Control and stablization of some hyperbolic and dispersive equations

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Contrôle et stabilisation pour des équations hyperboliques et dispersives

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Résumé

Dans cette thèse, nous étudions la contrôlabilité et la stabilisation pour des équation hyperboliques et dispersives.

La première partie de cette thèse est consacrée à la stabilisation du système de Stokes hyperbolique. La propagation des singularités pour le système de Stokes semi-classique est établie dans chapitre 1. La preuve repose sur la stratégie de Ivrii et Melrose-Sjöstrand. Cependant, par rapport à l'opérateur de Laplace, la difficulté est causée par la pression qui a un effet non trivial pour les solutions concentrées au bord. Nous utilisons la paramétrix des solutions près d'un point elliptique ou hyperbolique. En suite, on traite les solutions concentrées près de l'ensemble «glancing» par une décomposition micro-locale. L'effet de la pression est alors bien contrôlé grâce à la géométrie. Finalement on utilise un argument récurrence pour terminer la preuve. Par conséquent, nous prouvons la stabilisation du système de Stokes hyperbolique dans le chapitre 2 sous la condition de contrôle géométrique sur le support de l'amortissement.

La deuxième partie est consacrée à la contrôlabilité et la stabilisation de l'équation de Kadomtsev-Petviashvili (KP en bref). Dans le chapitre 3, en utilisant l'analyse semi-classique, nous avons prouvé la contrôlabilité verticale pour des données dans $L^2(\mathbb{T})$. De plus, un résultat négatif concernant la contrôlabilité horizontale est aussi obtenu. Dans le chapitre 4, nous considérons la contrôlabilité de l'équation de KP-I linéaire. C'est un modèle intéressant dans lequel la vitesse de groupe peut être dégénéré. Plus général, on a obtenu le plus petit ordre requis pour assurer l'observabilité des équations de KP-I fractionnaire linéaire. Finalement dans le chapitre 5, nous avons montré la contrôlabilité et la stabilisation des 'equations de KP-II et 5KP-II avec grandes données initiales dans l'espace de Sobolev, si la donnée initiale satisfait certaines hypothèses de compacité partielles. Ceci généralise la contrôlabilité des solutions de KP-II avec données petites dans le chapitre 3.

Abstract

In this thesis, we deal with the control and stabilization for certain hyperbolic and dispersive partial differential equations.

The first part of this work is devoted to the stabilization of hyperbolic Stokes equation. The propagation of singularity for semi-classical Stokes system is established in Chapter 1. This will be done by adpating the strategy of Ivrii and Melrose-Sjöstrand. However, compared to the Laplace operator, the difficulty is caused by the pressure term which has non-trivial impact to solutions concentrated near the boundary. We apply parametrix construction to resolve the issue in elliptic and hyperbolic regions. We next adapte a fine micro-local decomposition for solutions concentrated near the glancing set. The impact of pressure to the solution is then well controled by geometric considerations. As a consequence of the main theorem in Chapter 1, we prove the stabilization of hyperbolic Stokes equation under geometric control condition in Chapter 2.

The second part is devoted to the controllability of Kadomtsev–Petviashvili(KP in short) equations. In Chapter 3, the controllability in $L^2(\mathbb{T})$ from vertical strip is proved using semi-classical analysis. Additionally, a negative result for the controllability in $L^2(\mathbb{T})$ from horizontal strip is also showed. In Chapter 4, we prove the exact controllability of linear KP-I equation if the control input is added on a vertical domain. It is an interesting model in which the group velocity may degenerate. More generally, we have obtained the least dispersion needed to insure observability for fractional linear KP I equation. Finally in Chapter 5, we prove exact controllability and stabilization of KP-II equation and fifth order KP-II equation for any size of initial data in Sobolev spaces with additional partial compactness condition. This extends the exact controllability for small data obtained in Chapter 3.

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Contents

Chapter 0

Introduction en français

0.1 Vue d'ensemble de la théorie du contrôle d'EDP

Les objectives de cette thèse sont les problèmes de contrôlabilité et stabilisation pour des équations hyperboliques et dispersives. Ces équations peuvent être considérées comme les systèmes dynamiques de la forme

$$\frac{\partial U}{\partial t} + \mathcal{A}U = \mathcal{N}(U), \tag{0.1.1}$$

où l'opérateur linéaire $\mathcal{A}$ est anti-symmétrique sur certain espace hibertien $\mathcal{H}$, tandis que le terme non-linéaire $\mathcal{N}(U)$ est assez special pour que le système (0.1.1) possède des quantités conservées, comme la masse et l'énergie, etc. Cela constitue une grande famille de modèles mathématiques découlant de la physique et des sciences de l'ingénieur, qui décrivent les phénomènes de propagation des ondes.

0.1.1 Contrôlabilité, Observabilité et Stabilisation

Le problème de contrôle s'écrit sous la forme abstraite suivante:

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U = \mathcal{N}(U) + \mathcal{B}f, t \in [0,T] \\ U(0) = U_0 \in H \end{cases} \tag{0.1.2}$$

où H est un espace hibertien sur lequel le demi-groupe engendré par $\mathcal{A}$ est bien défini, $\mathcal{B}$ est l'opérateur de contrôle avec un certain contrôle f à choisir. Pour le problème de contrôle interne, $\mathcal{B}f$ est supporté dans un ouvert et le cas le plus fréquent est $\mathcal{B} = \mathbf{1}_\omega$.

La question de la contrôlabilité exacte est de savoir si étant donné un état initial U_0 au temps $t = 0$ et un état final U_1 au temps $t = T$, on peux trouver un contrôle $f \in L^2([0,T];;H)$ tel que la solution de (0.1.2) satisfait à $U(T) = U_1$. En particulier, lorsque l'état final est donné nul, nous parlons de contrôlabilité exacte à zéro.

Pour le système linéaire

$$\begin{cases} \dfrac{\partial U}{\partial t} + \mathcal{A}U = \mathcal{B}f, t \in [0,T] \\ U(0) = U_0 \in H \end{cases} \tag{0.1.3}$$

La méthode HUM de Jacques-Louis Lions [47] ramène la contrôlabilité exacte à zéro à la validité d'une inégalité d'observabilité pour le système adjoint. Plus précisément, pour le système adjoint

$$\begin{cases} \dfrac{\partial U}{\partial t} - \mathcal{A}^*U = 0, t \in [0,T] \\ U(T) = U_T \in H, \end{cases} \tag{0.1.4}$$

la contrôlabilité exacte à zero est equivalent à l'inégalité d'observabilité

$$\|U(0)\|_H^2 \leq C \int_0^T \|\mathcal{B}^*U(t)\|_H^2 dt.$$

Pour les systèmes réversibles, on a $\mathcal{A}^* = -\mathcal{A}$ (typiquement les équations hyperboliques et dispersives que l'on va considérer dans cette thèse), et donc la contrôlabilité exacte à zero implique la contrôlabilité exacte.

La stabilisation est une autre terminologie liée à la contrôlabilité. Il s'agit de remplacer le contrôle $\mathcal{B}f$ dans (0.1.2) ou (0.1.3) par un retour qui ne dépend que de la solution. Stabilisation signifie que l'on peux utiliser le retour pour stabiliser le système dans le sens que la solution converge vers certain équilibre. Ce problème est pratique dans des domaines de l'ingénieur, car en réalité, les équilibres sont souvent instables. Dans de nombreux cas, les retours servent de termes d'amortissement qui décroissent l'énergie des systèmes.

0.1.2 Quelques résultats sur des équations linéaires typiques

Dans cette section, nous allons rappeler quelques résultats sur la contrôlabilité de trois types d'équations d'évolution linéaires.

Equation de la chaleur

Equation de la chaleur

$$\partial_t u - \Delta u = 0$$

est une EDP de type parabolique qui décrit la distribution de la chaleur (ou variation de témperature) dans une région au fil du temps. La propriété de dissipation nous permet d'obtenir la contrôlabilité exacte à zéro en temps $T > 0$ quelconque par un contrôle supporté dans un ouvert $\omega \subset \Omega$ quelconque.

Théorème 0.1 (Lebeau-Robbiano[45],Fursikov-Imanuvilov[23])
Soit Ω un domaine borné régulier de $\mathbb{R}^d$ et $\omega \subset \Omega$ un ouvert non vide. Soient $T > 0$ et

$u_0 \in L^2(\Omega)$. *Alors il existe un contrôle* $g \in L^2([0,T] \times \omega)$ *tel que la solution de*

$$\partial_t u - \Delta u = \mathbf{1}_\omega g, u|_{t=0} = u_0$$

satisfait $u(T) = 0$.

Dans [45], les auteurs ont introduit une méthode appelée la stratégie de Lebeau-Robbiano, ce qui est très utile pour obtenir la contrôlabilité d'EDPs paraboliques. Essentiellement, il s'agit d'utiliser l'inégalité de Carleman pour construire le contrôle en une infinité d'étapes où l'on "tue" les basses fréquences, puis on utilise la forte dissipation à hautes fréquences. L'estimation plus générale

$$\|g\mathbf{1}_\omega\|_{L^2(\Omega)} \leq C(T)\|u_0\|_{L^2(\Omega)}$$

est établie. On appelée fonction de coût la meilleure constante $C(T)$ dans l'inégalité précédente. Dans [21], il s'avère que $C(T) \sim e^{\frac{C(\omega,\Omega)}{T}}$ quand $T \to 0$.

Bien que nous n'allons pas traiter les équations paraboliques dans cette thèse, une motivation du résultat de Chapitre 2 vient de comprendre la fonction de coût pour le système de Stokes, qui est parabolique.

Équation des ondes

Typiquement, l'équation des ondes

$$\partial_t^2 u - \Delta u = 0$$

décrit le phénomène de propagation d'ondes acoustiques, de la lumière, de la vibration des cordes et du champ électro-magnétique. Les hautes fréquences se propagent selon la loi d'optique géométrique. En mathématique, la propagation d'ondes est décrite par la propagation de singularité pour les équations hyperboliques. Plus précisément, le front d'onde, ou la mesure micro-locale(semi-classique) se propagent le long du flot géodésique dans l'espace de phase. Ceci a permis aux auteurs de [52] et [5] d'obtenir la condition optimale pour la contrôlabilité d'équation des ondes.

Avant énoncer le résultat célèbre, nous rappelons la définition de la condition de contrôle géométrique. Soit (M, g) une variété riemannienne compacte, sans bord ou à bord assez régulier. Dans le dernier cas, on suppose que les géodésiques de $\overline{M}$ n'ont pas de contact d'ordre infini avec ∂M. On dit que un ouvert non vide $\omega \subset M$ satisfait la condition de contrôle géométrique, s'il existe $T > 0$ tel que toute géodésique généralisée de M de longueur supérieure à T rencontre l'ouvert ω.

Théorème 0.2 ([5])
Soit M *une variété riemanienne compacte à bord lisse. On suppose que les géodésiques de* $\overline{M}$ *n'ont pas de contact d'ordre infini avec* ∂M. *Supposons que l'ouvert non vide* $\omega \subset M$ *satisfait la condition de contrôle géométrique. Alors pour tout* $(u_0, u_1) \in H_0^1(M) \times L^2(M)$,

il existe un contrôle $g \in L^2([0,T];L^2(M))$, supporté par $[0,T]\times\omega$, tel que la solution unique de

$$\partial_t^2 u - \Delta u = g, (u, \partial_t u)|_{t=0} = (u_0, u_1)$$

satisfait $(u, \partial_t u)|_{t=T} = (0,0)$. De plus, pour l'équation d'ondes amorties

$$\partial_t^2 u - \Delta u + a(x)\partial_t u = 0,$$

avec l'amortissement $0 \le a \in C(\overline{M})$ et $\omega = \{x : a(x) > 0\}$, on a stabilisation uniforme

$$\|(u(t), \partial_t u(t))\|_{H^1\times L^2} \le C_0 e^{-\gamma t}\|(u(0), \partial_t u(0))\|_{H^1\times L^2}.$$

Ce résultat est initialement montré par la propagation du front d'onde. La preuve peut être simplifiée en utilisant la mesure de défaut introduit indépendamment dans [25],[60]. Il y a plein de travaux pour les contrôlabilité et stabilisation dans ce cadre d'analyse microlocale. Par exemple, le meilleur taux de décroissance exponentielle en fonction du spectre et des moyennes de $a(x)$ sur les géodésiques de M est obtenu par Lebeau dans [43], la stabilisation uniforme pour un amortissement peu-régulier $a(x)$ est obtenu dans Burq-Gérard [9], le meilleur taux de décroissance polynomial est obtenu dans [2] lorsque la condition de contrôle géométrique n'est pas satisfaite, et les contrôlabilités et stabilisations des autres systèmes hyperboliques (le système de Lamé, l'équation d'ondes couplé) sont obtenus dans Burq-Lebeau[10], Dehman-Raymond[18], Dehman-Le Rousseau-Léautaud[19], etc.

Dans le chapitre 2, nous considérons la stabilisation d'un système de Stokes hyperbolique. Ce système est lié au système de Lamé au point de vue de la modélisation. Il mélange le phénomène de la propagation d'ondes et le problème aux limites d'opérateur elliptique.

Équation de Schrödinger

L'équation de Schrödinger

$$i\partial_t u = (-\Delta u + V(x))u$$

joue un rôle fondamental en mécanique quantique, comme l'équation de Newton en mécanique classique. La solution u s'interprète comme la fonction d'onde d'un système quantique.

Le résultat le plus général sur le contrôle de l'équation de Schrödinger libre est du à Lebeau.

Théorème 0.3 (Lebeau[42])
Soit M une variété riemannienne compacte et $\omega \subset M$ satisfaisant la condition de contrôle géométrique. Alors pour $T > 0$ et $u_0 \in L^2(M)$ quelconques, il existe un contrôle $g \in L^2([0,T];L^2(M))$, supporté dans $[0,T]\times\omega$ tel que la solution de

$$i\partial_t u + \Delta u = \mathbf{1}_\omega g, u(0) = u_0$$

satisfait $u(T) = 0$.

Dans ce résultat, la condition de contrôle géométrique n'est pas nécessaire. En raison de la vitesse infinie de propagation des équations dispersives, la condition géométrique sur ω pour avoir le contrôlabilité dépend des propriétés géométriques de la variété sous-jacente. Principalement, si les trajectoires sont instables(exemple typique: le tore), on pourrait avoir l'observabilité de la solution de l'équation de Schrödinger dans un ouvert ω non vide quelconque. En revanche, si les trajectoires sont stables (exemple typique: la sphère), il y a certains ouverts dans lesquels on ne pourrait pas avoir l'observabilité en temps fini. Ceci est due à la concentration à l'équateur des fonctions propres de hautes fréquences. La compréhension de l'effet de la géométrie sur la dispersion et l'observabilité est beaucoup plus compliquée. Le résultat suivant semble un point de départ de cette direction.

Théorème 0.4 (Jaffard[31],Burq-Zworski[12])
Soit $M = \mathbb{T}^2$. La conclusion du Théorème 0.3 est vraie pour un ouvert $\omega \subset \mathbb{T}^2$ non-vide quelconque.

La contrôlabilité exacte de l'équation de Schrödinger avec un potentiel $V \in C^\infty(\mathbb{T}^2)$ est aussi obtenu dans [12] en utilisant la méthode de forme normale semi-classique. Ce résultat est amélioré pour $V \in L^\infty$ dans Anantharaman-Macià [3] comme une conséquence de la structure fine de la mesure semi-classique. Dans Bourgain-Burq-Zowrski [7], le résultat dans [12] est amélioré au cas $V \in L^2(\mathbb{T}^2)$ par une méthode différente à l'aide de la dispersion.

0.1.3 Quelques résultats pour les équations des ondes et les équations dispersives nonlinéaires

Jusqu'à présent, Il semble que l'on ne s'ait pas sorti d'un petit voisinage des équations linéaires dans le domaine de l'analyse des EDP. Les méthodes pour traiter les problèmes nonlinéaires sont extrêmement limitées. Néanmoins, de nombreux phénomènes de physique modélisés par EDPs sont, soit avec de bonnes structures nonlinéaires, soit une petite pertubation des équations linéaires. Sous certain niveau de non-linéarité, ces EDPs se comprennent efficacement.

Beaucoup d'EDPs dispersives sont semi-linéaires, dans le sens où elles peuvent être résolues par la formule d'itération de Duhamel

$$U(t) = e^{-t\mathcal{A}}U_0 + \int_0^t e^{-(t-t')\mathcal{A}}\mathcal{N}(U(t'))dt'$$

dans des espaces fonctionnels adaptés. En effet, la résolution dépend de la propriété dispersive du groupe linéaire $e^{-t\mathcal{A}}$ ainsi que de la non-linéairité que l'on peut autoriser. D'un côté, l'analyse harmonique nous fournit divers types d'estimations de $e^{-t\mathcal{A}}$ avec lesquelles on a des flexibilités pour résoudre les équations dans les espaces fonctionnels plus petits. D'autre part, les non-linéarités raisonnables doivent respecter la règle: grande non-linéarité pour des données petites(sur-critique) tandis que non-linéarité petite pour des données grandes (sous-critique). D'un point de vue heuristique, les hautes fréquences de la solution des équations sous-critiques se comportent comme les équations linéaires à mesure que le temps passe. Ceci est le point clé pour avoir une bonne théorie de Cauchy.

La théorie du contrôle dépasse la théorie de Cauchy. Une fois qu'on a établi le caractère bien-posé et la contrôlabilité pour la partie linéaire, la contrôlabilité des équations nonlinéaires avec données initiales petites est prèsque gratuite. Ce type de résultat de s'appele contrôlabilité locale, ce qui est souvent suivi d'un argument de point fixe. Pour obtenir la contrôlabilité pour des données grandes, une façon possible est la stratégie de «stabilisation-contrôlabilité locale» que nous allons expliquer. Cette méthode s'effectue en deux étapes. La première étape consiste à établir la contrôlabilité locale et la deuxième étape consiste à obtenir la stabilisation pour les équations nonlinéaires. Alors le contrôle désiré est la combinaison d'un contrôle de retour et un contrôle local. Le désavantage de cette méthode est aussi claire. Pour réaliser le contrôle, on a besoin d'attendre longtemps, ce qui n'est pas pratique dans le domaine de l'ingénierie.

Dans ce qui suit, nous allons présenter des résultats sur la stabilisation des 'equations des ondes et dispersives.

Théorème 0.5 (Dehman-Lebeau-Zuazua[20],Dehman-Gérard[16])
Soient $a(x) \geq c_0 > 0$ pour $|x| \geq R$ et $1 < p \leq 5$. Étant donnée $E_0 > 0$, il existe des constantes $C > 0, \gamma > 0$, telles que la solution u de

$$\partial_t^2 u - \Delta u + a(x)\partial_t u + u + |u|^{p-1}u = 0, (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \quad (u, \partial_t u)|_{t=0} = (u_0, u_1)$$

satisfait

$$E[u](t) \leq Ce^{-\gamma t}E[u](0), \quad \forall t \geq 0,$$

à condition que $E[u](0) \leq E_0$, où la fonctionnelle d'énergie est

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla_{t,x}u(t,x)|^2 + |u(t,x)|^2)dx + \frac{1}{p+1}\int_{\mathbb{R}^3}|u(t,x)|^{p+1}dx.$$

Dans ce théorème, le cas d'énergie sous-critique $1 < p < 5$ est résolu dans [20] tandis que le cas critique $p = 5$ est obtenu dans [16] en utilisant la méthode de décomposition en profils introduit initialement par Bahouri-Gérard [4] pour les équations des ondes. La généralisation de ce résultat à une variété compacte riemannienne avec certaines conditions supplémentaires est due à Laurent[40], voir aussi Joly-Laurent [32] pour une méthode différente.

Pour l'équation de Schrödinger nonlinéaire sur une variété compacte, les contrôlabilités et stabilisations sont initialement obtenues sous certaines hypothèses raisonnables. Plus précisément, on a

Théorème 0.6 (Dehman-Gérard-Lebeau[17])
Soient M une variété riemannienne compacte de dimension 2(sans bord), $0 \leq a \in C^\infty(M)$ et $\omega = \{x : a(x) > 0\}$. Supposons que ω satisfait la condition de contrôle géométrique et la propriété de prolongement unique pour certain opérateur de Schrödinger linéaire particulier. Alors, étant donnée $R_0 > 0$, il existe constantes $C > 0, \gamma > 0$, telles que la solution u de

$$i\partial_t u + \Delta u - a(x)(1-\Delta)^{-1}a(x)\partial_t u = |u|^{p-1}u$$

satisfait

$$E[u](t) \leq Ce^{-\gamma t} E[u](0), \quad \forall t \geq 0$$

à condition que $\|u(0)\|_{H^1} \leq R_0$*, où la fonctionnelle d'énergie est*

$$E[u](t) = \frac{1}{2} \int_M |\nabla_x u(t,x)|^2 + \frac{1}{p+1} \int_M |u(t,x)|^{p+1} dx.$$

Pour NLS cubique en dimension 1, la stabilisation dans L^2 est obtenu dans Laurent [38]. L'analogie du Théorème 0.6 en dimension 3 est obtenue dans Laurent [39].

Un autre exemple typique d'équation dispersive est l'équation de Korteweg–de Vries(KdV bref). C'est un modèle mathématique pour décrire les surfaces d'eau peu profonde dans un canal. L'équation de KdV

$$\partial_t u + \partial_x^3 u + u \partial_x u = 0$$

a de riches propriétés mathématiques. En particulier, c'est un système intégrable avec une infinité de lois de conservation. Concernant la stabilisation de KdV, on a le résultat suivant:

Théorème 0.7 (Laurent-Rosier-Zhang[41])
Il existe $\gamma > 0$ *de sorte que pour tout* $R_0 > 0$*, il existe une constante* $C > 0$ *telle que pour* $u_0 \in L^2_0(\mathbb{T})$ *avec* $\|u_0\|_{L^2} \leq R_0$*, la solution* u *de*

$$\partial_t u + \partial_x^3 u + u \partial_x u = Au$$

avec donnée initiale u_0 *satisfait*

$$\|u(t)\|_{L^2} \leq Ce^{-\gamma t} \|u_0\|_{L^2}, \quad \forall t \geq 0,$$

où l'opérateur A *est définie par*

$$Af(x) = a(x) \left(f(x) - \int_{\mathbb{T}} a(x') f(x') dx' \right)$$

pour certaine fonction $a \in C^\infty(\mathbb{T})$ *fixé avec* $\int_{\mathbb{T}} a = 1$.

Dans la deuxième partie de cette thèse, nous allons montrer quelques résultats sur la contrôlabilité et la stabilisation de l'équation de Kadomtsev–Petviashvili equation, ce qui est la généralisation de KdV en dimension 2.

0.2 Résultats principaux obtenus par l'auteur

0.2.1 Partie I

La motivation des résultats dans la première partie de cette thèse est double. Considérons le système de contrôle de Stokes

$$\begin{cases} \partial_t v - \Delta v + \nabla p = f\mathbf{1}_\omega & \text{dans } \mathbb{R}\times\Omega, \\ \text{div } v = 0 & \text{dans } \mathbb{R}\times\Omega, \\ v = 0 & \text{sur } \mathbb{R}\times\partial\Omega, \\ v(0,x) = v_0 \in H, \end{cases} \tag{0.2.1}$$

où

$$H = \{u \in L^2(\Omega)^d : \text{div } u = 0, u\cdot\nu|_{\partial\Omega} = 0\}$$

et $\nu(x)$ est la normale sortent unitaire de Ω au point $x \in \partial\Omega$. Il est bien connu (voir par exemple [22]) que pour tout $v_0 \in H$, (0.2.1) est contrôlable exacte à zéro à temps $T > 0$ quelconque. De plus, on a

$$\|f\mathbf{1}_\omega\|_{L^2(\Omega)} \le C(T)\|v_0\|_{L^2(\Omega)}.$$

Une question naturelle consiste à le coût de contrôle $C(T)$. D'après le «**Control Transmutation Method**»(CTM) introduit par Miller[49], la contrôlabilité exacte à zéro du système de Stokes hyperbolique

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = f\mathbf{1}_\omega & \text{in } \mathbb{R}\times\Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{0.2.2}$$

à un certain temps $T_0 > 0$ entraîne que l'ordre de coût de contrôle du système de Stokes (0.2.1) est $e^{\frac{C}{T}}$ quand $T \to 0$, comme dans le cas de l'équation de la chaleur.

Le système de Stokes hyperbolique ((0.2.2) avec $f = 0$) est intéressant en soi. Il est lié à un modèle de l'élasticité pour des milieux incompressibles. Plus précisément, il découle du système de Lamé dans la théorie de l'élasticité linéaire lorsque un paramètre tend vers l'infini ([47]).

Le problème que nous allons considérer dans cette thèse est la stabilisation du système de Stokes hyperbolique

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p + a(x)\partial_t u = 0 & \text{dans } \mathbb{R}\times\Omega, \\ \text{div } u = 0 & \text{dans } \mathbb{R}\times\Omega, \\ u = 0 & \text{sur } \mathbb{R}\times\partial\Omega, \\ (u(0,x),\partial_t u(0,x)) = (u_0,v_0) \in V\times H, \end{cases} \tag{0.2.3}$$

où $V = \{u \in H_0^1(\Omega)^d : \text{div } u = 0\}$. Dans (0.2.3), l'amortissement $a \in L^\infty(\Omega)$ est satisfait $a(x) \ge 0$, pour tout $x \in \Omega$.

Si $u = u(t,x)$ est une (assez lisse) solution de ce système, on définit son énergie

$$E[u](t) = \frac{1}{2}\int_\Omega (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)dx, \quad \forall t \in \mathbb{R}.$$

Lorsque il n'y a pas d'amortissement, c'est-à-dire $a \equiv 0$, l'énergie est conservée, alors qu'en général elle est décroissante:

$$\frac{dE[u]}{dt} = -\int_\Omega a(x)|\partial_t u(t,x)|^2 dx \leq 0.$$

D'après un argument standard, (voir Chapitre 2 pour plus de détails), le taux de décroissance exponentielle de $E[u](t)$ est équivalent à l'inégalité d'observabilité

$$E[v](0) \leq C_{T_0}\int_0^{T_0}\int_\omega a(x)|\partial_t v(t,x)|^2 dxdt \tag{0.2.4}$$

pour le système de Stoke hyperbolique original

$$\begin{cases} \partial_t^2 v - \Delta v + \nabla p = 0 & \text{dans } \mathbb{R}\times\Omega, \\ \text{div } v = 0 & \text{dans } \mathbb{R}\times\Omega, \\ v = 0 & \text{sur } \mathbb{R}\times\partial\Omega, \\ (v(0,x), \partial_t v(0,x)) = (v_0, v_1) \in V \times H, \end{cases} \tag{0.2.5}$$

Ce système est bien-posé et la régularité cachée $\partial_\nu u|_{\partial\Omega} \in L^2(\partial\Omega)$ est vraie si Ω est étoilé (voir [47]). Notons que $\omega = \{x : a(x) > 0\}$, lorsque ω est intersection de Ω et d'un voisinage de $\partial\Omega$ dans $\mathbb{R}^d$, l'observabilité (0.2.4) est établie dans [14] par la méthode de multiplicateur élémentaire . Ceci permet d'obtenir (voir [14]) l'ordre de coût du contrôle du système de Stokes, si Ω est étoilé. Par rapport à l'équation des ondes, il est naturel de s'interroger si (0.2.4) est varie lorsque ω satisfait la condition de contrôle géométrique. Dans le chapitre 2 de cette thèse, nous allons donner une réponse affirmative.

Théorème 0.8
Soit $\Omega \subset \mathbb{R}^d$ est un domaine lisse, borné et étoilé. On suppose que les géodésiques de $\overline{\Omega}$ n'ont pas de contact d'ordre infini avec $\partial\Omega$. $a \in C(\overline{\Omega})$ est une fonction non-negative support dont le support satisfait la condition de contrôle géométrique. Alors, il existe les constantes C_0 et α telle que pour tout $(v_0, v_1) \in V \times H$, la solution $v(t)$ de (0.2.5) *satisfait* (0.2.4).

Par rapport au problème aux limites pour l'équation des ondes étudié dans [5], la difficulté est la présence du terme ∇p dont le comportement est assez compliqué au bord. On se ramène à l'observabilité du système de Stokes semi-classique.

Une réduction heuristique est la suivante. D'après l'équation, on note que $\nabla p \in H^{-1}_{\text{loc}}(\mathbb{R}\times\Omega)$ et pour t fixé, $p(t,\cdot)$ est une fonction harmonique dans Ω. Micro-localement près de la variété caractéristique $\tau^2 = |\xi|^2$, on a $\frac{1}{\tau}\nabla_x\left(\frac{1}{\tau}\widehat{p}\right) \in L^2(\Omega)$, où $\widehat{p}$ est la transformation de Fourier en temps t. Choisissons la paramètre semi-classique $h = \frac{1}{\tau} \gg 1$ et prenons la transformation de Fourier en variable de temps de l'équation, on a

$$(-h^2\Delta - 1)\widehat{u} + h\nabla_x q = 0, \quad h \text{ div } \widehat{u} = 0 \tag{0.2.6}$$

avec $q = \frac{1}{\tau}\widehat{p}$. Notons que la solution $(\widehat{u}, h^{-1}q)$ est une fonction propre du système de Stokes statique

$$-\Delta v + \nabla P = \lambda^2 v, \qquad \text{div } v = 0$$

avec valeur propre $\lambda = h^{-1}$. Plus général, nous considérons le système

$$\begin{cases} -h^2\Delta v - v + h\nabla q = f \text{ dans } \Omega \\ h \text{ div } v = 0 \text{ in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases} \tag{0.2.7}$$

avec $f \in H$ petite au certain sens. Il est prouvé dans le chapitre 2 que le Théorème 0.8 est une conséquence de l'observabilité pour le système (0.2.7):

$$\|v\|_{L^2} \leq C\left(\|a^{1/2}v\|_{L^2} + \frac{1}{h}\|f\|_{L^2}\right) \tag{0.2.8}$$

Pour montrer (0.2.8), nous allons établir un théorème de propagation des singularités dans le chapitre 1, dans le cadre de la mesure semi-classique, qui est intéressant en soi.

Théorème 0.9
Soit $\Omega \subset \mathbb{R}^d$ est un domaine lisse, borné et étoilé. On suppose que les géodésiques de $\overline{\Omega}$ n'ont pas de contact d'ordre infini avec $\partial\Omega$. Supposons que (u_k) et une suite de solutions du problème de quasi-mode (0.2.7) *avec paramètre semi-classique $h = h_k$. Supposons que $f_k \in H$, $\|f_k\|_{L^2(\Omega)} = o(h_k)$ et u_k convergent faiblement vers* 0 *dans $L^2(\Omega)$. Soit μ est une mesure semi-classique associé à une sous-suite de la paire (u_k, h_k), alors* suppμ *est invariante le long du flot de Melrose-Sjöstrand.*

Nous faisons quelques remarques sur ce résultat. Premièrement, la mesure μ est à valeur matricielle hermitienne, et on n'a pas obtenu l'invariance de la mesure μ mais celle de supp(μ). Deuxièmement, notre hypothèse que u_k tendent vers 0 dans $L^2(\Omega)$ est naturelle, car chaque suite de la fonction propre de l'opérateur de Stokes est un exemple important.

La démonstration est adaptée de la stratégie de V.Ivrii et Melrose-Sjöstrand [48]. Nous allons donner le schéma comme suit:

- Propagation à l'intérieur: Pour un opérateur pseudo-différentiel A à support compact dans $\Omega \times \mathbb{R}^d$, l'intégration par parties et le calculs symbolique impliquent

$$\frac{i}{h}\big([-h^2\Delta - 1, A]u, u\big)_{L^2(\Omega)} = o(1), \quad h \to 0.$$

- On construit une paramètrix standard et montre que le support de μ ne contient pas de point de la région elliptique $\mathcal{E} \subset T^*\partial\Omega$.

- On utilise la structure de l'équation et on montre que le support de μ se propage le long du rayon brisé .

- On utilise la méthode de commutateur positif pour traiter le cas où les rayons sont proche de $\mathcal{G}$. D'une part, pour un opérateur A bien choisi , on a

$$\frac{i}{h}\operatorname{Im}((-h^2\Delta-1)u,Au)_{L^2}=(B_0u,u)_{L^2}$$

où l'opérateur B a un symbole principal réel positif. D'autre part, en utilisant l'équation, on a, après micro-localisation, que

$$\frac{i}{h}\operatorname{Im}((-h^2\Delta-1)u,Au)_{L^2}=(B_1q,u)_{L^2}+o(1),\quad h\to 0.$$

Le but est de montrer que $(B_1q,u)_{L^2}=o(1)$. Pour ce faire, on fait une micro-localisation de la solution selon la distance à la surface «glancing» $\mathcal{G}$ que leur traite séparément. Pour la partie proche de $\mathcal{G}$, on utilise le fait que la pression p décroît rapidement dans le sens transverse tandis que les solutions, après micro-localisation, ne peuvent pas se concentrer trop près du bord. Pour la partie un peu loin de $\mathcal{G}$, elle est bien contrôlée grâce à la géométrie et la récurrence.

0.2.2 Partie II

La deuxième partie de cette thèse est consacrée à la contrôlabilité de l'équation de Kadomtsev–Petviashvili(KP bref)

$$\partial_x(\partial_t u+\partial_x^2u+u\partial_x u)+\lambda\partial_y^2u=0 \tag{0.2.9}$$

où $\lambda=\pm1$. Cette équation est nommée après Boris Borisovich Kadomtsev et Vladimir Iosifovich Petviashvili. Elle décrit la propagation des ondes nonlinéaires en eau peu profonde. Elle modélise une onde d'eau de grande longueur d'onde avec force de rappel faiblement nonlinéaire et dispersion en fréquences. Dans le cas où la tension superficielle est faible par rapport à la force gravitationnelle, on utilise $\lambda=1$(KP-II) ; dans le cas contraire, on utilise $\lambda=-1$(KP-I). En raison de l'asymétrie en variables x et y, les ondes décrites par l'équation de KP se comportent séparément dans la direction de propagation (direction x) et la direction transversale (direction y). L'oscillation en direction y tend à être plus douce (une petite déviation).

Ces équations se posent naturellement dans l'étude de la stabilité transversale des solitons d'équation de Kortewed de-Vries (KdV) et son analogie d'ordre supérieur. Ils sont largement étudiés dans des contextes nombreux et divers ([33],[51],[53]).

Pour des distributions avec moyenne horizontale nulle, c'est-à-dire $\int_{\mathbb{T}}u(x,\cdot)dx=0$, on peut réécrire (0.2.9) sous la forme non-locale

$$\partial_t u+\partial_x^3u+u\partial_x u\pm\partial_x^{-1}\partial_y^2u=0.$$

Nous allons étudier le problème de contrôlabilité pour l'équation de KP avec données périodiques et nous organisons le problème de contrôle comme

$$\partial_t u+\partial_x^3u+u\partial_x u\pm\partial_x^{-1}\partial_y^2u=\mathcal{G}f.$$

L'opérateur $\mathcal{G}$ est construit afin de maintenir la moyenne horizontale nulle. Plus précisément, pour un domaine de contrôle vertical de la forme $\omega = (a, b) \times \mathbb{T}$, on fixe une fonction réelle non-négative $g \in C_c^2(\omega)$ avec $\int_{\mathbb{T}} g = 1$. Puis, on définit le contrôle par

$$\mathcal{G}(h)(x, y) = \mathcal{G}_\perp(h)(x, y) := g(x) \left(h(x, y) - \int_{\mathbb{T}} g(x')h(x', y)dx' \right), \tag{0.2.10}$$

Si le domaine de contrôle est une bande horizontale $\omega = \mathbb{T} \times (a, b)$, on définit le contrôle via

$$\mathcal{G}h(x, y) := \mathcal{G}_\|(h)(x, y) = g(y) \left(h(x, y) - \int_{\mathbb{T}} h(x', y)dx' \right). \tag{0.2.11}$$

KP-II linéaire

Dans le chapitre 3, nous allons montrer la contrôlabilité exacte de l'équation de KP-II linéaire lorsque le domaine de contrôle est une bande verticale. Ce type de contrôle n'est pas artificiel puisque l'équation de KP modélise l'onde d'eau en dimension 2 dans un canal étroit, et la direction de propagation d'onde doit être proche de l'axe de x en réalité.

Théorème 0.10
Étant données $T > 0$, $u_0, u_1 \in L_0^2(\mathbb{T})$, il existe un contrôle $f \in L^2((0, T); L^2(\mathbb{T}))$, tel que la solution u de

$$\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = \mathcal{G}f, \quad u|_{t=0} = u_0$$

avec $\mathcal{G} = \mathcal{G}_\perp$ satisfait $u|_{t=T} = u_1$.

Lorsque le domaine de contrôle ω est une bande horizontale, on ne peut pas avoir la contrôlabilité de l'équation de KP-I linéaire et KP-II linéaire, pour $T > 0$ quelconque.

Proposition 0.1
Supposons que $\omega = \mathbb{T} \times (a, b)$. Pour $T > 0$ quelconque, la contrôlabilité déquation de KP-I linéaire et KP-II linéaire n'est pas vraie.

En utilisant l'estimation bilinéaire de KP-II introduit par J.Bourgain dans [6], la contrôlabilité locale en temps petit de l'équation de KP-II est aussi obtenue dans le chapitre 3.

Théorème 0.11
Soit $T > 0$. Il existe $r > 0$ telle que pour toute $u_0, u_1 \in L_0^2(\mathbb{T}^2)$ ayant $\|u_0\|_{L^2(\mathbb{T}^2)} \leq r$ et $\|u_1\|_{L^2(\mathbb{T}^2)} \leq r$, il existe un contrôle $f \in L^2((0, T); L_0^2(\mathbb{T}^2)$, tel que la solution u de

$$\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = \mathcal{G}f, \quad u|_{t=0} = u_0$$

avec $\mathcal{G} = \mathcal{G}_\perp$ satisfait $u|_{t=T} = u_1$.

L'esquisse de la démonstration du Théorème 0.10 se déroule comme suit. D'après la méthode HUM standard, on se ramène à montrer l'observabilité suivante

$$\|u(0)\|_{L^2}^2 \leq C_T \int_0^T \|\mathcal{G}u(t)\|_{L^2}^2 dt \tag{0.2.12}$$

pour toute solution u de l'équation de KP-II linéaire. On remarque que le symbole principal de l'équation est $p_+ = k^3 - \frac{m^2}{k}$. Contrairement à l'équation des ondes et à l'équation de Schrödinger, il n'y a pas d'homogénéité entre les variables x et y. Nous séparons les fréquences en plusieurs régimes:

- $|D_x|^2 \sim |D_y| \sim \frac{1}{h^2}$, la vitesse de groupe en direction de x est $\sim \frac{1}{h^2}$
- $|D_x|^2 \sim \frac{1}{h_1^2} \gg |D_y| \sim \frac{1}{h^2}$, la vitesse de groupe en direction de x est $\sim \frac{1}{h_1^2}$.
- $1 \ll |D_x|^2 \sim \frac{1}{h_1^2} \ll |D_y| \sim \frac{1}{h^2}$, la vitesse de groupe en direction de x est $\sim \frac{h_1}{h^2}$.
- $1 \sim |D_x|^2 \ll |D_y| \sim \frac{1}{h^2}$.
- $1 \sim |D_x|^2 \sim |D_y| \sim 1$.

Pour les premiers trois régimes, on utilise la forte propagation en direction x. Pour le quatrième régime, on utilise la dispersion en direction y. Le cinquième régime se traite par la compacité .

KP-I Linéaire

Dans le chapitre 4, nous étudions la contrôlabilité de l'équation de KP-I linéaire

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u = 0 \tag{0.2.13}$$

où le domaine de contrôle $\omega = (a, b) \times \mathbb{T}$ est une bande verticale. Contrairement à KP-II, la vitesse de groupe de (0.2.13) s'annule quelque part car la symbole $\mathcal{L}_- = p_- = k^3 + \frac{m^3}{k}$ nést pas monotone en variable k. Malgré cette dégénérescence, la dispersion forte en direction de x nous permet d'obtenir

Théorème 0.12
Étant données $T > 0$ et $u_0 \in L_0^2(\mathbb{T}^2), u_1 \in L_0^2(\mathbb{T}^2)$, il existe un contrôle $f \in L^2((0,T); L^2(\mathbb{T}^2))$ tel que la solution u de l'équation

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u = \mathcal{G}f, \quad u|_{t=0} = u_0 \tag{0.2.14}$$

satisfait $u|_{t=T} = u_1$.

Afin de comprendre l'effet entre la dispersion et la dégénérescence, nous étudions aussi la validité de l'observabilité pour des équations de KP-I linéaire fractionnaire de la forme

$$\partial_t u - |D_x|^\alpha \partial_x u - \partial_x^{-1}\partial_y^2 u = 0. \tag{0.2.15}$$

Nous allons montrer la dichotomie suivante qui donne le taux de dispersion nécessaire pour avoir l'observabilité (ou également, la contrôlabilité exacte).

Théorème 0.13 • *Si $\alpha \geq 1$, alors pour $T > 0$ quelconque, il existe $C_T > 0$, telle que*

$$\|u(0)\|_{L^2(\mathbb{T}^2)}^2 \leq C_T \int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u(t,x,y)|^2 dxdydt$$

est vraie pour toute la solution u de (0.2.15).

- *Si $0 < \alpha < 1$, alors pour $T > 0$ quelconque, il existe une suite de solutions (u_n) de* (0.2.15)*, telle que*

$$\lim_{n\to\infty} \frac{\int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u_n(t,x,y)|^2 dtdxdy}{\|u_n(0)\|_{L^2(\mathbb{T}^2)}^2} = 0.$$

On esquisse le point clé de la démonstration du Théorème 0.13. Pour se ramener au problème unidimensionnel, on est obligé de traiter les cinq régimes comme ceux pour KP-II. La seul différence intervient dans le régime $|D_x| \sim |D_y|^{\frac{2}{\alpha+2}} \sim \frac{1}{h}$, puisque le symbole

$$p = |\xi|^\alpha \xi + \frac{1}{\xi}$$

a deux points critiques $\xi_0^\pm = \pm\left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha+2}}$. Un effort supplémentaire est nécessaire pour l'analyse près de ces deux points.

On note $\phi_\alpha(\xi) = |\xi|^\alpha\xi + \frac{1}{\xi}$, $u = e^{i\lfloor \xi_0/h\rfloor x}v$, alors

$$h^{\alpha+1}D_t v - \Phi(hD_x)v = 0, \tag{0.2.16}$$

avec $\Phi(\xi) = \phi(\xi + h\lfloor\xi_0/h\rfloor)$. Le point critique de Φ près d'origine est $\sigma_h = \xi_0 - h\lfloor\xi_0/h\rfloor \in (0,h)$. De plus, $\Phi(\sigma_h) = \Phi'(\sigma_h) = 0, \Phi''(\sigma_h) \neq 0$. Remplaçant $v \mapsto v\exp\left(\frac{i\Phi(\sigma_h)t}{h^{\alpha+1}}\right)$, on peut supposer que $\Phi(\sigma_h) = 0$. Donc l'observabilité de u en fréquences localisées à ξ_0 est équivalent à l'observabilité de v en fréquences localisées à $\sigma_h \approx 0$. D'un point de vue heuristique, l'équation de KP-I linéaire se ramène à (0.2.16) $h^{\alpha+1}D_t v - \frac{\Phi''(0)}{2}h^2 D_x^2 v = O(h)$ par développement de Taylor, où v se localize en fréquence $|D_x| \lesssim \frac{1}{h}$.

Proposition 0.2
Soit $0 \leq \alpha < 1$. Étant donnée $T > 0$, il existe une suite w_n de solutions de l'équation

$$h_n^{\alpha+1}D_t w_n + (h_n D_x)^2 w_n = 0,$$

telle que

$$\frac{\int_0^T \int_{\mathbb{T}} |g(x)w_n(t,x)|^2 dxdt}{\|w_n(0)\|_{L^2}^2} \to 0, \quad n \to \infty.$$

D'après cette propriété,

- Si $0 \leq \alpha < 1$, l'observabilité localisée n'est pas vraie pour tout $T > 0$. Ceci dit que la dégénérescence bat la dispersion.
- Si $\alpha = 1$, la dispersion est la même que pour l'équation de Schrödinger. Si $\alpha > 1$, l'effet de dispersion est plus fort que l'équation de Schrödinger, donc dans ces deux cas, l'observabilité localisé est vraie pour tout $T > 0$. Ceci dit que la dispersion bat la dégénérescence.

KP-II avec donnée grande

Dans le chapitre 5, nous étudions la contrôllabilité et la stabilisation pour des équations de types de KP-II avec données initiales grandes. Les résultats principaux dans ce chapitre sont les suivants.

Théorème 0.14
Soient $s > 0$, $R_0 > 0$. Il existe $C > 0, \gamma > 0$ et $\delta_0 > 0$ tels que

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq Ce^{-\gamma t}\|u_0\|_{H^s(\mathbb{T}^2)}, \quad \forall t > 0$$

pour toute la solution u de l'équation de KP-II amortie

$$\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = -\mathcal{G}^*\mathcal{G}u$$

avec donné initiale qui satisfait $\|u_0\|_{H^s(\mathbb{T}^2)} \leq R_0, \|u_0\|_{H^{-1,0}(\mathbb{T}^2)} \leq \delta_0$.

En combinant la stabilisation et la contrôlabilité locale établies dans le chapitre 3, on déduit la contrôlabilité suivante

Corollaire 0.1
Soient $s > 0$, $R_0 > 0$. Il existe $T > 0$, $\delta_0 > 0$, telles que si $u_0, u_1 \in H_0^s(\mathbb{T}^2)$ satisferait

$$\|u_0\|_{H^s(\mathbb{T}^2)} \leq R_0, \|u_1\|_{H^s(\mathbb{T}^2)} \leq R_0, \|u_0\|_{H^{-1,0}(\mathbb{T}^2)} \leq \delta_0, \|u_1\|_{H^{-1,0}(\mathbb{T}^2)} \leq \delta_0,$$

il existe un contrôle $\mathcal{G}h$ avec $f \in L^2((0,T); H^s(\mathbb{T}^2))$ tel que l'équation

$$\partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = \mathcal{G}f$$

admet une solution $u \in C([0,T]; H_0^s(\mathbb{T}^2))$ qui satisfait

$$u(0) = u_0, u(T) = u_1.$$

On fait quelques remarques sur ces résultats. Premièrement, la condition $\|u_0\|_{H^{-1,0}} \leq \delta_0 \ll 1$ sera appelée l'hypothèse de petitesse en basse fréquences horizontales(en variable x). On a besoin de cette petitesse parce que dans notre preuve, nous ne somme pas capable de montrer l'observabilité pour le régime dans lequel les fréquences en x sont bornées tandis que les fréquences en y sont hautes. Essentiellement, dans ce régime, l'équation peut être considérée comme un système de Schrödinger nonlinéaire couplé en variable y. Dans ce cas, la propagation en direction x est négligeable et la propagation en variable y ne nous aide pas pour avoir l'observabilité dans une bande verticale. Dans le chapitre 3, on a utilisé l'effet de dispersion (l'inégalité d'Ingham) en variable y pour surmonter cette difficulté. Cependant, puisque $H^{-\frac{1}{2},0}$ est l'espace fonctionnel critique de KP-II par rapport au changement d'échelle, l'effet nonlinéaire ne pourrait plus être ignoré comme un système de Schrödinger nonlinéaire en y. Donc éliminer l'hypothèse de petitesse en basse fréquences horizontales est un problème compliqué.

On doit également mentionner que la régularité dont on a besoin pour obtenir la stabilisation d'équation de KP-II est H^s avec $s > 0$. Ceci est dû à la difficulté de prouver la propagation de petitesse en basse fréquences dans L^2. Dans le cadre général, afin de montrer la stabilisation, on a besoin de la propriété de semi-groupe nonlinéaire afin d'itérer. Dans chaque étape d'itération, il faut assurer la petitesse en basse fréquences horizontales pour continuer. Contrairement à la norme L^2, les basse fréquences, ou en équivalence, la norme de Sobolev avec indice negatif en x, n'est pas conservée le long du flot nonlinéaire. Donc un autre problème naturel apparaît:

Problème 0.1
Est-t-il existe $s < 0$ telle que pour tout $T > 0$, toute solution globale u de l'équation de KP-II satisfait

$$\sup_{t\in[0,T]} \|u\|_{H^{s,0}(\mathbb{T}^2)} \leq C(T, \|u(0)\|_{H^{s,0}(\mathbb{T}^2)})?$$

Toute réponse positive à ce problème avec une borne plus faible $C_T(\|u(0)\|_{L^2(\mathbb{T}^2)})$ à droite pourrait améliorer le théorème 0.14 à $s \geq 0$. On croit que ce problème 0.1 est intéressant en soi car il est relié à l'existence de solution faible de l'équation de KP-II au-dessous de $L^2(\mathbb{T}^2)$. Les analogies de ces problèmes dans le cadre de l'équation de Schrödinger nonlinéaire, l'équation de KdV (sans amortissement) ont bien été étudié, voir par exemple [36],[35] et leurs bibliographies.

0.3 Quelques problèmes ouverts liés à cette thèse

Nous fournirons quelques problèmes ouverts liés à cette thèse.

Concernant le système de Stokes hyperbolique, on conjecture que la contrôlabilité exacte de (0.2.2) à temps $T > T_{GCC}$ est vraie. Ici T_{GCC} est le temps avant lequel toute géodésique généralisée de la vitesse 1 rencontre l'ouvert ω.

Problème 0.2
Soit $\Omega \subset \mathbb{R}^d$ est un domaine lisse, borné et étoilé. On suppose que les géodésiques de $\overline{\Omega}$ n'ont pas de contact d'ordre infini avec $\partial\Omega$. Soit $\omega \subset \Omega$ un ouvert non vide qui satisfait la condition de contrôle géométrique. Supposons que $T > T_{GCC}$. Est-ce que pour tout

$(u_0, u_1) \in V \times H$, *il existe un contrôle* $f \in L^2([0,T]; L^2(\Omega))$, *supporté par* $[0,T] \times \omega$, *tel que la solution unique de* (0.2.2) *satisfait* $(u, \partial_t u)|_{t=T} = (0,0)$ *?*

Une autre direction intéressante est la contrôlabilité des équations dispersives «dégénérées». Le mot dégénéré peut signifier plusieurs choses. L'équation de KP-I linéaire est un exemple dont l'ensemble de points critiques de symbole est $\left\{3\xi^2 - \frac{\eta^2}{\xi^2} = 0\right\}$. Un autre exemple est l'opérateur hypoelliptique, typiquement l'opérateur de Grushin $\Delta_G = \partial_x^2 + x^2\partial_y^2$. La contrôlabilité de l'équation de la chaleur avec l'opérateur de Grushin a attiré des intérêts récemment , voir par exemple [34], [37].

Problème 0.3
Trouver des conditions sur ω *pour lesquelles on a la contrôlabilité exacte pour l'équation de Grushin Schrödinger*

$$i\partial_t u + \Delta_G u = \mathbf{1}_\omega f.$$

Concernant les équations de KP, en plus du problème mentionné précédemment, il est aussi intéressant de généraliser le résultat sur KP-I linéaire au cas non-linéaire.

Problème 0.4
Obtenir la contrôlabilité et la stabilisation pour l'équation de KP-I avec des données périodique.

Part I

Semiclassical analysis of Stokes-system and application

Chapter 1

Semi-classical propagation of singularity for Stokes system

1.1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded, star-shaped domain. Consider the eigenvalue problem of Stokes equation

$$\begin{cases} -\Delta u_k + \nabla P_k = \lambda_k^2 u_k \text{ in } \Omega \\ \text{div } u_k = 0 \text{ in } \Omega \\ u_k|_{\partial\Omega} = 0 \end{cases} \tag{1.1.1}$$

where $u_k \in (H^2(\Omega))^d \cap V$,$\|u_k\|_{L^2} = 1$, are $\mathbb{R}^d-$valued normalized eigenfunctions and

$$V = \{u \in (H_0^1(\Omega))^d : \text{div } u = 0\}.$$

We collect several facts which are well-known in functional analysis:

- u_k forms a orthonormal basis of

  $$H = \{u \in (L^2(\Omega))^d : \text{ div } u = 0, u \cdot \nu|_{\partial\Omega} = 0\}$$

  The canonical projector $\Pi : (L^2(\Omega))^d \to H$ is called Leray projector.
- The pressure $P_k \in L^2(\Omega)/\mathbb{R}$ satisfies $\int_\Omega P_k = 0$.
- $\|\nabla u_k\|_{L^2}^2 = \lambda_k^2$, $\|u_k\|_{H^2} \leq C\lambda_k^2$, $\|\nabla P_k\|_{L^2} \leq C\lambda_k^2$,$\|P_k\|_{L^2} \leq C\lambda_k^2$.

We rephrase the system (1.1.1) by semi-classical reduction. Taking $h_k = \lambda_k^{-1}$ and $q_k = \lambda_k^{-1} P_k$, dropping the sub-index, we obtain the following $h-$dependent quasi-mode system

$$\begin{cases} -h^2\Delta u - u + h\nabla q = f \text{ in } \Omega \\ h \text{ div } u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.1.2}$$

with the following conditions:

$$\|u\|_{L^2} = 1, \|h\nabla u\|_{L^2} = O(1), \|h^2\nabla^2 u\|_{L^2} = O(1),$$

$$\|h\nabla q\|_{L^2} = O(1), f \in H, \|f\|_{L^2} = o(h).$$

When h is small, the corresponding solution $u = u(h)$ can be interpreted as high-frequency quasi-mode as its mass, i.e., the L^2-norm, is essentially concentrated on the frequency scale h^{-1}.

Before stating the main result, it is worth mentioning the eigenvalue problem of Laplace operator in semi-classical version:

$$\begin{cases} -h^2\Delta u - u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1.3}$$

One method to capture the high-frequency behavior of the solutions of (1.1.3) is to use semi-classical defect measure associated to a bounded sequence (u_k) of $L^2(\Omega)$ and to a sequence of positive scales h_k converging to zero. This measure is aimed to describe quantitatively the oscillations of (u_k) at the frequency scale h_k^{-1}. More precisely, for any bounded sequence (w_k) of $L^2(\mathbb{R}^d)$, there exists a subsequence of (w_k) and a non-negative Radon measure μ on $T^*\mathbb{R}^d$ such that for any $a(x,\xi) \in \mathcal{S}(\mathbb{R}^{2d})$,

$$\lim_{k\to\infty} (a(x, h_k D_x)w_k|w_k)_{L^2(\mathbb{R}^d)} = \langle \mu, a \rangle.$$

When Ω is a bounded domain, the precise definition of defect measure corresponding to the boundary value problem will be described later.

Let us mention that a counterpart of semi-classical defect measure, micro-local defect measure, was introduced by P.Gérard [25] and L.Tartar [60] independently. These objects are widely used in the study of control and stabilization, scattering theory and quantum ergodicity, see for example [10], [8], [26].

In the context of semi-classical defect measure, the classical theorem of Melrose-Sjöstrand about propagation of singularity for hyperbolic equation can be rephrased as follows:

Theorem 1.1 ([26])
Assume that Ω is a smooth, bounded domain with no infinite order of contact on the boundary. Suppose μ is the semi-classical defect measure associated to the pair (u_k, h_k) where (u_k) is a sequence of solutions to (1.1.3) *(with $h = h_k$) which are bounded in $L^2(\Omega)$. Then μ is invariant under the Melrose-Sjöstrand flow.*

We will give the precise definition of Melrose-Sjöstrand flow and the associated concept of the order of contact in the second section. Intuitively, these flows are the generalization of geometric optics. No infinite order of contact means that the trajectory of the flow can not tangent to the boundary with an infinite order.

The main result of this paper is as follows.

Theorem 1.2
Assume that Ω is a smooth, bounded star-shaped domain with no infinite order of contact on the boundary. Suppose (u_k) is a sequence of solutions to the quasi-mode problem (1.1.2) *with semi-classical parameters $h = h_k$. Assume that $f_k \in H$, $\|f_k\|_{L^2(\Omega)} = o(h_k)$ and u_k converges weakly to* 0 *in $L^2(\Omega)$. Assume that μ is any semi-classical measure associated to some subsequence of (u_k, h_k), then* $\text{supp}(\mu)$ *is invariant under Melrose-Sjöstrand flow.*

We make some comments about the result. Firstly, the measure μ is Hermitian matrix-valued, and we have no information on the precise propagation for μ except for $\text{supp}(\mu)$. Secondly, since the eigenfunctions of Stokes operator converge weakly to 0 in $L^2(\Omega)$, our results includes this special case.

An application of Theorem 1.2 is the stabilization of a hyperbolic Stokes system under the geometric control condition. Let us consider the following hyperbolic-Stokes equation with damping

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p + a(x)\partial_t u = 0 & \text{in } \mathbb{R} \times \Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ (u(0,x), \partial_t u(0,x)) = (u_0, v_0) \in V \times H, \end{cases} \tag{1.1.4}$$

The energy

$$E[u](t) = \frac{1}{2}\int_\Omega (|\partial_t u|^2 + |\nabla u|^2)dx$$

is dissipative. This is the main objective in the next Chapter.

Let us describe briefly our strategy for the proof of Theorem 1.2. The pressure term q is harmonic and in heuristic, it can only have the influence to the solution near the boundary. We will prove that the measure is propagated in the same way as Laplace quasi-mode (semi-classical analogue of wave equation) along the rays inside the domain. When a ray touches the boundary, we need a more careful analysis between the wave-like propagation phenomenon and the impact of pressure. It is difficult to get a simple propagation formula near the boundary, comparing to the treatment of quasi-mode problem of Laplace operator as in [10],[26]. We partition the phase space into elliptic region $\mathcal{E}$, hyperbolic region $\mathcal{H}$ and glancing surface $\mathcal{G}$. It turns out that there is no singularity accumulated near elliptic region. For the hyperbolic region, the propagation argument is also standard, with an additional treatment when the incidence of the ray is right. Near the glancing surface, we will follow the arguments of V.Ivrii's and Melrose-Sjöstrand. The main difference is that we will encounter two new cross terms essentially of the form $(q|u)_{L^2}$ after certain micro-localization. To overcome this difficulty, we further micro-localize the solution according to the distance to the glancing surface $\mathcal{G}$ and treat them separately. For the part nearing $\mathcal{G}$, we use the fact that the pressure decays fast away from the boundary while the solution can not concentrate too much near the boundary, provided that it is micro-localized close enough to the glancing surface. For the part away from $\mathcal{G}$, it can be well-controlled by geometric considerations.

1.2 Preliminary

1.2.1 Notations

We will sometimes drop the sub-index k for a sequence of functions (u_k) and semi-classical parameters h_k. In this circumstance, the notion $\|u\|_X = O(1), o(1)$ as $h \to 0$ should be understood as $\|u_k\|_X = O(1), o(1)$ as $k \to \infty$ (thus $h_k \to 0$) up to certain subsequence.

As in the introduction, we always use

$$V = \{u \in H_0^1(\Omega)^N : \text{div } u = 0\}$$

and

$$H = \{u \in L^2(\Omega)^N : \text{div } u = 0, u \cdot \nu|_{\partial\Omega} = 0\}.$$

In this paper we always use ν to denote the outward normal vector on $\partial\Omega$.

For a manifold M, we let TM be its tangent bundle and T^*M be the cotangent bundle with canonical projection

$$\pi : TM(\text{ or } T^*M) \to M.$$

We will identify system (1.1.2) as a system on differential form

$$\begin{cases} h^2\Delta_H u - u + hdq = f \text{ in } \Omega \\ hd^*u = 0 \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.2.1}$$

where the unknown $u \in \Lambda^1(\Omega)$ is 1-form, and

$$d : \Lambda^p(\Omega) \to \Lambda^{p+1}(\Omega), d^* : \Lambda^{p+1}(\Omega) \to \Lambda^p(\Omega)$$

are exterior differential and divergence operator on forms, with respectively. Recall also that he Hodge Laplace operator is defined by

$$\Delta_H = dd^* + d^*d = (d + d^*)^2.$$

In the turbulent neighborhood of boundary, we can identify the Ω locally as one side of the turbulent neighborhood denoted by $Y_+ = [0, \epsilon_0) \times X$, $X = \{x' \in \mathbb{R}^{d-1} : |x'| < 1\}$. We denote by $\underline{\partial} Y_+ = Y_+|_{y=0}$ and $Y_+^0 = Y_+|_{y>0}$. For $x \in \overline{\Omega}$, we note $x = (y, x')$, where $y \in [0, \epsilon_0), x' \in X$, and $x \in \partial\Omega$ if and only if $x = (0, x')$. In this coordinate system, the Euclidean metric dx^2 can be written as matrices

$$\overline{g} = \begin{pmatrix} 1 & 0 \\ 0 & g(y, x') \end{pmatrix}, \overline{g}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & g^{-1}(y, x') \end{pmatrix},$$

with $|\xi'|^2_{g^{-1}(y,x'))} = \langle \xi', g^{-1}(y, x')\xi' \rangle_{\mathbb{C}^{d-1}} = g^{jk}\xi'_j\xi'_k$ be the induced metric on $T^*\partial\Omega$, parametrized by y. Note that $|\xi'|^2_{g^{-1}(0,x')} = \langle \xi', g^{-1}(0, x')\xi' \rangle_{\mathbb{C}^{d-1}} = g^{jk}\xi'_j\xi'_k$ is the natural norm on $T^*\partial\Omega$, dual of the norm on $T\partial\Omega$, induced by the canonical metric on $\overline{\Omega}$. Write $(x, \xi) = (y, x', \eta, \xi')$

and denote by $|\xi|$ the Euclidean norm on $T^*\mathbb{R}^d$. For $u, v \in \Lambda^1(Y_+)$ with support in the local chart of turbulence neighborhood, we define the L^2 norms and inner product on $[0, \epsilon_0) \times X$ via

$$\|u\|^2_{L^2(Y_+)} := \int_0^{\epsilon_0} \int_X \langle u|u\rangle \sqrt{\det(\overline{g})} dx' dy,$$

$$(u|v)_{L^2(Y_+)} := (u|v)_{Y_+} := \int_0^{\epsilon_0} \int_X \langle u|v\rangle \sqrt{\det(\overline{g})} dx' dy,$$

$$\|u(y,\cdot)\|^2_{L^2(\partial Y_+)} := \int_X \langle u(y,\cdot)|u(y,\cdot)\rangle \sqrt{\det(g)} dx',$$

$$(u|v)_{L^2(\partial Y_+)}(y) := \int_X \langle u(y,\cdot)|v(y,\cdot)\rangle \sqrt{\det(g)} dx',$$

where for $u = u_0 dy + u_j dx'^j, v = v_0 dy + v_j dx'^j$,

$$\langle u|v\rangle = u_0 \overline{v}_0 + u_j \overline{v_k} g^{jk}.$$

In certain situations we also use global notation for L^2 inner product:

$$(u|v)_\Omega := \int_\Omega u \cdot \overline{v} dx, \quad (f|g)_{\partial\Omega} := \int_{\partial\Omega} f \cdot \overline{g} d\sigma(x)$$

We will identify the unknown vector fields u, v, etc. and their dual 1-form. Formulation of differential form will simplify some calculations. In the turbulence neighborhood, we write a vector field

$$X = X_\parallel + X_\perp \frac{\partial}{\partial y}, \quad X_\parallel = \sum_{j=1}^{d-1} X_{\parallel,j} \frac{\partial}{\partial x'_j}$$

and we write $X = (X_\parallel, X_\perp)$. The normal component obeys the following convention: $(0, a) = -a\nu$.

As in [13], we will write down system (1.1.2) in the turbulent neighborhood. For $u = (u_\parallel, u_\perp)$, equation (1.1.2) can be written as:

$$\begin{cases} (-h^2\Delta_\parallel - 1)u_\parallel + h\nabla_{x'} q = f_\parallel, \\ (-h^2\Delta_g - 1)u_\perp + h\partial_y q = f_\perp, \\ h \text{ div }_\parallel u_\parallel + \dfrac{h}{\sqrt{\det g}} \partial_y(\sqrt{\det g} u_\perp) = 0 \end{cases} \tag{1.2.2}$$

where

$$h^2\Delta_\parallel = h^2\partial_y^2 - \Lambda^2(y, x', hD_{x'}) + hM_\parallel(y, x', hD'_x) + hM_1(y, x')h\partial_y,$$
$$h^2\Delta_g = h^2\partial_y^2 - \Lambda^2(y, x', hD_{x'}) + hM_\perp(y, x', hD'_x) + hN_1(y, x')h\partial_y,$$
$$h \text{ div }_\parallel u_\parallel = \frac{h}{\sqrt{\det g}} \sum_{j=1}^{N-1} \partial_{x'_j}(\sqrt{\det g} u_{\parallel,j}).$$

$h^2\Lambda^2(y, x', hD_{x'})$ has the symbol $\lambda^2 = |\xi'|^2_{\alpha(y,\cdot)}$, and $M_{\parallel,\perp}$ are both first-order matrix-valued semi-classical differential operators.

1.2.2 Geometric Preliminaries

Denote by ${}^bT\overline{\Omega}$ the vector bundle whose sections are the vector fields $X(p)$ on $\overline{\Omega}$ with $X(p) \in T_p\partial\Omega$ if $p \in \partial\Omega$. Moreover, denote by ${}^bT^*\overline{\Omega}$ the Melrose's compressed cotangent bundle which is the dual bundle of ${}^bT\overline{\Omega}$. Let

$$j : T^*\overline{\Omega} \to^b T^*\overline{\Omega}$$

be the canonical map. In our geodesic coordinate system near $\partial\Omega$, ${}^bT\overline{\Omega}$ is generated by the vector fields $\frac{\partial}{\partial x_1'}, \cdots, \frac{\partial}{\partial x_{d-1}'}, y\frac{\partial}{\partial y}$ and thus j is defined by

$$j(y, x'; \eta, \xi') = (y, x'; v = y\eta, \xi').$$

The principal symbol of operator $P_h = -(h^2\Delta + 1)$ is

$$p(y, x', \eta, \xi') = \eta^2 + |\xi'|^2_{g^{-1}(y,x')} - 1.$$

By $\mathrm{Car}(P)$ we denote the characteristic variety of p:

$$\mathrm{Car}(P) := \{(x, \xi) \in T^*\mathbb{R}^d|_{\overline{\Omega}} : p(x, \xi) = 0\}, Z := j(\mathrm{Car}(P)).$$

By writing in another way

$$p = \eta^2 - r(y, x', \xi'), r(y, x', \xi') = 1 - |\xi'|^2_{g^{-1}(y,x')},$$

we have the decomposition

$$T^*\partial\Omega = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G},$$

according to the value of $r_0 := r|_{y=0}$ where

$$\mathcal{E} = \{r_0 < 0\}, \mathcal{H} = \{r_0 > 0\}, \mathcal{G} = \{r_0 = 0\}.$$

The sets $\mathcal{E}, \mathcal{H}, \mathcal{G}$ are called elliptic, hyperbolic and glancing, with respectively.

For a symplectic manifold S with local coordinate (z, ζ), a Hamiltonian vector field associated with a real function f is given by

$$H_f = \frac{\partial f}{\partial \zeta}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \zeta}.$$

Now for $(x, \xi) \in \Omega$ far away from the boundary, the Hamiltonian vector field associated to the characteristic function p is given by

$$H_p = 2\xi\frac{\partial}{\partial x}.$$

We call the trajectory of the flow

$$\phi_s : (x, \xi) \mapsto (x + s\xi, \xi)$$

bicharacteristic or simply ray, provided that the point $x + s\xi$ is still in the interior.

To classify different situations as a ray approaching the boundary, we need more accurate decomposition of the glancing set $\mathcal{G}$. Let $r_1 = \partial_y r|_{y=0}$ and define

$$\mathcal{G}^{k+3} = \{(x', \xi') : r_0(x', \xi') = 0, H^j_{r_0}(r_1) = 0, \forall j \leq k; H^{k+1}_{r_0}(r_1) \neq 0\}, k \geq 0$$

$$\mathcal{G}^{2,\pm} := \{(x', \xi') : r_0(x', \xi') = 0, \pm r_1(x', \xi') > 0\}, \mathcal{G}^2 := \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-}.$$

No infinite order of contact means that we can decompose $\mathcal{G}$ into

$$\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j.$$

Given a ray $\gamma(s)$ with $\pi(\gamma(0)) \in \Omega$ and $\pi(\gamma(s_0)) \in \partial\Omega$ be the first point who attaches the boundary. If $\gamma(s_0) \in \mathcal{H}$, then $\eta_\pm(\gamma(s_0)) = \pm\sqrt{r_0(\gamma(s_0))}$ be the two different roots of $\eta^2 = r_0$ at this point. Notice that the ray starting with direction η_- will leave Ω, while the ray with direction η_+ will enter the interior of Ω. This motivates the following definition of broken bicharacteristic:

Definition 1.1 ([28])
A broken bicharacteristic arc of p is a map:

$$s \in I \setminus B \mapsto \gamma(s) \in T^*\Omega \setminus \{0\},$$

where I is an interval on $\mathbb{R}$ and B is a discrete subset, such that

1. *If J is an interval contained in $I \setminus B$, then $s \in J \mapsto \gamma(s)$ is a bicharacteristic of p over Ω.*

2. *If $s \in B$, then the limits $\gamma(s^+)$ and $\gamma(s^-)$ exist and belongs to $T^*_x\overline{\Omega} \setminus \{0\}$ for some $x \in \partial\Omega$, and the projections in $T^*_x\partial\Omega \setminus \{0\}$ are the same hyperbolic point.*

When a ray $\gamma(s)$ arrives at some point $\rho_0 \in \mathcal{G}$, there are several situations. If $\rho_0 \in \mathcal{G}^{2,+}$, then the ray passes transversally over ρ_0 and enters $T^*\Omega$ immediately. If $\rho_0 \in \mathcal{G}^{2,-}$ or $\rho_0 \in \mathcal{G}^k$ for some $k \geq 3$, then we can continue it inside $T^*\partial\Omega$ as long as it can not leave the boundary along the trajectory of the Hamiltonian flow of H_{-r_0}. We now give the precise definition.

Definition 1.2 ([28])
A generalized bicharacteristic ray of p is a map:

$$s \in I \setminus B \mapsto \gamma(s) \in (T^*\overline{\Omega} \setminus T^*\partial\Omega) \cup \mathcal{G}$$

where I is an interval on $\mathbb{R}$ and B is a discrete set of I such that $p \circ \gamma = 0$ and the following:

1. *$\gamma(s)$ is differentiable and $\frac{d\gamma}{ds} = H_p(\gamma(s))$ if $\gamma(s) \in T^*\overline{\Omega} \setminus T^*\partial\Omega$ or $\gamma(s) \in \mathcal{G}^{2,+}$.*

2. *Every $s \in B$ is isolated, $\gamma(s) \in T^*\overline{\Omega} \setminus T^*\partial\Omega$ if $s \neq t$ and $|s-t|$ is small enough, the limits $\gamma(s^\pm)$ exist and are different points in the same fibre of $T^*\partial\Omega$.*

3. *$\gamma(s)$ is differentiable and $\frac{d\gamma}{ds} = H_{-r_0}(\gamma(s))$ if $\gamma(s) \in \mathcal{G} \setminus \mathcal{G}^{2,+}$.*

Remark 1.1
The definition above does not depend on the choice of local coordinate, and in the geodesic coordinate system, the map

$$s \mapsto (y(s), \eta^2(s), x'(s), \xi'(s))$$

is always continuous and

$$s \mapsto (x'(s), \xi'(s))$$

is always differentiable and satisfies the ordinary differential equations

$$\frac{dx'}{dt} = -\frac{\partial r}{\partial \xi'}, \frac{d\xi'}{dt} = \frac{\partial r}{\partial x'},$$

the map $s \mapsto y(s)$ is left and right differentiable with derivative $2\eta(s^\pm)$ for any $s \in B$ (hyperbolic point).

Moreover, there is also the continuous dependence with the initial data, namely the map

$$(s, \rho) \mapsto (y(s,\rho), \eta^2(s,\rho), x'(s,\rho), \xi'(s,\rho))$$

is continuous. We denote the flow map by $\gamma(s,\rho)$.

Remark 1.2
Under the map $j : T^*\overline{\Omega} \to^b T^*\overline{\Omega}$, one could regard $\gamma(s)$ as a continuous flow on the compressed cotangent bundle ${}^bT^*\overline{\Omega}$, and it is called the Melrose-Sjöstrand flow. We will also call each trajectory generalized bicharacteristic or simply ray in the sequel.

It is well-known that if there is no infinite contact in $\mathcal{G}$, a generalized bicharacteristic is uniquely determined by any one of its points. In other words, the Melrose-Sjöstrand flow is globally well-defined. See [28] for more discussion.

1.2.3 definition of defect measure

We follow closely as in [8] and the one can find in [26] for a little different but comprehensive introduction.

We denote by S^m the usual symbol class. Define the partial symbol class S^m_∂ and the class of boundary h-pseudo-differential operators $\mathcal{A}^m_h$ as follows

$$S^m_\partial := \{a(y,x',\xi') : \sup_{\alpha,\beta,y\in[0,\epsilon_0]} |\partial^\alpha_{x'}\partial^\beta_{\xi'} a(y,x',\xi')| \leq C_{m,\alpha,\beta}(1+|\xi'|)^{m-\beta}\}.$$

$$\mathcal{A}^m_h =: \mathrm{Op}^{comp}_h(S^m) + \mathrm{Op}_h(S^m_{\xi'}) := \mathcal{A}^m_{h,i} + \mathcal{A}^m_{h,,\partial}.$$

Consider functions of the form $a=a_i+a_\partial$ with $a_i\in C_c^\infty(\Omega\times\mathbb{R}^d)$ which can be viewed as a symbol in S^0, and $a_\partial\in C_c^\infty(Y_+\times\mathbb{R}^{d-1})$ can be viewed as a symbol in $S^0_{\xi'}$. We quantize a as follows:Take $\varphi_i\in C_c^\infty(\Omega)$ equal to 1 near the $x-$projection of $\mathrm{supp}(a_i)$ and $\varphi_\partial\in C_c^\infty(\mathbb{R}^d)$ equal to 1 near the $x-$projection of $\mathrm{supp}(a_\partial)$. Define

$$
\begin{aligned}
\mathrm{Op}_h^{\varphi_i,\varphi_\partial}(a)f(y,x') =&\frac{1}{(2\pi h)^d}\int_{\mathbb{R}^{2d}} e^{\frac{i(x-z)\xi}{h}}a_i(x,\xi)\varphi_i(z)f(z)dzd\xi\\
&+\frac{1}{(2\pi h)^{d-1}}\int_{\mathbb{R}^{2(d-1)}} e^{\frac{i(x'-z')\xi'}{h}}a_\partial(y,x',\xi')\varphi_\partial(y,z')f(y,z')dz'd\xi'.
\end{aligned}
$$

According to the symbolic calculus, the operator $\mathrm{Op}_h^{\varphi_i,\varphi_\partial}(a)$ does not depend on the choice of functions $\varphi_i,\varphi_\partial$, modulo operators of norms $O_{L^2_{\mathrm{loc}}\to L^2_{\mathrm{comp}}}(h^\infty)$, and we will use the notion $\mathrm{Op}_h(a)$ in the sequel. Notice that the acting of tangential operator $\mathrm{Op}_h(a_\partial)$ can be viewed as pseudo-differential operator on the manifold $\partial\Omega$, parametrized by the parameter $y\in[0,\epsilon_0)$. The bounded family of operators $\mathcal{A}^m_{h,\partial}$ is defined uniquely up to a family of operators with norms uniformly dominated by Ch, as $h\to 0$. See [26] for more details. Moreover, for any family (A_h), such that

$$
\|A_h-\mathrm{Op}_h(a_\partial)\|_{L^2\to L^2}=O(h),
$$

the principal symbol $\sigma(A)$ is determined uniquely as a function on $T^*\partial\Omega$, smoothly depending on y, i.e. $\sigma(A)\in C^\infty([0,\epsilon_0)\times T^*\partial\Omega)$.

When we deal with vector-valued functions, we could require the symbol a to be matrix-valued. Now for any sequence of vector-valued function w_k, uniformly bounded in $L^2(\Omega)$, there exists a subsequence (still use w_k for simplicity), and a nonnegative definite Hermitian matrix-valued Radon measure μ_i on $T^*\Omega$ such that

$$
\lim_{k\to 0}(\mathrm{Op}_{h_k}(a_i)w_k|w_k)_{L^2}=\langle\mu_i,a\rangle:=\int_{T^*\Omega}\mathrm{tr}\ (ad\mu_i).
$$

For a proof, see for example [8], and the micro-local version was appeared in [25].

From now on the symbols and operators will be scalar-valued unless otherwise specified. Suppose u_k be a sequence of solutions to

$$
\begin{cases}-h_k^2\Delta u_k-u_k+h_k\nabla q_k=f_k,(u_k,f_k)\in(H^2(\Omega)\cap V)\times H,\\ h_k\mathrm{div}u_k=0,\mathrm{in}\ \Omega\end{cases}\tag{1.2.3}
$$

under the assumptions below:

$$
\begin{aligned}
&\|u_k\|_{L^2(\Omega)}=O(1),\quad f_k\in H\text{ and }\|f_k\|_{L^2(\Omega)}=o(h_k),\\
&\|h\nabla q_k\|_{L^2(\Omega)}=O(1),\quad \int_\Omega q_kdx=0.
\end{aligned}\tag{1.2.4}
$$

The following result shows that the interior measure μ_i is supported on the $\mathrm{Car}(P)$.

Proposition 1.1
Let $a_i \in C_c^\infty(\Omega \times \mathbb{R}^d)$ be equal to 0 near $\mathrm{Car}(P)$, *then we have*

$$\lim_{k\to\infty} (\mathrm{Op}_{h_k}(a_i)u_k|u_k)_{L^2} = 0.$$

Proof. Note that the symbol $b(x,\xi) = \frac{a_i(x,\xi)}{|\xi|^2-1} \in S^0$ is well-defined from the assumption on a_i. From symbolic calculus, we have

$$\mathrm{Op}_{h_k}(a_i) = B_{h_k}(-h_k^2\Delta - 1) + O_{L^2\to L^2}(h_k).$$

Therefore

$$\begin{aligned}(B_{h_k}(-h_k^2\Delta - 1)u_k|u_k)_{L^2} &= (B_{h_k}f_k|u_k)_{L^2} - (B_{h_k}h_k\nabla q_k|u_k)_{L^2}\\ &= o(1) + ([h_k\nabla, B_{h_k}]q_k|u_k)_{L^2} - (h_k\nabla B_{h_k}q_k|u_k)_{L^2}\\ &= o(1), \text{ as } k\to\infty,\end{aligned}$$

where in the last line we have used the symbolic calculus, integrating by part, and Lemma 1.3. ∎

Now we denote by $Z = j(\mathrm{Car}(P))$. Proposition 1.1 indicates that the interior defect measure μ_i is supported on Z. To define the defect measure near the boundary, we have to check that if $a_\partial \in C_c^\infty(U\times\mathbb{R}^{d-1})$ vanishing near Z (i.e. a_∂ is supported in the elliptic region for all y small) then

$$\lim_{k\to\infty}(\mathrm{Op}_{h_k}(a_\partial)u_k|u_k)_{L^2} = 0.$$

Indeed, this can be ensured by the analysis of boundary value problem in the elliptic region, which will be given later. Now for any family of operator $A_h \in \mathcal{A}_h^0$, let $a = \sigma(A_h)$ be the principal symbol of A_h and we define $\kappa(a) \in C^0(Z)$ via $\kappa(a)(\rho) := a(j^{-1}(\rho))$. Note that Z is a locally compact metric space and the set

$$\{\kappa(a) : a = \sigma(A_h), A_h \in \mathcal{A}_h^0\}$$

is a locally dense subset of $C^0(Z)$. We then have the following proposition, which guarantees the existence of a Radon measure on Z:

Proposition 1.2
There exists a subsequence of u_k, h_k and a nonnegative definite Hermitian matrix-valued Radon measure μ, such that

$$\lim_{k\to\infty}(A_{h_k}u_k|u_k)_{L^2} = \langle\mu, \kappa(a)\rangle, a = \sigma(A_h), \forall A_h \in \mathcal{A}_h^0.$$

The proof of this result can be found in [8], see also [10] and [25] for its micro-local counterpart. Notice that if we write $a = a_i + a_\partial$, then

$$(A_k u_k|u_k) \to \int_{T^*\Omega} \mathrm{tr}\ (a_i(\rho)d\mu_i(\rho)) + \int_Z \mathrm{tr}\ (a_\partial(\rho)d\mu(\rho)).$$

The following result shows that information about frequencies higher than the scale h_k^{-1} does not lost, and the measure μ contains the relevant information of the sequence (u_k).

Proposition 1.3
The sequence of solution (u_k) is h_k-oscillating in the following sense:

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int_{|\xi|\geq Rh_k^{-1}}|\widehat{\psi u_k}(\xi)|^2 d\xi = 0, \forall \psi\in C_c^\infty(\Omega),$$

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int_0^{\epsilon_0} dy\int_{|\xi'|\geq Rh_k^{-1}}|\widehat{\psi u_k}(y,\xi')|^2 d\xi' = 0, \forall \psi\in C_c^\infty(\overline{\Omega}),$$

where in the second formula, the Fourier transform is only taken for the x' direction.

The proof will be given in appendix.

1.3 A priori information about the system

1.3.1 Information about the trace

We consider the semi-classical Stokes system

$$\begin{cases} -h^2\Delta u - u + h\nabla q = f, (u,f)\in (H^2(\Omega)\cap V)\times H \\ h\ \mathrm{div} u = 0, \text{in } \Omega \end{cases} \tag{1.3.1}$$

Assume that $\|u\|_{L^2(\Omega)} = O(1)$, $\|f\|_{L^2(\Omega)} = o(h)$. Taking inner product with u and integrating by part, we have $\|h\nabla u\|_{L^2(\Omega)} = O(1)$. Since $q\in L^2(\Omega)/\mathbb{R}$, we may assume that $\int_\Omega q dx = 0$. From the regularity theory of steady Stokes system, (see [62], page 33), Poincaré inequality, we have

$$\|h^2\nabla^2 u\|_{L^2(\Omega)} = O(1), \|q\|_{L^2(\Omega)} = O(h^{-1}), \|h\nabla q\|_{L^2(\Omega)} = O(1).$$

The following is a direct consequence of trace theorem for $q_0 = q|_{\partial\Omega}$.

Lemma 1.1
$\|q_0\|_{H^{1/2}(\partial\Omega)} = O(h^{-1})$.

There is hidden regularity for the normal derivative.

Lemma 1.2
$h\partial_\nu u|_{\partial\Omega} = (h\partial_\nu u_\|, 0)$ *and* $\|h\partial_\nu u|_{\partial\Omega}\|_{L^2(\partial\Omega)} = O(1)$.

The proof of this lemma will be given in appendix A in which we use the fact that Ω is star-shaped. We will recover some information for low frequencies from the following lemma:

Lemma 1.3
Suppose $u \rightharpoonup 0$ in $L^2(\Omega)$. Then after extracting to subsequences, we have $h\nabla q\rightharpoonup 0$ weakly in $L^2(\Omega)$ and $hq\to 0$ strongly in $H^{1/2}(\Omega)$.

Proof. We may assume that $h\nabla q \rightharpoonup r$ weakly in $L^2(\Omega)$ and Rellich theorem implies that $hq \to P$ strongly in $L^2(\Omega)$, and thus $\nabla P = r$ with the property $\int_\Omega P = 0$. Moreover it is easy to see that $\Delta P = 0$ in Ω. Since the sequence $(h^2\nabla^2 u)$ is bounded in L^2, then up to a subsequence, $h^2\nabla^2 u \rightharpoonup W$ weakly in L^2. From Rellich theorem, the sequence $(h^2 u)$ converges strongly in L^2 and the strong limit must be 0 due to $u \rightharpoonup 0$ weakly in L^2. Thus $W = 0$ and this implies that $\nabla P = 0$. Finally, we must have $P = 0$ since it has zero mean value. The last assertion follows from Rellich theorem. ■

1.3.2 Semi-classical parametrix of the pressure term

In system (2.4.1), the family of pressures q satisfy the boundary value problem of Laplace equation

$$-h^2\Delta q = 0, \text{ in } \Omega, q|_{\partial\Omega} = q_0$$

with unknown boundary data q_0. We denote by $\mathrm{PI}(q_0)$ the Poisson integral of the corresponding harmonic function with trace q_0. Let $\mathcal{N}$ be the Dirichlet-Neumann operator satisfying

$$\mathcal{N}q_0 = \partial_\nu \mathrm{PI}(q_0)|_{\partial\Omega}.$$

Next we study the behaviour of the sequence of pressure q in the regime of frequency scale h^{-1}. We always fix the notation

$$\lambda(y, x', \xi') = |\xi'|_{g^{-1}(y,x')} \sim |\xi'|.$$

Let $Y = (-\epsilon_0, \epsilon_0)_y \times X_x$ and $Y_+ = [0, \epsilon_0)_y \times X_x$. We first have the L^2 bound of q, microlocally away from $\xi' = 0$.

Lemma 1.4
Let $(f_h)_{0<h<1}$ be a $h-$dependent distributions such that $\|f_h\|_{L^2(\mathbb{R}^n)} = O(h^{-N})$. Assume that for any $\chi \in C_c^\infty(\mathbb{R}^{2n})$, vanishing near $\xi = 0$, we have $\|\chi(x, hD_x)f_h\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} = O(h^{-1})$. Then

$$\|\chi(x, hD_x)f_h\|_{L^2(\mathbb{R}^n)} = O(h^{-\frac{1}{2}}).$$

Proof. Assume that $\{|\xi| \leq 2\delta_0\} \cap \mathrm{supp}(\chi) = \emptyset$. Take $\Phi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\Phi(\xi) = 1, |\xi| \leq \delta_0, \quad \Phi(\xi) = 0, |\xi| > 2\delta_0.$$

We write

$$\chi(x, hD_x)f = \Phi(hD_x)\chi(x, hD_x)f + (1 - \Phi(hD_x))\chi(x, hD_x)f.$$

From the support property we have $\Phi(hD_x)\chi(x, hD_x)f = O_{H^\infty}(h^\infty)$. Thus $(1-\Phi(hD_x))\chi(x, hD_x)f = O_{H^{\frac{1}{2}}}(h^{-1})$. Let $b(\xi) = |\xi|^{1/2}(1 - \Phi(\xi))$, and we have $b(hD_x)\chi(x, hD_x)f = O_{L^2}(h^{-\frac{1}{2}})$. Since $b(\xi) \neq 0$ on $\mathrm{supp}(\chi)$, we have

$$\|\chi(x, hD_x)f\|_{L^2(\mathbb{R}^n)} \leq C\|b(hD_x)\chi(x, hD_x)f\|_{L^2(\mathbb{R}^n)} + Ch\|\chi(x, hD_x)f\|_{L^2(\mathbb{R}^n)} \leq Ch^{-\frac{1}{2}}.$$

■

Lemma 1.5
Given $\delta_0 > 0$ and $\varphi, \widetilde{\varphi} \in C_c^\infty(Y_+)$. For any $\chi_{\delta_0} \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$ such that $\chi_{\delta_0}| \equiv 0$ if $\lambda(y, x', \xi') \leq 2\delta_0$, we have

$$\|\widetilde{\varphi}\mathrm{Op}_h(\chi_{\delta_0})(\varphi q)\|_{L^2(\mathbb{R}^d_+)} + h^{1/2}\| \left(\widetilde{\varphi}\mathrm{Op}_h(\chi_{\delta_0})(\varphi q)\right)|_{y=0}\|_{L^2(\mathbb{R}^{d-1})} \leq C_{\delta_0,\varphi,\widetilde{\varphi}}.$$

Proof. Write $D_j = \frac{1}{i}\frac{\partial}{\partial x'_j}$, we have $\|hD_j(\varphi q)\|_{L^2(\mathbb{R}^d_+)} = O(1)$. Note that

$$\frac{\xi'_j}{|\xi'|^2}\chi_{\delta_0}(y, x', \xi') \in S^0_\partial,$$

and if we let $\chi_j = \frac{\xi'_j}{|\xi'|^2}\chi_{\delta_0}$, then

$$\widetilde{\varphi}\chi_{\delta_0}(y, x', hD_{x'})(\varphi q) = \sum_{j=1}^{d-1}\widetilde{\varphi}\chi_j(y, x', hD_{x'})hD_j(\varphi q) + O_{L^2(\mathbb{R}^d_+)}(1) = O_{L^2(\mathbb{R}^d_+)}(1),$$

where the implicit bound in big O depends on $\delta_0, \varphi, \widetilde{\varphi}$. For the boundary term, we observe that $\widetilde{\varphi}\mathrm{Op}_h(\chi_{\delta_0})(\varphi q)|_{y=0} = O_{H^{1/2}(\mathbb{R}^{d-1})}(h^{-1})$ from trace theorem. Thus from Lemma **??**, $\widetilde{\varphi}\mathrm{Op}_h(\chi_{\delta_0})(\varphi q)|_{y=0} = O_{L^2(\mathbb{R}^{d-1})}(h^{-1/2})$. ■

We express semi-classical Laplace operator $h^2\Delta_g$ in the geodesic coordinates of turbulence neighborhood Y by

$$P_0 = h^2\partial_y^2 + \sum_{i,j} g^{ij}\partial_i\partial_j + h\sum_j M_j(y, x')h\partial_j + hH(y, x')h\partial_y$$

where $\partial_j = \partial_{x'_j}$. We make the ansatz

$$\widetilde{q}(y, x') := \frac{1}{(2\pi h)^{d-1}}\int a(y, h, x', \xi')e^{\frac{ix'\xi'}{h}}\theta(\xi')d\xi',$$

then we calculate

$$\begin{aligned}P_0(\widetilde{q})(y, x', \xi') =& \frac{1}{(2\pi h)^{d-1}}\int \left(h^2\partial_y^2 a + g^{jk}(h^2\partial_j\partial_k a - g^{jk}\xi'_j\xi'_k a)\right)e^{\frac{ix'\xi'}{h}}\theta(\xi')d\xi' \\ &+ \frac{1}{(2\pi h)^{d-1}}\int \left(ihg^{jk}\xi'_k\partial_j a\right)e^{\frac{ix'\xi'}{h}}\theta(\xi')d\xi' \\ &+ \frac{1}{(2\pi h)^{d-1}}\int \left((h^2M_j\partial_j a + ihM_j\xi'_j a) + h^2H\partial_y a\right)e^{\frac{ix'\xi'}{h}}\theta(\xi')d\xi'.\end{aligned}$$

We next look for the formal semi-classical expansion

$$a(y, h, x', \xi') \backsimeq \sum_{j\geq 0} h^j a_j(y, h, x', \xi')$$

with $a_j \in S_\partial^{-j}$ and $h^k\partial_y^k a_j \in S_\partial^{-j+k}$. We obtain

$$
\begin{aligned}
P_0\widetilde{q} \backsimeq \frac{1}{(2\pi h)^{d-1}} \int & ((h^2\partial_y^2 a_0 - g^{ij}\xi_i\xi_j a_0) \\
& + h(ig^{jk}\xi_k'\partial_j a_0 + iM_j\xi_j' a_0 + h^2 H\partial_y a_0) \\
& + h(h^2\partial_y^2 a_1 - g^{jk}\xi_j'\xi_k' a_1) \\
& + h^2(g^{jk}\partial_j\partial_k a_0 + M_j\partial_j a_0) \\
& + h^2(ig^{jk}\xi_k'\partial_j a_1 + iM_j\xi_j' a_1 + h^2 H\partial_y a_1) \\
& + h^2(h^2\partial_y^2 a_2 - g^{jk}\xi_j'\xi_k' a_2) \\
& + \cdots)e^{\frac{ix'\xi'}{h}}\theta(\xi')d\xi'.
\end{aligned}
$$

Pick $\varphi_{1,0} = \varphi_1|_{\partial\Omega}, \varphi_1 \in C_c^\infty(Y)$. For $\widetilde{q}_0 = \varphi_{1,0}q_0$, we put

$$
\theta(\xi') = \mathcal{F}_h(\widetilde{q}_0(\xi')) = (2\pi h)^{-(d-1)} \int_{\mathbb{R}^{d-1}} \widetilde{q}_0(x')e^{-ix'\xi'/h}dx',
$$

$$
a_0(0,\cdot) \equiv 1, \quad a_j(0,\cdot) \equiv 0, \forall j \geq 1,
$$

and we define the functions a_j inductively as follows: firstly we define a_0

$$
a_0(y,x',\xi') = e^{-\frac{y\lambda(y,x',\xi')}{h}}, \lambda(y,x',\xi') =: \sqrt{g^{ij}\xi_i'\xi_j'} \sim |\xi'|,
$$

and the quantity

$$
(h^2\partial_y^2 - \lambda^2)a_0 = h\left(\frac{h^2}{\lambda^2}\frac{y^2\lambda^2}{h^2}(\partial_y\lambda)^2 + \frac{2y\lambda}{h}\partial_y\lambda - 2\partial_y\lambda\right)e^{-\frac{y\lambda}{h}}
$$

can be viewed as of order h. Next we set $a_j, j \geq 1$ implicitly by solving a sequence of linear ODEs:

$$
\begin{aligned}
h^2\partial_y^2 a_1 - \lambda^2 a_1 = & - h^{-1}(h^2\partial_y^2 - \lambda^2)a_0 - (ig^{jk}\xi_k'\partial_j a_0 + iM_j\xi_j' a_0 + h^2 H\partial_y a_0). \\
h^2\partial_y^2 a_n - g^{ij}\xi_i\xi_j a_n = & - (g^{ij}\partial_i\partial_j a_{n-2} + M_j\partial_j a_{n-2}) \\
& - (ig^{jk}\xi_k'\partial_j a_{n-1} + iM_j\xi_j' a_{n-1} + h^2 H\partial_y a_{n-1}), n \geq 2.
\end{aligned}
$$

Unfortunately, the functions a_j constructed above are not symbols, since they have singularities when $\xi' = 0$. This indicates that we can only obtain information of $q(h)$ from such parametrix away from $\xi' = 0$. We modify the construction above by setting

$$
A_0(y,x',\xi') = e^{-\frac{y\lambda}{h}}\psi_{\delta_0}(\lambda)\varphi_2(y,x'), \quad (y,x',\xi') \in \mathbb{R}_+^d \times \mathbb{R}^{d-1},
$$

with $\psi_{\delta_0} = \psi(\delta_0^{-1}\cdot), \psi \in C^\infty(\mathbb{R}_+)$ satisfying $\psi(s) \equiv 1$ when $s \geq 1$ and $\psi(s) = 0$ when $0 < s \leq \frac{1}{2}$. We next modify other A_j in the same manner. Indeed, the ODEs which define A_j are linear ODEs in y variable. Thus for $j \geq 1$, $A_j(y,x'\xi') \equiv 0$ when $\lambda(y,x',\xi') \leq \frac{\delta_0}{2}$. We define the particular class of symbols in S_∂^j.

Definition 1.3

$$\mathcal{E}_{\partial}^{j} := \left\{ a \in S_{\partial}^{j} : |(h\partial_y)^l \partial_{x',\xi'}^{\alpha} a(y,x',\xi')| \leq C_{l,\alpha} e^{-\frac{C'_{l,\alpha} y}{h}} \right\}.$$

Lemma 1.6
The symbols constructed above can be chosen to satisfy $A_j \in \mathcal{E}_{\partial}^{-j}$ for all $j \in \mathbb{N}$.

The proof will be given in appendix.

In summary, we have $A \simeq \sum_{j\geq 0} h^j A_j$, and a tangential symbol $B_{\delta_0}(y,x',\xi')$ compactly supported in $\lambda(y,x',\xi') \leq \frac{\delta_0}{2}$, such that

$$\begin{aligned} \varphi P_0 A(y,x',hD_{x'})(\varphi_{1,0}q_0) =& \varphi B_{\delta_0}(y,x',hD_{x'})(\varphi_{1,0}q_0) + O_{H^\infty}(h^\infty), \\ \varphi_0 A(0,x',hD_{x'})(\varphi_{1,0}q_0) =& \varphi_0 \mathrm{Op}_h(\psi_\delta(\lambda))(\varphi_{1,0}q_0) + O_{H^\infty}(h^\infty). \end{aligned}$$

The following proposition states that the parametrix constructed above is an approximation of the pressure q in the relevant semi-classical scale.

Proposition 1.4
There exists $A \in S_{\partial}^0$ with principal symbol

$$A_0(y,x',\xi') = \exp\left(-\frac{y\lambda(y,x',\xi')}{h} \right) \psi_{\delta_0}(\lambda(y,x',\xi'))\varphi_1(y,x'),$$

which satisfies asymptotic expansion $A \sim \sum_{j\geq 0} h^j A_j, A_j \in \mathcal{E}_{\partial}^{-j}$. Moreover, for any

$$\varphi, \varphi_1 \in C_c^\infty(Y_+), \varphi_1|_{\mathrm{supp}(\varphi)} \equiv 1,$$

we have $\varphi \mathrm{Op}_h(\chi_{\delta_0} A_j)(\varphi_{1,0}q_0) = O_{L^2(\mathbb{R}_+^d)}(1)$ for all j, and

$$\begin{aligned} \varphi \mathrm{Op}_h(\chi_{\delta_0})(\varphi_1 q) &= \varphi \mathrm{Op}_h(\chi_{\delta_0} A)(\varphi_{1,0}q_0) + O_{L^2(\mathbb{R}_+^d)}(h^{3/4}), \\ \varphi \mathrm{Op}_h(\chi_{\delta_0}) h\partial_y(\varphi_1 q) &= \varphi \mathrm{Op}_h(\chi_{\delta_0}\lambda A)(\varphi_{1,0}q_0) + O_{L^2(\mathbb{R}_+^d)}(h^{3/4}), \\ \varphi \mathrm{Op}_h(\chi_{\delta_0}) h\partial_y(\varphi_1 q) &= \varphi \mathrm{Op}_h(\chi_{\delta_0}\lambda A)(\varphi_{1,0}q_0) + O_{H^{\frac{2}{3}}(\mathbb{R}_+^d)}(h^{1/4}), \end{aligned}$$

where $\varphi_0 = \varphi|_{\partial\Omega}, \varphi_{1,0} = \varphi_1|_{\partial\Omega}, \chi_{\delta_0,0} = \chi_{\delta_0}|_{y=0}$.

We postpone the proof of this proposition in appendix. A direct consequence is that the singularity of the pressure q must concentrate in a very thin strip near the boundary.

Lemma 1.7
With the same $\chi_{\delta_0} \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$ and $\varphi_1, \varphi \in C_c^\infty(Y_+)$ for any $0 < y_0 < \epsilon_0$, we have

$$\int_{y_0}^{\epsilon_0} \|\varphi \mathrm{Op}_h(\chi_{\delta_0})(\varphi_1 q)\|_{L^2(\mathbb{R}^{d-1})}^2 dy \leq C_{\delta_0}(e^{-\frac{cy_0}{h}} + h),$$

where the constant C_{δ_0} only depends on δ_0 and is independent of y_0 and h.

Proof. The second term appearing on the right hand side comes from all the possible remainder terms. It suffices to estimate the term

$$\int_{y_0}^{\epsilon_0} \|\varphi \mathrm{Op}_h(\chi_{\delta_0} A_0)(\varphi_1 q_0)\|_{L^2(\mathbb{R}^{d-1})}^2 dy.$$

Since $\varphi_{1,0} q_0 = O_{L^2(\mathbb{R}^{d-1})}(h^{-1/2})$, micro-locally, we have for each fixed $y > 0$ that

$$\begin{aligned}\|\varphi \mathrm{Op}_h(\chi_{\delta_0} A_0)(\varphi_{1,0} q_0)\|_{L^2(\mathbb{R}^{d-1})} \leq & Ch^{-1/2} \sum_{|\beta| \leq Cd} h^{\frac{|\beta|}{2}} \sup_{y>0,(x',\xi')} |\partial_{x',\xi'}^{\beta}(\chi_{\delta_0} A_0)| + O(h^\infty) \\ \leq & Ch^{-1/2} e^{-\frac{cy}{h}} \left(1 + \sum_{1 \leq m,n \leq Cd} h^{m/2} \left(\frac{y}{h}\right)^n\right) + O(h^\infty).\end{aligned}$$

Squaring and Integrating the right hand side in y variable yields the desired conclusion. ■

1.4 Main Steps of the Proof

The proof of Theorem 1.2 can be divided into several steps according to different geometric situations. We want to show that for any given point $\rho_0 \in^b T^*\overline{\Omega}$, if $\rho_0 \notin$ supp μ, then $\gamma(s, \rho_0) \notin$ supp μ for any $s > 0$. The first step is to show that if $\rho_0 \in T^*\Omega$, $\rho_0 \notin$ supp μ, then $\gamma(s, \rho_0) \notin$ supp μ for all $s > 0$ provided that $\pi(\gamma(\cdot, \rho_0)|_{[0,s]}) \cap \partial\Omega = \emptyset$. This can be summarized by the following proposition, in which we have stronger conclusion that the measure is also invariant under the flow.

Proposition 1.5
For any real-valued scalar function $a \in C_c^\infty(\Omega \times \mathbb{R}^d)$ vanishing near $\xi = 0$, we have

$$\frac{d}{ds}\langle \mu, a \circ \gamma(s, \cdot)\rangle = 0.$$

Proof. Let $A = \mathrm{Op}_h(a)$ and $P = -h^2\Delta - 1$. We apply equation and Lemma **??** to calculate

$$\begin{aligned}\frac{i}{h}\left([P, A]u|u\right)_\Omega &= \frac{i}{h}(Au|Pu)_\Omega - \frac{i}{h}(APu|u)_\Omega \\ &= \frac{i}{h}(Au|f - h\nabla q)_\Omega - \frac{i}{h}(A(f - h\nabla q)|u)_\Omega \\ &= -\frac{i}{h}(Au|h\nabla q)_\Omega + \frac{i}{h}(Ah\nabla q|u)_\Omega + o(1) \\ &= -\frac{i}{h}([A, h\mathrm{div}\ ]u|q)_\Omega + \frac{i}{h}([A, h\nabla]q|u)_\Omega + o(1) \\ &= i(\mathrm{Op}_h(\nabla a) \cdot u|q)_\Omega - i(\mathrm{Op}_h(\nabla a)q|u)_\Omega + o(1) \\ &= i(u|\mathrm{Op}_h(\nabla \overline{a})q)_\Omega - i(\mathrm{Op}_h(\nabla a)q|u)_\Omega + o(1).\end{aligned} \tag{1.4.1}$$

where we have used integrating by part freely, thanks to the fact that A has compact support in $x \in \Omega$. Now we claim that for any $\chi \in C_c^\infty(T^*\Omega)$, we have

$$(u|\mathrm{Op}_h(\nabla\chi)q)_\Omega = o(1).$$

We know that $q = O_{L^2(\Omega)}(1)$ micro-locally away from $\xi = 0$ since $h\nabla q = O_{L^2}(1)$. On the other hand, $h^2\Delta(\mathrm{Op}_h(\nabla\chi)q) = O_{L^2(\Omega)}(h)$ and this implies that $\mathrm{Op}_h(\nabla\chi)q = o_{L^2}(1)$ since the symbol of $h^2\Delta\mathrm{Op}_h(\nabla\chi)$ vanishes away near $\xi = 0$ as well as x near the boundary. In view of the definition of μ, this completes the proof. ∎

For the second step, we need prove that if $\rho_0 \in \mathcal{E}$, then $\mu = 0$ in a neighborhood of ρ_0.

Proposition 1.6
$\mu\mathbf{1}_{\mathcal{E}} = 0$. *If we let* ν *be the semi-classical defect measure of the sequence* $(h_k\partial_\nu u_k|_{\partial\Omega}, h_k)$, *then* $\nu\mathbf{1}_{\mathcal{E}} = 0$.

The third step consists of proving that after reflection near a hyperbolic point, the measure μ is still zero.

Proposition 1.7
Suppose $\rho_0 \notin \mathrm{supp}\mu$ *and there exists* $s_0 > 0$ *such that* $\gamma(s_0, \rho_0) \in \mathcal{H}$ *and* $\pi(\gamma(s, \rho_0)) \in \Omega$ *for all* $0 \leq s < s_0$. *Then there exists* $\delta > 0$ *such that*

$$\pi(\gamma(\cdot, \rho_0)|_{[s_0, s_0+\delta]}) \cap \mathrm{supp}\mu = \emptyset.$$

Next step is to prove the propagation near a diffractive point.

Proposition 1.8
Suppose $\rho_0 \notin \mathrm{supp}\mu$ *and there exists* $s_0 > 0$ *such that* $\gamma(s_0, \rho_0) \in \mathcal{G}^{2,+}$ *and* $\pi(\gamma(s, \rho_0)) \in \Omega$ *for all* $0 \leq s < s_0$. *Then* $\gamma(s_0, \rho_0) \notin \mathrm{supp}\mu$.

To deal with higher order contact, we will use induction. First let us introduce

Definition 1.4 ($k-$ propagation property)
For $k \geq 2$, *we say that* $k-$*propagation property holds, if along generalized ray* $\gamma(s, \rho_0)$, *the following statement is true: For some* $\sigma_0 > 0$, *if* $\gamma(\cdot, \rho_0)|_{[0,\sigma_0)} \cap \mathrm{supp}(\mu) = \emptyset$ *(or* $\gamma(\cdot, \rho_0)|_{(-\sigma_0,0]} \cap \mathrm{supp}(\mu) = \emptyset$*) and* $\gamma(\sigma_0, \rho_0) \in \bigcup_{2\leq j\leq k} \mathcal{G}^j$ *(or* $\gamma(-\sigma_0, \rho_0) \in \bigcup_{2\leq j\leq k} \mathcal{G}^j$*), then* $\gamma(\sigma_0, \rho_0) \notin \mathrm{supp}(\mu)$ *(or* $\gamma(-\sigma_0, \rho_0) \notin \mathrm{supp}(\mu)$ *).*

The last step for the proof of Theorem 1.2 can be reduced to

Proposition 1.9
$k-$*propagation property holds for all* $k \geq 2$.

1.5 Near $\mathcal{E}$

This section is devoted to the proof of Proposition 1.6. We set $Q(y, x', \xi') := \sqrt{\lambda^2 - 1}$ and define $Q_h = \varphi \mathrm{Op}_h(Q\chi_0)\varphi_1$ with $\chi_0 \in C_c^\infty(\mathbb{R}^{d-1}_{\xi'})$ with support near $\mathcal{E}$ in which $1 + \delta < \lambda < C$. With a bit abuse of notation, we refer q_0, q to be $\varphi \mathrm{Op}_h(\chi_0)\varphi_1 q_0, \varphi \mathrm{Op}_h(\chi_0)\varphi_1 q$ and u to be $\varphi \mathrm{Op}_h(\chi_0)\varphi_1 u$. In this manner, we can combine the parametrix in last section to write the system (1.1.2) as

$$\begin{cases} (-h^2\partial_y^2 + Q_h^2)u_{\parallel,j} + g^{jk}h\partial_{x'_k}(\mathrm{Op}_h(A_0)q_0) = R_{\parallel,j} = O_{L^2(\mathbb{R}^d_+)}(h), \\ (-h^2\partial_y^2 + Q_h^2)u_\perp + h\partial_y(\mathrm{Op}_h(A_0)q_0) = R_\perp = O_{L^2(\mathbb{R}^d_+)}(h). \end{cases} \tag{1.5.1}$$

Note that the symbol $A_0(y, x', \xi')$ is defined in last section which equals to $e^{-\frac{y\lambda}{h}}$ since $\lambda > 1$. Take $\psi \in C^\infty(\mathbb{R}_+)$, with $\psi|_{[0,\epsilon_0]} \equiv 1, \psi_{[2\epsilon_0,\infty)} \equiv 0$. Denoting the extended distributions of u by $w = (w_\parallel, w_\perp) = (u_\parallel, u_\perp)\psi(y)\mathbf{1}_{y\geq 0}$, we have from standard elliptic parametrix construction (see Appendix A) that modulo that $O_{H^\infty(\mathbb{R}^d_+)}(h^\infty)$,

$$\begin{cases} w_{\parallel,j} = E(y, x', hD_y, hD_{x'})(-\psi(y)g^{jk}h\partial_{x'_k}(\mathrm{Op}_h(A_0)q_0) + 2hv_j \otimes \delta_{y=0} + \psi(y)R_{\parallel,j}), \\ w_\perp = E(y, x', hD_y, hD_{x'})(-h\psi(y)\partial_y(\mathrm{Op}_h(A_0)q_0) + \psi(y)R_\perp). \end{cases} \tag{1.5.2}$$

where $v = h\partial_y u_\parallel|_{y=0} = O_{L^2_{x'}}(1)$. Recall that the principal symbol of E is given by

$$E^0 := \frac{\chi_0(\xi')\varphi(y, x')}{\eta^2 + \lambda(y, x', \xi')^2 - 1},$$

Now we need a lemma which deals with the trace of error terms:

Lemma 1.8
Assume that $R = \varphi \mathrm{Op}_h(\chi_0)\varphi_1 R + O_{H^\infty}(h^\infty)$, *then if* $\|\psi(y)R\|_{L^2(\mathbb{R}^d_+)} = O(h)$, *we have*

$$\|E(y, x', hD_y, hD_{x'})(\psi(y)R)|_{y=0}\|_{L^2(\mathbb{R}^{d-1}_{x'})} = O(h^{1/3}).$$

Proof. From the parametrix construction above, we know that

$$|\partial^\alpha_{y,x',\eta,\xi'} E(y, x'; \eta, \xi')| \leq \frac{C_\alpha}{\eta^2 + Q(y, x', \xi')^2}.$$

Therefore, the symbols $\eta E(y, x'; \eta, \xi')$ and $\lambda(y, x', \xi')E(y, x'; \eta, \xi')$ are uniformly bounded in S^0. Thus $E(y, x'; hD_y, hD_{x'})(\psi R) = O_{L^2(\mathbb{R}^d_+)}(h) = O_{H^1(\mathbb{R}^d_+)}(1)$, and from interpolation, we have $E(y, x'; hD_y, hD_{x'})(\psi R) = O_{H^{2/3}(\mathbb{R}^d_+)}(h^{1/3})$. The conclusion then follows from trace theorem that $H^s(\mathbb{R}^d_+) \to H^{s-1/2}(\mathbb{R}^{d-1})$ is bounded for $s > 1/2$. ∎

Proof of Proposition 1.6. Denote by $\mathcal{F}_h(q_0) = \theta$ the semi-classical Fourier transform of q_0, we calculate

$$
\begin{aligned}
&E(y, x', hD_y, hD_{x'})(\psi(y)h\partial_y \mathrm{Op}_h(A_0)q_0) \\
&= \frac{1}{(2\pi h)^d} \iint e^{\frac{i(y-z)\eta}{h}} dz d\eta \int \frac{e^{\frac{ix'\xi'}{h}}\theta(\xi')\varphi(y,x')\chi_0(\xi')}{\eta^2 + Q^2(y,x',\xi')} \psi(z) h\partial_z (e^{-\frac{z\lambda(z,x',\xi')}{h}}) d\xi' + R_1 \\
&= -\frac{h}{(2\pi h)^d} \iint \frac{\langle\xi'\rangle\theta(\xi') e^{\frac{i(y\eta + x'\xi')}{h}} B_1(\eta, x', \xi')\varphi(y,x')\chi_0(\xi')}{\eta^2 + Q^2(y,x',\xi')} d\eta d\xi' \\
&- \frac{h^2}{(2\pi h)^d} \int e^{\frac{ix'\xi'}{h}}\theta(\xi') d\xi' \int \frac{e^{\frac{iy\eta}{h}} B_0(\eta,x',\xi')\varphi(y,x')\chi_0(\xi')}{\eta^2 + Q(y,x',\xi')^2} d\eta + R_1,
\end{aligned}
\tag{1.5.3}
$$

with reminder term $R_1 = O_{L^2(\mathbb{R}^d)}(h)$, where $\lambda_0 = \lambda|_{y=0}$,

$$
B_1(\eta, x', \xi') = \int_0^\infty \psi(z) e^{-\left(\frac{i\eta + \lambda(z,x',\xi')}{h}\right)z} \frac{\lambda(z,x',\xi')}{\langle\xi'\rangle} \frac{1}{h} dz,
$$

and

$$
B_0(\eta, x', \xi') = \int_0^\infty \psi(hz) z (\partial_z \lambda)(hz, x', \xi') e^{-(i\eta + \lambda(hz,x',\xi'))z} dz.
$$

We notice that

$$
K_0(y, x', \xi') := \int \frac{e^{\frac{iy\eta}{h}} B_0(\eta,x',\xi')\varphi(y,x')\chi_0(\xi')}{\eta^2 + Q(y,x',\xi')^2} d\eta
$$

is a bounded symbol in $S^0_{\xi'}$. Thus the second term on the right hand side of (**??**) is equal to $R_2 = O_{C^0(\mathbb{R}_y; L^2(\mathbb{R}^{d-1}_{x'}))}(h)$ and we may concentrate on the first term.

Write

$$
B_1(\eta, x', \xi') = \int_0^\infty \psi(hz) e^{-(i\eta + \lambda(hz,x',\xi'))z} \frac{\lambda(hz,x',\xi')}{\langle\xi'\rangle} dz.
$$

Taylor expansion gives

$$
\begin{aligned}
e^{-\lambda(hz,x',\xi')z}\lambda(hz,x',\xi')\psi(hz) =& e^{-\lambda_0(x',\xi')z}\lambda_0(x',\xi') + \int_0^1 \frac{d}{dt}\left(e^{-\lambda(htz,x',\xi')z}\lambda(htz,x',\xi')\psi(htz)\right) dt \\
=& e^{-\lambda_0(x',\xi')z}\lambda_0(x',\xi') + h\int_0^1 P_t(z,x',\xi') e^{-\lambda(htz,x',\xi')z} dt
\end{aligned}
$$

with

$$
P_t(z,x',\xi') = -z^2(\lambda\partial_y\lambda)(htz,x',\xi')\psi(htz) + z(\partial_y\lambda)(htz,x',\xi')\psi(htz) + z\lambda(htz,x',\xi')\psi'(htz).
$$

Thus we have $B_1(\eta,x',\xi') = \dfrac{\lambda_0(x',\xi')}{(i\eta + \lambda_0(x',\xi'))\langle\xi'\rangle} + h\widetilde{B}_1(\eta,x',\xi')$, where

$$
\widetilde{B}_1(\eta,x',\xi') = \int_0^1 \int_0^\infty e^{-(i\eta + \lambda(htz,x',\xi'))z} \frac{1}{\langle\xi'\rangle} P_t(z,x',\xi') dz dt.
$$

Note that near a point in $\mathcal{E}$, $|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}P_t(z,x',\xi')| \leq C_{\alpha\beta}z^2$, independent of t,h, hence the symbol

$$\widetilde{K}_1(y,x',\xi') = \int \frac{e^{\frac{iy\eta}{h}}\varphi(y,x')\chi_0(\xi')}{\eta^2+Q(y,x',\xi')^2}\widetilde{B}_1(\eta,x',\xi')d\eta \in S^0_{\xi'}.$$

Therefore, the symbol in the principal term of $E(y,x',hD_y,hD_{x'})(\psi(y)h\partial_y \mathrm{Op}_h(A_0)q_0)$ equals to

$$\begin{aligned}
K_1(y,x',\xi') =&\lambda_0(x',\xi')\int \frac{e^{\frac{iy\eta}{h}}\varphi(y,x')\chi_0(\xi')}{(\eta^2+Q(y,x',\xi')^2)(i\eta+\lambda_0(x',\xi'))}d\eta + h\widetilde{K}_1(y,x',\xi')\\
=&2\pi\lambda_0\varphi(y,x')\chi_0(\xi')\left(\frac{e^{-\frac{yQ}{h}}}{2(\lambda_0-Q)Q}-\frac{e^{-\frac{y\lambda_0}{h}}}{\lambda_0^2-Q^2}\right)+h\widetilde{K}_1(y,x',\xi')\\
=&2\pi\lambda_0\varphi(y,x')\chi_0(\xi')\left(\frac{e^{-\frac{yQ}{h}}-e^{-\frac{y\lambda_0}{h}}}{2(\lambda_0-Q)Q}+\frac{e^{-\frac{y\lambda_0}{h}}}{2Q(\lambda_0+Q)}\right)+h\widetilde{K}_1(y,x',\xi').
\end{aligned}$$

Note that

$$E_1(y,x',\xi') = 2\pi\lambda_0\varphi(y,x')\chi_0(\xi')\left(\frac{e^{-\frac{yQ}{h}}-e^{-\frac{y\lambda_0}{h}}}{2(\lambda_0-Q)Q}+\frac{e^{-\frac{y\lambda_0}{h}}}{2Q(\lambda_0+Q)}\right) > 0$$

near $\mathcal{E}$, we have

$$E(y,x',hD_y,hD_{x'})(\psi(y)h\partial_y\mathrm{Op}_h(A_0)q_0) = E_1(y,x',hD_{x'})q_0 + R_1 + R_2,$$

with $R_2 = O_{C(\mathbb{R}_y;L^2(\mathbb{R}^{d-1}_{x'}))}(h)$. We claim that $R_1 = O_{H^1(\mathbb{R}^d)}(1)$ and thus by interpolation, $\|R_1\|_{H^{2/3}(\mathbb{R}^d)} = O(h^{1/3})$. Indeed, the reminder terms R_1 comes from symbols of the form hS^{-1} (in both η and ξ' variables), and the symbolic calculus yields $\partial_y R_1 = O_{L^2(\mathbb{R}^d)}(1)$, and $\partial_{x'}R_1 = O_{L^2(\mathbb{R}^d)}(1)$.

We next calculate the parallel component

$$\begin{aligned}
&\frac{1}{(2\pi h)^d}\iint \psi(z)e^{\frac{i(y-z)\eta}{h}}dzd\eta\int \frac{e^{\frac{ix'\xi'}{h}-\frac{z\lambda(z,x',\xi')}{h}}\theta(\xi')g^{jk}(z,x')\xi_k'\varphi(y,x')\chi_0(\xi')}{\eta^2+Q(y,x',\xi')^2}d\xi'\\
=&\frac{1}{(2\pi h)^d}\iint \frac{e^{\frac{i(x'\xi'+y\eta)}{h}}\xi_k'\theta(\xi')\varphi(y,x')\chi_0(\xi')}{\eta^2+Q(y,x',\xi')^2}d\eta d\xi'\int_0^\infty \psi(z)e^{-\frac{i\eta+\lambda(z,x',\xi')}{h}z}g^{jk}(z,x')dz\\
=&\frac{h}{(2\pi h)^d}\iint \frac{e^{\frac{i(x'\xi'+y\eta)}{h}}\xi_k'\theta(\xi')\varphi(y,x')\chi_0(\xi')}{(\eta^2+Q(y,x',\xi')^2)}B_{2,jk}(\eta,x',\xi')d\eta d\xi'\\
=:&\, E_2(y,x',hD_{x'})q_0.
\end{aligned}$$

where

$$B_{2,jk}(\eta,x',\xi') = \int_0^\infty \psi(z)e^{-\frac{i\eta+\lambda(z,x',\xi')}{h}z}g^{jk}(z,x')\frac{1}{h}dz.$$

Define

$$K_{2,jk}(y,x',\xi') = \int \frac{e^{\frac{iy\eta}{h}} B_{2,jk}(\eta,x',\xi')\varphi(y,x')\chi_0(\xi')}{\eta^2+Q(y,x',\xi')^2} d\eta,$$

and from similar argument we can write

$$K_{2,k}(y,x',\xi') = g^{jk}(0,x') \int \frac{e^{\frac{iy\eta}{h}}\varphi(y,x')\chi_0(\xi')}{(\eta^2+Q(y,x',\xi')^2)(i\eta+\lambda_0(x',\xi'))} + h\widetilde{K}_2(y,x',\xi')$$

and the principal symbol of $E_2(y,x',hD_{x'})$ is elliptic if $\lambda_0(\xi') > 1$ and y small enough. Finally,

$$\begin{aligned}
&E(y,x',hD_y,hD_{x'})(2hv\otimes\delta_{y=0}) \\
=&\frac{2h}{(2\pi h)^d}\iint \frac{\mathcal{F}_h(v)(\xi')e^{\frac{i(y\eta+x'\xi')}{h}}\varphi(y,x')\chi_0(\xi')}{\eta^2+Q(y,x',\xi')^2} d\xi' d\eta + O_{L^2(\mathbb{R}^d)}(h) \\
=&\frac{2h}{(2\pi h)^d}\int \mathcal{F}_h(v)(\xi')e^{\frac{ix'\xi'}{h}} \frac{\pi e^{-\frac{yQ(y,x',\xi')}{h}}\varphi(y,x')\chi_0(\xi')}{Q(y,x',\xi')} d\xi' + O_{L^2(\mathbb{R}^d)}(h) \\
=:& E_3(y,x',hD_{x'})v + O_{L^2(\mathbb{R}^d)}(h),
\end{aligned} \tag{1.5.4}$$

and again, $E_3(y,x',hD_{x'})$ is elliptic near $\lambda_0(\xi') > 1$. Moreover, the reminder terms are indeed of $O_{H^{2/3}(\mathbb{R}^d)}(h^{1/3})$ from the same argument as for R_1. Now the boundary condition $(w_\parallel, w_\perp)|_{y=0} = 0$ and trace theorem yields

$$E_1(0,x',hD_{x'})q_0 = O_{L^2(\mathbb{R}'^{d-1}_x)}(h^{1/3}),$$

$$E_2(0,x',hD_{x'})q_0 + E_3(0,x',hD_{x'})v = O_{L^2(\mathbb{R}^{d-1}_{x'})}(h^{1/3}).$$

Therefore, from the ellipticity of E_1, E_2, E_3, the measure of pressure at the elliptic region vanishes, so does the measure of v, namely $\sigma|_{\mathcal{E}} = \nu|_{\mathcal{E}} = 0$. In summary, the proof of Proposition 1.6 is complete.

∎

1.6 Near $\mathcal{H}$

We take $\varphi_1, \varphi \in C_c^\infty(Y_+)$ such that $\varphi_1|_{\mathrm{supp}(\varphi)} \equiv 1$. For any tangential symbol $b \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$, we define the pseudo-differential operator $B_h = \varphi \mathrm{Op}_h(b)\varphi_1$, with compact support in Y_+. We will change the notation of tangential variables (x',ξ') to (x,ξ). We always work in local coordinate (y,x) and sometimes abuse the notation $u = \varphi_1 u, q = \varphi_1 q$ as compactly supported functions in Y_+. Note that q_0, the trace of q is not bounded in L^2 in priori. Fortunately, it turns out that $q_0 = O_{L^2}(1)$, micro-locally near a point in $\mathcal{H}$.

1.6.1 L^2 bound of boundary datums

Take $b(y,x,\xi), b_1(y,x,\xi) \in C_c^\infty([0,\epsilon_0)\times\mathcal{H})$, such that $b_1|_{[0,\epsilon_0/2)\times supp(b)} \equiv 1$. Let $Q(y,x,\xi) = \sqrt{1-\lambda(y,x,\xi)^2}b_1(y,x,\xi)$. We will first factorize the operator $(-h^2\Delta-1)$ near a hyperbolic point.

Lemma 1.9
For $0 \le y < \epsilon_0$, we have

$$B_h(-h^2\Delta-1) = -(hD_y - Q_h^+)(hD_y - Q_h^-) + R' = -(hD_y - Q_h^-)(hD_y - Q_h^+) + R'',$$

where $R', R'' \in C^\infty([0,\epsilon_0], h^\infty\Psi^{-\infty}(\partial\Omega))$, and $Q_h^\pm$ have principal symbol $\pm Q(y,x,\xi)$.

Proof. The proof is quite standard, and we follow the construction in [10] by translating word by word to the semi-classical setting. In local coordinate, we have

$$B_h(-h^2\Delta-1) = h^2D_y^2 + R(y,x,hD_x) + hM_1(y,x')hD_y + hM_0(y,x)hD_x$$

with $\sigma(R) = Q^2$. Set $q_1^+ = \sqrt{Q}(y,x',\xi')$, $Q_1^+ = \mathrm{Op}_h(q_1^+)$ and $Q_1^- = -Q_1^+ - hM_1$. Direct calculation gives

$$\begin{aligned}(hD_y - Q_1^+)(hD_y - Q_1^-) &= h^2D_y^2 - (Q_1^+)^2 - hQ_1^+M_1 - (Q_1^+ + Q_1^-)hD_y - \frac{h}{i}\partial_y(Q_1^-)\\ &= h^2D_y^2 - (Q_1^+)^2 + hM_1hD_y - h(Q_1^+M_1 - i\partial_y(Q_1^-)).\end{aligned}$$

Thus $B_h(-h^2\Delta-1) - (hD_y - Q_1^+)(hD_y - Q_1^-) = hT_1$, with some operator T_1, bounded in L^2. Now for $j \ge 1$, suppose that we have

$$B_h(-h^2\Delta-1) - (hD_y - Q_j^+)(hD_y - Q_j^-) = h^jT_j,$$

then we set $Q_{j+1}^\pm := Q_j^\pm + h^jS_{j+1}^\pm$ with $S_{j+1}^+ + S_{j+1}^- = 0, \sigma(S_{j+1}^+) = \dfrac{\sigma(T_j)}{2\sigma(Q_j^+)}$ and obtain that

$$\begin{aligned}&B_h(-h^2\Delta-1) - (hD_y - Q_{j+1}^+)(hD_y - Q_{j+1}^-)\\ =&h^jT_j + h^j(S_{j+1}^+Q_j^- + Q_j^+S_{j+1}^-) - h^j(S_{j+1}^+ + S_{j+1}^-)hD_y - \frac{h^{j+1}}{i}\partial_y(S_{j+1}^-) + h^{2j}S_{j+1}^+S_{j+1}^-\\ =:&h^{j+1}T_{j+1},\end{aligned}$$

for some operator T_{j+1} bounded in L^2. The proof is then complete by induction.

■

Define $w = \varphi_1 u - h\nabla(\varphi_1 q)$, $w^\pm = B_h(hD_y - Q_h^\pm)w$ and its boundary values $w_0^\pm := w^\pm|_{y=0}$. Note that $\varphi P_h w = \varphi f$.

Proposition 1.10
$\|B_h h\partial_y w_\perp\|_{L^2(\mathbb{R}_+^d)} = O(1)$, *and consequently,* $\|w_\perp^\pm\|_{L^2(\mathbb{R}_+^d)} = O(1)$.

Proof. From h div $u = 0$, we have φh div $w = 0$, hence

$$\varphi(h\partial_y w_\perp + h\ \mathrm{div}_\| w_\|) = O_{L^2(\mathbb{R}^d_+)}(h),$$

where in local coordinates,

$$\mathrm{div}_\| w_\| = \frac{1}{\sqrt{\det(g)}}\sum_{j=1}^{d-1}\partial_{x_j}(\sqrt{\det(g)}w_{\|,j}).$$

Therefore,

$$\|B_h h\partial_y w_\perp\|_{L^2(\mathbb{R}^d_+)} \le O(h) + \|B_h h\mathrm{div}_\| w_\|\|_{L^2(\mathbb{R}^d_+)} = O(1).$$

■

Now we recall a semi-classical version of hyperbolic energy estimate.

Lemma 1.10
Suppose $A_h = \mathrm{Op}_h(a)$ is ellptic (with real-valued symbol a smoothly depending on t) of order 0 on a compact manifold M and w are solutions of the $h-$ dependent equations

$$(hD_t \pm A_h)w = g, \quad (t,x) \in \mathbb{R}\times M.$$

Assume that for any compact time interval I and small h,

$$\|w\|_{L^2(I\times M)} \le C(I), \quad \|g\|_{L^2(I\times M)} \le C(I)h,$$

then we have for all small h,

$$\sup_{t\in I'}\|w(t)\|_{L^2(M)} \le C(I'), \forall I' \subset I\, compact.$$

Proof. By symmetry, we only need to treat the case $hD_t - A_h$. Take $\chi(t) \in C_c^\infty(I')$, and we may assume that $0 \in I'$ with $\chi(0) = 1$. Multiplying by $\chi(t)$ to the equation, we have

$$(hD_t - A_h)(\chi w) = \chi g + [\chi, hD_t - A_h]w =: r = O_{L^2(\mathbb{R}\times M)}(h).$$

We now calculate

$$\begin{aligned} h\frac{d}{dt}(\chi w|\chi w)(t)_{L^2(M)} &= (ihD_t\chi w|\chi w)_{L^2(M)} + (\chi w|ihD_t\chi w)_{L^2(M)}\\ &= i(A_h(\chi w)+r|\chi w)_{L^2(M)} - i(\chi w|A_h(\chi w)+r)_{L^2(M)}\\ &= i((A_h - A_h^*)\chi w|\chi w)_{L^2(M)} + i(r|\chi w)_{L^2(M)} - i(\chi w|r)_{L^2(M)}\end{aligned}$$

Integrating the formula above from 0 to $\sup I'$, we finally have $\|w(0)\|^2_{L^2(M)} = O(1)$. ■

Lemma 1.11
$\|w_0^\pm\|_{L^2(\mathbb{R}^{d-1})} = O(1)$.

Proof. From Proposition 1.10, we have $(hD_y - Q_h^{\mp})w_{\perp}^{\pm} = O_{L^2(\mathbb{R}_+^d)}(h)$. Applying the previous lemma to $w_{\perp}^{\pm}$, we have$\|w_{0,\perp}^{\pm}\|_{L^2(\mathbb{R}_x^{d-1})} = O(1)$. Combining the boundary condition, we have

$$B_h(Q_h^+ - Q_h^-)(h\partial_y q)|_{y=0} = -B_h(Q_h^+ - Q_h^-)h\mathcal{N}q_0 = w_{0,\perp}^+ - w_{0,\perp}^- = O_{L^2(\mathbb{R}^{d-1})}(1).$$

Remark that in priori, $\mathcal{N}$ is a first order pseudo-differential operator, and we only have

$$\|B_h h\mathcal{N}q_0\|_{L^2(\mathbb{R}^{d-1})} \leq \|B_h\|_{H^{-1}\to L^2} h\|\mathcal{N}q_0\|_{H^{-1}(\mathbb{R}^{d-1})} = O(h^{-1}).$$

From the exact pricipal symbol of $Q_h^{\pm}$, we have $\|B_h h\mathcal{N}q_0\|_{L^2(\mathbb{R}^{d-1})} = O(1)$, and the constant in big O depends on the micro-local cut-off $b(y, x', \xi')$. As a consequence, $\|w_{0,\perp}^{\pm}\|_{L^2(\mathbb{R}^{d-1})} = O(1)$.

It remains to study $w_{\parallel}^{\pm}$. Notice that their boundary values are

$$w_{0,\parallel}^{\pm} = B_h(v - (hD_y h\nabla_{\parallel} q)|_{y=0}) - B_h Q_h^{\pm} h\nabla_{\parallel} q_0,$$

where $v = (h\partial_y u)|_{y=0} = O_{L^2(\mathbb{R}^{d-1})}(1)$. All terms are obviously bounded in $L^2(\mathbb{R}^{d-1})$ except the trace of $B_h h\nabla_{\parallel} hD_y q$. To bound it, we use the support property of b and Proposition 1.4, hence $B_h h\nabla_{\parallel} hD_y q|_{y=0} = -B_h h\nabla_{\parallel} h\mathcal{N}q_0 = O_{L^2(\mathbb{R}_x^{d-1})}(1)$. ■

Again by hyperbolic estimates, we can establish the following results:

Proposition 1.11
$\|w^{\pm}\|_{L^2(\mathbb{R}_+^d)} = O(1)$. *In particular,*

$$\|B_h hD_y w\|_{L^2(\mathbb{R}_+^d)} + \|B_h h\mathcal{N}q_0\|_{L^2(\mathbb{R}_x^{d-1})} + \|B_h h^2\Delta_0 q_0\|_{L^2(\mathbb{R}_x^{d-1})} = O(1),$$

where $\Delta_0 = \Delta_{\partial\Omega}$*, the Laplace-Beltrami operator on* $\partial\Omega$*.*

Proof. It remains to prove $\|B_h h^2\Delta_0 q_0\|_{L^2(\mathbb{R}_x^{d-1})} = O(1)$. Indeed,

$$\begin{aligned} B_h h\partial_y w_{\perp} =& B_h h\partial_y u_{\perp} - h^2 B_h \partial_y^2 q \\ =& h^2 B_h \partial_y u_{\perp} + h^2 B_h \left(\frac{1}{\sqrt{\det(g)}} \sum_{1\leq j,k\leq d-1} \partial_j(g^{jk}\partial_k q_0) + \frac{\partial_y\sqrt{\det(g)}}{\sqrt{\det(g)}}\partial_y q \right). \end{aligned}$$

Thus

$$B_h h\partial_y w_{\perp}|_{y=0} = B_h h^2\Delta_0 q_0 + O_{L^2(\mathbb{R}^{d-1})}(1),$$

thanks to $h\partial_y u_{\perp} = 0$ and $B_h h\mathcal{N}q_0 = O_{L^2(\mathbb{R}^{d-1})}(h^{-1})$. From $w_{0,\perp}^{\pm} = B_h hD_y w_{\perp}|_{y=0} + B_h Q_h^{\pm} h\mathcal{N}q_0 = O_{L^2(\mathbb{R}^{d-1})}(1)$, we deduce that $B_h hD_y w_{\perp}|_{y=0} = O_{L^2(\mathbb{R}^{d-1})}(1)$, and these yield $\|B_h h^2\Delta_0 q_0\|_{L^2(\mathbb{R}^{d-1})} = O(1)$.

■

Corollary 1.1
$\|B_h h\nabla q\|_{L^2(\mathbb{R}^d_+)} = o(1)$.

Proof. We will go back to the global notation in this calculation. It would be suffices to show that $B_h h\nabla q = \varphi\mathrm{Op}_h(b)\varphi_1 h\nabla q = o_{L^2(\Omega)}(1)$ since there are only change of bounded weight in the integral with respect to the measure $\sqrt{\det(g)}dydx$ and $dydx$ in local coordinate, and the former admit us to apply integrating by part and the structure of the equation in a simple way. We calculate

$$\begin{aligned}(B_h h\nabla q|B_h h\nabla q)_{L^2(\Omega)} =&([B_h, h\nabla]q|B_h h\nabla q)_{L^2(\Omega)} + (h\nabla B_h q|B_h h\nabla q)_{L^2(\Omega)}\\ =&o(1) - (A_h B_h q|h\mathrm{div} B_h h\nabla q)_{L^2(\Omega)}\\ &+(B_h hq_0|B_h(h\partial_\nu q)|_{\partial\Omega})_{L^2(\partial\Omega)}\\ =&o(1) + (B_h hq_0|B_h h\mathcal{N} q_0)_{L^2(\partial\Omega)},\end{aligned}$$

where we have used the fact that $hq = o_{L^2(\Omega)}(1)$ and $\Delta q = 0$ in the calculation. Now from Lemma 1.3, we know that $hq \rightharpoonup 0$ weakly in $H^1(\Omega)$ and $hq_0 \to 0$ strongly in $L^2(\partial\Omega)$. The last term is $o(1)$ since $B_h h\mathcal{N} q_0 = O_{L^2(\partial\Omega)}(1)$. ■

1.6.2 propagation of singularity

In this subsection, we prove Proposition 1.7.

We factorize $-h^2\Delta - 1$ as $(hD_y - Q_h^\pm)(hD_y - Q_h^\mp) + R^\pm$ near $z_0 \in \mathcal{H}$ and choose $Q_h^\pm$ with principal symbols $\pm Q(y, x, \xi) = \sqrt{1-\lambda^2} b_1(y, x, \xi)$, as in the previous subsection. Take $\psi \in C_c^\infty([0, \epsilon_0))$with $\psi \equiv 1$ in a neighborhood of $y = 0$. By an abuse of notation, we introduce

$$w^\pm = B_h^\pm(hD_y - Q_h^\mp)w,$$

with $B_h^\pm$ has principal symbols $\psi(y)b^\pm(y, x, \xi)$ where $b^\pm$are solutions of

$$\frac{\partial b^\pm}{\partial y} \mp H_{Q(y,x,\xi)} b^\pm = 0, b^\pm|_{y=0} = b_0, \tag{1.6.1}$$

b_0 is another micro-localization near z_0 with $b_1|_{\mathrm{supp}(b_0)} = 1$ and $H_Q b = \{Q, b\}$. Note that the compact support of $\psi(y)b^\pm$ can be chosen to be arbitrarily close to the semi-bicharacteristic curves $\gamma^\pm$ corresponding to the principal symbol p. Moreover, $b^\pm$ are invariant along $\gamma^\pm$. Under these notations, Proposition 1.7 can be rephrased as follows

Proposition 1.12
Let μ be the defect measure of u. If

$$b^+\mu\mathbf{1}_{0<y\leq\epsilon_0} = 0(b^-\mu\mathbf{1}_{0<y\leq\epsilon_0} = 0),$$

then we have

$$b^-\mu\mathbf{1}_{0<y\leq\epsilon_0} = 0(b^+\mu\mathbf{1}_{0<y\leq\epsilon_0} = 0).$$

Moreover, we have in fact $b^+\mu = b^-\mu = 0$ in this case.

The proof will be divided into several lemmas. First we calculate

$$(hD_y - Q_h^{\pm})w^{\pm} = [hD_y - Q_h^{\pm}, B_h^{\pm}](hD_y - Q_h^{\mp})w + B_h^{\pm}(hD_y - Q_h^{\pm})(hD_y - Q_h^{\mp})w,$$

and

$$[hD_y - Q_h^{\pm}, B_h^{\pm}] = \frac{h}{i}\mathrm{Op}_h(\partial_y b^{\pm} \mp H_Q b^{\pm})\psi(y) + \frac{h}{i}\psi'(y)B_h^{\pm} + R''.$$

The first operator vanishes thanks to the definition of $b^{\pm}$, and the remainder term $R'' = O_{L^2(\mathbb{R}^d_+)}(h^2)$. Therefore we have

$$\|R''(hD_y - Q_h^{\mp})w\|_{L^2(\mathbb{R}^d_+)} = O(h^2).$$

and consequently

$$(hD_y - Q_h^{\pm})w^{\pm} = \frac{h}{i}\psi'(y)w^{\pm} + g^{\pm},$$

with $g^{\pm} = o_{L^2(\mathbb{R}^d_+)}(h)$.

Lemma 1.12
Let $\mu^{\pm}$ be the semi-classical defect measure of $w^{\pm}$, b is defined as above. Suppose $b^{\pm}\mu^{\pm}\mathbf{1}_{0<y\leq\epsilon_0} = 0$, then we must have $b^{\pm}\mu^{\pm} \equiv 0$ and $\mu_0^{\pm} = 0$, where $\mu_0^{\pm}$ is the defect measure of $w_0^{\pm} = w^{\pm}|_{y=0}$.

Proof. Take $y_0 = \epsilon_0/2$, we first claim that $\|w^{\pm}(y_0)\|_{L^2_x} = o(1)$. Indeed, from the assumption and compactness, the measure $\mu^{\pm}$ vanishes in a small neighborhood of semi-bicharacteristic curve $\gamma^{\pm}$. Thus $\|w^{\pm}\|_{L^2([y_0,\epsilon_0]\times\mathbb{R}^{d-1})} = o(1)$, provided that we choose $\mathrm{supp}(b_0)$ small enough in the definition of $w^{\pm}$. Finally, repeat the argument in the proof of Lemma 1.10, we have

$$-h\|w^{\pm}(y_0)\|^2_{L^2(\mathbb{R}^{d-1})} = i\int_{y_0}^{\epsilon_0}((Q_h^{\pm} - (Q_h^{\pm})^*)\chi w^{\pm}|\chi w^{\pm})_{L^2(\mathbb{R}^{d-1})}(y)dy + o(h).$$

The claim then follows.

Integrating the identity

$$\begin{aligned} h\frac{d}{dy}(w^{\pm}|w^{\pm})_{L^2(\mathbb{R}^{d-1})} = {} & (i(Q_h^{\pm} - (Q_h^{\pm})^*)w^{\pm}|w^{\pm})_{L^2(\mathbb{R}^{d-1})} + 2h(\psi'(y)w^{\pm}|w^{\pm})_{L^2(\mathbb{R}^{d-1})} \\ & + 2\mathrm{Im}(w^{\pm}|g^{\pm})_{L^2(\mathbb{R}^{d-1})} \end{aligned}$$

from $y = z < y_0$ to $y = y_0$, we have

$$\|w^{\pm}(z)\|^2_{L^2(\mathbb{R}^{d-1})} \leq C\int_z^{y_0}\|w^{\pm}(y)\|^2_{L^2(\mathbb{R}^{d-1})}dy + o(1).$$

Using $\int_0^{y_0}\|w^{\pm}(y)\|^2_{L^2(\mathbb{R}^{d-1})}dy = o(1)$, we obtain that $\|w_0^{\pm}\|_{L^2(\mathbb{R}^{d-1})} = o(1)$. ■

Remark 1.3
Away from the boundary, the defect measure of u equals to the defect measure of w, and it propagates along the bi-characteristic curves $\gamma^{\pm}$. Since we can essentially decompose w into w^+ and w^- near a hyperbolic point, we call w^+(w^-)the incoming wave and the out-coming wave. Thus the above proposition asserts that if we have no singularity of w^+(w^-) along incoming wave(out-coming wave) near the boundary but strictly away from the boundary, then there is no singularity of the boundary data of incoming wave(out-coming wave).

Changing the role of $y = y_0$ and $y = 0$ in the proof of Lemma 1.12, we conclude that if $\mu_0^{\pm} = 0$, then $b^{\pm}\mu^{\pm} = 0$.

To finish the proof, we need understand how the singularity transfers form boundary data of in-coming wave to the boundary data of out-coming wave.

Lemma 1.13
$\mu_0^{\pm} = 0$ implies that $\mu_0^{\mp}\mathbf{1}_{\xi\neq 0} = 0$. Consequently, $\mu^{\mp}\mathbf{1}_{\xi\neq 0} = 0$.

Proof. By symmetry, we only need deduce $\mu_0^{-}\mathbf{1}_{\xi\neq 0} = 0$ from $\mu_0^{+} = 0$. For $\delta > 0$, we define

$$b_{0,\delta}(x,\xi) = b_0(x,\xi)\left(1 - \widetilde{\psi}\left(\frac{\lambda(0,x,\xi)}{\delta}\right)\right),$$

with some $\widetilde{\psi} \in C_c^{\infty}(\mathbb{R}), \widetilde{\psi}|_{[-2,2]} \equiv 1$. We define $b_{\delta}^{\pm}(y,x,\xi)$ by solving ODE (1.6.1) with initial data $b_{0,\delta}$.Let $B_{\delta,h}^{\pm}$ be the associated PdO with principal symbol $b_{\delta}^{\pm}$. From compactness and continuity dependence of the initial data, we have such that $\delta < \lambda(y,x,\xi) < c_0 < 1$ on $\mathrm{supp}(b_{\delta}(y))$ for $0 \leq y \leq \epsilon_0$. Since in $Y_+ \times \mathbb{R}^{d-1}$, $\frac{\lambda(y,x,\xi)}{|\xi|} \sim 1$. Since

$$b^{\pm}(y,x,\xi) = b_0 \circ \gamma^{\pm}(y)^{-1}(x,\xi), \quad b_{\delta}^{\pm}(y,x,\xi) = b_{0,\delta} \circ \gamma^{\pm}(y)^{-1}(x,\xi),$$

we have that $\frac{b_{\delta}^{\pm}}{b^{\pm}}$ is a smooth function with compact support in $Y_+ \times \mathbb{R}^{d-1}$. Denote by $w_{\delta}^{\pm} = B_{\delta,h}^{\pm}(hD_y - Q_h^{\mp})w$, associated with semi-classical defect measure $\mu_{\delta}^{\pm}$, we have $\mu_{\delta}^{\pm} = \mu^{\pm}\left(\frac{b_{\delta}^{\pm}}{b^{\pm}}\right)^2$. In particular, $\mu_{\delta,0}^{\pm} = \mu_0^{\pm}\left(1 - \widetilde{\psi}\left(\frac{\lambda_0}{\delta}\right)\right)^2$ and $\mathrm{supp}(\mu_{\delta}^{\pm}) \subset \mathrm{supp}(\mu^{\pm})$. On the boundary, $B_{\delta,h}^{+}$ and $B_{\delta,h}^{-}$ coincide and will be denoted by $B_{\delta,h}^{0}$. Taking the trace of $w_{\delta}^{\pm}$, we have

$$\begin{cases} w_{\delta,0,\parallel}^{+} = -iB_{\delta,h}^{0}v + ih^2 B_{\delta,h}^{0}\partial_y(\nabla q)_{\parallel}|_{y=0} + B_{\delta,h}^{0}Q_h^{+}h\nabla_{\parallel}q_0, \\ w_{\delta,0,\perp}^{+} = iB_{\delta,h}^{0}h^2\partial_y^2 q|_{y=0} + B_{\delta,h}^{0}Q_h^{+}h\partial_y q|_{y=0}, \end{cases}$$

where $v = h\partial_y u|_{y=0} = O_{L_x^2}(1)$. Similarly, we have

$$\begin{cases} w_{\delta,0,\parallel}^{-} = -iB_{\delta,h}^{0}v + ih^2 B_{\delta,h}^{0}\partial_y(\nabla q)_{\parallel}|_{y=0} + B_{\delta,h}^{0}Q_h^{-}h\nabla_{\parallel}q_0, \\ w_{\delta,0,\perp}^{-} = iB_{\delta,h}^{0}h^2\partial_y^2 q|_{y=0} + B_{\delta,h}^{0}Q_h^{-}h\partial_y q|_{y=0}. \end{cases}$$

Notice that $\sigma(Q_h^{+}) = -\sigma(Q_h^{-})$, and write $\alpha = -B_{\delta,h}^{0}h^2\Delta_0 q_0, \beta = B_{\delta,h}^{0}Q_h^{+}h\mathcal{N}q_0$, we have

$$w_{\delta,0,\perp}^{\pm} = i\alpha \mp \beta + O_{L_x^2}(h).$$

From assumption $\|w^+_{\delta,0,\perp}\|_{L^2} = o(1)$, we have $\|i\alpha - \beta\|^2_{L^2} = o(1)$, and this implies that $\|\alpha\|^2 + \|\beta\|^2 - 2\mathrm{Im}(\alpha|\beta) = o(1)$. We claim that $\mathrm{Im}(\alpha|\beta) = o(1)$.

Indeed, we first note that from Proposition 1.11 and the ellipticity of $\mathcal{N}$, we have that $q_0 = O_{L^2(\mathbb{R}^{d-1})}(1)$, micro-locally away from $\xi = 0$. Now from trace theorem and Proposition 1.4, we have

$$\beta = A_{\delta,h}q_0 + O_{L^2(\mathbb{R}^{d-1}_x)}(h^{1/3})$$

for some PdO with real-valued principal symbol a_δ, compactly supported and vanishing for $\lambda(y,x,\xi) \leq \delta/4$. Similarly,

$$\alpha = A'_{\delta,h}q_0 + o_{L^2(\mathbb{R}^{d-1})}(1)$$

for some PdO with real-valued principal symbol a'_δ. Thus $\mathrm{Im}(\alpha|\beta)_{L^2} = o(1)$, since all the principal symbols involved in the inner product are real-valued. Now from $\|\alpha\|_{L^2} = o(1), \|\beta\|_{L^2} = o(1)$, one can deduce that the terms on the righthand side of $w^\pm_{\delta,0,\|}$ involving pressure are also $o_{L^2_x}(1)$, and $v = o_{L^2_x}(1)$ follows since $w^-_{\delta,0,\|} = o_{L^2(\mathbb{R}^{d-1})}(1)$. Therefore $\mu^-_{\delta,0} = 0$ and consequently $\mu^-_\delta = 0$ from Lemma 1.12. This implies that $\mu^-_0\mathbf{1}_{\lambda>\delta} = \mu^-\mathbf{1}_{\lambda>\delta} = 0$. Since $\delta > 0$ is arbitrary, we have that $b^-\mu^-\mathbf{1}_{\xi\neq 0} = 0$. Moreover, Corollary 1.1 implies that $\mu\mathbf{1}_{\xi\neq 0} = 0$. ■

Now we finish the proof of Proposition 1.12 by showing the following lemma.

Lemma 1.14
$\mu^+ = 0$ implies that $\mu^- = 0$.

Proof. We need deal with $\xi = 0$. Take $\widetilde{\psi}$ be a cut-off function which equals 1 in a neighborhood of 0. Pick any $\epsilon > 0$, we define the operator

$$B^{\epsilon,\pm}_h = \mathrm{Op}_h(\widetilde{\psi}(\lambda(y,x,\xi)/\epsilon))B^\pm_h.$$

Applying divergence equation for $w^\pm$

$$B^{\epsilon,\pm}_h h\mathrm{div}_\| w^\pm_\| + B^{\epsilon,\pm}_h h\partial_y w^\pm_\perp = O_{L^2(\mathbb{R}^d_+)}(h),$$

we have

$$\|B^{\epsilon,\pm}_h h\partial_y w^\pm_\perp\|_{L^2(\mathbb{R}^d_+)} \leq \|B^{\epsilon,\pm}_h h\mathrm{div}_\| w^\pm_\|\|_{L^2(\mathbb{R}^d_+)} + R_\epsilon(h)$$

with $R_\epsilon(h) \to 0$, as $h \to 0$ for each fixed $\epsilon > 0$. By estimating the operator norm from its symbol, we have

$$\|B^{\epsilon,\pm}_h h\partial_y w^\pm_\perp\|_{L^2(\mathbb{R}^d_+)} \leq C\epsilon + R_\epsilon(h),$$

and

$$\limsup_{h\to 0^+} \|B^{\epsilon,\pm}_h h\partial_y w^\pm_\perp\|_{L^2(\mathbb{R}^d_+)} \leq C\epsilon.$$

Using the equation $hD_y w^\pm_\perp = Q^\pm_h w^\pm_\perp + O_{L^2(\mathbb{R}^d_+)}(h)$, we have

$$\limsup_{h\to 0^+} \|B^{\epsilon,\pm}_h Q^\pm_h w^\pm_\perp\|_{L^2(\mathbb{R}^d_+)} \leq C\epsilon$$

Finally let $\epsilon \to 0$, we have $\mu_{\perp}^{\pm}\mathbf{1}_{\xi=0} = 0$. Therefore $\mu_{\perp}^{-} = 0$. As a consequence of Lemma 1.12(more precisely, using the proof of Lemma 1.12), $\mu_{0,\perp}^{\pm}\mathbf{1}_{\xi=0} = 0$. Now let μ_α, μ_β be the defect measures of $\alpha = -B_h^0 h^2\Delta_0 q_0, \beta = B_h^0 Q_h h\mathcal{N} q_0$, and let $\mu_{i\alpha\pm\beta}$ be the defect measure of $i\alpha \pm \beta$. Denote also by $\mu_{\alpha\beta}$ the limit corresponding to the quadratic form $(A_h\alpha|\beta)$. Similarly for $\mu_{\beta\alpha}$. Note that $\mu_{\alpha\beta} = \overline{\mu_{\beta\alpha}}$. From

$$\langle \mu_{i\alpha+\beta}, \mathbf{1}_{\xi=0}\rangle = \langle \mu_\alpha, \mathbf{1}_{\xi=0}\rangle + \langle \mu_\beta, \mathbf{1}_{\xi=0}\rangle - \langle 2\,\mathrm{Im}\,\mu_{\alpha\beta}, \mathbf{1}_{\xi=0}\rangle = 0$$
$$\langle \mu_{i\alpha+\beta}, \mathbf{1}_{\xi=0}\rangle = \langle \mu_\alpha, \mathbf{1}_{\xi=0}\rangle + \langle \mu_\beta, \mathbf{1}_{\xi=0}\rangle + \langle 2\,\mathrm{Im}\,\mu_{\alpha\beta}, \mathbf{1}_{\xi=0}\rangle = 0,$$

we have $\mu_\alpha \mathbf{1}_{\xi=0} = \mu_\beta \mathbf{1}_{\xi=0} = 0$. Next we consider parallel components. The key claim is that the measure corresponding to $B_h^0 Q_h^{\pm} h\nabla_{\parallel} q_0$ vanishes on the set $\{\xi = 0\}$.

Indeed, from Lemma 1.3 and trace theorem, $hq_0 \to 0$ strongly in $L^2(\partial\Omega)$. From the ellipticity of $\mathcal{N}$, there exists a classical pseudo-differential operator E of order -1 such that $E\mathcal{N} = I + R$, where R is a non semi-classical smoothing operator.

We want to show that

$$\lim_{\epsilon\to 0}\limsup_{h\to 0} \|B_h^0 B_h^{\epsilon,0} Q_h^{\pm} h\nabla_{\parallel} q_0\|_{L^2(\mathbb{R}^{d-1})} = 0.$$

From symbolic calculus and the strong convergence of hq_0 in $L^2(\mathbb{R}_x^{d-1})$, it suffices to prove

$$\lim_{\epsilon\to 0}\limsup_{h\to 0} \|h\nabla_{\parallel} B_h^0 B_h^{\epsilon,0} Q_h^{\pm} q_0\|_{L^2(\mathbb{R}^{d-1})} = 0. \tag{1.6.2}$$

We write

$$\begin{aligned} h\nabla_{\parallel} B_h^0 B_h^{\epsilon,0} Q_h^{\pm} q_0 =& h\nabla_{\parallel} B_h^0 B_h^{\epsilon,0} Q_h^{\pm} E\mathcal{N} q_0 - h\nabla_{\parallel} B_h^0 B_h^{\epsilon,0} Q_h^{\pm} R q_0 \\ =& \nabla_{\parallel} E B_h^0 B_h^{\epsilon,0} Q_h^{\pm} h\mathcal{N} q_0 + h\nabla_{\parallel}[B_h^0 B_h^{\epsilon,0} Q_h^{\pm}, E]\mathcal{N} q_0 \\ & - h\nabla_{\parallel} B_h^0 B_h^{\epsilon,0} Q_h^{\pm} R q_0. \end{aligned} \tag{1.6.3}$$

Here we are taking the commutator between a semi-classical operator and a classical pseudo-differential operator, hence the semi-classical symbolic calculus is not applicable. Instead, it is not difficult to check that for any $a \in C_c^\infty(T^*\partial\Omega)$, $E \in S_{x,\xi}^{-1}$,

$$[a(x, hD_x), E(x, D_x)] = h\mathrm{Op}(S^{-1}) + \mathrm{Op}(S^{-2}),$$

where the implicit constants only depend on the semi-norms of the symbols $a(x,\xi)$ and $E(x,\xi)$. Notice that $h\nabla_{\parallel} B_h^0, B_h^{\epsilon,0}, Q_h^{\pm}$ are uniformly bounded operators in L_x^2 with respect to h, thus $\nabla_{\parallel} B_h^0 B_h^{\epsilon,0} Q_h^{\pm} R$, $\nabla_{\parallel} B_h^0 \mathrm{Op}(S^{-2})\mathcal{N}$, $h\nabla_{\parallel} B_h^0 \mathrm{Op}(S^{-1})\mathcal{N}$ are uniformly bounded operators in L_x^2 with respect to h. Thus from the strong convergence of hq_0, the last two terms on the right hand side of (1.6.3) are killed when we let $h \to 0$ first. Thus (1.6.2) follows from the vanishing of the measure of $\pm\beta = B_h^0 Q_h^{\pm} h\mathcal{N} q_0$ on the set $\{\xi = 0\}$. Combining the assumption that $\mu_{0,\parallel}^{+}\mathbf{1}_{\xi=0} = 0$, we deduce that $\mu_{0,\parallel}^{-}\mathbf{1}_{\xi=0} = 0$. The proof is now complete. ■

1.7 Near $\mathcal{G}^{2,+}$

In this section, we will follow the strategy of V.Ivrii (see [30] or [28]) to prove Proposition 1.8. Denote by $G = \det(\overline{g})$ and $P_h = h^2\Delta_H - 1$, we have

Lemma 1.15
In local coordinate Y_+, we have

$$P_h = -h^2 \frac{\overline{g}}{\sqrt{G}} \frac{\partial}{\partial y}\left(\sqrt{G}\overline{g}^{-1}\frac{\partial}{\partial y}\right) + R_h = h^2 D_y^2 + \mathrm{Op}_h(r) + O_{L^2\to L^2}(h).$$

Moreover, R_h is a matrix-valued second order differential operator in x with scalar principal symbol $r(y,x,\xi) = 1 - \lambda(y,x,\xi)^2$, which is self-ajoint with respect to the $(\cdot|\cdot)_{L^2(Y_+)}$.

The proof is direct calculation and will be given in the appendix.

In a fix local coordinate in Y_+, we identify $u = \varphi_1 u, q = \varphi_1 q$ and all the operators B by $\varphi B \varphi_1$.

Proposition 1.13
For any tangential operator B with scalar principal symbol $b(y,x,\xi)$ which vanishing near $\xi = 0$, we have

$$\limsup_{h\to 0} \|BhD_y u\|_{L^2(Y_+)} \leq \sup_{\rho\in supp\ (b)} |r(\rho)|^{1/2}|b(\rho)|.$$

Proof. We calculate

$$\begin{aligned}(BhD_yu|BhD_yu)_{Y_+} =&([B,hD_y]u|BhD_yu)_{Y_+} + (hD_yBu|BhD_yu)_{Y_+}\\ =&O(h) + (Bu|Bh^2D_y^2u)_{L^2(Y_+)}\\ =&O(h) - (Bu|BRu)_{L^2(Y_+)} + (Bu|BP_hu)_{Y_+}\\ =&O(h) - (Bu|BRu)_{Y_+} - (Bu|Bhdq)_{Y_+}.\end{aligned}$$

Integrating by part and using symbolic calculus, we have

$$\begin{aligned}(Bu|Bhdq)_{Y_+} =&(Bu|hdBq)_{Y_+} + (Bu|[B,hd]q)_{Y_+}\\ =&-([hd^*,B]u|Bq)_{Y_+} + (Bu|[B,hd]q)_{Y_+}\\ =&O(h),\end{aligned}$$

thanks to the fact that B has scalar-valued principal symbol. ■

The proof of Proposition 1.8 is based on the following integrating by part result.

Proposition 1.14
Given real scalar-valued tangential symbols a_0, a_1, there exist tangential operators A_0, A_1(constructed

in the local coordinate) with real, scalar-valued principal symbol a_0, a_1 and $A = A_1 hD_y + A_0$, such that for any 1-form w with compact support in Y_+, we have

$$\frac{2}{h}\operatorname{Im}(P_h w|Aw)_{Y_+} = (A_1 hD_y w|hD_y w)_{\underline{\partial} Y_+} + \operatorname{Re}\sum_{j=0}^{2}(C_j (hD_y)^j w|w)_{Y_+} + O(h),$$

where the tangential operators C_j have scalar-valued principal symbol $c_j(y,x,\xi)$ and

$$\sum_{j=0}^{2} c_j(y,x,\xi)\eta^j = \{p, a\}.$$

Proof. We first calculate

$$\begin{aligned}
I =& \frac{1}{ih}\left(-\frac{h^2}{\sqrt{G}}\overline{g}h\partial_y(\sqrt{G}\overline{g}^{-1}h\partial_y w)\middle| Aw\right)_{Y_+} - \frac{1}{ih}\left(Aw\middle| -\frac{h^2}{\sqrt{G}}\overline{g}h\partial_y(\sqrt{G}\overline{g}^{-1}h\partial_y w)\right)_{Y_+}\\
=& (hD_y w|A_1 hD_y w)_{\underline{\partial} Y_+} + (A_1 hD_y w|hD_y w)_{\underline{\partial} Y_+}\\
&+ \frac{1}{ih}(hD_y w|hD_y Aw)_{Y_+} - \frac{1}{ih}(hD_y Aw|hD_y w)_{Y_+}\\
=& (hD_y w|A_1 hD_y w)_{\underline{\partial} Y_+} + (A_1 hD_y w|hD_y w)_{\underline{\partial} Y_+}\\
&+ \frac{1}{ih}(hD_y w|[hD_y, A]w)_{Y_+} - \frac{1}{ih}([hD_y, A]w|hD_y w)_{Y_+}\\
&+ \frac{1}{ih}(hD_y w|AhD_y w)_{Y_+} - \frac{1}{ih}(AhD_y w|hD_y w)_{Y_+},
\end{aligned}$$

and the last two terms on the right hand side equal to

$$\frac{1}{ih}(A^* hD_y w|hD_y w) - (A_1^* hD_y w|hD_y w)_{\underline{\partial} Y_+} - \frac{1}{ih}(AhD_y w|hD_y w)_{Y_+}.$$

We want to construct operators A_0, A_1 such that $A_1^* = A_1 + O(h^2)$ and $A^* = A + O(h^2)$. Assume that

$$\widetilde{a_1} \backsimeq a_1^{(0)} + \frac{h}{i}a_1^{(1)}$$

with real-valued $a_1^{(j)}$ (not necessarily scalar-valued). From

$$\int_{Y_+}\langle A_1 u|v\rangle\sqrt{G}dydx = \int_{Y_+}\langle \overline{g}^{-1}A_1 u, v\rangle_{\mathbb{C}^{d-1}}\sqrt{G}dydx,$$

the symbol of A_1^* is equal to the symbol of $\frac{\overline{g}}{\sqrt{G}}\operatorname{Op}_h(\widetilde{a_1}^*)\sqrt{G}\overline{g}^{-1}$, which can be expressed by

$$b_1(y,x,\xi) \backsimeq \sum_{k\geq 0}\left(\frac{h}{i}\right)^k b_1^{(k)}(y,x,\xi),$$

with

$$b_1^{(k)}(y,x,\xi) = \sum_{j=0}^{1} \sum_{|\alpha|+|\beta|+j=k} (-1)^j \partial_\xi^\beta \left(\frac{\overline{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_1^{(j)} \right) \cdot \partial_x^\beta (\sqrt{G}\overline{g}^{-1}), \quad k \geq 1.$$

We have that

$$b_1^{(0)} = a_1^{(0)}, \quad b_1^{(1)} = -a_1^{(1)} + \sum_{|\alpha|+|\beta|=1} \partial_\xi^\beta \left(\frac{\overline{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_1^{(0)} \right) \cdot \partial_x^\beta (\sqrt{G}\overline{g}^{-1})$$

We set

$$a_1^{(0)} = a_1, \quad a_1^{(1)} = \frac{1}{2} \sum_{|\alpha|+|\beta|=1} \partial_\xi^\beta \left(\frac{\overline{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_1^{(0)} \right) \cdot \partial_x^\beta (\sqrt{G}\overline{g}^{-1}),$$

thus $A_1^* = A_1 + O(h^2)$. Note that $a_1^{(1)}$ is matrix-valued while $a_1^{(0)}$ is real and scalar-valued.

The construction of A_0 is similar. We observe that $(hD_y)^* = hD_y + h\frac{g}{\sqrt{G}} D_y(\sqrt{G}\overline{g}^{-1})$ and set

$$\widetilde{a_0} = a_0^{(0)} + \frac{h}{i} a_0^{(1)}.$$

A_0^* has symbol which can be expanded as

$$b_0 \backsimeq \sum_{k\geq 0} \left(\frac{h}{i}\right)^k b_0^{(k)}$$

with $b_0^{(0)} = a_0^{(0)}$ and

$$b_0^{(k)}(y,x,\xi) = \sum_{j=0}^{1} \sum_{|\alpha|+|\beta|+j=k} (-1)^j \partial_\xi^\beta \left(\frac{\overline{g}}{\sqrt{G}} \partial_\xi^\alpha \partial_x^\alpha a_0^{(j)} \right) \cdot \partial_x^\beta (\sqrt{G}\overline{g}^{-1}), \quad k \geq 1. \tag{1.7.1}$$

Note that

$$\begin{aligned}(hD_y)^* A_1^* - A_1^* hD_y =& [(hD_y)^*, A_1^*] + A_1^*(hD_y)^* - A_1^* hD_y \\ =& \frac{h}{i}(\partial_y A_1^*) + \frac{h}{i}\left[\frac{\overline{g}}{\sqrt{G}} \partial_y(\sqrt{G}\overline{g}^{-1}), A_1^* \right] + \frac{h}{i} A_1^* \frac{\overline{g}}{\sqrt{G}} \partial_y(\sqrt{G}\overline{g}^{-1}),\end{aligned}$$

and its symbol can be expanded as

$$\sum_{k\geq 0} \left(\frac{h}{i}\right)^k \kappa_k(y,x,\xi)$$

with $\kappa_0 = 0$ and

$$\begin{aligned}\kappa_1 =& \left(\partial_y b_1 + b_1 \frac{\overline{g}}{\sqrt{G}} \partial_y(\sqrt{G}\overline{g}^{-1}) \right), \\ \kappa_k =& \sum_{|\alpha|=k-1} \frac{1}{i^{|\alpha|+1}} \{\partial_\xi^\alpha, \partial_x^\alpha\} \left(\frac{\overline{g}}{\sqrt{G}} \partial_y(\sqrt{G}\overline{g}^{-1}), b_1 \right) \\ & \frac{h}{i}\left(\partial_y b_1 + b_1 \frac{\overline{g}}{\sqrt{G}} \partial_y(\sqrt{G}\overline{g}^{-1}) \right) + \frac{h}{i} \sum_{|\alpha|\geq 1} \frac{h^{|\alpha|}}{i^{|\alpha|}} \{\partial_\xi^\alpha, \partial_x^\alpha\} \left(\frac{\overline{g}}{\sqrt{G}} \partial_y(\sqrt{G}\overline{g}^{-1}), b_1 \right), \quad k \geq 2,\end{aligned}$$

where

$$\{\partial_\xi^\alpha, \partial_x^\alpha\}(f_1, f_2) = \partial_\xi^\alpha f_1 \partial_x^\alpha f_2 - \partial_\xi^\alpha f_2 \partial_x^\alpha f_1.$$

We set $b_0^{(0)} = a_0$ and $a_0^{(1)}$ such that $a_0^{(1)} = b_0^{(1)} + \kappa_1$(it has a solution thanks to (1.7.1)). Finally, we construct A_j by $\varphi_1 \mathrm{Op}_h(\widetilde{a_j})\varphi_1$ in local coordinate and one verify easily that

$$A_1^* = A_1 + O_{L^2\to L^2}(h^2), \quad A^* = A + O_{L^2\to L^2}(h^2).$$

Therefore

$$I = (hD_y w|A_1 hD_y w)_{\partial Y_+} + \frac{1}{ih}(hD_y w|[hD_y, A]w)_{Y_+} - \frac{1}{ih}([hD_y, A]w|hD_y w)_{Y_+} + O(h).$$

We next calculate

$$\begin{aligned}\frac{1}{ih}(Rw|Aw)_{Y_+} - \frac{1}{ih}(Aw|Rw)_{Y_+} =& \frac{1}{ih}((A^*R - R^*A)w|w)_{Y_+}\\ =& \frac{1}{ih}([A, R]w|w)_{Y_+} + O(h)\end{aligned}$$

since R is self-ajoint and $A^* - A = O_{L^2\to L^2}(h^2)$. Moreover, the principal symbol of $\frac{1}{ih}[A, R]$ is $\{r, a\}$. This completes the proof. ■

Now assume that we are working near a diffractive point $\rho \in \mathcal{G}^{2,+}$ in Y_+ where

$$\frac{\partial r}{\partial y} \geq c_0 > 0$$

The following lemma is a semi-classical version of Lemma 24.4.5 in [28]. The proof is a little more complicated, due to the different equation that we are considering.

Lemma 1.16
Let $B_j = \varphi B_j \varphi_1$, with real, scalar-valued tangential principal symbol $b_j, j = 0, 1, 2$, compactly supported and

$$\sum_{j=0}^{2} b_j(y, x, \xi)\eta^j = -\psi(y, x, \eta, \xi)^2 \textit{when } \eta^2 = r(y, x, \xi),$$

with some smooth function $\psi \in C_c^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$. Further assume that

$$dr \neq 0, \frac{\partial r}{\partial y} > 0, \textit{on } \{y = r = 0\} \cap \bigcup_{j=1}^{2} \mathrm{supp}(b_j).$$

Then one can chose compactly supported, tangential operators $\Psi_j, j = 0, 1$ with real, scalar-valued principal symbols $\psi_j, j = 0, 1$ which satisfy

$$\psi_0(y, x, \xi) = \psi(y, x, 0, \xi), \psi_1(y, x, \xi) = \partial_\eta(y, x, 0, \xi) \textit{ when } \eta = r(y, x, \xi) = 0,$$

so that for any solution u of $P_h u = f - h\nabla q, h\mathrm{div} u = 0$ with $u|_{y=0} = 0$ in trace sense, we have

$$\begin{aligned}&\mathrm{Re}\sum_{j=0}^{2}(B_j(hD_y)^j v|v)_{Y_+} + \|\Psi_0 v + \Psi_1 hD_y v\|^2_{L^2(Y_+)} + (\Theta P_h v|v)_{Y_+}\\ =&o(1)\end{aligned} \tag{1.7.2}$$

as $h \to 0$, where $v = \varphi \mathrm{Op}_h(\chi)\varphi_1 u$ and $\chi \in C_c^\infty(Y_+ \times \mathbb{R}^{d-1})$ has support near $\rho \in \mathcal{G}^{2,+}$. Θ is a tangential operator which depends on ψ_j, b_j, whose principal symbols are scalar-valued.

The proof is based on the following elementary lemma, for which the proof can be found as Lemma 24.4.3 in [28],

Lemma 1.17
Let X be an open subset of $\overline{\mathbb{R}^n_+} = \{x \in \mathbb{R}^n : x_1 \geq 0\}$, and let $r \in C^\infty(X)$. Assume that r is real-valued, that $dr \neq 0$ when $r = 0$ and that $\frac{\partial r}{\partial x_1} > 0$ when $r = x_1 = \frac{\partial r}{\partial x_j} = 0$ for $j \neq 1$. Let

$$F(t,x) = \sum_{j=0}^{2} f_j(x)t^j$$

be a quadratic polynomial in t with coefficients in $C^\infty(X)$ such that

$$F(t,x) = -\psi(t,x)^2 \text{ when } t^2 = r(x),$$

where $\psi \in C^\infty(\mathbb{R}\times X)$. Then one can find $\psi_0, \psi_1, \theta \in C^\infty(X)$ with $\psi_0(x) = \psi(0,x), \psi_1(x) = \frac{\partial \psi}{\partial t}(0,x)$ when $r(x) = 0$, and

$$F(t,x) + (\psi_0(x) + t\psi_1(x))^2 \leq \theta(x)(t^2 - r(x)); t \in \mathbb{R}, x \in X.$$

Proof of Lemma 1.16. Choose C^∞ functions $\psi_0(y,x,\xi)$ and $\psi_1(y,x,\xi)$ as in the previous Lemma with $\psi_j(y,x,\xi) = \partial_\eta^j \psi|_{y=0}, j = 0,1$ when $\eta = r(y,x,\xi) = 0$ and so that

$$\sum_{j=0}^{2} b_j \eta^j + (\psi_0 + \eta\psi_1)^2 \leq \theta(y,x,\xi)(\eta^2 - r).$$

Since ψ_0, ψ_1 and each b_j are compactly supported in variables (y,x,ξ), we may assume that θ is smooth and with compact support. Define $\Theta = \varphi \mathrm{Op}_h(\theta)\varphi_1$, $\Psi_j = \varphi \mathrm{Op}_h(\psi_j)\varphi_1, j = 0,1$ and consider the quantity

$$\mathrm{Re}\sum_{j=0}^{2}(B_j(hD_y)^j v|v)_{Y_+} + ((\Psi_0 + \Psi_1 hD_y)^2 v|v)_{Y_+} - (\Theta hD_y v|hD_y v)_{Y_+} + (\Theta R_h v|v)_{Y_+}.$$

The expression above can be written in the form

$$\sum_{j=0}^{2} \left(C_j(hD_y)^j v|v\right)_{Y_+},$$

where the tangential operators C_j have real, scalar-valued principle symbol. Moreover,

$$\sum_{j=0}^{2} c_j(y,x,\xi)\eta^j \leq 0.$$

However, since the symbol is not bounded in η and we can not apply sharp Gårding inequality directly. To resolve this issue, we extend each c_j to $\widetilde{c_j} \in C_c^m(\mathbb{R} \times \mathbb{R}^{2d-2})$ who agrees with c_j on $y \geq 0$ up to order m, any given order, of derivatives. This is possible since any order of y derivatives of all the symbols has continuous limit as $y \to 0$. We still use the notation c_j in what follows. Let $\underline{v} = v\mathbf{1}_{y\geq 0}$ and we use the boundary condition $v|_{y=0} = 0$ and calculate

$$\begin{aligned}
\left(\sum_{j=0}^{2} C_j(hD_y)^j v \middle| v\right)_{Y_+} &= \left(\sum_{j=0}^{2} C_j(hD_y)^j \underline{v} \middle| \underline{v}\right)_{Y_+} \\
&= \left(\psi\left(\frac{hD_y}{A}\right)\sum_{j=0}^{2} C_j(hD_y)^j \underline{v} \middle| \underline{v}\right)_{Y_+} \\
&+ \left(\left(1-\psi\left(\frac{hD_y}{A}\right)\right)\sum_{j=0}^{2} C_j(hD_y)^j \underline{v} \middle| \underline{v}\right)_{Y_+} \\
&=: I + II,
\end{aligned}$$

for any big number $A > 0$. Now we apply sharp Gåding inequality to the first term to get

$$I \leq C_A h,$$

with some constant C_A depending on A. The second term is essentially in the elliptic region and we define its symbol

$$\Xi(y,x,\eta,\xi) := \left(1-\psi\left(\frac{\eta}{A}\right)\right)\sum_{j=0}^{2}\frac{c_j(y,x,\xi)\eta^j}{\eta^2 - r(y,x,\xi)} \in S^0(\mathbb{R}^{2d}),$$

hence we can bound

$$\begin{aligned}
|II| \leq& O(h) + C\left(\Xi(y,x,hD_y,hD_x)\chi(y,x,hD_x)P_h\underline{u} \middle| \left(1-\psi\left(\frac{2hD_y}{A}\right)\underline{v}\right)\right)_{Y_+} \\
=& O(h) + C\left(\Xi(y,x,hD_y,hD_x)\chi(y,x,hD_x)(2hw\otimes\delta_{y=0}) \middle| \left(1-\psi\left(\frac{2hD_y}{A}\right)\underline{v}\right)\right)_{Y_+} \\
&+ C\left(\Xi(y,x,hD_y,hD_x)\chi(y,x,hD_x)(\mathbf{1}_{y\geq 0}h\nabla q) \middle| \left(1-\psi\left(\frac{2hD_y}{A}\right)\underline{v}\right)\right)_{Y_+},
\end{aligned}$$

with $w = hD_y u|_{y=0}$. Note that to obtain the expression above, one can not use symbolic calculus to deal with commutator between tangential symbol and usual symbol. However, since P_h is a differential operator, we can compute its commutator with $\chi(y,x,hD_x)$ directly.

Now from Proposition 1.22, the limsup of the third term on the right hand side when $h \to 0$ can be bounded by $\epsilon(A)$ with $\lim_{A\to\infty} \epsilon(A) = 0$. Here we can use the flat metric to estimate the L^2 norm. The second term on the right hand side can be bounded by

$$Ch\|(1-h^2\Delta_{y,x})^{-\frac{s}{2}}(w\otimes\delta_{y=0})\|_{L^2(\mathbb{R}^d)}\|(1-h^2\Delta_{y,x})^{\frac{s}{2}}\underline{v}\|_{L^2(\mathbb{R}^d)} \leq Ch^{1-s},$$

for any $s \in \left(\frac{1}{2}, 1\right)$. Here we have used the fact that $\delta_{y=0} \in H^{-s}(\mathbb{R}_y)$ for any $s > \frac{1}{2}$ and $h^s\underline{v}$ is bounded in $H^s(\mathbb{R}^d)$ since $\underline{v}|_{y=0} = 0$ and $h\nabla_{y,x'}v$ is bounded in $L^2(\mathbb{R}^d)$.

Therefore, for any $A > 0$, we have proved that

$$\limsup_{h\to 0} |II| \leq \epsilon(A),$$

and this completes the proof. ■

Adapting to the notations in this section, Proposition 1.8 can be rephrased as follows

Proposition 1.15
Suppose $\rho \in \mathcal{G}^{2,+}$, and $\rho_0 \in T^*\Omega$ approaching to ρ such that $\partial_y r(\rho_0) \geq \frac{1}{2}\partial_y r(\rho) \geq c_0$. Let $\gamma_- = [\rho_0, \rho]$ be a segment of the generalized ray issued from ρ_0 to ρ (the trajectory under the canonical projection is tangent to the boundary at ρ). Suppose $\rho_0 \notin \mathrm{supp}\mu$. Then we have $\rho \notin \mathrm{supp}\mu$.

Proof. Take a small neighborhood Γ_0 of ρ_0 such that $\Gamma_0 \cap \mathrm{supp}\mu = \varnothing$. Take a small neighborhood $W_0 \subset \overline{\Omega} \times \mathbb{R}^{d-1}$ such that $\frac{\partial r}{\partial y}(y, x, \xi) \geq c_0/4 > 0$. Shrinking W_0 if necessary, we assume that each point $(y, x, \xi) \in W_0$ with $r(y, x, \xi) \geq 0$ can be connected by a (possibly broken) ray issued from Γ_0 with at most one reflection or tangency at $\partial\Omega$. It suffices to prove the following statement:

For any $\chi \in C_c^\infty(\overline{\Omega} \times \mathbb{R}^{d-1})$ with supp $\chi \subset W_0$, small enough, we have

$$\varphi \mathrm{Op}_h(\chi)\varphi_1 u = o_{L^2}(1), h \to 0.$$

As in [28], we construct test functions which satisfy some properties as follows:

Lemma 1.18
There exists

$$a(y, x, \eta, \xi) = a_0(y, x, \xi) + a_1(y, x, \xi)\eta, a_j \in C_c^\infty(W_0)$$

with the following properties:

1. $a_1(0, x, \xi) = -t(x, \xi)^2$, *for some* $t \in C_c^\infty(T^*\underline{\partial}Y_+)$,
2. *For some large* $M \geq 0$, *when* $p = \eta^2 - r(y, x, \xi) = 0$, *we have*

$$\{p, a\} + aM = -\psi(y, x, \eta, \xi)^2 + \omega(y, x, \xi)(\eta - r^{1/2}(y, x, \xi)), a = s^2,$$

where $s \in C^\infty(Y_+ \times (\mathbb{R}^d \setminus \{0\}))$,$\psi \in C_c^\infty(Y_+ \times \mathbb{R}^d \setminus \{0\})$ *and* $\omega \in C_c^\infty(W_0)$. *Moreover,* $r|_{supp\ \omega} > 0$.

The construction is exactly the same as in [28] and will be given in the appendix D for the sake of completeness.

Now we take $\chi \in C_c^\infty(W_0)$ with $\chi \equiv 1$, in a neighborhood of supp $a_1\cup$ supp a_2. Let $v = \varphi\mathrm{Op}_h(\chi)\varphi_1 u$, and we calculate

$$\begin{aligned}(P_h v|Av)_{Y_+} =&(\varphi\mathrm{Op}_h(\chi)\varphi_1 P_h u|Av)_{Y_+} + ([P_h, \varphi\mathrm{Op}_h(\chi)\varphi_1]u|Av)_{Y_+}\\ =&(\varphi\mathrm{Op}_h(\chi)\varphi_1 f|Av)_{Y_+} - (\varphi\mathrm{Op}_h(\chi)\varphi_1 hdq|Av)_{Y_+}\\ &+([P_h, \varphi\mathrm{Op}_h(\chi)\varphi_1]u|Av)_{Y_+}.\end{aligned}$$

Here we have used the differential form formulation to calculate the inner product. Notice that $\{p,\chi\} = 0$ on supp a_j and $f = o_{L^2}(h)$, $hD_y u_\perp|_{y=0} = 0$, thus

$$\begin{aligned}\frac{2}{h}\,\mathrm{Im}(P_h v|Av)_\Omega = o(1) - \frac{2}{h}\,&\mathrm{Im}([\varphi\mathrm{Op}_h(\chi)\varphi_1, hd]q|A\varphi\mathrm{Op}_h(\chi)\varphi_1 u)_{Y_+}\\ +\frac{2}{h}\,&\mathrm{Im}(\mathrm{Op}_h(\chi)q|hd^*(A\varphi\mathrm{Op}_h(\chi)\varphi_1 u))_{Y_+}.\end{aligned}\tag{1.7.3}$$

From Proposition 1.14,

$$\begin{aligned}\sum_{j=0}^{2}(C_j(hD_y)^j v|v)_{Y_+} =& -(A_1 hD_y v|hD_y v)_{\underline{\partial} Y_+} - \frac{2}{h}\,\mathrm{Im}([\varphi\mathrm{Op}_h(\chi)\varphi_1, hd]q|Av)_{Y_+}\\ &+ \frac{2}{h}\,\mathrm{Im}(\varphi\mathrm{Op}_h(\chi)\varphi_1 q|hd^*(A\varphi\mathrm{Op}_h(\chi)\varphi_1 u))_{Y_+} + o(1).\end{aligned}\tag{1.7.4}$$

Since the principal symbol of A is scalar-valued, by using $d^*u = 0$, we could write

$$\frac{2}{h}\,\mathrm{Im}(\varphi\mathrm{Op}_h(\chi)\varphi_1 q|hd^*(A\varphi\mathrm{Op}_h(\chi)\varphi_1 u))_{Y_+} = (\varphi\mathrm{Op}_h(\chi)\varphi q|\Upsilon u)_{Y_+} + O(h)$$

where $\Upsilon = \Upsilon_0 + \Upsilon_1 hD_y$, Similarly,

$$-\frac{2}{h}\,\mathrm{Im}([\varphi\mathrm{Op}_h(\chi)\varphi_1, hd]q|Av)_{Y_+} = (\Upsilon_2 q|Av)_{Y_+} + O(h),$$

where Υ_j are matrix-valued tangential pseudo-differential operators with principal symbol supported in supp(χ) Using Lemma 1.16 for the function

$$\sum_{j=0}^{2} c_j\eta^j + aM - \omega(\eta - r^{1/2}) = -\psi^2,$$

we have

$$\begin{aligned}&\mathrm{Re}\left(\sum_{j=0}^{2} C_j(hD_y)^j u|u\right)_{Y_+} - \mathrm{Re}(\Phi(hD_y - \widetilde{Q}_+)v|\varphi(y,x,hD_x)v)_{Y_+}\\ &+ \mathrm{Re}(Mv|Av)_{Y_+} + (\Theta P_h v|v)_{Y_+} + \|\Psi_0 v + \Psi_1 hD_y v\|^2_{L^2(Y_+)}\\ \le& o(1) + Ch\|v\|^2_{L^2(Y_+)},\end{aligned}\tag{1.7.5}$$

where the compact supported tangential operator Φ has scalar-valued principal symbol $\phi \in C_c^\infty(W_0)$ and $r|_{\operatorname{supp}\phi} > 0$, $\phi = 1$ in a neighborhood of supp ω. $\widetilde{Q}_+$ is the operator constructed in the hyperbolic region with principal symbol $r^{1/2}$. This is possible since in the proof of Lemma 1.18, we indeed have $r \geq \delta^2|\xi|^2$ on the support of ω. Note that the principal symbol of A is positive on $\eta^2 - r = 0$, we can apply Lemma 1.16 again to the term $(Mv|Av)_\Omega$ and bound it from below by

$$o(1) - |(\Theta_1 P_h v|v)_{Y_+}|.$$

Thus we have

$$\begin{aligned}
&-(A_1 hD_y v|hD_y v)_{\partial Y_+} + \|\Psi_0 v + \Psi_1 hD_y v\|^2_{L^2(Y_+)} \\
&\leq o(1) + Ch\|v\|^2_{L^2(Y_+)} + C|(\Theta P_v|v)_{Y_+}| + C|(\Theta_1 P_h v|v)_{Y_+}| \\
&+ \left|\operatorname{Re}(\varphi \mathrm{Op}_h(\phi)\varphi_1(hD_y - \widetilde{Q}_+)v|\varphi \mathrm{Op}_h(\omega)\varphi_1 v)_{Y_+}\right| \\
&+ |(\Upsilon_2 q|Av)_{Y_+}| + |(\varphi \mathrm{Op}_h(\chi)\varphi_1 q|\Upsilon u)_{Y_+}|.
\end{aligned} \tag{1.7.6}$$

The terms on the left hind side are essentially positive from semi-classical sharp Gårding inequality, hence we only need to control the terms on the right hind side. The term $|(\Theta P_h v|v)_\Omega| + |(\Theta_1 P_h v|v)_\Omega| = o(1)$ follows from the equation and symbolic calculus since the principal symbols of Θ and Θ_1 are scalar-valued. We claim that

$$\left|\operatorname{Re}(\varphi \mathrm{Op}_h(\phi)\varphi_1(hD_y - \widetilde{Q}_+)v|\varphi \mathrm{Op}_h(\omega)\varphi_1 v)_{Y_+}\right| = o(1), \quad h \to 0. \tag{1.7.7}$$

Indeed, micro-locally on supp ϕ, $r \gtrsim \delta^2 > 0$, hence in the region where $\lambda^2(y,x,\xi) < 1$, we could construct $\widetilde{Q}_+, \widetilde{Q}_-$ micro-locally such that

$$P_h = (hD_y - \widetilde{Q}_-)(hD_y - \widetilde{Q}_+) + O(h^\infty)$$

as we have done in the hyperbolic case. From symbolic calculus and Corollary 1.1, we have

$$(hD_y - \widetilde{Q}_-)(hD_y - \widetilde{Q}_+)u = O_{L^2_{y,x}}(h) + h\nabla q = o_{L^2_{y,x}}(1), \text{ micro-locally on supp } \phi.$$

Therefore the measure μ concentrates on $\{\eta = -\sqrt{r}\} \cup \{\eta = \sqrt{r}\}$, on the support of ϕ. For any point $\rho_1 \in$ supp $\phi \cap$supp μ, with $\eta(\rho_1) = -\sqrt{r(\rho_1)} < 0$, the backward generalized ray issued from ρ_1 must enter Γ_0 without meeting any point in $\mathcal{G}^{2,+}$, since along the backward flow, η is decreasing. Consequently, away from the boundary, $u = o_{L^2(Y_+)}(1)$ and hence $(hD_y - \widetilde{Q}_+)u = o_{L^2(Y_+)}(1)$, micro-localized near $\eta = -\sqrt{r}$. From the fact that $hD_y - \widetilde{Q}_-$ is micro-locally elliptic near $\eta = \sqrt{r}$. Near the boundary and some point $\rho_1 \in \mathcal{H} \cap \operatorname{supp}(\phi)$, any point can be connected backwardly to Γ_0 by at most transversal reflection. Thus (1.7.7) holds true.

It remains to control the last two terms involving pressure. We just treat one of them, and the other can be treated in the same way.

Pick $\varphi_0 \in C_c^\infty((-2,2))$ which equals 1 on $(-1,1)$. Define

$$\chi_\epsilon(y,x,\xi) = \chi(y,x,\xi)\varphi_0\left(r(y,x,\xi)\epsilon^{-1}\right).$$

We fix any $\epsilon > 0$, small enough, and write

$$\begin{aligned}(\Upsilon_2 q|Av)_{Y_+} =&(\Upsilon_2 q|A\varphi \mathrm{Op}_h(\chi_\epsilon)\varphi_1 u)_{Y_+}\\ &+(\Upsilon_2 q|A\varphi \mathrm{Op}_h(\chi-\chi_\epsilon)\varphi_1 u)_{Y_+}\\ =&:I_{h,\epsilon}+II_{h,\epsilon}.\end{aligned} \tag{1.7.8}$$

We first deal with $I_{h,\epsilon}$. Notice that from Proposition 1.13, we have

$$\limsup_{h\to 0}\|hD_y\varphi \mathrm{Op}_h(\chi_\epsilon)\varphi_1 u\|_{L^2(Y_+)}\leq C\epsilon^{1/2}.$$

We then apply Cauchy Schwartz to estimate

$$\begin{aligned}\int_0^{y_0}\|\varphi \mathrm{Op}_h(\chi_\epsilon)\varphi_1 u\|^2_{L^2(\underline{\partial} Y_+,\sqrt{G}dx)}dy &\leq Ch^{-2}\int_0^{y_0}\int\left|\int_0^y hD_y\varphi \mathrm{Op}_h(\chi_\epsilon)\varphi_1 u(s,x)ds\right|^2 dxdy\\ &\leq \frac{Cy_0^2}{h^2}\|hD_y\varphi \mathrm{Op}_h(\chi_\epsilon\varphi_1 u)\|^2_{L^2_{x,y}}.\end{aligned}$$

Choose $\theta\in(0,1/2)$, and $y_0=h\epsilon^{-\theta}$, we estimate

$$\begin{aligned}|I_{h,\epsilon}| &\leq \left(\int_0^{y_0}+\int_{y_0}^{\epsilon_0}\right)|(\Upsilon_2 q|A\varphi \mathrm{Op}_h(\chi_\epsilon)\varphi_1 u)_{L^2(\underline{\partial} Y_+,\sqrt{G}dx)}|dy\\ &\leq C\frac{1}{\epsilon^{2\theta}}(\|hD_y \mathrm{Op}_h(\chi_\epsilon)u\|^2_{L^2_{x,y}}+O(h))+Ce^{-\frac{c}{\epsilon^\theta}},\end{aligned}$$

where we have used Lemma 1.7. Note that that lemma is applicable even when the micro-local cut-off χ_{δ_0} there is matrix-valued.

In summary we have

$$\limsup_{h\to 0}|I_{h,\epsilon}|\leq C(\epsilon^{1-2\theta}+e^{-\frac{c}{\epsilon^\theta}}).$$

We now turn to the estimates of $II_{h,\epsilon}$. This can be done from geometric argument.

Let

$$S_\epsilon:=\{(y,x,\xi):r(y,x,\xi)\geq\epsilon, y\leq 4\epsilon/c_0\}\cap W_0.$$

We claim that for any ray γ with $\gamma(0)\in\Gamma_0$ and $\Gamma(s_0)\in S_\epsilon$, $\gamma|_{[0,s_0]}\cap\mathcal{G}^{2,+}=\emptyset$.

Indeed, by contradiction, assume that for some γ and $s_1\in[0,s_0]$, we have $\rho_1\gamma(s_1)\in\mathcal{G}^{2,+}$. After time s_1, along γ we have

$$\dot{y}=2\eta, \dot{\eta}=\partial_y r\geq c_0/4,$$

with $y(s_1)=\eta(s_1)=0, \eta(s_0)\geq\sqrt{\epsilon}$. This implies that $s_0-s_1\geq 4\sqrt{\epsilon}/c_0$ and $y(s_0)\geq c_0T^2/4\geq\frac{4\epsilon\delta_0}{c_0}$. The claim follows.

Now we write

$$\begin{aligned}II_\epsilon =&(\varphi_0(c_0y/\epsilon)\Upsilon_2 q|A\varphi \mathrm{Op}_h(\chi-\chi_\epsilon)\varphi_1 u)_{Y_+}\\ &+((1-\varphi_0(c_0y/\epsilon))\Upsilon_2 q|A\varphi \mathrm{Op}_h(\chi-\chi_\epsilon)\varphi_1 u)_{Y_+}.\end{aligned}$$

From the discussion above, the first term on the right hand side above tends to 0 as $h \to 0$ for any fixed $\epsilon > 0$. while the second term is controlled from above by

$$\int_{\frac{\epsilon\delta_0}{4C}}^{\epsilon_0} \int |C(y, x, hD_x)q|^2 \, dxdy$$

for some zero order pseudo-differential operator with principal symbol $c(y, x, \xi)$, and supp $c \cap \{\xi = 0\} = \emptyset$. Using Lemma 1.7, we have $\limsup_{h\to 0} |II_\epsilon| = 0$ holds for any $\epsilon > 0$. Notice that the left hand side of (1.7.5) is independent of ϵ, we have

$$\limsup_{h\to 0}((-A_1 hD_y v|hD_y v)_{\underline{\partial} Y_+} + \|\Psi_0 v + \Psi_1 hD_y v\|^2_{L^2(Y_+)}) = 0.$$

From the construction of a_0, a_1 and the corresponding expression of ψ_0, ψ_1, we can choose another different $\widetilde{a_0}, \widetilde{a_1}$, such that the function $\widetilde{\psi_0} + \widetilde{\psi_1}\eta$ is independent of $\psi_0 + \psi_1\eta$ on supp χ(see appendix D). It follows then

$$\|v\|_{L^2(Y_+)} + \|hD_y v\|_{L^2(Y_+)} = o(1), \quad h \to 0,$$

and this completes the proof. ■

1.8 Near $\mathcal{G}^{2,-}$ and $\mathcal{G}^k$ for $k \geq 3$

This section is devoted to the proof of Proposition 1.9. Before proving it, we need some preparation. In what follows, we take tangential operator

$$A = \varphi \mathrm{Op}_h(a)\varphi_1, A^* = A + O_{L^2(\underline{\partial} Y_+)}(h^2).$$

Proposition 1.16

$$\frac{1}{h}(([P, A]u|u)_{Y_+} = \frac{1}{h}(Au|Pu)_{Y_+} - \frac{1}{h}(APu|u)_{Y_+} + O(h).$$

Proof. The proof goes in exactly the same way and much simpler than the diffractive case, and we omit it here. ■

Let $r_0 = r|_{y=0}$ and $r_1 = \partial_y r|_{y=0}$. Direct calculation gives

$$H_p a = 2\eta \frac{\partial a}{\partial y} + \frac{\partial r}{\partial y}\frac{\partial a}{\partial \eta} + H_{-r} a.$$

Pick $\rho_0 \in \mathcal{G}^{2,-} \subset T^*\partial\Omega \diagdown \{0\}$ and a small neighborhood $U \subset T^*\partial\Omega \diagdown \{0\}$ of ρ_0. Let $L \subset U$ be a co-dimension 1 hypersurface containing ρ_0 in $T^*\partial\Omega$ and transversal to the vector field H_{-r_0}. For small positive numbers $\delta, \tau > 0$, define

$$L^{\pm}(\delta, \tau; \rho_0) := \{\exp(tH_{-r_0})(\rho) \in U : \rho \in L, \mathrm{dist}\,(\rho, \rho_0) \leq \delta^2, 0 \leq \pm t \leq \tau\}.$$

When there is no risk of confusion, we write it simply as $L^{\pm}(\delta,\tau)$. Define also

$$\begin{aligned} &F^{\pm}(\delta,\tau) := \{(y,x,\xi) : 0 \leq y \leq \delta^2, (x,\xi) \in L^{\pm}(\delta,\tau)\}, \\ &F(\delta,\tau) = F^{+}(\delta,\tau) \cup F^{-}(\delta,\tau). \end{aligned}$$

Let $C_1 > 0$ sufficiently large and $\delta_0 > 0, \tau_0 > 0$ sufficiently small so that $\delta < \delta_0, \tau < \tau_0$

$$|r(y,x,\xi)| \leq \frac{1}{2}C_1^2\delta^2 \tag{1.8.1}$$

in $F(\delta,\tau)$ for all $0 < \delta \leq \delta_0, 0 < \tau \leq \tau_0$. With the same constant C_1, we further define the sets

$$\begin{aligned} V^{\pm}(\delta,\tau) :=& \{(y,x,\eta,\xi) : 0 \leq y \leq \delta^2/2, (x,\xi) \in L^{\pm}(\delta,\tau)\} \\ &\cup \{(y,x,\eta,\xi) : \delta^2/2 \leq y \leq \delta^2, (x,\xi) \in L^{\pm}(\delta,\tau), |\eta| \leq C_1\delta\}, \\ W^{\pm}(\delta,\tau) :=& \{(y,x,\eta,\xi) : 0 \leq y \leq \delta^2/2, (x,\xi) \in L^{\pm}(\delta,\tau)\} \\ &\cup \{(y,x,\eta,\xi) : \delta^2/2 \leq y \leq \delta^2, (x,\xi) \in L^{\pm}(\delta,\tau), |\eta| \leq 2C_1\delta\}, \end{aligned}$$

$$V(\delta,\tau) := V^{+}(\delta,\tau) \cup V^{-}(\delta,\tau), W(\delta,\tau) = W^{+}(\delta,\tau) \cup W^{-}(\delta,\tau).$$

We need test functions constructed in [48]:

Lemma 1.19 ([48])
Let $I = [0,\epsilon_0)_y$. There exist $\sigma > 0, \delta_0 > 0, \tau_0 > 0$ small enough with $\delta \ll \sigma \ll 1$ and families of smooth functions $a_\delta \in C_c^\infty(I \times U), g_\delta, h_\delta \in C^\infty(Y_+ \times \mathbb{R}_\eta \times \mathbb{R}_\xi^{d-1} \setminus \{0\})$ for any $0 < \delta \leq \delta_0$, with the following properties:

1. $a_\delta \geq 0$, $\mathrm{supp} a_\delta \subset F^{+}(\delta,\sigma\delta) \cup F^{-}(\delta,\delta^2)$.
2. $a_\delta(0,\exp(tH_{-r_0}(\rho_0))) \neq 0, \forall 0 \leq t < \delta\sigma$.
3. *$a_\delta > 0$ on $\mathrm{supp} a_{\delta'}$ if $0 < \delta' < \delta \leq \delta_0$ and a_δ is independent of y for $0 \leq y < \delta^2/2$.*
4. $g_\delta + h_\delta = -H_p a_\delta$.
5. *in $W(\delta,\tau)$, $g_\delta \geq 0$ and $g_\delta > 0$ when $a_\delta \neq 0$.*
6. *For any $m > 1$ and any multiple index $\alpha \in \mathbb{N}^d$, $|g_\delta^{-\frac{1}{m}}\partial^\alpha g_\delta| = O_\delta(1)$, locally uniformly on $W(\delta,\tau_0)$, where the implicit constant inside $O_\delta(1)$ depends on α, m and δ.*
7. *$\mathrm{supp} h_\delta \subset I \times L^{-}(\delta,\delta^2) \times \mathbb{R}_\eta$, and $\mathrm{supp} g_\delta \cup \mathrm{supp} h_\delta \subset \mathrm{supp} a_\delta$, g_δ, h_δ are independent of η for $0 \leq y \leq \delta^2/2$.*

For the convenience of the reader, we will recall the proof in the appendix D.

According to the lemma, we have $\partial(g_\delta^{1/2}) = 2g_\delta^{-1/2}\partial g_\delta = O(1)$, this implies that $g_\delta^{1/2} \in C^\infty(W(\delta,\tau))$. Set $b_\delta := g_\delta^{1/2} \in C^\infty(W(\delta,\tau))$. Note b_δ may not be smooth and with compact support. We need split it into two parts as follows: Let $\phi_1 \in C^\infty(\mathbb{R})$ such that $\phi_1 \equiv 1$ if $0 \leq y \leq \frac{\delta^2}{4}$ and $\phi_1 \equiv 0$ if $y > \frac{3\delta^2}{8}$. Let $\phi_2 \in C^\infty(\Omega \times \mathbb{R}^d \setminus \{0\})$ with compact support in x, ξ, η variables, such that $\phi_2 \geq 0$ and $\phi_2 \equiv 0$ whenever $y \leq \frac{\delta^2}{4}$ or $|\eta| > 2C_1\delta$. Indeed, we can choose $\kappa \in C_c^\infty(\mathbb{R})$, non-negative, smooth and with compact support, such that $\kappa(z) \equiv 0$ when $|z| > 2C_1\delta$ and $\kappa(z) \equiv 1$ when $|z| \leq \frac{3}{2}C_1\delta$. Now let $\phi_2(y,x,\eta,\xi)^2 = (1-\phi_1(y)^2)\kappa(\delta^{-1}\eta)\chi_\delta(y,x,\xi)$ with $\chi_\delta|_{\mathrm{supp}(a_\delta)} \equiv 1$, $\mathrm{supp}(\chi_\delta) \subset F^+(\delta,\sigma\delta) \cup F^-(\delta,\sigma\delta)$. We observe that

$$W^{\pm}(\delta,\tau) \cap \mathrm{supp}\ (1-\phi_1^2-\phi_2^2) \subset \left\{ (y,x,\eta,\xi) : \frac{\delta^2}{4} \leq y \leq \delta^2, |\eta| > \frac{3}{2}C_1\delta, \right\}.$$

We finally put $b_{\delta,j} := \phi_j b_\delta$, $j = 1,2$. Note that $b_{\delta,1} \in C_c^\infty(F(\delta,\tau))$ is a tangential symbol (since for $y \geq \frac{\delta^2}{2}$, $\mathrm{supp}(g_\delta) \subset \mathrm{supp}(a_\delta)$ is compact) while $b_{\delta,2} \in C_c^\infty(W(\delta,\tau))$ is a usual interior symbol with compact support in $T^*\Omega$.

1.8.1 Gliding case

The propagation of support of μ near a gliding point in $\mathcal{G}^{2,-}$ can be stated as follows:

Proposition 1.17
Suppose $\rho_0 \in \mathcal{G}^{2,-}$ and $L^+(\delta_0,\tau_0) \cup L^-(\delta_0,\tau_0) \subset \mathcal{G}^{2,-}$ for some $\delta_0, \tau_0 > 0$. Then for any $\sigma > 0$ with $\sigma\delta_0 < \tau_0$, such that if

$$\{(y,x,\eta,\xi) : 0 \leq y \leq \delta^2, (x,\xi) \in L^-(\delta,\delta^2;\rho_0)\} \cap \mathrm{supp}(\mu) = \varnothing$$

for some $0 < \delta \leq \delta_0$, then $\exp(tH_{-r_0})(\rho_0) \notin \mathrm{supp}(\mu)$ for any $t \in [0,\sigma\delta)$.

We need several lemmas.

Lemma 1.20
Suppose $a \in C_c^\infty(\mathbb{R}^{2d}), b \in C^\infty([0,1], C_c^\infty(\mathbb{R}^{2(d-1)}))$ with the following support property:

$$a(y,x,\eta,\xi) \equiv 0 \text{ if } y \leq c_0 < 1 \text{ or } |\eta| > C_0|\xi|.$$

Then the usual symbolic calculus for $a(y,x,hD_y,hD_x)b(y,x,hD_x)$ still valid. In particular,

$$a(y,x,hD_y,hD_x)b(y,x,hD_x) = c(y,x,hD_y,hD_x) + O_{L^2\to L^2}(h),$$

with

$$c(y,x,\eta,\xi) = a(y,x,\eta,\xi)b(y,x,\xi)$$

We postpone the proof in the appendix F.

Lemma 1.21
Given any $\rho_1 \in \mathcal{G}$, there exists $\delta_1 > 0, \tau_1 > 0, \sigma_1 > 0$ with $\delta_1 \ll \sigma_1$ and $\sigma_1\delta_1 < \tau_1$ such that if $\rho \in T^*\partial\Omega$ and $\mathrm{dist}(\rho,\rho_1) \leq \delta^2$ for some $0 < \delta \leq \delta_1$, then $\mathrm{dist}(\gamma(s,\rho),\gamma(s,\rho_1)) \leq C\delta^2$ for $|s| \leq \sigma_1\delta$. In particular, $\gamma(s,\rho) \in W(\delta,\tau_1)$ for all $|s| \leq \sigma_1\delta$.

Proof. Write $\gamma(s,\rho)$ and $\exp(sH_{-r_0})(\rho)$ in coordinate as

$$\gamma_1(s) = (y(s), \eta(s), x(s), \xi(s)) \text{ and } \gamma_2(s) = (\tilde{y}(s), \tilde{\eta}(s), \tilde{x}(s), \tilde{\xi}(s)).$$

From $\dot{y} = 2\eta, \dot{\eta} = O(1)$, we have $y(s) \leq Cs^2$ and the same estimate holds for $\tilde{y}(s)$. Let

$$d(s) = |x(s) - \tilde{x}(s)|^2 + |\xi(s) - \tilde{\xi}(s)|^2,$$

and then $\dot{d}(s) \leq C\sqrt{d(s)}$. This implies $d(s) \leq C_1\delta^2$ for all $|s| \leq \sigma_1\delta$. By the same argument, we have $\mathrm{dist}(\exp(sH_{-r_0})(\rho), \exp(sH_{-r_0})(\rho_1)) \leq C\delta^2$. The conclusion then follows from triangle inequality. ∎

We will see the crucial role of $\rho_0 \in \mathcal{G}^{2,-}$ in the following lemma:

Lemma 1.22
Assume that δ_1, τ_1 are parameters given in the previous lemma. Suppose that $-C_0 \leq \partial_y r(\rho) \leq -c_0 < 0$ for all $\rho \in W(\delta_1,\tau_1)$. Define $S_\epsilon = W(\delta_1,\tau_1) \cap \{r \geq \epsilon, y \leq \epsilon\}$ for sufficiently small $\epsilon > 0$. Then along any ray $\gamma(s,\rho_1)$ in $W(\delta_1,\tau_1)$ with $\rho_1 \in S_\epsilon$, if $y(\gamma(-t,\rho_1)) = 0$ for some $0 \leq t \leq \tau_1$, we have $r(y(\gamma(t,\rho_1)) \geq c_1\epsilon$, where c_1 depends only on $W(\delta_1,\tau_1)$.

Proof. Assume $\rho_1 = (y_1, x_1, \eta_1, \xi_1) \in S_\epsilon$ and $\gamma(s,\rho_1) = (y(s), x(s); \eta(s), \xi(s))$. Let $s_3 = \inf\{0 \leq s \leq \tau_1 : y(-s) = 0\}$. For $s \in [-s_3, 0]$, $\dot{y} = 2\eta, -C_0 \leq \dot{\eta} = \partial_y r \leq -c_0$.

There are two possibilities. If $\eta_1 \geq \sqrt{\epsilon}$, then $\eta(-s_3^+) \geq \eta_1 > 0$ since $\dot{\eta} < 0$. Otherwise, $\eta_1 \leq -\sqrt{\epsilon}$, and we denote by $s_2 = \inf\{s \in [0, s_1] : \eta(-s) = 0\}$. From $\eta_1 = \displaystyle\int_{-s_2}^{0} \dot{\eta} ds \geq -C_0 s_2$, we have $s_2 \geq \frac{|\eta_1|}{C_0}$. Moreover,

$$y_1 - y(-s_2) = 2\eta_1 s_2 - \int_{-s_2}^{0} ds \int_{s}^{0} \ddot{y} ds' \leq 2\eta_1 s_2 + C_0 s_2^2 \leq -\frac{|\eta_1|^2}{C_0}.$$

Now from

$$y(-s_2) = y(-s_2) - y(-s_3) = -\int_{-s_3}^{-s_2} ds \int_{s}^{-s_2} \ddot{y} ds' \leq C_0|s_3 - s_2|^2,$$

we have $|s_3 - s_2|^2 \geq \dfrac{y(-s_2)}{C_0} \geq \dfrac{(y(-s_2) - y_1)}{C_0} \geq \dfrac{|\eta_1|^2}{C_0^2}$ and finally

$$\eta(-s_3^+) = -\int_{-s_3}^{-s_2} \dot{\eta} ds \geq c_0|s_3 - s_2| \geq \frac{c_0\sqrt{\epsilon}}{C_0}.$$

∎

Proof of Proposition 1.17. For any δ', we define the operator

$$N_{\delta'} = \frac{1}{ih}[P, A_{\delta'}]$$

with principal symbol $n_{\delta'} = -H_p a_{\delta'} = g_{\delta'} + h_{\delta'}$. Define operators

$$B_{\delta',j} := \mathrm{Op}_h(b_{\delta',j}), j = 1,2, \quad N_{\delta,3} = \mathrm{Op}_h((1-\phi_1^2-\phi_2^2)n_{\delta'}).$$

Write $h_{\delta',j} = \phi_j^2 h_{\delta'}, H_{\delta',j} = \mathrm{Op}_h(h_{\delta',j}), j = 1,2$. The proposition will follow if we can show that for any $\delta' < \delta$,

$$\lim_{h\to 0}\sum_{j=1}^{2}\|B_{\delta',j}u\|^2_{L^2(Y_+)} = 0 \tag{1.8.2}$$

We remark that $h_{\delta',1}, b_{\delta',1}$ are both tangential symbols while $h_{\delta',2}, b_{\delta',2}$ are interior symbols vanishing near the boundary. Observe also that $N_{\delta',3}$ is interior pseudo-differential operator with symbol vanishing near the boundary as well as on $p^{-1}(0)$, thanks to the fact that in $W(\delta',\tau)$, $|r(y,x,\xi)| \leq \frac{1}{2}C_1^2\delta'^2$. Thus $N_{\delta',3}u = o_{L^2(Y_+)}(1)$ as $h \to 0$ for $\delta' > 0$ small enough. Moreover, from the assumption on the support of μ near the original point ρ_0 we have $H_{\delta',j}u = o_{L^2_{y,x}}(1)$. Now set

$$M_{\delta',j} = \phi_j^2 N_{\delta',j} - B^*_{\delta',j}B_{\delta',j} - H_{\delta',j}, j = 1,2.$$

From symbolic calculus, we have $M_{\delta',1} = O_{L^2\to L^2}(h)$ is a tangential operator. Note that definition of $M_{\delta',2}$, we will encounter the composition of tangential operator with interior operator $\mathrm{Op}_h(\phi_2^2)$. Since ϕ_2 has support far away form $y = 0$ and $\eta = 0$, the symbolic calculus still valid thanks to Lemma 1.20. Therefore $M_{\delta',2} = O_{L^2\to L^2}(h)$ is an interior operator. Finally, we have obtained

$$N_{\delta'} = N_{\delta',3} + \sum_{j=1}^{2}(B^*_{\delta',j}B_{\delta',j} + H_{\delta',j}) + O_{L^2(Y_+)\to L^2(Y_+)}(h).$$

Combining all the analysis above and the Proposition 1.16, we have

$$\begin{aligned}\sum_{j=1}^{2}\|B_{\delta',j}u\|^2_{L^2(Y_+)} \leq\, & o(1) + \frac{2}{h}\left|\mathrm{Im}([A_\delta, hd]q|u)_{Y_+}\right| + \frac{1}{h}\left|\mathrm{Im}(q|hd^*(A_\delta u))_{Y_+}\right| \\ =\, & o(1) + |(q|\Upsilon_1 u)_{Y_+}| + |(\Upsilon_2 q|u)_{Y_+}|\end{aligned} \tag{1.8.3}$$

where Υ_1, Υ_2 are compactly supported matrix-valued tangential operators with principal symbols vanishing outside $\mathrm{supp}(a_\delta)$.

To finish the proof, we need show that the right hand side of (1.8.3) is $o(1)$ as $h \to 0$. Pick $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(s) \equiv 1$ if $0 \leq s \leq \frac{1}{2}$ and $\chi(s) \equiv 0$ is $s \geq 1$. Let $\chi_\epsilon(y,x,\xi) = \chi(\epsilon^{-1}r(y,x,\xi))$. Denote by

$$I_{h,\epsilon} = \left|(\Upsilon_1\varphi\mathrm{Op}_h(\chi_\epsilon)\varphi_1 u|q)_{Y_+}\right|, II_{h,\epsilon} = \left|(\Upsilon_1(1-\varphi\mathrm{Op}_h(\chi_\epsilon)\varphi_1)u|q)_{Y_+}\right|.$$

The treatment of $I_{h,\epsilon}$ is exactly the same as in the diffractive case, so we have

$$\lim_{\epsilon\to 0}\limsup_{h\to 0} I_{h,\epsilon} = 0.$$

For $II_{h,\epsilon}$, we may assume that the the integral of y variable is only from $[0,\epsilon]$, since for $y \geq \epsilon$ we can use the rapid decreasing of q as in the treatment of $I_{h,\epsilon}$. According to Lemma 1.21 and Lemma 1.22, the measure of $\Upsilon_1(1-\varphi\mathrm{Op}_h(\chi_\epsilon)\varphi_1)u$ vanishes, since all the backward generalized rays starting from each point in S_ϵ will enter the small neighborhood of $\rho_0 \in \mathcal{G}^{2,-}$ by at most reflection at boundary. From the propagation theorem in the hyperbolic case(Proposition 1.7), the proof of Proposition 1.17 is complete. ∎

Remark 1.4
We remark that as a consequence of Proposition 1.17, the measure of q(or $h\nabla q$) also vanishes along $\exp(tH_{-r_0})$ *for* $t \in [0,\sigma\delta)$.

1.8.2 high order contact

In this subsection we will use a new coordinate system in a neighborhood $\widetilde{W}_k$ of $\rho_k \in \mathcal{G}^k$ in $[0,\epsilon_0] \times T^*\partial\Omega$:

$$(y,\eta,x,\xi) \mapsto (y,\eta,z,\zeta), z = (z_1, z'), \zeta = (\zeta_1,\zeta')$$

with $p = \eta^2 - r, r = \zeta_1 + yr_1(z,\zeta) + O(y^2)$, $\zeta_1 = r_0$, where $r_0 = r|_{y=0}, r_1 = \partial_y r|_{y=0}$. This is possible since $d_{x,\xi}r_0 \neq 0$, if $\xi \neq 0$. Along the generalized bicharacteristic flow $\gamma(s)$, (z,ζ) satisfies

$$\dot{z} = -\partial_\zeta r(y(s),z(s),\zeta(s)), \dot{\zeta} = \partial_z r(y(s),z(s),\zeta(s)).$$

This implies that in $\widetilde{W}_k$, $-\dot{z}_1 \sim 1 > 0$, as $y \to 0$, and thus $s \mapsto z_1(s)$ is strictly decreasing. Moreover, $\dot{\zeta}_1 \sim y\partial_{z_1} r_1$, as $y \to 0$.

Suppose now $k \geq 3$, we have locally that

$$\mathcal{G}^k := \{(z,\zeta) : \zeta_1 = 0, \partial_{z_1}^l r_1(z,\zeta) = 0, \forall l \leq k-3, \partial_{z_1}^{k-2} r_1(z,\zeta) \neq 0\}.$$

Define $\Sigma_k := \{(z,\zeta) : \partial_{z_1}^{k-3} r_1(z,\zeta) = 0, \partial_{z_1}^{k-2} r_1(z,\zeta) \neq 0\}$. From implicit function theorem, Σ_k is locally a hypersurface and we can write it as

$$\Sigma_k = \{(z,\zeta) : z_1 = \Theta_k(z',\zeta)\}.$$

$\mathcal{G}^k$ can be viewed locally as a closed subset of Σ_k.

Since the map $s \mapsto z_1(s)$ is bijective, we may assume that along each ray, $z_1(0) = \Theta_k(z'(0),\zeta(0))$, and

$$z_1(s) < \Theta_k(z'(s),\zeta(s)), s > 0,$$
$$z_1(s) > \Theta_k(z'(s),\zeta(s)), s < 0.$$

We see that all the generalized rays are transversal to the codimension 2 manifold(locally) Σ_k. Moreover, a ray passes Σ_k if and only if $y(0) = 0$ and $\zeta_1(0) = 0$. Now we define the set near ρ_k,

$$\Sigma_k^{\pm} := \{(y,\eta,z,\zeta) \in \mathrm{Car}(P) \cap \widetilde{W}_k : z_1 \mp \Theta_k(z',\zeta) > 0\}.$$

Note that the gliding rays $\exp(sH_{-r_0})$ intersect transversally to Σ_k and $H_{-r_0} = -\partial_{z_1}$ inside $T^*\partial\Omega$. Thus we can re-parametrize the gliding flows by z_1. Moreover, $\Sigma_k^\pm \cap \mathcal{G}^j = \emptyset, \forall j \geq k$, provided that we choose $\widetilde{W}_k$ small enough. In other word, z_1 gives a foliation of $T^*\partial\Omega$ near Σ_k for small $|z_1 - \Theta_k(z', \zeta)|$.

The following proposition is a long time extension of Proposition 1.17, adapted to the notations introduced above.

Proposition 1.18
Suppose $\rho_0 \in \mathcal{G}^{2,-}$ near $\rho_k \in \Sigma_k$ with coordinate (z, ζ), $z_1 > \Theta_k(z', \zeta)$. Then there exists $\delta_0 > 0$, sufficiently small such that if

$$\{(y, x, \eta, \xi) : 0 \leq y \leq \delta^2, (x, \xi) \in L^-(\delta, \delta^2; \rho_0)\} \cap \mathrm{supp}(\mu) = \varnothing$$

for $0 < \delta \leq \delta_0$, then $\exp(sH_{-r_0})(\rho_0) \notin \mathrm{supp}(\mu)$ for any $s < z_1 - \Theta_k(z', \zeta)$.

In other words, each generalized ray, issued from gliding set outside $\mathrm{supp}(\mu)$ does not carry any singularity until it touches some point in $\mathcal{G}^k$ for $k \geq 3$.

Proof. The proof is purely topological. For each $\rho_0 = (z, \zeta) \notin \mathrm{supp}(\mu)$ and $z_1 > 0$, let $s_1 := \sup\{s : s \leq z_1 - \Theta_k(z', \zeta), \exp(s' H_{-r_0}) \notin \mathrm{supp}\ (\mu), \forall s' \in [0, s)\}$. The existence of s_1 is guaranteed by Proposition 1.17. It remains to show that $s_1 = z_1 - \Theta_k(z', \zeta)$. By contradiction, suppose $s_1 < z_1 - \Theta(z', \zeta)$, then the point $\rho_1 = (z_1 - s_1, z', \zeta)$ is in $\mathcal{G}^{2,-}$. We can apply Proposition 1.17 again to obtain that for some small $\sigma_1 > 0$, $\exp(\theta\sigma H_{-r_0})(\rho_1) \notin \mathrm{supp}\ (\mu)$ for any $\theta \in [0, 1]$. This is a contradiction of the choice of s_1. ∎

As a consequence, we have

Corollary 1.2
Suppose $\rho_0 \in \mathcal{G}^{2,-}$ and $\rho_0 \notin \mathrm{supp}(\mu)$. Let $\gamma(s)$ be the generalized ray passing ρ_0 with $\gamma(0) = \rho_0$. Then $\gamma(s) \notin \mathrm{supp}(\mu)$ for any $s \in [-s_0, s_0]$, provided that $\gamma|_{[-s_0, s_0]} \subset \mathcal{G}^{2,-}$.

Combining the analysis near a diffractive point and a gliding point, we have already established the $k-$propagation property for $k = 2$. We will argue by induction to prove $k-$propagation property for all $k \geq 3$. To this end, we need an intermediate step. Let us first introduce a notation

$$\Gamma(\rho_0; \delta) := \{(y, x; z, \zeta) : 0 \leq y \leq \delta^2, (z, \zeta) \in L^-(\delta, \delta^2; \rho_0)\}$$

and a definition

Definition 1.5 (k-pre-propagation property)
For $k \geq 2$, we say that k-pre-propagation property holds, if the following statement is true:

For any $\rho_k \in \mathcal{G}^k$, there exists a neighborhood V_k of ρ_k in $T^*\partial\Omega$, and $\delta_k > 0, \sigma_k > 0, \delta_k\sigma_k \ll$, depending on V_k, such that for any $\rho_0 \in \big(\mathcal{G}^{2,-} \cup \bigcup_{3 \leq j \leq k} \mathcal{G}^j\big) \cap V_k$, if $\Gamma(\rho_0; \delta) \cap \mathrm{supp}(\mu) = \emptyset$ for some $0 < \delta < \delta_k$, then $\exp(sH_{-r_0})(\rho_0) \notin \mathrm{supp}(\mu)$ for all $s \in [0, \sigma_k\delta)$.

The key step is the following inductive proposition.

Proposition 1.19
Suppose $k \geq 3$ and $(k-1)$propagation property holds true, then k-pre-propagation property also holds true.

We do some preparation before proving this proposition. Select a neighborhood W_k of $\rho_k \in \mathcal{G}^k$ in $T^*\partial\Omega$ (and contained in $\widetilde{W_k}$) with compact closure such that in W_k such that $|\partial_{z_1}^{k-2} r_1(\rho)| \geq c_0 > 0$ for all $\rho \in \overline{W_k}$. By abusing the notation, we will refer $\mathcal{G}^k$ to be $\mathcal{G}^k \cap W_k$ in the sequel. According to the asymptotic behaviour of the flow $\exp(sH_{-r_0})$ as $s \to 0$, we have for any given $(z_1 = \Theta_k(z_0', \zeta_0), z_0', \zeta_0) \in \mathcal{G}^k$,

$$r_1 \circ \exp(sH_{-r_0})(z_0', \zeta_0) = b_k(z_0', \zeta_0)s^{k-2} + O(s^{k-1}),$$

where $b_k \neq 0$ can be viewed as a function of points in $\mathcal{G}^k$. From compactness, we can choose $\sigma > 0, \theta > 0$ depending only on $\overline{W}_k$ such that for all $\rho \in \mathcal{G}^k$,

$$|b_k(\rho)| \geq \theta > 0, |r_1 \circ \exp(s^{H_{-r_0}})(\rho)| \geq \frac{1}{2}|b_k s^{k-2}|, \quad \forall s \in [-\sigma, 0) \cup (0, \sigma].$$

Now we define a smaller neighborhood V_k of ρ_k such that for any $\rho_0 \in V_k$, and $\delta_k > 0, \sigma_k > 0$, $\exp(sH_{-r_0})(L^{\pm}(\delta_k, \delta_k^2; \rho_0)) \subset W_k$ for all $|s| \leq \sigma_k\delta_k$. Moreover, $|r_1 \circ \exp(sH_{-r_0})(\rho_0)| \leq \delta_k$. We also put $\widetilde{W_k} = [0, \delta_k^2] \times W_k$, $\widetilde{V_k} = [0, \delta_k^2] \times V_k$.

Choosing a cut-off $\tilde{a}_\delta \in C_c^\infty$ with $\tilde{a}_\delta \equiv 1$ near ρ_k, we define

$$S_{\delta,\epsilon} := \text{ supp } (\tilde{a}_\delta) \cap \{y \leq \epsilon, r \geq \epsilon\}$$

for any $0 < \epsilon \ll \delta$. Note that near $S_{\delta,\epsilon}$ (thus near $\rho_k \in \mathcal{G}^k, k \geq 3$) we have $|r_1| \leq \delta_k$, and this implies that $\zeta_1 \gtrsim \epsilon$, near $S_{\delta,\epsilon}$.

We divide the proof of Proposition 1.19 into several lemmas.

Lemma 1.23
Given any generalized ray $\gamma(s) = (y(s), \eta(s), \underline{z}(s), \zeta(s))$ with $\gamma(s_0) \in \Gamma(\rho_0; \delta) \cap \mathcal{G}^{2,-}$ and $\gamma(s_1) \in S_{\delta,\epsilon}$. Assume that $\gamma|_{[s_0,s_1]} \subset \mathrm{Car}(P) \cap \widetilde{W}_k$, then $\gamma(s) \notin \mathcal{G}^k$ for all $s \in [s_0, s_1]$.

Proof. Take $\Gamma^+(\rho_0; \delta) := \Gamma(\rho_0; \delta) \cap \Sigma_k^+$ and identify points in $\Sigma_k^{\pm}$ as their projection to (y, x, ξ). Let F_k(may be empty) be the union of generalized rays issued from $\Gamma^+(\rho_0; \delta)$ which meet $\mathcal{G}^k$. Note that along both real trajectories $\gamma(s)$ and $\exp(sH_{-r_0})$, $s \mapsto z_1$ is strictly decreasing, it suffices to show that $F_k \cap S_{\delta,\epsilon} \subset \Sigma_k^+$ since generalized rays intersect with Σ_k transversally,.

We argue by contradiction. Assume that some ray in F_k satisfies $\gamma(s_0) \in \Gamma^+(\rho_0; \delta)$, $\gamma(0) \in \mathcal{G}^k$, and $\gamma(s_1) \in S_{\delta,\epsilon}$ for $s_0 < 0 < s_1$. Write $\exp(sH_{-r_0})(\gamma(0)) = (\tilde{z}(s), \tilde{\zeta}(s))$, and

$$r_1 \circ \exp(sH_{-r_0})(z'(0), \zeta(0)) = r_1(\tilde{z}(s), \tilde{\zeta}(s)) = b_k s^{k-2} + O(s^{k-1}), s \to 0,$$

More precisely, we have

$$|b_k(z'(0),\zeta(0))| \geq \theta > 0, |r_1(\tilde{z}(s),\tilde{\zeta}(s))| \geq \frac{1}{2}|b_k s^{k-2}|, \forall s \in [-\sigma,0)\cup(0,\sigma].$$

After shrinking support of a_δ if necessary, we may assume that $s_1 < \sigma$. According to the parity of k and the sign of b_k, there are several situations.

If $b_k < 0$, then no longer k is $\gamma(s) \in \mathcal{G}^{2,-}$ for all $s \in (0,\sigma)$. This is impossible since $r \circ \gamma(s_1) \geq \epsilon$. Otherwise $b_k > 0$, in this case we have $r_1(\tilde{z}(s),\tilde{\zeta}(s)) \geq b_k s^{k-2}/2$, for all $s \in (0,\sigma)$, and

$$\begin{aligned}
&(\partial_{z_1} r_1)(\tilde{z}(s),\tilde{\zeta}(s)) \\
=&(\partial_{z_1} r_1)\circ \exp(sH_{-r_0}(z'(0),\zeta(0)) \\
=&-\partial_s\left(r_1 \circ \exp(sH_{-r_0}(z'(0),\zeta(0))\right) \\
=&-(k-2)b_k s^{k-3} + O(s^{k-2}) \leq 0, \forall s \in [0,\sigma),
\end{aligned} \tag{1.8.4}$$

thanks to $\partial_{\zeta'} r|_{y=0} = \partial_z r|_{y=0} = 0$. Taking the difference with real trajectory $\gamma(s) = (y(s),\eta(s);z(s),\zeta(s))$, we have

$$\begin{aligned}
&\partial_{z_1} r_1(y(s),z(s),\zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s),\tilde{\zeta}(s)) \\
=&(\partial_{z_1} r_1(0,z(s),\zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s),\zeta(s))) + (\partial_{z_1} r_1(\tilde{z}(s),\zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s),\tilde{\zeta}(s))) \\
&+(\partial_{z_1} r_1(y(s),z(s),\zeta(s)) - \partial_{z_1} r_1(0,z(s),\zeta(s)))
\end{aligned}$$

Using the fact that $(z(0),\zeta(0)) = (\tilde{z}(0),\tilde{\zeta}(0))$ and $y(s) = O(s^2)$, we have

$$\partial_{z_1} r_1(y(s),z(s),\zeta(s)) - \partial_{z_1} r_1(\tilde{z}(s),\tilde{\zeta}(s)) = O(s).$$

This together with (1.8.4) imply that

$$\begin{aligned}
&\dot{\zeta}_1 \leq y\partial_{z_1} r_1(y(s),z(s),\zeta(s)) + C_0 y^2 \leq C_0(y^2+ys), \dot{y} = 2\eta, \\
&\eta^2 = \zeta_1 + y r_1(z,\zeta) + O(y^2), (\zeta_1(0),y(0)) = (0,0),
\end{aligned} \tag{1.8.5}$$

where the constant C_0 and the implicit constant inside the big O only depend on supp a_δ.

Applying the formula $H_p^k y(0) = 2(H_{-r_0})^{k-2} r_1 = 2b_k(k-2)! > 0$ and Taylor expansion, we have

$$\begin{aligned}
y(s) &= \frac{2b_k}{k(k-1)} s^k + O(s^{k+1}) \geq \frac{b_k}{k(k-1)} s^k, s \in (0,\sigma), \\
\dot{y}(s) &= \frac{2b_k}{k-1} s^{k-1} + O(s^k) > \frac{b_k}{k-1} s^{k-1} > 0, s \in (0,\sigma).
\end{aligned} \tag{1.8.6}$$

Injecting in (1.8.5), we have $\dot{\zeta}_1(s) \leq C_0(\epsilon^2 + \epsilon s)$ for all $s > 0$ small as long as $y(s) \leq \epsilon$ and $\gamma(s) \notin S_{\delta,\epsilon}$. For these s,

$$\zeta_1(s) \leq C_0(\epsilon^2 s + \epsilon s^2/2).$$

Setting $s_2 = \inf\{0 \leq s \leq s_1 : \gamma(s) \in S_{\delta,\epsilon}\}$, we know that along the flow, $2\sqrt{\epsilon} = 2\eta(s_2) = \dot{y}(s_2)$, and this implies that $s_2 \sim \epsilon^{\frac{1}{2(k-1)}}$ since $y(s) > \epsilon$ if $s > \epsilon^{\frac{1}{2(k-1)}}$.

In summary, we have

$$\epsilon \leq r \circ \phi_{s_2} \leq 2C_0\epsilon^{1+\frac{1}{2(k-1)}} + \delta_k\epsilon + C_1\epsilon^2.$$

However, this contradicts to $r = \zeta_1 + yr_1 + O(y^2)$, provided that $\delta_k < 1$, $\epsilon \ll \delta_k < 1$.

■

Lemma 1.24
The conclusion of Proposition 1.19 holds if $\rho_0 \in \mathcal{G}^{2,-}$

Proof. Adapting the notations and argument in the proof of Proposition 1.17, we have

$$\sum_{j=1}^{2} \|B_{\delta,j}u\|^2_{L^2(Y_+)} \leq o(1) + \frac{2}{h}\left|\mathrm{Im}([A_\delta, hd]q|u)_{Y_+}\right| + \frac{1}{h}\left|\mathrm{Im}(q|hd^*(A_\delta u))_{Y_+}\right|. \qquad (1.8.7)$$

The goal is to show that the last two terms on the right hand side tend to 0 as $h \to 0$.

We denote by $\tilde{\gamma}(s)$ the gliding ray $\exp(sH_{-r_0})$ such that $\tilde{\gamma}(s_0) = \rho_0$ for some $s_0 < 0$. Suppose $\tilde{\gamma}(0) = \rho \in \mathcal{G}^k$ for some $k \geq 3$ and $\tilde{\gamma}(s) \in \mathcal{G}^{2,-}$ for $s \in (s_0, 0)$. In view of Corollary **??**, we may assume that ρ_0 is close enough to ρ, and $|s_0|$ is small. Pick $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi(s) \equiv 1$ if $0 \leq s \leq \frac{1}{2}$ and $\chi(s) \equiv 0$ if $s \geq 1$. For any $\epsilon > 0$, let $\chi_\epsilon(y, x, \xi) = \chi(\epsilon^{-1}r(y, x, \xi))$. Let

$$I_{h,\epsilon} = \frac{2}{h}\left|(hd^*(\varphi\mathrm{Op}_h(\chi_\epsilon)\varphi_1 u)|q)_{Y_+}\right|, II_{h,\epsilon} = \frac{2}{h}\left|(hd^*(1 - \varphi\mathrm{Op}_h(\chi_\epsilon)\varphi_1)u|q)_{Y_+}\right|.$$

The treatment of $I_{h,\epsilon}$ is exactly the same as in the diffractive case, we have

$$\lim_{\epsilon\to 0}\limsup_{h\to 0} I_{h,\epsilon} = 0.$$

For $II_{h,\epsilon}$, we only concern about the integral from $[0, \epsilon]$ in y variable.

From Lemma 1.23, any ray entering $S_{\delta,\epsilon}$ can at most pass $\mathcal{G}^j$ for $j < k$. Applying $(k-1)$ propagation property, we deduce that for any cut-off φ_ϵ with $\mathrm{supp}(\varphi_\epsilon) \subset S_{\delta,\epsilon}$, $\mathrm{supp}\,(\varphi_\epsilon) \cap \mathrm{supp}(\mu) = \emptyset$. Therefore

$$\lim_{h\to 0} II_{h,\epsilon} = 0$$

for any $\epsilon > 0$. This completes the proof.

■

Lemma 1.25
The conclusion of Proposition 1.19 holds if $\rho_0 \in \mathcal{G}^j$ *for some* $3 \leq j \leq k$.

Proof. Taking a micro-local cut-off $\psi(y,x,\xi)$ with support near ρ_0, we have

$$\|\varphi \mathrm{Op}_h(\psi)\varphi_1 u\|_{L^2(Y_+)} = o(1)$$

from the assumption that $\rho_0 \notin$ supp (μ). Note that along the flow of H_{-r_0} and on supp $(1-\psi)\cap V_k$ we have $|r_1(0,x,\xi)| \geq c(\psi,\delta) > 0$. Hence from Corollary 1.2, if $\exp(tH_{-r_0})(\rho_0) \in \mathcal{G}^{2,-}$ for all $t \in (0,\sigma\delta)$, and then $\exp(tH_{-r_0})(\rho_0) \notin \mathrm{supp}(\mu)$. Otherwise $\exp(tH_{-r_0})(\rho_0) \in \mathcal{G}^{2,+}$ for all $t \in (0,\sigma\delta)$, we claim that we still have $\exp(tH_{-r_0})(\rho_0) \notin \mathrm{supp}(\mu)$ from geometric consideration.

Indeed, by considering the backward generalized ray, we conclude that for any $s_0 \in (0,\sigma_k\delta)$, there exists $\rho \in \widetilde{W_k}$, so that $\gamma(s_0,\rho) = \exp(s_0 H_{-r_0})(\rho_0)$ where $\gamma(s,\rho)$ is the generalized ray issued from ρ. From this fact we must have $\gamma(s,\rho) \notin \mathcal{G}^k$ for $s \in (0,s_0)$, since any ray intersecting with $\mathcal{G}^k$ will enter $T^*\Omega$ or $\mathcal{G}^{2,-}$ immediately, provided that the neighborhood W_k is chosen to be small enough.

Therefore, to conclude, we only need to show that

$$\rho \in \{(y,z,\zeta) : 0 \leq y \leq \delta^2, |(z,\zeta)-\rho_0| \leq \delta^2\}.$$

We will prove this by comparing two flows $\exp(sH_{-r_0})(\rho_0) = (\tilde{z}(s),\tilde{\zeta}(s))$ and $\gamma(s,\rho) = (y(s),\eta(s),z(s),\zeta(s))$. Taking the difference the two dynamics, we have

$$\frac{d}{ds}\left(z_1(s)-\tilde{z}_1(s)\right) = -\,\partial_{\zeta_1} r(y(s),z(s),\zeta(s)) + \partial_{\zeta_1} r(0,\tilde{z}(s),\tilde{\zeta}(s)) = O(y(s)),$$

$$\frac{d}{ds}(z'(s)-\tilde{z}'(s)) = O(y(s)),\ \frac{d}{ds}(\zeta(s)-\tilde{\zeta}(s)) = O((y(s)),\ \frac{dy}{ds} = 2\eta(s).$$

Note that $|\eta|^2 = |r| = O(1)$ and $y(s_0) = 0, \tilde{z}(s_0) = z(s_0), \tilde{\zeta}(s_0) = \zeta(s_0)$, we have

$$y(s) \leq C(s-s_0)^2 \text{ for all } s \in [0,s_0].$$

Hence $y(0) \leq C\sigma_k^2\delta^2 < \delta^2$, provided that $\sigma_k^2 < 1/C$. Moreover,

$$|(z(0),\zeta(0))-\rho_0| \leq Cs_0^3 \leq C\sigma_k^3\delta^3 \leq \delta^2.$$

■

Proposition 1.20
Suppose that (k-1)-propagation property holds. Then k-pre-propagation property implies k-propagation property.

Proof. Up to re-parameter the flow, we may assume that $\rho_0 \in \mathcal{G}^k$ and $\gamma(s)$ is the generalized ray such that $\gamma(0) = \rho_0$. We also denote $\gamma(s)$ by $\gamma(s,\rho_0)$ in view of flow map. Suppose $\gamma(s_0) \notin$supp μ for some $s_0 < 0$ and $\phi|_{[s_0,0)}\cap$ supp $(\mu) = \emptyset$. Our goal is to show that $\rho_0 \notin$supp μ. Let $\sigma_{k-1} > 0$ be the required length in the definition of $(k-1)-$ propagation property.

Let $\delta_k > 0, \sigma_k > 0$ and V_k, neighborhood of $\rho_0 \in \mathcal{G}^k$ in $T^*\partial\Omega$ and $\widetilde{V}_k$, neighborhood of ρ_0 in $[0, \epsilon_0] \times T^*\partial\Omega$, as in the definition of k-pre-propagation property which satisfy the conditions in the paragraph in front of Lemma 1.23. Note in particular that we have $V_k \cap \mathcal{G}_j = \emptyset$ for all $j > k$. Without loss of generality, we may assume that $|s_0| < \min\{\sigma_{k-1}, \sigma_k\}$ and $\gamma(s_0) \in \widetilde{V}_k$, since otherwise we can choose $s_0' < 0, |s_0'|$ small enough and replace $\gamma(s_0)$ by $\gamma(s_0')$.

Let $\Gamma_0 \subset \widetilde{V}_k$ be a neighborhood of $\gamma(s_0)$ so that $\Gamma_0 \cap$ supp$(\mu) = \emptyset$. For $\delta_1 > 0$ small with $\delta_1 \ll \sigma_k$, we set $\rho_1 = \exp\left(-\frac{\sigma_k\delta_1}{2} H_{-r_0}\right)(\rho_0)$ and define

$$U_{\delta_1} := \{\rho = (y, \eta, z, \zeta) \in \mathrm{Car}(P) : 0 \leq y \leq \delta_1^2, |(z, \zeta) - \rho_1| \leq \delta_1^2\}.$$

From continuous dependence of the generalized bi-characteristic flow, we have

$$U_{\delta_1} \subset \gamma(s_0, \Gamma_0), \text{provided that } \delta_1 \text{ small enough }.$$

Now we claim that for possibly smaller $\delta_1 > 0$, we have

$$\gamma(s_1, U_{\delta_1}) \cap \bigcup_{j \geq k} \mathcal{G}^j = \emptyset, \forall s_1 \in (s_0, 0).$$

Indeed, it suffices to prove that $\gamma(s_1, U_{\delta_1}) \cap \mathcal{G}^k = \emptyset$ since there are no point of $\mathcal{G}^j$ in $\widetilde{V}_k$ for $j > k$. First from transversal intersection between the flow $\exp(sH_{-r_0})$ and Σ_k, we deduce that at ρ_1, $z_1 > \Theta_k(z', \zeta)$. By choosing δ_1 smaller, there is some constant $\epsilon_1 > 0$, such that for all $\rho \in U_{\delta_1}$, $z_1 > \Theta_k(z', \zeta) + \epsilon_1$ holds. In particular, $U_{\delta_1} \subset \Sigma_k^+$. We calculate

$$\begin{aligned} \frac{d}{ds}\Theta_k(z'(s), \zeta(s)) =& \frac{\partial\Theta_k}{\partial z'}\frac{dz'}{ds} + \frac{\partial\Theta_k}{\partial\zeta}\frac{d\zeta}{ds} \\ =& -\frac{\partial\Theta_k}{\partial z'}\frac{\partial r}{\partial\zeta'} + \frac{\partial\Theta_k}{\partial\zeta}\frac{\partial r}{\partial z}. \end{aligned}$$

Note that in $\widetilde{V}_k$, we can write $r = \zeta_1 + yr_1(z, \zeta) + O(y^2)$, hence

$$\frac{d}{ds}\Theta_k(z'(s), \zeta(s)) = O(y(s)).$$

Next we argue by contradiction, assume that for some $s_1 \in (s_0, 0)$ and $\rho \in U_{\delta_1}$ we have $\gamma(s_1, \rho) \in \mathcal{G}^k$ and $\gamma(s, \rho) \notin \mathcal{G}^k$ for any $0 > s > s_1$. In this case we have $|y(s)| \leq C|s - s_1|$ for all $s \in [s_1, 0]$. Therefore we must have

$$\left|\frac{d}{ds}\Theta_k(z'(s), \zeta(s))\right| \leq C|s - s_1|.$$

Combining with $\dot{z}_1 \sim -1$, we have

$$\begin{aligned}\Theta_k(z'(s_1),\zeta(s_1)) \leq & \Theta_k(z'(0),\zeta(0)) + C\int_{s_1}^{0} |s-s_1| ds \\ < & z_1(0) + Cs_1^2 \\ = & z_1(s_1) + \int_{s_1}^{0} \frac{dz_1}{ds} ds + Cs_1^2 \\ \leq & z_1(s_1) - C_1|s_1| + Cs_1^2 \\ \leq & z_1(s_1),\end{aligned}$$

provided that $|s_0|$ is chosen to be small enough. This implies that $\gamma(s_1,\rho) \in \Sigma_k^+$, which is a contradiction.

From $(k-1)$-propagation property, we know that $U_{\delta_1} \cap \operatorname{supp}(\mu) = \emptyset$. Therefore, applying k-pre-propagation property with respect to ρ_1 and U_{δ_1}, we deduce that $\rho_0 \notin \operatorname{supp}(\mu)$, and this completes the proof. ■

1.A Proof of Lemma 1.2

Proof of Lemma 1.2. The first assertion follows from $h\mathrm{div}u = 0$ and Dirichlet boundary condition, while we apply a multiplier method to prove the second. From the geometric assumption on Ω, we can find a vector field $L \in C^1(\overline{\Omega})$ such that $L|_{\partial\Omega} = \nu$(see [50], page 36). In global coordinate system, we write $L = L_j(x)\partial_{x_j}$. By using the equation, we have

$$\begin{aligned}\int_\Omega Lu\cdot f dx = & \int_\Omega Lu\cdot(-h^2\Delta u - u + h\nabla q)dx, \\ -\int_\Omega Lu\cdot u dx = & -\int_\Omega L_j(x)\partial_{x_j}u^i u^i dx \\ = & -\int_\Omega \partial_{x_j}\left(L_j(x)u^i\right)u^i dx + \int_\Omega \operatorname{div} L(x)|u|^2 dx \\ = & \int_\Omega L_j(x)u^i(x)\partial_{x_j}u^i dx + \int_\Omega \operatorname{div} L(x)|u|^2 dx \\ = & \int_\Omega Lu\cdot u dx + \int_\Omega \operatorname{div} L(x)|u|^2 dx,\end{aligned}$$

and thus $\int_\Omega Lu \cdot u dx = -\frac{1}{2}\int_\Omega \text{div } L(x)|u|^2 dx = O(1)$. Next we calculate

$$\begin{aligned} h\int_\Omega Lu \cdot \nabla q dx &= -h\int_\Omega u^i \partial_{x_j}\left(L_j \partial_{x_i} q\right) dx \\ &= -h\int_\Omega u \cdot L(\nabla q) dx - h\int_\Omega (\text{div } L(x)) u \cdot \nabla q dx \\ &= -h\int_\Omega u \cdot [L, \nabla] q dx - h\int_\Omega \text{div } L(x) u \cdot \nabla q dx \\ &= O(1), \end{aligned}$$

$$\begin{aligned} -h^2\int_\Omega Lu^i \Delta u^i dx = &- h^2\int_{\partial\Omega} \left|\partial_\nu u^i\right|^2 d\sigma + h^2\int_\Omega \nabla L(\nabla u^i, \nabla u^i) dx \\ &+ h^2\int_\Omega L_j(x)\partial^2_{x_j x_k} u^i \partial_{x_k} u^i \\ = &- h^2\int_{\partial\Omega} \left|\partial_\nu u^i\right|^2 d\sigma + h^2\int_\Omega \nabla L(x)(\nabla u^i, \nabla u^i) dx \\ &+ h^2\int_\Omega \partial_{x_j}\left(L_j \partial_{x_k} u^i\right)\partial_{x_k} u^i dx - h^2\int_\Omega \text{div } L(x)\nabla u^i \cdot \nabla u^i(x) dx, \end{aligned}$$

$$h^2\int_\Omega \partial_{x_j}\left(L_j \partial_{x_k} u^i\right)\partial_{x_k} u^i dx = h^2\int_{\partial\Omega} L\cdot\nu\left|\partial_\nu u^i\right|^2 d\sigma - h^2\int_\Omega L_j(x)\partial_{x_k} u^i \partial^2_{x_j x_k} u^i dx,$$

$$-h^2\int_\Omega Lu^i \Delta u^i dx = -\frac{h^2}{2}\int_{\partial\Omega}\left|\partial_\nu u^i\right|^2 d\sigma + \int_\Omega \nabla L(x)(h\nabla u^i, h\nabla u^i) dx - \frac{h^2}{2}\int_\Omega \text{div } L(x)|\nabla u^i|^2 dx.$$

Observing that $\int_\Omega Lu \cdot f dx = o(1)$, we have

$$\int_{\partial\Omega} |h\partial_\nu u|^2 d\sigma = O(1).$$

■

1.B Standard elliptic theory

The differential operator is given by

$$P_h = \text{Op}_h(\eta^2 + \lambda(y, x', \xi')^2 - 1 + hm(y, x', \eta, \xi')),$$

where the principaly symbol $p = \eta^2 + \lambda^2 - 1$ is scalar while m is matrix valued. When micro-locally near the region $p > 0$, we want to construct the parametrix of the inverse of P. Denote by U the turbulent neighborhood (two sided) of $\partial\Omega$. Take $\varphi \in C_c^\infty(U), \chi_0 \in$

$C^\infty(\mathbb{R}^{d-1})$ and the support of φ is contained in a coordinate patch near the boundary. We put

$$E^0 := \mathrm{Op}_h\left(\frac{\chi_0(\xi')\varphi(y,x')}{\eta^2+\lambda(y,x',\xi')^2-1}\right),$$

and we define matrix valued pdo E^l, $l\geq 1$ inductively via

$$\begin{aligned} E^1\times p &= -\sum_{|\alpha|=1}\frac{1}{i}\partial^\alpha_{\xi',\eta}E^0\times\partial^\alpha_{x',y}p - E^0\times m,\\ E^n\times p &= -\sum_{|\alpha|+k=n,k\neq n}\frac{1}{i^{|\alpha|}}\partial^\alpha_{\xi',\eta}E^k\times\partial^\alpha_{x',y}p - \sum_{|\alpha|+k=n-1}\frac{1}{i^{|\alpha|}}\partial^\alpha_{\xi',\eta}E^k\times\partial^\alpha_{x',y}m. \end{aligned} \tag{1.B.1}$$

For any $N\in\mathbb{N}$, we set

$$E_N=\sum_{j=0}^N h^jE^j,$$

and then

$$E_N\circ P=\chi_0(hD_{x'})\varphi(y,x')\mathrm{Id}+R_N,\quad \|R_N\|_{L^2\to L^2}=O(h^{N+1}).$$

Proposition 1.21
The sequence of solution (u_k) is h_k-oscillating in the following sense:

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int_{|\xi|\geq Rh_k^{-1}}|\widehat{\varphi u_k}(\xi)|^2d\xi=0,\forall\psi\in C_c^\infty(\Omega),$$

$$\lim_{R\to\infty}\limsup_{k\to\infty}\int_0^{\epsilon_0}dy\int_{|\xi'|\geq Rh_k^{-1}}|\widehat{\varphi u_k}(y,\xi')|^2d\xi'=0,\forall\psi\in C_c^\infty(\overline{\Omega}),$$

where in the second formula, the support of φ is contained in some local coordinate patch near the boundary, and the Fourier transform is only taken for the x' direction.

Proof. We drop the subindex k in the proof. For the first formula, one can use the equation of u to obtain

$$(-h^2\Delta-1)(\varphi u)=g=O_{L^2}(1),$$

and

$$\int_{|\xi|\geq Rh^{-1}}|\widehat{\varphi u}(\xi)|^2d\xi\leq\int_{h|\xi|\geq R}\frac{|\widehat{g}(\xi)|^2}{|h^2|\xi|^2-1|^2}d\xi\leq\frac{C}{(R^2-1)^2}.$$

For the second formula, it will be sufficient to show that

$$\lim_{R\to\infty}\limsup_{k\to\infty}\left\|\left(1-\chi\left(\frac{hD_{x'}}{R}\right)\right)(\varphi u)\right\|_{L^2}=0$$

for some $\chi\in C_c^\infty(-1,1)$. We write $w=u\mathbf{1}_{y\geq 0},\tilde{g}=g\mathbf{1}_{y\geq0},v=h\partial_yu|_{y=0}=O_{L^2(y=0)}(1)$. We apply the parametric construction above with $\chi_0(\xi')=1-\chi\left(\frac{\xi'}{R}\right)$. Let $e_N(y,x',\eta,\xi')$ be the symbol of the operator E_N, which is meromorphic in η with poles $\eta_0^\pm = \pm i\sqrt{\lambda^2(y,x',\xi')^2-1}$. Moreover,

$$|\partial^\alpha e_N(y,x',\eta,\xi')| \leq \frac{C_{N,\alpha}}{R}. \tag{1.B.2}$$

We take $\widetilde{\varphi}$ be a slight enlargement of φ such that $\widetilde{\varphi}\varphi=\varphi$. Then

$$E_N\circ\widetilde{\varphi}Pw = \left(1-\chi\left(\frac{hD_{x'}}{R}\right)\right)(\varphi w) + R_N w.$$

From the jump formula, $Pw=\widetilde{g}+2hv(x')\otimes\delta_{y=0}$. We have

$$\left(1-\chi\left(\frac{hD_{x'}}{R}\right)\right)(\varphi w) = E_N\left(\widetilde{\varphi}\widetilde{g}+\widetilde{\varphi}2hv\otimes\delta_{y=0}\right)+O_{L^2_{y,x'}}(h^{N+1}).$$

From symbolic calculus,

$$\|E_N(\widetilde{\varphi}\widetilde{g})\|_{L^2_{y,x'}} \leq \sum_{|\alpha|\leq Cd}\sup_{(y,x',\eta,\xi')\in\mathbb{R}^{2d}}|\partial^\alpha e_N(y,x',\eta,\xi')|+Ch,$$

and it vanishes after the limit procedure, thanks to (1.B.2). We next express

$$E_N(\widetilde{\varphi}2hv\otimes\delta_{y=0}) = \frac{\pi}{(2\pi h)^{d-1}}\int_{\mathbb{R}^{2(d-1)}} e^{\frac{i(x'-z')\cdot\xi'}{h}}\omega_N(y,x',\xi')\widetilde{\varphi}(0,z')v(z')dz'd\xi',$$

with

$$\omega_N(y,x',\xi')=\int_{-\infty}^{\infty}e_N(y,x',\eta,\xi')e^{\frac{iy\eta}{h}}d\eta.$$

To calculate ω_N for $y>0$, we deform the contour of integral in η in the half plane $\operatorname{Im}\eta>0$. From Residue formula, we have

$$\omega_N(y,x',\xi') = 2\pi i\operatorname{Res}(e_N(y,x',\eta,\xi');i\eta_0^+)e^{\frac{iy\eta_0^+}{h}}.$$

The principal symbol of ω_N is given by

$$\pi\exp\left(-\frac{yQ(y,x',\xi')}{h}\right)\frac{\varphi(y,x')\left(1-\chi\left(\frac{\xi'}{R}\right)\right)}{2Q(y,x',\xi')},\quad Q(y,x',\xi')=\sqrt{\lambda(y,x',\xi')^2-1}.$$

Therefore

$$\begin{aligned}&\limsup_{h\to 0}\|E_N(\widetilde{\varphi}2hv\otimes\delta_{y=0})\|_{L^2_{y,x'}}\\ \leq& C_{N,d}\int_0^\infty\sum_{|\alpha|\leq Cd}\sup_{(x',\xi')}|\partial^\alpha_{x',\xi'}\omega_N(y,x',\xi')|\|v\|_{L^2_{x'}}dy\leq\frac{C}{R},\end{aligned}$$

where we have used the point-wise estimate

$$|\partial^\alpha_{x',\xi'}\omega_N(y,x',\xi')|\leq\frac{C_\alpha e^{-\frac{y\sqrt{R^2-1}}{2Rh}}}{\sqrt{R^2-1}}\left(1+\left(\frac{y}{h}\right)^{|\alpha|}\right).$$

■

Given $\chi(y, x', \xi') \in C_c^\infty([0, \epsilon_0) \times \mathbb{R}^{2d-2})$, the following proposition can be deduced in the same manner.

Proposition 1.22
Let $w_k = \chi(y, x', hD_{x'})(\varphi u_k), \underline{w_k} = \mathbf{1}_{y\geq 0} u_k$. *Then for* $\chi_1 \in C_c^\infty(\mathbb{R}), 0 \leq \chi_1 \leq 1, \chi_1 = 1$ *near* 0, *we have*

$$\lim_{R\to\infty} \limsup_{k\to\infty} \left\| \left(1 - \chi_1\left(\frac{h_k D_y}{R}\right)\right) \underline{w_k} \right\|_{L^2(\mathbb{R}^d)} = 0.$$

Proof. We have $P\underline{w} = \underline{g} + 2hv(x') \otimes \delta_{y=0}$ with $\|\underline{g}\|_{L^2_{y,x'}} = O(1), \|v\|_{L^2_{x'}} = O(1)$. Note that the functions $\underline{g}, \underline{v}$ here may not coincide with the functions in the proof of Proposition 1.3. We define

$$E_N = \sum_{j=0}^{N} h^j E^j + R_N, \quad \|R_N\|_{L^2 \to L^2} = O(h^{N+1}),$$

with $E^0 = \mathrm{Op}_h\left(\dfrac{\widetilde{\varphi}(y, x')\left(1 - \chi_1\left(\frac{\eta}{R}\right)\right)}{\eta^2 + \lambda(y, x', \xi')^2 - 1}\right)\mathrm{Id}$ and $E^l, l \geq 1$ as in (1.B.1). This implies that

$$\left(1 - \chi_1\left(\frac{hD_y}{R}\right)\right)(\varphi\underline{w}) = E_N(\widetilde{\varphi}\underline{g} + 2h\widetilde{\varphi}v \otimes \delta_{y=0}) - R_N\underline{w}.$$

Consequently, we have

$$\lim_{R\to\infty} \limsup_{h\to 0} \|E_N(\widetilde{\varphi}\underline{g})\|_{L^2_{y,x'}} = 0.$$

$$\begin{aligned}
&E_N(2h\widetilde{\varphi}v \otimes \delta_{y=0})(y, x') \\
=&\frac{\pi}{(2\pi h)^{d-1}} \int_{\mathbb{R}^{2(d-1)}} e^{\frac{i(x'-z')\xi'}{h}} a(y, x', \xi')\widetilde{\varphi}(0, z')v(z')dz'd\xi', \\
&\text{with } a(y, x', \xi') = \int_{-\infty}^{\infty} e^{\frac{iy\eta}{h}} e_N(y, x', \eta, \xi')d\eta.
\end{aligned}$$

Observe that

$$\sup_{(x',\xi')} |\partial^\alpha_{x',\xi'} e_N(y, x', \eta, \xi')| \leq \frac{C_\alpha\left(1 - \chi_1\left(\frac{\eta}{R}\right)\right)}{(1 + \eta^2)(1 + y^2)},$$

and this implies that

$$\lim_{R\to\infty} \limsup_{h\to 0} \|E_N(2h\widetilde{\varphi}v \otimes \delta_{y=0})\|_{L^2_{y,x'}} = 0.$$

∎

1.C Proof of technical results in section 3

Proof of Lemma 1.6. The proof can be reduced to the point-wise estimate of the solution $F(y)$ of the ODE:

$$-h^2\frac{d^2F}{dy^2}+\lambda(y)^2F(y)=G(y), F(0)=F'(0)=0,$$

with $0<c_1\leq\lambda(y)^2\leq c_2$, $G\in C^\infty([0,\infty))$, and $|G(y)|\leq Ce^{-\frac{cy}{h}}$ for all $y\geq 0$. By rescaling $z=\frac{y}{h}$, it reduces to prove the exponential decay of the solution W of the ODE:

$$-\frac{d^2W}{dz^2}+V(z)W(z)=g(z), W(0)=W'(0)=0,$$

with $0<c_1\leq V(z)\leq c_2$, $g\in C^\infty([0,\infty))$ and $|g(z)|\leq Ce^{-cz}$ for all $z\geq 0$.

For this, we first notice that W is smooth and in $H^s(\mathbb{R}_+)$ for all $s\geq 0$. To prove the exponential decay, we pick $\theta_\epsilon(z)=e^{\frac{2\delta_0 z}{1+\epsilon z}}$ with $\delta_0>0$ to be chosen later. One observe easily that $0<\theta_\epsilon'(z)\leq 2\delta_0\theta_\epsilon(z)$ for $z\geq 0$. We multiply by $\theta_\epsilon W$ to the both sides of the equation and integrate it, then

$$\int_0^\infty\left((\theta_\epsilon W)'W'+\theta_\epsilon VW^2\right)dz=\int_0^\infty\theta_\epsilon WGdz.$$

Notice that $\theta_\epsilon' W'W\geq -2\delta_0\theta_\epsilon|W||W'|\geq -\delta_0\theta_\epsilon(|W|^2+|W'|^2)$, by choosing

$$\delta_0<\min\left\{\frac{c_1}{4},\frac{1}{4},\frac{c}{2}\right\},$$

we have that

$$\int_0^\infty\theta_\epsilon\left(|W'|^2+|W|^2\right)dz\leq 2\|Ge^{2\delta_0 z}\|_{L^2(\mathbb{R}_+)}\|W\|_{L^2(\mathbb{R}_+)},$$

thanks to Cauchy-Schwartz and the fact that $\theta_\epsilon\leq e^{2\delta_0 z}$, uniformly in ϵ. From dominating convergence theorem, we have $We^{\delta_0 z}\in L^2(\mathbb{R}_+)$ and $W'e^{\delta_0 z}\in L^2(\mathbb{R}_+)$. Finally, from theory of elliptic regularity, we have that $We^{\delta_0 z}\in L^\infty(\mathbb{R}_+)$,$W'e^{\delta_0 z}\in L^\infty(\mathbb{R}_+)$. ■

Proof of Proposition 1.4. We choose $\varphi_2\in C_c^\infty(Y_+)$ such that

$$\varphi_1|_{\mathrm{supp}(\varphi_2)}=1, \varphi_2|_{\mathrm{supp}(\varphi)}=1.$$

We first claim that

$$\varphi_2\mathrm{Op}_h(\chi_{\delta_0}A_j)(\varphi_{1,0}q_0)=O_{L^2(\mathbb{R}^d_+)}(1). \tag{1.C.1}$$

Indeed, we can write

$$\varphi_2\mathrm{Op}_h(A_j\chi_{\delta_0})(\varphi_{1,0}q_0)=\varphi_2\mathrm{Op}_h(A_j)\varphi_1\varphi\mathrm{Op}_h(\chi_{\delta_0})(\varphi_{1,0}q_0)+h\varphi_2\mathrm{Op}_h(B_j\widetilde{\chi}_{\delta_0})(\varphi_{1,0}q_0)$$

with $B_j \in \mathcal{E}_\partial^{-j}$ and $\widetilde{\chi}_{\delta_0}$ has similar support property as χ_{δ_0}. Thus from symbolic calculus, we have for each fixed $y > 0$,

$$\|\varphi_2 \mathrm{Op}_h(A_j(y)\chi_{\delta_0}(y))(\varphi_{1,0}q_0)\|^2_{L^2(\mathbb{R}^{d-1})} \leq C_j e^{-\frac{c_j y}{h}} h^{-1} \left(1 + \left(\frac{y}{h}\right)^{n_j}\right),$$

thanks to Lemma 1.5. Integrating for $y > 0$ yields (1.C.1).

By taking $\mathrm{supp}(\chi_{\delta_0})$ small such that $\varphi_2\chi_{\delta_0} = \chi_{\delta_0}$, we have that

$$\varphi_2 \left(\mathrm{Op}_h(\chi_{\delta_0})(\varphi_1 q) - \mathrm{Op}_h(\chi_{\delta_0}A)(\varphi_{1,0}q_0)\right) = w + O_{L^2(\mathbb{R}^d_+)}(h)$$

with

$$w = \varphi_2 \left(\mathrm{Op}_h(\chi_{\delta_0})(\varphi_1 q) - \mathrm{Op}_h(\chi_{\delta_0})\varphi_1\varphi_2\mathrm{Op}_h(A)(\varphi_{1,0}q_0)\right)$$

Lemma 1.5 implies that $w = O_{L^2(\mathbb{R}^d_+)}(1)$ and $hD_y w = O_{L^2(\mathbb{R}^d_+)}(1)$. The trace of w satisfies

$$w|_{y=0} = \varphi_{2,0}\mathrm{Op}_h(\chi_{\delta_0}(1 - \psi_{\delta_0}(\lambda_0)))(\varphi_{1,0}q_0) = O_{H^\infty(\mathbb{R}^{d-1})}(h^\infty).$$

w satisfies the equation (we use $\varphi_2 = \varphi_1\varphi_2$ here)

$$P_0 w = \varphi_1 \left[P_0, \varphi_2\mathrm{Op}_h(\chi_{\delta_0})\right] \left(\varphi_1 q - \varphi_2\mathrm{Op}_h(A)(\varphi_{1,0}q_0)\right) + O_{H^\infty(\mathbb{R}^d)}(h^\infty). \tag{1.C.2}$$

Notice that $\varphi_1 q - \varphi_2\mathrm{Op}_h(A)(\varphi_{1,0}q_0) = O_{L^2(\mathbb{R}^d_+)}(1)$, micro-locally for $\lambda \geq \frac{\delta_0}{2}$, the right hand side of (1.C.2) is equal to $O_{L^2(\mathbb{R}^d_+)}(h)$ as well as $O_{H^1(\mathbb{R}^d_+)}(1)$. Multiply by $w = \varphi_1 w$ to the both sides of (1.C.2) and integrate it, we have

$$\int_0^\infty \int_{\mathbb{R}^{d-1}} w(y,x')P_0 w dx'dy = O(h).$$

The left hand side can be integrated by part as

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^{d-1}} wP_0 w dx'dy = &- \int_0^\infty \int_{\mathbb{R}^{d-1}} |h\partial_y w|^2 dx'dy - \int_{\mathbb{R}^{d-1}} h^2 (w\partial_y w)|_{y=0} dx' \\ &- \int_0^\infty \int_{\mathbb{R}^{d-1}} \sum_{1\leq j,k\leq d-1} g^{jk} h\partial_j w h\partial_k w dx'dy + O(h).\end{aligned}$$

This implies that

$$\|hD_{y,x'}w\|_{L^2(\mathbb{R}^d_+)} + \|w\|_{L^2(\mathbb{R}^d_+)} = O(h^{1/2}).$$

Using this smallness and redo the integrating by part argument, we can improve each bound in the procedure above and obtain that

$$\|hD_{y,x'}w\|_{L^2(\mathbb{R}^d_+)} + \|w\|_{L^2(\mathbb{R}^d_+)} = O(h^{3/4}).$$

To conclude, we observe that

$$\begin{aligned}\varphi h\partial_y w = &\varphi h\partial_y \mathrm{Op}_h(\chi_{\delta_0})(\varphi_1 q) - \varphi h\partial_y\mathrm{Op}_h(\chi_{\delta_0})\varphi_2\mathrm{Op}_h(A)(\varphi_{1,0}q_0) \\ = &\varphi\mathrm{Op}_h(\chi_{\delta_0})h\partial_y(\varphi_1 q) - \varphi\mathrm{Op}_h(\chi_{\delta_0})\varphi_2\mathrm{Op}_h(h\partial_y A)(\varphi_{1,0}q_0) \\ &+ O_{L^2(\mathbb{R}^d_+)}(h) \\ = &\varphi\mathrm{Op}_h(\chi_{\delta_0})h\partial_y(\varphi_1 q) - \varphi\mathrm{Op}_h(\chi_{\delta_0}\lambda A)(\varphi_{1,0}q_0) \\ &+ O_{L^2(\mathbb{R}^d_+)}(h),\end{aligned} \tag{1.C.3}$$

where we have used symbolic calculus and Lemma 1.5 several times. Plugging into (1.C.2), we have that $P_0 w = O_{L^2(\mathbb{R}^d_+)}(h)$, $w|_{y=0} = O_{H^\infty(\mathbb{R}^{d-1})}(h^\infty)$. We decompose $w = w_1 + w_2$ with $P_0 w_1 = P_0 w, w_1|_{y=0} = 0$ and $P_0 w_2 = 0, w_2|_{y=0} = w|_{y=0}$. From elliptic regularity of boundary value problem, we have $h^2 w_1 = O_{H^2(\mathbb{R}^d_+)}(h)$ and $h^2 w_2 = O_{H^2(\mathbb{R}^d_+)}(h^\infty)$ and thus $h\partial_y w = O_{H^1(\mathbb{R}^d_+)}(1)$. From interpolation, we have $h\partial_y w = O_{H^{\frac{2}{3}}(\mathbb{R}^d_+)}(h^{\frac{1}{4}})$. Observe that the error terms on the right hand side of (1.C.3) can be also bounded by $O_{H^1(\mathbb{R}^d_+)}(1)$ and thus $O_{H^{\frac{2}{3}}(\mathbb{R}^d_+)}(h^{\frac{1}{4}})$ by interpolation. This completes the proof.

■

1.D Construction of test functions

We first give the detailed construction of $a = a_0 + a_1\eta$ used in the first step of the proof of Proposition1.15, which closely follows from [28].

Proof of Lemma 1.18 . : Given $\chi_1 \in C_c^\infty(-2,2)$ with $\chi_1|_{(-1,1)} = 1$ and $\chi_2 \in C_c^\infty(-3,3)$ such that $\chi_2|_{(-2,2)} = 1$. Consider $\chi_0(t) = e^{-1/t}\mathbf{1}_{t>0}$ with smoothness. We work in the local coordinate (y, x, ξ), and assume $(0, x_0, \xi_0) \in \mathcal{G}^{2,+}$ with $|\xi_0| \sim 1$. Set $\phi = \phi_0 + \phi_1\eta$ with

$$\phi_1(y,x,\xi) = \frac{1}{|\xi|}, \phi_0(y,x,\xi) = y^2 + |x-x_0|^2 + |\xi-\xi_0|^2.$$

We calculate

$$H_p\phi = \eta\left(2\partial_y\phi_0 - \{r,\phi_1\}\right) + \phi_1\partial_y r - \{r,\phi_0\} \geq 2c > 0$$

provided that $|\eta| \leq c_0$ for some $c_0 > 0$ and W_0 is choosing small enough such that $\frac{\partial r}{\partial y} \geq 4c$ in it. The positivity then follows from the direct calculation:

$$\begin{aligned}\{r,\phi_0\} =& 2\partial_\xi r\cdot(x-x_0) - 2\partial_x r\cdot(\xi-\xi_0),\\ \partial_y\phi_0 =& 2y, \{r,\phi_1\} = \partial_x r\cdot\frac{\xi}{|\xi|^3}.\end{aligned}$$

We next take

$$f(y,x,\eta,\xi) := \chi_2\left(\frac{\phi_0}{\delta}\right)^2\chi_0\left(1-\frac{\phi}{\delta}\right).$$

The desired functions a_0, a_1 are chosen to be the remainder when f is divided by $p = \eta^2 - r(y,x,\xi)$ thanks to the Malgrange preparation theorem:

$$f(y,x,\eta,\xi) = (\eta^2 - r(y,x,\xi))g(y,x,\eta,\xi) + a_1(y,x,\xi)\eta + a_0(y,x,\xi).$$

On the support of f, we observe that

$$\phi_0(y,x,\xi) = |(y,x,\xi)-(0,x_0,\xi_0)|^2 \leq 3\delta, \frac{\eta}{|\xi|} + \phi_0 \leq \delta,$$

which implies $\eta \leq \delta|\xi|$. Moreover, on supp $f \cap \mathrm{supp}\partial\chi_2(\delta^{-1}\phi_0)$, we have $\phi_0 \geq 2\delta, \phi_0+\phi_1\eta \leq \delta$, and these imply $\eta \leq -\delta|\xi|$, hence $r(y,x,\xi) = \eta^2 \geq \delta^2|\xi|^2$, when $p = \eta^2 - r = 0$.

Direct calculation yields

$$H_p f + fM|\xi| + \psi^2 = \chi_0\left(1 - \frac{\phi}{\delta}\right) H_p\left(\chi_2\left(\frac{\phi_0}{\delta}\right)^2\right) - \left(1 - \chi_1\left(\frac{\eta}{\delta|\xi|}\right)^2\right) N,$$

with

$$N = \chi_2\left(\frac{\phi_0}{\delta}\right)^2\left(\chi_0'\left(1-\frac{\phi}{\delta}\right)\frac{H_p\phi}{\delta} - \chi_0\left(1-\frac{\phi}{\delta}\right)M|\xi|\right) \in C^\infty, \psi = \chi_1\left(\frac{\eta}{\delta|\xi|}\right)N^{1/2}.$$

Here $N \geq 0$ on supp(ψ) if we choose $\delta > 0$ small enough. Observe that when $\eta = r^{1/2} \geq 0$, we have $\chi_0(1-\delta^{-1}\phi)H_p\left(\chi_2\left(\delta^{-1}\phi_0\right)^2\right) = 0$, $\left(1-\chi_1\left(\frac{\eta}{\delta|\xi|}\right)^2\right)N = 0$. We then define a function

$$\varphi(y,x,\xi) = -\frac{\chi_0\left(1-\frac{\phi}{\delta}\right)H_p\left(\chi_2\left(\frac{\phi_0}{\delta}\right)^2\right) - \left(1-\chi_1\left(\frac{\eta}{\delta|\xi|}\right)^2\right)N}{2r^{1/2}}|_{\eta=-r^{1/2}}\mathbf{1}_{r(y,x,\xi)>0}$$

and then

$$H_p f + fM|\xi| + \psi^2 = \varphi(\eta - r^{1/2}), \text{when } p = \eta^2 - r = 0.$$

Therefore, on $p = 0$, we have

$$H_p a + aM|\xi| + \psi^2 = \varphi(\eta - r^{1/2}).$$

It is left to check the smoothness of functions φ, ψ and ρ. Indeed, on the support of ψ,$|\phi_1\eta| \leq 2\delta, \phi_0 \leq 3\delta$, and then$1 - \frac{\phi}{\delta} \leq 3$. Notice that $\frac{\chi_0(t)}{\chi_0'(t)} = t^2$, we have

$$N^{1/2} = \chi_2\left(\frac{\phi_0}{\delta}\right)\sqrt{\chi_0'\left(1-\frac{\phi}{\delta}\right)\frac{H_p\phi}{\delta}}G\left(\frac{M|\xi|\delta}{H_p\phi}, 1-\frac{\phi}{\delta}\right) \in C^\infty,$$

since the function $G(a,t) = \sqrt{1-at^2} \in C^\infty$ for $t \leq 3, |a| \ll 1$. This implies that $\psi \in C^\infty$ provided that δ is chosen small enough.

For φ, the smoothness comes from the fact that on the support of

$$\chi_0\left(1-\frac{\phi}{\delta}\right)H_p\left(\chi_2\left(\frac{\phi_0}{\delta}\right)^2\right)|_{\eta^2=r} - \left(1-\chi_1\left(\frac{\eta}{\delta|\xi|}\right)^2\right)N|_{\eta^2=r},$$

we have $r \geq \delta^2|\xi|^2$. Moreover, φ has compact support.

Finally, from the definition of a, we have

$$a_1(y,x,\xi) = \frac{f(y,x,\eta,\xi) - f(y,x,-\eta,\xi)}{2\eta}|_{\eta=\sqrt{r(y,x,\xi)}} \in C_c^\infty,$$

$$a_0(y,x,\xi) = \frac{f(y,x,\eta,\xi)+f(y,x,-\eta,\xi)}{2}\Big|_{\eta=\pm\sqrt{r(y,x,\xi)}} \in C_c^\infty.$$

we deduce that $\frac{\partial f}{\partial \eta} = -\frac{\phi_1}{\delta}\chi_2\left(\frac{\phi_0}{\delta}\right)^2\chi_0'\left(1-\frac{\phi}{\delta}\right) < 0$, hence

$$\frac{f(y,x,\eta,\xi)-f(y,x,-\eta,\xi)}{2\eta} = \frac{1}{2}\int_{-1}^{1}\frac{\partial f}{\partial \eta}(y,x,s\eta,\xi)ds < 0.$$

Define

$$t(y,x,\xi) = \left(-\frac{1}{2}\int_{-1}^{1}\frac{\partial f}{\partial \eta}(y,x,s\eta,\xi)ds\Big|_{\eta=\sqrt{r(y,x,\xi)}}\right)^{1/2},$$

one can show that t is a smooth function with compact support.

The last observation is that $a = f > 0$ on $p = 0$, hence $s = f^{1/2}|_{p=0} \in C_c^\infty$.

We give some more calculations: Let $\psi_0 = \psi|_{y=0}, \psi_1 = \frac{\partial \psi}{\partial \eta}|_{y=0}$, when $\eta = r = 0$. Thus at (x_0,ξ_0),

$$\begin{aligned}
t(x_0,\xi_0) =& \sqrt{\frac{\chi_0'(1)\phi_1(x_0,\xi_0)}{\delta}},\\
\psi_0(x_0,\xi_0)^2 =& \chi_0'(1)\frac{\partial r}{\partial y}(0,x_0,\xi_0)\frac{\phi_1(0,x_0,\xi_0)}{\delta} - \chi_0(1)M|\xi| > 0,\\
2\psi_0\psi_1(x_0,\xi_0) =& -\chi_0''(1)\frac{\partial r}{\partial y}(x_0,\xi_0)\frac{\phi_1(x_0,\xi_0)^2}{\delta^2} - \chi_0'(1)\{r,\phi_1\}(0,x_0,\xi_0)\frac{1}{\delta}\\
&+ \frac{\chi_0'(1)M\phi_1(0,x_0,\xi_0)}{\delta} > 0, \text{for } \delta \text{ small enough since } \chi_0''(1) < 0.
\end{aligned}$$

Observe that near (x_0,ξ_0), we have $\frac{\psi_1}{\psi_0} \sim -\frac{\phi_1(x_0,\xi_0)\chi_0''(1)}{2\chi_0'(1)\delta}$, provided that δ is small enough. Now if we make a different choice of $\widetilde{\delta} > 0$, the difference between two ratios $\frac{\psi_1}{\psi_0}$ and $\frac{\widetilde{\psi}_1}{\widetilde{\psi}_0}$ is non-zero. This implies that we can choose a further cut-off χ near $(0,x_0,\xi_0)$ such that $\|\varphi\mathrm{Op}_h(\chi)\varphi_1 u\|_{L^2(Y_+)} = o(1)$ and $\|\varphi\mathrm{Op}_h(\chi)hD_y\varphi_1 u\|_{L^2(Y_+)} = o(1)$ from $\|\varphi\mathrm{Op}_h(\psi_0 + \psi_1\eta)\varphi_1 u\|_{L^2(Y_+)} = o(1)$ and $\|\varphi\mathrm{Op}_h(\widetilde{\psi}_0 + \widetilde{\psi}_1\eta)\varphi_1 u\|_{L^2(Y_+)} = o(1)$. ■

Next we recall the proof of Lemma 1.19, which is essentially given in [48].

Proof of Lemma 1.19. : From the transversal property, we can choose a new coordinate (s,t) in U such that $\rho_0 = (0,0)$ and $H_{-r_0} = \partial_t$ in this coordinate.

Step 1. Consider the function $\chi(u) = e^{\frac{1}{u-3/4}}\mathbf{1}_{u<3/4}$. It is easy to check that χ is smooth and non-increasing with the property:

$$\partial^N\chi(u) = O((-\chi')^{1/m}), \forall N \in \mathbb{N}, m > 1, \text{locally uniformly}.$$

Step 2. Next we choose $\beta \in C^\infty(\mathbb{R})$ such that $\beta \geq 0$ vanishing on $(-\infty,-1)$ and be strictly increasing on $(-1,-\frac{1}{2})$ and be equal to 1 on $(-\frac{1}{2},\infty)$. We modify β such that

$$\partial^N\beta = O(\beta^{1/m}), \forall N \in \mathbb{N}, m > 1, \text{locally uniformly}.$$

Step 3. Choose $f \in C^\infty(\mathbb{R})$ so that f vanishes on $(-\infty, 1/2)$ and is strictly increasing and convex on $(1/2, \infty)$ with $f(1) > 1$.

Now we set

$$a_\delta = \beta\left(\frac{3t}{4\delta^2}\right)\chi\left(\frac{t}{\sigma\delta} + \frac{|s|^2}{\delta^4} + f\left(\frac{y}{\delta^2}\right)\right),$$

and

$$g_\delta = -\beta\left(\frac{3t}{4\delta^2}\right)H_p(\chi(u))$$

with $u = \dfrac{3t}{4\sigma\delta} + \dfrac{|s|^2}{\delta^4} + f\left(\dfrac{y}{\delta^2}\right)$. Finally we define $h_\delta = -H_p a_\delta - g_\delta$. Note that

$$\text{supp } (a_\delta) = \left\{(y, s, t) : -\delta^2 \leq t, \frac{t}{\sigma\delta} + \frac{|s|^2}{\delta^4} + f\left(\frac{y}{\delta^2}\right) \leq \frac{3}{4}\right\},$$

hence it is clear that (1)(2)(3)(4) in Lemma 1.19 are satisfied.

Since $r = r_0 + O(y)$, when H_{-r} acts on functions independent of η we have

$$H_p = \partial_t + O(y)\partial_s + O(y)\partial_t + O(\delta)\partial_y,$$

since $|\eta| = O(\delta)$. Therefore, we have

$$\begin{aligned}
-g_\delta &= \beta\left(\frac{t}{\delta^2}\right)\left(\left(\frac{1}{\delta\sigma} + O(\delta^2)\frac{|s|}{\delta^4} + O(\delta^2)\frac{1}{\delta\sigma} + O(\delta)\frac{1}{\delta^2}\right)\chi'(u) + O(1)\chi(u)\right) \\
&= \beta\left(\frac{t}{\delta^2}\right)\left(\frac{3}{4\delta\sigma} + O\left(\frac{\delta}{\sigma}\right) + O\left(\frac{1}{\delta}\right) + O(1)))\chi'(u)\right) \\
&\sim \beta\left(\frac{t}{\delta^2}\right)\frac{1}{\delta\sigma}\chi'(u).
\end{aligned}$$

provided that $\delta, \sigma \ll 1$. In the calculation above, we have used the fact that $\chi(u) = O(\chi'(u))$ on the support of g_δ. Thus (5) in Lemma 1.19 follows.

(6) follows from the construction of χ.

To check (7), we observe that supp $g_\delta \cup$supp h_δ $\subset$supp a_δ. Moreover, from the construction, g_δ, h_δ is independent of η whenever $0 \leq y < \frac{\delta^2}{2}$. Finally, to check the support of h_δ, we write $h_\delta = -H_p\left(\beta\left(\frac{t}{\delta^2}\right)\right)\chi(u)$. Since β is independent of y, η, we have $H_p\left(\beta\left(\frac{t}{\delta^2}\right)\right) = H_{-r}\left(\beta\left(\frac{t}{\delta^2}\right)\right)$, which is supported on $I \times L^-(\delta, \delta^2) \times \mathbb{R}_\eta$, thanks to supp $\beta' \subset [-1, -\frac{1}{2}]$. ∎

1.E Proof of Lemma 1.15

Lemma 1.26
In local coordinate Y_+, we have

$$P_h = -h^2\frac{\overline{g}}{\sqrt{G}}\frac{\partial}{\partial y}\left(\sqrt{G}\overline{g}^{-1}\frac{\partial}{\partial y}\right) + R_h = h^2D_y^2 + \text{Op}_h(r) + O_{L^2\to L^2}(h).$$

Moreover, R_h is a matrix-valued second order differential operator in x with scalar principal symbol $r(y,x,\xi) = 1-\lambda(y,x,\xi)^2$, which is self-ajoint with respect to the $(\cdot|\cdot)_{L^2(Y_+)}$.

Proof. Denote by $y = x^0$, $\partial_0 = \partial_y, \partial_j = \partial_{x_j}, j = 1,2,\cdots d-1$. Let $u \in \Lambda^1(Y_+)$ and $w \in \Lambda^2(Y_+)$ written in the form

$$u = u_0 dx^0 + u_j dx^j, \quad w = w_{0j}dx^0 \wedge dx^j + w_{jk}dx^j \wedge dx^k.$$

We have from direct calculation that

$$\begin{aligned} du =&(\partial_0 u_j - \partial_j u_0)dx^0 \wedge dx^j + \partial_k u_j dx^j \wedge dx^k, \\ d^*u =&-\frac{1}{\sqrt{G}}\partial_0(u_0\sqrt{G}) - \frac{1}{\sqrt{G}}\partial_j(g^{jk}u_k\sqrt{G}), \\ d^*w =&\frac{1}{\sqrt{G}}\partial_k(w_{0j}g^{jk}\sqrt{G})dx^0 \\ &-\frac{1}{\sqrt{G}}g_{jl}\left(\partial_0(w_{0k}g^{kl}\sqrt{G}) + \partial_m(\sqrt{G}w_{pk}(g^{pl}g^{km}-g^{pm}g^{kl})\right)dx^j \end{aligned}$$

From direct calculation, $h^2\Delta_H u = h^2(dd^*+d^*d)u = v_0 dx^0 + v_j dx^j + R_h u$ with

$$\begin{aligned} v_0 =&-h^2\partial_0^2 u_0 - h^2\frac{\partial_0(\sqrt{G})}{\sqrt{G}}\partial_0 u_0, \\ v_j =&-h^2\partial_0^2 u_l - \frac{h^2}{\sqrt{G}}g_{jl}\partial_0(g^{kl}\sqrt{G})\partial_0 u_k \end{aligned}$$

and the $R_h u$ consists only the tangential derivatives ∂_j. Hence in the matrix form,

$$v = L_h u := -h^2\frac{\overline{g}}{\sqrt{G}}\frac{\partial}{\partial y}\left(\sqrt{G}\overline{g}^{-1}\frac{\partial u}{\partial y}\right).$$

Moreover, one easily verified that $L_h^* = L_h$, thus $R_h^* = R_h$. ■

1.F Proof of Lemma 1.20

Proof of Lemma 1.20. For our need, it suffices to prove the last assertion. We first let $A_h = a(y,x,hD_y,hD_x)$ and $B_h = b(y,x,hD_x)$, then

$$A_h B_h u(y,x) = \frac{1}{(2\pi h)^d}\iint e^{\frac{i(x-x')\xi + i(y-y')\eta}{h}}\varphi(y',y,x,\eta,\xi)u(y',x')dy'dx'd\xi d\eta,$$

where

$$\varphi(y',y,x,\eta,\xi) = \frac{1}{(2\pi h)^{d-1}}\iint e^{\frac{i(x-z)(\xi'-\xi)}{h}}a(y,x,\eta,\xi')b(y',z,\xi)d\xi' dz.$$

Talor expansion gives

$$\varphi(y',y,x,\eta,\xi)=\varphi(y,y,x,\eta,\xi)+(y'-y)\int_0^1\partial_{y'}\varphi(ty'+(1-t)y,y,x,\eta,\xi)dt.$$

Denote by $c(y,x,\eta,\xi)=\varphi(y,y,x,\eta,\xi)$, it is obvious that c is an interior symbol, since it can be viewed as a tangential symbol for fixed η, and we have $(1+|\xi|)^m\lesssim(1+|\xi|+|\eta|)^m$ for all $m\in\mathbb{R}$ on the support of c, thanks to the supporting property of a. Now we note $C_h=c(y,x,hD_y,hD_x)$, and write $A_hB_hu=C_hu+R_h'u$, where

$$\begin{aligned}R_h'u(y,x)&=\int_0^1dt\frac{1}{(2\pi h)^d}\iint e^{\frac{i(x-x')\xi+i(y-y')\eta}{h}}(y-y')\partial_{y'}c_t(y',y,x,\eta,\xi)u(y',x')dy'dx'd\xi d\eta\\&=ih\int_0^1dt\frac{1}{(2\pi h)^d}\iint e^{\frac{i(x-x')\xi+i(y-y')\eta}{h}}\partial_\eta\partial_{y'}c_t(y',y,x,\eta,\xi)u(y',x')dy'dx'd\xi d\eta\\&=:ih\int_0^1C_tu(y,x)dt.\end{aligned}$$

with $c_t(y',y,x,\eta,\xi)=\varphi(ty'+(1-t)y,y,x,\eta,\xi)$. Notice that

$$\partial_\eta\partial_{y'}c_t(y',y,x,\eta,\xi)=\frac{1}{(2\pi h)^{d-1}}\iint e^{\frac{i(x-z)(\xi'-\xi)}{h}}\partial_\eta a(y,x,\eta,\xi')(\partial_yb)(ty'+(1-t)y,z,\xi)d\xi'dz.$$

We need to be careful here since ∂_yb only exists for $y>0$ and at the point $y=0$, the right derivative $(\partial_y^m)^+b(0):=\lim_{y\to0^+}\partial^mb(y)$ exists for any order m. Since we are dealing with Dirichlet boundary condition, we always apply a tangential operator $B(y,x,hD_x)$ to functions $u(y,x)$ with $u|_{y=0}=0$ in the trace sense. We could thus extend $u(y',x')$ by $u(y',x')\mathbf{1}_{y'\geq0}$in y' in the expression of the form

$$\frac{1}{(2\pi h)^d}\iint e^{\frac{i(x-x')\xi'+i(y-y')\eta}{h}}\varphi(y',y,x,\eta,\xi)u(y',x')dy'dx'd\xi d\eta.$$

Therefore, we have

$$\sup_{y,y'\geq0,0<t<1,z,\xi}|\partial_z^\alpha\partial_\xi^\beta b(ty'+(1-t)y,z,\xi)|\leq C_{m,\alpha,\beta},\forall m\in\mathbb{N},\alpha,\beta\in\mathbb{N}^{d-1}.$$

Now it is reduced to prove the uniform L^2 boundness of the operator

$$T_hu=\int_{\mathbb{R}^d}K_h(y',x',y,x)u(y',x')dy'dx',$$

with kernel

$$K_h(y',x',y,x)=\frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d}e^{\frac{i(x-x')\xi+i(y-y')\eta}{h}}H_t(y',y,x,\eta,\xi)d\eta d\xi$$

where

$$H_t(y', y, x, \eta, \xi) = \mathbf{1}_{y', y \geq 0} \frac{1}{(2\pi h)^{d-1}} \int e^{\frac{iz\zeta}{h}} a_1(y, x, \eta, \xi + \zeta) b_1(ty' + (1-t)y, x - z, \zeta) dz d\zeta.$$

From Schur test lemma, we need to show

$$\sup_{(y,x) \in \mathbb{R}^d_+} \int_{\mathbb{R}^d_+} |K_h(y', x', y, x)| dy' dx' \leq C_1 < \infty,$$
$$\sup_{(y',x') \in \mathbb{R}^d_+} \int_{\mathbb{R}^d_+} |K_h(y', x', y, x)| dy dx \leq C_2 < \infty,$$

with C_1, C_2 independent of h and t.

We first define

$$k_h(y, x, w, v) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iv\xi + iw\eta} H_t(y - hw, y, x, \eta, \xi) d\eta d\xi,$$

hence,

$$T_h u(y, x) = \frac{1}{h^d} \int_{\mathbb{R}^d_+} k_h \left(y, x, \frac{y - y'}{h}, \frac{x - x'}{h} \right) u(y', x') dy' dx'.$$

Notice that $H_t(y', y, x, \eta, \xi)$ are tangential symbol, parametrized by (y', y, η). Moreover, it is compactly supported in (y, x, η, ξ) variables, uniformly in the first variable y'. Thus, $\partial_\eta^m \partial_\xi^\alpha H_t(y - hw, y, x, \eta, \xi)$ has compact support in (η, ξ) and

$$|\partial_\eta^m \partial_\xi^\alpha H_t(y - hw, y, x, \eta, \xi)| \leq C_{m,\alpha}$$

for any $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{d-1}$. Thus, integrating by part in the expression of k_h yields

$$\sup_{(y,x)} |k_h(y, x, w, v)| \leq C(1 + |w| + |v|)^{-(d+1)}.$$

Therefore, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d_+} |K_h(y', x', y, x)| \, dy' dx' &= \frac{1}{h^d} \int_{\mathbb{R}^d_+} \left| k_h \left(y, x, \frac{y - y'}{h}, \frac{x - x'}{h} \right) \right| dy' dx' \\
&= \int_{\mathbb{R}^d} |k_h(y, x, w, v)| dw dv \\
&\leq C_1,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathbb{R}^d_+} |K_h(y, x, y', x')| dy dx &= \frac{1}{h^d} \int_{\mathbb{R}^d_+} \left| k_h \left(y, x, \frac{y - y'}{h}, \frac{x - x'}{h} \right) \right| dy dx \\
&\leq \int_{\mathbb{R}^d} \sup_{(y,x)} |k_h(y, x, w, v)| dw dv \\
&\leq C_2.
\end{aligned}$$

■

Chapter 2

Stabilization of Hyperbolic-Stokes System

This chapter is based on a joint-work with Felipe Chaves-Silva.

2.1 Introduction and Main Results

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a smooth, bounded star-shpaed domain whose boundary $\partial\Omega$ is smooth, ω be a small subset of Ω and let $T > 0$.

In this paper, we are interested in the stabilization problem for the following hyperbolic Stokes system:

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p + a(x)\partial_t u = 0 & \text{in } \mathbb{R} \times \Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ (u(0,x), \partial_t u(0,x)) = (u_0, v_0) \in V \times H, \end{cases} \tag{2.1.1}$$

where V and H are the usual spaces in the context of fluid mechanics, i.e.,

$$V = \{u \in H_0^1(\Omega)^d : \text{div } u = 0\}$$

and

$$H = \{u \in L^2(\Omega)^d : \text{div } u = 0, u \cdot \nu|_{\partial\Omega} = 0\},$$

and $\nu(x)$ is the outward normal to Ω at the point $x \in \partial\Omega$. In (2.1.1), the damping term $a \in L^\infty(\Omega)$ and satisfies $a(x) \geq 0$, for all $x \in \Omega$.

If $u = u(t,x)$ is a (sufficiently smooth) solution of the system, we define its energy as

$$E[u](t) = \frac{1}{2}\int_\Omega (|\partial_t u(t,x)|^2 + |\nabla u(t,x|^2)dx, \quad \forall t \in \mathbb{R},$$

and when there is no damping, namely $a \equiv 0$, the energy is conserved, while in general we only have that $E[u](t)$ is non-increasing:

$$\frac{dE[u]}{dt} = -\int_\Omega a(x)|\partial_t u(t,x)|^2 dx \leq 0.$$

As for other hyperbolic systems, the stabilization problem for (**??**) concerns about the decay rate in time of the energy $E[u](t)$ under appropriate assumptions on the damping term.

It is well-known that stabilization problems are closely related to observability and exact controllability problems in abstract settings. In fact, if we consider the undamped system

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = 0 & \text{in } \mathbb{R}\times\Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ (u(0,x), \partial_t u(0,x)) = (u_0, v_0) \in V \times H, \end{cases} \tag{2.1.2}$$

we say that (2.1.2) is observable at time T with observation in ω if there exists $C > 0$ such that

$$||u_0||_V^2 + ||v_0||_H^2 \leq C \int_0^T \int_\omega |\partial_t u(t,x)|^2 dx dt, \tag{2.1.3}$$

for every $(u_0, v_0) \in V \times H$.

When (2.1.3) holds, one can show that for any $(u_0, v_0) \in V \times H$ there exists $f \in L^2((0,T)\times\omega)^d$ such that the solution of

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = f\mathbf{1}_\omega & \text{in } \mathbb{R}\times\Omega, \\ \text{div } u = 0 & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ (u(0,x), \partial_t u(0,x)) = (u_0, v_0) \in V \times H, \end{cases} \tag{2.1.4}$$

satisfies

$$u(T,x) = 0, \quad \partial_t u(T,x) = 0,$$

that is to say, system (2.1.4) is exact controllable at T with control localized in ω. Nevertheless, it is important to mention that a complete characterization of the sets ω for which (2.1.3) is true remains open. A partial answer to this question was given by the first author in [14].

The motivation for studying the stabilization of system (2.1.1) is two folded. First, system (2.1.2) is the hyperbolic counterpart of Stokes system, which is the linearized version of the well-known Navier-Stokes equation in fluid mechanics. In fact, if we know that system (2.1.2) is exact controllable at some time $T > 0$, with control applied to some control region ω, then the so-called Control Transmutation Method can be applied to obtain the null controllability at any time and the optimal cost of controllability (in time) for the Stokes system (for more details, see [14]). On the other hand, system (2.1.2) comes from simple models of dynamical elasticity for incompressible materials. More precisely, it can be derived as a limit model of Lamé system in linear elastic theory when one parameter tends to infinity ([47]). For the sake of completeness, in the Appendix we give a derivation of system (2.1.2) from Lamé system. It is important to remark that the stabilization problem for the Lamé system has been already studied in [10].

To state our main results, let us introduce several concepts. Some terminologies and notation will be clear in the next section.

Definition 2.1
We say that the support of a non-negative function $a \in C(\overline{\Omega})$ satisfies the geometric control condition (GCC in short) if there exists $T > 0$, such that each generalized bicharacteristic ray $\gamma(t)$ with speed 1 issued from a point $\rho \in^b T^*\overline{\Omega}$ enters the set $\{x \in \overline{\Omega} : a(x) > 0\}$ in a time $t < T$.

We recall that an open set Ω has no infinite order of contact, if in the decomposition

$$T^*\partial\Omega = \mathcal{E} \cup \mathcal{H} \cup \mathcal{G},$$

we have

$$\mathcal{G} = \bigcup_{j=2}^{\infty} \mathcal{G}^j.$$

Here, the sets $\mathcal{E}, \mathcal{H}, \mathcal{G}$ are called elliptic zone, hyperbolic zone and glancing zone, respectively, and $\mathcal{G}^j$ are the sets of points with $j-$th order of contact. The precise definition of this sets will be given in the next section.

Our first main result is as follows.

Theorem 2.1
Suppose $\Omega \subset \mathbb{R}^d$ is a smooth, bounded star-shaped domain with no infinite order of contact and $a \in C(\overline{\Omega})$ is a non-negative function whose support satisfies the geometric control condition. Then, there exist positive constants C_0 and α such that for any $(u_0, v_0) \in V \times H$, the corresponding solution $u(t)$ to (2.1.1) *has the exponential decay:*

$$E[u](t) \leq C_0 E[u](0) e^{-\alpha t}, \forall t \geq 0. \tag{2.1.5}$$

In what follows, we say that the stabilization of (2.1.1) holds if (2.1.5) holds true.

Remark 2.1
As a byproduct of the proof of Theorem 2.1, we obtain the null (exact) controllability at some time T of system (2.1.4)*. Namely, there exists $T > 0$ and a control $f \in L^2([0, T] \times \omega)$ such that the corresponding solution u to* (2.1.4) *satisfies $(u(T), \partial_t u(T)) = (0, 0)$. However, we do not know the control time T explicitly, since we prove the observability inequality* (2.1.3) *by reducing it to a quasi-mode estimate.*

Let us mention that if a is supported in a neighborhood of boundary $\partial\Omega$, the same result is true by adapting the strategy in [14], where the author has proved the exact controllability of the system (2.1.4) with ω be a neighborhood of $\partial\Omega$. Our result is a great generalization of the result obtained in [14].

The pioneering work of J.Rauch and M.Taylor [52] related the exponential decay of damped wave equation to geometric control condition (GCC) of damped region on compact Riemannian manifold without boundary.

Until the celebrated work of C. Bardos, G. Lebeau, and J. Rauch [?], the presence of the boundary has been understood and the exactly controllability for wave equation as well as the exponential stabilization are obtained under (GCC). The proof mainly relies on the propagation of singularity under Melrose-Sjöstrand flow. Later on, the tool of micro-local defect measure, introduced by P.Gérard and L.Tartar independently, has been used to simplify the proof of these results and adapt to many other problems, see for example [10] for Lamé systems and [19] for a coupled wave system. The key ingredient of the measure-based proof is the propagation formula, which can be viewed as a transport equation for defect measure. As a consequence, the propagation of singularity can be derived as a special case of measure invariance under bicharacteristic flow.

For the present system (2.1.1), the presence of the pressure term ∇p introduces nontrivial difficulties if we want to adapt the strategy in [10] directly, due to the rough regularity of time-dependent harmonic function $p(t,x)$. However, follow the semi-classical reduction in [9], it turns out that the exponentially stabilization of (2.1.1) can be reduced to the following semi-classical version observability estimate:

Proposition 2.1
Assume that $a \in C^0(\overline{\Omega})$ and $\int_\Omega a dx > 0$. Suppose the following statement holds true:

$$\exists h_0 > 0, C > 0 \text{ such that } \forall 0 < h < h_0, \forall (u,q,f) \in H^2(\Omega) \cap V \times H^1(\Omega) \times H$$

solves the equation

$$-h^2\Delta u - u + h\nabla q = f, \tag{2.1.6}$$

implies

$$\|u\|_{L^2(\Omega)} \leq C\left(\|a^{1/2}u\|_{L^2(\Omega)} + \frac{1}{h}\|f\|_{L^2(\Omega)}\right). \tag{2.1.7}$$

Then we have the stabilization of (2.1.1).

Note that the system (2.1.7) is just a quasi-mode equation of stationary Stokes system, and in particular, if $f = 0$, the solution $u(h)$ is a eigenfunction of Stokes operator corresponding to eigenvalues h^{-2}.

The proof of (2.1.7) is based on the propagation of semi-classical measure μ treated in the previous chapter. We give a brief recall here. The sequence of pressure q are harmonic, and their impact on the solution only occurs at the boundary. It has been shown that the measure is propagated along bi-characteristic rays which is invariant under the flow. When a ray touches the boundary, more careful analysis between the wave-like propagation phenomenon and the impact of the pressure yield the propagation of the support of the measure μ along generalized bi-characteristic ray defined in [48].

We organize this paper as follows. In section 2, we recall the propagation result given in the previous chapter. In section 3, we follow the strategy in [9] to reduce the stabilization to semi-classical observability (2.1.7). In section 4, we prove the semi-classical observability by adapting the propagation result. Finally in Appendix, we give the derivation from Lamé system to system (2.1.1).

2.2 Review of Semi classical propagation of singularity

Now let us recall the several results proved in the last chapter.

$$\begin{cases} -h^2\Delta u - u + h\nabla q = f, (u,f) \in (H^2(\Omega)\cap V)\times H \\ h \ \mathrm{div} u = 0, \text{in } \Omega \end{cases} \tag{2.2.1}$$

Assume that $\|u\|_{L^2(\Omega)} = O(1), \|f\|_{L^2(\Omega)} = o(h)$. and $u \rightharpoonup 0$ in $L^2(\Omega)$. Let μ be a semi-classical defect measure associated with (u,h) (some subsequence of it). The precise definition and some properties of μ is given in the last chapter and we will not recall it here for simplicity.

In the interior, the full transport property of defect measure is proved.

Proposition 2.2
For any real-valued scalar function $a \in C_c^\infty(\Omega\times\mathbb{R}^d)$ vanishing near $\xi = 0$, we have

$$\frac{d}{ds}\langle \mu, a\circ\gamma(s,\cdot)\rangle = 0.$$

The following proposition illustrates that near a elliptic point on the boundary, there is no accumulate of singularity.

Proposition 2.3
$\mu\mathbf{1}_{\mathcal{E}} = 0$*. If we let ν be the semi-classical defect measure of the sequence $(h_k\partial_\nu u_k|_{\partial\Omega}, h_k)$, then $\nu\mathbf{1}_{\mathcal{E}} = 0$.*

When a ray travels near a hyperbolic point or point in the glancing surface, the knowledge of the singularity is much less. Nevertheless, we have

Theorem 2.2
Assume that Ω is a smooth, bounded star-shpaed domain with no infinite order of contact on the boundary. Suppose (u_k) is a sequence of solutions to the quasi-mode problem (2.1.1) *with semi-classical parameters $h = h_k$. Assume that $f_k \in H$, $\|f_k\|_{L^2(\Omega)} = o(h_k)$ and u_k converges weakly to* 0 *in $L^2(\Omega)$. Assume that μ is any semi-classical measure associated to some subsequence of (u_k,h_k), then* suppμ *is invariant under Melrose-Sjöstrand flow.*

2.3 Reduction to Semi-classical observability

This section is devoted to the proof of Proposition2.1. In fact, It is classical from [24] that stabilization or observability of a self adjoint evolution system is equivalent to resolvent estimates. See also [9], [11].

Recall that the damped system is given by

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)\partial_t u + \nabla p = 0, (t,x) \in \mathbb{R} \times \Omega \\ \mathrm{div} u = 0, \text{in } \Omega \\ u(t,.)|_{\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = (u_0, v_0) \in V \times H \end{cases} \tag{2.3.1}$$

We always assume that $\Omega \subset \mathbb{R}^d$ is a bounded domain (open, connected set). We use ν to denote the outward normal vector on $\partial\Omega$ and the damping term $a \in L^\infty(\Omega)$ with $a(x) \geq 0$.

We also consider the undamped system

$$\begin{cases} \partial_t^2 u - \Delta u + \nabla p = 0, (t,x) \in \mathbb{R} \times \Omega \\ \mathrm{div} u = 0, \text{in } \Omega \\ u(t,.)|_{\partial\Omega} = 0 \\ (u(0), \partial_t u(0)) = (u_0, v_0) \in V \times H \end{cases} \tag{2.3.2}$$

2.3.1 Some functional analysis preliminaries

We work with a Hilbert space $\mathcal{H} := V \times H$, equipped with the norm

$$\|(f,g)^t\|_{\mathcal{H}}^2 := \|\nabla f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2.$$

and denote $\Pi : L^2(\Omega)^N \to H$ be the orthogonal projector(Leray-projector) and $A = \Pi\Delta$ be the Stokes operator. We consider the operator:

$$\mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ A & -\Pi a \end{pmatrix} \tag{2.3.3}$$

with domain

$$D(\mathcal{A}) = (V \cap H^2(\Omega)) \times V.$$

In order to use semi-group theory, we first show that for some $\lambda > 0$, the operator $(\mathcal{A} - \lambda)$ is invertible: Take $(f,g) \in V \times H$, and consider the system

$$\begin{cases} v - \lambda u = f \\ Au - (\Pi a + \lambda)v = g \end{cases} \tag{2.3.4}$$

We consider the bilinear form

$$B(u_1, u_2) = \int_\Omega \nabla u_1 \cdot \nabla u_2 dx + \int_\Omega (\lambda^2 + \lambda a(x)) u_1 \cdot u_2 dx,$$

defined on $V \times V$. We then conclude from Lax-Milgram that for $\lambda > 0$, there exists $u \in V$ such that for any $w \in V$, we have

$$B(u, w) = -\int_\Omega (g \cdot w + (a(x) + \lambda) f \cdot w) dx.$$

Set $v = \lambda u + f$, we have solved the system (2.3.4) in weak sense. Standard regularity argument gives that for $\lambda > 0$.

$$(\mathcal{A} - \lambda)^{-1} : \mathcal{H} \to D(\mathcal{A}),$$

is a bounded, and $(\mathcal{A} - \lambda)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. Moreover, if $\lambda \in \mathrm{Spec}(\mathcal{A})$, we must have $\mathrm{Re}\, \lambda < 0$. This will be clear in the proof of Proposition 2.6.

However, since the operator $\mathcal{A}$ is not maximal dissapative, the Hille-Yoshida theorem is not applicable. A slightly general modification ensures the existence of semi-group $e^{t\mathcal{A}}$ which evolves the initial data in $D(\mathcal{A})$ and solves the equation (2.3.1) with more regular data.

For solution $u, \partial_t u$ to (2.3.1), we consider the energy functional

$$E[u](t) := \frac{1}{2} \int_\Omega (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2) dx,$$

and we calculate

$$\begin{aligned}
\frac{d}{dt} E[u](t) &= \int_\Omega \partial_t u \cdot (\partial_t^2 u - \Delta u) dx \\
&= -\int_\Omega \partial_t u \cdot \nabla p dx - \int_\Omega a(x) |\partial_t u|^2 dx \\
&\leq 0,
\end{aligned}$$

thus

$$E[u](t) \leq E[u](s), \forall s \leq t. \tag{2.3.5}$$

By density argument, we can solve (2.3.1) with initial data in $\mathcal{H}$ such that the energy dissapation (2.3.5) still holds.

2.3.2 Observability and Stabilization

In this section, we will prove the stabilization for damped system is equivalent to observability for undamped system. For this part, we follow closely in the appendix of [9] in which the authors have sketched the standard argument for damped wave equation.

We first introduce the quantity

$$D[u](T) = \int_0^T \int_\Omega a(x)|\partial_t u(t,x)|^2 dxdt,$$

and it quantifies the dissipation of the energy:

$$E[u](T) = E[u](0) - D[u](T).$$

Proposition 2.4
The following assertions are equivalent:

1. *Stabilization: There exists $C_0, \alpha > 0$, such that for every solution $u \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to the damped system* (2.3.1)*, we have*

$$E[u](t) \leq C_0 E[u](0) e^{-\alpha t}, \forall t \geq 0.$$

2. *Observability: There exists $C > 0$ and $T > 0$, such that, for every solution $v \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to the undamped system* (2.3.2)*, the observability inequality holds:*

$$E[v](0) \leq CD[v](T).$$

Proof. We first claim that the stabilization of damped system is equivalent to the observability of damped system.

It is clear that

$$E[u](0) = E[u](t) - D[u](t).$$

Let us first assume the stabilization of damped system. Argue by contradiction, suppose the observability of damped system does not hold. We first choose $T_0 > 0$ large enough such that $C_0 e^{-\alpha T_0} < \frac{1}{2}$. We can select a sequence of solutions (u_k) and such that

$$E[u_k](0) = 1, D[u_k](T_0) \to 0, \text{ as } k \to \infty.$$

We thus have

$$\frac{1}{2} > C_0 e^{-\alpha T_0} \geq E[u_k](T_0) = E[u_k](0) - D[u_k](T_0) = 1 + o(1), \text{ as } k \to \infty,$$

which leads to a contradiction.

Let us now assume the observability for damped system, i.e.

$$E[u](0) \leq CD[u](T),$$

We may assume that $C > 1$, from the energy dissapation and observability, we have

$$E[u](2T) = E[u](0) - D[u](2T) \leq \left(1 - \frac{1}{C}\right) E[u](0).$$

For any $t > 0$, we write $m = \left\lfloor \frac{t}{2T} \right\rfloor$, therefore we have

$$E[u](t) \leq E[u](m) \leq \left(1 - \frac{1}{C}\right)^m E[u](0),$$

after choosing C_0, α appropriately, we have the stabilization of damped system.

Our second step is to justify the equivalence between observability of damped system (2.3.1) and undamped system (2.3.2). To do this, we denote u and v be solutions of the damped and of the undamped system, respectively, with the same initial data at $t = 0$. Let $w = u - v$, and simple calculations yield

$$\partial_t^2 w - \Delta w = -a\partial_t u - \nabla q,$$
$$\partial_t^2 w - \Delta w + a\partial_t w = -a\partial_t v - \nabla q,$$

with some pressure function q.

We calculate

$$\frac{d}{dt}E[w](t) = -\int_\Omega a(x)|\partial_t u|^2 dx + \int_\Omega a(x)\partial_t u \cdot \partial_t v dx - \int_\Omega \partial_t w \cdot \nabla q dx,$$

and the last term of left hand side vanishes, thanks to $\partial_t w \in C(\mathbb{R}; V)$. Thus we can write

$$\frac{d}{dt}E[w](t) = -\int_\Omega a(x)|\partial_t u|^2 dx + \int_\Omega a(x)\partial_t u \cdot \partial_t v dx$$

or equivalently

$$\frac{d}{dt}E[w](t) = -\int_\Omega a(x)\partial_t u \cdot \partial_t w dx.$$

Integrating the two expressions above and using the inequality of the type

$$|ab| \leq \epsilon|a|^2 + C(\epsilon)|b|^2, \forall \epsilon > 0,$$

one easily get

$$E[w](T) \leq B \min\left(D[u](T), D[v](T)\right), \forall T > 0, \tag{2.3.6}$$

where B is another absolute constant.

Now suppose we have observability for the damped system (2.3.1), if $D[u](T) \leq D[v](T)$, the observability of undamped system (2.3.2) is trivial. Now assume that $D[u](T) > D[v](T)$, we deduce from (2.3.6) that

$$\begin{aligned} E[v](0) = E[u](0) &\leq CD[u](T) \leq CD[W](T) + CD[v](T) \\ &\leq C(E[w](T) + D[v](T)) \leq CD[v](T). \end{aligned}$$

The derivation of observability from undamped system to the damped follows in the same way, and we omit the details. ■

Remark 2.2
Since the domain $D(\mathcal{A})$ is dense in $\mathcal{H}$ and the observability and energy decay only involves the L^2 norm of ∇u and $\partial_t u$, thus the same result of proposition 2.4 holds if we replace $u \in C(\mathbb{R}; V \cap H^2(\Omega)) \cap C^1(\mathbb{R}; V)$ to $u \in C(\mathbb{R}; V) \cap C^1(\mathbb{R}; H)$.

2.3.3 Resolvent estimates and stabilization

Recall that from the previous sections, the study of damped system (2.3.1) is equivalent to the project system

$$
\begin{aligned}
&\frac{d}{dt}\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & \mathring{Id} \\ A & -\Pi a \end{pmatrix}\begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \\
&\begin{pmatrix} u \\ \partial_t u \end{pmatrix} \in C(\mathbb{R}; V \times H).
\end{aligned} \tag{2.3.7}
$$

We will use the notation $U = (u, \partial_t u)^t$ in the sequel.

In this part, we follows almost the same way as in the appendix of [9], only to pay attention to the changing of working spaces (appearance of the pressure term and divergence free structure). Moreover, we add some technical details which may seems standard to experts in analysis but not disposable for many applied people.

From last section, we know that the observability of undamped system (2.3.2) is equivalent to the stabilization of damped system (2.3.1), therefore we will concentrate ourselves to the study of stabilization of (2.3.1). The following result is standard in semigroup theory:

Proposition 2.5
Consider a semi-group e^{tL} on a Hilbert space $\mathcal{X}$, with infinitesimal generator L defined on $D(L)$. Then if there exists $C > 0, \delta > 0$ such that the resolvent of L, $(L - \lambda)^{-1}$ exists for $\operatorname{Re}\lambda \geq -\delta$ *and satisfies*

$$\forall \lambda \in \mathbb{C}^{\delta} := \{z \in \mathbb{C} : \operatorname{Re} z > -\delta\}, \|(L-\lambda)^{-1}\|_{\mathcal{L}(\mathcal{X})} \leq C.$$

Then there exists $M > 0$ such that for any $t > 0$,

$$\|e^{tL}\|_{\mathcal{L}(\mathcal{X})} \leq M e^{-\frac{\delta t}{2}}.$$

We need a lemma from complex analysis. We temporarily use the convention of Fourier transform

$$\widehat{u}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\tau} u(t) dt.$$

Lemma 2.1
Let u, v be two continuous functions with support in $\mathbb{R}_+ = (0, \infty)$. Assume moreover that $u, v \in L^2(\mathbb{R}_+)$ and v has compact support. From Winer-Paley theory, we know that the Fourier transform $\widehat{v}$ admits a holomorphic extension to $\mathbb{C}$ and of exponential type. Given $a_0 > 0$, suppose that the Fourier transform $\widehat{u}$ is also holomorphic in $S_{a_0} = \{z \in \mathbb{C} : \operatorname{Im} z < a_0\}$ and satisfies

$$|\widehat{u}(z)| \leq C|\widehat{v}(z)|, \forall z \in S_{a_0}.$$

Then for any $a < a_0$, $e^{at}u(t) \in L^2(\mathbb{R}_+)$ and

$$\int_0^{\infty} e^{2at}|u(t)|^2 dt = \int_{-\infty}^{\infty} |\widehat{u}(\tau + ia)|^2 d\tau.$$

Proof. We first claim that

$$\int_0^\infty e^{2at}|v(t)|^2dt = \int_{-\infty}^\infty |\widehat{v}(\tau+ia)|^2 d\tau, \forall a\in\mathbb{R}. \tag{2.3.8}$$

Indeed, since v is compactly supported,

$$\widehat{v}(\tau+ia) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{at}e^{-it\tau}v(t)dt$$

which is analytic in a and rapidly decreasing in τ for each fixed $a\in\mathbb{R}$. Thus one easily deduce from the Plancherel (or calculate the integral directly) that (2.3.8) is true.

As a consequence, $\widehat{u}(.+ia)\in L^2(\mathbb{R})$ for each $a<a_0$. Notice also that $u\in L^2(\mathbb{R}_+)$, thus for each a with $\operatorname{Re} a<0$, the formula

$$\int_0^\infty e^{2at}|u(t)|^2dt = \int_{-\infty}^\infty |\widehat{u}(\tau+ia)|^2 d\tau \tag{2.3.9}$$

holds true and analytic with respect to a. In particular, $|\widehat{u}(z)|\leq C|\widehat{v}(z)|$,$z\in S_{a_0}$ implies that $\widehat{u}(\tau+ia)$ is rapidly decreasing in τ for each fixed $a<a_0$. For $z=\tau+ia$ with $a<a_0$, consider the integral

$$F(a,t) = \frac{e^{at}}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{itz}\widehat{u}(z)d\tau = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{it\tau}\widehat{u}(z)d\tau \in L^2(\mathbb{R}).$$

From Cauchy integral theorem, we have that

$$F(a,t) = \frac{e^{at}}{\sqrt{2\pi}}\int_{-\infty}^\infty \widehat{u}(\tau)e^{it\tau}d\tau = e^{at}u(t)\in L^2(\mathbb{R}).$$

From this, we conclude that (2.3.9) follows for all $a<a_0$. ∎

Remark 2.3
In the previous lemma, the same results hold true if we replace u,v to be Hilbert-space valued functions.

proof of proposition 2.5. The basic tool to prove this proposition is the Fourier-Laplace transform in time variable. From the property of strongly continuous semi-group, we know that there exists $\omega_0>0$ such that (see [61])

$$\|e^{tL}\|_{\mathcal{L}(\mathcal{X})}\leq e^{\omega_0 t}, \forall t\geq 1.$$

Take $u_0\in D(L)$, and pick a nonnegative cut-off $\chi\in C^\infty(\mathbb{R})$ such that $\chi\equiv 0,\forall t\leq 1$ and $\chi\equiv 1,\forall t>2$. We define $u(t):=\chi(t)e^{tL-\omega t}u_0$ for some $\omega>\omega_0$ and thus $u\in L^\infty(\mathbb{R};\mathcal{X})$. Moreover, we have the equation

$$(\partial_t+\omega-L)u = \chi'(t)e^{tL-\omega t}u_0 =: v(t).$$

By taking Fourier transform we get

$$(i\tau+\omega-L)\widehat{u}=\widehat{v}(\tau).$$

Since v is compactly supported in positive axis in time variable, the $\widehat{v}(\tau)$ has a holomorphic and bounded extension in any domain of the form

$$S_\alpha=\{\tau\in\mathbb{C}:\operatorname{Im}\tau<\alpha\},\alpha>0.$$

From the assumption on the resolvent, we deduce that $(i\tau+\omega-L)$ is invertible if $\tau\in S_{\delta+\omega}$ and thus $\widehat{u}(\tau)$ admits a bounded holomorphic extension to $S_{\delta+\omega}$ which satisfies

$$\|\widehat{u}(\tau)\|_{\mathcal{X}}\leq C\|\widehat{v}(\tau)\|_{\mathcal{X}}.$$

Apply Lemma **??**, we deduce that

$$\begin{aligned}\int_{-\infty}^{\infty}\|e^{(\omega_0+\delta)t}u\|_{\mathcal{X}}^2dt&=\int_{-\infty}^{\infty}\|\widehat{u}(\xi+i(\omega_0+\delta))\|_{\mathcal{X}}^2d\xi\\&\leq C\int_{-\infty}^{\infty}\|\widehat{v}(\xi+i(\omega_0+\delta))\|_{\mathcal{X}}^2d\xi\\&\leq C\int_{-\infty}^{\infty}\|e^{(\omega_0+\delta)t}v\|_{\mathcal{X}}^2dt\leq C\|u_0\|_{\mathcal{X}}^2.\end{aligned}$$

We remark that one need use various types of Winer-Paley theorems to justify the above calculations, thanks to the fact that $u(t),v(t)$ is supported on $[1,\infty)$ and furthermore $v(t)$ has compact support. Take $\omega<\omega_0+\frac{\delta}{2}$ in the definition of u, we have that

$$\|e^{\frac{\delta t}{2}}e^{tL}u_0\|_{L^2(\mathbb{R}_+;\mathcal{X})}\leq C_1\|u_0\|_{\mathcal{X}}.$$

Thanks to the semi-group structure and uniform bound principal, we have that there exists $M_0>0$, such that for any interval $I\subset(0,+\infty)$ of length 1,

$$\sup_{t\in I,s>0,t+s\in I}\frac{|f(t+s)|}{|f(t)|}\leq M_0.$$

with $f(t)=\|e^{tL}u_0\|_{\mathcal{X}}$. Therefore, for any $T>0$,

$$\int_T^{T+1}e^{\delta t}|f(t)|^2dt\geq e^{\delta T}\min_{t\in[T,T+1]}|f(t)|^2.$$

Therefore,

$$|f(T+1)|^2\leq M_0^2\min_{t\in[T,T+1]}|f(t)|^2\leq e^{-\delta T}\int_T^{T+1}e^{\delta u}|f(t)|^2dt.$$

This implies the exponential decay

$$\|e^{tL}u_0\|_{\mathcal{X}}\leq Me^{-\frac{\delta t}{2}}\|u_0\|_{\mathcal{X}}.$$

∎

Now we can introduce the semi-classical observability

Proposition 2.6
Assume that $a \in L^\infty(\Omega) \cap C^0(\overline{\Omega})$ and $\int_\Omega a dx > 0$. Then the stabilization of system (??) is implied by the following statement:

$$\exists h_0 > 0, C > 0 \text{ such that } \forall 0 < h < h_0, \forall (u, q, f) \in H^2(\Omega) \cap V \times H^1(\Omega) \times H$$

solves the equation

$$-h^2 \Delta u - u + h \nabla q = f,$$

we have

$$\|u\|_{L^2(\Omega)} \leq C \left(\|a^{1/2} u\|_{L^2(\Omega)} + \frac{1}{h} \|f\|_{L^2(\Omega)} \right). \tag{2.3.10}$$

For the proof, we need two lemmas.

Lemma 2.2
Let L be a linear operator on Hilbert space $\mathcal{X}$ with a compact resolvent $(L - \mathrm{Id})^{-1}$. Suppose the spectrum Spec$(L) \subset \{z : \operatorname{Re} z < 0\}$ and satisfies that for any $\sigma \in \mathbb{R}$, $L - i\sigma$ is invertible and satisfies the uniform bound

$$\sup_{\sigma \in \mathbb{R}} \|(L - i\sigma)^{-1}\| < \infty.$$

Then there exists $\delta > 0$, such that

$$\sup_{\lambda \in \mathbb{C}^\delta} \|(L - \lambda)^{-1}\| < \infty,$$

where $\mathbb{C}^\sigma := \{z \in \mathbb{C} : \operatorname{Re} z > -\sigma\}$ for any $\sigma \in \mathbb{R}$.

Proof. Write

$$\sup_{\sigma \in \mathbb{R}} \|(L - i\sigma)^{-1}\| = C$$

We denote $R(z) = (L - z)^{-1}$ for $z \in \rho(L) := \{z : z \in \mathbb{C} \setminus \text{ Spec}(L)\}$. Take $z_0 \in \rho(L)$, we write

$$L - z = (L - z_0)(\mathrm{Id} + (L - z_0)^{-1}(z_0 - z)),$$

and for $|z - z_0| < \frac{1}{\|(L - z_0)^{-1}\|}$, we have

$$\|R(z)\| \leq \|R(z_0)\| \sum_{n=0}^{\infty} |z - z_0|^n \|(L - z_0)^{-1}\|^n \leq R(z_0).$$

Therefore, for λ with $|\operatorname{Re} \lambda| \leq \delta$, where $0 < \delta < \frac{1}{2C}$, we have $\|R(\lambda)\| \leq C$. To conclude, we only need show that there exists $C_1 > 0$, such that

$$\sup_{\operatorname{Re} z > \delta} \|(L - z)^{-1}\| \leq C_1.$$

Consider the holomorphic equivalence $\varphi : \mathbb{C}^0 \to \mathbb{D}, \psi = \varphi^{-1}$.

$$\varphi(z) = \frac{z-1}{z+1}, \psi(\zeta) = \frac{1+\zeta}{1-\zeta},$$

where $\mathbb{D} := \{\zeta : |\zeta| < 1\}$ be the unit disk. One easily verifies that the operator-valued function

$$\Phi(\zeta) = R(\psi(\zeta)) : \mathbb{D} \to \mathcal{L}(\mathcal{X})$$

is analytic and satisfies the Cauchy integral formula

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\Phi(\zeta)}{\zeta - \zeta_0} d\zeta, \forall \zeta_0 \in \mathbb{D}.$$

Since dist $(\partial\mathbb{D}, \varphi(\mathbb{C}^{-\delta})) \geq \epsilon_0 > 0$ for some ϵ_0 depends only on δ, we deduce that for any $z \in \mathbb{C}^{-\delta}$,

$$\|R(z)\| \leq \left\| \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\Phi(\zeta)}{\zeta - \varphi(z)} d\zeta \right\| \leq \frac{C}{\epsilon_0}.$$

■

Lemma 2.3 (Unique Continuation of Stoke Operator)
Let $\sigma > 0$ and $u \in V$ satisfies that

$$Au = \sigma^2 u.$$

Then if $u|_\omega \equiv 0$, we must have $u \equiv 0$.

Proof. It is equivalent to write

$$-\Delta u + \nabla p = \sigma^2 u, \text{ div } u = 0, u \in V, \int_\Omega p dx = 0.$$

Take divergence of the equation, we have $\Delta p = 0$. The vanishing of u in ω implies that p equals to a constant in a component of ω. Now since Ω is connected, the maximum principal implies that $p \equiv 0$ in Ω. Therefore we have reduced to unique continuation of eigenfunction of Laplace operator, and this implies that $u \equiv 0$ in Ω. ■

proof of proposition 2.1. We need show that the semi-classical observability implies the stabilization.

Note that the operator $(\mathcal{A} - \lambda)$ is invertible for any $\lambda > 0$. One write

$$\mathcal{A} - z = (\text{Id} + (1-z)(\mathcal{A} - \text{Id})^{-1})(\mathcal{A} - 1), \forall z \in \mathbb{C}.$$

Since $\text{Id} + (1-z)(\mathcal{A} - \text{Id})^{-1}$ is Fredholm with index 0, we infer that $\mathcal{A} - z$ is invertible iff it is injective. In light of the previous lemmas and the Proposition 2.5, we only have to prove the fact that

$$\exists C > 0, \text{ such that } \forall \sigma \in \mathbb{R}, U \in D(\mathcal{A}), F \in V \times H, (\mathcal{A} - i\sigma)U = F$$
$$\text{implies } \|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}.$$

We argue by contradiction. If it is not true then we can find sequences $(\sigma_n), (U_n)$, and (F_n) such that

$$(\mathcal{A} - i\sigma_n)U_n = F_n, \|U_n\|_{\mathcal{H}} = 1, \|F_n\|_{\mathcal{H}} < \frac{1}{n}.$$

After extracting subsequences we may assume that $\sigma_n \to \sigma$, and we write

$$U_n = (u_n, v_n)^t, F_n = (f_n, g_n)^t.$$

We have several cases to analyse, according to the limit value σ.

1. $\sigma = 0$: In this case, we have $\mathcal{A}U_n = o(1)_{\mathcal{H}}$, which is equivalent to

$$v_n = o(1)_{H_0^1}, Au_n - \Pi a v_n = o(1)_{L^2},$$

thus $Au_n = o(1)_{L^2}$. Taking inner product with u_n and integrating by part we have

$$\int_\Omega |\nabla u_n|^2 dx = o(1).$$

This contradicts to $\|U_n\|_{\mathcal{H}} = 1$.

2. $0 < |\sigma| < \infty$: In this case we have $\mathcal{A}U_n - i\sigma U_n = o(1)_{\mathcal{H}}$, or equivalently,

$$v_n - i\sigma u_n = o(1)_{H_0^1}, Au_n - (i\sigma + \Pi a)v_n = o(1)_{L^2}.$$

Thanks to Poincaré inequality, we deduce that

$$Au_n - i\sigma(i\sigma + \Pi a)u_n = o(1)_{L^2}.$$

Applying Rellich compact embedding theorem followed by extracting to suitable subsequences, we may assume that

$$u_n \to u, \text{ in } L^2(\Omega), u_n \rightharpoonup u, \text{ in } V.$$

Taking inner product with u_n, we have

$$-\int_\Omega |\nabla u_n|^2 dx = -\sigma \int_\Omega |u_n^2| dx + i\sigma \int_\Omega a(x)|u_n|^2 dx + o(1),$$

which implies that $au \equiv 0$ in Ω. Thus we can conclude that u is an eigenfunction of Stokes operator A and vanishes in a non trivial open subset of Ω. The unique continuation property for the system

$$-\Delta u + \nabla p = \sigma^2 u, \text{ div } u = 0$$

implies that $u \equiv 0$. As a consequence, we have that $u_n = o(1)_{H_0^1}, v_n = o(1)_{L^2}$. This contradicts to the original assumptions.

3. $|\sigma| = \infty$: We only study the case $\sigma_n \to +\infty$ (the other one is obtained by considering $\overline{U_n}$).

   Let $h_n = \sigma_n^{-1}$, and we deduce from the system $\mathcal{A}U_n - i\sigma_n U_n = o(1)_{\mathcal{H}}$:

$$h_n^2 A u_n + u_n - ih_n \Pi a u_n = h_n^2 \Pi a f_n + ih_n f_n + h_n^2 g_n = o_{L^2}(h_n)$$

$$h_n v_n - iu_n = h_n f_n = o(h_n)_{H_0^1},$$

$$h_n^2 A v_n + v_n - ih_n \Pi a v_n = ih_n g_n - h_n^2 A f_n = o_{L^2}(h_n) + o_{H^{-1}}(h_n).$$

   Define the operator $P_h = h^2 A + \mathrm{Id} - ih\Pi a$ on H with domain $H^2(\Omega) \cap V$, we have (dropping the subindex n for the moment)

$$(P_h u|u)_{L^2} = \|u\|_{L^2(\Omega)}^2 - \|h\nabla u\|_{L^2(\Omega)}^2 - ih\|a^{1/2}u\|_{L^2(\Omega)}^2.$$

   Taking imaginary part, we have

$$\|a^{1/2}u\|_{L^2(\Omega)}^2 \leq C\frac{\|P_h u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}}{h}.$$

   Applying the semi-classical observability to the equation

$$h^2 A u + u = ih\Pi a u + \tilde{f}$$

   with $\tilde{f} = o_{L^2}(h)$, we have

$$\begin{aligned} \|u\|_{L^2(\Omega)}^2 \leq & C\left(\|a^{1/2}u\|_{L^2(\Omega)}^2 + \frac{1}{h^2}(\|f\|_{L^2(\Omega)}^2 + h^2\|a^{1/2}u\|_{L^2(\Omega)}^2)\right) \\ \leq & \frac{C}{h}\|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \frac{C}{h^2}\|f\|_{L^2(\Omega)}^2. \end{aligned} \tag{2.3.11}$$

   This implies that

$$\|u_n\|_{L^2(\Omega)} \leq \frac{C}{h_n}\|f_n\|_{L^2(\Omega)} = o(1).$$

   To conclude, observe that v_n satisfies

$$h_n^2 A v_n + v_n = o_H(h_n) + o_{H^{-1}(\Omega)}(h_n^2),$$

   and we claim that if $(h^2 A + 1)v = f_1 + f_2$, then

$$\begin{aligned} &\|v\|_{L^2(\Omega)} + \|h\nabla v\|_{L^2(\Omega)} \\ \leq & C\left(\|a^{1/2}v\|_{L^2} + \frac{\|f_1\|_{L^2(\Omega)}}{h} + \frac{\|f_2\|_{H^{-1}(\Omega)}}{h^2}\right). \end{aligned} \tag{2.3.12}$$

   Assume the claim for the moment, we thus have $\|h_n \nabla v_n\|_{L^2(\Omega)} = o(1)$, and $\|\nabla u_n\|_{L^2(\Omega)} = o(1)$, thanks to $u_n + ih_n v_n = ih_n f_n$. This contradicts to the original assumption.

Now we turn to the proof of the claim. By density, (2.3.10) still valid when $v \in V$. Taking inner product of v with $P_h v$, we have

$$(P_h v|v)_{L^2} = \|v\|^2_{L^2(\Omega)} - \|h\nabla v\|^2_{L^2(\Omega)} - ih\|a^{1/2}v\|^2_{L^2(\Omega)}.$$

Therefore, by taking real part and injecting (2.3.10), we have

$$\|h\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq C\left(\|a^{1/2}v\|_{L^2(\Omega)} + \frac{\|P_h v\|_{L^2(\Omega)}}{h}\right). \tag{2.3.13}$$

By taking real and imaginary part of $(P_h v|v)_{L^2}$, we have

$$\|a^{1/2}v\|^2_{L^2(\Omega)} \leq \frac{\|P_h v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}}{h}, \tag{2.3.14}$$

Substituting (2.3.14) into (2.3.13), we obtain that

$$\|h\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \leq C\left(\frac{\|P_h v\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}}{h} + \frac{\|P_h v\|^2_{L^2(\Omega)}}{h^2}\right),$$

and this implies that

$$\|h\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq C\frac{\|P_h v\|_{L^2(\Omega)}}{h}.$$

Thus P_h is bijective from $H^2(\Omega) \cap V$ to H and hence invertible. From the fact that

$$P_h = (1 + (2 - ih\Pi a)(h^2 A - 1)^{-1})(h^2 A - 1),$$

P_h can be written as composition of a positive operator and a Fredholh operator of index 0. From the estimate above, we conclude that

$$\|P_h^{-1}\|_{L^2\to L^2} \leq \frac{C}{h}, \|P_h^{-1}\|_{L^2\to H^1} \leq \frac{C}{h^2}.$$

Now come back to the equation $(h^2 A + 1)v = f_1 + f_2$. Taking $g \in H$, and letting $w = P_h^{-1}g$, we have

$$\begin{aligned}(v|g)_{L^2} =&((h^2 A + 1)v|w)_{L^2} + ih(v|\Pi a w)_{L^2}\\ =&(f_1 + f_2|w)_{L^2} + ih(av|w)_{L^2}\\ \leq&\|f_1\|_{L^2(\Omega)}\|P_h^{-1}g\|_{L^2()\Omega)}\\ &+\|f_2\|_{H^{-1}(\Omega)}\|P_h^{-1}g\|_V + h\|av\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}\\ \leq&C\left(\|a^{1/2}v\|_{L^2(\Omega)} + \frac{\|f_1\|_{L^2(\Omega)}}{h} + \frac{\|f_2\|_{H^{-1}(\Omega)}}{h^2}\right)\|g\|_{L^2(\Omega)}.\end{aligned} \tag{2.3.15}$$

This completes the proof.

■

2.4 Apriori Estimates for the quasi-mode system

Now we consider the quasi-modes of Stokes system

$$\begin{cases} -h_k^2\Delta u_k - u_k + h_k\nabla q_k = f_k, (u_k, f_k) \in (H^2(\Omega)\cap V)\times H, \\ h_k \mathrm{div} u_k = 0, \text{in } \Omega \end{cases} \tag{2.4.1}$$

To simplify the notation, we drop the sub index k and just keep the semi-classical parameter h everywhere. Note that the functions u, v, etc. should be understood as $u(h), v(h)$, etc.

Assume that

$$\|u\|_{L^2(\Omega)} = O(1), \|f\|_{L^2(\Omega)} = o(h).$$

Taking inner product with u and integrate by part, we have

$$\|h\nabla u\|_{L^2(\Omega)} = O(1).$$

One can always assume that $\int_\Omega q dx = 0$, since $q \in L^2(\Omega)/\mathbb{R}$. From the regularity theory of steady Stokes system, (see [62], page 33), and Poincaré inequality, we have

$$\|h^2\nabla^2 u\|_{L^2(\Omega)} = O(1), \|q\|_{L^2(\Omega)} = O(h^{-1}), \|h\nabla q\|_{L^2(\Omega)} = O(1).$$

We now give some estimates on the trace. Write $q_0 = q|_{\partial\Omega}$,

Lemma 2.4
$\|q\|_{L^2(\Omega)} = O(h^{-1}), \|q_0\|_{H^{1/2}(\partial\Omega)} = O(h^{-1}), \|q_0\|_{L^2(\partial\Omega)} = O(h^{-1})$.

Proof. Since q is harmonic function, then one can apply trace theorem $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ for any $s \in \mathbb{R}$. Hence the conclusions follows from these and interpolations. ∎

Lemma 2.5
$h\partial_\nu u|_{\partial\Omega} = (h\partial_\nu u_\parallel, 0)$, and $\|h\partial_\nu u|_{\partial\Omega}\|_{L^2(\partial\Omega)} = O(1)$.

Proof. The first assertion follows from $h\mathrm{div}u = 0$ and Dirichlet boundary condition, while we apply a multiplier method to prove the second. From the geometric assumption on Ω, we can find a vector field $L \in C^1(\overline{\Omega})$ such that $L|_{\partial\Omega} = \nu$(see [50], page 36). In global coordinate system, we write $L = L_j(x)\partial_{x_j}$. By using the equation, we have

$$\begin{aligned} \int_\Omega Lu\cdot f dx &= \int_\Omega Lu\cdot(-h^2\Delta u - u + h\nabla q)dx, \\ -\int_\Omega Lu\cdot u dx &= -\int_\Omega L_j(x)\partial_{x_j} u^i u^i dx \\ &= -\int_\Omega \partial_{x_j}\left(L_j(x)u^i\right)u^i dx + \int_\Omega \mathrm{div}\ L(x)|u|^2 dx \\ &= \int_\Omega L_j(x)u^i(x)\partial_{x_j}u^i dx + \int_\Omega \mathrm{div}\ L(x)|u|^2 dx \\ &= \int_\Omega Lu\cdot u dx + \int_\Omega \mathrm{div}\ L(x)|u|^2 dx, \end{aligned}$$

and thus $\displaystyle\int_\Omega Lu \cdot u dx = -\frac{1}{2}\int_\Omega \text{div } L(x)|u|^2 dx = O(1)$. We next calculate

$$\begin{aligned}
h\int_\Omega Lu \cdot \nabla q dx &= -h\int_\Omega u^i \partial_{x_j}\left(L_j \partial_{x_i} q\right) dx \\
&= -h\int_\Omega u \cdot L(\nabla q) dx - h\int_\Omega (\text{div } L(x)) u \cdot \nabla q dx \\
&= -h\int_\Omega u \cdot [L, \nabla] q dx - h\int_\Omega \text{div } L(x) u \cdot \nabla q dx \\
&= O(1),
\end{aligned}$$

$$\begin{aligned}
-h^2\int_\Omega Lu^i \Delta u^i dx =& -h^2\int_{\partial\Omega} \left|\partial_\nu u^i\right|^2 d\sigma + h^2\int_\Omega \nabla L(\nabla u^i, \nabla u^i) dx \\
&+ h^2\int_\Omega L_j(x)\partial^2_{x_j x_k} u^i \partial_{x_k} u^i \\
=& -h^2\int_{\partial\Omega} \left|\partial_\nu u^i\right|^2 d\sigma + h^2\int_\Omega \nabla L(x)(\nabla u^i, \nabla u^i) dx \\
&+ h^2\int_\Omega \partial_{x_j}\left(L_j \partial_{x_k} u^i\right)\partial_{x_k} u^i dx - h^2\int_\Omega \text{div } L(x)\nabla u^i \cdot \nabla u^i(x) dx,
\end{aligned}$$

$$h^2\int_\Omega \partial_{x_j}\left(L_j \partial_{x_k} u^i\right)\partial_{x_k} u^i dx = h^2\int_{\partial\Omega} L\cdot\nu \left|\partial_\nu u^i\right|^2 d\sigma - h^2\int_\Omega L_j(x)\partial_{x_k} u^i \partial^2_{x_j x_k} u^i dx,$$

$$-h^2\int_\Omega Lu^i \Delta u^i dx = -\frac{h^2}{2}\int_{\partial\Omega}\left|\partial_\nu u^i\right|^2 d\sigma + \int_\Omega \nabla L(x)(h\nabla u^i, h\nabla u^i) dx - \frac{h^2}{2}\int_\Omega \text{div } L(x)|\nabla u^i|^2 dx.$$

Observing that $\int_\Omega Lu \cdot f dx = o(1)$, we have

$$\int_{\partial\Omega} |h\partial_\nu u|^2 d\sigma = O(1).$$

■

Lemma 2.6
Under additional assumption that

$$\|a^{1/2} u_k\|_{L^2(\Omega)} = o(1),$$

after extracting to subsequences, we have $h_k \nabla q_k \rightharpoonup 0$ $L^2(\Omega)$ and $u_k \rightharpoonup 0$ weakly in $L^2(\Omega)$. Therefore from Rellich theorem, we have $hq \to 0$, strongly in $L^2(\Omega)$.

Proof. We may assume that $h\nabla q \rightharpoonup r$, weakly in $L^2(\Omega)$, and Rellich theorem implies that $hq \to P$, strongly in $L^2(\Omega)$, and thus $\nabla P = r$, with the property $\int_\Omega P = 0$. Now we claim that $\Delta P = 0$ in Ω.

Indeed, take any $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \nabla P \cdot \nabla\varphi = \lim_{h\to 0}\int_\Omega h\nabla q \cdot \nabla\varphi = 0.$$

Now suppose $u_k \to U$, weakly in $L^2(\Omega)$, $w_k = h_k^2 u_k \to W$, weakly in $H^2(\Omega)$, by taking the weak limit in the equation, we have

$$-\Delta W - U + \nabla P = 0, \text{ in } L^2(\Omega).$$

Notice that $a^{1/2}u_k \to 0, a^{1/2}w_k \to 0$, strongly in $L^2(\Omega)$, and this implies that $U|_\omega = W|_\omega = 0$. Therefore, in a connect component ω' of ω, we have $\nabla P \equiv 0$. However, P is a harmonic function, then $P \equiv$const., thanks to the fact that Ω is connected. Note that $\int_\Omega P = 0$, hence $P \equiv 0$. Moreover, from Rellich theorem that $w_k \to W$ strongly in $L^2(\Omega)$, and on the other hand $\|h_k^2 u_k\|_{L^2(\Omega)} = o(1)$ we must have $W = 0$. Therefore $U = 0$. ■

2.5 Proof of the Observability Estimates

In this part, we will prove the Proposition 2.1 under the assumption in Theorem 2.1 on Ω and ω.

We argue by contradiction, suppose (**??**) is not true, we can then choose a sequence $(u_n, h_n, q_n, f_n) \in H^2(\Omega) \cap V \times \mathbb{R}_+ \times H^1(\Omega) \times H$ satisfies equation

$$-h_n^2 \Delta u_n - u_n + h_n \nabla q_n = f_n \tag{2.5.1}$$

with the following properties:

$$\|u_n\|_{L^2(\Omega)} = 1, \|f_n\|_{L^2(\Omega)} = o(h_n), \|a^{1/2}u_n\|_{L^2(\Omega)} = o(1), n \to \infty.$$

Up to extracting to subsequence, we can associate (u_n, h_n) with a semi-classical defect measure μ. From the h_n-oscillating(see the previous chapter) of the measure μ and $a^{1/2}u_k \to 0$ in $L^2(\Omega)$ we deduce that $\langle \mu, a\rangle = 0$, namely $\omega \cap \pi(\text{supp}(\mu)) = \emptyset$, where we denote $\pi : T^*\overline{\Omega} \to \overline{\Omega}$ be the canonical projection.

Denote $\phi(s,\rho)$ be the globally defined generalized bicharacteristic flow, thanks to the geometric assumption that Ω has no infinite contact. Pick any point ρ_0 with $\pi(\rho_0) \in \omega$.For any time segment $[0, s_0]$, there are several situations:

Either $\phi([0,s_0],\rho_0) \subset \Omega$, or there exist $\pi(\phi([0,s_0],\rho_0)) \cap \partial\Omega \neq \emptyset$, then from the assumption on Ω, all points $\phi(s,\rho_0)$ with $s \in [0,s_0]$ and $\pi(\phi(s,\rho_0)) \in \partial\Omega$ must lie in $\mathcal{H} \cup \mathcal{G}^{2,+} \cup \mathcal{G}^{2,-} \cup \bigcup_{k\geq 3}\mathcal{G}^k$. Now Theorem 2.2 implies that

$$\text{supp }(\phi(s,\cdot)_*\mu) \subset \text{ supp }(\mu).$$

Therefore, we have

$$\phi([0,s_0],\rho_0) \cap \text{ supp }(\mu) = \emptyset.$$

We now invoke the geometric control condition to deduce that

$$\overline{\Omega} \subset \pi \left(\bigcup_{\rho_0 \in \omega} \phi([0, T_0], \rho_0) \right)$$

for some $T_0 > 0$ and thus $\mu = 0$. This contradicts to the assumption that

$$\int_\Omega |u_n(x)|^2 dx = 1.$$

2.A Formal derivation of hyperbolic Stokes system

We will derive the hyperbolic Stokes system (2.1.2) from certain limit procedure of Lamé system from elastic theory:

$$\begin{cases} \partial_t^2 w - \mu \Delta w - (\lambda + \mu) \nabla \mathrm{div} w = 0, (t, x) \in [0, T] \times \Omega \\ w(t, .)|_{\partial\Omega} = 0 \\ (w(0), \partial_t w(0)) = (w_0, z_0) \in (H_0^1(\Omega) \times L^2(\Omega))^d \end{cases} \tag{2.A.1}$$

where the solution $w(t, x)$ is vector-valued.

Define $u(t, x) := w(t/\sqrt{\mu}, x)$, then we find that

$$\partial_t^2 u - \Delta u - \frac{\lambda + \mu}{\mu} \nabla \mathrm{div} u = 0.$$

We let $\epsilon = \frac{\mu}{\mu + \lambda} \ll 1$, in the case that $\lambda \gg \mu > 0$. Thus we obtain a family of equations

$$\begin{cases} \partial_t^2 u_\epsilon - \Delta u_\epsilon + \nabla p_\epsilon = 0, (t, x) \in [0, T] \times \Omega \\ u_\epsilon(t, .)|_{\partial\Omega} = 0 \\ (u_\epsilon(0), \partial_t u_\epsilon(0)) = (u_{0,\epsilon}, v_{0,\epsilon}) \in (H_0^1(\Omega) \times L^2(\Omega))^d \end{cases} \tag{2.A.2}$$

where $p_\epsilon = -\frac{1}{\epsilon} \mathrm{div} u_\epsilon$ and satisfies $\int_\Omega p_\epsilon dx = 0$.

We make further assumption on the family of initial data $(u_{0,\epsilon}, v_{0,\epsilon})$ so that

$$\|(u_{0,\epsilon}, v_{0,\epsilon}) - (u_0, v_0)\|_{H^1 \times L^2} \leq C\epsilon$$

for some divergence free data $(u_0, v_0) \in V \times H$. In particular, we have

$$\|\mathrm{div} u_{0,\epsilon}\|_{L^2(\Omega)} \leq C\epsilon.$$

From the well-posedness of Lamé system, we have that $u_\epsilon \in C([0, T]; H_0^1(\Omega)), \partial_t u_\epsilon \in C([0, T]; L^2(\Omega))$, and $p_\epsilon \in C([0, T]; L^2(\Omega))$. Moreover, we have the conservation of energy

$$E[u_\epsilon] = \frac{1}{2} \int_\Omega \left(|\partial_t u_\epsilon|^2 + |\nabla u_\epsilon|^2 + \epsilon |p_\epsilon|^2 \right) dx$$

and therefore
$$E[u_\epsilon] = \frac{1}{2}\int_\Omega \left(|u_{0,\epsilon}|^2 + |v_{0,\epsilon}|^2 + \frac{1}{\epsilon}|\mathrm{div} u_{0,\epsilon}|^2 \right) dx.$$
From this, we have, up to some subsequence of $(u_\epsilon, \partial_t u_\epsilon)$

$$\begin{aligned}
&\mathrm{div} u_\epsilon \to 0, \text{ in } L^\infty([0,T]; L^2(\Omega)),\\
&u_\epsilon * \rightharpoonup u, *\text{weakly in } L^\infty([0,T]; H_0^1(\Omega)),\\
&\partial_t u_\epsilon * \rightharpoonup \partial_t u, *\text{weakly in } L^\infty([0,T]; L^2(\Omega)).
\end{aligned}$$

From the uniform bound of $\|\partial_t u_\epsilon\|_{L^\infty([0,T];L^2(\Omega))}$, and apply Ascoli theorem, we have that (up to some subsequence)
$$u_\epsilon \to u, \text{ in } C([0,T]; L^2(\Omega)).$$
Using the equation, we conclude that $\|\nabla p_\epsilon\|_{L^\infty([0,T];H^{-1}(\Omega))}$ is uniformly bounded. Combine with the fact $\int_\Omega p_\epsilon = 0$, we have that $\|p_\epsilon\|_{L^\infty([0,T];L^2(\Omega))}$ is uniformly bounded, thus up to some subsequence, we may assume that

$$p_\epsilon * \rightharpoonup p, *\text{weakly in } L^\infty([0,T]; L^2(\Omega)).$$

Now it is not difficult to verify that (u,p) is a weak solution to (2.1.1). Moreover, p satisfies the zero mean condition
$$\int_\Omega p dx = 0.$$

Part II

Control and stabilization of KP type equations

Chapter 3

Exact Controllability for linear KP-II equation

This chapter is based on a joint-work with Ivonne Rivas.

3.1 Introduction

The Kadomtsev-Petviashvili equation better known as KP is

$$\partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) \pm \partial_y^2 u = 0 \tag{3.1.1}$$

and it was introduced by Kodomtsev and Petviashvili (see[33]) in 1970 from the study of transverse stability of the solitary wave solution of the Kortewed de-Vrie (KdV) equation. The KP equations are completely integrable and can be solved by inverse scattering transform. Moreover, the equation (3.1.1) has been studied separatyly depending the on the sign is used, with negative sign is known as KP-I equation, otherwise is the KP-II equation, these propagation of the trajectories behave very differently from one equation to another one and do not allow us to study at the same time. In this paper, we concentrate on the KP-II equation.

Concerning about the Cauchy problem, the KP-II equation has been well studied. In the pioneering work of J.Bourgain [6], he proved the global well-posedness of KP-II equation in $L^2(\mathbb{T}^2)$ by using the Fourier restriction norm introduced by himself. For non-periodic setting, Takaoka and Tzvetkov in [57] demonstrated local well-posedness in anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$ with $s_1 > -\frac{1}{3}$ and $s_2 \geq 0$. Hadac, Kerr and Koch in [27] proved global well-posedness and scattering for small data in critical functional space $H^{-\frac{1}{2},0}(\mathbb{R}^2)$. Molinet, Saut and Tzvetkov in [51] showed the local and global well-posedness for partially periodic data.

We will address the exact controllability problem for KP-II equation. Before getting into the problem, we observe that (3.1.1) can be written as

$$\partial_t u + \partial_x^3 u + u\partial_x u \pm \partial_x^{-1}\partial_y^2 u = 0,$$

where the Fourier multiplier ∂_x^{-1} is defined by

$$\widehat{\partial_x^{-1} v}(k,\eta) = \frac{1}{ik}\widehat{v}(k,\eta)$$

for all functions

$$v \in \mathcal{D}_0'(\mathbb{T}^2) := \{v \in \mathcal{D}'(\mathbb{T}^2) : \widehat{v}(0,l) = 0 \text{ for all } l \in \mathbb{Z}\}.$$

For any $s \in \mathbb{R}$, we denote by $H_0^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \cap \mathcal{D}_0'(\mathbb{T}^2)$, a closed subspace of $H^s(\mathbb{T}^2)$.

The internal control problem that we are interested in studying in this paper is as follows: Given $T > 0$ and $u_0, u_1 \in L_0^2$, does there exist a control input $h \in L^2((0,T); L^2(\mathbb{T}^2)$ in order to make the solution of

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = \mathcal{G}(h), & (t,x,y) \in \mathbf{R} \times \mathbb{T} \times \mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2), \end{cases} \tag{3.1.2}$$

satisfy $u(T,\cdot) = u_1$?

The first step is to consider the internal control problem for linearized KP-II equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = \mathcal{G}(h), & (t,x,y) \in \mathbf{R} \times \mathbb{T} \times \mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T} \times \mathbb{T}). \end{cases} \tag{3.1.3}$$

In order to keep the solution $u(t)$ in L_0^2, we need to define the control input $\mathcal{G}h$ to keep it in the space $\mathcal{D}_0'(\mathbb{T}^2)$. In this paper, we only consider the case where the control region ω is either a vertical strip or a horizontal strip.

For a vertical control region of the form $\omega = (a,b) \times \mathbb{T}$, we fix a non-negative real function $g \in C_c^2(\mathbb{T}^2)$ with $\int_{\mathbb{T}} g = 1$. In this case, we define the control input by

$$\mathcal{G}(h)(x,y) = \mathcal{G}_\perp(h)(x,y) := g(x)\left(h(x,y) - \int_{\mathbb{T}} g(x')h(x',y)dx'\right), \tag{3.1.4}$$

when the control region is a horizontal strip of the form $\omega = \mathbb{T} \times (a,b)$, we put the control input as

$$\mathcal{G}h(x,y) := \mathcal{G}_\|(h)(x,y) = g(y)\left(h(x,y) - \int_{\mathbb{T}} g(y')h(x,y')dy'\right). \tag{3.1.5}$$

Our first result, gives a positive answer to the internal controllability of the linearized KP-II equation on vertical region:

Theorem 3.1
Given $T > 0$. For any $u_0, u_1 \in L_0^2(\mathbb{T})$, there exists a control $h \in L^2((0,T); L^2(\mathbb{T}))$, such that the solution u of (3.1.3) with $\mathcal{G} = \mathcal{G}_\perp$ satisfies $u(T) = u_1$.

For vertical region, once the exact controllability for linearized KP-II is established, we can adapt the technique in the Cauchy theory of KP-II equation to prove exact controllability for KP-II in local sense.

Theorem 3.2
Given $T > 0$. There exists $R > 0$ such that for any $u_0, u_1 \in L_0^2(\mathbb{T}^2)$ satisfying $\|u_0\|_{L^2(\mathbb{T}^2)} \leq R$ and $\|u_1\|_{L^2(\mathbb{T}^2)} \leq R$, there exists a control $h_2 \in L^2((0,T); L_0^2(\mathbb{T}^2)$, such that the solution u of (3.1.2) with $\mathcal{G} = \mathcal{G}_\perp$ satisfies $u(T) = u_1$.

Remark 3.1
In [6], KP-II equation is globally well-posed in $H_0^s(\mathbb{T}^2)$ for all $s \geq 0$. Our results Theorem 3.1 and 3.2 also hold for any data in $H_0^s(\mathbb{T}^2)$. The reason for working in $L^2(\mathbb{T}^2)$ is that the quantity

$$\int_{\mathbb{T}^2} |u(t,x,y)|^2 dxdy$$

is conserved along KP flow (3.1.1) and hence L^2 is the natural space to address the control problem.

On the contrary, for the controllability on horizontal region, we have a negative answer which shows that the exact controllability for linearized KP-II equation can not hold at any time $T > 0$ when the control region is a horizontal strip.

Theorem 3.3
Given $T > 0$ there exists $u_1 \in L^2(\mathbb{T}^2)$ and there does not exist $h \in L^2((0,T); L_0^2(\mathbb{T}^2))$ such that the solution u of (3.1.3) with $\mathcal{G} = \mathcal{G}_{\|}$ satisfies $u(T) = u_1$.

The proof and disproof of controllability for linear equation rely on the propagation of singularity for KP-II flow. Because of the asymmetry in the horizontal,x, and vertical,y, coordinate, the waves described by the KP-II equation behave differently in the direction of propagation (x-direction) and transverse (y- direction). It turns out that the propagation on the horizontal direction is KdV like and much stronger than the propagation on the vertical direction. The heuristic is that any singularity will travel into some vertical control region in a very short time while the singularity cannot travel vertically into the horizontal control region in finite time. For this reason, we believe that the following formal criteria for the exact controllability is valid, although further efforts are needed to proved it:

1. If the control region ω satisfies that any horizontal geodesic will enter it before some time $T_0 > 0$, then (3.1.3) is exact controllable for any time $T > 0$.

2. If there is a horizontal geodesic which does not intersect with ω, then the exact controllability for (3.1.3) cannot hold for any time $T > 0$.

In fact, the setting of the control problem, namely the good definition of the operator $\mathcal{G}$, for general control region should different from what we have done for vertical and horizontal strip. It seems that there is no obvious way to keep the control input to be localized and simultaneously have zero horizontal mean. This observation suggests that we should look for the control problem directly for the equation

$$\partial_x(\partial_t u + \partial_x^3 u) + \partial_y^2 u = \mathcal{G}h$$

instead of the non local version (3.1.3).

The paper is organized as follows. In section 2, some results of well-posedness are mentioned, they will recover importance in the proof of the controllability of the full control system. In section 3, the linear controllability is established by proving the observability inequality. In section 4, the local controllability of the nonlinear equation is proved by fixed point argument. In section 5, we construct a counterexample to complete the proof of Theorem 3.3.

3.2 Notations and Preliminaries

Throughout this article, we use the identification $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) = [-\pi, \pi]/\mathbb{Z}_2$. We need the following classical inequality

Proposition 3.1 (Ingham inequality [29])
Suppose $\lambda_{k+1} - \lambda_k \geq \gamma$ for all $k \in \mathbb{Z}$. Then for all $T > \frac{2\pi}{\gamma}$, there exists two positive constants C_1, C_2 depending only on γ and T such that

$$C_1 \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_0^T \left| \sum_{k\in\mathbb{Z}} a_k e^{it\lambda_k} \right|^2 dt \leq C_2 \sum_{k\in\mathbb{Z}} |a_k|^2.$$

Now we briefly review the Cauchy theory of KP-II and we mainly follow the material in [51]. The initial value problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, & (t,x) \in \mathbf{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2), \end{cases} \tag{3.2.1}$$

is proved in [6] by Bourgain to be globally well-posed when $u_0 \in H^s(\mathbb{T}^2)$ for $s \geq 0$. Bourgain introduced a Fourier restriction norm by

$$\|u\|_{X^{s,b,b_1}}^2 = \int_{\mathbb{R}} \sum_{(k,l)\in\mathbb{Z}^2} \left\langle \frac{\langle \sigma(\tau,k,l)\rangle}{\langle k\rangle^3} \right\rangle^{2b_1} \langle \sigma(\tau,k,l)\rangle^{2b} \langle (k,l)\rangle^{2s} |\widehat{u}(\tau,k,l)|^2 d\tau$$

where $\sigma(\tau,k,l) = \tau - k^3 + \frac{l^2}{k}$ and $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$. For $T>0$, the norm in the localized time interval $[0,T]$ is defined by

$$\|u\|_{X_T^{s,b,b_1}} := \inf\{\|w\|_{X^{s,b,b_1}} : w(t) = u(t)\text{on}(0,T)\}.$$

Define by the linear evolution flow $S(t) = e^{-it\left(\partial_x^3+\partial_x^{-1}\partial_y^2\right)}$. We have the following linear estimate

Proposition 3.2
For $s \geq 0$, $-\frac{1}{2} < b' \leq 0 < \frac{1}{2} < b \leq b'+1$, $b_1 \in \mathbf{R}$ and $T \leq 1$, we have

$$\left\|\int_0^t S(t-t')F(t')dt'\right\|_{X_T^{s,b,b_1}} \leq CT^{1-(b-b')}\|F\|_{X_T^{s,b',b_1}}.$$

The proposition above is false for the end points $b' = -\frac{1}{2}$ and $b = \frac{1}{2}$. However, for periodic problem, it seems that we can not avoid to use these end points. To compromise, we need to use another norm

$$\|u\|_{Z^{s,b}} := \|\langle\sigma\rangle^{b-\frac{1}{2}}\langle(k,l)\rangle^s\widehat{u}\|_{l^2_{(k,l)}L^1_\tau}$$

and the restricted spaces $Z_T^{b,s}$ defined in the same manner. With these auxiliary norms, the linear estimate now holds true.

Proposition 3.3
Under the same conditions as in Proposition (3.2)

$$\left\|S(t)u_0 + \int_0^t S(t-t')F(t')dt'\right\|_{X_T^{s,\frac{1}{2},b_1}\cap Z_T^{s,\frac{1}{2}}} \leq C\|u_0\|_{H^s} + C\|F\|_{X_T^{s,-\frac{1}{2},b_1}\cap Z_T^{s,-\frac{1}{2}}}.$$

The essential of the proof can be found in [58]. To show that the equation (3.2.1) is locally well-posed in the spaces with the Fourier restriction norm through the integral form of the solution

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)u\partial_x u d\tau, \tag{3.2.2}$$

the following bilinear estimate is crucial

Proposition 3.4
([51]) There exist $\frac{1}{4} < b_1 < \frac{3}{8}$ and $\nu > 0$ such that for all functions $u, v \in X^{s,\frac{1}{2},b_1}$ with

$$\int_{\mathbb{T}} u(t,x,y)dx = \int_{\mathbb{T}} v(t,x,y)dx = 0,$$

the following bilinear estimate holds

$$\|\partial_x(uv)\|_{X_T^{s,-\frac{1}{2},b_1}\cap Z_T^{s,-\frac{1}{2}}} \leq CT^\nu\|u\|_{X_T^{s,\frac{1}{2},b_1}}\|v\|_{X_T^{s,\frac{1}{2},b_1}},$$

provided that $s \geq 0$.

We remark that this bilinear estimate is essentially established by J.Bourgain in [6]. We adapt to the statement in [51] here, in which the authors dealt with partially periodic data.

3.3 Linear controllability on vertical strip

In this section, the study of the internal controllability of linear system (3.1.3) is addressed by defining a linear operator in Proposition 3.10, which characterize the control input of the linear system and drives the solution from an initial state u_0 to a final state u_1. Notice that from reversability, the exact controllability is equivalent to null controllability: given any initial state $u_0 \in L_0^2$, find a function $h \in L^2((0,T)\times\mathbb{T}^2)$ so that the equation satisfies $u(0,\cdot) = u_0$ and $u(T,\cdot) = 0$. Hence, we will study the null controllability.

The classical strategy to study the null controllability is to show the observability inequality for the adjoint system associated to the equation, in the KP-II case, it matches with the homogeneous linearized KP-II equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = 0, & (t,x,y) \in \mathbf{R}\times\mathbb{T}\times\mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}),\ u|_{t=T} = u_1 \in L_0^2(\mathbb{T}), \end{cases} \tag{3.3.1}$$

From classical Hilbert Uniqueness Method (HUM), one can deduce that the null controllability is equivalent to the observability for its adjoint system.

Proposition 3.5
Given $T > 0$, the system (3.1.3) *is null controllable at T if and only if there exists a constant $C = C(T) > 0$ such that*

$$\|u_0\|_{L^2(\mathbb{T}^2)}^2 \le C\int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u(t,x)|^2 dxdt, \quad \forall u \in L^2(\mathbb{T}^2). \tag{3.3.2}$$

The region where the control will be placed is a vertical strip given by

$$\omega := (a,b)\times\mathbb{T}$$

and the operator $\mathcal{G} = \mathcal{G}_\perp$ is given by (3.1.4). The region ω will allow us to get a reduction of the KP-II equation (3.3.1) in one dimension. As it is stated in the following Remark:

Remark 3.2
Expanding the solution $u(t,x)$ to (3.3.1) *in Fourier series in y variable*

$$u(t,x,y) = \sum_{l\in\mathbb{Z}} a_l(t,x)e^{ily},$$

we find that for each $l \in \mathbb{Z}$, a_l satisfies the equation

$$\partial_t a_l + \partial_x^3 a_l - l^2\partial_x^{-1}a_l = 0$$

Therefore, by changing notations, the equation (3.3.1) *can be reduced to the study of following λ-dependent equation*

$$\begin{cases} \partial_t u + \partial_x^3 u - \lambda^2\partial_x^{-1} u = 0, & (t,x) \in \mathbf{R}\times\mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}), \end{cases} \tag{3.3.3}$$

3.3.1 Observability inequality

Thanks to Proposition 3.5, the proof of Theorem 3.1 is reduced to the proof of (3.3.2). From the previous remark and Plancherel's theorem, we can further reduce the observability (3.3.2) to the following uniform observability for the family of system (3.3.3).

Theorem 3.4
Given $T > 0$. There exists $C = C(T) > 0$ such that for all $\lambda > 0$,

$$\|u_0\|^2_{L^2(\mathbb{T})} \leq C \int_0^T \int_{\mathbb{T}} |\mathcal{G}u(t,x)|^2 dxdt \tag{3.3.4}$$

holds for all solution u of (3.3.3).

Now we concentrate to the proof of this theorem. The strategy is as follows. First we reduce (3.3.4) to a weaker one, which on the one hand is the observability for high frequencies and on the other hand gets rid of the normalization part. Next we use time-scaling and semi-classical reduction, inspired by the work of Lebeau in [42], to reduce this weak observability for system (3.3.3) to an inequality of the same form but for another semi-classical system. The third step is to reduce the inequality in the previous step to a frequency-localized one. Finally, we use propagation argument to prove the frequency-localized semi-classical observability.

Reduction to weak observability

The weak observability takes the form, uniformly in $\lambda \geq 0$,

$$\|u_0\|^2_{L^2(\mathbb{T})} \leq C \int_0^T \int_{\mathbb{T}} |g(x)u(t,x)|^2 dxdt + C\|u_0\|^2_{H^{-1}(\mathbb{T})}. \tag{3.3.5}$$

First, we prove a lemma concerning about the commutator of a high frequency cut-off and the operator $\mathcal{G}$

Lemma 3.1
Take $1-\chi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\chi) \subset \{|\xi| > 1\}$ and $\chi|_{|\xi|\geq 2} = 1$. Then, we have

$$\int_0^T \|[\chi(hD_x), \mathcal{G}]u(t,\cdot)\|^2_{L^2(\mathbb{T})} dt \leq Ch^2 \|u(0)\|^2_{L^2(\mathbb{T})}.$$

Proof.

$$\int_0^T \|[\chi(hD_x), \mathcal{G}]u(t,\cdot)\|^2_{L^2(\mathbb{T})} dt \leq C(I + II),$$

$$I = \int_0^T \int_{\mathbb{T}} |[g(x), \chi(hD_x)]u(t,x,y)|^2 dxdt,$$

$$II = \int_0^T \int_{\mathbb{T}} \left| g(x) \int_{\mathbb{T}} g(x')\chi(hD_x)u(t,x')dx' - \chi(hD_x)\left(g(x)\int_{\mathbb{T}} g(x')u(t,x')dx'\right)\right|^2 dxdt,$$

Symbolic calculus yields (though g is not assumed to be smooth, the following estimate still valid)

$$\|[g(x), \chi(hD_x)]\|_{L^2\to L^2} \leq Ch,$$

and hence due to the conservation of L^2 norm

$$I \leq Ch^2 \int_0^T \|u(t)\|^2_{L^2(\mathbb{T})} dt = Ch^2 T \|u(0)\|^2_{L^2(\mathbb{T})}.$$

For II, we first calculate (for simplicity the variable t is omitted in here)

$$\begin{aligned}
&\left(g(x)\int_{\mathbb{T}} g(x')(\chi(hD_x)u)(x')dx'\right)^{\widehat{}}(l) - \chi(hD_x)\left(g(x)\int_{\mathbb{T}} g(x')u(x')dx'\right)^{\widehat{}}(l) \\
=&\widehat{g}(l)\sum_{l_1\neq 0}(\chi(hl_1)-\chi(hl))\,\widehat{g}(l_1)\widehat{u}(l).
\end{aligned}$$

Since,

$$|\chi(hl_1)-\chi(hl)| \leq \|\chi'\|_{L^\infty} h|l_1 - l|,$$

we have

$$\begin{aligned}
II \leq& Ch^2 \sum_l |\widehat{g}(l)|^2 \left|\sum_{l_1\neq 0} |l_1 - l|\widehat{g}(l_1)\widehat{u}(l_1)\right|^2 \\
\leq& Ch^2 \sum_l |\widehat{g}(l)|^2 \left(\sum_{l_1\neq 0} |l_1-l|^2|\widehat{g}(l_1)|^2\right)\left(\sum_{l_1\neq 0}|\widehat{u}(l_1)|^2\right) \\
\leq& Ch^2\|u\|^2_{L^2(\mathbb{T})} \sum_{l,l_1\neq 0} |l_1-l|^2|\widehat{g}(l_1)|^2|\widehat{g}(l)|^2 \\
=& Ch^2\|u\|^2_{L^2(\mathbb{T})},
\end{aligned}$$

due to $g \in C_c^2(\mathbb{T})$. ■

Proposition 3.6
(3.3.5) *implies the following full observability inequality*

$$\|u_0\|^2_{L^2(\mathbb{T})} \leq C\int_0^T\int_{\mathbb{T}} |\mathcal{G}u(t,x)|^2 dxdt. \tag{3.3.6}$$

Proof. The proof is essentially a unique continuation argument. However, by the λ-dependence family of equations, we will divide the proof in two steps.

First, we prove that for any given $\lambda > 0$, (3.3.6) holds with constant $C > 0$ which may depend on λ. We argue by contradiction, assuming that (3.3.6) is not true, then we can select a sequence u_n of solutions to (3.3.3) so that

$$\|u_n(0)\|_{L^2(\mathbb{T})} = 1 \quad \text{and} \quad \lim_{n\to\infty}\int_0^T\int_{\mathbb{T}} |\mathcal{G}u_n(t,x)|^2 dxdt = 0.$$

After extracting to some subsequence, we may assume that $u_n(0) \rightharpoonup u_0$, weakly in $L^2(\mathbb{T})$. One can easily verify that $u_0 \in L^2_0$. Moreover, from semi-group property, $u_n(t) \rightharpoonup u(t)$ weakly in $C([0,T];L^2(\mathbb{T}))$ and $u(t)$ is the distributional solution to (3.3.3) with initial data u_0. Since $\mathcal{G} : L^2_0 \to L^2_0$ is a bounded operator, we have that $\mathcal{G}u(t,\cdot) = 0$ in $L^2_0(\mathbb{T})$ for a.e. $t \in [0,T]$. This means that $u(t,x)|_\omega = C(t)$ in $\mathcal{D}'(\omega)$ for a.e. $t \in [0,T]$. Moreover, from the strong continuity of the semi-group on L^2_0,

$$C(t) = \int_{\mathbb{T}} g(x)u(t,x)dx, \forall t \in [0,T].$$

and $C(t)$ is a continuous function in t. Therefore we have that

$$g(x)\left(u(t,x) - C(t)\right) = 0, \text{ in } C([0,T];L^2(\mathbb{T}))$$

and thus $u(t,x)|_{x\in\omega} = C(t)$ in $\mathcal{D}'(\omega)$ for all $t \in [0,T]$. Now, if we rewrite the equation (3.3.3) as $\partial_x(\partial_t u + \partial_x^3 u) + \lambda u = 0$ and evaluate u for $x \in \omega$, we have that $u|_\omega = 0$ in $\mathcal{D}'(\omega)$.

We claim that $u \equiv 0$. Indeed, consider the following set:

$$\mathcal{N} := \{u_0 \in L^2_0(\mathbb{T}) : u(t,\cdot)|_\omega = 0, \forall t \in [0,T]\}.$$

Apply inequality (3.3.5) and we have that

$$\|u_0\|_{L^2} \leq C\|u_0\|_{H^{-1}(\mathbb{T})}$$

for all $u_0 \in \mathcal{N}$. This implies that the subspace $\mathcal{N}$ in $L^2_0(\mathbb{T})$ is finite dimensional.

Thus, for any $u_0 \in \mathcal{N}$, we can write the solution in the form

$$u(t,x) = \sum_{1\leq|l|\leq M} a_l e^{it(l^3-\lambda^2 l^{-1})}e^{ilx},$$

this trigonometric polynomial is smooth and it vanishes in ω. From classical result (see for instance [44]), $a_l \equiv 0$ for all $1 \leq |l| \leq M$. This implies that $u \equiv 0$.

Since the weak limit of $u_n(0)$ is 0, we have $\int_{\mathbb{T}} g(x)u_n(t,x)dx \to 0$ and hence $\|gu_n\|_{L^2([0,T]\times\mathbb{T})} \to 0$. Moreover, up to a subsequence, we have $\|u_n(0)\|_{H^{-1}(\mathbb{T})} \to 0$, due to Rellich theorem. This is a contradiction to the assumption that $\|u_n(0)\|_{L^2(\mathbb{T})} = 1$.

For the second step, we need to prove (3.3.6) uniformly in λ. Again, we assume that (3.3.6) is not true. Then there are a sequence of positive numbers $\lambda_n > 0$ and a sequence solutions u_n to (3.3.3) with parameters λ_n such that

$$\|u_n(0)\|_{L^2(\mathbb{T})} = 1 \quad \text{and} \quad \lim_{n\to\infty}\int_0^T\int_{\mathbb{T}} |\mathcal{G}u_n(t,x)|^2 dxdt = 0.$$

Up to a subsequence, we may assume that $\lambda_n \to \lambda_\infty \in [0, \infty]$. Suppose $\lambda_\infty < \infty$, similar argument as in Step 1 will lead to contradiction.

The possibility that is left to study is $\lambda_\infty = \infty$. We write

$$u_n(0) = \sum_{l \neq 0} a_{n,l} e^{ilx}$$

and the corresponding solution to (3.3.3) satisfies

$$u_n(t,x) = \sum_{l \neq 0} a_{n,l} e^{it\left(l^3 - \frac{\lambda_n^2}{l}\right)} e^{ilx}.$$

For any $\epsilon_0 > 0$, we set

$$u_n^{(\epsilon_0)} := \sum_{|l| \geq \frac{1}{\epsilon_0}} a_{n,l} e^{it\left(l^3 - \frac{\lambda_n^2}{l}\right)} e^{ilx}, \quad v_n^{(\epsilon_0)} = u_n - u_n^{(\epsilon_0)}.$$

From Lemma 3.1, we have

$$\int_0^T \|\mathcal{G} u_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2 \|u_n(0)\|_{L^2(\mathbb{T})}^2 + C \int_0^T \|(\mathcal{G} u_n)^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt.$$

Thus, there exists $C > 0$ such that for any $\epsilon_0 > 0$, we have

$$\begin{aligned} \limsup_{n \to \infty} \int_0^T \|\mathcal{G} u_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2, \\ \limsup_{n \to \infty} \int_0^T \|\mathcal{G} v_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2. \end{aligned} \tag{3.3.7}$$

For any $\epsilon > 0$ small, we can take $\epsilon_0 > 0$ small enough such that

$$\sum_{|l| \geq \frac{1}{\epsilon_0}} |\widehat{g}(l)|^2 \leq \epsilon^2,$$

and then

$$\left\| g(x) \int_{\mathbb{T}} g(x') u_n^{(\epsilon_0)}(t, x') dx' \right\|_{L^2(\mathbb{T})}^2 \leq \epsilon^2 \|g\|_{L^2(\mathbb{T})}^2 \|u_n^{(\epsilon_0)}(0)\|_{L^2(\mathbb{T})}^2.$$

Thus, from (3.3.5),

$$\begin{aligned} \|u_n^{(\epsilon_0)}(0)\|_{L^2(\mathbb{T})}^2 &\leq C\epsilon^2 + C\epsilon_0^2 + \|u_n^{(\epsilon_0)}(0)\|_{H^{-1}(\mathbb{T})}^2 \\ &\leq C(\epsilon^2 + \epsilon_0^2), \end{aligned}$$

holds true for n large enough.

On the other hand, direct calculation yields

$$\begin{aligned}\int_0^T \|\mathcal{G}v_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt =& \int_0^T \sum_l \left| \sum_{1\leq |l_1|\leq 1/\epsilon_0} (\widehat{g}(l-l_1)-\widehat{g}(l)\widehat{g}(l_1))a_{n,l_1} e^{it\left(l_1^3-\frac{\lambda_n^2}{l_1}\right)} \right|^2 dt \\ \geq& C\sum_l \sum_{1\leq |l_1|\leq 1/\epsilon_0} |\widehat{g}(l-l_1)-\widehat{g}(l)\widehat{g}(l_1)|^2 |a_{n,l_1}|^2 \\ =& C\sum_{1\leq |l_1|\leq 1/\epsilon_0} c_{l_1}|a_{n,l_1}|^2\end{aligned}$$

with $c_{l_1} = \sum_l |\widehat{g}(l-l_1)-\widehat{g}(l)\widehat{g}(l_1)|^2$, where we have used the Ingham inequality, due to the assumption that $\lambda_n \to \infty$. Notice that the constant C can be chosen independent of n and ϵ_0, provided that we choose n large enough such that

$$\sum_{1\leq |l_1|\leq 1/\epsilon_0} \left| (l_1+1)^3 - l_1^3 - \frac{\lambda_n^2}{l_1} + \frac{\lambda_n^2}{l_1+1} \right| \geq \gamma > 0 \quad \text{and} \quad T > \frac{2\pi}{\gamma}.$$

Note that $c_{l_1} \geq |\widehat{g}(0) - \widehat{g}(l_1)^2|^2$ and $\widehat{g}(0) = 1$, hence there exists a constant $c_0 > 0$, independent of ϵ_0, ϵ and n, so that $c_{l_1} \geq c_0$ for all $1 \leq |l_1| \leq 1/\epsilon_0$. Thus, for sufficiently large n,

$$\|v_n^{(\epsilon_0)(0)}\|_{L^2(\mathbb{T})}^2 \leq \frac{C}{c_0}\int_0^T \|\mathcal{G}v_n^{(\epsilon_0)}(t)\|_{L^2(\mathbb{T})}^2 dt \leq C\epsilon_0^2.$$

Therefore,

$$1 = \limsup_{n\to\infty} \|u_n(0)\|_{L^2(\mathbb{T})}^2 = \|u_n^{\epsilon_0}(0)\|_{L^2(\mathbb{T})}^2 + \|v_n^{\epsilon_0}(0)\|_{L^2(\mathbb{T})}^2 \leq C(\epsilon_0^2+\epsilon^2) < 1,$$

which cannot happen. ■

3.3.2 Reduction to semi-classical observability

Now, we consider the semi-classical equation of the following form:

$$\begin{cases} h\partial_t u + (h\partial_x)^3 u - (h\partial_x)^{-1} u = 0, & (t,x) \in \mathbf{R}\times\mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}), \end{cases} \tag{3.3.8}$$

Proposition 3.7
Assume that there exists $T_0 > 0$ such that the following semi-classical observability

$$\|u_0\|_{L^2(\mathbb{T})}^2 \leq C\int_0^{T_0}\int_{\mathbb{T}} |g(x)u(t,x)|^2 dxdt + C\|u_0\|_{H^{-1}(\mathbb{T})}^2 \tag{3.3.9}$$

holds for any solution u to (3.3.8) *with initial data $u_0 \in L_0^2$, uniformly for $0 < h < 1$. Then for any $T > 0$, observability inequality* (3.3.5) *holds true.*

Proof. It sufficient to prove (3.3.5) when $\lambda > 1$ is large enough since for bounded $\lambda \geq 0$, it can be viewed as a pertubation of linear KdV equation and the constant C in front of the right hand side can be chosen to be continuously depended on λ. For λ large enough, we write $\lambda^2 = \frac{1}{h^4}$ and (3.3.3) becomes

$$h^3\partial_t u + (h\partial_x)^3 u - (h\partial_x)^{-1} u = 0.$$

Consider the scaling in time variable $w(t,x) = u(h^2 t, x)$, we have

$$h\partial_t w + (h\partial_x)^3 w - (h\partial_x)^{-1} w = 0.$$

Now from (3.3.9), we have

$$\|w(0)\|^2_{L^2(\mathbb{T})} \leq C \int_0^{T_0} \int_{\mathbb{T}} |g(x)w(t,x)|^2 dxdt + C\|w(0)\|^2_{H^{-1}(\mathbb{T})},$$

changing back to the variable $u(t,x)$,

$$\|u(0)\|^2_{L^2(\mathbb{T})} \leq \frac{C}{h^2} \int_0^{h^2 T_0} \int_{\mathbb{T}} |g(x)u(s,x)|^2 dxds + C\|u(0)\|^2_{H^{-1}(\mathbb{T})}.$$

Due to the time-translation invariant and conservation of H^s-norm under the semi-classical flow, we have for any $M \in \mathbb{N}$,

$$\begin{aligned}
\|u(Mh^2T_0)\|^2_{L^2\mathbb{T}} &= \|u(0)\|^2_{L^2(\mathbb{T})} \\
&\leq \frac{C}{h^2} \int_{Mh^2T_0}^{(M+1)h^2T_0} \int_{\mathbb{T}} |g(x)u(s,x)|^2 dxds + C\|u(Mh^2T_0)\|^2_{H^{-1}(\mathbb{T})} \\
&= \frac{C}{h^2} \int_{Mh^2T_0}^{(M+1)h^2T_0} \int_{\mathbb{T}} |g(x)u(s,x)|^2 dxds + C\|u(0)\|^2_{H^{-1}(\mathbb{T})}
\end{aligned}$$

Summing M from 0 to $\epsilon_0 h^{-2}$ with $\epsilon_0 T_0 \leq T$,

$$\|u(0)\|^2_{L^2(\mathbb{T})} \leq \frac{C}{\epsilon_0} \int_0^T \int_{\mathbb{T}} |g(x)u(t,x)|^2 dxdt + \frac{C}{\epsilon_0}\|u(0)\|^2_{H^{-1}(\mathbb{T})}.$$

This completes the proof. ∎

Reduction to frequency localized semi-classical observability

We use a standard homogeneous Littlewood-Paley decomposition. Take $\chi \in C_c^\infty(\mathbb{R})$ with support supp$\psi \subset \{1/2 \leq |\xi| \leq 2\}$ and $\psi_k \in C_c^\infty(\mathbb{R})$ such that

$$\sum_{k\in\mathbb{Z}} \psi_k(\xi) = 1, \forall \xi \neq 0,$$

where $\psi_k(\xi) = \psi(2^k \xi)$.

Proposition 3.8
There exists $\epsilon_0 > 0, h_0 > 0$, *small and* $T_0 > 0$, $C_0 = C_0(\epsilon_0) > 0$ *such that for all* $k \in \mathbb{Z}$, *with* $2^k h \leq \epsilon_0$,

$$\|\psi_k(hD_x)u(0)\|_{L^2(\mathbb{T})}^2 \leq C_0 \int_0^{T_0} \int_{\mathbb{T}} |g(x)\psi_k(hD_x)u(t,x)|^2 dxdt \tag{3.3.10}$$

holds true for all solutions $u(t,x)$ *of* (3.3.8), *uniformly in* $h \in (0, h_0)$.

We will prove this proposition in the next subsection. In fact, from the proof, we can deduce that if Proposition 3.8 holds true for some $\epsilon_0 > 0, h_0 > 0$, it is also true for any other parameter ϵ_1, h_1 such that $\epsilon_1 < \epsilon_0$ and $h_1 < h_0$ without change of the constant $C_0 = C_0(\epsilon_0)$.

Lemma 3.2
Proposition 3.8 implies (3.3.9).

Indeed, applying Lemma (3.1), we have

$$\begin{aligned}\|g\psi_k(hD_x)u\|_{L^2(\mathbb{T})}^2 \leq & 2\|\psi_k(hD_x)(gu)\|_{L^2(\mathbb{T})}^2 + 2\|[\psi(2^k hD_x), g]u\|_{L^2(\mathbb{T})}^2 \\ \leq & 2\|\psi_k(hD_x)(gu)\|_{L^2(\mathbb{T})}^2 + C_0(2^k h)^2 \|u(t)\|_{L^2(\mathbb{T})}^2,\end{aligned}$$

where the constant $C_0 = C_0(\epsilon_0)$ for some fixed but small $\epsilon_0 > 0$. Now for any $\epsilon_1 > 0$, we have

$$\begin{aligned}\sum_{k \leq \log_2(\epsilon_1/h)} \|\psi_k(hD_x)u(0)\|_{L^2(\mathbb{T})}^2 \leq & C_0 \sum_{k \leq \log_2(\epsilon_1/h)} \int_0^{T_0} \|\psi_k(hD_x)(gu(t))\|_{L^2(\mathbb{T})}^2 \\ & + C_0 T_0 \sum_{k \leq \log_2(\epsilon_1/h)} (2^k h)^2 \|u(0)\|_{L^2(\mathbb{T})}^2 \\ \leq & C_0 \int_0^{T_0} \|gu(t)\|_{L^2(\mathbb{T})}^2 dt + C_0 T_0 \epsilon_1^2 \|u(0)\|_{L^2(\mathbb{T})}^2.\end{aligned}$$

Thus, from Littlewood-Paley decomposition,

$$\|u(0)\|_{L^2(\mathbb{T})}^2 \leq C_0 \int_0^{T_0} \int_{\mathbb{T}} |g(x)u(t,x)|^2 dxdt + C_0 T_0 \epsilon_1^2 \|u(0)\|_{L^2(\mathbb{T})}^2 + C_0 \|u(0)\|_{H^{-1}(\mathbb{T})}^2.$$

We then choose $\epsilon_1^2 < \frac{C_0 T_0}{2}$ to obtain (3.3.9).

In summary, we have showed that in order to prove the uniform observability inequality (3.3.6) for all solutions of (3.3.3), it suffices to prove the observability (3.3.10) for all solutions of (3.3.8), uniformly in $0 < h \ll 1$ and $k \in \mathbb{Z}$ such that $2^k h < \epsilon_0$ holds.

Propagation estimate with parameter dependence symbol

This section is devoted to the proof of Proposition 3.8. We need some preparation about $\tilde{h}-$pseudo-differential calculus. For $m \in \mathbb{R}$, let S^m be the set of $\tilde{h}$-dependent functions $a(x,\xi,\tilde{h})$ with parameter $\tilde{h} \in (0,1)$ such that for any indices α, β,

$$\sup_{(x,\xi,\tilde{h})\in\mathbb{R}^{2d}\times(0,1)} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi,\tilde{h})| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}.$$

For $a \in S^m$, we denote by $\mathrm{Op}_{\tilde{h}}(a)$ the $\tilde{h}-$pseudo-differential operator acting on Schwartz functions via

$$\mathrm{Op}_{\tilde{h}}(a)f(x) := \frac{1}{(2\pi\tilde{h})^d}\int_{\mathbb{R}^{2d}} e^{\frac{i(x-y)\cdot\xi}{\tilde{h}}} a(x,\xi,\tilde{h})f(y)dyd\xi.$$

We refer [63] for symbolic calculus and another basic properties about $\tilde{h}-$pseudo-differential operator. For functions on a compact Riemannian manifold, $\tilde{h}-$ we can also define $\tilde{h}-$pseudo-differential operator by using local coordinate and partition of unity.

Now let us consider the following $\epsilon-$dependence symbols:

$$p_\epsilon(x,\xi) = \left(\frac{\epsilon^4}{\xi} - \xi^3\right)\chi(\xi), \quad q_\epsilon(x,\xi) = \left(\frac{1}{\xi} - \epsilon^4\xi^3\right)\chi(\xi),$$

where $\chi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\xi) \subset \{\alpha < |\xi| < \beta\}$ for some $0 < \alpha < \frac{1}{2}$, $\beta > 2$ and $\chi \equiv 1$ in a neighborhood of $\{1/2 \leq |\xi| \leq 2\}$. Denote $P_\epsilon = \mathrm{Op}_{\tilde{h}}(p_\epsilon)$ and $Q_\epsilon = \mathrm{Op}_{\tilde{h}}(q_\epsilon)$. Denote $U_\epsilon(t), V_\epsilon(t)$ solution operators to the equations

$$\begin{cases} \frac{\tilde{h}}{i}\partial_t U_\epsilon(t) + U_\epsilon(t)P_\epsilon = 0, \\ U_\epsilon(0) = I, \end{cases} \tag{3.3.11}$$

$$\begin{cases} \frac{\tilde{h}}{i}\partial_t V_\epsilon(t) + V_\epsilon(t)Q_\epsilon = 0, \\ V_\epsilon(0) = I \end{cases} \tag{3.3.12}$$

The flow associated to the vector fields $H_{p_\epsilon}, H_{q_\epsilon}$ is explicitly given by

$$\phi_{\epsilon,t}(x_0,\xi_0) = \left(x_0 - \left(\frac{\epsilon^4}{\xi_0^2} + 3\xi_0^2\right)\chi(\xi_0)t + \left(\frac{\epsilon^4}{\xi_0} - \xi_0^3\right)\chi'(\xi_0)t, \xi_0\right),$$

$$\varphi_{\epsilon,t}(x_0,\xi_0) = \left(x_0 - \left(\frac{1}{\xi_0^2} + 3\epsilon^4\xi_0^2\right)\chi(\xi_0)t + \left(\frac{1}{\xi_0} - \epsilon^4\xi_0^3\right)\chi'(\xi_0)t, \xi_0\right)$$

with respectively.

From Egorov's theorem (see [63]), we know that for any symbol $a(x,\xi) \in C_c^\infty(T^*M)$,

$$U_\epsilon(-t)\mathrm{Op}_{\tilde{h}}(a)U_\epsilon(t) = \mathrm{Op}_{\tilde{h}}(a\circ\phi_{\epsilon,t}) + O_{L^2\to L^2}(\tilde{h}),$$

$$V_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)V_\epsilon(t) = \mathrm{Op}_{\widetilde{h}}(a\circ\varphi_{\epsilon,t}) + O_{L^2\to L^2}(\widetilde{h}).$$

We remark that the bound $O_{L^2\to L^2}(\widetilde{h})$ is independent of $\epsilon \leq 1$ since all the semi-norms of the symbol p_ϵ, q_ϵ can be chosen continuously depending on ϵ.

Now we prove the following localized observability estimates:

Proposition 3.9
There exists $C_0 > 0, T_0 > 0, \widetilde{h}_0 > 0$ such that for all $u_0 \in L_0^2(\mathbb{T})$, all $\widetilde{h} \leq \widetilde{h}_0$

$$\|\psi(\widetilde{h}D_x)u_0\|_{L^2(\mathbb{T})}^2 \leq C_0 \int_0^{T_0} \|gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|_{L^2(\mathbb{T})}^2 dt, \tag{3.3.13}$$

$$\|\psi(\widetilde{h}D_x)u_0\|_{L^2(\mathbb{T})}^2 \leq C_0 \int_0^{T_0} \|gV_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|_{L^2(\mathbb{T})}^2 dt. \tag{3.3.14}$$

Proof. Here we only prove the first inequality, and the second one will follow in the same manner. Consider the symbol $a(x,\xi) = g(x)^2\widetilde{\psi}(\xi)$ (strictly speaking, g is not smooth and we need approximate it by smoothing functions) and its quantization $\mathrm{Op}_{\widetilde{h}}(a) = (g(x))^2\widetilde{\psi}(\widetilde{h}D_x)$, where $\widetilde{\psi}$ is a slight enlargement of ψ wo that $\widetilde{\psi}\psi = \psi$ and $\mathrm{supp}\widetilde{\psi} \subset \{\alpha < |\xi| < \beta\}$. From Egorov's theorem, we have

$$U_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)U_\epsilon(t) = \mathrm{Op}_{\widetilde{h}}(a\circ\phi_{\epsilon,t}) + O_{L^2\to L^2}(\widetilde{h}), \text{ uniformly in } \epsilon \leq 1.$$

Note that on the support of a, $\chi'(\xi) = 0$, and thus we have

$$\varphi_{\epsilon,t}(x_0,\xi_0) = \left(x_0 - \left(\frac{\epsilon^4}{\xi_0^2} + 3\xi_0^2\right)t, \xi_0\right).$$

Notice that $\left|\frac{\epsilon^4}{\xi_0^2} + 3\xi_0^2\right| \geq c_0 > 0$, uniformly in ϵ, on the $\xi-$support of $\widetilde{\psi}$. Therefore, for some $T_0 = T_0(c_0) > 0$, and $c_1 > 0$, we have

$$\int_0^{T_0} a\circ\phi_{\epsilon,t}dt \geq c_1 > 0.$$

Now we calculate

$$\begin{aligned}
&\int_0^{T_0} \|gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|_{L^2(\mathbb{T})}^2 dt \\
=&\int_0^{T_0} \Big(gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0, gU_\epsilon(t)\widetilde{\psi}(\widetilde{h}D_x)\psi(\widetilde{h}D_x)u_0\Big)_{L^2(\mathbb{T})} dt \\
=&\int_0^{T_0} \Big(U_\epsilon(-t)\widetilde{\psi}(\widetilde{h}D_x)g^2U_\epsilon(t)u_0, \psi(\widetilde{h}D_x)u_0\Big)_{L^2(\mathbb{T})} dt \\
=&\Big(\mathrm{Op}_{\widetilde{h}}(b_{T_0})\psi(\widetilde{h}D_x)u_0, \psi(\widetilde{h}D_x)u_0\Big)_{L^2(\mathbb{T})},
\end{aligned}$$

with $b_{T_0}(x,\xi)=\int_0^{T_0} a\circ\phi_{\epsilon,t}dt$ modulo $\widetilde{h}S^0$. Thus, from Sharp Gårding inequality,

$$\left(\mathrm{Op}_{\widetilde{h}}(b_{T_0})\psi(\widetilde{h}D_x)u_0,\psi(\widetilde{h}D_x)u_0\right)_{L^2(\mathbb{T})}\geq\frac{c_1}{2}\|\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})}-C\widetilde{h}\|\psi(\widetilde{h}D_x)u_0\|^2_{L^2(\mathbb{T})}.$$

To conclude the proof, we just need choose $\widetilde{h}_0<\min\{\frac{c_1}{4C},1\}$. ■

Now we prove Proposition 3.8

Proof. Fox fixed $h\ll 1$, we divide $k\in\mathbb{Z}$ into three regimes:

Regime I: $|k|\leq N_0$ for some large natural number N_0

This regime corresponds to the case $|\xi|\sim 1$. Let $u_k=\psi_k(hD_x)u$, the equation satisfied by u_k is simple (3.3.8). In this case, we can either use (3.3.13) or (3.3.14) with parameter $\epsilon=1$ to obtain that (note that $\widetilde{h}=2^k\widetilde{h}\sim h$ in this regime)

$$\|\psi_k(hD_x)u_0\|^2_{L^2(\mathbb{T})}\leq C_0\int_0^{T_0}\int_0^{T_0}\|g\psi_k(hD_x)u(t)\|^2_{L^2(\mathbb{T})}dt.$$

Regime II: $k\leq -N_0$ for some large constant N_0

Look back to our first micro-localization, this case corresponds to $|\xi|\sim 2^{-k}\gg 1$. Define a new semi-classical parameter $\widetilde{h}_k=2^kh\ll 1$ and rescale the time variable by setting $w_k(t,x):=\psi(\widetilde{h}_kD_x)u(2^{2k}t,x)$, $u_k=\psi(\widetilde{h}_kD_x)u$, we find the equation satisfied by w_k:

$$\widetilde{h}_k\partial_t w_k+(\widetilde{h}_k\partial_x)^3w_k+2^{4k}(\widetilde{h}\partial_x)^{-1}w_k=0.$$

then by applying (3.3.13) to w_k with $\epsilon=2^k\ll 1$ and $\widetilde{h}=\widetilde{h}_k$ we obtain

$$\|w_k(0)\|^2_{L^2(\mathbb{T})}\leq C\int_0^{T_0}\|gw_k(t)\|^2_{L^2(\mathbb{T})}dt.$$

From conservation of L^2-norm along the flow, we apply the inequality above $2^{-2k}-1$ times to obtain

$$\begin{aligned}\frac{1}{2^{2k}}\|u_k(0)\|^2_{L^2(\mathbb{T})}&\leq\frac{C}{2^{2k}}\sum_{M=0}^{2^{-2k}-1}\int_{M2^{2k}T_0}^{(M+1)2^{2k}T_0}\|gu_k(t)\|^2_{L^2(\mathbb{T})}dt\\&=\frac{C}{2^{2k}}\int_0^{T_0}\|gu_k(t)\|^2_{L^2(\mathbb{T})}dt,\end{aligned}$$

and this is exactly

$$\|\psi_k(hD_x)u(0)\|^2_{L^2(\mathbb{T})}\leq C\int_0^{T_0}\|g\psi_k(hD_x)u(t)\|^2_{L^2(\mathbb{T})}dt.$$

Regime III: $k \geq N_0$

This case corresponds to $|\xi| \sim 2^{-k} \ll 1$. Define the new small semi-classical parameter $\tilde{h}_k = 2^k h$, thanks to the restriction that $2^k h \leq \epsilon_0 \ll 1$.

Denote $u_k = \psi(\widetilde{h}_k D_x)u$ and define $v_k(t,x) = u_k(2^{-2k}t, x)$. v_k solves the equation

$$\widetilde{h}_k \partial_t v_k + 2^{-4k}(\widetilde{h}_k \partial_x)^3 v_k + (\widetilde{h}_k \partial_x)^{-1} v_k = 0.$$

Applying (3.3.14) with $\widetilde{h} = \widetilde{h}_k, \epsilon = 2^{-k}$, we obtain that

$$\|v_k(0)\|^2_{L^2(\mathbb{T})} \leq C \int_0^{T_0} \|g v_k(t)\|^2_{L^2(\mathbb{T})} dt.$$

Again by conservation of L^2-norm as in the argument of regime II, we finally have

$$\|u_k(0)\|^2_{L^2(\mathbb{T})} \leq C \int_0^{T_0} \|g u_k(t)\|^2_{L^2(\mathbb{T})} dt.$$

■

Once the observability inequality (3.3.5) has been established the internal controllability for the linear KP II is obtained, we conclude this section by summarizing it in the following proposition:

Proposition 3.10
For $T > 0$ given. There exists a bounded linear operator

$$\Upsilon : (L^2(\mathbb{T}^2))^2 \to L^2(0,T; L^2(\mathbb{T}^2))$$

such that for any $u_0, u_T \in L^2(\mathbb{T}^2)$, the control defined by $h := \Upsilon(u_0, u_1)$ drives the solution of

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = \mathcal{G}h, \quad (t,x) \in \mathbf{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0, \end{cases} \tag{3.3.15}$$

to $u(T) = u_1$.
Moreover, there exists $C > 0$ such that for any $u_0, u_1 \in L^2(\mathbb{T}^2)$, we have

$$\|\Upsilon(u_0,u_1)\|_{L^2(0,T;L^2(\mathbb{T}^2))} \leq C\|(u_0,u_1)\|_{(L^2(\mathbb{T}^2))^2}.$$

3.4 Local controllability of Nonlinear equation

For the full KP-II control system

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = \mathcal{G}h, \quad (t,x) \in \mathbf{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0,\ u|_{t=T} = u_1, \end{cases} \tag{3.4.1}$$

in order to prove the existence of a $u \in L^2(0,T;L^2(\mathbb{T}^2))$ solving $u|_{t=0} = u_0$, $u|_{t=T} = u_1$, we will reduce it to a fix point problem by standard argument. The solution defined by (3.4.1) with control input h is given by

$$u(t) = S(t)u_0 + \upsilon(t,u) + \int_0^t S(t-t')\mathcal{G}h(t')dt'$$

with

$$\upsilon(t,u) = \int_0^t S(t-t')u\partial_x u dt'.$$

It must satisfy

$$u_1 = S(T)u_0 + v(T,u) + \int_0^T S(T-t')\mathcal{G}h(t')dt.$$

Choosing the control input of the form $h = \Upsilon(u_0, w)$, this implies that

$$S(T)u_0 + \int_0^T S(T-t')\mathcal{G}h(t')dt' = w.$$

This indicates that $w = u_1 - \upsilon(T,u)$. In summary, define the nonlinear map Γ by

$$\Gamma(u) = S(t)u_0 + \upsilon(t,u) + \int_0^t S(t-t')\mathcal{G}h_u(t')dt'$$

with

$$h_u = \Upsilon(u_0, u_1 - v(T,u)),$$

and we need to find a fix point of Γ.

From (3.2.2) we define the map $\Gamma : X_T^{0,\frac{1}{2},b_1} \cap Z_T^{0,\frac{1}{2}} \to X_T^{0,\frac{1}{2},b_1} \cap Z_T^{0,\frac{1}{2}}$ as

$$\Gamma(u) = S(t)u_0 + \int_0^t S(t-\tau)\mathcal{G}h_u d\tau + \int_0^t S(t-\tau)u\partial_x u d\tau. \tag{3.4.2}$$

From the bilinear estimates and linear estimate, we have

$$\begin{aligned}
\|\Gamma(u)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} &\leq C\left(\|u_0\|_{L^2(\mathbb{T})} + \|\mathcal{G}h_u\|_{X_T^{0,-\frac{1}{2},b_1}} + \|u\|^2_{X_T^{0,\frac{1}{2},b_1}}\right) \\
&\leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)} + \|u_1\|_{L^2(\mathbb{T}^2)} + \|\upsilon(T,u)(T)\|_{L^2(\mathbb{T})} + \|u\|^2_{X_T^{0,\frac{1}{2},b_1}}\right) \\
&\leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)} + \|u_1\|_{L^2(\mathbb{T}^2)} + \|u\|^2_{X_T^{0,\frac{1}{2},b_1}}\right)
\end{aligned}$$

and $C > 0$ does not depend on u_0. For $R > 0$, let $B_R = B_R(0)$ be the ball center at cero with radio R, that is

$$B_R := \{u \in X_T^{0,\frac{1}{2},b_1} \cap Z_T^{0,\frac{1}{2}} : \|u\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} < R\}$$

and

$$\|\Gamma(u)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} \leq C\left(\|u_0\|_{L^2(\mathbb{T}^2)} + \|u_1\|_{L^2(\mathbb{T}^2)} + R^2\right). \tag{3.4.3}$$

Additionally, for $u, v \in B_R$ we have

$$\begin{aligned}
\|\Gamma(u)-\Gamma(v)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} \leq & C\left\|\int_0^t S(t-\tau)(\mathcal{G}h_u - \mathcal{G}h_v)dt'\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} \qquad (3.4.4)\\
&+\left\|\int_0^t S(t-t')(u\partial_x u - v\partial_x v)dt'\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
\leq & C\left\|\Upsilon(u_0, u_1 - v(T,u)) - \Upsilon(u_0, u_1 - v(T,v)))\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
&+C\left\|\int_0^t S(t-t')(u\partial_x u - v\partial_x v)dt'\right\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}}\\
\leq & C\|v(T,u)-v(T,v)\|_{X_T^{0,\frac{1}{2},b_1}\cap Z_T^{0,\frac{1}{2}}} \qquad (3.4.5)\\
&+\|u-v\|_{X_T^{0,\frac{1}{2},b_1}}\|u+v\|_{X_T^{0,\frac{1}{2},b_1}}\\
\leq & C\|u-v\|_{X_T^{0,\frac{1}{2},b_1}}\|u+v\|_{X_T^{0,\frac{1}{2},b_1}} \qquad (3.4.6)\\
\leq & \frac{1}{2}\|u-v\|_{X_T^{0,\frac{1}{2},b_1}} \qquad (3.4.7)
\end{aligned}$$

by using properties of the bounded linear operator Υ. Choosing $\delta > 0$ and $R > 0$ such that $2C\delta + CR^2 \leq R$ and $CR < \frac{1}{2}$ with $\|u_0\|_{L^2(\mathbb{T}^2)} < \delta$ and $\|u_1\|_{L^2(\mathbb{T}^2)} < \delta$. We can conclude from (3.4.3) that the image of B_R through Γ stays in the ball B_R and from (3.4.4) that Γ is a contraction.

3.5 Non Controllability in horizontal strip

In this section, we will address the exact control problem of linearized KP-II equation (3.1.3) with $\mathcal{G} = \mathcal{G}_{\|}$ and prove Theorem 3.3.

We first construct a counterexample of observability inequality for 1D semi-classical Schrödinger equation for short time.

Lemma 3.3
Assume that $\omega = (-\pi, \alpha) \cup (\alpha, \pi]$. Then for any $T > 0$, there exists a sequence of solutions u_n to

$$\begin{cases} ih_n\partial_t u_n + h_n^2\partial_x^2 u_n = 0, \\ u_n|_{t=0} = u_{n,0} \in L^2(\mathbb{T}) \end{cases} \tag{3.5.1}$$

such that

$$\liminf_{n\to\infty} \|u_{n,0}\|_{L^2(\mathbb{T})} > 0$$

and

$$\lim_{n\to\infty} \int_0^T \int_\omega |u_n(t,x)|^2 dxdt = 0.$$

Proof. Take $G(x) = e^{-\frac{x^2}{4}}$ and define $G^{\epsilon_n}(x) = \frac{1}{\sqrt{\epsilon_n}} G\left(\frac{x}{\epsilon_n}\right)$. Denote the Fourier coefficient of G^{ϵ_n} by

$$g^{\epsilon_n}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G^{\epsilon_n}(x) e^{-ikx} dx = \frac{\sqrt{\epsilon_n}}{2\pi} \int_{-\frac{\pi}{\epsilon_n}}^{\frac{\pi}{\epsilon_n}} G(s) e^{-i\epsilon_n ks} ds.$$

The coefficient function $g^{\epsilon_n}(z)$ satisfies the following estimates:

$$\|g^{\epsilon_n}\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{1/2}), \|(g^{\epsilon_n})'\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{3/2}), \|(g^{\epsilon_n})''\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{5/2}). \tag{3.5.2}$$

Take an even cut-off function$\psi \in C_c^\infty(\mathbb{R})$ with supp $\psi \subset [-B, B]$ and $0 \le \psi \le 1$, $\psi|_{[-b,b]} \equiv 1$. We define

$$u_{n,0} = \sum_{k\in\mathbb{Z}} g^{\epsilon_n}(k)\psi(h_n k) e^{ikx},$$

and then the corresponding solution to (3.5.1) is given explicitly by

$$u_n(t,x) = \sum_{k\in\mathbb{Z}} g^{\epsilon_n}(k)\psi(h_n k) e^{i(kx - k^2 h_n t)}.$$

We need estimate the mass of initial data. Firstly,

$$\|G^{\epsilon_n}\|_{L^2(\mathbb{T})}^2 = \sum_{k\in\mathbb{Z}} |g^{\epsilon_n}(k)|^2 \sim 1$$

holds from Plancherel theorem and the definition of $g^{\epsilon_n}(k)$. We next estimate the mass away from the frequency scale h_n^{-1}, that is

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |(1-\psi(h_n k)) g^{\epsilon_n}(k)|^2 &\le \sum_{|k|>h_n^{-1}b} |g^{\epsilon_n}(k)|^2 \\
&\le \sum_{|k|>h_n^{-1}b} \frac{\epsilon_n}{4\pi^2} \left| \int_{\mathbb{R}} G(z) e^{-ik\epsilon_n z} dz \right|^2 \\
&= \sum_{|k|>h_n^{-1}b} \frac{\epsilon_n}{4\pi^2} \left| \int_{\mathbb{R}} G(z) \frac{1}{-ik\epsilon_n} \frac{d}{dz} e^{-ik\epsilon_n z} dz \right|^2 \\
&\le \sum_{|k|>h_n^{-1}b} \frac{1}{4k^2\pi^2\epsilon_n} \|G'\|_{L^1(\mathbb{R})}^2.
\end{aligned}$$

By setting $\epsilon_n = \sqrt{h_n} \ll 1$, we have $\|(1-\psi(h_nD_x))G^{\epsilon_n}\|_{L^2(\mathbb{T})} \ll 1$ and then $\|u_{n,0}\|_{L^2(\mathbb{T})} \sim 1$. It remains to estimate the term on the right hand side of observability inequality.

Observe that $u_{n,0}$ is localized by $|k| \leq \frac{B}{h_n}$ in frequency and by $|x| \leq \epsilon_n$ in space obeying uncertain principal ($\epsilon_n h_n^{-1} \gtrsim 1$). Since the wave packet of the frequency scale smaller than Bh_n^{-1} moves at velocity $\leq 2Bh_n^{-1}$, it will remain small for $|t| < T$ in ω. More precisely, we need dispersive estimate for $|u_n(t,x)|$ when $x \in \omega$ and $|t| < T$.

Now we choose $B > 0$ so that $|x - 2Bt| \geq c_0 > 0 \mod 2\pi$ for all $x \in \omega$ and $|t| \leq T$.

Write

$$u_n(t,x) = \sum_{k\in\mathbb{Z}} K_{t,x}^{(n)}(k)$$

with

$$K_{t,x}^{(n)}(z) = g^{\epsilon_n}(z)\psi(h_nz)e^{i(zx-h_nz^2t)}.$$

From Poisson summation formula, we have

$$u_n(t,x) = \sum_{m\in\mathbb{Z}} \widehat{K_{t,x}^{(n)}}(2\pi m).$$

For fixed $m \in \mathbb{Z}$,

$$\begin{aligned}\widehat{K_{t,x}^{(n)}}(2\pi m) &= \int_{\mathbb{R}} g^{\epsilon_n}(z)\psi(h_nz)e^{i\varphi_{t,x}(z)}dz \\ &= \int_{\mathbb{R}} g^{\epsilon_n}(z)\psi(h_nz)\mathcal{L}^2(e^{i\varphi_{t,x}(z)})dz\end{aligned}$$

with $\mathcal{L} = \dfrac{1}{i\varphi_{t,x}'(z)}\dfrac{d}{dz}$ and $\varphi_{t,x}(z) = (x-2\pi m)z - h_nz^2t$. We then integrate by part to get

$$\widehat{K_{t,x}^{(n)}}(2\pi m) = \int_{\mathbb{R}} \frac{d}{dz}\left(\frac{1}{i\varphi_{t,x}'(z)}\frac{d}{dz}\left(\frac{g^{\epsilon_n}(z)\psi(h_nz)}{i\varphi_{t,x}'(z)}\right)\right)e^{i\varphi_{t,x}(z)}dz.$$

After tedious calculation, we have

$$\begin{aligned}&\frac{d}{dz}\left(\frac{1}{i\varphi_{t,x}'(z)}\frac{d}{dz}\left(\frac{g^{\epsilon_n}(z)\psi(h_nz)}{i\varphi_{t,x}'(z)}\right)\right) \\ =&\frac{(g^{\epsilon_n})''\psi(h_nz) + 2h_n(g^{\epsilon_n})'\psi'(h_nz) + h_n^2\psi''(h_nz)g^{\epsilon_n}}{(\varphi_{t,x}')^2} \\ -&\frac{3((g^{\epsilon_n})'\psi(h_nz) + h_n\psi'(h_nz)g^{\epsilon_n})\varphi_{t,x}''}{(\varphi_{t,x}')^3} \\ -&\frac{3g^{\epsilon_n}\psi(h_nz)(\varphi_{t,x}'')^2}{(\varphi_{t,x}')^4}.\end{aligned}$$

From (3.5.2) and the choice $\epsilon_n = \sqrt{h_n}$, we have

$$|\widehat{K_{t,x}^{(n)}}(2\pi m)| \leq \sup_{|h_nz|\leq B} \frac{C\epsilon_n^{1/2}\|\psi\|_{W^{2,1}(\mathbb{R})}}{|(x-2h_nzt)-2\pi m|^2}.$$

For any $x \in 2\pi p + (-\pi, -\alpha) \cup (\alpha, \pi]$, $|x - 2h_n zt| \geq c_0 > 0$ module 2π, it holds

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |\widehat{K_{t,x}^{(n)}}(2\pi m)| \leq& C \sum_{m\in\mathbb{Z}} \frac{C\epsilon_n^{1/2}}{|c_0 - 2\pi(m-p)|^2} \\ \leq& C\epsilon_n^{1/2}.\end{aligned}$$

Therefore,

$$\int_0^T \int_\omega |u_n(t,x)|^2 dxdt \leq C\epsilon_n^{1/2} T|\omega| \to 0, \text{ as } n \to \infty.$$

■

Remark 3.3
Uniform observability estimate for semi-classical Schrödinger equation only holds for frequency scale $\frac{1}{h}$.

Now we prove Theorem 3.3 by disproving the horizontal observability.

Theorem 3.5
Suppose $\omega = \mathbb{T} \times ((-\pi, \alpha) \cup (\alpha, \pi]) \subset \mathbb{T}^2$. *Then for any* $T > 0$, *the observability inequality*

$$\|u(0)\|_{L^2(\mathbb{T}^2)}^2 \leq C \int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u(t,x)|^2 dxdydt$$

can not hold.

Proof. For any $T > 0$, we will construct a sequence of solutions u_n to the linearized KP-II equation such that

$$\|u_n(0)\|_{L^2(\mathbb{T}^2)} \sim O(1) \quad \text{and} \quad \lim_{n\to\infty} \int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u_n(t,x,y)|^2 dxdydt = 0.$$

Denote by $v_n(t,y)$ the sequence of solutions to the semi-classical Schrodinger equation which satisfies the conditions in Lemma 3.3. Define

$$u_n(t,x,y) = v_n(t,y) e^{\frac{it}{h_n^3}} e^{\frac{ix}{h_n}} = \sum_{k\in\mathbb{Z}} \widehat{v_n}(k) e^{i(ky - h_n k^2 t)} e^{i\left(\frac{x}{h_n} + \frac{t}{h_n^3}\right)}.$$

Then u_n solves linearized KP-II equation. Moreover,

$$\|u_n(0)\|_{L^2(\mathbb{T}^2)} = \|v_n(0)\|_{L^2(\mathbb{T})} \sim O(1),$$

and

$$\int_0^T \int_\omega |u_n(t,x,y)|^2 dxdydt = \int_0^T \int_{(-\pi,\alpha)\cup(\alpha,\pi]} |v_n(t,y)|^2 dtdy \to 0, \text{as } n \to \infty.$$

Now we claim that

$$\lim_{n\to\infty}\int_{\mathbb{T}} g(y')v_n(t,y')dy' \to 0 \text{ in } L^\infty([0,T];L^2(\mathbb{T})).$$

Indeed,

$$\begin{aligned}\left|\int_{\mathbb{T}} g(y')v_n(t,y')dy'\right| &= \left|\sum_{k\in\mathbb{Z}} \overline{\widehat{g}(k)} g^{\epsilon_n}(k)\psi(h_n k)e^{-ik^2 t}\right| \\ &= \left|\left(\sum_{|k|\leq M} + \sum_{|k|>M}\right)\overline{\widehat{g}(k)} g^{\epsilon_n}(k)\psi(h_n k)e^{-ik^2 t}\right| \\ &\leq \epsilon_n^{1/2}\|g\|_{L^2(\mathbb{T})}M^{1/2} + \|G^{\epsilon_n}\|_{L^2(\mathbb{T})}\left(\sum_{|k|>M}|\widehat{g}(k)|^2\right)^{1/2}\end{aligned}$$

and the right hand side tends to 0 as $n\to\infty$ since we can choose M to be arbitrarily large before taking the limit in n. The validaty of the claim implies that $g(y)\displaystyle\int_{\mathbb{T}} g(y')u_n(t,x,y')dy' \to 0$ in $L^2([0,T]\times\mathbb{T}^2)$. This completes the proof. ∎

Chapter 4

Exact controllability of linear KP-I equation

4.1 Introduction

The precise model considered here is the linear KP-I equation

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u = 0, \tag{4.1.1}$$

where the Fourier multiplier ∂_x^{-1} is defined by

$$\widehat{\partial_x^{-1} v}(k,) = \frac{1}{ik}\widehat{v}(k,)$$

for all functions

$$v \in \mathcal{D}_0'(\mathbb{T}^2) := \{v \in \mathcal{D}_0'(\mathbb{T}^2) : \widehat{v}(0, l) = 0 \text{ for all } l \in \mathbb{Z}\}.$$

We denote by $L_0^2(\mathbb{T}^2) = L^2(\mathbb{T}^2) \cap \mathcal{D}_0'(\mathbb{T}^2)$. For a vertical control region of the form $\omega = (a, b)_x \times \mathbb{T}_y$, we fix a non-negative real function $g \in C_c^2(\mathbb{T})$ with $\int_{\mathbb{T}} g = 1$. In this case, we define the control input by

$$\mathcal{G}(h)(x, y) = \mathcal{G}_\perp(h)(x, y) := g(x)\left(h(x, y) - \int_{\mathbb{T}} g(x')h(x', y)dx'\right). \tag{4.1.2}$$

The main result of this note is the observability from a vertical region.

Theorem 4.1
For any $T > 0$, there exists $C_T > 0$, such that for any solution $u \in C(\mathbb{R}; L_0^2(\mathbb{T}^2))$ of (4.1.1)*, we have*

$$\|u(0)\|_{L^2(\mathbb{T}^2)}^2 \le C_T \int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u(t, x, y)|^2 dxdydt. \tag{4.1.3}$$

As explained in chapter 3, a consequence of HUM method of Lions [47] is the exact controllability of linear KP-I equation from vertical domain.

Theorem 4.2
Given any $T > 0$ and $u_0 \in L^2_0(\mathbb{T}^2), u_1 \in L^2_0(\mathbb{T}^2)$, there exists $f \in L^2((0,T); L^2(\mathbb{T}^2)$ such that the solution of the equation

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u = \mathcal{G}f, u|_{t=0} = u_0 \tag{4.1.4}$$

satisfies $u|_{t=T} = u_1$.

Remark 4.1
When the control region is a horizontal strip of the form $\omega = \mathbb{T}_x \times (a,b)_y$ and we put the control input as

$$\mathcal{G}_{\|}h(x,y) := g(y)\left(h(x,y) - \int_{\mathbb{T}} g(y')h(x,y')dy'\right). \tag{4.1.5}$$

Then for any given time $T > 0$, the similar observability (4.1.3)*, replacing $\mathcal{G}u$ by $\mathcal{G}_{\|}u$, does not hold true. The argument is the same as the treatment for linear KP-II in chapter 3.*

The main part of this note is devoted to the proof of Theorem 4.2. In appendix, we discuss the validity of the observability for fractional linear KP I of the form

$$\partial_t u - |D_x|^\alpha \partial_x u - \partial_x^{-1}\partial_y^2 u = 0. \tag{4.1.6}$$

We will prove the following dichotomy result which asserts the least dispersion needed for the observability.

Theorem 4.3 *1. If $\alpha \geq 1$, then for any $T > 0$, there exists $C_T > 0$, such that*

$$\|u(0)\|^2_{L^2(\mathbb{T}^2)} \leq C_T \int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u(t,x,y)|^2 dxdydt$$

holds for any solution u of (4.1.6)*.*

2. If $0 < \alpha < 1$, then for any $T > 0$, there exist a sequence of solution (u_n) of (4.1.6)*, such that*

$$\lim_{n\to\infty} \frac{\int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u_n(t,x,y)|^2 dtdxdy}{\|u_n(0)\|^2_{L^2(\mathbb{T}^2)}} = 0.$$

4.2 Notations and Preliminaries

4.2.1 Notation

We identify $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ with fundamental domain $[-\pi,\pi]^d$. The Fourier transform on $\mathbb{T}^d$ is denoted by

$$\widehat{f}(\xi) = (2\pi)^{-d}\int_{\mathbb{T}^d} f(z)e^{-iz\cdot\xi}dz, \quad \xi \in \mathbb{Z}^d.$$

In the case where there is no risk of confusing, we will also use $\widehat{f}$ to note the Fourier transform of one variable. For the derivative, we sometimes use the notation $D_t = \frac{1}{i}\partial_t, D_x = \frac{1}{i}\partial_x$.

We will only use L^2 based norms for this linear problem, hence we denote by

$$\|v\| := \|v\|_{L^2(\mathbb{T}^d)}, \|v\|_s := \|v\|_{H^s(\mathbb{T}^d)}, \|f\|_T := \|f\|_{L^2(0,T;L^2(\mathbb{T}^d))}.$$

We will also use the inner product notations

$$(u,v) := \int_{\mathbb{T}^d} u(x)\cdot\overline{v}(x)dx, \quad (f,w)_T := \int_0^T\int_{\mathbb{T}^d} f(t,x)\overline{w}(t,x)dxdt,$$

where $d = 1$ or 2, which will be clear in the context.

Symbols and quantization on Torus

We briefly review the h pseudo-differential calculus on torus. For $m \in \mathbb{R}$, let S^m be the set of h-dependent functions $a(x,\xi,h)$ with parameter $h \in (0,1)$ such that for any indices α,β,

$$\sup_{(x,\xi,h)\in\mathbb{R}^{2d}\times(0,1)} |\partial_x^\alpha\partial_\xi^\beta a(x,\xi,h)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}. \tag{4.2.1}$$

For $a \in S^m$, we denote by $\mathrm{Op}_h(a)$ the h pseudo-differential operator acting on Schwartz functions via

$$\mathrm{Op}_h(a)f(x) := \frac{1}{(2\pi h)^d}\int_{\mathbb{R}^{2d}} e^{\frac{i(x-y)\cdot\xi}{h}}a(x,\xi,h)f(y)dyd\xi.$$

We refer [63] for symbolic calculus and another basic properties about h pseudo-differential operator. For functions on a compact Riemannian manifold, we can define h pseudo-differential operator by using local coordinate and partition of unity. On the torus, we can also use the global definition of pseudo-differential calculus. Denote by S^m_{per} be symbols in $S^m(\mathbb{R}^{2d})$ which are 2π-periodic in $x \in \mathbb{R}^d$, namely

$$a(x+2\pi k,\xi) = a(x,\xi), \quad \forall(x,\xi)\in\mathbb{R}^{2d}, k\in\mathbb{Z}^d.$$

Symbols in S^m_{per} can depend on h with uniform estimate (4.2.1),though the dependence is not displayed in our notation. We quantize $a \in S^m_{per}$ as an operator on $\mathcal{S}'(\mathbb{T}^d)$ via the formula

$$\mathrm{Op}_h(a)f(x) := \sum_{k\in\mathbb{Z}^d}\frac{1}{(2\pi h)^d}\int_{[-\pi,\pi]^d}\int_{\mathbb{R}^d} a(x,\xi)e^{\frac{i(x-y+2k\pi)\cdot\xi}{h}}f(y)dyd\xi \tag{4.2.2}$$

From Poisson summation formula, we have

$$\mathrm{Op}_h(a)f(x) = \sum_{k\in\mathbb{Z}^d} a(x, h_1 k)\widehat{f}(k)e^{ik\cdot x}. \tag{4.2.3}$$

The globally defined quantization via (4.2.3) is equivalence to (modulo hS^{m-1}) the usual definition via partition of unity, see the exercise in the book [1].

4.2.2 Quick review of 1D semi-classical reduction

Expanding the solution $u(t, x, y)$ to (4.1.1) in Fourier series in y variable

$$u(t, x, y) = \sum_{l\in\mathbb{Z}} a_l(t, x)e^{ily},$$

we find that for each $l \in \mathbb{Z}$, a_l satisfies the equation

$$\partial_t a_l + \partial_x^3 a_l - l^2\partial_x^{-1} a_l = 0$$

Therefore, by changing notations, the equation (4.1.1) can be reduced to the study of the following λ dependent equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \lambda^2\partial_x^{-1} u = 0, & (t, x) \in \mathbf{R}\times\mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}), \end{cases} \tag{4.2.4}$$

We take $\lambda = \frac{1}{h^2}$ and rewrite (4.2.4) as

$$\begin{cases} h^3 D_t u - (hD_x)^3 u - (hD_x)^{-1} u = 0, & (t, x) \in \mathbf{R}\times\mathbb{T}, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}), \end{cases} \tag{4.2.5}$$

The solution u depends on the parameter h and we will drop the dependence in the sequel. From the same proof of Proposition 3.6 in chapter 3, we reduce the proof of Theorem 4.1 to the following weak observability.

Theorem 4.4
$T > 0$ be given. There exist a constant $C_T > 0$ and a sufficiently small number $h_0 > 0$, such that for all $h \in (0, h_0)$, the solution u of the h dependent equation (4.2.5) satisfies

$$\|u_0\|^2 \le C_T \int_0^T \int_{\mathbb{T}} |g(x)u(t, x)|^2 dxdt + C_T\|u_0\|_{-1}^2. \tag{4.2.6}$$

We use a standard homogeneous Littlewood-Paley decomposition. Take $\psi \in C_c^\infty(\mathbb{R})$ with support $\mathrm{supp}\psi \subset \{1/2 \le |\xi| \le 2\}$ and $\psi_n \in C_c^\infty(\mathbb{R})$ such that

$$\sum_{n\in\mathbb{Z}} \psi_n(\xi) = 1, \quad \forall \xi \neq 0,$$

where $\psi_n(\xi) = \psi(2^n\xi)$. With this notation, we further reduce the proof of Theorem 4.4 to the following frequency-localized estimate.

Proposition 4.1
Let $T > 0$ and $\epsilon_0 > 0$ be given. There exist $h_0 > 0$, small and $C_0 = C_0(\epsilon_0, T) > 0$ such that for all $n \in \mathbb{Z}$ which subject to $2^n h \leq \epsilon_0$,

$$\|\psi_n(hD_x)u(0)\|^2 \leq C_0 \int_0^{T_0} \int_{\mathbb{T}} |g(x)\psi_n(hD_x)u(t,x)|^2 dxdt \tag{4.2.7}$$

holds true for all solutions $u(t,x)$ of (4.2.5).

The derivation from Proposition 4.1 to Theorem 4.4 is simple and can be found in chapter 3. The remaining part of this note is devoted to the proof of (4.2.7). We summarize the path of the proof as follows:

- Regimes $n \geq N_0$ and $n \leq -N_0$: $n \leq -N_0$ corresponds to the very low frequency regime in which the term $(hD_x)^{-1}$ dominates the dispersion. $n \geq N_0$ corresponds to the very high frequency regime in which the term $(hD_x)^3$ dominates the dispersion. The arguments are similar as for linear KP-II.
- Regime $|n| \leq N_0$: This is the essential difference between KP-I and KP-II. The group velocity of KP-I could be very small in this regime.

4.3 The proof of Proposition 4.1

4.3.1 Regimes far from critical points

Let us consider the following $\epsilon-$dependence symbols:

$$p_\epsilon(x,\xi) = \left(\frac{\epsilon^4}{\xi} + \xi^3\right)\chi(\xi), \quad q_\epsilon(x,\xi) = \left(\frac{1}{\xi} + \epsilon^4\xi^3\right)\chi(\xi),$$

where $\chi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\chi) \subset \{\mu < |\xi| < \nu\}$ for some $0 < \mu < \frac{1}{2}$, $\nu > 2$ and $\chi \equiv 1$ in a neighborhood of $\{1/2 \leq |\xi| \leq 2\}$. Denote by $P_\epsilon = \mathrm{Op}_{\widetilde{h}}(p_\epsilon)$ and $Q_\epsilon = \mathrm{Op}_{\widetilde{h}}(q_\epsilon)$.We use the notations $U_\epsilon(t), V_\epsilon(t)$ to represent solution operators to the following two equations

$$\begin{cases} \frac{\widetilde{h}}{i}\partial_t U_\epsilon(t) + U_\epsilon(t)P_\epsilon = 0, \\ U_\epsilon(0) = I, \end{cases} \tag{4.3.1}$$

$$\begin{cases} \frac{\widetilde{h}}{i}\partial_t V_\epsilon(t) + V_\epsilon(t)Q_\epsilon = 0, \\ V_\epsilon(0) = I \end{cases} \tag{4.3.2}$$

The flows associated to the vector fields $H_{p_\epsilon}, H_{q_\epsilon}$ are explicitly given by

$$\phi_{\epsilon,t}(x_0,\xi_0) = \left(x_0 + \left(-\frac{\epsilon^4}{\xi_0^2} + 3\xi_0^2\right)\chi(\xi_0)t + \left(\frac{\epsilon^4}{\xi_0} + \xi_0^3\right)\chi'(\xi_0)t, \xi_0\right),$$

$$\varphi_{\epsilon,t}(x_0,\xi_0)=\left(x_0+\left(-\frac{1}{\xi_0^2}+3\epsilon^4\xi_0^2\right)\chi(\xi_0)t+\left(\frac{1}{\xi_0}+\epsilon^4\xi_0^3\right)\chi'(\xi_0)t,\xi_0\right)$$

with respectively.

From Egorov's theorem (see [63]), we know that for any symbol $a(x,\xi)\in C_c^\infty(T^*M)$,

$$U_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)U_\epsilon(t)=\mathrm{Op}_{\widetilde{h}}(a\circ\phi_{\epsilon,t})+O_{L^2\to L^2}(\widetilde{h}),$$

$$V_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)V_\epsilon(t)=\mathrm{Op}_{\widetilde{h}}(a\circ\varphi_{\epsilon,t})+O_{L^2\to L^2}(\widetilde{h}).$$

We remark that the bound $O_{L^2\to L^2}(\widetilde{h})$ is independent of $\epsilon\leq 1$ since all the semi-norms of the symbol p_ϵ,q_ϵ can be chosen continuously depending on ϵ.

Now we prove the following localized observability estimates:

Proposition 4.2
There exists $C_0>0,T_0>0,\widetilde{h}_0>0$ and $\delta_0>0$ such that for all $u_0\in L_0^2(\mathbb{T})$, and all $\widetilde{h}\leq\widetilde{h}_0$

$$\|\psi(\widetilde{h}D_x)u_0\|^2\leq C_0\int_0^{T_0}\|gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|^2dt, \tag{4.3.3}$$

$$\|\psi(\widetilde{h}D_x)u_0\|^2\leq C_0\int_0^{T_0}\|gV_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|^2dt, \tag{4.3.4}$$

uniformly in $\epsilon<\delta_0$.

Proof. Here we only prove the first inequality, and the second one will follow in the same manner. Consider the symbol $a(x,\xi)=g(x)^2\widetilde{\psi}(\xi)$ (strictly speaking, g is not smooth and we need approximate it by smoothing functions) and its quantization $\mathrm{Op}_{\widetilde{h}}(a)=(g(x))^2\widetilde{\psi}(\widetilde{h}D_x)$, where $\widetilde{\psi}$ is a slight enlargement of ψ so that $\widetilde{\psi}\psi=\psi$ and $\mathrm{supp}\chi|_{\mathrm{supp}(\widetilde{\psi})}=1$. From Egorov's theorem, we have

$$U_\epsilon(-t)\mathrm{Op}_{\widetilde{h}}(a)U_\epsilon(t)=\mathrm{Op}_{\widetilde{h}}(a\circ\phi_{\epsilon,t})+O_{L^2\to L^2}(\widetilde{h}),\text{ uniformly in }\epsilon\leq 1.$$

Note that on the support of a, $\chi'(\xi)=0$, and thus we have

$$\varphi_{\epsilon,t}(x_0,\xi_0)=\left(x_0+\left(-\frac{\epsilon^4}{\xi_0^2}+3\xi_0^2\right)t,\xi_0\right).$$

We choose $\delta_0=\delta_0(\mu,\nu)$, sufficiently small, such that $\left|-\frac{\epsilon^4}{\xi_0^2}+3\xi_0^2\right|\geq c_0>0$, uniformly in $\epsilon<\delta_0$ on the $\xi-$support of $\widetilde{\psi}$. Therefore, for some $T_0=T_0(c_0)>0$, and $c_1>0$, we have

$$\int_0^{T_0}a\circ\phi_{\epsilon,t}dt\geq c_1>0.$$

Now we calculate

$$\begin{aligned}
&\int_0^{T_0} \|gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0\|^2 dt \\
&= \int_0^{T_0} \Big(gU_\epsilon(t)\psi(\widetilde{h}D_x)u_0, gU_\epsilon(t)\widetilde{\psi}(\widetilde{h}D_x)\psi(\widetilde{h}D_x)u_0\Big)\, dt \\
&= \int_0^{T_0} \Big(U_\epsilon(-t)\widetilde{\psi}(\widetilde{h}D_x)g^2U_\epsilon(t)u_0, \psi(\widetilde{h}D_x)u_0\Big)\, dt \\
&= \Big(\mathrm{Op}_{\widetilde{h}}(b_{T_0})\psi(\widetilde{h}D_x)u_0, \psi(\widetilde{h}D_x)u_0\Big),
\end{aligned}$$

with $b_{T_0}(x,\xi) = \int_0^{T_0} a \circ \phi_{\epsilon,t} dt$ modulo $\widetilde{h}S^0$. Thus, from Sharp Gårding inequality,

$$\Big(\mathrm{Op}_{\widetilde{h}}(b_{T_0})\psi(\widetilde{h}D_x)u_0, \psi(\widetilde{h}D_x)u_0\Big) \geq \frac{c_1}{2}\|\psi(\widetilde{h}D_x)u_0\|^2 - C\widetilde{h}\|\psi(\widetilde{h}D_x)u_0\|^2.$$

To conclude the proof, we just need choose $\widetilde{h}_0 < \min\{\frac{c_1}{4C}, 1\}$. ■

As a consequence, we can proof Proposition 4.1 in the easy regimes:

Corollary 4.1
There exist $h_0 > 0$ and a integer N_0 which depends on δ_0 in Proposition 4.2, such that for all $h < h_0$, $|n| \geq N_0$, and $2^n h \leq \epsilon_0$, the inequality (4.2.7) holds true.

Proof. Take $N_0 \in \mathbb{N}$ such that $2^{-N_0} < 2^{-10}\delta_0$. Fix $h_0 < \min\{2^{-N_0}\epsilon_0, \widetilde{h_0}\} > 0$. We first consider the case $n \leq -N_0$. Define a new semi-classical parameter $\widetilde{h}_n = 2^n h \ll 1$ and rescale the time variable by setting $w_n(t,x) := \psi(\widetilde{h}_n D_x)u(2^{2n}t, x)$. w_n satisfies the following equation:

$$\widetilde{h_n}\partial_t w_n + (\widetilde{h_n}\partial_x)^3 w_n + 2^{4n}(\widetilde{h_n}\partial_x)^{-1} w_n = 0.$$

Applying (4.3.3) to w_n with $\epsilon = 2^n \leq \delta_0$ and $\widetilde{h} = \widetilde{h}_n$ we obtain that

$$\|w_n(0)\|^2 \leq C\int_0^{T_0} \|gw_n(t)\|^2 dt.$$

From conservation of L^2 norm along the flow, we apply the inequality above $2^{-2n} - 1$ times to obtain

$$\frac{1}{2^{2n}}\|u_n(0)\|^2 \leq \frac{C}{2^{2n}} \sum_{M=0}^{2^{-2n}-1} \int_{M2^{2n}T_0}^{(M+1)2^{2n}T_0} \|gu_k(t)\|^2 dt = \frac{C}{2^{2n}}\int_0^{T_0} \|gu_n(t)\|^2 dt,$$

and this is exactly

$$\|\psi_n(hD_x)u(0)\|^2 \leq C\int_0^{T_0} \|g\psi_n(hD_x)u(t)\|^2 dt.$$

We next consider the case $n \geq N_0$ and $2^n h \leq \epsilon_0$. Define the new small semi-classical parameter $\widetilde{h}_n = 2^n h$, thanks to the restriction that $2^n h \leq \epsilon_0 \ll 1$. Denote by $u_n = \psi(\widetilde{h}_n D_x)u$ and define $v_n(t,x) = u_n(2^{-2k}t, x)$. Thus v_n solves the equation

$$\widetilde{h}_n \partial_t v_n + 2^{-4n}(\widetilde{h}_n \partial_x)^3 v_n + (\widetilde{h}_n \partial_x)^{-1} v_n = 0.$$

Applying (4.3.4) with $\widetilde{h} = \widetilde{h}_n, \epsilon = 2^{-n}$, we obtain that

$$\|v_n(0)\|^2 \leq C \int_0^{T_0} \|g v_n(t)\|^2 dt.$$

Again from conservation of L^2 norm as in the previous argument, we finally have

$$\|u_n(0)\|^2 \leq C \int_0^{T_0} \|g u_n(t)\|^2 dt.$$

$\blacksquare$

4.3.2 Near the critical points

Now we prove inequality (4.2.7) for $|n| \leq N_0$. Since N_0 only depends on $\mu, \nu > 0$ which is chosen in a priori, it would be suffices to prove the inequality for $n = 0$ only. Rewriting (4.2.5) as

$$h^3 D_t u - \phi(hD_x)u = 0, u = \psi(hD_x)u,$$

with Fourier multiplier $\phi(\xi) = \xi^3 + \frac{1}{\xi}$. There are only two zeros of $\phi'(\xi) = 3\xi^2 - \frac{1}{\xi^2}$, say $\xi_0 = \pm\frac{1}{\sqrt[4]{3}}$. Splitting $\psi(\xi) = \psi_+(\xi) + \psi_-(\xi)$ with $\psi_+ = \psi \mathbf{1}_{\xi > 0}$, and $\psi_- = \psi \mathbf{1}_{\xi < 0}$, it would be sufficient to prove (4.2.7) for $u = \psi_+(hD_x)u$. For $\delta > 0$, we take another cut-off $\chi_\delta \in C_c^\infty(\mathbb{R})$ such that

$$\chi_\delta(\xi)|_{|\xi - \xi_0| \leq \delta} \equiv 1, \quad \chi_\delta(\xi)|_{|\xi - \xi_0| > 2\delta} \equiv 0.$$

Taking $\delta > 0$ smaller, we may assume that $\chi_\delta(\xi)\psi(\xi) = \chi_\delta(\xi)$. On the support of $(1 - \chi_\delta)\psi_+$, we have $|\psi'(\xi)| \geq c_\delta > 0$, thus the same propagation argument as in the previous subsection yields

$$\|(1 - \chi_\delta(hD_x))\psi_+(hD_x)u(0)\|^2 \leq C_\delta \int_0^{T_\delta} \|g(1 - \chi_\delta(hD_x))\psi_+(hD_x)u(t)\|^2 dt \tag{4.3.5}$$

for some C_δ, T_δ depending on $\delta > 0$. To complete the proof, it remains to prove the similar inequality for $\chi_\delta(hD_x)u$. Indeed, the sum of the two frequency pieces on the right hand side can be bounded by $\|g\psi_+(hD_x)u\|_{T_\delta}^2 + C_\delta T h \|u(0)\|^2$ in which the error term comes from the commutator $[g, \chi_\delta(hD_x)]$.

Before treating the term $\chi_\delta(hD_x)u$, we make a further simplification. Denote by $v = \chi_\delta(hD_x)u$, $v = e^{i\lfloor\frac{\xi_0}{h}\rfloor x}w$, and then $\widehat{w}(k) = \widehat{v}\left(k + \left\lfloor\frac{\xi_0}{h}\right\rfloor\right)$. We see that

$$h^3 D_t w - \Phi(hD_x)w = 0, w(0) = \theta_\delta(hD_x)w_0, \tag{4.3.6}$$

with

$$\Phi(\xi) = \phi\left(\xi + h\left\lfloor\frac{\xi_0}{h}\right\rfloor\right), \theta_\delta(\xi) = \chi\left(\xi + h\left\lfloor\frac{\xi_0}{h}\right\rfloor\right).$$

Note that the support of θ_δ is now near the origin and

$$\phi'(\sigma_h) = 0, \Phi''(\sigma_h) = \frac{12}{\sqrt[4]{3}} > 0, \sigma_h = h\left(\frac{\xi_0}{h} - \left\lfloor\frac{\xi_0}{h}\right\rfloor\right).$$

We are now ready to close the demonstration of Proposition 4.1 by proving the following, which is the main ingredient of this note:

Proposition 4.3
There exist constants $\delta > 0, h_0 > 0$ small and $C_T > 0$ such that for all $0 < h < h_0$ and $w = \theta_\delta(hD_x)w$, h dependent solution to (4.3.6), *we have*

$$\|w(0)\|^2 \leq C_T \int_0^T \int_{\mathbb{T}} \|g(x)w(t)\|^2 dt.$$

The proof is down by splitting the frequency into two parts. One part contains cluster of relatively low frequencies and we control it by spectral inequality. The other part contains relatively high frequencies and will be controlled by propagation estimate after rescaling the time variable. First we notice that the inequality would not change if we replace w by $w\exp\left(\frac{i\Phi(\sigma_h)t}{h^3}\right)$. We may assume that $\Phi(\sigma_h) = 0$. Denote by $r_h = h^{-1}\sigma_h \in [0, 1)$. For any $n_0 \in \mathbb{N}$, we define the sharp frequency truncation

$$\Pi_{\geq n_0} f := \sum_{|k| \geq n_0} \widehat{f}(k)e^{ikx}.$$

We divide the proof into several lemmas.

Lemma 4.1
Given $T > 0$, there exist $N_0 \in \mathbb{N}$, $h_0 > 0$, $C_T > 0$, such that for any integer $n_0 \geq N_0, h < h_0$ and $T > 0$,

$$\|\Pi_{\geq n_0} w(0)\|^2 \leq C_T \int_0^T \|g(x)w(t)\|^2 dt \tag{4.3.7}$$

holds true for all solutions of (4.3.6).

Proof. We rewrite

$$\Pi_{\geq n_0} w(0) = \sum_{l_0 \leq l \leq L_0} \psi(2^l hD_x)\Pi_{\geq n_0} w(0),$$

with $2^{-l_0}=4\delta, 2^{-L_0}=\dfrac{n_0h}{4}$. From almost orthogonal inequality

$$\|\Pi_{\geq n_0}w(0)\|^2\leq 4\sum_{l\leq l\leq L_0}\|\psi(2^lhD_x)w(0)\|^2,$$

we need estimate each term in the summation. By choosing $n_0\geq N_0$ large, we denote by $h_1=2^lh\leq\frac{4}{n_0}$ a new semi-classical parameter. We put $\omega=\psi(h_1D_x)w$, and then ω solves

$$2^{-l}h_1^3D_t\omega-\Phi_l(h_1D_x)\omega=0,$$

with $\Phi_l(\xi)=2^{2l}\Phi(2^{-l}\xi)$. Note that Φ_l is a symbol with uniform bound in l for $|\xi|\leq 2$ as well as all of its derivatives. We rescale the time by setting $v(t,x):=\omega(2^{-l}h_1^2t,x)$, hence

$$h_1D_tv-\Phi_l(h_1D_x)v=0.$$

Notice that $|\partial_\xi\Phi_l(\xi)|=|2^l\Phi'(2^{-l}\xi)|\sim 1$ for $\xi\in\mathrm{supp}\psi$. From the same argument as in the proof of Proposition 4.2, there exist $T_0>0$ and $C_{T_0}>0$ such that

$$\|\psi(h_1D_x)v(0)\|_{L^2}^2\leq C_{T_0}\int_0^{T_0}\int_{\mathbb{T}}|g(x)\psi(h_1D_x)v(t,x)|^2dxdt$$

holds true for all $h_1=2^lh$, provided that $h<h_0$ is small enough and $n_0\geq N_0$ is large(while keeping the relation $hn_0\ll 1$). Back to the function w, we have

$$\begin{aligned}\|\psi(h_1D_x)w(0)\|_{L^2}^2=\|\psi(h_1D_x)v(0)\|_{L^2}^2\leq&C_{T_0}\int_0^{T_0}\int_{\mathbb{T}}|g(x)\psi(h_1D_x)v(t,x)|^2dxdt\\ =&\frac{C_{T_0}}{h_1h}\int_0^{h_1hT_0}\int_{\mathbb{T}}|g(x)\psi(h_1D_x)w(t,x)|^2dxdt.\end{aligned}$$

Thanks to the conservation of L^2 norm, we have for all $q\in\mathbb{N}$,

$$\|\psi(h_1D_x)w(0)\|_{L^2}^2=\|\psi(h_1D_x)w(qh_1hT_0)\|_{L^2}^2\leq\frac{C_{T_0}}{h_1h}\int_{qh_1hT_0}^{(q+1)h_1hT_0}\int_{\mathbb{T}}|g(x)\psi(h_1D_x)w(t,x)|^2dxdt.$$

Summing q from 0 to $\lfloor\frac{T}{h_1hT_0}\rfloor$, we have

$$\|\psi(h_1D_x)w(0)\|_{L^2}^2\leq C_T\int_0^T\int_{\mathbb{T}}|g(x)\psi(h_1D_x)w(t,x)|^2dxdt. \tag{4.3.8}$$

Thus

$$\begin{aligned}\sum_{l_0\leq l\leq L_0}\|\psi(2^lhD_x)w(0)\|_{L^2}^2\leq&C_T\sum_{l_0\leq l\leq L_0}\int_0^T\int_{\mathbb{T}}|g(x)\psi(2^lhD_x)w(t,x)|^2dxdt\\ \leq&\sum_{l_0\leq l\leq L_0}C_T\int_0^T\int_{\mathbb{T}}\left(|\psi(2^lhD_x)(gw)(t,x)|^2+2^{2l}h^2|w(t,x)|^2\right)dxdt\\ \leq&C_T\int_0^T\int_{\mathbb{T}}|gw(t,x)|^2dxdt+\frac{C_T}{n_0^2}\|w(0)\|_{L^2}^2,\end{aligned}$$

where we have used the simple commutator estimate $\|[g,\psi(2^lhD_x)]\|_{L^2\to L^2}\leq C2^lh$ in the second inequality. This completes the proof by choosing n_0 sufficiently large. ■

We need the following spectral inequality, and the proof is classical and can be found in [44].

Lemma 4.2
There exists an positive increasing function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\sum_{|k|\leq m_0} |c_k|^2 \leq \kappa(m_0) \int_{\mathbb{T}} \left| g(x) \sum_{|k|\leq m_0} c_k e^{ikx} \right|^2 dx. \tag{4.3.9}$$

The following elementary lemma is needed in the final argument.

Lemma 4.3
For any $r \in [0,1)$, there exist $\mu_1, \mu_2 \in \left[\frac{1}{8}, \frac{7}{8}\right]$, such that

$$\mu_1 + \mu_2 = 2r \ \ mod\ \mathbb{Z}. \tag{4.3.10}$$

Proof. We denote by $\{x\} := x - \lfloor x \rfloor$, the fractional part of a real number. If $\frac{1}{4} \leq \{2r\} \leq \frac{3}{4}$, then there exist $\mu_1, \mu_2 \in \left[\frac{1}{8}, \frac{7}{8}\right]$, such that (4.3.10) holds true. If $\frac{3}{4} < \{2r\} < 1$, we can choose $\mu_1 = \mu_2 \in \left(\frac{3}{8}, \frac{1}{2}\right)$, and then $2\mu = 2r \mod \mathbb{Z}$. For $0 \leq \{2r\} < \frac{1}{4}$, we choose $\mu_1 = \mu_2 = \mu \in \left[\frac{1}{2}, \frac{5}{8}\right)$. ■

Now we finish the proof.

Proof of Proposition 4.3. Let h_0, N_0 as in Lemma 4.1. Fix $h > 0$ and $n_0 > 2N_0$ while keeping $hn_0^3 \ll 1$ and $\kappa(n_0+2)h \ll 1$. From Lemma 4.3.10, for any $n_0 \in \mathbb{N}$ satisfies $hn_0^3 \ll 1$, there exist $\mu_1, \mu_2 \in \left[\frac{1}{8}, \frac{7}{8}\right]$ and $m_0 \in -\mathbb{N}$, such that

$$(n_0 + \mu_2) + (m_0 + \mu_1) = 2r_h. \tag{4.3.11}$$

Put $M_0 := h^{-1}\sqrt{\Phi(h(n_0+\mu_2))}$. Recall that Φ is strictly increasing for $\xi \in [\sigma_h, \delta)$ and strictly decreasing for $\xi \in (-\delta, \sigma_h]$. Thus there exist $\xi^* < r_h$, such that $\Phi(\xi^*) = h^2 M_0^2$. We fix $\delta > 0$ small such that $2 < \Phi''(\xi) < 12$ for all $|\xi| \leq \delta$. Thus $\frac{1}{4} \leq \frac{M_0^2}{(n_0+\mu_2)^2} \leq 9$. We claim that for sufficiently small $h > 0$, we have

$$\left\lfloor \frac{\xi^*}{h} \right\rfloor = m_0, \quad \left| \left\{ \frac{\xi^*}{h} \right\} - \mu_1 \right| < \frac{1}{16}.$$

Indeed, Taylor expansion gives

$$\frac{\Phi''(\sigma_h)h^2}{2}\left(\frac{\xi^*}{h} - r_h\right)^2 = \frac{\Phi''(\sigma_h)h^2}{2}(n_0+\mu_2-r_h)^2 + O(h^3 m_0^3),$$

with implicit constant in big O depending only on $\sup\limits_{\xi\in(-\delta,\delta)} |\Phi'''(\xi)|$. As a consequence, we have

$$\left| m_0 + \mu_1 - \frac{\xi^*}{h} \right| = O(hM_0^2). \tag{4.3.12}$$

The claim follows easily by choosing h small enough.

Define a slightly different frequency truncation

$$w_L(t,x) = \sum_{k:|\Phi(hk)|\le h^2M_0^2} \widehat{w_0}(k)e^{ikx+\frac{it\Phi(hk)}{h^3}}, \quad w_H = w - w_L.$$

From (4.3.9) and the property of Φ, we have

$$\|w_L(0)\|^2 \le \kappa(n_0+2)T\int_0^T\int_{\mathbb{T}} |g(x)w_L(t,x)|^2dxdt. \tag{4.3.13}$$

Note that $\Pi_{\ge\frac{n_0}{2}}w_H = w_H$, then from Lemma 4.1 we have

$$\|w_H(0)\|^2 \le C_T\int_0^T \|g(x)w(t)\|^2dt. \tag{4.3.14}$$

We next calculate

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}} g(x)w_H(t,x)\cdot\overline{g(x)w_L(t,x)}dxdt\right| \\
=&\left|\int_0^T\int_{\mathbb{T}} g(x)^2\sum_{|\Phi(hk_1)|\le h^2M_0^2}\ \sum_{|\Phi(hk_2)|>h^2M_0^2}\overline{\widehat{w_0}(k_1)}\widehat{w_0}(k_2)e^{-i(k_1-k_2)x}e^{\frac{it}{h^3}(\Phi(hk_2)-\Phi(hk_1))}dxdt\right| \\
=&\left|\sum_{|\Phi(hk_1)|\le h^2M_0^2}\ \sum_{|\Phi(hk_2)|>h^2M_0^2} 2\pi\widehat{G}(k_1-k_2)\overline{\widehat{w_0}(k_1)}\widehat{w_0}(k_2)h^3\frac{e^{\frac{iT}{h^3}(\Phi(hk_2)-\Phi(hk_1))}-1}{\Phi(hk_2)-\Phi(hk_1)}\right| \\
\le& Ch^3\sum_{|\Phi(hk_1)|\le h^2M_0^2}\ \sum_{|\Phi(hk_2)|>h^2M_0^2} |\widehat{G}(k_1-k_2)\widehat{w_0}(k_1)\widehat{w_0}(k_2)|\frac{1}{\Phi(hk_2)-\Phi(hk_1)}
\end{aligned}$$

with $G(x) = g(x)^2$. If $(k_1-r_h)(k_2-r_h)\ge 0$, we have

$$|\Phi(hk_2)-\Phi(hk_1)| = \left|\int_{hk_1}^{hk_2}\Phi'(t)dt\right| \ge h|\Phi'(hk_1)||k_2-k_1| \ge ch^2|k_2-k_1|,$$

in the case $|k_1-r_h|\ge 1$. If otherwise $|k_1-r_h|<1$, we directly estimate

$$|\Phi(hk_2)|-|\Phi(hk_1)| \ge h^2\left(M_0^2 - \sup_{\xi\in[-2h,2h]}|\Phi'(\xi)|\right) > \frac{h^2M_0^2}{2}$$

by taking n_0 reasonable. There are two possibilities in the case of $(k_1-r_h)(k_2-r_h)<0$:

either $k_2 \geq n_0 + 1, \dfrac{\xi^*}{h} \leq k_1 < r_h$, or $k_2 < \left\lfloor \dfrac{\xi^*}{h} \right\rfloor, r_h < k_1 \leq n_0$. For the first case, we have

$$\begin{aligned}
\Phi(hk_2) - \Phi(hk_1) \geq & \Phi(hk_2) - \Phi(\xi^*) \\
= & \frac{\Phi''(\sigma_h)h^2}{2} \left(k_2 - \frac{\xi^*}{h}\right) \left(k_2 + \frac{\xi^*}{h} - 2r_h\right) + O(h^3 M_0^3) \\
\geq & \frac{\Phi''(\sigma_h)h^2}{2} (k_2 - k_1)(k_2 + m_0 + \mu_1 - 2r_h) + O(h^3 M_0^3) \\
\geq & \frac{\Phi''(\sigma_h)h^2}{2} |k_2 - k_1|(n_0 + 1 + m_0 + \mu_1 - 2r_h) + O(h^3 M_0^2) \\
\geq & \frac{|k_2 - k_1|h^2}{16}
\end{aligned}$$

by choosing h small enough, thanks to (4.3.11), (4.3.10) and (4.3.11). In the case that $k_2 < \left\lfloor \dfrac{\xi^*}{h} \right\rfloor$, we have

$$\begin{aligned}
\Phi(hk_2) - \Phi(hk_1) \geq & \frac{\Phi''(\sigma_h)h^2}{2} (k_1 - k_2)(-k_2 - k_1 + 2r_h) + O(h^3 M_0^3) \\
\geq & h^2 |k_2 - k_1|(-m_0 - n_0 + 2r_h) + O(h^3 M_0^2) \\
\geq & \frac{|k_1 - k_2|h^2}{16}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\left| \int_0^T \int_{\mathbb{T}} g(x)^2 w_H(t,x) \cdot \overline{w_L}(t,x) dx dt \right| \leq & Ch \sum_{k \neq k_2} |\widehat{G}(k_1 - k_2)| |\widehat{w_0}(k_1)| |\widehat{w_0}(k_2)| \\
\leq & Ch \left(\sum_{k \in \mathbb{Z}} |\widehat{G}(k)| \right) \|w_0\|^2,
\end{aligned}$$

where we have used Young's convolution inequality. From this, we could improve the estimate of $\|w_L(0)\|^2$ as follows.

$$\begin{aligned}
\|w_L(0)\|^2 \leq & \kappa(n_0 + 2)T \int_0^T \int_{\mathbb{T}} |g w_L(t,x)|^2 dx dt \\
= & \kappa(n_0 + 2)T \int_0^T \int_{\mathbb{T}} |g(x) w(t,x)|^2 dx dt - \kappa(n_0 + 2)T \int_0^T \int_{\mathbb{T}} |g(x) w_H(t,x)|^2 dx dt \\
& -2\kappa(n_0 + 2)T \operatorname{Re} \int_0^T \int_{\mathbb{T}} g(x) w_H(t,x) \cdot \overline{g(x) w_L(t,x)} dx dt \\
\leq & \kappa(n_0 + 2)T \int_0^T \|g w(t)\|^2 dt + Ch\kappa(n_0 + 2)T \|w(0)\|^2,
\end{aligned}$$

and

$$\begin{aligned}\|w(0)\|^2 =&\|w_L(0)\|^2 + \|w_H(0)\|^2 \\ \leq& (C_T + \kappa(n_0+2)T)\int_0^T \|gw(t)\|^2 dt + Ch\kappa(n_0+2)\|w(0)\|^2.\end{aligned}$$

The last term on the right hand side can be absorbed to the left, and this completes the proof. ■

4.A On the observability of fractional linear KP I

In this appendix, we will give a proof of Theorem 4.3 for fractional KP I equation

$$\partial_t u - |D_x|^\alpha \partial_x u - \partial_x^{-1}\partial_y^2 u = 0. \tag{4.A.1}$$

When $\alpha \geq 1$, the proof of observability can be reduced to the 1D uniform observability of

$$h^{\alpha+1}D_t v - \Phi_\alpha(hD_x)v = 0, v = \chi_\delta(hD_x)v,$$

with $\Phi_\delta(\xi) = \phi_\alpha\left(\xi + h\left\lfloor\frac{\xi_0}{h}\right\rfloor\right), \phi_\alpha(\xi) = |\xi|^\alpha\xi + \frac{1}{\xi}$, after doing the same reduction as for the linear KP-I. Thus it would be sufficient to prove Proposition 4.3 for solutions of (4.A.1). Actually, the proof of Proposition 4.3 works also in the case $\alpha > 1$. For $\alpha = 1$, we need a little more argument.

Taylor expansion gives

$$\Phi_1(\xi) = \frac{\Phi_1''(\sigma_h)}{2}(\xi - \sigma_h)^2 + \frac{\Phi_1'''(\theta)}{6}(\xi-\sigma_h)^3.$$

Note that $\Phi_1''(\sigma_h) = \phi_1''(\xi_0) = 2A_0$ is independent of h, and we have

$$\Phi_1(hD_x) = h^2 A_0(D_x - r_h)^2 + O(\|\Phi_1'''\|_{L^\infty})((hD_x-\sigma_h)^3).$$

with $r_h = \frac{\sigma_h}{h} \in [0,1)$. For $0 < \delta \ll 1$, we decompose

$$v = v_1 + v_2, \quad v_1 = \chi_{A\delta}(h^{1/3}D_x)\chi_\delta(hD_x)v.$$

Then v_1 solves

$$D_t v_1 - A_0(D_x - \sigma_h)^2 v_1 = O_{L^2\to L^2}(A\delta)v_1.$$

We denote by $S_{\sigma_h}(t)$ the semi-group associated with the evolution Schrödinger operator $D_t - A_0(D_x-\sigma_h)^2$. From observability for classical Schrödinger equation, we have

$$\|v_1(0)\|^2 \leq C_T\int_0^T \|gS_{\sigma_h}(t)v_1(0)\|^2 dt$$

with constant C_T independent of $\sigma_h \in (0,1)$. For this independence assertion, we refer to Lemma 2.4 of [12]. Therefore, we have from Duhamel formula that

$$\begin{aligned}\|v_1(0)\|^2 \leq & C_T \int_0^T \left\| gv_1(t) - g\int_0^t O_{L^2\to L^2}(\delta)v_1(t')dt' \right\|^2 dt \\ \leq & C_T \int_0^T \|gv_1(t)\|^2 dt + AC_T\delta\|v_1\|_T^2 \\ \leq & C_T \int_0^T \|gv_1(t)\|^2 dt + AC_T\delta\|v_1(0)\|^2,\end{aligned} \tag{4.A.2}$$

where we have used the conservation of L^2 norm in the last step. For given $T>0$, we take $\delta > 0$ sufficiently small in a priori, and thus

$$\|v_1(0)\|^2 \leq C_T \int_0^T \|gv_1(t)\|^2 dt.$$

The estimate of v_2 follows in the same way as in the proof of Lemma 4.1. Therefore we have

$$\|v_2(0)\|^2 \leq C_T \int_0^T \|gv_2(t)\|^2 dt.$$

Finally, from the commutator estimate $\left\| \left[\chi_{A\delta}(h^{1/3}D_x)\chi_\delta(hD_x), g\right] \right\|_{L^2\to L^2} \leq Ch^{1/3}$, the proof is complete.

We now construct the conterexample of observability for the case $\alpha < 1$. The construction is in the same spirit as in [56].

Proposition 4.4
Suppose $0 < \alpha < 1$. *Then for any* $T > 0$, *there exists a sequence* v_n, *solutions of*

$$h_n^{1+\alpha} D_t v_n + \Phi_1(h_n D_x)v_n = 0,$$

such that

$$\lim_{n\to\infty} \frac{\int_0^T \int_\omega |v_n(t,x)|^2 dxdt}{\int_{\mathbb{T}} |v_n(0,x)|^2 dx} = 0.$$

Proof. We may assume that $\omega = (-\pi, -\beta) \cup (\beta, \pi]$. Take $G(x) = e^{-\frac{x^2}{2}}$ and define $G^{\epsilon_n}(x) = \frac{1}{\sqrt{\epsilon_n}} G\left(\frac{x}{\epsilon_n}\right)$. Denote the Fourier coefficient of G^{ϵ_n} by

$$g^{\epsilon_n}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G^{\epsilon_n}(x)e^{-ikx}dx = \frac{\sqrt{\epsilon_n}}{2\pi}\int_{-\frac{\pi}{\epsilon_n}}^{\frac{\pi}{\epsilon_n}} G(z)e^{-i\epsilon_n kz}dz.$$

The coefficient function $g^{\epsilon_n}(z)$ satisfies the following estimates:

$$\|g^{\epsilon_n}\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{1/2}), \|(g^{\epsilon_n})'\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{3/2}), \|(g^{\epsilon_n})''\|_{L^\infty(\mathbb{R})} = O(\epsilon_n^{5/2}). \tag{4.A.3}$$

Take an even cut-off function $\psi \in C_c^\infty(\mathbb{R})$ with supp $\psi \subset [-B, B]$ and $0 \le \psi \le 1$, $\psi|_{[-b,b]} \equiv 1$. We define

$$v_{n,0} = \sum_{k\in\mathbb{Z}} g^{\epsilon_n}(k)\psi(\widetilde{h}_n k)e^{ikx},$$

with $\widetilde{h_n} = h_n^{1-\alpha}$. The corresponding solution is given explicitly by

$$v_n(t,x) = \sum_{k\in\mathbb{Z}} g^{\epsilon_n}(k)\psi(\widetilde{h}_n k)\exp\left(ikx - \frac{it\Phi_1(\widetilde{h}_n^{\frac{1}{1-\alpha}}k)}{\widetilde{h}_n^{\frac{1+\alpha}{1-\alpha}}}\right).$$

We first estimate the lower bound of the mass of initial data.

$$\|G^{\epsilon_n}\|_{L^2(\mathbb{T})}^2 = \sum_{k\in\mathbb{Z}} |g^{\epsilon_n}(k)|^2 \sim 1$$

holds from Plancherel theorem and the definition of $g^{\epsilon_n}(k)$. We next estimate the mass away from the frequency scale h_n^{-1}, that is

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \left|(1-\psi(\widetilde{h}_n k))g^{\epsilon_n}(k)\right|^2 &\le \sum_{|\widetilde{h}_n k|>b} |g^{\epsilon_n}(k)|^2 \\
&\le \sum_{|\widetilde{h}_n k|>b} \frac{\epsilon_n}{4\pi^2}\left|\int_{\mathbb{R}} G(z)e^{-ik\epsilon_n}zdz\right|^2 \\
&= \sum_{|\widetilde{h}_n k|>b} \frac{\epsilon_n}{4\pi^2}\left|\int_{\mathbb{R}} G(z)\frac{1}{-ik\epsilon_n}\frac{d}{dz}e^{-ik\epsilon_n}zdz\right|^2 \\
&\le \sum_{|\widetilde{h}_n k|>b} \frac{1}{4k^2\pi^2\epsilon_n}\|G'\|_{L^1(\mathbb{R})}^2.
\end{aligned}$$

By setting $\epsilon_n = \sqrt{\widetilde{h}_n} \ll 1$, we have $\|(1-\psi(\widetilde{h}_n D_x))G^{\epsilon_n}\|_{L^2(\mathbb{T})} \ll 1$ and then $\|v_{n,0}\|_{L^2(\mathbb{T})} \sim 1$.

Now we choose $B > 0$ so that $|x - \|\Phi_1''\|_{L^\infty([-\delta,\delta])}Bt| \ge 2c_0 > 0 \mod 2\pi$ for all $x \in \omega = (-\pi, -\beta) \cup (\beta, \pi)$ and $|t| \le T$. Write

$$v_n(t,x) = \sum_{k\in\mathbb{Z}} K_{t,x}^{(n)}(k)$$

with

$$K_{t,x}^{(n)}(z) = g^{\epsilon_n}(z)\psi(\widetilde{h}_n z)\exp\left(izx - i\widetilde{h}_n^{-\frac{1+\alpha}{1-\alpha}}\Phi_1(\widetilde{h}_n^{\frac{1}{1-\alpha}}z)t\right).$$

From Poisson summation formula, we have

$$v_n(t,x) = \sum_{m\in\mathbb{Z}} \widehat{K_{t,x}^{(n)}}(2\pi m).$$

For fixed $m \in \mathbb{Z}$,

$$
\begin{aligned}
\widehat{K_{t,x}^{(n)}}(2\pi m) &= \int_{\mathbb{R}} g^{\epsilon_n}(z)\psi(h_n z) e^{i\varphi_{t,x}(z)} dz \\
&= \int_{\mathbb{R}} g^{\epsilon_n}(z)\psi(h_n z)\mathcal{L}^2(e^{i\varphi_{t,x}(z)}) dz
\end{aligned}
$$

with $\mathcal{L} = \dfrac{1}{i\varphi'_{t,x}(z)}\dfrac{d}{dz}$ and $\varphi_{t,x}(z) = (x - 2\pi m)z - \widetilde{h}_n^{-\frac{1+\alpha}{1-\alpha}}\Phi_1(\widetilde{h}_n^{\frac{1}{1-\alpha}} z)t$. Thus

$$
\widehat{K_{t,x}^{(n)}}(2\pi m) = \int_{\mathbb{R}} \frac{d}{dz}\left(\frac{1}{i\varphi'_{t,x}(z)}\frac{d}{dz}\left(\frac{g^{\epsilon_n}(z)\psi(\widetilde{h}_n z)}{i\varphi'_{t,x}(z)}\right)\right) e^{i\varphi_{t,x}(z)} dz.
$$

After tedious calculation, we have

$$
\begin{aligned}
&\frac{d}{dz}\left(\frac{1}{i\varphi'_{t,x}(z)}\frac{d}{dz}\left(\frac{g^{\epsilon_n}(z)\psi(\widetilde{h}_n z)}{i\varphi'_{t,x}(z)}\right)\right) \\
=&\frac{(g^{\epsilon_n})''\psi(\widetilde{h}_n z) + 2\widetilde{h}_n (g^{\epsilon_n})'\psi'(\widetilde{h}_n z) + \widetilde{h}_n^2 \psi''(\widetilde{h}_n z) g^{\epsilon_n}}{(\varphi'_{t,x})^2} \\
-&\frac{3((g^{\epsilon_n})'\psi(\widetilde{h}_n z) + \widetilde{h}_n \psi'(\widetilde{h}_n z) g^{\epsilon_n})\varphi''_{t,x}}{(\varphi'_{t,x})^3} \\
-&\frac{3g^{\epsilon_n}\psi(\widetilde{h}_n z)(\varphi''_{t,x})^2}{(\varphi'_{t,x})^4}.
\end{aligned}
$$

From (4.A.3), we have

$$
|\widehat{K_{t,x}^{(n)}}(2\pi m)| \leq \sup_{|\widetilde{h}_n z| \leq B} \frac{C\epsilon_n^{1/2}\|\psi\|_{W^{2,1}(\mathbb{R})}}{\left|(x - 2\pi m) - \widetilde{h}_n^{\frac{-\alpha}{1-\alpha}}\Phi_1'(\widetilde{h}_n^{\frac{1}{1-\alpha}} z)t\right|^2}.
$$

Note that from Taylor expansion,

$$
\widetilde{h}_n^{\frac{-\alpha}{1-\alpha}}\Phi_1'(\widetilde{h}_n^{\frac{1}{1-\alpha}} z)t = \Phi_1''(\theta_n)\widetilde{h}_n z t - \Phi_1''(\theta_n)\sigma_{h_n}\widetilde{h}_n^{\frac{-\alpha}{1-\alpha}} t
$$

for some $\theta_n \in (\sigma_{h_n}, \widetilde{h}_n^{\frac{1}{1-\alpha}} z)$. Therefore, for sufficiently large n, and for any $x \in 2\pi\mathbb{Z} + (-\beta, -\alpha) \cup (\beta, \pi]$,

$$
\left|x - \widetilde{h}_n^{\frac{-\alpha}{1-\alpha}}\Phi_1'(\widetilde{h}_n^{\frac{1}{1-\alpha}} z)t\right| \geq c_0 > 0 \text{ module } 2\pi,
$$

thus

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}} |\widehat{K_{t,x}^{(n)}}(2\pi m)| \leq& C\sum_{m\in\mathbb{Z}} \frac{C\epsilon_n^{1/2}}{|c_0 - 2\pi(m-p)|^2} \\
\leq& C\epsilon_n^{1/2}
\end{aligned}
$$

holds for any $p \in \mathbb{Z}$. Therefore,

$$\int_0^T \int_\omega |v_n(t,x)|^2 dxdt \le C\epsilon_n^{1/2} T|\omega| \to 0, \text{ as } n \to \infty.$$

■

Corollary 4.2
Suppose $0 < \alpha < 1$*, then for any* $T > 0$*, the observability for* u_n*, solutions of* (4.A.1) *does not hold true.*

Proof. We take h_n, v_n as in Proposition 4.4. Define

$$u_n(t,x,y) = w_n(t,x) \exp\left(iyh_n^{-\frac{\alpha+2}{2}} \right),$$

where $(h_n^{-\frac{\alpha+2}{2}})$ is a sequence of positive integers which converges to infinity. u_n solves (4.A.1) means that

$$h_n^{\alpha+1}\partial_t w_n - |h_n D_x|^\alpha h_n \partial_x w_n - h_n^{-1}\partial_x^{-1} w_n = 0.$$

Now we set $w_n = v_n e^{\lfloor \frac{i\xi_0}{h_n} \rfloor x}$, and we have from Proposition 4.4 that

$$\lim_{n\to\infty} \frac{\int_0^T \int_{\mathbb{T}^2} |g(x)u_n(t,x,y)|^2 dxdydt}{\int_{\mathbb{T}^2} |u_n(0,x,y)|^2 dxdy} = 0.$$

We finally need replace $\displaystyle\int_0^T \int_{\mathbb{T}^2} |g(x)u_n(t,x,y)|^2 dxdydt$ by $\displaystyle\int_0^T \int_{\mathbb{T}^2} |\mathcal{G}u_n(t,x,y)|^2 dxdydt$. This is guranteed by

$$\int_{\mathbb{T}} g(x')u_n(t,x',y)dx' \to 0, \quad \text{ in } L^2((0,T)\times \mathbb{T}_y)$$

from our construction. This completes the proof. ■

Chapter 5

Control and Stabilization of Nonlinear KP-II type equations

5.1 Introduction

The precise models considered here are KP-II equation

$$\partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) + \partial_y^2 u = 0, (t, x, y) \in \mathbb{R} \times \mathbb{T}^2 \tag{5.1.1}$$

and fifth order KP-II equation, or 5KP-II in short,[1]

$$\partial_x(\partial_t u - \partial_x^5 u + u\partial_x u) + \partial_y^2 u = 0, (t, x, y) \in \mathbb{R} \times \mathbb{T}^2. \tag{5.1.2}$$

These equations arise naturally in the study of transverse stability of the solitary wave solutions of the Kortewed de-Vries (KdV) equation and its higher order analogues. They have been widely studied in many different contexts([33],[51],[53])

In the periodic setting, both KP-II and 5KP-II are known to be global well-posed for L^2 initial data (see [6], [55]). The two models are infinite dimensional Hamiltonian system and the first conserved quantity for them are

$$M[u](t) = \int_{\mathbb{T}^2} u(t, x, y)^2 dxdy.$$

Moreover, the solutions to (5.1.1) and (5.1.2) must satisfy

$$\int_{\mathbb{T}} u(t, x, y) dx = 0$$

in the distributional sense, we may rewrite the equations in a non-local version

$$\partial_t u + (-1)^{l-1}\partial_x^{2l+1} u + u\partial_x u + \partial_x^{-1}\partial_y^2 u = 0,$$

[1] More generally, one could add the third order linear term $a\partial_x^3 u$ to the equation.

where the Fourier multiplier ∂_x^{-1} is defined by

$$\widehat{\partial_x^{-1} v}(k,\eta) = \frac{1}{ik}\widehat{v}(k,\eta)$$

for all functions

$$v \in \mathcal{D}_0'(\mathbb{T}^2) := \{v \in \mathcal{D}'(\mathbb{T}^2) : \widehat{v}(0,m) = 0 \text{ for all } m \in \mathbb{Z}\}. \tag{5.1.3}$$

We will study (5.1.3) in control point of view with a forcing term $f(t,x,y)$ added as control input:

$$\partial_t u + (-1)^{l-1}\partial_x^{2l+1} u + u\partial_x u + \partial_x^{-1}\partial_y^2 u = f, \tag{5.1.4}$$

where f is assumed to be supported in a given region $\omega \subset \mathbb{T}^2$. In control theory, the following two problems are essential:
Exact controllability: Given initial state u_0 and final state u_1, can one find a control input f such that the equation (5.1.4) admits a solution u which satisfies $u(0) = u_0$ and $u(T) = u_1$?
Stabilization: Can one find a feedback control law $f = \mathcal{K}u$ such that the resulting system

$$\partial_t u + (-1)^{l-1}\partial_x^{2l+1} u + u\partial_x u + \partial_x^{-1}\partial_y^2 u = \mathcal{K}u$$

is asymptotically stable as $t \to \infty$? Those problems are firstly studied by Russel-Zhang for KdV equation (see [54]) and widely extended to other dispersive models in last decades. Most notably, [20],[16],[40], for nonlinear Klein-Gordon(wave) equation, [17],[39] for nonlinear Schrödinger equation, [41] for KdV equation and [46] for Benjamin-Ono equation .

The set up of control problems for KP type equations is a little different. We observe that there is no obvious way to keep the control input f to be localized and simultaneously have zero horizontal mean, unless the region $\omega \subset \mathbb{T}^2$ is a vertical strip of the form $(a,b)\times\mathbb{T}$ or a horizontal strip $\mathbb{T}\times(a,b)$. When ω is prescribed as a horizontal strip, it is proved in [56] that the observability estimate can not hold for linear KP-II equation, due to the weak propagation phenomenon in y direction. Therefore, we address the problems of exact controllability and stabilization only for $\omega = (a,b)\times\mathbb{T}$ in this paper. In this case, the control input f should be of the form

$$f = \mathcal{G}h := g(x)\left(h(x,y) - \int_{\mathbb{T}} g(x')h(x',y)dx'\right),$$

where

$$g \in C_c^\infty(\mathbb{T}), \{x \in \mathbb{T} : g(x) > 0\} = (a,b) \text{ and } \int_{\mathbb{T}} g(x)dx = 1. \tag{5.1.5}$$

We fix the assumption above throughout this paper. For the stabilization, the feedback control law is given by

$$f = -\mathcal{G}^*\mathcal{G}u.$$

The resulting closed-loop system is called damped KP-II equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = -\mathcal{G}^*\mathcal{G}u, & (t,x) \in \mathbf{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2), \end{cases} \tag{5.1.6}$$

and damped 5KP-II equation

$$\begin{cases} \partial_t u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = -\mathcal{G}^*\mathcal{G}u, & (t,x) \in \mathbf{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2). \end{cases} \tag{5.1.7}$$

To state our main results, we define the δ_0- partial compact subset of $L_0^2(\mathbb{T}^2)$ by

$$S_{\delta_0} := \left\{ u_0 \in L_0^2(\mathbb{T}^2) : \|u_0\|_{H^{-1,0}(\mathbb{T}^2)} \leq \delta_0 \right\}.$$

The main results in this paper for KP-II equation are as follows.

Theorem 5.1
Let $s > 0$, $R_0 > 0$ be given. Then, there exist $C > 0, \gamma > 0$ and $\delta_0 > 0$ such that the inequality

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq Ce^{-\gamma t}\|u_0\|_{H^s(\mathbb{T}^2)}, t > 0$$

holds for every solution u of damped KP-II equation (5.1.6) *with initial data in S_{δ_0} such that $\|u_0\|_{H^s(\mathbb{T}^2)} \leq R_0$.*

From the usual arguments combining with the stabilization theorem and exact control result for small-data established in chapter 3, we have the exact control result for large data as follows:

Corollary 5.1
Let $s > 0$, $R_0 > 0$ be given. There exist $T > 0$, $\delta_0 > 0$, such that if $u_0, u_1 \in H_0^s(\mathbb{T}^2) \cap S_{\delta_0}$ satisfy

$$\|u_0\|_{H^s(\mathbb{T}^2)} \leq R_0, \|u_1\|_{H^s(\mathbb{T}^2)} \leq R_0,$$

then one can find a control input $\mathcal{G}h$ with $h \in L^2((0,T); H^s(\mathbb{T}^2))$ such that the system (5.1.4) *with $l = 1$ admits a solution $u \in C([0,T]; H_0^s(\mathbb{T}^2))$ satisfying*

$$u(0) = u_0, u(T) = u_1.$$

For 5KP-II, less regularity of the initial data is needed for stabilization and exact controllability. More precisely, we have

Theorem 5.2
Let $s > 0$, $R_0 > 0$ be given. Then, there exist $C > 0, \gamma > 0$ and $\delta_0 > 0$ such that the inequality

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq Ce^{-\gamma t}\|u_0\|_{H^s(\mathbb{T}^2)}, t > 0$$

holds for every solution u of damped KP-II equation (5.1.7) *with initial data in S_{δ_0} such that $\|u_0\|_{H^s(\mathbb{T}^2)} \leq R_0$.*

Corollary 5.2
Let $s \geq 0$, $R_0 > 0$ be given. There exist $T > 0$, $\delta_0 > 0$, such that if $u_0, u_1 \in H_0^s(\mathbb{T}^2) \cap S_{\delta_0}$ with

$$\|u_0\|_{H^s(\mathbb{T}^2)} \leq R_0, \|u_1\|_{H^s(\mathbb{T}^2)} \leq R_0,$$

then one can find a control input $\mathcal{G}h$ with $h \in L^2((0,T); H^s(\mathbb{T}^2))$ such that the system (5.1.4) *with $l = 2$ admits a solution $u \in C([0,T]; H_0^s(\mathbb{T}^2))$ satisfying*

$$u(0) = u_0, u(T) = u_1.$$

Let us make some comments on the results. Firstly, we call the condition that data in S_{δ_0} the smallness assumption of low horizontal frequency(in x variable). For both KP-II and 5KP-II, we need this smallness assumption because in our argument in section 5, we do not prove the observability for the regime where the frequency in x remains bounded while the frequency in y is high. Heuristically, we can view this regime as a coupled system of nonlinear Schrödinger equation in y variable. In this case, the propagation in x-direction is negligible and the propagation estimate in y variable could not help us to observe the solution from the vertical control region. In chapter 3, for linear KP-II, we exploit certain dispersive effect (Ingham inequality) in y variable to resolve this difficulty. However, since $H^{-\frac{1}{2},0}$ is the scaling-critical functional space for KP-II equation, the non-linear effect can no longer be ignored when we treat it as a system of nonlinear Schödinger equation in y variable. Therefore, How to remove this low horizontal frequency condition is a challenging nonlinear problem.

We also mention that the result we have obtained for KP-II equation is slightly weaker than 5KP-II equation. It is attributed to *a priori* estimate for Sobolev norms of solutions below L^2. In general framework of proving stabilization, we need semi-group property of the equation to iterate. In each step of iterating, our argument need the smallness of low horizontal frequency portion of the solution. However, low frequency portion, or equivalently, Sobolev norm with negative index in x, is not conserved along the nonlinear flow. This gives rise to the following problem:

Problem 5.1
Does there exist $s < 0$ such that for any given $T > 0$, any global solution u of equation (5.1.6) *satisfies*

$$\sup_{t\in[0,T]} \|u\|_{H^{s,0}(\mathbb{T}^2)} \leq C(T, \|u(0)\|_{H^{s,0}(\mathbb{T}^2)})?$$

Any positive answer of the problem above with the weaker bound $C(T, \|u(0)\|_{L^2(\mathbb{T}^2)})$ on the right hand side will improve the Theorem 5.1 to all $s \geq 0$. We believe that Problem 5.1 has its own interest since it relates to the existence of weak solution of KP-II below $L^2(\mathbb{T}^2)$. Similar problems in the context of nonlinear Schrödinger equation, KdV equation without damping term have been studied intensively, see for example [36],[35] and the references therein. Nevertheless, partially due to the local-well posedness of periodic 5KP-II equation below L^2(see [55]), we are able to apply almost-conservation trick, or I-method, to achieve *a priori* estimate of negative Sobolev norms of solutions of 5KP-II equation.

We organise the paper as follows. In the second section, we introduce some notations and state several basic propositions of Bourgain type functional spaces related to both KP-II and 5KP-II equation. In the third section, we prove the global well-posedness of damped KP-II and damped 5KP-II equation. Next, we use energy estimate to prove that the smallness condition of low horizontal frequencies of KP-II and 5KP-II equation is persistent along the nonlinear flow. In the fifth section, we prove the observability estimate for high horizontal frequencies of both inhomogeneous KP-II type equations by adapting the dyadic propagation estimate for linear equation in chapter 3. The final section devotes to the proof of stabilization. We also add an appendix to give a proof of stabilization of linear KP-II type equations by using observability inequality established in chapter 3.

5.2 Notations and Preliminaries

5.2.1 Notations

We begin by some notations which are standard in Partial Differential Equations. We use the notation $A \lesssim B$ or $A = O(B)$ to denote the estimate that $|A| \leq CB$ for certain uniform constant. We also use the notation $|A| \leq C_{A_1,A_1,\cdots}B$ of $|A| \leq C(A_1, A_2, \cdots)B$ to denote various constants C, depending on the quantities $A_1, A_2, \cdots$, which may change from line to line. We will use Japanese bracket $\langle\cdot\rangle := \sqrt{1+|\cdot|^2}$.

We identify $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. For $2\pi-$periodic function f on $\mathbb{R}$, we define its Fourier transform on Z by

$$\mathcal{F}_x f(k) = \widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx.$$

We identify function f on $\mathbb{T}^2$ by $2\pi-$periodic function f on $\mathbb{R}^2$. The Fourier transform on $\mathbb{Z}^2$ is denoted by

$$\widehat{f}(n) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(z)e^{-iz\cdot n}dxdy$$

with $z = (x, y) \in \mathbb{T}^2$ and $n = (k, m) \in \mathbb{Z}^2$

We introduce some notations about one-dimensional Little-wood Paley projection. Fix a cutoff function η such that

$$\eta \in C_c^\infty(\mathbb{R}), 0 \leq \eta \leq 1, \eta|_{[-1,1]} = 1, \mathrm{supp}(\eta) \subset [-2, 2].$$

We define

$$\phi(\xi) = \eta(\xi) - \eta(2\xi) \text{ and } \phi_N(\xi) = \phi\left(\frac{\xi}{N}\right)$$

Throughout the paper, capitalized variables such as $N, M, N_1, M_1, \cdots$ are assumed to be dyadic and large or equal than 1. Summations over these capitalized variables are presumed to be dyadic summation. We define Littlewood-Paley projector

$$P_N f := \mathcal{F}_x^{-1}\phi_N\mathcal{F}_x f, Q_M f := \mathcal{F}_y^{-1}\phi_N\mathcal{F}_y f$$

for $N \geq 1$ and

$$P_0 f = \mathcal{F}_x^{-1}\eta(2\cdot)\mathcal{F}_x f, Q_0 f = \mathcal{F}_y^{-1}\eta(2\cdot)\mathcal{F}_y f.$$

We have

$$P_0 + \sum_N P_N = 1, Q_0 + \sum_M Q_M = 1.$$

We will also use

$$P_{>N}(\text{ or } Q_{>N}) := \sum_{K>N} P_N(\text{ or } Q_N), P_{\leq N}(\text{ or } Q_{\leq N}) = 1 - P_{>N}(\text{ or } Q_{>N}).$$

We need the notations of non-dyadic spectral projection. We denote by

$$\Pi_{k_0 0} f = \mathcal{F}_x^{-1}\mathbf{1}_{|k|=k_0 0}\mathcal{F}_x f, \Pi_{\neq 0} := \mathcal{F}_x^{-1}\mathbf{1}_{k\neq 0}\mathcal{F}_x, \pi_{m_0} f = \mathcal{F}_y^{-1}\mathbf{1}_{|m|=m_0}\mathcal{F}_y f,$$

$$\Pi_{\geq k_0} = \sum_{k\geq k_0}\Pi_k, \Pi_{>k_0} = \sum_{k>k_0}\Pi_k, \Pi_{\leq k_0} = 1 - \Pi_{>k_0}, \Pi_{<k_0} = 1 - \Pi_{\geq k_0}$$

and similarly for $\pi_{\geq m_0}, \pi_{>m_0}, \pi_{\leq m_0}, \pi_{<m_0}$.

5.2.2 Functional spaces

Anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{T}^2)$ is defined via the norm

$$\|f\|^2_{H^{s_1,s_2}(\mathbb{T}^2)} := \sum_{(k,m)\in\mathbb{Z}^2} \langle k\rangle^{2s_1}\langle m\rangle^{2s_2}|\widehat{f}(k,m)|^2$$

The free groups associated with the linearized KP-II and 5KP-II equation are defined as

$$\widehat{S(t)f}(k,m) = e^{it\left(k^3-\frac{m^2}{k}\right)}\widehat{f}(k,m), \widehat{U(t)f}(k,m) = e^{it\left(k^5-\frac{m^2}{k}\right)}\widehat{f}(k,m)$$

with respectively. Anisotropic Bourgain spaces adapted to periodic KP-II and 5KP-II equations are defined via the norms

$$\|u\|^2_{X_3^{s_1,s_2;b,b_1}} := \sum_{(k,m)\in\mathbb{Z}^2}\int_{\mathbb{R}}\left\langle\frac{\tau - p_3(k,m)}{\langle k\rangle^3}\right\rangle^{2b_1}\langle\tau - p_3(k,m)\rangle^{2b}\langle k\rangle^{2s_1}\langle m\rangle^{2s_2}|\widehat{u}(\tau,k,m)|^2 d\tau,$$

$$\|u\|^2_{X_5^{s_1,s_2;b,b_1}} := \sum_{(k,m)\in\mathbb{Z}^2}\int_{\mathbb{R}}\left\langle\frac{\tau - p_5(k,m)}{\langle k\rangle^4}\right\rangle^{2b_1}\langle\tau - p_5(k,m)\rangle^{2b}\langle k\rangle^{2s_1}\langle m\rangle^{2s_2}|\widehat{u}(\tau,k,m)|^2 d\tau,$$

with respectively, where

$$p_3(\tau,k,m) = k^3 - \frac{m^2}{k}, p_5(\tau,k,m) = k^5 - \frac{m^2}{k}.$$

We make some conventions here. If there is no risk of confusing, we denote by $X^{s_1,s_2;b,b_1}$ the Bourgain spaces of KP-II or 5KP-II. We also denote by $X^{s_1,s_2,b}$ if $b_1=0$ and $X^{s,b}$ simply if $s_1=s_2=s, b_1=0$.

The norm in the localized time interval I is defined by

$$\|u\|_{X^{s_1,s_2,b,b_1}(I)} := \inf\{\|w\|_{X^{s_1,s_2,b,b_1}} : w(t)|_I = u(t)|_I\}.$$

When $I=[0,T]$ we also denote by $X^{s_1,s_2;b_1,b_2}(T)$. In general, we estimate the norm $X^{s_1,s_2;b,b_1}(I)$ by estimating $\|\psi_T u\|_{X^{s_1,s_2;b,b_1}}$ for time localization version of u, where $\psi_T = \psi(T^{-1}\cdot)$, $\psi \in C_c^\infty(\mathbb{R})$, $\psi(t)\equiv 1$, $|t|\le \frac{1}{2}$ and $\psi(t)\equiv 0$ for $|t|\ge 1$.

Proposition 5.1
For $-\frac{1}{2} < b' \le b < \frac{1}{2}$ and $0<T<1$, we have

$$\|\psi_T u\|_{X^{s_1,s_2;b'}} \le CT^{b-b'}\|u\|_{X^{s_1,s_2;b}}.$$

We have the following linear estimate for 5KP-II.

Proposition 5.2
For $-\frac{1}{2} < b' \le 0 < \frac{1}{2} < b \le b'+1, T\le 1$, we have

1. $$\|\psi_T(t)U(t)u_0\|_{X_5^{s_1,s_2;b}} \le CT^{\frac{1}{2}-b}\|u_0\|_{H^s(\mathbb{T}^2)}.$$

2. $$\left\|\psi_T \int_0^t U(t-t')F(t')dt'\right\|_{X_5^{s_1,s_2,b}} \le CT^{1-(b-b')}\|F\|_{X_5^{s_1,s_2,b'}}$$

For KP-II, it seems that we can not avoid to use the end points $b'=-\frac{1}{2}$ and $b=\frac{1}{2}$. However, the analogue of the linear estimate above is false. We need an auxiliary norm

$$\|u\|_{Y^{s,b}} := \|\langle \tau - p_3(n)\rangle^{b-\frac{1}{2}}\langle n\rangle^s \widehat{u}\|_{l_n^2 L_\tau^1}$$

and the restricted spaces $Y^{b,s}(I)$ defined in the same manner. We let

$$Z^{s,b}(I) := X_3^{s,s;b,b_1}(I)\cap Y^{s,b}(I).$$

With this axillary norm, the linear estimate now holds true.

Proposition 5.3 *1.*
$$\|\psi_T(t)S(t)u_0\|_{Z^{s,\frac{1}{2}}} \le C\|u_0\|_{H^s(\mathbb{T}^2)}.$$

2.
$$\left\|\psi_T(t)\int_0^t S(t-t')F(t')dt'\right\|_{Z^{s,\frac{1}{2}}} \le C\|F\|_{Z^{s,-\frac{1}{2}}}.$$

3.

$$\sup_{t\in[0,T]} \left\| \int_0^t S(t-t')F(t')dt' \right\|_{H^s(\mathbb{T}^2)} \leq C\langle T\rangle \|F\|_{Z^{s,-\frac{1}{2}}}.$$

All the constants C are independent of T.

We summarize several basic properties of these spaces:

1. $X^{s_1,s_2;b,b_1}$ is a Hilbert space. For $b' \leq b, s_1' \leq s_1, s_2' \leq s_2, b_1' \leq b_1$, $X^{s_1,s_2;b,b_1}$ is continuously imbedded in the space $X^{s_1',s_2';b',b_1'}$.
2. $Z^{s,\frac{1}{2}}(I) \subset C(I; H^s(\mathbb{T}^2))$ for any $s \in \mathbb{R}$.
3. $X^{s,b+\epsilon} \subset Y^{s,b}$ holds for any $\epsilon > 0$.
4. Given any finite interval I, if $b' < b, b_1' < b_1, s_1' < s_1, s_2' < s_2$, then the space $X^{s_1,s_2;b,b_1}(I)$ is compactly imbedded into $X^{s_1',s_2';b',b_1'}(I)$.

We note that bounded sequence may not have weak limit in the space $Z^{s,\frac{1}{2}}(I)$ since it is L^1 based space and Banach-Alaoglu theorem is not applicable.

We now recall the bilinear estimates needed for Cauchy problem and energy estimate. For KP-II, we have

Proposition 5.4 ([6],[51])
There exist $\frac{1}{4} < b_1 < \frac{3}{8}$ and $\nu > 0$ such that for all functions $u, v \in X^{s,\frac{1}{2},b_1}$ with

$$\int_{\mathbb{T}} u(t,x,y)dx = \int_{\mathbb{T}} v(t,x,y)dx = 0,$$

the following bilinear estimate holds

$$\|\partial_x(uv)\|_{Z^{s,-\frac{1}{2}}} \leq C\|u\|_{X_3^{s,s;\frac{1}{2},b_1}} \|v\|_{X_3^{s,b}} + C\|v\|_{X_3^{s,s;\frac{1}{2},b_1}} \|u\|_{X_3^{s,b}},$$

with some $0 < b < \frac{1}{2}$, for all $s \geq 0$. As a consequence,

$$\left\| \psi_1(t) \int_0^t S(t-t')\psi_T(t')\partial_x(uv(t'))dt' \right\|_{Z^{s,\frac{1}{2}}} \leq CT^\nu \|u\|_{X_3^{s,s;\frac{1}{2},b_1}} \|v\|_{X_3^{s,s;\frac{1}{2},b_1}}$$

for some $\nu > 0$. All the constants C are independent of T.

For 5KP-II, we have

Proposition 5.5 ([55])
There exist $\frac{1}{4} \leq b_1 < \frac{3}{8}$ $b > \frac{1}{2}, b' < \frac{1}{2}, b + b' < 1$, then

$$\|\partial_x(uv)\|_{X_5^{s_1,s_2;-b',\frac{1}{4}}} \leq C\|u\|\|_{X_5^{s_1,s_2;b,b_1}} \|v\|\|_{X_5^{s_1,s_2;b,b_1}}.$$

Moreover,

$$\left\|\psi_1(t)\int_0^t U(t-t')\psi_T(t')\partial_x(uv(t'))dt'\right\|_{X_5^{s,s;-b',b_1}} \le CT^\nu\|u\|_{X_5^{s,s;b,b_1}}\|v\|_{X_5^{s_1,s_2;b,b_1}}$$

for some $\nu>0$.All the constants C are independent of T.

5.3 Cauchy-Problem for nonlinear system with damping

First we prove global well-posedness of damped KP-II equation

$$\begin{cases}\partial_t u+\partial_x^3 u+\partial_x^{-1}\partial_y^2 u+u\partial_x u=-\mathcal{G}^*\mathcal{G}u, & (t,x)\in\mathbf{R}\times\mathbb{T}^2,\\ u|_{t=0}=u_0\in H_0^s(\mathbb{T}^2),\end{cases} \tag{5.3.1}$$

Theorem 5.3
Fix any $s\ge 0$. For any given $T>0$ and any $u_0\in H_0^s(\mathbb{T}^2)$, there exists a unique solution $u\in Z^{s,\frac12}(T)\cap C([0,T];H_0^s(\mathbb{T}^2))$ of (5.3.1). Moreover, we have the following estimate

$$\|u\|_{Z^{s,\frac12}(T)}\le\alpha(T,\|u_0\|_{L^2(\mathbb{T}^2)})\|u_0\|_{H^s(\mathbb{T}^2)}.$$

The positive function $\alpha(T,R)$ depends polynomial in T.

Proof. Given $u_0\in H_0^s(\mathbb{T}^2)$. For each $v\in Z^{s,\frac12}(T)$, we extend v to a function on $\mathbb{R}\times\mathbb{T}^2$ and still denote by v. For any given $u_0\in H_0^s(\mathbb{T}^2)$, define the map

$$\Phi_{u_0}(v)(t)=S(t)v_0-\int_0^t S(t-t')(v\partial_x v)(t')dt'-\int_0^t S(t-t')\mathcal{G}^*\mathcal{G}v(t')dt.'$$

For $t\le T\le 1$, we rewrite it as

$$\Phi_{u_0}(v)(t)=\psi_1(t)S(t)v_0-\frac12\psi_1(t)\int_0^t S(t-t')\partial_x(\psi_T^2v^2)(t')dt'-\psi_1(t)\int_0^t S(t-t')\mathcal{G}^*\mathcal{G}\psi_T v(t')dt'.$$

Define the ball

$$B_T(r)=\{u\in Z_T^{s,\frac12}:\|u\|_{Z^{s,\frac12}(T)}\le r\}.$$

From linear and bilinear estimates, we have

$$\begin{aligned}\|\Phi_{u_0}(u)\|_{Z^{s,\frac12}}\le& C\left(\|u_0\|_{H^s(\mathbb{T}^2)}+\|\partial_x(\psi_T^2u^2)\|_{Z^{s,-\frac12}}+\|\mathcal{G}^*\mathcal{G}\psi_T u\|_{Z^{s,-\frac12}}\right)\\ \le& C\left(\|u_0\|_{H^s(\mathbb{T}^2)}+T^\nu\|\psi_T u\|^2_{X_3^{s,s;\frac12,b_1}}+\|\mathcal{G}^*\mathcal{G}\psi_T u\|_{Z^{s,-\frac12}}\right).\end{aligned}$$

We estimate

$$\|\mathcal{G}^*\mathcal{G}\psi_T u\|_{Y^{s,-\frac12}}\le C\|\mathcal{G}^*\mathcal{G}\psi_T u\|_{X^{s,-\frac12+\epsilon}}\le CT^{\frac12-\epsilon}\|\psi_T u\|_{X^{s,0}}.$$

$$\|\mathcal{G}^*\mathcal{G}\psi_T u\|_{X^{s,s;-\frac{1}{2},b_1}} \leq \|\mathcal{G}^*\mathcal{G}\psi_T u\|_{X^{s,s;-\frac{1}{2}+\epsilon,b_1}}.$$

For $-\frac{1}{2} < b' < 0$, we calculate

$$\begin{aligned}\|\psi_T v\|_{X^{s,s;b',b_1}} \leq& \|\psi_T v\|_{X^{s,b'}} + \left\| \frac{\langle D_t - p_3(D)\rangle^{b_1}}{\langle D_x^3\rangle^{b_1}} \psi_T v \right\|_{X^{s,b'}} \\ \leq& CT^{-b'}\|v\|_{X^{s,0}} + \|\psi_T v\|_{X^{s,b'+b_1}} \\ \leq& CT^{-b'-b_1}\|v\|_{X^{s,0}},\end{aligned}$$

provided that $b_1 + b' \leq 0$, thanks to Proposition 5.1. Applying the estimate above to $b' = -\frac{1}{2} + \epsilon$ and $v = \mathcal{G}^*\mathcal{G}\psi_T u$, we have

$$\|\Phi_{u_0}(u)\|_{Z^{s,\frac{1}{2}}} \leq C\left(\|u_0\|_{H^s(\mathbb{T}^2)} + T^{\nu'}\|\psi_T u\|^2_{X^{s,s;\frac{1}{2},b_1}} + T^{\nu'}\|\psi_T u\|_{X^{s,s;\frac{1}{2},b_1}}\right)$$

for some $\nu' > 0$, thanks to the condition $b_1 < \frac{3}{8}$. This implies that

$$\|\Phi_{u_0}(u)\|_{Z^{s,\frac{1}{2}}(T)} \leq C\left(\|u_0\|_{H^s(\mathbb{T}^2)} + T^{\nu'}\|\psi_T u\|^2_{Z^{s,\frac{1}{2}}(T)} + T^{\nu'}\|\psi_T u\|_{Z^{s,\frac{1}{2}}(T)}\right).$$

Similar manipulation yields

$$\|\Phi_{u_0}(u_1) - \Phi_{u_0}(u_2)\|_{Z^{s,\frac{1}{2}}(T)} \leq CT^{\nu'}\left(1 + \|u_1\|_{Z^{s,\frac{1}{2}}(T)} + \|u_2\|_{Z^{s,\frac{1}{2}}(T)}\right)\|u_1 - u_2\|_{Z^{s,\frac{1}{2}}(T)}$$

Therefore, by taking $r > 2C\|u_0\|_{H^s(\mathbb{T}^2)}$ and $T = T(\|u_0\|) > 0$, the map $\Phi_{u_0} : B_T(r) \to B_T(r)$ is well-defined and contraction. This implies that For any given $u_0 \in H_0^s(\mathbb{T}^2)$, there exists $T_0 = T_0(\|u_0\|_{H^s(\mathbb{T}^2)})$ such that the solution $u \in Z^{s,\frac{1}{2}}(T_0) \cap C([0, T_0]; H_0^s(\mathbb{T}^2))$ exists and unique. Moreover, $\|u\|_{Z^{s,\frac{1}{2}}(T_0)} \leq 2C_0\|u_0\|_{H^s(\mathbb{T}^2)}$.

We next extend this local solution to global one. First we consider the case $s = 0$. From the mass dissipation

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} = \|u(t)\|^2_{L^2(\mathbb{T}^2)} + \int_0^t \|\mathcal{G}u(t')\|^2_{L^2(\mathbb{T}^2)}dt',$$

we have a priori bound

$$\|u(t)\|_{L(\mathbb{T}^2)} \leq \|u_0\|_{L^2(\mathbb{T}^2)}.$$

This implies the global existence for $s = 0$.

Now we want to prove the global existence for all $s \geq 0$. Suppose that $u_0 \in H^3(\mathbb{T}^2)$, let $v = \partial_t u, w = \partial_y u$, then v, w satisfy the same equation

$$\partial_t v + \partial_x^3 v + \partial_x^{-1}\partial_y^2 v = -\partial_x(uv) - \mathcal{G}^*\mathcal{G}v.$$

We have that

$$\|v\|_{Z^{0,\frac{1}{2}}(T)} \leq C_0\|v_0\|_{L^2(\mathbb{T}^2)} + C_0T^{\nu'}\left(1 + \|u\|_{Z^{0,\frac{1}{2}}(T)}\right)\|v\|_{Z^{0,\frac{1}{2}}(T)}.$$

For $T < T_1(\|u_0\|_{L^2(\mathbb{T}^2)}) = \min\left\{2T_0(\|u_0\|_{L^2(\mathbb{T}^2)}), \left(2C_0(1 + \|u_0\|_{L^2(\mathbb{T}^2)})\right)^{-\frac{1}{\nu'}}\right\}$, we have

$$\|v\|_{L^\infty([0,T];L^2(\mathbb{T}^2))} \leq C'\|v\|_{Z^{0,\frac{1}{2}}(T)} \leq C_1\|v_0\|_{L^2(\mathbb{T}^2)}.$$

Now since $\|u(t)\|_{L^2(\mathbb{T}^2)} \leq \|u_0\|_{L^2(\mathbb{T}^2)}$ for all $t \geq 0$, we can extend v globally. The same argument holds true for $w = \partial_y u$. Taking $w_1 = \partial_y^2 u, w_2 = \partial_y^3 u$, we have that $u \in C(\mathbb{R}_+; H_0^{0,3}(\mathbb{T}^2))$. From the equation

$$\partial_x^3 u = -v - \partial_x^{-1}\partial_y^2 u - u\partial_x u - \mathcal{G}^*\mathcal{G}u,$$

we obtain that for any $T_0 > 0$ and $0 \leq t < T_0$,

$$\begin{aligned}\|\partial_x^3 u(t)\|_{L^2(\mathbb{T}^2)} \leq& \|v(t)\|_{L^2(\mathbb{T}^2)} + \|\partial_x^{-1}\partial_y^2 u(t)\|_{L^2(\mathbb{T}^2)}\\ &+C\left(\|u(t)\|_{L^2(\mathbb{T}^2)} + \|u(t)\|_{L_x^2(\mathbb{T};L_y^\infty(\mathbb{T}))}\|\partial_x u(t)\|_{L_x^\infty(\mathbb{T};L_y^2(\mathbb{T}))}\right).\end{aligned}$$

Using one-dimensional Sobolev imbedding and interpolation, we finally have

$$\sup_{t\in[0,T_0]}\|\partial_x^3 u(t)\|_{L^2(\mathbb{T}^2)} \leq C(T_0, \|u_0\|_{H^3(\mathbb{T}^2)})$$

and $u \in C(\mathbb{R}_+; H^3(\mathbb{T}^3))$. From induction, if $u_0 \in H_0^{3k}(\mathbb{T}^2)$ for $k \in \mathbb{N}$, we have that $u \in C(\mathbb{R}_+; H^{3k}(\mathbb{T}^2))$. Other values of s can be obtained from interpolation result. This completes the proof. The assertion that $\alpha(T, R_0)$ is polynomial in T follows from inductio and partition of $[0, T]$ for large $T > 0$. ■

For damped 5KP-II

$$\begin{cases}\partial_t u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = -\mathcal{G}^*\mathcal{G}u, & (t, x) \in \mathbf{R} \times \mathbb{T}^2,\\ u|_{t=0} = u_0 \in H_0^s(\mathbb{T}^2),\end{cases} \tag{5.3.2}$$

we have

Theorem 5.4
Fix any $s \geq 0$. For any given $T > 0$ and any $u_0 \in H_0^s(\mathbb{T}^2)$, there exists a unique solution $u \in X_5^{s,s;b,b_1}(T) \cap C([0, T]; H_0^s(\mathbb{T}^2))$ of (5.4), where $\frac{1}{2} < b < \frac{5}{8}$ and $\frac{1}{4} \leq b_1 < \frac{3}{8}$. Moreover, we have the following estimate

$$\|u\|_{X_5^{s,s;b,b_1}(T)} \leq C(T, \|u_0\|_{L^2(\mathbb{T}^2)})\|u_0\|_{H^s(\mathbb{T}^2)}.$$

Proof. The proof is similar as the proof of Theorem 5.3, with the minor difference of the choice of functional spaces. We just write down the contraction step here. Define the map

$$\Phi_{u_0}(v) = \psi_1(t)U(t)v_0 - \frac{1}{2}\psi_1(t)\int_0^t U(t-t')\partial_x(\psi_T^2 v^2)(t')dt' - \psi_1(t)\int_0^t U(t-t')\mathcal{G}^*\mathcal{G}\psi_T v(t')dt'.$$

and the ball

$$B_T(r) = \{u \in X_5^{s,s;b,b_1}(T) : \|u\|_{X_5^{s,s;b,b_1}(T)} \leq r\}.$$

Applying Proposition 5.2 and Proposition 5.5, we have

$$\begin{aligned}\|\Phi_{u_0}(u)\|_{X^{s,s;b,b_1}} \leq & C\left(\|u_0\|_{H^s(\mathbb{T}^2)} + T^\nu\|\psi_T u\|^2_{X_5^{s,s;b,b_1}} + \|\mathcal{G}^*\mathcal{G}\psi_T u\|_{X_5^{s,s;b-1,b_1}}\right) \\ \leq & C\left(\|u_0\|_{H^s(\mathbb{T}^2)} + T^\nu\|\psi_T u\|^2_{X_5^{s,s;b,b_1}} + T^{\nu'}\|\psi_T u\|_{X_5^{s,s,0,0}}\right),\end{aligned}$$

where to the last inequality, we have used the estimate

$$\begin{aligned}\|\psi_T v\|_{X_5^{s,s;b-1,b_1}} \leq & \|\psi_T v\|_{X_5^{s,s;b-1,0}} + \left\|\frac{\langle\sigma(D_{t,x,y})\rangle^{b_1}}{\langle D_x^4\rangle^{b_1}}\psi_T v\right\|_{X_5^{s,s;b-1,0}} \\ \leq & CT^{1-b}\|v\|_{X_5^{s,s;0,0}} + \|\psi_T v\|_{X_5^{s,s;b'+b_1,0}} \\ \leq & CT^{1-b-b_1}\|v\|_{X_5^{s,s;0,0}},\end{aligned}$$

thanks to $b + b_1 < 1$. This implies that

$$\|\Phi_{u_0}(u)\|_{X_5^{s,s;b,b_1}(T)} \leq C\left(\|u_0\|_{H^s(\mathbb{T}^2)} + T^{\nu'}\|\psi_T u\|^2_{X_5^{s,s;b,b_1}(T)} + T^{\nu'}\|\psi_T u\|_{X_5^{s,s;b,b_1}(T)}\right).$$

Similar manipulation yields

$$\|\Phi_{u_0}(u_1) - \Phi_{u_0}(u_2)\|_{X_5^{s,s;b,b_1}(T)} \leq CT^{\nu'}\left(1 + \|u_1\|_{X_5^{s,s;b,b_1}(T)} + \|u_2\|_{X_5^{s,s;b,b_1}(T)}\right)\|u_1 - u_2\|_{X_5^{s,s;b,b_1}(T)}.$$

By choosing $r > 0$ large and $T > 0$ small enough, Φ_{u_0} is a contraction map on $B_T(r)$. ■

5.4 Long time a priori estimate

5.4.1 KP-II

Next we prove a lemma which ensures the smallness of low frequency portion as well as $H^{-1,0}$ norm along KP-II flow.

Proposition 5.6
Assume $R_0 > 0, T_0 > 0$ are given. For any $\epsilon > 0$, there exists a small number $\delta_0 > 0$, such that for all solutions u of (5.1.6) *with*

$$\|u(0)\|_{H^s(\mathbb{T}^2)} \leq R_0, \|u(0)\|_{H^{-1,0}(\mathbb{T}^2)} \leq \delta_0,$$

the solution satisfies

$$\sup_{t\in[0,T_0]} \|u(t)\|_{H^{-1,0}(\mathbb{T}^2)} \leq \epsilon.$$

Proof. First we note that from Cauchy theory,

$$\|u\|_{Z_T^{s,\frac{1}{2}}} \leq C(T_0, R_0).$$

Since for any N_0,

$$\|u\|^2_{H^{-1,0}(\mathbb{T}^2)} \leq \|P_{\leq N_0} u\|^2_{L^2(\mathbb{T}^2)} + \frac{1}{N_0^2}\|u\|^2_{L^2(\mathbb{T}^2)},$$

it would be suffice to estimate $\|P_{\leq N_0} u(t)\|^2_{L^2(\mathbb{T}^2)}$. We perform a simple energy estimate.

Let $v = P_{\leq N_0} u$, and $w = u - v$. v solves the equation

$$(\partial_t + \partial_x^3 + \partial_x^{-1}\partial_y^2)v = P_{\leq N_0}\left(-\frac{1}{2}\partial_x(v^2) - \partial_x(vw) - \frac{1}{2}\partial_x(w^2) - \mathcal{G}^*\mathcal{G}(v+w)\right).$$

We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{T}^2)} =& -\|\mathcal{G}v(t)\|^2_{L^2(\mathbb{T}^2)} - \int_{\mathbb{T}^2} v\partial_x(vw) - \frac{1}{2}\int_{\mathbb{T}} v\partial_x(w^2)\\ &+\frac{1}{2}\int_{\mathbb{T}^2} P_{>N_0} v\partial_x(v^2) - \int_{\mathbb{T}^2}\mathcal{G}v\cdot\mathcal{G}w.\end{aligned}$$

From bilinear estimate, we have

$$\begin{aligned}\|v(t)\|^2_{L^2(\mathbb{T}^2)} - \|v(0)\|^2_{L^2(\mathbb{T}^2)} \leq& C\|v\|_{X_{T_0}^{0,\frac{1}{2}}}\left(\|v\|_{X^{0,0;\frac{1}{2},b_1}}\|w\|_{X^{0,0;\frac{1}{2},b_1}} + \|w\|^2_{X^{0,0;\frac{1}{2},b_1}}\right)\\ &+C\|v\|_{L^\infty([0,T_0];L^2(\mathbb{T}^2))}\|w\|_{L^\infty([0,T_0];L^2(\mathbb{T}^2))}\\ \leq& C\frac{\|u\|_{X_{T_0}^{0,\frac{1}{2}}}}{N_0^s}\left(\|u\|_{X^{0,0;\frac{1}{2},b_1}}\|w\|_{X^{s,s;\frac{1}{2},b_1}} + \|w\|^2_{X^{s,s;\frac{1}{2},b_1}}\right)\\ &+\frac{C}{N_0^s}\|u\|_{L^\infty([0,T_0];L^2(\mathbb{T}^2))}\|w\|_{L^\infty([0,T_0];H^s(\mathbb{T}^2))}\\ \leq& \frac{C(T_0,R_0)}{N_0^s}.\end{aligned}$$

We complete the proof by choosing N_0 large enough and δ_0 small such that

$$\delta_0^2 + C(T_0,R_0)N_0^{-s} < \frac{\epsilon}{2}.$$

■

5.4.2 5KP-II

Fix $s_0 \in \left(0, \frac{1}{4}\right)$. The Stronger dispersion of 5KP-II enable us to prove a finer energy estimate in $H^{-s_0,0}(\mathbb{T}^2)$ for bounded L^2 solutions.

Proposition 5.7
Assume that $R_0 > 0, T_0 > 0$ are given. Then there exists a positive increasing function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, such that for all solutions u of (5.3.2) with $\|u(0)\|_{L^2(\mathbb{T}^2)} \leq R_0$, we have

$$\sup_{t\in[0,T_0]} \|u(t)\|_{H^{-s_0,0}(\mathbb{T}^2)} \leq \beta(T_0, R_0)^{\frac{8s_0}{1+8s_0}} \|u(0)\|_{H^{-s_0,0}(\mathbb{T}^2)}^{\frac{1}{1+8s_0}}.$$

Furthermore, for any fixed $R_0 > 0$, $\beta(T_0, R_0)$ depends polynomial on T_0.

Corollary 5.3
Assume $R_0 > 0, T_0 > 0$ are given. For any $\epsilon > 0$, there exists a small number $\delta_0 > 0$, such that for all solutions u of (5.3.2) with

$$\|u(0)\|_{L^2(\mathbb{T}^2)} \leq R_0, \|u(0)\|_{H^{-1,0}(\mathbb{T}^2)} \leq \delta_0,$$

the solution satisfies

$$\sup_{t\in[0,T_0]} \|u(t)\|_{H^{-1,0}(\mathbb{T}^2)} \leq \epsilon.$$

Remark 5.1
The choice of δ_0 will be the power of ϵ divided by power of T_0. One can also use local Cauchy theory in $H^{-s_0,0}(\mathbb{T}^2)$ for 5KPII to prove Corollary 5.3 directly, which will be enough for the final proof. However, the small number δ_0 depend exponentially on T_0 in this way. In what follows, we choose to prove Proposition 5.7 via energy estimate, both for relaxing the restriction on δ_0 and its own interest.

For the proof, we will perform a slightly different energy estimate as in the last section. Inspired by the "I-method", firstly introduced by C-K-S-T-T in [15], we define the operator

$$I_N u = \mathcal{F}_x^{-1}\theta(k)\mathcal{F}_x u,$$

where the Fourier multiplier

$$\theta(\xi) = 1, |\xi| \leq N, \theta(\xi) = \frac{N^{s_0}}{|\xi|^{s_0}}, |\xi| > 2N.$$

for some $0 < s_0 < \frac{1}{4}$ and a large dyadic number $N \gg 1$ to be chosen later. In what follows, we denote by I the operator I_N.

Lemma 5.1

$$\|[I, \mathcal{G}]\|_{L^2\to L^2} \leq \frac{C}{N},$$

where the constant C only depend on the operator $\mathcal{G}$.

Proof. Since all the operators are acting on functions of $L^2(\mathbb{T})$, it suffices to estimate for functions of one variable. For any $f \in L^2(\mathbb{T})$, we calculate

$$\|[I, \mathcal{G}]f\|_{L^2(\mathbb{T})}^2 = \sum_{k\in\mathbb{Z}} \left| \sum_{k_1\in\mathbb{Z}} (g_{k-k_1} - g_k\overline{g}_{k_1})(\theta(k) - \theta(k_1))\widehat{f}(k_1) \right|^2.$$

We estimate the sum according to $kk_1 \geq 0$ and $kk_1 < 0$. If k and k_1 have the same sign, say positive, we use the fundamental theorem of calculus

$$|\theta(k) - \theta(k_1)| = \left|(k - k_1)\int_0^1 \theta'(\lambda k_1 + (1-\lambda)k)d\lambda\right| \leq \frac{C|k-k_1|}{N}.$$

Thus

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}\left|\sum_{k_1\neq 0, k_1k\geq 0}(g_{k-k_1} - g_k\overline{g}_{k_1})(\theta(k) - \theta(k_1))\widehat{f}(k_1)\right|^2\\
\leq&\frac{C^2}{N^2}\sum_{k\in\mathbb{Z}}\left|\sum_{k_1\neq 0}|k-k_1|(g_{k-k_1} - g_k\overline{g}_{k_1})\widehat{f}(k_1)\right|^2\\
\leq&\frac{C^2}{N^2}
\end{aligned}$$

from Young's convolution inequality and Cauchy-Schwartz, since $g \in C_c^\infty(\mathbb{T})$ yields $(kg_k)_{k\in\mathbb{Z}} \in l_k^1$. If k and k_1 have different sign, the only contributions of the sum comes from:

1. $-N \leq k \leq 0, k_1 > N$ or $0 \leq k \leq N, k_1 < -N$.
2. $k < -N, 0 \leq k_1 \leq N$ or $k > N, -N \leq k_1 \leq 0$.
3. $|k| > N, |k_1| > N, kk_1 < 0$.

In both situations we could bound the summation by

$$\|g_k\|^2_{l^1(|k>N|)}\|\widehat{f}(k)\|^2_{l^2(k)} + \|g_k\|^2_{l^2}\|g_k\|^2_{l^2(|k>N|)}\|\widehat{f}(k)\|^2_{l^2(k)} \leq \frac{C}{N^2},$$

thanks to Young's convolution inequality, Cauchy-Schwartz and the fact that $g \in C_c^\infty(\mathbb{T})$. ■

We need a dyadic version of bilinear estimate proved in [55].

Lemma 5.2 ([55])
Let u_1 and u_2 be two functions defined on $\mathbb{R}\times\mathbb{T}^2$ such that

$$\text{supp}(\widehat{u_j}) = \left\{(\tau_j, k_j, m_j) : |k_j| \sim N_j, \left\langle \tau_j - k_j^5 + \frac{m_j^2}{j_j}\right\rangle \sim L_j\right\}, j = 1, 2.$$

Then

$$\begin{aligned}
&\|P_N(u_1u_2)\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\\
\leq&C(L_1\wedge L_2)^{\frac12}(L_1\vee L_2)^{\frac14}(N_1\wedge N_2)^{\frac12}\frac{(N_1N_2)^{\frac14}}{N^{\frac14}}\|u_1\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\|u_2\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}.
\end{aligned}$$

Corollary 5.4
We have that for any u_1, u_2, any $0 < \epsilon \ll 1$,

$$\|(P_{N_1}u_1)(P_{N_2}u_2)\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} \leq C_\epsilon (N_1 \wedge N_2)^{\frac{1}{2}} (N_1N_2)^{\frac{1}{4}} \|u_1\|_{X^{0,0;\frac{1}{2}+\epsilon,0}} \|u_2\|_{X^{0,0;\frac{1}{4}+\epsilon,0}}.$$

The proof of Proposition 5.7 follows the I-method firstly introduced by the authors of [15]. In order to take advantage of several cancellation relations, we first introduce some notations briefly. For any two triples $(N_1, N_2, N_3), (N_1', N_2', N_3') \in (2^{\mathbb{N}})^3$, we define an equivalence relation

$$(N_1, N_2, N_3) \sim (N_1', N_2', N_3')$$

iff there exists a permutation σ, such that $(N_{\sigma(1)}, N_{\sigma(2)}, N_{\sigma(3)}) = (N_1', N_2', N_3')$. We denote by $\mathcal{P}$ all the equivalence classes and $[N_1, N_2, N_3]$, the elements in $\mathcal{P}$. $\mathcal{P}$ is the quotient set of $(2^{\mathbb{N}})^3/S_3$ and it gives a partition of $(2^{\mathbb{N}})^3$.

Proof of Proposition 5.7.

$$\begin{aligned}\frac{d}{dt}\|Iu(t)\|^2_{L^2(\mathbb{T}^2)} =& -2\,\mathrm{Re}\int_{\mathbb{T}^2} I(u\partial_x u)(t)\cdot\overline{Iu}(t) - 2\,\mathrm{Re}\int_{\mathbb{T}^2} I\mathcal{G}^*\mathcal{G}u(t)\cdot\overline{Iu}(t)\\ \leq& -2\,\mathrm{Re}\int_{\mathbb{T}^2} I(u\partial_x u)(t)\cdot\overline{Iu}(t) - 2\,\mathrm{Re}\int_{\mathbb{T}^2} [I,\mathcal{G}^*\mathcal{G}]u(t)\cdot\overline{Iu}(t).\end{aligned}$$

We split

$$\int_{\mathbb{T}^2} I\partial_x(u^2)(t)\cdot\overline{Iu}(t) = \int_{\mathbb{T}^2}\sum_{N_1,N_2,N_3} I\partial_x(P_{N_1}uP_{N_2}u)(t)\overline{IP_{N_3}u}(t).$$

where the dyadic summation is taken over all dyadic numbers $N_1, N_2, N_3 \geq 1$. Moreover, since $IP_{\leq N} = 1$, we can only take the summation above over all triples (N_1, N_2, N_3) such that $N_{\max} := N_1 \vee N_2 \vee N_3 \geq \frac{N}{4}$.

Indeed,

$$\begin{aligned}&\int_{\mathbb{T}^2}\sum_{N_1\leq\frac{N}{4},N_2\leq\frac{N}{4},N_3\leq\frac{N}{4}} I\partial_x(P_{N_1}uP_{N_2}u)(t)\overline{IP_{N_3}u}(t)\\ =&\int_{\mathbb{T}^2} I\partial_x((P_{\leq\frac{N}{4}}u)^2)(t)\overline{IP_{\leq\frac{N}{4}}u}(t) = 0.\end{aligned}$$

To estimate the increasing of $\|Iu(t)\|^2_{L^2}$, we need integrate it over the time interval $[0, T]$ and we denote by I_{N_1,N_2,N_3}. Using the symmetric notation, we rewrite the summation we want to estimate as

$$\sum_{\mathcal{P},N_{\max}>\frac{N}{4}} I_{[N_1,N_2,N_3]} := \sum_{\mathcal{P},N_{\max}>\frac{N}{4}}\ \sum_{(N_1',N_2',N_3')\in[N_1,N_2,N_3]} I_{N_1',N_2',N_3'}.$$

To remove the annoyed non-smooth time localization $\mathbf{1}_{[0,T_0]}$, we perform as in [15]. For any fixed term $[N_1, N_2, N_3] \in \mathcal{P}$, we pick a smooth approximation to the identity $G(t)$ of width

$N_{\max}^{-100}$ and define $a_{[N_1,N_2,N_3]}(t) = \mathbf{1}_{[0,T_0]} * G_{[N_1,N_2,N_3]}$ and $b_{[N_1,N_2,N_3]}(t) = \mathbf{1}^3_{[0,T_0]} - a_{[N_1,N_2,N_3]}(t)^3$. We first estimate the contribution of $b_{[N_1,N_2,N_3]}(t)$ to $I_{[N_1,N_2,N_3]}$. From Corollary 5.4, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}\times\mathbb{T}^2} b_{[N_1,N_2,N_3]}(t) I\partial_x(P_{N_1}uP_{N_2}u)(t)\overline{IP_{N_3}u}(t)\right| \\
=&\|b_{[N_1,N_2,N_3]}(t)P_{N_3}u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\|I\partial_x(P_{N_1}uP_{N_2}u)\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} \\
\leq& CN_{\max}\|P_{N_1}uP_{N_2}u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}\|b_{[N_1,N_2,N_3]}\|_{L^2(\mathbb{R})}\|P_{N_3}u\|_{L^\infty(\mathbb{R};L^2(\mathbb{T}^2))} \\
\leq& C_\epsilon N_{\max}^{-48}\|b_{[N_1,N_2,N_3]}\|_{L^2(\mathbb{R})}\|u\|^3_{X^{0,0;\frac{1}{2}+\epsilon,0}},
\end{aligned}$$

where to the last inequality we have used the property that $\|b_{[N_1,N_2,N_3]}\|_{L^2} \leq N_{\max}^{-50}$. We thus write $I_{[N_1,N_2,N_3]} = J_{[N_1,N_2,N_3]} + r_{[N_1,N_2,N_3]}$, with

$$J_{[N_1,N_2,N_3]} := \sum_{(N_1',N_2',N_3')\in[N_1,N_2,N_3]} J_{N_1',N_2',N_3'}, r_{[N_1,N_2,N_3]} = I_{[N_1,N_2,N_3]} - J_{[N_1,N_2,N_3]}$$

and

$$J_{N_1,N_2,N_3} = \int_{\mathbb{R}\times\mathbb{T}^2} I\partial_x(P_{N_1}vP_{N_2}v)(t)\overline{IP_{N_3}v}(t), v = a_{[N_1,N_2,N_3]}(t)u.$$

since $a_{[N_1,N_2,N_3]}$ is uniformly bounded in L^∞ with respect to the equivalence class $[N_1,N_2,N_3]$. We next investigate the contribution $a_{[N_1,N_2,N_3]}(t)$. From the trivial estimate for the approximation of identity and the Sobolev estimate

$$\|(f_1f_2)\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})} \leq C_\epsilon\|f_1\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})}\|f_2\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})},$$

we have that

$$\|P_{N_j}v\|_{X^{0,0;\frac{1}{2}+\epsilon,-}} \leq C_\epsilon\|P_{N_j}u\|_{X^{0,0;\frac{1}{2}+\epsilon,0}}N_{\max}^{100\epsilon}.$$

To estimate $J_{[N_1,N_2,N_3]}$, we use Plancherel in space and time to calculate

$$\begin{aligned}
&J_{[N_1,N_2,N_3]} \\
=&\int_{\tau_3+\tau_4=0}\sum_{n_3+n_4=0} ik_4\theta(k_3)\theta(k_4)\sum_{(N_1',N_2',N_3')\in[N_1,N_2,N_3]}\widehat{v_{N_1'}v_{N_2'}}(\tau_4,n_4)\widehat{v_{N_3'}}(\tau_3,n_3) \\
=&\int_{\tau_1+\tau_2+\tau_3=0}\sum_{n_1+n_2+n_3=0} i(k_1+k_2)\theta(k_1+k_2)\theta(k_3)\sum_{(N_1',N_2',N_3')\in[N_1,N_2,N_3]}\prod_{j=1}^3\widehat{v_{N_j'}}(\tau_j,n_j) \\
=&-i\int_{\tau_1+\tau_2+\tau_3=0}\sum_{n_1+n_2+n_3=0} k_3\theta(k_3)^2\sum_{(N_1',N_2',N_3')\in[N_1,N_2,N_3]}\prod_{j=1}^3\widehat{v_{N_j'}}(\tau_j,n_j) \\
=&-\frac{i}{3}\int_{\tau_1+\tau_2+\tau_3=0}\sum_{n_1+n_2+n_3=0}\sum_{j=1}^3 k_j\theta(k_j)^2\sum_{(N_1',N_2',N_3')\in[N_1,N_2,N_3]}\prod_{j=1}^3\widehat{v_{N_j'}}(\tau_j,n_j),
\end{aligned}$$

where to the last step, we have used the fact that the permutation of indices of (τ_j, n_j) keeps the symmetric summation

$$\sum_{(N_1', N_2', N_3') \in [N_1, N_2, N_3]} \prod_{j=1}^{3} \widehat{v_{N_j'}}(\tau_j, n_j)$$

invariant. For fixed $[N_1, N_2, N_3]$, it would be suffice to estimate the term in which $N_1 \leq N_2 \leq N_3$. We further decompose the modulation of each v, and we have

$$J_{[N_1,N_2,N_3]} = \sum_{L_1,L_2,L_3} J_{N_1,N_2,N_3}^{L_1,L_2,L_3},$$

where

$$J_{N_1,N_2,N_3}^{L_1,L_2,L_3} = \int_{\tau_1+\tau_2+\tau_3=0} \sum_{n_1+n_2+n_3=0} \left(\sum_{j=1}^{3} k_j \theta(k_j)^2 \right) \prod_{j=1}^{3} \widehat{v_{L_j N_j}}(\tau_j, n_j),$$

and

$$\widehat{v_{L_j N_j}}(\tau_j, n_j) = \phi_{L_j}\left(\langle \tau_j - p_5(n_j) \rangle\right) \phi_{N_j}(k_j) \widehat{v}(\tau_j, n_j).$$

Case 1: $N_1 \sim N_2 \sim N_3$: In this case, we may assume that $L_1 \leq L_2 \leq L_3$. From Lemma 5.2, we estimate

$$J_{N_1,N_2,N_3}^{L_1,L_2,L_3} \leq C_\epsilon \frac{N_{\max} N^{2s_0}}{N_{\max}^{2s_0}} \frac{N_{\max}^{\frac{3}{4}}}{L_1^\epsilon L_2^{\frac{1}{4}} L_3^{\frac{1}{2}}} \prod_{j=1}^{3} \|v_{L_j N_j}\|_{X^{0,0;\frac{1}{2}+\epsilon,0}}. \tag{5.4.1}$$

The modulation condition for fifth order KP-II

$$\sum_{j=1}^{3} (\tau_j - p_5(n_j)) = -5 k_1 k_2 k_3 (k_3^2 + k_3 k_1 + k_1^2) - \frac{(k_1 m_3 - k_3 m_1)^2}{k_1 k_2 k_3}$$

implies that $L_3 \geq C N_{\max}^4 N_{\min}$. Thus

$$\sum_{L_1,L_2,L_3} J_{N_1,N_2,N_3}^{L_1,L_2,L_3} \leq \frac{C_\epsilon}{N_{\max}^{\frac{3}{4}-100\epsilon}} \|u\|_{X^{0,0;\frac{1}{2}+\epsilon,0}}^3,$$

provided that we take $\epsilon < \frac{3}{400}$.

Case 2: $N_1 \ll N_2 \sim N_3$: By symmetry, we only need estimate $J_{N_1,N_2,N_3}^{L_1,L_2,L_3}$ separately according to $L_1 = L_{\max}$ or $L_1 \ll L_3 = L_{\max}$. From modulation condition, we have $L_{\max} \gtrsim N_2^4 N_1$. Using mean value theorem, we observe that

$$\begin{aligned} \sum_{j=1}^{3} k_j \theta(k_j)^2 =& k_1 \theta(k_1^2) + k_2 \theta(k_2)^2 + k_3 \left(\theta(k_2)^2 + 2 k_1 \theta(k_2 + \lambda k_1) \theta'(k_2 + \lambda k_1) \right) \\ =& O\left(N_1\right) \end{aligned}$$

We apply Lemma 5.2 to estimate

$$\sum_{L_1,L_2,L_3,L_1=L_{\max}} J^{L_1,L_2,L_3}_{N_1,N_2,N_3} \leq \sum_{L_1,L_2,L_3,L_1=L_{\max}\gtrsim N_2^4 N_1} C_\epsilon \frac{N_1N_2}{L_2^\epsilon L_3^\epsilon L_1^{\frac{1}{2}} N_1^{\frac{1}{4}}} \prod_{j=1}^{3} \|v_{L_jN_j}\|_{X^{0,0;\frac{1}{2}+\epsilon,0}}$$
$$\leq \frac{C_\epsilon}{N_{\max}^{\frac{3}{4}-100\epsilon}} \|u\|^3_{X^{0,0;\frac{1}{2}+\epsilon,0}}.$$

Similar estimate yields

$$\sum_{N\ll N_1\sim N_2, N_1\gtrsim N} \sum_{L_1,L_2,L_3,L_3=L_{\max}} J^{L_1,L_2,L_3}_{N_1,N_2,N_3} \leq \frac{C_\epsilon}{N_{\max}^{\frac{3}{4}-100\epsilon}} \|u\|^3_{X^{0,0;\frac{1}{2},0}}.$$

In summary, we have

$$I_{[N_1,N_2,N_3]} \leq \frac{C_\epsilon}{N_{\max}^{\frac{3}{4}-100\epsilon}} \|u\|^3_{X^{0,\frac{1}{2}+\epsilon}}.$$

. We fix $\epsilon < \frac{1}{400}$ and thus

$$\left|\int_0^T \int_{\mathbb{T}^2} I\partial_x(u^2)(t)\overline{Iu}(t)dtdxdy\right|$$
$$= \left|\sum_{\mathcal{P},N_{\max}>\frac{N}{4}} I_{[N_1,N_2,N_3]}\right| \leq \sum_{\mathcal{P},N_{\max}>\frac{N}{4}} \frac{C_\epsilon \|u\|^3_{X^{0,\frac{1}{2}+\epsilon}}}{N_{\max}^{\frac{1}{2}}} \leq \frac{C_\epsilon \|u\|^3_{X^{0,\frac{1}{2}+\epsilon}} \log(N)^2}{N^{\frac{1}{2}}}$$

since for any fixed dyadic number N_0,

$$\#\{[N_1,N_2,N_3] : N_{\max} = N_0\} \leq 3\log_2(N_0)^2.$$

Combining with Lemma 5.1, we have

$$\sup_{t\in[0,T_1]} \|Iu(t)\|^2_{L^2(\mathbb{T}^2)} \leq \|Iu(0)\|^2_{L^2(\mathbb{T}^2)} + \frac{C_\epsilon\left(\|u\|^3_{X_{T_0}^{0,0;\frac{1}{2}+\epsilon,0}} + \|u\|^2_{X_{T_0}^{0,0;\frac{1}{2}+\epsilon,0}}\right)}{N^{\frac{1}{4}}}.$$

Notice that

$$\|u(t)\|_{H^{-s_0,0}(\mathbb{T}^2)} \leq \|Iu(t)\|_{L^2(\mathbb{T}^2)} \leq N^{s_0}\|u(t)\|_{H^{-s_0,0}(\mathbb{T}^2)},$$

we have that for any $0 < t \leq T_0$,

$$\|u(t)\|_{H^{-s_0,0}(\mathbb{T}^2)} \leq N^{s_0}\|u(0)\|_{H^{-s_0,0}(\mathbb{T}^2)} + \frac{\beta(T_0,R_0)}{N^{\frac{1}{4}}},$$

with $\beta(T_0,R_0) = C_\epsilon(\alpha(T_0,R_0)^3R_0^3 + 1)$. We complete the proof by optimally choosing $N = \left(\frac{\beta(T_0,R_0)}{\|u(0)\|_{H^{-s_0,0}(\mathbb{T}^2)}}\right)^{\frac{8}{1+8s_0}}$. ■

5.5 Observability for high frequencies

In this section, we will consider the inhomogeneous linearized KP-II type equations in a uniform way

$$\partial_t u + (-1)^{l-1}\partial_x^{2l+1} u + \partial_x^{-1}\partial_y^2 u = F, \tag{5.5.1}$$

where the system is KP-II if $l = 1$ and 5-KP-II if $l = 2$. Each Fourier mode in y, u_m, satisfies

$$\partial_t u_m + (-1)^{l-1}\partial_x^{2l+1} u_m - m^2\partial_x^{-1} u_m = F_m. \tag{5.5.2}$$

To capture the high frequency information $|m| \gg 1$, we write $|m| = \frac{1}{h^{l+1}}$ and reduce the study of (5.5.2) to the semi-classical equation

$$h^{2l+1}\partial_t w + (-1)^{l-1}(h\partial_x)^{2l+1} w - (h\partial_x)^{-1} w = h^{2l+1} f. \tag{5.5.3}$$

5.5.1 Semi-classical estimate

To perform the one dimensional analysis and simplify the notation, we introduce semi-classical Bourgain space via the norm

$$\|u\|_{X_h^{s,b}} := \left\| \langle \tau - h^{-2l-1}p(hk)\rangle^b \langle k\rangle^s \widehat{u}(\tau,k)\right\|_{L^2_{\tau,k}}$$

where

$$p(k) = k^{2l+1} - \frac{1}{k}.$$

We need a lemma. In what follows, we pick a family of parameter-dependent cutoff function $(\chi_{c_0})_{0<c_0<1} \in C_c^\infty(\mathbb{R})$ with $\chi(r) = 1, c_0 < |r| \leq \frac{1}{c_0}$ and $\chi(r) = 0$ for $|r| \leq \frac{c_0}{2}$ or $|r| > \frac{2}{c_0}$.

Lemma 5.3
Fix $\varphi \in C^\infty(\mathbb{T})$. For any given $c_0 > 0$, there exists a constant $C_{c_0} = C_{c_0}(\varphi) > 0$ such that

$$\left\|\Pi_{\neq 0}\left[(\epsilon^{2l+2}(h_1D_x)^{2l+1} - (h_1D_x)^{-1})\Pi_{\neq 0}, \varphi\right]\chi_{c_0}(h_1D_x)\right\|_{L^2\to L^2} \leq C_{c_0}h_1, \tag{5.5.4}$$

$$\left\|\Pi_{\neq 0}\left[((h_1D_x)^{2l+1} - \epsilon^{2l+2}(h_1D_x)^{-1})\Pi_{\neq 0}, \varphi\right]\chi_{c_0}(h_1D_x)\right\|_{L^2\to L^2} \leq C_{c_0}h_1, \tag{5.5.5}$$

uniformly for any $0 < \epsilon \leq 1$ and $0 < h_1 \ll 1$.

Proof. The proof follows from direct calculation. We only prove (5.5.4) and (5.5.5) will follow in the same way. Take $v = \chi_{c_0}(h_1D_x)u$, we have

$$\mathcal{F}\left(\epsilon^{2l+2}(h_1D_x)^{2l+1} - (h_1D_x)^{-1}\right)\Pi_{\neq 0}(\varphi v)(k) = \sum_{k_1}\left(\epsilon^{2l+2}(h_1k)^{2l+1} - \frac{1}{h_1k}\right)\mathbf{1}_{k\neq 0}\widehat{\varphi}(k-k_1)\widehat{v}(k_1),$$

$$\mathcal{F}\left(\varphi\left(\epsilon^{2l+2}(h_1D_x)^{2l+1} - (h_1D_x)^{-1}\right)\Pi_{\neq 0}v\right)(k) = \sum_{k_1}\left(\epsilon^{2l+2}(h_1k_1)^{2l+1} - \frac{1}{h_1k_1}\right)\mathbf{1}_{k_1\neq 0}\widehat{\varphi}(k-k_1)\widehat{v}(k_1).$$

Taking the difference of the two expressions, we have (note that $\widehat{v}(0) = 0$)

$$\begin{aligned}
&\mathcal{F}\left(\Pi_{\neq 0}\left[(\epsilon^{2l+2}(h_1 D_x)^{2l+1} - (h_1 D_x)^{-1})\Pi_{\neq 0}, \varphi\right] v\right)(k) \\
=&\mathbf{1}_{k\neq 0}\sum_{k_1\neq 0}\left(\epsilon^{2l+2}h_1^{2l+1}(k^{2l+1} - k_1^{2l+1}) - \frac{1}{h_1}\left(\frac{1}{k} - \frac{1}{k_1}\right)\right)\varphi(k-k_1)\widehat{v}(k_1)
\end{aligned}$$

We conclude by the algebraic manipulation

$$\begin{aligned}
&\left(\epsilon^{2l+2}(h_1 k)^{2l+1} - \frac{1}{h_1 k}\right) - \left(\epsilon^{2l+2}(h_1 k_1)^{2l+1} - \frac{1}{h_1 k_1}\right) \\
=&h_1(k-k_1)\left(\sum_{j=0}^{2l}\begin{pmatrix} 2l+1 \\ j \end{pmatrix}(\epsilon^{2l+2}(h_1 k_1)^j (h_1(k-k_1))^{2l-j} + \frac{1}{h_1^2}\left(\frac{1}{k_1^2} + \frac{(k_1-k)}{k k_1^2}\right)\right)
\end{aligned}$$

since all the factors $(k-k_1)^j$ will fall on $\widehat{\varphi}$ as derivatives and all the power $(h_1 k_1)^\alpha$ is a bounded Fourier multiplier. ■

For dyadic number $N \gg 1$, we denote by $w_N = P_N w$, $f_N = P_N f$. Pick $\psi \in C_c^\infty(\mathbb{R})$ with $0 \leq \psi \leq 1$, $\psi|_{[-1/2,1/2]} \equiv 1$, $\psi(t) = 0, |t| > 1$. For any $T > 0$, we define $\psi_T(\cdot) := \psi(T^{-1}\cdot)$.

Lemma 5.4
Given $T > 0$, there exist a small parameter $0 < h_0 \ll 1$, such that for all $0 < h < h_0$, $\frac{1}{2^{100}h} \leq N \leq \frac{2^{100}}{h}$, any solution w to (5.5.3) satisfies

$$\begin{aligned}
\|\psi_T(t)w_N\|_{L^2_{t,x}}^2 \leq &C_T h^l \|\psi_T(t)w_N\|_{X_h^{0,\frac{1}{2}}}\|\psi_T(t)f_N\|_{X_h^{0,-\frac{1}{2}}} \\
&+ C_T\|\psi_T(t)g w_N\|_{L^2_{t,x}}^2,
\end{aligned} \tag{5.5.6}$$

where the constant C_T depends only on $T > 0$.

Proof. In the calculation below, all the implicit constants depend on T. Let

$$A(x, hD_x) = \varphi(x)p'(hD_x)^{-1}\chi(hD_x)$$

where the cutoff $\chi = \chi_{2^{-100}}$. Denote by $v_N = \psi_T(t)w_N$ and $P_h = h^{2l+1}\partial_t - ip(hD_x)$. On the one hand, we calculate

$$\begin{aligned}
\left(\frac{1}{h}[A(x,hD_x), P_h]v_j, v_j\right)_{L^2_{t,x}} =& -\left(\frac{i}{h}[A(x,hD_x), p(hD_x)]v_N, v_N\right)_{L^2_{t,x}} \\
=&(\varphi'(x)v_N, v_N)_{L^2_{t,x}} + O(h)\|v_N\|_{L^2_{t,x}}^2.
\end{aligned} \tag{5.5.7}$$

In the calculus above, we have used symbolic calculus and the Poisson bracket(see [63]) that $\left\{\varphi(x)\chi(\xi)p'(\xi)^{-1}, p(\xi)\right\} = \varphi'(x)\chi(\xi)$. On the other hand, we calculate the commutator by using the equation $P_h w_N = h^{2l+1} f_N$ that

$$\begin{aligned}
\frac{1}{h}\left([A(x,hD_x),P_h]v_N,v_N\right)_{L^2_{t,x}} =&\frac{1}{h}\Big((A(x,hD_x)P_h(v_N),v_N)_{L^2_{t,x}}-(P_hA(x,hD_x)(v_N),v_N)_{L^2_{t,x}}\Big)\\
&=\left(h^{2l}A(x,hD_x)f_N\psi_T,w_N\right)_{L^2_{t,x}}+\left(h^{2l}A(x,hD_x)\psi_T'w_N,v_N\right)_{L^2_{t,x}}\\
&+\left(h^{2l}A(x,hD_x)v_N,\psi_T'w_N\right)_{L^2_{t,x}}+\left(h^{2l}A(x,hD_x)\psi_Tw_N,\psi_Tf_N\right)_{L^2_{t,x}}\\
&=h^{2l}(A(x,hD_x)\psi_Tf_N,\psi_Tw_N)_{L^2_{t,x}}+h^{2l}\left(A(x,hD_x)\psi_Tw_N,\psi_Tf_N\right)_{L^2_{t,x}}\\
&+O(h^{2l})\|\psi_Tw_N\|^2_{L^2_{t,x}}.
\end{aligned}\tag{5.5.8}$$

From Cauchy-Schwartz, we have

$$h^{2l}\left|(A(x,hD_x)\psi_Tw_N,\psi_Tf_N)_{L^2_{t,x}}\right|\le h^{2l}\left\|\Pi_{\neq 0}\varphi(x)p'(hD_x)^{-1}\chi(hD_x)\psi_Tw_N\right\|_{X_h^{0,\frac12}}\|\psi_Tf_N\|_{X_h^{0,-\frac12}}.\tag{5.5.9}$$

We need estimate the two terms on the right hand side. For the first term, though the Fourier multiplier $p'(hD_x)^{-1}\chi(hD_x)$ is uniformly bounded on $X_h^{s_1,s_2;b,b_1}$, multiplication by $\varphi(x)$ is not. We observe that for any space-time function $G_N = P_N G_N$,

$$\begin{aligned}
\|\Pi_{\neq0}\varphi(x)G_N\|^2_{X_h^{s,0}}\lesssim&\|G_N\|^2_{X_h^{s,0}},\\
\|\Pi_{\neq0}\varphi(x)G_N\|^2_{X_h^{s,1}}=&\|\Pi_{\neq0}\varphi(x)G_N\|^2_{X_h^{s,0}}+h^{-2(2l+1)}\left\|\Pi_{\neq0}(h^{2l+1}D_t-p(hD_x))(\varphi(x)G_N)\right\|^2_{X_h^{s,0}}\\
\lesssim&\|G_N\|^2_{X_h^{s,0}}+h^{-2(2l+1)}|\left\|\Pi_{\neq0}\varphi(x)(h^{2l+1}D_t-p(hD_x))G_N\right\|^2_{X_h^{s,0}}\\
&+h^{-2(2l+1)}\left\|\Pi_{\neq0}\left[(h^{2l+1}D_t-p(hD_x))\Pi_{\neq0},\varphi\right]G_N\right\|^2_{X_h^{s,0}}\\
\lesssim&\|G_N\|^2_{X_h^{s,1}}+\frac{1}{h^{2(2l+1)}}\left\|\Pi_{\neq0}\left[p(hD_x)\Pi_{\neq0},\varphi\right]G_N\right\|^2_{X_h^{s,0}}\\
\lesssim&\frac{1}{h^{4l}}\|G_N\|^2_{X_h^{s,1}},
\end{aligned}$$

thanks to Lemma 5.3. From interpolation, we have

$$\|\Pi_{\neq0}\varphi(x)G_N\|_{X_h^{s,\frac12}}\lesssim\frac{1}{h^l}\|G_N\|_{X_h^{s,\frac12}}.$$

Therefore,

$$h^{2l}\left|(A(x,hD_x)\psi_Tw_N,\psi_Tf_N)_{L^2_{t,x}}\right|\lesssim h^l\|\psi_Tw_N\|_{X_h^{0,\frac12}}\|\psi_Tf_N\|_{X_h^{0,-\frac12}}.$$

Similarly,

$$h^{2l}\left|(A(x,hD_x)\psi_Tf_N,\psi_Tw_N)_{L^2_{t,x}}\right|\lesssim h^l\|\psi_Tw_N\|_{X_h^{0,\frac12}}\|\psi_Tf_N\|_{X_h^{0,-\frac12}}.$$

Combining (5.5.7),(5.5.8), we have obtained that

$$\left|(\varphi'(x)\psi_T w_N, \psi_T w_N)_{L^2_{t,x}}\right| \leq C_{T,\varphi} h^l \|\psi_T(t) w_N\|_{X_h^{0,\frac{1}{2}}} \|\psi_T(t) f_N\|_{X_h^{0,-\frac{1}{2}}}$$

for any function $\varphi \in C^\infty(\mathbb{T})$. Fix any $x_0 \in \mathbb{T}$ and any smooth function θ, there exists $\varphi \in C^\infty(\mathbb{T})$ such that

$$\varphi'(x) = \theta(x) - \theta(x - x_0).$$

Now we take $\theta \geq 0$ such that $\mathrm{supp}(\theta) \subset \left\{x \in \mathbb{T} : g(x) > \frac{1}{4}\right\}$ and this implies that

$$\begin{aligned}\left|(\theta(\cdot - x_0)\psi_T w_N, \psi_T w_N)_{L^2_{t,x}}\right| \leq & C_T h^l \|\psi_T(t) w_N\|_{X_h^{0,-\frac{1}{2}}} \|\psi_T(t) f_N\|_{X_h^{0,-\frac{1}{2}}} \\ & + C_T \|g\psi_T w_N\|^2_{L^2_{t,x}}.\end{aligned}$$

Now we conclude by choosing a finite set S of x_0 such that

$$\bigcup_{x_0 \in S} \mathrm{supp}(\theta(\cdot - x_0)) = \mathbb{T}.$$

■

Lemma 5.5
Given $T > 0$. There exists $0 < \epsilon_0 \ll 1$ and $0 < h_0 \ll 1, h_0 \ll \epsilon_0$, such that for any $\frac{1}{\epsilon_0} < N \leq \frac{1}{2^{100}h}$ and any solution w_N of (5.5.3), we have

$$\begin{aligned}\|\psi_T(t) w_N\|^2_{L^2_{t,x}} \leq & C_{\epsilon_0,T} h^{l+1} N \|\psi_T(t) w_N\|_{X_h^{0,\frac{1}{2}}} \|\psi_T(t) f_N\|_{X_h^{0,-\frac{1}{2}}} \\ & + C_{\epsilon_0,T} \|\psi_T(t) g w_N\|^2_{L^2_{t,x}},\end{aligned} \tag{5.5.10}$$

where the constant $C_{\epsilon_0,T}$ depends only on ϵ_0, T.

Proof. We first chose $\epsilon_0 > 0$, small enough to fit the semi-classical pseudo-differential calculus, which will be made precise implicitly later. Define a new semi-classical parameter $h_1 := \frac{h}{\epsilon} = \frac{1}{N} \leq \epsilon_0$, we rewrite the equation (5.5.3) as

$$h_1^{2l+1} \epsilon^{2l+2} \partial_t w_N - i p_\epsilon(h_1 D_x) w_N = \epsilon^{2l+2} h_1^{2l+1} f_N \tag{5.5.11}$$

with

$$p_\epsilon(\xi) = \epsilon^{2l+2} \xi^{2l+1} - \frac{1}{\xi}.$$

Define the operator $A(x, h_1 D_x) = \varphi(x) p'_\epsilon(h_1 D_x)^{-1} \chi(h_1 D_x)$. As in the proof of previous lemma, we compute the commutator

$$\frac{1}{h_1}\left(\left[A(x, h_1 D_x), h_1^{2l+1}\epsilon^{2l+2}\partial_t - i p_\epsilon(h_1 D_x)\right] \psi_T w_N, \psi_T w_N\right)_{L^2_{t,x}}$$

in two ways. From symbolic calculus, we have

$$\begin{aligned}
&\frac{1}{h_1}\left(\left[A(x,h_1D_x),h_1^{2l+1}\epsilon^{2l+2}\partial_t-ip_\epsilon(h_1D_x)\right]\psi_Tw_N,\psi_Tw_N\right)_{L^2_{t,x}}\\
=&(\varphi'(x)\psi_Tw_N,w_N)_{L^2_{t,x}}+O(h_1)\|\psi_Tw_N\|^2_{L^2_{t,x}}.
\end{aligned}\tag{5.5.12}$$

We remark that the implicit constant in big O is independent of ϵ. Using equation satisfied by w_N, we have

$$\begin{aligned}
&\frac{1}{h_1}\left(\left[A(x,h_1D_x),h_1^{2l+1}\epsilon^{2l+2}\partial_t-ip_\epsilon(h_1D_x)\right]\psi_Tw_N,\psi_Tw_N\right)_{L^2_{t,x}}\\
=&h_1^{2l}\epsilon^{2l+2}\left(A(x,h_1D_x)\psi_Tf_N,\psi_Tw_N\right)_{L^2_{t,x}}+h_1^{2l}\epsilon^{2l+2}\left(A(x,h_1D_x)\psi_Tw_N,\psi_Tf_N\right)_{L^2_{t,x}}\\
&+O(h_1^{2l}\epsilon^{2l+2})\|\psi_Tw_N\|^2_{L^2_{t,x}}.
\end{aligned}\tag{5.5.13}$$

Combining (5.5.12),(5.5.13), we have

$$\begin{aligned}
\left|(\varphi'(x)\psi_Tw_N,\psi_Tw_N)_{L^2_{t,x}}\right|\lesssim&h_1^{2l}\epsilon^{2l+2}\left|(A(x,h_1D_x)\psi_Tf_N,\psi_Tw_N)_{L^2_{t,x}}\right|\\
&+h_1^{2l}\epsilon^{2l+2}\left|(A(x,h_1D_x)\psi_Tw_N,\psi_Tf_N)_{L^2_{t,x}}\right|
\end{aligned}\tag{5.5.14}$$

The two terms on the right hand side are of the same type. Hence we only detail the estimate for the second term. For $N\sim\frac{1}{h_1}$ and any frequency localized space-time function $G_N=P_NG_N$, we have$\|\Pi_{\neq 0}\varphi(x)G_N\|^2_{X_h^{s,0}}\lesssim\|G_N\|^2_{X_h^{s,0}}$, and

$$\begin{aligned}
\|\Pi_{\neq 0}\varphi(x)G_N\|^2_{X_h^{s,1}}\lesssim&\|G_N\|^2_{X_h^{s,0}}+\left\|\Pi_{\neq 0}\varphi(x)\left(D_t-D_x^{2l+1}+h^{-(2l+2)}D_x^{-1}\right)G_N\right\|^2_{X_h^{s,0}}\\
&+\left\|\Pi_{\neq 0}\left[(D_t-D_x^{2l+1}+h^{-(2l+2)}D_x^{-1})\Pi_{\neq 0},\varphi\right]G_N\right\|^2_{X_h^{s,0}}\\
\lesssim&\|G_N\|^2_{X_h^{s,1}}+\left\|\Pi_{\neq 0}\left[(D_x^{2l+1}-h^{-(2l+2)}D_x^{-1})\Pi_{\neq 0},\varphi\right]G_N\right\|^2_{X_h^{s,0}}
\end{aligned}$$

$$\begin{aligned}
&\left\|\Pi_{\neq 0}\left[(D_x^{2l+1}-h^{-(2l+2)}D_x^{-1})\Pi_{\neq 0},\varphi\right]G_N\right\|_{X_h^{s,0}}\\
=&\frac{1}{h_1^{2l+1}}\left\|\Pi_{\neq 0}\left[((h_1D_x)^{2l+1}-\epsilon^{-(2l+2)}(h_1D_x)^{-1})\Pi_{\neq 0},\varphi\right]G_N\right\|_{X_h^{s,0}}\\
=&\frac{1}{h_1^{2l+1}\epsilon^{2l+2}}\left\|\Pi_{\neq 0}\left[(\epsilon^{2l+2}(h_1D_x)^{2l+1}-(h_1D_x)^{-1})\Pi_{\neq 0},\varphi\right]G_N\right\|_{X_h^{s,0}}\\
=&\frac{1}{h_1^{2l}\epsilon^{2l+2}}\|G_N\|_{X_h^{s,0}},
\end{aligned}$$

thanks to Lemma 5.3. Thus from interpolation, we have

$$\|\varphi(x)G_N\|_{X_h^{s,\frac12}}\lesssim\frac{1}{h_1^l\epsilon^{l+1}}\|G_N\|_{X_h^{s,\frac12}}.$$

Therefore,

$$h_1^{2l}\epsilon^{2l+2}\left|(A(x,h_1D_x)\psi_T w_N,\psi_T f_N)_{L^2_{t,x}}\right|\lesssim h_1^l\epsilon^{l+1}\|\psi_T w_N\|_{X_h^{0,\frac{1}{2}}}\|\psi_T f_N\|_{X_h^{0,-\frac{1}{2}}}$$

The rest argument is the same as in the proof of Lemma 5.4.

■

Lemma 5.6
Given $T>0$. There exists $0<\epsilon_0\ll 1$ such that for any $h>0$(not necessarily small) and any $N\geq\frac{1}{\epsilon_0}\vee\frac{2^{100}}{h}$, all solution w to (5.5.3) *satisfies*

$$\begin{aligned}\|\psi_T(t)w_N\|^2_{L^2_{t,x}}\leq & C_{\epsilon_0,T}N^{-l}\|\psi_T(t)w_N\|_{X_h^{0,\frac{1}{2}}}\|\psi_T(t)f_N\|_{X_h^{0,-\frac{1}{2}}}\\ &+C_{\epsilon_0,T}\|\psi_T(t)gw_N\|^2_{L^2_{t,x}},\end{aligned}\tag{5.5.15}$$

where the constant $C_{\epsilon_0,T}$ only depends on ϵ_0,T.

Proof. We first choose $\epsilon_0>0$ as in Lemma 5.5. Set $h_1=\epsilon h\sim\frac{1}{N}\leq\epsilon_0$ and we rewrite the equation 5.5.3 as

$$h_1^{2l+1}\partial_t w_N-iq_\epsilon(h_1D_x)w_N=h_1^{2l+1}f_N\tag{5.5.16}$$

with

$$q_\epsilon(\xi)=\xi^{2l+1}-\frac{\epsilon^{2l+2}}{\xi}.$$

Define $A(x,h_1D_x)=\varphi(x)q_\epsilon'(h_1D_x)^{-1}\chi(h_1D_x)$ and we calculate on the one hand that

$$\begin{aligned}&\frac{1}{h_1}\left([A(x,h_1D_x),h_1^{2l+1}\partial_t-iq_\epsilon(h_1D_x)]\psi_T w_N,\psi_T w_N\right)_{L^2_{t,x}}\\ =&(\varphi'(x)\psi_T w_N,\psi_T w_N)_{L^2_{t,x}}+O(h_1)\|\psi_T w_N\|^2_{L^2_{t,x}}\end{aligned}$$

where the implicit constant in the big O does not depend on h_1,h and ϵ. On the other hand, we use the equation to write

$$\begin{aligned}&\frac{1}{h_1}\left([A(x,h_1D_x),h_1^{2l+1}\partial_t-iq_\epsilon(h_1D_x)]\psi_T w_N,\psi_T w_N\right)_{L^2_{t,x}}\\ =&h_1^{2l}\left(A(x,h_1D_x)\psi_T f_N,\psi_T w_N\right)_{L^2_{t,x}}+h_1^{2l}\left(A(x,h_1D_x)\psi_T w_N,\psi_T f_N\right)_{L^2_{t,x}}\\ &+O(h_1^2)\|\psi_T w_N\|^2_{L^2_{t,x}}.\end{aligned}$$

We estimate

$$\left|h_1^{2l}\left(A(x,h_1D_x)\psi_T w_N,\psi_T f_N\right)_{L^2_{t,x}}\right|\lesssim h_1^{2l}\|\Pi_{\neq 0}A(x,h_1D_x)\psi_T w_N\|_{X_h^{0,\frac{1}{2}}}\|\psi_T f_N\|_{X_h^{0,-\frac{1}{2}}}.$$

For $G_N = P_N G_N$, we have $\|\Pi_{\neq 0}\varphi(x)G_N\|_{X_h^{s,0}} \lesssim \|G_N\|_{X_h^{s,0}}$, and

$$\begin{aligned}
\|\Pi_{\neq 0}\varphi(x)G_N\|_{X_h^{s,1}} \lesssim & \|G_N\|_{X_h^{s,1}} + \left\|\Pi_{\neq 0}\left[(D_t + D_x^{2l+1} - h^{-2l+2}D_x^{-1})\Pi_{\neq 0}, \varphi\right] G_N\right\|_{X_h^{s,0}} \\
\lesssim & \|G_N\|_{X_h^{s,1}} + \frac{1}{h_1^{2l+1}} \|\Pi_{\neq 0}\left[q_\epsilon(h_1 D_x)\Pi_{\neq 0}, \varphi\right] G_N\|_{X_h^{s,0}} \\
\lesssim & \frac{1}{h_1^{2l}}\|G_N\|_{X_h^{s,1}}.
\end{aligned}$$

Therefore, from interpolation and Lemma 5.3, we have

$$\|\varphi(x)G_N\|_{X_h^{s,\frac{1}{2}}} \lesssim \frac{1}{h_1^l}\|G_N\|_{X_h^{s,\frac{1}{2}}}.$$

Applying the estimate above to $G_N = q'_\epsilon(h_1 D_x)^{-1}\chi(h_1 D_x)w_N$, we have

$$\left|h_1^{2l}\left(A(x,h_1D_x)\psi_T w_N, \psi_T f_N\right)_{L^2_{t,x}}\right| \lesssim h_1^l \|\psi_T w_N\|_{X_h^{0,\frac{1}{2}}} \|\psi_T f_N\|_{X_h^{0,-\frac{1}{2}}}.$$

The rest argument is the same as in the proof of Lemma 5.4 and we omit the details. ■

Proposition 5.8
Fix $\epsilon_0 > 0, T > 0$. There exists $0 < h_0 \ll 1$, $C_{\epsilon_0,T} > 0$, $\theta > 0$, and a dyadic number $N_0 \geq 1$, such that for all $h < h_0$, all $N_1 \geq N_0$, and all solution w of (5.5.3),

$$\begin{aligned}
\|\psi_T P_{\geq N_1} w\|^2_{L^2_{t,x}} \leq & C_{\epsilon_0,T}\frac{1}{N_1^\theta}\|\psi_T P_{\geq N_1/2}w\|_{X_h^{0,\frac{1}{2}}}\|\psi_T P_{\geq N_1/2}f\|_{X_h^{0,-\frac{1}{2}}} \\
& + C_{\epsilon_0,T}\|(gP_{\geq N_1/2}w)\|^2_{L^2_{t,x}} + \frac{C_{\epsilon_0,T}}{N_1^2}\|\psi_T P_{\geq N_1/2}w\|^2_{L^2_{t,x}}.
\end{aligned}$$

Proof. Denote by $w_{\geq N_1, \leq N_2} = P_{\geq N_1}P_{\leq N_2}w$. Fix any $\epsilon_0 \ll 1$ as in Lemma 5.5 and 5.6. We also denote by $N_0 = A\epsilon_0^{-1}$ with the big constant $A > 0$ to be chosen later such that

$$hA < 2^{-100}\epsilon_0 \ll 1. \tag{5.5.17}$$

From Lemma 5.5, we have

$$\begin{aligned}
\sum_{\frac{A}{\epsilon_0}\leq N\leq\frac{1}{2^{100}h}} \|\psi_T(t)w_N\|^2_{L^2_{t,x}} \leq & C_{\epsilon_0,T}\sum_{\frac{A}{\epsilon_0}\leq N\leq\frac{1}{2^{100}h}} N^{-l}\|\psi_T(t)w_N\|_{X_h^{0,\frac{1}{2}}}\|\psi_T(t)f_N\|_{X_h^{0,-\frac{1}{2}}} \\
& + C_{\epsilon_0,T}\sum_{\frac{A}{\epsilon_0}\leq N\leq\frac{1}{2^{100}h}} \|\psi_T(t)gw_N\|^2_{L^2_{t,x}}.
\end{aligned}$$

We note that here and in the sequel, the summations taken over N, M and another captical alphabets are all dyadic summations. Using the commutator estimate

$$\|[g, P_N]\|_{L^2\to L^2} \leq \frac{C}{N},$$

we have

$$\begin{aligned}
&\|\psi_T(t)w_{\geq N_0,\leq 2^{-100}h^{-1}}\|^2_{L^2_{t,x}}\\
\leq& C_{\epsilon_0,T}\frac{\epsilon_0^l}{A^l}\ \|\psi_T(t)w_{\geq N_0/2,\leq 2^{-99}h^{-1}}\|_{X_h^{0,\frac{1}{2}}}\|\psi_T(t)f_{\geq N_0/2,\leq 2^{-99}h^{-1}}\|_{X_h^{0,-\frac{1}{2}}}\\
&+C_{\epsilon_0,T}\|\psi_T P_{\geq N_0/2}P_{\leq 2^{-99}h^{-1}}(gw_{\geq N_0/2})\|^2_{L^2_{t,x}}+\frac{\epsilon_0^2 C_{\epsilon_0,T}}{A^2}\|\psi_T(t)w_{\geq N_0/2}\|^2_{L^2_{t,x}},
\end{aligned}\tag{5.5.18}$$

where the constants $C_{\epsilon_0,T}$ may change while keeping the dependence of ϵ_0 and T.

From Lemma 5.4, we have

$$\begin{aligned}
\sum_{\frac{1}{2^{100}h}\leq N\leq\frac{2^{100}}{h}}\|\psi_T(t)w_N\|^2_{L^2_{t,x}}\leq& C_T\sum_{\frac{1}{2^{100}h}\leq N\leq\frac{2^{100}}{h}}h^l\|\psi_T(t)w_N\|_{X_h^{0,\frac{1}{2}}}\|\psi_T(t)f_N\|_{X_h^{0,-\frac{1}{2}}}\\
&+C_T\sum_{\frac{1}{2^{100}h}\leq N\leq\frac{2^{100}}{h}}\|\psi_T(t)gw_N\|^2_{L^2_{t,x}}.
\end{aligned}$$

From the commutator estimate, we have

$$\sum_{\frac{1}{2^{100}h}\leq N\leq\frac{2^{100}}{h}}\|\psi_T(t)gw_N\|^2_{L^2_{t,x}}\leq C_T\left\|\psi_T P_{\geq\frac{1}{2^{100}h}}P_{\leq\frac{2^{100}}{h}}(gw_{\geq N_0/2})\right\|^2_{L^2_{t,x}}+C_Th^2\|\psi_T w_{\geq N_0/2}\|^2_{L^2_{t,x}}$$

and thus

$$\begin{aligned}
&\left\|\psi_T w_{\geq\frac{1}{2^{100}h},\leq\frac{2^{100}}{h}}\right\|^2_{L^2_{t,x}}\\
\leq& C_Th^l\left\|\psi_T w_{\geq\frac{1}{2^{101}h},\leq\frac{2^{101}}{h}}\right\|_{X_h^{0,\frac{1}{2}}}\left\|\psi_T f_{\geq\frac{1}{2^{101}h},\leq\frac{2^{101}}{h}}\right\|_{X_h^{0,-\frac{1}{2}}}\\
&+C_T\left\|\psi_T P_{\geq\frac{1}{2^{100}h}}P_{\leq\frac{2^{100}}{h}}(gw_{\geq N_0/2})\right\|^2_{L^2_{t,x}}+C_Th^2\|\psi_T w_{\geq N_0/2}\|^2_{L^2_{t,x}}.
\end{aligned}\tag{5.5.19}$$

Finally from Lemma 5.6 and similar arguments, we have

$$\begin{aligned}
\left\|\psi_T w_{\geq\frac{2^{100}}{h}}\right\|^2_{L^2_{t,x}}\leq& C_{\epsilon_0,T}\frac{\epsilon_0^l}{A^l}\left\|\psi_T w_{\geq\frac{2^{99}}{h}}\right\|_{X_h^{0,\frac{1}{2}}}\left\|\psi_T f_{\geq\frac{2^{99}}{h}}\right\|_{X_h^{0,-\frac{1}{2}}}\\
&+C_{\epsilon_0,T}\left\|\psi_T P_{\geq\frac{2^{100}}{h}}(gw_{\geq N_0/2})\right\|^2_{L^2_{t,x}}+C_{\epsilon_0,T}\frac{\epsilon_0^2}{A^2}\|\psi_T w_{\geq N_0/2}\|^2_{L^2_{t,x}}.
\end{aligned}\tag{5.5.20}$$

Finally, by choosing $A^2>2^{100}\epsilon_0^2C_{\epsilon_0,T}+2^{100}$ and $h_0<\frac{\epsilon_0}{2^{100}A}$, we have that for all $h<h_0$,

$$\begin{aligned}
\|\psi_T w_{\geq N_0}\|^2_{L^2_{t,x}}\leq& C_{\epsilon_0,T}\frac{1}{N_0^\theta}\|\psi_T w_{\geq N_0/2}\|_{X_h^{0,\frac{1}{2}}}\|\psi_T f_{\geq N_0/2}\|_{X_h^{0,-\frac{1}{2}}}+C_{\epsilon_0,T}\|gw_{\geq N_0/2}\|^2_{L^2_{t,x}}\\
&+\frac{C_{\epsilon_0,T}}{N_0^2}\|\psi_T w_{\geq N_0/2}\|^2_{L^2_{t,x}}.
\end{aligned}$$

for some $\theta>0$. The estimate still holds true for any $N_1\geq N_0$.

■

5.5.2 high-frequency estimate for inhomogeneous linear equation

Proposition 5.9
Given $T > 0$, there exist a large dyadic number $N_0 > 0$ and $C = C(T, N_0) > 0$ such that for any solutions u to (5.5.1) *and any $N_1 \geq N_0$, we have*

$$\begin{aligned}\|\psi_T(t)P_{\geq N_1}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} \leq & \frac{C}{N_1^\theta}\|\psi_T(t)u\|_{X^{0,\frac{1}{2}}}\|\psi_T(t)F\|_{X^{0,-\frac{1}{2}}} + \frac{C}{N_1^2}\|\psi_T(t)P_{\geq N_1/2}u\|_{L^2(\mathbb{R}\times\mathbb{T}^2)} \\ & + C\|\psi_T(t)gP_{\geq N_1}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}.\end{aligned} \tag{5.5.21}$$

Moreover, the same inequality holds if the second term on the right hand side is replaced by $\|\psi_T(t)(\mathcal{G}P_{\geq N_1/2}u)\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}$.

Proof. Take an integer $m_0 > 0$ such that $m_0^{-1} \leq h_0$, where $h_0 > 0$ is given in Proposition 5.8. For any integer m, $|m| \geq m_0$, we define a semi-classical parameter $h = |m|^{-\frac{1}{l+1}}$, and from Proposition 5.8, we have

$$\begin{aligned}\|\psi_T(t)P_{\geq N_1}\pi_m u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} \leq & \frac{C}{N_1^\theta}\|\psi_T(t)\pi_m u\|_{X^{0,\frac{1}{2}}}\|\psi_T(t)\pi_m F\|_{X^{0,-\frac{1}{2}}} \\ & + \frac{C}{N_1^2}\|\psi_T(t)\pi_m P_{\geq N_1/2}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \|\psi_T(t)g\pi_m P_{\geq N_1/2}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}.\end{aligned} \tag{5.5.22}$$

Summing over all integers $|m| \geq m_0$, we obtain that

$$\begin{aligned}\|\psi_T(t)P_{\geq N_1}\pi_{\geq m_0}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} \leq & \frac{C}{N_1^\theta}\|\psi_T(t)\pi_{\geq m_0}u\|_{X^{0,\frac{1}{2}}}\|\psi_T(t)\pi_{\geq m_0}F\|_{X^{0,-\frac{1}{2}}} \\ & + \|\psi_T(t)g\pi_{\geq m_0}P_{\geq N_1}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \frac{C}{N_1^2}\|\psi_T(t)\pi_{\geq m_0}P_{\geq N_1/2}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)},\end{aligned}$$

where we have used that fact that π_m is interchangeable with the multiplication by $g(x)$. Now we claim that the inequality (5.5.22) also holds true Indeed if we change $\pi_m u$ by $\pi_{\leq m_0}u$. This fact can be deduced easily by repeating the proof of Lemma 5.6.

Indeed, in the proof of Proposition 5.8, the semi-classical parameter is chosen so that $h < h_0 \sim \frac{1}{2^{100}N_0}$. This means that $m_0 \sim 2^{100(l+1)}N_0^{l+1}$. The small number $\epsilon_0 > 0$ can be firstly fixed to fit the h_1 pseudo-differential calculus (in x variable). The dyadic number $N_0 \gg \frac{1}{\epsilon_0}$ has been fixed after fixing ϵ_0. We can then choose $N_1 \geq N_0$, large enough, such that $N_1 \gg \frac{N_0}{\epsilon_0^2}$. Therefore, we can apply the proof of Lemma 5.6 to gain a factor N_1^{-l} in front of $\|\psi(t)\pi_m F\|_{X^{0,-\frac{1}{2}}}$, uniformly in $0 < m \leq m_0$ (In the proof of Lemma 5.6, we do the semi-classical analysis with respect to $h_1 \sim \epsilon h$ instead of h). Summing over all $0 < m \leq m_0$ yields

$$\begin{aligned}\|\psi_T(t)P_{\geq N_1}\pi_{\neq 0}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} \leq & \frac{C}{N_1^\theta}\|\psi_T(t)\pi_{\neq 0}u\|_{X^{0,\frac{1}{2}}}\|\psi_T(t)\pi_{\neq 0}F\|_{X^{0,-\frac{1}{2}}} \\ & + \|\psi_T(t)g\pi_{\neq 0}P_{\geq N_1/2}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \frac{C}{N_1^2}\|\psi_T(t)\pi_{\neq 0}P_{\geq N_1/2}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}\end{aligned}$$

since g commutes with $\pi_{\leq m_0}\pi_{\neq 0}$. To complete the proof of (5.5.21), it remains to treat the zero Fourier mode $\pi_0 u$. Notice that $v = P_{\geq N_1}\pi_{\neq 0}u$ satisfies the inhomogeneous KdV

$$\partial_t v + (-1)^{l-1}\partial_x^{2l+1}v = P_{\geq N_1}\pi_{\neq 0}F$$

equation and the argument in the proof of Lemma 5.6 still valid.

Finally, In order to replace $\|\psi_T(t)P_{\geq N_1/2}(gu)\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}$ by $\|\psi_T(t)P_{\geq N_1/2}(\mathcal{G}u)\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}$, we write

$$\mathcal{G}P_{\geq N_1/2}u = g(x)P_{\geq N_1/2}u - g(x)\left(\int_{\mathbb{T}} P_{\geq N_1/4}g(x')P_{\geq N_1/2}u dx'\right).$$

Since g is smooth, we have $\|P_{\geq N_1/4}g\|_{L^2(\mathbb{T})} \leq \frac{C}{N_1}$, and we finally obtain that

$$\begin{aligned}\|\psi_T(t)P_{\geq N_1}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} \leq & \frac{C}{N_1^{\theta}}\|\psi_T(t)u\|_{X^{0,\frac{1}{2}}}\|\psi_T(t)F\|_{X^{0,-\frac{1}{2}}} \\ & + \|\psi_T(t)\mathcal{G}P_{\geq N_1/2}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)} + \frac{C}{N_1^2}\|\psi_T P_{\geq N_1/2}u\|^2_{L^2(\mathbb{R}\times\mathbb{T}^2)}.\end{aligned}$$

∎

5.6 Stabilization

5.6.1 Conditional observability for data in $L^2(\mathbb{T}^2)$

We first prove the conditional observability for both general KP-II type equations

$$\begin{cases}\partial_t u + (-1)^{l-1}\partial_x^{2l+1}u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = -\mathcal{G}^*\mathcal{G}u, \quad (t,x) \in \mathbf{R}\times\mathbb{T}^2, \\ u|_{t=0} = u_0 \in H_0^s(\mathbb{T}^2),\end{cases} \tag{5.6.1}$$

Proposition 5.10
Given $R_0 > 0, T > 0$, there exist $\epsilon_0 > 0$ and $C_T > 0$, such that for all $u_0 \in L_0^2(\mathbb{T}^2)$ satisfying $\|u_0\|_{L^2(\mathbb{T}^2)} \leq R_0$, if the solution u to (5.6.1) with $l = 1$ or $l = 2$ satisfies

$$\sup_{t\in[0,T]}\|u(t)\|_{H^{-1,0}(\mathbb{T}^2)} \leq \epsilon_0,$$

then

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C_T\int_0^T \|\mathcal{G}u(t)\|^2_{L^2(\mathbb{T}^2)}dt.$$

The constant C_T depends only on T, R_0 and ϵ_0.

Proof. We argue by contradiction. Suppose there exists a sequence $u_{n,0} \in L_0^2(\mathbb{T}^2)$ with

$$\|u_{n,0}\|_{L^2(\mathbb{T}^2)} \leq R_0, \lim_{n\to\infty}\|u_n\|_{L^\infty([0,T];H^{-1,0}(\mathbb{T}^2))} = 0,$$

and

$$\lim_{n\to\infty} \frac{\int_0^T \|\mathcal{G}u_n(t)\|_{L^2(\mathbb{T}^2)}^2 dt}{\|u_{n,0}\|_{L^2(\mathbb{T}^2)}^2} = 0.$$

Denote by $\alpha_n := \|u_{n,0}\|_{L^2(\mathbb{T}^2)}$ and $\beta_n := \sup_{t\in[0,T]} \|u_n(t)\|_{H^{-1,0}(\mathbb{T}^2)}$. From Cauchy theory, we have $\|u_n\|_{Z^{0,\frac{1}{2}}(T)} \leq \alpha(T, R_0)$, if $l = 1$ and $\|u_n\|_{X_5^{0,0;b,b_1}(T)} \leq \alpha(T, R_0)$ if $l = 2$.

Case 1: $\alpha_n \to \alpha > 0$.

After normalization, we may assume that $\|u_{n,0}\|_{L^2(\mathbb{T}^2)} = 1$ for all n and

$$\|u_n\|_{L^2((0,T/2);L^2(\mathbb{T}^2)}^2 \geq \frac{T}{10}$$

since

$$\|u_n(t)\|_{L^2(\mathbb{T}^2)}^2 = \|u_{n,0}\|_{L^2(\mathbb{T}^2)}^2 - \int_0^t \|\mathcal{G}u_n(t')\|_{L^2(\mathbb{T}^2)}^2 dt'.$$

Thanks to Corollary 5.9, for any large number N_1, we have

$$\begin{aligned}\|P_{\geq N_1}u_n\|_{L^2([0,T/2]\times\mathbb{T}^2)}^2 \leq & \frac{C_1}{N_1^\theta}\|u_n\|_{X_T^{0,\frac{1}{2}}} \|(u_n\partial_x u_n + \mathcal{G}^*\mathcal{G}u_n)\|_{X_T^{0,-\frac{1}{2}}} \\ & + C_1\int_0^T \|\mathcal{G}P_{\geq N_1}u_n(t)\|_{L^2(\mathbb{T}^2)}^2 dt + \frac{C_1}{N_1^2}\|P_{\geq N_1/2}u_n\|_{L^2([0,T]\times\mathbb{T}^2)},\end{aligned} \tag{5.6.2}$$

where the constant C_1 is independent of N_1. From Cauchy-theory, commutator estimate and bilinear-estimate, we have

$$\|P_{\geq N_1}u_n\|_{L^2([0,T/2]\times\mathbb{T}^2)}^2 \leq \frac{C_2(T,R_0)}{N_1^\theta} + \delta_n, \tag{5.6.3}$$

with $\delta_n \to 0$ as $n \to \infty$. Since

$$\|P_{\leq N_1}u_n\|_{L^2([0,T/2]\times\mathbb{T}^2)}^2 \leq N_1^2\|u_n\|_{L^2([0,T/2];H^{-1,0})}^2 \leq N_1^2\delta_n,$$

we have that

$$\|u_n\|_{L^2([0,T/2]\times\mathbb{T}^2)}^2 \leq \frac{C_2(T,R_0)}{N_1^\theta} + N_1^2\beta_n.$$

Now we take N_1 large enough such that

$$\frac{C_2(T,R_0)}{N_1^\theta} < \frac{T}{20}.$$

Therefore,

$$\frac{T}{10} \leq \limsup_{n\to\infty} \|u_n\|_{L^2((0,T/2)\times\mathbb{T}^2)} \leq \frac{C_2(T,R_0)}{N_1^\theta} < \frac{T}{20},$$

which is a contradiction.

case 2: $\alpha_n \to \alpha = 0$.

In this case, we make the substitution $\widetilde{u_n} := \frac{u_n}{\alpha_n}$. $\widetilde{u_n}$ satisfies

$$(\partial_t + \partial_x^3 + (-1)^{l-1}\partial_x^{2l+1}\partial_y^2)\widetilde{u_n} = -\alpha_n \widetilde{u_n}\partial_x \widetilde{u_n} - \mathcal{G}^*\mathcal{G}\widetilde{u_n}.$$

The classical boot-strap argument yields $(\widetilde{u_n})$ is a bounded sequence in $E(T)$, where $E(T) := Z^{0,\frac{1}{2}}(T)$ if $l = 1$ and $E(T) = X_5^{0,0;b,b_1}(T)$ if $l = 2$. Observe that $\|\widetilde{u_n}(0)\|_{L^2(\mathbb{T}^2)} = 1$, and

$$\int_0^T \|\mathcal{G}\widetilde{u_n}(t)\|^2_{L^2(\mathbb{T}^2)}dt \to 0.$$

Note that in this case, the nonlinear term $\alpha_n \widetilde{u_n}\partial_x\widetilde{u_n} \to 0$ strongly in $E(T)$ and we do not need the smallness of $\|\widetilde{u_n}\|_{H^{-1,0}(\mathbb{T}^2)}$ (it is not necessarily small). We need the following lemma from linear analysis.

Lemma 5.7
Denote by $S_{2l+1}(t) = e^{-t\left(\partial_x^{2l+1} - \partial_x^{-1}\partial_y^2\right)}$. Then

$$\|P_{\leq N_0}Q_{\geq M_0}v(0)\|^2_{L^2(\mathbb{T}^2)} \leq C_T \int_0^T \|\mathcal{G}P_{\leq N_0}S_{2l+1}(t)v(0)\|^2_{L^2(\mathbb{T}^2)}dt,$$

in which $M_0 \gg N_0$ and the constant C_T only depends on $T > 0$.

Proof. The proof is explicit calculation. We write

$$v(t,x,y) = P_{\leq N_0}Q_{\geq M_0}S_{2l+1}(t)v(0) = \sum_{0<|k|\leq 2N_0, |m|\geq M_0/2} a_{k,m}e^{ikx+imy}e^{-it\left(k^{2l+1} - \frac{m^2}{k}\right)}$$

$$\mathcal{G}v(t,x,y) = \sum_{k,|m|\geq M_0/2} e^{ikx+imy} \sum_{0<|k_1|\leq 2N_0} (g_{k-k_1} - g_k g_{k_1})a_{k_1,m}e^{-it\left(k_1^{2l+1} - \frac{m^2}{k_1}\right)}$$

$$\begin{aligned}\int_0^T \|\mathcal{G}v(t)\|^2_{L^2(\mathbb{T}^2)}dt &= \sum_{k,|m|\geq M_0/2}\int_0^T \left|\sum_{0<|k_1|\leq 2N_0}(g_{k-k_1} - g_k g_{k_1})a_{k_1,m}e^{-it\left(k_1^{2l+1} - \frac{m^2}{k_1}\right)}\right|^2 dt \\ &\geq C_1 \sum_{k_1,|m|\geq M_0/2}\left(\sum_k |g_{k-k_1} - g_k g_{k_1}|^2\right)|a_{k_1,m}|^2 \\ &\geq C_2\|v(0)\|^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

For the first inequality, we have used Ingham inequality since for $M_0 \gg N_0$ and the spectral gap

$$\inf_{k_1\neq k_2, 0<|k_1|,|k_2|\leq N_0}\left|k_1^{2l+1} - k_2^{2l+1} + m^2\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\right| \geq \gamma_0 > 0 \tag{5.6.4}$$

and γ_0 can be chosen independent of N_0, provided that we fix $M_0 \gg N_0$ large enough. This implies that the constant C_1 is independent of N_0.

For the second inequality, we use the fact that

$$g_k = \frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-ikx}dx$$

which satisfies $g_0 = 1$ and $|g_k| < 1$ for all $k \neq 0$. Therefore,

$$\inf_{0<|k_1|\leq N_0} \sum_k |g_{k-k_1} - g_k g_{k_1}|^2 \geq c_0 > 0$$

independent of N_0. Hence the constant C_2 does not depend on N_0. ■

We apply this lemma to $\widetilde{u}_n(0)$. Take $M_0 \gg N_0$ and we have

$$\begin{aligned}
&\|P_{\leq N_0} Q_{\geq M_0}\widetilde{u_n}(0)\|^2_{L^2(\mathbb{T}^2)}\\
\leq &C_T\int_0^T \|\mathcal{G}P_{\leq N_0}\widetilde{u_n}(t)\|^2_{L^2(\mathbb{T}^2)}dt + C_T\alpha_n^2\left\|\mathcal{G}P_{\leq N_0}\int_0^t S_{2l+1}(t-t')\widetilde{u_n}\partial_x\widetilde{u_n}(t')dt'\right\|^2_{L^2([0,T]\times\mathbb{T}^2)}\\
+&C_T\left\|\mathcal{G}P_{\leq N_0}\int_0^t S_{2l+1}(t-t')\mathcal{G}^*\mathcal{G}\widetilde{u_n}(t')dt'\right\|^2_{L^2([0,T]\times\mathbb{T}^2)}\\
\leq &C_T\int_0^T \|\mathcal{G}\widetilde{u_n}(t)\|^2_{L^2(\mathbb{T}^2)}dt + C_T\alpha_n^2\|\widetilde{u_n}\|^2_{E(T)} + \frac{C_T}{N_0^2}\|\widetilde{u_n}\|^2_{E(T)}.
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty} \|P_{\leq 2N_0} Q_{\geq M_0/2}\widetilde{u_n}(0)\|^2_{L^2(\mathbb{T}^2)} \leq \frac{C_T}{N_0^2}. \tag{5.6.5}$$

Applying Corollary 5.9, we have

$$\|P_{\geq N_0}\widetilde{u}_n\|_{L^2([0,T/2]\times\mathbb{T}^2)} \leq \frac{C_2(T,R_0)}{N_0^\theta} + \delta_n$$

as in Case 1. Notice that

$$\frac{1}{2}\frac{d}{dt}\|P_{\geq N_0}\tilde{u}_n\|^2_{L^2(\mathbb{T}^2)} = -\alpha_n\int_{\mathbb{T}^2} P_{\geq N_0}\widetilde{u}_n \cdot P_{\geq N_0}\partial_x\left(\frac{\widetilde{u}_n^2}{2}\right)dx - \int_{\mathbb{T}^2} P_{\geq N_0}\widetilde{u}_n P_{\geq N_0}\mathcal{G}^*\mathcal{G}\widetilde{u}_n dx.$$

Integrating in time, bilinear estimate yield

$$\|P_{\geq N_0}\widetilde{u}_n(0)\|_{L^2(\mathbb{T}^2)} \leq C_T\left(\frac{1}{N_0^\theta} + \delta_n\right),$$

thanks to $\alpha_n \to 0$. By choosing N_0 large enough and $M_0 \gg N_0$, for sufficiently large n, we have $\|P_{\leq 2N_0}Q_{\leq 2M_0}\widetilde{u}_n(0)\|_{L^2(\mathbb{T}^2)} \geq \frac{1}{2}$. Up to a subsequence, we assume that $\widetilde{u_n}(0)$ converges to weakly to $\widetilde{u}_0$ in $L^2(\mathbb{T}^2)$. Thus from Rellich theorem, we have that

$$\|P_{\leq 2N_0}Q_{\leq 2M_0}\widetilde{u}_0\|_{L^2(\mathbb{T}^2)} \geq \frac{1}{2} \tag{5.6.6}$$

by taking N_0 and M_0 large enough. Now since $\alpha_n \widetilde{u_n} \partial_x \widetilde{u_n} \to 0$, strongly in $L^2([0,T] \times \mathbb{T}^2)$ as well as $\mathcal{G}^* \mathcal{G} \widetilde{u_n}$, there exists a subsequence of $\widetilde{u_n}$ (still denoted by $\widetilde{u_n}$) which converges to $\widetilde{u}$ weakly in $L^2([0,T] \times \mathbb{T}^2)$ and $\widetilde{u}$ is the solution of linear KP-II equation

$$\begin{cases} \partial_t \widetilde{u} + (-1)^{l-1} \partial_x^{2l+1} \widetilde{u} + \partial_x^{-1} \partial_y^2 \widetilde{u} = 0 \\ \widetilde{u}|_{t=0} = \widetilde{u}_0 \in L_0^2(\mathbb{T}^2), \\ \mathcal{G} \widetilde{u}| \equiv 0, \text{ in } \mathcal{D}_0'(\mathbb{T}^2). \end{cases} \tag{5.6.7}$$

On the support of g, we have $\widetilde{u}(t,x,y) = C(t,y)$. Rewriting the equation as

$$\partial_x(\partial_t \widetilde{u} + (-1)^{l-1} \partial_x^{2l+1} \widetilde{u}) + \partial_y^2 \widetilde{u} = 0,$$

we have that $\partial_y^2 C(t,y) = 0$ and thus $\widetilde{u}_{[0,T] \times \omega} = C(t)$, in the distributional sense. From the equation, we also have $\partial_x^{-1} \partial_y^2 \widetilde{u}|_{[0,T] \times \omega} = -C'(t)$ in $\mathcal{D}'([0,T] \times \omega)$. Take non-zero function $\chi \in C_c^\infty(\omega)$, depending only on x variable and $\chi \geq 0$, we have

$$-\frac{1}{2\pi} \int_0^T C'(t) \psi(t) dt \int_{\mathbb{T}} \chi(x) dx = \int_0^T \psi(t) dt \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \chi(x) \partial_x^{-1} \partial_y^2 \widetilde{u}(t,x,y) dx dy = 0$$

from integrating by part on y variable. This implies that $\int_0^T C'(t) \psi(t) dt = 0$ for any $\psi \in C_c^\infty((0,T))$. Thus $C'(t) = 0, a.e.$ and $\widetilde{u}|_\omega \equiv C_0$ in the distributional sense. The unique continuation argument for linear KP-II (see [56]) yields $\widetilde{u} = C_0 = 0$ since $\widetilde{u} \in \mathcal{D}_0'$. This contradicts to (5.6.6). ∎

As a consequence, we have the L^2 stabilization.

Corollary 5.5
Let $s > 0$, $R_0 > 0$ be given. There exist $C > 0, \gamma > 0$ and $\delta_0 > 0$ such that the inequality

$$\|u(t)\|_{L^2(\mathbb{T}^2)} \leq C e^{-\gamma t} \|u_0\|_{L^2(\mathbb{T}^2)}, t > 0$$

holds for every solution u of damped KP-II equation (5.1.6) *with initial data in S_{δ_0} such that $\|u_0\|_{H^s(\mathbb{T}^2)} \leq R_0$.*

Proof. The proof follows from a standard argument by using semi-group property. Let $T > 0, \epsilon_0 > 0$ as in Proposition 5.10. Let

$$n_0 = \left\lfloor \frac{\log \epsilon_0 - \log R_0}{\log(C_T - 2) - \log C_T} \right\rfloor + 1, T_0 = n_0 T.$$

From Corollary 5.6, there exists $\delta_0 > 0$ such that if $u(0) \in S_{\delta_0}$, we have

$$\sup_{t \in [0,T_0]} \|u(t)\|_{H^{-1,0}(\mathbb{T}^2)} \leq \epsilon_0.$$

Thus we can apply Proposition 5.10 for time interval $[nT, (n+1)T]$ with $0 \le n \le n_0 - 1$. Thus

$$\|u((n+1)T)\|_{L^2(\mathbb{T}^2)}^2 = \|u(nT)\|_{L^2(\mathbb{T}^2)} - \int_{nT}^{(n+1)T} \|\mathcal{G}u(t)\|_{L^2(\mathbb{T}^2)}^2 dt \le \left(1 - \frac{1}{C_T}\right) \|u(nT)\|_{L^2(\mathbb{T}^2)}^2.$$

This implies that

$$\|u(nT)\|_{L^2(\mathbb{T}^2)}^2 \le \left(1 - \frac{1}{C_T}\right)^n \|u(0)\|_{L^2}^2 \tag{5.6.8}$$

for $n \le n_0 - 1$ and $\|u(T_0)\|_{L^2(\mathbb{T}^2)}^2 \le \epsilon_0^2$. Since the L^2 norm is non increasing, (5.6.8) holds true for all n. This completes the proof. ■

The same argument yields the stabilization for 5KP-II.

Corollary 5.6
Let $R_0 > 0$ be given. There exist $C > 0, \gamma > 0$ and $\delta_0 > 0$ such that the inequality

$$\|u(t)\|_{L^2(\mathbb{T}^2)} \le Ce^{-\gamma t}\|u_0\|_{L^2(\mathbb{T}^2)}, t > 0$$

holds for every solution u of damped 5KP-II equation (5.1.7) *with initial data in S_{δ_0} such that $\|u_0\|_{L^2(\mathbb{T}^2)} \le R_0$.*

5.6.2 Stabilization for in $H^s(\mathbb{T}^2)$

Proof of Theorem 5.1. Now we turn to the proof of stabilization for general $s \ge 0$. This is a consequence of L^2- stabilization and a stability argument. We mainly follow the strategy in [41], in which the authors dealt with the stabilization for KdV equation.

Define the semi-group $S_{\mathcal{G}}(t)$ be the solution map associated with the dissipative equation

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x^{-1}\partial_y^2 u = -\mathcal{G}^*\mathcal{G}u, \quad (t, x) \in \mathbf{R} \times \mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2), \end{cases} \tag{5.6.9}$$

We need the following lemma about the bilinear estimate for the semi-group $S_{\mathcal{G}}(t)$.

Lemma 5.8
Let $I \subset \mathbb{R}$ be any finite time interval. Then the bilinear estimate

$$\|S_{\mathcal{G}}(t)u_0\|_{Z^{s,\frac{1}{2}}(I)} \le C(I)\|u_0\|_{H^s(\mathbb{T}^2)}. \tag{5.6.10}$$

$$\left\|\int_0^t S_{\mathcal{G}}(t-t')\partial_x(uv)(t')dt'\right\|_{Z^{s,\frac{1}{2}}(I)} \le C(I)\|u\|_{Z^{s,\frac{1}{2}}(I)}\|v\|_{Z^{s,\frac{1}{2}}(I)} \tag{5.6.11}$$

holds true.

The proof is similar as in [41]. For the sake of completeness, we briefly recall the proof here.

Proof. We may assume that $I = [0, T]$. From Duhamel furlmula, we have

$$S_{\mathcal{G}}(t)u_0 = S(t)u_0 - \int_0^t S(t-t')\mathcal{G}^*\mathcal{G}(S_{\mathcal{G}}(t')u_0)dt'.$$

For the first inequality, we use the usual bilinear estimate and the fact(see the proof of Theorem 5.3)

$$\|\mathcal{G}^*\mathcal{G}v\|_{Z^{s,-\frac{1}{2}}(T)} \leq CT^{\nu'}\|v\|_{Z^{s,\frac{1}{2}}(T)} \tag{5.6.12}$$

to get

$$\|S_{\mathcal{G}}(t)u_0\|_{Z^{s,\frac{1}{2}}(T)} \leq C\|u_0\|_{H^s(\mathbb{T}^2)} + CT^{\nu'}\|S_{\mathcal{G}}(t)u_0\|_{Z^{s,\frac{1}{2}}(T)}.$$

Then (5.6.10) follows for all $T < T_0$ with $T_0 \leq (2C)^{-\frac{1}{\nu'}}$. For $T \geq T_0$, we can partition $[0, T]$ into finite many intervals of length at most T_0 and conclude by induction. Now we prove (5.6.11). Observe that

$$\begin{aligned}
\int_0^t S_{\mathcal{G}}(t-t')F(t')dt' =& \int_0^t S(t-t')F(t')dt' \\
&- \int_0^t S(t-t')\left(\int_0^{t-t'} S(-t'')\mathcal{G}^*\mathcal{G}S_{\mathcal{G}}(t'')F(t')dt''\right)dt' \\
=& \int_0^t S(t-\tau)F(\tau)d\tau \\
&- \int_0^t S(t-\tau)\left(\int_0^{\tau} \mathcal{G}^*\mathcal{G}S_{\mathcal{G}}(\tau-\sigma)F(\sigma)d\sigma\right)d\tau,
\end{aligned}$$

where in the last step, we have used the change of variable $\tau = t'' + t', \sigma = t'$. Put $F = -\partial_x(uv)$. Applying Lemma 5.4, the usual bilinear estimate, we have that

$$\left\|\int_0^t S(t-\tau)\left(\int_0^{\tau} \mathcal{G}^*\mathcal{G}S_{\mathcal{G}}(\tau-\sigma)F(\sigma)d\sigma\right)d\tau\right\|_{Z^{s,\frac{1}{2}}(T)} \leq CT^{\nu'}\left\|\int_0^t S_{\mathcal{G}}(t-\tau)\partial_x(uv)(\tau)d\tau\right\|_{Z^{s,\frac{1}{2}}(T)}.$$

Thus there exists $T_0 > 0$ such that $CT_0^{\nu'} < \frac{1}{2}$ and (5.6.11) is true for all $T \leq T_0$. The case $T > T_0$ follows from (5.6.10) and induction. ∎

Now we prove the stabilization for any $s \geq 0$. We first deal with the case $s = 3$. From L^2- stabilization and Cauchy theory, for any $T > 0$, there exists a number $C_T = C(T, \|u_0\|_{L^2(\mathbb{T}^2)})$ such that for all $t \geq 0$,

$$\|u\|_{Z^{0,\frac{1}{2}}([t,t+T])} \leq C_T e^{-c_0 t}\|u_0\|_{L^2(\mathbb{T}^2)}.$$

In particular, given any $T > 0$ and $\epsilon > 0$, there exists $T_1 = T_1(\epsilon, \|u_0\|_{L^2(\mathbb{T}^3)}) > 0$ such that for any $t > T_1$,

$$\|u\|_{Z^{0,\frac{1}{2}}([t,t+T])} \leq \epsilon.$$

Let $v = \partial_t u$ or $\partial_y u$ as in the proof of Cauchy problem, thus

$$\partial_t v + \partial_x^3 v + \partial_x^{-1}\partial_y^2 v = -\partial_x(uv) - \mathcal{G}^*\mathcal{G}v.$$

We first assume that Suppose that $u_0 \in H_0^3(\mathbb{T}^2)$. Fixing $t^* \geqq T_1$, we write

$$v(t) = S_{\mathcal{G}}(t)v(0) - \int_0^t S_{\mathcal{G}}(t-t')\partial_x(uv)(t')dt'.$$

From stabilization result for linear KP-II equation (see Appendix), we have that for any $T > 0$,

$$\|S_{\mathcal{G}}(t)v(0)\|_{L^2(\mathbb{T}^2)} \leq C_0 e^{-c_0 t}\|v(0)\|_{L^2(\mathbb{T}^2)}$$

where the constants C_0, c_0 are independent of T and t^*. From the proof of Cauchy problem, we have

$$\begin{aligned}\|v((n+1)T)\|_{L^2(\mathbb{T}^2)} \leq & C_0 e^{-c_0 T}\|v(nT)\|_{L^2(\mathbb{T}^2)} + C_T\|u\|_{Z^{0,\frac{1}{2}}([nT,(n+1)T])}\|v\|_{Z^{0,\frac{1}{2}}([nT,(n+1)T])} \\ \leq & C_0 e^{-c_0 T}\|v(nT)\|_{L^2(\mathbb{T}^2)} + C_T\epsilon\|v\|_{Z^{0,\frac{1}{2}}([nT,(n+1)T])}.\end{aligned}$$

We also have

$$\|v\|_{Z^{0,\frac{1}{2}}([nT,(n+1)T])} \leq C_T\|v(nT)\|_{L^2(\mathbb{T}^2)} + C_T\epsilon\|v\|_{Z^{0,\frac{1}{2}}([nT,(n+1)T])}.$$

Now fix any $c_1 \in (0, c_0)$, we take $T > 0$ large enough such that

$$C_0 e^{-c_0 T} < \frac{1}{2}e^{-c_1 T}.$$

Then we fix T and choose $\epsilon > 0$ small enough so that

$$C_T\epsilon < \frac{1}{2}, C_T^2\epsilon < \frac{1}{4}e^{-c_1 T}.$$

Thus we have

$$\begin{aligned}\|v((n+1)T)\|_{L^2(\mathbb{T}^2)} \leq & \left(C_0 e^{-c_0 T} + 2C_T^2\epsilon\right)\|v(nT)\|_{L^2(\mathbb{T}^2)} \\ \leq & e^{-c_1 T}\|v(nT)\|_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Therefore,

$$\|v(t)\|_{L^2(\mathbb{T}^2)} \leq C(\|v(0)\|_{L^2(\mathbb{T}^2)})e^{-c_1 t}\|v(0)\|_{L^2(\mathbb{T}^2)}, \text{ for all } t \geq 0.$$

By taking $v = \partial_y^2 u, \partial_y^3 u$, the same arguments yields

$$\|\partial_t u(t)\|_{L^2(\mathbb{T}^2)} + \|u(t)\|_{H^{0,3}(\mathbb{T}^2)} \leq C(\|u_0\|_{H^3(\mathbb{T}^2)})e^{-c_1 t}\|u(0)\|_{H^3(\mathbb{T}^2)}.$$

From the equation

$$-\partial_t u = \partial_x^3 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u + \mathcal{G}^*\mathcal{G}u,$$

we have that

$$\begin{aligned}\|\partial_x^3 u(t)\|_{L^2(\mathbb{T}^2)} \leq & \|\partial_t u(t)\|_{L^2(\mathbb{T}^2)} + \|\partial_x^{-1}\partial_y^2 u(t)\|_{L^2(\mathbb{T}^2)} \\ & + C\|u(t)\|_{L^2(\mathbb{T}^2)}\left(1 + \|u(t)\|_{L^2(\mathbb{T}^2)}^{\frac{1}{2}}\|\partial_x^3 u(t)\|_{L^2(\mathbb{T}^2)}^{\frac{1}{2}}\right).\end{aligned}$$

This implies that

$$\|u(t)\|_{H^3(\mathbb{T}^2)} \leq C(\|u_0\|_{H^3(\mathbb{T}^2)})e^{-c_1' t}$$

for some $0 < c_1' < c_1$. Now by induction, the stabilization holds for any $s \in 3\mathbb{N}$. To complete the proof, it would be suffice to deal with the case $s \in (0, 3)$. Observe that $w = u - v$ satisfies

$$\partial_t w + \partial_x^3 w + \partial_x^{-1}\partial_y^2 w = -\partial_x(w(u+v)) - \mathcal{G}^*\mathcal{G}w,$$

we can repeat the argument above to obtain

$$\|u(t) - v(t)\|_{L^2(\mathbb{T}^2)} \leq C_{R_0} e^{-c_2 t}\|u(0) - v(0)\|_{L^2(\mathbb{T}^2)},$$

for all $\|u(0)\|_{L^2(\mathbb{T})^2} \leq R_0, \|v(0)\|_{L^2(\mathbb{T}^2)} \leq R_0$. Applying interpolation theorem 5.5, the H^s stabilization holds true. ■

Finally, the proof of Theorem 5.2 follows the same way as for KP-II by changing the functional spaces adapted to the Cauchy problem of 5KP-II equation. We thus omit the details.

5.A Stabilization for linear equation

Consider the damped general linear KP-II

$$\begin{cases}\partial_t u + (-1)^{l-1}\partial_x^{2l+1} u + \partial_x^{-1}\partial_y^2 u = -\mathcal{G}^*\mathcal{G}u, & (t,x) \in \mathbf{R}\times\mathbb{T}^2, \\ u|_{t=0} = u_0 \in L_0^2(\mathbb{T}^2), \end{cases} \tag{5.A.1}$$

Proposition 5.11
Let $s \geq 0$. There exists constants $C' = C_s', \gamma = \gamma_s > 0$ such that for all solutions u to (5.A.1) *with initial data $u_0 \in H_0^s(\mathbb{T}^2)$,*

$$\|u(t)\|_{H^s(\mathbb{T}^2)} \leq C' e^{-\gamma t}\|u_0\|_{H^s(\mathbb{T}^2)}.$$

Proof. First we assume that $s = 0$. From semi-group property, it suffices to show that there exists $T > 0$ and $C_T > 0$ such that for all solutions $u_0 \in L_0^2(\mathbb{T}^2)$,

$$\|u_0\|_{L^2(\mathbb{T}^2)}^2 \leq C_T \int_0^T \|\mathcal{G}S_{\mathcal{G}}(t)u(0)\|_{L^2(\mathbb{T}^2)}^2 dt, \tag{5.A.2}$$

where we denote by $S_{\sqcup\rceil\S\mathcal{G}}(t)$ the linear semi-group associated to (5.A.1). In chapter 3 [2], we know that the observability for undamped linear KP-II holds true, namely

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C_T \int_0^T \|\mathcal{G}S_{2l+1}(t)u_0\|^2_{L^2(\mathbb{T}^2)}dt. \tag{5.A.3}$$

We denote by $u(t) = S_{\mathcal{G}}(t)u_0$ and $v(t) = S(t)u_0$ and $w(t) = v(t) - u(t)$. w solves

$$(\partial_t + (-1)^{l-1}\partial_x^{2l+1} + \partial_x^{-1}\partial_y^2)w = \mathcal{G}^*\mathcal{G}u.$$

Multiplying by w and integrating on $\mathbb{T}^2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{T}^2)} =& \int_{\mathbb{T}^2} \mathcal{G}v(t)\cdot \mathcal{G}u(t)dxdy - \|\mathcal{G}u(t)\|^2_{L^2(\mathbb{T}^2)}\\ \leq& C(\epsilon)\|\mathcal{G}v(t)\|^2_{L^2(\mathbb{T}^2)} - (1-\epsilon)\|\mathcal{G}u(t)\|^2_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Thus

$$\|w(t)\|^2_{L^2(\mathbb{T}^2)} \leq C(\epsilon)\int_0^t \|\mathcal{G}v(t')\|^2_{L^2(\mathbb{T}^2)}dt'. \tag{5.A.4}$$

If $\int_0^T \|\mathcal{G}v(t)\|^2_{L^2(\mathbb{T}^2)}dt \leq \int_0^T \|\mathcal{G}u(t)\|^2_{L^2(\mathbb{T}^2)}dt$, then

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} = \|v(0)\|^2_{L^2(\mathbb{T}^2)} \leq C_T\int_0^T \|\mathcal{G}v(t)\|^2_{L^2(\mathbb{T}^2)}dt \leq C_T\int_0^T \|\mathcal{G}u(t)\|^2_{L^2(\mathbb{T}^2)}dt.$$

Otherwise, we have

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C_T\int_0^T \|\mathcal{G}v(t)\|^2_{L^2(\mathbb{T}^2)}dt \leq C_T\left(\int_0^T \|\mathcal{G}u(t)\|^2_{L^2(\mathbb{T}^2)}dt + \int_0^T \|\mathcal{G}w(t)\|^2_{L^2(\mathbb{T}^2)}dt\right).$$

From (5.A.4) we have

$$\|\mathcal{G}w\|_{L^2([0,T]\times\mathbb{T}^2)} \leq C\|w\|_{L^2([0,T]\times\mathbb{T}^2)} \leq C(\epsilon)T^{\frac{1}{2}}\int_0^T \|\mathcal{G}u\|_{L^2([0,T]\times\mathbb{T}^2)},$$

and this gives (5.A.2). To finish the proof of general $s\geq 0$, we first observe that

$$\|u(t)\|_{H^{0,s_2}(\mathbb{T}^2)} \leq C'e^{-\gamma t}\|u_0\|_{H^{0,s_2}(\mathbb{T}^2)}$$

for any integer s_2. Next we replace u by $\partial_t u$ and deduce that the stabilization holds for any $u_0\in H^{3s_1,s_2}$, $s_1, s_2\in\mathbb{N}$. Other values of s follows from interpolation. This completes the proof. ∎

[2]Though the we only treat KP-II equation, namely $l=1$, in chapter 3, the generalization to any integer $l>1$ is trivial.

5.B Nonlinear Interpolation result

Let (A_0, A_1) and (B_0, B_1) are two pair of Banach spaces. $A_0 \subset A_1, B_0 \subset B_1$. Suppose that the map (nonlinear) $T : A_0 \to B_0$ and $T : A_1 \to B_1$ satisfies

- $\|Ta - Tb\|_{B_1} \leq f(\|a\|_{A_1}, \|b\|_{A_1})\|a - b\|_{A_1}^{\alpha}, \forall a, b \in A_1,$
- $\|Ta\|_{B_0} \leq g(\|a\|_{A_1})\|a\|_{A_0}^{\beta}, \forall a \in A_0,$

where f, g are continuous functions.

Theorem 5.5 ([59])
For any $0 < \theta < 1, 1 \leq p \leq \infty$, we have

$$\|Ta\|_{(B_0,B_1)_{\eta,q}} \leq Cg(2\|a\|_{A_1})^{1-\eta} f(\|a\|_{A_1}, 2\|a\|_{A_1})^{\eta} \|a\|_{(A_0,A_1)_{\theta,p}}^{(1-\eta)\beta+\eta\alpha},$$

where η and q are given by

$$\frac{1-\eta}{\eta} = \frac{1-\theta}{\theta}\frac{\alpha}{\beta},$$
$$q = \max\left\{1, \frac{p}{(1-\eta)\beta + \eta\alpha}\right\}.$$

In our applications, we take $A_0 = B_0 = L^2(\mathbb{T}^2), A_1 = B_1 = H^3(\mathbb{T}^2)$, $\theta = \frac{s}{3}$,$p = 2$. Thus $(H^3, L^2)_{\theta,2} = H^s$ for $0 < s < 3$. T be the solution map $u(0) \mapsto u(t)$ and $\alpha = \beta = 1, \eta = \theta$.

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