Relative hyperbolicity of suspensions of free products

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relative hyperbolicity of suspensions of free products.

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Relative hyperbolicity of suspensions of free products

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Abstract

In this thesis, we are interested in the study of the relative hyperbolicity of the suspensions of free products, as well as the conjugacy problem of certain automorphisms of free products.

To be more precise, given a free product

$$G = G_1 * \cdots * G_p * F_k$$

an automorphism ϕ is said atoroidal if no power fixes the conjugacy class of an hyperbolic element. It is called fully irreducible if the given free factor system $[G_1], \ldots, [G_p]$ is the largest one that is fixed by every power of the automorphism. It is said toral if for all i, there exists $g_i \in G$ such that $\mathrm{ad}_{g_i} \circ \phi|_{G_i}$ is the identity on the free factor G_i. It is said to have central condition if for each i, there exists $g_i \in G$ conjugating $\phi(G_i)$ to G_i, and if there exists a non-trivial element of $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$ that is central in $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$.

We prove, in Theorem 4.34, that if ϕ is atoroidal and fully irreducible, and if the free product is non-elementary ($k \geq 2$ or $p + k \geq 3$), the group $G \rtimes_\phi \mathbb{Z}$ is relatively hyperbolic (relative to the mapping torus of each G_i). Then in Theorem 6.10, we prove the same result holds if ϕ is atoroidal with central condition. We also prove in Theorem 7.22, that if all G_i are abelian, the conjugacy problem is solvable for toral atoroidal automorphisms. These are analogue of the result of Brinkmann [7] (which gave the hyperbolicity result for free groups) and the result of Dahmani [12] (which solved the conjugacy problem of hyperbolic automorphisms).

Key words: Relative hyperbolicity, Conjugacy problem, Free product, Atoroidality, Irreducibility.

Résumé

Dans la thèse présente, nous nous intéressons à l'étude de la relative hyperbolicité des produits semi-direct des produits libres, ainsi que le problème de conjugaison pour certains automorphismes de ces produits libres.

Plus précisement, pour un produit libre

$$G = G_1 * \cdots * G_p * F_k$$

un automorphisme ϕ est intitulé atoroidal s'il ne fixe pas (ni aucune de ses puissances) la classe de conjugaison d'un élément hyperbolique de G. Cet automorphisme est appelé completement irréductible si le système de facteurs libres est le plus grand qui est fixé par toutes les puissances de cet automorphisme. Il est appelé toral si pour tous les i, il existe $g_i \in G$ tel que $\mathrm{ad}_{g_i} \circ \phi|_{G_i}$ est identité sur le facteur libre G_i. Nous disons qu'il a la condition centrale si pour chaque i, il existe $g_i \in G$ conjugue $\phi(G_i)$ à G_i, et s'il existe un 'elément non trivial de $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$ qui est central dans $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$.

Nous prouvons, dans le Théorème 4.34, que si ϕ est atoroidal et completement irréductible, et si le produit libre est non-elementaire ($k \geq 2$ ou $p + k \geq 3$), le groupe $G \rtimes_\phi \mathbb{Z}$ est relativement hyperbolique (relativement a des suspensions de chaque G_i). Après, dans le Théorème 6.10, nous prouvons le même résultat si ϕ est atoroidal avec la condition centrale. Nous prouvons aussi dans le Théorème 7.22, que si tous les G_i sont abelien, le problème de conjugaison est solvable pour les automorphismes atoroidaux, toraux. Ces sont des analogues du résultat de Brinkmann [7] (celui qui a donné le résultat d'hyperbolicité pour les groupes libres), et du résultat de Dahmani [12] (celui qui a résolu le problème de conjugaison des automorphismes hyperboliques).

les mots clés: Relative hyperbolicité, Problème de conjugaison, produit libre, Atoroidalité, Irreducibilité.

Contents

1 Introduction

1.1 Automorphisms

For any given group G, let $Aut(G)$ be its automorphism group. The *inner automorphism group* $Inn(G)$ is defined as $Inn(G) = \{ad_h : G \to G, g \mapsto hgh^{-1}; h \in G\}$, it is a normal subgroup of $Aut(G)$. The quotient, $Aut(G)/Inn(G)$ is the *outer automorphism group* of G, denoted by $Out(G)$.

These groups are easy to define, but automorphisms are usually not so easy to deal with. A natural problem arises to determine whether two given automorphisms have the same image in the outer automorphism group (i.e. they differ only by a conjugation in G). This problem involves the so-called "*simultaneous conjugacy problem* in G": given the generating set of G, determine whether or not images of all the generators by two given automorphisms are conjugate by a same element in G. For some groups, say free groups, this problem is easy to solve, as the centraliser of any element in the free group is the maximal cyclic group containing such an element.

Other natural problems are to determine whether two given automorphisms are conjugate by an element in $Aut(G)$ (the conjugacy problem in $Aut(G)$), and to determine whether two given automorphisms are conjugate by an element in $Out(G)$ (the conjugacy problem in $Out(G)$). For example, take $G = \mathbb{Z}^n$ (where n is a positive integer), its automorphism group is $GL_n(\mathbb{Z})$. Its inner automorphism group is trivial, so $Out(\mathbb{Z}^n)$ is also $GL_n(\mathbb{Z})$. The conjugacy problem of $GL_n(\mathbb{Z})$ can be related to a theorem by Latimer and MacDuffee in 1933 (see [33]). The theorem says that for a given irreducible (over $\mathbb{Z}$) monic polynomial P (with $P(0) = 1$), the conjugacy classes of matrices in $GL_n(\mathbb{Z})$ with characteristic polynomial P are in bijection with the ideal classes of $\mathbb{Z}[X]/(P)$. This sheds light on the conjugacy problem of $Aut(\mathbb{Z}^n)$, but still, the arithmetic approach is complicated, even in the case when the characteristic polynomials are irreducible.

In order to survive this complexity, one may look for hyperbolicity.

1.2 Hyperbolicity

We first come back to the example of $G = \mathbb{Z}^n$, but this time, we take a simple case where $n = 2$ and consider the subgroup $SL_2(\mathbb{Z})$ (subgroup of $GL_2(\mathbb{Z})$). It acts on the hyperbolic plane $\mathbb{H}_2$ in the following way (expressed in the upper half plane model):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

This action has a kernel of $\{\pm Id\}$. Hence we can identify $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm Id\}$ with the group of all linear fractional transformations of $\mathbb{H}_2$.

There is a fundamental domain $\mathbb{M} = \{z : |z| > 1, -\frac{1}{2} < Re(z) \leq \frac{1}{2}\} \cup \{e^{i\theta} : \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}$ of $PSL_2(\mathbb{Z})$ on $\mathbb{H}_2$: each point in the hyperbolic plane is equivalent (by the action of the group $PSL_2(\mathbb{Z})$) to some point in $\mathbb{M}$, while no point in $\mathbb{M}$ (except for itself) is equivalent to another point in $\mathbb{M}$.

In addition, one can show that the union of all images of the arc $\{e^{i\theta} : \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}\}$ by elements of $SL_2(\mathbb{Z})$ form a tree, whose edges are precisely the different images of the arc. The group $SL_2(\mathbb{Z})$ thus acts on this tree, one can check that the action is without inversion (stabilizer of any edge is the stabilizer of both end points), and without global fixed point. It follows, by Bass-Serre theory that $SL_2(\mathbb{Z})$ has the familiar presentation as an amalgamated free product: $SL_2(\mathbb{Z}) \simeq \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Hence every element has a normal form, and by comparing the cyclic reduced normal form (up to cyclic permutation) one can tell whether or not two given elements in this group are conjugate.

Hyperbolic geometry gives us the tree which provides us a solution to the conjugacy problem.

But, in the case of $SL_3(\mathbb{Z})$, things are more difficult, as there is no hyperbolic space for this group, and very little hyperbolic behaviour for this group. Hence the method above (for $SL_2(\mathbb{Z})$) does not work in this case.

Take another example, a genus g closed surface (denoted by Σ_g). Its universal cover is $\mathbb{H}_2$, the hyperbolic plane. Moreover, $\pi_1(\Sigma_g)$ is an analogue of $\mathbb{Z}^2$: $\mathbb{Z}^2$ corresponds to a tessellation of an Euclidean plane by squares, and $\pi_1(\Sigma_g)$ corresponds to a tessellation of the hyperbolic plane by $(4g)$gons. In other words, $\mathbb{Z}^2$ is an euclidean lattice, while $\pi_1(\Sigma_g)$ is a hyperbolic lattice.

By a theorem of Dehn and Nielsen, the mapping class group (the group of isotopy classes of diffeomorphisms) of Σ_g is isomorphic to a subgroup of index 2 of $Out(\pi_1(\Sigma_g))$. And Thurston (see [38]) gave a classification of all the elements in the mapping class group by classifying into 3 kinds: periodic, reducible (i.e, the element preserving some finite union of disjoint simple closed curves on the surface up to isotopy), and pseudo-Anosov.

In 1986, Thurston gave an important theorem (see [39]), revealing that, for a genus g (where $g \geq 2$) surface Σ_g, an element ϕ in the mapping class group of Σ_g is pseudo-Anosov if and only if the fundamental group of its suspension (or, in other words, $\pi_1(\Sigma_g) \rtimes_\phi \mathbb{Z}$, see Subsection "Automorphisms, suspensions, semidirect product" 2.4 for formal definition) is Gromov-hyperbolic.

One can revisit the case of $SL_2(\mathbb{Z})$, using the viewpoint that $SL_2(\mathbb{Z})$ is the mapping class group of the punctured torus. Elements that are loxodromic in the modular tree are pseudo Anosov. Although the punctured torus is not a closed surface, Thurston's theorem says that, for these elements, the suspension is a geometrically finite hyperbolic manifold, and its fundamental group is toral relatively hyperbolic. If elements are conjugate, they give isomorphic suspensions. Although it seems unnecessarily complicated for the case of $SL_2(\mathbb{Z})$, one can argue that all the hyperbolicity needed to solve the conjugacy problem is in the suspensions.

1.3 Free groups

Replacing $\pi_1(S)$ by another group G, and letting ϕ be an automorphism of G, it is interesting to know whether $G \rtimes_\phi \mathbb{Z}$ is hyperbolic or relatively hyperbolic. One kind of interesting example is free groups, as it is an analogue of $\pi_1(\Sigma_g)$ (in the sense that, $\pi_1(\Sigma_g)$ is a hyperbolic lattice, and free groups are lattices in trees).

Bestvina, Feighn, Handel investigated an analogue of Thurston's result on the hyperbolicity of the semi-direct product (see [2]). In his thesis, Brinkmann also approached the problem, giving an account in a broader generality (see [7]) and providing a characterisation. Brinkmann's result is that for a free group G, and for any $\phi \in Aut(G)$, $G \rtimes_\phi \mathbb{Z}$ is hyperbolic if and only if ϕ is an atoroidal automorphism (here *atoroidal* means that no non-trivial conjugacy class of element in the group can be preserved by some non-zero power of ϕ). The general method for this theorem is to:
- use train track maps that represent topologically on a specific graph (or on its universal cover, a tree), the automorphisms that we consider
- analyse how edges and paths are folded and stretched by this train track map
- deduce the growth of conjugacy classes of the elements of the group under the action of the automorphism
- use the Bestvina-Feighn combination theorem that indicates when a HNN extension is a word-hyperbolic group.

Still, there are plenty examples of non-hyperbolic groups that need to be analysed:

The group $\mathbb{Z}^n \rtimes \mathbb{Z}$ is never hyperbolic or relatively hyperbolic, because $\mathbb{Z}^n$ is normal in it, and normal parabolic groups must be finite.

Take $\phi \in Aut(F_n)$, where F_n is a free group of rank n (where $n > 2$), if ϕ is a toroidal automorphism (for example, $G = F_2 * F_{n-2}$, $\phi \in Aut(G)$ such that $\phi|_{F_2} = Id$, and $\phi|_{F_{n-2}}$ is atoroidal), then $F_n \rtimes_\phi \mathbb{Z}$ is not hyperbolic as it contains $\mathbb{Z}^2$. But, in some cases, it can be relatively hyperbolic (see Definition 4.3 for formal description). In this example, it is hyperbolic relative to $F_2 \times \mathbb{Z}$.

In [12] Dahmani gave a solution to the conjugacy problem for atoroidal automorphisms of free groups, using this hyperbolicity. In [11] he also gave a solution to the conjugacy problem for more cases of automorphisms, which produce relatively hyperbolic suspensions.

1.4 Free products

In this thesis, we assume that G is a free product, and show the relative hyperbolicity of $G \rtimes_\phi \mathbb{Z}$ under some condition on ϕ. Our main results are Theorem 4.34 and Theorem 6.10.

The first study of outer automorphism group of free products goes back to pioneering work of Fouxe-Rabinovitch (1941) ([19], [20]) where she found generators and relations. Golowin and Szadowsky ([23]) worked out the case of 2 factors in 1938.

We will use a more modern approach. Any free product G acts on Bass-Serre trees (vertex stabilizers are either conjugate to some of G_i's or trivial), and $Out(G)$ acts on the space of these Bass-Serre trees, formalised by Guirardel and Levitt in [30] as outer spaces of free products. Francaviglia and Martino (in [21]) identified that for each so-called irreducible automorphism ϕ, there is a family of preferred trees on which ϕ induces a train track map (a map with good cancellation properties).

We first restrict to the fully irreducible case, for non-elementary free product(i.e. either $k \geq 2$ or $p + k \geq 3$). To achieve relative hyperbolicity of the suspension $G \rtimes_\phi \mathbb{Z}$, we simply need to prove the relative hyperbolicity of the automorphism ϕ, thanks to a combination theorem proved by Gautero and Weidmann (see Corollary 7.3 in [22], cited in this thesis as Theorem 4.8). The relative hyperbolicity of the automorphism, in this case, is shown by the exponential growth of length of fundamental segments by a map representing such an automorphism. Such a map we use in this thesis, is the train track map, which is similar to the case of free group. The study of train track maps in the free group case was developed by Bestvina and Handel, and was introduced into the free product case by Francaviglia and Martino in [21].

The technique we use in this thesis is analogous to that in the free group case. In free product, things are going different and more complicated, as we are dealing with locally infinite trees instead of finite graphs. But still, some similar result can be achieved in the fully irreducible and atoroidal case (where we have train track maps that every edge stretch by a same factor strictly larger than 1). Then by analyzing the growth of each fundamental segment, and the exponential growth of its subpaths, we achieve the relative hyperbolicity of $G \rtimes_\phi \mathbb{Z}$ in the case when ϕ is atoroidal and fully irreducible, under the assumption that the free product decomposition is non-elementary.

For the reducible case, we first show that there is a larger free factor system such that the automorphism is still fully irreducible and atoroidal. And by a method of descent, assuming that the a certain property that we call the "central property" holds (i.e., for each $i, g_i \in G$ conjugates $\phi(G_i)$ to G_i, and there exists a non-trivial element of $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$ that is central in $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$), we can achieve relative hyperbolicity of $G \rtimes_\phi \mathbb{Z}$ if ϕ is atoroidal (see Theorem 6.10 for detailed information).

1.5 Conjugacy Problem of toral atoroidal automorphisms of free products

Conjugacy problem is another big problem in group theory. In this thesis, we focus on the conjugacy problem of toral atoroidal automorphisms of free products. An automorphism is *toral* if for all i, there is $g_i \in G$ such that $\mathrm{ad}_{g_i} \circ \phi|_{G_i}$ is the identity on the free factor G_i; it is *atoroidal* if no non-trivial conjugacy class of hyperbolic element is fixed by some power of ϕ.

We will use a basic, well-known fact concerning the conjugacy problem: two automorphisms $\phi_1, \phi_2 \in Aut(G)$ are conjugate in $Aut(G)$ if and only if there is an isomorphism from $G \rtimes_{\phi_1} \langle t_1 \rangle$ to $G \rtimes_{\phi_2} \langle t_2 \rangle$ such that it sends G to G and t_1 to t_2; they are conjugate in $Out(G)$ if and only if there is an isomorphism from $G \rtimes_{\phi_1} \langle t_1 \rangle$ to $G \rtimes_{\phi_2} \langle t_2 \rangle$ such that it sends G to G and t_1 to $t_2 G$. Also, similar results hold for the abelianisation (see Lemma 7.7 for detail).

Dahmani and Groves proved the solvability of isomorphism problem of toral relatively hyperbolic groups (see Theorem 7.1 of [13]). What remains to us to check is, whether or not the element t_1 is sent to the corresponding element (depending on Out or Aut) and G is sent to G. This is referred to as the "Orbit problem of uprighting hyperplanes".

Guirardel and Levitt proved an interesting result (see Theorem 1.4 of [29]), saying that there is a finite index subgroup

$Out^1(H)$ of $Out(H)$ for any toral relatively hyperbolic one-ended group H fitting in an exact sequence

$$1 \to \mathbb{T} \to Out^1(H) \to \prod_{i=1}^{P} MCG^0(\Sigma_i) \times \prod_{j=1}^{m} GL_{r_j,n_j}(\mathbb{Z}) \to 1$$

in which $\mathbb{T}$ is an abelian well-identified group (generated by algebraic Dehn-twists).

Computing coset representatives of $Out^1(H)$ in $Out(H)$ is made possible (using Guirardel and Levitt) by computing a primary JSJ-decomposition of the relatively hyperbolic group H. This is done in the work of Dahmani and Groves ([13]). Thus conjugacy problem can be turned into the orbit problem in $Out(H)$ of uprighting hyperplanes in $G \rtimes_{\phi_2} \langle t_2 \rangle$, and the image of the solution in each GL_{r_j,n_j} satisfies the Condition 1. This Condition 1 is actually directly induced by Lemma 7.7, by the restriction to each GL_{r_j,n_j} (see Section 7.4 for formal definition).

The method we use in the rest of the work is an analogue of the work of Dahmani (see [12]) in the case of hyperbolic automorphisms. Because the action of $Out^1(G \rtimes \langle t \rangle)$ on the abelianisation of $G \rtimes \langle t \rangle$ is generated by transvections, we turn the orbit problems to the corresponding Diophantine equation problems, and prove the solvability of conjugacy problem of toral atoroidal automorphisms of free products (see Subsection "Conjugacy Problem for Toral Relatively Hyperbolic Automorphisms" for details). Our main result in this direction is Theorem 7.22 that we state here: For any given free product G, and two toral atoroidal automorphisms, there is an algorithm to decide whether ϕ_1 and ϕ_2 are conjugate in $Aut(H)$ (and in $Out(H)$).

1.6 Further perspectives

1. It is still not clear what would happen if the condition on central elements is removed. There are counterexamples: Consider two copies $\mathbb{Z}_a^2, \mathbb{Z}_b^2$ of $\mathbb{Z}^2$, and make their free product $G = \mathbb{Z}_a^2 * \mathbb{Z}_b^2$. Consider the automorphism $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ (with obvious meaning). ϕ preserves both factors $\mathbb{Z}_a^2, \mathbb{Z}_b^2$ and in the suspension, the stable letter t normalises both groups $\mathbb{Z}_a^2, \mathbb{Z}_b^2$ and produces semi-direct products $\mathbb{Z}_a^2 \rtimes \langle t\rangle$ and $\mathbb{Z}_b^2 \rtimes \langle t\rangle$. If G is relatively hyperbolic both subgroups $\mathbb{Z}_a^2 \rtimes \langle t\rangle$ and $\mathbb{Z}_b^2 \rtimes \langle t\rangle$ must be parabolic (because they contain a normal $\mathbb{Z}^2$). But their intersection $\langle t\rangle$ is infinite. Therefore, they must be in the same maximal parabolic subgroup, but since they generate the whole semi-direct product $(\mathbb{Z}_a^2 * \mathbb{Z}_b^2) \rtimes \langle t\rangle$, it means that the whole group must be parabolic. Thus there is no proper relative hyperbolic structure on it. What can be a necessary and sufficient condition, then?

One potential approach is to see the filtration of the tree (stratify the tree into zero-strata, polynomial growing strata, and exponentially growing strata, similar to the free group case), and to analyse the possible "improved relative train track map". But the existence of which, in the free product case, is unclear to me yet.

2. Also, in free group case, atoroidal irreducible automorphisms are fully irreducible, it is unknown in the free product case. I would like to have some investigation on this in the near future. Recent work of Guirardel and Horbez ([26]) might be relevant.

3. Given the work of Dahmani-Touikan[16], it seems legitimate to try to extend our main result on the conjugacy problem (Theorem 7.22)to the case where $A_1, \dots, A_k$ are nilpo-

tent and ϕ_i induce the identity on them. One can also possibly hope to extend the result to the case where ϕ_i induces a nilpotent automorphism, i.e. an automorphism of an abelian or a nilpotent group, such that the semi-direct product is still nilpotent.

1.7 Ackowledgements

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2 Graphs, trees, groups acting on them

2.1 Conventions on graphs

Let us recall some conventions regarding graphs, trees, metrics, and group actions.

A *graph* X is a pair (V, E) where V is a non-empty set, and E is a set, that is endowed with three applications $i : E \to V$, $t : E \to V$ and $b : E \to E$ such that b is a fix-point free involution and such that $t = i \circ b$. The elements of V are called the *vertices* of the graph, the elements of E are called the *oriented edges* of the graph. The map b is the reversing of the orientation of an edge. In the above notation, we denote by $V(X)$ the set of vertices of X, and by $E(X)$ the set of edges in X. We will from now denote by $\bar{e}$ the edge $b(e)$.

Consider X a graph. The *degree* of a vertex v is the cardinality of the set $\{e : i(e) = v\}$.

A *finite path* in X is a finite sequence of edges e_i, $i = 1, \ldots, n$, such that for all i, $t(e_i) = i(e_{i+1})$. We say that such a path starts at $i(e_1)$ and ends at $t(e_n)$ (or is from $i(e_1)$ to $t(e_n)$).

A graph is *connected* if for every pair of vertices, there is a path starting at one and ending at the other.

We say that a path is *reduced* if for all i, $e_i \neq \bar{e}_{i+1}$. Clearly if there is a path from x to y, there is a reduced path from x to y. An *infinite path* is a sequence $(e_i)_{i \in \mathbb{N}}$ such that $t(e_i) = i(e_{i+1})$ for all i. Again it can be reduced, or not. A *bi-infinite path* is a sequence $(e_i)_{i \in \mathbb{Z}}$ such that $t(e_i) = i(e_{i+1})$ for all i. A path is said to be *closed* if its starting point and its ending point are identical. The shift of parameter $k \in \mathbb{Z}$ on a bi-infinite path $(e_i)_{i \in \mathbb{Z}}$ is the bi-infinite path $(e_{i+k})_{i \in \mathbb{Z}}$. Slightly more formally, the shift of parameter $k \in \mathbb{Z}$ is a map from the set of all bi-infinite paths to itself defined by the above association.

A *tree* is a connected graph for which, given any pair of vertices, there is a unique reduced path between them. A *forest* is a disjoint union of trees.

An *automorphism of a graph* $X = (V, E)$ is a pair of bijections $\phi_V : V \to V, \phi_E : E \to E$ such that $i(\phi(e)) = \phi(i(e))$ and $\phi(\bar{e}) = \overline{\phi(e)}$.

The automorphisms of a tree are of two kinds. An automorphism is said to be *elliptic* if it fixes a vertex or a pair of vertices; otherwise it is said to be *hyperbolic*. An elementary result in the geometry of trees justifies this terminology: any hyperbolic automorphism preserves a unique bi-infinite path (up to shifting), and in restriction to it, acts as a shift on the indices of the edges, or a translation. See Theorem 2.1, Chapter 2 of [5].

Consider a group G and its action on a tree T, by automorphisms. We say that the action is *trivial* if there exists a vertex of T that is fixed by all elements of G. We say that it is *without inversion* if every element of G that fixes a pair of vertices actually fixes both vertices of the pair. Recall Serre's lemma: if G is finitely generated, and if its action on T is non-trivial, then G contains hyperbolic elements (see [37]).

Given a tree T, we call a vertex $v \in VT$ *redundant*, if it has degree two, and if any $g \in G$ fixing v fixes edges adjacent to v. It is called *terminal* if $T - \{v\}$ is connected (although $T - \{v\}$ is no longer a graph).

2.2 Digression on metrics

Let us put a metric on a graph. A *length function* on a graph $X = (V, E)$ is a function $l_X : E \to \mathbb{R}_+ \setminus \{0\}$. It is *symmetric* if $l_X(e) = l_X(\bar{e})$ for all $e \in E$. Given a length function, one defines the length distance on X (more precisely on V) as follows.

First, the length of a path $p = (e_1, \ldots, e_k)$ is $\ell_X(p) = \sum_i l_X(e_i)$. Second, the length distance between $v_1, v_2 \in V$, denoted by $d_X(v_1, v_2)$, is the infimum of the lenghts of paths that starts at v_1 and end at v_2. It is symmetric as soon as l_X is symmetric. It obviously satisfy the triangular inequality. In general one does not have $(d_X(v_1, v_2) = 0) \implies (v_1 = v_2)$. However this is satisfied in many cases, for example: when X is locally finite, when there is $\epsilon > 0$ such that all edges have $l_X(e) > \epsilon$ (in particular when there are only finitely many possible values), when the graph is a tree, etc. In all our examples, our graphs will satisfy one or several of these conditions.

An elementary example is when $l_X(e) = 1$ for all e. However we will use more examples.

2.3 Free groups

The universal cover of a rose is a regular tree, which is the Cayley graph for the free group.

Let X be a subset of a group F. Assume that $X^{-1}(= \{x^{-1}, x \in X\})$ is disjoint from X. Recall that F is a *free group* of basis X if every non-trivial element of F can be represented uniquely as a product $f = x_1 \dots x_n$, for some $n \geq 1$, and $x_i \in X \cup X^{-1}$, with the only constraint that $x_i x_{i+1} \neq 1$ for all i. See [5] for instance.

Free groups are very important in geometric and combinatorial group theory.

Here are a few elementary important properties. A group is free if and only if it acts freely on a simplicial tree. A group is free if and only if it is the fundamental group of a graph. Any group is a quotient of a free group. Any map from X to a group extends uniquely in a group homomorphism from the free group of basis X to that group.

Let us describe free groups as fundamental groups of graphs more precisely. In this thesis, a *rose*, following a well established convention, is a graph with a single vertex. Given a rose R, let $e_1, \dots e_n, \bar{e}_1, \dots, \bar{e}_n$ its oriented edges, so that $e_1, \dots e_n$ is a certain choice of orientation of the set of edges. Then, denoting by v the vertex of R, the group $\pi_1(R, v)$ is naturally isomorphic to the free group of basis $\{e_1, \dots e_n\}$.

The universal cover of R is a $2n$-regular tree, and can be naturally identified with the Cayley graph of $\pi_1(R, v)$, over the generating set $\{e_1, \dots e_n\}$, by choosing a base point.

2.4 Automorphisms, suspensions, semidirect product

Consider a topological space S, which is assumed to be path-connected. Let $f : S \to S$ be a continuous map.

One can construct the *suspension* of S along f to be the quotient topological space $\Sigma(S, f) = S \times [0, 1]/ \sim$ where $(s, 0) \sim (f(s), 1)$ for all s. In general, one needs to take the transitive closure of the relation, but if f is injective, this is not necessary and the equivalence classes have one or two point.

For example, if $S = S^1$, and $f = Id$, one gets the torus; if $f = -Id$, the Klein bottle. Given $s_0 \in S$, the homotopy group $\pi_1(\Sigma(S, f), s_0)$ has its isomorphic type depending only on the homotopy class of f. Therefore, since S is assumed path connected, to realise the isomorphism class of $\pi_1(\Sigma(S, f), s_0)$ we may freely assume that $f(s_0) = s_0$. In that case, f realises an endomorphism of $\pi_1(S, s_0)$. Note that if one uses another representative f' of the homotopy class of f, then f' defines an endomorphism that differs from f by the composition with an inner automorphism of $\pi_1(S, s_0)$ (that is, a conjugation by an element of $\pi_1(S, s_0)$).

Assume also that f is a homotopy equivalence of S. Then f defines an automorphism of $\pi_1(S, s_0)$. Let us denote it by ϕ_f. Recall again that only the outer class of ϕ_f (its class in the outer automorphism group of $\pi_1(S, s_0)$) is well defined. Seifert van Kampen theorem indicates that $\pi_1(\Sigma(S, f), s_0)$ is isomorphic to the semidirect product $\pi_1(S, s_0) \rtimes_{\phi_f} \mathbb{Z}$. Actually one should note that the theorem gives a presentation of $\pi_1(S, s_0)$ as an HNN-extension, but in the case of an automorphism, it is a semidirect product with $\mathbb{Z}$.

A famous example (that we recalled in the introduction) is given for S a closed orientable surface of genus $g \geq 2$. This, in particular, illustrates our interest in finding hyperbolic properties of maps f through hyperbolic properties of the group $\pi_1(S, s_0) \rtimes_{\phi_f} \mathbb{Z}$.

2.5 Basics of Bass-Serre theory

We recall the definition of graph-of-groups, and the duality with trees. We refer to [5].

Let S be a set. Denote here by F_S the free group with basis S.

A *graph of groups*

$$(Y, \{G_v, v \in V(Y)\}, \{G_e, e \in E(Y)\}, \{\alpha_e, e \in E(Y)\})$$

consists of a graph Y (called the *underlying graph*), a group (referred to as *vertex group*) G_v for each vertex v of $V(Y)$, a group G_e (called *edge group*) G_e for each edge e of $E(Y)$, such that $G_e = G_{\bar{e}}$ for all $e \in E(Y)$, and a monomorphism (an injective homomorphism) $\alpha_e : G_e \to G_{i(e)}$ for each edge e in $E(Y)$.

For a given graph of groups

$$\mathbb{Y} = (Y, \{G_v, v \in V(Y)\}, \{G_e, e \in E(Y)\}, \{\alpha_e, e \in E(Y)\})$$

let $(*_{v \in V(Y)} G_v) * F_{E(Y)}$ be the free product of all vertex groups, and of the free group over the set of edges of Y.

We define the *Bass group* $B(\mathbb{Y})$ to be the quotient group of $(*_{v \in V(Y)} G_v) * F_{E(Y)}$ by the normal closure of

$$\{t_e^{-1} \alpha_e(g) t_e (\alpha_{\bar{e}}(g))^{-1}, t_e t_{\bar{e}} : e \in E(Y), g \in G_e\}.$$

An element in the form $g_1 t_{e_2} g_2 ... t_{e_n} g_n$ (where $e_2 ... e_n$ is a closed path in Y, and where for all j, $g_j \in G_{t(e_j)}$) is called a *path element in the Bass group.* We say that it starts at $i(e_2)$ and ends at $t(e_n)$.

Given a vertex V_0 in the graph Y, we define the *fundamental group* $\pi_1(\mathbb{Y}, V_0)$ of the graph of groups $\mathbb{Y}$ *with the base point* V_0, to be the subgroup of $B(\mathbb{Y})$ of all path elements starting and ending at V_0.

If now τ is a maximal subtree of the graph Y, the *fundamental group* $\pi_1(\mathbb{Y}, \tau)$ of the graph of groups $\mathbb{Y}$ *with maximal subtree* τ is the quotient group of $B(\mathbb{Y})$ by the normal closure of $\{t_e : e \in E(\tau)\}$.

An important result of Bass-Serre theory, reported as Corollary 16.7 of [5], states that $\pi_1(\mathbb{Y}, V_0)$ and $\pi_1(\mathbb{Y}, \tau)$ are isomorphic (therefore, both $\pi_1(\mathbb{Y}, V_0)$ and $\pi_1(\mathbb{Y}, \tau)$ are regardless of the choice of vertex and maximal subtree). More precisely, the quotient map restricted to $\pi_1(\mathbb{Y}, V_0)$ is an isomorphism onto $\pi_1(\mathbb{Y}, \tau)$.

Another central point of Bass-Serre theory is the duality between graphs of groups and trees. First, there exists a tree T on which $\pi_1(\mathbb{Y}, \tau)$ acts by automorphisms, without inversion, and in which the vertex stabilizers are exactly the conjugates of the images of the groups $G_v, v \in V(Y)$ in $\pi_1(\mathbb{Y}, \tau)$, and in which the edge stabilizers are exactly the conjugates of the images of the groups $G_e, e \in E(Y)$ in $\pi_1(\mathbb{Y}, \tau)$. Second, or conversely, anytime a group G acts on a tree T non-trivially, and without inversion, then G is the fundamental group of a graph of groups constructed on the graph $G\backslash T$ where the vertex groups and the edge groups are copies of the vertex stabilizers in G and of the edge stabilizers in G. We refer to [37] and to [5] for a comprehensive treatment.

Example. The group obtained as the free product $G = \mathbb{Z}^2 * \mathbb{Z}^3 * \mathbb{Z}^4 * F_{\{a,b,c\}}$ is isomorphic to the fundamental group of the graphs of groups (and many more) shown in Figure 1. Each of them is associated to a dual Bass-Serre tree on which G acts. To deal with the multiplicity of trees, on which G can

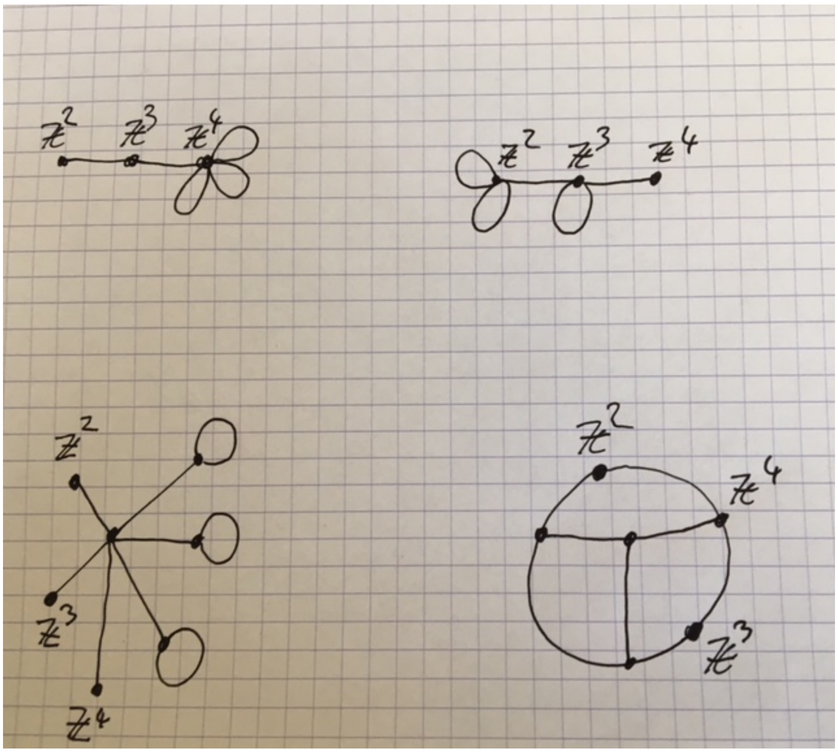

Figure 1: Four graphs of groups whose fundamental groups are isomorphic to $G = \mathbb{Z}^2 * \mathbb{Z}^3 * \mathbb{Z}^4 * F_{\{a,b,c\}}$ (edge groups are all trivial).

act, let us gather some vocabulary.

We first define *equivariant maps* in the most common way. Given a group G with group actions $G \times X \to X, G \times Y \to Y$ (acting on the left), a map $f : X \to Y$ is referred to as *G-equivariant*, if $f(gx) = gf(x)$ for each $g \in G$ and $x \in X$.

Given a group G, we say T is a *G-tree* if T is a tree endowed with a simplicial G-action $\alpha : G \times T \to T$ by tree automorphisms. A G-tree is a metric G-tree if it carries a metric (as above, in the sense of graphs) such that the action of G is by isometries. Denote by $T(G)$ the set of metric G-trees. For a G-tree T, we have a quotient graph $G\backslash T$. Denote by π_T the projection map from T to $G\backslash T$.

Two G-trees T, T' are said to be *equivalent*, denoted by $T \sim T'$, if there is a G-equivariant isometry f from T to T'.

2.6 Grushko decompositions, free product decompositions

Grushko Decomposition Theorem states that for any nontrivial finitely generated group there exists a decomposition into a free product of the form

$$G \simeq G_1 * ... * G_p * F_k \tag{GD}$$

where G_i is freely indecomposable and not infinite cyclic for each $i = 1, ..., p$, and F_k is a free group of rank k. We say that this decomposition is *non-trivial* if either $(k > 0, p \geq 1)$ or $p > 1$.

Moreover, Grushko's unicity theorem states that if moreover

$$G \simeq G'_1 * ... * G'_{p'} * F_{k'}$$

where G_i is freely indecomposable and F_k free, then $p' = p, k' = k$ and for any $i \in \{1, ..., p\}$, there is some $j \in \{1, ..., p\}$, such that G'_i is conjugate to G_j.

This unicity needs to be understood to be rather weak in the context of this thesis. We will call Grushko decomposition of G any graph-of-group decomposition for which the vertex groups are the conjugates of the G_i, $i = 1, ..., p$, and the trivial group, and the edge groups are trivial. This leaves many possibilities as we illustrated in the Figure 1.

Hence in the following parts of this thesis, we fix a group G and a non-trivial free product decomposition $G \simeq G_1 * ... * G_p * F_k$. However, we will not always assume that each G_i is freely indecomposable. Given a free product decomposition with the above notation, by Bass-Serre theory, G acts on a tree T with the following properties:

- 1. G acts on tree T without edge-inversions, and without fixed point;
- 2. all edge stabilizers are trivial;
- 3. any vertex stabilizer is either $\{1\}$ or conjugate to G_i for some $i = 1, ..., p$, and each G_i fixes a vertex;
- 4. this action is minimal: no proper subtree of T is invariant under G.

2.7 Outer space of free products

Following Guirardel and Levitt, we recall the definition of outer space. We refer to [30].

Recall that $T(G)$ is the set of metric G trees.

Definition 2.1 *Let $\mathbb{G} = \mathbb{G}(G, (G_i)_{i=1}^{p}, F_k)$ be the subset of $T(G)/\sim$ of simplicial, metric G-trees T, up to equivarient isometry, such that*

- *(C0) T has no redundant vertices;*
- *(C1) the G-action of T is minimal (i.e. there is no proper invariant subtree), with trivial edge stabilizers;*
- *(C2) for each $i = 1, ..., p$, there is one orbit of vertices with stabilizer conjugate to G_i, and each G_i fixes a vertex;*
- *(C3) the stabilizers of all other vertices are trivial. These vertices are referred to as* free *vertices.*

Definition 2.2 ([21], Definition 3.2) *Let G be a free product in the form of $G_1 * ... * G_p * F_k$, we define $Aut(G, \mathbb{G})$ (also denoted by $Aut(G, ([G_1], \dots, [G_p]))$, or $Aut(G; (G_i)_{i=1}^p)$) to be the subset of $Aut(G)$ satisfies the following:*

for any automorphism $\Phi \in Aut(G)$, Φ is in the subset $Aut(G, \mathbb{G})$ if and only if for all $i \in \{1, \dots, p\}$, there is a element $g_i \in G$ and $j \in \{1, \dots, p\}$ such that, $\Phi(G_i) = g_i^{-1} G_j g_i$.

Since $Inn(G)$ is a normal subgroup of $Aut(G, \mathbb{G})$, we define $Out(G, \mathbb{G})$ (also denoted by $Out(G, ([G_1], \dots, [G_p]))$, or $Out(G; (G_i)_{i=1}^p)$) to be $Aut(G, \mathbb{G}))/Inn(G)$.

The multiplication of all lengths of the edges by the same positive number is a transformation from $\mathbb{G}$ to $\mathbb{G}$. And therefore one can define a space $P\mathbb{G} = P\mathbb{G}(G, (G_i)_{i=1}^p, F_k)$ from the space $\mathbb{G}$, to be the projective space for this rescaling process. It is called the *outer space* of G relative to $G_1, ..., G_p$.

If G is free (the collection of G_i is empty), this is isomorphic to Culler-Vogtmann's outer space for free groups, about which there is abundant literature (for instance, [4]). Note that Culler and Vogtmann's outer space is a space of marked metric graphs, where the space Guirardel and Levitt defined is a space of metric G-trees, but the correspondence is clear, through passing to universal covers, with deck transformation actions.

If G is finitely generated, and each of the G_i is freely indecomposable, we obtain the space of all Grushko decompositions.

Consider the case when $G = F_n$, and G_i are free factors, but not necessarily freely indecomposable. For instance, take $G = F_n$ and select some free factors that, for some reason, we want to see preserved. Let $G_1, \ldots, G_k$ be these free factors. Then, these G_i are free groups (whose sum of ranks is less than n) and the space $\mathbb{G}$ is the relative outer space associated to the free factor system as studied for instance by Radhika Gupta in her thesis ([32], [31]).

Let us notice that $Out(G; \{(G_i)_{i=1}^p\})$ acts on $\mathbb{G}$ in the following way:

If G acts on a tree T in $\mathbb{G}$, as $\alpha : G \times T \to T$, and if $\Phi \in Out(G)$, choose $\phi \in Aut(G; (G_i)_{i=1}^p)$ a representative of Φ, then G acts on T in the following way: $(g, x) \mapsto (\phi(g)x) \in T$. First, the new G-tree T (endowed with the new action) is in $\mathbb{G}$ since the new stabilizers of vertices are the images of the conjugates of the G_i by ϕ^{-1}, and therefore are the conjugates of the G_i, and the stabilizers of edges are still trivial since they are intersection of different conjugates of the G_i. Second, the new G-tree T does not depend on the choice of ϕ in the class of Φ, because precomposing the action by a conjugation in G amounts to precomposing the action by an equivarient isometry of T, and therefore remains in the same $\sim$-class.

One of the main results of [30] is the description of a geodesic between any two points in $P\mathbb{G}$, in terms of a folding path. This allows to describe a geodesic between a tree and its image by an outer automorphism. In particular, since every folding move is rather simple, this decomposes any automorphism into simple moves.

2.8 From automorphisms to maps between trees

Definition 2.3 *(representative map of an automorphism) [[21], Section 8.1] For $T \in \mathbb{G}$ and $\Phi \in Aut(G, \mathbb{G})$, we say that a map $f : T \to T$* represents Φ *on T if for any $g \in G$ and $t \in T$, the equation $f(gt) = \Phi(g)(f(t))$ holds.*

If moreover the action is minimal on the tree T, and f continuous, it implies that f is surjective.

All the maps we consider in this thesis will be continuous and even Lipschitz.

The existence of such a map is shown in the following lemma, which is proven by Francaviglia and Martino.

Lemma 2.4 ([21], Lemma 4.2) *Given any $T \in \mathbb{G}$ and $\Phi \in Aut(G, \mathbb{G})$, there exist a map $f : T \to T$ representing Φ on T. Moreover, if $f_1 : T \to T, f_2 : T \to T$ are 2 different maps representing Φ on T, they coincide on non-free vertices.*

2.9 Deformation spaces

We recall the definition of deformation spaces. We refer to [27].

By Theorem 5.12 of [21], for any $A, B \in \mathbb{G}$, and for any $\Phi \in Aut(G)$, there exists a G-equivariant map $f : A \to B$ that is piecewise linear with minimal Lipschitz constant that represents Φ.

It also follows that (by reparametrising the map), for any $A, B \in \mathbb{G}$, and for any $\Phi \in Aut(G)$, there exists a map $f : A \to B$ representing Φ that is piecewise linear.

Consider two trees T_1 and T_2, we say that T_1 *dominates* T_2 if there exist an equivariant map $T_1 \to T_2$ (see [27]). It is equivalent to say that, if a group is elliptic in T_1, then it is elliptic in T_2.

The *deformation space* $DF(T)$ of a tree T is the set of metric trees T' such that T and T' dominates each other, up to equivariant isometry. Equivalently, it is a set of trees T' such that T' have the same elliptic subgroups as with T. For example, if we fix G with a free product decomposition like (GD), and if we see $\mathbb{G} = \mathbb{G}(G, (G_i)_{i=1}^p, F_k)$ as a space of G-trees, then $\mathbb{G}$ is the deformation space for T_0 the tree of the free product.

Note that for $g \in G$, g is not hyperbolic for some $T \in \mathbb{G}$ if and only if g lies in a G-conjugate of some G_i, $i \in \{1, ..., p\}$ (see also section 4.1, [21]). Therefore $g \in G$ is hyperbolic for $T \in \mathbb{G}$ if and only if it is hyperbolic for every element in $\mathbb{G}$. So we denote the set of hyperbolic elements of G for some (and for all) $T \in \mathbb{G}$ by $Hyp(\mathbb{G})$.

3 Train tracks on marked graphs, and on trees

In this section, we define, and recall results about train tracks on marked graphs, or on G-trees. We refer to [21], where Francaviglia and Martino described the train track maps on the G-trees and proved some result of irreducible automorphisms of free product. We will also define irreducibility, and atoroidality in the context of automorphisms of free products.

3.1 Irreducibility

Recall that an automorphism of a free group F is *irreducible* if it does not preserve the conjugacy class of any proper free factor. It is *fully irreducible* if all its positive powers are irreducible.

In the case of automorphisms of free product, Francaviglia and Martino have adapted the definition of irreducibility.

We recall two definitions of irreducibility of free product automorphisms from [21], and prove that these definitions are equivalent (as was stated in [21]).

For $g \in G$, we denote by $[g]$ its conjugacy class in G; likewise, for a subgroup H of the group G, denote by $[H]$ the conjugacy class of H in G.

Given a group $G = G_1 * ... * G_p * G'$, the set $\{[G_1], ..., [G_p]\}$ is called a *free factor system* for G. If there is another free factor system $\{[G'_1], ..., [G'_{p'}]\}$ for G such that, for each $i \in \{1, \dots, p'\}$ there is some $j \in \{1, \dots, p\}$ satisfying $G'_i \leq gG_jg^{-1}$ for some $g \in G$, then we declare $\{[G'_1], ..., [G'_{p'}]\} \prec \{[G_1], ..., [G_p]\}$.

Proposition 3.1 *The relation $\prec$ defined as above is a partial order.*

Proof: If $\{[G_1], ..., [G_p]\} \prec \{[G'_1], ..., [G'_{p'}]\}$, and if $\{[G'_1], ..., [G'_{p'}]\} \prec \{[G_1], ..., [G_p]\}$, then for any i, there is some j, g, g' such that $g'G_jg'^{-1} \leq G'_i \leq gG_jg^{-1}$, thus $g^{-1}g \in G$, $g'G_jg'^{-1} = G'_i = gG_jg^{-1}$. Hence we have $[G'_i] = [G_j]$, then $\{[G_1], ..., [G_p]\} = \{[G'_1], ..., [G'_{p'}]\}$. And since the reflexivity and transitivity is obvious from the definition of the relation $\prec$, it follows that the relation $\prec$ is a partial order relation. □

Definition 3.2 *(Irreducible and fully irreducible maps) Let Φ be an automorphism of a group G, $T \in \mathbb{G}$ and $f : T \to T$ represent $\Phi \in Aut(G)$. We call f* irreducible*, if for every* *proper subgraph W of the tree T that is G-invariant and f-invariant, the quotient graph $G\backslash W$ is a collection of isolated subtrees with at most one non-free vertex. We say f is* fully irreducible *if for any integer $i > 0$, f^i is irreducible.*

Definition 3.3 *($\mathbb{G}$-irreducible and $\mathbb{G}$-fully-irreducible automorphisms) We say $\Phi \in Aut(G, \mathbb{G})$ is $\mathbb{G}$-irreducible if for any $T \in \mathbb{G}$ and for any $f : T \to T$ representing Φ, f is irreducible. Likewise, we refer Φ to as* fully irreducible *if f is $\mathbb{G}$-fully-irreducible (or fully irreducible for short) for any $T \in \mathbb{G}$ and for any $f : T \to T$ representing Φ.*

Definition 3.4 *(Irreducible automorphisms relative to a free factor system) Given a group G with a free product decomposition $G = G_1 * ... * G_p * F_k$, an automorphism $\Phi \in Aut(G)$ is said to be* irreducible relative to the free factor system $\{[G_1], ..., [G_p]\}$ *if $\{[G_1], ..., [G_p]\}$ is a maximal (under the order $\prec$) proper free factor system that is invariant under Φ.*

Lemma 3.5 *Given a group G with a free product decomposition $G = G_1 * ... * G_p * F_k$ and given $\Phi \in Aut(G, \mathbb{G})$. The following two statements are equivalent:*

- *(1). Φ is $\mathbb{G}$-irreducible;*
- *(2). Φ is irreducible relative to the free factor system $\{[G_1], ..., [G_p]\}$.*

We first give the following lemma, the proof of which is standard (because edge stabilizers are trivial), thus we do not show the proof in this thesis.

Lemma 3.6 *Let G be a group with a given free product decomposition $G = G_1 * ... * G_p * F_k$. For any tree $T \in \mathbb{G}$, if*

$g_1 \in G$ fixes vertex v_1, $g_2 \in G$ fixes vertex v_2, and if $v_1 \neq v_2$, then $g_1 g_2$ is hyperbolic in G relative to this free product decomposition.

Proof: [of Lemma 3.5]

Assume that Φ is not irreducible relative to the free factor system $\{[G_1], ..., [G_p]\}$, then there is another Φ-invariant free factor system $\{[H_1], ...[H_{p'}]\}$ such that $\{[G_1], ..., [G_p]\} \subseteq \{[H_1], ...[H_{p'}]\}$. Without loss of generality, we may assume that $G_1 < hH_1h^{-1}$ for some $h \in G$. Denote here $H_1' = hH_1h^{-1}$.

Define W to be a minimal H_1' invariant subgraph in T, and let $f : T \to T$ representing Φ such that $f|_W : W \to W$ represents $\Phi|_{H_1'}$.

From the previous lemma, we know that there is a hyperbolic element $g \notin H_1'$ relative to $\{[G_1], ..., [G_p]\}$(if we find an elliptic element $g_2 \notin H_1'$, let $g_1 \in G_1$, since they fix different vertices, $g = g_1 g_2$ is a hyperbolic element not in H_1'). Notice that the quotient $H_1' \backslash W$ is finite (because H_1' is finitely generated). Let A_g be the axis of g, A_g is not contained in W (otherwise, $H_1' A_g \subset W$, A_g maps to the quotient by a finite path. By decomposition we have that $g \in H_1'$, a contradiction). Therefore, W is a proper subgraph of T.

By Kurosh Subgroup theorem, H_1 either contains a subgroup that is conjugate to a subgroup of G_i $(i > 1)$ or contains a free group as a subgroup. So the quotient $G \backslash W$ is not a disjoint union of trees each of which consists at most 1 non-free vertex.

Thus we have proved that Φ is not $\mathbb{G}$-irreducible if it is not irreducible relative to the free factor system $\{[G_1], ..., [G_p]\}$.

To prove that (2) implies (1): suppose now that Φ is irreducible relative to the free factor system $\{[G_1], ..., [G_p]\}$. Let $T \in \mathbb{G}$ be a tree, and let $f : T \to T$ be a map representing

$\Phi \in Aut(G)$. If there is a proper G-subset W in T that is invariant under f, by collapsing each connected component of W to a point G equivariantly, we have another tree $T' \in \mathbb{G}$ in the different deformation space from T. In the quotient graph, this point corresponds to a free factor of G. As Φ is irreducible relative to $\{[G_1], ..., [G_p]\}$, we conclude that the quotient is a disjoint union of trees each of which contains at most 1 non-free vertex.

□

3.2 Atoroidality and Nielsen paths

Recall that $[g]$ is the conjugacy class of g in G for $g \in G$.

Recall that an automorphism ϕ of a free group F is said to be *atoroidal* if for any non trivial conjugacy class $[x]$ of F, $[\phi^n(x)] \neq [x]$ for all positive integer n. The explanation of the terminology is more clear when one considers the negation. An automorphism ϕ is *toroidal* (i.e. not atoroidal) if there is $n > 0$ and $g \neq 1$ in F, and $h \in F$ such that $\phi^n(g) = hgh^{-1}$. In the semidirect product $F \rtimes_\phi \mathbb{Z}$, the two elements g and $t^n h$ (where t is the element of $\mathbb{Z}$ inducing ϕ on F) generate an abelian group, which is easily seen to be free of rank 2. That is the fundamental group of a torus, embedded in $F \rtimes_\phi \mathbb{Z}$, as discussed in the section "Automorphisms, suspensions, semidirect product". Topologically, the suspension of a rose (whose fundamental group is F) by a map realising ϕ also contains the π_1-injective image of a 2-torus, namely the suspension of a loop representing g in the rose.

In the free product case, we cannot define atoroidality like that. We need to consider only elements that are not conjugate in one of the free factors $G_1, \ldots, G_p$.

Definition 3.7 *(Atoroidal automorphisms) Let* $\Phi \in Aut(G, \mathbb{G})$.

We say that Φ *is* atoroidal*, if for any* $g \in Hyp(\mathbb{G})$*, and for any positive integer* n*,* $[\Phi^n(g)] \neq [g]$*.*

It is necessary to allow n to be different from 1, for instance in a free group $F_{\{a,b\}}$, the automorphism defined by $a \mapsto b$ and $b \mapsto ab$ sends the commutator $[a, b]$ to its inverse, and therefore is toroidal (its square preserve the commutator), but does not preserve any conjugacy class.

Related, we need the following,

Definition 3.8 *(Nielsen and pre-Nielsen paths) Let* G *be a group,* T *be a* G*-tree, and* Φ *be an automorphism of* G*,* $f : T \to T$ *be a map representing* Φ*. A path* ρ *is called a* Nielsen *path if there exist an exponent* $n > 0$ *and an element* $g \in G$ *such that* $f^n(\rho) = g\rho$ *after reduction. A path* ρ *is called* pre-Nielsen *if there exist an exponent* $M > 0$ *such that* $f^M(\rho)$ *is Nielsen.*

Observe that even if Φ is atoroidal , there can be Nielsen paths: they do not map on closed loops in $G \backslash T$.

3.3 Train Tracks Maps

Definition 3.9 *(Train track structure, legal turn, and legal paths)*

Given a graph X*, an ordered pair* (e_1, e_2) *of oriented edges such that* $i(e_1) = i(e_2)$ *is called a* turn *(at the vertex* $i(e_1)$*). A trivial turn is a turn of the form* (e, e)*.*

A train track structure *(or a gate structure) on a* G*-tree* T *is a* G*-invariant equivalence relation on the set of oriented edges at each vertex of* T*, with at least two equivalence classes at each vertex.*

Each equivalence class of oriented edges is referred to as a gate*.*

In a gate structure, a turn is said to be legal *if the two oriented edges are in different equivalent classes. A reduced path is said to be legal if every turn of it are legal.*

To describe a gate structure, it is enough to specify which turns are legal (or illegal).

An important example of gate structure is the one given as follows (and this is the one we will use). Consider T and T' two G-trees, as well as a map $f : T \to T'$ which is equivariant, and piecewise linear (linear, non constant, on edges). Define the gate structure on T induced by f as follows. Declare that a turn (e_1, e_2) is illegal if $f(e_1)$ and $f(e_2)$ share their first edge in T'. It is easy to check that this defines an equivalence relation on the oriented edges issued from a same vertex, and that it is invariant for G, by equivariance of f.

In this construction, it is obvious that any legal turn is send by f on a pair of paths whose first edges define non-trivial turn (by abuse of language we say that any legal turn is send by f on a non-trivial turn). However, if $T' = T$, in principle, a legal turn could be sent on an illegal turn, and in that case f^2 would send a legal turn to a trivial turn. This is not a pleasant situation, and motivates the following.

Definition 3.10 *(Train track maps) Given $T \in \mathbb{G}, \Phi \in Aut(G, \mathbb{G})$, and given $f : T \to T$ a piecewise linear G-equivariant map (linear, non constant, on edges) representing Φ, we say that f is a* train track map *if, for the gate structure it defines,*

- *f maps edges to legal paths;*
- *if $f(v)$ is a vertex, then f maps legal turns at v to legal turns at $f(v)$.*

One of the main results of [21], due to Francaviglia and Martino is the following, and will be important for us.

Theorem 3.11 *(see Theorem 8.18 of [21]) If* $\Phi \in Aut(G, \mathbb{G})$ *is irreducible, then there exist* $T \in \mathbb{G}$ *and* $f : T \to T$ *representing* $\Phi \in Aut(G)$*, such that* f *is a train track map.*

4 Relative Hyperbolicity of an Automorphism in Fully Irreducible Case

We review vocabulary and materials from Gautero and Weidmanns' contribution ([22]). However we begin with definition of relatively hyperbolic groups that goes back to Bowditch ([6]) and Farb ([18]).

Let G be a finitely generated group with a generating set S and a Cayley graph $\Gamma_S(G)$. Let Λ be a set and let $\mathbb{H} = \{H_i\}_{i\in\Lambda}$ be a family of subgroups H_i of G.

Definition 4.1 *($\mathbb{H}$-coned graph) The* $\mathbb{H}$-coned graph $\Gamma_S^{\mathbb{H}}(G)$ *is a graph obtained from* $\Gamma_S(G)$ *by adding a vertex* $v(gH_i)$ *for each left coset* gH_i *and adding an edge of length* $\frac{1}{2}$ *between* $v(gH_i)$ *and each* v *for* $v \in \Gamma_S(G) \cap gH_i$*.*

Definition 4.2 *Given a group* G *and* $\phi \in Aut(G)$*, we say that* ϕ *is a* hyperbolic automorphism *of* G*, if* $G \rtimes_\phi \mathbb{Z}$ *is Gromov-hyperbolic.*

Definition 4.3 *($\mathbb{H}$-word metric, relatively hyperbolic group) The* $\mathbb{H}$-word metric $|\cdot|_{\mathbb{H}}$ *is the word-metric for* G *equipped with generating set* $S_{\mathbb{H}} = S \cup (\cup H_i)$ *where* $\cup H_i$ *is the union of all* H_i *in* $\mathbb{H}$*;*

The group G *is* hyperbolic relative to $\mathbb{H}$ *if the following holds:*

- *$\Gamma_S^{\mathbb{H}}(G)$ is Gromov hyperbolic;*
- *for any positive integer n, any edge in $\Gamma_S^{\mathbb{H}}(G)$ is contained in finitely many embedded loops of length n (called "fineness" property).*

We say that a group G is relatively hyperbolic *if there is a family $\mathbb{H}$ of subgroups of G such that G is hyperbolic relative to $\mathbb{H}$.*

Definition 4.4 *(Mapping torus of a subgroup) Let H be a subgroup of a group G, and $\phi \in Aut(G)$. Suppose that n_H is the smallest positive integer such that there exist $g_H \in G$ satisfying $\phi^{n_H}(H) = g_H^{-1} H g_H$.*

Then the semidirect product $H \rtimes_{\mathrm{ad}_{g_H} \circ \phi^{n_H}|_H} \mathbb{Z}$ is referred to as the mapping torus of H.

Likewise, for $\mathbb{H}$ a set of subgroup of G, the mapping torus of $\mathbb{H}$ *is defined as $\{H \rtimes_{\mathrm{ad}_{g_H} \circ \phi^{n_H}|_H} \mathbb{Z} : H \in \mathbb{H}\}$.*

Note that for the free product $G = G_1 * \cdots * G_p * F_k$, and for $\phi \in Aut(G, \mathbb{G})$, the mapping torus of each G_i is well-defined since ϕ does preserve the free factor system $\{[G_1], ..., [G_p]\}$ (set wise).

We may now define what it means to be a relatively hyperbolic automorphism.

Definition 4.5 *(Relatively hyperbolic automorphisms) Let G be a group. Let Λ be a set and let $\mathbb{H} = \{H_i\}_{i \in \Lambda}$ be a family of subgroups H_i of G such that each H_i is its own normalizer. An automorphism $\Phi \in Aut(G, \mathbb{H})$ is* hyperbolic relative to $\mathbb{H}$ *(or in short,* relatively hyperbolic*) if it satisfies the following:*

there exist $\lambda > 1$, $M, N \geq 1$, such that for any $g \in G$ with $|g|_{\mathbb{H}} \geq M$, the inequality holds:

$$\lambda |g|_{\mathbb{H}} \leq \max\{|\Phi^N(g)|_{\mathbb{H}}, |\Phi^{-N}(g)|_{\mathbb{H}}\}$$

Recall that for a path ρ in a metric tree T, $l_T(\rho)$ is its length. Assume now that G is a free product.

Definition 4.6 *Let T, T' be a pair of metric trees in $\mathbb{G}$. Let $\alpha : T \to T'$ and $\alpha' : T' \to T$ be G-equivariant Lipschitz homotopy equivalences between these trees.*

Let $\Phi \in Aut(G, \mathbb{G})$, and consider $f : T \to T$, $f' : T' \to T'$ homotopy equivalences representing $\Phi \in Aut(G, \mathbb{G})$ and $\Phi^{-1} \in Aut(G, \mathbb{G})$ respectively.

If there exist a natural number $M > 0$ and a real number $\lambda > 1$ for trees T, T' and maps α, α', f, f', such that for all $\mathbb{G}$-hyperbolic element $g \in G$, for all fundamental segment σ for g in T, and σ' the reduced path of $\alpha(\sigma)$ (hence a fundamental segment for g in T') one has:

$$\lambda l_T(\sigma) \leq \max\{l_T[f^M(\sigma)], l_T([f'^M(\sigma')])\}$$

then Φ is refered to as a hyperbolic automorphism *of $(G, \{G_1, \ldots, G_p\}, T, T'f, f')$, and any path (does not required to be a fundamental segment of a hyperbolic element) satisfying the above inequality is said to have the* desired growth.

Remark 4.7 *Notice that, given G and $G \simeq G_1 * ... * G_p * F_k$ a free product decomposition, let $\mathbb{H}_0 = \{G_1, ..., G_p\}$,let T be in $\mathbb{G}$, then by Bass-Serre Theory, the $\mathbb{H}_0$-word metric is quasi-isometric to the length in the Bass-Serre tree. Therefore, if Φ is a hyperbolic automorphism of $(G, \{G_1, \ldots, G_p\}, T, T'f, f')$, then it is relatively hyperbolic.*

The relevance of hyperbolic automorphisms is through the following combination theorem , which is proven by Gautero and Weidmann:

Theorem 4.8 *(Corollary 7.3, [22]) Let G be a finitely generated group, and α be an automorphism of G. Let $\mathbb{H}_0$ be a finite family of infinite subgroups of G and α be hyperbolic relative to $\mathbb{H}_0$. If G is hyperbolic relative to $\mathbb{H}_0$, then the semi-direct product $G \rtimes_\alpha \mathbb{Z}$ is relatively hyperbolic (more precisely, $G \rtimes_\alpha \mathbb{Z}$ is hyperbolic relative to the mapping-torus of $\mathbb{H}_0$) .*

Our main result in this section is the following. It is an analogue of Brinkman's first result in [7], and of a result of Bestvina-Feighn-Handel for free groups in [2]

Theorem 4.9 *Let G be a finitely generated group with a given free product decomposition $G \simeq G_1 * ... * G_p * F_k$ (where $k \geq 2$ or $p + k \geq 3$), and let $\mathbb{H}_0 = \{G_1, ..., G_p\}$. Assume that $\Phi \in Aut(G, \mathbb{H}_0)$ is fully irreducible and atoroidal. Then the semi-direct product $G \rtimes_\Phi \mathbb{Z}$ is relatively hyperbolic.*

Actually we can say that $G \rtimes_\Phi \mathbb{Z}$ is hyperbolic relative to the mapping-torus of $\mathbb{H}_0$.

4.1 Growth of Edges

We will use several technical results of Francaviglia and Martino in [21] in this subsection, and we will give each precise result from their work.

Lemma 4.10 ([21], Lemma 8.16) *If $f : T \to T$ is a train track map representing an irreducible automorphism Φ, then there is a constant λ such that $l_T(f(e)) = \lambda l_T(e)$ for all edge e in T after a re-scaling of edges.*

The authors gave a proof of this lemma in their paper, but we still would like to show another proof of this lemma.

Proof: Let l_i be the length of the edge e_i (and so the length of ge_i for any $g \in G$) in the tree T, and consider the non-negative matrix M with the element $M_{i,j}$ denoting the total time of $f(e_i)$ passing through the G-orbit of e_j. This matrix, and its transpose,${}^tM$, are irreducible because of the irreducibility of Φ. By Perron-Frobenius Theorem, there is a unique maximal eigenvalue of ${}^tM$, denoted by λ that is strictly larger than 1. Moreover, an eigenvector of this eigenvalue has positive entries (and this eigenvector is unique up to multiplication of a real number). We scale it such that its smallest entry is 1, denote by v_0 such an eigenvector. Set l_i to be the corresponding entry of the eigenvector. Hence, for every vector v, we have that

$$^tv_0 \cdot (Mv) = (^tv_0 \cdot M)v =^t (^tM \cdot v_0)v = \lambda^t v_0 \cdot v$$

Take $v = (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ respectively, and for this re-scaling of edges, every edge is stretched by the same factor λ. □

By the proof of the previous lemma, the constant λ is unique (which is the eigenvalue corresponding to the unique eigenvector). Hence the following definition becomes natural:

Definition 4.11 *(Growth rate of a train track map) Let $\Phi \in Aut(G, \mathbb{G})$ be irreducible, $f : T \to T$ be a train track map representing Φ. Rescale the edge such that every edge in the tree stretch with the same factor λ. Such a factor is called the* growth rate *of f.*

Lemma 4.12 (Lemma8.20, section 8.3, [21]) *If $f : T \to T$ is a piecewise linear map representing Φ, and if f is a train track map, then so is f^k which represents Φ^k.*

For the following lemma, in order to explain the assumption, notice that in a graph-of-group decomposition of $G = G_1 * ... * G_p * F_k$, giving a free product decomposition in the same deformation space, the number of edges is at least $\max\{p-1, 0\} + k$. Indeed, the Euler Characteristic of the underlying graph is $-(k-1)$ (because the rank of the free group factor is k) and the number of vertices is at least $\max\{p, 1\}$. It follows that the number of edges is at least $\max\{p, 1\} + k - 1$. This clarifies that if either $k \geq 2$ or $p + k \geq 3$, then T has at least two orbits of edges.

Lemma 4.13 *Let G be a group with free product decomposition $G = G_1 * ... * G_p * F_k$, Φ be an automorphism of G relative to the free factor system $\{[G_1], \dots, [G_p]\}$. Assume that either $k \geq 2$ or $p + k \geq 3$ and that Φ is fully irreducible. Let $T \in \mathbb{G}$, and $f : T \to T$ be a train track representative for ϕ on T. Then the growth rate of f is strictly larger than 1.*

Proof: By the assumption there are at least two orbits of edges.

Fix a shortest fundamental segment σ in T of a hyperbolic element in G. Due to the finiteness of the orbits of edges, there are only finitely many paths (up to translation) in T that are not longer than this length, and hence there are only finitely many (up to translation) fundamental segments $\sigma = \sigma_1, \dots, \sigma_s$ (denote by $t_1, \dots, t_s$ the corresponding hyperbolic elements in G) in T that has the same length as σ.

If the growth rate is 1, then each fundamental segment σ_i of t_i is sent to a segment of the same length (which is a fundamental segment of $\Phi(t_i)$). Thus some power of f sends σ_i to the translate of $\sigma_{w(i)}$. As Φ is an automorphism, we can assume w to be a permutation of $\{1, \dots, s\}$ so that $f^n(\sigma_i) = h_i \sigma_i$ for some i, n and $h_i \in G$. Denote by v_0 the ending vertex of

σ_i, notice that $h_i A_i = f^n(A_i) = f^n(t_i A_i) = \Phi^n(t_i) f^n(A_i) = \Phi^n(t_i) h_i A_i$, where A_i is the axis of t_i. Thus $h_i^{-1}\Phi^n(t_i)h_i$ fixes the axis of t_i. It follows that $h_i^{-1}\Phi^n(t_i)h_i = t_i g_{v_0}$, where g_{v_0} fixes v_0. Denote by G_v the stabilizer of v_0, we also have that $h_i^{-1}\Phi^n(G_v)h_i = G_v$. Therefore, the free factor $\langle t_i, G_v \rangle$ is invariant under some power of Φ, by the assumption of p and k, this is a proper free factor. This contradicts the irreducibility, and thus the growth rate of f is strictly larger than 1. $\square$

4.2 Angle Analysis on each vertex

When a graph is not locally finite, it is often helpful to have a notion of angle between adjacent edges, that brings back some local finiteness. In the context of fine graphs, angles are defined metrically (see Bowditch [6], Dahmani-Yaman [17], and Dahmani [8]). Although trees are fine, this is not useful for us here (as all angles would be infinite). We thus proceed differently.

Definition 4.14 *For each i, fix a word metric $|\cdot|_i$ on G_i, let v_{G_i} be a vertex of a tree T that is fixed by G_i. Choose once and for all a transversal set of adjacent edges (a minimal set of a choice of adjacent edges, where every other adjacent edge is the translation of one of the edge in the set) of v_{G_i} for the action of its stabiliser $\varepsilon_0, ...\varepsilon_r$. For each pair of edges e, e' adjacent to v_{G_i}, we define the angle $Ang_{v_{G_i}}(e, e')$ to be the word length of $g^{-1}g'$, where g, g' satisfies $ge_1 \in \{\varepsilon_0, ...\varepsilon_r\}, g'e_2 \in \{\varepsilon_0, ...\varepsilon_r\}$.*

Remark 4.15 *We notice that the angle $Ang_{v_{G_i}}(e, e')$ as above is well-defined: the choice of g and g' is unique, because stabilizer of each edge (and thus of $\varepsilon_0, ...\varepsilon_r$) is trivial.*

We now define angles on other non-free vertices:

Definition 4.16 *(Angles at a vertex) For each i, for each $v \in Gv_{G_i}$, and for each pair of edges e_1, e_2 adjacent to v, define the angle $Ang_v(e_1, e_2)$ to be $Ang_{v_{G_i}}(ge_1, ge_2)$ where g is the element in G such that $gv = v_{G_i}$.*

Remark 4.17 *In general, the choice of g in the definition is not unique, but only differs from an element in G_i, and by the definition of angles at the vertex whose stabilizer is G_i, element in G_i preserves the angle (to be more exact, for any $g' \in G_i$, $Ang_{v_{G_i}}(ge_1, ge_2) = Ang_{v_{G_i}}(g'ge_1, g'ge_2)$), so the angle $Ang_v(e_1, e_2)$ here is still well-defined.*

We also notice that the angle is locally finite: for a given edge e_1 with starting vertex v and a given number $C > 0$, there are only finitely many possible e_2 satisfying $Ang_v(e_1, e_2) < C$. This is easy to see as there are only finitely many edges (up to G-orbit) adjacent to v and that there are only finitely many elements in G_i whose word length is bounded by C.

Finally, we notice that the angle is invariant under any translation by any element $h \in G$. More precisely, if e, e' are adjacent to vertex v, then $Ang_v(e, e') = Ang_{hv}(he, he')$. This is obvious as the angle equals the corresponding angle at v_{G_i} (which is well-defined).

Definition 4.18 *(Angles between paths) For any free vertex v, and for each pair of edges e_1, e_2 adjacent to v, define the angle $Ang_v(e_1, e_2)$ to be 1.*

Let v be a vertex in a tree T, ρ_1, ρ_2 be two paths starting from v, and let e_1, e_2 be the starting edge of ρ_1, ρ_2 respectively (hence these two edges also start from v). We define $Ang_v(\rho_1, \rho_2)$ to be $Ang_v(e_1, e_2)$.

Definition 4.19 *("Boundedness after iteration" of angle of a vertex on a path) Let $\Phi \in Aut(G, \mathbb{G})$ be an automorphism of*

G, $f : T \to T$ be a map representing Φ, let ρ be a path, v be a vertex on ρ that is neither the starting nor the ending vertex. Denote by e_1, e_2 be the adjacent edges on ρ containing v (hence the ending vertex of e_1 is the starting vertex of e_2, which is v). We say that angle at v is bounded after iteration*, if there is a constant C such that $Ang_{f^n(v)}(\overline{f^n(e_1)}, f^n(e_2)) < C$ for all integer $n > 0$.*

Definition 4.20 *(Θ-straight paths, Θ-patterns) Let ρ be a (reduced) path in the tree T.*

A subdivision *of the path is a choice of reduced subpaths $\rho_1, \ldots, \rho_k$ such that ρ is the concatenation of $\rho_1 - \cdots - \rho_k$, where each subpath ρ_i ($i \in \{1, \ldots, k\}$) is called a* component *of the path ρ.*

We say that ρ is Θ-straight *if all angles between consecutive edges are bounded by Θ.*

Two paths ρ, ρ' with subdivisions $\rho_1, \ldots, \rho_k$ and $\rho'_1, \ldots, \rho'_k$ respectively are G-equivalent *if for all i, ρ'_i is in the G-orbit of ρ_i.*

A Θ-pattern *is an equivalence class for this relation where all components are Θ-straight.*

Lemma 4.21 *Given a tree $T \in \mathbb{G}$, with any given choice of measure of angle, for all $C > 0$ and for all $\Theta > 0$, there are only finitely many Θ-patterns not longer than C.*

Proof: For any given C, there are only finitely many possible subdivisions of a path ρ not longer than C such that each component is Θ-straight, and denote by $\rho_1, \ldots, \rho_k$ any choice of subdivision of this kind. Since there are only finitely many possible Θ-straight paths of the length equal to ρ_i up to G-action for all $i \in \{i, ..., k\}$, there are finitely many choice of ρ' (up to G-action) with some subdivision $\rho'_1, \ldots, \rho'_k$ that is not G-equivalent to ρ with subdivision $\rho_1, \ldots, \rho_k$. Therefore, there are only finitely many Θ-patterns not longer than C. $\square$

Lemma 4.22 *Let $\Phi \in Aut(G, \mathbb{G})$ be an automorphism of G, $f : T \to T$ be a map representing Φ. Then for any $\Theta_1 > 0$, there exists $\Theta_2 > 0$, such that for any vertex v and for any pair of edges e_1, e_2 starting from vertex v with $Ang_v(e_1, e_2) > \Theta_2$, we have that $Ang_{f(v)}(f(e_1), f(e_2)) > \Theta_1$.*

Proof: Assume that the stabilizer of v in G is H (and $H = h_1 G_i h_1^{-1}$ for some G_i and $h_1 \in G$). Since f represents Φ, $f(v)$ is fixed by $\Phi(H)$, assume that $\Phi(H)$ is conjugate to G_j. Then there is an induced quasi-isometric map from the Cayley Graph of G_i to the Cayley Graph of G_j. Thus we can find $\Theta_2 > 0$, such that the pre-image of the subset of the Cayley Graph of G_i whose vertices are all the elements with word-length not larger than Θ_1 is contained in the subset of the Cayley Graph of G_j whose vertices are all the elements with word-length not larger than Θ_2. Equivalently, the subset of the Cayley Graph of G_j with word metric strictly larger than Θ_2 is sent to subset of the Cayley Graph of G_i with word metric strictly larger than Θ_1. For such a Θ_2, $Ang_f(v)(f(e_1, f(e_2))) > \Theta_1$. $\square$

4.3 Growth of paths

Directly from the above lemma, we have:

Lemma 4.23 *Let G be a group with free product decomposition $G = G_1 * \ldots * G_p * F_k$, Φ be an automorphism of G relative to the free factor system $\{[G_1], \ldots, [G_p]\}$. Assume that either $k \geq 2$ or $p+k \geq 3$ and that Φ is fully irreducible and atoroidal. Let $T \in \mathbb{G}$, $f : T \to T$ be a train track representative for ϕ*

on T. Then for any given element h hyperbolic in G, any fundamental segment τ of h in T, and for any $C > 0$, there is an integer $N > 0$ such that $l_T(f^N(\tau)) > C$.

Proof: Suppose otherwise, that the length of a fundamental segment τ of a hyperbolic element h is bounded after iteration. As there are only finitely many Θ-patterns of length with such a upper bound for any given $\Theta > 0$, we divide by angle at each vertex after iteration of f into two cases:

Case 1. Angle at each vertex is bounded after iteration:

In this case, there are finitely possible $f^n(\tau)$ up to the action of G. This means that there exist $n_2 > n_1 > 0$ and $g \in G$ such that $f^{n_2}(\tau) = gf^{n_1}(\tau)$, which contradicts the fact that Φ is atoroidal.

Case 2. There are vertices at which angles are unbounded after iteration:

By the assumption, there are subsequence of $f^n(\tau)$ that has the same length, in short, we write τ_i such a sequence. Consider τ_0 a segment of this length in $\mathbb{R}$, and the embedding map $p_i : \tau_0 \to \tau_i$. Extract a subsequence such that there are (finitely many) points $a_1, \ldots, a_q$ in τ_0 ($q - 1 \leq l_T(\tau_0)$) such that for all i, angles at $p_i(a_j)$ go to infinity, while angles at $p_i(a)$ for any other points a are bounded, and denote this upper bound by Θ_0.

If $q > 1$:

Again, apply Lemma 4.21 to each component, segment $p_i([a_1, a_2])$ has only finitely many possible images up to G-action. Assume that $p_{i_t}([a_1, a_2])$ are in the same G-orbit, and that the angles at other vertices in these subsequence of segments are bounded by Θ_0. For t sufficiently large, angles at $p_{i_t}(a_1)$ and $p_{i_t}(a_2)$ become sufficiently large.

By our assumption, $p_{i_{t+1}}([a_1, a_2])$ is obtained from $p_{i_t}([a_1, a_2])$ by action of some power of f after cancellation of possible illegal turns. And by Lemma 4.22, we have that each $p_{i_t}(a_j)$ is the image of $p_{i_1}(a_j)$ by the corresponding power of f $(t \geq 1)$, for $j = 1, 2$. Thus there are some $n_1 > 0$ such that

$$f^{n_1}([p_{i_2}(a_1), p_{i_2}(a_2)]) = g'([p_{i_1}(a_1), p_{i_1}(a_2)])$$

for some $g' \in G$. Thus $[p_{i_2}(a_1), p_{i_2}(a_2)] - \cdots - [p_{i_2}(a_{q-1}), p_{i_2}(a_q)]$ is a Θ_0-pattern, and Φ^{n_1} fixes the conjugacy class of $g_1 g_2$ where g_1 fixes $p_{i_1}(a_1)$, and g_2 fixes $p_{i_1}(a_2)$. By the lemma 3.6, $g_1 g_2$ is hyperbolic, which contradicts the atoroidality.

If $q = 1$:

Denote by a and b the starting and ending point of τ_0 respectively. As $p_i(a_1)$ is fixed by some power of f, we can assume that the segments $[p_i(a), p_i(a_1)]$ (respectively $[p_i(a_1), p_i(b)]$) are in the same G-orbit. If one of them contains a non-free vertex, the angle at this vertex becomes unbounded, and by the previous argument (in $q > 1$) this assumption violates the atoroidality. If none of them contains a non-free vertex, h can be written in the form of tg_1 after conjugation, where t is conjugate in F_k, and g_1 is in some G_m (assume without loss of generality that $g_1 \in G_1$). Since by a power of Φ, t is sent on tg_1'. Thus the element $h' = g_1' t g_1' t^{-1}$ is invariant under Φ (after some iteration). And h', as composition of g_1' and $t g_1' t^{-1}$, is hyperbolic. Again, this violates atoroidality. $\square$

Recall that for a path $\alpha \subset T$, $[\alpha]$ is defined to be the reduced path of α.

Lemma 4.24 (Bounded cancellation lemma) *Let Φ be an automorphism of G in $Aut(G, \mathbb{G})$, $T \in \mathbb{G}$, $f : T \to T$ be piecewise linear representing Φ. Then there exist a constant $C_f > 0$, depending only on f, such that for any path $\rho \subset T$ obtained by concatenating two legal paths α, β without cancel-*

lation, we have

$$l_T([f(\rho)]) \geq l_T(f(\alpha)) + l_T(f(\beta)) - C_f$$

Proof: Let f' be a homotopic inverse of f representing Φ^{-1}, and let $g = f' \circ f : T \to T$. Clearly, g represents $Id \in Aut(G)$. Assume that two vertices $A, B \in VT$ have the same image under f, then $g(A) = g(B)$. Then we have a homotopy $H : [0,1] \times T \to T$ such that $H|_0 \equiv g, H|_1 \equiv Id$. As H can be chosen to be piecewise linear (because H is a homotopy, we can re-scale it on each edge such that it is piecewise linear), we may assume that H is Lipschitz. Thus there is $D > 0$ such that $d(H(0,x), H(1,x)) \leq D$ for all x, therefore $d(A,B) \leq 2D$. Let $v_1 \in \alpha, v_2 \in \beta$ such that the subpath of ρ between v_1 and v_2 is exactly the reduced subpath by the image of f (i.e., $f(v_1) = f(v_2)$ is the ending point of $f(\alpha) \cap [f(\rho)]$, and the starting point of $f(\beta) \cap [f(\rho)]$), then $d(v_1, v_2)$ is bounded by $2D$, hence there exist a uniform upper bound for $d(f(v_1), f(v_2))$, which is the C_f in this lemma. $\square$

Assume that $f : T \to T$ is a train track map representing $\Phi \in Aut(G, \mathbb{G})$. Denote by λ the growth rate of f. By bounded cancellation lemma, we can easily come to the following conclusion (in fact, this is a computation by induction):

Lemma 4.25 (Lemma 5.2 of [7]) *If β is a legal path in T with $\lambda l_T(\beta) - 2C_f > l_T(\beta)$ (i.e. $l_T(\beta) > \frac{2C_f}{\lambda - 1}$), and α, γ are paths such that the concatenation $\alpha - \beta - \gamma$ is locally injective, then there is a constant $\nu > 0$ (independent of β) such that the length of legal leaf segment (the length of the maximal legal segment) of $f^i(\alpha - \beta - \gamma)$ corresponding to β is at least $\nu \lambda^i l_T(\beta)$ for all integer $i > 0$.*

Definition 4.26 *(Critical constant of a train track map) Let G be a free product, $\Phi \in Aut(G, \mathbb{G})$ be irreducible, $f : T \to T$ be a train track map representing Φ, λ be the growth rate of f. The constant $\frac{2C_f}{\lambda - 1}$ is called* the critical constant *of f, where C_f is the constant defined in the Bounded Cancellation Lemma.*

4.4 Legal Control in Iteration

Similar to Bestvina, Feighn and Handel in [2], we have

Lemma 4.27 *[analogue of [2], Lemma 2.9] Let G be a group with free product decomposition $G = G_1 * ... * G_p * F_k$, $\Phi \in Aut(G, \mathbb{G})$ be fully irreducible and atoroidal, $f : T \to T$ be a train track map representing Φ. Assume that either $k \geq 2$ or $p + k \geq 3$. Then for every $C > 0$, there exists an exponent $M > 0$, such that for any path ρ in T, one of the three following holds:*

- *(A) the length of the longest legal segment of $[f^M(\rho)]$ is greater than C;*
- *(B) $[f^M(\rho)]$ has strictly less illegal turns than ρ;*
- *(C) ρ is a concatenation of $\gamma_1 - \alpha_1 - \cdots - \alpha_s - \gamma_2$, where γ_1, γ_2 has at most 1 illegal turn with length at most $2C$, and that each α_i is a pre-Nielsen path with at most 1 illegal turn.*

Proof: Assume (B) fails for all integer $M > 0$, then no illegal turn becomes legal after iteration. In addition, let us assume that (A) fails as well. As f is a train track map, none of the legal turns become illegal, the total number of illegal turns (and henceforth the number of legal segments) remains the same after iteration. While each legal segment has a uniformly bounded length after iteration, then there is an exponent N such that

$$\pi_T(\rho) = \pi_T(f^N(\rho)) = ... = \pi_T(f^{iN}(\rho)) = ...$$

for all $i \in \mathbb{Z}$ (remind that π_T is the quotient map from the tree T to its quotient graph of groups).

We classify ρ in the following cases:

Case 1. Angles at every vertex in ρ are bounded after iteration:

Since statements (A) and (B) fail, the length of ρ is bounded after iteration. By Lemma 4.21, there are only finitely many possible $f^n(\rho)$ (up to G-action), i.e., there exist $N_0 > 0, n > 0, g \in G$ such that $[f^n(f^{N_0}(\rho))] = g[f^{N_0}(\rho)]$ (in other word, ρ is pre-Nielsen).

Denote by $v_{i,1}, v_{i,2}$ the starting and ending vertices of the maximal legal segment ρ_i. By the action of f^n, each maximal legal segment ρ_i in $f^{N_0}(\rho)$ grows and cancels at the possible illegal turns with ρ_{i-1} or ρ_{i+1} (or possibly more), and obtain $g\rho_i$. In addition, the legal segment between $gv_{i,1}$ and $f(v_{i,1})$ (if they are different vertices) and the legal segment between $gv_{i,2}$ and $f(v_{i,2})$ (if they are different vertices) are canceled. Hence there is a subsegment (which is legal) ζ_i of ρ_i such that $f(\zeta_i) \subset g\rho_i$. For this reason, there is a vertex v_i in each ζ_i (thus it is in ρ_i) such that $[f^n(f^{N_0}(v_i))] = g[f^{N_0}(v_i)]$. Call these v_i "pre-periodic points", we have that ρ is a concatenation of $\gamma_1 - \alpha_1 - \cdots - \alpha_s - \gamma_2$, where γ_1, γ_2 has at most 1 illegal turn with length at most $2C$, and that each α_i is a pre-Nielsen path with at most 1 illegal turn. In addition, as ρ pre-Nielsen, γ_1, γ_2 are also pre-Nielsen.

Case 2. There exists a vertex in ρ where angles become unbounded after iteration:

Assume that a subsequence $f^N(\rho)$ has the same length with ρ, denote by a_N the isometries between ρ and each $f^N(\rho)$.

Let v_1, v_2 be two such vertices that angles at $a_N(v_1), a_N(v_2)$ are unbounded and that angles at $a_N(v)$ are bounded for all vertices v between v_1 and v_2, and let q be the total number of vertices in ρ where angles are bounded (this is restricted by the length of ρ). If $q > 1$, we assume that $v_1 \neq v_2$. Choose $g_1 \in G$ preserves v_1, $g_2 \in G$ preserves v_2, then $g_1 g_2$ is a hyperbolic element in G, but this is impossible as the conjugacy class of $g_1 g_2$ is fixed by Φ, thus contradicts atoroidality.

If there is only one such vertex v_1 in ρ, i.e. $q = 1$, denote the starting and ending vertex of ρ by v_a, v_b respectively, subdivide the path ρ into 3 segments ρ_1, ρ_2, ρ_3 such that $\rho_1 \subset [v_a, v_1], \rho_3 \subset [v_1, v_b]$, and that $v_1 \in \rho_2$ such that ρ_2 contains only one edge in $[v_a, v_1]$ and one edge in $[v_1, v_b]$. By the same argument of the subcase where $q > 1$, $f^N(v_1) = g_2 v_1, f^N(\rho_2) = g_2 \rho_2$ (where $g_2 \in G$), and since angles at every vertex in ρ_1, ρ_2 are bounded, we have that ρ_1, ρ_2 are pre-Nielsen, thus $\rho = \rho_1 - \rho_2 - \rho_3$ is a Θ-pattern for some $\Theta > 0$. As in Case 1, we further subdivide ρ_1 and ρ_3 such that statement (C) of the lemma holds. □

Lemma 4.28 *Let G be a group with a free product decomposition $G = G_1 * ... * G_p * F_k$, $T \in \mathbb{G}$, Φ be an automorphism of G relative to the free factor system $\{[G_1], \ldots, [G_p]\}$. Assume that either $k \geq 2$ or $p + k \geq 3$ and that Φ is fully irreducible. If $f : T \to T$ is a train track representative for ϕ on T, then there exists a constant M_0 such that it is impossible to concatenate more than M_0 Nielsen paths.*

Proof: We first observe that the concatenation of two Nielsen paths is still Nielsen: Let $\rho_1 - \rho_2$ be the resulting path, by the assumption, there exist some N, g_1, g_2 such that $[f^N(\rho_1)] = g_1 \rho_1, [f^N(\rho_2)] = g_2 \rho_2$. Due to the fact that the terminal vertex of ρ_1 is exactly the initial vertex of ρ_2, we deduce that $g_1 = g_2$, which implies that $\rho_1 - \rho_2$ is still Nielsen. A basic induction shows that any finitely many concatenation of Nielsen path would be still Nielsen.

In order to prove the lemma, it suffices to prove that any path concatenating n_0 (where $n_0 \leq M_0 + 1$) Nielsen paths contains such a subpath, which leads to a contradiction: on one hand, this subpath is still a concatenation of Nielsen paths (thus its length remains the same after iteration); on the other hand, this subpath will eventually grow after iteration of f.

Since the quotient $G\backslash T$ is finite, we have that there is m such that the image in the quotient of the concatenation of at most m paths contains a loop (which consists of a concatenation of these paths). This means that the concatenation of at most m paths contains a fundamental segment of a hyperbolic element (this statement also holds for a concatenation of these Nielsen paths). Choose $M_0 = m - 1$, then after concatenate n_0 ($n_0 \leq M_0 + 1$) Nielsen paths $\alpha_1 - \cdots - \alpha_M$, the whole path contain a subpath $\alpha_I - \cdots - \alpha_{I+J}$ (where I, J are positive integers such that $1 \leq I < I + J \leq M$) which is a fundamental segment of a hyperbolic element, which eventually grows after iteration. But this is impossible as it is still Nielsen. Thus it is impossible to concatenate more than M_0 Nielsen paths. $\square$

As an application of Lemma 4.27, we have

Lemma 4.29 (analogue of [2], Lemma 2.10) *Let Φ be an irreducible and atoroidal automorphism in $Aut(G, \mathbb{G})$, $f : T \to T$ and $f' : T' \to T'$ be train track maps representing Φ and Φ^{-1} respectively. And let $\alpha : T \to T'$ and $\alpha' : T' \to T$ be G-equivariant, Lipschitz homotopy equivalences. Assume that either $k \geq 2$ or $p + k \geq 3$. Then for any $C > 0$ there exist an exponent $N > 0$ and $L_0 > 0$, such that if ρ is a path of* *length* $\geq L_0$, *and if* $\rho' = [\alpha(\rho)]$, *then either* $[f^N(\rho)]$ *or* $[f'^N(\rho')]$ *contains a legal segment of length greater than* C.

Proof: By the above lemma, there exist a constant M_0 such that it is impossible to concatenate more than M_0 pre-Nielsen paths.

Fix $C > 0$ such that it is larger than the critical constant for both f and f'. Suppose that $[f^N(\rho)]$ does not contain a legal segment of length greater than C for all $N > 0$. Then (A) of Lemma 4.27 fails. In addition, fix $L_0 > (2M_0 + 4)C$, where M_0 is the bound of Lemma 4.28, thus (C) of Lemma 4.27 also fails (otherwise, by Lemma 4.28, (A) of Lemma 4.27 holds, a contradiction). Then (B) of Lemma 4.27 must hold. Let M be the greater one of the integers according to Lemma 4.27 when we apply it to f, C and f', C.

We then represent ρ as a concatenation of segments $\beta \subset \rho$ such that $f^M(\partial\beta) \subset [f^M(\rho)]$ with uniformly bounded length (this bound is independent on ρ). Thus the upper bound to the number of illegal turns in each segment exists, we denote it by P. Fix Q such that $\frac{P-1}{P} < Q < 1$. For a path τ we denote by $NIT(\tau)$ the number of illegal turns in τ. Then for long enough subsegment γ of ρ we have

$$\frac{NIT([f^M(\gamma)])}{NIT(\gamma)} \leq Q$$

We do the same construction to $f^M(\rho), f^{2M}(\rho), \ldots.$ Then for a given large integer $s > 0$ and long enough segment γ,

$$\frac{NIT([f^{sM}(\gamma)])}{NIT(\gamma)} \leq Q^s$$

Since we require that any legal segment in each $[f^N(\rho)]$ is bounded by C, and it is obviously not less than the length of shortest edge, there is some constant $K = K(f, C) > 0$, such that

$$\frac{l_T([f^{sM}(\gamma)])}{l_T(\gamma)} \leq KQ^s \tag{1}$$

Apply the same discussion to $[\alpha f^{sM}(\rho)]$ as we did to ρ, and consider f' instead of f. If we assume that $[f'^N(\rho')]$ does not contain a legal segment of length greater than C either for $N = sM$, then we have

$$\frac{l_{T'}([f'^{sM}\alpha f^{sM}(\gamma)])}{l_{T'}([\alpha f^{sM}(\gamma)])} \leq K'Q^s \tag{2}$$

where $K' = K'(f', C)$ is a constant.

Notice that $f'^{sM}\alpha f^{sM}$ is conjugate to α, hence there is some constant $\mu > 1$ such that for long L,

$$\frac{1}{\mu} \leq \frac{l_{T'}([f'^{sM}\alpha f^{sM}(\gamma)])}{l_{T'}([\alpha(\gamma)])} \leq \mu \tag{3}$$

Multiply (1), (2) and the inverse of (3) we have

$$\frac{l_T([f^{sM}(\gamma)])}{l_{T'}([\alpha f^{sM}(\gamma)])}\frac{l_{T'}([\alpha(\gamma)])}{l_T(\gamma)} \leq \mu KK'Q^{2s}$$

Notice that $\frac{l_T([f^{sM}(\gamma)])}{l_{T'}([\alpha f^{sM}(\gamma)])}\frac{l_{T'}([\alpha(\gamma)])}{l_T(\gamma)} \geq \frac{1}{\mu}Lip(\alpha)Lip(\alpha')$, where $Lip(\alpha), Lip(\alpha')$ is the Lipschitz constant of α, α' respectively, we have

$$\frac{1}{\mu}Lip(\alpha)Lip(\alpha') \leq \frac{l_T([f^{sM}(\gamma)])}{l_{T'}([\alpha f^{sM}(\gamma)])}\frac{l_{T'}([\alpha(\gamma)])}{l_T(\gamma)} \leq \mu KK'Q^{2s}$$

Since s is large enough, and Q is a constant that $0 < Q < 1$, the right part goes to 0 as s goes to infinity, while the left part is a positive constant. Thus this inequality fails, and proved the lemma. $\square$

Definition 4.30 *(C-legality of a path) Given a $T \in \mathbb{G}$ and a constant C, for any immersed path $\rho \subset T$, the C*-legality *of ρ ,denoted by $LEG_{T,C}(\rho)$, is the ratio of the sum of lengths of all the maximal legal segments in ρ that are longer than C over the total length of ρ.*

Lemma 4.31 (see [2], Lemma 5.6) *Let $\Phi \in Aut(G, \mathbb{G})$ be fully irreducible and atoroidal, $f : T \to T$, $f' : T' \to T'$ be train track maps representing Φ and Φ^{-1} respectively. And let $\alpha : T \to T'$ and $\alpha' : T' \to T$ be G-equivariant, Lipschitz homotopy equivalences. Assume that C is the larger one of the critical constant of f and f' and that either $k \geq 2$ or $p+k \geq 3$. Then there is $\epsilon > 0$ and an integer $N_1 > 0$ such that for every hyperbolic element $x \in G$, if σ is a fundamental segment of x in T, σ' is an isometric immersion of $[\alpha(\sigma)]$, then for every $N > N_1$, either $LEG_{T,C}(f^N(\sigma)) \geq \epsilon$ or $LEG_{T',C}(f'^N(\sigma')) \geq \epsilon$.*

Proof: By Lemma 4.23, there is an integer N', such that $l_T(f^{N'}(\sigma)) > L_0$ and $l_{T'}(f'^{N'}(\sigma')) > L_0$, where L_0 is defined according to Lemma 4.29. And by Lemma 4.29, there is N_1 such that either $[f^{N_1}(\sigma)]$ or $[f'^{N_1}(\sigma')]$ contains a legal segment of length greater than C.

Suppose the result does not hold, then there are sequences $\{\sigma_i\}$ and $\{\sigma'_i\}$ in T and T' respectively such that their C-legality converges to 0. Then there exists arbitrarily long segments in $\{\sigma_i\}$ and $\{\sigma'_i\}$ that do not contain a legal segment of length $\geq C$. Thus contradicts the Lemma 4.29. $\square$

We wish to show the growth of every fundamental segment of every hyperbolic element in the free product (to be more precise, we want to show the hyperbolicity of the automorphism in the sense of Definition 4.6). To achieve this, it suffices to show that after some iteration of f, the image of any fundamental segment of any hyperbolic element contains such a subset of segments, that

- 1. the total length of the segments in this subset amounts certain (say ϵ, which is independent of the choice of fundamental segment and hyperbolic element) ratio of the length of the total length of whole path (after iteration);
- 2. it has the desired growth (in the sense of Definition 4.6).

Remind in Definition 4.6 we defined an automorphism is a hyperbolic automorphism of $(G, \{G_1, \ldots, G_p\}, T, T'f, f')$ if it satisfies the desired growth in Definition 4.6.

By Lemma 4.29, after a certain times of iteration (forward or backward) of f, there is a set of leagal segment that is longer than C (where C is the larger one of the critical constant of f and f'), and each of these segments has the desired growth (in the sense of Definition 4.6) since it is leagal. And by the analysis of the C-legality of each fundamental segment (i.e. Lemma 4.31), the following theorem becomes natural:

Theorem 4.32 *Let $G = G_1 * \cdots * G_p * F_k$ (where $k \geq 2$ or $p + k \geq 3$), $\Phi \in Aut(G, \mathbb{G})$ be fully irreducible and atoroidal, $f : T \to T$, $f' : T' \to T'$ be train track maps representing Φ and Φ^{-1} respectively. Then Φ is a hyperbolic automorphism of $(G, \{G_1, \ldots, G_p\}, T, T'f, f')$.*

Proof: Let $\alpha : T \to T'$ and $\alpha' : T' \to T$ be G-equivariant Lipschitz homotopy equivalences. For each hyperbolic element $x \in G$, let σ be a fundamental segment of x in T, σ' be an isometric immersion of $[\alpha(\sigma)]$.

Choose C be the larger one of the critical constant of f and f'. By Lemma 4.31, there is $\epsilon > 0$ and integer $N_1 > 0$

such that for every $N > N_1$, either $LEG_{T,C}(f^N(\sigma)) \geq \epsilon$ or $LEG_{T',C}(f'^N(\sigma')) \geq \epsilon$.

Denote here by $S_C(\sigma)$ the set of legal leaf segments (maximal legal segments) in σ whose length is longer than C.

We assume that $LEG_T(f^N(\sigma)) \geq \epsilon$ (the other case is similar), by our introduction of critical length, let λ be the growth rate of f, there is $\nu > 0$ such that

$$l_T([f^i(\sigma)]) \geq \nu\lambda^i l_T(S_C(\sigma)) \geq \nu\epsilon\lambda^i l_T(\sigma)$$

holds for all $i > N_1$. □

4.5 Relative Hyperbolicity in Fully Irreducible Case

Recall that in the previous sections, we defined the relatively hyperbolic automorphism in Definition 4.5 in terms of the growth of conjugacy classes of hyperbolic elements. And we have explained that relatively hyperbolicity of an automorphism can be shown by the growth of fundamental segments of hyperbolic elements in a tree.

From Theorem 4.32, it follows that

Corollary 4.33 *Let G be a finitely generated group with a given free product decomposition $G \simeq G_1 * ... * G_p * F_k$ (where $k \geq 2$ or $p + k \geq 3$), $\mathbb{H}_0 = \{G_1, ..., G_p\}$. If $\Phi \in Aut(G, \mathbb{G})$ is fully irreducible and atoroidal, then Φ is hyperbolic relative to $\mathbb{H}_0$.*

Gautero and Weidmann proved a relevant combination theorem (see Corollary 7.3 of [22]), recalled as Theorem 4.8 of this thesis). By the combination theorem, the relative hyperbolicity of $G \rtimes_\Phi \mathbb{Z}$ is true by the fact that Φ is hyperbolic relative to $\mathbb{H}_0$. Thus our result of Theorem 4.9 holds. We give a more precise version of the theorem below:

Theorem 4.34 *Let G be a finitely generated group with a given free product decomposition $G \simeq G_1 * ... * G_p * F_k$ (where $k \geq 2$ or $p+k \geq 3$), let $\mathbb{H}_0 = \{G_1, ..., G_p\}$, assume $\Phi \in Aut(G, \mathbb{G})$ is fully irreducible and atoroidal. Then $G \rtimes_\phi \mathbb{Z}$ is hyperbolic relative to the mapping-torus of $\mathbb{H}_0$.*

From the above theorem, we come to a relative hyperbolicity result of a kind of toroidal automorphism of free group:

Corollary 4.35 *Let G be a free group and ϕ be an automorphism of G such that there exist a free group decomposition $G = F_n \simeq G_1 * ... * G_p * F_k$ with the following properties:*

- *1. each $[G_i]$ ($i \in \{1, \ldots, p\}$) is preserved by some power of ϕ (i.e. there exist some positive integer n_i such that $\phi^{n_i}(G_i) = g_i^{-1} G_i g_i$ for some $g_i \in G$);*
- *2. ϕ is fully irreducible and atoroidal in $Aut(G, \mathbb{H}_0)$, where $\mathbb{H}_0 = \{G_1, ..., G_p\}$;*
- *3. $k \geq 2$ or $p+k \geq 3$.*

Then there is some positive integer n such that $G \rtimes_{\phi^n} \mathbb{Z}$ is hyperbolic relative to the mapping-torus of $\mathbb{H}_0$.

Indeed, choose n to be the least common multiple of (n_i)s, by definition, ϕ^n is fully irreducible and atoroidal in $Aut(G, \mathbb{H}_0)$. We can obtain this corollary from Theorem 4.34.

5 Detecting fully irreducible automorphisms

Theorem 4.34 gives the relative hyperbolicity result of $G \rtimes_\phi \mathbb{Z}$ for the free product $G \simeq G_1 * ... * G_p * F_k$ under the assumption that Φ is fully irreducible and atoroidal. We give in this section a condition for full irreducibility. We first give a definition of Whitehead graph for a given train track map.

Definition 5.1 *(Whitehead graphs) Let G be a group with free product Decomposition $G = G_1 * ... * G_p * F_k$, $\Phi \in Aut(G, \mathbb{G})$, $f : T \to T$ be a train track map representing Φ. For any vertex $v \in V(T)$ in the tree, we define the* Whitehead graph *$Wh_T(v, f)$ as follows:*

The set of vertices in $Wh_T(v, f)$ is the set of oriented edges in T originating from v; two vertices v_1, v_2 (corresponding to e_1, e_2 respectively in T) in $Wh_T(v, f)$ are adjacent if there exist some $g_1, g_2 \in G$, $n \geq 1$ and $e \in E(T)$ such that the segment $g_1\bar{e_1} - g_2e_2$ is a subsegment of $f^n(e)$ or a reducible segment in the form of $e' - \bar{e'}$ (for some edge e').

Definition 5.2 *(Transition matrix of a map) Let G be a group with free product Decomposition $G = G_1 * ... * G_p * F_k$, $\Phi \in Aut(G, \mathbb{G})$, $f : T \to T$ be a map representing Φ. Fix a choice of edges $\{e_1, \ldots, e_s\}$ as the basis of T such that every edge in the tree is in the G-orbit of some e_i and that e_i and e_j are not in the same G-orbit for $i \neq j$ (where $i, j \in \{1, \ldots, s\}$). The transition matrix $M(f)$ of f is a non-negative matrix with the element $M_{i,j}$ denoting the total time of $f(e_i)$ intersecting the G-orbit of e_j.*

Lemma 5.3 *Let $G = G_1 * ... * G_p * F_k$ with either $k \geq 2$ or $p + k \geq 3$, $\Phi \in Out(G, \mathbb{G})$, and let $f : T \to T$ be a train track map representing Φ such that $l_T(f^n(e))$ goes to infinity as n goes to infinity for some $e \in E(T)$. Assume that Φ is irreducible. Then for every vertex $v \in V(T)$ the Whitehead graph $Wh_T(v, f)$ is connected.*

Proof: Suppose otherwise, that there exist some $u \in V(T)$ such that $Wh_T(u, f)$ is disconnected.

We first define a new graph Γ based on T to be as follows: For every vertex $v \in T$, we define $\hat{v}$ in Γ correspondingly (call it the *central vertex* corresponding to v), and introduce in Γ some "*attached vertices*" $v_1, \ldots, v_k$, where k is the number of connected components of $Wh_T(v, f)$. For each central vertex $\hat{v}$, we add an (oriented) edge between $\hat{v}$ and each of its attached vertex for every $v \in V(T)$ (such edges are called *attached edges* attached to v). For edge e in T from v to w, we add an edge e' in the graph Γ from v_i to w_j, where v_i and w_j are attached vertices to v and w respectively, such that v_i and w_j correspond to the connected components of $Wh_T(v, f)$ and of $Wh_T(w, f)$ containing e.

We then define a map $f' : \Gamma \to \Gamma$ based on f as follows: First we set $f'(\hat{v}) = \widehat{f(v)}$ and $f'(e') = e'_1 - \cdots - e'_n$ (where $f(e) = e_1 - \cdots - e_n$) for all $v \in V(T)$ and for all $e \in E(T)$ (the latter makes sense by the definition of the Whitehead graph). For an attached vertex v_i that is attached to v, denote by e_i an edge in T starting from v that represents v_i, and set $f'(v_i)$ to be the attached vertex w_i (attached to $f(v)$) that corresponds to the starting edge of $f(e)$. It follows that the starting edge of $f(e)$ and the starting edge of $f(e')$ are adjacent in $Wh_T(v, f)$ if e, e' are two edges starting from the same vertex v that are adjacent in $Wh_T(v, f)$. Hence the definition of f' on the attached vertex is well-defined. And for an attached edge $\breve{e}$ (attached to v, from $\hat{v}$ to v_i), we set $f'(\breve{e})$ to be the attached edge starting from $f'(\hat{v})$ to $f'(v_i)$. We can see from this construction that f' is a continuous map, and that contracting all attached edges to points in Γ is a homotopy equivalence between Γ and T. Hence the map $f' : T' \to T'$ still represents Φ.

Consider a subgraph W of Γ by removing all central vertices and interiors all attached edges (for all vertices). This subgraph W is f'-invariant and G-invariant. By the disconnectedness of $Wh_T(u,f)$, the inclusion map from W to Γ is not a homotopy equivalence. In addition, by the expanding property of f, there is some edge e and some large integer n such that $f^n(e)$ contains a subpath σ (where σ is a fundamental segment of a hyperbolic element). It follows from the construction that, as the image of $f'^n(e')$, σ' is such a path whose image in the quotient graph is a loop. Thus we found W to be a f'-invariant and G-invariant subgraph that its image in quotient graph of groups is non-trivial (topologically) and such that the inclusion map from W to Γ is not a homotopy equivalence. This contradicts the fact that Φ is irreducible. $\square$

By the above lemma, we come to the following proposition, which serves as a condition for full irreducibility:

Proposition 5.4 *Let $G = G_1 * ... * G_p * F_k$ with either $k \geq 2$ or $p + k \geq 3$, $\Phi \in Out(G, \mathbb{G})$. If Φ is fully irreducible and atoroidal, then there exists a train track map $f : T \to T$ representing Φ such that the following holds:*

- *1. the transition matrix $M(f)$ is irreducible;*
- *2. for every vertex $v \in V(T)$ the Whitehead graph $Wh_T(v,f)$ is connected.*

6 On the reducible case

Let G be a group with free product decomposition $G \simeq G_1 * ... * G_p * F_k$ (as GD).

In this section, we consider an automorphism Φ such that $\{[G_1], \ldots, [G_p]\}$ is preserved by Φ, and such that it is reducible with respect to the free factor system $\{[G_1], \ldots, [G_p]\}$

(in this case, there is a larger free factor system that is also Φ-invariant). We deduce that, if Φ is an atoroidal automorphism of G such that the free factor system $\{[G_1], \ldots, [G_p]\}$ is invariant under Φ^n for all integer $n > 0$, then there is a larger free factor system to which Φ is fully irreducible and atoroidal, and we can even use the result of the previous subsection to obtain the relative hyperbolicity of the HNN extension of G relative to Φ.

Lemma 6.1 *Let $\{[G'_1], \ldots, [G'_{p'}]\}$ be a Φ-invariant free factor system larger than $\{[G_1], \ldots, [G_p]\}$, $\mathbb{G}' = \mathbb{G}'(G, (G'_i)_{i=1}^{p'}, F_{k'})$ be the set of Bass-Serre Trees defined in Definition 2.1. If $\Phi \in Aut(G, \mathbb{G})$ is an atoroidal automorphism, then $\Phi \in Aut(G, \mathbb{G}')$ is also atoroidal.*

Proof: It suffices to prove that, for any trees $T \in \mathbb{G}$, $T' \in \mathbb{G}'$, any hyperbolic element in $(G, \mathbb{G}')$ (on T') is a hyperbolic element in $(G, \mathbb{G})$ (on T). Equivalently, we only have to prove that any elliptic element in $(G, \mathbb{G})$ is also elliptic in $(G, \mathbb{G}')$, which is obvious by $(C2), (C3)$ of Definition 2.1 and by the assumption that $\{[G'_1], \ldots, [G'_{p'}]\}$ is a larger free factor system than $\{[G_1], \ldots, [G_p]\}$. $\square$

Lemma 6.2 *Let $G \simeq G_1 * \cdots * G_p * F_k \simeq G'_1 * \cdots * G'_{p'} * F_{k'}$, $\{[G'_1], \ldots, [G'_{p'}]\}$ be a strictly larger free factor system than $\{[G_1], \ldots, [G_p]\}$. Then $(k', p') < (k, p)$ in the lexicographical order, i.e. if $k' \geq k$ then $k' = k$ and $p' < p$.*

Proof: First, we prove that $k' \leq k$. Suppose otherwise, then F_k is a proper subgroup of $F_{k'}$, this violates the fact that $G_1 * \cdots * G_p * F_k \simeq G'_1 * \cdots * G'_{p'} * F_{k'}$ as $\{[G'_1], \ldots, [G'_{p'}]\}$ is a larger free factor system than $\{[G_1], \ldots, [G_p]\}$.

The relation between two free factor system can be divided into the following two cases:

Case 1: there exist some $x \in F_k$, $g \in G$ and $q \in \{1, \ldots, p'\}$ such that $x \in gG'_q g^{-1}$.

In this case, $x \notin F_{k'}$ (because x fixes some vertex in the Bass-Sree Tree in $\mathbb{G}'(G, (G'_i)_{i=1}^{p'}, F_{k'})$, while every element in $F_{k'}$ act on the Bass-Serre Tree by translation). Denote by $< x, F_{k'} >$ the subgroup of G that is generated by x and $F_{k'}$, which is actually $F_{k'+1}$. By Gurosh Subgroup Theorem (apply to the decomposition $G \simeq G_1 * \cdots * G_p * F_k$), $< x, F_{k'} >$ is a subgroup of F_k. Hence $k' + 1 \leq k$, equivalently, $k' < k$. Thus $(k', p') < (k, p)$.

Case 2: for all $x \in F_k$ and for all $q \in \{1, \ldots, p'\}$, $x \notin gG'_q g^{-1}$ for all $g \in G$.

We define a map $\chi : \{G_1, \ldots, G_p\} \to \{G'_1, \ldots, G'_{p'}\}$ in the following natural way: for all $i \in \{1, \ldots, p\}$, if there exist $j \in \{1, \ldots, p'\}$ such that G_i is a subgroup of $gG'_j g^{-1}$ for some $g \in G$, then define $\chi(G_i) = G'_j$. It is easy to check that this map is well-defined. By the assumption of this case, every G'_j has a pre-image of χ in $\{G_1, \ldots, G_p\}$ (i.e. χ is surjective), we have that $p' \leq p$.

Let T be a Bass-Serre Tree in $\mathbb{G}(G, (G_i)_{i=1}^{p}, F_k)$. By the assumption that the free factor system $\{[G'_1], \ldots, [G'_{p'}]\}$ is strictly larger than $\{[G_1], \ldots, [G_p]\}$, there exist $r \in \{1, \ldots, p\}$, $r' \in \{1, \ldots, p'\}$ and $y \in G'_{r'}$ such that $gG_r g^{-1}$ is a subgroup of $G'_{r'}$, $y \notin gG_r g^{-1}$. Moreover, we can assume that y is elliptic in $(G, \mathbb{G})$ (because for any such hyperbolic element $h \in G'_{r'}$, $hgG_r g^{-1}h^{-1} < G'_{r'}$, and $hgG_r g^{-1}h^{-1} \neq gG_r g^{-1}$ as $g^{-1}hg \notin G_r$, we have element in $hgG_r g^{-1}h^{-1}$ that is elliptic in $(G, \mathbb{G})$). Consider the subgroup $< y >$ generated by y, it fixes some vertex v_y in the Tree T. Denote by H_y the stabilizer of v_y in G. H_y is conjugate to one of G_is (which is a subgroup of one of G'_js), As $y \in H_y$ and $y \in G'_{r'}$, H_y is a subgroup of $G'_{r'}$. By the choice of y and v_y, H_y is not conjugate to G_r (otherwise, there exist some $h \in G$ different from g such that hG_rh^{-1} is a subgroup of $G'_{r'}$, so G_r is a subgroup of the intersection of $g^{-1}G'_{r'}g$ and $h^{-1}G'_{r'}h$, which is empty). Thus there exist $s \neq r$ and $g' \in G$ such that $g'G_sg'^{-1}(= H_y)$ is a subgroup of $G'_{r'}$. This implies that $\chi(G_r) = \chi(G_s) = G'_{r'}$. And since χ is surjective, $p' < p$. Therefore, $(k', p') < (k, p)$. □

Remark 6.3 *1. The condition that $\{[G'_1], \ldots, [G'_{p'}]\}$ is a strictly larger free factor system than $\{[G_1], \ldots, [G_p]\}$ does not imply that $k' + p' < k + p$. One counterexample is to let $G = G_1 * G_2 * F_4$ where $F_4 =< a, b, c, d >$, consider these two decomposition $G = G_1 * G_2 * F_4 = G'_1 * G_2 * G'_3 * F_3$ (where $G'_1 = G_1, G'_2 = G_2$, $G'_3 =< a >$), in this case, $p + k = p' + k' = 6$.*

2. The condition that $\{[G'_1], \ldots, [G'_{p'}]\}$ is a strictly larger free factor system than $\{[G_1], \ldots, [G_p]\}$ also does not imply that $(p', k') < (p, k)$ in the lexicographical order. Consider $G = G_1 * G_2 * F_5$ where $F_5 =< a, b, c, d, e >$ with the following two decomposition $G = G_1 * G_2 * G_3 * F_3 = G'_1 * G'_2 * G'_3 * G'_4 * F_2$ (where $G'_1 = G_1, G'_2 = G_2, G_3 =< a, b >, G'_3 =< c >, G'_4 =< a, b >$), in this case, $p' > p$.

Directly from Lemma 6.2, we have

Lemma 6.4 *Let $G \simeq G_1 * \cdots * G_p * F_k$, $\Phi \in Aut(G)$ such that the free factor system $\{[G_1], \ldots, [G_p]\}$ is invariant under Φ^n for all integer $n > 0$. Then there exist another free factor system $\{[\hat{G}_1], \ldots, [\hat{G}_q]\}$ of G, such that Φ is fully irreducible relative to $\{[\hat{G}_1], \ldots, [\hat{G}_q]\}$.*

Proof: By the Lemma 6.2, for any given free factor system, a strictly larger free factor system has a less (k, p). And there are only finitely many (k', p')s that are smaller than (k, p). This implies that, by finitely many step (each step we find a strictly larger free factor system that is invariant under Φ^n), one can find a maximal free factor system $\{[\hat{G}_1], \dots, [\hat{G}_q]\}$ preserved by Φ, such a free factor system is what we are looking for. $\square$

Directly from Lemma 6.1, Lemma 6.4, and Theorem 4.34, we have

Theorem 6.5 *Let G be a finitely generated group with a given free product decomposition $G \simeq G_1 * ... * G_p * F_k$, and let $\Phi \in Aut(G, \mathbb{G})$ be an atoroidal, reducible automorphism. Then there exist another free product decomposition $G \simeq \hat{G}_1 * ... * \hat{G}_q * F_r$ such that Φ is atoroidal and fully irreducible. Moreover, if $r \geq 2$ or $q + r \geq 3$, let $\hat{\mathbb{H}}_0 = \{\hat{G}_1, ..., \hat{G}_q\}$, then $G \rtimes_\phi \mathbb{Z}$ is hyperbolic relative to the mapping-torus of $\mathbb{H}_0$.*

Proposition 6.6 *Let $G \simeq G_1 * \cdots * G_p * F_k$, and let $\phi \in Aut(G, ([G_1], \dots, [G_p]))$ be atoroidal. Then there exist a free product decomposition of G, $G \simeq \hat{G}_1 * \cdots * \hat{G}_q * F_r$, such that each $[\hat{G}_i]$ is ϕ-invariant, on which ϕ is irreducible and atoroidal.*

6.1 Descent and the Theorem 6.10

Now that we have proved Theorem 4.34 about fully irreducible atoroidal automorphisms, and that we have Proposition 6.6, we want a descent argument to show that, in some cases at least, beyond irreducibility, $G \rtimes_\phi \mathbb{Z}$ is relatively hyperbolic, relative to the mapping tori of the G_i.

A first part of the argument is the following well known fact. For instance see the work of Osin [34].

Proposition 6.7 *(see Theorem 2.40 of [34]) If G is hyperbolic relative to a family of subgroups $H_1, \dots, H_s$, and if each* *subgroup H_i is hyperbolic relative to subgroups $H_{i,1}, \ldots, H_{i,r_i}$, then, G is hyperbolic relative to $\{H_{i,j}, i \leq s, j \leq r_i\}$.*

This allows, in many cases, to reduce the study of the relative hyperbolicity of $G \rtimes_\phi \mathbb{Z}$ to that of $G_i \rtimes_{\phi_i} \mathbb{Z}$, for ϕ_i the composition of the restriction of ϕ to G_i with a conjugation by an element of G sending $\phi(G_i)$ on G_i.

Indeed, if one applies Proposition 6.6, and if the pair (q, r) thus obtained is such that $r \geq 2$ or $q+r \geq 3$, then one can use Theorem 4.34 in conjunction with Proposition 6.7 in order to reduce our problem to that of $\hat{G}_i \rtimes_{\phi_i} \mathbb{Z}$.

However there are two particular cases in this descent (the case $r = 1, q = 0$ and the case $r = 0, q = 2$), where we cannot apply Theorem 4.34. We thus focus on them in the following two statements.

Proposition 6.8 *If $G = G_1 * G_2$ and if $\phi \in Aut(G, ([G_1], [G_2]))$ is atoroidal, and assume that each $G_i \rtimes_{\phi_i} \mathbb{Z}$ is hyperbolic relative to the mapping tori of certain free factors of G_i. Assume also that each maximal parabolic subgroup of $G_i \rtimes_{\phi_i} \mathbb{Z}$ has a non-trivial central element.*

Then $G \rtimes_\phi \mathbb{Z}$ is hyperbolic relative to the peripheral subgroups of $G_1 \rtimes_{\phi_1} \mathbb{Z}$ and $G_2 \rtimes_{\phi_2} \mathbb{Z}$.

Proposition 6.9 *If $G = G_1 * \mathbb{Z}$ and if $\phi \in Aut(G, ([G_1]))$ is atoroidal, and assume that $G_1 \rtimes_{\phi_1} \mathbb{Z}$ is hyperbolic relative to the mapping tori of certain free factors of G_1. Assume also that each maximal parabolic subgroup of $G_1 \rtimes_{\phi_1} \mathbb{Z}$ has a non-trivial central element.*

Then $G \rtimes_\phi \mathbb{Z}$ is hyperbolic relative to the peripheral subgroups of $G_1 \rtimes_{\phi_1} \mathbb{Z}$.

We will prove these two Proposition shortly, but let us present the main application here.

We consider a descent of free product systems by applying telescopically Proposition 6.6 to each G_i with ϕ_i. Precisely, we equip them with the free product decomposition obtained by their action on the original Bass-Serre tree of G. And we extract by Proposition 6.6 another free factor system of each G_i on which ϕ_i is fully irreducible. We do again on the factors of this later one. Note that the descent terminates by Grushko's uniqueness theorem for G.

Thus, using telescopically Proposition 6.6, and using either Theorem 4.34 with Proposition 6.7, or directly one of the two previous Propositions 6.8, 6.9, we obtain theorem 6.10, which contains Brinkmann's original result.

Theorem 6.10 *Let $G \simeq G_1 * \cdots * G_p * F_k$, and let $\phi \in Aut(G, ([G_1], \dots, [G_p]))$ be atoroidal.*

Assume that, for each i, there exists $g_i \in G$ conjugating $\phi(G_i)$ to G_i. Assume that for each i there exists a non-trivial element of $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$ that is central in $G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}$.

Then $G \rtimes_\phi \mathbb{Z}$ is relatively hyperbolic, relative to the collection $\{G_i \rtimes_{\mathrm{ad}_{g_i} \circ \phi|_{G_i}} \mathbb{Z}, i \leq p\}$.

It is not so clear to us what can happen if one drops the condition on central elements. Here is an embarrassing example. $G = F_3 = F_{a,b} * \langle c \rangle$ and ϕ sends a on b, b on ab, and c on $caba^{-1}b^{-1}$. We see $F_{a,b}$ as G_1, so ϕ is atoroidal (for the given free factor decomposition). The restriction of ϕ on G_1 is well known (fully irreducible, pseudo-Anosov), the element t representing ϕ will have its square commute with $[a,b] = aba^{-1}b^{-1}$, thus necessarily parabolic in any relative hyperbolic structure, but as well $t^{-2}ct^2 = c$ which means that c is also in the same parabolic group. The legitimate parabolic group seems to be $\langle t, [a,b], c \rangle$ which is not a mapping torus of a free factor.

We now need to prove the two proposition 6.8 and 6.9.

6.2 The two elementary cases

We begin by Proposition 6.8. We may freely change our automorphism ϕ whithin its outer automorphism class, since it does not change neither the assumptions, nor the isomorphism class of the group obtained.

Thus, we may assume that $\phi(G_1) = G_1$, and that $\phi(G_2) = G_2$ at the same time.

A look at presentations of the groups reveals that

$$G \rtimes_\phi \mathbb{Z} \simeq (G_1 \rtimes_{\phi_1} \langle t_1 \rangle) *_{\langle t_1 \rangle = \langle t_2 \rangle} (G_2 \rtimes_{\phi_2} \langle t_2 \rangle)\,.$$

Lemma 6.11 *Either t_1 or t_2 (or both) is not in a parabolic subgroup of respectively $G_1 \rtimes_{\phi_1} \langle t_1 \rangle$ and $G_2 \rtimes_{\phi_2} \langle t_2 \rangle$.*

Proof: Assume that both t_1 and t_2 are in parabolic subgroups of $(G_1 \rtimes_{\phi_1} \langle t_1 \rangle)$ and $(G_2 \rtimes_{\phi_2} \langle t_2 \rangle)$. Then, there are non-trivial elements $a \in G_1$ and $b \in G_2$ that respectively commute with t_1 and t_2 (either because they are in the center of a maximal parabolic subgroups, or because t_i is in the center of a maximal parabolic subgroups).

Thus, ab is preserved by conjugation by t, therefore is preserved by ϕ. But ab is easily seen to be a hyperbolic element of the free product $G_1 * G_2$, thus contradicting that ϕ is atoroidal. This proves the lemma. $\square$

Moreover, each t_i is a generator of the maximal cyclic subgroup that it contains. Therefore one can use the Combination Theorem [9, Main theorem 0.1, case 3], (possibly with a classical [9, Lemma 4.4] in order to turn a hyperbolic t_i into a parabolic element) to get the desired result.

Therefore Proposition 6.8 is proved.

We proceed and prove Proposition 6.9. Let $G = G_1 * \langle f \rangle$.

The argument is rather similar. One may first assume that $\phi(G_1) = G_1$, and that there is $g \in G_1$ such that $\phi(f) = f^{\pm 1}g$. Notice that we may assume that $g \neq 1$, otherwise, the previous case applies. Up to taking the square of ϕ (thus passing to a finite index subgroup, we may assume that $\phi(f) = fg$.

Then a presentation of the groups reveals that, for g such that $\phi(f) = f^{\pm 1}g$,

$$G \rtimes_\phi \langle t \rangle \simeq (G_1 \rtimes_{\phi_1} \langle t \rangle) *_{\langle t \rangle, \langle tg^{-1} \rangle}$$

Indeed, one can write $G \rtimes_\phi \langle t \rangle \simeq (G_1 * \langle f \rangle) \rtimes_\phi \langle t \rangle$ which has presentation

$$\begin{aligned} G \rtimes_\phi \langle t \rangle &\simeq \langle G_1, f, t \,|\, t^{-1}g_1 t = \phi(g_1), t^{-1}ft = fg \rangle \\ &\simeq \langle G_1, f, t \,|\, t^{-1}g_1 t = \phi(g_1), f^{-1}t^{-1}f = gt^{-1} \rangle \end{aligned}$$

We now check:

Lemma 6.12 *Either t or tg^{-1} is not parabolic in $G_1 \rtimes_{\phi_1} \langle t \rangle$.*

Proof: If both are, then by the the proof of the previous lemma, there are two free factors of G_1, say A, B, such that there exist non trivial element $a \in A$ and $b \in B$ with a normalized by t and b by tg^{-1}. Hence $\phi(a) = a$. Since $f^{-1}tf = tg^{-1}$, it means that f conjugates $\langle t, A \rangle$ into $\langle tg^{-1}, B \rangle$.

Consider the element $x = afbf^{-1}$, this is a hyperbolic element in G (because a and fbf^{-1} fixes two different vertices in the Bass Serre tree), but $\phi(x) = a(fg)(t^{-1}bt)(g^{-1}f^{-1}) = af(tg^{-1})^{-1}b(tg^{-1})f^{-1} = x$. This violates the assumption that ϕ is atoroidal. Thus the lemma is proved. $\square$

Finally, one can use again the Combination Theorem [9, Main theorem 0.1, case 4], (again possibly with a classical [9, Lemma 4.4]) to conclude. Proposition 6.9 is proved.

7 Conjugacy problem

In this section we recall what is the conjugacy problem in a group, and in particular in a group of automorphisms.

7.1 Two cases from the literature

7.1.1 The linear case

In all this thesis we have considered automorphism groups of some particular groups (free groups, free products). We could discuss a little the case of automorphisms of free abelian groups. Let G be $\mathbb{Z}^m$. Then $Aut(G)$ is actually $GL(m, \mathbb{Z})$.

The conjugacy problem in this case (i.e in $GL(m, \mathbb{Z})$) is a natural problem. It can be understood in two different ways. The first way, is to classify when matrices in $GL(m, \mathbb{Z})$ are conjugate by a matrix in $GL(m, \mathbb{Z})$. The second way is to find an algorithm that answer the question whether two given matrices are thus conjugate.

One recognise a classical situation where one tries to find out whether two matrices are conjugate. However, we are not working over a field (like $\mathbb{R}, \mathbb{C}$ or $\mathbb{Q}$) but over a ring $\mathbb{Z}$, and the tools of linear algebra (often relying on the principality of $K[X]$) are not helping us.

The classification of conjugacy classes in $GL(m, \mathbb{Z})$ (for arbitrary m) is settled by a theorem of Latimer and MacDuffee of 1936. It states the following. Given an irreducible (over $\mathbb{Z}$) monic polynomial $P \in \mathbb{Z}[X]$ (with $P(0) = 1$), the conjugacy classes of matrices in $GL(m, \mathbb{Z})$ with characteristic polynomial P are in bijection with the ideal classes of $\mathbb{Z}[X]/(P)$. A bijection is given by the following. First, one chooses a root ξ of P in $\overline{\mathbb{Q}}$ and for every matrix M with characteristic polynomial P, one selects an eigenvector v_M of M for ξ, whose entries are in $\mathbb{Z}[\xi]$, and one selects the ideal class of the ideal generated by the entries of v_M. The theorem of Latimer MacDuffee says that this is well defined as an ideal class (i.e. up to multiplication by an element of the ring), two matrices give the same class if and only if they are conjugate, and every ideal of $\mathbb{Z}[\xi]$ is thus obtained.

For non-irreducible characteristic polynomial, the discussion is slightly more involved. However, it is not clear at all how to use this for algorithmic purpose, even in the case of irreducible polynomials.

The second way, is the algorithmic way. Let us mention that an algorithmic solution to the conjugacy problem in $GL(m,\mathbb{Z})$ was given by Grunewald in 1977 (see [24]).

For the sake of illustration, let us repeat here our abstract way to tackle with the conjugacy problem in automorphisms groups, even if , in this case, it is not conclusive.

If G is a group, two automorphisms ϕ_1 and ϕ_2 are conjugate in $Aut(G)$ if and only if the two semi-direct products $G \rtimes_{\phi_1} \mathbb{Z}$ and $G \rtimes_{\phi_2} \mathbb{Z}$ are isomorphic by an isomorphism sending G on G and the generator t_1 of $\mathbb{Z}$ acting as ϕ_1, to the generator t_2 of $\mathbb{Z}$ acting as ϕ_2. (See Lemma 7.7)

Let us try to apply it to the case of a group G that is $\mathbb{Z}^m$.

We should consider the semidirect products $G_M = \mathbb{Z}^m \rtimes_M \mathbb{Z}$ (for $M \in GL(m,\mathbb{Z})$, and $m \geq 2$) for the two matrices. We certainly cannot expect any relatively hyperbolic structure, since these groups contain normal abelian subgroups (by definition). However, we can see that, if the matrix M has at least one eigenvalue of modulus different of 1, then the group G_M posseses only one normal subgroup isomorphic to $\mathbb{Z}^m$ by which the quotient is infinite cyclic. This applies in particular if the matrix is irreducible, and of infinite order in $GL(m,\mathbb{Z})$.

Indeed here is an argument: consider H_1, H_2 two different normal subgroups of G_M abelian of rank $m \geq 2$, such that $G_M/H_i \simeq \mathbb{Z}$. Then $H_1 \cap H_2$ must be isomorphic to $\mathbb{Z}^{m-1}$ (otherwise H_2 injects in the quotient by H_1). Since $H_1 \neq H_2$, we have an element outside H_1 that centralises a hyperplane (of rank $m-1$) H' in H_1. This element has the form vt^r where $v \in \mathbb{Z}^m$ and $r \geq 1$. But for all $w \in H'$, v and w commute (because they are in H_1 which is abelian), so it means that $M^r(w) = w$, which means that M^r is the identity on H'. So M^r has eigenvalue 1 with multiplicity $m-1$. Thus all eigenvalues (in $\overline{\mathbb{Q}}$) of M, except at most one, have modulus 1. Since M has determinant 1, it must have all its eigenvalues of modulus 1.

However, even in such a situation, it does not seem easier to check whether M_1 and M_2 are conjugate, because the isomorphism problem for the groups G_{M_i} falls in the metabelian case, and nothing seems gained. Notice that the isomorphism problem for polycyclic groups was solved by Segal after heavy development of the theory of orbit problems for arithmetic groups by Grunewald and Segal (see [25]), after the solution of Segal of the conjugacy problem in $GL(m, \mathbb{Z})$.

7.1.2 The case of $Out(F_n)$

In this section, we introduce the solvability of the conjugacy problem of some automorphisms of free group, which is solved by Dahmani in [12]. To be more exact, let $F = F_n$ be a free group of rank n, ϕ_1, ϕ_2 be two atoroidal automorphisms of F (namely, $\phi_i^k([g]) \neq [g]$ for all $i \in \{1, 2\}, k \in \mathbb{Z}^+$ and for all $g \in F \setminus \{1\}$), we show that the problem is decidable whether or not ϕ_1 and ϕ_2 is conjugate in $Out(F)$ (The full proof is shown in [12] by Dahmani).

To work out this conjugacy problem, we recall that these two automorphisms are conjugate in $Out(F)$ if and only if there is an isomorphism $\Psi : F \rtimes_{\phi_1} \langle t_1 \rangle \to F \rtimes_{\phi_2} \langle t_2 \rangle$ that sends F on F and t_1 on t_2F (see Lemma 3.1 of [12]).

In [7], Brinkmann proved that for any automorphism ϕ of a free group F, $F \rtimes_\phi \mathbb{Z}$ is Gromov hyperbolic if and only if ϕ is atoroidal. In order to check atoroidality, one needs the following two parallel steps. The first step is to check the preserved conjugacy class by enumerating elements in F and their images by ϕ^k, which stops when ϕ is found to be atoroidal by definition. The second step is to check the hyperbolicity of $F \rtimes_\phi \mathbb{Z}$ by the procedure given by Papasoglu in [35], which stops when $F \rtimes_\phi \mathbb{Z}$ is found to be hyperbolic. By Brinkmann, exactly one of these two step will stop, and thus one can decide whether or not the given automorphism is indeed atoroidal.

In [36], [13] and [15], the isomorphism problem of hyperbolic groups is proved to be solvable. If there is no isomorphism $\Psi : F \rtimes_{\phi_1} \langle t_1 \rangle \to F \rtimes_{\phi_2} \langle t_2 \rangle$, ϕ_1 and ϕ_2 are certainly not conjugate. Otherwise, we still needs to check the *orbit problem* of automorphism group of the semi-direct product: to verify whether or not an automorphism $\Psi : F \rtimes_{\phi_1} \langle t_1 \rangle \to F \rtimes_{\phi_2} \langle t_2 \rangle$, ϕ_1 and ϕ_2 satisfies $\Psi(F) \subset F$ and $\Psi(t_1) \in t_2F$.

In [12], Dahmani proved that if G is a finitely generated group, and if $G \rtimes_{\phi_1} \mathbb{Z}$, $G \rtimes_{\phi_2} \mathbb{Z}$ are hyperbolic groups, then this orbit problem in $Out(G)$ can be turned into Diophantine equation problems (see Section 2.1 of [12]), which is decidable. Then in the proof of Theorem 3.2 of [12], Dahmani gave a method to determine whether or not two given automorphisms of G are conjugate in $Out(G)$, if G is a free group and the given automorphisms are atoroidal.

7.2 The atoroidal-toral case

In this section, we fix the setting of a free product $H = A_1 * \cdots * A_k * F_p$, where F_p is a free group of rank p, and where each A_i is a finitely generated group. Let $Aut(H, (A_1, \ldots, A_k))$ be the subgroup of $Aut(H)$ of all automorphisms ϕ such that $\phi(A_i)$ is conjugate to A_i for all i.

Lemma 7.1 *If each A_i is freely abelian, whose rank is at least 2, and $rank(A_i) \neq rank(A_j)$ for all $i \neq j$, then*

$$Aut(H, (A_1, \ldots, A_k)) = Aut(H)$$

Proof: We only have to check that $Aut(H, (A_1, \ldots, A_k)) \supset Aut(H)$. For any $\phi \in Aut(H)$, From Kurosh subgroup theorem, $\phi(A_i)$ can be written in the form of free product of the conjugate of subgroups of A_j and a free group. Since the rank of each A_i is at least 2, it does not split into free products, which implies that $\phi(A_i)$ is conjugate to some A_j. As ϕ is an automorphism, $\phi(A_i)$ are conjugate to A_i in Out(G), equivalently, $\phi \in Aut(H, (A_1, \ldots, A_k))$. □

Lemma 7.2 *One can define a map $\rho_i : Aut(H, (A_1, \ldots, A_k)) \to Out(A_i)$ as follows: if $\phi \in Aut(H)$ is an automorphism such that $\phi(A_i) = h_i^{-1} A_i h_i$, then $\rho_i(\phi) = [ad_{h_i} \circ \phi|_{A_i}]$, where $ad_{h_i}(x) = h_i x h_i^{-1}$.*

Proof: One can check by computation of the definition that $ad_{h_i} \circ \phi|_{A_i}$ maps A_i to A_i. What remains is to check that definition is irrelevant to the choice of h_i. Suppose there is another $h_i' \neq h_i$ such that $\phi(A_i) = h_i'^{-1} A_i h_i'$, then $ad_{h_i} \circ \phi|_{A_i}$ and $ad_{h_i'} \circ \phi|_{A_i}$ only differ by conjugation of $h_i' h_i^{-1}$, which is in the normalizer of A_i (because $\phi \in Aut(H, (A_1, ..., A_k))$).

We now claim that $h_i' h_i^{-1} \in A_i$, for this , we only have to prove that $A_i = N_H(A_i)$, where $N_H(A)$ is the normaliser of A_i in H: consider the Bass-Serre Tree T of H, let x_i be the vertex that is fixed by A_i. For any element $h \in N_H(A_i)$, $A_i h x_i = h A_i x_i = h x_i$, meaning A_i also fixes hx_i. But in Bass-Serre Tree, x_i is the only vertex that is fixed by A_i, thus $hx_i = x_i$. Since the stabilizer of x_i is A_i, we have $h \in A_i$, $A_i \supset N_H(A_i)$. By the definition of normalizer, one can verify that $A_i \subset N_H(A_i)$. Hence $A_i = N_H(A_i)$.

By previous analysis, $h'_i h_i^{-1} \in A_i$, then $ad_{h_i} \circ ad_{h'_i}^{-1} \in Inn(A_i)$, therefore $[ad_{h_i} \circ \phi|_{A_i}] = [ad_{h'_i} \circ \phi|_{A_i}]$ in $Out(A_i)$. The map is well-defined. □

Remark 7.3 *By the proof of the previous lemma, the conjugation element h_i is unique (up to a left-multiplication by an element of A_i), hence we have the following definition.*

Definition 7.4 *(Toral automorphisms) Let A_i be abelian, $\phi \in Aut(H, (A_1, \ldots, A_k))$ is said to be* toral *if for all i, $[\mathrm{ad}_{g_i} \circ \phi|_{G_i}]$ is the identity on the free factor G_i, where g_i is the element in H such that $\phi(A_i) = g_i^{-1} A_i g_i$.*

Definition 7.5 *(Toral relatively hyperbolic group) A group H is said to be* toral relatively hyperbolic *if the group is torsion free, relatively hyperbolic, and with abelian parabolic subgroups.*

Recall that we have already define relatively hyperbolic automorphisms (see Definition 4.5), thus we have the following:

Proposition 7.6 *If each A_i is abelian, H is torsion free, and the automorphism $\phi \in Aut(H, (A_1, \ldots, A_k))$ is a* toral relatively hyperbolic automorphism, *then $H \rtimes_\phi \mathbb{Z}$ is toral relatively hyperbolic.*

Proof: Since H is torsion free, so is $H \rtimes_\phi \mathbb{Z}$. And $H \rtimes_\phi \mathbb{Z}$ is relatively hyperbolic as the automorphism is relatively hyperbolic. We only have to find the abelian parabolic subgroups.

Because ϕ is toral, $[\mathrm{ad}_{g_i} \circ \phi|_{G_i}]$ is trivial for all i (where g_i is the element such that $\phi(A_i) = g_i^{-1} A_i g_i$). Thus there exist some $b_i \in H \rtimes_\phi \mathbb{Z}$, such that b_i commutes with every element in A_i in the semidirect product for every $i \in \{1, \dots, k\}$. Since each A_i is abelian, we have that the subgroup generated by A_i and b_i is a abelian parabolic subgroup of $H \rtimes_\phi \mathbb{Z}$. $\square$

By the definition, the set of toral automorphisms forms a subgroups of $Aut(H, (A_1, \dots, A_k))$.

But, the set of toral relatively hyperbolic automorphisms does not form a subgroup in $Aut(H, (A_1, \dots, A_k))$: this set does not contain the identity automorphism (let $\mathbb{Z} = \langle t \rangle$, then $H \rtimes_{Id} \mathbb{Z}$ is not relatively hyperbolic, because $t \in Z(H \rtimes_{Id} \mathbb{Z})$, which violates the fineness property of relative hyperbolicity).

In the previous sections (more precisely, see Theorem 4.34, Theorem 6.5, and Theorem 6.10), we have shown that if ϕ is atoroidal with some other good property (for example, full irreduciblility, or more generally, with central condition), then it is relatively hyperbolic.

Given an automorphism ϕ in $Aut(H, (A_1, \dots, A_k))$, by computation of $ad_{h_i} \circ \phi|_{A_i}$ one can decide whether it is toral. By the important result worked by Dahmani and Guirardel (see Corollary 0.2 of [10] and Theorem 2 of [14]), one can decide whether or not it is relatively hyperbolic.

In the following part of the thesis, we mainly focus on the toral relatively hyperbolic automorphisms, and deal with the conjugacy problem between two such given automorphisms.

7.2.1 Basic Properties on Conjugacy Problem

We first denote here by $H_1(G)$ the abelianisation of G for any given group G, and for an element (or a subgroup) g in G, denote by $\bar{g}$ the corresponding element (or subgroup) of g in $H_1(G)$. We introduce a lemma for the beginning, in which the 1.1 and 2.1 are well-known facts.

Lemma 7.7 *(see Lemma 3.1 of [12]) Given H a finitely presented group, we have:*

1.Each of these two statements is equivalent to say that $\phi_1, \phi_2 \in Aut(H)$ are conjugate in $Aut(H)$:

- *1.1.there exist an isomorphism $\Psi : H \rtimes_{\phi_1} \langle t_1 \rangle \to H \rtimes_{\phi_2} \langle t_2 \rangle$ such that $\Psi(H) \subset H, \Psi(t_1) = t_2$;*
- *1.2.there is an isomorphism $\Psi : H \rtimes_{\phi_1} \langle t_1 \rangle \to H \rtimes_{\phi_2} \langle t_2 \rangle$ and an automorphism $\eta \in Aut(H \rtimes_{\phi_2} \langle t_2 \rangle)$, such that the image $\bar{\eta}$ in $Aut(H_1(H \rtimes_{\phi_2} \langle t_2 \rangle))$ sends $\overline{\Psi(H)}$ in $\bar{H}$ and sends $\overline{\Psi(t_1)}$ to $\bar{t_2}$.*

2.Each of these two statements is equivalent to say that $\phi_1, \phi_2 \in Aut(H)$ are conjugate in $Out(H)$:

- *2.1.there exist an isomorphism $\Psi : H \rtimes_{\phi_1} \langle t_1 \rangle \to H \rtimes_{\phi_2} \langle t_2 \rangle$ such that $\Psi(H) \subset H, \Psi(t_1) \in t_2 H$;*
- *2.2.there is an isomorphism $\Psi : H \rtimes_{\phi_1} \langle t_1 \rangle \to H \rtimes_{\phi_2} \langle t_2 \rangle$ and an automorphism $\eta \in Aut(H \rtimes_{\phi_2} \langle t_2 \rangle)$, such that the image $\bar{\eta}$ in $Aut(H_1(H \rtimes_{\phi_2} \langle t_2 \rangle))$ sends $\overline{\Psi(H)}$ in $\bar{H}$ and sends $\overline{\Psi(t_1)}$ to $\bar{t_2}\bar{H}$.*

Proof: Statement 2 of this lemma is concluded and proven in the Lemma 3.1 of [12]. And by the proof in Lemma

3.1 of [12] also shows that statement 1.1 and 1.2 are equivalent.

Here we show that statement 1.1 is equivalent to say that $\phi_1, \phi_2 \in Aut(H)$ are conjugate in $Aut(H)$: if $\phi_1 = \alpha^{-1} \circ \phi_2 \circ \alpha$ for some $\alpha \in Aut(H)$, we extend α to $\tilde{\alpha} : H * \langle t_1 \rangle \to G_2$ by sending t_1 to t_2. This induces a map $\bar{\alpha} : H \rtimes_{\phi_1} \langle t_1 \rangle \to H \rtimes_{\phi_2} \langle t_2 \rangle$ which map $h \in H$ to $\alpha(h)$ and map t_1 to t_2, as $\alpha(h) = \bar{\alpha}(h) = \bar{\alpha}(t_1 \phi_1(h) t_1^{-1}) = t_2 \phi_2 \alpha(h) t_2^{-1}$, the map is well-defined (and the similar analysis show that it is injective). This $\bar{\alpha}$ is the isomorphism we are looking for.

Conversely, if there is $\Psi : G_1 \to G_2$ sending H to H and t_1 to t_2. Denote by $\alpha = \Psi|_H$. For all $h \in H$, in G_1, we have $t^{-1} h t = \phi_1(h)$, thus in G_2, $t_2^{-1} \alpha(h) t_2 = \alpha(\phi_1(h))$, meaning, $\phi_2 \circ \alpha = \alpha \circ \phi_1$, hence ϕ_1 and ϕ_2 are conjugate in $Aut(H)$.

$\square$

In the paper of Dahmani and Groves (see Theorem 7.1 of [13]), the isomorphism problem of toral relatively hyperbolic groups is proven to be solvable.

7.3 Complements on Bass-Serre theory: the automorphisms

We will describe more precisely how automorphisms of a group can be read on a graph-of-group decomposition. We refer to [5].

Given two graphs of groups $(Y, \{G_v, v \in V(Y)\}, \{G_e, e \in E(Y)\})$ and $(Y', \{G_{v'}, v' \in V(Y')\}, \{G_{e'}, e' \in E(Y')\})$, an isomorphism of graph of groups

$$\begin{aligned}\Psi :&(Y, \{G_v, v \in V(Y)\}, \{G_e, e \in E(Y)\}) \\ &\to (Y', \{G_{v'}, v' \in V(Y')\}, \{G_{e'}, e' \in E(Y')\})\end{aligned}$$

is a tuple $(\Psi_Y, \psi_v, \psi_e, \gamma_e)$ satisfying the following conditions:

- 1. $\Psi_Y : Y \to Y'$ is an isomorphism between two underlying graphs;
- 2. for every vertex v, and for every edge e, $\psi_v : G_v \to G_{\Psi_Y(v)}$, $\psi_e = \psi_{\bar{e}} : G_e \to G_{\Psi_Y(e)}$ are isomorphisms;
- 3. for each e, $\gamma_e \in G_{\Psi_Y(t(e))}$,

  $$\psi_v \circ i|_e = ad_{\gamma_e^{-1}} \circ i|_{\Psi_Y(e)} \circ \psi_e$$

  where $ad_{\gamma_e} : x \to \gamma_e x \gamma_e^{-1}$ is an inner automprohism.

If we denote by $\beta_e : G_e \to G_{t(e)}$, then the third point implies that:

$$\begin{aligned}\psi_v \circ \beta_e(G_e) = \psi_v(G_v) = G_{\Psi_Y(v)} \\ &= ad_{\gamma_e^{-1}}(G_{\Psi_Y(v)}) = ad_{\gamma_e^{-1}} \circ \beta_{\Psi_Y(e)}(G_{\Psi_Y(e)}) \\ &= ad_{\gamma_e^{-1}} \circ \beta_{\Psi_Y(e)} \circ \psi_e(G_e)\end{aligned}$$

Assume that $(Y, \{G_v, v \in V(Y)\}, \{G_e, e \in E(Y)\})$ equals $(Y', \{G_{v'}, v' \in V(Y')\}, \{G_{e'}, e' \in E(Y')\})$. Then isomorphism between them can be composed naturally (see [1], 2.11), which gives the automorphism group of $(Y, \{G_v, v \in V(Y)\}, \{G_e, e \in E(Y)\})$, denoted by $Aut(Y, \{G_v, v \in V(Y)\}, \{G_e, e \in E(Y)\})$.

7.3.1 The JSJ decompositions and JSJ Theorem

We recall the situation of a JSJ decomposition in a hyperbolic group, or a relatively hyperbolic group. We refer to [28].

Definition 7.8 *Given a group G, a family of subgroups $\mathbb{A}$ of G, an* $\mathbb{A}$-tree *is a tree together with a G action on it with edge stabilizers in $\mathbb{A}$.*

Definition 7.9 *An $\mathbb{A}$-tree is referred to as* universally elliptic *if it is such a tree whose edge stabilizers are elliptic in every $\mathbb{A}$-tree.*

Recall that in the previous Section 2.9, we have already defined the tree domination to be a relation such that a subgroup elliptic in dominating tree is also elliptic in the dominated trees.

Definition 7.10 *(see the introduction of [28])(JSJ Decomposition) A* JSJ decomposition *(also called a* JSJ tree*) of a group H over a family of subgroup $\mathbb{A}$ is an $\mathbb{A}$-tree T satisfying the following:*

- *1. T is universally elliptic;*
- *2. For any other universally elliptic tree T', T dominates T'.*

Using the above notations, the quotient, $H\backslash T$ is called a *JSJ-splitting* (also see [28]).

Given a JSJ tree T of the group $G = H \rtimes \mathbb{Z}$, there are 3 kinds of vertices:

- 1. rigid vertices: that cannot be further splitted, for such a vertex v, the vertex group G_v satisfies $|Out(G_v)| < \infty$;
- 2. (QH) vertices: vertex groups are surface groups with boundaries;

- 3. elementary vertices: each vertex group G_v is either parabolic or (virtually) cyclic, in toral relatively hyperbolic case, each G_v is abelian.

Recall that the $\mathbb{A}$-JSJ decomposition of a group is a JSJ decomposition of this group over a family of subgroup $\mathbb{A}$.

For a vertex v in a graph of groups $\mathbb{X}$, denote by $\hat{v}$ the corresponding vertex group. Let E_v be the choice of an order on (oriented) edges adjacent to v, $S(E_v)$ be the choice of a generating set of the edge groups. A *marked peripheral structure* of $\hat{v}$, denoted by $MPS(v)$ is the tuple of conjugacy classes of the images of these generating sets by the attaching maps. Denote by $Out(\hat{v}, MPS(v))$ the subgoup of $Out(\hat{v})$ that leaves $MPS(v)$ invariant.

Definition 7.11 *(Hanging bounded Fuchsian groups) A group H is called a* hanging bounded Fuchsian group *if there is a finite normal subgroup K of H such that H/K is isomorphic to the fundamental group of a non-elementary hyperbolic compact 2-orbifold with boundary, by an isomorphism sending the images of the adjacent edge stabilizers on the boundary subgroups of the orbifold group.*

In the following part of this thesis, we assume that the choice of order and generating sets are done.

If $\mathbb{A}$ is the family of infinite cyclic subgroups of G, we can consider $\mathbb{A}$-trees, that we call $\mathbb{Z}$-trees (where every edge stabilizers are $\mathbb{Z}$).

For hyperbolic group, the JSJ theorem (Bowditch, Sella) states that any one-ended hyperbolic group admits a canonical $\mathbb{Z}$-JSJ tree (or a $\mathbb{Z}$-JSJ decomposition).

We then have the following theorem, which is proven by Dahmani and Guirardel.

Proposition 7.12 *(see Prop.3.1 of [15] and Lemma 2.13 of [12]) Let G be a hyperbolic group with one end. Let $(\hat{v}, MPS(v))$ be a vertex group of the $\mathbb{Z}$-JSJ decomposition $\mathbb{X}$ of G, with the marked peripheral structure induced by $\mathbb{X}$. If $Out(\hat{v}, MPS(v))$ is infinite, then G admits a splitting with a hanging bounded Fuchsian vertex group.*

For torsion-free relatively hyperbolic groups, we take $\mathbb{A}$ as the infinite elementary subgroups (infinite cyclic, or infinite parabolic). The JSJ theorem of Guirardel and Levitt (Corollary 9.20 of [28]) says that any one-ended toral relatively hyperbolic group admits a canonical $\mathbb{A}$-JSJ tree (or $\mathbb{A}$-JSJ decomposition).

A similar result as Proposition 7.12 holds for a relatively hyperbolic group G if one replace "$\mathbb{Z}$-JSJ decomposition" by "$\mathbb{A}$-JSJ decomposition" and replace "G admits a splitting with a hanging bounded Fuchsian vertex group" with "G admits a splitting vertex group that is either hanging bounded fuchsian or parabolic". The proof is the same.

By Dahmani and Groves (see [13]), the canonical JSJ decomposition for toral relatively hyperbolic groups is proven to be computable.

The following Theorem, proven by Guirardel and Levitt (see Theorem 1.4 of [29]), turns out to be instrumental in the later analysis of conjugacy problem in this thesis.

Theorem 7.13 *Let H be toral relatively hyperbolic and one-ended. Then some finite index subgroup $Out^1(H)$ of $Out(H)$ fits in the exact sequence:*

$$1 \to \mathbb{T} \to Out^1(H) \to \prod_{i=1}^{P} MCG^0(\Sigma_i) \times \prod_{j=1}^{m} GL_{r_j,n_j}(\mathbb{Z}) \to 1 \tag{ES}$$

where $\mathbb{T}$ is finitely generated free abelian, $MCG^0(\Sigma_i)$ is the group of isotopy classes of homeomorphisms of compact surface Σ_i mapping each boundary component to itself in an orientation-preserving way; $GL_{r,n}(\mathbb{Z})$ is the group of automorphisms of $\mathbb{Z}^{r+n}$ fixing the first n generators.

Remark: In the above theorem, each $MCG^0(\Sigma_i)$ is the automorphism group of the corresponding (QH) vertex; each $GL_{r,n}(\mathbb{Z})$ is the automorphism of the corresponding elementary vertex, whare r is the rank of G_v, and n is the rank of group generated by edges.

If one take element $(M_1, ...M_m) \in \prod_{j=1}^{m} GL_{r_k,n_k}(\mathbb{Z})$, and take each $MCG^0(\Sigma_i) = 1$, then there is $\phi \in Out^1(G)$ with a surjective map $\phi \to (M_1, ..., M_m)$, such that ϕ acts on each elementary vertex G_{v_i} by M_i and that ϕ acts on each edge by identity.

From Theorem 7.13, in order to study an element $\phi \in Out^1(G)$, we only need to know the twist in $\mathbb{T}$, and how it acts on the vertex groups.

Lemma 7.14 *If $\mathbb{Y}$ is a graph of groups with a finitely generated fundamental group G, such that every edge group is finitely generated, then every vertex group of it is also finitely generated.*

Proof: Take $x_1, ..., x_m$ generators of the fundamental group G, and take their normal forms in the Bass group.

For any vertex group G_v, denote by S_v the set of all elements in G_v that occurs in the normal forms, define $H_v = \langle S_v \rangle$. By this definition, $H_v \leq G_v$ and H_v is finitely generated.

Replace now each G_v in $\mathbb{Y}$ with H_v, we obtain a new graph of groups $\mathbb{Y}'$, which is a sub graph of groups. Then the fundamental group of $\mathbb{Y}'$ is the subgroup of G, but since it contains all generators $x_1, ..., x_m$ of G, we can deduce that the fundamental group of $\mathbb{Y}'$ is exactly G (which is the fundamental group of $\mathbb{Y}$). This implies that the vertex group of $\mathbb{Y}'$ (which is H_v) equals the vertex group of $\mathbb{Y}$ (which is G_v). As H_v is finitely generated, so is G_v. $\square$

Lemma 7.15 *Let $H = A_1 * ... * A_k * F_p$ be finitely generated, each A_i be abelian, and B be a non-cyclic abelian subgroup of H. Then B is a subgroup of some A_i for $i = 1, ..., k$.*

Proof: Let T' be the Bass-Serre Tree of H. Consider two different non-trivial elements $b, b' \in B$ and their actions on the tree T'. There are several cases of these two actions:

- 1. b fixes a vertex and b' fixes a line;
- 2. Both b and b' fix a vertex;
- 3. b' fixes a vertex and b fixes a line;
- 4. Both b and b' fix a line.

For Case 1. b fixes a vertex and b' fixes a line: In this case, we may assume without loss of generality that $b \in g^{-1}A_1g$ for some $g \in H$ (because the stabilizer of the vertex fixed by b is conjugate to one of the A_is). Since B is abelian, $b'^{-1}bb' = b$, and hence it is also in $g^{-1}A_1g$. Thus $b \in g^{-1}A_1g \cap b'g^{-1}A_1gb'^{-1}$. Due to the assumption that b fixes a vertex, and the fact that both $g^{-1}A_1g$ and $b'g^{-1}A_1gb'^{-1}$ are the stabilizers of a unique vertex in the Bass-Serre Tree, it follows that $g^{-1}A_1g = b'g^{-1}A_1gb'^{-1}$. If $g = 1$, then we can deduce that $b' \in A_1$, which violates the assumption that b' fixes a line, hence $g \neq 1$. Let v be the vertex fixed by A_1, then $g^{-1}A_1g$ fixes $g^{-1}v$, while $b'g^{-1}A_1gb'^{-1}$ fixes $b'g^{-1}v$. As $b' \neq 1$ acts on the tree by translation, $g^{-1}v \neq b'g^{-1}v$, again contradicts the fact that $g^{-1}A_1g = b'g^{-1}A_1gb'^{-1}$. This analysis shows that Case 1 (and the same for Case 3) cannot be true, and similarly, if both b and b' fix a line (or a vertex), these lines (or vertices) that are fixed by them cannot be different (by the similar argument). Thus we can deduce that there are only two possibilities:

- (i). b and b' fix the same vertex;
- (ii). b and b' fix the same axis.

If possibility (ii) holds, then by Euclid algorithm, one can find a segment L on the axis and integer n_1, n_2 such that n_1L and n_2L are fundamental segment of b and b' respectively. In terms of the group B, we find an element $\tilde{b}$ (that is generated by b and b') such that b and b' are multiples of $\tilde{b}$, which implies that B is cyclic.

Now the only possibility for b and b' is that they fix the same vertex. Notice that H is the free product and that each A_i is abelian, we conclude that B is a subgroup of some A_i for $i = 1, ..., k$. $\square$

Lemma 7.16 *(see Lemma 2.6 and Lemma 2.7 of of [12]) Let $G = H \rtimes \mathbb{Z}$, let T be the Bass-Serre G-tree of a reduced splitting of G. Then the stabilizer in G of any given vertex in T is a suspension of its stabilizer in H, and the stabilizer in G of any given edge in T is a suspension of its stabilizer in H.*

Lemma 7.17 *(see Lemma 2.10 of of [12]) For any finitely generated group H that has infinitely many ends, any finitely generated normal subgroup has finite index.*

Lemma 7.18 *Assume that each A_i is abelian and $G = H \rtimes \mathbb{Z}$ toral relatively hyperbolic. Then there is no (QH)-vertex in the JSJ-splitting (equivalently, no vertex group is free).*

Proof: Let T be a $\mathbb{Z}$-JSJ tree, let $\mathbb{X}, \mathbb{Y}$ be the quotient $\mathbb{X} = G\backslash T$, $\mathbb{Y} = H\backslash T$ respectively which are graphs of groups, and let $H = A_1 * \cdots * A_k * F_p$. Since edge groups in $\mathbb{X}$ are abelian, edge groups in $\mathbb{Y}$ (which are normal subgroups of the edge groups in $\mathbb{X}$) are also abelian. From Lemma 7.15, any non-cyclic abelian subgroup in H is a subgroup of a free factor of H, the edge group in $\mathbb{Y}$ are subgroups of some A_i. Note that A_i is isomorphic to $\mathbb{Z}^N$ for some N, and a subgroup of $\mathbb{Z}^N$ has finite basis. This implies that edge group in $\mathbb{Y}$ are finitely generated, so is vertex group in $\mathbb{Y}$ by Lemma 7.14. While the vertex group in $\mathbb{Y}$ is a infinite index normal subgroup of the corresponding vertex group of $\mathbb{X}$, by Lemma 7.17, no vertex group in $\mathbb{X}$ is free. $\square$

7.4 Conjugacy Problem for Toral Relatively Hyperbolic Automorphisms

In this section we analyse and give a complete answer to the conjugacy problem of two given toral relatively hyperbolic automorphisms of $H = A_1 * ... * A_k * F_p$.

The isomorphism problem of toral relatively hyperbolic groups is proven to be solvable (see Theorem 7.1 of [13]). Assume that there is such an isomorphism Ψ sending $G_1 = H \rtimes_{\phi_1} \langle t_1 \rangle$ to $G_2 = H \rtimes_{\phi_2} \langle t_2 \rangle$ (otherwise, if there is no such isomorphism, then ϕ_1 and ϕ_2 are not conjugate). Thus by Lemma 7.7, if Ψ satisfies statement 1.1 (or, respectively, statement 2.1) in Lemma 7.7, then ϕ_1 and ϕ_2 are conjugate in $Aut(H)$ (or respectively in $Out(H)$).

Suppose from now that Ψ does not satisfy statement 1.1 (or, respectively, statement 2.1) in Lemma 7.7, in order to solve the conjugacy problem, we can convert the problem into the following problem, which we call the *Orbit Problem of uprighting hyperplanes*:

- (for conjugacy problem in $Out(H)$, called *Problem* $\mathbb{O}_{Out}$) whether there is such an automorphism $\eta \in Aut(G_2)$ such that $\eta(\Psi(H)) \subset H$ and $\eta(\Psi(t_1)) \in t_2 H$
- (for conjugacy problem in $Aut(H)$, called *Problem* $\mathbb{O}_{Aut}$) whether there is such an automorphism $\eta \in Aut(G_2)$ such that $\eta(\Psi(H)) \subset H$ and $\eta(\Psi(t_1)) = t_2$

In addition, from Theorem 7.13, coset representatives of the subgroup $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ in $Out(H \rtimes_{\phi_2} \langle t_2 \rangle)$ are computable, the conjugacy problem can be turned into the Orbit Problem of uprighting hyperplanes in $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$. If τ is a solution in $Out^1(G_2)$ to the Orbit Problem of uprighting hyperplanes of $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$, then its image M_j of ϕ in each GL_{r_j,n_j} satisfies the condition (named *Condition 1*):

- (for conjugacy problem in $Out(H)$):

$$M_j(\Psi(H) \cap G_{v_j}) = H \cap G_{v_j}, M_j(\Psi(t_1 h_{vj})) \subset t_2 h_{vj} H$$

- (for conjugacy problem in $Aut(H)$):

$$M_j(\Psi(H) \cap G_{v_j}) = H \cap G_{v_j}, M_j(\Psi(t_1 h_{vj})) = t_2 h_{vj}$$

Thus we give the following lemma:

Lemma 7.19 *Let $H = A_1 * ... * A_k * F_p$, ϕ_1, ϕ_2 be two toral relatively hyperbolic automorphisms. Assume that there is an* *isomorphism* $\Psi : H_1 \rtimes_{\phi_1} \langle t_1 \rangle \to H_2 \rtimes_{\phi_2} \langle t_2 \rangle$ *. Then in the exact sequence (ES) of Theorem 7.13, one can decide whether there are matrices* $M_j \in GL_{r_j,n_j}(\mathbb{Z})$ *such that* $M_j(\Psi(H) \cap G_{v_j}) \subset (H \cap G_{v_j})$ *for all* $j = 1, ..., m$*. And if there exist such matrices, these matrices are computable.*

Proof: As each vertex group G_{v_j} in the JSJ-splitting $\mathbb{X}$ is finitely generated and abelian, we can find a basis for each G_{v_j}. Since $\Psi(H)$ is the kernel of a cyclic quotient of G, $\Psi(H) \cap G_{v_j}$ is the kernel of a cyclic quotient of G_{v_j}. This gives a linear equation of integer coefficient, whose result is the generating set S_j of $\Psi(H) \cap G_{v_j}$ (which does not generates other elements). Now the question can be turned into the existence of M_j such that $M_j S_j \subset (H \cap G_{v_j})$. Quotient both sides of the equation by $H \cap G_{v_j}$, the question turns to the equation $\overline{M_j S_j} = 1$, which is another linear equation problem of integer coefficient. Since there exist an algorithm to determine whether any given Diophantine equation is solvable, this Diophantine equation has a solution if and only if such a matrix M_j exist, and by the solution to the last equation (if it does has a solution), each M_j is computed. □

By the above Lemma and the same analysis, we come easily to the statement:

Proposition 7.20 *Let* $H = A_1 * ... * A_k * F_p$, ϕ_1, ϕ_2 *be two toral relatively hyperbolic automorphisms. Assume that there is an isomorphism* $\Psi : H_1 \rtimes_{\phi_1} \langle t_1 \rangle \to H_2 \rtimes_{\phi_2} \langle t_2 \rangle$ *. Then in the exact sequence (ES) of Theorem 7.13, one can decide whether there are matrices* $M_j \in GL_{r_j,n_j}(\mathbb{Z})$ *such that Condition 1 holds. And if there exist such matrices, these matrices are computable.*

From the previous analysis, if there are no matrices matrices $M_j \in GL_{r_j,n_j}(\mathbb{Z})$ satisfying Condition 1, then there is no solution in $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ to the orbit problem of $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$, and consequentely ϕ_1 and ϕ_2 are not conjugate. When such matrices exist and are computable, in order to find out whether there is some $\eta' \in Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ satisfying the orbit problem in $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$, we need to know whether there is a twist $\xi \in \mathbb{T}$ such that some $\eta' \in Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ (η' ir relevant to ξ) satisfying the orbit problem in $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$, here we give a lemma.

Lemma 7.21 *Let $H = A_1 * ... * A_k * F_p$, ϕ_1, ϕ_2 be two toral relatively hyperbolic automorphisms. Assume that there is an isomorphism $\Psi : (H_1 \rtimes_{\phi_1} \langle t_1 \rangle) \to (H_2 \rtimes_{\phi_2} \langle t_2 \rangle)$. Assume also in the exact sequence (ES) of Theorem 7.13, matrices $M_j \in GL_{r_j,n_j}(\mathbb{Z})$ satisfy Condition 1.*

Then one can decide whether or not there exists some $\eta' \in Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ satisfying the orbit problem in $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$. To be more specific, one can decide whether or not there exists some $\eta' \in Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ satisfying statement 1.2 in Lemma 7.7 (when consider conjugacy problem in $Aut(H)$), or satisfying statement 2.2 in Lemma 7.7 (when consider conjugacy problem in $Out(H)$).

Proof: From Lemma 7.18, there is no (QH)-vertex in the JSJ-splitting, so each $MCG^0(\Sigma_i)$ in the exact sequence (ES) of Theorem 7.13 is trivial. Let $\{h_1, ...h_q\}$ be a generating set of H. Since $\mathbb{T}$ is generated by Dehn twist, and since $\mathbb{T}$ is abelian, $\{DT_e | e \in NSE\}$ (in which NSE denotes the set of non-separating edges) forms a generating set of $\mathbb{T}$.

Denote by $(M_1, ..., M_m)$ the tuple of matrices satisfying Condition 1, which is in $\prod_{j=1}^{m} GL_{r_j,n_j}(\mathbb{Z})$ in the exact sequence (ES) of Theorem 7.13, denote by M a pre-image (lift) of $(M_1, ..., M_m)$ in $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$, and denote by M' the image of M in the abelianisation. If the abelianisation of η' fits the orbit problem, denote by ξ the corresponding the solution in $\mathbb{T}$. We can write $\bar{\xi}$ in the form $DT_{e_1}^{n_1}...DT_{e_s}^{n_s}$. Then the corresponding element $\bar{\eta'} \in Out^1(H_1(H \rtimes_{\phi_2} \langle t_2 \rangle))$ can be written as $M'DT_{e_1}^{n_1}...DT_{e_s}^{n_s}$. The question is turned into the existence problem of $n_1, ...n_s$ such that

- (for conjugacy problem in $Out(H)$): $\bar{\eta}(\bar{h_i}) \in \bar{H}$ for all $i = 1, ..., q$, $\bar{\eta}(\bar{t_1}) \in \bar{t_2}\bar{H}$
- (for conjugacy problem in $Aut(H)$): $\bar{\eta}(\bar{h_i}) \in \bar{H}$ for all $i = 1, ..., q$, $\bar{\eta}(\bar{t_1}) = \bar{t_2}$

Once again, quotient each side of above equations by $\bar{H}$, one can get Diophantine equations. And Diophantine equation problem is decidable. These Diophantine equations have a solution if and only if there is $\xi \in \mathbb{T}$ such that some $\eta' \in Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ satisfies the orbit problem in $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$. $\square$

The previous lemma provides the solvability for conjugation problem if $\eta' \in Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$: if there is such a solution, then we are done; if not, we still need to proceed as follows.

As $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ is a subgroup of $Out(H \rtimes_{\phi_2} \langle t_2 \rangle)$ of finite index, and that coset representatives of $Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ in $Out(H \rtimes_{\phi_2} \langle t_2 \rangle)$ are computable, denote by $\{h_1, \dots, h_u\}$ the set of these cosets ($h_i \neq h_j$ for $i \neq j$). Then we can reproduce the process in the proof of last lemma to decide whether or not there is $\xi \in \mathbb{T}$ such that some $\eta \in h_i Out^1(H \rtimes_{\phi_2} \langle t_2 \rangle)$ satisfies statement 1.2 in Lemma 7.7 (when consider conjugacy problem in $Aut(H)$) or satisfies statement 2.2 in Lemma 7.7 (when consider conjugacy problem in $Out(H)$). Once again, we obtain Diophantine equations. If for some h_i, the Diophantine equations have a solution, then the two given automorphisms are conjugate in $Out(H)$ (or in $Aut(H)$ respectively); if the Diophantine equations do not have a solution for any h_i, then the two given automorphisms are not conjugate in $Out(H)$ (or in $Aut(H)$ respectively).

If we assume that $H = A_1 * ... * A_k * F_p$, with each A_i free abelian, and that ϕ_1 and ϕ_2 are two toral atoroidal automorphisms of H, then ϕ_1, ϕ_2 satisfies the central condition, and by Theorem 6.10, $H \rtimes_{\phi_1} \mathbb{Z}, H \rtimes_{\phi_2} \mathbb{Z}$ are relatively hyperbolic. Furthermore, ϕ_1 and ϕ_2 are toral relatively hyperbolic.

From above Lemmas and analyses, one can conclude the following:

Theorem 7.22 *Given $H = A_1 * ... * A_k * F_p$, with each A_i free abelian, there is an algorithm to decide, given ϕ_1 and ϕ_2 two toral atoroidal automorphisms, whether they are conjugate in $Aut(H)$ (and in $Out(H)$).*

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