

We begin by detailing in which sense $(f_d)_{d\geq 1}$ converges toward $f : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$. For any sequence $(m_d)_{d\geq 1}$, we define $(\tilde{m}_d)_{d\geq 1} \in (L^2(\mathbb{T}^2))^{\mathbb{N}}$ by

$$\tilde{m}_d(x,y) = (m_d)_{i,j} \text{ if } \begin{cases} i-1 \leq x \times d < i, \\ j-1 \leq y \times d < j. \end{cases}$$

We assume that if $(\tilde{m}_d)_{d\geq 1}$ converges toward m in $L^2(\mathbb{T}^2)$, then $(\tilde{f}_d(m_d))_{d\geq 1}$ converges toward $f(m)$ in $L^2(\mathbb{T}^d)$. We also assume, using the same notations, that $(\tilde{\rho}_d)_{d\geq 1}$ converges toward ρ in $L^2(\mathbb{T}^2)$. We now start by proving a lemma which gives the main idea for the convergence of the finite problem.

Lemma 4.3.1. *Let us assume that $f_d : \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ is α monotone and that (u,m) is the unique solution of (4.18). For any $\epsilon_1, \epsilon_2 > 0$ and (v,μ) such that*

$$\begin{cases} < f_d(\mu) - Av, \mu' - \mu > \geq -\epsilon_1, \forall \mu' \in \mathbb{R}^{d^2}, \mu' \geq 0; \\ < Av - Av', \mu > + < v - v', \rho > \leq \epsilon_2; \forall v' \in \mathbb{R}^{d^2}, v' \leq 0; \\ v \leq 0; \mu \geq 0; \end{cases} \tag{4.21}$$

the following holds :

$$||m - \mu||^2 \leq \frac{\epsilon_1 + \epsilon_2}{\alpha}.$$

Proof. Because of the inequalities verified by (v,μ),

$$\begin{cases} < f_d(\mu) - Av, m - \mu > \geq -\epsilon_1; \\ < Av - Au, \mu > + < v - u, \rho > \leq \epsilon_2. \end{cases}$$

Therefore, using the fact that (u,m) is the solution of (4.18), we deduce that

$$\begin{cases} < f_d(\mu) - f_d(m), m - \mu > + < Au - Av, m - \mu > \geq -\epsilon_1; \\ < Av - Au, \mu - m > \leq \epsilon_2. \end{cases}$$

Thus we obtain that

$$< f_d(\mu) - f_d(m), \mu - m > \leq \epsilon_1 + \epsilon_2.$$

Using the α monotonicity of f_d the result is proved. □

Remark 4.3.1. *If (4.21) is not satisfied for all μ', v' but only for m, u, the results of the lemma still holds.*

We now show an exemple of result of convergence.

Proposition 4.3.1. *Let us assume that $f : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ is α monotone and let us assume that there exists (u^*, m^*), unique solution of (4.6). We also assume that for every $d \in \mathbb{N}, d \geq 1$, $f_d : \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ is α monotone and that there exists a unique solution (u_d, m_d) of (4.18). Then the following holds*

$$||m_d - m_d^*||_{\mathbb{R}^{d^2}} \underset{d\to\infty}{\longrightarrow} 0;$$

where $m_d^* \in \mathbb{R}^{d^2}$ is such that defining $\tilde{m^*}$ by

$$\tilde{m^*}(x, y) = (m_d^*)_{i,j} \text{ if } \begin{cases} i - 1 \leq x \times d < i, \\ j - 1 \leq y \times d < j; \end{cases}$$

we have the convergence :

$$||m^* - \tilde{m^*}||_{L^2(\mathbb{T}^2)} \underset{d\to\infty}{\longrightarrow} 0.$$

Proof. We denote for all $d \geq 1$ (u_d, m_d) the unique solution of (4.18) and by (u, m) the unique solution of (4.6). Our aim is to build for all $d \geq 1$, $v_d, \mu_d \in \mathbb{R}^{d^2}$, an approximate solution of (4.18) using (u, m). We define $x_{i,j} = ((i-1)h, (j-1)h)$ for $1 \leq i, j \leq d$. And we consider a C^∞ partition of the unity $(\phi_d^{i,j})_{1\leq i,j\leq d}$ subordinate to the cover $(B(x_{i,j}, h\sqrt{2}))_{1\leq i,j\leq d}$, where $B(x, \delta)$ denotes the open ball of center x and radius δ. We define $(\varphi_d^{i,j})_{1\leq i,j\leq d}$ by $\varphi_d^{i,j} = (\int \phi_d^{i,j})^{-1}\phi_d^{i,j}$. We then define $v_d, \mu_d \in \mathbb{R}^{d^2}$ by :

$$(v_d)_{i,j} = (u * \varphi_d^{i,j})(x_{i,j})$$
$$(\mu_d)_{i,j} = (m * \varphi_d^{i,j})(x_{i,j})$$

It is easy to verify that there exists $(\epsilon_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}}$ and $(\epsilon'_n)_{n\geq 0}, (\tilde{\epsilon}_n)_{n\geq 0}$ such that $\epsilon'_n, \tilde{\epsilon}_n \in \mathbb{R}^{n^2}$ for every $n \geq 1$ and the three sequences converge to zero together with

$$< f_d(\mu_d) - Av_d, \mu_d > \leq \epsilon_d$$
$$< f_d(\mu_d) - Av_d, m_d > \geq - < \epsilon'_d, m_d >$$
$$< Av_d, \mu_d > - < v_d, \rho_d > \leq \epsilon_d$$
$$< Au_d, \mu_d > - < u_d, \rho_d > \geq - < \tilde{\epsilon}_d, Au_d >$$

Thus using lemma 4.3.1 (and the remark following) we deduce that

$$< f(\mu_d) - f(m_d), \mu_d - m_d > \leq < \epsilon'_d, m_d > + < \tilde{\epsilon}_d, Au_d > +2\epsilon_d.$$

Using estimates on $(Au_n)_{n\geq 0}$ and $(m_n)_{n\geq 0}$ (which are easy to obtain) we deduce, using the α monotonicity of f_d that

$$< m_d - \mu_d, m_d - \mu_d > \underset{d\to\infty}{\longrightarrow} 0.$$

The result then follows. □

4.3.3 Numerical results

The optimal stopping case

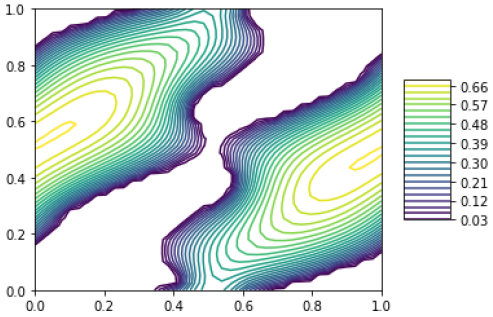
In figure 4.1, we give the density m and its Lagrange multiplier u obtained after 20 Uzawa's iterations. We use a standard Uzawa's algorithm to perform numerically the projection which updates the Lagrange multiplier at each step. The parameters of the model are

ν	0.02
λ	1
d	40
$f(m)$	$f_0 + m + (-\Delta + Id)^{-1}m$
f_0	$\cos(2\pi x) + 2\cos(2\pi(y-x)) + \cos(6\pi x)$
δ	0.5
ρ	1

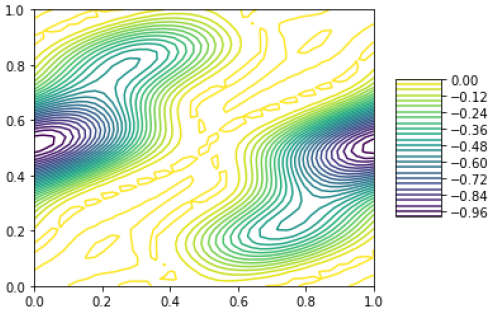
The impulse control case

In figure 4.2 we give the density m and its Lagrange multiplier u obtained after 40 Uzawa's iterations. We use at each step Uzawa's algorithm to perform numerically the projection which updates the Lagrange multiplier in our Uzawa's iterations. The parameters of the model are

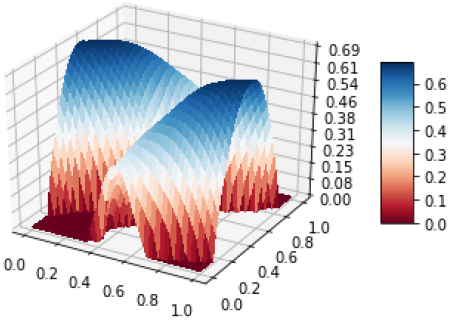
ν	0.02
λ	1
d	40
$f(m)$	$f_0 + m + (-\Delta + Id)^{-1}m$
f_0	$\cos(2\pi x) + 2\cos(2\pi(y-x)) + \cos(6\pi x)$
k_0	0.5
ξ	$(20/7, 0)$
δ	0.5
ρ	1



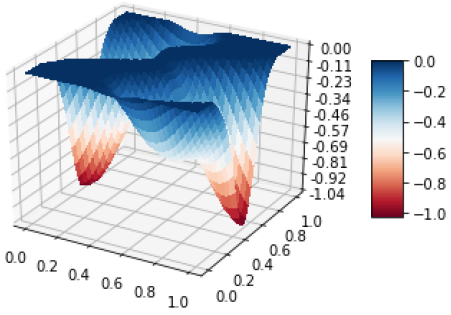
(a) Contours of m



(b) Contours of u

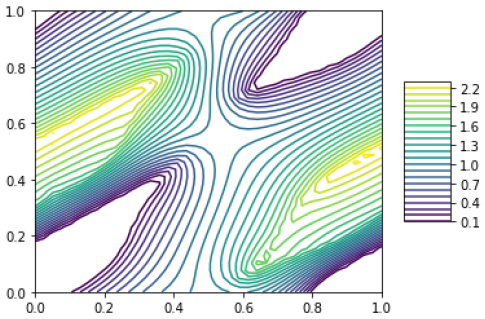


(c) Graph of m

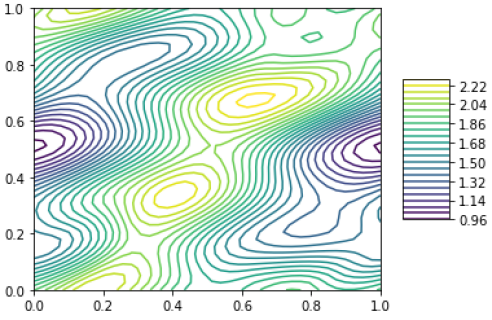


(d) Graph of u

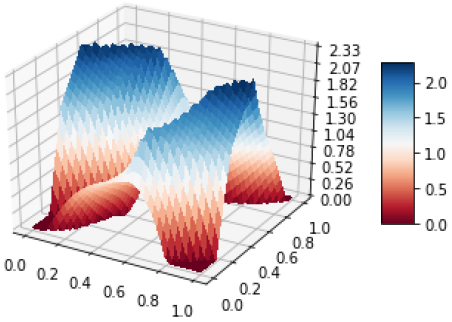
FIGURE 4.1 – Uzawa's iterations for (4.18)



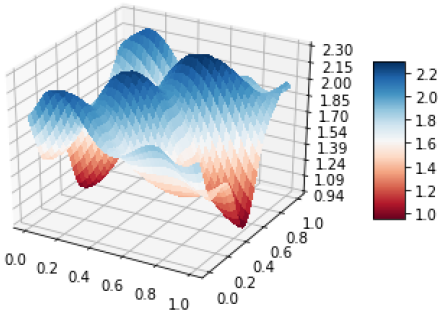
(a) Contours of m



(b) Contours of u



(c) Graph of m



(d) Graph of u

FIGURE 4.2 – Uzawa's iterations for (4.19)

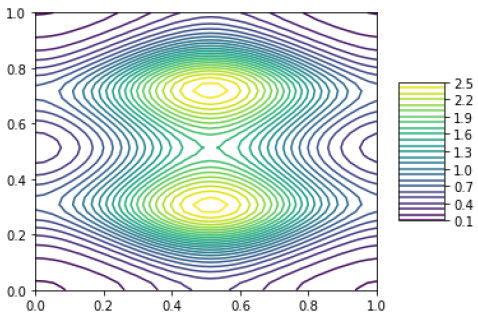
The continuous control case

In figure 4.3 we give the density m and its Lagrange multiplier u obtained after 3000 Uzawa's iterations. We use a standard Newton method on the finite differences scheme at each step to solve the Hamilton-Jacobi-Bellmann equation which updates the Lagrange multiplier. To compute at each iteration n the value of $L_{u_n}^{*-1}\rho$, we use a biconjugate gradient stabilized method. The parameters of the model are

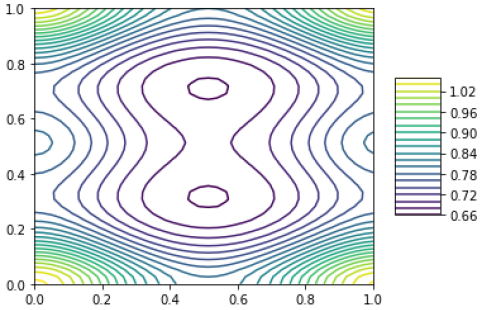
ν	0.05
λ	1
d	40
$f(m)$	$f_0 + m + (-\Delta + Id)^{-1}m$
$f_0(x, y)$	$\cos(2\pi x) + \cos(2\pi y) + \cos(4\pi x)$
$H(p)$	$\sqrt{1 + \lvert p\rvert^2}$
$(\delta_n)_{n\geq 0}$	$0.05\ \forall n$
ρ	1

Acknowledgments

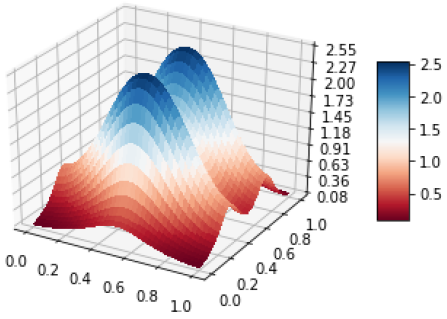
I would like to thank Pr. Yves Achdou (Université Paris Descartes) for his helpful advices on MFG and numerical simulations.



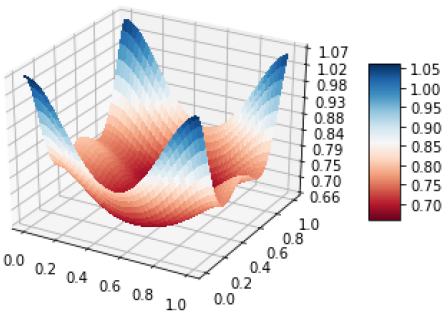
(a) Contours of m



(b) Contours of u



(c) Graph of m



(d) Graph of u

FIGURE 4.3 – Uzawa's iterations for (4.20)

Chapitre 5

Stratégies de transmission basées sur les jeux à champ moyen pour des communications massives de type machine

Les communications massives de type machine sont l'une des trois principaux types d'applications de communications dans le futur réseau sans fil 5G. Dans ce type de communications, le réseau doit gérer un grand nombre d'appareils qui transmettent de l'information à la même station receveur, de manière non-coordonnée. Dans ce contexte, le problème de minimiser l'utilisation d'énergie tout en achevant une contrainte de qualité de service est un problème d'optimisation stochastique faisant intervenir un très grand nombre de variables. Dans ce chapitre, on propose un modèle de MFG pour ce problème qui réduit grandement la complexité du problème de départ, suffisamment pour pouvoir obtenir des résultats numériques.

Ce chapitre est issu d'un travail en collaboration avec S. Vassilaras, G. Paschos, M. Debbah, J.-M. Lasry et P.-L. Lions. Ce travail a été accepté pour publication dans les Proceedings of the International Symposium on Wireless Communications Systems (ISWCS) 2018.

5.1 Introduction

With the proliferation of Internet of Things (IoT), one of the key requirements for wireless networks of the future will be their ability to serve a huge number of wireless devices. It is foreseen that in 5G (and beyond) networks, the density of Machine-Type Communications (MTC) networks may surpass 1 million of devices per km^2 [1]. Thus, despite network densification, an extremely large number of machines will desire to communicate with a serving base station (BS) each with a small traffic requirement. Despite the low traffic per device, the massive MTC (mMTC) scenario is one of the most challenging of 5G and beyond, since we desire a communication protocol that at the same time is (i) decentralized (overheads for millions of devices cannot be tolerated), (ii) energy-efficient (machines may need to be unattended for long time), and (iii) guarantees QoS performance (such as low average service time).

A typical barrier in this case is the *curse of dimensionality*, whereby the large system state-space makes the discovery of the optimal protocol extremely challenging and computationally infeasible. To overcome this problem, and transform the dimensionality from a curse into a blessing, we propose a novel model based on Mean Field Games (MFG) [54] [52] [53]. In MFG, the optimal action of each device depends only on the average behavior of other devices, which allows to efficiently compute the optimal solution.

More specifically, we focus on the mMTC uplink of a single cell containing one BS and a very large number of transmitting devices using a Non-Orthogonal Medium Access (NOMA) scheme which has been proposed as a candidate for 5G wireless networks mMTC. This type of medium access is basically a Code Division Multiple Access (CDMA) scheme with the difference that the transmitting devices do not have codes that are completely orthogonal to each other. Although orthogonal codes would be desirable for nearby devices, offline assignment of orthogonal codes would require an impossibly large number of orthogonal codes, while online assignment would require a high degree of coordination, which is also prohibitive in our setting.

NOMA resolves this issue by an a priori assignment of codes that are quasi-orthogonal thus requiring a reduced number of codes to serve all devices, allowing some small degree of overlap due to randomness. Yet, when any subset of the devices transmit simultaneously, the transmitted information can still be decoded at the receiver, albeit at a lower data rate than if all devices transmitted with orthogonal codes.

In addition, NOMA can be combined with transmit power control to improve both throughput and energy efficiency. Power control has always been recognized as an important problem for multiuser communications. In particular distributed power control policies, where mobile terminals can freely choose their transmit

power level $p_i(t)$ and do not need to be controlled from central nodes, are of special importance as they avoid the complexity, signaling overhead and delay of centralized solutions. Notice that this is a very general scenario where for example $p_i(t)$ can be chosen between two extremes to emulate TDMA, or with a probability to emulate CSMA.

In our model, each device is given an energy budget and a number of bits, and has to choose its own transmit power in order to optimize a local utility increasing in remaining energy and decreasing in transmission time. Since the uplink channels are governed by a Markovian process, the design of the optimal protocol naturally leads to a stochastic differential game of large dimensions, a notoriously difficult problem to solve. We propose a novel modeling via the Mean-Field Game approximation, where the number of users is taken to infinity and their actions remain coupled only via average entities, allowing us to recover a set of equations which can be solved numerically to provide the optimal protocol.

5.2 Related work

The problem of distributed power control in the uplink of a single cell using a CDMA MAC has been modeled as a stochastic differential game in [59]. The players in this game are the transmitters who adapt their power level to the quality of their time-varying link with the receiver, their battery level, and the strategy updates of the other transmitters. The paper is of theoretical nature providing a simple sufficient condition for the existence of a Nash equilibrium in this game. As the uniqueness and determination of equilibria are difficult issues in general, especially when the number of players goes large, the paper addresses two special cases : the single player case and the large number of players case. The latter case is treated with a MFG approach for which reasonable sufficient conditions for convergence and uniqueness are provided. Illustrative numerical results which indicate that this MFG approach can lead to significant gains in terms of energy efficiency are shown in [58].

In this paper, we follow a similar model as in [59] and [58]. However, we shift our attention to Quality of Service considerations and more specifically the need to transmit a given number of information bits with a small delay. Latency requirements are important in a wide range of applications and receive considerable attention in 5G networks design. The problem of meeting latency requirements over wireless links is particularly challenging due to channel quality fluctuations which require adjustments to the transmission parameters (transmit power, modulation and coding scheme, etc) to achieve an optimum trade-off between transmission rate, packet error rate and energy efficiency [62] [37].

A recent paper [51] applies MFG techniques to develop a transmit power se-

lection policy with the goal to optimize a combined cost of average queue length and energy consumption. Although the problem is formulated in the general case, in order to keep the solution mathematically tractable, a bufferless queue and a two-state channel are considered. In our paper we tackle the bufferless problem and consider a more general channel model which is similar to the channel model adopted in [59] [58]. We keep the mathematical solution tractable by applying an appropriate optimization objective which leads to a stationary solution, thus removing the dependence on time.

5.3 System model and problem formulation

We consider a large number K of wireless devices which need to transmit information to the same base station receiver using a NOMA scheme. Each device receives at random times a number of bits that it needs to transmit to the common receiver and a certain energy budget for transmitting those bits. To make the model as generic as possible, the number of bits and energy budget are randomly drawn from a known probability distribution on $[0, B_{max}] \times [0, E_{max}]$. At any time t the i-th device can choose its own transmit power $p_i(t) \in [0, P_{max}]$. The achievable instantaneous transmission rate is given by

$$r_i(t) = Rf(\gamma_i(p_1(t), p_2(t), \ldots, p_K(t))) \tag{5.1}$$

where R is a constant (in bits/s) and γ_i is the Signal to Interference plus Noise Ratio (SINR) after Matched-Filter detection at the receiver of a random CDMA scheme [35, Sec. 12.2] expressed as

$$\gamma_i(p_1(t), p_2(t), \ldots, p_K(t)) = \frac{p_i(t)H_i(t)}{\sigma^2 + \frac{1}{M}\sum\limits_{j\in\mathcal{K}, j\neq i} p_j(t)H_j(t)} \tag{5.2}$$

where $H_i(t)$ is a parameter representing the quality of the channel between the i-th transmitter and the receiver, σ^2 is a constant which models the communication noise effects at the receiver, M is the length of each CDMA spreading code (with the common assumption that as K grows large, $\frac{K}{M} \to c$, with c being a finite constant [35]), and $f(\cdot)$ is a function which depends on the transmitter's capabilities and knowledge of the current SINR. Sophisticated transmitters which can use many modulations and coding schemes and adapt the used scheme to the current SINR can approximate the Shannon capacity limit. In this case $f(x) = \log_2(1 + x)$. For simpler transmitters employing a single modulation and coding scheme, $f : \mathbb{R}^+ \to [0, 1]$ is a sigmoidal function which represents the bit success rate. In general, other choices of smooth increasing functions can be used.

The channel quality $H_i(t)$ evolves according to a random process which depends on many factors. A vast literature on modeling the evolution of the channel as a random process exists. We are going to adopt a commonly used model which is both representative of practical scenarios and amenable to mathematical analysis : different channel qualities are assumed to be independent and identically distributed (i.i.d) and each $H_i(t)$ is assumed to follow a reflected Brownian motion in the interval $[H_{min}, H_{max}]$. Note however, that our problem formulation and solution does not depend on the nature of the random process $H_i(t)$ and can be easily applied to a vast range of different random processes.

The dynamics of the system can be modeled as follows :

$$\begin{aligned} dE_i(t) &= -p_i(t)dt \\ dB_i(t) &= -r_i(t)dt \\ dH_i(t) &= \sqrt{2\nu}d\mathbb{W}_i(t) \end{aligned} \tag{5.3}$$

where $E_i(t)$ is the amount of remaining energy budget for device i and $B_i(t)$ is the amount of information (in number of bits [1]) that device i has still to transmit, both at time t. $\mathbb{W}_i(t)$ denotes a Wiener process on a given probability space modeling the fluctuations of the channel quality H_i (fading) and the parameter $\nu > 0$ describes the intensity of the fading (it can be used to capture slow or fast fading). Note that in the above dynamics, the actions of each device affect the states of all other devices through the dependence of $r_i(t)$ to all the $p_i(t)$, $i = 1, \ldots, K$.

The goal of each device is to transmit all its information bits within its energy budget and with an optimal trade-off between energy consumption and the time needed to complete the transmission. We can model this phenomenon as a Stochastic Differential Game (SDG), that is, K coupled stochastic optimal control problems. We assume that device i will exit the game once it has transmitted all the information, i.e., when $B_i(t) = 0$ or when it is out of energy, i.e., $E_i(t) = 0$ (whichever comes first). We denote by τ_i the first time at which one of these conditions is satisfied. Each device faces the stochastic optimal control problem :

$$\sup_{0 \leq p_i(0 \to \tau_i) \leq P_{max}} \mathbb{E}\left[\int_0^{\tau_i} -\theta e^{-\lambda t}dt + e^{-\lambda \tau_i}\psi(E_i(\tau_i), B_i(\tau_i))\right] \tag{5.4}$$

where $\psi\,()$ is an appropriate terminal utility which is an increasing function of the amount of remaining energy and a decreasing function in the number of information bits not transmitted at the exit time, e.g., $\psi(E_i(\tau_i), B_i(\tau_i)) = E_i(\tau_i) - B_i(\tau_i)$, $\theta > 0$ is a parameter of the model that puts appropriate weights on the utility depending on the exit time (the first term) and depending on the exit cost (the second term), and λ is the inter-temporal preference rate of the devices (i.e., the

1. we assume a fluid model under which the number of bits is seen as a continuous variable

weight they put on the present versus the future). The intuition behind the utility function in (5.4) is that it penalizes devices the longer they stay in the system (through the first term), and the more energy they have consumed and the less bits they have transmitted when they exit the system (through the second term). The multiplicative factor $e^{-\lambda t}$ guarantees that the average cost or utility function stays finite. Let us note that τ_i is not a control but it is a function of the whole trajectory of device i, thus a function of p_i and of $(p_j)_{j\neq i}$ (because of the coupling).

In this infinite horizon SDG, the control of a device i is its transmit power $p_i(t)$ which is allowed to depend not only on time, but on its own state (E_i, B_i, H_i) and on the states of all other devices in the system. This is an extremely complicated problem not only because of the huge number of variables and couplings involved but also because in practical implementations the communication overhead of relaying all the state of each device to each other device will be prohibitively large. In order to arrive at a practical and efficient scheme, we propose below to use a MFG limit for this game.

In practical situations, the same devices can enter and exit the game over and over again as their transmitters get some data to transmit at random time instances from the applications running on the device. As we want to keep the model simple, without considering the need for buffering data received in more than one round, we assume that devices that are in the game (having still data to transmit) cannot receive new data from their application layer. We assume that such a data-blocking event happens with an extremely small probability as the mean exit time (during which the device is in a blocking-additional-data mode) is much smaller than the minimum inter-arrival time of new data.

5.4 Mean Field Game analysis

It is well known [59][58] that this type of SDG can be simplified by considering the MFG regime as $K \longrightarrow \infty$. In this setting, a player has interactions with the other players only through mean field terms (averaged quantities), which in our problem is the SINR experienced by the player. Because the coupling between the players only appears through an averaged quantity, the optimization problem faced by a generic player only depends on the distribution m of other players.

Moreover, let us note that in the SDG with K players, as devices reach their exit time, they exit the game. Thus the number of players in the game is decreasing over time, and will eventually reach 0, making the game trivial. We assume in this MFG that new players arrive randomly in the game according to a Poisson process and that the state of a newly arriving player is randomly drawn from a distribution $m_s(E, B, H)$. Because the problem has an infinite horizon and the distribution of arriving player m_s as well as the terminal utility $\psi(E, B)$ do not depend on time,

the Nash equilibrium is stationary, i.e., it does not depend on time. If we denote by u the value function of the MFG (and by m the non-normalized density of the devices), then a stationary Nash equilibrium is described by a solution of the following stationary system of PDE (the first PDE is the infinitesimal dynamic programming principle and the second one the infinitesimal local conservation of the number of devices) on $\Omega = (0, E_{max}) \times (0, B_{max}) \times (H_{min}, H_{max})$:

$$\lambda u - \nu \partial_{HH} u + p^* \partial_E u + Rf(\frac{Hp^*}{\Gamma}) \partial_B u + \theta = 0 \tag{5.5}$$

$$-\nu \partial_{HH} m - \partial_E (p^* m) - R \partial_B \left(f(\frac{Hp^*}{\Gamma}) m \right) = m_s \tag{5.6}$$

where, p^* is the optimal strategy, i.e., the solution of

$$p^*(E, B, H) = \text{argmax}_{0 \leq p \leq P_{max}} \left\{ -p \partial_E u - Rf(\frac{Hp}{\Gamma}) \partial_B u \right\} \tag{5.7}$$

and Γ is representing the mean field coupling of the two PDE through the interference experienced by a generic player which is given by :

$$\Gamma = \sigma^2 + c \int \int \int_{\Omega} h \, p^*(E, B, H) \, m(E, B, H) \, dB \, dE \, dH \tag{5.8}$$

The above system of PDE is subject to the following boundary conditions :

$$\begin{aligned} u(0, B, H) &= -B \\ u(E, 0, H) &= E \\ \partial_H u(E, B, H_{min}) &= 0 \\ \partial_H u(E, B, H_{max}) &= 0 \end{aligned} \tag{5.9}$$

Most of the boundary conditions above are self-evident. The last two boundary conditions are a direct consequence of the fact that the channel state H is modeled as a reflected Brownian motion in $[H_{min}, H_{max}]$.

The above system of PDE ((5.5) - (5.9)) is typical in MFG. We refer the reader to [54] [52] for more details on this system. Equation (5.5) is known as the Hamilton-Jacobi-Bellman (HJB) equation and given a fixed mean field Γ, its solution is the value function of a generic player facing (5.4) whose dynamics, under a choice of control p, evolve according to :

$$\begin{aligned} dE(t) &= -p dt \\ dB(t) &= -Rf(\frac{pH}{\Gamma}) dt \\ dH(t) &= \sqrt{2\nu} d\mathbb{W}(t) \end{aligned}$$

where $(\mathbb{W}(t))_{t\geq 0}$ is a reflected brownian motion in $[H_{min}, H_{max}]$. It is classical to check by a verification argument that such a trajectory is optimal for the players. The Fokker-Planck (FP) equation (5.6) is solved by the non-normalized density of devices, provided that with energy E, data left to transmit B and quality of channel H, all the devices use the control $p^*(E, B, H)$. Moreover let us note that we can solve the problem of a given device by solving this system. Indeed if a device is in the state (E, B, H) it uses the optimal control $p^*(E, B, H)$. Thus, the state of a device being at (E_0, B_0, H_0) at time t_0 evolves according to :

$$\begin{aligned}
dE(t) &= -p^*(E(t), B(t), H(t))dt; \\
dB(t) &= -Rf(h\frac{p^*(E(t), B(t), H(t))}{\Gamma})dt; \\
dH(t) &= \sqrt{2\nu}d\mathbb{W}(t); \\
E(t_0) &= E_0; B(t_0) = B_0; H(t_0) = H_0;
\end{aligned}$$

while $\min(E(t), B(t)) > 0$. Hence, knowing the solution of the MFG system, the players can compute their optimal control depending only on their own state, provided that all the other devices use the control $p^*(E, B, H)$ while in the state (E, B, H), making their anticipation of the mean field (the value of Γ) correct. We recall that we are looking for Nash equilibria of the game.

This system of equations has no closed form solution and must be solved numerically. We provide below some explanations on how to solve the MFG system numerically, see [2] for more information. In order to solve the PDE problem at hand we propose the following iterative method :

We start from a given value of Γ and we fix a parameter $1 \geq \eta > 0$. We then proceed as follows :

— Solve the two PDE (5.5) and (5.6) using a finite differences scheme. Note that for a fixed Γ, (5.5) does not depend on the solution of (5.6), thus we solve first (5.5) and then (5.6).

— Calculate Γ' using equation (5.8), and update Γ according to :

$$\Gamma = (1 - \eta)\Gamma + \eta\Gamma' \tag{5.10}$$

— Iterate until $|\Gamma - \Gamma'| < \epsilon$, where ϵ is a very small positive number setting the convergence criterion.

Although we do not have any formal proof of convergence, the proposed iterative method seems to always converge in the simulation experiments we have run.

To solve the system (5.5), (5.6), (5.7) we work on a uniform grid G_l of l^3 points on $[0, E_{max}] \times [0, B_{max}] \times [H_{min}, H_{max}]$. A function on G_l is associated to an element of $(\mathbb{R}^N)^3$, where $N = \frac{1}{l}$. The steps on the grid are of size $dE = \frac{E_{max}}{N}; dB = \frac{B_{max}}{N}; dH = \frac{H_{max} - H_{min}}{N}$. First, we find a zero of the function $F(u)$ defined on $(\mathbb{R}^N)^3$

by :

$$\begin{aligned} F(u)_{e,b,h} =& \lambda u_{e,b,h} + \theta + p^*_{e,b,h} \frac{u_{e,b,h} - u_{e-1,b,h}}{dE} + \\ & Rf(\frac{H_h p^*_{e,b,h}}{\Gamma}) \frac{u_{e,b,h} - u_{e,b-1,h}}{dB} - \\ & \nu \frac{u_{e,b,h+1} - 2u_{e,b,h} + u_{e,b,h-1}}{dH^2} \end{aligned} \tag{5.11}$$

where $1 \leq e, b, h \leq N$; $p^*_{e,b,h}$ is given by :

$$\begin{aligned} p^*_{e,b,h} = \text{argmax}_{0 \leq p \leq P_{max}} \Big\{ & -p \frac{u_{e,b,h} - u_{e-1,b,h}}{dE} - \\ & Rf(\frac{H_h p}{\Gamma}) \frac{u_{e,b,h} - u_{e,b-1,h}}{dB} \Big\} \end{aligned} \tag{5.12}$$

and H_h is given by :

$$H_h = H_{min} + \frac{H_{max} - H_{min}}{N} h.$$

We use as boundary conditions the standard conventions that for any $1 \leq e, b, h \leq N$:

$$\begin{aligned} u_{0,b,h} &= -B_{max} \frac{b}{N} \\ u_{e,0,h} &= E_{max} \frac{e}{N} \\ u_{e,b,0} &= u_{e,b,2} \\ u_{e,b,N+1} &= u_{e,b,N-1} \end{aligned}$$

We then use a Newton algorithm to find a solution of $F(u) = 0$. The function F is the discretization of the HJB equation, hence solving $F(u) = 0$ yields a numerical solution of the HJB equation.

In order to solve the Fokker-Planck equation (5.6), we use the newly found solution of the HJB equation $u \in (\mathbb{R}^N)^3$, such that $F(u) = 0$. Then we define $\tilde{F}$ by $\tilde{F}(u)_{e,b,h} = F(u)_{e,b,h} - \lambda u_{e,b,h} - \theta$, and we look at the linear problem :

$$(D\tilde{F}(u))^T m = m_s^h$$

where $D\tilde{F}(u)$ stands for the differential of $\tilde{F}$ at u, the unknown is $m \in (\mathbb{R}^N)^3$ and $m_s^l \in (\mathbb{R}^N)^3$ is the numerical approximation of m_s. This linear problem is the direct analogous of the FP equation.

5.5 Numerical Results

In this section we provide numerical results that illustrate the control strategies obtained as a solution to our MFG formulation of the optimal transmit power problem. In order to obtain these results we have used the following parameters :

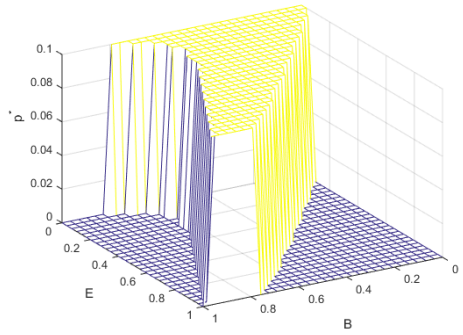
— $R = 8$ Mbps ($= 8 \times 10^6$ bps)
— $P_{max} = 0.1$ Watt
— $\sigma^2 = 1$
— $c = 500$
— $\nu = 0.28$
— $E_{max} = 1$ Joule
— $B_{max} = 1$ Kbit ($= 1000$ bits)
— $H_{min} = 0.2$
— $H_{max} = 3.0$
— $\theta = 0.101$
— $\lambda = 0.1$

We have also assumed a uniform source rate $m_s(E, B, H) = 0.1$ everywhere, a simple function $\psi(E_i(\tau_i), B_i(\tau_i)) = E_i(\tau_i) - B_i(\tau_i)$ and the Shannon capacity approximation for the achievable rate, i.e., that $f(x) = \log_2(1 + x)$.

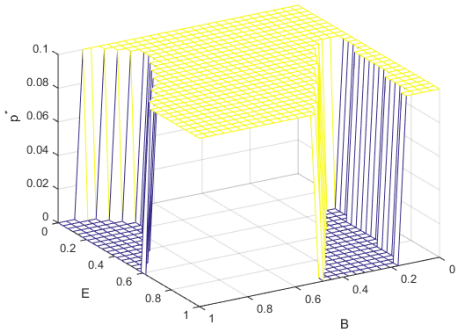
Using the above parameters we have solved numerically the system of PDE characterizing the MFG as described in the previous section for $N = 30$. In Figure 5.1, we plot the optimal transmit power as a function of E and B for different values of h. Let us note that because we choose the parameters such that $\lambda^{-1}\theta > B_{max}$, there is no device which has an interest not to exit the game. Indeed, $-\lambda^{-1}\theta$ is the cost to stay in the game forever and $-B_{max}$ is the worst possible terminal cost.

A first important observation is that the optimal policy takes the values 0 or P_{max} at almost all grid points. We can see that as the quality of the channel grows, the number of points in the (E, B) plane for which $p^*(E, B, H) = P_{max}$ grows as well (we don't plot p^* for high values of H, as for $h \geq 21$ it is $p^*(E, B, H) = P_{max}$ for all (E, B)). When a device experiences very low channel quality (Figure 5.1a), it doesn't want to spend energy to get some bits transmitted in two cases : (i) when it has a lot of energy left and few bits to transmit and (ii) when it has little to medium energy left and many bits to transmit. In the first case, the device has the luxury to wait for better channel conditions as it only has a few bits to transmit and hence it can complete their transmission in a small amount of time. In the latter case, the device has little energy left and prefers to wait for safer channel conditions. As the quality of the channel improves, both the areas corresponding to cases (i) and (ii) above shrink in size with the area corresponding to case (i) shrinking and disappearing faster (Figures 5.1b, 5.1c, and 5.1d).

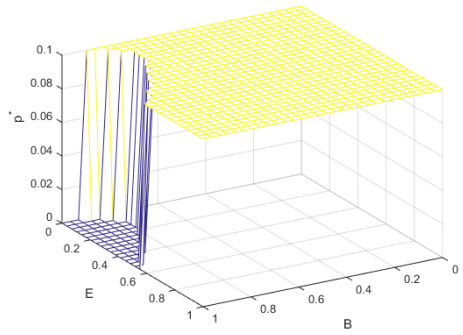
In Figure 5.2, we plot the expected exit time $\mathbb{E}[\tau_i(E, B)]$ averaged over channel



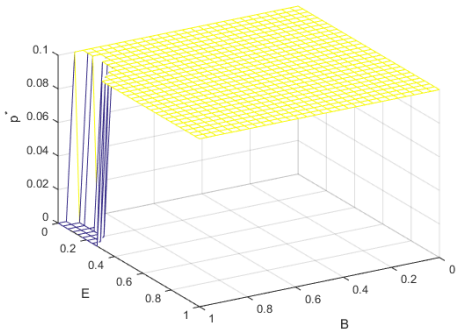
(a) $h = 1 (H = 0.2)$



(b) $h = 5 (H = 0.5862)$



(c) $h = 6 (H = 0.6828)$



(d) $h = 11 (H = 1.1655)$

FIGURE 5.1 – Optimal transmit power as a function of remaining energy E and remaining bits to transmit B.

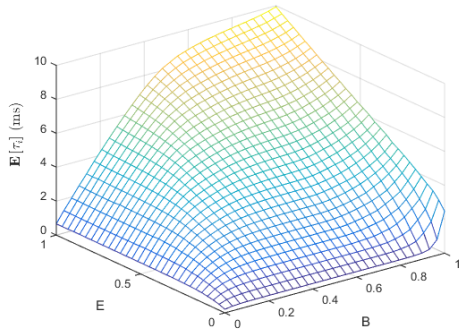


FIGURE 5.2 – Expected exit time (in milliseconds) as a function of energy budget E and received bits to transmit B. This is identical to the expected remaining time in the game as a function of remaining energy and remaining bits to transmit.

quality, for a device that has received B bits to transmit with an energy budget of E Joules. Note that this is also the expected remaining time in the game for a device that at some point in time has B remaining bits to transmit and E Joules of remaining energy. As expected, $\mathbb{E}[\tau_i(E, B)]$ is an increasing function in both E and B.

5.6 Conclusions and directions for future work

The problem of minimizing energy usage while achieving QoS requirements in uncoordinated uplink transmissions for MTC users is a very complex stochastic control problem with multiple agents. We have proposed a MFG model for this type of problems that reduces the complexity by a great deal and is thus amenable to numerical solution. Our model is general enough to include generic rate functions, arbitrary energy and QoS requirements per user, different channel fading models, and design knobs for determining the importance of different performance goals.

A lot of extensions to the model presented in this paper seem possible and easy to compute. We are particularly interested to the case of non-identical and correlated channels which is both encountered in practice and of theoretical interest in MFG. Indeed, in MFG literature the correlated channels case is known as MFG with common noise and its solution involves the so-called master equation. This will be the subject of a future work. Other potential extensions include the case where the QoS quantity of interest is not the mean transmission delay but the probability that a given delay threshold is exceeded. Another interesting case is when arriving data from the application layer are stored in a buffer forming a FIFO

queue and the objective is to control the queue length. Finally, we believe practical questions can be solved by applying MFG inspired policies in actual systems. Such practical questions include the robustness of the control policies to various errors, including errors in estimating the channel quality at the transmitter and MFG approximation errors, i.e., how good is the control policy derived for the MFG limit ($K \longrightarrow \infty$) when applied by a finite (yet large) number of transmitters.

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Résumé

Cette thèse porte sur l'étude de nouveaux modèles de jeux à champ moyen. On étudie dans un premier temps des modèles d'arrêt optimal et de contrôle impulsionnel en l'absence de bruit commun. On construit pour ces modèles une notion de solution adaptée pour laquelle on prouve des résultats d'existence et d'unicité sous des hypothèses naturelles. Ensuite, on s'intéresse à plusieurs propriétés des jeux à champ moyen. On étudie la limite de ces modèles vers des modèles d'évolution pures lorsque l'anticipation des joueurs tend vers 0. On montre l'unicité des équilibres pour des systèmes fortement couplés (couplés par les stratégies) sous certaines hypothèses. On prouve ensuite certains résultats de régularités sur une "master equation" qui modélise un jeu à champ moyen avec bruit commun dans un espace d'états discret. Par la suite on présente une généralisation de l'algorithme standard d'Uzawa et on l'applique à la résolution numérique de certains modèles de jeux à champ moyen, notamment d'arrêt optimal ou de contrôle impulsionnel. Enfin on présente un cas concret de jeu à champ moyen qui provient de problèmes faisant intervenir un grand nombre d'appareils connectés dans les télécommunications.

Abstract

This thesis is concerned with new models of mean field games. First, we study models of optimal stopping and impulse control in the case when there is no common noise. We build an appropriate notion of solutions for those models. We prove the existence and the uniqueness of such solutions under natural assumptions. Then, we are interested with several properties of mean field games. We study the limit of such models when the anticipation of the players vanishes. We show that uniqueness holds for strongly coupled mean field games (coupled via strategies) under certain assumptions. We then prove some regularity results for the master equation in a discrete state space case with common noise. We continue by giving a generalization of Uzawa's algorithm and we apply it to solve numerically some mean field games, especially optimal stopping and impulse control problems. The last chapter presents an application of mean field games. This application originates from problems in telecommunications which involve a huge number of connected devices.

Mots Clés

EDP. Jeux à champ moyen. Inéquations variationnelles.

Keywords

PDE. Mean field games. Variational inequalities.