Vector bundles on hyperelliptic curves

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Introduction

Les espaces de modules sont des constructions fondamentales de la géométrie algébrique. Ils apparaissent naturellement dans des problèmes de classification. De façon générale, un espace de modules pour une famille X d'objets et une relation d'équivalence $\sim$ est un espace dont chaque point correspond exactement à une classe d'équivalence de ces objets. Les espaces de modules sont donc, par définition, des espaces de classification. Dans notre contexte, les objets qu'on cherche à classifier sont des variétés algébriques. Ainsi, une question naturelle est si on peut aussi munir ces espaces de modules d'une structure algébrique. On souhaite aussi que notre espace de modules soit unique (à isomorphisme près).

On considère une classe particulière d'objets : les fibrés vectoriels sur une variété algébrique. Plus précisement, un espace de modules de fibrés vectoriels stables sur une variété algébrique lisse X est un schéma dont les points sont en bijection *naturelle* avec les classes d'isomorphisme de fibrés vectoriels stables sur X.

Ces espaces peuvent être construits grâce à la théorie géométrique des invariants (GIT) comme quotients de certains schémas par une certaine action de groupe naturelle. Néanmoins, cette construction ne nous donne pas des informations sur la géométrie de l'espace de modules en question. Ainsi, une fois qu'on a montré l'existence d'un tel espace, il est naturel de chercher à donner une description de sa structure locale et globale. Dans certains cas, lorsqu'on analyse la structure algébrique de certains espaces de modules, on retrouve des variétés projectives issues de la géométrie algébrique classique. En outre, les automorphismes de ces variétés peuvent être parfois interpretés comme des transformations des objets classifiés. Dans ce cas ces automorphismes sont dits *modulaires.*

Cette thèse est dédiée à l'étude des espaces de modules de fibrés (respectivement de fibrés quasi-paraboliques) sur une courbe algébrique et lisse sur le corps des nombres complexes. Le texte est composé de deux parties bien différenciées :

Dans la première partie, je m'intéresse à la géométrie liée aux classifications de fibrés quasi-paraboliques de rang 2 sur une courbe elliptique 2-pointée, à isomorphisme près. Les notions d'indécomposabilité, simplicité et stabilité de fibrés donnent lieu à des espaces de modules qui classifient ces objets. La structure projective de ces espaces est décrite explicitement, et on prouve un théorème de type Torelli qui permet de retrouver les informations de départ. Cet espace de modules est aussi mis en relation avec l'espace de modules de fibrés quasi-paraboliques sur $\mathbb{P}^1$, qui apparaît naturellement comme revêtement de l'espace de modules sur la courbe elliptique. Finalement, on démontre explicitement la modularité des automorphismes de cet espace de modules. Cette partie correspond à l'article de recherche [28].

Dans la deuxième partie, j'étudie l'espace de modules $\mathcal{SU}_C(2)$ de fibrés semistables de rang 2 et déterminant trivial sur une courbe hyperelliptique C. Plus précisément, je m'intéresse à l'application naturelle $i_{\mathcal{L}} : \mathcal{SU}_C(2) \to |\mathcal{L}|^*$ induite par le système linéaire $|\mathcal{L}|$, où $\mathcal{L}$ est le fibré déterminant, générateur du groupe de Picard de $\mathcal{SU}_C(2)$. Cette application peut être interpretée en termes du groupe de Picard de C. En effet, considérons la variété de Picard $\mathrm{Pic}^{g-1}(C)$ de fibrés en droites de degré $g-1$. Le *diviseur canonique de C* $\Theta \subset \mathrm{Pic}^{g-1}(C)$ est défini, en tant qu'ensemble, par

$$\Theta := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C, L) \neq 0\}.$$

Alors, l'application $i_{\mathcal{L}}$ s'identifie à l'application $\theta : \mathcal{SU}_C(2) \to |2\Theta|$ définie par

$$\theta(E) := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C, E \otimes L) \neq 0\},$$

et elle est de degré 2 dans notre cas.

Dans le Chapitre 3, on définit une fibration $\mathcal{SU}_C(2) \dashrightarrow \mathbb{P}^g$ dont la fibre générique est birationnelle à l'espace de modules $\mathcal{M}_{0,2g}$ de courbes rationnelles $2g$-épointées, et on cherche à décrire la restriction de θ aux fibres de cette fibration. On montre que cette restriction est, à une transformation birationnelle près, une projection osculatoire centrée en un point. En utilisant une description due à Kumar, on démontre que la restriction de l'application θ à cette fibration ramifie sur la variété de Kummer d'une certaine courbe hyperelliptique de genre $g-1$. Cette partie a été développée dans l'article [29].

1.1 Fibrés quasi-paraboliques sur une courbe elliptique

1.1.1 Définitions

Soit C une courbe complexe, compacte et lisse de genre $g \geq 0$ et soit $T = t_1 + \cdots + t_n$ un diviseur réduit sur C. Un *fibré quasi-parabolique de rang 2* $(E, \boldsymbol{p} = (p_1, \ldots, p_n))$ sur (C, T) est la donnée d'un fibré vectoriel

E de rang 2 sur C, et d'un sous-espace linéaire $p_i \subset E_{t_i}$ de dimension 1 pour chaque $i = 1, \dots, n$, où E_{t_i} est la fibre de t_i. Le sous-espace $p_i \subset E_{t_i}$ est appelé la *structure quasi-parabolique* au dessus de t_i. Ces objets sont souvent appellés *fibrés quasi-quasi-paraboliques* dans la literature. Les fibrés quasi-paraboliques ont été introduits par Seshadri pour construire une désingularisation [53] de l'espace de modules de fibrés de rang 2 et degré zéro.

Si la courbe C est hyperelliptique, on peut étudier la relation entre fibrés quasi-paraboliques de rang 2 sur C et sur son quotient hyperelliptique. Le cas $g = 0$ a été étudié dans l'article [39], et le cas $g = 2$ a été développé dans [30]. Dans mon travail [28], je m'intéresse au cas $g = 1$.

Notation. Dans ce qui suit, les fibrés quasi-paraboliques sont supposés de rang 2, sauf si le contraire est indiqué, notamment s'il s'agit de fibrés en droites.

1.1.2 Indécomposabilité, simplicité, stabilité

On considère trois notions associées aux fibrés quasi-paraboliques : indécomposabilité, simplicité et stabilité. Un fibré quasi-parabolique est *indécomposable* s'il ne peut pas être écrit comme somme de deux fibrés quasi-paraboliques linéaires, et *simple* s'il n'admet pas d'automorphismes non scalaires (qui préservent les directions quasi-paraboliques). Soit $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \in [0, 1]^n$ un vecteur de *poids*. Par définition, un fibré quasi-parabolique $\mathcal{E}$ est *$\boldsymbol{\mu}$-semistable* si, pour tout sous-fibré en droites $L \subset E$, on a

$$\deg E - 2\deg L + \sum_{p_i \not\subset L} \mu_i - \sum_{p_i \subset L} \mu_i \geq 0.$$

On dit que $\mathcal{E}$ est $\boldsymbol{\mu}$-stable si l'inégalité est stricte pour tout L.

Ces trois notions coïncident quand C est la droite projective [40], mais elles diffèrent pour les courbes de genre supérieur [30]. Dans le travail [28], je m'intéresse au cas (C, T) d'une courbe elliptique C et d'un diviseur $T = t_1 + t_2$ de degré 2. On montre que, dans cette situation, les implications suivantes ne sont pas des équivalences :

$$\boldsymbol{\mu}\text{-stable} \implies \text{simple} \implies \text{indécomposable}.$$

On montre aussi que tout fibré quasi-parabolique simple est $\boldsymbol{\mu}$-stable pour un certain choix de $\boldsymbol{\mu}$.

1.1.3 L'espace de modules de fibrés $\boldsymbol{\mu}$-semistables

On s'intéresse maintenant aux espaces de modules de fibrés $\boldsymbol{\mu}$-semistables pour $\boldsymbol{\mu}$ fixé, au sens GIT. Ces espaces ont été étudiés dans le cas $g = 0$ dans [6], [40] et [21]. Bauer [6] a prouvé que l'espace de modules correspondant est rationnel. Casagrande [21] construit cet espace comme une certaine variété de sous-espaces linéaires contenues dans l'intersection de deux quadriques. Plus généralement, tous les espaces de modules de fibrés quasi-paraboliques connus sont rationnellement connexes. On sait que, pour les surfaces complexes lisses, connexité rationnelle implique rationalité. Il est donc naturel de s'intérroger sur la rationalité de ces espaces de modules en genre supérieur. Notre premier résultat est le suivant :

Théorème A. *Soit C une courbe de genre 1 et T un diviseur réduit de degré 2. Alors, l'espace de modules $\mathrm{Bun}^{\boldsymbol{\mu}}(C,T)$ de fibrés quasi-paraboliques $\boldsymbol{\mu}$-semistables avec déterminant fixé est isomorphe à $\mathbb{P}^1\times\mathbb{P}^1$.*

Pour fixer les notations, considérons maintenant et dans ce qui suit l'espace de modules $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ des fibrés quasi-paraboliques de déterminant trivial. Si $\mu_1\neq\mu_2$, tout fibré $\boldsymbol{\mu}$-semistable est $\boldsymbol{\mu}$-stable. Quand $\mu_1=\mu_2$, on a un lieu strictement $\boldsymbol{\mu}$-semistable $\Gamma\subset\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$: c'est une courbe plongée dans $\mathbb{P}^1\times\mathbb{P}^1$ comme une courbe de bidegré $(2,2)$. Soit $\mathbf{G}$ le groupe d'automorphismes de $\mathbb{P}^1\times\mathbb{P}^1$. On a l'isomorphisme suivant :

$$\mathbf{G}\cong(\mathrm{PGL}(2,\mathbb{C})\times\mathrm{PGL}(2,\mathbb{C}))\ltimes\mathbb{Z}/2\mathbb{Z}.$$

Le facteur $\mathbb{Z}/2\mathbb{Z}$ du produit semi-direct corresponde à l'involution $(z,w)\to(w,z)$ de $\mathbb{P}^1\times\mathbb{P}^1$.

Dans la section 2.5.1, on montre le théorème de type Torelli suivant :

Théorème B. *Soit C une courbe de genre 1 et T un diviseur réduit de degré 2. Alors, la courbe Γ est isomorphe à C, et on peut retrouver le diviseur T à partir du plongement $\Gamma\subset\mathbb{P}^1\times\mathbb{P}^1$. De plus, il existe des bijections canoniques entre les ensembles suivants :*

$$\left\{\begin{matrix}(2,2)\textit{-courbes}\\ \Gamma\subset\mathbb{P}^1\times\mathbb{P}^1\end{matrix}\right\}\Big/\mathbf{G}\overset{1:1}{\longleftrightarrow}\left\{\begin{matrix}\textit{courbes}\\ \textit{elliptiques}\\ 2\textit{-pointées}\\ (C,T)\end{matrix}\right\}\Big/\sim\overset{1:1}{\longleftrightarrow}\left\{\begin{matrix}\textit{courbes}\\ \textit{rationnelles}\\ (4+1)\text{-}\\ \textit{pointées}\\ (\mathbb{P}^1,\underline{D}+t)\end{matrix}\right\}\Big/\sim$$

1.1.4 L'espace de modules de fibrés simples

L'espace des poids $\boldsymbol{\mu}$ est divisé en deux chambres $C_<$ et $C_>$ avec espaces de modules respectifs associés $\mathrm{Bun}^{<}_{\mathcal{O}}(C,T)$ et $\mathrm{Bun}^{>}_{\mathcal{O}}(C,T)$. Dans chaque chambre, les espaces de modules $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ sont *constants*, i. e. les points de $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ représentent les mêmes fibrés indépendamment des poids $\boldsymbol{\mu}$. Sur la courbe Γ on retrouve un phénomène de *wall-crossing* : un point dans Γ représente un fibré différent selon que $\boldsymbol{\mu}$ est dans $C_<$ or $C_>$, En revanche, les fibrés représentés dans le complémentaire $\mathrm{Bun}^{<}_{\mathcal{O}}(C,T)\setminus\Gamma$ et $\mathrm{Bun}^{>}_{\mathcal{O}}(C,T)\setminus\Gamma$

sont les mêmes. En identifiant les fibrés identiques, on construit le schéma non-séparé

$$\mathrm{Bun}_{\mathcal{O}}(C,T) = X_< \coprod X_> \Big/ \sim$$

qui paramètre l'ensemble de fibrés quasi-paraboliques simples sur (C,T).

1.1.5 Une application entre deux espaces de modules

Soit $\underline{D} = 0 + 1 + \lambda + \infty + t$ un diviseur réduit de degré 5 sur $\mathbb{P}^1$. Dans [40], les auteurs construisent l'espace de modules grossier $\mathrm{Bun}(\mathbb{P}^1, \underline{D})$ de fibrés quasi-paraboliques indécomposables de rang 2 avec déterminant fixé sur $(\mathbb{P}^1, \underline{D})$ comme un recollement de cartes projectives. Ces cartes sont elle mêmes des espaces de modules $\mathrm{Bun}^{\boldsymbol{\nu}}(\mathbb{P}^1, \underline{D})$ de fibrés $\boldsymbol{\nu}$-semistable pour des poids $\boldsymbol{\nu}$ spécifiques, et chaque $\boldsymbol{\nu}$-chambre correspond à une de ces cartes.

L'une de ces cartes est isomorphe au plan projectif $\mathbb{P}^2$. De plus, on a cinq points fixés dans cette carte : quatre points D_i pour $i = 0, 1, \lambda, t$ et un point *spécial* D_t. Ces points correspondent à des configurations quasi-paraboliques particulières sur un certain fibré sur $\mathbb{P}^1$. La deuxième carte $\mathcal{S}$ est définie comme l'éclatement de $\mathbb{P}^2$ dans ces cinq points. La carte $\mathcal{S}$ est donc par définition une surface de del Pezzo de degré 4.

Soit C la courbe elliptique définie par le revêtement double $\pi : C \to \mathbb{P}^1$ ramifié sur les 4 points $0, 1, \lambda$ et ∞. Le point $t \in \mathbb{P}^1$ se relève par ce revêtement en deux points t_1 et t_2 dans C.

On construit un morphisme modulaire

$$\Phi : \mathcal{S} \cong \mathrm{Bun}(\mathbb{P}^1, \underline{D}) \to \mathrm{Bun}(C,T) \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Ce morphisme est essentiellement la composition du tiré en arrière par l'application π, suivi de *transformations élémentaires* de fibrés, qui correspondent projectivement aux transformations élémentaires classiques des surfaces réglées.

Théorème C. *Le morphisme Φ est le revêtement double de $\mathbb{P}^1 \times \mathbb{P}^1$ ramifié sur la courbe Γ.*

1.1.6 La géométrie du revêtement Φ

On cherche maintenant à donner une interprétation modulaire à la géométrie du revêtement Φ. Soit τ l'involution de $\mathcal{S}$ induite par l'application de degré 2 Φ. On montre que τ est le relèvement à $\mathcal{S}$ d'un automorphisme de de Jonquières de degré 3 du plan projectif. Plus précisément, τ est le relèvement d'une transformation birationnelle de $\mathbb{P}^2$ préservant le pinceau de droites qui passent par le point spécial D_t et le pinceau de coniques qui passent par les quatre points D_i. On montre ensuite que le groupe $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ d'automorphismes de $\mathbb{P}^1 \times \mathbb{P}^1$ qui préservent Γ est engendré par trois involutions

$\widetilde{\sigma_0}$, $\widetilde{\sigma_1}$ et ϕ_T qui commutent deux à deux. Ces involutions sont modulaires, dans le sens où elles correspondent à des transformations naturelles de fibrés quasi-paraboliques. Elles se relèvent respectivement en des automorphismes σ_0, σ_1 et φ_T de $\mathcal{S}$.

Ces involutions nous permettent de réinterpréter le groupe $\mathrm{Aut}(\mathcal{S})$ d'automorphismes de $\mathcal{S}$ comme un groupe de transformations modulaires :

Théorème D. *Le groupe* $\mathrm{Aut}(\mathcal{S})$ *est engendré par les involutions* σ_k, φ_T *et* τ.

Les automorphismes de $\mathcal{S}$ sont complètement caractérisés par leur action sur l'ensemble suivant de 16 éléments de $\mathbb{P}^2$: les cinq points D_i, les dix droites Π_{ij} passant par D_i et D_j et la conique Π qui passe par tous les points D_i. On calcule explicitement cette action et l'expression en coordonnées projectives de chaque involution.

1.2 Fibrés de rang 2 sur une courbe hyperelliptique

1.2.1 Les espaces de modules de fibrés vectoriels $\mathcal{SU}_C(r)$

Soit C une courbe complexe, compacte, lisse et de genre $g \geq 2$. Soit $\mathcal{SU}_C(r)$ l'espace de modules de fibrés semistables de rang r et déterminant trivial sur C. Cet espace de modules est une variété normale, projective et unirationnelle de dimension $(r^2-1)(g-1)$.

L'étude de la structure projective des espaces de modules de fibrés pour rang et genre bas a conduit à des belles descriptions géométriques, souvent à la rencontre de la géométrie classique. Par exemple, dans le cas d'une courbe hyperelliptique C, Desale et Ramanan [23] ont étudié le quotient $\mathcal{SU}_C(2)/i^*$ de l'espace de modules de fibrés par l'application i^* induite par l'involution hyperelliptique i. Ils ont montré qu'il existe deux quadriques Q_1 et Q_2 dans l'espace projectif de dimension $2g+1$ telles que le quotient $\mathcal{SU}_C(2)/i^*$ est isomorphe à la variété des sous-espaces linéaires de dimension g contenus dans Q_1, qui appartiennent à un système fixé des espaces isotropes maximaux, et qui intersectent Q_2 en quadriques de rang au plus 4. On peut trouver d'autres résultats sur la structure projective de $\mathcal{SU}_C(r)$ dans [50] et [48].

L'application theta

Le groupe de Picard de $\mathcal{SU}_C(r)$ est isomorphe à $\mathbb{Z}$, et il est engendré par un fibré en droites $\mathcal{L}$ qu'on appelle fibré déterminant [7, 8]. Le diviseur canonique $\Theta \subset \mathrm{Pic}^{g-1}(C)$ est défini en tant qu'ensemble comme

$$\Theta := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C,L) \neq 0\}.$$

L'application naturelle $\alpha_{\mathcal{L}} : \mathcal{SU}_C(r) \dashrightarrow |\mathcal{L}|^*$ induite par le système linéaire $|\mathcal{L}|$ peut être étudiée grâce aux séries linéaires $r\Theta$ sur la variété jacobienne $\mathrm{Jac}(C)$. Plus précisément, soit $\mathrm{Pic}^{g-1}(C)$ la variété jacobienne de diviseurs de degré $g-1$ sur C. Pour chaque fibré $E \in \mathcal{SU}_C(r)$, on définit le lieu

$$\theta(E) := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C, E \otimes L) \neq 0\}.$$

Si $\theta(E)$ n'est pas égal à $\mathrm{Pic}^{g-1}(C)$, on a que $\theta(E)$ est un diviseur dans $\mathrm{Pic}^{g-1}(C)$. Ce diviseur appartient au système linéaire $|r\Theta|$, où Θ est le diviseur canonique de $\mathrm{Pic}^{g-1}(C)$. On obtient donc une application rationnelle

$$\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|.$$

Le résultat clé est le suivant :

Théorème 1.2.1 (Beauville, Narasimhan, Ramanan [10]). *Il existe un isomorphisme canonique $|\mathcal{L}|^* \cong |r\Theta|$ qui identifie $\alpha_{\mathcal{L}}$ à θ.*

Si θ est un morphisme, alors c'est un morphisme fini. En effet, puisque le système linéaire $|\mathcal{L}|$ est ample, θ ne peut pas contracter des courbes. Il est connu que l'application θ est un morphisme dans les trois cas suivants : $r = 2$; $r = 3$ et $g = 2$ ou 3 ; et $r = 3$ et C générique [51]. En outre, l'application θ n'est pas un morphisme si $r \gg 0$ [51, 22], et elle est génériquement injective pour C générale et $g \gg r$ [20].

Le cas $r = 2$

Si C est une courbe de genre $g = 2$, Narasimhan et Ramanan [46] ont prouvé que l'application θ est un isomorphisme entre $\mathcal{SU}_C(2)$ et $|2\Theta| \cong \mathbb{P}^3$. Pour les courbes de genre supérieur, on a le résultat suivant :

Théorème 1.2.2 ([23, 7, 19, 55]). *Soit C une courbe de genre $g \geq 3$. Alors :*

1. *Si C n'est pas hyperelliptique, l'application $\theta : \mathcal{SU}_C(2) \to |2\Theta|$ est un plongement.*
2. *Si C est hyperelliptique, l'application θ se factorise par l'involution hyperelliptique i et plonge le quotient $\mathcal{SU}_C(2)/i^*$ dans $|2\Theta|$.*

En particulier, si C est une courbe hyperelliptique avec $g \geq 3$, l'application θ est un morphisme de degré 2.

1.2.2 La géométrie de l'application theta

Le but principal de mon travail [29] est d'expliquer la géométrie associée à l'application θ dans le cas $r = 2$ et lorsque C est une courbe hyperelliptique. Dans le cas non hyperelliptique, l'article [1] décrit un lien entre l'espace de modules $\mathcal{SU}_C(2)$ et l'espace de modules $\mathcal{M}_{0,2g}$ de courbes rationnelles avec $2g$ points marqués. Dans le travail [29], la relation avec l'espace de modules $\mathcal{M}_{0,2g}$ présente une nouvelle description du morphisme θ si C est hyperelliptique.

Notation. Dans le texte qui suit, une forme F de degré r dans $\mathbb{P}^n$ dénote un élément de l'espace vectoriel $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \mathrm{Sym}^r(\mathbb{C}^{n+1})^*$. Si l'on fixe une base $x_0, \ldots, x_n$ de $(\mathbb{C}^{n+1})^*$, F est simplement un polynôme homogène de degré r dans les variables $x_0, \ldots, x_n$.

Une fibration en $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$

Soit C une courbe hyperelliptique de genre $g \geq 3$. Soit $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$ le quotient GIT $(\mathbb{P}^1)^{2g+2} // \mathrm{PGL}(2, \mathbb{C})$ de l'espace $(\mathbb{P}^1)^{2g+2}$ par l'action diagonale du groupe $\mathrm{PGL}(2, \mathbb{C})$ pour la $\mathrm{PGL}(2, \mathbb{C})$-linéarisation naturelle du fibré en droites $\boxtimes_{i=1}^{2g+2} \mathcal{O}_{\mathbb{P}^1}(1)$. C'est donc une compactification de l'espace de modules $\mathcal{M}_{0,2g}$.

Mon premier résultat est une extension de [1] au cas hyperelliptique.

Proposition 1.2.1. *Soit D un diviseur effectif général de degré g sur C. Il existe une fibration $p_D : \mathcal{SU}_C(2) \dashrightarrow |2D| \cong \mathbb{P}^g$ dont la fibre générale est birationnelle à $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$. De plus, on a*

1. *Pour tout diviseur général $N \in |2D|$, il existe une application rationnelle dominante $h_N : \mathbb{P}_N^{2g-2} \dashrightarrow p_D^{-1}(N)$ et $2g$ points fixés sur $\mathbb{P}_N^{2g-2}$, tels que les fibres de h_N sont les courbes rationnelles normales qui passent par ces points.*
2. *La famille de courbes définie par h_N est la famille universelle de courbes rationnelles normales sur la fibre générique $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$.*

La construction de l'espace $\mathbb{P}_N^{2g-2}$ est explicite : on considère le diviseur $K + 2D$ dans C, où K est le diviseur canonique. Ce diviseur apparaît naturellement quand on regarde l'espace $\mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D)) = |K + 2D|^*$ qui classifie les extensions de la forme

$$(e) \quad 0 \to \mathcal{O}(-D) \to E_e \to \mathcal{O}(D) \to 0.$$

Le diviseur $K + 2D$ est très ample, en particulier l'application rationnelle associée au système linéaire $|K + 2D|$ plonge la courbe C dans un espace projectif $\mathbb{P}_D^{3g-2}$. L'espace $\mathbb{P}_N^{2g-2}$ est définit comme le sous-espace projectif engendré par le support du diviseur N, vu comme ensemble de points dans $C \subset \mathbb{P}_D^{3g-2}$. Cet ensemble est constitué de $2g$ points $p_1, \ldots, p_{2g}$, ce sont exactement les points marqués dans $\mathbb{P}_N^{2g-2}$ énoncés dans la Proposition 1.2.1.

Une courbe rationnelle normale spéciale

L'objectif maintenant est de décrire l'application θ restreinte aux fibres génériques de la fibration p_D. Pour cela, la construction qui suit est cruciale :

Soient deux points p, $i(p)$ dans C en involution hyperelliptique, et considérons la droite $l \subset \mathbb{P}_D^{3g-2}$ sécante à C et passant par p et $i(p)$. On montre que la droite l intersecte toujours l'espace $\mathbb{P}_N^{2g-2}$ en un point exactement, et que le lieu $\Gamma \subset \mathbb{P}_N^{2g-2}$ de ces intersections quand on fait varier $p, i(p)$ est une courbe rationnelle normale qui passe par les points $p_1, \ldots, p_{2g}$. Par la Proposition 1.2.1, l'application h_N contracte les courbes rationnelles normales de $\mathbb{P}_N^{2g-2}$ passant par $p_1, \ldots, p_{2g}$. En particulier, h_N contracte la courbe Γ en un point $P \in p_D^{-1}(N) \underset{bir.}{\sim} \mathcal{M}_{0,2g}^{\mathrm{GIT}}$.

Dans l'article [35], Kumar définit le système linéaire Ω de $(g-1)$-formes sur $\mathbb{P}^{2g-3}$ qui s'annulent avec multiplicité $g-2$ sur $2g-1$ points généraux. Il montre que Ω induit une application birationnelle $i_\Omega : \mathbb{P}^{2g-3} \dashrightarrow \mathcal{M}_{0,2g}^{\mathrm{GIT}}$ sur le quotient GIT de $\mathcal{M}_{0,2g}$. Le système linéaire partiel $\Lambda \subset \Omega$ des formes qui s'annulent avec multiplicité $g-2$ dans un point général additionnel $e \in \mathbb{P}^{2g-3}$ induit une projection rationnelle $\kappa : \mathcal{M}_{0,2g}^{\mathrm{GIT}} \dashrightarrow |\Lambda|^*$. En particulier, κ est une projection osculatoire centrée au point $w = i_\Omega(e)$. Kumar montre aussi que cette application est de degré 2. Mon deuxième résultat décrit birationnellement les restrictions de l'application θ aux fibres $p_D^{-1}(N)$ à l'aide de l'application de Kumar :

Théorème E. *L'application θ restreinte à la fibre générique $p_D^{-1}(N)$ est la projection osculatoire κ centrée au point $P = h_N(\Gamma)$, modulo composition avec une application birationnelle.*

Kumar démontre que l'image de κ est une composante connexe de l'espace de modules $\mathcal{SU}_{C_w}(2)$, où C_w est la courbe hyperelliptique de genre $g-1$ obtenue comme revêtement ramifié de $\mathbb{P}^1$ qui ramifie exactement sur les $2g$ points définis par $w \in \mathcal{M}_{0,2g}^{\mathrm{GIT}}$. Il montre aussi que le lieu de ramification du morphisme κ est la variété de Kummer $\mathrm{Sing}(\mathcal{SU}_C(2)) = \mathrm{Kum}(C_w) \subset \mathcal{SU}_{C_w}(2)$, i. e. le lieu des fibrés décomposables $L \oplus L^{-1}$, avec $L \in \mathrm{Jac}(C_w)$.

Ces résultats de Kumar et le Théorème E nous permettent de décrire le lieu de ramification de l'application θ :

Théorème F. *Le lieu de ramification de l'application θ est birationnel à une fibration sur $|2D| \cong \mathbb{P}^g$ en variétés de Kummer de dimension $g-1$.*

1.2.3 Les applications de classification

Un outil fondamental dans les arguments de [29] est l'étude des applications de classification

$$f_L : \mathbb{P}\,\mathrm{Ext}^1(L^{-1}, L) \dashrightarrow \mathcal{SU}_C(2),$$

où L est un fibré en droites. Plus précisément, l'application f_L associe à la classe d'équivalence d'une extension

$$(e) \quad 0 \to L \to E_e \to L^{-1} \to 0.$$

le fibré vectoriel $E_e \in \mathcal{SU}_C(2)$. Les applications f_L ne sont pas toujours bien définies : en effet, le lieu de base de f_L est le lieu de classes d'extensions instables dans $\mathbb{P}\operatorname{Ext}^1(L^{-1}, L)$. Ce lieu correspond a une certaine variété sécante de la courbe $C \subset \mathbb{P}_D^{3g-2} = \mathbb{P}\operatorname{Ext}^1(L^{-1}, L)$.

L'emploi de ce type d'applications de classification est une approche classique à l'étude des espaces de modules de fibrés. Par exemple, ils ont été utilisés par Atiyah [3] pour étudier les fibrés vectoriels sur les courbes elliptiques, et par Newstead [47] pour étudier l'espace de modules de fibrés vectoriels semistables de rang 2 avec déterminant impair fixé dans le cas $g = 2$. Dans [1] et [15], ils ont été utilisés pour étudier l'espace de modules $\mathcal{SU}_C(2)$ quand C est une courbe de genre $g \geq 2$, non hyperelliptique si $g > 2$.

Notons $\varphi_L := \theta \circ f_L$. Bertram [11] montre l'existence d'un isomorphisme

$$H^0(\mathcal{SU}_C(2), \mathcal{L}) \cong H^0(\mathbb{P}_D^{3g-2}, \mathcal{I}_C^{g-1}(g)), \tag{1.1}$$

qui se traduit en la caractérisation suivante, clé dans les preuves de [29] :

Théorème 1.2.3 (Bertram [11]). *L'application φ_L est donné par le système linéaire $|\mathcal{I}_C^{g-1}(g)|$ de formes de degré g qui s'annulent avec multiplicité au moins $g - 1$ sur C.*

Les applications de classification φ_L apparaissent dans [29] dans le contexte suivant :

Proposition 1.2.2. *Soit D un diviseur effectif de degré g sur C et posons $L = \mathcal{O}(D)$. Alors, l'application $\kappa \circ h_N$ coïncide avec la restriction $\varphi_L|_{\mathbb{P}_N^{2g-2}}$, modulo composition avec une application birationnelle.*

La restriction de φ_L en genre bas

Une étude précise du lieu de base de l'application de restriction $\varphi_L|_{\mathbb{P}_N^{2g-2}}$ conduit au résultat suivant :

Théorème G. *Soit C une courbe hyperelliptique de genre 3, 4, ou 5. Alors, pour N générique, la restriction à $\mathbb{P}_N^{2g-2}$ de l'application φ_L est exactement la composition $\kappa \circ h_N$.*

Chapitre 2

Geometry of the moduli of parabolic bundles on genus 1 curves

2.1 Introduction

Let C be a smooth compact projective curve over $\mathbb{C}$ and $T = t_1 + \cdots + t_n$ an effective reduced divisor on C of degree n. A *rank 2 quasi-parabolic bundle* $(E, \boldsymbol{p} = (p_1, \ldots, p_n))$ over (C, T) consists in a rank 2 vector bundle E over C together with 1-dimensional linear subspaces p_i of the fiber E_{t_i} of t_i for $i = 1, \ldots, n$. The linear subspaces p_i are called the *quasi-parabolic directions.*

If C is hyperelliptic, we can study the relation between quasi-parabolic bundles over C and over its hyperelliptic quotient. The case $g = 0$ is explored in [39], and the case $g = 2$, building on [15] and [14], is developed in [30]. Here we investigate the case $g = 1$.

We are interested in three notions associated to these objects: indecomposability, simplicity and stability. A quasi-parabolic bundle is indecomposable if it cannot be written as a sum of two quasi-parabolic line bundles, and simple if it does not admit non-scalar automorphisms (preserving quasi-parabolic directions). Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in [0, 1]^n$ be a vector of weights. By definition, a quasi-parabolic bundle $\mathcal{E} = (E, \boldsymbol{p})$ is $\boldsymbol{\mu}$-semistable if, for every line subbundle $L \subset E$, we have

$$\mathrm{ind}_{\boldsymbol{\mu}}(L) := \mathrm{deg}E - 2\mathrm{deg}L + \sum_{p_i \not\subset L} \mu_i - \sum_{p_i \subset L} \mu_i \geq 0.$$

We say that the bundle $\mathcal{E}$ is $\boldsymbol{\mu}$-stable when the inequality is strict for every L.

These three notions coincide when the curve is the projective line (see [40]), but they are different for curves of higher genus. In this paper, we study the case of an genus 1 curve C with a divisor $T = t_1 + t_2$ of degree 2.

We show that, in this situation, $\boldsymbol{\mu}$-stability implies simplicity and simplicity implies indecomposability:

$$\boldsymbol{\mu}\text{-stable} \implies \text{simple} \implies \text{indecomposable}.$$

We also show that every simple bundle is $\boldsymbol{\mu}$-stable for some $\boldsymbol{\mu}$. Thus, the set of simple quasi-parabolic bundles is the union of the different sets of $\boldsymbol{\mu}$-stable bundles when $\boldsymbol{\mu}$ varies.

We are interested in the moduli space of $\boldsymbol{\mu}$-semistable bundles, for fixed $\boldsymbol{\mu}$. These spaces have been studied in the genus 0 case in [6], [40] and [21]. It has been shown by Bauer [6] that the corresponding moduli space is rational. Furthermore, Casagrande [21] constructs this space as the variety of linear subspaces of a projective space contained in the intersection of two quadrics. More generally, all the known moduli spaces of quasi-parabolic bundles are rationally connected. For complex smooth surfaces, this implies rationality. Thus, it is natural to ask about rationality of these moduli spaces in higher genus. In our case, this question is answered positively in Section 2.4, where we prove the following Theorem:

Theorem A. *Let C be a smooth genus 1 curve and $T = t_1 + t_2$ a reduced divisor of degree 2. Then, the moduli space $\mathrm{Bun}_L^{\boldsymbol{\mu}}(C,T)$ of $\boldsymbol{\mu}$-semistable quasi-parabolic bundles with fixed determinant L is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.*

For $\mu_1 \neq \mu_2$, every $\boldsymbol{\mu}$-semistable bundle in this moduli space is $\boldsymbol{\mu}$-stable. For $\mu_1 = \mu_2$, we have a strictly $\boldsymbol{\mu}$-semistable locus $\Gamma \subset \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C,T)$, which is a bidegree $(2,2)$ genus 1 curve in $\mathbb{P}^1 \times \mathbb{P}^1$. The curve Γ does not depend on the weights μ_1 and μ_2. We show that this locus contains the initial data (C,T), equivalent to the data of a degree 4 reduced divisor $\underline{D}$ and a point $t \notin \underline{D}$ in $\mathbb{P}^1$. This is proved in the following Theorem of Torelli type, where we consider all curves smooth:

Theorem B. *Let C be a smooth genus 1 curve and $T = t_1 + t_2$ a reduced divisor of degree 2. Then, the curve Γ is isomorphic to C, and we can recover the divisor T from the embedding $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, let $\mathbf{G} = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ be the group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ preserving the curve Γ. Then, there exist one-to-one correspondences between the following sets:*

$$\left\{\begin{matrix}(2,2)\text{-}\\ \textit{curves}\\ \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1\end{matrix}\right\} \Big/ \mathbf{G} \xleftrightarrow{1:1} \left\{\begin{matrix}2\text{-}\textit{punctured}\\ \textit{genus 1}\\ \textit{curves}\\ (C,T)\end{matrix}\right\} \Big/ \sim \xleftrightarrow{1:1} \left\{\begin{matrix}(4+1)\text{-}\\ \textit{punctured}\\ \textit{rational}\\ \textit{curves}\\ (\mathbb{P}^1, \underline{D}+t)\end{matrix}\right\} \Big/ \sim$$

where the relations $\sim$ in the second and third sets are defined as follows:

— *In the second set, $(C,T) \sim (C',T')$ if there exists an isomorphism between C and C' sending the divisor T to T'.*
— *In the third set, $(\mathbb{P}^1, \underline{D}+t) \sim (\mathbb{P}^1, \underline{D}'+t')$ if there exists an automorphism of $\mathbb{P}^1$ sending the divisor $\underline{D}$ to $\underline{D}'$ et t to t'.*

The space of weights $\boldsymbol{\mu}$ is divided in two chambers $C_<$ and $C_>$ with associated moduli spaces $\mathrm{Bun}^{<}_{\mathcal{O}}(C,T)$ and $\mathrm{Bun}^{>}_{\mathcal{O}}(C,T)$. Inside each chamber, the points in $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ are represented by the same bundles, regardless of the weight. Along the curve Γ occurs a *wall-crossing phenomena*: a point in Γ is represented by a different bundle for $\boldsymbol{\mu}$ in $C_<$ or $C_>$. In contrast, the bundles appearing in the spaces $\mathrm{Bun}^{<}_{\mathcal{O}}(C,T) \setminus \Gamma$ and $\mathrm{Bun}^{>}_{\mathcal{O}}(C,T) \setminus \Gamma$ are the same. Identifying identical bundles, we construct the non-separated scheme

$$\mathrm{Bun}_{\mathcal{O}}(C,T) = C_< \coprod C_> \Big/ \sim$$

which parametrizes the set of simple quasi-parabolic bundles on (C,T).

Let $\underline{D}$ be a reduced divisor on $\mathbb{P}^1$ of degree 5. In [40], the authors construct the full coarse moduli space $\mathrm{Bun}_{-1}(\mathbb{P}^1, \underline{D})$ of rank 2 indecomposable quasi-parabolic vector bundles over $(\mathbb{P}^1, \underline{D})$ as a patching of projective charts. These charts are moduli spaces $\mathrm{Bun}^{\boldsymbol{\nu}}_{-1}(\mathbb{P}^1, \underline{D})$ of $\boldsymbol{\nu}$-semistable bundles for specific weights $\boldsymbol{\nu}$. Thus, when we move weights from one chamber to another, we change between the charts.

One of these charts is isomorphic to the projective plane $\mathbb{P}^2$. We have five points in this chart, namely four points D_i for $i = 0, 1, \lambda, t$; and a special point D_t. Moving weights, we change to the chart $\mathcal{S}$, which is the blow-up of $\mathbb{P}^2$ in these five points. This chart is by definition a del Pezzo surface of degree 4 (recall that a del Pezzo surface of degree d is a blow-up of the plane in $9-d$ points).

We construct a degree 2 modular map

$$\Phi : \mathcal{S} \cong \mathrm{Bun}^{\boldsymbol{\mu}}_{0}(\mathbb{P}^1, \underline{D}) \to \mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T) \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

This map is the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over the curve Γ. We give a modular interpretation of the geometry of this covering. We start by studying the automorphism τ of $\mathcal{S}$ induced by Φ. This map is the lift of a de Jonquières automorphism of the projective plane. More precisely, it is the lift of a birational transformation of $\mathbb{P}^2$ preserving the pencil of lines passing through D_t and the pencil of conics passing through the four points D_i. Then, we show that the group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ preserving Γ is generated by three involutions $\tilde{\sigma_0}$, $\tilde{\sigma_1}$ and ϕ_T commuting pairwise. These involutions are modular, in the sense that correspond to natural quasi-parabolic bundle transformations. They lift to automorphisms σ_0, σ_1 and ψ_T of $\mathcal{S}$.

These involutions allow us to reinterpret the group $\mathrm{Aut}(\mathcal{S})$ of automorphisms of $\mathcal{S}$ in modular terms:

Theorem C. *The group* $\mathrm{Aut}(\mathcal{S})$ *is generated by the involutions* σ_k*,* ψ_T *and* τ*.*

These automorphisms are completely characterized by their action on a set Ω, which is the union of the following geometric elements in $\mathbb{P}^2$: the five points D_i, the ten lines Π_{ij} passing by D_i and D_j and the conic Π passing by all the points D_i. We compute explicitely this action and the coordinate expression for each involution.

2.2 GIT moduli spaces and elementary transformations

In this Section, we recall the definitions of the main objects and maps that appear in this paper. These are the moduli spaces of rank 2 quasi-parabolic vector bundles over (C, T) and elementary transformations of quasi-parabolic vector bundles, where C is a genus g curve and T is an effective reduced divisor.

2.2.1 The GIT moduli space of quasi-parabolic vector bundles

Let C be a genus $g \geq 0$ curve. Let $T = t_1 + \cdots + t_n$ be an effective reduced divisor of degree n on C. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in [0,1]^n$ be a vector of weights.

The coarse moduli space $\mathrm{Bun}_L^{\boldsymbol{\mu}}(C, T)$ of $\boldsymbol{\mu}$-semistable quasi-parabolic bundles of determinant L over (C, T) is a separated projective variety. The subset of $\boldsymbol{\mu}$-stable points is Zariski open and smooth, and the boundary is the $\boldsymbol{\mu}$-strictly semistable locus, containing the singular part (see [45], [13], [43], [52], [35] and [12]).

Let $(E, \boldsymbol{p})$ be a rank 2 quasi-parabolic bundle over (C, T) and $T' \subset T$ a subdivisor. A quasi-parabolic line subbundle of $(E, \boldsymbol{p})$ is a quasi-parabolic line bundle $(L, \boldsymbol{p}')$ over (C, T') such that $L \subset E$ and L does not pass through the quasi-parabolic directions supported by $T'' = T - T'$ (here, $\boldsymbol{p}'$ is the unique quasi-parabolic structure over (C, T')). The quotient quasi-parabolic bundle is the quasi-parabolic line bundle $(E/L, \boldsymbol{p}'')$ over (C, T'').

The slopes of $\mathcal{L} = (L, \boldsymbol{p}')$ and $\mathcal{E} = (E, \boldsymbol{p})$ are respectively the quantities

$$\mathrm{slope}(\mathcal{L}) := \deg L + \sum_{t_i \in T'} \mu_i \quad , \quad \mathrm{slope}(\mathcal{E}) := \frac{1}{2}\left(\deg E + \sum_{t_i \in T} \mu_i\right)$$

With this notation, we have that

$$\mathrm{ind}_{\boldsymbol{\mu}}(L) = 2\ \mathrm{slope}(\mathcal{E}) - 2\ \mathrm{slope}(\mathcal{L}).$$

In particular, $\mathcal{E}$ is $\boldsymbol{\mu}$-semistable if, and only if, $\mathrm{slope}(\mathcal{E}) \geq \mathrm{slope}(\mathcal{L})$ for every quasi-parabolic line subbundle $\mathcal{L}$, the strict inequality corresponding to $\boldsymbol{\mu}$-stability.

Let $\mathcal{E}$ be $\boldsymbol{\mu}$-semistable of rank 2. A Jordan-Hölder filtration is a sequence

$$0 \subset \mathcal{L} \subset \mathcal{E}$$

such that $\mathrm{slope}(\mathcal{L}) = \mathrm{slope}(\mathcal{E})$. This sequence is not canonical, but the graded bundle $\mathrm{gr}\mathcal{E} = \mathcal{L} \oplus (\mathcal{E}/\mathcal{L})$ is canonical. The moduli space $\mathrm{Bun}_L^{\boldsymbol{\mu}}(C,T)$ is constructed by identifying bundles $\mathcal{E}$ and $\mathcal{E}'$ if $\mathrm{gr}\mathcal{E} \cong \mathrm{gr}\mathcal{E}'$ as holomorphic quasi-parabolic bundles. We say then that $\mathcal{E}$ and $\mathcal{E}'$ are in the same S-equivalence class (see [45] and [13]).

From now on, we will write $\mathrm{Bun}_B^{\boldsymbol{\mu}}(C,T)$ instead of $\mathrm{Bun}_{\mathcal{O}(B)}^{\boldsymbol{\mu}}(C,T)$ for a divisor B.

2.2.2 Elementary transformations

Let C be a genus $g \geq 0$ curve. Let $T = t_1 + \cdots + t_n$ be an effective reduced divisor on C of degree n. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in [0,1]^n$ be a vector of weights.

We recall here the fundamental properties of elementary transformations of quasi-parabolic vector bundles. For a more complete reference, see [41], [30], [39] or [27].

Let E be a rank 2 vector bundle over C. Let $t \in C$, and let us denote by E_t the fiber of t. The projective space $\mathbb{P}(E_t)$ of the fiber is the space of 1-dimensional vector subspaces of E_t. The *projectivization* $\mathbb{P}(E)$ of E is the projective bundle given by taking the projective spaces $\mathbb{P}(E)_t = \mathbb{P}(E_t)$ for every $t \in C$. Hence, $\mathbb{P}(E)$ is a ruled surface.

Let $(E, \boldsymbol{p})$ be a quasi-parabolic bundle over (C,T). The *projectivization* $\mathbb{P}(E, \boldsymbol{p})$ of $(E, \boldsymbol{p})$ consist of the projective bundle $\mathbb{P}(E)$ together with the line $p_i \in \mathbb{P}(E_{t_i})$ for each t_i.

The *elementary transformation* elm_{t_i} of $\mathbb{P}(E, \boldsymbol{p})$ is a birational transformation of the total space $\mathrm{tot}(\mathbb{P}(E))$: it is the blow-up of the point $p_i \in \mathbb{P}(E)$ followed by the contraction of the total transform $\hat{F}$ of the fibre F. The point resulting from this contraction gives the new quasi-parabolic direction p_i'.

In the vectorial setting $(E, \boldsymbol{p})$, we have two transformations which coincide projectively with the above definition: the positive elementary transformation $\mathrm{elm}_{t_i}^+$ and the negative elementary transformation $\mathrm{elm}_{t_i}^-$. We recall their properties in the following Proposition:

Proposition 2.2.1. *Let $(E, \boldsymbol{p})$ be a quasi-parabolic bundle over (C,T). Then, the quasi-parabolic bundle $(E', \boldsymbol{p}') = \mathrm{elm}_{t_i}^+(E, \boldsymbol{p})$ satisfies the following properties:*

— $\det(E', \boldsymbol{p}') = \det(E, \boldsymbol{p}) \otimes \mathcal{O}(t_i)$.

— *If $L \subset E$ is a line subbundle passing by p_i, its image by* $\mathrm{elm}^+_{t_i}$ *is a subbundle* $L' \cong L \otimes \mathcal{O}(t_i)$ *of* $\mathrm{elm}^+_{t_i}(E)$ *not passing by* p'_i.
— *If $L \subset E$ is a line subbundle not passing by p_i, its image by* $\mathrm{elm}^+_{t_i}$ *is a subbundle* $L' \cong L$ *of* $\mathrm{elm}^+_{t_i}(E)$ *passing by* p'_i.

For the negative elementary transformation, the quasi-parabolic bundle $(E'', \boldsymbol{p}'') = \mathrm{elm}^-_{t_i}(E, \boldsymbol{p})$ *satisfies:*
— $\det(E'', \boldsymbol{p}'') = \det(E, \boldsymbol{p}) \otimes \mathcal{O}(-t_i)$.
— *If $L \subset E$ is a line subbundle passing by p_i, its image by* $\mathrm{elm}^-_{t_i}$ *is a subbundle* $L' \cong L$ *of* $\mathrm{elm}^+_{t_i}(E)$ *not passing by* p''_i.
— *If $L \subset E$ is a line subbundle not passing by p_i, its image by* $\mathrm{elm}^-_{t_i}$ *is a subbundle* $L' \cong L \otimes (-t_i)$ *of* $\mathrm{elm}^-_{t_i}(E)$ *passing by* p''_i.

From this Proposition, we obtain

$$(E, \boldsymbol{p}) \text{ is } (\boldsymbol{\mu})\text{-semistable} \implies \mathrm{elm}^+_{t_i}(E, \boldsymbol{p}) \text{ is } (\boldsymbol{\mu}_i)\text{-semistable.}$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ and $\boldsymbol{\mu}_i = (\mu_1, \dots, 1 - \mu_i, \dots, \mu_n)$. Therefore, an elementary transformation is a class of isomorphisms, well-defined between the corresponding moduli spaces:

$$\mathrm{Bun}^{\boldsymbol{\mu}}_L(C, T) \xrightarrow{\mathrm{elm}^+_{t_i}} \mathrm{Bun}^{\boldsymbol{\mu}_i}_{L(t_i)}(C, T).$$

The inverse map is $\mathrm{elm}^-_{t_i}$, and the composition $\mathrm{elm}^+_{t_i} \circ \mathrm{elm}^+_{t_i}$ is the tensor product by $\mathcal{O}(t_i)$. If $t_i \neq t_j$, $\mathrm{elm}^*_{t_i} \circ \mathrm{elm}^*_{t_j} = \mathrm{elm}^*_{t_j} \circ \mathrm{elm}^*_{t_i}$. We denote by $\mathrm{elm}^*_{T'}$ the composition $\mathrm{elm}^*_{T'} := \mathrm{elm}^*_{t_{i_1}} \circ \cdots \circ \mathrm{elm}^*_{t_{i_m}}$, where $T' = t_{i_1} + \cdots + t_{i_m} \subset T$ is a subdivisor.

Tensoring by a line bundle M gives the *twist automorphism*

$$\mathrm{Bun}^{\boldsymbol{\mu}}_L(C, T) \xrightarrow{\otimes M} \mathrm{Bun}^{\boldsymbol{\mu}}_{L \otimes M^2}(C, T).$$

If the genus of C is 1, the line bundles of even degree over C are always of the form M^2 for a suitable line bundle M. Consequently, the moduli spaces $\mathrm{Bun}^{\boldsymbol{\mu}}_L(C, T)$ and $\mathrm{Bun}^{\boldsymbol{\mu}}_{L'}(C, T)$ are isomorphic if the degrees of L and L' have the same parity. Thus, if $g = 1$ it is enough to study the moduli spaces $\mathrm{Bun}^{\boldsymbol{\mu}}_L(C, T)$ for $L = \mathcal{O}$ and $L = \mathcal{O}(w_\infty)$, for a fixed point $w_\infty \in C$.

Notation. Let us denote by $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C, T)$ the moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}(w_\infty)}(C, T)$.

2.3 Rank 2 indecomposable vector bundles over genus 1 curves

Let C be a genus 1 curve. Let $\mathrm{Jac}(C) = \mathrm{Pic}^0(C)$ be the Jacobian of C. Recall that the variety $\mathrm{Jac}(C)$ is a curve isomorphic to C. Moreover, $\mathrm{Jac}(C)$ is a group, and the 2-torsion subgroup $\mathrm{Jac}(C)[2]$ consists of four torsion bundles L in $\mathrm{Jac}(C)$, thus satisfying $L^2 = \mathcal{O}$.

2.3.1 The group law in C

Let us now fix an embedding $C \subset \mathbb{P}^2$ such as C is given by the equation

$$y^2 = x(x-1)(x-\lambda)$$

in an affine chart, with $\lambda \in \mathbb{C} \setminus \{0,1\}$. Let $\pi : C \to \mathbb{P}^1$ be the double cover $(x,y) \mapsto x$. We consider the usual group law in C given by collinearity. Let w_i be the four Weierstrass points, namely the preimages of the ramification points $i = 0, 1, \lambda, \infty$ via π. The point at infinity w_∞ is an inflection point on the curve, and it is thus the identity element for the group action.

Every element L of $\mathrm{Jac}(C)$ is of the form $L = \mathcal{O}(p - w_\infty)$. The elements of the 2-torsion subgroup $\mathrm{Jac}(C)[2]$ are the four line bundles

$$L_k = \mathcal{O}(w_k - w_\infty)$$

corresponding to the Weierstrass points w_k on C, for $k \in \Delta = \{0, 1, \lambda, \infty\}$.

Let $q \in C$ be a point. We define an involution $\iota_q : C \to C$ as follows: for every $p \in C$, $\iota_q(p)$ is the unique point in C satisfying the following equation of linear equivalence:

$$p + q + \iota_q(p) \sim 3w_\infty.$$

The subgroup $\mathrm{Jac}(C)[2]$ acts on $\mathrm{Jac}(C)$ by tensor product of line bundles. Each element L_k of $\mathrm{Jac}(C)[2]$ thus induces an involution on $\mathrm{Jac}(C)$, and therefore also on C.

The double transpositions of $\mathbb{P}^1$

Consider the set $\Delta = \{0, 1, \lambda, \infty\} \subset \mathbb{P}^1$. For each subset $\Omega \subset \Delta$ of two points, there exists un automorphism of $\mathbb{P}^1$ that exchanges the points in Ω, and the points in $\Delta \setminus \Omega$. More precisely, let $k \in \Delta$, and consider the following maps:

$$\beta_0(z) = \frac{\lambda}{z}, \quad \beta_1(z) = \frac{z-\lambda}{z-1}, \quad \beta_\lambda(z) = \frac{\lambda z - \lambda}{z - \lambda}$$

For $k \in \Delta \setminus \{\infty\}$, the map β_k is the unique automorphism of $\mathbb{P}^1$ that permutes the points in the pairs $\{k, \infty\}$ and $\Delta \setminus \{k, \infty\}$. We adopt the notation β_∞ for the identity map of $\mathbb{P}^1$.

Let us also remark that the maps β_k are involutions, and that we have the following relation:

$$\beta_0 \circ \beta_1 = \beta_\lambda.$$

The double transpositions are closely related to the action of the subgroup $\mathrm{Jac}(C)[2]$:

Proposition 2.3.1. *Let $\lambda \in \mathbb{C} \setminus \{0,1\}$, and let C be the elliptic curve given by the equation $y^2 = x(x-1)(x-\lambda)$. Let $k \in \Delta = \{0, 1, \lambda, \infty\}$. Then, the involution $C \xrightarrow{\otimes L_k} C$ coincides with the composition $\iota_{w_k} \circ \iota_{w_\infty}$. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} C & \xrightarrow{\otimes L_k} & C \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\beta_k} & \mathbb{P}^1 \end{array}$$

Proof. Let $M = \mathcal{O}(p - w_\infty)$ be an arbitrary line bundle of degree 0. Then $M \otimes L_k = \mathcal{O}(p - w_k) = \mathcal{O}(q - w_\infty)$ for a suitable $q \in C$. The group law on C implies that q, w_k and $\iota_{w_\infty}(p)$ are collinear in the projective model $\mathbb{P}^2$ of affine coordinates (x, y). This implies $q = \iota_{w_k} \circ \iota_{w_\infty}(p)$. Since the hyperelliptic involution ι_{w_∞} and the twist $\iota_{w_k} \circ \iota_{w_\infty}$ commute for all k, the map β_k is well defined.

The set $W := \{w_0, w_1, w_\lambda, w_\infty\}$ is invariant under the map $\iota_{w_k} \circ \iota_{w_\infty}$. Since points w_k in C are projected by π to points k in $\mathbb{P}^1$ for $k \in \Delta$, the map β_k is as described. □

2.3.2 Indecomposable rank 2 vector bundles over C

The classification of indecomposable vector bundles E of rank 2 over genus 1 curves C was achieved by Atiyah in [3]. In this Section we recall some of his results. For a fixed determinant, the set of these bundles is parametrized by the Jacobian of C.

Let E_0, E_1 be the unique non trivial extensions given by exact sequences

$$0 \to \mathcal{O} \to E_1 \to \mathcal{O}(w_\infty) \to 0 \qquad \text{and} \qquad 0 \to \mathcal{O} \to E_0 \to \mathcal{O} \to 0.$$

The following results are due to Atiyah (1-5) and to Maruyama (6):

Theorem 2.3.2 (Atiyah [3], Maruyama [42]). *The bundles E_1 and E_0 satisfy the following properties:*

1. *The vector bundle E_1 is the unique indecomposable rank 2 bundle of determinant $\mathcal{O}(w_\infty)$ up to isomorphism.*
2. *Let $L \in Jac(C)$. Then, there is a unique inclusion $L \subset E_1$.*
3. *The bundle E_1 does not admit non-scalar automorphisms.*
4. *The indecomposable rank 2 vector bundles of trivial determinant are exactly those of the form $E_0 \otimes L_k$, where L_k is a torsion line bundle.*
5. *There is a unique inclusion $\mathcal{O} \subset E_0$.*
6. *Let $T = t_1 + t_2$ be a reduced divisor in C. Consider the following set of quasi-parabolic bundles over (C, T):*

$$P_1 = \{(E_0, \boldsymbol{p} = (p_1, p_2)) \mid p_1 \subset \mathcal{O} \text{ and } p_2 \not\subset \mathcal{O}\}$$

The group of automorphisms of E_0 modulo scalar automorphisms acts transitively and freely on this set.

Let us sketch an argument for (6). By Theorem 2 and Remark 3 of [42], the group of automorphisms of $\mathbb{P}(E_0)$ is $\mathbb{C}$. This group acts by translation in each fiber and fixes the section corresponding to the line subbundle $\mathcal{O} \subset E_0$. Since p_1 is contained in $\mathcal{O}$, the action of this group is clearly transitive and free on the set P_1.

Remark. By (3), E_1 only admits scalar automorphisms. On the other hand, multiplication by torsion line bundles give the automorphisms of the projectivisation bundle $\mathbb{P}(E_1)$. In contrast, E_0 admits non-scalar automorphisms. In fact, for every line bundle L of degree -1 and every quasi-parabolic bundle in P_1, there is a unique inclusion $L \subset E_1$ such that L passes by both quasi-parabolic directions p_1 and p_2.

The geometry of $\mathbb{P}(E_1)$

Let us fix a point $p \in C$ and a line bundle $L \in \mathrm{Jac}(C)$. By (3) in Theorem 2.3.2, L is a subbundle of E_1 and the inclusion is unique (up to scalar multiplication). This inclusion thus defines a single quasi-parabolic direction $m_p(L)$ in the fiber $(E_1)_p$ of p. Hence, we have a map

$$m_p : \mathrm{Jac}(C) \to \mathbb{P}(E_1)|_p.$$

The line subbundle $L \subset E_1$ correspond to a cross-section s_L of self-intersection $+1$ of the ruled surface $\mathbb{P}(E)$. The following Proposition describes the geometrical situation of these cross-sections: for every point m in the fiber $\mathbb{P}(E_1)|_p$ there are generically two $+1$ cross-sections of $\mathbb{P}(E_1)$ passing through m.

Proposition 2.3.3. *Let $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and let C be the elliptic curve given by the equation $y^2 = x(x-1)(x-\lambda)$. Let p be a point of C. Then, the map $m_p : Jac(C) \to \mathbb{P}(E_1)|_p$ is of degree 2 ramified in four points. If $z \in \mathbb{P}(E_1)|_p$ is a non-ramification point, we have*

$$m_p^{-1}(z) = \{\mathcal{O}(q - w_\infty), \mathcal{O}(\iota_p(q) - w_\infty)\}$$

for $q \in C$. The preimages of ramification points are the four line bundles $M = \mathcal{O}(q - w_\infty)$ with $\iota_p(q) = q$. They satisfy $M^2 = \mathcal{O}(w_\infty - p)$.

Proof. Let $L = \mathcal{O}(q - w_\infty) \in \mathrm{Jac}(C)$ be a non-ramification point of m_p. Let $L' = \mathcal{O}(\tilde{q} - w_\infty) \neq L$ such that $m_p(L) = m_p(L') = m$. The associated cross-sections s_L and $s_{L'}$ only intersect in m. In particular, the underlying bundle E of the image of the quasi-parabolic bundle $(E_1, m_p(L))$ over (C, p) via elm_p^+ is decomposable:

$$E = \mathcal{O}(p + q - w_\infty) \oplus \mathcal{O}(p + \tilde{q} - w_\infty)$$

By the properties of elementary transformations, we have that $\det E = \mathcal{O}(w_\infty + p)$, which implies $\tilde{q} = \iota_p(q)$. Suppose that there is a third distinct line bundle L'' passing through m. Then, the projective bundle $\mathbb{P}(E)$ has three non crossing $+0$ sections s_L, $s_{L'}$, $s_{L''}$, implying that E is trivial. But then $E_1 = \mathrm{elm}_p^-(E)$ is decomposable, which is absurd. This proves that the map m_p is of degree 2, and the four ramification points are the fixed points of the involution ι_p. □

2.3.3 The Tu isomorphism

Let C be a curve of genus 1. We say that a rank 2 (non quasi-parabolic) vector bundle E over C is semistable (resp. stable) if it is $\boldsymbol{\mu}$-semistable (resp. stable) for $\boldsymbol{\mu} = (0, \ldots, 0)$. The moduli space $\mathrm{Bun}_{\mathcal{O}}(C)$ of semistable rank 2 vector bundles over C with trivial determinant is constructed in [54].

The bundles appearing in $\mathrm{Bun}_{\mathcal{O}}(C)$ are in fact all strictly semistable. They are the decomposable bundles $L \oplus L^{-1}$, together with the indecomposable bundles $E_0 \otimes L_k$, with L_k a torsion line bundle.

Theorem 2.3.4 (Tu [54])**.** *Let $\lambda \in \mathbb{C} \setminus \{0, 1\}$, and let C be the elliptic curve given by the equation $y^2 = x(x-1)(x-\lambda)$. Then, the moduli space* $\mathrm{Bun}_{\mathcal{O}}(C)$ *is isomorphic to the quotient of Jac(C) by the hyperelliptic involution. More precisely, there is an isomorphism*

$$\mathrm{Bun}_{\mathcal{O}}(C) \xrightarrow{\mathrm{Tu}} \mathbb{P}^1$$

such that $\mathrm{Tu}(\mathcal{P}) = \pi(p)$, *where $\mathcal{P} = [L \oplus L^{-1}]$ for $L = \mathcal{O}(p - w_\infty)$ and π is the hyperelliptic cover. Moreover, we have*

- *If $p \in \mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$, $\mathcal{P}$ is represented by a single isomorphism class $[L \oplus L^{-1}]$.*
- *If $p = k \in \{0, 1, \lambda, \infty\}$, $\mathcal{P}$ is represented by two different isomorphism classes $[L_k \oplus L_k]$ and $[E_0 \otimes L_k]$.*

For a classification of more general principal bundles over genus 1 curves, we refer to [38].

The forgetful map

Let $\boldsymbol{\mu} \in [0, 1]^n$. The forgetful map Forget : $\mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C, T) \to \mathrm{Bun}_{\mathcal{O}}(C)$ is defined in the obvious way:

$$\mathrm{Forget}[E, \boldsymbol{p}] = [E].$$

Proposition 2.3.5. *The map* Forget *is an algebraic map.*

Proof. First, let us remark that the map Forget is set-theoretically a well-defined map. Indeed, the underlying bundles of the quasi-parabolic bundles appearing in the classification of Proposition 2.4.5 are either of the form $L \oplus L^{-1}$, for $L \in \mathrm{Jac}(C)$ or of the form $E_0 \otimes L$, for $L \in \mathrm{Jac}(C)[2]$. These are all strictly semistable by Theorem 2.3.4.

Let $\boldsymbol{\mu} \in [0,1]^2 \setminus W'$. According to Proposition 2.4.7, the quasi-parabolic bundles in $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ are all stable. By the universal property of the moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$, there exists an universal family defined over the set $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$, i.e. a vector bundle $\mathfrak{E}$ over $C \times \mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ together with a line subbundle $\mathfrak{F}$ of $\mathfrak{E}|_{D\times T}$. The map Forget transforms this family on the family given by the vector bundle $\mathfrak{E}$. This map is algebraic.

If $\boldsymbol{\nu} \in W'$, we have a canonical isomorphism

$$\varphi : \mathrm{Bun}^{\boldsymbol{\nu}}_{\mathcal{O}}(C,T) \to \mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T),$$

with $\boldsymbol{\mu} \in [0,1]^2 \setminus W'$, such that the diagram

$$\begin{array}{ccc} \mathrm{Bun}^{\boldsymbol{\nu}}_{\mathcal{O}}(C,T) & \xrightarrow{\varphi} & \mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T) \\ & {\scriptstyle \mathrm{Forget}_{\boldsymbol{\nu}}} \searrow & \downarrow {\scriptstyle \mathrm{Forget}_{\boldsymbol{\mu}}} \\ & & \mathrm{Bun}_{\mathcal{O}}(C) \end{array}$$

commutes set-theoretically. Since φ and $\mathrm{Forget}_{\boldsymbol{\mu}}$ are algebraic, the composition $\mathrm{Forget}_{\boldsymbol{\nu}}$ is also algebraic. □

2.4 The coarse moduli space of quasi-parabolic bundles over a genus 1 curve

Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. In this Section, we describe the GIT moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{L}(C,T)$ of $\boldsymbol{\mu}$-semistable rank 2 quasi-parabolic bundles with quasi-parabolic directions over T and fixed determinant L. This moduli space depends on the choice of weights. More precisely, there is a hyperplane cutting out $[0,1]^2$ into two chambers, and strictly $\boldsymbol{\mu}$-semistable bundles only occur along this wall. The moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{L}(C,T)$ is constant in each chamber, i. e. the set of quasi-parabolic bundles representing each of its points does not vary.

While we are mostly interested in the trivial determinant case, it turns out that the computations are easier in the odd degree case. Therefore, we will first study the odd degree case and then translate our results to the even degree setting.

2.4.1 The odd degree moduli space

Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. Let w_∞ be a point of C. We remark that we do not specify an embedding of C in $\mathbb{P}^2$ in this Section, but we keep the notation w_∞ for coherence with the upcoming sections.

In this Section, we describe the moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$, subsequently giving a proof of Theorem A. Let us define the *wall* $W \subset [0,1]^2$ as the hyperplane $\mu_1+\mu_2 = 1$. We will see that semistable bundles arise only when weights are in W. Let us start by listing $\boldsymbol{\mu}$-semistable and $\boldsymbol{\mu}$-stable bundles in W:

Proposition 2.4.1. *Let C be a genus 1 curve, and let $T = t_1 + t_2$ be an effective reduced divisor on C. Let w_∞ be a point of C. Let $\boldsymbol{\mu}$ be weights in the wall W, with $\mu_i \neq 0$ for $i = 1,2$. Then, the quasi-parabolic bundles representing points in $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ are exactly the following:*

- *$\boldsymbol{\mu}$-stable bundles:*
  - *$(E_1,\boldsymbol{p})$ with quasi-parabolic directions $\boldsymbol{p}$ not lying on the same subbundle $L \in Jac(C)$.*
- *Strictly $\boldsymbol{\mu}$-semistable bundles:*
  - *$(E_1,\boldsymbol{p})$ with quasi-parabolic directions $\boldsymbol{p}$ lying on the same subbundle $L \in Jac(C)$.*
  - *$E = L \oplus L^{-1}(w_\infty)$ with $L \in Jac(C)$ and no quasi-parabolic directions lying on $L^{-1}(w_\infty)$.*

If $\mu_i = 0$, we also find the bundles $E = L \oplus L^{-1}(w_\infty)$ with m_i lying on $L^{-1}(w_\infty)$.

Proof. By Theorem 2.3.2, the underlying vector bundle of an element of $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ is either decomposable or E_1. The result follows from direct computation of the $\boldsymbol{\mu}$-quasi-parabolic degree of each bundle. □

Now we describe the S-equivalence classes in $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$. Let Γ be the strictly $\boldsymbol{\mu}$-semistable locus. We have the following result:

Theorem 2.4.2. *Let C be a genus 1 curve, and let $T = t_1+t_2$ be an effective reduced divisor on C. Let w_∞ be a point of C. Let $\boldsymbol{\mu}$ be weights in the wall W. Then, the locus $\Gamma \subset \mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ is a curve parametrized by the Jacobian of C. More precisely, if $0 < \mu_1, \mu_2 < 1$, for each point $\mathcal{L}$ in Γ there exists a unique $L \in Jac(C)$ such that $\mathcal{L}$ is represented by precisely the following three different isomorphism classes of quasi-parabolic vector bundles:*

- *$\mathcal{E}_<(L) = (E_1,\boldsymbol{p})$ with both quasi-parabolic directions lying on $L \subset E_1$.*
- *$\mathcal{E}_>(L) = (L \oplus L^{-1}(w_\infty),\boldsymbol{p})$ with quasi-parabolic directions outside of $L^{-1}(w_\infty)$, not on the same L.*
- *$\mathcal{E}_=(L) = (L\oplus L^{-1}(w_\infty),\boldsymbol{p})$ with both quasi-parabolic directions on the same L.*

Proof. We claim that the three bundles above are all of the same S-equivalence class, i. e. they are all identified in a single point in the moduli space. Indeed, a Jordan-Hölder filtration for the first configuration is $0 \subset (L,\boldsymbol{p}') \subset \mathcal{E}_<(L)$, where $(L,\boldsymbol{p}')$ is the unique quasi-parabolic structure over (C,T). The bundles $(L,\boldsymbol{p}')$ and $\mathcal{E}_<(L)$ have equal slope 1. This gives the graded bundle

$$\mathrm{gr}_{\boldsymbol{\mu}}\mathcal{E}_<(L) = (L,\boldsymbol{p}') \oplus (L^{-1}(w_\infty),\phi).$$

For the second configuration, we choose the filtration $0 \subset (L^{-1}(w_\infty), \phi) \subset \mathcal{E}_>(L)$ with associated graded bundle

$$\mathrm{gr}_{\boldsymbol{\mu}}\mathcal{E}_>(L) = (L^{-1}(w_\infty), \phi) \oplus (L, \boldsymbol{p}).$$

Since clearly $\mathrm{gr}_{\boldsymbol{\mu}}\mathcal{E}_<(L) \cong \mathrm{gr}_{\boldsymbol{\mu}}\mathcal{E}_>(L)$, the quasi-parabolic bundles $\mathcal{E}_<(L)$ and $\mathcal{E}_>(L)$ are in the same S-equivalence class. The second filtration works also for the bundle $\mathcal{E}_=(L)$, hence this bundle is also identified with the previous two.

From the description of strictly $\boldsymbol{\mu}$-semistable bundles in Proposition 2.4.1, it is clear that no other bundle belongs to the same S-equivalence class. Consequently, the map $\Gamma \to \mathrm{Jac}(C)$ given by $[\mathcal{E}_*(L)] \mapsto L$ is an isomorphism, where $[\mathcal{E}_*(L)]$ is the S-equivalence class of the three quasi parabolic bundles $\mathcal{E}_<(L)$, $\mathcal{E}_>(L)$ and $\mathcal{E}_=(L)$. □

A map onto $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$

Fix $\boldsymbol{\mu}$ in the wall W. By Proposition 2.4.1, $\boldsymbol{\mu}$-stable bundles are of the form $(E_1, \boldsymbol{m})$. Let us consider the vector bundle

$$\mathfrak{E} = E_1 \times \{0\} \xrightarrow{p\times 0} X = C \times (\mathbb{P}(E_1)_{t_1} \times \mathbb{P}(E_1)_{t_2})$$

where $p : E_1 \to C$ is the Atiyah bundle. The subvariety

$$Y = T \times (\mathbb{P}(E_1)_{t_1} \times \mathbb{P}(E_1)_{t_2}) \subset X$$

is of codimension one. Let $\mathfrak{F}$ be the subbundle of $\mathfrak{E}|_Y$ defined by

$$\mathfrak{F}|_{(t_k,p_1,p_2)} = \begin{cases} p_1 \times \{0\} & \text{if } k = 1 \\ p_2 \times \{0\} & \text{if } k = 2 \end{cases}$$

where $t_k \in T$ and $p_l \subset (E_1)|_{t_l}$ is the line passing by the origin of $(E_1)|_{t_l}$ corresponding to the point $p_l \in \mathbb{P}(E_1)_{t_l}$. The pair $(\mathfrak{E}, \mathfrak{F})$ is an algebraic family of quasi-parabolic vector bundles over (C,T) over $\mathbb{P}(E_1)_{t_1} \times \mathbb{P}(E_1)_{t_2}$. Moreover, the fibers of this family are quasi-parabolic bundles of the form $(E_1, \boldsymbol{p} = (p_1, p_2))$ over (C,T). This family induces an algebraic map

$$M : \mathbb{P}(E_1)_{t_1} \times \mathbb{P}(E_1)_{t_2} \to \mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$$

given by $M(p_1,p_2) = [E_1, \boldsymbol{p} = (p_1,p_2)]$. We are now in position to prove Theorem A when weights are in the wall.

Theorem 2.4.3. *Let C be a genus 1 curve, and let $T = t_1 + t_2$ be an effective reduced divisor on C. Let w_∞ be a point of C. Fix a vector of weights $\boldsymbol{\mu}$ in the wall W. Then, the map M is an isomorphism. Furthermore, the curve Γ is of bidegree (2,2) in $\mathbb{P}(E_1)_{t_1} \times \mathbb{P}(E_1)_{t_2} \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let $\mathcal{E} = (E_1, \boldsymbol{p})$ be a quasi-parabolic bundle. If $\mathcal{E}$ is stable, then it is the only quasi-parabolic bundle in its S-equivalence class. If $\mathcal{E}$ is semistable, Theorem 2.4.2 shows that it is the only quasi-parabolic bundle in its S-equivalence class having E_1 as underlying bundle. This shows injectivity, and surjectivity is also clear from Theorem 2.4.2.

Strict $\boldsymbol{\mu}$-semistability occurs when there is a degree 0 line subbundle L passing through both quasi-parabolic directions. For a generic choice of m_1, there are two of these line subbundles passing through m_1, thus defining generically two quasi-parabolic directions on the fiber of t_2 (see Proposition 2.3.3). Hence, the locus of strictly $\boldsymbol{\mu}$-semistable quasi-parabolic configurations is a curve of bidegree (2,2). Theorem 2.4.2 shows that this locus is exactly Γ. □

When we move the weights $\boldsymbol{\mu}$ outside the wall and inside the chambers, the family of $\boldsymbol{\mu}$-semistable bundles changes. More precisely, some of the formerly strictly $\boldsymbol{\mu}$-semistable bundles become stable, while some others become unstable. Therefore, points in Γ are represented by different isomorphism classes of bundles depending on the choice of weights. The following Proposition summarizes the situation:

Proposition 2.4.4. *Let C be a genus 1 curve, and let $T = t_1 + t_2$ be an effective reduced divisor on C. Let w_∞ be a point of C. Let $\boldsymbol{\mu}$ be a vector of weights outside the wall. Then, the moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Each point in this space is represented by a single $\boldsymbol{\mu}$-stable bundle. Points outside Γ are represented by a single quasi-parabolic configuration $(E_1, \boldsymbol{p})$ with $\boldsymbol{p}$ not lying on the same line subbundle $L \in Jac(C)$. Each point $\mathcal{L} \in \Gamma$ is represented by:*
- *The bundle $\mathcal{E}_<(L)$, if $\boldsymbol{\mu} \in I_<$.*
- *The bundle $\mathcal{E}_>(L)$, if $\boldsymbol{\mu} \in I_>$.*

Proof. It is sufficient to show that the bundle $\mathcal{E}_<(L)$ is $\boldsymbol{\mu}$-stable when $\mu_1 + \mu_2 < 1$ and $\boldsymbol{\mu}$-unstable when $\mu_1 + \mu_2 > 1$, and that the bundle $\mathcal{E}_>(L)$ is respectively $\boldsymbol{\mu}$-stable and $\boldsymbol{\mu}$-unstable in the opposite chambers. This is proved by computing the quasi-parabolic $\boldsymbol{\mu}$-degree of L in each case. □

We have proven that $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ for every $\boldsymbol{\mu}$. The same result holds for $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ applying an elementary isomorphism. This completes the proof of Theorem A.

Consider the set of quasi-parabolic bundles representing points in the moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ (resp. in $\mathrm{Bun}^{\boldsymbol{\nu}}_{w_\infty}(C,T)$). These sets are the same if, and only if, both weights $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ belong to the same set among the following three:
- The *chamber* $I_< = \{(\mu_1,\mu_2) \in (0,1)^2 \mid \mu_1 + \mu_2 < 1\}$.
- The *chamber* $I_> = \{(\mu_1,\mu_2) \in (0,1)^2 \mid \mu_1 + \mu_2 > 1\}$.
- The *wall* $W = \{(\mu_1,\mu_2) \in (0,1)^2 \mid \mu_1 + \mu_2 = 1\}$.

We will thus adopt the notations

$$\mathrm{Bun}^{<}_{w_\infty}(C,T),\ \mathrm{Bun}^{>}_{w_\infty}(C,T) \text{ and } \mathrm{Bun}^{=}_{w_\infty}(C,T)$$

for the moduli spaces $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ and $\boldsymbol{\mu}$ in $I_<$, $I_>$ and W respectively.

2.4.2 The even degree moduli space

Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. Let w_∞ be a point of C. As in the previous Section, we do not specify an embedding of C in $\mathbb{P}^2$.

In this Section we use the description of the odd degree moduli space to describe the moduli space $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$. The map

$$\mathrm{Elm}^{+}_{t_2} := \mathrm{elm}^{+}_{t_2} \otimes R$$

provides an isomorphism between $\mathrm{Bun}^{\boldsymbol{\mu}}_{w_\infty}(C,T)$ and $\mathrm{Bun}^{\boldsymbol{\mu}_2}_{\mathcal{O}}(C,T)$, where R is a convenient line bundle and $\boldsymbol{\mu}_2 = (\mu_1, 1-\mu_2)$ (see Table 2.2). The wall and chambers change in even degree. The sets of bundles representing points in $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ and $\mathrm{Bun}^{\boldsymbol{\nu}}_{\mathcal{O}}(C,T)$ are the same if, and only if, both weights $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ belong to the same set among the following three:

— The *chamber* $J_< = \{(\mu_1,\mu_2) \in (0,1)^2 \mid \mu_1 < \mu_2\}$.
— The *chamber* $J_> = \{(\mu_1,\mu_2) \in (0,1)^2 \mid \mu_1 > \mu_2\}$.
— The *wall* $W' = \{(\mu_1,\mu_2) \in (0,1)^2 \mid \mu_1 = \mu_2\}$.

Recall that we have two kinds of semistable rank 2 vector bundles E of trivial determinant: the decomposable case, i. e. where $E = L \oplus L^{-1}$ with $\deg L = 0$, and the indecomposable case $E_0 \otimes L_k$ with L_k a torsion line bundle. Let us start by fixing weights $\boldsymbol{\mu}$ in the wall, as in the odd degree case.

Proposition 2.4.5. *Let C be a genus 1 curve, and let $T = t_1 + t_2$ be an effective reduced divisor on C. Let w_∞ be a point of C. If $\boldsymbol{\mu} \in W' \setminus \{0\}$, the quasi-parabolic bundles representing points in $\mathrm{Bun}^{=}_{\mathcal{O}}(C,T)$ are the following:*

— *$\boldsymbol{\mu}$-stable bundles:*
  — *$(L \oplus L^{-1}, \boldsymbol{p})$, $L \neq L_k$, with no quasi-parabolic directions on L nor on L^{-1}.*
  — *$(E_0 \otimes L_k, \boldsymbol{p})$ with no quasi-parabolic directions on $L_k \subset E_0 \otimes L_k$.*
— *Strictly $\boldsymbol{\mu}$-semistable bundles:*
  — *$(L \oplus L^{-1}, \boldsymbol{p})$, $L \neq L_k$, with one or two quasi-parabolic directions on L or L^{-1}, but not both on the same.*
  — *$(E_0 \otimes L_k, \boldsymbol{p})$ with exacly one quasi-parabolic on $L_k \subset E_0 \otimes L_k$.*
  — *$(L_k \oplus L_k, \boldsymbol{p})$ with quasi-parabolic directions not lying on the same L_k.*

Proof. By applying the elementary transformation $\mathrm{Elm}^{+}_{t_2}$ to the bundles of Proposition 2.4.1, we obtain the described bundles. The different cases $L =$

L_k and $L \neq L_k$ correspond in the odd degree setting to E_1 having one or two degree 0 bundles passing through m_2. $\square$

Let us keep the notation Γ for the strictly $\boldsymbol{\mu}$-semistable locus in $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$. This locus is a curve parametrized by $\mathrm{Jac}(C)$ by Theorem 2.4.3, and the description of the bundles representing points in Γ is as follows:

Theorem 2.4.6. *Let C be a genus 1 curve, and let $T = t_1 + t_2$ be an effective reduced divisor on C. Let w_∞ be a point of C. Let $0 < \mu_1 = \mu_2 < 1$. For each point $\mathcal{L}$ in $\Gamma \subset \mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ there exists a unique $L \in Jac(C)$ such that the point $\mathcal{L}$ is represented by precisely the following three different isomorphism classes of quasi-parabolic vector bundles:*

- *If L is a 2-torsion line bundle:*
  - *$\mathcal{F}_<(L) = (E_0 \otimes L, \boldsymbol{p})$ with m_1 on $L \subset E_0 \otimes L$ and m_2 not on L.*
  - *$\mathcal{F}_>(L) = (E_0 \otimes L, \boldsymbol{p})$ with m_2 on $L \subset E_0 \otimes L$ and m_1 not on L.*
  - *$\mathcal{F}_=(L) = (L \oplus L, \boldsymbol{p})$ with quasi-parabolic directions not lying on the same L.*
- *If L is not a 2-torsion line bundle:*
  - *$\mathcal{F}_<(L) = (L \oplus L^{-1}, \boldsymbol{p})$ with $m_1 \in L$ and m_2 out of L and L^{-1}.*
  - *$\mathcal{F}_>(L) = (L \oplus L^{-1}, \boldsymbol{p})$ with $m_2 \in L^{-1}$ and m_1 out of L and L^{-1}.*
  - *$\mathcal{F}_=(L) = (L \oplus L^{-1}, \boldsymbol{p})$ with $m_1 \in L$ and m_2 in L^{-1}.*

Proof. One way to prove this result is to translate the situation of Theorem 2.4.6, computing the images of the bundles $\mathcal{E}_<(L)$, $\mathcal{E}_>(L)$ and $\mathcal{E}_=(L)$ in $\mathrm{Bun}^{=}_{w_\infty}(C,T)$ by the elementary transformation $\mathrm{Elm}^+_{t_2}$. Let $M \in \mathrm{Jac}(C)$ be an arbitrary line bundle and $L = M(t_2 + R)$, with $2R = -t_1 - w_\infty$. We claim that

$$\mathrm{Elm}^+_{t_2}(\mathcal{E}_*(M)) = \mathcal{F}_*(L)$$

for $* \in \{<, >, =\}$. Let us prove the assertion for the first case. The bundle $\mathcal{E}_<(M) = (E_1, \boldsymbol{p})$ has quasi-parabolic directions $\boldsymbol{p}$ lying on $M \subset E_1$. If L is a 2-torsion line bundle, say $L = L_k$, Proposition 2.3.3 states that M is the unique degree 0 subbundle of E_1 passing through p_2 . This implies that the underlying bundle of $\mathrm{Elm}^+_{t_2}(\mathcal{E}_<(M))$ is indecomposable, and thus equal to $E_0 \otimes L$. If L is not a 2-torsion line bundle, the underlying bundle of $\mathrm{Elm}^+_{t_2}(\mathcal{E}_<(M))$ is decomposable, therefore it is $L \oplus L^{-1}$. The position of quasi-parabolic directions is given by the configuration in $\mathcal{E}_<(M)$. For $* \in \{>, =\}$, the proof is similar. $\square$

Alternatively, one can also show that strictly semistable configurations in the even degree case are given by Jordan-Hölder filtrations

$$\begin{aligned} 0 \subset (L, p_1') \subset & \mathcal{F}_*(L) \qquad \text{for } * \in \{<, >\} \\ 0 \subset (L^{-1}, p_2') \subset & \mathcal{F}_=(L) \end{aligned}$$

where (L, p_1') (resp. (L, p_2') is the line subbundle over (C, t_1) (resp. (C, t_2)). These give isomorphic graded bundles. Therefore, $\mathcal{F}_<(L)$, and $\mathcal{F}_>(L)$ are identified in the moduli space. Moving weights outside the wall, we find the same situation as in the odd degree case: some of the strictly semistable bundles $\mathcal{F}_*(L)$ become stable, while some become unstable.

Proposition 2.4.7. *Let C be a genus 1 curve, and let $T = t_1 + t_2$ be an effective reduced divisor on C. Let w_∞ be a point of C. Let $\boldsymbol{\mu}$ be weights outside the wall and $[E, \boldsymbol{p}] \in \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C, T) \setminus \Gamma$. Then, one of the following holds:*

- *$E = L \oplus L^{-1}$, with $L \neq L_k$, and quasi-parabolic directions outside L and L^{-1}.*
- *$E = E_0 \otimes L_k$, with quasi-parabolic directions outside L_k.*

Each point $\mathcal{L} \in \Gamma$ is represented by one of the following bundle:

- *The bundle $\mathcal{F}_<(L)$ if $\boldsymbol{\mu} \in J_<$.*
- *The bundle $\mathcal{F}_>(L)$ if $\boldsymbol{\mu} \in J_>$.*

We follow the notation of the previous section: for $\boldsymbol{\mu}$ in $J_<$ (resp. in $J_>$), we will denote by $\mathrm{Bun}_{\mathcal{O}}^{<}(C, T)$ (resp. $\mathrm{Bun}_{\mathcal{O}}^{>}(C, T)$) the moduli space $\mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C, T)$.

2.4.3 Coordinate systems

Let C be a genus 1 curve, and let $T = t_1 + t_2$ be an effective reduced divisor on C. In this Section we define coordinate systems for the moduli spaces $\mathrm{Bun}_L^{=}(C, T)$. These coordinate systems will be automatically defined on $\mathrm{Bun}_L^{<}(C, T)$ and $\mathrm{Bun}_L^{>}(C, T)$ since every bundle appearing in these spaces is already in $\mathrm{Bun}_L^{=}(C, T)$.

The coordinate system for $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$

We start by defining an automorphism ϕ_T of $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$. Let $\mathcal{P} = [E, \boldsymbol{l}]$ be an element of $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$. The bundle $\mathrm{elm}_T^+(E, \boldsymbol{l})$ has determinant $\mathcal{O}(2w_\infty)$. Define the bundle $\phi_T(E, \boldsymbol{l}) := \mathrm{elm}_T^+(E, \boldsymbol{l}) \otimes \mathcal{O}(-w_\infty)$, with trivial determinant. Using the classification result of Proposition 2.4.5, we check that $\phi_T(E, \boldsymbol{l})$ is an element of $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$, and that its S-class does not depend on the representative bundle of $\mathcal{P}$. We thus obtain an automorphism

$$\phi_T : \mathrm{Bun}_{\mathcal{O}}^{=}(C, T) \to \mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$$

We define the coordinate system on $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$ as follows. Consider the moduli space $\mathrm{Bun}_{\mathcal{O}}(C, T)$ defined in Section 2.3.3: by composing the map

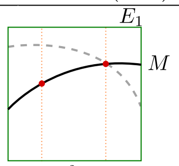
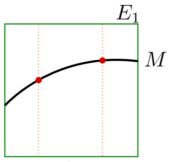
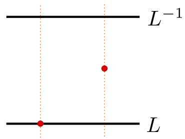
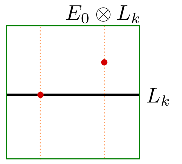
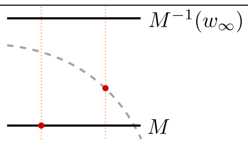
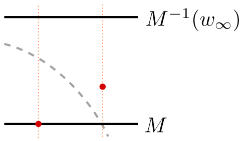
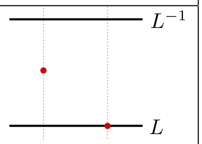
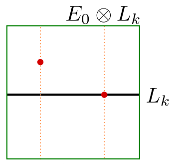
Simple parabolic bundles	
$\det E = \mathcal{O}(w_\infty)$	$\det E = \mathcal{O}$
$\mathcal{E}_<(M)$  with $M^2(t_2) \neq \mathcal{O}(w_\infty)$  with $M^2(t_2) = \mathcal{O}(w_\infty)$	$\mathcal{F}_<(L)$  with $L^2 \neq \mathcal{O}$  with $L_k^2 = \mathcal{O}$
$\mathcal{E}_>(M)$  with $M^2(t_2) \neq \mathcal{O}(w_\infty)$  with $M^2(t_2) = \mathcal{O}(w_\infty)$	$\mathcal{F}_>(L)$  with $L^2 \neq \mathcal{O}$  with $L_k^2 = \mathcal{O}$

Table 2.1 – List of simple quasi-parabolic bundles over (C, T). Each figure represent the corresponding projective bundle, and each quasi-parabolic bundles on the right side a obtained by applying the elementary transformation $\mathrm{Elm}^+_{t_2}$ to the corresponding bundle on the left side.

ϕ_T with the forgetful and Tu maps we obtain the diagram

$$\begin{array}{ccc} \mathrm{Bun}^{=}_{\mathcal{O}}(C,T) & \xrightarrow{\phi_T} & \mathrm{Bun}^{=}_{\mathcal{O}}(C,T) \\ \text{Forget}\downarrow & & \downarrow\text{Forget} \\ \mathrm{Bun}_{\mathcal{O}}(C) & & \mathrm{Bun}_{\mathcal{O}}(C) \\ \text{Tu}\downarrow\cong & & \cong\downarrow\text{Tu} \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

The *coordinate map* in $\mathrm{Bun}^{=}_{\mathcal{O}}(C,T)$ is the mapping

$$\mathrm{Bun}^{=}_{\mathcal{O}}(C,T) \xrightarrow{\varepsilon} \mathbb{P}^1 \times \mathbb{P}^1$$

defined by $\varepsilon = (\mathrm{Tu} \circ \mathrm{Forget}) \times (\mathrm{Tu} \circ \mathrm{Forget} \circ \phi_T)$. This map is algebraic since

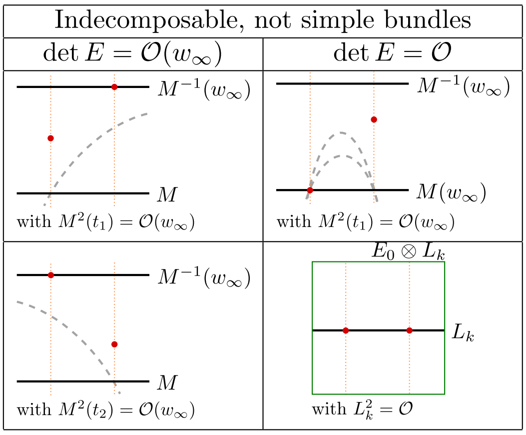


Table 2.2 – List of indecomposable, not simple quasi-parabolic bundles over (C,T). Each figure represent the corresponding projective bundle, and each quasi-parabolic bundles on the right side a obtained by applying the elementary transformation $\mathrm{Elm}^+_{t_2}$ to the corresponding bundle on the left side.

it is given by linear projections. We will prove the following Proposition using the coordinate system in the odd degree case:

Proposition 2.4.8. *The mapping ε is an isomorphism.*

The coordinate system for $\mathrm{Bun}^{=}_{w_\infty}(C,T)$

Recall that $T = t_1 + t_2$. Let us write $t_1 = (t,s)$ and $t_2 = (t,-s)$, where $C = \{y^2 = x(x-1)(x-\lambda)\}$. Consider the projections $\varepsilon_1 : C \to \mathbb{P}^1$ and $\varepsilon_2 : C \to \mathbb{P}^1$ with center the points t_1 and t_2 respectively, and satisfying $\varepsilon_i(w_k) = k$ for $k \in \{0,1,\lambda\}$. A straightforward calculation shows that these maps are defined by the equations

$$\varepsilon_1(x,y) = \frac{ty - sx}{y - s}\ , \quad \varepsilon_2(x,y) = \frac{ty + sx}{y + s}\ .$$

The preimages of this map are thus of the form $\{p, \iota_{t_j}(p)\}$, where ι is the involution defined in Section 2.3.1. In particular, for $k \in \{0,1,\lambda,\infty\}$, we have that the image of the point $\mathcal{L}_k = [\mathcal{E}_=(L_k)]$ by ε is the point $(k,k) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Now we are in position to define our coordinate system ε on $\mathrm{Bun}^{=}_{w_\infty}(C,T)$. Let $\mathcal{P}$ be a point in $\mathrm{Bun}^{=}_{w_\infty}(C,T)$. By theorem 2.4.2, $\mathcal{P}$ is represented by a unique bundle of the form $(E_1, \boldsymbol{m})$. By Proposition 2.3.3, the quasi-parabolic direction m_1 (resp. m_2) corresponds to a pair $\{p, i_{t_1}(p)\}$ (resp. $\{q, i_{t_2}(q)\}$) for p, $q \in C$. The *coordinate map* in $\mathrm{Bun}^{=}_{w_\infty}(C,T)$ is the mapping

$$\varepsilon : \mathrm{Bun}^{=}_{w_\infty}(C,T) \to \mathbb{P}^1 \times \mathbb{P}^1$$

defined by $\varepsilon(E_1, \boldsymbol{m}) = (\varepsilon_1(p), \varepsilon_2(q))$.

Proposition 2.4.9. *The coordinate map* $\varepsilon : \mathrm{Bun}^{=}_{w_\infty}(C,T) \to \mathbb{P}^1 \times \mathbb{P}^1$ *is an isomorphism, and its inverse is the map* M *defined in Theorem 2.4.3.*

Proof. Let $(m_1, m_2) \in (E_1)_{t_1} \times (E_1)_{t_2}$. Let $p, q \in C$ such that $m_1 = \varepsilon_1(p)$ and $m_2 = \varepsilon_2(q)$. By definition of the maps M and ε, we have that

$$\varepsilon \circ M(m_1, m_2) = (\varepsilon_1(p), \varepsilon_2(q)) = (m_1, m_2)$$

The same argument shows that, for every $\mathcal{P} = [E_1, \boldsymbol{m}] \in \mathrm{Bun}^{=}_{w_\infty}(C,T)$, $M \circ \varepsilon(\mathcal{P}) = \mathcal{P}$. □

Proof of Proposition 2.4.8. Let $R = \mathcal{O}_C(r - w_\infty)$ be a divisor such that $2R = w_\infty - t_1 = t_2 - w_\infty$. Then, the following diagram is commutative:

$$\begin{array}{ccccc}
 & & \mathrm{Bun}^{=}_{w_\infty}(C,T) & & \\
 & {\scriptstyle \mathrm{elm}^+_{t_1} \otimes \mathcal{O}(R)} \nearrow & & \searrow {\scriptstyle \mathrm{elm}^+_{t_2} \otimes \mathcal{O}(R+w_\infty)} & \\
\mathrm{Bun}^{=}_{\mathcal{O}}(C,T) & & \xrightarrow{\ \phi_T\ } & & \mathrm{Bun}^{=}_{\mathcal{O}}(C,T) \\
\downarrow {\scriptstyle \mathrm{Tu} \circ \mathrm{Forget}} & & & & \downarrow {\scriptstyle \mathrm{Tu} \circ \mathrm{Forget}} \\
\mathbb{P}^1 & & & & \mathbb{P}^1
\end{array}$$

Elements of $\mathrm{Bun}^{=}_{w_\infty}(C,T)$ are of the form $[E_1, \boldsymbol{m} = (m_1, m_2)]$. Fix the second quasi-parabolic direction m_2. For each $m_1 = \varepsilon_1(p)$, we define a point $z(m_1) \in \mathbb{P}^1$ as follows:

$$\begin{aligned} z(m_1) &:= \left(\mathrm{Tu} \circ \mathrm{Forget} \circ \mathrm{elm}^-_{t_1} \otimes \mathcal{O}(-R)\right)[E_1, \boldsymbol{m}] \\ &= \mathrm{Tu}\left(\mathcal{O}(p - r) \oplus \mathcal{O}(\iota_{t_1}(p) - r)\right). \end{aligned}$$

We see that the map $m_1 \mapsto z(m_1)$ is bijective. We also define $w(m_2)$ fixing this time m_1, the map $m_2 \mapsto w(m_2)$ is also bijective. Since the map $\mathrm{elm}^+_{t_1} \otimes \mathcal{O}(R)$ is an isomorphism of moduli spaces, the coordinate map

$$\varepsilon : \mathrm{Bun}^{=}_{\mathcal{O}}(C,T) \to \mathrm{Bun}^{=}_{w_\infty}(C,T)$$

is an isomorphism.

□

The relation between even and odd coordinates

Here we describe the coordinate change between even and odd degree moduli spaces. Remark that we have seen in the proof of Proposition 2.4.8 that this coordinate change is not canonical, since we have to choose a root of t_1. Let $R = \mathcal{O}(r - w_\infty)$ be a divisor such that $2R = \mathcal{O}(t_1 - w_\infty)$. The elementary map

$$\mathrm{Bun}^{=}_{\mathcal{O}}(C,T) \xrightarrow{\mathrm{elm}^+_{t_1} \otimes \mathcal{O}(R)} \mathrm{Bun}^{=}_{w_\infty}(C,T)$$

yields a coordinate transformation $\theta : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ satisfying

$$\theta \circ \varepsilon = \varepsilon \circ \left(\mathrm{elm}_{t_1}^{+} \otimes \mathcal{O}(R)\right).$$

For $k \in \Delta = \{0, 1, \lambda, \infty\}$, define the points $p_k := \iota_{w_k}(r)$ and $q_k := \iota_{w_\infty}(p_k)$.

Proposition 2.4.10. *The map θ is given by $\theta = \theta_1 \times \theta_2$, where θ_1 and θ_2 are the unique automorphisms of $\mathbb{P}^1$ satisfying*

$$\theta_1(\pi(p_k)) = \theta_2(\pi(q_k)) = k$$

for every $k \in \Delta$.

Proof. Let $k, l \in \Delta$ and $[L \oplus L^{-1}, \boldsymbol{m}] = \varepsilon^{-1}(\pi(p_k), \pi(q_l))$. In the generic situation, there exists M in $\mathrm{Jac}(C)$ such that $\boldsymbol{m}$ is defined by the intersection of unique subbundles $M(-w_\infty)$ and $M^{-1}(-w_\infty)$ of $L \oplus L^{-1}$. By the definition of ε, we have that $L = \mathcal{O}(p_k - w_\infty)$ and $M = \mathcal{O}(q_l - w_\infty)$.

The image of $[L \oplus L^{-1}, \boldsymbol{m}]$ by $\mathrm{elm}_{t_1}^{+} \otimes \mathcal{O}(R)$ is a quasi-parabolic bundle $[E_1, \boldsymbol{n} = (n_1, n_2)]$, where n_1 is the intersection of $L(R)$ and $L^{-1}(R)$ and n_2 is the intersection of $M(-R)$ and $M^{-1}(-R)$. By our choice of p_k and q_l, we have also that $L(R) = \mathcal{O}(w_k - w_\infty)$ and $M(-R) = \mathcal{O}(w_l - w_\infty)$. By the definition of ε, we have that $\varepsilon[E_1, \boldsymbol{n}] = (k, l)$.

We can repeat the arguments in this proof fixing L and varying M, or viceversa. In particular, $\theta = \theta_1 \times \theta_2$. □

Remark. It would be interesting to describe the moduli space when the divisor is not reduced, i. e. when $t_1 = t_2$. However, it would be important to clarify the notion of quasi-parabolic bundle in this setting. One approach could be to consider two quasi-parabolic directions over the multiple point in C. Nevertheless, this proposal does not seem suitable under the point of view of the study of connections on quasi-parabolic bundles, developed for instance in in [40] and [30]. On the other hand, we can think of quasi-parabolic directions as sections over each point of the divisor T. Consequently, when the divisor is not reduced, we may consider k-jets of sections over a point of multiplicity k.

2.5 The group of automorphisms and the Torelli result

Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. Let $\pi : C \to \mathbb{P}^1$ be the double cover of $\mathbb{P}^1$ satisfying $\pi(t_1) = \pi(t_2) = t \in \mathbb{P}^1$.

We are now interested in simple quasi-parabolic bundles over (C, T). The following Proposition holds:

Proposition 2.5.1. *Let $(E, \boldsymbol{p})$ be a rank 2 quasi-parabolic bundle over (C, T). Then $(E, \boldsymbol{p})$ is simple if, and only if, it is $\boldsymbol{\mu}$-stable for some $\boldsymbol{\mu}$.*

Proof. Since simplicity is preserved by elementary transformations and twists, we can assume $\det E = \mathcal{O}(w_\infty)$. Bundles in $\mathrm{Bun}^{<}_{w_\infty}(C,T) \bigcup \mathrm{Bun}^{>}_{w_\infty}(C,T)$ are listed in Proposition 2.4.4: E_1 is simple as a vector bundle by Theorem 2.3.2, thus $(E_1, \boldsymbol{p})$ is simple for every $\boldsymbol{p}$. The remaining bundles are of the form $(L \oplus L^{-1}(w_\infty), \boldsymbol{p})$ with exactly one quasi-parabolic direction on L. The automorphism group of $(L \oplus L^{-1}(w_\infty))$ is of projective dimension 1 acting on the complementary of L and L^{-1}. Hence, only the identity fixes the quasi-parabolic directions $\boldsymbol{p}$ and $(L \oplus L^{-1}(w_\infty), \boldsymbol{p})$ is simple.

Conversely, again using the classification of Proposition 2.4.5, $(E, \boldsymbol{p})$ simple implies $(E, \boldsymbol{p})$ is $\boldsymbol{\mu}$-semistable for some $\boldsymbol{\mu}$. Parabolic $\boldsymbol{\mu}$-semistable bundles which are non-$\boldsymbol{\mu}$-stable for any $\boldsymbol{\mu}$ are of the form $\mathcal{E}_=(L)$ by Theorem 2.4.2 and Proposition 2.4.4. These are not simple. □

Simple bundles are then parametrized by the non-separated scheme

$$\mathrm{Bun}_{\mathcal{O}}(C,T) = C_< \coprod C_> \Big/ \sim$$

constructed by patching the two charts $C_< = \mathrm{Bun}^{<}_{\mathcal{O}}(C,T)$ and $C_> = \mathrm{Bun}^{>}_{\mathcal{O}}(C,T)$, where we identify identical quasi-parabolic bundles along $C_< \setminus \Gamma$ and $C_> \setminus \Gamma$.

In this Section, we will study some of the automorphisms of this moduli space and in Section 2.7 we will prove that there are no more. Automorphisms of $\mathrm{Bun}_{\mathcal{O}}(C,T)$ consist of pairs of maps (ψ_1, ψ_2) that coincide on the gluing locus and that satisfy one of the following two conditions:

- Each ψ_k is an automorphism of C_* leaving invariant Γ, or
- The maps $\psi_1 : C_< \to C_>$ and $\psi_2 : C_> \to C_<$ are isomorphisms leaving invariant Γ.

Moreover, it is sufficient to explicit only one of the ψ_k to completely define the corresponding automorphism.

Twist automorphisms

The twist by a torsion line bundle L_k induces an automorphism between the charts

$$C_< \xrightarrow{\otimes L_k} C_< \ , \qquad C_> \xrightarrow{\otimes L_k} C_>.$$

It is clear by construction that these isomorphisms coincide on the complement $C_k \setminus \Gamma$ of the strictly semistable loci, thus they extend to a global isomorphism

$$\mathrm{Bun}_{\mathcal{O}}(C,T) \xrightarrow{\otimes L_k} \mathrm{Bun}_{\mathcal{O}}(C,T).$$

Since the $\boldsymbol{\mu}$-stability index is left unchanged under $\otimes L_k$, this automorphism preserves Γ. We will call this mappings *twist automorphisms.* Remark that $\otimes L_\infty$ is the identity map and $\otimes L_0 \circ \otimes L_1 = \otimes L_\lambda$.

Our aim now is to compute these twists in terms of the coordinate system ε defined in Section 2.4.3.

Proposition 2.5.2. *Let $k \in \{0, 1, \lambda, \infty\}$. Then, in the coordinate system given by ε, the mapping $\otimes L_k$ is expressed as follows:*

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\xrightarrow{\otimes L_k} \mathbb{P}^1 \times \mathbb{P}^1 \\ (z, w) &\mapsto (\beta_k(z), \beta_k(w)) \end{aligned}$$

with β_k the map defined in Section 2.3.1.

Proof. Let $(E, \boldsymbol{l})$ be a quasi-parabolic bundle in $C_<$. We will now compute the images

$$E = \mathrm{Forget}(E, \boldsymbol{l}) \quad \text{and} \quad F = \mathrm{Forget} \circ \phi_T(E, \boldsymbol{l}).$$

The twist by L_k in $\mathrm{Bun}_{\mathcal{O}}^{<}(C, T)$ corresponds to tensoring each of these bundles by L_k. Hence, the first (resp. the second) argument of the mapping $\otimes L_k$ depends only on the first (resp. the second) coordinate. Finally, these maps are the twists appearing in Proposition 2.3.1, i. e. those making the diagram

$$\begin{array}{ccc} \mathrm{Bun}_{\mathcal{O}}(C) & \xrightarrow{\otimes L_k} & \mathrm{Bun}_{\mathcal{O}}(C) \\ {\scriptstyle \mathrm{Tu}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{Tu}} \\ \mathbb{P}^1 & \xrightarrow{\beta_k} & \mathbb{P}^1 \end{array}$$

commute. □

The elementary automorphism

Consider the map ϕ_T introduced in Section 2.4.3. Since $C_<$ is an open set of $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$, ϕ_T is well defined on $C_<$. Recall that elm_T^+ changes the weights μ_i into $1 - \mu_i$. It follows that ϕ_T permutes $C_<$ and $C_>$. We will hence consider the map $\phi_T : C_< \to C_>$. The description of this map in our setting is the following:

Proposition 2.5.3. *In the coordinate system defined by ε, the mapping ϕ_T is the mapping that exchanges coordinates between the two factors, i. e. it is the map*

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\phi_T} \mathbb{P}^1 \times \mathbb{P}^1$$

defined by $\phi_T(z, w) = (w, z)$.

Proof. Let $(z, w) = ([E = L \oplus L^{-1}], [M \oplus M^{-1}])$ be a point in $\mathbb{P}^1 \times \mathbb{P}^1$. In the generic situation, $L \neq L^{-1}$ and there are unique subbundle inclusions $M(-w_\infty) \subset E$ and $M^{-1}(-w_\infty) \subset E$ defining a quasi-parabolic configuration $(E, \boldsymbol{m})$ over (C, T) with both quasi-parabolic directions outside L and L^{-1}.

Then, by the properties of elementary transformations, we have

$$\phi_T([L \oplus L^{-1}], [M \oplus M^{-1}]) = ([M \oplus M^{-1}], [L \oplus L^{-1}])$$

as stated. □

In particular, we have that the curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ is invariant under the transformation $(z, w) \mapsto (w, z)$.

The odd degree case

The former automorphisms can also be defined on the odd degree case, namely for the moduli space $\mathrm{Bun}_{w_\infty}(C, T)$. We will show that they have the same coordinate expression than in the even degree case.

Proposition 2.5.4. *The twist automorphism* $\mathrm{Bun}_{w_\infty}(C, T) \xrightarrow{\otimes L_k} \mathrm{Bun}_{w_\infty}(C, T)$ *is expressed as follows in the coordinate system* ε*:*

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\xrightarrow{\otimes L_k} \mathbb{P}^1 \times \mathbb{P}^1 \\ (\tilde{z}, \tilde{w}) &\mapsto (\beta_k(\tilde{z}), \beta_k(\tilde{w})) \end{aligned}$$

with β_k *the map defined in Section 2.3.1.*

Proof. It is enough to prove the assertion on $C_<$. Let $[E_1, \boldsymbol{m}] \in C_<$ be an arbitrary point. By Proposition 2.3.3, the direction m_j is defined by two line subbundles $\mathcal{O}(p_j - w_\infty)$ and $\mathcal{O}(\iota_{t_j}(p_j) - w_\infty)$, for $j \in \{1, 2\}$. Twisting by L_k yields the involution of Proposition 2.3.1 on each of these subbundles. □

The map ϕ_T is defined in the same way as in the even degree case: it is the elementary transformation elm_T^+ followed by the twist by $\mathcal{O}(-w_\infty)$.

Proposition 2.5.5. *In the coordinate system* ε*, the mapping* ϕ_T *is the map*

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\phi_T} \mathbb{P}^1 \times \mathbb{P}^1$$

defined by $\phi_T(\tilde{z}, \tilde{w}) = (\tilde{w}, \tilde{z})$.

2.5.1 The Torelli result

Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. By Theorem 2.4.2, the strictly $\boldsymbol{\mu}$-semistable locus Γ in the moduli space $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$ is an embedding of $\mathrm{Jac}(C)$, which is itself isomorphic to C. Hence, the curve C is embedded in the moduli space $\mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$. In this Section we show that this embedding also contains the information about the divisor T.

Let ε be the coordinate system defined in Section 2.4.3. Let $(\tilde{z}, \tilde{w}) \in \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$ be a point in the moduli space. Let us consider the vertical and horizontal lines passing by $(\tilde{z}, \tilde{w})$:

$$V_{\tilde{z}} = \{(\tilde{z}, w) \mid w \in \mathbb{P}^1\} \ , \ H_{\tilde{w}} = \{(z, \tilde{w}) \mid z \in \mathbb{P}^1\}$$

The following Lemma characterizes the vertical and horizontal lines tangent to $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$:

Lemma 2.5.6. *Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. Let $\Delta = \{0, 1, \lambda, \infty\} \subset \mathbb{P}^1$. Then, the vertical line $V_{\tilde{z}}$ is tangent to Γ if, and only if, $\tilde{z} \in \Delta$. Similarly, the horizontal line $H_{\tilde{w}}$ is tangent to Γ if, and only if, $\tilde{w} \in \Delta$.*

Proof. For $\tilde{z} = [L \oplus L^{-1}] \in \mathbb{P}^1$ fixed, there are generically two points in Γ with first coordinate $\tilde{z}$, namely $[\mathcal{F}_*(L)]$ and $[\mathcal{F}_*(L^{-1})]$. Therefore, $V_{\tilde{z}}$ is tangent if, and only if L is a 2-torsion bundle, which correspond to $\tilde{z} \in \Delta$ according to Theorem 2.3.4.

Since Γ is invariant under the transformation $(z, w) \mapsto (w, z)$, $H_{\tilde{w}}$ is a tangent horizontal line if, and only if, $V_{\tilde{w}}$ is a tangent vertical line. □

Let $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projections with respect to the first and second factors. The restriction of each of these maps to Γ is a double cover of $\mathbb{P}^1$ that ramify in the points of the set Δ according to Lemma 2.5.6. Thus, from the embedding $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ we recover directly the set Δ as the set of ramification points of the projections π_1 and π_2. The point t associated to the degree 2 divisor 2 is also given by the embedding:

Proposition 2.5.7. *Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. Let $(\infty, \tilde{w}) \in \Gamma \cap V_\infty$ be the tangency point of the curve Γ with the vertical tangent V_∞. Then, $\tilde{w} = t$.*

More generally, let $k \in \Delta$ and V_k be the corresponding vertical tangent to Γ. Let $(k, \tilde{w}) \in \Gamma \cap V_k$ be the tangency point. Then, $\beta_k(\tilde{w}) = t$.

Proof. Let $\mathcal{P} = (\infty, \tilde{w})$. By definition of ε, we have that $\infty = \mathrm{Tu} \circ \mathrm{Forget}(\mathcal{E})$, where $\mathcal{E} \in \mathrm{Bun}_{\mathcal{O}}^{=}(C, T)$ represents the point $\mathcal{P}$. Thus, by Theorem 2.4.6 and Theorem 2.3.4, the underlying vector bundle of $\mathcal{E}$ is $\mathcal{O} \oplus \mathcal{O}$ or E_0. Since $\mathcal{P}$ is a point in the curve Γ, we have that $\mathcal{P} = [\mathcal{F}_*(\mathcal{O})]$.

Consider the bundle $\mathcal{F}_<(\mathcal{O}) = (E_0, \boldsymbol{p})$, where $p_1 \subset \mathcal{O} \subset E_0$ and $p_2 \not\subset \mathcal{O}$. According to the definition of ε, we have to show that $\mathrm{Tu} \circ \mathrm{Forget} \circ \phi_T(\mathcal{E}) = t$.

Let $q \in C$ be the unique point such that the subbundle $L = \mathcal{O}(-q)$ passes through both p_1 and p_2. Let us apply the mapping ϕ_T to $\mathcal{F}_<(\mathcal{O})$. After the first elementary transformation $\mathrm{elm}_{t_1}^+$, we get the bundle $\mathcal{E} = (\mathcal{O}(t_1) \oplus L(t_1), \boldsymbol{p}')$, where p_1' is outside both factors and p_2' lies on $L(t_1)$. Therefore, the underlying bundle of $\mathrm{elm}_{t_2}^+(\mathcal{E})$ is $\mathcal{O}(t_1) \oplus L(t_1 + t_2)$. Twisting by $\mathcal{O}(-w_\infty)$ gives the bundle $F = \mathcal{O}(t_1 - w_\infty) \oplus L(w_\infty)$. Since $\det F = \mathcal{O}$, it follows that $q = t_1$ and $\tilde{w} = \mathrm{Tu}[F] = t$ as we wanted to show.

For a $k \in \Delta \setminus \{0\}$, the above discussion holds if we multiply every bundle by L_k. The final underlying bundle is $F_k = F \otimes L_k$. By Proposition 2.3.1, $\tilde{w} = \mathrm{Tu}[F_k] = \beta_k(t)$. □

The next Lemma will be useful to show Theorem B.

Lemma 2.5.8. *Let Γ be a smooth curve of genus 1 embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a $(2,2)$-curve. Then, there exists an automorphism δ of $\mathbb{P}^1 \times \mathbb{P}^1$ such that the vertical (resp. horizontal) tangents to the image $\Gamma' = \delta(\Gamma) \subset \mathbb{P}^1 \times \mathbb{P}^1$ are the lines V_k (resp. H_k), for $k \in \Delta = \{0, 1, \lambda, \infty\}$ and some $\lambda \in \mathbb{C} \setminus \{0, 1\}$.*

Proof. Let us consider the first projection $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Since Γ is a $(2,2)$-curve, the restriction map $\pi_1|_\Gamma$ is a double cover of $\mathbb{P}^1$. Furthermore, by Riemann-Hurwitz formula, the map $\pi_1|_\Gamma$ ramifies in four distinct points $x_0, x_1, x_\lambda, x_\infty$ in $\mathbb{P}^1$. Similarly, the map $\pi_2|_\Gamma$ ramifies in four distinct points $y_0, y_1, y_\lambda, y_\infty$ in $\mathbb{P}^1$. Let δ_1 (resp. δ_2) be the unique automorphism of $\mathbb{P}^1$ such that $\delta_1(x_k) = k$ (resp. $\delta_2(y_k) = k$), for $k \in \Delta \setminus \{\lambda\}$. It suffices to take $\delta = \delta_1 \times \delta_2$. □

Let Γ be a smooth curve of genus 1 embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a $(2,2)$-curve. By Lemma 2.5.8, there exists $\lambda \in \mathbb{C} \setminus \{0,1\}$ such that the vertical and horizontal tangents of Γ are respectively the lines V_k and H_k, with $k \in \Delta$. In this setting, the curve Γ is symmetric with respect to the line $z = w$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 2.5.9. *Let $\lambda, t \in \mathbb{C} \setminus \{0, 1\}$, $\lambda \neq t$. Then, there exists a unique smooth irreducible bidegree $(2,2)$ curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that*

- *The vertical and horizontal tangents to Γ are precisely the lines V_k and H_k with $k \in \Delta$.*
- *The tangency point between Γ and V_∞ is (∞, t).*

Moreover, the curve Γ is symmetric with respect to the line $z = w$.

Proof. A bidegree $(2,2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is given by a polynomial

$$P(z, w) = \sum_{0 \leq i,j \leq 2} a_{ij} z^i w^j$$

In particular, for fixed $k \in \Delta$, the polynomial $P_k(w) = P(k, w)$ is a degree 2 polynomial in w. The curve Γ is tangent to the vertical V_k if and only if the polynomial $P_k(w)$ has a double root, i.e. its discriminant vanishes. This condition gives a degree 2 equation on the a_{ij}. Thus, tangency with the lines V_k and H_k gives 8 conditions, (one of them is actually redundant: since Γ is a smooth bidegree $(2,2)$ curve, thus the projections π_{t_1} and π_{t_2} cannot simultaneously ramify at arbitrary points). Vanishing at the point (∞, t) gives a linear equation on the a_{ij}. By direct computation, we check that (up to scalar multiplication), there is a unique choice of the a_{ij} such the polymial P is irreducible. This choice gives the polynomial

$$P(z,w) = t^2z^2 - 2tz^2w + t^2w^2 - 2tzw^2 + z^2w^2 \\ - 2\lambda tz - 2\lambda tw + 2\big(2(\lambda+1)t - t^2 - \lambda\big)zw + \lambda^2,$$

which is symmetric in the variables (z, w). □

Let $\mathbf{G} = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ be the group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ preserving the curve Γ. Remark that the involutions

$$\beta_k \times \beta_k, \phi \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1), \quad \text{for } k \in \Delta,$$

where $\phi(z, w) = (w, z)$, preserve the tangents V_k and H_k.

Proposition 2.5.10. *Let Γ be a smooth curve of genus 1 embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a $(2, 2)$-curve, and satisfying the conditions of Lemma 2.5.8. Then, the group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ is generated by the maps of the form $\gamma \times \gamma$, where γ is an automorphism of $\mathbb{P}^1$ preserving the set Δ, and by the involution ϕ.*

Proof. Recall that the we have the group isomorphism

$$\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \cong (\mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C})) \ltimes \mathbb{Z}/2\mathbb{Z},$$

where the semi-direct factor $\mathbb{Z}/2\mathbb{Z}$ corresponds to the involution ϕ.

By Lemma 2.5.9, the curve Γ is symmetric with respect to the line $z = w$, hence $\phi \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$.

Let $\varphi = \varphi_1 \times \varphi_2 \in \mathrm{PGL}(2, \mathbb{C}) \times \mathrm{PGL}(2, \mathbb{C})$ be an automorphism preserving Γ. Then, φ must preserve the set T of vertical and horizontal tangents to Γ. In particular, φ_1 and φ_2 must preserve the set Δ, and hence there is a finite number of possible candidates. By using the expression of the polynomial $P(z, w)$, we check that if φ preserves Γ, then $\varphi = \gamma \times \gamma$ with $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ preserving Δ. □

If $\lambda \notin \{-1, 1/2, 2\}$, the set of automorphisms of $\mathbb{P}^1$ preserving Δ is precisely the set of double transpositions β_k. Since $\beta_0 \circ \beta_1 = \beta_\lambda$, we have the following result:

Corollary 2.5.11. *Let Γ be a smooth curve of genus 1 embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a $(2, 2)$-curve, and satisfying the conditions of Lemma 2.5.8. If $\lambda \notin \{-1, 1/2, 2\}$, the group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, and it is generated by the involutions $\beta_k \times \beta_k$ and ϕ_T, for $k \in \Delta$.*

Recall that if $\lambda \in \{-1, 1/2, 2\}$, the group of automorphisms of the elliptic curve (Γ, w_∞) is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ (and not to $\mathbb{Z}/2\mathbb{Z}$). In this case, the group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ is slightly bigger. The reason is that there are automorphisms of $\mathbb{P}^1$ that exchange the elements of the set $\{k, \lambda\}$ while leaving invariant the elements of $\Delta \setminus \{k, \lambda\}$, for some $k \in \Delta \setminus \{\lambda\}$.

Proof of Theorem B. We have to show that there are one-to-one correspondences between the sets

$$\left\{ \begin{array}{c} (2,2)\text{-curves} \\ \Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1 \end{array} \right\} \Big/ \mathbf{G} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{2-punctured} \\ \text{genus 1} \\ \text{curves } (C,T) \end{array} \right\} \Big/ \sim \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} (4+1)\text{-} \\ \text{punctured} \\ \text{rational} \\ \text{curves} \\ (\mathbb{P}^1, \underline{D}+t) \end{array} \right\} \Big/ \sim$$

For the first bijection, let $\mathcal{A}$ be the set of $(2,2)$-curves $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$ modulo the group $\mathbf{G}$, and let $\mathcal{B}$ be the set of 2-punctured genus 1 curves (C,T) modulo the group of isomorphism of C preserving T.

Consider the embedding $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$. By Lemma 2.5.8, we can assume that the vertical and horizontal tangents with respect to Γ are respectively the lines V_k and H_k with $k \in \Delta = \{0,1,\lambda,\infty\}$. Let $t \in \mathbb{P}^1$ such that $(\infty,t) \in \mathbb{P}^1 \times \mathbb{P}^1$ is the tangent point of Γ and V_∞. Let T be the divisor given by the preimages of t by the restriction of the second projection $\pi_2|_\Gamma : \Gamma \to \mathbb{P}^1$. If $\sigma \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$, the restriction $\sigma|_\Gamma$ yields un automorphism $(\Gamma, \sigma(t_1) + \sigma(t_2))$. Thus, we have defined a map $\varphi : \mathcal{A} \to \mathcal{B}$.

Conversely, let (C,T) be a 2-puctured genus 1 curve. Let $\pi : C \to \mathbb{P}^1$ be a double cover of $\mathbb{P}^1$ such that $\pi(t_1) = \pi(t_2) = t$ and such π ramifies in the points in $\Delta = \{0,1,\lambda,\infty\} \in \mathbb{P}^1$ for a certain $\lambda \in \mathbb{C}$. By Lemma 2.5.9, there exists un unique embedding $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ such the vertical and horizontal tangents with respect to Γ are respectively the lines V_k and H_k with $k \in \Delta = \{0,1,\lambda,\infty\}$. If $\rho : (C,T) \to (C,T')$ is an isomorphism with $T' = t'_1 + t'_2$, let us consider the double cover $\pi : C \to \mathbb{P}^1$ such that $\pi(t'_1) = \pi(t'_2) = t'$ and such π ramifies in the points $0,1,\lambda,\infty \in \mathbb{P}^1$. The automorphism ρ induces an automorphism γ of $\mathbb{P}^1$ preserving the set Δ and such that $t' = \gamma(t)$. According to Proposition 2.7.2, the map $\gamma \times \gamma$ preserves the curve Γ. Hence, this construction defines a map $\psi : \mathcal{B} \to \mathcal{A}$. The maps φ and ψ are well-defined and inverses.

The second correspondance is given by the double cover $\pi : C \to \mathbb{P}^1$ such that $\pi(t_1) = \pi(t_2) = t$, and its ramification divisor $\underline{D}$. □

This type of result has been also found by Balaji, Biswas and Del Baño Rollin [5], where the authors study the moduli of quasi-parabolic bundles in the higher genus case, subsequently proving a theorem of Torelli type.

2.6 A map between moduli spaces

Let $\underline{W} = 0 + 1 + \lambda + \infty$ and $\underline{D} = \underline{W} + t$ be reduced divisors on $\mathbb{P}^1$. Let C be the curve defined by the equation $y^2 = x(x-1)(x-\lambda)$. Consider the divisors $W = w_0 + w_1 + w_\lambda + w_\infty$ and $T = t_1 + t_2$ on C respectively supported by the Weierstrass points and by the two preimages of t under the hyperelliptic cover π. Let $D = W + T$.

We study in this Section a map Φ between the moduli spaces $\mathrm{Bun}_0^{\underline{\boldsymbol{\mu}}}(\mathbb{P}^1, \underline{D})$ and $\mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C, T)$, with fixed weights $\underline{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}$. The first space is the moduli space of $\underline{\boldsymbol{\mu}}$-semistable rank 2 quasi-parabolic vector bundles of degree 0 over $(\mathbb{P}^1, \underline{D})$. This moduli space is described and constructed in [40].

More precisely, the authors consider the full coarse moduli space $\mathrm{Bun}_{-1}(\mathbb{P}^1, \underline{D})$ of degree -1 indecomposable quasi-parabolic bundles ($P_{-1}(\boldsymbol{t})$ in their notation). They construct this space by patching projective charts consisting of moduli spaces $\mathrm{Bun}_{-1}^{\boldsymbol{\nu}}(\mathbb{P}^1, \underline{D})$ of $\boldsymbol{\nu}$-semistable bundles.

Let us describe two of these charts. The chart V corresponds to the moduli space $\mathrm{Bun}_{-1}^{\boldsymbol{\nu}}(\mathbb{P}^1, \underline{D})$, for «democratic»weights $\nu_i = \nu$, with $\frac{1}{5} < \nu < \frac{1}{3}$, and is isomorphic to $\mathbb{P}^2$. It consists of those indecomposable quasi-parabolic bundles of the form $E = (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \boldsymbol{n})$ with no n_i lying in $\mathcal{O}_{\mathbb{P}^1}$ and not all n_i lying in $\mathcal{O}_{\mathbb{P}^1}(-1)$ (see Proposition 3.7 of [40]). There are 16 special geometric objects in V, namely:

— Five points D_i for $i \in \{0, 1, \lambda, \infty, t\}$.

— Ten lines Π_{ij} joining D_i and D_j.

— The conic Π passing through all D_i.

Let $\tilde{\Omega}$ be the set of these objects. The chart $\mathcal{S}$ corresponds to the moduli space with democratic weights $\boldsymbol{\nu}$ such that $\frac{1}{3} < \nu < \frac{3}{5}$. As a projective surface, it is isomorphic to the blow-up of the five points D_i in $\mathbb{P}^2$. This is by definition a del Pezzo surface of degree 4 (see [25]). In particular, the exceptional divisors Π_i of D_i and the total transforms of Π_{ij} and Π constitute 16 (-1)-curves in $\mathcal{S}$. We will keep the notation Π_{ij} and Π for the total transforms when there is no risk of confusion. There are exactly five (-1)-curves intersecting Π, namely the Π_i's, and the chart V is obtained as the blow-down of these 5 curves. Similarly, there are 5 (-1)-curves intersecting Π_i, namely Π and Π_{ij} for $j \neq i$. The other charts are isomorphic to projective surfaces obtained by blow-downs of some of these (-1)-curves.

The open set

$$U_{-1} = V \setminus \{D_i, \Pi_{ij}, \Pi\}$$

of generic quasi-parabolic bundles is common to every chart. The final non-separated patching is made via the blow-down birational maps between the charts (see Theorem 1.3 in [40]).

2.6.1 Defining the map Φ

Let us start by fixing weights $\underline{\boldsymbol{\mu}} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \mu)$, where the free weight μ corresponds to the point t. Consider the associated moduli space $\mathrm{Bun}_0^{\underline{\boldsymbol{\mu}}}(\mathbb{P}^1, \underline{D})$. The isomorphism

$$\mathrm{elm}_0^+ : \mathrm{Bun}_{-1}^{\underline{\boldsymbol{\mu}}}(\mathbb{P}^1, \underline{D}) \to \mathrm{Bun}_0^{\underline{\boldsymbol{\mu}}}(\mathbb{P}^1, \underline{D})$$

is well-defined by the properties of elementary transformations. Let U_0 be the image of U_{-1} by this map. We have the following result:

Proposition 2.6.1. *The moduli space* $\mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D})$ *is isomorphic to* $\mathcal{S}$ *for all* $\mu \in [0,1]$. *Every bundle in* U_0 *has trivial underlying bundle.*

Proof. The map elm_0^+ is an isomorphism of moduli spaces preserving $\underline{\boldsymbol{\mu}}$-stability. A straightforward calculation shows that all bundles in the families Π, Π_i, and $\Pi_{i,j} \subset \mathrm{Bun}_{-1}^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D})$ are $\underline{\boldsymbol{\mu}}$-semistable (stable when $0 < \mu < 1$). Hence, $\mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D}) \cong \mathcal{S}$.

For the second assertion, remark that every quasi-parabolic bundle $\mathcal{E}$ in U_{-1} can be written as $(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \boldsymbol{l})$ with $l_0 \subset \mathcal{O}_{\mathbb{P}^1}(-1)$. By properties of elementary transformations, $\mathrm{elm}_0^+(\mathcal{E})$ has trivial underlying bundle. □

Let us now define the mapping Φ. Let $\mathcal{E}$ be a quasi-parabolic bundle in $\mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D})$. Consider the pullback bundle $\pi^*\mathcal{E}$ of $\mathcal{E}$: it is a quasi-parabolic bundle over (C, D). The bundle $\pi^*\mathcal{E}$ is $\underline{\boldsymbol{\mu}}'$-semistable, where $\underline{\boldsymbol{\mu}}' = (1,1,1,1,\mu,\mu)$.

Consider the following composition of maps:

$$\mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}'}(C,D) \xrightarrow{\mathrm{elm}_W^+} \mathrm{Bun}_{4w_\infty}^{\boldsymbol{\mu}''}(C,D) \xrightarrow{\mathrm{Forget}_W} \mathrm{Bun}_{4w_\infty}^{\boldsymbol{\mu}}(C,T) \xrightarrow{\otimes M} \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C,T).$$

The weights here are $\underline{\boldsymbol{\mu}}'' = (0,0,0,0,\mu,\mu)$ and $\boldsymbol{\mu} = (\mu,\mu)$. The first map is the positive elementary transformation over W. The second map forgets quasi-parabolic directions over W and keeps those over T. Because of the nullity of the weights over W, this map preserves the stability notion. The last map is the twist automorphism, with $M = \mathcal{O}_C(-2w_\infty)$.

Let ϕ_W be the composition of these four maps. The map Φ is the composition of π^* and ϕ_W:

$$\begin{array}{ccc} \mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1,\underline{D}) & \xrightarrow{\quad\Phi\quad} & \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C,T) \\ & \searrow^{\pi^*} \quad {}^{\phi_W}\nearrow & \\ & \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}'}(C,D) & \end{array}$$

Notice that the role of $t \in \mathbb{P}^1$ in the definition of this map is different from the roles of the other points in $\underline{D}$. We have defined a morphism $\Phi : \mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D}) \to \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C,T)$ between our moduli spaces.

2.6.2 Computing the map Φ

First, we will compute the map $\Phi|_{U_0}$. For $c, l \in \mathbb{P}^1$, consider the set

$$U_C := \left\{ (\mathcal{O}_C \oplus \mathcal{O}_C, \underline{\boldsymbol{m}}) \mid c, l \in \mathbb{P}^1 \right\} \subset \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}'}(C,D).$$

of quasi-parabolic bundles on (C,D), where $\underline{\boldsymbol{m}} := (0,1,c,\infty,l,l)$. Since $\pi(t_1) = \pi(t_2)$, the quasi-parabolic directions over t_1 and t_2 of $\pi^*\mathcal{E}$ are the same. Thus, π^* is a birational map between U_0 and U_C.

We will use the coordinate chart $U_C \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the coordinate system ε for $\mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C,T)$ defined in Section 2.4.3.

The map $\phi_W|_{U_C}$

Let $(E,\boldsymbol{m})$ be a quasi-parabolic bundle over (C,D) and $D_0 \subset D$ a subdivisor. We say that a line subbundle L of E passes through D_0 if L passes through every point of D_0.

Lemma 2.6.2. *Let $\mathcal{E} = (\mathcal{O}_C \oplus \mathcal{O}_C, \boldsymbol{m})$ be a quasi-parabolic bundle over (C,D). Let $S^{-2}_W(\mathcal{E})$ be the set of degree -2 line subbundles of $\mathcal{O}_C \oplus \mathcal{O}_C$ passing through $\boldsymbol{m}|_W$. Then, there exists a point $p \in C$ such that*

$$S^{-2}_W(\mathcal{E}) = \{L_W, L'_W\} \text{ with } L_W \cong \mathcal{O}_C(-p - w_\infty),\ L'_W \cong \mathcal{O}_C(-\iota_{w_\infty}(p) - w_\infty)$$

Similarly, let $S^{-3}_D(\mathcal{E})$ be the set of degree -3 line subbundles of $\mathcal{O}_C \oplus \mathcal{O}_C$ passing through $\boldsymbol{m}|_D$. Then, there is a point $q \in C$ such that

$$S^{-3}_D(\mathcal{E}) = \{M_D, M'_D\} \text{ with } M_D \cong \mathcal{O}_C(-q - 2w_\infty),\ M'_D \cong \mathcal{O}_C(-\iota_{w_\infty}(q) - 2w_\infty).$$

Proof. Let us prove the assertion for $S^{-2}_W(\mathcal{E})$. Consider a line bundle $L = \mathcal{O}_C(-\tilde{p} - w_\infty)$, for $\tilde{p}$ in C. The vector space $\mathrm{Hom}(L, \mathcal{O}_C \oplus \mathcal{O}_C)$ is isomorphic to $\mathrm{Hom}(\mathcal{O}_C, L^{-1} \oplus L^{-1})$. The projective dimension of this space is 3 by Riemann-Roch. This means that there is an inclusion $L \hookrightarrow \mathcal{O}_C \oplus \mathcal{O}_C$ such that L passes through three quasi-parabolic directions, say m_0, m_1 and m_λ. By moving $\tilde{p}$ in C we move on the fiber of w_∞. This constitutes a double covering of $\mathbb{P}^1$, hence there are generically two points p and p' in C such that $S^{-2}_W(\mathcal{E}) = \{L, L' = \mathcal{O}_C(-p' - w_\infty)\}$. To see that $p' = \iota_{w_\infty}(p)$, consider the case $p \neq p'$. We have that $\mathrm{Forget} \circ \mathrm{elm}^+_W(\mathcal{E}) \otimes \mathcal{O}_C(-2w_\infty) = \mathcal{O}_C(w_\infty - p) \oplus \mathcal{O}_C(w_\infty - p')$ has trivial determinant, implying $p' = \iota_{w_\infty}(p)$. The proof for $S^{-3}_D(\mathcal{E})$ is similar. □

Proposition 2.6.3. *The map $\phi_W|_{U_C}$ is given by*

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\phi_W|_{U_C}} \mathbb{P}^1 \times \mathbb{P}^1 \quad , \quad \phi_W|_{U_C}(c,l) = \left(\phi_W|^1_{U_C}(c,l), \phi_W|^2_{U_C}(c,l)\right)$$

where

$$\phi_W|^1_{U_C}(c,l) = \lambda\frac{c-1}{c-\lambda} \quad \text{and} \quad \phi_W|^2_{U_C}(c,l) = \frac{\lambda l(\lambda(l-1) + t(1-c))}{\lambda\left(t(l-c) + l(c-1)\right) + ct(1-l)}.$$

Proof. Consider the trivial rank 2 vector bundle $\mathcal{O}_C \oplus \mathcal{O}_C$ and its projectivization $C \times \mathbb{P}^1$. Let $\boldsymbol{m} = (0,1,c,l,l,\infty)$ be a quasi-parabolic configuration

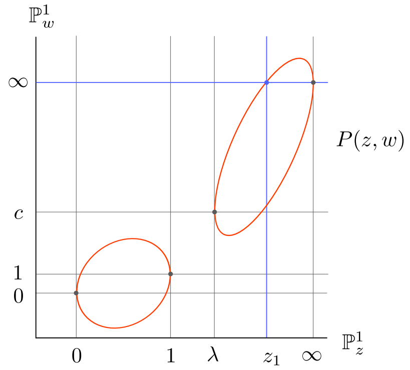


Figure 2.1 – The curve of bidegree $(2,2)$ defined by $P(z,w)=0$. Here, $z_1=\lambda\frac{c-1}{c-\lambda}$.

on $\mathcal{O}_C\oplus\mathcal{O}_C$. Let us write $(E,\boldsymbol{n}) := \phi_W|_{U_C}(\mathcal{O}_C\oplus\mathcal{O}_C,\boldsymbol{m})$. We want to compute the ε-coordinates of $(E,\boldsymbol{n})$ in terms of (c,l).

There exists $p\in C$ such that $[E]=[L\oplus L^{-1}]$ in $\mathrm{Bun}_{\mathcal{O}}(C)$, with $L=\mathcal{O}(p-w_\infty)$. By the properties of elementary transformations, the preimages of the line bundles L and L^{-1} via $\phi_W|_{U_C}$ are the following subbundles of $\mathcal{O}_C\oplus\mathcal{O}_C$:

$$L_W=\mathcal{O}_C(-p-w_\infty) \quad , \quad L'_W=\mathcal{O}_C(-\iota_{w_\infty}(p)-w_\infty)$$

By construction, L_W and L'_W pass through the quasi-parabolic directions over W. Moreover, these are the unique subbundles of degree -2 passing through these quasi-parabolic directions by Lemma 2.6.2.

Consider the $(+4)$-cross-sections s_W and s'_W of the trivial projective bundle $C\times\mathbb{P}^1$ corresponding to L_W and L'_W respectively. The section s_W (resp. s'_W) intersects the constant section $y=\infty$ in two points with base coordinates w_∞ and p (resp. w_∞ and $\iota_{w_\infty}(p)$) in C. We claim that $\pi(p)=\lambda\frac{c-1}{c-\lambda}$. This will imply

$$\phi_W|^1_{U_C}(c,l)=T\circ\mathrm{Forget}\circ\phi_W|_{U_C}(\mathcal{O}_C\oplus\mathcal{O}_C,\boldsymbol{m})=\lambda\frac{c-1}{c-\lambda}$$

as desired. In order to prove the claim, consider the pullback of the section s_W by π. It is a curve of bidegree $(2,2)$ in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})=\mathbb{P}^1\times\mathbb{P}^1$ having vertical tangencies over the ramification points $0,1,\lambda,\infty\in\mathbb{P}^1$ and intersecting the horizontal $w=\infty$ in two points with base coordinates $\tilde{z}=\infty$ and $z_1=\pi(p)$ respectively (see Figure 2.1). This curve is defined by a bidegree (2,2) polynomial $P(z,w)$ satisfying the former conditions, which translate to 8 polynomial equations on the coefficients of P. Solving this polynomial system yields $z_1=\lambda\frac{c-1}{c-\lambda}$.

For the second coordinate, we have that

$$\phi_W|^2_{U_C}(c,l)=T\circ\mathrm{Forget}\circ\phi_T\circ\phi_W|_{U_C}(\mathcal{O}_C\oplus\mathcal{O}_C,\boldsymbol{p})=T[M\oplus M^{-1}],$$

with $M = \mathcal{O}(q - w_\infty)$ for $q \in C$. The preimages of M and M^{-1} by $\phi_T \circ \phi_W|_{U_C}$ are

$$M_D = \mathcal{O}_C(-\iota_{w_\infty}(p) - 2w_\infty) \quad , \quad M'_D = \mathcal{O}_C(-p - 2w_\infty)$$

These correspond to two (+3)-cross-sections s_D and s'_D of the projective bundle passing through every quasi-parabolic point over $W + T$. The section s_D (resp. s'_D) intersects the constant section $y = l$ in three points having base coordinates t_1, t_2 and $\iota_{w_\infty}(p)$ (resp. t_1, t_2 and p). The pullback of these cross-sections by π is a curve of bidegree $(3, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ having vertical tangencies over the ramification points and crossing in a node over t. Applying these conditions to a polynomial of degree (3,2) on z, w yields the desired formula for $\phi_W|^2_{U_C}(c, l)$. □

The map Φ

Let $\mathbb{P}^2_{\boldsymbol{b}}$ be the projective line with homogeneous coordinates $\boldsymbol{b} = (b_0 : b_1 : b_2)$. The birational coordinate change $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2_{\boldsymbol{b}} \supset U_0$ is explicited in Section 6 of [40]. Composing with the coordinates of $\phi_W|_{U_C}$ of Proposition 2.6.3 of [40], we find that the mapping

$$\Phi|_{U_0} : U_0 \to \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathrm{Bun}^{\boldsymbol{\mu}}_{\mathcal{O}}(C, T)$$

is given in U_0 by the expression

$$\Phi|_{U_0}(b_0 : b_1 : b_2) = \left(\frac{b_1 t - b_2}{b_0 t - b_1}, -b_1 \frac{b_0 \lambda - b_1 \lambda - b_1 + b_2}{b_1^2 - b_0 b_2} \right).$$

This map extends to a degree 2 rational map $\tilde{\Phi} : \mathbb{P}^2_{\boldsymbol{b}} \to \mathbb{P}^1 \times \mathbb{P}^1$ by analytic continuation.

2.6.3 The special locus of $\tilde{\Phi}$ and the geometric configuration on $\mathbb{P}^2_{\boldsymbol{b}}$

In this Section we will relate the set $\tilde{\Omega}$ of 16 geometric objects in $\mathbb{P}^2_{\boldsymbol{b}}$ with the undeterminacy and the ramification locus of $\tilde{\Phi}$.

Special locus of $\tilde{\Phi}$

The undeterminacy locus of $\tilde{\Phi}$ consists of the five special points $D_i \in \mathbb{P}^2_{\boldsymbol{b}}$. Their projective coordinates are

$$D_0 = (1:0:0), \quad D_1 = (1:1:1), \quad D_\lambda = (1:\lambda:\lambda^2),$$
$$D_t = (1:t:t^2), \quad D_\infty = (0:0:1).$$

The conic Π passing by all the D_i is given by the equation $\Pi : b_1^2 - b_0 b_1$. A calculation shows that the map $\tilde{\Phi}$ ramifies over the cubic $\Sigma \subset \mathbb{P}^2_{\boldsymbol{b}}$ defined by the equation

$$\begin{aligned}\Sigma : &-b_0^2 b_1 \lambda t^2 + b_0^2 b_2 \lambda t + \big(\lambda t^2 + \lambda t + t^2\big) b_0 b_1^2 - \big(t^2 + \lambda\big) b_1^3 \\ &-2\,(\lambda t + t) b_0 b_1 b_2 + b_1^2 b_2 (\lambda + t + 1) + b_0 b_2^2 t - b_1 b_2^2 = 0.\end{aligned}$$

The cubic Σ passes through the 5 points D_i and is tangent to Π in D_t. This cubic is precisely the preimage of $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$. The curve Γ has equation

$$\begin{aligned}\Gamma : = &\, t^2 z^2 - 2tz^2 w + t^2 w^2 - 2tzw^2 + z^2 w^2 \\ &- 2\lambda t z - 2\lambda t w + 2\big(2(\lambda+1)t - t^2 - \lambda\big) zw + \lambda^2\end{aligned}$$

The action of $\tilde{\Phi}$ on the 16 objects

There are four vertical lines V_i and four horizontal lines tangent to Γ (see Section 2.5.1):

$$V_i = \{(z,w) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid z = i\}, \quad H_i = \{(z,w) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid w = i\}$$

with $i \in \{0, 1, \lambda, \infty\}$. The map $\tilde{\Phi}$ sends each curve of $\tilde{\Omega}$ to one of these tangents lines to Γ in the following way:

- For $i \neq t$, the line Π_{it} is sent to V_i.
- The conic Π is sent to H_∞.
- For $\{i,j,k\} = \{0,1,\lambda\}$, the lines $\Pi_{i\infty}$ and Π_{jk} are sent to H_i.

The desingularisation map Φ

The map $\tilde{\Phi}$ is birational with base points D_i. Hence, there exists a morphism Φ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S} \cong \mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D}) & \xrightarrow{\Phi} & \mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C, T) \cong \mathbb{P}^1 \times \mathbb{P}^1 \\ \big\downarrow \text{blow-up} & \nearrow_{\tilde{\Phi}} & \\ \mathbb{P}^2_{\boldsymbol{b}} & & \end{array}$$

The vertical map is the blow-up of the 5 points D_i in $\mathbb{P}^2_{\boldsymbol{b}}$. The map Φ defined in Section 2.6.1 is the desingularisation map of $\tilde{\Phi}$. From the previous discussion, we have the following result:

Theorem 2.6.4. *The map Φ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified over the curve $\Gamma \subset \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let us prove that $\mathcal{S}$ is the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along Γ. Consider the standard Segre embedding $i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3_u$ given by

$$i((z_0 : z_1), (w_0 : w_1)) = (z_0 w_0 : z_0 w_1 : z_1 w_0 : z_1 w_1).$$

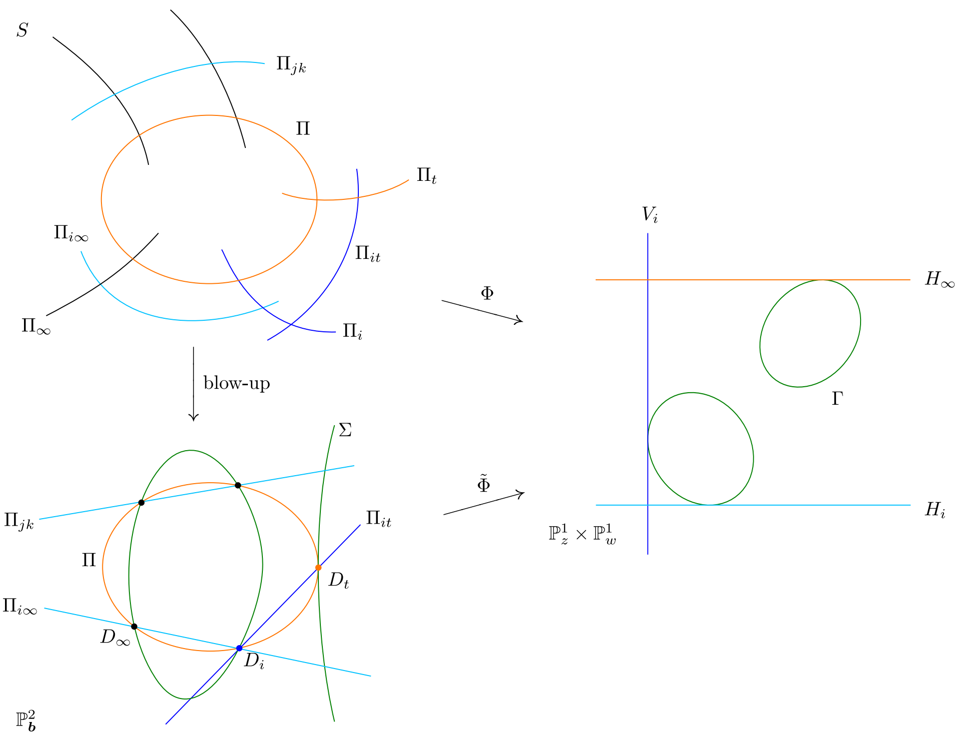


Figure 2.2 – The geometry of the map Φ.

Its image is the quadric given by the equation $f = u_0u_3 - u_1u_2$. The curve Γ is the restriction of a quadric in $\mathbb{P}^3_u$ of equation $g = 0$. The ramified cover of $\mathbb{P}^3_u$ along $g = 0$ is then given by the equation $v^2 = g$ in $\mathbb{P}^4_{u,v}$, and the covering morphism is given by the projection onto u.

The double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ is thus given by the quadratic forms $t^2 = g$ and $f = 0$ in $\mathbb{P}^4_{u,v}$. Since the intersection is smooth, the covering is a del Pezzo surface of degree 4 (see for example [25], Section 8.6). Since the map Φ is also of degree 2 with ramification locus Γ, it is the covering map. □

Let $\Omega = \{\Pi_i, \Pi_{ij}, \Pi\}$. From the description of $\tilde{\Phi}$ on $\tilde{\Omega}$, we get that the elements of Ω are sent by Φ to vertical and horizontal tangents to Γ as follows (see Figure 2.2):

— For $i \neq t$, the exceptional divisor Π_i and the line Π_{it} are sent to V_i.

— The exceptional divisor Π_t and the conic Π are sent to H_∞.

— For $\{i, j, k\} = \{0, 1, \lambda\}$, the lines $\Pi_{i\infty}$ and Π_{jk} are sent to H_i.

Each (-1)-curve is mapped by Φ bijectively onto its image.

2.6.4 The involution τ

Let $(E, \boldsymbol{l}) \in \mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D})$ be a quasi-parabolic bundle. Define

$$\tau(E, \boldsymbol{l}) := \mathrm{elm}^+_{\underline{W}}(E, \boldsymbol{l}) \otimes \mathcal{O}_{\mathbb{P}^1}(-2).$$

The bundle $\tau(E, \boldsymbol{l})$ is again an element of $\mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D})$, therefore the map τ is an automorphism of this moduli space.

Proposition 2.6.5. *The mapping τ is the involution induced by Φ. More precisely, τ satisfies $\Phi \circ \tau = \Phi$ and it is not the identity.*

Proof. Locally, $\mathbb{P}(E)$ is a product $U_x \times \mathbb{P}^1$. Consider the quasi-parabolic point $p = (0, 0) \in U_x \times \mathbb{P}^1$. The pullback map π^* is given by

$$(x, y) \leftarrow (z, y),$$

where z is a local coordinate in C with $z^2 = x$. An elementary transformation centered in $(z = 0, y = 0)$ corresponds to a coordinate transformation (z, y') with $y' = y/z$.

Let us now apply an elementary transformation to (E, p). We obtain the coordinate transformation $(x, y'' = y/x)$. The pullback gives

$$(x, y'') \leftarrow (z, \tilde{y} = y/z^2).$$

The elementary transformation of $\pi^*(E, p)$ is locally given by $(z, \tilde{y}z = y/z = y')$ (notice that the quasi-parabolic point is now at $y'' = \infty$).

We have shown that the quasi-parabolic bundles (E, p) and $\mathrm{elm}_p(E, p)$ have the same image by pullback followed by elementary transformation on a point. Since the arguments are local, the result holds also for Φ. □

The map τ is a degree 3 birational transformation of $\mathbb{P}^2_{\boldsymbol{b}}$ (see Table 2.3). The vanishing locus of its Jacobian determinant is exactly the union of the geometric objects in $\tilde{\Omega}$.

The de Jonquières automorphism

Let P be a point in $\mathbb{P}^2_{\boldsymbol{b}}$. The line Π_{P,D_t} passing through P and D_t and the conic Π_P passing through the five points D_i for $i = 0, 1, \lambda, \infty$ and P intersect generically in two points, which are P and $\tau(P)$.

The involution τ is given by the intersection of two foliations on $\mathbb{P}^2_{\boldsymbol{b}}$: the pencil of lines passing through D_t and the pencil of conics passing through D_i, for $i \neq t$. The ramification locus $\Sigma \subset \mathbb{P}^2_{\boldsymbol{b}}$ of $\tilde{\Phi}$ is the tangency locus of these two foliations. The involution τ leaves invariant Σ pointwise.

Each point in Σ is represented by a quasi-parabolic bundle $(\mathcal{O}_C \oplus \mathcal{O}_C, \boldsymbol{p})$ fixed by τ. Following Lemma 2.6.2, this condition means exactly that

$$\{L_W, L'_W\} = \{M_W, M'_W\}.$$

This is equivalent to L_W passing through at least one of the quasi-parabolic directions over t_i. Thus, Σ is exactly the locus of semistable bundles in $\mathrm{Bun}_0^{\boldsymbol{\mu}}(\mathbb{P}^1, \underline{D})$. Since $\boldsymbol{\mu}$-stability is preserved by elementary transformations, we obtain again that the image $\Phi(\Sigma)$ is the curve Γ of Section 2.4.

The action of τ on Ω is summarized in Table 2.4.

2.7 The del Pezzo geometry and the group of automorphisms

Let $\mathrm{Aut}(\mathcal{S})$ be the group of automorphisms of the del Pezzo surface $\mathcal{S}$. We have the following Theorem:

Theorem 2.7.1. *Let $0+1+\lambda+\infty+t$ be a reduced degree 5 divisor in $\mathbb{P}^1$. Let $D_k \in \mathbb{P}^2$ be the points defined in Section 2.6.3, $k \in \{0,1,\lambda,\infty,t\}$, and let $\mathcal{S}$ be the blow-up of $\mathbb{P}^2$ in the points D_k. If λ and t are general, the group $\mathrm{Aut}(\mathcal{S})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ and acts transitively and freely on Ω.*

Proof. See, for example, Chapter 8 of [25]. □

In particular, if λ and t are general, every automorphism of S is an involution and it is uniquely defined by the image of a (-1) curve in Ω. This set is invariant under $\mathrm{Aut}(\mathcal{S})$.

2.7.1 The group of automorphisms $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$

Let C be a genus 1 curve. Let $T = t_1 + t_2$ be an effective reduced degree 2 divisor on C. Let $\pi : C \to \mathbb{P}^1$ be the double cover of $\mathbb{P}^1$ satisfying $\pi(t_1) = \pi(t_2) = t \in \mathbb{P}^1$.

In Section 2.5 we defined five automorphisms of the moduli space $\mathrm{Bun}_{\mathcal{O}}^{\boldsymbol{\mu}}(C, T)$: the four twists $\tilde{\sigma_k} := \otimes L_k$, for $k \in \Delta = \{0,1,\lambda,\infty\}$, and the map ϕ_T. These maps are involutions preserving the bidegree $(2,2)$ curve Γ. Recall that $\tilde{\sigma_\lambda} = \tilde{\sigma_0} \circ \tilde{\sigma_1}$.

These involutions act transitively on the set of vertical and horizontal tangents to Γ: the involution ϕ_T exchanges verticals and horizontals, and the twists $\tilde{\sigma_k}$ act separately on horizontal and vertical tangents as double transpositions. More precisely, for $k \in \{0,1,\lambda\}$,

— $\tilde{\sigma_k}(H_\infty) = H_k$ and $\tilde{\sigma_k}(V_\infty) = V_k$.
— $\tilde{\sigma_k}(H_i) = H_j$ and $\tilde{\sigma_k}(V_i) = V_j$ for $\{i,j,k\} = \{0,1,\lambda\}$.
— $\phi_T(H_i) = V_i$.

Let $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ be the group of automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ leaving invariant Γ. Recall by Corollary 2.5.11 that we have:

Proposition 2.7.2. *If $\lambda \notin \{-1, 1/2, 2\}$, the group $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, and it is generated by the involutions $\tilde{\sigma_k}$ and ϕ_T.*

τ	$\tau_0 = ((\lambda t)b_0b_1 + (t^2 - t\lambda - t)b_0b_2 + (-t^2 - \lambda)b_1^2 + (t + \lambda + 1)b_1b_2 - b_2^2)(b_0t - b_1)$ $\tau_1 = (b_0\lambda - b_1\lambda - b_1 + b_2)(b_0t - b_1)(b_1t - b_2)t$ $\tau_2 = \left(b_0^2 + (-t^2\lambda - t^2 - t\lambda)b_0b_1 + (t\lambda + t - \lambda)b_0b_2 + (t^2 + \lambda)b_1^2 + (-t)b_1b_2\right)(b_1t - b_2)t$
σ_0	$\sigma_{00} = (\lambda b_0 + (-1 - \lambda + t)b_1)(tb_1 - b_2)$ $\sigma_{01} = \lambda(tb_0 - b_1)(tb_1 - b_2)$ $\sigma_{02} = \lambda(tb_0 - b_1)((-\lambda + t + \lambda t)b_1 - tb_2)$
σ_1	$\sigma_{10} = (tb_0 - (1 + t)b_1 + b_2)((\lambda - t - 1)b_1 + tb_0)$ $\sigma_{11} = t(\lambda b_0 - b_1)(tb_0 + (-1 - t)b_1 + b_2)$ $\sigma_{12} = t\left(\lambda^2 tb_0^2 + (\lambda^2 + t)b_1^2 + (-\lambda^2 - \lambda t - \lambda^2 t)b_0b_1 + (\lambda - t + \lambda t)b_0b_2 - \lambda b_1b_2\right)$
σ_λ	$\sigma_{\lambda 0} = (\lambda tb_0 + (1 - \lambda - t)b_1)(\lambda tb_0 - (\lambda + t)b_1 + b_2)$ $\sigma_{\lambda 1} = \lambda t(b_0 - b_1)(\lambda tb_0 - (\lambda + t)b_1 + b_2)$ $\sigma_{\lambda 2} = \lambda t\left(\lambda tb_0^2 + (1 + \lambda t)b_1^2 - (\lambda + t + \lambda t)b_0b_1 + (\lambda + t - \lambda t)b_0b_2 - b_1b_2\right)$
ψ_T	$\psi_{T0} = (-\lambda - t^2)b_1^2 - b_2^2 + \lambda tb_0b_1 + (t^2 - t - \lambda t)b_0b_2 + (\lambda + t + 1)b_1b_2$ $\psi_{T1} = t(\lambda b_0 - (1 + \lambda)b_1 + b_2)(tb_1 - b_2)$ $\psi_{T2} = t((t - \lambda + \lambda t)b_1 - tb_2)(\lambda b_0 - (1 + \lambda)b_1 + b_2)$

Table 2.3 – The involutions τ, σ_k and ψ_T in the projective coordinates $\boldsymbol{b}$. Here $\tau = (\tau_0 : \tau_1 : \tau_2)$, $\sigma_k = (\sigma_{k0} : \sigma_{k1} : \sigma_{k2})$ and $\psi_T = (\psi_{T0} : \psi_{T1} : \psi_{T2})$.

Each involution $\tilde{\sigma_k}$ lifts to two automorphisms σ_k and $\sigma_k \circ \tau$ of S. These involutions act transitively on the set Ω. Choose σ_k to be the lift of $\tilde{\sigma_k}$ such that $\sigma_k(\Pi_{t\infty}) = \Pi_k$ (and thus $\sigma_k \circ \tau(\Pi_{t\infty}) = \Pi_{tk}$, and $\sigma_\infty = \tau$). Finally, let ψ_T be the lift of ϕ_T such that $\psi_T(\Pi_{t\infty}) = C$. The action of these involutions on the set Ω is summarized in Table 2.4.

τ	Π_t	$\longleftrightarrow$	Π	
	Π_i	$\longleftrightarrow$	Π_{it}	for $\{i \neq t\}$
	$\Pi_{i\infty}$	$\longleftrightarrow$	Π_{jk}	for $\{i, j, k\} = \{0, 1, \lambda\}$
σ_k $(k \neq \infty)$	Π_k	$\longleftrightarrow$	$\Pi_{t\infty}$	
	Π_{kt}	$\longleftrightarrow$	Π_∞	
	$\Pi_{i\infty}$	$\longleftrightarrow$	Π_{jt}	for $\{i, j, k\} = \{0, 1, \lambda\}$
	Π_{ik}	$\longleftrightarrow$	Π_i	
	Π	$\longleftrightarrow$	Π_{ij}	
ψ_T	$\Pi_{t\infty}$	$\longleftrightarrow$	Π	
	Π_t	$\longleftrightarrow$	Π_∞	
	Π_{ij}	$\longleftrightarrow$	Π_k	for $\{i, j, k\} = \{0, 1, \lambda\}$

Table 2.4 – The action of the involutions τ, σ_k and ψ_T on the set Ω.

By Proposition 2.7.2, the subgroup of $\mathrm{Aut}(\mathcal{S})$ generated by σ_0, σ_1 and ψ_T is isomorphic to $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$. Since τ is not an automorphism of

$\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1, \Gamma)$, we directly obtain by cardinality of $\mathrm{Aut}(\mathcal{S})$ that

$$\mathrm{Aut}(\mathcal{S}) = \langle \sigma_0, \sigma_1, \psi_T, \tau \rangle$$

This completes the proof of Theorem C.

Chapitre 3

Involutions on moduli spaces of vector bundles and GIT quotients

3.1 Introduction

Let C be a complex compact smooth curve of genus $g \geq 3$. Let $\mathcal{SU}_C(r)$ be the moduli space of semistable vector bundles of rank r with trivial determinant on C. This moduli space is a normal, projective, unirational variety of dimension $(r^2 - 1)(g - 1)$. The study of the projective structure of the moduli spaces of vector bundles in low rank and genus has produced some beautiful descriptions, frequently meeting constructions issued in the context of classical geometry.

For example, in the case of a hyperelliptic curve C, Desale and Ramanan [23] characterize the quotient $\mathcal{SU}_C(2)/i^*$ of the moduli space of rank 2 vector bundles by the map i^* induced by the hyperelliptic involution i. They show that there exists two quadrics Q_1 and Q_2 in a $(2g+1)$-dimensional projective space such that the quotient $\mathcal{SU}_C(2)/i^*$ is isomorphic to the variety of g-dimensional linear subspaces contained in Q_1, belonging to a fixed system of maximal isotropic spaces, and intersecting Q_2 in quadrics of rank ≤ 4. Some other beautiful results regarding the projective structure of $\mathcal{SU}_C(r)$ can be found in [50] and [48].

The natural map $\alpha_{\mathcal{L}} : \mathcal{SU}_C(r) \dashrightarrow |\mathcal{L}|^*$ induced by the determinant line bundle $\mathcal{L}$ on $\mathcal{SU}_C(r)$ is determined by the $r\Theta$ linear series on the Jacobian variety Jac(C). More precisely, let $\mathrm{Pic}^{g-1}(C)$ be the Picard variety of divisors of degree $g-1$ over C. For every $E \in \mathcal{SU}_C(r)$, let us define

$$\theta(E) := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C, E \otimes L) \neq 0\}.$$

If $\theta(E)$ is not equal to $\mathrm{Pic}^{g-1}(C)$, we have that $\theta(E)$ is a divisor in $\mathrm{Pic}^{g-1}(C)$ lying in the linear system $|r\Theta|$, where Θ is the canonical divisor in $\mathrm{Pic}^{g-1}(C)$.

This way, we obtain a rational map

$$\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$$

which is canonically identified to $\alpha_{\mathcal{L}}$ [10].

Let us now fix $r = 2$. In this setting, the map θ is a finite morphism [51]. When $g = 2$, the map θ is an isomorphism onto $\mathbb{P}^3$ [46]. For $g \geq 3$, the map θ is an embedding if C is non-hyperelliptic, and it is a map of degree 2 if C is hyperelliptic [23, 7, 19, 55] (see Section 3.2.1 for more details). If $g \geq 3$, the singular locus $\mathrm{Sing}(\mathcal{SU}_C(2))$ is the locus of decomposable bundles $L \oplus L^{-1}$, with $L \in \mathrm{Jac}(C)$. The map $\mathrm{Jac}(C) \to \mathcal{SU}_C(2)$ defined by $L \to L \oplus L^{-1}$ identifies the Kummer variety of $\mathrm{Jac}(C)$ with the singular locus of $\mathcal{SU}_C(2)$.

The goal of this paper is to describe the geometry associated to the map θ in the case $r = 2$ and C hyperelliptic. In the non-hyperelliptic case, the paper [1] outlines a connection between the moduli space $\mathcal{SU}_C(2)$ and the moduli space $\mathcal{M}_{0,n}$ of rational curves with n marked points. A generalization of [1] has been given in [17]. In the present work, the link with the moduli space of curves offers also a new description of the θ-map if C is hyperelliptic.

From now on, let C be a hyperelliptic curve of genus $g \geq 3$. The first result of the present work is an extension of [1, Theorem 1.1] to the hyperelliptic setting:

Proposition 3.1.1. *Let D be a general effective divisor of degree g on C. Then, there exists a surjective fibration $p_D : \mathcal{SU}_C(2) \dashrightarrow |2D| \cong \mathbb{P}^g$ whose general fiber is birational to $\mathcal{M}_{0,2g}$. More precisely, we have:*

1. *For every general divisor $N \in |2D|$, there exists a $2g$-pointed projective space $\mathbb{P}^{2g-2}_N$ and a rational dominant map $\sigma_N : \mathbb{P}^{2g-2}_N \dashrightarrow p_D^{-1}(N)$ such that the fibers of h_N are rational normal curves passing by the $2g$ marked points.*
2. *The family of rational normal curves defined by σ_N is the universal family of rational curves over an open subset of the general fiber $\mathcal{M}_{0,2g}$.*

The $2g$-pointed space $\mathbb{P}^{2g-2}_N$ appears naturally as a classifying space for certain extension classes. More precisely, consider extensions

$$(e) \quad 0 \to \mathcal{O}(-D) \to E_e \to \mathcal{O}(D) \to 0.$$

These are classified by the projective space

$$\mathbb{P}^{3g-2}_D := \mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D)) = |K + 2D|^*,$$

where K is the canonical divisor on C. Since the divisor $K+2D$ is very ample, the linear system $|K + 2D|$ embeds the curve C in $\mathbb{P}^{3g-2}_D$. The projective space $\mathbb{P}^{2g-2}_N$ is defined as the span in $\mathbb{P}^{3g-2}_D$ of the $2g$ marked points $p_1, \ldots, p_{2g}$ on C defined by the effective divisor $N \in |2D|$.

Our aim is to describe the map θ restricted to the general fibers of the fibration p_D. To this end, the following construction is crucial:

Let i be the hyperelliptic involution of C. Let $p, i(p)$ be two involution-conjugate points in C; and consider the line $l \subset \mathbb{P}_D^{3g-2}$ secant to C and passing through p and $i(p)$. We show that this line intersects the subspace $\mathbb{P}_N^{2g-2}$ in a point. Moreover, the locus $\Gamma \subset \mathbb{P}_N^{2g-2}$ of these intersections as we vary p is a rational normal curve passing through the points $p_1, \ldots, p_{2g}$. We show that the map σ_N factors by a map $h_N : \mathbb{P}_N^{2g-2} \dashrightarrow \mathcal{M}_{0,2g}^{\mathrm{GIT}}$ that contracts the rational normal curves passing by the $2g$ marked points. In particular, the map h_N contracts the curve Γ onto a point $w \in \mathcal{M}_{0,2g}$.

Notation. Let $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ denote the n-dimensional complex projective space of lines through the origin of $\mathbb{C}^{n+1}$. Throughout this paper, a *form* F of degree r on $\mathbb{P}^n$ will denote element of the vector space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \mathrm{Sym}^r(\mathbb{C}^{n+1})^*$. If we fix a basis $x_0, \ldots, x_n$ of $(\mathbb{C}^{n+1})^*$, F is simply a homogeneous polynomial of degree r on $x_0, \ldots, x_n$.

In the article [35], Kumar defines the linear system Ω of $(g-1)$-forms on $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ at $2g-1$ general points. He shows that Ω induces a birational map $i_\Omega : \mathbb{P}^{2g-3} \dashrightarrow \mathcal{M}_{0,2g}^{\mathrm{GIT}}$ onto the GIT compactification of the moduli space $\mathcal{M}_{0,2g}$. The partial linear system $\Lambda \subset \Omega$ of forms vanishing with multiplicity $g-2$ at an additional general point $e \in \mathbb{P}^{2g-3}$ induces an osculating rational projection $\kappa : \mathcal{M}_{0,2g}^{\mathrm{GIT}} \dashrightarrow |\Lambda|^*$. More precisely, the center of the projection κ is the point $w = i_\Omega(e)$. Kumar also shows that the map κ is of degree 2 for all g. We describe birationally the restrictions of θ to the fibers $p_D^{-1}(N)$ using Kumar's map:

Theorem D. *The map θ restricted to the fibers $p_D^{-1}(N)$ is Kumar's osculating projection κ centered at the point $w = h_N(\Gamma)$, up to composition with a birational map.*

Furthermore, Kumar shows that the image of κ is a connected component of the moduli space $\mathcal{SU}_{C_w}(2)$, where C_w is the hyperelliptic double cover of $\mathbb{P}^1$ ramifying over the $2g$ points defined by w. He also proves that the ramification locus of the map κ is the Kummer variety $\mathrm{Sing}(\mathcal{SU}_C(2)) = \mathrm{Kum}(C_w) \subset \mathcal{SU}_{C_w}(2)$. These results due to Kumar combined with Theorem D allow us to describe the ramification locus of the map θ:

Theorem E. *The restriction of the map θ to the general fiber of $p_{\mathbb{P}_c}$ ramifies on the Kummer variety $\mathrm{Kum}(C_w)$ of dimension $g-1$, obtained from the hyperelliptic curve C_w which is the double cover of $\mathbb{P}^1$ ramified along the $2g$ points identified by $P = h_N(\Gamma)$.*

The fundamental tools in our arguments are the classification maps

$$f_L : \mathbb{P}\,\mathrm{Ext}^1(L^{-1}, L) \dashrightarrow \mathcal{SU}_C(2),$$

where L is a line bundle (in this paper, we will use $L = \mathcal{O}(-D)$). More precisely, the map f_L associates to the equivalence class of a extension (e) the vector bundle $E_e \in \mathcal{SU}_C(2)$ sitting in the sequence. Consequently, the base locus of f_L is the locus of unstable classes in $\mathbb{P}\operatorname{Ext}^1(L^{-1}, L)$.

The use of classification maps constitutes a classical approach to the study of moduli spaces of vector bundes. For example, they have been used by Atiyah [3] to study vector bundles over elliptic curves, and by Newstead [47] to study the moduli space of rank 2 semistable vector bundles with odd determinant in the case $g = 2$. In [1] and [15], they have been used to study the moduli space $\mathcal{SU}_C(2)$ when C is a curve of genus $g \geq 2$, non-hyperelliptic if $g > 2$.

Finally, we conclude this chapter by giving account of the situation in low genus. Let $\varphi_L := \theta \circ f_L$. We make a precise examination of the base locus of the restriction map $\varphi_L|_{\mathbb{P}^{2g-2}_N}$ in genus 3, 4, 5 and 6.

Theorem F. *Let C be a hyperelliptic curve of genus 3, 4 or 5. Then, for general N, the restriction of φ_L to the subspace $\mathbb{P}^{2g-2}_N$ is exactly the composition $\kappa \circ h_N$.*

3.2 Moduli of vector bundles

We briefly recall here some results about moduli of vector bundles. For a more detailed reference, see [9].

3.2.1 Moduli of vector bundles and the map θ

Let C be a smooth genus $g \geq 2$ algebraic curve (not necessarily hyperelliptic). Let us denote by $\operatorname{Pic}^d(C)$ the Picard variety of degree d line bundles on C. The Jacobian $\operatorname{Pic}^0(C)$ of C will also be denoted by $\operatorname{Jac}(C)$. The canonical divisor $\Theta \subset \operatorname{Pic}^{g-1}(C)$ is defined set-theoretically as

$$\Theta := \{L \in \operatorname{Pic}^{g-1}(C) \mid h^0(C, L) \neq 0\}.$$

Let $\mathcal{SU}_C(2)$ be the moduli space of semistable rank 2 vector bundles on C with trivial determinant. This variety parametrizes S-equivalence classes of such vector bundles, where the S-equivalence relation is defined as follows: every strictly semistable vector bundle E admits a *Jordan-Hölder filtration*

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 = E$$

such that the quotients $E_1 = E_1/E_0$ and E_2/E_1 are stable of slope equal to the slope of E. The graded object

$$\operatorname{gr}(E) := E_1 \oplus (E_2/E_1)$$

does not depend on the choice of the Jordan-Hölder filtration. By definition, two strictly semistable vector bundles E and E' on C are S-equivalent if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$, and two stable bundles are S-equivalent if and only if they are isomorphic.

The Picard group $\mathrm{Pic}(\mathcal{SU}_C(r))$ is isomorphic to $\mathbb{Z}$, and it is generated by a line bundle $\mathcal{L}$ called the *determinant bundle* [26]. For every $E \in \mathcal{SU}_C(r)$, let us define the *theta divisor*

$$\theta(E) := \{L \in \mathrm{Pic}^{g-1}(C) \mid h^0(C, E \otimes L) \neq 0\}.$$

In the rank 2 case, $\theta(E)$ is a divisor in the linear system $|2\Theta| \cong \mathbb{P}^{2^g-1}$ (this is not true in general in higher rank), which leads to the definition of the theta map

$$\theta : \mathcal{SU}_C(2) \longrightarrow |2\Theta|.$$

If θ is a morphism, then we have that θ is finite. Indeed, since the linear system $|\mathcal{L}|$ is ample, the map θ cannot contract any curve. It is known that the map θ is a morphism if $r = 2$; $r = 3$ and $g = 2$ or 3; or $r = 3$ and C is general [51]. The map θ is not a morphism if $r \gg 0$ [51, 22], and it is generically injective for C general and $g \gg r$ [20].

Let us now fix $r = 2$. In particular, the map θ is a finite morphism. If C is not hyperelliptic, θ is known to be an embedding [19, 55]. This is also the case in genus 2, where θ is actually an isomorphism onto $\mathbb{P}^3$ [46]. If C is hyperelliptic of genus $g \geq 3$, we have that θ factors through the involution

$$E \mapsto i^*E$$

induced by the hyperelliptic involution i, embedding the quotient $\mathcal{SU}_C(2)/i^*$ into $|2\Theta|$ [23, 7].

3.2.2 The classifying maps

Let C be a smooth genus $g \geq 2$ algebraic curve (not necessarily hyperelliptic). Let D be a general degree g effective divisor on C. Let (e) be an extension of the form

$$(e) \quad 0 \to \mathcal{O}(-D) \to E_e \to \mathcal{O}(D) \to 0.$$

The classes of isomorphism of these extensions are classified by the $(3g-2)$-dimensional projective space

$$\mathbb{P}^{3g-2}_D := \mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D)) = |K + 2D|^*,$$

where K is the canonical divisor of C. The divisor $K + 2D$ is very ample and embeds C as a degree $4g - 2$ curve in $\mathbb{P}^{3g-2}_D$. Let us define the rational surjective *extension map*

$$f_D : \mathbb{P}^{3g-2}_D \dashrightarrow \mathcal{SU}_C(2)$$

which sends the extension class (e) to the vector bundle E_e. The composition map

$$\varphi_D := \theta \circ f_D : \mathbb{P}_D^{3g-2} \dashrightarrow |2\Theta|$$

has been described by Bertram in [11].

Let us denote by $\mathrm{Sec}^n(C)$ the variety of $(n+1)$-secant n-planes on C. Notice that $\mathrm{Sec}^n(C) \subset \mathrm{Sec}^{n+1}(C)$. Moreover, we have that the singular locus of $\mathrm{Sec}^{n+1}(C)$ is the secant variety $\mathrm{Sec}^n(C)$ for every n. Proposition 1.1 of [37] implies:

Proposition 3.2.1. *Let C be a smooth genus $g \geq 2$ curve, and D a degree g effective divisor on C. Then, we have:*
— *The bundle E_e is not semistable if and only if $e \in \mathrm{Sec}^{g-2}(C)$.*
— *The bundle E_e is not stable if and only if $e \in \mathrm{Sec}^{g-1}(C)$.*

In particular, the base locus of the map φ_D is precisely the secant variety $\mathrm{Sec}^{g-2}(C)$. Theorem 2 of [11] gives an isomorphism

$$H^0(\mathcal{SU}_C(2), \mathcal{L}) \cong H^0(\mathbb{P}_D^{3g-2}, \mathcal{I}_C^{g-1}(g)), \tag{3.1}$$

where $\mathcal{I}_C$ is the ideal sheaf of C. This isomorphism yields:

Theorem 3.2.2 (Bertram [11])**.** *The map φ_D is given by the linear system $|\mathcal{I}_C^{g-1}(g)|$ of forms of degree g vanishing with multiplicity at least $g-1$ on C.*

The relation between the linear system $|\mathcal{I}_C^{g-1}(g)|$ and the secant variety $\mathrm{Sec}^{g-2}(C)$ is as follows:

Proposition 3.2.3 ([1, Lemma 2.5])**.** *The linear systems $|\mathcal{I}_C^{g-1}(g)|$ and $|\mathcal{I}_{\mathrm{Sec}^{g-2}(C)}(g)|$ on $\mathbb{P}_D^{3g-2}$ are the same.*

Proof. We reproduce here the proof for the reader's convenience. The elements of both linear systems can be seen as symmetric g-linear forms on the vector space $H^0(C, K+2D)^*$. Let F, G be such forms. Then, F belongs to $|\mathcal{I}_C^{g-1}(g)|$ (resp. G belongs to $|\mathcal{I}_{\mathrm{Sec}^{g-2}(C)}(g)|$) if and only if

$$F(p_1, \ldots, p_g) = 0 \quad \text{for all } p_k \in C \text{ such that } p_i = p_j \text{ for some } 1 \leq i, j \leq g$$

$$G(p, \ldots, p) = 0 \quad \text{for any linear combination } p = \sum_{k=1}^{g-1} \lambda_i p_i, \text{ where } p_i \in C.$$

One can show that these conditions are equivalent by exhibiting appropriate choices of λ_i. □

3.2.3 The exceptional fibers of f_D

Let C be a smooth genus $g \geq 2$ algebraic curve (not necessarily hyperelliptic). Let D be a general degree g effective divisor on C. Since $\dim \mathcal{SU}_C(2) = 3g-3$, the general fiber of the map f_D has dimension one. The set of stable bundles for which $\dim(f_D^{-1}(E)) > 1$ is a proper subset of $\mathcal{SU}_C(2)$. In order to study this subset, and following [1], we introduce the "Serre dual" divisor

$$B := K - D$$

with $\deg(B) = g-2$. As in the previous paragraphs, the isomorphism classes of extensions

$$0 \to \mathcal{O}(-B) \to E \to \mathcal{O}(B) \to 0$$

are classified by the projective space

$$\mathbb{P}_B^{3g-6} := \mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(B), \mathcal{O}(-B)) = |K+2B|^*.$$

We also have the rational classifying map $f_B : \mathbb{P}_B^{3g-6} \dashrightarrow \mathcal{SU}_C(2)$ defined in the same way as f_D.

Proposition 3.2.4. *Let $E \in \mathcal{SU}_C(2)$ be a stable bundle. Then*

$$\dim(f_D^{-1}(E)) \geq 2 \qquad \text{if and only if} \qquad E \in \overline{f_B(\mathbb{P}_B^{3g-6})}.$$

Proof. Let E be a stable bundle. Then, by Riemann-Roch and Serre duality theorems, the dimension of $f_D^{-1}(E)$ is given by

$$\begin{aligned} h^0(C, E \otimes \mathcal{O}(D)) &= h^0(C, E \otimes \mathcal{O}(B)) + 2g - 2(g-1) \\ &= h^0(C, E \otimes \mathcal{O}(B)) + 2 \end{aligned}$$

Thus, $\dim(f_D^{-1}(E)) \geq 2$ if and only if there exists a non-zero sheaf morphism $\mathcal{O}(-B) \to E$. This is equivalent to $E \in \overline{f_B(\mathbb{P}_B^{3g-6})}$. □

If $g > 2$, the divisor $|K+2B|$ embeds C as a degree $4g-6$ curve in $\mathbb{P}_B^{3g-6}$ (recall that $\mathbb{P}\,\mathrm{Ext}^1(\mathcal{O}(B), \mathcal{O}(-B)) = |K+2B|^*$). Again by Theorem 3.2.2, the map φ_B is given by the linear system $|\mathcal{I}_C^{g-3}(g-2)|$. Moreover, Pareschi and Popa [49, Theorem 4.1] proved that this linear system has projective dimension $\left(\sum_{i=0}^{g-2} \binom{g}{i}\right) - 1$.

Let us denote by $\mathbb{P}_c$ the linear span of $\theta(\overline{f_B(\mathbb{P}_B^{3g-6})})$ in $|2\Theta|$. Since the map θ is finite, $\mathbb{P}_c$ has projective dimension $\left[\sum_{i=0}^{g-2} \binom{g}{i}\right] - 1$, and Proposition 3.2.4 also applies to φ_D: the fibers of φ_D with dimension ≥ 2 are those over $\mathbb{P}_c$.

3.3 A linear projection in $|2\Theta|$

Let C be a smooth genus $g \geq 3$ algebraic curve (not necessarily hyperelliptic). Let D be a general degree g effective divisor on C. In this Section, we describe the projection with center $\mathbb{P}_c$, seen as a linear subspace of $|2\Theta|$.

Let $p_{\mathbb{P}_c}$ be the linear projection in $|2\Theta|$ with center $\mathbb{P}_c$. Recall that $\dim \mathbb{P}_c = \left[\sum_{i=0}^{g-2} \binom{g}{i}\right] - 1$. We can check that the supplementary linear subspaces of $\mathbb{P}_c$ in $|2\Theta|$ are of dimension g. Thus, the image of $p_{\mathbb{P}_c}$ is a g-dimensional projective space. Let us write

$$\mathcal{SU}_C^{gs}(2) := \mathcal{SU}_C(2) \setminus (\mathrm{Kum}(C) \cup \overline{\varphi_D(\mathbb{P}_B^{3g-6})})$$

where gs stands for *general stable* bundles. Recall that the space $H^0(C, E \otimes \mathcal{O}(D))$ has dimension 2 for $E \in \mathcal{SU}_C^{gs}(2)$. Consequently, we can pick two sections s_1 and s_2 that constitute a basis for this space.

Theorem 3.3.1. *The image of the projection $p_{\mathbb{P}_c}$ can be identified with the linear system $|2D|$ on C, in such a way that the restriction of the projection $p_{\mathbb{P}_c}$ to $\theta(\mathcal{SU}_C^{gs}(2))$ coincides with the map*

$$\begin{aligned} \theta(\mathcal{SU}_C^{gs}(2)) &\to |2D| \\ \theta(E) &\mapsto \mathrm{Zeroes}(s_1 \wedge s_2) \end{aligned}$$

Proof. This result was proved in [1] for C non hyperelliptic, but the proof extends harmlessly to the hyperelliptic case. The Picard variety $\mathrm{Pic}^{g-1}(C)$ contains a model $\widetilde{C}$ of C, made up by line bundles of type $\mathcal{O}(B+p)$, with $p \in C$. The span of $\widetilde{C}$ inside $|2\Theta|^*$ corresponds to the complete linear system $|2D|^*$. Moreover, the linear span of $\widetilde{C}$ is the annihilator of $\mathbb{P}_c$. In particular, the projection $p_{\mathbb{P}_c}|_{\theta(\mathcal{SU}_C^{gs}(2))}$ determines a hyperplane in the annihilator of $\mathbb{P}_c$, which is a point in $|2D|$. We have that this projection is given as

$$\begin{aligned} p_{\mathbb{P}_c}|_{\theta(\mathcal{SU}_C^{gs}(2))} : \theta(\mathcal{SU}_C^{gs}(2)) &\to |2D| \\ \theta(E) &\mapsto \Delta(E), \end{aligned}$$

where $\Delta(E)$ is the divisor defined by

$$\Delta(E) := \{p \in C \mid h^0(C, E \otimes \mathcal{O}(B+p)) \neq 0\}. \tag{3.2}$$

Equivalently, we have that $\Delta(E) = \theta(E) \cap \widetilde{C}$. Since $\theta(E) = \theta(i^*E)$, we directly obtain from this equation that $\Delta(E) = \Delta(i^*E)$. Finally, an easy Riemann-Roch argument shows that $\Delta(E)$ is the divisor of zeroes of $s_1 \wedge s_2$. $\square$

Recall that the linear system $|K+2D|$ embeds the curve C in the projective space $\mathbb{P}_D^{3g-2}$. Let $N \in |2D|$ and consider the linear span $\langle N \rangle \subset \mathbb{P}_D^{3g-2}$.

The annihilator of $\langle N \rangle$ is the vector space $H^0(C, 2D+K-N)$, which has dimension g. In particular, the linear span $\langle N \rangle$ has dimension $(3g-2)-g = 2g-2$. Let us write

$$\mathbb{P}_N^{2g-2} := \langle N \rangle \subset \mathbb{P}_D^{3g-2}.$$

We will study the classifying map φ_D by means of the restriction maps $\varphi_D|_{\mathbb{P}_N^{2g-2}}$ when N vary in the linear system $|2D|$.

Notation. For simplicity, let us write $\varphi_{D,N}$ for the restriction map $\varphi_D|_{\mathbb{P}_N^{2g-2}}$.

Proposition 3.3.2. *Let N in $|2D|$ be a general divisor on $C \subset \mathbb{P}_D^{3g-2}$. Then, the image of*

$$\varphi_{D,N} : \mathbb{P}_N^{2g-2} \dashrightarrow \theta(\mathcal{SU}_C(2))$$

is the closure in $\theta(\mathcal{SU}_C(2))$ of the fiber over $N \in |2D|$ of the projection $p_{\mathbb{P}_c}$.

Proof. Let $(e) \in \mathbb{P}_D^{3g-2}$ be an extension

$$(e) \quad 0 \to \mathcal{O}(-D) \xrightarrow{i_e} E_e \xrightarrow{\pi_e} \mathcal{O}(D) \to 0.$$

By [37, Proposition 1.1], we have that $e \in \mathbb{P}_N^{2g-2}$ if and only if there exists a section

$$\alpha \in H^0(C, \mathrm{Hom}(\mathcal{O}(-D), E))$$

such that $\mathrm{Zeroes}(\pi_e \circ \alpha) = N$. This means that α and i_e are two independent sections of $E_e \otimes \mathcal{O}(D)$ with $\mathrm{Zeroes}(\alpha \wedge i_e) = N$. Consequently, $\theta(E_e) = \varphi_{D,N}(e)$ is projected by $p_{\mathbb{P}_c}$ on $N \in |2D|$ by Theorem 3.3.1. Hence, the image of $\varphi_{D,N}$ is contained in $\overline{p_{\mathbb{P}_c}^{-1}(N)}$.

Conversely, by the proof of Theorem 3.3.1, we have that for every bundle $E \in \mathcal{SU}_C^{gs}(2)$, $\theta(E)$ is projected by $p_{\mathbb{P}_c}$ to a divisor $\Delta(E) \in |2D|$. The argument used above implies that the fiber $\varphi_D^{-1}(\theta(E)) = f_D^{-1}(E)$ of such a bundle is contained in $\mathbb{P}_{\Delta(E)}^{2g-2}$. Consequently, the fiber of a general divisor $N \in |2D|$ by $p_{\mathbb{P}_c}$ is contained in the image of $\varphi_{D,N}$. □

3.4 The restriction map

Let C be a smooth genus $g \geq 3$ curve (not necessarily hyperelliptic). Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$. Consider the span $\mathbb{P}_N^{2g-2}$ in $\mathbb{P}_D^{3g-2}$ of the $2g$ marked points $p_1, \ldots, p_{2g}$. In this Section, we will recall results about the restriction map

$$\varphi_{D,N} = \varphi_D|_{\mathbb{P}_N^{2g-2}} : \mathbb{P}_N^{2g-2} \dashrightarrow \mathcal{SU}_C(2).$$

These results can be found in [16] and [1].

3.4.1 Linear systems in $\mathbb{P}_N^{2g-2}$

Recall that the secant variety $\mathrm{Sec}^{g-2}(C)$ is the base locus for φ_D (see Theorem 3.2.2 and Proposition 3.2.3). Therefore, it is also natural to define the following secant varieties in $\mathbb{P}_N^{2g-2}$:

$$\mathrm{Sec}^N := \mathrm{Sec}^{g-2}(C) \cap \mathbb{P}_N^{2g-2} \ ,$$
$$\mathrm{Sec}^{g-2}(N) := \bigcup_{\substack{M \subset N \\ \# M = g-1}} \mathrm{span}\{M\} \ .$$

Note that, since the points of N are already in $\mathbb{P}_N^{2g-2}$, we have the inclusion $\mathrm{Sec}^{g-2}(N) \subset \mathrm{Sec}^N$.

Let us also define the associated linear systems in $\mathbb{P}_N^{2g-2}$

$$\mathcal{S} = |\mathcal{I}_{\mathrm{Sec}^N}(g)| \ , \quad \mathcal{T} = |\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$$

of forms of degree g vanishing in the corresponding secant varieties. The previous inclusion of secant varieties implies that $\mathcal{S}$ is a linear subsystem of $\mathcal{T}$.

Lemma 3.4.1. *The restriction map $\varphi_{D,N}$ is given by a linear subsystem $\mathcal{R}$ of $\mathcal{S}$.*

Proof. This is a direct consequence of Theorem 3.2.2 and Proposition 3.2.3. □

3.4.2 Moduli spaces of pointed rational curves

We will outline in this Section the relation between the restriction map $\varphi_{D,N}$ and the moduli spaces of pointed rational curves. Let us start by recalling some facts about these moduli spaces. For a more complete discussion, we refer to [16].

Let $n \geq 3$. We will say that n points $e_1, \ldots, e_n \in \mathbb{P}^{n-2}$ *are in general position* if no $n-2$ of them lie in a hyperplane.

Two compactifications of $\mathcal{M}_{0,n}$

The moduli space $\mathcal{M}_{0,n}$ of ordered configurations of n disctinct points on the projective line is not compact, since it does not include the limit configurations where two disctinct points «collide». We will consider the two main compactifications of $\mathcal{M}_{0,n}$. The first one is the GIT quotient

$$\mathcal{M}_{0,n}^{\mathrm{GIT}} := (\mathbb{P}^1)^n // \mathrm{PGL}(2, \mathbb{C})$$

of $(\mathbb{P}^1)^n$ by the diagonal action of $G = \mathrm{PGL}(2, \mathbb{C})$ for the natural G-linearization of the line bundle $L = \mathcal{O}_{\mathbb{P}^1}(1)^{\otimes n}$ (see [24]). Moreover, the quotient $\mathcal{M}_{0,n}^{\mathrm{GIT}}$

is naturally embedded in the projective space $\mathbb{P}(H^0((\mathbb{P}^1)^n, L)^G)$ of invariant sections. In this compactification, if n is even, up to $n/2$ points are allowed to coincide.

The second one is the Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$, constructed by Knudsen [34]. The points of $\overline{\mathcal{M}}_{0,n}$ are isomorphism classes of *stable curves*. A stable curve is a connected (possibly reducible) curve C together with n disctinct points $x_1, \ldots, x_n \in C \setminus \mathrm{Sing}(C)$ such that the following conditions are satisfied:

1. The curve C has only double points, and every irreducible component of C is isomorphic to $\mathbb{P}^1$.
2. The arithmetic genus of C is equal to 0.
3. On each component of C there are at least three points wich are either marked or double.

Limit configurations in $\overline{\mathcal{M}}_{0,n}$ arise as follows: when two different marked smooth points x and y in C merge at a smooth point $p \in C$, the limit in $\overline{\mathcal{M}}_{0,n}$ is a curve with a new component glued at p, and x y will be two distinct points of this component. More details on these constructions can be found in [32].

Both $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$ and $\overline{\mathcal{M}}_{0,n}$ contain $\mathcal{M}_{0,n}$ as an open set. However, the Mumford-Knudsen compactification is somehow finer: there exists a *contraction* dominant morphism

$$c_n : \overline{\mathcal{M}}_{0,n} \to \mathcal{M}^{\mathrm{GIT}}_{0,n}$$

contracting some components of the boundary of $\overline{\mathcal{M}}_{0,n}$, that restricts to the identity on $\mathcal{M}_{0,n}$ [4].

Knudsen also proves [34] that there exists n forgetful morphisms

$$\zeta_k : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1}$$

for $k = 1, \ldots, n$. The morphism ζ_k *forgets* the labelling of the k-th point. Furthermore, the morphism $\zeta_k : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1}$ is the universal curve over $\overline{\mathcal{M}}_{0,n-1}$ for every k.

A variety of rational normal curves

Let $e_1, \ldots e_n \in \mathbb{P}^{n-2}$ be n points in general position. Let $\mathcal{H}$ be the Hilbert scheme of subschemes of $\mathbb{P}^{n-2}$. Let $V_0(e_1, \ldots, e_n) \subset \mathcal{H}$ be the subvariety of rational normal curves in $\mathbb{P}^{n-2}$ passing through the points $e_1, \ldots, e_n$, and let $V(e_1, \ldots, e_n)$ be the closure of $V_0(e_1, \ldots, e_n)$ inside the Hilbert scheme of subschemes of $\mathbb{P}^{n-2}$. The boundary $V(e_1, \ldots, e_n) \setminus V_0(e_1, \ldots, e_n)$ consists on reducible rational normal curves, i.e. reducible non-degenerate curves of degree n such that each component is a rational normal curve in its projective span.

Theorem 3.4.2 (Kapranov [32]). *Let $e_1, \dots e_n \in \mathbb{P}^{n-2}$ be n points in general position. Then, there exists an isomorphism $V_0(e_1, \dots, e_n) \cong \mathcal{M}_{0,n}$. Moreover, this isomorphism extends to an isomorphism between $V(e_1, \dots, e_n)$ and $\overline{\mathcal{M}}_{0,n}$.*

The first part of this theorem is classical (see, for example, [24]). The isomorphism $V_0(e_1, \dots, e_n) \cong \mathcal{M}_{0,n}$ associates to a rational normal curve passing by $e_1, \dots, e_n$ the corresponding ordered configuration of n points in $\mathbb{P}^1$. The extension to $V(e_1, \dots, e_n) \cong \overline{\mathcal{M}}_{0,n}$ is due to Kapranov [32].

The blow-up construction

The following construction is due to Kapranov [31]: let $e_1, \dots, e_{n-1} \in \mathbb{P}^{n-3}$ be $(n-1)$ points in general position. Consider the following sequence of blow-ups:

1. Blow-up the points $e_1, \dots, e_{n-1}$.
2. Blow-up the proper transforms of lines spanned by pairs of points in $\{e_1, \dots, e_{n-1}\}$.
3. Blow-up the proper transforms of planes spanned by triples of points in $\{e_1, \dots, e_{n-1}\}$.

$\vdots$

$(n-4)$. Blow-up the proper transforms of linear subspaces spanned by $(n-4)$-ples of points in $\{e_1, \dots, e_{n-1}\}$.

Let $\mathrm{bl}(\mathbb{P}^{n-3})$ be the $(n-3)$-variety obtained in this way, and $b : \mathrm{bl}(\mathbb{P}^{n-3}) \to \mathbb{P}^{n-3}$ the composition of this sequence of blow-ups. We will call this map the *Kapranov blow-up map centered in the points* $e_1, \dots, e_{n-1}$.

Theorem 3.4.3 (Kapranov [31]). *Let $n \geq 4$. Then, the moduli space $\overline{\mathcal{M}}_{0,n}$ is isomorphic to* $\mathrm{bl}(\mathbb{P}^{n-3})$.

Moreover, the images by b of the fibres of the map ζ_k over the points in the open set $\mathcal{M}_{0,n-1} \subset \overline{\mathcal{M}}_{0,n-1}$ are the rational normal curves in $\mathbb{P}^{n-3}$ passing through the $n-1$ points $e_1, \dots, e_{n-1}$ (see [33], Proposition 3.1).

The Cremona inversion

Let $e_1, \dots e_{n-1} \in \mathbb{P}^{n-3}$ in general position. Without loss of generality, we may assume $e_k = [0 : \dots : 1 : \dots : 0]$ for $k = 1, \dots, n-2$; and $e_{n-1} = [1 : \dots : 1]$. The *Cremona inversion with respect to* $e_1, \dots, e_{n-1}$ is the birational involutive map

$$\begin{aligned} \mathrm{Cr}_{n-1} : \mathbb{P}^{n-3} &\dashrightarrow \mathbb{P}^{n-3} \\ [x_0 : \ldots : x_{n-3}] &\mapsto [1/x_0 : \ldots : 1/x_{n-3}] \,. \end{aligned}$$

By abuse of notation, we will use only the index $n-1$ to identify this map. The Cremona inversion has the following property: any non-degenerate rational normal curve passing through the points $e_1, \dots, e_{n-1}$ is transformed into a line passing by the point $\mathrm{Cr}_{n-1}(e_{n-1})$. Let $\tau_{n-1} : \mathbb{P}^{n-2} \dashrightarrow \mathbb{P}^{n-3}$ be the linear projection with center $\mathrm{Cr}_{n-1}(e_{n-1})$. From the previous property, we obtain that the composition $\tau_{n-1} \circ \mathrm{Cr}_{n-1}$ contracts non-degenerate rational normal curves passing through $e_1, \dots, e_{n-1}$.

Let $k \in \{1, \dots, n\}$. By making e_k play the role of e_{n-1} in the definition of Cr_{n-1}, we define similarly the Cremona inversion Cr_k. Let $\tau_k : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-4}$ be the linear projection with center $\mathrm{Cr}_k(e_k)$. By the property given in the previous paragraph, we conclude that he map Cr_k satisfies the following Lemma:

Lemma 3.4.4. *Let $e_1, \dots e_{n-1} \in \mathbb{P}^{n-3}$ in general position. Then, the composition $\tau_k \circ \mathrm{Cr}_k$ contracts the non-degenerate rational normal curves passing through $e_1, \dots, e_n$.*

For every $k \in \{1, \dots, n-1\}$, we define

$$H_k = \langle e_1, \dots, \widehat{e_k}, \dots, e_{n-2} \rangle \subset \mathbb{P}^{n-3}$$

as the hyperplane of $\mathbb{P}^{n-3}$ spanned by all the points e_i excluding e_k and e_{n-1}. These hyperplanes are contracted to a point by the map Cr_n, and the same results holds for the composition $\tau_n \circ \mathrm{Cr}_n$. Let $\widetilde{e_k}$ be the image of H_k by the rational map $\tau_n \circ \mathrm{Cr}_n$ for every $k \in \{1, \dots, n-1\}$.

Theorem 3.4.2 and the fact that $\overline{\mathcal{M}}_{0,n}$ is the universal curve over $\overline{\mathcal{M}}_{0,n-1}$ imply the following Proposition:

Proposition 3.4.5 (Kapranov [32])**.** *Let $e_1, \dots e_{n-1} \in \mathbb{P}^{n-3}$ in general position. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n} & \xrightarrow{\zeta_{n-1}} & \overline{\mathcal{M}}_{0,n-1} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b_{n-1}} \\ \mathbb{P}^{n-3} & \overset{\tau_{n-1} \circ \mathrm{Cr}_{n-1}}{\dashrightarrow} & \mathbb{P}^{n-4} \end{array}$$

Here, the map b_{n-1} is the Kapranov blow-up map centered in the points $\widetilde{e_k}$ for $k = 1, \dots, n-2$.

The Kumar birational map

Let $g \geq 3$, and let $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ in general position. Let Ω be the linear system of $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$. The following result holds:

Theorem 3.4.6 (Kumar [35]). *Let $g \geq 3$, and let $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ in general position. Then, the rational map*

$$i_\Omega : \mathbb{P}^{2g-3} \dashrightarrow \Omega^*$$

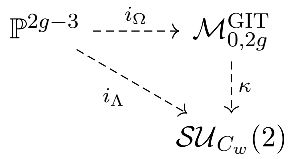
induced by the linear system Ω maps $\mathbb{P}^{2g-3}$ birationally onto $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$.

The map is well-defined even without specifying an ordering. The choice of an ordering defines an isomorphism $i_\Omega \cong \mathcal{M}_{0,2g}^{\mathrm{GIT}}$. A different ordering operates as an automorphism of $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$ (for more details, see [18]).

Let $e_0 \in \mathbb{P}^{2g-3}$ such that $w = i_\Omega(e_0)$ lies in the open set $\mathcal{M}_{0,2g} \subset \mathcal{M}_{0,2g}^{\mathrm{GIT}}$. The point w represents a hyperelliptic genus $(g-1)$ curve C_w (namely the double cover of $\mathbb{P}^1$ ramifying in the $2g$ marked points) together with an ordering of the Weierstrass points. Let $\mathcal{SU}_{C_w}(2)$ be the moduli space of rank 2 semistable vector bundles with trivial determinant over the curve C_w.

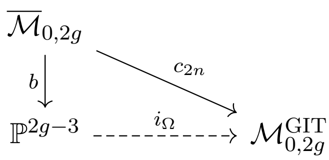
Consider the partial linear system Λ of Ω consisting of the $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in all the points $e_0, e_1, \dots, e_{2g-1}$. Let $\kappa : \mathcal{M}_{0,2g}^{\mathrm{GIT}} \dashrightarrow \Lambda^*$ be the rational projection induced by the linear system Λ.

Theorem 3.4.7 (Kumar [35]). *Let $g \geq 3$, and let $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ be $2g-1$ points in general position. Let $e_0 \in \mathbb{P}^{2g-3}$ such that $w = i_\Omega(e_0)$ lies in the open set $\mathcal{M}_{0,2g} \subset \mathcal{M}_{0,2g}^{\mathrm{GIT}}$. Then, the map i_Λ induced by the linear system Λ factors as $\kappa \circ i_\Omega$, where κ is a degree 2 map onto a connected component of the moduli space $\mathcal{SU}_{C_w}(2)$. Furthermore, the map κ ramifies along the Kummer variety* $\mathrm{Kum}(C_w) \subset \mathcal{SU}_{C_w}(2)$.



The map c_{2n} can also be described in terms of Kumar's linear system Ω:

Lemma 3.4.8 (Bolognesi [16]). *Let $g \geq 3$, and let $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ in general position. Then, the following diagram is commutative:*



Here, the map b is the Kapranov blow-up map centered in the general points $p_1, \dots, p_{2g-1}$.

3.4.3 The map h_N

Let C be a smooth genus $g \geq 3$ curve (not necessarily hyperelliptic). Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$. Consider the span $\mathbb{P}^{2g-2}_N$ in $\mathbb{P}^{3g-2}_D$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

We are now in situation to apply the discussion of Section 3.4.2 to the general points $p_1, \ldots, p_{2g}$ in the projective space $\mathbb{P}^{2g-2}_N$, taking $n = 2g+1$. Proposition 3.4.5 and Lemma 3.4.8 imply that we have a commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{0,2g+1} & \xrightarrow{\zeta_{2g-1}} & \overline{\mathcal{M}}_{0,2g} & & \\ \big\downarrow b & & \big\downarrow b_{2g-1} & \overset{c_{2g}}{\searrow} & \\ \mathbb{P}^{2g-2}_N & \overset{\tau_{2g-1} \circ \mathrm{Cr}_{2g-1}}{\dashrightarrow} & \mathbb{P}^{2g-3} & \overset{i_\Omega}{\dashrightarrow} & \mathcal{M}^{\mathrm{GIT}}_{0,2g} \end{array}$$

where Ω is the linear system of $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ at the $2g-1$ points $\tilde{e}_1, \ldots, \tilde{e}_{2g-1} \in \mathbb{P}^{2g-3}$ defined in Proposition 3.4.5.

Let us define the rational map

$$h_N : \mathbb{P}^{2g-2}_N \dashrightarrow \mathcal{S}^*$$

given by the complete linear system $\mathcal{S} = |\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$.

Proposition 3.4.9 (Bolognesi [16]). *Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$. Then, the map h_N coincides with the composition $i_\Omega \circ \tau_{2g-1} \circ \mathrm{Cr}_{2g-1}$ for every $k = 1, \ldots, 2g$.*

Let us summarize the results of Lemma 3.4.4, Theorem 3.4.6 and Proposition 3.4.9 in the following Proposition:

Proposition 3.4.10. *The image of h_N is isomorphic to the GIT moduli space $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$ of ordered configurations of $2g$ points in $\mathbb{P}^1$. The map h_N is dominant and its general fiber is of dimension 1. More precisely, h_N contracts every rational normal curve Z passing through the $2g$ points N to a point z in $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$. This point represents an ordered configuration of the $2g$ points N on the rational curve Z.*

Since $\mathcal{R}$ is a linear subsystem of $\mathcal{S}$ by Lemma 3.4.1, we have that the map $\varphi_{D,N}$ factors by h_N:

$$\begin{array}{ccc} \mathbb{P}^{2g-2}_N & \overset{h_N}{\dashrightarrow} & \mathcal{M}^{\mathrm{GIT}}_{0,2g} \\ & \overset{\varphi_{D,N}}{\searrow} & \big\downarrow \\ & & \theta(\mathcal{SU}_C(2)) \end{array} \tag{3.3}$$

The natural locus Sec^N

Let us now study the natural locus Sec^N obtained by intersecting the base locus of φ_D with $\mathbb{P}_N^{2g-2}$. By definition, the points in Sec^N are given by the intersections $\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}$, where L_{g-1} is an effective divisor of degree $g-1$ and $\langle L_{g-1}\rangle$ is its linear span in $\mathbb{P}_D^{3g-2}$. If L_{g-1} is contained in N, it is clear that $\langle L_{g-1}\rangle \subset \mathrm{Sec}^{g-2}(N) \subset \mathbb{P}_N^{2g-2}$.

Lemma 3.4.11. *Let L_{g-1} be an effective divisor on C of degree $g-1$, not contained in N. Then,*

$$\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2} \neq \phi \qquad \textit{if and only if} \qquad \dim |L_{g-1}| \geq 1.$$

Moreover, if the intersection is non-empty, we have that

$$\dim(\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}) = \dim |L_{g-1}| - 1.$$

Proof. First, let us suppose that L_{g-1} and N have no points in common. The vector space $V := H^0(C, 2D + K - L_{g-1})$ is the annihilator of the span $\langle L_{g-1}\rangle$ in $\mathbb{P}_D^{3g-2}$. By the Riemann-Roch theorem, we see that V has dimension $2g$, hence

$$\dim\langle L_{g-1}\rangle = (3g-2) - 2g = g - 2.$$

Let d be the dimension of the span $\langle L_{g-1}, N\rangle$ of the points of L_{g-1} and N. Since the dimension of $\mathbb{P}_N^{2g-2} = \langle N\rangle$ is $2g-2$, we have that $d \leq (g-2) + (2g-2) + 1 = 3g-3$, where the equality holds iff $\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}$ is empty.

In particular, this intersection is non-empty iff $d \leq 3g-4$. Since $\dim |K + 2D|^* = \dim \mathbb{P}_D^{3g-2} = 3g-2$, this is equivalent to the annihilator space

$$W := H^0(C, 2D + K - L_{g-1} - N) = H^0(C, K - L_{g-1})$$

being of dimension ≥ 2. By Riemann-Roch and Serre duality, we obtain that this condition is equivalent to $\dim |L_{g-1}| \geq 1$.

More precisely, let us suppose that $\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}$ is non-empty and let $e := \dim(\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2})$. Then, we have that

$$d = 3g - 3 - (e+1),$$

and the annihilator space W is of dimension $2+e$. Again by a Riemann-Roch computation, we conclude that $e = \dim |L_{g-1}| - 1$.

Finally, if L_{g-1} and N have some points in common, we have to count them only once in the vector space W to avoid requiring higher vanishing multiplicity to the sections.

□

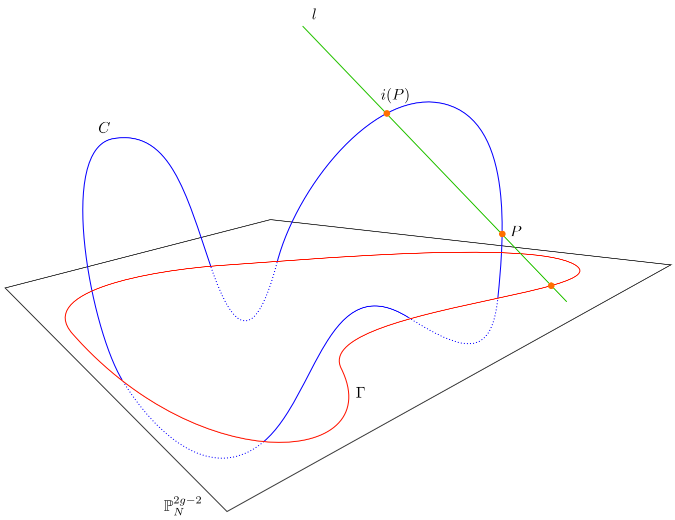


Figure 3.1 – The situation in genus 3. The curves Γ and C intersect along the divisor D, of degree 6. The secant lines l cutting out the hyperelliptic pencil define the curve Γ.

From this Lemma, we conclude that $\mathrm{Sec}^{g-2}(N)$ is a proper subset of Sec^N if and only if there exists a divisor L_{g-1} not contained in N with $\dim |L_{g-1}| \geq 1$. By the Existence Theorem of Brill-Noether theory (see [2, Theorem 1.1, page 206]) this is equivalent to $g \geq 4$ in the non-hyperelliptic case, whereas such a linear system exists also for $g = 3$ if C is hyperelliptic. We will discuss the first low genera cases for C hyperelliptic in Section 3.7.

3.5 The hyperelliptic case

From now on, C will be a *hyperelliptic* curve of genus $g \geq 3$. This situation is thus different from the non-hyperelliptic case examined in [1], and the results presented are new.

According to Proposition 3.2.1, the base locus of the map $\varphi_{D,N}$ is contained in Sec^N. We have seen that the secant variety $\mathrm{Sec}^{g-2}(N)$ is contained in Sec^N and that this inclusion is strict for $g \geq 4$ in the non-hyperelliptic case.

In the hyperelliptic case, we have an additional base locus for every $g \geq 3$, which appears due to the hyperelliptic nature of the curve. This locus arises as follows: for each pair $P = \{p, i(p)\}$ of involution-conjugate points in C, consider the hyperelliptic secant line l in $\mathbb{P}^{3g-2}_D$ passing through the points p and $i(p)$. Let Q_P be the intersection of the line l with $\mathbb{P}^{2g-2}_N$. Let us define $\Gamma \subset \mathbb{P}^{2g-2}_N$ as the locus of intersection points Q_P when we vary the pair P.

Lemma 3.5.1. *The locus $\Gamma \subset \mathbb{P}_N^{2g-2}$ is a non-degenerate rational normal curve in $\mathbb{P}_N^{2g-2}$. Moreover, Γ passes through the $2g$ points $N \subset C$.*

Proof. Let us start by showing that the intersection is non-empty for every pair $\{p, i(p)\}$. Since $\dim |p + i(p)| = \dim |h| = 1$, the intersection $l \cap \mathbb{P}_N^{2g-2}$ is non-empty by Lemma 3.4.11.

Let us show that this intersection is a point, i.e. that the line l is not contained in $\mathbb{P}_N^{2g-2}$. Recall that $\mathbb{P}_D^{3g-2} = |2D + K|^*$. If the points p and $i(p)$ are both not contained in the divisor N, the vector space

$$V := H^0(C, 2D + K - N - (p + i(p)) = H^0(C, 2D + K - N - h)$$

is exactly the annihilator of the span $\langle l, \mathbb{P}_N^{2g-2} \rangle$ in $\mathbb{P}_D^{3g-2}$. In particular, the codimension of $\langle l, \mathbb{P}_N^{2g-2} \rangle$ in $\mathbb{P}_D^{3g-2}$ is the dimension of V. By Riemann-Roch and Serre duality, we get that $\dim V = g - 2$, thus $\dim \langle l, \mathbb{P}_N^{2g-2} \rangle = 3g - 2 - (g - 2) = 2g$. This means that the intersection $l \cap \mathbb{P}_N^{2g-2}$ is a point.

For the case $p \in N$ and $i(p) \notin N$, let us remark that the the annihilator of the span $\langle l, \mathbb{P}_N^{2g-2} \rangle$ is now the vector space $H^0(C, 2D + K - N - i(p))$. Since

$$h^0(C, 2D + K - N - i(p)) < h^0(C, 2D + K - N),$$

we conclude that the line l is not contained in $\mathbb{P}_N^{2g-2}$. The case $\{p, i(p)\} \subset N$ is excluded by our genericity hypotheses on N. To summarize, we deduce that the locus Γ is a curve in $\mathbb{P}_N^{2g-2}$.

Let q be a point of N. Then, q is a point of the plane $\mathbb{P}_N^{2g-2}$. Consequently, the line passing through q and $i(q)$ intersects the plane $\mathbb{P}_N^{2g-2}$ at q. Thus, we have that Γ passes through the points of N. Moreover, it is clear that N is the only intersection of Γ and C, i.e. $\Gamma \cap C = N$.

Let us prove now that Γ is a rational normal curve. Since Γ is defined by the hyperelliptic pencil, it is clear that Γ is rational. Moreover, since the divisor D is general, the span of any subset of $2g - 1$ points of D is $\mathbb{P}_N^{2g-2}$. Thus, it suffices to show that the degree of $\Gamma \subset \mathbb{P}_N^{2g-2}$ is precisely $2g - 2$. Let us write

$$N = q_1 + \cdots + q_{2g}$$

with $q_1, \ldots, q_{2g} \in C$. By the previous paragraph, Γ passes by these $2g$ points. Let us consider a hyperplane H of $\mathbb{P}_N^{2g-2}$ spanned by $2g - 2$ points of N. Without loss of generality, we can suppose that these points are the first $2g - 2$ points $q_1, \ldots, q_{2g-2}$. To show that the degree of Γ is $2g - 2$, we have to show that the intersection of Γ with H consists exactly of the points $q_1, \ldots, q_{2g-2}$.

The intersection $l \cap H$ is empty if and only if the linear span $\langle l, H \rangle$ of l and H in $\mathbb{P}_D^{3g-2}$ is of maximal dimension $2g - 1$, i.e. of codimension $g - 1$ in $\mathbb{P}_D^{3g-2}$. Consider the divisors

$$D_H = q_1 + \cdots + q_{2g-2} \qquad \text{and} \qquad D_l = q + i(q) \ .$$

As before, if $\{p, i(p)\} \cap \{q_1, \ldots, q_{2g-2}\}$ is empty, the vector space $W = H^0(C, 2D + K - D_H - D_l)$ is the annihilator of the span $\langle l, H\rangle$ in $\mathbb{P}_D^{3g-2}$. In particular, the codimension of $\langle l, H\rangle$ in $\mathbb{P}_D^{3g-2}$ is given by the dimension of W. Again by Riemann-Roch and Serre duality theorems, we can check that

$$\dim W = h^0(C, -2D + D_H + D_l) + g - 1.$$

Thus, the codimension of $\langle l, H\rangle$ in $\mathbb{P}_D^{3g-2}$ is greater than $g - 1$ if and only if $h^0(C, -2D + D_H + D_l) > 0$. Since $\deg(-2D + D_H + D_l) = 0$, this is equivalent to $-2D + D_H + D_l \sim 0$. Since $N = q_1 + \cdots + q_{2g} \sim 2D$, we have that

$$\begin{aligned} -2D + D_H + D_l \sim 0 &\iff p + i(p) \sim q_{2g-1} + q_{2g} \\ &\iff h \sim q_{2g-1} + q_{2g} \\ &\iff i(q_{2g-1}) = q_{2g}. \end{aligned}$$

By our genericity hypothesis on N, the last condition is not satisfied. Consequently, we conclude that the line l intersects the hyperplane H iff $\{p, i(p)\} \cap \{q_1, \ldots, q_{2g-2}\}$ is non-empty, i.e. iff p or $i(p)$ is one of the q_k for $k = 1, \ldots, 2g - 2$. In particular,

$$\Gamma \cap H = \{q_1, \ldots, q_{2g-2}\}$$

as we wanted to show. □

Hence, the curve Γ is contracted by the map h_N to a point $w \in \mathcal{M}_{0,2g}^{\mathrm{GIT}}$ by Proposition 3.4.10. The point w represents a hyperelliptic curve C_w of genus $g - 1$ together with an ordering of the Weierstrass points N on the rational curve Γ.

3.5.1 The classifying map φ_D restricted to $\mathbb{P}_N^{2g-2}$

Let C be a hyperelliptic genus $g \geq 3$ curve. Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$, and consider the span $\mathbb{P}_N^{2g-2}$ in $\mathbb{P}_D^{3g-2}$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

In this Section, we describe birationally the restrictions of θ to the fibers $p_D^{-1}(N)$ by means of the maps presented in Section 3.4. This restriction happens to have a strikingly simple description in terms of basic projective geometry of the GIT quotients.

The global factorization

Recall that the base locus of the map φ_D is the secant variety $\mathrm{Sec}^{g-2}(C)$ by Proposition 3.2.1. In [11], the author constructs the resolution $\widetilde{\varphi_D}$ of the map φ_D as a sequence of blow-ups

$$\begin{array}{ccc}
\widetilde{\mathbb{P}_D^{3g-2}} & & \\
{\scriptstyle \mathrm{bl}_{g-1}}\downarrow & \searrow{\scriptstyle \widetilde{\varphi_D}} & \\
\vdots & & \\
{\scriptstyle \mathrm{bl}_1}\downarrow & & \\
\mathbb{P}_D^{3g-2} & \overset{\varphi_D}{\dashrightarrow} & |2\Theta|
\end{array}$$

along the secant varieties

$$C = \mathrm{Sec}^0(C) \subset \mathrm{Sec}^1(C) \subset \cdots \subset \mathrm{Sec}^{g-1}(C) \subset \mathbb{P}_D^{3g-2}.$$

This chain of morphisms is defined inductively as follows: the center of the first blow-up bl_1 is the curve $C = \mathrm{Sec}^0(C)$. For $k = 2, \ldots, g-1$, the center of the blow-up bl_k is the strict transform of the secant variety $\mathrm{Sec}^{k-1}(C)$ under the blow-up bl_{k-1}.

The map φ_D is, by definition, the composition of the classifying map f_D defined in Section 3.2.2 and the degree 2 map θ. Thus, the map f_D lifts to a map $\widetilde{f_D}$ which makes the following diagram commute:

$$\begin{array}{ccc}
\widetilde{\mathbb{P}_D^{3g-2}} & \xrightarrow{\widetilde{f_D}} & \mathcal{SU}_C(2) \\
& {\scriptstyle \widetilde{\varphi_D}}\searrow & \downarrow{\scriptstyle \theta} \\
& & |2\Theta|
\end{array} \tag{3.4}$$

Osculating projections

We recall here a generalization of linear projections. For a more complete reference, see for example [44]. Let $X \subset \mathbb{P}^N$ be an integral projective variety of dimension n, and $p \in X$ a smooth point. Let

$$\begin{aligned}
\phi : \mathcal{U} \subset \mathbb{C}^n &\longrightarrow \mathbb{C}^N \\
(t_1, \ldots, t_n) &\longmapsto \phi(t_1, \ldots, t_n)
\end{aligned}$$

be a local parametrization of X in a neighborhood of $p = \phi(0) \in X$. For $m \geq 0$, let O_p^m be the affine subspace of $\mathbb{C}^N$ passing through $p \in X$ and generated by the vectors $\phi_I(0)$, where ϕ_I is a partial derivative of ϕ of order $\leq m$.

By definition, the *m-osculating space* $T_p^m X$ of X at p is the projective closure in $\mathbb{P}^N$ of O_p^m. The *m-osculating projection*

$$\Pi_p^m : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_m}$$

is the corresponding linear projection with center T_p^m.

3.5.2 Further base locus of $\varphi_{D,N}$ and an osculating projection

Let C be a hyperelliptic genus $g \geq 3$ curve. Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$, and consider the span $\mathbb{P}_N^{2g-2}$ in $\mathbb{P}_D^{3g-2}$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

We define the *further base locus* of $\varphi_{D,N}$ as the set

$$\mathrm{Sec}^{N'} := \mathrm{Sec}^N \setminus \{\Gamma \cup \mathrm{Sec}^{g-2}(N)\}.$$

This locus is non-empty for $g \geq 4$ due to the existence of effective divisors L_{g-1} in the conditions of Lemma 3.4.11, as we will see in Section 3.7.

Lemma 3.5.2. *Let Q be a r-form in $\mathbb{P}^n$ vanishing at the points P_1 and P_2 with multiplicity l_1 and l_2 respectively. Then, Q vanishes on the line passing through P_1 and P_2 with multiplicity at least $l_1 + l_2 - r$.*

Proof. See, for example, [36, page 2]. □

Let $\mathcal{S} = |\mathcal{I}_{\mathrm{Sec}^N}(g)|$ be the natural linear system associated to $\varphi_{D,N}$ (see Section 3.4). The forms in $\mathcal{S}$ vanish with multiplicity $g-1$ along the points of C (see Lemma 3.2.3). By Lemma 3.5.2, these forms vanish then with multiplicity $(g-1)+(g-1)-g = g-2$ along the secant lines l cutting out the hyperelliptic pencil. Thus, these forms vanish with multiplicity $g-2$ on the curve Γ.

Let us also consider the linear system $\mathcal{T} = |\mathcal{I}_{\mathrm{Sec}^{g-2}(N)}(g)|$. Let $\mathcal{T}(\Gamma) \subset \mathcal{T}$ be the partial linear system of forms vanishing (with multiplicity 1) along $\mathrm{Sec}^{g-2}(N)$, and with multiplicity $g-2$ in Γ. By our previous observation, we have the following inclusions of linear systems:

$$\mathcal{S} \subset \mathcal{T}(\Gamma) \subset \mathcal{T}.$$

These inclusions yield a factorization

$$\begin{array}{ccccc} \mathbb{P}_N^{2g-2} & \overset{h_N}{\dashrightarrow} & \mathcal{M}_{0,2g}^{\mathrm{GIT}} \subset |\mathcal{T}|^* & \overset{\pi_N}{\dashrightarrow} & |\mathcal{T}(\Gamma)|^* \\ & & & \searrow^{\varphi_{D,N}} & \downarrow l_N \\ & & & & \theta(\mathcal{SU}_C(2)) \end{array} \tag{3.5}$$

The first map h_N is the one defined in Section 3.4.3, its image is the moduli space $\mathcal{M}_{0,2g}^{\mathrm{GIT}}$. According to Proposition 3.4.10, this map contracts the curve Γ to a point $h_N(\Gamma)$.

Proposition 3.5.3. *The map π_N is the $(g-3)$-osculating projection Π_P^{g-3} with center the point $w = h_N(\Gamma)$.*

Proof. From the definition of the linear systems $\mathcal{T}(\Gamma)$ and $\mathcal{S}$, the base locus of the map π_N is the point $w = h_N(\Gamma)$. In particular, the map π_N is an osculating projection of some order with respect to this point. Since the forms in $\mathcal{T}(\Gamma)$ vanish with multiplicity $g-2$ along Γ, the order the projection π_N is $g-3$. □

We will show in the next Section that the map l_N is actually birational, and that the map π_N coincides with the restriction of the map θ.

3.5.3 The Kumar factorization

Let C be a hyperelliptic genus $g \geq 3$ curve. Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$, and consider the span $\mathbb{P}^{2g-2}_N$ in $\mathbb{P}^{3g-2}_D$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

According to Proposition 3.4.10, the map h_N contracts the curve Γ to a point w in $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$ representing an ordered configuration of the $2g$ marked points N. This point in turn corresponds to a hyperelliptic genus $(g-1)$ curve C_w together with an ordering of the Weierstrass points.

Let Ω be the linear system of $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ at the $2g-1$ points

$$e_1 = \tau_{2g} \circ \mathrm{Cr}_{2g}(p_1),\ \ldots\ , e_{2g-1} = \tau_{2g} \circ \mathrm{Cr}_{2g}(p_{2g-1})$$

in $\mathbb{P}^{2g-3}$. Theorem 3.4.6 states that the map i_Ω induced by Ω maps birationally $\mathbb{P}^{2g-3}$ onto the GIT quotient $\mathcal{M}^{\mathrm{GIT}}_{0,2g}$. Since Γ is non degenerate, there exists $e_0 \in \mathbb{P}^{2g-3}$ such that $w = i_\Omega(e_0)$. Let Λ be the partial linear system of Ω consisting of the $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in all the points $e_0, e_1, \ldots, e_{2g-1}$. Let $\kappa : \mathcal{M}^{\mathrm{GIT}}_{0,2g} \dashrightarrow \Lambda^*$ be the rational projection induced by the linear system Λ.

Theorem 3.5.4. *The map π_N coincides with the Kumar map κ. In particular, the map π_N is of degree 2.*

Proof. Consider the moduli space $\mathcal{V} = \mathcal{M}^{\mathrm{GIT}}_{0,2g}$ in its natural embedding. The osculating projection π_N is given by the linear system $|\mathcal{O}_{\mathcal{V}}(1) - (g-2)w|$ of hyperplanes vanishing in w with multiplicity $g-2$. By definition of Ω, this linear system pulls back via i_Ω to the linear system of $(g-1)$-forms in $\mathbb{P}^{2g-3}$ vanishing with multiplicity $g-2$ in $e_1, \ldots, e_{2g-1}$, and also with multiplicity $g-2$ in e_0, which is precisely Λ. Hence, the map π_N is the map induced by the same linear system as κ.

□

3.5.4 The global description

Let C be a hyperelliptic genus $g \geq 3$ curve. Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$, and consider the span $\mathbb{P}_N^{2g-2}$ in $\mathbb{P}_D^{3g-2}$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

The resolution map $\widetilde{\varphi_D}$ of φ_D factors through the map θ as shown in Diagram 3.4. When restricted to $\mathbb{P}_N^{2g-2}$ for a divisor N, we have also shown that the restriction map $\varphi_{D,N}$ factors through the degree 2 map π_N. Now we link these two factorizations:

Theorem 3.5.5. *Let $N \in |2D|$ be a general effective divisor. Then, the restriction map $\theta|_{f_D(\mathbb{P}_N^{2g-2})}$ is the map π_N modulo composition with a birational map.*

Proof. Let us place ourselves on the open set $\mathcal{SU}_C^{gs}(2) \subset \mathcal{SU}_C(2)$ of general stable bundles. One observes that the factorization $\widetilde{\widetilde{\varphi_D} = \theta \circ \widetilde{f_D}}$ of Diagram 3.4 is the Stein factorization of the map $\widetilde{\varphi_D}$ along $\mathbb{P}_D^{3g-2}$. Indeed, the map θ is of degree 2 as explained in Section 3.1. Moreover, the preimage of a general stable bundle E by the map f_D is the projectivisation of the space of extensions of the form

$$e: \quad 0 \to \mathcal{O}(-D)) \to E \to \mathcal{O}(D) \to 0.$$

By Riemann-Roch, this projectivisation is in fact a $\mathbb{P}^1$, since $h^0(E(D))$ is generically 2. In particular, the fibers of $\widetilde{f_D}$ over $\mathcal{SU}_C^{gs}(2)$ are connected.

The restriction of $\widetilde{\varphi_D}$ to $\mathbb{P}_N^{2g-2}$ factors through the maps h_N and π_N (see Diagram 3.5), followed by the map l_N. According to Proposition 3.4.10, the fibers of h_N are rational normal curves, thus connected. Moreover, the map π_N is of degree 2 by Theorem 3.5.4. By unicity of the Stein factorization, we have our result.

Comparing with the factorization $\widetilde{\varphi_D} = \theta \circ \widetilde{f_D}$, we see that l_N cannot have relative dimension > 0. Hence, l_N is a finite map. Since the degree of the map θ in the Stein factorization is 2, which is equal to the degree of π_N, we have that l_N cannot have degree > 1. In particular, we have that the map l_N is a birational map. $\square$

From this description and the arguments of Section 3.5.3 yields the following result:

Theorem 3.5.6. *The restriction of the map θ to the general fiber of $p_{\mathbb{P}_c}$ ramifies on the Kummer variety of dimension $g-1$, obtained from the Jacobian of the hyperelliptic curve obtained as a double cover of $\mathbb{P}^1$ ramified along the $2g$ points identified by $P = h_N(\Gamma)$.*

3.6 The case $g = 3$

Let us now illustrate the geometric situation by explaining in detail the first case in low genus. Let C be a hyperelliptic curve of genus 3. In this setting, we have that the map θ factors through the involution i^*, and embeds the quotient $\mathcal{SU}_C(2)/\langle i^* \rangle$ in $\mathbb{P}^7 = |2\Theta|$ as a quadric hypersurface (see [7] and [23]). Let D be a general effective divisor of degree 3. The projective space $\mathbb{P}^7_D$, as defined in Section 3.1, parametrizes the extension classes in $\mathrm{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D))$. The classifying map φ_D is given in this case by the complete linear system $|\mathcal{I}^2_C(3)|$ of cubics vanishing on C with multiplicity 2. According to Proposition 3.2.3, this linear system coincides with the linear system $|\mathcal{I}_{\mathrm{Sec}^1}(3)|$ of cubics vanishing along the secant lines of C. Recall from Section 3.5 that among these lines we have the secant lines l passing through involution-conjugate points. These form a pencil given by the linear system $|h|$.

The image of the projection of $\theta(\mathcal{SU}^{gs}_C(2))$ from $\mathbb{P}_c = \mathbb{P}^3 \subset |2\Theta|$ is also a $\mathbb{P}^3$, that is identified with $|2D|$ by Theorem 3.3.1. Let $N \in |2D|$ be a general reduced divisor. By Proposition 3.3.2, the closure of the fiber $p^{-1}_{\mathbb{P}_c}(N)$ is the image via φ_D of the $\mathbb{P}^4_N$ spanned by the six points of N.

3.6.1 The restriction to $\mathbb{P}^4_N$

The base locus of the restriction map $\varphi_{D,N} = \varphi_D|_{\mathbb{P}^4_N}$ contains $\mathrm{Sec}^N = \mathrm{Sec}^1(C) \cap \mathbb{P}^4_N$ by Lemma 3.4.1. The secant variety $\mathrm{Sec}^1(N) \subset \mathrm{Sec}^N$ is the union of the 15 lines passing through pairs of the 6 points of N. According to Lemma 3.4.11, the further base locus $\mathrm{Sec}^N \setminus \mathrm{Sec}^1(N)$ is given by the intersections of $\mathbb{P}^4_N$ with the lines spanned by degree 2 divisors L_2 on C not contained in N satisfying $\dim |L_2| \geq 1$. By Brill-Noether theory, there is only one linear system of such divisors on a genus 3 curve, namely the hyperelliptic linear system $|h|$ (see, for example, [2], Chapter V). We will review these ideas in Section 3.7). This linear system defines, by the intersections with $\mathbb{P}^{2g-2}_N$ of the lines spanned by the hyperelliptic pencil, the curve Γ that we introduced in Section 3.5. Hence, we have that $\mathrm{Sec}^N = \{15 \text{ lines}\} \cup \Gamma$, and the restriction map $\varphi_{D,N}$ factors as

$$\begin{array}{ccc} \mathbb{P}^4_N & \overset{h_N}{\dashrightarrow} & \mathcal{M}^{\mathrm{GIT}}_{0,6} \subset \mathbb{P}^4 \\ & \underset{\varphi_{D,N}}{\searrow} & \downarrow p \\ & & \mathbb{P}^3 \end{array}$$

where h_N is the map defined by the complete linear system $|\mathcal{I}_{\mathrm{Sec}^1(N)}(3)|$ of cubics vanishing along the 15 lines defined by the points of N, and p is the projection with center the image via h_N of the further base locus Γ.

The image of $\varphi_{D,N}$ is a $\mathbb{P}^3$. Indeed, this image cannot have higher dimension, since the map factors through the projection of a point of $\mathcal{M}^{\mathrm{GIT}}_{0,6} \subset \mathbb{P}^4$. Also, it cannot have dimension < 3 since otherwise the relative dimension of $\varphi_{D,N}$ would be > 1, or equivalently the global map φ_D would not surjet onto $\mathcal{SU}_C(2)$. Hence, in this case the map $\varphi_{D,N}$ is defined exactly by the system of cubics in $\mathbb{P}^4_N$ vanishing on Sec^N.

According to Proposition 3.4.10, the image of h_N is the GIT moduli space $\mathcal{M}_{0,6}^{\mathrm{GIT}}$ if N is general and reduced. It is a classic result that this moduli space is embedded in $\mathbb{P}^4$ as the Segre cubic S_3 (see for instance [24]). This 3-fold arises by considering the linear system i_Ω of quadrics in $\mathbb{P}^3$ that pass through five points in general position. It is also isomorphic to the blow-up of $\mathbb{P}^3$ at these points, followed by the blow-down of all lines joining any two points. The curve $\Gamma \subset \mathbb{P}^4_N$ is a rational normal curve by Lemma 3.5.1, hence Γ is contracted to a point P by h_N again by Proposition 3.4.10.

By [11] and Lemma 3.4.1, the linear system $|\mathcal{O}_{S_3}(1)|$ of hyperplanes in S_3 is pulled back by h_N to $|\mathcal{I}_{\mathrm{Sec}^N}(3)|$ on $\mathbb{P}^4_N$. The linear system $|\mathcal{O}_{S_3}(1) - P|$ of hyperplanes in S_3 passing through P is pulled back to the complete linear system $|\mathcal{I}_{\mathrm{Sec}^N}(C)|$ defining $\varphi_{D,N}$. Hence, the map p is the linear projection with center P. Since S_3 is a cubic, the projection p is a degree 2 map.

The point P in $\mathcal{M}_{0,6}^{\mathrm{GIT}}$ represents a rational curve with 6 marked points. Let C' be the hyperelliptic genus 2 curve constructed as the double cover of this rational curve ramified in these 6 points. According to Theorem 4.2 of [35], the Kummer variety $\mathrm{Kum}(C')$ is contained in the image of p, and it is precisely the branching locus of π.

3.6.2 The global map φ_D

In the genus 3 setting, the linear system $|2D|$ is a $\mathbb{P}^3$. By Proposition 3.3.2, the image of $\mathbb{P}^4_N$ by φ_D is the closure of the fiber $p_{\mathbb{P}_c}^{-1}(N)$. For each point N in $|2D|$, this image is $\mathbb{P}^3 = |\mathcal{I}_{\mathrm{Sec}^N}(3)|^*$, which is the image of the Segre variety $\mathcal{M}_{0,6}^{\mathrm{GIT}}$ under the projection with center P. Thus, the image of the global map φ_D birational to a $\mathbb{P}^3$-blundle over $|2D| = \mathbb{P}^3$. In fact, this image is also a quadric hypersurface in $\mathbb{P}^7$ [23].

3.7 Further base locus in low genera

Let C be a hyperelliptic genus $g \geq 3$ curve. Let D be a general degree g effective divisor on C. Let $N = p_1 + \cdots + p_{2g}$ be a general divisor in the linear system $|2D|$, and consider the span $\mathbb{P}^{2g-2}_N$ in $\mathbb{P}^{3g-2}_D$ of the $2g$ marked points $p_1, \ldots, p_{2g}$.

Recall from Section 3.4 that the intersection $\mathrm{Sec}^N = \mathrm{Sec}^{g-2}(C) \cap \mathbb{P}^{2g-2}_N$ arises naturally as part of the base locus of the restriction map $\varphi_{D,N}$. The subvarieties $\mathrm{Sec}^{g-2}(N)$ and Γ of Sec^N yield the factorization of $\varphi_{D,N}$ through the maps h_N and π_N of Proposition 3.5.3. Let us now describe the set

$$\mathrm{Sec}^{N\prime} = \mathrm{Sec}^N \setminus \{\Gamma \cup \mathrm{Sec}^{g-2}(N)\}.$$

This set is empty for $g = 3$, hence the map $\varphi_{D,N}$ is exactly the composition of h_N and π_N, as described in Section 3.6. In higher genus, the existence of non-empty additional base locus Sec^N corresponds to the fact that, in higher genus, the map $\varphi_{D,N}$ is not exactly the composition of the maps h_N and π_N. In other words, the map l_N is non-trivial in higher genus.

This supplementary base locus is given by the intersections of $(g-2)$-dimensional $(g-1)$-secant planes of C in $\mathbb{P}_D^{3g-2}$ with $\mathbb{P}_N^{2g-2}$ out of $\mathrm{Sec}^{g-2}(N)$ and Γ. According to Lemma 3.4.11, these intersections are given by effective divisors L_{g-1} on C of degree $g-1$, not contained in $\mathbb{P}_N^{2g-2}$, and satisfying $\dim |L_{g-1}| \geq 1$. Also by Lemma 3.4.11, we obtain $\dim(\langle L_{g-1}\rangle \cap \mathbb{P}_N^{2g-2}) = \dim |L_{g-1}| - 1$.

We will now give account of the situation in low genera.

Case $g = 4$

In this case, the divisor N is of degree 8 and the map

$$\varphi_D|_N : \mathbb{P}_N^6 \subset \mathbb{P}_D^{10} \dashrightarrow |2\Theta| = \mathbb{P}^{15}$$

is given by the linear system $|\mathcal{I}_C^3(4)|$. This map factors through the map π_N which coincides with the 1-osculating projection Π_w^1, where $w = h_N(\Gamma)$.

We are looking for degree 3 divisors L_3 with $\dim |L_3| \geq 1$. These satisfy all $\dim |L_3| = 1$ and are of the form

$$L_3 = h + q \qquad \text{for } q \in C,$$

where h is the hyperelliptic divisor. Let p be a point of C. Then $L_3 = p + i(p) + q$. Since $\dim |L_3| = 1$, the secant plane $\mathbb{P}_{L_3}^2$ in $\mathbb{P}_D^{10}$ spanned by p, $i(p)$ and q intersects $\mathbb{P}_N^6$ in a point. But this point necessarily lies in Γ, since the line passing through p and $i(p)$ is already contained in this plane. Hence, we do not obtain any additional locus.

Case $g = 5$

In this case, the divisors L_4 of degree 4 are all of the form

$$L_4 = h + q + r \qquad \text{for } q, r \in C,$$

and satisfy $\dim |L_4| = 1$. Thus, the corresponding secant $\mathbb{P}_{L_4}^3$ spanned by p, $i(p)$, q and r intersects $\mathbb{P}_N^8$ in a point. As before, this point lies in Γ, thus we do not obtain any additional locus.

Case $g = 6$

Here we have, as in the genus 5 case, the divisors of the form

$$L_3 = h + q \qquad \text{for } q \in C,$$

which do not give rise to any additional base locus. But there is a new family of divisors

$$L_5 = 2h + r \qquad \text{for } r \in C.$$

These divisors satisfy $\dim |L_5| = 2$. In particular, the intersection of the $\mathbb{P}^4_{L_5}$ spanned by p, $i(p)$, q, $i(q)$ and r, for $p, q \in C$ with $\mathbb{P}^{10}_N$ is a line m in $\mathbb{P}^{10}_N$. The line l_1 (resp. l_2) spanned by p and $i(p)$ (resp. q, $i(q)$) intersects Γ in a point $\widetilde{p}$ (resp. $\widetilde{q}$). In particular, the line m is secant to Γ and passes through $\widetilde{p}$ and $\widetilde{q}$. Since every point of Γ comes as an intersection of a secant line in C with $\mathbb{P}^{10}_N$, we obtain the following description of the base locus of $\varphi_{D,N}$:

Proposition 3.7.1. *Let C be a curve of genus $g = 6$. Then, the base locus of the restriction map $\varphi_{D,N}$ is the ruled 3-fold* $\mathrm{Sec}^1(\Gamma)$.

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