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Supergravities in Superspace

Supergravités en Superespace

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Résumé

Les corrections d'ordre supérieur en dérivées applicables à la théorie de supergravité à onze dimensions constituent un puissant outil pour étudier la structure microscopique de la théorie M. Plus particulièrement, l'invariant supersymétrique à l'ordre huit en dérivées est nécessaire à la cohérence quantique de la théorie, mais il n'en existe à ce jour aucune expression complète. Dans cette thèse, après une introduction formelle aux théories de supergravité, nous présentons une technique appelée principe d'action, dont le but est de générer le superinvariant complet associé au terme de Chern-Simons d'ordre huit en faisant usage du formalisme superespace. Bien que ce résultat ne soit pas encore atteint, nous en déterminons certaines caractérisiques, et ouvrons la voie à une résolution systématiques des étapes de calcul à venir. Dans le chapitre suivant, nous présentons les principales fonctionnalités du programme informatique élaboré pour gérer les imposants calculs liés au principe d'action. Ce programme est particulièrement adapté au traitement des matrices gamma, des tenseurs et des spineurs tels qu'ils surviennent en superespace. Enfin, à l'aide de cet outil, nous abordons un autre sujet calculatoire : la condensation fermionique en supergravité IIA massive. En utilisant la formulation superespace des supergravités IIA, nous dérivons les termes de l'action quartiques en fermions, puis en imposant une valeur moyenne dans le vide non-nulle, nous montrons qu'il est possible de construire une solution de géométrie de Sitter dans deux cas simples.

Abstract

High-order derivative terms in eleven dimensional supergravity are a powerful tool to probe the microscopic structure of M-theory. In particular, the superinvariant at order eight in number of derivatives is required for quantum consistency, but has not been completely constructed to this day. In this thesis, after a formal introduction to supergravity, we focus on a technique called the action principle, with the aim of generating the full superinvariant associated to the Chern-Simons term at order eight, using the superspace formalism of supergravity. Although we do not construct the superinvariant, we determine some of its characteristics, and pave the way for a systematic treatment of the computations leading to the desired result. Then we present the main features of the computer program we built for dealing with the calculations encountered in the application of the action principle. It is specifically designed to deal with gamma matrices, tensors and spinors as they appear in superspace. Finally, with the help of this program, we tackle another computationally intensive subject : the fermionic condensation in IIA massive supergravity. We use the superspace formulations of IIA supergravities to find the quartic fermion term of the action, and by imposing a non-vanishing vacuum expectation value for this term, we realize a de Sitter solution in two simple cases.

Résumé étendu

La supergravité regroupe un ensemble de théories ayant pour particularité d'être invariantes par transformation de supersymétrie locale. Autrement dit, l'algèbre de supergravité inclut, en plus des translations et des transformations de SO(1,d), un ensemble de générateurs fermioniques anti-commutants, appelés supercharges. L'algèbre ainsi formée, dite de super-Poincaré, intègre les translations locales, et décrit donc des théories incluant la relativité générale. Le nombre de supercharges, dénoté N, est contraint par la dimension de l'espace-temps d.

Le supergravité N =1 à onze dimensions, formulée en 1978 par E. Cremmer, B. Julia, et J. Scherk (CJS) est particulière à plus d'un titre. D'abord, onze est la dimension maximale de l'espace-temps pour laquelle il est possible de décrire une théorie de supergravité pertinente du point de vue physique (en effet, la présence de plus de 32 supercharges conduirait à inclure des champs de spin supérieur à 2, considérés non-physiques). Ensuite, la théorie est unique, et joue un rôle particulier dans le réseau de dualités liant les différentes théories de cordes, les supergravités, et la théorie M. Plus particulièrement, en plus d'être liée à la supergravité IIA par réduction dimensionnelle, la supergravité CJS est comprise comme étant la limite à basse énergie de la théorie M. En ce sens, il est attendu que son Lagrangien admette des corrections d'ordre supérieur à deux en dérivées (en tant que développement autour de ce point de basse énergie) afin de rendre compte du comportement microscopique de la théorie M. Ces corrections constituent un moyen détourné de sonder la structure de la théorie M, dont la formulation microscopique n'est pas établie.

Le premier chapitre de la thèse consiste en une introduction générale aux théories de supergravité, ainsi qu'une présentation des principaux outils nécessaires à la compréhension des chapitres suivants. Tout d'abord y sont définis les champs impliqués dans différentes théories de supergravité (le graviton et son superpartenaire le gravitino, les flux, etc.), et les structures mathématiques qui les supportent. Ensuite, l'algèbre de super-Poincaré est introduite, et particularisée aux deux cas étudiés par la suite : la supergravité CJS à onze dimensions, et la supergravité IIA à dix dimensions. Enfin, le superespace est brièvement décrit, et quelques outils de théorie des groupes appliquée au calcul tensoriel sont présentés.

Le deuxième chapitre porte sur le calcul des superinvariants de la supergravité CJS. En premier lieu, la formulation classique de la supergravité à onze dimensions est présentée en détails, puis la formulation en superespace, sur laquelle se basent les parties à venir. Le principe d'action est alors introduit : en se basant sur la formulation de la théorie en superespace, et en utilisant le terme de Chern-Simons, il est possible de générer l'invariant supersymétrique à un ordre donné, via la résolution d'une série d'équations en superespace. Ce principe, aussi appelé supersymétrisation du terme de Chern-Simons, est ensuite appliqué à la génération des invariants à l'ordre 2, puis 5 en dérivées. L'application du principe à l'invariant d'ordre 8, dont la présence est requise pour la cohérence quantique de la théorie, est inachevée. Toutefois, le développement effectué permet tout de même à déterminer certaines caractéristiques probables de l'invariant, et ouvre la voie à une résolution systématique des équations à venir.

Le troisième chapitre présente en détail le programme informatique construit pour gérer les calculs liés à la supersymétrisation du terme de Chern-Simons. En effet, les équations de superespace rencontrées dans le second chapitre ont des particularités qui rendent difficile leur manipulation : elles contiennent des (anti-)symétrisations sur de grands nombres d'indices, font intervenir des tenseurs dont les indices spinoriels doivent être traités explicitement, et impliquent de nombreuses matrices gamma requérant des transformations non-triviales. Les principales fonctions du programme y sont listées, et quelques applications pratiques y sont données à titre d'exemple.

Le quatrième chapitre se concentre sur les théories de supergravité à dix dimensions, et plus particulièrement la théorie IIA massive de Romans. Il existe deux théories massives de supergravité IIA à dix dimensions, nommées Romans et HLW, qui dans leur limite de masse nulle redonnent la supergravité IIA sans masse (qui elle est obtenue par réduction dimensionnelle de la théorie à onze dimensions). Ces trois théories admettent une description unifiée, qui prend forme dans la résolution des identités des Bianchi en superespace, après avoir imposé une contrainte conventionnelle sur la torsion. Cette méthode, en plus de regrouper les trois supergravités, offre une manière systématique de trouver les équations du mouvement complètes des théories à dix dimensions (incluant les termes fermioniques). Ainsi, en se concentrant sur la supergravité IIA de Romans, et en faisant usage des outils informatiques développés dans le troisième chapitre, il est possible de déterminer les termes quartiques en fermions intervenant dans les équations du mouvement (et la procédure peut être facilement généralisée aux autres supergravités). En imposant une valeur moyenne dans le vide non nulle pour le terme quartique ainsi obtenu, on montre qu'il est possible, dans deux cas simples, de réaliser des solution de géométrie « de Sitter » se basant sur la valeur non-nulle du condensat de fermions.

Enfin, en plus de définitions et de conventions, les annexes de la thèse contiennent une partie des calculs du premier chapitre qui ne sont pas développés dans le corps de texte, ainsi qu'un bref répertoire des fonctions accessibles dans le programme informatique du troisième chapitre.

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Publications

The following work led to three publications :

- *Action principle and the supersymmetrization of Chern-Simons terms in eleven-dimensional supergravity*
  B. Soueres D. Tsimpis, Phys. Rev., **D 95** 026013 (2017)

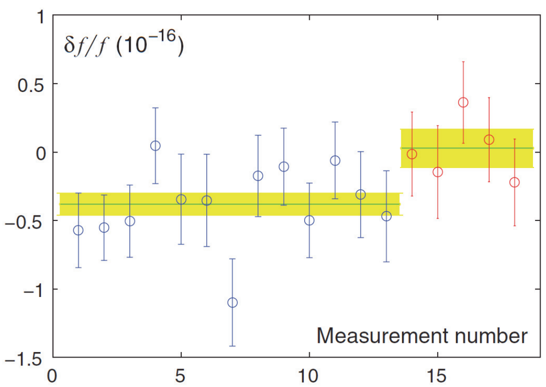
- *De Sitter space from dilatino condensates in (massive) IIA*
  B. Soueres D. Tsimpis, Phys. Rev., **D 97** 046005 (2018)

- *Superspace Gamma Package : Tensors, Spinors and Gamma matrices*
  B. Soueres, Work in progress (2018)

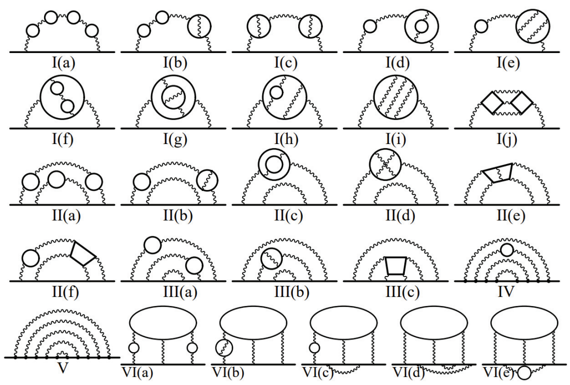
Introduction

THE STANDARD MODEL [1–3] is a theory aiming to describe elementary particles and their interactions at microscopic level. With the use of relatively few empirical adjustments [1], it provides a mathematical framework which allows to predict the behavior of particles with astounding accuracy. On the other hand, the theory of General Relativity [4] describes gravitation at large scale, i.e. the dynamics of objects at the level of planets, stars, and galaxies. Although it is formulated in very distinct terms, its predictions are just as precise as the standard model.

Both theories have made bold predictions which have been successfully verified, from the existence of the Higgs boson in 2012 [5, 6], to the detection of gravitational waves in 2016 [7]. Both have been extensively tested, and provide some of the most precise agreement with empirical results in physics. In 2010, researchers managed to measure the time dilatation due to a 33 cm difference in height of an atomic clock (figure A, [8]), and found a good agreement with general relativity. In 2014, the fine structure constant was computed using the 12 672 10th order Feynman diagrams associated to the electron propagation (figure B, [9]), and agrees with experimental value by more than 10 digits [2].



A. Relative shift in frequency of an atomic clock at the surface of the earth, then lifted by 33 cm.



B. Sample of the Feynman diagrams calculated to obtain the anomalous magnetic moment of the electron $a_e = 0.001156652181643(764)$

1. Depending on what is taken into account, the standard model is usually said to have 19 to 29 free parameters. Their value is directly adjusted from experiments, and mainly contains the particle masses and the coupling constants

2. Currently, those results might not be the most precise tests of the theories, but they are particularly striking for grasping the level of accuracy mentioned

The standard model is a relativistic quantum field theory whose last bricks were laid in the late 1970's. It incorporates field theory, quantum mechanics, and special relativity :

— it is a field theory : the basic objects are spacetime-dependent fields of various kinds. The theory makes use of a Lagrangian density, from which the dynamics of the fields can be deduced. Particles are represented by excited states of those fields ;

— it is a quantum mechanical theory : the fields are operators acting on a Fock space, and their dynamics can be described using the path integral formulation, for a given action and initial state. The results must be understood as a probability of transition from an initial state to a final state ;

— it is a special relativistic theory : its formulation is the same in every inertial reference frame, and spacetime is endowed with a Minkowski metric, which allows for Lorentz transformations.

General Relativity is a classical geometric theory of gravity formulated in 1915 :

— it is a geometric theory of gravity : instead of describing this interaction through particles or forces, it describes how the shape of spacetime itself is affected by mass and energy, through the metric (which intuitively defines how spacetime bends) ;

— it is classical : the theory can be described using field theory, but solutions are found using the least action principle, and the usual quantum apparatus cannot be applied.

A symmetry is a transformation that can be applied on a physical object (spacetime, or any field), which does not alter the observable content of the theory. In the Standard Model, symmetries dictates a lot about the nature and properties of particles. They are usually split in two groups : spacetime and internal symmetries. Spacetime (or external) symmetries, obviously, are symmetries that can be applied on spacetime. Using some group theoretical arguments, those symmetries can be shown to attach properties to the particles of the theory. Internal symmetries, or gauge symmetries, are spacetime-dependent transformations that apply on the fields themselves. They give rise to the fundamental interactions between particles, and require the existence of mediating bosons.

In the Standard Model (and in most other theories), the external symmetries include Lorentz transformations and spacetime translations, thus forming the Poincaré group of transformations. The internal symmetries belong to $SU(3) \times SU(2) \times U(1)$, and give rise to the corresponding fundamental force : strong, weak and electromagnetic interactions. General Relativity is only diffeomorphism invariant, meaning its formulation is unchanged under any coordinate transformation. Via the curvature of spacetime, it describes the gravitational force alone.

Most of the time (actually, in all current physical experiments), the dissociation between gravity and the other forces does not prevent researchers from doing extremely precise predictions, for gravity is orders of magnitude weaker than the other forces [3]. At the surface of the earth, all

3. One simple way to perceive this is to consider two electrons at the surface of the earth, 1 centimeter apart from each other. In this configuration, the two particles are several hundred thousand times more attracted to each other (by electromagnetic effects) than they are to the entire earth (by gravity).

standard model calculations make use of a flat Minkowski metric, neglecting the effect of gravity on the particles. On the other hand, when dealing with computations at cosmological scale, one rarely cares about weak quantum effects happening at microscopic scale.

There are some places in the universe, though, where one can encounter strong spacetime curvature at microscopic scale, namely near black holes. As out of reach as they are, a complete theory should be able to handle them, and describe how gravity behaves with the other forces, at microscopic scale. Moreover, the standard model has some shortcomings which might suggest it is incomplete : its mathematical coherence remains to be proven, and it lacks a few features that are now thought to exist, such as dark matter or neutrino masses. But more importantly, it is now believed that the standard model is an effective theory, only valid below a certain energy scale.

There is indeed a canonical way to formulate gravity as a quantum field theory, but all attempts to quantize it in the conventional way (as a quantum field theory) have failed at renormalization, and inescapably lead to infinite quantities. Physicists attempted to widen the standard model, by looking for a group of transformations that contains internal and spacetime symmetries in a non trivial way. However, the Coleman-Mandula theorem, published in 1967 [10], stated that in a coherent and physically relevant quantum field theory, the internal and external symmetries could only be combined as a direct product. However, it was soon realized in [11] that the underlying algebra could include anti-commuting generators of transformations, or fermionic charges, that led to a new class of theories escaping the reach of the premises of [11].

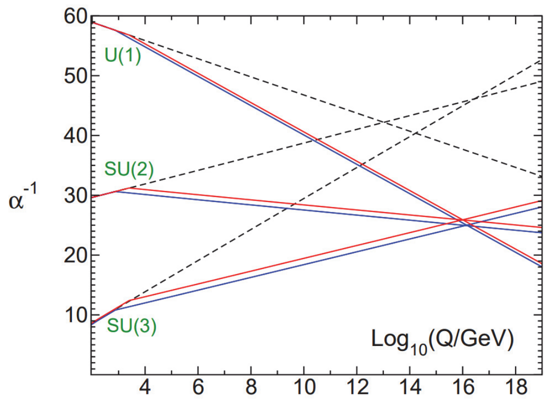
Supersymmetry is a theory in which the Poincaré algebra is extended to a $\mathbb{Z}_2$-graded lie algebra including the anti-commuting fermionic generators F (and the usual bosonic generators B). The super-algebra thus formed is schematically,

$$[B, B] = B, \qquad [B, F] = F, \qquad \{F, F\} = B.$$

Several supersymmetry transformations can be included in the algebra ; their number is denoted by $\mathcal{N}$. To this new algebra of external transformations can be attached other internal gauge symmetries, and thus construct a theory with interactions. The first rudimentary 4D supersymmetric theory is the Wess-Zumino model [12], and was soon extended to include internal symmetries.

Of course, supersymmetry has heavy phenomenological implications. First of all, each particle is required to have a superpartner of equal mass, with a 1/2 difference in spin. For example, in the Minimal Supersymmetric Standard Model, the Z boson of spin 1 is paired with the Zino fermion of spin 1/2, the bottom quark of spin 1/2 is paired with a bottom squark of spin 0, etc. Since none of those new particles has been detected, a realistic supersymmetry must be broken to allow particles and their partners to have different masses (conveniently, those hypothetical heavy partners provide a candidate for dark matter).

Most notably, this new symmetry allows the three coupling constants of the standard model to meet at high energies (see figure above), encouraging the elegant idea of a unifying gauge group



Inverse coupling constants evolution obtained by a two-loop renormalization performed in the Standard Model (dashed lines) and in the Minimal Supersymmetric Standard Model (solid lines). The masses of the heavy superpartners is varied between 750 GeV and 2.5 TeV. Supersymmetric coupling constants meet at the energy of 10^{16} GeV, whereas their usual equivalent never intersect. This figure is taken from [13]

comprising $SU(3) \times SU(2) \times U(1)$ [14]. However, although supersymmetric theories are promising and diverse, they do not address the issue of gravity, and cannot qualify as completely unifying theories.

Supergravity is a local (or gauged) supersymmetry, meaning that what previously was a global transformation (from a boson to a fermion, and vice versa) now acquires a spacetime dependency. Equipped with this unconventional transformation, the algebra of symmetries can be shown to include local translations, which are the infinitesimal versions of diffeomorphisms, thus describing gravity.

The first model with a local supersymmetry in 4 dimensions, containing four scalars and a spinor field, was built in 1976. It was soon extended to proper supergravity theories (involving a spin-2 graviton and a spin-3/2 superpartner, the gravitino) [15, 16], generalized to higher dimensions, and higher number of supergravity transformations ($\mathcal{N} > 1$). Even though at first, those theories were considered viable models for unifying all fundamental forces, their poor behavior under renormalization discarded this possibility. Supergravities are now mainly studied with regard to their relation to string theory.

String theory is currently considered to be one of the most promising frameworks in which our current description of all forces and particles can be unified. In string theory, fundamental objects are not described by point particles, but strings propagating through spacetime. This approach has the advantage of describing gravity naturally, without running into the divergences previously mentioned. It was first built with bosonic strings in 26 spacetime dimensions, and then extended to supersymmetric strings living in a ten-dimensional (10d) spacetime. At low energy, the string extension can be ignored, and one recovers a theory describing point-like objects propagating in ten dimensions, with local supersymmetry invariance. Thus, 10d supergravities can be viewed as low-energy effective theories [4], and in that sense, they admit high energy corrective terms due to the extended character of strings.

4. In this regard, their non-renormalizability is acceptable, since they would require an infinite series of corrections to be “UV-complete”.

All known 10d string theories are related by dualities (cf. figure below, where the arrows represent the dualities connecting different theories), meaning that a particular limit of one of these leads to the same observable quantities in another.

$$\begin{array}{ccccccccccc}
\text{Type I} & \overset{S}{\longleftrightarrow} & \underset{SO(32)}{\text{Heterotic}} & \overset{T}{\longleftrightarrow} & \underset{E_8\times E_8}{\text{Heterotic}} & & & & \text{Type IIA} & \overset{T}{\longleftrightarrow} & \text{Type IIB} \\
 & & & & & \nwarrow\!\!\!\searrow & & \swarrow\!\!\!\nearrow & & & \\
 & & & & & & \text{M-Theory} & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & \text{11d SuGra} & & & &
\end{array}$$

This led to the conjecture that all of them emerge as particular limits of an eleven-dimensional (11d) theory whose low-energy limit is 11d supergravity [17]. This theory, called M-theory, is still lacking a complete microscopic description. In this regard, supergravities can be studied as limits of string theories (or M-theory), and could lead to low energy effective 4D theories upon dimensional compactification [18].

Anomalies are violations of gauge symmetries of the classical action at the quantum level (i.e. when computing loop diagrams). In order for a theory to be consistent, it must be anomaly free. This requirement can either lead to the introduction of new terms, or new constraints on the theory in order to preserve classical symmetries. One notable example of this is the Green-Schwarz mechanism [19], which cancels different types of anomalies in $\mathcal{N} = (1,0)$ string theory by adding a well-chosen term in the classical action (and also requires the gauge group of the theory to be $SO(32)$ or $E_8 \times E_8$).

In the same way, 11d supergravity is subject to anomalies that should vanish for the consistency of its quantum limit. However, since M-theory is not fully formulated, the necessary corrections to 11d supergravity cannot be constructed "microscopically", and must be computed using methods intrinsic to supergravity. Although difficult, building those terms is a powerful tool to probe M-theory, check its consistency, and learn about its microscopic structure. Several methods have been used so far, but they only lead to a fraction of the terms that should constitute the full correction.

The thesis is structured as follow :

- Chapter 1 establishes the formal framework of supergravities in any dimension : it briefly describes the fields of supergravity, its algebra, the superspace formalism, and a few computational tools that will be useful in the following;
- Chapter 2 focuses on 11d supergravity : it presents the original formulation of the theory, its casting in superspace, and most importantly describes how to compute the superinvariant using the supersymmetrization of the Chern-Simons term;
- Chapter 3 is more technical : it presents the computational package that was built when trying to deal with the calculations of the previous chapter. Since it is particularly well suited for the purpose of superspace computation and gamma matrices gymnastic, it can be useful more generally;

— Chapter 4 makes use of the various tools for superspace manipulations assembled in the preceding chapters to tackle another computationally intensive work related to supergravity : the determination of quartic fermionic terms in 10d massive IIA supergravity, for the purpose of generating fermionic condensates that can alter the cosmological solutions of the theory.

1
Supergravities

This chapter is devoted to the presentation of some of the general features of supergravity. The geometric structures and the usual fields that will play a role in the following chapters are introduced, along with the basic principles of superspace. Finally, some techniques and definitions of group theory widely used in chapter 2 and 4 are defined.

1.1 Formal framework of supergravity

Defining the fields of supergravity in a relatively precise way requires the introduction of a few mathematical structures. Inevitably, those theories contain a graviton and a gravitino, whose structure and properties can only be grasped adequately with the help of some formalism.

A *fiber bundle* generalizes the notion of direct product between two spaces, to make it applicable for manifold, whose coordinates can only be defined locally (and accordingly, a fiber bundle can only be locally identified to a product space). Formally, it is a structure written (E, B, F, π), where

- E is a smooth manifold called the total space (the local equivalent of $B \times F$);
- B and F are smooth manifolds called the base and the fiber;
- Elements of E can be projected on B using the surjection π. For every $x \in E$, there is a neighborhood $U \subset B$ around $\pi(x)$ whose inverse image by π can be trivialized by the diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$, such that $\varphi(x) = \pi(x)$.

This is often summed up by the requirement that the following diagram should commute,

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U & \end{array}$$

(where p_1 is the projection onto the first element) i.e. over each U in B, $\pi^{-1}(U)$ is diffeomorphic to $U \times F$, through φ. However, this decomposition is restricted to an open set U, and cannot in general be extended to the whole space B. Besides, different trivializations φ_1, φ_2, on U have to be compatible, in the sense that, for each point $u \in \pi^{-1}(p)$,

$$\left.\begin{array}{l} \varphi_1(u) = (p, f_1) \\ \varphi_2(u) = (p, f_2) \end{array}\right\} \quad \Rightarrow \qquad f_2 = g_{21} \cdot f_1 \ ,$$

where, g_{12} belong to a group G, called the *structure group*.

A *principal bundle* is a special case of fiber bundle that includes a few more requirement. It is a fiber bundle (P, B, G, π) in which the fiber (previously called F) is the same as the Lie group G. This structure is equipped with a smooth and free group action of G on the total space $P \times G \to P$, and this action is compatible with the local trivialization : for $u \in P$ such that $\varphi(u) = (p, h)$, $\varphi(ug) = \varphi(u)g = (p, h)g = (p, hg)$. Although we shall not use it explicitly, the definition of the principal bundle is at the center of the formalism of gauge theories, in which the local transformations from G are encoded in the fiber over every point p of spacetime.

A *section of a fiber bundle* associates to every point in B a specific element of the fiber in F, thus forming an element of E. A section of a fiber bundle (E, B, F, π) is a smooth map $s : B \to E$ such that $\pi \circ s = \mathbb{I}_B$, meaning that for and element $p \in B$, $s(p)$ gives an element of E which is indeed over p. In practice, the section is defined locally : for $U \subset B$, s is a smooth map $U \to \pi^{-1}(U)$ such that for $p \in B$, $\varphi(s(p)) = (p, f)$ with $f \in F$.

Every smooth manifold is naturally equipped with a vector bundle called the *tangent bundle*. It is the bundle that hosts the gradient of functions defined from B to $\mathbb{R}$. The tangent bundle of a d-dimensional manifold B is the $2d$-dimensional bundle $(TB, B, \mathbb{R}^d, \pi)$, where the fiber at a point p in B is called T_pB. A local set of coordinates $\{x_m,\ m = 1 \dots d\}$ on $U \subset B$ defines the *coordinate basis* $\left\{\frac{\partial}{\partial x_m},\ m = 1 \dots d\right\}$: for $p \in U$, an element V of T_pB is written,

$$V = v^m \frac{\partial}{\partial x_m} = v^m \partial_m \ .$$

This vector bundle has a dual, simply constructed by attaching to each point p the dual of T_pB, T_p^*B. This dual vector bundle is called T^*B, and its element can be expressed in a basis $\{dx^m,\ m = 1 \dots d\}$ complementary to the coordinate basis, i.e. $\langle dx^m, \partial_n \rangle = \delta^m_n$, such that elements of the dual bundle are written :

$$L = l_m dx^m \; .$$

A *pseudo-Riemannian manifold* is a smooth manifold is equipped with an inner product on its tangent space : for every $p \in B$, there is a non-degenerate [1] bilinear form $g_p : \; T_pB \times T_pB \to \mathbb{R}$ (such that if V and W are smooth vector fields, $p \mapsto g_p(V_p, W_p)$ is smooth). This inner product allows to talk about the length of vectors, angles, etc.

Every smooth manifold comes equipped with a principal bundle, called the *frame bundle*. For each $p \in B$, the basis of the tangent vector space T_pB can be changed via an action of the group $GL(d, \mathbb{R})$, such that a vector V can be written [2]

$$V \;=\; v^m \frac{\partial}{\partial x^m} \;=\; v^m \underbrace{\frac{\partial y^n}{\partial x^m}}_{\in\, GL(d,\mathbb{R})} \frac{\partial}{\partial y^n}$$

for two basis $\frac{\partial}{\partial x^m}$ and $\frac{\partial}{\partial y^m}$, related by the invertible Jacobian matrix $\frac{\partial y^n}{\partial x^m}$. This frame bundle can be written $(P, B, GL(d, \mathbb{R}), \pi)$, and represents all the basis transformations allowed over each point p.

Besides, if the manifold is pseudo-Riemannian, it can be associated to a more specific principal bundle called the *orthonormal frame bundle*. In most physical cases, pseudo-Riemannian manifolds have a metric of signature $(1, d{-}1)$, meaning that, at each point $p \in M$, there exists a basis in T_pM in which the metric is diagonal with values $\pm(1, -1, \ldots, -1)$. Since there is a vast range of basis which preserve this property, one can define the principal bundle composed, at each point, of the change of basis that preserve the diagonal metric. Such elements are members of $O(1, d{-}1)$ (actually restricted to $SO^+(1, d{-}1)$, since we are interested in basis with the same orientation and time direction).

The *spin bundle* is a complex vector bundle that can be built from the orthonormal frame bundle. One first has to lift the $SO(1, d{-}1)$ bundle to its double cover, the $Spin(1, d{-}1)$ principal bundle P_S, and make the $Spin(1, d{-}1)$ group act on a $2^{[d/2]}$-dimensional complex vector space V through a representation ρ. The quotient $(P_S \times V)/G$ is a complex vector bundle, with the equivalence,

$$(u, v) \sim \big(ug, \rho(g^{-1})v\big) \quad \text{with} \quad \begin{cases} (u, v) \in P_S \times V \\ \quad\; g \in Spin(1, d{-}1) \end{cases}$$

This complex vector bundle is equipped with a Clifford action form TM verifying

$$v_1, v_2 \in TM, \quad \Gamma(v_1)\Gamma(v_2) + \Gamma(v_2)\Gamma(v_1) = 2g(v_1, v_2) \tag{1.1}$$

1. A Riemannian manifold requires the bilinear form to be a proper scalar product, i.e. it must be definite positive.
2. The notation $\frac{\partial}{\partial x^m}$ comes from the fact that vectors can be viewed as operators action on functions $f : B \to \mathbb{R}$, as $V[f]|_p = v^m \frac{\partial f}{\partial x^m}\big|_p$. Conversely, derivatives of those functions form a vector space over each point of the manifold.

which can be written in components with gamma matrices, as

$$\Gamma_m \Gamma_n + \Gamma_n \Gamma_m = 2 g_{mn} \mathbb{I} \, .$$

Finally, the action of $Spin(1, d\text{–}1)$ over V is built using this representation of the Clifford algebra : upon a local Lorentz transformation parameterized by the function $\epsilon_{mn}(x)$, elements of V transform using the representation $\frac{1}{2}\Gamma_{mn}$ of $\mathfrak{spin}(1, d\text{–}1)$,

$$\Gamma_{mn} = \Gamma_{[m}\Gamma_{n]} \, , \qquad \psi \longrightarrow e^{\frac{1}{2}\epsilon_{mn}(x)\Gamma^{mn}} \, \psi \, .$$

1.2 Geometric objects & Fields of supergravity

We now have the required structures to define the physically relevant objects that arise in supergravity theories :

- spacetime is a smooth pseudo-Riemannian manifold M whose metric has signature $\pm(1, \text{–}1 \ldots \text{–}1)$;
- a real (complex) scalar field is a functions from M to $\mathbb{R}$ ($\mathbb{C}$) ;
- a vector field is a section of the vector bundle over M ;
- a tensor field is a section of tensor products of the vector bundle and its dual : $TM \otimes \cdots \otimes TM^* \otimes \cdots$;
- a spinor field is a section of the spin bundle over M.

The fields defined in this list are abstractly defined globally, over the whole spacetime manifold. Of course, in most cases those fields are considered through local sections, and depend on a local coordinate chart in M.

Vielbein

At any point p of a pseudo-Riemannian manifold, consider two vectors V and W, locally written $v^m\partial_m$ and $w^n\partial_n$. The bilinear form g can be expressed as an element of $T^*M \otimes T^*M$ acting on vectors :

$$g = g_{mn} \, dx^m \otimes dx^n \qquad \text{and} \qquad g(v^m\partial_m, w^n\partial_n) = g_{mn} \, v^m w^n := v^m w_m \, .$$

However, at each point p, one can define a basis of $\{e^a, a = 1 \ldots d\}$ in T_p^*M such that g is diagonal, and accordingly, a basis $\{e^a, a = 1 \ldots d\}$ of T_p^*M such that

$$g = \eta_{ab} \, e^a \otimes e^b \qquad \text{and} \qquad g(v^a e_a, w^b e_b) = \eta_{ab} \, v^a w^b := v^a w_a \, .$$

The dual basis of e^a is denoted e_a, and verifies $\langle e^a, e_b \rangle = \delta^a_b$. The coordinate and flat basis are related by the vielbein e^a_m and the inverse vielbein e^m_a :

$$e^a = e^a_m \, dx^m \qquad \text{and} \qquad e_a = e^m_a \, \partial_m$$

Usually, when tensors are used without reference to the basis, indices from *a* to *l* refer to the flat basis of the tangent space, while letters from *m* to *z* refer to the coordinate basis. For example, the relation 1.1 defining gamma matrices in curved spacetime can be express in the flat basis as,

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\eta_{ab}\mathbb{I}$$

where the matrices Γ_a are related to Γ_m through the relation $\Gamma_m = e_m^a \, \Gamma_a$.

It must be remarked that each vielbein over a point of the manifold is defined up to a Lorentz transformation. Indeed, the transformation $e_m^a \to \Lambda^a{}_b \, e_m^b$ induces a vielbein which still diagonalize the metric,

$$g_{mn} = e_m^a \, \eta_{ab} \, e_n^b \quad \longrightarrow \quad e_m^c \, \underbrace{\Lambda^a{}_c \, \eta_{ab} \, \Lambda^b{}_d}_{\eta_{cd}} \, e_n^d = g_{mn} \ ,$$

since the Lorentz group is defined as the set of metric preserving transformations. Thus, the vielbein field has a gauge freedom consisting of local Lorentz transformations.

Spin connection

On any Riemannian manifold, one can define a connection Γ^m_{np}, and define the covariant derivative of vectors, covectors and tensors through,

$$D_m T_n{}^p \ = \ \partial_m T_n{}^p + \Gamma^q_{np} \, T_q{}^p - \Gamma^p_{nq} \, T_n{}^q \ ,$$

which is easily generalized for any type of tensors. In the flat basis, it is possible to define another object, called the spin connection ω, so that it is compatible with the connection Γ by imposing the condition $D_m V^n = e^n_b D_m V^b$ (where the derivative on the lhs uses Γ, whereas the derivative on the rhs uses the spin connection). This leads to,

$$\omega_m{}^a{}_b = e_n^a \, \Gamma^n_{mp} \, e_b^p \ - \ e_b^n \, \partial_m e_n^a \ ,$$

where ω is the connection one-form over the $SO(1, d\text{–}1)$ orthonormal frame bundle previously defined. It can be viewed as a the gauge field of the local Lorentz transformations that can be applied on the vielbein, and is a $\mathfrak{so}(1, d\text{–}1)$ valued one-form,

$$\omega^a{}_b = (\omega_m)^a{}_b \, dx^m, \qquad\qquad \omega_{ab} = -\omega_{ba} \ .$$

which defines a covariant derivative acting on vectors components expressed in the flat basis (while the covariant derivative defined with gamma acts with vectors with coordinate indices),

$$D_m T_a{}^b = \partial_m T_a{}^b + \omega_m{}^c{}_a \, T_c{}^b - \omega_m{}^b{}_c \, T_a{}^c \ .$$

Spinors, which are naturally defined in flat spacetime, can be covariantly derivated using the spin connection as well, as

$$D_m\psi^\alpha \;=\; \partial_m\psi^\alpha + \frac{1}{4}\,\omega_m{}^{ab}\,(\Gamma_{ab})^\alpha{}_\beta\,\psi^\beta \tag{1.2}$$

Torsion

In general relativity, the connection is required to be metric compatible, meaning $D_p\, g_{mn} = 0$ (such that the squared norm of a covariantly constant vector $V^m V^n g_{mn}$ is preserved upon parallel transport). This imposes the form,

$$\Gamma^m_{np} = \underbrace{\frac{1}{2}g^{mr}\big(\partial_n g_{rp} + \partial_p g_{rn} - \partial_r g_{np}\big)}_{\hat{\Gamma}^m_{np}} + \underbrace{\frac{1}{2}\big(T^m{}_{np} + T_n{}^m{}_p + T_p{}^m{}_n\big)}_{C^m{}_{np}}\,,$$

where $\hat{\Gamma}$ is the Levi-Civita connection, and $C^m{}_{np}$ is called the contorsion. The torsion T is an unconstrained tensor defined in terms of the connection as $T^m{}_{np} = 2\Gamma^m_{[np]}$. Generally, the torsion is defined on any basis of the tangent space as the tensor associated to the application $T_pM^2 \to T_pM, T(u,v) = \nabla_u v - \nabla_v u - [u,v]$, where u, v and $T(u,v)$ are vectors in TM. Form the principal bundle point of view, the torsion 2-form can be defined using the vielbein, as

$$\begin{aligned} T^a = De^a &= \mathrm{d}e^a + \omega^a{}_b \wedge e^b \\ &= \frac{1}{2}T_{mn}{}^a\, dx^m \wedge dx^n\,. \end{aligned}$$

Classical general relativity imposes the torsion to be zero, but as we will see in the next chapters, non-vanishing torsions are extensively used in higher-dimensional supergravities.

Curvature

Similarly, the curvature can be defined in the coordinate basis as a tensor

$$R_{mn,p}{}^q = 2\big(\partial_{[m}\Gamma^q_{\;n]p} + \Gamma^q_{[m|s}\Gamma^s_{\;|n]p}\big)$$

or more generally, in any set of coordinates by the tensor associated to the application $T_pM^3 \to T_pM$, $R(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$, where u, v, w, and $R(u,v)w$ are vectors in TM. From the principal bundle point of view, it can be defined as the curvature 2-form, using the $\mathfrak{g}$-valued connection,

$$\begin{aligned} R^a{}_b = D\omega^a{}_b &= \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \\ &= R_{mn}{}^a{}_b\, dx^m \wedge dx^n\,. \end{aligned}$$

Both the curvature and the torsion can be expressed in the coordinate or the flat basis. In any case, they must respect some geometrical relations, called the Bianchi identities (BI),

$$
\begin{aligned}
\mathrm{d}T^a + \omega^a{}_b \wedge T^b &= e^b \wedge R^a{}_b \\
\mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - \omega^c{}_b \wedge R^a{}_c &= 0 \,.
\end{aligned}
\tag{1.3}
$$

The Ricci tensor and the scalar curvature, that appear in the gravitational part of the action and equations of motions, are obtained via the following contractions,

$$
\begin{aligned}
R_{mn} &= g^{nq} R_{mn,pq} = e^m_a R_{mn}{}^{ab} \\
R &= g^{mp} g^{nq} R_{mn,pq} = e^m_a e^n_b R_{mn}{}^{ab}
\end{aligned}
$$

Form fields

Higher dimensional supergravities also include fields called fluxes, or p-form fields, whose behavior mimic those of F and A in Maxwell's electromagnetism. Generally, a form is a smooth section of the totally anti-symmetrized tensor product of the dual tangent bundle. In practice, a p-form ω is written

$$
\omega = \frac{1}{p!} \begin{cases} dx^{m_1} \wedge \ldots \wedge dx^{m_p} \; \omega_{m_1 \ldots m_p} & \text{on the coordinate basis,} \\ e^{a_1} \wedge \ldots \wedge e^{a_p} \; \omega_{a_1 \ldots a_p} & \text{on the flat basis,} \end{cases} \qquad \omega \in \bigwedge^p T^*_p M \,.
$$

Symmetry transformations

Let's consider a symmetry transformation applied on the coordinates, of the form $x \to y$, with $y = t(x)$. Any tensor is abstractly defined without reference to the coordinate system, and can be expressed in both. For a tensor from $TM \otimes T^*M$, it reads,

$$
\mathbf{T} = T_a{}^b(x)\, dx^a \otimes \frac{\partial}{\partial x^b} = \mathcal{T}_a{}^b(y)\, dy^a \otimes \frac{\partial}{\partial y^b} \,, \tag{1.4}
$$

where $T_a{}^b(x)$ and $\mathcal{T}_a{}^b(y)$ are different functions of different coordinates. Being only interested in the local variation of the field, one can express the rhs of (1.4) in the $(dx, \frac{\partial}{\partial x})$ basis as

$$
\mathcal{T}_a{}^b(y) \; \frac{\partial y^a}{\partial x^c} \frac{\partial x^d}{\partial y^b} \; dx^c \otimes \frac{\partial}{\partial x^d} \,,
$$

and recover the variation $\delta T_c{}^d$ at x, caused by the transformation $x \to y$, as :

$$
\delta T_c{}^d(x) = \mathcal{T}(t^{-1}(y))_a{}^b \; \frac{\partial y^a}{\partial x^c}(x) \frac{\partial x^d}{\partial y^b}(x) - T_c{}^d(x) \,. \tag{1.5}
$$

In the following chapter, we will be concerned about the variation of forms. In that case, and with the transformation $x \to y$ being infinitesimal ($y(x) = x + \xi$), the variation of a k-form ϕ is expressed using (1.4), and reads,

$$\delta_\xi(x)\phi_{m_1\ldots m_k} \;=\; \xi^p\,\partial_p\phi_{m_1\ldots m_k} + n\,(\partial_{[m_1|}\xi^p)\;\phi_{p|m_1\ldots m_k]}\;, \tag{1.6}$$

which corresponds to the *Lie derivative* $\mathcal{L}$ of ϕ. Although Lie derivatives can be applied on any type of tensors, they take a particularly simple form when applied on differential forms, using the *exterior derivative* d and the *interior product* ι,

$$\mathcal{L}\phi = (\mathrm{d}\,\iota_\xi + \iota_\xi\,\mathrm{d})\phi\,, \qquad \text{where} \;\begin{cases}\iota_\xi\phi = \frac{1}{(n-1)!}\;\xi^p\phi_{pm_2\ldots m_k}\;dx^{m_2}\wedge\ldots\wedge dx^{m_k}\\ \mathrm{d}\phi = \frac{1}{(n+1)!}\;\partial_p\,\phi_{m_1\ldots m_k}\;dx^p\wedge dx^{m_1}\wedge\ldots\wedge dx^{m_k}\;.\end{cases} \tag{1.7}$$

1.3 Algebra of supersymmetry

The algebra of transformations in supersymmetric theories is composed of the generators of Lorentz transformations M_{nr}, translations P_m, and supersymmetry transformations Q, with a hidden spinor index . It can also host what's called central charges denoted Z (which commute with all other members), and R-symmetry charges R acting on the Q's. The bosonic part of the algebra has the same structure in all dimensions :

$$[M_{nr}, M^{pq}] = 4\,\delta^{[p}_{[n}M^{q]}_{r]}$$
$$[P_m, M_{pq}] = \eta_{m[p}P_{q]}$$
$$[P_m, P_n] = 0$$

The spinorial operators Q_α, called supercharges, have the same structure as the corresponding spinors of the theory, and change accordingly with the dimension of spacetime. The two example below are the algebras of 11d and 10d IIA supergravity. In 10d, there are 2×16 Majorana-Weyl supercharges, of chirality $(+)$ and $(-)$, whereas in 11d, there are 32 real Majorana supercharges :

$$\{Q_\pm, Q^T_\pm\} = \frac{1}{2}(1\pm\Gamma_{11})\Gamma^m C P_m$$
$$\{Q_\pm, M_{nr}\} = \frac{1}{2}(\Gamma_{nr})\,Q_\pm$$
$$\{Q_+, Q^T_+\} = 0$$

$$\{Q, Q^T\} = (\Gamma^m C)\,P_m$$
$$\{Q, M_{nr}\} = \frac{1}{2}(\Gamma_{nr})\,Q$$

Some fundamental properties can be derived only from examination of this algebra. For example, $[P_m, Q_\alpha] = 0$ implies $[P^2, Q_\alpha] = 0$, with P^2 being one of the Casimir invariant of the Poincaré group that represent the invariant mass of a state. Thus, all fields related by a supersymmetric transformation have the same mass (this requires the symmetry to be broken for this theory to be phenomenologically relevant).

Basic features of 11d supergravity can also be derived with a few manipulations. If we consider a massless representation ($P^mP_m = 0$), one can always describe such a state by the impulsion $(E, E, 0, \ldots, 0)$ by going to the suitable reference frame. According to Wigner's classification, those states will be indexed by the irreducible representations of their little group. The bosonic part of the group is reduced from $SO(1, d-1)$ to $SO(d-2)$, and spinors, supercharges, and gamma matrices also have to be decomposed in a less trivial way : if the matrices γ_m verify $\{\gamma_m, \gamma_n\} = \delta_{mn}$ in $(d-2)$ Euclidean dimensions, then the matrices Γ_m

$$\Gamma_0 = i\sigma^2 \otimes \mathbb{I}, \qquad \Gamma_1 = \sigma^1 \otimes \mathbb{I}, \qquad \Gamma_m = \sigma^3 \otimes \gamma_{m-1} \quad (m = 2, \ldots, d-1),$$

verify $\{\Gamma_m, \Gamma_n\} = \eta_{mn}$ in d-dimensional Minkowski spacetime. In this particular representation, going from $(d-2)$ to $(1, d-1)$ dimensions splits the spinors (and the supercharges) in two components of the same size. For example, in $d = 11$,

$$\psi = \begin{pmatrix} \psi^{(+)} \\ \psi^{(-)} \end{pmatrix} \qquad Q = \begin{pmatrix} Q^{(+)} \\ Q^{(-)} \end{pmatrix} ,$$

where ψ and Q contain 32 components, while the quantities with $(+)$ or $(-)$ contain 16. The conjugation matrix $C_{(11)}$ can also be split as $i\sigma^2 \otimes C_9$, such that the anti-commutator of the supercharges becomes, in matrix form,

$$\{Q, Q\} \;=\; E\,(\Gamma_0 - \Gamma_1)\,C_{(11)} \;=\; 2E \begin{pmatrix} \mathbb{I}_{16} & 0 \\ 0 & 0 \end{pmatrix} .$$

Written in terms of $Q^{(+)}$ and $Q^{(-)}$, one has

$$\{Q^{(+)}_\alpha, Q^{(+)}_\beta\} = 2E\,\mathbb{I}_{16} , \qquad \{Q^{(-)}_\alpha, Q^{(-)}_\beta\} = \{Q^{(+)}_\alpha, Q^{(-)}_\beta\} = 0 .$$

The anti-commutation of $Q^{(+)}_\alpha$ allows to build a Clifford algebra in a canonical way : half of the charges are set to be creation operators, the other half are destruction operators, whose action cancel the Clifford vacuum state $|0\rangle$. Thus, in our 11-dimensional example, the physical states are created by the 8 charges $Q^{(+)}$ acting on the vacuum, and add up to a total of 256 states corresponding to the total degrees of freedom of 11d supergravity (cf. chapter 2). Through dimensional reduction to 4 dimension (cf. chapter 4), the existence of more supercharges would have led to helicities greater than 2, leading to physical inconsistencies [4].

Helicity [3]	state	Multiplicity
-2	$\|0\rangle$	1
-3/2	$Q_{i_1}\|0\rangle$	8
-1	$Q_{i_1}Q_{i_2}\|0\rangle$	28
-1/2	$Q_{i_1}Q_{i_2}Q_{i_3}\|0\rangle$	56
0	$Q_{i_1}\ldots Q_{i_4}\|0\rangle$	70
1/2	$Q_{i_1}\ldots Q_{i_5}\|0\rangle$	56
1	$Q_{i_1}\ldots Q_{i_6}\|0\rangle$	28
3/2	$Q_{i_1}\ldots Q_{i_7}\|0\rangle$	8
2	$Q_1\ldots Q_8\|0\rangle$	1

3. Helicity in not defined in dimensions other than 4. This number refers to the helicity of the representations once compactified to 4d, where it is easy to prove that the operators Q raise helicity.

4. A supersymmetry theory involving a spin-2 and a spin-5/2 partner was considered in [20]. It was discarded as a physically relevant theory due its inability to describe a non-zero graviton-graviton scattering amplitude.

1.4 Superspace formulation

Superspace can be viewed as an extension of usual space in which supergravity transformations arise in a more natural way, and where the spinors are treated on an equal footing with vectors and tensors. Besides, as we will see in the next chapter, it is also a powerful computational tool in which all the calculations of this thesis are implemented.

The basic idea of superspace consist in supplementing the usual space, whose points are labeled by coordinates $\{x^m, m = 0 \ldots d{-}1\}$, with anti-commuting coordinates $\{\theta^\mu, \mu = 1 \ldots 2^{\left[\frac{d}{2}\right]}\}$. Introducing anti-commuting variables allows to write the superalgebra in terms of commutators only, and a general transformation corresponding to an element of the superalgebra can be written as an exponential with those variables θ as parameters. A finite supertranslation then takes the form,

$$G\left(x, \theta, \bar{\theta}\right) = e^{-i\left(-x^m P_m + \theta \cdot Q + \bar{\theta} \cdot \bar{Q}\right)} ,$$

where the gray part can be discarded when dealing with Majorana spinors. Thus, these anti-commuting variables must be as numerous as the spinor components, and must be of the same type (Majorana, Weyl, etc.). In 11d, since spinors are Majorana, one needs 32 real θ^α, while in 10d IIA, the Majorana-Weyl spinors require 2×16 variables of opposed chirality, θ^α and θ_α. To distinguish the two parts of the superspace, the usual commutative part is often called *even part*, or the *body*, and the anti-commutative part is called the *odd part*, or more exotically, the *soul*. Of course, those coordinates give rise to their own part of the tangent space, dual tangent space, and frame bundle. Tensors can now belong to both parts of the tangent space, and can take the form, for example,

$$T = T_{a\alpha}{}^{b\beta}\, dx^a \otimes \mathrm{d}\theta^\alpha \otimes \frac{\partial}{\partial x^b} \otimes \frac{\partial}{\partial \theta^\beta} .$$

Using the generalized coordinates $Z^M = (x^m, \theta^\mu)$, we extend the flat basis (and its dual) as,

$$E^A = E^A_M\, dZ^M , \qquad E_A = E^M_A\, \frac{\partial}{\partial Z^M} ,$$

where E^A_M is the supervielbein, verifying,

$$E^A_M\, E^N_A = \delta^N_M = \begin{pmatrix} \delta^n_m & 0 \\ 0 & \delta^\nu_\mu \end{pmatrix} .$$

A general superfield is written with capital indices, each of which can take values in both parts of the superspace. In the following, we will mainly consider superforms written in flat superspace with the following convention,

$$H = \frac{1}{k!}\, E^{A_k} \wedge \ldots \wedge E^{A_1}\, H_{A_1 \ldots A_k} ,$$

where the supervielbein inherit the anti-commutation property of the coordinates,

$$E^A{}_\wedge E^B = \left\{ \begin{array}{ll} (-1) & \text{if A or B is even} \\ (+1) & \text{if A and B are odd} \end{array} \right\} \times E^B{}_\wedge E^A \ ,$$

and, conversely commuting $\partial_A \partial_B$ brings the opposite sign. A superfield hides a huge number of degrees of freedom : each capital index can be particularized to an even or odd value, e.g.

$$E^A{}_\wedge E^B \ H_{AB} = E^b{}_\wedge E^a \ H_{ab} + 2 \, E^a{}_\wedge E^\alpha \ H_{\alpha a} + E^\beta{}_\wedge E^\alpha \ H_{\alpha\beta} \tag{1.8}$$

and each of those H depend on θ and x, such that, in 11d superspace, the Taylor expansion of H_{AB} is,

$$H_{AB}(x,\theta) = H_{AB}(x) + \sum_{i=1}^{32} \frac{1}{i!} \, \theta^{\mu_1} \ldots \theta^{\mu_i} \, (H_{\mu_1 \ldots \mu_2})_{AB} \ . \tag{1.9}$$

In the remainder of the thesis, we are not so much interested in the θ Taylor expansion of the field as we are in the index expansion. In the following, we will mainly be interested in the forms at $\theta = 0$, meaning that the development (1.8) is valid, but the last part of (1.9) vanishes.

All the geometric objects defined above are generalized to superspace : the spin connection $\Omega^a{}_b$, the torsion tensor T^a, and the curvature two-form $R^a{}_b$ become,

$$\Omega^A{}_B \ = \ dZ^M (\Omega_M)^A{}_B \quad \longrightarrow \quad \left\{ \begin{array}{lllll} R^A{}_B & = & D\Omega^A{}_B & = & \mathrm{d}\Omega^A{}_B + \Omega^A{}_{C\wedge} \Omega^C{}_B \\ T^A & = & DE^A & = & \mathrm{d}E^A + E^B{}_\wedge \Omega^B{}_A \ , \end{array} \right. \tag{1.10}$$

and all of them can be expanded as in (1.8). In the following, we shall establish some restrictions on the structure of the supermanifold, by imposing the Lorentz manifold condition, i.e.

$$\Omega^\alpha{}_\beta \ = \ \frac{1}{4} (\Gamma_a{}^b)^\alpha{}_\beta \, \Omega^a{}_b, \qquad \Omega^\alpha{}_b \ = \ \Omega^a{}_\beta \ = \ 0 \ . \tag{1.11}$$

The last condition of (1.11) can be viewed as the requirement that there be no mixing between the odd and even part of the manifold upon parallel transport, while the first condition simply states that the supercovariant derivative of a spinor,

$$D_M \psi^\alpha \ = \ \partial_M \psi^\alpha + \omega_M{}^\alpha{}_\beta \, \psi^\beta$$

corresponds to the derivative (1.2) for $M = m$.

Superspace diffeomorphism transformation of the coordinates, $Z^M \to Z'^M = Z^M + \xi^M$, acts on superforms as in (1.6) and (1.7), where the Lie Derivative is generalized to superspace. As an example, the supersymmetry transformations of the gravitino and the spinor can be found in (D.4) and (D.5) of appendix D.2.2, in ten-dimensional IIA superspace. In the remaining of the thesis we shall use the conventions of [21] where all forms are given in "superspace conventions",

$$\Phi_{(p)} = \frac{1}{p!}\Phi_{m_1\ldots m_p}\mathrm{d}x^{m_p}\wedge\ldots\wedge\mathrm{d}x^{m_1}\ , \qquad \mathrm{d}\left(\Phi_{(p)}\wedge\Psi_{(q)}\right) = \Phi_{(p)}\wedge\mathrm{d}\Psi_{(q)} + (-1)^q\,\mathrm{d}\Phi_{(p)}\wedge\Psi_{(q)}\ .$$

These are better suited for the discussions of the next chapters, in which we mainly manipulate elements in the superspace formulation (of 11d and IIA supergravities), where these conventions are the natural ones.

Integration of actions in superspace are defined using either the canonical volume form (i.e. the Hodge dual of 1), or the wedge product of a form and its Hodge dual. The Hodge star is defined as follows,

$$\star(\mathrm{d}x^{a_1}\wedge\ldots\wedge\mathrm{d}x^{a_p}) = \frac{1}{(d\text{–}p)!}\varepsilon^{a_1\ldots a_p}{}_{b_1\ldots b_{10-p}}\ \mathrm{d}x^{b_1}\wedge\ldots\wedge\mathrm{d}x^{b_{10-p}}\ ,$$

so that,

$$\Phi_{(p)}\wedge\star\,\Phi_{(p)} = (\star 1)\frac{1}{p!}\Phi_{m_1\ldots m_p}\Phi^{m_1\ldots m_p}\ .$$

1.5 A word about irreducible representations

In the next chapters, we make use of some group theoretic notions that we briefly recall in this section. A representation of a group G is a map ρ from G to a vector space V, such that the group structure is preserved :

$$\rho : G \to GL(V)\ \Big|\ g_1, g_2 \in G,\ \ \rho(g_1)\rho(g_2) = \rho(g_1 g_2) \tag{1.12}$$

with a matrix product on the lhs, and a group product on the rhs. Such a representation is *reducible* if one can find a subset H of V such that H is stable by the action of G :

$$\forall h \in H,\ \forall g \in G,\ \ \rho(g)h \in H\ .$$

On the contrary, an *irreducible representation* (or *irrep*) only has trivial stable subsets (0 and V itself), meaning it transforms "as one" under the group action. Irreducible representations of groups play an important role in physics, and in this thesis, where it is extensively used in tensorial calculations.

Let's focus on the two groups that will appear in the following : the symmetric group S_n and the d-dimensional Lorentz group $SO(1, d\text{–}1)$.

1.5.1 Symmetric group S_n

The symmetric group S_n consists of all permutation of n distinguishable elements. It is finite, contains $n!$ members, and it is possible to make it act on tensors with n indices. For example, a member $s = \begin{Bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{Bmatrix}$ of S_3 acts on a 3-indices tensor T as

$$T_{a_1a_2a_3} \quad \xrightarrow{s} \quad T_{a_3a_1a_2} \ .$$

Actually, defining this action only require T to be a repertory of ordered indices, rather than a true tensor. If those objects can be summed, it becomes possible to define invariant quantities. One of the simplest example is a fully anti-symmetrized tensor :

$$T_{[a_1a_2a_3]} = \frac{1}{6}T_{a_1a_2a_3} - \frac{1}{6}T_{a_2a_1a_3} + \cdots - \frac{1}{6}T_{a_3a_2a_1} \quad \xrightarrow{S_n} \quad \pm T_{[a_1a_2a_3]} \ .$$

which will lead to $\pm$ itself when acted upon by any member of S_3. Young diagrams (and young tableaux) are useful to generalize this, and find all subsets of invariant tensors for S_n.

First, let's consider a representation of S_n on $\mathbb{R}^{n!}$, provided with the basis consisting of all the tensor product of the basis vectors $\{e_1, \cdots, e_n\}$ of $\mathbb{R}^n$:

$$\{e_1 \otimes \cdots \otimes e_n, \cdots, e_n \otimes \cdots \otimes e_1\} := \{e_{1\cdots n}, \cdots, e_{n\cdots 1}\} \ .$$

In this basis, a vector $(-4, \ldots, 7)$ is then $-4\, e_{1\ldots n} + \ldots + 7\, e_{n\ldots 1}$. For example, on S_3, the symmetric group acts on $\mathbb{R}^6$, and the element $g = \begin{Bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{Bmatrix}$ corresponds to the $GL(\mathbb{R}^6)$ matrix,

$$\rho(g) = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} e_{123} \\ e_{132} \\ e_{213} \\ e_{231} \\ e_{312} \\ e_{321} \end{matrix} \end{array}$$

where the shaded elements correspond to the basis vectors. For example, equipped with this representation the action $\rho(g) \cdot (\,2\, e_{123} - 3\, e_{231}) \;=\; (\,2\, e_{312} - 3\, e_{123})$ corresponds to,

$$\rho(g) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} & \begin{matrix} e_{123} \\ e_{132} \\ e_{213} \\ e_{231} \\ e_{312} \\ e_{321} \end{matrix} \end{array}$$

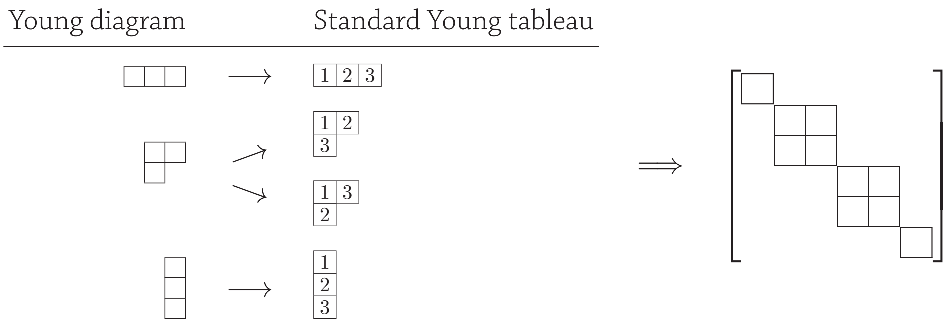
Whether or not this representation of S_3 is reducible can then be stated clearly, since it corresponds to finding a basis of $\mathbb{R}^6$ for which the whole group action acts as a block diagonal matrix. For example, the case of S_2 is very simple : the basis is $\{e_{12}, e_{21}\}$, and the two elements of the group act as,

$$\rho\left(\begin{Bmatrix} 1 & 2 \\ 1 & 2 \end{Bmatrix}\right) = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} e_{12} \\ e_{21} \end{matrix} \end{array} \qquad \rho\left(\begin{Bmatrix} 1 & 2 \\ 2 & 1 \end{Bmatrix}\right) = \begin{array}{cc} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{matrix} e_{12} \\ e_{21} \end{matrix} \end{array} .$$

It is trivial to express this group action in the basis $\{e_{12}+e_{12}, e_{21}-e_{12}\}$, on which the group acts diagonally,

$$\rho\left(\begin{Bmatrix}1 & 2\\ 1 & 2\end{Bmatrix}\right) = \begin{array}{cc}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} & \begin{array}{l} e_{12}+e_{21}\\ e_{12}-e_{21}\end{array}\end{array} \qquad \rho\left(\begin{Bmatrix}1 & 2\\ 2 & 1\end{Bmatrix}\right) = \begin{array}{cc}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} & \begin{array}{l} e_{12}+e_{21}\\ e_{12}-e_{21}\end{array}\end{array}.$$

Thus, the action of S_2 on $\mathbb{R}^2$ can be split on two irreducible separate subspaces of dimensions 1. The case of S_3 is less obvious, and higher dimensions are absolutely non-trivial. That's were Young diagrams come into play : each diagram composed of n boxes represents an irreducible representation of S_n, and the multiplicity of the associated *standard Young tableau* give both the multiplicity of the representation, and the dimension of the invariant subspace [5] on $\mathbb{R}^{n!}$. Going back to S_3 and writing all the associated standard Young tableaux allows to determine the diagonal form of the group action (even-though the ways of finding the subspaces is still unspecified),



(1.13)

The basis on which this configuration is realized can be built from the standard tableaux, using a simple procedure : Each element of the "naive" basis must be (1) successively symmetrized over the indices numbers specified in each row of the tableau, (2) anti-symmetrized over the indices specified in each column, (3) permute the indices of the $e_i's$ to span the subspace. Applied on the first element e_{123}, with the tableau $\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}$, it gives,

$$e_{123} \xrightarrow{(1)} e_{123}+e_{213} \xrightarrow{(2)} e_{123}-e_{321}+e_{213}-e_{231} \xrightarrow{(3)} \begin{cases} e_{123}-e_{321}+e_{213}-e_{231}\\ e_{213}-e_{312}+e_{123}-e_{132}\\ e_{132}-e_{231}+e_{312}-e_{321}\\ \vdots \end{cases}.$$

From the six possible indices permutations of the last step, only 2 will be independent, as expected, since the subspace associated to this diagram is 2-dimensional. Doing the same for the second standard tableau $\begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}$ will lead to an isomorphic (but different) subspace, while $\begin{smallmatrix}1 & 2 & 3\end{smallmatrix}$ and $\begin{smallmatrix}1\\ 2\\ 3\end{smallmatrix}$ both lead to a 1-dimensional subspace corresponding to the fully symmetric and anti-symmetric configurations. The new basis is composed of independent elements, span the whole space, and indeed leads to a group action of the form (1.13).

5. A standard young tableau is a Young diagram with boxes labeled by numbers strictly increasing from left to right, and top to bottom. The links between Young diagrams/tableaux and the irreps of S_n is a non-trivial statement. The proof can be found in [22].

By doing so for each standard Young diagram of n boxes, one can find all the irreps of S_n. What have been done on the $e_{i_1} \otimes \cdots \otimes e_{i_n}$ can be easily translated for a tensor $T_{i_1 \ldots i_n}$, by applying on the indices all that has been applied to the basis. Thus, there is a systematic way to decompose all tensors according to the irreps of the symmetric group. Practical examples of decomposition can be found in Appendix (C.1).

1.5.2 Lorentz group $SO(1, d - 1)$

The Lorentz group $SO(1, d–1)$ is the group obtained after reduction of the frame bundle to the orthonormal frame bundle. Thus, tensors expressed in the non-coordinate basis are $SO(1, d–1)$-tensors, expressed in the vielbein basis. As such, they form a space on which $SO(1, d–1)$ acts,

$$T_{a_1 \ldots a_n} \longrightarrow \Lambda_{a_1}{}^{b_1} \ldots \Lambda_{a_1}{}^{b_1} \; T_{b_1 \ldots b_n} \qquad (n < d) \, ,$$

and they can be decomposed according to irreducible representations of this group. Irreducible representations of simple Lie algebras are indexed by *Dynkin labels*, whose general form is $(i_1 \ldots i_n)$, where each i is an integer, and n depends on the dimension of the algebra [6]. The Dynkin labels containing 1 as single non-zero entry represent the *fundamental representation*, from which the other can arise through tensor products. In the following, we will restrict the discussion to groups used in the thesis : $SO(1, 10)$ and $SO(1, 9)$, whose corresponding complex algebras are respectively $B5$ and $D5$.

The Dynkin labels of $B5$ and $D5$ are of the form $(i_1 \ldots i_5)$. Of course, it possible to establish a standard correspondence between fundamental Dynkin labels and some usual representations :

$SO(1,10)$	dim	
(00000)	1	scalar
(10000)	11	vector
(01000)	55	2-form
(00100)	165	3-form
(00010)	330	4-form
(00001)	32	spinor

$SO(1,9)$	dim	
(00000)	1	scalar
(10000)	10	vector
(01000)	45	2-form
(00100)	120	3-form
(00010)	16	chiral spinor $(+)$
(00001)	16	chiral spinor $(-)$

.

The tensor product of two irreducible representations leads to another representation, which is in general reducible. Fortunately, it is possible to systematically decompose representations obtained by product of irreps into direct sums of irreps [22]. For example, a 2-tensor without symmetry in its indices (i.e. living in a tensor product of a vector space with itself $V \otimes V$) is decomposed as :

$$\begin{array}{ccccccccc} (10000) \otimes (10000) & = & (20000) & \oplus & (01000) & \oplus & (00000) \\ T^{mn} & = & S^{mn} & + & A^{mn} & + & I \\ 121 & = & 65 & + & 55 & + & 1 \, , \end{array}$$

6. More precisely, it depends on the dimension of its Cartan subalgebra. For a complete review of the classification of irreducible representation of semi-simple Lie algebras, see [22].

where S is traceless symmetric, A is anti-symmetric, and I is a scalar. Those decompositions into irreducible representations are extremely useful when trying to solve equations involving tensors, for they allow to separate the expression into unrelated subparts. Let's consider the equation $F_{abc}G_d = H_{ab}H_{cd}$, involving tensors in 11 dimensions, whose rhs and lhs decompose into irreducible representations of $B5$ as,

$$\begin{aligned}(00100) \otimes (10000) &= \mathbf{(00010)} \oplus (01000) \oplus (10100) \\ (01000)^{2\otimes_S} &= (00000) \oplus \mathbf{(00010)} \oplus (02000) \oplus (20000)\,.\end{aligned}$$

The only common irrep shared on both sides is (00010), meaning that this equation can only constrain the 4-form part of its constituents. Hence it can be equivalently written as $F_{[abc}G_{d]} = H_{[ab}H_{cd]}$. The same principle can be applied on arbitrarily large cases, and is widely use for the resolution of Bianchi identities in superspace.

2

Supersymmetrization of the Chern-Simons term in eleven-dimensional supergravity

2.1 Supergravity in 11 dimensions

As mentioned in the introduction, supergravity in eleven dimensions plays a particular role in the collection of supergravities. Eleven is the highest spacetime dimension allowing for a physically relevant supergravity theory, since a supersymmetry in higher dimensions would exceed the maximal number of supercharges, and lead to compactified 4d theories with spins greater than 2. Besides, this d$=11, \mathcal{N}=1$ supergravity can withstand only very few modifications : its connection form can be slightly generalized (cf. section 4.1.1, [23]), but it is unique, does not admit a cosmological constant [24], and only has an on-shell formulation [23, 25].

2.1.1 Cremmer-Julia-Scherk (CJS) supergravity

The eleven dimensional supergravity action [26] was built in 1978, starting with an analysis of the degrees of freedom available to find the physical objects involved. It contains only three fields : the graviton g_{mn}, the gravitino ψ_m and a 3 form field C_{mnp}. On-shell degrees of freedom are found by analyzing the little group representations, which is $SO(9)$ in the case of massless fields in 11 dimensions.

The graviton g_{mn} — The first field that is required to describe a gravity theory is the graviton g_{mn}. It had to be present in 11d supergravity, since it was known that 10 is the highest spacetime dimension in which there cannot be a symmetric tensor field [27]. The graviton is either represented by the metric g_{mn} itself, or its associated vielbein e^a_m. Either way, each of them must contain the same number of degrees of freedom when all the non-relevant ones are removed :

$$\begin{array}{lcll} g_{mn} & \longrightarrow & \overbrace{(1/2)\cdot 9\cdot(9+1)}^{\substack{SO(9)\text{ symm.}\\ \text{2-tensor}}} \; - \overbrace{1}^{det(g)\neq 0} & = \; 44 \, , \\ e^a_m & \longrightarrow & \underbrace{9\cdot 9}_{\substack{SO(9)\\ \text{2-tensor}}} \; \underbrace{-\,(1/2)\cdot 8\cdot 9}_{\text{Lorentz gauge}} \; \underbrace{-\;1}_{det(e)\neq 0} & = \; 44 \, . \end{array}$$

The gravitino ψ_m — The superpartner of the graviton is the gravitino ψ^μ_m. It is a spin 3/2 spinor bearing a spacetime index, and a spinor index often implicit. It is a Rarita-Schwinger field, living in the (10001) representation of $SO(1,10)$ (it is traceless : $\Gamma^m\psi_m = 0$), and its degrees of freedom are counted as follows :

$$\psi^\mu_m \;\longrightarrow\; \underbrace{2^{[\frac{9}{2}]}\cdot 9}_{SO(9)\text{ spinor}} \;-\; \underbrace{2^{[\frac{9}{2}]}}_{\text{traceless}} \;=\; 128\, .$$

The supersymmetric transformation of the graviton and the gravitino have a common part in all supergravities, namely

$$\delta_\epsilon\,\psi_m \;\propto\; D_m\epsilon \, , \qquad\qquad \delta_\epsilon\, e^a_m \;\propto\; \bar\epsilon\,\Gamma^a\psi_m \, . \tag{2.1}$$

The 3-form C — The existence of the 3-form C was deduced via a counting of the remaining degrees of freedom : it contains exactly the right number to equate the bosonic and fermionic ones. A 3-indices anti-symmetric tensor of $SO(9)$ represents $\binom{9}{3} = 84$ free components. With this field, there are 128 fermionic and $84+44=128$ bosonic degrees of freedom (on-shell).

The action — Eleven dimensional supergravity action was originally formulated by E. Cremmer, B. Julia and J. Scherk in [26]. With a few cosmetic changes to suit our conventions, it has the form :

$$\mathcal{S} = \frac{1}{\kappa^2}\int d^{11}x\,\sqrt{g}\,\Big[-\frac{1}{4}\,R(\omega) \tag{2.2}$$

$$-\frac{i}{2}\,\psi_m\Gamma^{mnr}D_n\left(\frac{\omega+\hat\omega}{2}\right)\psi_r \tag{2.3}$$

$$-\frac{1}{48}\,G_{mnrs}\,G^{mnrs} \tag{2.4}$$

$$+\frac{1}{192}\,\left(\bar\psi_m\,\Gamma^{mnrspq}\,\psi_n + 12\,\bar\psi^r\,\Gamma^{sp}\,\psi^q\right)\left(G_{rspq}+\hat G_{rspq}\right) \tag{2.5}$$

$$-\frac{2}{\sqrt{g}(12)^4}\,\epsilon^{r_1\ldots r_8 mnp}\,G_{r_1\ldots r_4}G_{r_5\ldots r_8}C_{mnp}\,\Big] \tag{2.6}$$

where the first 3 terms are familiar : the Einstein-Hilbert action (2.2), the Rarita-Schwinger action (2.3), and the kinetic term for the 3-form field C (2.4). The field $\hat G$ is the supercovariant version of G, the field-strength of C. It is a 4-form whose expression in components is

$$\hat{G}_{mnpq} = G_{mnpq} + 3\,\bar{\psi}_{[m}\Gamma_{np}\psi_{q]}$$
$$G_{mnpr} = 4\,\partial_{[m}C_{npq]}\ .$$

$R(\omega)$ is the scalar curvature, and the term $\hat{\omega}$ is the spin connection one-form, equal to $(\omega_m^0)_a{}^b + (K_m)_a{}^b$, where K is the contorsion tensor :

$$\hat{\omega}_{mab} = \omega_{mab} + i\,\frac{\kappa^2}{4}\,\psi^c\,\Gamma_{mabcd}\,\psi^d$$
$$\hat{\omega}_{mab} = \omega^{(0)}_{mab} + i\,\frac{\kappa^2}{2}(\bar{\psi}_m\,\Gamma_b\,\psi_a - \bar{\psi}_m\,\Gamma_a\,\psi_b + \bar{\psi}_b\,\Gamma_m\,\psi_a)$$

and $\omega^{(0)}_{mab}$ is the usual connection defined in terms of the vielbein. In the last term, $\epsilon^{r_1\dots r_{11}}$ is the Levi-Civita tensor, related to the Levi-Civita symbol by

$$\epsilon^{r_1\dots r_{11}} = \sqrt{g}\,\epsilon^{a_1\dots a_{11}}\,e^{r_1}_{a_1}\dots e^{r_{11}}_{a_{11}}\ ,$$

itself defined as

$$-\epsilon^{a_1\dots a_{11}} = \epsilon_{a_1\dots a_{11}} = \begin{cases} +1 & \text{if } (a_1\dots a_{11}) \text{ is an even permutation of } (0\dots 10) \\ -1 & \text{if } (a_1\dots a_{11}) \text{ is an odd permutation of } (0\dots 10) \\ 0 & \text{if an index is repeated} \end{cases}\ .$$

Local supersymmetry — The supersymmetry that mixes the three fields, but leaves the action invariant acts with the local spinorial parameter $\epsilon(x)$ as,

$$\begin{aligned} \delta_\epsilon\,C_{mnr} &= \frac{3}{2}\,\bar{\epsilon}\,\Gamma_{[mn}\psi_{r]} \\ \delta_\epsilon\,e^a_m &= -\,i\,\bar{\epsilon}\,\Gamma^a\psi_m \\ \delta_\epsilon\,\psi_m &= D_m(\hat{\omega})\,\epsilon + \frac{i}{12^2}(\Gamma^{abcd}{}_m - 8\,\Gamma^{abc}\delta^d_{m)})\,F_{abcd} \end{aligned} \tag{2.7}$$

and the superalgebra closes, as can be seen from the table below, where the transformation $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ is applied on the three fields, giving :

	Diffeomorphism		Supersymmetry		Lorentz		Gauge	
e^a_m :	$(\partial_m\xi_1^n)\,e^a_n + \xi_1^n\,(\partial_n e^a_m)$	$+$	$i\kappa\bar{\xi}_2\,\Gamma^a\psi_m$	$+$	$(\xi_3)^a{}_b\,e^b_m$	$+$	0	
ψ_m :	$(\partial_m\xi_1^n)\,\psi_n + \xi_1^n\,(\partial_n\psi_m)$	$+$	$\frac{1}{\kappa}\hat{D}_m\xi_2$	$+$	$\frac{1}{4}(\xi_3)_{ab}\,\Gamma^{ab}\psi_m$	$+$	0	$+\,\mathcal{K}_m$
C_{mnr} :	$3(\partial_{[m\vert}\xi_1^p)C_{p\vert nr]} + \xi_1^p(\partial_p C_{mnr})$	$+$	$\frac{3}{2}\xi_2\,\Gamma_{[mn}\psi_{r]}$	$+$	0	$+$	$\partial_{[m}(\xi_4)_{nr]}$	

where $\xi_{(i)}$'s are the emerging parameters of transformations, which depend on ϵ_1, ϵ_2, and the fields themselves. The term $\mathcal{K}_m$ is proportional to the equation of motion of ψ_m, thus making the algebra close only on-shell (once the equations of motion are imposed).

The action was found by postulating the existence of first three terms ((2.2), (2.3) and (2.4) without the hatted fields), with the supersymmetric proto-transformation rules (2.1). Invariance was imposed by steps, first by adding the last two counter-terms ((2.5) and (2.6), without the hatted fields), and modifying the transformation rule of the gravitino. Then, in order to cancel the remaining 3- and 4-fermion terms, supercovariant hatted fields were defined and added to the action.

In the following sections, and the manipulations presented therein, we rely on the use of the last term of the action called the *Chern-Simons (CS) term*. Unlike the other terms, the Chern-Simons term is integrated without the canonical volume form $d^{11}x\sqrt{g}$, and should rather be expressed as an integral over the top form,

$$\frac{2}{(12)^4}\int G_\wedge\, G_\wedge\, C\ .$$

Since C is subject to a transformation $C \to C + \mathrm{d}\Lambda$, it is worth verifying that this action is indeed gauge invariant. Using integration by parts, and the Bianchi identity $\mathrm{d}G = 0$, the general variation of C reads,

$$\frac{2}{(12)^4}\,3\int G_\wedge\, G_\wedge\, \delta C\ .$$

The fact that a single term is obtained allowed to use the Chern-Simons term as the last keystone to make the action supersymmetric. Replacing δC by $\mathrm{d}\Lambda$ cancels the whole term (via integration by parts, and up to border terms), and makes this term gauge invariant, despite the appearance of C.

2.1.2 Eleven-dimensional supergravity in superspace formalism

In order to establish conventions, and present the main objects that will be useful in the remaining of the chapter, this section briefly presents 11d supergravity formulated in superspace [28, 29]. As previously stated in (1.4), eleven-dimensional superspace consists of eleven even and thirty-two odd dimensions, with structure group the eleven-dimensional spin group. Let $A = (a, \alpha)$ be flat tangent superindices, where $a = 0, \ldots 10$ is a Lorentz vector index and $\alpha = 1, \ldots 32$ is a Majorana spinor index. Curved superindices will be denoted by $M = (m, \mu)$, with the corresponding supercoordinates denoted by $Z^M = (x^m, \theta^\mu)$. The supercoframe is denoted by $E^A = (E^a, E^\alpha)$ while its dual is denoted by $E_A = (E_a, E_\alpha)$.

We shall assume the existence of a connection one-form $\Omega_A{}^B$ with values in the Lie algebra of the Lorentz group. The associated supertorsion and supercurvature tensors, defined in (1.10), are :

$$T^A = \frac{1}{2}E^C{}_\wedge E^B\ T_{BC}{}^A \qquad\qquad R_A{}^B = \frac{1}{2}E^D{}_\wedge E^C\ R_{CDA}{}^B\ , \tag{2.8}$$

The assumption of a Lorentzian structure group implies that the components of the curvature two-form obey the Lorentz condition defined in (1.11). The super-Bianchi identities (BI) for the torsion and the curvature,

$$DT^A = E^B{\scriptstyle\wedge} R_B{}^A \,, \qquad\qquad DR_A{}^B = 0 \,, \tag{2.9}$$

follow from the definitions (2.8). Moreover, a theorem due to Dragon [30] ensures that for a Lorentz structure group the second BI above follows from the first and need not be considered separately.

As was shown in [28, 29], very few constraints on the superfields, together with the Bianchi identities, lead to the equations of motion of supergravity. The Bianchi identities are usually geometrical identities that must be automatically satisfied. The first step for finding the superspace formulation of supergravity is to enforce a particularly well-chosen value for one of the low dimensional components of the torsion, called the *rigid constraint*,

$$T_{\alpha\beta}{}^c \;=\; i\,(\Gamma^c)_{\alpha\beta} \,. \tag{2.10}$$

With this restriction imposed, the Bianchi identities cease to be automatically satisfied, and must be solved. In the first two articles [28, 29], the superspace generalization of G and C,

$$E^D{\scriptstyle\wedge}\ldots{\scriptstyle\wedge} E^A\, G_{ABCD} \,, \qquad E^C{\scriptstyle\wedge} E^B{\scriptstyle\wedge} E^A\, C_{ABC} \,, \qquad G_{ABCD} \;=\; 4\,\partial_{[A}C_{BCD]} \,,$$

are introduced by hand before the computation, accompanied by their corresponding BI : $\mathrm{d}G = 0$. However, it was shown in [31] that the resolution of the BI 2.9 require the existence of a closed four-form, which naturally brings back G and C [1]. Remarkably then, solving the torsion Bianchi identity with the sole constraint (2.10) is enough to recover ordinary eleven-dimensional supergravity, and to define all mixed components of the superfields involved. All the non-zero mixed components, and spinorial derivative of fields are reported in appendix (B.1.2).

The four-form arising during the process can also be used to introduce a seven-form G_7, although the four-form G_4 is still needed. The identities force those forms to obey [31, 32], [2]

$$\mathrm{d}G_4 = 0 \,, \qquad \mathrm{d}G_7 + \frac{1}{2} G_4{\scriptstyle\wedge} G_4 = 0 \,, \tag{2.11}$$

where the bosonic components correspond to the eleven-dimensional supergravity four-form and its Hodge-dual, respectively :

$$(G_7)_{m_1\ldots m_7} = (\star G_4)_{m_1\ldots m_7} \,.$$

The superspace resolution of the BI finally leads to the equations of motion (B.1.2), whose inte-

1. It was also proven in [31] that there exists a dimension one-half spinor field, that was set to zero. It was later shown in [] that this term could be interpreted as the dimension one-half of the scale connection, and that this corresponding connection vanishes on a topologically trivial manifold.
2. The G_7 BI receives a correction at the eight-derivative order, cf. (2.44) below.

gration leads to the superspace action,

$$S = \int \left(R \star 1 - \frac{1}{2}\, G_4{}_\wedge \star G_4 - \frac{1}{6}\, C_3{}_\wedge G_4{}_\wedge G_4\right)\Big| \ , \tag{2.12}$$

where it is understood that only the bosonic $(11, 0)$ components of the forms enter this formula (the vertical bar denotes the evaluation of a superfield at $\theta^\mu = 0$).

Reduced units ($\hbar = c = 1$) make all physical dimensions of the action coalesce into a single one, called length of mass dimensions (with $-[X]_{\text{mass}} = [X]_{\text{length}}$). In terms of mass dimension, the requirement that the action be dimensionless imposes,

$$\begin{array}{llll} [\kappa] = \frac{9}{2} & [dx] = -1 & [G_{mnpq}] = 1 & [R_{mnpq}] = 2 \\ & \left[\frac{\partial}{\partial x}\right] = 1 & [C_{mnp}] = 0 & [\psi_m] = \frac{1}{2} \ . \end{array} \tag{2.13}$$

In superspace, consistency requires to define the following mass dimensions,

$$\begin{array}{ll} [d\theta^\alpha] = -\frac{1}{2} & [E^\alpha_m] = \frac{1}{2} \\ \left[\frac{\partial}{\partial\theta^\alpha}\right] = \frac{1}{2} & [E^m_\alpha] = -\frac{1}{2}. \end{array}$$

Accordingly, the mixed superspace components of the fields will have a mass dimension increasing by $1/2$ with each switching from a fermionic index to a bosonic index,

$$[G_{abcd}] = 1 \ , \qquad [G_{\alpha bcd}] = \frac{1}{2} \ , \qquad [G_{\alpha\beta cd}] = 0 \ , \qquad \text{etc.}$$

2.2 Supersymmetrization of the Chern-Simons term

Eleven-dimensional supergravity is believed to be the low-energy limit of M-theory [17], the conjectured non-perturbative completion of string theory. As such it is expected to receive an infinite tower of higher-order corrections in an expansion in the Planck length or, equivalently, in the derivative expansion. At present such higher-order corrections cannot be systematically constructed within M-theory, so one must resort to indirect approaches.

One such approach is to calculate the higher-order corrections within perturbative string theory, in particular type IIA in ten dimensions, which is related to eleven-dimensional supergravity by dimensional reduction (cf. 4.1.1). The effective action of string theory can be systematically constructed perturbatively in a loop expansion in the string coupling,

$$S_{\text{eff}} = \sum_{g=0}^{\infty} g_s^{2g-2} \int \mathrm{d}^{10}x\sqrt{G}\mathcal{L}_g \ ,$$

where g is the loop order, g_s is the string coupling constant, G is the spacetime metric and $\mathcal{L}_g$ is the effective action at order g. Besides, each $\mathcal{L}_g$ admits a perturbative expansion in an infinite series of higher-order derivative terms. Moreover it is expected that each $\mathcal{L}_g$ should correspond to an independent superinvariant in ten dimensions (see e.g. [33]).

The bosonic part of the tree-level effective action takes schematically the following form,

$$\mathcal{L}_0 = \mathcal{L}_{\rm IIA} + \alpha'^3\Big(I_0(R) - \frac{1}{8}I_1(R) + \cdots\Big) + \mathcal{O}(\alpha'^4)\ , \tag{2.14}$$

where $\mathcal{L}_{\rm IIA}$ is the (two-derivative) Lagrangian of ten-dimensional IIA supergravity, and the ellipses stand for terms which have not been completely determined yet. The first higher-derivative correction starts at order α'^3 (eight derivatives), and the I_0, I_1 in (2.14) are defined as follows,

$$\begin{aligned} I_0(R) &= t_8 t_8 R^4 + \frac{1}{2}\varepsilon_{10} t_8 B R^4\ , \\ I_1(R) &= -\varepsilon_{10}\varepsilon_{10} R^4 + 4\varepsilon_{10} t_8 B R^4\ . \end{aligned} \tag{2.15}$$

These were constructed by directly checking invariance under part of the supersymmetry transformations, without using superspace, in [34] (Those two terms, with the tensors ϵ_{10} and t_8, stand for large expressions quartic in the Riemann tensor). The terms in (2.15) linear in B are, up to a numerical coefficient, Hodge-dual to the Chern-Simons term $B{\wedge}X_8$ [35, 36]. The eight-form X_8, see (2.43) below, is related by descent to the M5-brane anomaly polynomial and is a linear combination of $({\rm tr}R^2)^2$ and ${\rm tr}R^4$. Note that the Chern-Simons term drops out of (2.14).

The superinvariant I_0 can be further decomposed into two separate $\mathcal{N}=1$ superinvariants in ten dimensions [34], $I_0 = -\,6\,I_{0a} + 24\,I_{0b}$, where,

$$\begin{aligned} I_{0a} &= \Big(t_8 + \frac{1}{2}\,\varepsilon_{10}B\Big)\Big({\rm tr}R^2\Big)^2 + \cdots \\ I_{0b} &= \Big(t_8 + \frac{1}{2}\,\varepsilon_{10}B\Big)\,{\rm tr}R^4 + \cdots\ , \end{aligned}$$

correspond to the supersymmetrization of the $B{\wedge}({\rm tr}R^2)^2$ and $B{\wedge}{\rm tr}R^4$ Chern-Simons terms respectively. As we show in the following, if the uplift of I_{0a}, I_{0b} gives rise to two separate superinvariants in eleven dimensions, they will necessarily have to be cubic or lower in the fields.

The one-loop effective action takes the following form [37, 38],

$$\mathcal{L}_1 = \alpha'^3\Big(I_0(R) + \frac{1}{8}I_1(R) + \cdots\Big) + \mathcal{O}(\alpha'^4)\ . \tag{2.16}$$

In particular we see that in this case the Chern-Simons term does not drop out, cf. (2.15). The ellipses above indicate terms which are not completely known, although partial results exist thanks to five- and six-point amplitude computations [39–42]. Contrary to the tree-level superinvariant $\mathcal{L}_0$ which is suppressed at strong coupling, the uplift of the one-loop superinvariant $\mathcal{L}_1$ is expected to survive in eleven dimensions, and thus to be promoted to an eleven-dimensional superinvariant. We will refer to the latter as *the supersymmetrization of the Chern-Simons term* $C{\wedge}X_8$, the uplift of the ten-dimensional Chern-Simons term, where C is the three-form potential introduced above.

An argument of [43], which we review in the following, guarantees that if the supersymmetrization of the Chern-Simons term is quartic or higher in the fields, then it is unique at the eight-derivative order [3]. The uniqueness of this superinvariant is also supported by the results of [44–46] which uses the Noether procedure to implement part of the supersymmetry transformations of eleven-dimensional supergravity. The results of these references constrain the supersymmetrization of the Chern-Simons term to be of the form,

$$\Delta\mathcal{L} = l^6\Big(t_8 t_8 R^4 - \frac{1}{4!}\varepsilon_{11}\varepsilon_{11}R^4 - \frac{1}{6}\varepsilon_{11}t_8 CR^4 + R^3G^2 + \cdots\Big) + \mathcal{O}(l^7)\,, \tag{2.17}$$

where l is the Planck length. The ellipses indicate terms which were not determined by the analysis of [44–46], while the R^3G^2 terms were only partially determined. The reduction of the above to ten dimensions is consistent, as expected, with the one-loop IIA superinvariant (2.16). In addition the quartic interactions $R^2(\partial G)^2$ and $(\partial G)^4$ were determined in [47] by eleven-dimensional superparticle one-loop computations in the light cone, and in [48–50] by a different method which uses tree amplitudes instead. [4] The $t_8t_8R^4$ terms have also been obtained by four-graviton one-loop amplitudes in eleven dimensions [51, 52], while it can be shown [53] that higher loops do not contribute to the superinvariant (2.17).

In this chapter, we reexamine the problem of calculating supersymmetric higher-order derivative corrections to eleven-dimensional supergravity from the point of view of the action principle approach. This method relies on the superspace formulation of the theory and is particularly well suited to the supersymmetrization of Chern-Simons terms.

2.2.1 De Rham cohomology and Weil triviality

The calculations involved in the supersymmetrization of the Chern-Simons terms require the use of cohomology groups in superspace. In this section, before presenting the action principle, we review the various superspace cohomology groups that will be useful in the following. Let Ω be the space of superforms of any degree, and Ω^n be the space of n-superforms. Thanks to the nilpotency of the exterior superderivative, one can define de Rham cohomology groups in superspace in the same way as in the case of bosonic space, as the ensemble of non-exact closed forms,

$$H^n = \{\omega \in \Omega^n | \mathrm{d}\omega = 0\} \,/\, \big\{\omega \sim \omega + \mathrm{d}\lambda, \lambda \in \Omega^{n-1}\big\}\ .$$

The fact that the topology of the odd directions is trivial means that the de Rham cohomology of a supermanifold coincides with the de Rham cohomology of its underlying bosonic manifold. We shall also assume that the body is trivial, thus making the supermanifold trivial i.e. every d-closed superform is d-exact, and the cohomology groups defined above should all be empty.

There is an important caveat to the previous statement : it is only valid when the cohomology

3. The existence of independent superinvariants starting at order higher than eight in the derivative expansion will of course spoil the uniqueness of the superinvariant at higher orders.

4. There is disagreement between [47] and [49] concerning part of the $(\partial G)^4$ terms.

is computed on the space of unconstrained superfields. Once physical constraints are imposed it ceases to be automatically satisfied. Adopting the terminology of [54], we shall call *Weil-trivial* those d-closed superforms which are also d-exact on the space of constrained (also referred to as "on-shell", or "physical") superfields. The cohomology groups computed on the space of constrained superfields will be denoted by $H^n(\mathrm{phys})$, as in [43]. There is no a priori reason why $H^n(\mathrm{phys})$ should coincide with the cohomology of the body of the supermanifold, meaning that $H^n(\mathrm{phys})$ may very well be non-empty.

2.2.2 τ-cohomology

The space of superforms Ω can be further graded according to the even, odd degrees of the forms. We denote the space of forms with p even and q odd components by $\Omega^{p,q}$ so that,

$$\Omega^n = \oplus \sum_{p+q=n} \Omega^{p,q} \ .$$

A (p,q)-superform $\omega \in \Omega^{p,q}$ can be expanded as follows (in the flat basis),

$$\omega = \frac{1}{p!q!} E^{\beta_q} \wedge \ldots \wedge E^{\beta_1} \wedge E^{a_p} \wedge \ldots \wedge E^{a_1} \ \omega_{a_1 \ldots a_p \beta_1 \ldots \beta_q} \ . \tag{2.18}$$

In the following we will use the notation $\Phi_{(p,q)} \in \Omega^{p,q}$ for the projection of a superform $\Phi \in \Omega^n$ onto its (p,q) component.

The exterior superderivative d, when written out in this basis will give rise to components of the torsion as it acts on the coframe. Thus, $\mathrm{d}\omega$ can be divided according to four bigradings, $\mathrm{d} : \Omega^{p,q} \longrightarrow \Omega^{p+1,q} + \Omega^{p,q+1} + \Omega^{p-1,q+2} + \Omega^{p+2,q-1}$. Following [55] we split d into its various components with respect to the bigrading,

$$\mathrm{d} = \mathrm{d}_b + \mathrm{d}_f + \tau + t \ , \tag{2.19}$$

where d_b, d_f are even, odd derivatives respectively, such that $\mathrm{d}_b : \Omega^{p,q} \to \Omega^{p+1,q}$, $\mathrm{d}_f : \Omega^{p,q} \to \Omega^{p,q+1}$. The operators τ and t are purely algebraic and can be expressed in terms of the torsion. Explicitly, for any $\omega \in \Omega^{p,q}$ we have,

$$(\mathrm{d}_b\omega)_{a_1 \ldots a_{p+1} \beta_1 \ldots \beta_q} = (p+1) \times \tag{2.20}$$
$$\Big(D_{[a_1} \omega_{a_2 \ldots a_{p+1}] \beta_1 \ldots \beta_q} + \frac{p}{2} T_{[a_1 a_2|}{}^c \ \omega_{c|a_3 \ldots a_{p+1}] \beta_1 \ldots \beta_q} + q\,(-1)^p \ T_{[a_1|(\beta_1|}{}^\gamma \ \omega_{|a_2 \ldots a_{p+1}]\gamma|\beta_2 \ldots \beta_q)} \Big)$$

$$(\mathrm{d}_f\omega)_{a_1 \ldots a_p \beta_1 \ldots \beta_{q+1}} = (q+1) \times \tag{2.21}$$
$$\Big((-1)^p D_{(\beta_1|} \omega_{a_1 \ldots a_p|\beta_2 \ldots \beta_{q+1})} + \frac{q}{2} T_{(\beta_1 \beta_2|}{}^\gamma \ \omega_{a_1 \ldots a_p \gamma|\beta_3 \ldots \beta_{q+1})} + p\,(-1)^p \ T_{(\beta_1|[a_1|}{}^c \ \omega_{c|a_2 \ldots a_p]|\beta_2 \ldots \beta_{q+1})} \Big)$$

$$(\tau\omega)_{a_1 \ldots a_{p-1} \beta_1 \ldots \beta_{q+2}} = \frac{1}{2}(q+1)(q+2) \ T_{(\beta_1 \beta_2|}{}^c \ \omega_{c a_1 \ldots a_{p-1}|\beta_3 \ldots \beta_{q+2})} \tag{2.22}$$

$$(t\omega)_{a_1 \ldots a_{p+2} \beta_1 \ldots \beta_{q-1}} = \frac{1}{2}(p+1)(p+2) \ T_{[a_1 a_2}{}^\gamma \ \omega_{a_3 \ldots a_{p+2}]\gamma \beta_1 \ldots \beta_{q-1}} \ . \tag{2.23}$$

When the exterior derivative is reapplied on the terms above, and the terms gathered according to their bigradings, the nilpotency of the exterior derivative, $\mathrm{d}^2 = 0$, implies the following identities :

$$\begin{aligned}
&(p{-}2,\, q{-}4) \qquad && 0 = \tau^2 \\
&(p{-}1,\, q{+}3) && 0 = \mathrm{d}_f\tau + \tau\mathrm{d}_f \\
&(p,\, q{+}2) && 0 = \mathrm{d}_f^2 + \mathrm{d}_b\tau + \tau\mathrm{d}_b \\
&(p{+}1,\, q{+}1) && 0 = \mathrm{d}_b\mathrm{d}_f + \mathrm{d}_f\mathrm{d}_b + \tau t + t\tau \\
&(p{+}2,\, q) && 0 = \mathrm{d}_b^2 + \mathrm{d}_f t + t\mathrm{d}_f \\
&(p{+}3,\, q{-}1) && 0 = \mathrm{d}_b t + t\mathrm{d}_b \\
&(p{+}4,\, q{-}2) && 0 = t^2 \,.
\end{aligned} \tag{2.24}$$

The first and the last of these equations are algebraic identities and are always satisfied. On the other hand, as a consequence of the splitting of the tangent bundle into even and odd directions, the remaining identities are only satisfied provided the torsion tensor obeys its Bianchi identity.

The first of the equations in (2.24), the nilpotency of the τ operator, implies that we can consider the cohomology of τ, as first noted in [55] (see also [32] for some related concepts). Explicitly we set the (p, q) graded τ-*cohomology* group,

$$H_\tau^{p,q} = \{\omega \in \Omega^{p,q} \,|\, \tau\omega = 0\} \,/\, \left\{\omega \sim \omega + \tau\lambda, \lambda \in \Omega^{p+1,q-2}\right\} \,.$$

As in the case of de Rham cohomology, one could make a distinction between cohomology groups computed on the space of unconstrained superfields and those computed on the space of physical fields.

Suppose now that the rigid constraint (2.10) is imposed so that τ reduces to a gamma matrix. It was conjectured in [43], consistently with the principle of maximal propagation of [56], that in this case the only potentially nontrivial τ-cohomology appears as a result of the so-called M2-brane identity,

$$(\Gamma^a)_{(\alpha_1\alpha_2}(\Gamma_{ab})_{\alpha_3\alpha_4)} = 0 \,. \tag{2.25}$$

Explicitly, for $p = 0, 1, 2$, one may form only the following τ-closed (p, q)-superforms,

$$\begin{aligned}
\omega_{\alpha_1\ldots\alpha_q} &= S_{\alpha_1\ldots\alpha_q} \\
\omega_{a\alpha_1\ldots\alpha_q} &= (\Gamma_{ab})_{(\alpha_1\alpha_2}P^b{}_{\alpha_3\ldots\alpha_q)} \\
\omega_{ab\alpha_1\ldots\alpha_q} &= (\Gamma_{ab})_{(\alpha_1\alpha_2}U_{\alpha_3\ldots\alpha_q)} \,,
\end{aligned} \tag{2.26}$$

with S, P, U, arbitrary non-τ-exact superfields. It can be seen using (2.25) that the forms ω above correspond to nontrivial elements of $H_\tau^{p,q}$ with $p = 0, 1, 2$. The conjecture of [43] means that all nontrivial cohomology is thus generated, and that all $H_\tau^{p,q}$ groups are trivial for $p \geq 3$. This was subsequently proven in [57] and [58–61].

2.2.3 Spinorial cohomology

Following [43], let us now define a spinorial derivative d_s which acts on elements of τ-cohomology, $\mathrm{d}_s : H_\tau^{p,q} \to H_\tau^{p,q+1}$. For any $\omega \in [\omega] \in H_\tau^{p,q}$ we set,

$$\mathrm{d}_s[\omega] := [\mathrm{d}_f \omega] \ . \tag{2.27}$$

To check that this is well-defined, one first shows that $\mathrm{d}_f\omega$ is τ-closed,

$$\tau \mathrm{d}_f \omega = -\mathrm{d}_f \tau \omega = 0 \ ,$$

where we used the second equation in (2.24). Moreover $\mathrm{d}_s[\omega]$ is independent of the choice of representative,

$$[\mathrm{d}_f(\omega + \tau\lambda)] = [\mathrm{d}_f\omega - \tau \mathrm{d}_f \lambda] = [\mathrm{d}_f\omega] \ .$$

Furthermore it is simple to check that $\mathrm{d}_s^2 = 0$,

$$\mathrm{d}_s^2[\omega] = \mathrm{d}_s[\mathrm{d}_f\omega] = [\mathrm{d}_f^2\omega] = -[(\mathrm{d}_b\tau + \tau \mathrm{d}_b)\omega] = 0 \ ,$$

where we took into account the third equation in (2.24). We can therefore define the corresponding spinorial cohomology groups $H_s^{p,q}$ as follows,

$$H_s^{p,q} = \{\omega \in H_\tau^{p,q} \,|\, \mathrm{d}_s\omega = 0\} \,/\, \left\{\omega \sim \omega + \mathrm{d}_s\lambda, \lambda \in H_\tau^{p,q-1}\right\} \ . \tag{2.28}$$

The notion of spinorial cohomology was originally introduced in [56, 62] and applied in a series of papers with the aim of computing higher-order corrections to supersymmetric theories [63–67], and more recently in [68–70]. The spinorial cohomology as presented above was introduced in [43] and is independent of the value of the dimension-zero torsion. It reduces to the spinorial cohomology of [56, 62] upon imposing the rigid constraint (2.10).

2.3 The action principle

The *action principle*, also known as *ectoplasmic integration*, is a way of constructing superinvariants in D spacetime dimensions as integrals of closed D-superforms [71, 72]. Indeed if L is a closed D-superform, the following action is invariant under supersymmetry,

$$S = \frac{1}{D!} \int \mathrm{d}^D x \, \varepsilon^{m_1 \dots m_D} L_{m_1 \dots m_D}| \ , \tag{2.29}$$

where a vertical bar denotes the evaluation of a superfield at $\theta^\mu = 0$. This can be seen as follows. Consider an infinitesimal super-diffeomorphism generated by a super-vector field ξ. The corresponding transformation of the action reads,

$$\delta L = \mathcal{L}_\xi L = (\mathrm{d} i_\xi + i_\xi \mathrm{d})L = \mathrm{d} i_\xi L \ , \tag{2.30}$$

where we took into account that L is closed. On the other hand, local supersymmetry transformations and spacetime diffeomorphisms are generated by $\xi|$ and, in view of (2.30), the integrand in (2.29) transforms as a total derivative under such transformations. The action is thus invariant assuming boundary terms can be neglected.

This method is particularly well-suited to actions with Chern-Simons terms and indeed has been used to construct all Green-Schwarz brane actions [73, 74], see [75, 76] for more recent applications to other theories and [77] for applications to higher-order corrections.

The idea is as follows : let Z_D be the CS term and $W_{D+1} = \mathrm{d}Z_D$ its exterior derivative. Obviously W_{D+1} is a closed form. On the other hand one might be led to conclude that the de Rham cohomology group of rank $D+1$ must be trivial on a supermanifold whose body is D-dimensional, hence W_{D+1} must also be exact. However, since we are dealing with physical fields, we are interested in the Weil-triviality of W_{D+1}, and its exactness has to be checked. If it is indeed Weil-trivial, it can be written as $W_{D+1} = \mathrm{d}K_D$ where now, contrary to Z_D, K_D is a globally-defined (gauge-invariant) superform. It follows that $L_D := Z_D - K_D$ is a closed superform, and can therefore be used to construct a supersymmetric action as in (2.29). To summarize,

$$W_{12} \ := \ \underbrace{\mathrm{d}\,\overbrace{Z_{11}}^{\substack{\text{non-gauge}\\ \text{invariant}\\ \uparrow}} \ = \ \mathrm{d}\,\overbrace{K_{11}}^{\substack{\text{gauge}\\ \text{invariant}\\ \uparrow}}}_{\substack{Z_{11}-K_{11}\\ \text{non-zero and closed}}} \tag{2.31}$$

We shall parameterize the derivative expansion in terms of the Planck length l, so that the CJS two-derivative action corresponds to zeroth order in l. In section (2.3.1), we show that if Z_{11} is taken to be the usual Chern-Simons term of CJS supergravity, applying this method only recovers the $\mathcal{O}(l^0)$ action. For higher-derivative invariant to appear, some sort of modified Chern-Simons term has to be chosen as a starting point. Concerning the five derivative or $\mathcal{O}(l^3)$ case of section (2.3.2), the superinvariant was already computed in [66] and contained a modified Chern-Simons term which can be used as a Z_{11} for our method. In the 8 derivative or $\mathcal{O}(l^6)$ case of section (2.4), as was shown in [36, 78], the requirement that the M5-brane gravitational anomaly is cancelled by inflow from eleven dimensions requires the existence of certain Chern-Simons terms at the eight-derivative order in the eleven-dimensional theory. This is the modified Chern-Simons term Z_{11}, that should lead to the complete $\mathcal{O}(l^6)$ invariant.

As we will see in details in the following, in practice one solves for the flat components of the superform K_D in a stepwise fashion in increasing engineering dimension. Once all flat components of K_D have been determined, L_D follows, and the explicit form of the action (2.29) can be extracted using the formula,

$$
\begin{aligned}
L_{m_1 \ldots m_D}\big| = \;& e_{m_D}^{a_D} \cdots e_{m_1}^{a_1} \, L_{a_1 \ldots a_D}\big| \\
& + D \, e_{m_D}^{a_D} \cdots e_{m_2}^{a_2} \, \psi_{m_1}^{\alpha_1} \, L_{\alpha_1 a_2 \ldots a_D}\big| \\
& \quad \vdots \\
& + D \, e_{m_D}^{a_D} \, \psi_{m_{D-1}}^{\alpha_{D-1}} \cdots \psi_{m_1}^{\alpha_1} \, L_{\alpha_1 \ldots \alpha_{D-1} a_D}\big| \\
& + \psi_{m_D}^{\alpha_D} \cdots \psi_{m_1}^{\alpha_1} \, L_{\alpha_1 \ldots \alpha_D}\big| \, ,
\end{aligned} \tag{2.32}
$$

where $\psi_m^\alpha := E_m{}^\alpha|$ and $e_m{}^a := E_m{}^a|$ are identified as the gravitino and the vielbein of (bosonic) spacetime respectively. In particular the bosonic terms of the Lagrangian can be read off immediately from $L_{a_1 \ldots a_D}$.

2.3.1 CJS supergravity in the action principle formulation

As a reminder, the eleven-dimensional supergravity action reads,

$$
S = \int \big(R \star 1 - \frac{1}{2} \, G_4 \wedge G_7 - \frac{1}{6} \, C_3 \wedge G_4 \wedge G_4 \big)\big| \ , \tag{2.33}
$$

where the CS terms is in last position. From the point of view of the action principle we have,

$$
Z_{11} = -\frac{1}{6} C_3 \wedge G_4 \wedge G_4 \ , \qquad W_{12} := \mathrm{d}Z_{11} = -\frac{1}{6} G_4 \wedge G_4 \wedge G_4 \ .
$$

As mentioned before, a first consistency test would be to verify that the action principle applied on this CS term actually leads back to the regular action. Most of the computation is included in appendix (A.1), but the first few steps are sketched below. According to (2.31), K_{11} defined as

$$
\mathrm{d}K_{11} = \mathrm{d}Z_{11} = -\frac{1}{6} \, G_4 \wedge G_4 \wedge G_4 \ , \tag{2.34}
$$

should lead to the expected $\mathcal{O}(l^0)$ Lagrangian. This equation can be developed on the vielbein basis, and gives the equation in superspace components,

$$
D_{[A_1} K_{A_2 \ldots A_{12})} + \frac{11}{2} \, T_{[A_1 A_2|}{}^F \, K_{F|A_3 \ldots A_{12})} = -\frac{11!}{6(4!)^3} \, G_{[A_1 \ldots A_4} \, G_{A_5 \ldots A_8} \, G_{A_9 \ldots A_{12})} \ , \tag{2.35}
$$

where all 12 capital indices can either belong to the odd or even part of the basis. Thus, it can be split in several expressions graded according to their number of even/odd indices (and similarly, according to their mass dimension). We then have a total of thirteen equations, ranging from $(12, 0)$ to $(0, 12)$ odd/even indices. Since building the Lagrangian only requires the top bosonic component $K_{a_1 \ldots a_{11}}$ of the superform K, it is tempting to look to directly at the last equation, namely [5]

$$D_{a_1} K^{(2)}_{a_2\ldots a_{12}} + \frac{11}{2} T_{a_1 a_2}{}^{\gamma}\ K^{(3/2)}_{\gamma a_3 \ldots a_{12}} = -\frac{11!}{6(4!)^3}\ G_{[a_1\ldots a_4}\ G_{a_5\ldots a_8}\ G_{a_9\ldots a_{12})}\ ,$$

where the desired top term $K^{(2)}$ appears only once. However, it happens to be accompanied by a term $K^{(3/2)}$ which must be determined by the previous equation, itself containing the term $K^{(1)}$, etc. The 13 equations are nested in such a way that they can only be solved from bottom to top, i.e. from $K^{(-7/2)}$ to $K^{(2)}$. Let's examine the first few equations, from mass dimension –3/2 to –1/2 :

$$\begin{array}{c|l} \dim -\frac{3}{2} & 0 = D_{\alpha_1} K^{(-7/2)}_{\alpha_2\ldots\alpha_{12}} + \frac{11}{2} T_{\alpha_1\alpha_2}{}^{f}\ K^{(-3)}_{f\alpha_3\ldots\alpha_{12}} \\ \vdots & \qquad\qquad \vdots \\ \dim -\frac{1}{2} & 0 = \frac{7}{12} D_{\alpha_1} K^{(-1)}_{\alpha_2\ldots\alpha_7 a_1\ldots a_5} - \frac{5}{12} D_{a_1} K^{(-3/2)}_{a_2\ldots a_5\alpha_1\ldots\alpha_7} + \frac{11}{2}\left(\frac{5}{33} T_{\alpha_1\alpha_2}{}^{f} K^{(-1/2)}_{f\alpha_3\ldots\alpha_7 a_1\ldots a_5} + \ldots\right) . \end{array} \tag{2.36}$$

For all of them, the G^4 part cancels, and one has to find the right values for the K's of dimension –7/2 to –1, such that the equations are verified. The mass dimensions of the physical fields which can be involved in the construction of K_{11} are,

$$\begin{array}{lll} [D_{a_1}] = 1 & [T_{a_1 a_2}{}^{\alpha}] = 3/2 & [G_{ab\alpha\beta}] = [T_{\alpha\beta}{}^{a}] = 0 \\ [D_{\alpha_1}] = 1/2 & [T_{a\alpha}{}^{\beta}] = 1 & [G_{abcd}] \ = 1 \end{array}$$

However, since the dimensions of the first K are negative, none of them can be expressed as a combination of fields or gamma matrices, and must be set to zero. The equation in (2.36) are trivially satisfied.

The following equation, of dimension 0, has a non-zero rhs and involves $K^{(0)}$ to $K^{(-2)}$,

$$\begin{aligned} &\frac{1}{2}\ D_{\alpha_1} \overbrace{K_{\alpha_2\ldots\alpha_6 a_1\ldots a_6}}^{0} + \frac{1}{2} D_{a_1} \overbrace{K_{a_2\ldots a_6\alpha_1\ldots\alpha_6}}^{0} + \\ &\frac{11}{2}\left(\frac{5}{22}\ T_{\alpha_1\alpha_2}{}^{f}\ K_{f\alpha_3\ldots\alpha_6 a_1\ldots a_6} + \frac{5}{22}\ T_{a_1 a_2}{}^{\gamma}\ \underbrace{K_{\gamma a_3\ldots a_6\alpha_1\ldots\alpha_6}}_{0} + \frac{12}{22}\ T_{a_1\alpha_1}{}^{\gamma}\ \underbrace{K_{\gamma\alpha_2\ldots\alpha_6 a_2\ldots a_6}}_{0}\right) \\ &= -\frac{11!}{6(4!)^3}\ \frac{18}{77}\ G_{a_1a_2\alpha_1\alpha_2} G_{a_3a_4\alpha_3\alpha_4} G_{a_5a_6\alpha_5\alpha_6}\ . \end{aligned}$$

Since all the previous K's are zero, it takes the simple form,

$$(\Gamma^f)_{\alpha_1\alpha_2} K_{f a_1\ldots a_6\alpha_3\ldots\alpha 6} = 90\ (\Gamma_{a_1a_2})_{\alpha_1\alpha_2} (\Gamma_{a_3a_4})_{\alpha_3\alpha_4} (\Gamma_{a_5a_6})_{\alpha_5\alpha_6}\ ,$$

where the right hand side comes from the value of $G_{\alpha\beta ab}$ that was computed in the superspace formulation of CJS supergravity, in section (2.1.2). Using the M2-brane identity as well as the

5. The superscript in parenthesis over K indicates the mass dimension of the component. This should not be confused with one of the other superscripts used elsewhere.

so-called M5-brane identity,

$$(\Gamma^e)_{\alpha_1\alpha_2}(\Gamma_{ea_1\ldots a_4})_{\alpha_3\alpha_4} = 3\ (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}(\Gamma_{a_3a_4})_{\alpha_3\alpha_4}\ ,$$

it is easy to check that the solution is given by,

$$K_{a_1\ldots a_7\alpha_1\ldots\alpha_4} = 42\ (\Gamma_{a_1\ldots a_5})_{\alpha_1\alpha_2}(\Gamma_{a_6a_7})_{\alpha_3\alpha_4}\ , \tag{2.37}$$

plus any τ-exact term, of the form $K_{a_1\ldots a_7\alpha_1\ldots\alpha_4} = \beta(\Gamma_f)_{\alpha_1\alpha_2}S_{a_2\ldots a_7\alpha_3\alpha_4}$, where S can be anything of the right dimension. However, these sort of solutions, besides having the potential to be quite large, come with a numerical coefficient that cannot be constrained, and most of all, lead to exact terms in K_{11} (that have no effect on the superinvariant).

The expression (2.37) is the first non-zero component of K, with mass dimension 0. It will be re-injected in the next equation, on dimension $1/2$, and so on. The remaining steps are included in the appendix (A.1). The top component $K^{(2)}$ first appears in the equation of dimension 2, and must be

$$K_{a_1\ldots a_{11}} = \frac{1}{72}\ \epsilon_{a_1\ldots a_{11}}G_{d_1\ldots d_4}G^{d_1\ldots d_4}\ . \tag{2.38}$$

The equation of dimension $5/2$ does not bring any new component of K, but serves as a consistency check for the expressions we found for the last few K's. At this point, we have constructed the explicit expression of the 12 components of K_{11} and have seen that they are unique up to τ-exact terms. In particular, its purely bosonic component takes the following form,

$$K^{(2)} = \frac{1}{11!}e^{a_{11}}{\scriptstyle\wedge}\ldots{\scriptstyle\wedge}\, e^{a_1}\left(\frac{1}{72}\ \epsilon_{a_1\ldots a_{11}}G_{d_1\ldots d_4}\ G^{d_1\ldots d_4}\right) = \frac{1}{3}\ G{\scriptstyle\wedge}G_7\ . \tag{2.39}$$

The result, however, does not yet resemble the CJS action (2.33) : taking $L_{11} = Z_{11} - K_{11}$ we get,

$$S = \int\ \big(-\frac{1}{3}G_4{\scriptstyle\wedge}G_7 - \frac{1}{6}\ C_3{\scriptstyle\wedge}G_4{\scriptstyle\wedge}G_4\big)\Big|\ .$$

Actually, we have to use the equation of motion of CJS supergravity in order to retrieve the correct form of the action. It may seem paradoxical to use the equation of motion of an action that is not yet established, but this section is a consistency test, not a true derivation of the CJS action. The volume element is defined as,

$$\mathrm{d}V = \star 1 = -\frac{1}{11!}\ \epsilon_{a_1\ldots a_{11}}\ e^{a_1}\ldots e^{a_{11}}\ ,$$

and by taking the trace of the third relation of the Einstein equation (B.4), it follows,

$$R\star 1 = \frac{1}{144}\ G_{d_1\ldots d_4}G^{d_1\ldots d_4}\ \mathrm{d}V = \frac{1}{6}\ G{\scriptstyle\wedge}\star G\ .$$

$K^{(2)}$ can thus be split in two part : $-R\star 1 + \frac{1}{2}\ G{\scriptstyle\wedge}\star G$, and taking $Z_{11} - K_{11}$ lead to the expected action.

2.3.2 The $\mathcal{O}(l^3)$ correction (five derivatives)

It was shown in [66], by directly computing the relevant spinorial cohomology group, that there is a unique superinvariant at the five derivative level (order l^3 in the Planck length). The modified eleven-dimensional action to order l^3 reads,

$$S = \int \left(R \star 1 - \frac{1}{2}\, G_4{\wedge} \star G_4 - \frac{1}{6} C_3{\wedge}G_4{\wedge}G_4 + l^3 \big(C_3{\wedge}G_4{\wedge}\mathrm{tr}R^2 + 2\mathrm{tr}R^2{\wedge} \star G_4 \big) \right)\Big| \ , \quad (2.40)$$

where an arbitrary numerical coefficient has been absorbed in the definition of l and $\mathrm{tr}R^2 := R_a{}^b{\wedge}R_b{}^a$; it is understood that only the bosonic $(11,0)$ components of the forms enter the formula above. This action can also be easily understood from the point of view of the action principle as follows. Consider the twelve-form corresponding to the CS term at order l^3,

$$Z_{11} = C_3{\wedge}G_4{\wedge}\mathrm{tr}R^2\ , \qquad W_{12} = \mathrm{d}Z_{11} = G_4{\wedge}G_4{\wedge}\mathrm{tr}R^2\ .$$

Applying the action principle in this non-trivial case requires a solution K to the equation,

$$\mathrm{d}K_{11} = G_4{\wedge}G_4{\wedge}R^{ab}{\wedge}R_{ba}\ . \quad (2.41)$$

or, in components,

$$D_{[A_1}K_{A_2\ldots A_{12})} + \frac{11}{2}\, T_{[A_1A_2|}{}^F\, K_{F|A_3\ldots A_{12})} = \frac{11!}{4(4!)^2}\, R_{[A_1A_2|c_1c_2}\, R_{|A_3A_4|}{}^{c_2c_1}\, G_{|A_5\ldots A_8}\, G_{A_9\ldots A_{12})}$$

The dimensions of the physical fields are the same as before, with the addition of $[R_{abcd}] = 2$. The dimensions of the various components of K range from $-1/2$ ($K_{\alpha_1\ldots\alpha_{11}}$) to 5 ($K_{a_1\ldots a_{11}}$). Just like in the previous case, the first few Ks vanish for dimensional (or group theoretical) reasons. However, the remaining steps are far less direct, and are only described in appendix (A.2). For the sake of having at least one example in the main text, we shall skip to the second-to-last equation, of dimension 5, leading to the expression of the top component of K,

$$\begin{aligned}
&\frac{2}{12}\, D_{\alpha_1}K_{\alpha_2a_1\ldots a_9} \;+\; \frac{10}{12}\, D_{a_1}K_{a_2\ldots a_{10}\alpha_1\alpha_2}\\
&+\frac{11}{2}\left(\frac{1}{66}\, T_{\alpha_1\alpha_2}{}^f\, K_{fa_1\ldots a_{10}} \;+\; \frac{15}{22}\, T_{a_1a_2}{}^\gamma\, K_{\gamma\alpha_1\alpha_2a_3\ldots a_{10}} \;+\; \frac{10}{33}\, T_{a_1\alpha_1}{}^\gamma\, K_{\gamma\alpha_2a_2\ldots a_{10}}\right)\\
&= \frac{11!}{4(4!)^2}\Bigg(2\,\frac{1}{66}\, R_{\alpha_1\alpha_2}{}^{c_1c_2}\, R_{a_1a_2c_2c_1}\, G_{a_3\ldots a_6}\, G_{a_7\ldots a_{10}}\\
&\qquad\qquad - 1\,\frac{2}{33}\, R_{\alpha_1a_1}{}^{c_1c_2}\, R_{\alpha_2a_2c_2c_1}\, G_{a_3a_4a_5a_6}\, G_{a_7\ldots a_{10}}\\
&\qquad\qquad\qquad + 2\,\frac{1}{11}\, R_{a_1a_2}{}^{c_1c_2}\, R_{a_3a_4c_2c_1}\, G_{a_5a_6\alpha_1\alpha_2}\, G_{a_7a_8\alpha_3\alpha_4}\Bigg)\ .
\end{aligned}$$

It involves several of the previous Ks, and requires a large amount of correspondences between different groups of terms to be consistent. Using the equations of motion (B.1.2), a lot of gamma matrix algebra, and the BI (2.9), (2.11) it is possible to find a solution in components,

$$K^{(5)}_{a_1\dots a_{11}} = -165\, \epsilon_{a_1\dots a_7}{}^{b_1\dots b_4}\, G_{b_1\dots b_4}\, R_{a_8 a_9}{}^{c_1 c_2}\, R_{a_{10}a_{11}c_2 c_1}$$

or,

$$K_{11}\big| = \big(-2\, G_7{\wedge}\mathrm{tr}R^2\big)\big| \,.$$

Taking $L_{11} = Z_{11} - K_{11}$ we obtain the following superinvariant at order l^3,

$$\Delta S = \int \big(C_3{\wedge}G_4{\wedge}\mathrm{tr}R^2 + 2\, G_7{\wedge}\mathrm{tr}R^2\big)\big| \ ,$$

which precisely correspond to the order-l^3 corrective terms in (2.40). Contrary to the previous case, this is a non-trivial application of the action principle, and it leads to a correct superinvariant.

As explained in [66], on a topologically trivial spacetime manifold this superinvariant can be removed by an appropriate field redefinition of the 3-form superpotential. However on a spacetime with non-vanishing first Pontryagin class, the superinvariant cannot be redefined away. Since we placed ourselves in a trivial space, the correction at this order is not physically relevant, and only constitutes a proof of concept.

2.4 The $\mathcal{O}(l^6)$ correction (eight derivatives)

As was shown in [36, 78], the requirement that the M5-brane gravitational anomaly is cancelled by inflow from eleven dimensions implies the existence of certain CS terms Z_{11} at the eight-derivative order in the eleven-dimensional theory. The corresponding twelve-form reads,

$$Z_{11} = l^6 C_3{\wedge}X_8 \,, \qquad W_{12} = \mathrm{d}Z_{11} = l^6 G_4{\wedge}X_8 \,, \tag{2.42}$$

where X_8 is related to the M5-brane anomaly polynomial by descent,

$$X_8 = \mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2 \,, \tag{2.43}$$

and we have set $(\mathrm{tr}R^2)^2 := \mathrm{tr}R^2{\wedge}\mathrm{tr}R^2$ and $\mathrm{tr}R^4 := R_a{}^b{\wedge}R_b{}^c{\wedge}R_c{}^d{\wedge}R_d{}^a$. This Chern-Simons term can also be viewed as a $\mathcal{O}(l^6)$ modification of the four- and seven-form Bianchi identities :

$$\mathrm{d}G_4 = 0 \,, \qquad \mathrm{d}G_7 + \frac{1}{2}G_4{\wedge}G_4 = l^6 X_8 \,, \tag{2.44}$$

where a numerical coefficient has been absorbed in the definition of l. We expand the forms perturbatively in l,

$$G_4 = G_4^{(0)} + l^6 G_4^{(1)} + \cdots \,, \qquad G_7 = G_7^{(0)} + l^6 G_7^{(1)} + \cdots \,, \tag{2.45}$$

and similarly for the supercurvature $R_A{}^B$. Note that in the expansion above the bosonic components of the lowest-order fields, $G^{(0)}_{m_1\dots m_4}$ etc, are identified with the fieldstrengths of the supergravity multiplet, while the higher-order fields $G^{(1)}_4$ etc, are composite higher-derivative fields which are polynomial in the fieldstrengths of the supergravity fields. Solving perturbatively the BI at each order in l, taking into account that the exterior superderivative $\mathrm{d} = \mathrm{d}z^M \partial_M$ is zeroth-order in l, implies,

$$\begin{aligned} \mathrm{d}G^{(0)}_4 &= 0\,, & \mathrm{d}G^{(0)}_7 + \tfrac{1}{2}G^{(0)}_4{\wedge}\, G^{(0)}_4 &= 0\,, \\ \mathrm{d}G^{(1)}_4 &= 0\,, & \mathrm{d}G^{(1)}_7 \;+\; G^{(1)}_4{\wedge}\, G^{(0)}_4 &= X^{(1)}_8\,, \end{aligned} \tag{2.46}$$

where we have set $l^6 X_8 = l^6 X^{(1)}_8 + \cdots$. Note that $X^{(1)}_8$ only involves the lowest-order curvature $R^{(0)}$. Let us expand the twelve-form W_{12} perturbatively in l, $W_{12} = l^6\, W^{(1)}_{12} + \cdots$, so that,

$$W^{(1)}_{12} = X^{(1)}_8{\wedge}\, G^{(0)}_4 = \mathrm{d}Z_{11}\,, \qquad Z_{11} = X^{(1)}_8{\wedge}\, C^{(0)}_3\,. \tag{2.47}$$

It then follows from (2.46) that this can also be written in a manifestly Weil-trivial form as follows,

$$W^{(1)}_{12} = \mathrm{d}K_{11}\,, \qquad K_{11} = G^{(1)}_7{\wedge}\, G^{(0)}_4 - 2G^{(0)}_7{\wedge}\, G^{(1)}_4\,. \tag{2.48}$$

In particular we see that it suffices to solve the four- and seven-form BI in order to determine the order-l^6 superinvariant corresponding to $L_{11} = Z_{11} - K_{11}$,

$$\Delta S = l^6 \int \Big(X^{(1)}_8{\wedge}\, C^{(0)}_3 - G^{(1)}_7{\wedge}\, G^{(0)}_4 + 2G^{(0)}_7{\wedge}\, G^{(1)}_4 \Big)\Big|\,, \tag{2.49}$$

where it is understood that only the bosonic $(11,0)$ components of the forms enter. This is the superinvariant corresponding to the supersymmetrization of the CS term (2.42). The action would then read to this order,

$$\begin{aligned} S = \int \Big(& R^{(0)} \star 1 - \frac{1}{2}\, G^{(0)}_4{\wedge} \star G^{(0)}_4 - \frac{1}{6}\, C^{(0)}_3{\wedge}\, G^{(0)}_4{\wedge}\, G^{(0)}_4 \\ & + l^6 \big(X^{(1)}_8{\wedge}\, C^{(0)}_3 - G^{(1)}_7{\wedge}\, G^{(0)}_4 + 2G^{(0)}_7{\wedge}\, G^{(1)}_4 \big) \Big)\Big|\,, \end{aligned} \tag{2.50}$$

where $R^{(0)}$, $G^{(0)}$ are identified with the fieldstrengths of the physical fields in the supergravity multiplet, while the first-order fields $R^{(1)}$, $G^{(1)}$ should be thought of as gauge-invariant functions of the physical fields.

To conclude this part, there are two paths that lead to the superinvariant : one can either solve $\mathrm{d}K_{11} = G_4{\wedge}\, X_8$ directly, and find a Lagrangian as in the first two cases, or solve simultaneously the two BIs for G_4 and G_7, including the X_8 term. As we will see, for the first few terms, those two approaches are equivalent, and lead to the exact same difficulty.

We see that the action above is in agreement with the expectation that the bosonic part of the derivative-corrected supergravity action should be of the form,

$$S = \int \Big(R \star 1 - \frac{1}{2}\, G_4{\wedge} \star G_4 - \frac{1}{6} C_3{\wedge} G_4{\wedge} G_4 + l^6 \big(X_8{\wedge} C_3 + \Delta L \star 1 \big) \Big) \,, \tag{2.51}$$

with ΔL a function of R, G and their derivatives. Since ΔL is gauge invariant, we see in particular that the CS terms do not receive higher-order corrections beyond eight derivatives : varying (2.51) with respect to C_3 implies,

$$\mathrm{d} \star G_4 + \frac{1}{2} G_4{\wedge} G_4 = X_8 + \frac{\delta}{\delta C_3} (\Delta L \star 1) \,. \tag{2.52}$$

It is straightforward to see that the second term on the right-hand side above is exact by virtue of the fact that ΔL is gauge invariant and thus only depends on C_3 through G_4. Indeed the variation of the C_3 dependent terms in the ΔL part of the action (2.51) can be written (possibly up to integration by parts) in the form $\int \Phi_7{\wedge} \mathrm{d}\delta C_3$, for some seven-form Φ_7. Therefore by appropriately correcting the lowest-order duality relation by higher-derivative terms, $G_7 = \star G_4 + \mathcal{O}(l^6)$, one arrives at the modified BI (2.44).

2.4.1 Argument for the existence of $\mathcal{O}(l^6)$ invariant

Let's start by trying to solve the seven-form BI at $\mathcal{O}(l^6)$. Since the first non-zero component of $G_4^{(0)}$ is $G_{\alpha\beta ab}$, the term $G_4^{(1)} {\wedge} G_4^{(0)}$ drops out of the fully fermionic equation in components,

$$D_{(\alpha_1|}\, G^{(1)}_{|\alpha_2 \dots \alpha_8)} - \frac{7}{2}\, i\, (\Gamma^f)_{(\alpha_1 \alpha_2|}\, G^{(1)}_{f|\alpha_3 \dots \alpha_8)} \;=\; \big(X_8^{(1)}\big)_{\alpha_1 \cdots \alpha_8} \,.$$

Based on what is known about superinvariants in $D < 11$ dimensions [79], it is plausible to assume that the superinvariant (2.49) corresponding to the supersymmetrization of the CS term (2.42) should be quartic or higher in the fields. As pointed out in [43], a necessary condition for the superinvariant to be quartic is that the order-l^6 sevenform should be quartic or higher in the fields. Since $G^{(1)}_{0,7}$ cannot be quartic or higher in the fields, as can be seen by dimensional analysis, the order-l^6 seven-form BI (2.46) must be solved for $G^{(1)}_{0,7} = 0$. Then, the equation above simplifies to,

$$(\Gamma^f)_{(\alpha_1 \alpha_2|}\, G^{(1)}_{f|\alpha_3 \dots \alpha_8)} \;=\; \big(X_8^{(1)}\big)_{\alpha_1 \cdots \alpha_8} \,. \tag{2.53}$$

It then follows, for consistency, that the purely spinorial component of the M5-brane anomaly 8-form $X_{0,8}$ must be τ-exact.

If, instead of solving the G_7 BI, we apply the action principle as in the first to case of (2.3.2), the equation for K_{11} in components reads,

$$\begin{aligned} &D_{[A_1}\ K_{A_2\ldots A_{12})} + \frac{11}{2}\ T_{[A_1A_2|}{}^{F}\ K_{F|A_3\ldots A_{12})} = \\ &\quad \frac{11!}{(4!)4^2}\Big(G_{[A_1\ldots A_4|}\ R_{|A_5A_6|c_1c_2}\ R_{|A_7A_8|}{}^{c_2c_3}\ R_{|A_9A_{10}|c_3c_4}\ R_{|A_{11}A_{12})}{}^{c_4c_1} \\ &\qquad\qquad - \frac{1}{4}\ G_{[A_1\ldots A_4|}\ R_{|A_5A_6|c_1c_2}\ R_{|A_7A_8|}{}^{c_1c_2}\ R_{|A_9A_{10}|d_1d_2}\ R_{|A_{11}A_{12})}{}^{d_1d_2}\Big)\ , \end{aligned} \tag{2.54}$$

Where the mass dimension of the various components of K_{11} involved range from $[K_{\alpha_1\ldots\alpha_{12}}] = \frac{5}{2}$ to $[K_{a_1\ldots a_{11}}] = 8$ (and the dimension of the other fields are unchanged). As usual, let's proceed in increasing dimension.

Dimensions 3 **and** $7/2$ **—** If we assume that the superinvariant at $\mathcal{O}(l^6)$ is quartic or higher in fields, the first potentially non-vanishing component of K_{11} appears at dimension 4 (it is of the form G^4), thus forcing :

$$K^{(5/2)}_{\alpha_1\ldots\alpha_{11}} = K^{(3)}_{a_1\alpha_1\ldots\alpha_{10}} = K^{(7/2)}_{a_1a_2\alpha_1\ldots\alpha_9} = 0\ ,$$

which make the equations of dimension 2, $5/2$ and 3 trivially satisfied (since the rhs also vanishes for the three first equations).

Dimension 4 **—** At the next dimension, after simplification eq. (2.54) takes the form,

$$\begin{aligned} (\Gamma^f)_{\alpha_1\alpha_2}\ K^{(4)}_{fa_1a_2\alpha_3\ldots\alpha_{10}} &= 2520\,(\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\Big(R_{\alpha_3\alpha_4}{}^{c_1c_2}\ R_{\alpha_5\alpha_6c_2c_3}\ R_{\alpha_7\alpha_8}{}^{c_3c_4}\ R_{\alpha_9\alpha_{10}c_4c_1} \\ &\qquad - \frac{1}{4}\ R_{\alpha_3\alpha_4}{}^{c_1c_2}\ R_{\alpha_5\alpha_6c_1c_2}\ R_{\alpha_7\alpha_8}{}^{d_1d_2}\ R_{\alpha_9\alpha_{10}d_1d_2}\Big) \\ &= (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\ X^{(8)}_{\alpha_3\ldots\alpha_{10}}. \end{aligned} \tag{2.55}$$

Explicitly, the term $(\mathrm{tr}R^2)^2$ reads (omitting the factor – $1/4$),

$$\begin{aligned} &\tfrac{1}{6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{u_4u_5})(\Gamma^{u_6u_7})\ \ G_{u_0u_1}{}^{y_0y_1}G_{u_2u_3y_0y_1}G_{u_4u_5z_0z_1}G_{u_6u_7}{}^{z_0z_1} \\ &\tfrac{4}{24\cdot 6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{u_4u_5})(\Gamma^{v_0\ldots v_5})\ \ G_{u_0u_1}{}^{y_0y_1}G_{u_2u_3y_0y_1}G_{u_4u_5v_0v_1}G_{v_2\ldots v_5} \\ &\tfrac{2}{24^2\,6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{v_0\ldots v_3x_0x_1})(\Gamma^{w_0\ldots w_3}{}_{x_0x_1})\ \ G_{u_0u_1}{}^{y_1y_2}G_{u_2u_3y_1y_2}G_{v_0\ldots v_3}G_{w_0\ldots w_3} \\ &\tfrac{4}{24^2\,6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{v_0\ldots v_5})(\Gamma^{w_0\ldots w_5})\ \ G_{u_0u_1v_0v_1}G_{u_2u_3w_0w_1}G_{v_2\ldots v_5}G_{w_2\ldots w_5} \\ &\tfrac{4}{24^3\,6^4}\ (\Gamma^{u_0u_1})(\Gamma^{v_0\ldots v_5})(\Gamma^{w_0\ldots w_3y_0y_1})(\Gamma^{x_0\ldots x_3}{}_{y_0y_1})\ \ G_{u_0u_1v_0v_1}G_{v_2\ldots v_5}G_{w_0\ldots w_3}G_{x_0\ldots x_3} \\ &\tfrac{1}{24^4\,6^4}\ (\Gamma^{u_0\ldots u_3y_0y_1})(\Gamma^{v_0\ldots v_3}{}_{y_0y_1})(\Gamma^{w_0\ldots w_3z_0z_1})(\Gamma_{x_0\ldots x_3\,z_0z_1})\ \ G_{u_0\ldots u_3}G_{v_0\ldots v_3}G_{w_0\ldots w_3}G_{x_0\ldots x_3}\ , \end{aligned}$$

while the term $\mathrm{tr}R^4$ reads,

$$\begin{aligned}
&\tfrac{1}{6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{u_4u_5})(\Gamma^{u_6u_7})\ \ G_{u_0u_1}{}^{y_0y_1}G_{u_2u_3y_0z_0}G_{u_4u_5y_1z_1}G_{u_6u_7}{}^{z_0z_1}\\
&\tfrac{4}{24\cdot 6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{u_4u_5})(\Gamma^{v_0\dots v_5})\ \ G_{u_0u_1}{}^{y_0y_1}G_{u_2u_3v_0y_0}G_{u_4u_5v_1y_1}G_{v_2\dots v_5}\\
&\tfrac{2}{24^2\ 6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{v_0\dots v_5})(\Gamma^{w_0\dots w_5})\ \ G_{u_0u_1v_0w_0}G_{u_2u_3v_1w_1}G_{v_2\dots v_5}G_{w_2\dots w_5}\\
&\tfrac{4}{24^2\ 6^4}\ (\Gamma^{u_0u_1})(\Gamma^{u_2u_3})(\Gamma^{v_0\dots v_4x_0})(\Gamma^{w_0\dots w_4}{}_{x_0})\ \ G_{u_0u_1v_0}{}^{y_0}G_{u_2u_3w_0y_0}G_{v_1\dots v_4}G_{w_1\dots w_4}\\
&\tfrac{4}{24^3\ 6^4}\ (\Gamma^{u_0u_1})(\Gamma^{v_0\dots v_4y_0})(\Gamma^{w_0\dots w_4y_1})(\Gamma^{x_0\dots x_3}{}_{y_0y_1})\ \ G_{u_0u_1v_0w_0}G_{x_0\dots x_3}G_{v_1\dots v_4}G_{w_1\dots w_4}\\
&\tfrac{1}{24^4\ 6^4}\ (\Gamma^{u_0\dots u_3}{}_{y_0y_1})(\Gamma^{v_0\dots v_3y_0}{}_{z_0})(\Gamma^{w_0\dots w_3y_1}{}_{z_1})(\Gamma_{x_0\dots x_3}{}^{z_0z_1})\ \ G_{u_0\dots u_3}G_{v_0\dots v_3}G_{w_0\dots w_3}G_{x_0\dots x_3}\,.
\end{aligned}$$

If the purely fermionic component of X_8 can be cast in the τ-exact form of eq. (2.53), this equation becomes,

$$\begin{aligned}
(\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\ X^{(8)}_{\alpha_3\dots\alpha_{10}} &= (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\ (\Gamma^f)_{\alpha_3\alpha_4}\ G_{f\alpha_5\dots\alpha_{10}}\\
&= (\Gamma^f)_{\alpha_1\alpha_2}\Big(3\ (\Gamma_{[a_1a_2|})_{\alpha_3\alpha_4}\ G_{|f]\alpha_5\dots\alpha_{10}} - 2\ (\Gamma_{fa_1})_{\alpha_3\alpha_4}\ G_{a_2\alpha_5\dots\alpha_{10}}\Big)\\
&= (\Gamma^f)_{\alpha_1\alpha_2}\ \Big(3\ (\Gamma_{[a_1a_2|})_{\alpha_3\alpha_4}\ G_{|f]\alpha_5\dots\alpha_{10}}\Big)\,,
\end{aligned}$$

which yields,

$$K_{fa_1a_2\alpha_3\dots\alpha_{10}} = \ 3\ (\Gamma_{[a_1a_2|})_{\alpha_3\alpha_4}\ G_{|f]\alpha_5\dots\alpha_{10}}\,.$$

Both methods previously evoked lead to the same requirement : the fermionic X_8 must be τ-exact.

τ-exactness of X_8 — Since (2.55) contains many different types of terms, it is useful to reduce this expression by simplifying every pair of gamma matrices whose bosonic indices contain contractions, using the decompositions in appendix B.1. Applying those Fierzings several times (until the whole expression reaches a fixed point), and using the $\Gamma^{(5)}/\Gamma^{(6)}$ hodge duality, we arrive at an expression with terms of the form $\Gamma^{(2)}\Gamma^{(2)}\Gamma^{(2)}\Gamma^{(2)}$, $\Gamma^{(2)}\Gamma^{(2)}\Gamma^{(2)}\Gamma^{(6)}$ or $\Gamma^{(2)}\Gamma^{(2)}\Gamma^{(5)}\Gamma^{(5)}$, contracted with G^4 (without any contractions among gamma matrices),

$$(\Gamma^{a_1a_2})(\Gamma^{a_3a_4})(\Gamma^{a_5a_6})(\Gamma^{a_7a_8})\ G^4_{a_1a_2;a_3a_4;a_5a_6;a_7a_8} \tag{2.56}$$
$$(\Gamma^{a_1a_2})(\Gamma^{a_3a_4})(\Gamma^{a_5a_6})(\Gamma^{a_7\dots a_{12}})\ G^4_{a_1a_2;a_3a_4;a_5a_6;a_7\dots a_{12}} \tag{2.57}$$
$$(\Gamma^{a_1a_2})(\Gamma^{a_3a_4})(\Gamma^{a_5\dots a_9})(\Gamma^{a_{10}\dots a_{14}})\ G^4_{a_1a_2;a_3a_4;a_5\dots a_9;a_{10}\dots a_{14}}\,, \tag{2.58}$$

and we have suppressed spinorial indices for simplicity of notation. In the above, $G^4_{a_1a_2;\dots;a_7a_8}$, $G^4_{a_1a_2;\dots;a_7\dots a_{12}}$, $G^4_{a_1a_2;\dots;a_{10}\dots a_{14}}$, denote certain sums of G^4 terms with 8,4,2 indices contracted respectively. More explicitly,

$$\begin{aligned}
G^4_{a_1a_2;a_3a_4;a_5a_6;a_7a_8} &= \frac{7}{2^7 3^3}\, G_{a_1}{}^{efg} G_{a_2a_7a_8e} G_{a_3a_5a_6}{}^{h} G_{a_4fgh} + \cdots \\
G^4_{a_1a_2;a_3a_4;a_5a_6;a_7\ldots a_{12}} &= \frac{25}{2^9 3^4}\, G_{a_1a_7}{}^{fg} G_{a_2a_{11}a_{12}f} G_{a_3a_4a_9a_{10}} G_{a_5a_6a_8g} + \cdots \\
G^4_{a_1a_2;a_3a_4;a_5\ldots a_9;a_{10}\ldots a_{14}} &= \frac{1}{2^{11} 3^6}\, G_{a_1a_2a_{10}}{}^{f} G_{a_3a_5a_{11}f} G_{a_4a_{12}a_{13}a_{14}} G_{a_6a_7a_8a_9} + \cdots ,
\end{aligned}$$

where the ellipses stand for more than a hundred terms of this form (and summed indices are shaded). No obvious cancellations appear between these three types of terms, at this point. Let us further analyze how $X_{0,8}$ is decomposed into irreducible components. First, the product of four gamma matrices contains a symmetric product of eight spinor indices which can be decomposed as follows, in irreps of B_5,

$$(00001)^{\otimes_S 8} = \underbrace{1(00000) \oplus \cdots \oplus 1(40000)}_{\text{45 terms with multiplicity 1}} \oplus \underbrace{2(00004) \oplus 2(10002) \oplus 2(01002)}_{\text{3 terms with multiplicity 2}} . \qquad (2.59)$$

Each irrep on the right-hand side above corresponds to a gamma-structure which can be thought of as a Clebsch-Gordan coefficient form the product $(00001)^{\otimes_S 8}$ to all the particular irreps. For example, the gamma-structure corresponding to (00002) is $(\Gamma^{e_1})(\Gamma_{e_1})(\Gamma^{e_2})(\Gamma_{e_2a_1\ldots a_5})$, since it allows, roughly speaking, to make a 5-form out of 8 symmetrized spinor indices.

Next, the product of four four-forms G can be decomposed as follows in irreps of B_5,

$$(00010)^{\otimes_S 4} = \underbrace{4(00000) \oplus \cdots \oplus 6(00004) \oplus \cdots \oplus 3(40000)}_{\text{95 terms, various multiplicites}} , \qquad (2.60)$$

and all 95 terms except (00006), (00008), (01006), and (10006) can be found in $(00001)^{\otimes_S 8}$. This analysis implies that the contraction of four gamma matrices with four 4-forms G can be decomposed into 51 gamma-structures, each contracted with (multiple) G^4 terms corresponding to the same irrep of B_5.

For example, the term (00000) in the decomposition of $(00001)^{\otimes_S 8}$ gives rise to a single gamma-structure contracted with the four possible G^4 terms giving rise to a scalar. Explicitly we have,

$$(\Gamma^{e_1})(\Gamma_{e_1})(\Gamma^{e_2})(\Gamma_{e_2}) \Big(\alpha_1\, G^{(4)} G_{(4)}\, G^{(4')} G_{(4')} + \cdots + \alpha_4\, G_{(1)}{}^{(3)}\, G_{(1')(3)}\, G^{(1)}{}_{(3')}\, G^{(3')(1')} \Big) ,$$

for some constants $\alpha_1, \ldots, \alpha_4$, and where the dots stand for the two remaining independent combinations of G^4 that form a scalar. Similarly, the (00004) gives rise to the following term,

$$\begin{aligned}
&\Big(\beta_1\, (\Gamma^{e_1})(\Gamma_{e_1})(\Gamma^{a_1\ldots a_6})(\Gamma^{b_1\ldots b_6}) + \beta_2\, (\Gamma^{[a_1})(\Gamma^{a_2\ldots a_6]})(\Gamma^{[b_1})(\Gamma^{b_2\ldots b_6]}) \Big) \\
&\times \Big(\alpha_1\, G_{a_1a_2b_1}{}^{e_1}\, G_{a_3b_2b_3e_1}\, G_{a_4a_5a_6}{}^{e_2}\, G_{b_4b_5b_6e_2} + \cdots + \alpha_6\, G_{a_1a_2a_3b_3}\, G_{a_4a_5a_6b_2}\, G_{b_1b_2}{}^{e_1e_2}\, G_{b_5b_6e_1e_2} \Big) ,
\end{aligned}$$

for some constants $\beta_1, \beta_2, \alpha_1, \ldots, \alpha_6$, and where the dots stand for the 4 remaining combinations of G^4 that are in (00004). The 51 gamma-structures involved in the decomposition of $X^{(8)}$ can all be found explicitly, and only three of them are not τ-exact : (04000), (03002), and (02004). In other words, except for the structures corresponding to these three irreps all other gamma-structures appearing in $X_{0,8}$ involve at least one contraction with a $\Gamma^{(1)}$.

Going back to (2.56) : the $G^4_{a_1a_2;\ldots;a_7a_8}$ term, by virtue of its contraction with the four gamma matrices, transforms in the symmetrized product of four Young diagrams $\ydiagram{1,1}$, cf. appendix C.1. Decomposing in irreducible representations of S_8,

$$\ydiagram{1,1}^{\otimes_S 4} = \ydiagram{2,2,2,2} \oplus \ydiagram{1,1,1,1,1,1,1,1} \oplus \ydiagram{2,2,1,1,1,1} \oplus \ydiagram{3,3,1,1} \oplus \overset{YT1}{\ydiagram{4,4}} \qquad \text{(5 terms)} \tag{2.61}$$

At the same time $G^4_{a_1a_2;\ldots;a_7a_8}$ admits a decomposition into modules of $B_5 \times S_8$, $\sum_R V_R \times R$, where V_R is the plethysm of the module $V = (10000)$ of B_5 with respect to the Young diagram R of S_8. Moreover only the plethysms corresponding to the right-hand side of (2.61) will appear in the decomposition of $G^4_{a_1a_2;\ldots;a_7a_8}$ under $B_5 \times S_8$. On the other hand we can compute the module V_R corresponding to each R on the right-hand side of (2.61), using [80], with the result that only the plethysm corresponding to $YT1$ contains (04000), while neither (02004) nor (03002) is contained in any of the plethysms corresponding to the Young diagrams on the right-hand side of (2.61).

The $G^4_{a_1a_2;\ldots;a_7\ldots a_{12}}$ term of (2.57) admits the following decomposition in irreps of S_{12},

$$\ydiagram{1,1}^{\otimes_S 3} \otimes \ydiagram{1,1,1,1,1} = \overset{YT2}{\ydiagram{4,4,1,1,1,1}} \oplus \cdots \qquad \text{(16 terms)} \tag{2.62}$$

Only the plethysms corresponding to the Young diagrams on the right-hand side of (2.62) will appear in the decomposition of $G^4_{a_1a_2;\ldots;a_7\ldots a_{12}}$ under $B_5 \times S_{12}$. On the other hand it can be shown that only the plethysm corresponding to $YT2$ contains (03002), while neither (04000) nor (02004) is contained in any of the plethysms corresponding to the Young diagrams on the right-hand side of (2.62).

Finally, the $G^4_{a_1a_2;\ldots;a_{10}\ldots a_{14}}$ term of (2.58) admits the following decomposition in irreps of S_{14},

$$\ydiagram{1,1}^{\otimes_S 2} \otimes \ydiagram{1,1,1,1,1}^{\otimes_S 2} = \overset{YT3}{\ydiagram{4,4,2,2,2}} \oplus \overset{YT4}{\ydiagram{4,4,2,1,1,1,1}} \oplus \cdots \qquad \text{(23 terms)} \tag{2.63}$$

Moreover only the plethysms corresponding to the Young diagrams on the right-hand side of (2.63) will appear in the decomposition of $G^4_{a_1a_2;\dots;a_{10}\dots a_{14}}$ under $B_5\times S_{14}$. On the other hand it can be shown that only the plethysm corresponding to $YT3$ contains (02004) ; only the plethysm corresponding to $YT4$ contains (03002), while (04000) is not contained in any of the plethysms corresponding to the Young diagrams on the right-hand side of (2.63).

Using the method of appendix C.1, the gamma matrices in (2.56) and (2.58) can be projected respectively onto $YT1$ and $YT3$. The terms (2.56), (2.58) can thus be shown to vanish. Moreover, it can be seen that the cancellations are sensitive to the relative coefficient between $(\mathrm{tr}R^2)^2$ and $\mathrm{tr}R^4$ inside X_8. In other words, it can be shown that $(\mathrm{tr}R^2)^2$ and $\mathrm{tr}R^4$ are not separately τ-exact.

Next equations — Moreover, it can be shown that all higher components of K_{11} solving $W_{12}=\mathrm{d}K_{11}$ are automatically guaranteed to exist. To see this, let us define the twelve-form,

$$I_{12}:=W_{12}-\mathrm{d}K_{11}\ ,$$

which is closed by construction,

$$0=(\mathrm{d}I)_{p,13-p}=\tau I_{p+1,11-p}+\mathrm{d}_f I_{p,12-p}+\mathrm{d}_b I_{p-1,13-p}+tI_{p-2,14-p}\ . \tag{2.64}$$

On the other hand, as we saw above, provided (2.53) holds, condition $W_{12}=\mathrm{d}K_{11}$ is solved up to dimension 4, i.e. $I_{p,12-p}=0$ for $p=0,1,2$. Setting $p=2$ in (2.64) then gives $\tau I_{3,9}=0$, which implies $I_{3,9}=0$ up to a τ-exact piece that can be absorbed in $K_{4,7}$, since all τ-cohomology groups $H^{p,12-p}_\tau$ are trivial for $p\geq 3$, cf. section 2.2.2. By induction we easily see that $I_{p,12-p}=0$, for all $p\geq 3$. In other words, provided (2.53) holds, the Weil-triviality condition $W_{12}=\mathrm{d}K_{11}$ is guaranteed to admit a solution.

2.4.2 How many superinvariants ?

We have seen that provided the modified BI (2.44) are satisfied, there will be at least one superinvariant at eight derivatives, cf. (2.49). A second independent superinvariant can also be similarly constructed as follows. Consider the twelve-form,

$$W'_{12}=\frac{1}{6}G_4{\wedge}G_4{\wedge}G_4\ . \tag{2.65}$$

Expanding perturbatively to order l^6 we obtain,

$$W'^{(1)}_{12}=\frac{1}{2}G^{(0)}_4{\wedge}G^{(0)}_4{\wedge}G^{(1)}_4=\mathrm{d}Z_{11}\ ,\qquad Z_{11}=\frac{1}{2}\,G^{(1)}_4\wedge G^{(0)}_4\wedge C^{(0)}_3\ , \tag{2.66}$$

The above can also be written in a manifestly Weil-trivial form using (2.46),

$$W'^{(1)}_{12}=\mathrm{d}K_{11}\ ,\qquad K_{11}=-G^{(0)}_7{\wedge}G^{(1)}_4\ . \tag{2.67}$$

The order-l^6 superinvariant corresponding to $Z_{11} - K_{11}$ then reads,

$$\Delta S' = l^6 \int G_4^{(1)} {\scriptstyle\wedge} \left(\frac{1}{2} G_4^{(0)} {\scriptstyle\wedge} C_3^{(0)} + G_7^{(0)} \right) , \tag{2.68}$$

where it is understood that only the bosonic $(11, 0)$ components of the forms enter. The above superinvariant does not contain the correct CS terms required by anomaly cancelation, cf. (2.51), and should therefore be excluded by the requirement of quantum consistency of the theory. However if one is only interested in counting superinvariants at order l^6 in the classical theory, the above superinvariant is perfectly acceptable and its existence is guaranteed provided the BI are obeyed to order l^6.

Dropping the requirement of quantum consistency, relying on classical supersymmetry alone, one may also consider the following two 12-forms,

$$U_{12} = l^6 G_4 {\scriptstyle\wedge} \mathrm{tr} R^4 , \qquad V_{12} = l^6 G_4 {\scriptstyle\wedge} (\mathrm{tr} R^2)^2 , \tag{2.69}$$

so that $U - \frac{1}{4} V$ is the Weil-trivial 12-form corresponding to the CS terms of eleven-dimensional supergravity required for anomaly cancellation, cf. (2.42). It follows that either U, V are both Weil-trivial, or neither U nor V is Weil-trivial. If the former is true, there would exist gauge-invariant 11-forms K_U, K_V so that at order l^6 we have $U^{(1)} = \mathrm{d}K_U$ and $V^{(1)} = \mathrm{d}K_V$. One can then construct two corresponding superinvariants using the action principle,

$$\Delta S_U = l^6 \int \left(\mathrm{tr} R^4 {\scriptstyle\wedge} C_3^{(0)} - K_U \right) , \qquad \Delta S_V = l^6 \int \left((\mathrm{tr} R^2)^2 {\scriptstyle\wedge} C_3^{(0)} - K_V \right) . \tag{2.70}$$

By the argument at the end of the last section, ΔS_U, ΔS_V should correspond to a modified BI obtained by replacing the right-hand side of the second equation in (2.44) by $\mathrm{tr} R^4$, $(\mathrm{tr} R^2)^2$ respectively. Then K_U, K_V would still be given by (2.48) but with $G_4^{(1)}$, $G_7^{(1)}$ solutions of the new modified BI.

Together with the superinvariant $\Delta S'$ of (2.68), we would then have a total of at least three independent superinvariants at the eight-derivative order, with only one linear combination thereof, ΔS of (4.34), corresponding to the quantum-mechanically consistent eight-derivative correction. As we saw in section 2.4.1, if ΔS_U, ΔS_V exist they must necessarily be cubic or lower in the field (this requirement is also proven independently of all the development of section (2.4.1) in the appendix (A.3). It is shown that attempting to supersymmetrize V fails if the condition that the superinvariant ΔV must be quartic is not dropped.)

2.4.3 Integrability

The perturbative expansion of the curved components following from (2.45) reads,

$$G_{M_1 \dots M_4} = G^{(0)}_{M_1 \dots M_4} + l^6 \, G^{(1)}_{M_1 \dots M_4} + \cdots ,$$

and similarly for G_7 and $R_A{}^B$. Note that in terms of flat components there is a mixing between zeroth order and order l^6 due to,

$$\Phi \;=\; E^A\Phi_A \;=\; E^{(0)A}\Phi_A^{(0)} \;+\; l^6(E^{(0)A}\Phi_A^{(1)} \;+\; E^{(1)A}\Phi_A^{(0)}) \;+\; \cdots\;,$$

where we have expanded the coframe, $E^A = E^{(0)A} + l^6E^{(1)A} + \cdots$, and we have considered an arbitrary one-form Φ for simplicity. However, if one restricts to the top bosonic component of a superform at $\theta = 0$ as in (2.29), then there is no mixing :

$$\Phi_m^{(0)}| \;=\; e_m{}^a\,\Phi_a^{(0)}| \;+\; \psi_m^\alpha\,\Phi_\alpha^{(0)}|\;, \qquad\qquad \Phi_m^{(1)}| \;=\; e_m{}^a\,\Phi_a^{(1)}| \;+\; \psi_m^\alpha\,\Phi_\alpha^{(1)}|\;,$$

where $e_m{}^a$, ψ_m^α were defined below (2.32). Indeed the $\mathcal{O}(l^6)$ corrections to the coframe E^A only start at higher orders in the θ-expansion and could be systematically determined as in e.g. [81] once the $\mathcal{O}(l^6)$ corrections to the torsion components have been determined.

In practice the BI are solved for the flat components of the superforms involved, $G^{(0)}_{A_1\ldots A_4}$, $G^{(1)}_{A_1\ldots A_4}$ etc, at each order in l. Consequently the corresponding BI, $\mathrm{d}G_4 = 0$ etc, are only shown to be satisfied up to terms of the next order in l. In principle there may be an integrability obstruction to the solution of the BI at next-to-leading order in the derivative corrections, although that would most probably be prohibitively difficult to check in practice. In the following we shall see that the integrability of a certain superinvariant is guaranteed provided the BI admit solutions to all orders in l. Note however that all-order integrability need not be a consequence of the BI.

The phenomenon of inducing a higher-order correction at next-to-leading order is also well understood at the level of the component action, $S = S^{(0)} + l^6S^{(1)} + \cdots$. The condition of invariance of the action under supersymmetry transformations $\delta = \delta^{(0)} + l^6\delta^{(1)} + \cdots$ reads,

$$\delta^{(0)}S^{(0)} = 0\;, \qquad\qquad \delta^{(0)}S^{(1)} + \delta^{(1)}S^{(0)} = 0\;, \tag{2.71}$$

and similarly at higher orders. The term $\delta^{(1)}S^{(0)}$ in the second equation above is proportional to the lowest-order equations of motion. Therefore in constructing $S^{(1)}$ we only need to check its invariance with respect to the lowest-order supersymmetry transformations $\delta^{(0)}$ and only up to terms which vanish by virtue of the lowest-order equations of motion. This corresponds, in the superspace approach, to the fact that in solving the first-order BI one uses the zeroth-order equations for the various superfields. Once $S^{(1)}$ is thus constructed, the correction $\delta^{(1)}$ to the supersymmetry transformations can be read off. Since $\delta^{(1)}S^{(1)} \neq 0$ in general, this induces a correction $S^{(2)}$ to the action and a corresponding correction $\delta^{(2)}$ to the supersymmetry transformations, and so on.

The existence of an intergrability obstruction can also be understood in the context of the Noether procedure. Indeed at next-to-leading order we have,

$$\delta^{(2)}S^{(0)} + \delta^{(1)}S^{(1)} + \delta^{(0)}S^{(2)} = 0\;.$$

Therefore there must exist an action $S^{(2)}$ such that its variation with respect to lowest-order supersymmetry transformations is equal to $-\delta^{(1)}S^{(1)}$, up to terms that vanish by virtue of the lowest-order equations of motion. This condition will not be automatically satisfied for every $S^{(1)}$.

In particular one would like to know how many of the independent superinvariants at order l^6 presented in section 2.4.2 survive to all orders in the derivative expansion. Assuming M-theory is a non-perturbatively consistent theory, we expect the superinvariant (2.49), corresponding to the supersymmetrization of the CS term required for anomaly cancellation, to be integrable to all orders. Moreover, assuming this superinvariant is at least quartic in the fields, a similar argument as the one detailed below (2.64) shows that it must be unique at order l^6 [43].

In addition, if one assumes that the BI admit a solution to all orders in a perturbative expansion in l, then there is one linear combination of the superinvariants presented in section 2.4.2 that is guaranteed to exist to all orders in l. Indeed in that case the twelve-form,

$$W_{12} = \big(l^6 X_8 - \frac{1}{2} G_4 \wedge G_4\big) \wedge G_4 = \mathrm{d}\big(G_7 \wedge G_4\big) \, ,$$

is Weil-trivial by virtue of (2.44), which should now be considered valid to all orders in l. However this is *not* the superinvariant which corresponds to the supersymmetrization of the anomaly term, cf. (2.49). Indeed by the usual action principle procedure the twelve-form above would give rise to the superinvariant,

$$\Delta S = \int \left(l^6 X_8 \wedge C_3 - \frac{1}{2} G_4 \wedge G_4 \wedge C_3 - G_7 \wedge G_4 \right) \Big| \, . \tag{2.72}$$

Expanding to order l^6 and assuming G_4 receives a non-vanishing correction at this order, we see that (2.72) does not coincide with (2.49) and the corresponding l^6-corrected action is different from (2.50).

In conclusion, under the aforementioned assumptions, we would then expect (at least) two independent superinvariants to exist to all orders in a perturbative expansion in l. Only one of these, the one corresponding to the supersymmetrization of the CS anomaly term, will be quantum-mechanically consistent.

2.5 A systematic approach to the $\mathcal{O}(l^6)$ invariant

2.5.1 Overview of the equations to come

The developments of the previous section, although providing information about X_8 and the superinvariant, do not offer a practical path to actually solve the equations leading to the superform K. The solution of $K^{(4)}_{a_1\ldots a_3\alpha_1\ldots\alpha_8}$ involves four tensors G contracted with four gamma matrices, where the non-τ exact combinations of matrices, (04000), (03002) and (02004) should vanish (cf. (2.59) and (2.60)). And all the other 45 irreps have to be determined, and belong to $K^{(4)}_{a_1\ldots a_3\alpha_1\ldots\alpha_8}$ (with a particular coefficient for each one).

With the help of the computer program described in chapter 3, decomposing such expressions into irreducible components is in principle a simple procedure. However, in this case the large number of spinor indices $(00001)^{\otimes S8}$, combined with the four tensors G $(00010)^{\otimes S4}$ leads to some difficulty. Although this is still an unfinished work, the methods described in this sections might be useful for finding a systematic treatment of the superspace equations leading to the desired action.

First, we notice that all the equations of higher dimensions will be of the form

$$(\Gamma^f)_{(\alpha_1\alpha_2|}K_{fa_1\ldots a_k|\alpha_3\ldots\alpha_{12-k})} = T_{[a_1\ldots a_k](\alpha_1\ldots\alpha_{12-k})} \,, \tag{2.73}$$

where T is any expression involving gamma matrices and gauge-invariant fields of the theory. For example, at dimension $9/2$, it will involve at most terms like TDR, GRT, RDT, T^3, $GTDG$, G^3T, $DTDG$, and G^2DT. A way of dealing with equation (2.73) would be to decompose the rhs in irreps of $SO(1,10)$, and check that all terms either vanish or belong to the solution for K. This possibility in examined in the following paragraphs.

2.5.2 An ansatz for the decomposition

Let's go back to the equation (2.53) that states the τ-exactness of X_8 (and also defines the solution for K_{11} of dimension 4),

$$(\Gamma^f)_{(\alpha_1\alpha_2|}G^{(1)}_{f|\alpha_3\ldots\alpha_8)} = X^{(1)}_{\alpha_1\ldots\alpha_8} \,.$$

The irreps shared between $(00001)^{\otimes_S 8}$ and $(00010)^{\otimes_S 4}$ represent all the terms that can enter the decomposition of X_8. They both come with a particular multiplicity, and represent a certain contraction of the four G tensors or gamma matrices. It is partially reported in the table below,

irrep	gamma multiplicity	gamma structure	tensor multiplicity	tensor structure
(00014)	1	$\Gamma^e\Gamma_{ea_1\ldots a_4}\Gamma_{b_1\ldots b_6}\Gamma_{c_1\ldots c_6}$	1	$G_{a_1\ldots a_4}G_{b_1\ldots b_4}G_{b_5b_6c_1c_2}G_{c_3\ldots c_6}$
(31000)	1	$\Gamma^{a_1}\Gamma_{b_1}\Gamma_{c_1}\Gamma_{d_1\ldots d_9}$	1	$G_{a_1d_1d_2}{}^{e_1}G_{b_1d_3d_4e_1}G_{c_1d_5d_6}{}^{e_2}G_{d_7d_8d_9e_2}$
⋮	⋮	⋮	⋮	⋮
(10102)	1	$\Gamma^{a_1}\Gamma_{[b_1}\Gamma_{b_2b_3]}\Gamma_{c_1\ldots c_5}$	19	(?)
(01002)	2	$\left\{\begin{matrix}\Gamma^e\Gamma_e\Gamma_{a_1a_2}\Gamma_{b_1\ldots b_6}\\ \Gamma^{a_1a_2}\Gamma_{[b_1b_2}\Gamma_{b_3b_4}\Gamma_{b_5b_6]}\end{matrix}\right\}$	23	(?)

The main problem in finding the terms with (?) is that the multiplicity of the irreps makes it complicated to find the right non-redundant G^4 terms. For example the irrep (10102) has a multiplicity of 19 : it is very difficult to write all of them by hand. When taking into account all the multiplicities of the irreps above (for both gamma matrices and G^4), we obtain a total of 405 term for the decomposition in irreps. The algorithm allowing to generate this ansatz for one particular irrep is as follows,

— Write all non-equivalent permutations of the 16 indices on the expression $G_{a_1\ldots a_4}\, G_{a_5\ldots a_8}\, G_{b_1\ldots b_4}\, G_{b_5\ldots b_8}$, leading to $16!/(4!)^4/4! = 2\,627\,625$ terms.[6]

— Contract certain indices from all the terms above, according to the number of free indices required for projecting into the desired representation. This leads to a large number of redundant terms.

— Remove the duplicate terms by finding all non-trivially identical contractions (cf. section 3.2.4)

— Project all the terms in the Young diagram corresponding to the $SO(1,10)$ irrep. Attribute to each of them a coefficient z_i.

— Find the condition on the z_i required to make the whole sum vanish. The number of z_i in the solution (whose value is expressed in terms of the other coefficients) should be at least equal to multiplicity of the $SO(1,10)$ irrep, and constitute an acceptable basis for the decomposition in the desired irrep.

For example, let's build the term (01102) coming from $(00001)^{\otimes_S 8}$ with a multiplicity 1, and from $(00010)^{\otimes_S 4}$ with a multiplicity 14. One of the possible gamma matrices configuration (among many others) is

$$(\Gamma_{[a_1})(\Gamma_{a_2 a_3]})(\Gamma_{a_4 a_5})(\Gamma_{a_6 a_7 a_8 b_1 b_2})\,. \tag{2.74}$$

The irrep (01102) is a "2-3-5-form" requiring 10 free indices. Among the 16 indices of G^4, six have to be contracted with each other. Since the first step of the algorithm leads to all the possible position of indices, we can arbitrarily choose to contract $(b_3,\, b_4)$, $(b_5,\, b_6)$ and $(b_7\, b_8)$. The 641 different terms obtained go down to 267 when the trivially and non-trivially equivalent terms are removed. Then, projecting all those candidates on the Young tableau corresponding to (2.74)

$$\begin{array}{|c|c|c|}\hline a_6 & a_1 & a_4 \\ \hline a_7 & a_2 & a_5 \\ \hline a_8 & a_3 & \multicolumn{1}{c}{} \\ \cline{1-2} b_1 & \multicolumn{2}{c}{} \\ \cline{1-1} b_2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

and requiring the sum (of the 267 projections) to vanish leads exactly to 14 terms, expressed as linear combinations of the others. All of them have the same free indices as (2.74), and contracting them leads to the (01102) part of the ansatz. The same process has to be repeated for all other irreducible representations available.

This method actually leads to an ansatz of 438 terms, instead of 405, since it sometimes leads to a non-minimal set for the G^4 part. All the coefficients have to be determined using contractions with all possible sets of four gamma matrices (since there are 8 symmetrized indices). This adds

6. Generating such a high number of terms can be quite challenging in itself, and requires to proceed with care.

up to 15 different cases, which all require to explicitly develop the symmetry of the eight fermionic indices. For each case, we must compute $438\frac{8!}{2^4} = 1\,103\,760$ traces which contain at most eight gamma matrices. The simplest of this contractions, with $\Gamma^{(1)}\Gamma^{(1)}\Gamma^{(1)}\Gamma^{(1)}$ have already been computed, and the other should soon follow. However, given the substantial amount of computational power required, the higher contractions might be prohibitively long, or too memory consuming for a standard computer.

In principle, this method of decomposition allows to compute systematically the ansatz for the decomposition of any expression involving gamma matrices and tensors, and can be used to solve all the equations of the form (2.73). The next few equations might involve expressions that are too large to be computed in practice, but since the amount of spinorial indices decreases with the mass dimension, there shall be less and less room for gamma matrices algebra, which greatly simplifies computations.

3

The SSGamma package : Tensors, spinors and gamma matrices

The computations presented in the previous chapter crucially depend on the use of a computer program. Most of them involve a large amount of terms, gamma matrices in high dimensions, (anti-)symmetrizations over numerous indices, etc. A Mathematica package, specifically designed to fit our needs was developed and extensively used throughout chapter 2 and 4. It is named SSGamma, shorthand for SuperSpace Gamma, and this chapter aims at introducing its main features.

In this chapter, basic knowledge of the Mathematica syntax is required. The main introduction to Mathematica programming is available in [82].

3.1 Motivations and requirements

The package presented in this section is derived from the already very useful Mathematica package called *Gamma* [83]. *Gamma* was built to deal with gamma-matrix algebra, Fierz transformations, and some simple calculations regarding tensors and spinors. It is remarkable in many ways : it's easy to use, works in all spacetime dimensions, and allows to make use of the rich Mathematica ecosystem to define new functions effortlessly. However, when dealing with the superspace related issues of the first chapter, several shortcomings had to be addressed in order to handle some of the computations. Let's use as an example a typical superspace equation encountered in chapter 2 :

$$D_{[A_1} K_{A_2 \ldots A_{12})} + \frac{11}{2} T_{[A_1 A_2|}{}^F K_{F|A_3 \ldots A_{12})} = \frac{11!}{4(4!)^2} R_{[A_1 A_2|c_1 c_2} R_{|A_3 A_4|}{}^{c_2 c_1} G_{|A_5 \ldots A_8} G_{A_9 \ldots A_{12})} \, .$$

This expression contains 12 (anti-)symmetrized superspace indices, each of which can be either fermionic or bosonic. It also contains a derivative applied to a tensors K whose mixed components (i.e. with both types of superspace indices) probably have a definition in terms of gamma matrices and other fields. Finally, it contains two superspace R and G tensors, which give rise to four G tensors when evaluated with the right indices. Dealing with this simple expression thus directly leads to a few requirements :

— Handle large (anti-)symmetrizations efficiently, i.e. by taking into account the symmetries of the tensors on which they are applied.
— Operate with tensors with mixed symmetries, i.e. with symmetric and anti-symmetric parts, and with both bosonic and fermionic indices.
— Be able to deal with large expressions involving gamma-matrix algebra that appear when symmetrizations are made explicit (traces, expansions, etc.)
— Manage to simplify and merge non-trivially identical terms containing several contracted identical tensors (like the four G tensors above).
— Be flexible enough to allow an easy implementation of new functions.

Several existing programs already have some of the required features listed above. The closest existing program is Cadabra [84], but for the purposes of this work, it lacks flexibility : since it is programmed in C++, any additional function requires a substantial amount of effort to be implemented.

3.2 Main features and functions

The five following sections present most of the functions that can be used in the package. Several of them are very similar to what already existed in Gamma [83]. However, as I mention before, all of them have been completely adapted, either to broaden their range of application, or to improve them in a particular way. The nature of the modifications is not always mentioned, for it often consists of technical details about Mathematica programming.

3.2.1 Spacetime dimensions and conventions

The spacetime dimension must be specified using the instruction SetDim : `SetDim[11]` (for 11 dimensions). The package was built for 11-dimensional Majorana, and 10-dimensional Majorana-Weyl spinors. For now, it cannot deal with Dirac or symplectic spinors, although a generalization might be set up without too much effort. The practical manipulation of spinors is presented in the section (3.2.3).

Secondly, one can chose the convention associated to the Levi-Civita tensor, namely,

$$\epsilon_{a_1 \dots a_d} = \pm 2^{[d/2]} \operatorname{tr}\left(\gamma_{a_1 \dots a_d}\right) ,$$

by using the function SetEpsilonConvention, with the value 1 or −1 : `SetEpsilonConvention[±1]`. This choice will also affect the Hodge duality relation of section (3.2.2).

3.2.2 Gamma matrices

Basic gamma-matrix manipulations — A symbolic gamma matrix $(\gamma^a)_\alpha{}^\beta$ is represented by the function TG (for Tensor Gamma), where both bosonic and spinor indices must be specified as arguments. The program do not explicitly consider the $2^{[d/2]} \times 2^{[d/2]}$ matrix elements, but simply defines symbolic matrices verifying the Clifford relation $\gamma^{(a}\gamma^{b)} = \eta^{ab}$. The up or down position of spinor indices must be indicated using $\{1,\alpha\}$ or $\{0,\alpha\}$:

```
In[1]:= TG[{a},{{0,α},{1,β}}].
```

By default, the matrices are denoted γ, but other names like Γ can be specified using `TG[Γ,{a},{{0,α},{1,β}}]`. As usual, those matrices can be easily combined into anti-symmetrized blocks $(\gamma^{abc})_\alpha{}^\beta = (\gamma^{[a}\gamma^b\gamma^{c]})_\alpha{}^\beta$ as

```
In[1]:= TG[{a,b,c},{{0,α},{1,β}}].
```

Any product of these block of matrices, for example $(\gamma^{abc}\gamma^{de}\gamma^{fghi})_\alpha{}^\beta$ is written as

```
In[1]:= TG[{a,b,c},{d,e},{f,g,h,i},{{0,α},{1,β}}].
```

The product can be explicitly computed by using the function GME (for Gamma Matrices Expansion) :

```
In[1]:= GME @ TG[{a,b},{c},{{0,α},{1,β}}]
```
Out[1]= $\delta^b_c\ (\gamma_a)_\alpha{}^\beta - \delta^a_b\ (\gamma_c)_\alpha{}^\beta + (\gamma_{abc})_\alpha{}^\beta$

which corresponds to the expansion given by

$$\gamma^{a_1\ldots a_n}\gamma_{b_1\ldots b_k} = \sum_{i=0}^{\mathrm{Min}(n,k)} \binom{n}{i}\binom{k}{i}\ i!\ (-1)^{\frac{i}{2}(i-1)+i(n-i)}\ \delta^{[a_1\ldots a_i}_{[b_1\ldots b_i}\ \gamma^{a_{i+1}\ldots a_n]}{}_{b_{i+1}\ldots b_n]}. \tag{3.1}$$

The generalized Kronecker δ defined by $\delta^{a_1\ldots a_n}_{b_1\ldots b_n} = \delta^{[a_1}_{[b_1}\ldots\delta^{a_n]}_{b_n]}$ is represented by the function TD[1] (for Tensor Delta) :

TD[{$a_1\ldots a_n$},{$b_1\ldots b_n$}]

Similarly, the generalized Levi-Civita symbol is called with TE (for Tensor Epsilon) :

In[1]:= TE[{$a_1,\ldots,a_d$}]
Out[1]= $\epsilon_{a_1\ldots a_d}$

When there is more than one product, the function GME has to be applied several times. In the special case where the product contains repeated indices in adjacent matrices, the function GMC (for Gamma Matrices Contraction) is automatically applied

```
In[1]:= GMC @ TG[{a,b,e},{e,c},{{0,α},{1,β}}]
```

1. In TD and TE, instead of δ and ϵ, custom name can be specified by TD[Δ, ...] and TE[ε, ...]

```
Out[1]= 
```
$- 9\,\delta^b_c\,(\gamma_a)_\alpha{}^\beta + 9\,\delta^a_b\,(\gamma_b)_\alpha{}^\beta - 8\,(\gamma_{abc})_\alpha{}^\beta$

using the formula

$$\gamma^{a_1\ldots a_n e_1\ldots e_p}\gamma_{e_p\ldots e_1 b_1\ldots b_k} = \sum_{i=0}^{\mathrm{Min}(n,k)}\left(\prod_{l=0}^{p-1}(D-n-k-l+i)\right)\binom{n}{i}\binom{k}{i}\; i!\;(-1)^{\frac{i}{2}(i-1)+i(n-i)}\;\delta^{[a_1\ldots a_i}_{[b_1\ldots b_i}\;\gamma^{a_{i+1}\ldots a_n]}{}_{b_{i+1}\ldots b_n]}$$

When the contracted indices are separated by a gamma matrix, like $(\gamma^{a_1a_2e}\gamma_{b_1b_2}\gamma_{c_1c_2e})_\alpha{}^\beta$, the product is developed using 3.1 in reverse :

$$(\gamma^{a_1a_2e}\gamma_{b_1b_2}\gamma_{c_1c_2e})_\alpha{}^\beta = (\gamma^{a_1a_2}\gamma^{e}\gamma_{b_1b_2}\gamma_{c_1c_2e})_\alpha{}^\beta - \delta^{[a_1}_e(\gamma^{a_2]}\gamma_{b_1b_2}\gamma_{c_1c_2e})_\alpha{}^\beta \qquad (3.2)$$

where we only get terms without contracted indices between gamma matrices (the second term in the rhs of 3.2), and terms of the form

$$(\gamma^{e_1\ldots e_l}\gamma_{b_1\ldots b_k}\gamma_{c_1\ldots c_n e_1\ldots e_l})_\alpha{}^\beta,$$

where the first matrix only contains indices $(e_1 \ldots e_l)$ that appear in the last matrix. It is then possible to expand this product to find,

$$(\gamma^{e_1\ldots e_l}\gamma_{b_1\ldots b_k}\gamma_{c_1\ldots c_n e_1\ldots e_l})_\alpha{}^\beta = \sum_{i=0}^{\substack{\mathrm{Min}\\(l,k)}}\;\sum_{p=0}^{\substack{\mathrm{Min}\\(k-i,n+i)}} i!\;p!\binom{l}{i}(-1)^{\frac{i}{2}(i-1)+i(l-i)}\,(-1)^{\frac{p}{2}(p-1)+p(k-i-p)}$$
$$\times\left(\prod_{m=0}^{p}(D-k-n-m+p)\right)\binom{k-i}{p}\binom{n+i}{p}\;\delta^{[y_1\ldots y_p}_{[z_1\ldots z_p}\,(\gamma^{y_{p+1}\ldots y_{k-i}]}{}_{z_{p+1}\ldots z_{n+i}]})_\alpha{}^\beta + (\cdots)$$

where $y_1 \ldots y_{k-i} = b_{i+1} \ldots b_k$ and $z_1 \ldots z_{n+i} = b_1 \ldots b_i\; c_1 \ldots c_n$, and $(\ldots)$ represent all the permutations over $b_1 \ldots b_k$.

If the spinor indices of a gamma matrix are contracted, the trace can be explicitly computed using the function GMT (for Gamma Matrix Trace). The following three examples are computed for $D = 11$, with Dirac matrices :

```
In[1]:= GMT @ TG[{a,b},{c},{{0,α},{1,α}}]
```
$\longrightarrow (\gamma^{ab}\gamma_c)_\alpha{}^\alpha$

```
Out[1]= 0
```

```
In[2]:= GMT @ TG[{a,b},{c,d},{{0,α},{1,α}}]
```
$\longrightarrow (\gamma^{ab}\gamma_{cd})_\alpha{}^\alpha$

```
Out[2]= 
```
$-64\,\delta^{ab}_{cd}$

```
In[3]:= GMT @ TG[{a1...a3},{a4...a11},{{0,α},{1,α}}]
```
$\longrightarrow (\gamma_{a_1\ldots a_3}\gamma_{a_4\ldots a_{11}})_\alpha{}^\alpha$

```
Out[3]= 
```
$-32\,\epsilon_{a_1\ldots a_d}$

In some cases, two gamma matrices can appear with contracted spinorial indices. In order to make the contraction explicit (and be able to apply the previous function), the matrices must be gathered using GMS (for Gamma Matrices Simplification) :

Out[1]= $(\gamma_{ab})_\alpha{}^\beta\ (\gamma_{cd})_\beta{}^\delta$

In[2]:= GMS[%]
Out[2]=$(\gamma_{ab}\gamma_{cd})_\alpha{}^\delta$

It is sometimes useful to use the hodge duality for gamma matrices, in order to quicken a computation, or simply to regroup identical terms. The function GMD (for Gamma Matrix Duality) will transform a gamma matrix into its hodge dual via the formula,

$$\gamma_{a_1\ldots a_k} = \frac{1}{(d-k)!}\,(i)^{[d/2]+1}\,(-1)^{\frac{1}{2}d(d-1)}\,(-1)^{\frac{1}{2}k(k-1)}\,\epsilon_{a_1\ldots a_k}{}^{b_1\ldots b_{d-k}}\,\gamma_{b_1\ldots b_{d-k}}\,\gamma^*\,,$$

where γ^* is the chirality matrix. The matrices that have to be transformed can be specified by calling the function with a list of integers as a second argument. It represents what order of gamma matrix must be dualized :

Out[1]= $(\gamma_{a_1a_2})_\alpha{}^\beta(\gamma_{b_1\ldots b_9})_{\beta\epsilon}$

In[2]:= GMD[%,{9}]
Out[2]= $\frac{1}{2}\,\epsilon_{b_1\ldots b_9e_1e_2}\,(\gamma_{a_1a_2})_\alpha{}^\beta\,(\gamma_{e_1e_2})_{\beta\epsilon}$

A word about computation time — Those manipulations can be extremely demanding in terms of computational resources, mainly due to the inevitable symmetrizations of indices that have to be made explicit. To get a sense of the numbers involved, let's consider the two following examples :

$$(\gamma^{a_1\ldots a_5}\,\gamma^{b_1\ldots b_5}\,\gamma_{c_1\ldots c_n})_\alpha{}^\beta \quad \text{and} \quad (\gamma_{a_1b_1c_1}\,\gamma_{a_2b_2c_2}\,\cdots\,\gamma_{a_nb_nc_n})_\alpha{}^\beta$$

and look at the number of terms contained in the product expansion, and the time it took Mathematica to compute them :

n	Number of terms	Time(s)
3	12 206	1.0
5	94 428	7.5
7	383 283	28.6

and

n	Number of terms	Time(s)
3	490	< 0.1
4	13 562	1.4
5	539 041	51.1

Luckily, the products one encounters in real-life situations often appear contracted with tensors, whose symmetries can be used to simplify the computation. If T and H are two 5-indices and 2-indices anti-symmetric tensors contracted with some indices of the gamma matrices, the computation is significantly faster, and the number of terms is greatly reduced :

$$(\gamma^{a_1\ldots a_5}\gamma^{b_1\ldots b_5}\gamma_{c_1\ldots c_n})_\alpha{}^\beta\, T_{a_1b_1c_1a_2b_2}H_{a_3b_3c_3} \quad \text{and} \quad (\gamma_{a_1b_1c_1}\gamma_{a_2b_2c_2}\ldots\gamma_{a_nb_nc_n})_\alpha{}^\beta\, T_{a_1b_1c_1a_2b_2}H_{a_3b_3c_3}$$

n	Number of terms	Time(s)
3	1 138	0.2
5	7 654	0.9
7	31 332	2.9

and

n	Number of terms	Time(s)
3	26	< 0.1
4	465	0.3
5	16 058	3.0

However, those results are given with non-evaluated Kronecker deltas, and still need a few further simplifications. For instance, the last expression, containing 16 058 terms looks like,

$$\left(\delta^{c_2}_{a_1}\,\delta^{b_5c_5}_{b_4c_4}\,\delta^{a_4a_5b_3c_3}_{a_2b_1b_2c_1}\,(\gamma_{a_3})_\alpha{}^\beta + \delta^{a_3a_4}_{a_1a_2}\,(\gamma_{a_5b_1b_2b_3b_4b_5c_1c_2c_3c_4c_5})_\alpha{}^\beta + \ldots\right) T_{a_1a_2b_1b_2c_1}\, H_{a_3b_3c_3},$$

where all the Kronecker can be eliminated using the function DeltaSim (for Delta Simplification), whose role is to apply the Kronecker deltas, one by one, on the adjacent tensors :

Out[1]= $\delta^{c_2}_{a_1}\,\delta^{b_4c_3}_{a_5c_5}\,\delta^{a_4b_3b_5c_4}_{a_2b_1b_2c_1}\,(\gamma_{a_3})_\alpha{}^\beta\, H_{a_3b_3c_3}\, T_{a_1a_2b_1b_2c_1}$

In[2]:=DeltaSim[%]

Out[2]= $\delta^{b_4c_3}_{a_5c_5}\,\delta^{a_4b_3b_5c_4}_{a_2b_1b_2c_1}\,(\gamma_{a_3})_\alpha{}^\beta\, H_{a_3b_3c_3}\, T_{a_2b_1b_2c_1c_2}$

In[3]:=DeltaSim[%]

Out[3]= $\delta^{c_5}_{b_4}\,\delta^{a_4b_3b_5c_4}_{a_2b_1b_2c_1}\,(\gamma_{a_3})_\alpha{}^\beta\, H_{a_3a_5b_3}\, T_{a_2b_1b_2c_1c_2} + \delta^{b_4}_{a_5}\,\delta^{a_4b_3b_5c_4}_{a_2b_1b_2c_1}\,(\gamma_{a_3})_\alpha{}^\beta\, H_{a_3b_3c_5}\, T_{a_2b_1b_2c_1c_2}$

In[3]:=DeltaSim[%]

Out[3]= $\delta^{c_5}_{b_4}\,(\gamma_{a_3})_\alpha{}^\beta\, H_{a_3a_5b_3}\, T_{a_4b_3b_5c_2c_4} + \delta^{b_4}_{a_5}\,(\gamma_{a_3})_\alpha{}^\beta\, H_{a_3b_3c_5}\, T_{a_4b_3b_5c_2c_4}$

This part of the computation can sometimes be even more demanding than the actual computation of the traces. In order to control its behavior, it is preferable not to call this function automatically. For large expressions, when trying to compute expansions of gamma matrices by recursive application of GME, it might be fruitful to check whether or not DeltaSim must be called in between the GMEs, or if it must be called at the end of the expansion.

3.2.3 Tensors and spinors

SSGamma allows to manipulate spinors and tensors of any symmetries, with any number of bosonic and fermionic indices, with various indices symmetries.

Defining tensors and spinors — A tensor is declared using the function TensDef, specifying the name as first argument, then all the indices (in a list), the anti-symmetric indices, the symmetric indices, and if needed the group-swapping symmetry.

— A 6-indices tensor T, anti-symmetric in its first, second and fourth indices, symmetric in its third and fifth indices, without symmetry in its sixth index :

```
In[1]:= TensDef[T,{a1,a2,b1,a3,b2,c1},{{a1,a2,a3}},{{b1,b2}}]
Out[1]= Canonical formmm of the tensor: T_{a1 a2 b1 a3 b2 c1}
   ASym positions: {1,2,4} Sym positions: {3,5}
```

— A tensor anti-symmetric in its first and second indices, in its third and fourth indices, with a symmetry by exchange of the two pairs :

```
In[1]:= TensDef[R,{a,b,c,d},{{a,b},{c,d}},{{}},ASymSwap->{1,2}]
Out[1]= Canonical form of the tensor: R_{abcd}
   ASym positions: {1,2}{3,4} Sym positions: {}
```

— A three-indices tensor without any symmetries :

```
In[1]:= TensDef[G,{a,b,c},{{}},{{}}]
Out[1]= Canonical form of the tensor: G_{abc}
   ASym positions: {} Sym positions: {}
```

— A three-indices tensor without any symmetries :

```
In[1]:= TensDef[ψ,{a,b,{1,\alpha}},{{}},{{a,b}}]
Out[1]= Canonical form of the tensor: ψ_{ab}^α
   ASym positions: {} Sym positions: {a,b}
```

The tensors declared in this way will automatically sort their indices according to their symmetries, and produce the right ± 1 factor if needed. The tensors already defined can be displayed by calling `TensDef[ ]` (without arguments) :

```
In[1]:= TensDef[]
Out[1]= Canonical form of the tensor: G_{abcd}
     ASym positions: {a,b,c,d} Sym positions: {}
   Canonical form of the tensor: T_{ab}^α
     ASym positions: {a,b} Sym positions: {}
   Tensors with other names are assumed anti-symmetric
```

As noted, all tensors whose name has not been defined are considered fully anti-symmetric by default. If a tensor has been declared, its name should not be used for anything else (and it must keep the same number of indices as in the declaration). The list of defined tensors can be cleared with the instruction `TensClear[]`, and a specific tensor can be removed from it using `TensClear[T]` (for a tensor named T).

Spinors in 11 dimensions — In 11 dimensions, the spinors are Majorana with 32 components. The properties of the conjugation matrix are,

$$C_{\alpha\beta} = -C_{\beta\alpha}, \qquad C_{\alpha\beta}C^{\beta\gamma} = \delta_\alpha^\gamma, \qquad \psi^\alpha = C^{\alpha\beta}\psi_\beta,$$

and the spinor bilinears are written as,

$$\begin{aligned}\bar{\psi}\gamma^{(i)}\psi &= \psi^{\alpha}(\gamma^{(i)})_{\alpha}{}^{\beta}\chi_{\beta}\\ &= -\psi_{\gamma}C^{\gamma\alpha}(\gamma^{(i)})_{\alpha}{}^{\beta}\chi_{\beta}\\ &= -\psi_{\gamma}(\gamma^{(i)})^{\gamma\beta}\chi_{\beta}\end{aligned}$$

where the gamma matrix with both indices upward is implicitly contracted with the conjugation matrix. In SSGamma, the conjugation matrix is defined to be the gamma matrix without bosonic index, such that $\bar{\psi}\psi = \psi_{\alpha}C^{\alpha\beta}\psi_{\beta}$ must be written,

```
In[1]:= TS[ψ, {{1,α}}]**TG[{},{{0,α},{0,β}}]**TS[ψ, {{1,β}}]
Out[1]= ψ^α ** ψ_α
```

Equivalently, a bilinear with a $\gamma^{(2)}$ matrix can be written,

```
In[1]:= TS[ψ, {{1,α}}]**TG[{n,p},{{0,α},{0,β}}]**TS[ψ, {{1,β}}]
Out[1]= (γ_np)_αβ ψ^α**ψ^β
```

or with any other position of the fermionic indices, as long as they are contracted in NW-SE or NE-SW position. Two identical fermionic indices at the same up/down position must never occur : their contraction is not defined, and will result in errors.

Spinors in 10 dimensions — In 10 dimensions, spinors are Majorana-Weyl with 32 components. They can be split in two chiral parts, denoted using the up/down position of the fermionic indices, i.e. `TS[ψ,{{0,α}}]` is right-chiral, while `TS[ψ,{{1,α}}]` is left-chiral.

The conjugation and gamma matrices split accordingly (cf. appendix D.2.1), and we can choose,

$$C = \begin{bmatrix} 0 & -\delta^{\alpha}{}_{\beta} \\ \delta_{\alpha}{}^{\beta} & 0 \end{bmatrix}, \qquad \Gamma^{m} = \begin{bmatrix} 0 & (\gamma^{m})_{\alpha\beta} \\ (\gamma^{m})^{\alpha\beta} & 0 \end{bmatrix}, \qquad \Gamma^{mn} = \begin{bmatrix} (\gamma^{mn})^{\alpha}{}_{\beta} & 0 \\ 0 & (\gamma^{mn})_{\alpha}{}^{\beta} \end{bmatrix}, \quad \text{etc.}$$

Thus, in 10 dimensions, fermionic indices have positions allowed depending on the order of the gamma matrix. Except this particular rule, all other manipulations are the same as in the previous case, except that we manipulate half matrices (16x16). Bilinears are declared, for example, as

```
In[1]:= TS[ψ, {0,α}}]**TG[{n,p},{{1,α},{0,β}}]**TS[ψ, {{1,β}}]
Out[1]= (γ_np)^α_β ψ_α**ψ^β
```

Note that the matrix $(\gamma_{np})^{\alpha}{}_{\beta}$ can only be involved in a bilinear with two opposite chirality spinors, since the contrary would require $(\gamma_{np})^{\alpha\beta}$, which is not defined.

Manipulation of tensors & spinors — Derivatives can be applied on any expression using the function Der, deployed using the Leibniz rule. The process can be repeated several times :

```
In[1]:= 2*TS[B,{a,b}]*TS[F,{a,b,c}]*TS[R,{c,d,e,f}]
```
Out[1]= $2B_{ab}\,F_{abc}\,R_{cdef}$

```
In[2]:=Der[%,d]
```
Out[2]= $2\,(D_dB_{ab})\,F_{abc}\,R_{cdef}\;+\;2B_{ab}\,(D_dF_{abc})\,R_{cdef}\;+\;2B_{ab}\,F_{abc}\,(D_dR_{cdef})$

```
In[3]:=Der[%,e]
```
Out[3]= $2(D_eD_dB_{ab})\,F_{abc}\,R_{cdef}\;+\;2(D_dB_{ab})\,(D_eF_{abc})\,R_{cdef}\;+\;\ldots$

SSGamma also has a non-commutative product that can be useful for spinors and superspace fields with mixed indices. It is based on the built-in function NonCommutativeMultiply, which was vastly modified to suit our particular needs. Just like the function Times is implicitly called using $*$, NonCommutativeMultiply is called via $**$. For example, two anti-commuting spinors are declared using,

```
In[1]:= 11*TS[μ,{{1,α}}]**TS[λ,{{0,α}}]
```
Out[1]= $11\;\mu^{\alpha}{**}\lambda_{\alpha}$

The anti-commutation only cares for tensors : constants and gamma matrices are automatically excluded,

```
In[1]:= 2**TS[μ,{{1,α}}]**(z1+z2)**TS[λ,{{0,α}}]**TG[{a},{{0,β},{0,δ}}]
```
Out[1]= $11\,(z_1+z_2)(\gamma_a)_{\beta\delta}\;\mu^{\alpha}{**}\lambda_{\alpha}$

Contrary to the usual product, NonCommutativeMultiply will not automatically re-order the tensors on which it is applied. This role is played by another function, NCS (for NonCommutativeSimplify). NCS has two effects : it sorts the tensors linked by a non-commutative product,

```
In[1]:= TS[ρ,{{1,α}}]**TS[ν,{{1,β}}]**TS[μ,{{0,β}}]
```
Out[1]= $\rho_{\beta}{**}\nu^{\beta}{**}\mu^{\alpha}$

```
In[2]:=NCS[%]
```
Out[2]= $-\mu^{\alpha}{**}\nu^{\beta}{**}\rho_{\beta}$

(notice that this sorting does not necessarily gather contracted spinors) and it takes out of the product all tensors that should not anti-commute with the rest. More precisely, the package considers that anti-commutation comes from spinor indices : objects with an even number of such indices (including zero) must commute. However, since it sometimes happens that a bosonic tensor in introduced to be replaced later by an object with spinors indices, tensors are not automatically excluded from the non-commutative product. Calling NCS on an expression with a bosonic tensor acts as,

```
In[1]:= TS[μ,{{0,β}}]**TS[H,{a,b,c}]**TS[λ,{{0,α}}]
```
Out[1]= $\mu_{\beta}{**}H_{abc}{**}\lambda_{\alpha}$

```
In[2]:=NCS[%]
```
Out[2]= $-H_{abc}\ \lambda_\alpha$**μ_β

Delaying the exclusion of H_{abc} allows to replace it by, for example, $\lambda_\delta(\gamma_{abc})^{\delta\epsilon}\lambda_\epsilon$, which is composed of two spinors (whose place is in the non-commutative product).

In addition to the usual derivative, there exists a non-commutative derivative, called NCDer. It behaves like Der, but obeys the non-commutative Leibniz rule when it crosses a spinor index. For example,

---

Out[1]= μ_β**H_{abc}**λ_α

In[2]:= NCDer[%,{1,ϵ}]
Out[2]= $(D^\epsilon\lambda_\alpha)$**$H_{abc}$**$\mu_\beta$ $-$ λ_α**$(D^\epsilon H_{abc})$**μ_β $+$ λ_α**H_{abc}**$(D^\epsilon\mu_\beta)$

When working with all the non-commutative functions, one restriction must be kept in mind : there must be only one anti-commutative product in each expressions. For example, the following term

---

Out[1]= 11 μ_β**H_{abc}*μ_β**λ_α

---

will ultimately lead to errors at some point. If this sort of expression comes out, it is possible to transform the Time product into a NonCommutativeProduct using the replacement rule `% /. Times->NonCommutativeMultiply`. All tensors will then be gathered into a single non-commutative group.

The function FactorTens regroups stricly identical tensorial expressions (the dummy indices need to be properly renamed for the function to work) :

Out[1]= $z_1\ (\gamma_a)_\alpha{}^\beta\ G_{ab} + 2\ z_2\ (\gamma_a)_\alpha{}^\beta\ G_{ab}$

In[2]:= FactorTens[%]
Out[2]= $(z_1 + 2\ z_2)(\gamma_a)_\alpha{}^\beta\ G_{ab}$

Finally, the function LatexReady attempts to convert a Mathematica expression to the Latex syntax (although a few rearrangements might still be necessary) :

Out[1]= $\frac{12}{7}\ z1\ \delta_{a_1}^{b_1}\ (\gamma_{a_1a_2}\gamma_{c_1})_{\alpha\beta}\ G_{b_1b_2}$

```
In[2]:= LatexReady[%]
Out[2]= \frac{12}{7} \; z1 \; \delta_{a_{1}}^{b_{1}} \; (\gamma_{a_{1}
    a_{2}}\gamma_{c_{1}})_{{\alpha}{\beta}} \; G{}_{b_{1}b_{2}}
```

3.2.4 General functions

Renaming summed indices — The function RD (for RenameDummy) rename indices in a simple way, in order to merge tensorial expressions that only differ by the names of contracted indices. The new indices are taken from the six last letters of the alphabet, from $u_0, \ u_1, \ \ldots$ to $\ldots \ z_8, \ z_9$:

```
Out[1]= 
```
$$G_{a_1a_3}\,G_{a_1a_4}\,G_{a_2a_3}\,G_{a_2a_4}+G_{a_1a_2}\,G_{a_1a_4}\,G_{a_2a_3}\,G_{a_3a_4}+G_{a_1a_2}\,G_{a_1a_3}\,G_{a_2a_4}\,G_{a_3a_4}$$

```
In[2]:= RD[%]
Out[2]= 
```
$$3\,G_{u_0u_1}\,G_{u_0u_2}\,G_{u_1u_3}\,G_{u_2u_3}$$

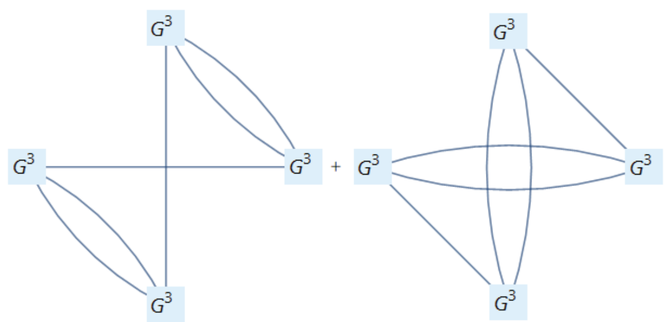
RD actually calls two functions, RDR and RDL, renaming the dummy indices form right to left, then from left to right. When RD fails to merge two identical set of tensors, those two functions can be applied independently for better results (for example, `RDL@RDL@RDR@RDR[...]` might sometimes help). This algorithm behind this function is exactly the same that is used in the original package GAMMA. The function was only adapted to better correspond to the "Mathematica style" of programming, relying heavily on pattern matching ; those simple changes allowed to speed up the execution by a factor 10 to 50.

In general, the canonical simplification of contracted tensorial expression is a notoriously hard problem (although in most cases, the function RD is sufficient). Of course, when all contracted tensors are distinguishable, their order can be determined easily, and there is no renaming problem (in this program, they are sorted by alphabetical order, then by number of indices, and finally, by alphabetical order of the list of indices). But when several contracted tenors are identical, renaming contracted indices quickly become a tricky problem.

Let's consider a problematic case where four 2-indices tensors G are contracted. The function SumGraph allows to represent the contractions in the form of graphs, where each vertex is a tensor, and each edge is a contracted index shared by the two vertex concerned :

```
Out[1]= 
```
$$G_{u_0u_1u_2}\,G_{u_0u_3u_4}\,G_{u_1u_2u_5}\,G_{u_3u_4u_5}+G_{u_0u_1u_2}\,G_{u_0u_1u_3}\,G_{u_2u_4u_5}\,G_{u_3u_4u_5}$$

```
In[2]:= SumGraph[%]
Out[2]=
```



With this sort of graphic representation, one clearly realize that those two expressions are actually very similar : swapping the position of two vertex in one of the graphs leads to the other one. However, no matter how RD, RDR or RDL are applied, those two tensors cannot be renamed to the same form (in order for Mathematica to merge them). This example is one of the simpler cases where this sort of complication can appear, but it becomes more and more frequent when manipulating larger expressions.

The function CanonicalRD is made to overcome this issue. It requires a much greater analysis, and is therefore considerably slower than RD. But as can be seen in the following example, it successfully simplifies the expression :

`Out[1]=` $G_{u_0 u_1 u_2}\, G_{u_0 u_3 u_4}\, G_{u_1 u_2 u_5}\, G_{u_3 u_4 u_5} + G_{u_0 u_1 u_2}\, G_{u_0 u_1 u_3}\, G_{u_2 u_4 u_5}\, G_{u_3 u_4 u_5}$

```
In[2]:= RD[%] // AbsoluteTiming
        CanonicalRD[%%] // AbsoluteTiming
```

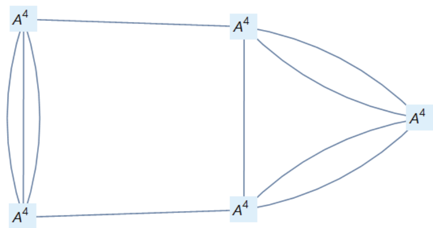
`Out[2]=` `{0.00051,` $G_{u_0 u_1 u_2}\, G_{u_0 u_3 u_4}\, G_{u_1 u_2 u_5}\, G_{u_3 u_4 u_5} + G_{u_0 u_1 u_2}\, G_{u_0 u_1 u_3}\, G_{u_2 u_4 u_5}\, G_{u_3 u_4 u_5}$`}`
`{0.0041,` $2\, G_{u_0 u_1 u_2}\, G_{u_0 u_1 u_3}\, G_{u_2 u_4 u_5}\, G_{u_3 u_4 u_5}$`}`

CanonicalRD works by finding a canonical form for the adjacency matrix associated to the tensorial expressions. In order to understand the scope and the limits of the function, let's review its inner details by considering the following expression [1] :

$$A_{a_0 a_1 a_3 a_8}\; A_{a_0 a_5 a_7 a_9}\; A_{a_1 a_3 a_6 a_8}\; A_{a_2 a_4 a_5 a_9}\; A_{a_2 a_4 a_6 a_7}$$

whose graph and adjacency matrix are

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \qquad \sim$$



Finding a canonical tensor ordering is equivalent to finding a canonical basis permutation for the adjacency matrix. A simple way to assign an order to different matrices is to flatten it, and then ordering them numerically. This method defines an order between any matrices (whether or not they are isomorphic). For example, among the following two matrices, the first one is the greater :

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow [1\ 2\ 1\ 2\ 0\ 1\ 1\ 1\ 0] \\ \vee \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow [0\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 0] \end{array} \tag{3.3}$$

1. The algorithm that is presented in this section is freely inspired from [85] and [86].

Finding the right vertex order (or tensor order) can simply be made by determining the greatest adjacency matrix among all the basis permutations. This method is by far the easier to implement, but has to order $n!$ matrices corresponding to the basis permutations (for n identical tensors). In most cases, $n!$ remains small, since one rarely deals with expressions involving 12 contracted identical tensors.

However, in an attempt to improve this scaling property, the algorithm first splits the identical tensors into groups, distinguishing them by their number of links with other vertex. The adjacency matrix of our five tensors A can be split in three groups,

$$\left.\begin{array}{l} \text{vertex 1} \\ \text{vertex 2} \\ \text{vertex 3} \\ \text{vertex 4} \\ \text{vertex 5} \end{array}\begin{bmatrix} 0 & 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \begin{array}{cc} \longrightarrow & 13 \\ \longrightarrow & 112 \\ \longrightarrow & 13 \\ \longrightarrow & 22 \\ \longrightarrow & 112 \end{array}\right\} \text{ 3 distinguishable groups (13, 112, 22)}$$

respectively containing two, two and one tensor. It is then possible to assign a pre-order to the matrix, by arranging the basis according to a simple numerical order $31 > 22 > 112$. A valid pre-ordered matrix is shown at the lhs of 3.4, where the two first basis vectors can still be exchanged, as well as the third and fourth. The last one is alone in its group, and will remain at the last position.

Now is the time to determine the global order of the matrix using the criteria 3.3. Since the matrix is pre-ordered, the algorithm has to review only $2!\ 2!\ 1!$ configurations, instead of $5!$. The highest matrix available is shown in the rhs of 3.4 :

$$\begin{array}{l} \text{vertex 3} \\ \text{vertex 1} \\ \hline \text{vertex 2} \\ \text{vertex 5} \\ \hline \text{vertex 4} \end{array} \left[\begin{array}{cc|cc|c} 0 & 3 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 2 & 2 & 0 \end{array}\right] \xrightarrow{\text{global ordering}} \begin{array}{l} \text{vertex 1} \\ \text{vertex 3} \\ \hline \text{vertex 2} \\ \text{vertex 5} \\ \hline \text{vertex 4} \end{array} \begin{bmatrix} 0 & 3 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix} \tag{3.4}$$

At this point, a canonical tensor order is fixed, and the indices can be renamed from left to right. Our example above gives

$$-A_{u_0u_1u_2u_3}\ A_{u_0u_4u_5u_6}\ A_{u_1u_7u_8u_9}\ A_{u_2u_3u_4u_5}\ A_{u_6u_7u_8u_9}\ .$$

This is the canonical renamed form of this expression, and every non-trivially isomorphic term would have been projected in this exact same term.

Some significant complications arise when one considers tensors with symmetries by exchange of groups of indices (like the Riemann tensor), because it is then necessary to decompose the tensor into all its exchangeable parts. The basic principle remains the same : pre-order the matrix as much as possible according to base-permutation invariant criteria, and determine the greatest matrix among the remaining permutations.

Symmetry and Anti-Symmetry — Most of the time, when tensorial expressions involve (anti)symmetrized indices, it is beneficial, in terms of efficiency, to keep the symmetry implicit. In this sense, the two following function are critically useful :

— AC (for Anti-symmetric Canonical Order) regroups a set of given indices, and order them in alphabetical order, with a minus sign for odd permutations.

— SC (for Symmetric Canonical Order) regroups a set of given indices, and order them in alphabetical order.

Both AC and SC make the expression vanish if they are applied on obviously symmetric and anti-symmetric indices, respectively. Those functions can be applied on any expression containing all the required indices.

`Out[1]=` $2\,\delta^{b_1 b_2}_{a_1 a_3}\;A_{a_2 a_4} + \delta^{b_1 b_2}_{a_1 a_2}\;A_{a_3 a_4}$

`In[2]:=` `SC[%,{`a_2, a_3`}]`
`AC[%%,{`a_2, a_3`}]`
`Out[2]=` $3\,\delta^{b_1 b_2}_{a_1 a_2}\;A_{a_3 a_4}$
$-\delta^{b_1 b_2}_{a_1 a_2}\;A_{a_3 a_4}$

The function ASym and Sym do the opposite :

— ASym makes anti-symmetry explicit on a given list of indices,

— Sym makes symmetry explicit on a given list of indices.

These functions work on bosonic and fermionic indices as well. Applied on a simple expression, this gives,

---

`Out[1]=`$(\gamma_{a_1 a_2 a_3})_\alpha{}^\beta\;A_{a_4 a_5}$

`In[2]:=` `ASym[%,{`a_1, a_2, a_3`}]`
`Sym[%%,{`a_3, a_4`}]`

`Out[2]=` $\frac{1}{3}\Big(-A_{a_4 a_5}(\gamma_{a_1 a_2 a_3})_\alpha{}^\beta + A_{a_2 a_5}(\gamma_{a_1 a_3 a_4})_\alpha{}^\beta - A_{a_1 a_5}(\gamma_{a_2 a_3 a_4})_\alpha{}^\beta\Big)$
$\frac{1}{2}\Big(A_{a_4 a_5}(\gamma_{a_1 a_2 a_3})_\alpha{}^\beta + A_{a_3 a_5}(\gamma_{a_1 a_2 a_4})_\alpha{}^\beta\Big)$

The functions Sym and ASym quickly become rather demanding, since anti-symmetrizing n indices can lead to $n!$ terms. Fortunately, one can take into account the preexisting symmetries of the tensors involved in the (anti)symmetrization, and considerably reduce the number of relevant permutations. The following example, although anti-symmetrized over 14 indices, only contains 3 432 terms, computed in a quarter of a second :

`Out[1]=`$\mathtt{A}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7}\;\mathtt{B}_{b_1 b_2 b_3 b_4 b_5 b_6 b_7}$,

`In[2]:=` `ASym[%,{`$a_1, a_2, a_3, a_4, a_5, a_6, a_7, b_1, b_2, b_3, b_4, b_5, b_6, b_7$`}]`

```
Out[2]= 1/3432 (A_{a1a2a3a4a5a6a7} B_{b1b2b3b4b5b6b7} + A_{a1a2a3a4a5a6b1} B_{a7b2b3b4b5b6b7} + 3430 terms)
```

while in the next case, the same anti-symmetrization leads to $14!$ terms, which is far too much to be computed and stored in a regular computer :

```
Out[1]=A_{a1} B_{a2} C_{a3} D_{a4} E_{a5} F_{a6} G_{a7} H_{b1} I_{b2} J_{b3} K_{b4} L_{b5} M_{b6} N_{b7}

In[2]:= ASym[%,{a1,a2,a3,a4,a5,a6,a7,b1,b2,b3,b4,b5,b6,b7}]
Out[2]= Error
```

The function SymCheck is designed to identify particular symmetries by exchanges of tensors that make an expression vanish. For example, the following term, with H being an anti-symmetric tensor, is annihilated by SymCheck

```
Out[1]= H_{ab}T_{ea}T_{eb}

In[2]:= SymCheck[%]
Out[2]= 0
```

since (ab) is anti-symmetric on H_{ab}, and symmetric on $T_{ea}T_{eb}$. This function is general enough to handle the vast majority of real-life case. However, it is not exhaustive, in the sense that highly non-trivial cancellations might not be detected.

Gamma tracelessness of tensors — The function GMET allows to simplify an expression involving contracted gamma matrices and gamma-traceless tensors. A gamma-traceless tensors verifies $(\gamma_a)_{\alpha\beta}\, T_{a_1\ldots a_n}{}^{\beta} = 0$, which also implies simplification of expressions of the form $(\gamma_{a_1\ldots a_n})_{\alpha}{}^{\beta}\, T_{a_1\ldots a_k}{}^{\beta}$ for $k < n$. Using the expansion 3.1, one can apply the following relation

$$(\gamma^{a_1\ldots a_n b_1\ldots b_k})_{\alpha}{}^{\beta}\; T_{b_1\ldots b_k} = (-1)^{\frac{1}{2}k(k-1)}\; k! \binom{n}{k}\; (\gamma_{[a_1\ldots a_k|})_{\alpha}{}^{\beta}\; T_{|a_{k+1}\ldots a_n]}\;,$$

in order to simplify some expressions. Note that the name of the traceless tensor must be specified in the second argument of GMET :

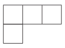
```
Out[1]= (γ_{abcde})_α^β J_{beβ}

In[2]:= GMET[%,J]
Out[2]= −6 (γ_a)_α^β J_{cdβ}
```

3.2.5 Group theory functions

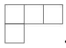
A explained in section (1.5.1), tensors can host representations of the symmetric group (or equivalently, the group $SU(N)$). There are several ways to project a tensor into a corresponding Young diagram, using the function IrrTens [2]. The projection of a 4 indices tensor T into the Young di-

2. The standard way of projecting a tensor in irreps of S_n can be found in [87], or more generally in [22]

gram  is called by

```
In[1]:= IrrTens[T,{{a,b},{c},{d}}]
```

Out[1]= $\frac{3}{2}\Big(\frac{1}{6}T_{abcd}+\frac{1}{6}T_{abdc}+\frac{1}{2}\big(\frac{1}{6}T_{acbd}-\frac{1}{6}T_{bcad}\big)+\frac{1}{2}\big(\frac{1}{6}T_{acdb}-\frac{1}{6}T_{bcda}\big)+\frac{1}{2}\big(\frac{1}{6}T_{adbc}-\frac{1}{6}T_{bdac}\big)+\frac{1}{2}\big(\frac{1}{6}T_{adcb}-\frac{1}{6}T_{bdca}\big)\Big)$

The output has all the symmetries expected of the irrep . When the function is called, the tensor T, anti-symmetric in its two first indices, is added to the list of defined tensors (and then cannot be used for anything else unless it is manually removed). In simple case, when the Young Diagram just has two columns, it is possible to remove all traces by specifying `NoTrace→True` as argument. IrrTens then projects the tensors into an irreducible representation of $SO(1,d-1)$ instead of a representation of the permutation group.

```
In[1]:= IrrTens[T,{{a,b},{c}}, NoTraces→True]
```

Out[1]= $\frac{2}{3}\big(T_{abc}+\frac{1}{2}(T_{acb}-T_{bca})\big)+\frac{1}{10}\big(-\delta^c_b\,T_{ar_1r_1}+\delta^c_a\,T_{br_1r_1}\big)$

The function IrrTens can also be applied on more general expressions, involving several types of tensors or gamma matrices, as long as the indices involved in the Young diagram are free to be (anti)symmetrized :

```
In[1]:= IrrTens[2*TG[{a,b,c},{{0,α},{1,β}}]*TS[H,{d,e}],{{a,b},{d}}]
```

Out[1]= $\frac{4}{3}\Big(\frac{1}{2}\big((\gamma_{bcd})_\alpha{}^\beta H_{ae}-(\gamma_{acd})_\alpha{}^\beta H_{be}\big)+(\gamma_{abc})_\alpha{}^\beta H_{de}\Big)$

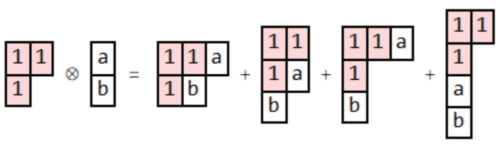
The general factor appearing on the left makes sure that IrrTens is indeed a projector, i.e. that $\text{IrrTens}^2=\text{IrrTens}$.

The function IrrYT allows to compute the tensor product of two (or more) Young diagrams. For example, the product  $\otimes$  is

```
In[1]:= IrrYT[{2,1},{2}]
Out[1]= {{3,2},{4,1},{2,2,1},{3,1,1}}
```

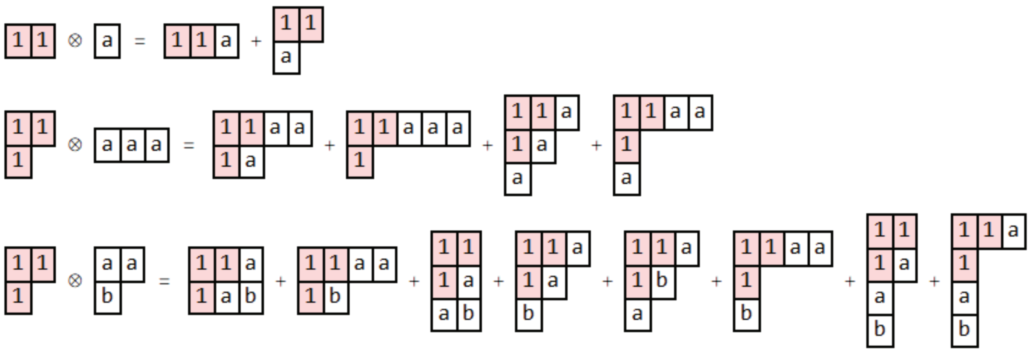
By default, the result is given as several lists of numbers representing the Young diagrams inside the product. For a visual representation, the user has to specify the optional argument `show→True` :

```
In[1]:= IrrYT[{2,1},{2}, show→True]
Out[1]= {{3,2},{2,2,1},{3,1,1},{2,1,1,1}}
```



If more than two diagrams are sent in arguments, the function returns the irreps corresponding to the whole product, but the diagrams drawn show the steps leading to the complete result :

```
In[1]:= IrrYT[{2,1},{2},{1} show→True]
Out[1]= {{{4,2},{5,1},{3,2,1},{4,1,1}}, ... }
```



Since this algorithm behaves very badly for large Young diagrams, it is recommended to use this function with care (it quickly leads to interminable calculations).

Finally, the function AllStandardYT returns all the standard Young tableaux associated to a specific arrangement of indices specified as argument. For example, all the standard Young tableaux associated to $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d \\ \cline{1-1}\end{array}$ can be displayed using,

```
In[1]:= AllStandardYT[{{a,b,c},{d}}]
Out[1]= {{{a,b,c},{d}}, {{a,b,d},{c}}, {{a,c,d},{b}}}
```

3.3 Some neat examples

This section presents in some details a few well-chosen examples, which show how to handle functions defined above in real life situations. Moreover, most of those examples are closely related to manipulations that are mentioned in chapter 1 and 3.

3.3.1 gamma-matrix identities and Fierzing

The program does not automatically computes Fierz identities, but provides a framework in which it is relatively easy to find and apply all sorts of identities related to gamma matrices. Let's start with a simple example in 11 dimensions, where one wants to simplify (or decompose into simpler parts) the expression

$$(\gamma^{e_1 e_2})_{(\alpha_1\alpha_2|}(\gamma_{ae_1e_2})_{|\alpha_3)\alpha_4}\,, \tag{3.5}$$

which contains one free bosonic index, and four spinor indices, three of which are symmetrized. This set of matrices can be viewed as a Clebsch-Gordan coefficient from the product of irreps $(00001)\otimes(00001)^{\otimes 3S}$ to the irrep (10000). Since $(00001)\otimes(00001)^{\otimes 3S} = 2(10000)+\ldots$, there are two independents irreps that can be found having the same indices content as 3.5. One can then assume that 3.5 can be split into two simpler terms (with unknown coefficients). In order to determine them, one defines the quantity

$$I^a_{\alpha_1\alpha_2\alpha_3,\alpha_4} = (\gamma_{e_1e_2})_{(\alpha_1\alpha_2|}(\gamma_a{}^{e_1e_2})_{|\alpha_3)\alpha_4} - z_1\,(\gamma^{e_1})_{(\alpha_1\alpha_2|}(\gamma_{ae_1})_{|\alpha_3)\alpha_4} - z_2\,(\gamma_a)_{(\alpha_1\alpha_2|}C_{|\alpha_3)\alpha_4} \stackrel{!}{=} 0$$

and try to contract it with several sets of gamma matrices to obtain enough constraints on z_1 and z_2. First, one has to make the symmetry over the spinor indices explicit, in order to contract with two well-chosen gamma matrices, here $\gamma^{p_1p_2}\gamma^{q_1}$:

In[1]:= GMS[Sym[$I^a_{\alpha_1\alpha_2\alpha_3,\alpha_4}$,{$\alpha_1,\alpha_2,\alpha_3$}]*
    TG[{p_1,p_2},{{1,α_1},{1,α_1}}]*TG[{q_1,q_2},{{1,α_3},{1,α_4}}]]
Out[1]= $-\frac{1}{6}\,z\,(\gamma_{a_1a_2}\gamma_{q_1})_{\alpha_3}{}^{\alpha_3}\,(\gamma_{a_3a_4}\gamma_{p_1p_2})_{\alpha_1}{}^{\alpha_1}\,G_{a_1a_2a_3a_4}$ + 10 other traces

---

Finally, one can launch the actual computation of all the traces, apply the Kroneckers, and factorize the result to obtain

---

In[3]:= FactorTens@FullDeltaSim@FullGMT[%]
Out[3]= $\frac{128}{3}\,(54+8z_1-z_2)\,\delta^{p_1p_2}_{aq_1}$

This first condition in not enough to determine the two coefficients. A second contraction, with $\gamma^{p_1\ldots p_5}\gamma^{q_1\ldots q_5}$ leads to

---

Out[4]= $\frac{64}{3}\,(10+z_2)\,\epsilon_{ap_1\ldots p_5q_1\ldots q_5}$

The two coefficients must now verify

$$\begin{cases} 10+z_2 & =0 \\ 54+8z_1-z_2 & =0 \end{cases} \quad\Rightarrow\quad \begin{cases} z_1=-8 \\ z_2=-10 \end{cases}$$

The example above is one of the simplest form of identities that can be shown, and does not reflects the potential of the program. Let's consider a less trivial example of identity, and suppose one has to simplify the expression

$$(\gamma^{a_1a_2e_1\ldots e_4})_{(\alpha_1\alpha_2|}\,(\gamma^{a_3a_4}{}_{e_2e_3e_4})_{|\alpha_3\alpha_4|}\,(\gamma^{b_1}{}_{e_1})_{|\alpha_5\alpha_6)}$$

containing six symmetrized spinorial indices, and where one forces the bosonic indices $(a_1\ldots a_4,b_1)$ to be in the irrep (10010). The decomposition of a symmetrized product of six spinors leads to several irreps, among which only one (10010) representation : $(00001)^{\otimes 6S} = 1(10010)+\ldots$. As above, one can expect a relation

$$I^{a_1\ldots a_4,b_1}_{\alpha_1\ldots\alpha_4} =$$
$$(\gamma^{a_1a_2e_1\ldots e_4})_{(\alpha_1\alpha_2|}\,(\gamma^{a_3a_4}{}_{e_2e_3e_4})_{|\alpha_3\alpha_4|}\,(\gamma^{b_1}{}_{e_1})_{|\alpha_5\alpha_6)} - z_1\,(\gamma^{[a_1a_2|})_{(\alpha_1\alpha_2|}\,(\gamma^{|a_3a_4]})_{|\alpha_3\alpha_4|}\,(\gamma^{b_1})_{|\alpha_5\alpha_6)} = 0$$

In this case, the free indices have to be projected into the (10010) representation by contracting them with the irreducible tensor $T^{(10010)}_{a_1a_2a_3a_4,b_1}$,

```
In[1]:= I^{a1...a4,b1}_{α1...α4}*IrrTens[T,{{a1,a2,a3,a4},{b1}}]
```

and then only, the symmetrization of the spinor indices and the contraction with three gamma matrices can be made. The first non-zero contraction is obtained with $(\gamma^{p_1p_2})^{\alpha_1\alpha_2}(\gamma^{q_1q_2})^{\alpha_3\alpha_4}(\gamma^{r_1})^{\alpha_5\alpha_6}$, and leads to

$$\frac{8192}{45}\,(1+15z_1)\Big(16\,T_{p_1p_2q_1q_2r_1}+8\,T_{p_1p_2q_1r_1q_2}+5\,\delta^{r_1}_{q_1}T_{p_1p_2q_2u_0u_0}+8\,T_{p_1q_1q_2r_1p_2}+5\,\delta^{r_1}_{p_1}T_{p_2q_1q_2u_0u_0}\Big)\,.$$

The only value z_1 can take is then $-1/15$. Of course, all the equations on z_1 found via other contractions (there are 10 of them) must be consistent with this one. Those can be checked for consistency, although in principle the first equation is enough the determine z_1 without ambiguity.

3.3.2 Replacement rules

Let's consider a completely different setup, where one is dealing with derivatives in superspace. For example, consider a computation leading to a spinorial and bosonic derivative acting on a superspace tensor T with three indices

```
Out[1]= 2(D_α D_a T_{bβ}^ε) .
```

Usually, mixed components of superspace tensors can be expressed using bosonic tensors, spinors and gamma matrices. Moreover, in all examples considered in this thesis, spinorial derivatives applied on tensors can be expressed in terms of other tensors. In this case, we will assume that $T_{b\beta}{}^{\epsilon}=(\gamma_a)_\beta{}^\epsilon\,H^a{}_b$, and that we already know the effect of a spinorial derivative on H.

In order to develop and simplify the previous expression, it is necessary to replace the tensor T by its expression involving H, and then use the derivative (anti)commutation relations to apply the spinor derivative on H.

By default, the program considers a derivative applied on a tensor as a single object that cannot be broken. In order to modify T itself, one first has to use the command ExplicitDer :

```
In[2]:= ExplicitDer[%]
Out[2]= 2 Der[T_{bβ}^ε,{a,{0,α}}]
```

which separates the derivative from the tensor, and displays it as a deactivated function applied on T (hence the coloring), with the indices of the derivatives as arguments. At this point, it is perfectly possible to replace the function Der by any user-defined operator with a replacement rule `% /.Der->Operator` (thus using the function Der only for its Leibniz rule property), but it does not concern our current goal. Now the derivative is separated from the T, one can simply replace T by its expression in terms of H,

```
In[3]:= % /. TS[T,{b,f1_,f2_}] :> TG[{c},{f1,f2}]*TS[H,{bc}]
Out[3]= 2 Der[(γ_c)_β^ε H_{bc},{a,{0,α}}]
```

and then use the Mathematica function ACTIVATE to re-apply DER on its new argument. The gamma matrix automatically goes through the derivations, and one simply gets

```
In[4]:= Activate[%]
Out[4]= 2(γc)β^ε(Dα Da Hbc).
```

To further expand the expression, on can use the rule that specifies the commutation relation of superspace derivatives,

$$[D_A, D_B)\, V_C \;=\; -T_{AB}{}^E D_E + R_{AB,C}{}^E V_E$$

to reverse the order of D_α and D_a (since we made the assumption that $D_\alpha H$ is known),

```
In[5]:= % /. Der[x_,{d1_,d2_}]:> Der[x,{d2,d1}]
     + TS[T,{d1,d2,e}]*Der[x,e] + TS[T,{d1,d2,{1,δ}}]*Der[x,{0,δ}]+ ...
Out[5]= 2(γc)β^ε(Da Dα Hbc + Taα^δ Hδc + Taαe Hec + ...)
```

where the dots stand for the superspace Riemann tensor involved in the relation. Other replacement rules can then be applied to substitute $D_\alpha H_{bc}$ by its non-superspace value, or replace any other part of the expression that requires simplification. Once the basic syntax of TS and TG is known, it become quite handy to replace or modify terms inside large expressions.

Of course, all the usual apparatus of Mathematica is available, like the repeated replacement (using the operator `//.`), or the conditional replacement (using `/;`). For example, the following instruction

```
In[4]:= % /. TS[s_,l_List]/;(s==H||s==G):> 0
```

replaces all tensors from the previous expression by zero, only if their name is H or G.

3.3.3 Superspace & non-commutative fields

Finally, we shall now consider a simple example of non-commutative spinor manipulation. Let's place ourselves in 10 dimensions, with two anti-commuting Majorana-Weyl spinors denoted λ_α and μ^α, involved in the following expression,

$$(\gamma^{ab}\lambda)^\alpha(\gamma_{ab}\lambda)^\beta\,, \tag{3.6}$$

that we want to simplify by Fierzing. In components, given the conventions adapted in (3.2.3), the expression has the form,

$$I^{\alpha\beta} \;=\; (\gamma^{abc})^{\alpha\epsilon_1}(\gamma_{ab})^{\beta\epsilon_2}\,\lambda_{\epsilon_1}\,\lambda_{\epsilon_2}\,,$$

The superspace manipulations of chapter 4 often lead to large expressions containing terms very similar to this one (with several non-contracted spinor indices, sometimes (anti)symmetrized).

Most of the time, equivalent configurations are mixed up in several different forms, and some Fierzing transformation have to be performed in order to get a manageable expression. In SSGAMMA, our example will be displayed in the form

Out[1]= $\lambda_{\epsilon_1} ** \lambda_{\epsilon_2} \ (\gamma^{abc})_{\alpha\epsilon_1}(\gamma_{abc})^{\beta\epsilon_2}$

As in the previous cases, possible all possible decompositions are found using irreducible representation. Here, the term (3.6) has two free fermionic indices (α, β) of the same chirality, that decompose as,

$$(00001)^{\otimes 2} \ = \ (10000) \oplus (00100) \oplus (00002) \ ,$$

i.e. a vector, a 3-form, and a self-dual 5-form (in this product, we ignored on purpose the anti-commutation property of the spinors). The product can then be cast on the basis,

$$\begin{aligned} &z_1 \ (\lambda\gamma^{h_1}\mu)(\gamma_{h_1})^{\alpha\beta} \\ &z_3 \ (\lambda\gamma^{h_1h_2h_3}\mu)(\gamma_{h_1h_2h_3})^{\alpha\beta} \\ &z_5 \ (\lambda\gamma^{h_1\ldots h_5}\mu)(\gamma_{h_1\ldots h_5})^{\alpha\beta} \ . \end{aligned}$$

The z_i are coefficients to be determined by enforcing the cancellation of $-I^{\alpha\beta} + z1(\cdots) + z3(\cdots) + z5(\cdots)$. The free indices can be contracted with $\gamma^{(1)}$, $\gamma^{(3)}$ and $\gamma^{(5)}$ (with the function GMS, GMT, GME and DELTASIM), respectively leading to,

Out[1]= 16 (18 + z_1) λ_{u_0}**λ_{u_0} $(\gamma_{h_1})^{u_0u_1}$
48 (-1 + 2 z_3) λ_{u_0}**λ_{u_0} $(\gamma_{h_1h_2h_3})^{u_0u_1}$
1920 z_5 λ_{u_0}**λ_{u_0} $(\gamma_{h_1\cdots h_5})^{u_0u_1}$ + 16 z_5 λ_{u_0}**λ_{u_0} $\epsilon_{h_1\cdots h_5u_2\cdots u_6}$ $(\gamma_{u_2\cdots u_6})^{u_0u_1}$

The parameters seem sufficiently constrained, but applying the function SYMMETRYCHECK on the first and the third make them vanish, by virtue of the symmetry properties of the anti-commuting product (alternatively, a gamma matrix of the third term can be dualized using GMD, and cancels against the other). The remaining constraint only applies to $z3$, and requires $z_3 = 1/2$, while the others must be zero (of course, in this case, z_1 and z_5 could be discarded from the beginning, but some cases involving bigger identities are less trivial).

As can be seen from the last few examples, there is no systematic way of dealing with Fierzing in the package. This choice is deliberate : it turns out that gamma-matrix identities are so diverse that we preferred to keep a slightly less automatized process, allowing the user to deal with a far broader range of identities.

4

Fermionic condensation in IIA supergravity

4.1 IIA supergravities

4.1.1 Basic features of IIA supergravities

Contrary to the eleven-dimensional case, several types of supergravity exist in ten dimensions. The maximal number of supercharges is 32, and since the dimension is even, different types are distinguished by the chirality of their supercharges,

Type	I	IIA	IIB
$\mathcal{N}$	$(1,0)$	$(1,1)$	$(2,0)$

The subject of the present chapter only concerns IIA supergravity, and particularly, one of the massive deformations of it (which includes massless IIA as the $m \to 0$ limit).

Compactification from eleven-dimensional supergravity — Supergravity in eleven dimensions has been compactified in several ways, thus leading to several other supergravity theories. The main purpose of finding a $\mathcal{N} = 1$ eleven-dimensional supergravity was to reduce it to the four-dimensional $\mathcal{N} = 8$ maximal supergravity [88], and it was later used to build the $\mathcal{N} = (1,1)$ theory in ten dimensions [89]. The two compactifications above follow the prescription of Kaluza-Klein theory [90] : going from an 11 to a k-dimensional space requires to split the manifold as follows,

$$\begin{aligned} M_{11} &\longrightarrow M_k \times T_{11-k} \qquad\qquad (1 \leq k \leq 10) \\ (X_0, X_1, \ldots, X_{10}) &\longrightarrow (x_0, x_1, \ldots, x_{k-1}, z_1, \ldots, z_{11-k}) \,, \end{aligned}$$

and separate the fields accordingly, depending on whether they carry indices from M_k of T_{11-k}. Besides, all fields are assumed to loose their dependency on the compactified dimensions.

Let's take $M_{11} \to M_{10} \times S_1$ as an illustration, and denote indices from TM_{11} (TM_{10}) with capital (lowercase) letters. Fields defined on M_{11} are denoted with a bold symbol. The usual partitioning of the vielbein will dictate the splitting of the other bosonic fields, and reads,

$$\boldsymbol{e}_A^M \;=\; e^{\frac{2}{3}\phi} \cdot \left[\begin{array}{c|c} e^{-\frac{3}{4}\phi}\, e_a^m & \sqrt{2}A_a \\ \hline 0 & 1 \end{array}\right] ,$$

where two new fields were introduced : a vector A and a scalar field ϕ, the dilaton (the triangular form can be preserved upon supersymmetric transformation using the Lorentz gauge freedom). This splitting of the vielbein lead to the metric splitting $\boldsymbol{g}_{11} \;=\; e^{-\frac{1}{6}\phi} + e^{\frac{4}{3}\phi}(dz + A)^{\otimes 2}$. Form fields on M_{10} are decomposed using the rule

$$\begin{aligned}
\boldsymbol{F}_{a_1 \dots a_n} &= \boldsymbol{e}_{a_1}^{M_1} \dots \boldsymbol{e}_{a_n}^{M_n}\; \boldsymbol{F}_{M_1 \dots M_n} \\
&= e^{-\frac{n-1}{12}\phi} \left(e^{-\frac{1}{12}\phi}\, e_{a_1}^{m_1} \dots e_{a_n}^{m_n} + \sqrt{2}\, n\, e^{\frac{2}{3}\phi}\; \delta_{11}^{m_1} A_{[a_1} e_{a_2}^{m_2} \dots e_{a_n]}^{m_n} \right) F_{m_1 \dots m_n} \\
&= e^{-\frac{n}{12}\phi} F_{a_1 \dots a_n} + \sqrt{2}\, n\, e^{\left(\frac{2}{3} - \frac{n-1}{12}\right)\phi}\; A_{[a_1|} F_{11\,|a_2 \dots a_n]} \;.
\end{aligned}$$

The last term of this expression, $F_{11\,a_2 \dots a_n}$, can be renamed as a new $(n{-}1)$ form since its first index is now fixed. If this form field in 11 dimensions can be written as $n\,\partial_{[M_1} C_{M_2 \dots M_n]}$, then the same expansion has to be applied on the rhs, thus leading to the introduction of a $(n-2)$ form, $C_{11\,a_3 \dots a_n}$ by the same process. In order to reconstruct a supersymmetric action in 10d, the gravitino must be decomposed as

$$\boldsymbol{\psi}_M \longrightarrow \begin{cases} \boldsymbol{\psi}_m &= e^{-\frac{1}{24}\phi}\left(\psi_m + \frac{\sqrt{2}}{12} \Gamma_{11} \Gamma_m \Lambda \right) + \sqrt{2} A_m \psi_{11} \\ \boldsymbol{\psi}_{11} &= \frac{2\sqrt{2}}{3} e^{\frac{17}{24}\phi}\, \Lambda \end{cases} ,$$

where a spinor Λ have been introduced. Since spinors in 10d are Majorana-Weyl, both ψ_m and Λ can be projected on a right and left chiral parts using $\frac{1}{2}\,(1 \pm \Gamma_{11})$.

The transformations rules in 11 dimensions are also decomposed in a similar fashion. The parameters are split, and give rise to new symmetries in the reduced theory. For example, the transformation of the 11d vielbein under diffeomorphism $\delta x_m = \xi_m(x)$,

$$\delta_\xi E_M^A \;=\; \xi^P \left(\partial_P E_M^A \right) \;+\; \left(\partial_M \xi^P \right) E_P^A,$$

will impose the gauge transformation $A_m \;\to\; A_m + \frac{-1}{\sqrt{2}} \partial_m \xi_{11}$ (when restricted to the A_m part of the vielbein). The remaining 10d supersymmetric transformations are computed straightforwardly from 11d, with the addition of a 10d Lorentz rotation with parameter $\bar{\epsilon}\, \Gamma_{11} \Gamma_{mn} \Lambda$. The 10d theory thus obtained contains 10 fields :

$$\boldsymbol{\psi}_M^{(11)} \to \begin{cases} \psi_m^{\pm} \\ \Lambda^{\pm} \end{cases} \qquad \boldsymbol{e}_M^A \to \begin{cases} e_m^a \\ C_a \\ \phi \end{cases} \qquad \boldsymbol{G}_4 = \mathrm{d}\boldsymbol{C}_3 \to \begin{cases} G_4 = \mathrm{d}C_3 \\ G_3 = \mathrm{d}C_3 + G_2 \wedge C_2 \\ G_2 = \mathrm{d}C \end{cases} ,$$

and the Lagrangian, with the supersymmetric transformation rules can be found in ([89]).

Massive deformations of ten-dimensional supergravity — The compactification of the previous section leads to a supergravity where all fields are massless. However, there are two ways to deform this theory to include a massive object (these were found in [91] and [92], and it was later proven in [93] that no other deformations are allowed). The first one involves a compactification from a generalized 11d supergravity, while the second one is based on a mechanism taking place within IIA massless supergravity :

— HLW supergravity (for Howe, Lambert, and West [92]) was built by dimensional compactification from a slightly generalized version of 11d supergravity, where the $Spin(1,10)$ connection is extended to a $Spin(1,10) \times \mathbb{R}^+$ connection,

$$\omega_A{}^B \longrightarrow \omega_A{}^B + \delta_A^B K \, .$$

With this adjustment, re-deriving the equations of motion using the superspace Bianchi identities leads to a theory that is equivalent to the original (up to a super-Weyl transformation) if the space M_{11} is simply connected [23]. However, when considering a compactification on (the non-simply connected space) $M_{10} \times S_1$, one obtains a different theory of supergravity that includes a massive vector field.

— Romans supergravity [91] was the first massive deformation built from IIA supergravity, and does not rely on 11d supergravity. In this theory, the mass is introduced by hand, by merging the fields C_2 and the field strength G_2 in a modified field-strength $G_2^{'} = G_2 + mC_2$. Then, gauging the theory and adding suitable terms in the Lagrangian to preserve supersymmetry leads to a Lagrangian where C_1 can be absorbed in C_2, thus leading to a massive but non-gauge-invariant C_2 [94].

4.2 Dilatino condensates in massive IIA supergravity

In this chapter, we are interested in giving a non-zero expectation value to the dilatino condensate in (massive) IIA supergravity. For that purpose we need to know the terms both quadratic and quartic in the dilatino.

4.2.1 Fermionic condensation in IIA supergravity

Fermionic condensates have been considered in the past mostly in the context of heterotic theory [95–102] and, to a lesser extent, in eleven-dimensional supergravity [103, 104]. Of course, spinor vacuum expectation values (vevs) must vanish in a Lorentz-invariant vacuum, however scalar quadratic- and quartic-fermion condensates are allowed by the symmetry of the vacuum and may be generated by quantum effects.

In type IIA theory there is a single scalar that can be constructed in ten dimensions out of four dilatini, as can be seen by the decomposition of the possible arrangement of right- and left-chiral anti-commuting $Spin(1,9)$ spinors,

$$\begin{aligned} (00001)^{\otimes_A 4} &\not\subset (00000) \\ (00010) \otimes (00001)^{\otimes_A 3} &\not\subset (00000) \\ (00010)^{\otimes_A 2} \otimes (00001)^{\otimes_A 2} &\subset 1 \times (00000) \end{aligned}$$

The presence of a unique quartic-dilatino term in the action thus gives a simple and interesting possibility to generate a positive cosmological constant via fermionic condensation. For this to be actually possible, the quartic dilatino term has to be positive. Since massless IIA supergravity was derived in [89, 105, 106] with quartic fermions terms, and in [107] for massive IIA, one could therefore use these references to provide the "missing" quartic-fermion terms of Romans supergravity. However, previous attempts failed to conclude whether the quartic-fermion terms in [89, 105–107] agree with each other.

On the other hand all IIA supergravities admit a unified superspace formulation, given in [93], in which the quartic-fermion terms are given implicitly (unfortunately their explicit form was not worked out in this paper). With the help of SSGamma, our strategy will be to first determine the fermionic action, S_f, up to gravitino terms. This action S_f, that we call the *dilatino-condensate action* is obtained from the full fermionic action by setting the gravitino to zero.

4.2.2 Massive IIA in superspace, with fermionic fields

As mentioned above, all IIA supergravities emerge from a common superspace derivation, starting not from a correspondence between physical fields and superfields, but directly by the resolution of the super-Bianchi identities. This set up provides more freedom, as it allows a progressive identification of the emerging structures with the physical fields. The geometrical superfields involved, and the corresponding Bianchi identities are,

$$\begin{array}{rl|ll} DE^A = & T^A & DT^A & = E^B\, R_A{}^B \\ \mathrm{d}\Omega_A{}^B + \Omega_A{}^C{}_\wedge\, \Omega_C{}^B = & R_A{}^B & DR_A{}^B & = 0\,. \end{array} \tag{4.1}$$

The first step to resolve the Bianchi identities is to use the redefinition freedom of the vielbein and the spin connection to define the most simple, yet general starting point for the low dimensional fields involved in the equations. For example, the most general value of the mass-dimension zero torsion is given by the following product of irreps,

$$(10000) \otimes (00001)^{\otimes_S 2} = 1(00000) \oplus \ldots \quad \Rightarrow \quad T_{\alpha\beta}{}^c = f(\phi)(\gamma^c)_{\alpha\beta}\,,$$

where $f(\phi)$ could be any function of the dimension zero dilaton. However, it is possible to start with a simpler version of this, defining all the possible shifts of E, Ω and T as,

$$\left.\begin{array}{rl} h_A{}^B = & E_A^M\,\delta E_M^B \\ d_{AB}{}^C = & E_A^M\,\delta\Omega_{MB}{}^C \end{array}\right\} \;\Rightarrow\; \delta T_{AB}{}^C = 2d_{[AB)}{}^C + 2D_{[A}h_{B)}{}^C - 2h_{[A|}{}^D\,T_{D|B)}{}^C + T_{AB}{}^D\,h_D{}^C\ .$$

It is then possible to use all the degrees of freedom of h and d to redefine the components of T as it suits us best. In the example above, using one of the h components among ($h_\alpha{}^\beta$, $h^\alpha{}_\beta$, and $h_a{}^b$) allows to shift to a more familiar ansatz for the torsion,

$$T_{\alpha\beta}{}^c: \quad f(\phi)(\gamma^c)_{\alpha\beta} \xrightarrow{\text{gauge fixing}} -i(\gamma^c)_{\alpha\beta}$$

Once all the components of h and d are exhausted, the proper resolution of the Bianchi identities can begin with the values,

$$\begin{array}{lll} T_{\alpha\beta}{}^c = -i(\gamma^c)_{\alpha\beta} & T_{\alpha b}{}^c = 0 & \\ T^{\alpha\beta c} = -i(\gamma^c)^{\alpha\beta} & T^\alpha{}_b{}^c = 0 & T_{ab}{}^c = 0 \end{array} \tag{4.2}$$

Then follows a rather tedious process, where the Bianchi identities are systematically written (from mass dimension $1/2$ to 2), and solved. Of course, since only a few torsion are already determined (4.2), this requires the introduction of fields that are identified with those of IIA supergravities. Let's consider as an example the first identity on the torsion, of dimension $1/2$:

$$D_{(\alpha}T_{\beta\gamma)}{}^e + T_{(\alpha\beta|}{}^\epsilon T_{\epsilon|\gamma)}{}^e \;=\; R_{(\alpha\beta\gamma)}{}^e \;=\; 0\ ,$$

where the Riemann tensor vanishes due to the Lorentz condition, and where the symmetrized indices can take both up and down position, depending on the chirality they represent (thus leading to 2^3 equations). All indices in lower position, with the ansatz (4.2), reads,

$$T_{(\alpha\beta|}{}^\epsilon\,(\gamma^e)_{\epsilon|\gamma)} = 0\ , \tag{4.3}$$

The mass dimension one half imposes the use of a spinor, that is taken to be the dilatino λ_α of IIA supergravity. Then we have to find the most general expression involving λ, that obeys equation (4.3). Since

$$(00010)^{\otimes_S 2} \otimes (00001) \;=\; 2(00001) \oplus \dots\ ,$$

there are two ways of building an expression with λ that have the index content of the torsion $T_{\alpha\beta}{}^\epsilon$. It should have the following structure,

$$z_1\ \lambda_{(\alpha}\delta_{\beta)} + z_2\ (\gamma^f)_{\alpha\beta}(\gamma_f\lambda)^\gamma\ ,$$

where the two coefficients can be constrained by imposing the BI with this particular ansatz. It leads $z_2 = \frac{1}{2}z_1$. The other seven versions of this BI partially define the other lower components of the torsion, and all involve one of the two chiral spinors λ_α or μ^α. The next BI will involve several objects that require the introduction of new fields. For example, it will contain unconstrained terms like $D_\alpha \mu_\beta$, and $T_{a\alpha}{}^\beta$ that have to be decomposed as,

$$\begin{array}{r|ccccc} {\scriptstyle (00001)\otimes(00010)} & {\scriptstyle (00000)} & \oplus & {\scriptstyle (01000)} & \oplus & {\scriptstyle (00002)} \\ D_\alpha \mu^\beta & L\,\delta_\alpha^\beta & + & L_{ef}\,(\gamma^{ef})_\alpha{}^\beta & + & L_{e_1\dots e_4}\,(\gamma^{e_1\dots e_4})_\alpha{}^\beta \end{array}$$

$$\begin{array}{r|ccccccc} {\scriptstyle (10000)\otimes(00001)\otimes(00010)} & {\scriptstyle (00000)} & \oplus & {\scriptstyle (01000)} & \oplus & {\scriptstyle (00002)} & \oplus & {\scriptstyle (00002)} \\ T_{a\alpha}{}^\beta & V_a^{(1)}\,\delta_\alpha^\beta & + & V_e^{(2)}\,(\gamma_a{}^e)_\alpha{}^\beta & + & (\gamma_a{}^{efg})_\alpha{}^\beta H^{(1)}_{efg} & + & (\gamma_a{}^{ef})_\alpha{}^\beta H^{(2)}_{aef}\,, \end{array}$$

where the tensors H and V can potentially be interpreted as other physical fields of the theory. Similarly to $D_\alpha \mu^\beta$, the expression $D^\alpha \lambda_\beta$ is decomposed as $L'\,\delta^\alpha_\beta + \dots$. The value of the two dimension 1 parameters (L, L') will later turn out to determine which supergravity is obtained by all this development. Indeed, the equations of motion for the dilatini (obtained at dimension $3/2$) and the bosonic fields (obtained at dimension 2) are similar to one of the three known supergravities, depending on the value of L and L', as

$$\begin{array}{ccc} \text{Massless} & \text{Romans} & \text{HLW} \\ \hline L\ =\ \frac{3}{4}(\mu\lambda) & L\ =\ \frac{1}{2}me^{2\phi}+\frac{3}{4}(\mu\lambda) & L\ =\ \frac{3}{2}m+\frac{3}{4}(\mu\lambda) \\ L'=-\frac{3}{4}(\mu\lambda) & L'=-\frac{1}{2}me^{2\phi}-\frac{3}{4}(\mu\lambda) & L'=\frac{3}{2}m-\frac{3}{4}(\mu\lambda)\,, \end{array} \tag{4.4}$$

where, as expected, the massless case is retrieved from both massive supergravities. In the following, the discussion is restricted to the Romans case, bearing in mind that all can be adapted effortlessly for all three theories. The equations of motion for the spinors are obtained at the BI of dimension $3/2$, and are written in (4.25) of [93]. Finally, the usual Bianchi identities and the equation of motion for the bosonic fields all emerge from the dimension 2 superspace Bianchi identity,

$$\frac{1}{3}D_\alpha T_{ab}{}^\delta + \frac{2}{3}D_{[a|}T_{|b]\alpha}{}^\delta + \frac{2}{3}T_{\alpha[a|}{}^E T_{E|b]}{}^\delta + \frac{1}{3}T_{[ab|}{}^E T_{E\alpha}{}^\delta \ = \ R_{ab\alpha}{}^\delta\,,$$

where the index E span the whole superspace, and the indices (α, δ) can both be in up/down position, thus leading to four inequivalent equations. At that point, all the relevant quantities have been defined, and those identities are just constraints on the physical fields. The 2×2 possible positions of the fermionic indices lead to four identities, that are explicitly written in (3.91) of [93], as $\mathcal{A} = \mathcal{B} = \mathcal{C} = \mathcal{D} = 0$ [1]. Setting all fermionic superfields to zero and restricting to the x-space component of the bosonic superfields (i.e. the lowest-order term in the theta-expansion), the content of the $\mathcal{B} = \mathcal{C} = 0$ equations can be seen to be equivalent to the following set of

1. As mentioned above, in the reference [93], fermionic fields are set to zero. Our present computation does not make this assumption, so that each expression $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ now contains all the fermionic terms. We could therefore find the exact Bianchi identities and equations of motion for the fields using these four expressions. However, we shall set the fermionic fields to zero, and use a simpler method that we explain below.

equations,

$$
\begin{aligned}
0 &= \mathrm{d}L_2 \ + \ \tfrac{18i}{5}\, m e^{2\phi} K_3 \\
0 &= i\mathrm{d}L_4 \ - \ \tfrac{2}{3}\, K_3{\wedge}L_2 \ + \ 4\, K_1{\wedge}L_4 \\
0 &= i\mathrm{d}\star L_4 \ + \ 8\, K_1{\wedge}\star L_4 \ + \ 24\, K_3{\wedge}L_4 \\
0 &= i\mathrm{d}\star L_2 \ + \ 12\, K_1{\wedge}\star L_2 \ + \ 864\, K_3{\wedge}\star L_4 \, .
\end{aligned} \tag{4.5}
$$

The $\mathcal{A} = \mathcal{D} = 0$ equations can be seen to be equivalent to,

$$
\begin{aligned}
&0 = \mathrm{d}K_1 \\
&0 = i\mathrm{d}K_3 \ - \ 4\, K_1{\wedge}K_3 \qquad (4.6)\\
&0 = i\mathrm{d}\star K_3 \ - \ 8\, K_1{\wedge}\star K_3 \ - \ \tfrac{128}{3}\, L_2{\wedge}\star L_4 \ - \ 768\, L_4{\wedge}L_4 \ - \ \tfrac{8}{45}\, m e^{2\phi}\star L_2 \\
&0 = i\mathrm{d}\star K_1 \ - \ 12\, K_1{\wedge}\star K_1 \ - \ \tfrac{32}{3}\, L_2{\wedge}\star L_2 \ - \ 144\, K_3{\wedge}\star K_3 \ - \ 4608\, L_4{\wedge}\star L_4 \ + \ \tfrac{2}{5}\, m^2 e^{4\phi} \, ,
\end{aligned}
$$

together with the Einstein equation,

$$
\begin{aligned}
R_{mn} = g_{mn}&\Big(\frac{3i}{2}\nabla\cdot K_{(1)} + 18K^2_{(1)} - \frac{1}{25}m^2 e^{4\phi}\Big) \\
&+ 12i\nabla_{(m}K_{n)} - 16K_m K_n - \frac{64}{9}\Big(2L^2_{(2)mn} - \frac{1}{8}g_{mn}L^2_{(2)}\Big) \\
&+ 48\Big(3K^2_{(3)mn} - \frac{1}{4}g_{mn}K^2_{(3)}\Big) - 768\Big(4L^2_{(4)mn} - \frac{3}{8}g_{mn}L^2_{(4)}\Big) \, ,
\end{aligned} \tag{4.7}
$$

where we have set $\Phi^2_{(p)} := \Phi_{m_1\dots m_p}\Phi^{m_1\dots m_p}$, $\Phi^2_{(p)mn} := \Phi_{mm_2\dots m_p}\Phi_n{}^{m_2\dots m_p}$, for any p-form Φ. Moreover in order to put the Einstein equation in the form (4.7) we have made use of the last equation in (4.6). Note that the latter can be obtained by acting on the equations of motion of the fermionic superfield, cf. (4.16) below, with a spinor derivative and contracting the free spinor indices with each other.

The first equation in (4.6) above can be solved by introducing a scalar field ϕ,

$$
K_{(1)} = \frac{i}{2}\mathrm{d}\phi \, , \tag{4.8}
$$

where the normalization has been chosen so that ϕ is identified with the dilaton. The equations above are not automatically expressed in the Einstein frame in ten dimensions. To transform to the Einstein frame we define a new Weyl-rescaled metric,

$$
\hat{g}_{mn} = e^{\frac{3}{2}\phi} g_{mn} \, . \tag{4.9}
$$

The Einstein equation then takes the form,

$$\hat{R}_{mn} = -\frac{1}{2}\partial_m\phi\partial_n\phi - \frac{1}{25}m^2e^{4\phi}g_{mn} - \frac{64}{9}\Big(2L^2_{(2)mn} - \frac{1}{8}g_{mn}L^2_{(2)}\Big) + 48\Big(3K^2_{(3)mn} - \frac{1}{4}g_{mn}K^2_{(3)}\Big) - 768\Big(4L^2_{(4)mn} - \frac{3}{8}g_{mn}L^2_{(4)}\Big)\,, \tag{4.10}$$

where $\hat{R}_{mn}$ is the Ricci tensor of $\hat{g}$; the contractions on the right-hand side are taken with respect to the unrescaled metric g.

The equations above can be recognized as the bosonic equations of Romans supergravity. For example one can readily make contact with the formulation of [21] by using the following dictionary,

Here	L_2	L_4	K_3	$\hat{g}_{mn}$	R	m
There	$-\frac{3}{16}F$	$-\frac{i}{24}e^{-2\phi}H$	$\frac{1}{192}e^{-2\phi}G$	g_{mn}	$-R$	$\frac{5}{2}m$,

(4.11)

up to fermionic bilinear terms which will be determined in the following, cf. (4.19) below. Using these more familiar fields, the equations of motion read,

$$\begin{aligned}
0 &= \hat{\nabla}^2\phi - \frac{3}{8}e^{3\phi/2}F^2 + \frac{1}{12}e^{-\phi}H^2 - \frac{1}{96}e^{\phi/2}G^2 - \frac{4}{5}m^2e^{5\phi/2}\\
0 &= \mathrm{d}(e^{3\phi/2}\hat{\star}F) + e^{\phi/2}H{\wedge}\hat{\star}G\\
0 &= \mathrm{d}(e^{-\phi}\hat{\star}H) + e^{\phi/2}F{\wedge}\hat{\star}G - \frac{1}{2}G{\wedge}G + \frac{4}{5}me^{3\phi/2}\hat{\star}F\\
0 &= \mathrm{d}(e^{\phi/2}\hat{\star}G) - H{\wedge}G\,,
\end{aligned} \tag{4.12}$$

where the covariant derivative $\hat{\nabla}$ and the Hodge star $\hat{\star}$ are taken with respect to the rescaled metric $\hat{g}$, and,

$$0 = \hat{R}_{mn} + \frac{1}{2}\partial_m\phi\,\partial_n\phi + \frac{1}{25}m^2e^{5\phi/2}\hat{g}_{mn} + \frac{1}{4}e^{3\phi/2}\Big(2F^2_{(2)mn} - \frac{1}{8}\hat{g}_{mn}F^2_{(2)}\Big) + \frac{1}{12}e^{-\phi}\Big(3H^2_{(3)mn} - \frac{1}{4}\hat{g}_{mn}H^2_{(3)}\Big) + \frac{1}{48}e^{\phi/2}\Big(4G^2_{(4)mn} - \frac{3}{8}\hat{g}_{mn}G^2_{(4)}\Big)\,, \tag{4.13}$$

where the contractions on the right-hand side are computed using $\hat{g}$. Moreover the forms obey the following Bianchi identities,

$$\mathrm{d}F = \frac{4}{5}mH\,, \qquad \mathrm{d}H = 0\,, \qquad \mathrm{d}G = H{\wedge}F\,. \tag{4.14}$$

It can also be checked that the equations of motion integrate to the following bosonic action in the Einstein frame, cf. (2.1) of [21],

$$S_b = \int \mathrm{d}^{10}x\sqrt{\hat{g}}\Big(\hat{R} + \frac{1}{2}(\partial\phi)^2 + \frac{8}{25}m^2e^{5\phi/2} + \frac{1}{2!2}e^{3\phi/2}F^2 + \frac{1}{3!2}e^{-\phi}H^2 + \frac{1}{4!2}e^{\phi/2}G^2\Big) + \mathrm{CS}\,, \tag{4.15}$$

where contractions are taken with respect to the rescaled metric and CS denotes the Chern-Simons term.

The dilatino terms — As explained in the introduction, we are interested in determining the fermionic action up to gravitino terms. The fermionic equations of motion appear at dimension 3/2 and are given in (4.25) of [93],

$$
\begin{aligned}
i\nabla\!\!\!/\lambda &= -\frac{24}{5}me^{2\phi}\mu - \frac{36}{5}(\mu\lambda)\mu - \frac{16}{3}L_{(2)}(\gamma^{(2)}\mu) \\
&\quad - 12K_{(1)}(\gamma^{(1)}\lambda) + 3K_{(3)}(\gamma^{(3)}\lambda) + \frac{3}{40}(\mu\gamma_{(3)}\mu)(\gamma^{(3)}\lambda) \\
i\nabla\!\!\!/\mu &= \frac{24}{5}me^{2\phi}\lambda + \frac{36}{5}(\mu\lambda)\lambda - \frac{16}{3}L_{(2)}(\gamma^{(2)}\lambda) \\
&\quad - 12K_{(1)}(\gamma^{(1)}\mu) - 3K_{(3)}(\gamma^{(3)}\mu) + \frac{3}{40}(\lambda\gamma_{(3)}\lambda)(\gamma^{(3)}\mu)\ .
\end{aligned}
\tag{4.16}
$$

These are exact superfield equations, i.e. valid to all orders in the theta-expansion. In order to identify the fermionic part of the action giving rise to these equations we must first address the following two issues : **Firstly**, once the fermionic superfields are turned on, the bosonic equations (4.5), (4.6), (4.7) will be violated by terms quadratic and quartic in the fermion superfields. In other words, these equations are not valid as full-fledged superspace equations for superfields. In particular, the superspace Bianchi identities for the superforms at mass dimension 1, read,

$$
\begin{aligned}
0 &= \mathrm{d}\hat{K}_1 \\
0 &= \mathrm{d}\hat{L}_2 + \frac{18}{5}\, me^{2\phi}\, \hat{K}_3 \\
0 &= \mathrm{d}\hat{K}_3 + 4i\ \hat{K}_1{\wedge}\hat{K}_3 \\
0 &= \mathrm{d}\hat{L}_4 + \frac{2i}{3}\ \hat{L}_2{\wedge}\hat{K}_3 - 4i\ \hat{K}_1{\wedge}\hat{L}_4\ ,
\end{aligned}
\tag{4.17}
$$

where the hatted superfields differ in general from the unhatted ones by spinor superfield bilinears. Explicitly in components the Bianchi identities read :

$$
\begin{aligned}
0 &= D_{[A}\hat{K}_{B)} + \frac{1}{2}T_{AB}{}^{F}\hat{K}_F \\
0 &= D_{[A}\hat{L}_{BC)} + T_{[AB|}{}^{F}\hat{L}_{F|C)} + \frac{6}{5}\, me^{2\phi}\, \hat{K}_{ABC} \\
0 &= D_{[A}\hat{K}_{BCD)} + \frac{3}{2}T_{[AB|}{}^{F}\hat{K}_{F|CD)} + 4i\ \hat{K}_{[A}\hat{K}_{BCD)} \\
0 &= D_{[A}\hat{L}_{BCDE)} + 2T_{[AB|}{}^{F}\hat{L}_{F|CDE)} + \frac{4i}{3}\ \hat{L}_{[AB}\hat{K}_{CDE)} - 4i\ \hat{K}_{[A}\hat{L}_{BCDE)}\ .
\end{aligned}
\tag{4.18}
$$

These can be solved following the standard procedure, taking into account the expressions for the superfield components of [93]. The solution reads,

$$\begin{aligned}
\hat{K}_a &= K_a \\
\hat{L}_{ab} &= L_{ab} + \frac{3}{8}\,\mu\gamma_{ab}\lambda \\
\hat{K}_{abc} &= K_{abc} - \frac{1}{8}\,\mu\gamma_{abc}\mu + \frac{1}{8}\,\lambda\gamma_{abc}\lambda \\
\hat{L}_{abcd} &= L_{abcd} + \frac{1}{32}\,\mu\gamma_{abcd}\lambda\,,
\end{aligned} \tag{4.19}$$

for the top (bosonic) components and

$$\begin{aligned}
&\hat{K}_\alpha = \frac{i}{2}\,\lambda_\alpha &&\hat{L}_\alpha{}^\beta = -\frac{3}{16}\,\delta_\alpha^\beta &&\hat{K}_{ab\alpha} = \frac{i}{12}\,(\gamma_{ab}\lambda)_\alpha &&\hat{L}_{abc\alpha} = \frac{i}{96}\,(\gamma_{abc}\mu)_\alpha \\
&\hat{K}^\alpha = \frac{i}{2}\,\mu^\alpha &&\hat{L}^\alpha{}_\beta = -\frac{3}{16}\,\delta^\alpha_\beta &&\hat{K}_{ab}{}^\alpha = -\frac{i}{12}\,(\gamma_{ab}\mu)^\alpha &&\hat{L}_{abc}{}^\alpha = -\frac{i}{96}\,(\gamma_{abc}\lambda)^\alpha \\
& && &&\hat{K}_{a\alpha\beta} = -\frac{1}{24}\,(\gamma_a)_{\alpha\beta} &&\hat{L}_{ab\alpha}{}^\beta = -\frac{1}{192}\,(\gamma_{ab})_\alpha{}^\beta \\
& && &&\hat{K}_a{}^{\alpha\beta} = \frac{1}{24}\,(\gamma_a)^{\alpha\beta} &&\hat{L}_{ab}{}^\alpha{}_\beta = \frac{1}{192}\,(\gamma_{ab})^\alpha{}_\beta\,,
\end{aligned}$$

for the remaining components. *The ordinary bosonic forms are identified with the lowest-order components in the theta-expansion of the hatted superfields in* (4.19).

Secondly, solving the Bianchi identities (4.1) at dimension $3/2$, one can show that the torsion must be decomposed as $T_{ab} = \widetilde{T}_{ab} + \gamma_{[a}\widetilde{T}_{b]} + \gamma_{ab}\widetilde{T}$, where the $\widetilde{T}$ are

$$\tilde{T}^\alpha = \frac{344}{225}L\mu + \frac{8}{9}L_{(2)}(\gamma^{(2)}\mu) + \frac{8}{45}L_{(4)}(\gamma^{(4)}\mu) + \frac{8}{9}K_{(1)}(\gamma^{(1)}\lambda) - \frac{16}{45}K_{(3)}(\gamma^{(3)}\lambda) - \frac{11}{450}(\mu\gamma_{(3)}\mu)(\gamma^{(3)}\lambda) \tag{4.20}$$

$$\tilde{T}_\alpha = -\frac{344}{225}L\lambda + \frac{8}{9}L_{(2)}(\gamma^{(2)}\lambda) - \frac{8}{45}L_{(4)}(\gamma^{(4)}\lambda) + \frac{8}{9}K_{(1)}(\gamma^{(1)}\mu) + \frac{16}{45}K_{(3)}(\gamma^{(3)}\mu) - \frac{11}{450}(\lambda\gamma_{(3)}\lambda)(\gamma^{(3)}\mu) \tag{4.21}$$

$$\widetilde{T}_a^\alpha = -\frac{3i}{20}(\gamma_a^{(1)}\nabla_{(1)}\mu) - \frac{1}{5}L_{(2)}(\gamma_a^{(2)}\lambda) + \frac{2}{5}L_{(4)}(\gamma_a^{(4)}\lambda) + \frac{1}{5}K_{(1)}(\gamma_a^{(1)}\mu) - \frac{3}{20}K_{(3)}(\gamma_a^{(3)}\mu) + \frac{3}{160}(\lambda\gamma_{(3)}\lambda)(\gamma_a^{(3)}\mu) \tag{4.22}$$

$$\widetilde{T}_{a\alpha} = -\frac{3i}{20}(\gamma_a^{(1)}\nabla_{(1)}\lambda) - \frac{1}{5}L_{(2)}(\gamma_a^{(2)}\mu) - \frac{2}{5}L_{(4)}(\gamma_a^{(4)}\mu) + \frac{1}{5}K_{(1)}(\gamma_a^{(1)}\lambda) + \frac{3}{20}K_{(3)}(\gamma_a^{(3)}\lambda) + \frac{3}{160}(\mu\gamma_{(3)}\mu)(\gamma_a^{(3)}\lambda) \tag{4.23}$$

Then the two terms $\Delta T^\alpha := -\tilde{T}^\alpha + (\text{rhs } 4.20)$ and $\Delta T_\alpha := -\tilde{T}_\alpha + (\text{rhs } 4.21)$ vanish on-shell (cf. (4.9),(4.10) of [93]). Hence we are free to add to the right-hand sides of equations (4.16) terms proportional to ΔT above. When integrated to a fermionic action, they induce terms proportional to $\tilde{T}^\alpha \lambda_\alpha$, $\tilde{T}_\alpha \mu^\alpha$. Given that $\tilde{T}$ is the trace of the dimension $3/2$ torsion, these are gravitino terms which we set to zero here. [2]

Let us take as our starting point the fermionic equations (4.16), adding to the right-hand sides the terms $c_1 \Delta T^\alpha$, $c_2 \Delta T_\alpha$, as explained in the previous paragraph, for some coefficients c_1, c_2. Provided we take $c_2 = c_1$, the resulting equations can be integrated into the following fermionic action :

$$
\begin{aligned}
S_f = \int \mathrm{d}^{10}x \sqrt{\hat{g}}\, e^{(6-8c_1/9)\phi} \Big[&(\bar{\Lambda}\Gamma^m \nabla_m \Lambda) - \frac{4}{225}(270 - 43c_1) e^{5\phi/4} m (\bar{\Lambda}\Lambda) \\
&- (1 - \frac{1}{6}c_1)\, e^{3\phi/4} F_{mn} (\bar{\Lambda}\Gamma^{mn}\Gamma_{11}\Lambda) + (\frac{1}{8} - \frac{2}{135}c_1)\, e^{-\phi/2} H_{mnp} (\bar{\Lambda}\Gamma^{mnp}\Gamma_{11}\Lambda) \\
&+ \frac{1}{1080} c_1\, e^{\phi/4} G_{mnpq} (\bar{\Lambda}\Gamma^{mnpq}\Lambda) + \frac{2}{5}(15 - c_1)(\bar{\Lambda}\Lambda)^2 \Big] ,
\end{aligned}
$$

where the Dirac gamma matrices Γ^m and the Majorana fermions Λ are given in (D.10), (D.11) respectively; we have expressed the final result in terms of the rescaled metric (4.9) and the bosonic forms in (4.11), with the understanding that the unhatted forms therein are now replaced by the corresponding hatted ones given in (4.19) :

$$
F := -\frac{16}{3}\, \hat{L}_{(2)} , \qquad H := 24i\, e^{2\phi} \hat{K}_{(3)} , \qquad G := 192\, e^{2\phi} \hat{L}_{(4)} . \tag{4.24}
$$

The total action (up to gravitino terms) is thus given by : $S = S_b + \alpha S_f$ for some coefficient α to be determined. Next consider the dilaton equation of motion,

$$
\begin{aligned}
0 = &-2i\nabla \cdot K_{(1)} - 24K_{(1)}^2 - \frac{4}{5}m^2 e^{4\phi} - \frac{32}{3}\hat{L}_{(2)}^2 - 48\hat{K}_{(3)}^2 - 384\hat{L}_{(4)}^2 - \frac{16}{5}m e^{2\phi}(\lambda\mu) \\
&- 8(\lambda\gamma^{(2)}\mu)\hat{L}_{(2)} + 8\big((\lambda\gamma^{(3)}\lambda) - (\mu\gamma^{(3)}\mu)\big)\hat{K}_{(3)} - 32(\lambda\gamma^{(4)}\mu)\hat{L}_{(4)} + 144(\lambda\mu)^2 ,
\end{aligned} \tag{4.25}
$$

which is an exact superfield equation obtained from the Bianchi identities at dimension 2; it reduces to the bosonic dilaton equation given in (4.12) upon setting to zero the fermionic superfields, and transforming to the Einstein-frame metric. As explained above, we can modify equation (4.25) by adding on the right hand-side a term of the form $c_3\lambda_\alpha \Delta T^\alpha + c_4 \mu^\alpha \Delta T_\alpha$, which vanishes on-shell. This will generate gravitino terms $\lambda_\alpha \tilde{T}^\alpha$, $\mu^\alpha \tilde{T}_\alpha$, which we can then set to zero. Demanding that the resulting equation of motion coincides with the dilaton equation coming from $S_b + \alpha S_f$, gives an overdetermined system of equations for the unknown coefficients (α, $c_1, \ldots, c_4$). The solution reads,

$$
\alpha = -80 , \qquad c_1 = c_2 = \frac{27}{4} , \qquad c_3 = c_4 = -45 .
$$

2. The precise relation between T_{ab}^α and the gravitino can be derived using the procedure described in detail in e.g. [81] and it is of the form : $e_m{}^a e_n{}^b T_{ab}^\alpha = \nabla_{[m}\psi_{n]}^\alpha + \mathcal{O}(\psi)$. In particular it vanishes upon setting $\psi_m^\alpha \equiv 0$.

Plugging back the above into the action we obtain,

$$
\begin{aligned}
S = S_b - 80 \int \mathrm{d}^{10}x\sqrt{\hat{g}}\Big[&(\bar{\Lambda}\Gamma^m\nabla_m\Lambda) + \frac{9}{25}\,e^{5\phi/4}m(\bar{\Lambda}\Lambda) \\
&+ \frac{1}{8}\,e^{3\phi/4}F_{mn}(\bar{\Lambda}\Gamma^{mn}\Gamma_{11}\Lambda) + \frac{1}{40}e^{-\phi/2}H_{mnp}(\bar{\Lambda}\Gamma^{mnp}\Gamma_{11}\Lambda) \\
&+ \frac{1}{160}\,e^{\phi/4}G_{mnpq}(\bar{\Lambda}\Gamma^{mnpq}\Lambda) + \frac{33}{10}(\bar{\Lambda}\Lambda)^2\Big]\,,
\end{aligned}
\tag{4.26}
$$

where the bosonic part of the action S_b was given in (4.15). The Einstein equation can be used as a further consistency check. The dimension 2 superspace Bianchi identities give,

$$
\begin{aligned}
R_{bc} = \eta_{bc}\Big(&-\frac{1}{25}m^2e^{4\phi} + \frac{3i}{2}\nabla\cdot K_{(1)} + 18K_{(1)}^2 + \frac{8}{9}\hat{L}_{(2)}^2 - 12\hat{K}_{(3)}^2 + 288\hat{L}_{(4)}^2 \\
&- \frac{36}{5}(\lambda\mu)me^{2\phi} - \frac{16}{3}(\lambda\gamma^{(2)}\mu)\hat{L}_{(2)} + 6\big[(\lambda\gamma^{(3)}\lambda) - (\mu\gamma^{(3)}\mu)\big]\hat{K}_{(3)} \\
&+ 24(\lambda\gamma^{(4)}\mu)\hat{L}_{(4)} - 108(\lambda\mu)^2\Big) \\
&+ 12i\nabla_{(b}K_{c)} - 16K_bK_c - \frac{128}{9}\hat{L}_{(2)bc}^2 + 144\hat{K}_{(3)bc}^2 - 3072\hat{L}_{(4)bc}^2 \\
&+ 4i(\lambda\gamma_{(b}\nabla_{c)}\lambda) + 4i(\mu\gamma_{(b}\nabla_{c)}\mu) - \frac{32}{3}(\lambda\gamma_{(b}{}^{i}\mu)\hat{L}_{c)i} - 36\big[(\lambda\gamma_{(b}{}^{ij}\lambda) - (\mu\gamma_{(b}{}^{ij}\mu)\big]\hat{K}_{c)ij} \\
&- 192(\lambda\gamma_{(b}{}^{ijk}\mu)\hat{L}_{c)ijk}\,.
\end{aligned}
\tag{4.27}
$$

Proceeding as before, we note that the two terms $\Delta T_a^\alpha := -\tilde{T}^\alpha + (\text{rhs } 4.22)$ and $\Delta T_\alpha := -\tilde{T}_{a\alpha} + (\text{rhs } 4.23)$ vanish on-shell (cf. (4.9), (4.10) of [93]). Therefore the right-hand side of the Einstein equation (4.27) can be modified by a term of the form, $c_5(\Delta T_{(b}\gamma_{c)}\lambda) + c_6(\Delta T_{(b}\gamma_{c)}\mu) + c_7\,\eta_{bc}(\Delta T\lambda) + c_8\,\eta_{bc}(\Delta T\mu)$. Demanding that the Einstein equation thus modified agrees with the Einstein equation coming from (4.26) leads once again to a highly overdetermined system of equations. As required for consistency, a unique solution exists and is given by,

$$
c_5 = c_6 = -24\,, \qquad c_7 = c_8 = -\frac{81}{4}\,.
$$

4.2.3 General dilatonic vacua

Since the superspace formalism presented above turns out to be in a frame different from the Einstein frame, the dilatino-condensate action (4.26) cannot be compared to the actions in [89, 105, 106], which are also in the Einstein frame, by simply setting the gravitino in those references to zero.

The dilatino ψ_m of the superspace formulation is canonically related (through the suspersymmetry transformations) to the metric g_{mn}, whereas the dilatino Ψ_m of appendix D.12 is canonically related to the rescaled Einstein-frame metric $\hat{g}_{mn}$, cf. (4.9). The action (4.26) is obtained by setting the superspace gravitino to zero which thus corresponds to,

$$
\psi_m \equiv 0 \quad \longleftrightarrow \quad \Psi_m \equiv -\frac{3}{4}\Gamma_m\Lambda\,,
\tag{4.28}
$$

as can be seen from the dictionary (D.11). More generally, setting the gravitino to zero is a frame-dependent statement. This can be seen directly from the supersymmetry transformation for the vielbein (D.3) which, when evaluated at the lowest order in the θ-expansion, gives $\delta_\xi e_m{}^a = -i(\epsilon\gamma^a\psi_m) - i(\zeta\gamma^a\psi_m)$, up to a Lorentz transformation. More generally, it canonically associates the vielbein of the metric $g^{(\beta)}$ with the gravitino $\psi^{(\beta)}$, where,

$$g^{(\beta)}_{mn} := e^{2\beta\phi}\hat{g}_{mn}\,, \qquad \psi^{(\beta)}_m := \Psi_m - \beta\,\Gamma_m\Lambda\,, \qquad \beta\in\mathbb{R}\,, \tag{4.29}$$

and we have used, $\delta_\xi\phi = \xi\cdot\nabla\phi = (\epsilon\lambda) + (\zeta\mu)$. It follows that setting the gravitino $\psi^{(\beta)}$ to zero corresponds to,

$$\psi^{(\beta)}_m \equiv 0 \quad \longleftrightarrow \quad \Psi_m \equiv \beta\Gamma_m\Lambda\,, \tag{4.30}$$

which generalizes (4.28) to an arbitrary frame. In particular, we distinguish the following cases,

$$\beta = \begin{cases} -\frac{3}{4}, & \text{vanishing superspace-frame gravitino} \\ 0, & \text{vanishing Einstein-frame gravitino} \\ \frac{1}{4}, & \text{vanishing string-frame gravitino}\,. \end{cases}$$

The four-fermion part of the IIA Lagrangian in [89] is given as a sum of 24 terms expressed in terms of $\widehat{\Psi}^{GP}_m$, λ^{GP} (where GP stands for Giani-Pernici, cf. appendix D.2.3). Substituting (4.30) in [89] corresponds to setting,

$$\psi^{GP}_m \equiv \beta\sqrt{2}\,\Gamma_{11}\Gamma_m\lambda^{GP}\,, \qquad \widehat{\Psi}^{GP}_m \equiv c\,\Gamma_{11}\Gamma_m\lambda^{GP}\,, \tag{4.31}$$

where $c := \sqrt{2}(\beta + 1/12)$, with $\beta\in\mathbb{R}$. We thus obtain the following expression for the $(\bar\lambda\lambda)^2$ term in [89],

$$\begin{aligned}&(\bar\lambda\Gamma_{mn}\Gamma_{11}\lambda)^2\Big(\frac{26\sqrt{2}}{3}c^3 - \frac{29}{4}c^4\Big) + (\bar\lambda\Gamma_{mnpq}\lambda)^2\Big(\frac{1}{\sqrt{2}}c^3 - \frac{21}{8}c^4\Big)\\ &\quad + (\bar\lambda\Gamma_{mnp}\lambda)^2\Big(\frac{7}{3\sqrt{2}}c^3 - 5c^4\Big) + (\bar\lambda\Gamma_{mnp}\Gamma_{11}\lambda)^2\Big(-\frac{2}{3}c^2 + \frac{7}{\sqrt{2}}c^3 + \sqrt{2}c^3 - 6c^4\Big)\\ &\quad = \Big(32c^2 - 276\sqrt{2}c^3 + \frac{1773}{2}c^4\Big)(\bar\lambda\lambda)^2\,,\end{aligned} \tag{4.32}$$

where in the last equality we used the following Fierz identities,

$$\begin{aligned}(\bar\lambda\Gamma_{mn}\Gamma_{11}\lambda)^2 &= 6\,(\bar\lambda\lambda)^2 \qquad & (\bar\lambda\Gamma_{mnp}\Gamma_{11}\lambda)^2 &= -48\,(\bar\lambda\lambda)^2\\ (\bar\lambda\Gamma_{mnp}\lambda)^2 &= 48\,(\bar\lambda\lambda)^2 & (\bar\lambda\Gamma_{mnpq}\lambda)^2 &= -336\,(\bar\lambda\lambda)^2\,.\end{aligned} \tag{4.33}$$

Furthermore substituting (4.30) in the massive IIA action of [91], completing it with the quartic-fermion term (4.32) and normalizing to our conventions, cf. appendix D.2.3, we obtain the one-parameter family of dilatonic-condensate pseudoactions,

$$
\begin{aligned}
S = S_b + \int \mathrm{d}^{10}x\sqrt{\hat{g}}\Big\{&(1-144\beta^2)\bar{\Lambda}\Gamma^m\nabla_m\Lambda) - (36\beta^2 - 10\beta + \frac{21}{20})e^{5\phi/4}m(\bar{\Lambda}\Lambda) \\
&- \frac{1}{2}(29\beta^2 - \frac{9}{2}\beta + \frac{5}{16})e^{3\phi/4}F_{mn}(\bar{\Lambda}\Gamma^{mn}\Gamma_{11}\Lambda) - (4\beta^2 + \frac{1}{3}\beta)e^{-\phi/2}H_{mnp}(\bar{\Lambda}\Gamma^{mnp}\Gamma_{11}\Lambda) \\
&- \frac{1}{24}(21\beta^2 - \frac{1}{2}\beta - \frac{3}{16})e^{\phi/4}G_{mnpq}(\bar{\Lambda}\Gamma^{mnpq}\Lambda) - (8c^2 - 69\sqrt{2}c^3 + \frac{1773}{8}c^4)(\bar{\Lambda}\Lambda)^2\Big\}\,,
\end{aligned}
$$

where S_b is given in (4.15), and c was defined below (4.31). To sum up, this action is obtained from the action of [91] completed with the quartic-fermion terms of [89], by imposing (4.30) with arbitrary parameter β. Of course, setting the gravitino to zero is in general inconsistent, since the gravitino couples linearly to terms of the form (flux)$\times$ (dilatino) and (dilatino)^3. However, in a Lorentz-invariant vacuum, where linear and cubic fermion vevs vanish, this does not lead to an inconsistency. These dilatino-condensate actions of the present paper should thus be regarded as *pseudoactions* : book-keeping devices whose variation with respect to the bosonic fields gives the correct bosonic equations of motion in the presence of dilatino condensates. Moreover the fermionic equations of motion are trivially satisfied in the Lorentz-invariant vacuum.

Setting $\beta = -3/4$ in (4.34) we recover the action (4.26). The dilatonic-condensate pseudoactions S^E, S^{st} obtained by setting the Einstein-frame, string-frame gravitino to zero ($\beta = 0, 1/4$ respectively) read,

$$
\begin{aligned}
S^E = S_b + \int \mathrm{d}^{10}x\sqrt{\hat{g}}\Big[&(\bar{\Lambda}\Gamma^m\nabla_m\Lambda) - \frac{21}{20}e^{5\phi/4}m(\bar{\Lambda}\Lambda) + \frac{3}{512}(\bar{\Lambda}\Lambda)^2 \\
&- \frac{5}{32}e^{3\phi/4}F_{mn}(\bar{\Lambda}\Gamma^{mn}\Gamma_{11}\Lambda) + \frac{1}{128}e^{\phi/4}G_{mnpq}(\bar{\Lambda}\Gamma^{mnpq}\Lambda)\Big]\,,
\end{aligned}
\tag{4.34}
$$

and

$$
\begin{aligned}
S^{st} = S_b + \int \mathrm{d}^{10}x\sqrt{\hat{g}}\Big[&-8(\bar{\Lambda}\Gamma^m\nabla_m\Lambda) - \frac{4}{5}e^{5\phi/4}m(\bar{\Lambda}\Lambda) - \frac{5}{2}(\bar{\Lambda}\Lambda)^2 \\
&- \frac{1}{2}e^{3\phi/4}F_{mn}(\bar{\Lambda}\Gamma^{mn}\Gamma_{11}\Lambda) - \frac{1}{3}e^{-\phi/2}H_{mnp}(\bar{\Lambda}\Gamma^{mnp}\Gamma_{11}\Lambda) - \frac{1}{24}e^{\phi/4}G_{mnpq}(\bar{\Lambda}\Gamma^{mnpq}\Lambda)\Big]\,.
\end{aligned}
\tag{4.35}
$$

Note that the quartic-dilaton term in S^E can potentially generate a positive cosmological constant, contrary to the quartic-dilaton term in S^{st}, which is negative. The dilaton and Einstein equations following from action (4.34) are written in appendix (D.1), and the form equations read,

$$
\begin{aligned}
0 &= \mathrm{d}\Big(\hat{\star}\big[e^{3\phi/2}F - (29\beta^2 - \frac{9}{2}\beta + \frac{5}{16})e^{3\phi/4}(\bar{\Lambda}\Gamma_{(2)}\Gamma_{11}\Lambda)\big]\Big) + e^{\phi/2}H{\wedge}\hat{\star}G \\
0 &= \mathrm{d}\Big(\hat{\star}\big[e^{-\phi}H - (24\beta^2 + 2\beta)e^{-\phi/2}(\bar{\Lambda}\Gamma_{(3)}\Gamma_{11}\Lambda)\big]\Big) + e^{\phi/2}F{\wedge}\hat{\star}G - \frac{1}{2}G{\wedge}G + \frac{4}{5}me^{3\phi/2}\hat{\star}F \\
0 &= \mathrm{d}\Big(\hat{\star}\big[e^{\phi/2}G - (21\beta^2 - \frac{1}{2}\beta - \frac{3}{16})e^{\phi/4}(\bar{\Lambda}\Gamma_{(4)}\Lambda)\big]\Big) - H{\wedge}G\,,
\end{aligned}
\tag{4.36}
$$

where we have defined : $(\bar{\Lambda}\Gamma_{(p)}\Lambda) := \frac{1}{p!}(\bar{\Lambda}\Gamma_{m_1\dots m_p}\Lambda)\,\mathrm{d}x^{m_p}{\wedge}\dots{\wedge}\mathrm{d}x^{m_1}$, similarly to our definition for the bosonic forms. In addition to the equations above, the forms obey the Bianchi identities given in (4.14).

4.2.4 de Sitter vacua

Having obtained the general dilatino-condensate action (4.34), we can look for de Sitter solutions supported by non-vanishing dilatino condensates. Let us be clear that these are *formal* solutions of IIA supergravity, obtained by simply assuming non-vanishing values of the dilatino condensates of the theory. Our approach is similar to e.g. [103], in that we do not offer any concrete scenario or mechanism for the generation of the dilatino condensate. We will use for that purpose the dilatino-condensate pseudoaction (4.34), obtained by setting the Einstein-frame gravitino to zero ($\beta = 0$), although the analysis can be easily extended to a general value of the parameter β.

dS_{10} — In this section we show that the massless IIA theory admits ten-dimensional de Sitter vacua supported by the quartic-dilatino condensate, with constant dilaton and vanishing flux. The only potentially non-vanishing condensates in the ten-dimensional Lorentz-invariant vacuum are the scalar condensates $(\bar{\Lambda}\Lambda)$ and $(\bar{\Lambda}\Lambda)^2$. Note in particular that these vevs are a priori independent. [3]

With these assumptions, setting $m, \beta = 0$, we see that the Bianchi identities (4.14), the form equations in (4.36) and the dilaton equation (D.1) are trivially satisfied. Moreover the Einstein equation (D.2) reduces to,

$$-\hat{R}_{mn} = \frac{3}{2^{12}}(\bar{\Lambda}\Lambda)^2 \hat{g}_{mn} .$$

For a non-vanishing quartic-dilatino condensate we thus obtain a simple realization of dS_{10} in massless IIA theory. [4] The de Sitter radius is set by the value of the condensate.

$dS_4 \times M_6$ without flux — Let us now consider compactifications, on six-dimensional Kähler-Einstein manifolds M_6, of massless IIA supergravity to a maximally-symmetric Lorentzian manifold $M_{1,3}$ with vanishing flux, $F, H, G = 0$, and constant dilaton which we also set to zero for simplicity, $\phi = 0$. More specifically, we assume that the ten-dimensional spacetime is of direct product form $M_{1,3} \times M_6$,

$$\mathrm{d}s^2 = \mathrm{d}s^2(M_{1,3}) + \mathrm{d}s^2(M_6) . \tag{4.37}$$

Moreover,

$$-R_{\mu\nu} = \Omega \, g_{\mu\nu} , \qquad -R_{mn} = \omega g_{mn} , \tag{4.38}$$

3. Strictly-speaking these vevs should be denoted by $\langle\bar{\Lambda}\Lambda\rangle$ and $\langle(\bar{\Lambda}\Lambda)^2\rangle$ respectively, where $\langle(\bar{\Lambda}\Lambda)^2\rangle \neq \langle\bar{\Lambda}\Lambda\rangle^2$ in general. Omitting the brackets should hopefully not lead to confusion.

4. Note that in our "superspace" conventions for the forms, $\hat{R} < 0$, $\hat{R} > 0$ corresponds to de Sitter, anti-de Sitter space respectively.

where $g_{\mu\nu}$, g_{mn} are the components of the metric in the external, internal space respectively; we have chosen the parameterization so that positive Ω corresponds to de Sitter space, and similarly for ω (cf. footnote 4).

The internal manifold being Kähler-Einstein, it admits a nowhere-vanishing spinor, η, of positive chirality, which we take to be commuting. Moreover the spinor obeys,

$$\nabla_m \eta = i\mathcal{P}_m \eta \ ,$$

where $\mathrm{d}\mathcal{P}$ is proportional to J, the Kähler form of M_6. Furthermore J can be expressed as an η bilinear,

$$i\eta^\dagger \gamma_{(2)} \eta = J \ .$$

We decompose the chiral and anti-chiral components of the dilatino, λ and μ respectively, cf. (D.11), as follows,

$$\lambda = \chi_+ \otimes \eta + c.c. \ , \qquad \mu = \chi_- \otimes \eta + c.c. \ , \tag{4.39}$$

where χ_+ (χ_-) is a chiral (anti-chiral) anti-commuting Weyl spinor of $M_{1,3}$. The rationale for this decomposition is that, in the effective action describing the compactification on M_6, (4.39) should give rise to "light" four-dimensional spinors $\chi_\pm$;[5] it generalizes to the Kähler-Einstein case the decomposition of [95], where M_6 is taken to be a Calabi-Yau. Similar decompositions were adopted in e.g. [100].

It follows from (4.39) that, for a Lorentz-invariant four-dimensional vacuum, the dilatino bilinear condensates take the form,

$$(\bar{\Lambda}\Lambda) = \Re(A) \ , \qquad (\bar{\Lambda}\Gamma_{(2)}\Lambda) = \Re(A)\ J \ , \qquad (\bar{\Lambda}\Gamma_{(4)}\Lambda) = \Im(A)\ \mathrm{vol}_4 + \Re(A)\ \frac{1}{2} J^2 \ ,$$

where the complex number $A := 4\,(\bar{\chi}_+\chi_-)$ is the four-dimensional quadratic-dilatino condensate, and vol_4 is the volume element of $M_{1,3}$. Furthermore, setting $m, \beta = 0$, we see that the Bianchi identities (4.14), the form equations (4.36) and the dilaton equation (D.1) are all automatically satisfied. The mixed (μ, m) components of the Einstein equations (D.2) are automatically satisfied, while the internal and external components of the Einstein equations reduce to,

$$\Omega = \omega = \frac{3}{2^{12}} (\bar{\Lambda}\Lambda)^2 \ , \tag{4.40}$$

where we have used that vevs of the form $(\bar{\Lambda}\Gamma_{(m}\nabla_{n)}\Lambda)$ vanish. For a non-vanishing quartic-dilatino condensate we thus obtain a simple realization of $dS_4 \times M_6$ in massless IIA theory. The curvature of de Sitter space and the internal manifold are both set by the value of the condensate.

5. Although certainly plausible, this is hard to show in general beyond the Calabi-Yau case.

A few concluding remarks can be made. First, those two simple examples of de Sitter background are supported by the non-vanishing vev of the quartic fermions terms computed in (4.2.3), and the existence of those configurations depends on the coefficient of the $(\bar{\Lambda}\Lambda)^2$ term in the action (4.34) (we emphasize, however, that we do not provide a concrete scenario that would establish the existence of this non-vanishing fermionic background). Besides, throughout this chapter, we only focused on the dilatini terms of Romans supergravity. However, the superspace formalism from which it emerges, together with the package of chapter 3 [6] can also be used to study HLW supergravity, or gravitini terms that were discarded in our analysis.

6. Actually, given the very small differences between HLM and Romans in the superspace formalism (cf. (4.4)), going from one to the other requires a very reasonable amount of work.

Conclusions

AS A CONCLUSION, of this thesis, let's briefly review the subjects covered and examine some future directions.

As explained in the introduction, the quantum consistency required in eleven dimensional supergravity calls for particular terms of higher order in the Planck length, or in derivative order. If the conjectured M-theory is a microscopic completion of this supergravity, it is plausible to expect the existence of a full supersymmetric invariant completing the anomaly-canceling term considered in chapter 2. Finding superinvariants is a notoriously hard problem, and several techniques, based on IIA string computations or supersymmetry have been developed to this end. However, only a few terms were computed, and the full superinvariant remains to be found. We tackled this problem from the point of view of the action principle, which relies on the superspace formulation of supergravity to generate supersymmetric invariants. In principle, this technique leads to the full correction associated to the Chern-Simons term.

Although this program has not been completed, several conclusions were made about the structure of the expected superinvariants. Our results strongly support the τ-exactness of X_8. Besides, the existence of the superinvariant is ensured if this condition is fulfilled. Finally, a systematic approach to the resolution was initiated, and might be applied in the next steps of the resolution (provided there is no computational barrier).

Dealing with these rather lengthy computations leads to the development of a Mathematica package called SSGAMMA. It is designed to handle with ease the type of algebra we encounter when working with tensors and superspace equations : gamma gymnastic and Fierzing, spinors and tensors with explicit fermionic and bosonic indices, large (anti-)symmetrizations, group theoretic functions, etc. Even though it still needs some polishing, we hope this program will be useful for future manipulations in eleven- or ten- dimensional supergravity, and that it will contribute to further developments in the domain of superinvariants. The package was also extended to deal with ten-dimensional spinors and gamma matrices, to tackle another computationally intensive problem presented in chapter 4.

All versions of IIA supergravity were proven to emerge from the same superspace origin : solving Bianchi identities with well-chosen conventional constraints on the torsion leads to both massive supergravities (with the massless case as a zero-mass limit), via particular choices for two parameters emerging from the calculation. This approach is interesting in that it gives a systematic way of computing the complete fermion terms (which were only implicitly given in [93]). With the much-needed help of SSGamma, we generated the equations of motion including the quadratic and quartic fermion terms of Romans supergravity. Then, in two simple cases, we generated de Sitter backgrounds made possible by the non-zero vev of the quartic fermion term. This superspace approach was employed to study the quartic dilatino terms in Romans supergravity, but could be extended to all IIA supergravities, where fermion terms are often left aside due to the lengthy calculations they entail.

Bibliographie

[1] S. L. Glashow, "Partial-symmetries of weak interactions," *Nuclear Physics*, vol. 22, no. 4, pp. 579–588, 1961.

[2] S. Weinberg, "A model of leptons," *Physical review letters*, vol. 19, no. 21, p. 1264, 1967.

[3] A. Salam, "Elementary particle theory," *Ed. N. Svartholm, Stockholm,\Almquist and Wiksell*, vol. 367, 1968.

[4] A. Einstein, "Erklarung der perihelionbewegung der merkur aus der allgemeinen relativitatstheorie," *Sitzungsber. preuss. Akad. Wiss., vol. 47, No. 2, pp. 831-839, 1915*, vol. 47, pp. 831–839, 1915.

[5] S. Chatrchyan, V. Khachatryan, A. M. Sirunyan, A. Tumasyan, W. Adam, E. Aguilo, T. Bergauer, M. Dragicevic, J. Erö, C. Fabjan, *et al.*, "Observation of a new boson at a mass of 125 gev with the cms experiment at the lhc," *Physics Letters B*, vol. 716, no. 1, pp. 30–61, 2012.

[6] G. Aad, T. Abajyan, B. Abbott, J. Abdallah, S. A. Khalek, A. Abdelalim, O. Abdinov, R. Aben, B. Abi, M. Abolins, *et al.*, "Observation of a new particle in the search for the standard model higgs boson with the atlas detector at the lhc," *Physics Letters B*, vol. 716, no. 1, pp. 1–29, 2012.

[7] B. P. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, *et al.*, "Observation of gravitational waves from a binary black hole merger," *Physical review letters*, vol. 116, no. 6, p. 061102, 2016.

[8] C.-W. Chou, D. Hume, T. Rosenband, and D. Wineland, "Optical clocks and relativity," *Science*, vol. 329, no. 5999, pp. 1630–1633, 2010.

[9] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, "Tenth-order qed contribution to the electron g- 2 and an improved value of the fine structure constant," *Physical Review Letters*, vol. 109, no. 11, p. 111807, 2012.

[10] S. Coleman and J. Mandula, "All possible symmetries of the s matrix," *Physical Review*, vol. 159, no. 5, p. 1251, 1967.

[11] R. Haag, J. T. Łopuszański, and M. Sohnius, "All possible generators of supersymmetries of the s-matrix," in *Supergravities in Diverse Dimensions : Commentary and Reprints (In 2 Volumes)*, pp. 49–66, World Scientific, 1989.

[12] J. Wess and B. Zumino, "Supergauge transformations in four dimensions," in *Supergravities in Diverse Dimensions : Commentary and Reprints (In 2 Volumes)*, pp. 24–35, World Scientific, 1989.

[13] S. P. Martin, "A supersymmetry primer," in *Perspectives on supersymmetry II*, pp. 1–153, World Scientific, 2010.

[14] U. Amaldi, W. de Boer, and H. Fürstenau, "Comparison of grand unified theories with electroweak and strong coupling constants measured at lep," *Physics Letters B*, vol. 260, no. 3-4, pp. 447–455, 1991.

[15] S. Deser and B. Zumino, "Consistent supergrav1ty," in *Supergravities in Diverse Dimensions : Commentary and Reprints (In 2 Volumes)*, pp. 517–519, World Scientific, 1989.

[16] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, "Progress toward a theory of supergravity," *Physical Review D*, vol. 13, no. 12, p. 3214, 1976.

[17] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.*, vol. B443, pp. 85–126, 1995.

[18] L. Castellani, L. Romans, and N. P. Warner, "A classification of compactifying solutions for d= 11 supergravity," *Nuclear Physics B*, vol. 241, no. 2, pp. 429–462, 1984.

[19] M. B. Green and J. H. Schwarz, "Anomaly cancellations in supersymmetric D = 10 gauge theory require SO(32)," *Phys. Lett.*, vol. 149B, pp. 117–122, 1984.

[20] M. T. Grisaru, H. Pendleton, and P. Van Nieuwenhuizen, "Supergravity and the s matrix," *Physical Review D*, vol. 15, no. 4, p. 996, 1977.

[21] D. Lüst and D. Tsimpis, "Supersymmetric ads4 compactifications of iia supergravity," *Journal of High Energy Physics*, vol. 2005, no. 02, p. 027, 2005.

[22] J. F. Cornwell, *Group theory in physics, Vol III*, vol. 10 of *Techniques in Physics*. Academic Press, 1989.

[23] P. S. Howe, "Weyl superspace," *Phys. Lett.*, vol. B415, pp. 149–155, 1997.

[24] K. Bautier, S. Deser, M. Henneaux, and D. Seminara, "No cosmological d= 11 supergravity," *Physics Letters B*, vol. 406, no. 1-2, pp. 49–53, 1997.

[25] H. Nishino and S. J. Gates Jr, "Toward an off-shell 11d supergravity limit of m-theory," *Physics Letters B*, vol. 388, no. 3, pp. 504–511, 1996.

[26] E. Cremmer, B. Julia, and J. Scherk, "Supergravity theory in 11 dimensions," *Phys. Lett.*, vol. B76, pp. 409–412, 1978.

[27] W. Nahm, "Supersymmetries and their representations," in *Supergravities in Diverse Dimensions : Commentary and Reprints (In 2 Volumes)*, pp. 86–103, World Scientific, 1989.

[28] E. Cremmer and S. Ferrara, "Formulation of eleven-dimensional supergravity in superspace," *Phys. Lett.*, vol. B91, p. 61, 1980.

[29] L. Brink and P. Howe, "Eleven-dimensional supergravity on the mass shell in superspace," *Phys. Lett.*, vol. B91, pp. 384–386, 1980.

[30] N. Dragon, "Torsion and curvature in extended supergravity," *Z. Phys.*, vol. C2, pp. 29–32, 1979.

[31] A. Candiello and K. Lechner, "Duality in supergravity theories," *Nucl. Phys.*, vol. B412, pp. 479–501, 1994.

[32] R. D'Auria and P. Fre, "Geometric Supergravity in d = 11 and Its Hidden Supergroup," *Nucl. Phys.*, vol. B201, pp. 101–140, 1982. [Erratum : Nucl. Phys.B206,496(1982)].

[33] K. Peeters, P. Vanhove, and A. Westerberg, "Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace," *Class. Quant. Grav.*, vol. 18, pp. 843–890, 2001.

[34] M. de Roo, H. Suelmann, and A. Wiedemann, "The supersymmetric effective action of the heterotic string in ten-dimensions," *Nucl. Phys.*, vol. B405, pp. 326–366, 1993.

[35] C. Vafa and E. Witten, "A one loop test of string duality," *Nucl. Phys.*, vol. B447, pp. 261–270, 1995.

[36] M. J. Duff, J. T. Liu, and R. Minasian, "Eleven-dimensional origin of string / string duality : A one-loop test," *Nucl. Phys.*, vol. B452, pp. 261–282, 1995.

[37] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory. 2. Vertices and Trees," *Nucl. Phys.*, vol. B198, pp. 252–268, 1982.

[38] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory. 3. Loops and Renormalization," *Nucl. Phys.*, vol. B198, pp. 441–460, 1982.

[39] K. Peeters, P. Vanhove, and A. Westerberg, "Chiral splitting and world-sheet gravitinos in higher- derivative string amplitudes," *Class. Quant. Grav.*, vol. 19, pp. 2699–2716, 2002.

[40] D. M. Richards, "The One-Loop Five-Graviton Amplitude and the Effective Action," *JHEP*, vol. 10, p. 042, 2008.

[41] D. M. Richards, "The One-Loop H^2R^3 and $H^2(\nabla H)R^2$ Terms in the Effective Action," *JHEP*, vol. 10, p. 043, 2008.

[42] J. T. Liu and R. Minasian, "Higher-derivative couplings in string theory : dualities and the B-field," *Nucl. Phys.*, vol. B874, pp. 413–470, 2013.

[43] P. S. Howe and D. Tsimpis, "On higher-order corrections in M theory," *JHEP*, vol. 09, p. 038, 2003.

[44] Y. Hyakutake and S. Ogushi, "R^4 corrections to eleven dimensional supergravity via supersymmetry," *Phys. Rev.*, vol. D74, p. 025022, 2006.

[45] Y. Hyakutake and S. Ogushi, "Higher derivative corrections to eleven dimensional supergravity via local supersymmetry," *JHEP*, vol. 02, p. 068, 2006.

[46] Y. Hyakutake, "Toward the Determination of R^3F^2 Terms in M-theory," *Prog. Theor. Phys.*, vol. 118, p. 109, 2007.

[47] K. Peeters, J. Plefka, and S. Stern, "Higher-derivative gauge field terms in the M-theory action," *JHEP*, vol. 08, p. 095, 2005.

[48] S. Deser and D. Seminara, "Counterterms / M theory corrections to D = 11 supergravity," *Phys. Rev. Lett.*, vol. 82, pp. 2435–2438, 1999.

[49] S. Deser and D. Seminara, "Tree amplitudes and two loop counterterms in D = 11 supergravity," *Phys. Rev.*, vol. D62, p. 084010, 2000.

[50] S. Deser and D. Seminara, "Graviton-form invariants in D=11 supergravity," *Phys. Rev.*, vol. D72, p. 027701, 2005.

[51] M. B. Green, M. Gutperle, and P. Vanhove, "One loop in eleven dimensions," *Phys. Lett.*, vol. B409, pp. 177–184, 1997.

[52] J. G. Russo and A. A. Tseytlin, "One-loop four-graviton amplitude in eleven-dimensional supergravity," *Nucl. Phys.*, vol. B508, pp. 245–259, 1997.

[53] M. B. Green, H. h. Kwon, and P. Vanhove, "Two loops in eleven dimensions," *Phys. Rev.*, vol. D61, p. 104010, 2000.

[54] L. Bonora, P. Pasti, and M. Tonin, "Chiral Anomalies in Higher Dimensional Supersymmetric Theories," *Nucl. Phys.*, vol. B286, pp. 150–174, 1987.

[55] L. Bonora, K. Lechner, M. Bregola, P. Pasti, and M. Tonin, "A Discussion of the constraints in N=1 SUGRA-SYM in 10-D," *Int. J. Mod. Phys.*, vol. A5, pp. 461–477, 1990.

[56] M. Cederwall, B. E. W. Nilsson, and D. Tsimpis, "Spinorial cohomology and maximally supersymmetric theories," *JHEP*, vol. 02, p. 009, 2002.

[57] M. V. Movshev, A. Schwarz, and R. Xu, "Homology of Lie algebra of supersymmetries and of super Poincare Lie algebra," *Nucl. Phys.*, vol. B854, pp. 483–503, 2012.

[58] F. Brandt, "Supersymmetry algebra cohomology I : Definition and general structure," *J. Math. Phys.*, vol. 51, p. 122302, 2010.

[59] F. Brandt, "Supersymmetry Algebra Cohomology : II. Primitive Elements in 2 and 3 Dimensions," *J. Math. Phys.*, vol. 51, p. 112303, 2010.

[60] F. Brandt, "Supersymmetry algebra cohomology III : Primitive elements in four and five dimensions," *J. Math. Phys.*, vol. 52, p. 052301, 2011.

[61] F. Brandt, "Supersymmetry algebra cohomology IV : Primitive elements in all dimensions from D=4 to D=11," *J. Math. Phys.*, vol. 54, p. 052302, 2013.

[62] M. Cederwall, B. E. W. Nilsson, and D. Tsimpis, "The structure of maximally supersymmetric Yang-Mills theory : Constraining higher-order corrections," *JHEP*, vol. 06, p. 034, 2001.

[63] M. Cederwall, B. E. W. Nilsson, and D. Tsimpis, “D = 10 super-Yang-Mills at $\mathcal{O}(\alpha'^2)$,” *JHEP*, vol. 07, p. 042, 2001.

[64] M. Cederwall, B. E. W. Nilsson, and D. Tsimpis, “Spinorial cohomology of abelian d = 10 super-Yang-Mills at O(α'^3),” *JHEP*, vol. 11, p. 023, 2002.

[65] P. S. Howe, S. F. Kerstan, U. Lindstrom, and D. Tsimpis, “The deformed M2-brane,” *JHEP*, vol. 09, p. 013, 2003.

[66] D. Tsimpis, “11D supergravity at $\mathcal{O}(l^3)$,” *JHEP*, vol. 10, p. 046, 2004.

[67] M. Cederwall, U. Gran, B. E. W. Nilsson, and D. Tsimpis, “Supersymmetric corrections to eleven-dimensional supergravity,” *JHEP*, vol. 05, p. 052, 2005.

[68] R. Xu, A. Schwarz, and M. Movshev, “Integral invariants in flat superspace,” *Nucl. Phys.*, vol. B884, pp. 28–43, 2014.

[69] C.-M. Chang, Y.-H. Lin, Y. Wang, and X. Yin, “Deformations with Maximal Supersymmetries Part 1 : On-shell Formulation,” 2014.

[70] C.-M. Chang, Y.-H. Lin, Y. Wang, and X. Yin, “Deformations with Maximal Supersymmetries Part 2 : Off-shell Formulation,” *JHEP*, vol. 04, p. 171, 2016.

[71] R. D’Auria, P. Fre, P. K. Townsend, and P. van Nieuwenhuizen, “Invariance of Actions, Rheonomy and the New Minimal $N = 1$ Supergravity in the Group Manifold Approach,” *Annals Phys.*, vol. 155, no. CERN-TH-3495, p. 423, 1984.

[72] S. J. Gates, Jr., M. T. Grisaru, M. E. Knutt-Wehlau, and W. Siegel, “Component actions from curved superspace : Normal coordinates and ectoplasm,” *Phys. Lett.*, vol. B421, pp. 203–210, 1998.

[73] P. S. Howe, O. Raetzel, and E. Sezgin, “On brane actions and superembeddings,” *JHEP*, vol. 08, p. 011, 1998.

[74] I. A. Bandos, D. P. Sorokin, and D. Volkov, “On the generalized action principle for superstrings and supermembranes,” *Phys. Lett.*, vol. B352, pp. 269–275, 1995.

[75] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, “Conformal supergravities as Chern-Simons theories revisited,” *JHEP*, vol. 03, p. 113, 2013.

[76] S. M. Kuzenko and J. Novak, “On supersymmetric Chern-Simons-type theories in five dimensions,” *JHEP*, vol. 02, p. 096, 2014.

[77] N. Berkovits and P. S. Howe, “The Cohomology of superspace, pure spinors and invariant integrals,” *JHEP*, vol. 06, p. 046, 2008.

[78] D. Freed, J. A. Harvey, R. Minasian, and G. W. Moore, “Gravitational anomaly cancellation for M theory five-branes,” *Adv. Theor. Math. Phys.*, vol. 2, pp. 601–618, 1998.

[79] P. S. Howe and U. Lindstrom, “Higher Order Invariants in Extended Supergravity,” *Nucl. Phys.*, vol. B181, pp. 487–501, 1981.

[80] A. M. Cohen, M. van Leeuwen, and B. Lisser, "`LiE v.2.2` A computer algebra package for Lie group computations (1998),"

[81] D. Tsimpis, "Curved 11D supergeometry," *JHEP*, vol. 11, p. 087, 2004.

[82] Wolfram, "An elementary introduction to the wolfram language," 2017.

[83] U. Gran, "GAMMA : A Mathematica package for performing Gamma-matrix algebra and Fierz transformations in arbitrary dimensions," 2001.

[84] K. Peeters, "Introducing cadabra : A symbolic computer algebra system for field theory problems," *arXiv preprint hep-th/0701238*, 2007.

[85] R. Portugal, "An algorithm to simplify tensor expressions," *Computer physics communications*, vol. 115, no. 2-3, pp. 215–230, 1998.

[86] N. Obeid, "On the simplification of tensor expressions," 2001.

[87] S. A. Fulling, R. C. King, B. G. Wybourne, and C. J. Cummins, "Normal forms for tensor polynomials. 1 : The Riemann tensor," *Class. Quant. Grav.*, vol. 9, pp. 1151–1197, 1992.

[88] E. Cremmer and B. Julia, "The so (8) supergravity," *Nuclear Physics B*, vol. 159, no. 1-2, pp. 141–212, 1979.

[89] F. Giani and M. Pernici, "N= 2 supergravity in ten dimensions," *Physical Review D*, vol. 30, no. 2, p. 325, 1984.

[90] M. J. Duff, B. E. Nilsson, and C. N. Pope, "Kaluza-klein supergravity," *Physics Reports*, vol. 130, no. 1-2, pp. 1–142, 1986.

[91] L. Romans, "Massive n= 2a supergravity in ten dimensions," in *Supergravities in Diverse Dimensions : Commentary and Reprints (In 2 Volumes)*, pp. 242–248, World Scientific, 1989.

[92] P. S. Howe, N. D. Lambert, and P. C. West, "A new massive type iia supergravity from compactification," *Phys. Lett.*, vol. B416, pp. 303–308, 1998.

[93] D. Tsimpis, "Massive iia supergravities," *Journal of High Energy Physics*, vol. 2005, no. 10, p. 057, 2005.

[94] H. Ruegg and M. Ruiz-Altaba, "The stueckelberg field," *International Journal of Modern Physics A*, vol. 19, no. 20, pp. 3265–3347, 2004.

[95] M. Dine, R. Rohm, N. Seiberg, and E. Witten, "Gluino condensation in superstring models," *Physics Letters B*, vol. 156, no. 1-2, pp. 55–60, 1985.

[96] G. L. Cardoso, G. Curio, G. Dall'Agata, and D. Lüst, "Heterotic string theory on non-kähler manifolds with h-flux and gaugino condensate," *Fortschritte der Physik*, vol. 52, no. 6-7, pp. 483–488, 2004.

[97] J.-P. Derendinger, C. Kounnas, and P. M. Petropoulos, "Gaugino condensates and fluxes in n= 1 effective superpotentials," *Nuclear Physics B*, vol. 747, no. 1-2, pp. 190–211, 2006.

[98] P. Manousselis, N. Prezas, and G. Zoupanos, "Supersymmetric compactifications of heterotic strings with fluxes and condensates," *Nuclear Physics B*, vol. 739, no. 1-2, pp. 85–105, 2006.

[99] A. Chatzistavrakidis, O. Lechtenfeld, and A. D. Popov, "Nearly kähler heterotic compactifications with fermion condensates," *Journal of High Energy Physics*, vol. 2012, no. 4, p. 114, 2012.

[100] K.-P. Gemmer and O. Lechtenfeld, "Heterotic g 2-manifold compactifications with fluxes and fermionic condensates," *Journal of High Energy Physics*, vol. 2013, no. 11, p. 182, 2013.

[101] C. Quigley, "Gaugino condensation and the cosmological constant," *Journal of High Energy Physics*, vol. 2015, no. 6, p. 104, 2015.

[102] R. Minasian, M. Petrini, and E. E. Svanes, "On heterotic vacua with fermionic expectation values," *Fortschritte der Physik*, vol. 65, no. 3-4, 2017.

[103] M. J. Duff and C. Orzalesi, "The cosmological constant in spontaneously compactified d= 11 supergravity," *Phys. Lett. B*, vol. 122, no. CERN-TH-3473, pp. 37–40, 1982.

[104] B. Jasinschi and A. Smith, "Fermionic mass and cosmological-constant generation fromn= 1, d= 11 supergravity theory," *Il Nuovo Cimento A (1965-1970)*, vol. 96, no. 1, pp. 107–123, 1986.

[105] I. Campbell and P. C. West, "N= 2, d= 10 non-chiral supergravity and its spontaneous compactification," *Nuclear Physics B*, vol. 243, no. 1, pp. 112–124, 1984.

[106] M. Huq and M. Namazie, "Kaluza–klein supergravity in ten dimensions," in *Supergravities in Diverse Dimensions : Commentary and Reprints (In 2 Volumes)*, pp. 225–241, World Scientific, 1989.

[107] R. Nicoletti and E. Orazi, "Geometric iia supergravity theories in the string frame," *International Journal of Modern Physics A*, vol. 26, no. 26, pp. 4585–4602, 2011.

A

Weil triviality in eleven dimensional supergravity

A.1 Weil triviality at $\mathcal{O}(l^0)$

In this section we give the details of the solution of the superspace equation $W_{12} = \mathrm{d}K_{11}$ at lowest order in the Planck length. As a byproduct, we shall see that the solution for K_{11} given in section (2.3.1) is unique up to exact terms. We look for the solution of the equation,

$$\mathrm{d}K_{11} = -\frac{1}{6}\, G_4{\scriptstyle\wedge} G_4{\scriptstyle\wedge} G_4\ , \tag{A.1}$$

where K_{11} must be gauge-invariant, i.e. function of the field strengths of the physical fields. The explicit construction of K_{11} in flat components proceeds by solving the equation at each mass dimension in a stepwise fashion, from dimension -3 to 2 (i.e. from $K_{\alpha_1\ldots\alpha_{11}}$ to $K_{a_1\ldots a_{11}}$). In components the equation (A.1) reads,

$$D_{[A_1}K_{A_2\ldots A_{12})} + \frac{11}{2}\, T_{[A_1A_2|}{}^{F}\, K_{F|A_3\ldots A_{12})} = -\frac{11!}{6(4!)^3}\, G_{[A_1\ldots A_4}\, G_{A_5\ldots A_8}\, G_{A_9\ldots A_{12})}, \tag{A.2}$$

where the torsion term arises from the the action of the exterior derivative on the supervielbein. The notation $[ABC)$ represents the graded anti-symmetrization (i.e. symmetrization or anti-symmetrization, depending on the bosonic or fermionic nature of the indices). *In the following, anti-symmetrization of the indices a_i and symmetrization of the indices α_i is always implied.*

The mass dimensions of the physical fields which will be involved in the construction of K_{11} are,

$$[D_{a_1}] = 1 \qquad [T_{a_1a_2}{}^{\alpha}] = 3/2 \qquad [G_{ab\alpha\beta}] = [T_{\alpha\beta}{}^{a}] = 0$$
$$[D_{\alpha_1}] = 1/2 \qquad \left[T_{a\alpha}{}^{\beta}\right] = 1 \qquad [G_{abcd}] = 1$$

From dimension -3 **to** -1/2

From dimension -3 (12 odd indices) to $-1/2$ (7 odd and 5 even indices), the right hand side of (A.1) always vanishes. Given the dimensions of the field strengths of the physical fields, the first non-vanishing component of K_{11} is $K_{\alpha_1\ldots\alpha_4a_1\ldots a_7}$, appearing for the first time in the 0-dimensional equation (6 fermionic indices and 6 bosonic indices). For example, the equation (A.2) at dimension $-1/2$ reads,

$$\frac{7}{12}\,D_{\alpha_1}K_{\alpha_2\ldots\alpha_7a_1\ldots a_5} \;-\; \frac{5}{12}D_{a_1}K_{a_2\ldots a_5\alpha_1\ldots\alpha_7}\;+$$
$$\frac{11}{2}\left(\frac{5}{33}\,T_{\alpha_1\alpha_2}{}^{f}\,K_{f\alpha_3\ldots\alpha_7a_1\ldots a_5} \;-\; \frac{7}{22}\,T_{a_1a_2}{}^{\epsilon}\,K_{\epsilon a_3\ldots a_5\alpha_1\ldots\alpha_7} \;-\; \frac{35}{66}\,T_{a_1\alpha_1}{}^{\epsilon}\,K_{\epsilon\alpha_2\ldots\alpha_7a_2\ldots a_5}\right) = 0\,,$$

and involves [1] $K^{(-1/2)}_{\alpha_1\ldots\alpha_5a_1\ldots a_6}$, $K^{(-1)}_{\alpha_1\ldots\alpha_6a_1\ldots a_5}$, $K^{(-3/2)}_{\alpha_1\ldots\alpha_7a_1\ldots a_4}$ and $K^{(-2)}_{\alpha_1\ldots\alpha_8a_1\ldots a_3}$, which cannot be expressed in terms of the physical fields : the equation is thus trivially satisfied.

Dimension 0 **- (**$A_1\ldots A_6 \to \alpha_1\ldots\alpha_6$**,** $A_7\ldots A_{12}\to a_1\ldots a_6$**)**

At dimension 0, eq. (A.2) reads :

$$\frac{1}{2}\,D_{\alpha_1}\,\overbrace{K_{\alpha_2\ldots\alpha_6a_1\ldots a_6}}^{0} + \frac{1}{2}D_{a_1}\,\overbrace{K_{a_2\ldots a_6\alpha_1\ldots\alpha_6}}^{0} +$$
$$\frac{11}{2}\left(\frac{5}{22}\,T_{\alpha_1\alpha_2}{}^{f}\,K_{f\alpha_3\ldots\alpha_6a_1\ldots a_6} \;+\; \frac{5}{22}\,T_{a_1a_2}{}^{\epsilon}\,\underbrace{K_{\epsilon a_3\ldots a_6\alpha_1\ldots\alpha_6}}_{0} + \frac{12}{22}\,T_{a_1\alpha_1}{}^{\epsilon}\,\underbrace{K_{\epsilon\alpha_2\ldots\alpha_6a_2\ldots a_6}}_{0}\right)$$
$$= -\frac{11!}{6(4!)^3}\,\frac{18}{77}\,G_{a_1a_2\alpha_1\alpha_2}\;G_{a_3a_4\alpha_3\alpha_4}\;G_{a_5a_6\alpha_5\alpha_6}\;.$$

Most terms vanish and the equation simplifies as follows,

$$(\Gamma^f)_{\alpha_1\alpha_2}K_{fa_1\ldots a_6\alpha_3\ldots\alpha_6} = 90\,(\Gamma_{a_1a_2})_{\alpha_1\alpha_2}(\Gamma_{a_3a_4})_{\alpha_3\alpha_4}(\Gamma_{a_5a_6})_{\alpha_5\alpha_6}.$$

Using the M2-brane identity as well as the so-called M5-brane identity,

$$(\Gamma^e)_{\alpha_1\alpha_2}(\Gamma_{ea_1\ldots a_4})_{\alpha_3\alpha_4} = 3\,(\Gamma_{a_1a_2})_{\alpha_1\alpha_2}(\Gamma_{a_3a_4})_{\alpha_3\alpha_4}\;, \tag{A.3}$$

1. In the following we will use superscripts to indicate the dimension. This should not be confused with the notation in the main text, e.g. (2.45) where the superscript denotes the order in the derivative expansion.

it is easy to check that the solution is given by,

$$K_{a_1\ldots a_7\alpha_1\ldots\alpha_4} = 42\,(\Gamma_{a_1\ldots a_5})_{\alpha_1\alpha_2}(\Gamma_{a_6a_7})_{\alpha_3\alpha_4}\,. \tag{A.4}$$

Dimension $1/2$ **- (**$A_1\ldots A_5 \to \alpha_1\ldots\alpha_5$**,** $A_6\ldots A_{12}\to a_1\ldots a_7$**)**

At dimension $1/2$, eq. (A.2) reads,

$$\begin{aligned}
&\frac{5}{12}\,\overbrace{D_{\alpha_1}K_{\alpha_2\ldots\alpha_5a_1\ldots a_7}}^{D_\alpha\left(\Gamma^{(5)}\Gamma^{(2)}\right)=0} - \frac{7}{12}\,D_{a_1}\,\overbrace{K_{a_2\ldots a_7\alpha_1\ldots\alpha_5}}^{0} + \\
&\frac{11}{2}\left(\frac{5}{33}\,T_{\alpha_1\alpha_2}{}^{f}\,K_{f\alpha_3\ldots\alpha_5a_1\ldots a_7} - \frac{35}{66}\,T_{a_1a_2}{}^{\epsilon}\,\underbrace{K_{\epsilon a_3\ldots a_7\alpha_1\ldots\alpha_5}}_{0} - \frac{7}{22}\,T_{a_1\alpha_1}{}^{\epsilon}\,\underbrace{K_{\epsilon\alpha_2\ldots\alpha_6a_2\ldots a_6}}_{0}\right) = 0\,,
\end{aligned}$$

which simplifies to,

$$(\Gamma^f)_{\alpha_1\alpha_2}K_{fa_1\ldots a_7\alpha_3\ldots\alpha_5} = 0\,.$$

Since $[K_{fa_1\ldots a_7\alpha_3\ldots\alpha_5}] = 1/2$ and there is no gauge-invariant field with that dimension, we conclude,

$$K_{a_1\ldots a_8\alpha_1\ldots\alpha_3} = 0\,. \tag{A.5}$$

Dimension 1 **- (**$A_1\ldots A_4 \to \alpha_1\ldots\alpha_4$**,** $A_5\ldots A_{12}\to a_1\ldots a_8$**)**

At dimension 1, eq. (A.2) reads,

$$\begin{aligned}
&\frac{4}{12}\,D_{\alpha_1}\,\overbrace{K_{\alpha_2\ldots\alpha_4a_1\ldots a_8}}^{0} + \frac{8}{12}\,\overbrace{D_{a_1}K_{a_2\ldots a_8\alpha_1\ldots\alpha_4}}^{\mathrm{d}_{a_1}\left(\Gamma^{(5)}\Gamma^{(2)}\right)=0} + \\
&\frac{11}{2}\left(\frac{1}{11}\,T_{\alpha_1\alpha_2}{}^{f}\,K_{f\alpha_3\alpha_4a_1\ldots a_8} + \frac{14}{33}\,T_{a_1a_2}{}^{\gamma}\,\underbrace{K_{\gamma a_3\ldots a_8\alpha_1\ldots\alpha_4}}_{0} + \frac{16}{33}\,T_{a_1\alpha_1}{}^{\gamma}\,K_{\gamma\alpha_2\ldots\alpha_4a_2\ldots a_8}\right) \\
&= -\frac{11!}{6(4!)^3}\,\frac{12}{55}\,G_{a_1a_2a_3a_4}G_{a_5a_6\alpha_1\alpha_2}G_{a_7a_8\alpha_3\alpha_4}\,,
\end{aligned}$$

which becomes, using (B.2),

$$\begin{aligned}
(\Gamma^f)_{\alpha_1\alpha_2}K_{fa_1\ldots a_8\alpha_3\alpha_4} = &-\frac{56}{3}\,i\,G_{a_1a_2a_3f}\,(\Gamma^f)_{\alpha_1\alpha_2}\,(\Gamma_{a_4\ldots a_8})_{\alpha_3\alpha_4} \\
&+\frac{7}{18}\,i\,G_{fghi}\,(\Gamma_{a_1\ldots a_6}{}^{fghi})_{\alpha_1\alpha_2}\,(\Gamma_{a_7a_8})_{\alpha_3\alpha_4} \\
&+70\,i\,(\Gamma_{a_1a_2})_{\alpha_1\alpha_2}(\Gamma_{a_3a_4})_{\alpha_3\alpha_4}G_{a_5\ldots a_8}\,.
\end{aligned} \tag{A.6}$$

The last term above can be expanded as,

$$
\begin{aligned}
70\, i\, (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\, (\Gamma_{a_3a_4})_{\alpha_3\alpha_4}\, G_{a_5\dots a_8} &= \frac{70}{3}\, i\, (\Gamma^f)_{\alpha_1\alpha_2}\, (\Gamma_{fa_1\dots a_4})_{\alpha_3\alpha_4}\, G_{a_5\dots a_8} \\
&= 42\, i\, (\Gamma^f)_{\alpha_1\alpha_2}\, (\Gamma_{[fa_1\dots a_4|})_{\alpha_3\alpha_4}\, G_{|a_5\dots a_8]} - \frac{56}{3}\, i\, (\Gamma^f)_{\alpha_1\alpha_2}\, (\Gamma_{[a_1\dots a_5|})_{\alpha_3\alpha_4}\, G_{|a_6a_7a_8]f}\ .
\end{aligned}
$$

Similarly, the second term on the right-hand side of (A.6) can be written in a manifestly τ-exact form,

$$
\begin{aligned}
\frac{7}{18}\, (\Gamma_{a_1\dots a_6}{}^{fghi})_{\alpha_1\alpha_2}\, (\Gamma_{a_7a_8})_{\alpha_3\alpha_4} &= -\frac{7}{18}\, \epsilon_{ja_1\dots a_6}{}^{fghi}\, (\Gamma^j)_{\alpha_1\alpha_2}\, (\Gamma_{a_7a_8})_{\alpha_3\alpha_4} \\
&= -\frac{1}{2}\, \epsilon_{[ja_1\dots a_6|}{}^{fghi}\, (\Gamma^j)_{\alpha_1\alpha_2}\, (\Gamma_{|a_7a_8]})_{\alpha_3\alpha_4} + \frac{1}{9}\, \epsilon_{a_1\dots a_7}{}^{fghi}\, \underbrace{(\Gamma^j)_{\alpha_1\alpha_2}\, (\Gamma_{a_8j})_{\alpha_3\alpha_4}}_{0}\ .
\end{aligned}
$$

Then eq. (A.6) takes the following form,

$$
(\Gamma^j)_{\alpha_1\alpha_2} K_{ja_1\dots a_8\alpha_3\alpha_4} = (\Gamma^j)_{\alpha_1\alpha_2}\Big(42\, i\, (\Gamma_{[ja_1\dots a_4})_{\alpha_3\alpha_4}\, G_{a_5\dots a_8]} - \frac{1}{2}\, i\, \epsilon_{[ja_1\dots a_6|}{}^{i_1\dots i_4}\, (\Gamma_{|a_7a_8]})_{\alpha_3\alpha_4}\, G_{i_1\dots i_4}\Big)\ .
$$

Since the cohomology group $H^{9,2}_\tau$ is trivial, the solution to the above equation reads,

$$
K_{a_1\dots a_9\alpha_1\alpha_2} = 42\, i\, (\Gamma_{a_1\dots a_5})_{\alpha_1\alpha_2}\, G_{a_6\dots a_9} - \frac{1}{2}\, i\, \epsilon_{a_1\dots a_7}{}^{i_1\dots i_4}\, (\Gamma_{a_8a_9})_{\alpha_1\alpha_2}\, G_{i_1\dots i_4}\ ,
$$

up to τ-exact terms.

Dimension $3/2$ **-** $(A_1 \dots A_3 \to \alpha_1 \dots \alpha_3,\ A_4 \dots A_{12} \to a_1 \dots a_9)$

At dimension $3/2$, eq. (A.2) reads,

$$
\begin{aligned}
&\frac{3}{12}\, D_{\alpha_1} K_{\alpha_2\alpha_3a_1\dots a_9} \; - \; \frac{9}{12} D_{\alpha_1}\, \overbrace{K_{a_2\dots a_9\alpha_1\dots\alpha_3}}^{0} \; - \\
&\frac{11}{2}\Big(\frac{1}{22}\, T_{\alpha_1\alpha_2}{}^f\, K_{fa_1\dots a_9\alpha_3} \; + \; \frac{6}{11}\, T_{a_1a_2}{}^\gamma\, K_{\gamma a_3\dots a_9\alpha_1\dots\alpha_3} \; + \; \frac{9}{22}\, T_{a_1\alpha_1}{}^\gamma\, \underbrace{K_{\gamma\alpha_2\alpha_3a_2\dots a_9}}_{0}\Big) = 0\ ,
\end{aligned}
$$

which becomes, using (B.2),

$$
\begin{aligned}
(\Gamma^f)_{\alpha_1\alpha_2} K_{fa_1\dots a_9\alpha_3} = &+ 252\, (\Gamma_{a_1\dots a_5})_{\alpha_2\alpha_3}\, (\Gamma_{a_6a_7})_{\alpha_1\Gamma}\, T_{a_8a_9}{}^\gamma \\
&- 3\, \epsilon_{a_1\dots a_7}{}^{i_1\dots i_4} (\Gamma_{a_8a_9})_{\alpha_1\alpha_2}\, (\Gamma_{i_1i_2})_{\alpha_3\Gamma}\, T_{i_3i_4}{}^\gamma \\
&+ 504\, (\Gamma_{a_1a_2})_{(\alpha_1\alpha_2|}\, (\Gamma_{a_3\dots a_7})_{|\alpha_3\Gamma)}\, T_{a_8a_9}{}^\gamma\ . \qquad \text{(A.7)}
\end{aligned}
$$

The decomposition of $K_{fa_1\dots a_9\alpha_3}$ in irreducible components is given by

$$
(10000) \otimes (00001) = (10001) \oplus (00001)\ ,
$$

whereas $T_{ab}{}^{\alpha}$ is in the representation (01001). It follows that,

$$K_{a_1\dots a_{10}\alpha_1} = 0 \,, \tag{A.8}$$

and moreover the right-hand side of (A.7) must vanish identically. This can be verified by e.g. taking the Hodge dual of $(\Gamma_{i_1 i_2})_{\alpha_3\Gamma}$ in the second term of (A.7), and using the gamma-tracelessness of $T_{ab}{}^{\gamma}$, cf. (B.4).

Dimension 2 **-** $(A_1A_2 \to \alpha_1\alpha_2$**,** $A_3\dots A_{12} \to a_1\dots a_{10})$

At dimension 2, eq. (A.2) reads,

$$\begin{aligned}
&\frac{2}{12}\, D_{\alpha_1} K_{\alpha_2 a_1\dots a_{10}} \;+\; \frac{10}{12} D_{a_1} K_{a_2\dots a_{10}\alpha_1\alpha_2} \,+ \\
&\frac{11}{2}\left(\frac{1}{66}\, T_{\alpha_1\alpha_2}{}^{f}\, K_{f a_1\dots a_{10}} \;+\; \frac{10}{33}\, T_{a_1 a_2}{}^{\gamma}\, K_{\gamma a_3\dots a_{10}\alpha_1\alpha_2} \;+\; \frac{15}{22}\, T_{a_1\alpha_1}{}^{\gamma}\, K_{\gamma\alpha_2 a_2\dots a_{10}}\right) \\
&= -\frac{11!}{6(4!)^3}\,\frac{18}{66}\, G_{a_1a_2a_3a_4} G_{a_5a_6a_7a_8} G_{a_9a_{10}\alpha_1\alpha_2} \,,
\end{aligned}$$

which becomes, using (B.2),

$$\begin{aligned}
&(\Gamma^{f})_{\alpha_1\alpha_2} K_{f a_1\dots a_{10}} = \\
&-10\, i \left(42\, i\, (\Gamma_{a_2\dots a_6})_{\alpha_1\alpha_2}\, \mathrm{d}_{a_1} G_{a_7\dots a_{10}} - \frac{1}{2}\, i\, \epsilon_{a_2\dots a_8}{}^{i_1\dots i_4}\, (\Gamma_{a_9 a_{10}})_{\alpha_1\alpha_2}\, \mathrm{d}_{a_1} G_{i_1\dots i_4}\right) \\
&-20\, i\, T_{a_1\alpha_1}{}^{\epsilon}\left(42\, i\, (\Gamma_{a_2\dots a_6})_{\epsilon\alpha_2}\, G_{a_7\dots a_{10}} - \frac{1}{2}\, i\, \epsilon_{a_2\dots a_8}{}^{i_1\dots i_4}\, (\Gamma_{a_9 a_{10}})_{\epsilon\alpha_2}\, G_{i_1\dots i_4}\right) \\
&-1575\, G_{a_1\dots a_4} G_{a_5\dots a_8} (\Gamma_{a_9 a_{10}})_{\alpha_1\alpha_2} \,.
\end{aligned} \tag{A.9}$$

Multiplying by $\Gamma^{(1)}$ and taking the trace leads to,

$$K_{a_1\dots a_{11}} = \frac{1}{72}\, \epsilon_{a_1\dots a_{11}} G_{d_1\dots d_4} G^{d_1\dots d_4} \,. \tag{A.10}$$

On the other hand contracting (A.9) with $\Gamma^{(2)}$ or $\Gamma^{(5)}$ imposes that the contraction of the right-hand side must be identically zero. This can indeed be straightforwardly verified using (B.4).

Dimension $5/2$ **-** $(A_1 \to \alpha_1$**,** $A_2\dots A_{12} \to a_1\dots a_{11})$

The equation at dimension 5/2 does not contain any additional information, but serves as a consistency check for the expressions we found for $K_{a_1\dots a_{11}}$. It reads,

$$\frac{1}{12}\, D_{\alpha_1} K_{a_1\dots a_{11}} \;-\; \frac{11}{12}\, D_{a_1} K_{a_2\dots a_{11}\alpha_1} \;-\; \frac{11}{2}\left(\frac{2}{12}\, T_{a_1 a_2}{}^{\gamma}\, K_{\gamma a_3\dots a_{11}\alpha_1} \;-\; \frac{10}{12}\, T_{a_1\alpha_1}{}^{\gamma}\, K_{\gamma a_2\dots a_{11}}\right) = 0 \,,$$

which becomes, using (B.2) and (A.10),

$$\frac{1}{72}\,\epsilon_{a_1\ldots a_{11}}\,D_{\alpha_1}G_{abcd}G^{abcd} - 330\,i\,G_{a_1a_2gh}\,(\Gamma^{gh}{}_{a_3\ldots a_9}\,T_{a_{10}a_{11}})_{\alpha_1} + 2310\,i\,G_{a_1\ldots a_4}\,(\Gamma_{a_5\ldots a_9}\,T_{a_{10}a_{11}})_{\alpha_1}\,.$$

Using (B.3), (B.1) we then obtain the constraint,

$$\begin{aligned}0 = \epsilon_{a_1\ldots a_{11}}\,T_{d_1d_2}{}^{\delta}\,(\Gamma_{d_3d_4})_{\delta\alpha_1}\,G^{d_1\ldots d_4} &+ \frac{77}{4}\,\epsilon_{a_1\ldots a_5}{}^{b_1\ldots b_6}\,(\Gamma_{b_1\ldots b_6}\,T_{a_6a_7})_{\alpha_1}\,G_{a_8\ldots a_{11}}\\ &- 990\,\epsilon_{a_1\ldots a_7}{}^{b_1b_2gh}\,(\Gamma_{b_1b_2}\,T_{a_8a_9})_{\alpha_1}\,G_{a_{10}a_{11}gh}\,,\end{aligned} \tag{A.11}$$

which can be seen to be automatically satisfied by contracting (A.11) with $\epsilon_{a_1\ldots a_{11}}$. The next equation (of dimension 3) is trivially satisfied, since the purely bosonic component of a 12-form vanishes automatically in eleven dimensions.

A.2 Weil triviality at $\mathcal{O}(l^3)$

In this section we are looking for the solution to the equation,

$$\mathrm{d}K_{11} = G_4{\wedge}G_4{\wedge}R^{ab}{\wedge}R_{ba}\,. \tag{A.12}$$

We will construct all components of K_{11} explicitly and confirm that the solution of section 2.3.2 is unique up to exact terms. The resolutions of the first few equations are entirely developed, but the last ones are only sketched. In components the equation above takes the following form,

$$D_{[A_1}\,K_{A_2\ldots A_{12})} + \frac{11}{2}\,T_{[A_1A_2|}{}^{F}\,K_{F|A_3\ldots A_{12})} = \frac{11!}{4(4!)^2}\,R_{[A_1A_2|c_1c_2}\,R_{|A_3A_4|}{}^{c_2c_1}\,G_{|A_5\ldots A_8}\,G_{A_9\ldots A_{12})}\,. \tag{A.13}$$

The dimensions of the physical fields are the same as before, with the addition of $[R_{abcd}] = 2$. The dimensions of the various components of K range from – $1/2$ ($K_{\alpha_1\ldots\alpha_{11}}$) to 5 ($K_{a_1\ldots a_{11}}$).

Dimension 0 to $3/2$

Since the dimension of $K_{\alpha_1\ldots\alpha_{11}}$ is $-1/2$, it must be set to zero as it cannot be expressed in terms of the physical fields. The equation of dimension 0 then takes the form,

$$D_{\alpha_1}\,\underbrace{K_{\alpha_2\ldots\alpha_{12}}}_{0} + \frac{11}{2}\,T_{\alpha_1\alpha_2}{}^{f}\,K_{f\alpha_3\ldots\alpha_{12}} = \frac{11!}{4(4!)^2}\,R_{\alpha_1\alpha_2c_1c_2}\,R_{\alpha_3\alpha_4}{}^{c_2c_1}\,\underbrace{G_{\alpha_5\ldots\alpha_8}\,G_{\alpha_9\ldots\alpha_{12}}}_{0}\,,$$

which simplifies to,

$$(\Gamma^f)_{\alpha_1\alpha_2}\,K_{f\alpha_3\ldots\alpha_{12}} = 0\,. \tag{A.14}$$

Since $[K_{f\alpha_3\ldots\alpha_{12}}] = 0$ and $H^{1,10}_{\tau}$ is non-trivial, a τ-non-exact solution involving only gamma matrices could exist. In that case $K_{f\alpha_3\ldots\alpha_{12}}$ would necessary transform as a scalar, since the only available gauge-invariant superfield of zero dimension is a constant. On the other hand,

$$(10000) \otimes (00001)^{\otimes_S 10} = 1 \times (00000) + \cdots \ ,$$

i.e. the decomposition of $K_{f\alpha_3\ldots\alpha_{12}}$ contains a unique scalar combination. It follows that,

$$K_{f\alpha_3\ldots\alpha_{12}} \propto (\Gamma_f)_{\alpha_3\alpha_4}(\Gamma^a)_{\alpha_5\alpha_6}(\Gamma_a)_{\alpha_7\alpha_8}(\Gamma^b)_{\alpha_9\alpha_{10}}(\Gamma_b)_{\alpha_{11}\alpha_{12}} \ .$$

However it can be verified that this expression does not satisfy eq. (A.14), unless $K_{a_1\alpha_3\ldots\alpha_{12}} = 0$.

The right-hand side of eq. (A.13) vanishes from dimension 0 to dimension $3/2$, and the equations to solve are all similar to (A.14) : The component $K^{(1/2)}_{a_1a_2\alpha_1\ldots\alpha_9}$ will be set to zero because there is no gauge-invariant field of dimension $1/2$. The components $K^{(1)}_{a_1a_2a_3\alpha_1\ldots\alpha_8}$, $K^{(3/2)}_{a_1a_2a_3a_4\alpha_1\ldots\alpha_7}$ will be set to zero, up to exact terms, as a consequence of the triviality of $H^{3,8}_\tau$, $H^{4,7}_\tau$.

Dimension 2 **-** $(A_1 \ldots A_8 \to \alpha_1 \ldots \alpha_8$**,** $A_9 \ldots A_{12} \to a_1 \ldots a_4)$

This is the first equation with a non-zero right-hand side,

$$\begin{aligned}
&\frac{8}{12} D_{\alpha_1} \overbrace{K_{\alpha_2\ldots\alpha_8 a_1\ldots a_4}}^{0} + \frac{4}{12} D_{a_1} \overbrace{K_{a_2\ldots a_4\alpha_1\ldots\alpha_8}}^{0} \\
&+\frac{11}{2}\left(\frac{14}{33} T_{\alpha_1\alpha_2}{}^{f} K_{fa_1\ldots a_4\alpha_3\ldots\alpha_8} + \frac{1}{11} T_{a_1a_2}{}^{\gamma} \underbrace{K_{\gamma\alpha_1\ldots\alpha_8 a_3a_4}}_{0} + \frac{16}{33} T_{a_1\alpha_1}{}^{\gamma} \underbrace{K_{\gamma\alpha_3\ldots\alpha_8 a_2\ldots a_4}}_{0}\right) \\
&=3\,\frac{4}{55}\,\frac{11!}{4(4!)^2} R_{\alpha_1\alpha_2c_1c_2} R_{\alpha_3\alpha_4}{}^{c_2c_1} G_{a_1a_2\alpha_5\alpha_6} G_{a_3a_4\alpha_7\alpha_8} \ ,
\end{aligned}$$

which becomes, using (B.2),

$$(\Gamma^f)_{\alpha_1\alpha_2} K_{fa_1\ldots a_4\alpha_3\ldots\alpha_8} = -180\, i\, (\Gamma^f)_{\alpha_1\alpha_2}(\Gamma_{fa_1\ldots a_4})_{\alpha_3\alpha_4} R_{\alpha_5\alpha_6}{}^{c_1c_2} R_{\alpha_7\alpha_8c_2c_1} \ .$$

Since $H^{5,6}_\tau$ is trivial, the solution reads,

$$K^{(2)}_{a_1\ldots a_5\alpha_1\ldots\alpha_6} = -180\, i\, (\Gamma_{a_1\ldots a_5})_{\alpha_1\alpha_2} R_{\alpha_3\alpha_4}{}^{c_1c_2} R_{\alpha_5\alpha_6c_2c_1} \ ,$$

up to τ-exact terms.

Dimension $5/2$ **- (**$A_1 \ldots A_7 \to \alpha_1 \ldots \alpha_7$**,** $A_8 \ldots A_{12} \to a_1 \ldots a_5$**)**

At dimension $5/2$, eq. (A.13) reads,

$$
\begin{aligned}
&\frac{7}{12}\, D_{\alpha_1} K_{\alpha_2 \ldots \alpha_7 a_1 \ldots a_5} \;-\; \frac{5}{12}\, D_{a_1} \overbrace{K_{a_2 \ldots a_5 \alpha_1 \ldots \alpha_7}}^{0} \\
&- \frac{11}{2}\Big(\frac{7}{22}\, T_{\alpha_1\alpha_2}{}^{f}\, K_{f a_1 \ldots a_5 \alpha_3 \ldots \alpha_7} \;+\; \frac{5}{33}\, T_{a_1 a_2}{}^{\gamma}\, \underbrace{K_{\gamma \alpha_1 \ldots \alpha_7 a_3 \ldots a_5}}_{0} + \frac{35}{66}\, T_{a_1 \alpha_1}{}^{\gamma}\, \underbrace{K_{\gamma \alpha_3 \ldots \alpha_7 a_2 \ldots a_5}}_{0}\Big) \\
&= \frac{11!}{4(4!)^2}\Big(2\, \frac{1}{11}\, R_{\alpha_1\alpha_2 c_1 c_2}\, R_{\alpha_3 a_1}{}^{c_2 c_1}\, G_{a_2 a_3 \alpha_4 \alpha_5}\, G_{a_4 a_5 \alpha_6 \alpha_7}\Big),
\end{aligned}
$$

which becomes, using (B.2),

$$
\begin{aligned}
&(\Gamma^{f})_{\alpha_1\alpha_2}\, K_{f a_1 \ldots a_5 \alpha_3 \ldots \alpha_7} = \\
&- 120\, i\, (\Gamma_{a_1 \ldots a_5})_{\alpha_1\alpha_2} R_{\alpha_3\alpha_4}{}^{c_1 c_2} \left((\Gamma^{e_1 e_2})_{\alpha_5\alpha_6} (\Gamma_{[c_1 c_2}\, T_{e_1 e_2]})_{\alpha_7} + \frac{1}{24} (\Gamma_{c_1 c_2}{}^{e_1 \ldots e_4})_{\alpha_5\alpha_6} (\Gamma_{e_1 e_2}\, T_{e_3 e_4})_{\alpha_7} \right) \\
&+ 1800\, i\, (\Gamma_{a_1 a_2})_{\alpha_1\alpha_2}\, (\Gamma_{a_3 a_4})_{\alpha_3\alpha_4} R_{\alpha_5\alpha_6}{}^{c_1 c_2} R_{\alpha_7 a_5 c_2 c_1}\ . \qquad \text{(A.15)}
\end{aligned}
$$

The second term in (A.15) can be written,

$$
\begin{aligned}
&1800\, i\, (\Gamma_{[a_1 a_2|})_{\alpha_1\alpha_2}\, (\Gamma_{|a_3 a_4|})_{\alpha_3\alpha_4}\, R_{\alpha_5\alpha_6}{}^{c_1 c_2} R_{\alpha_7 |a_5] c_2 c_1} \qquad \text{(A.16)} \\
&= 600\, i\, (\Gamma^{g})_{\alpha_1\alpha_2}\, (\Gamma_{g[a_1 a_2 a_3 a_4|})_{\alpha_3\alpha_4}\, R_{\alpha_5\alpha_6}{}^{c_1 c_2} R_{\alpha_7 |a_5] c_2 c_1} \\
&= 600\, i\, (\Gamma^{g})_{\alpha_1\alpha_2} \left(\frac{6}{5} (\Gamma_{[g a_1 a_2 a_3 a_4|})_{\alpha_3\alpha_4}\, R_{\alpha_5\alpha_6}{}^{c_1 c_2} R_{\alpha_7 |a_5] c_2 c_1} \;+\; \frac{1}{5} (\Gamma_{a_1 a_2 a_3 a_4 a_5})_{\alpha_3\alpha_4}\, R_{\alpha_5\alpha_6}{}^{c_1 c_2} R_{\alpha_7 g c_2 c_1} \right).
\end{aligned}
$$

One can then verify that the second term on the right-hand side of (A.16) cancels with the first term on the right-hand side of (A.15). Since the first term on the right-hand side of (A.16) is in a τ-exact form and $H^{6,5}_{\tau}$ is trivial, the solution reads,

$$
K^{(5/2)}_{a_1 \ldots a_6 \alpha_1 \ldots \alpha_5} = 720\, i\, (\Gamma_{a_1 \ldots a_5})_{\alpha_1\alpha_2}\, R_{\alpha_3\alpha_4}{}^{c_1 c_2} R_{\alpha_5 a_6 c_2 c_1}\ ,
$$

up to τ-exact terms.

Dimension 3 **-** $(A_1 \ldots A_6 \to \alpha_1 \ldots \alpha_6,\, A_7 \ldots A_{12} \to a_1 \ldots a_6)$

At dimension 3, eq. (A.13) reads,

$$\begin{aligned}
&\frac{1}{2}\,D_{\alpha_1} K_{\alpha_2\ldots\alpha_6 a_1\ldots a_6} + \frac{1}{2}\,D_{a_1}\,\overbrace{K_{a_2\ldots a_6\alpha_1\ldots\alpha_6}}^{0} \\
&+\frac{11}{2}\left(\frac{5}{22}\,T_{\alpha_1\alpha_2}{}^{f}\,K_{f a_1\ldots a_6\alpha_3\ldots\alpha_6} + \frac{5}{22}\,T_{a_1 a_2}{}^{\gamma}\,\underbrace{K_{\gamma\alpha_1\ldots\alpha_6 a_3\ldots a_6}}_{0} + \frac{6}{11}\,T_{a_1\alpha_1}{}^{\gamma}\,\underbrace{K_{\gamma\alpha_3\ldots\alpha_6 a_2\ldots a_6}}_{0}\right) \\
&=\frac{11!}{4(4!)^2}\Bigg(-\frac{12}{77}\,R_{\alpha_1 a_1}{}^{c_1c_2}\,R_{\alpha_2 a_2 c_2 c_1}\,G_{a_3a_4\alpha_3\alpha_4}\,G_{a_5a_6\alpha_5\alpha_6} \\
&\qquad\qquad + 2\,\frac{1}{154}\,R_{\alpha_1\alpha_2}{}^{c_1c_2}\,R_{\alpha_3\alpha_4 c_2 c_1}\,G_{a_1a_2\alpha_5\alpha_6}\,G_{a_3\ldots a_6} \\
&\qquad\qquad + 2\,\frac{3}{77}\,R_{a_1 a_2}{}^{c_1c_2}\,R_{\alpha_1\alpha_2 c_2 c_1}\,G_{a_3a_4\alpha_3\alpha_4}\,G_{a_5a_6\alpha_5\alpha_6}\Bigg)\,,
\end{aligned}$$

which becomes, using (B.2),

$$\begin{aligned}
-\frac{5}{4}\,i\,(\Gamma^f)_{\alpha_1\alpha_2}\,K_{f a_1\ldots a_6\alpha_3\ldots\alpha_6} = &-\frac{1}{2}\,D_{\alpha_1}\,K_{\alpha_2\ldots\alpha_6 a_1\ldots a_6} \\
&-\frac{1}{2}\,D_{a_1}\,K_{a_2\ldots a_6\alpha_1\ldots\alpha_6} \\
&-3\,T^{\epsilon}_{a_1\alpha_1}\,K_{\epsilon\alpha_2\ldots\alpha_6 a_2\ldots a_6} \\
&-2700\,R_{\alpha_1 a_1}{}^{c_1c_2}\,R_{\alpha_2 a_2 c_2 c_1}\,G_{a_3a_4\alpha_3\alpha_4}\,G_{a_5a_6\alpha_5\alpha_6} \\
&+225\,R_{\alpha_1\alpha_2}{}^{c_1c_2}\,R_{\alpha_3\alpha_4 c_2 c_1}\,G_{a_1a_2\alpha_5\alpha_6}\,G_{a_3\ldots a_6} \\
&+1350\,R_{a_1a_2}{}^{c_1c_2}\,R_{\alpha_1\alpha_2 c_2c_1}\,G_{a_3a_4\alpha_3\alpha_4}\,G_{a_5a_6\alpha_5\alpha_6}\,.
\end{aligned}$$

Let us now examine separately each group of terms in the equation above with the same type of field content. There are four G^3 terms which read,

$$\begin{aligned}
&-225\,i\,G_{a_1\ldots a_4}\,(\Gamma_{a_5a_6})_{\alpha_1\alpha_2}\,R_{\alpha_3\alpha_4}{}^{c_1c_2}\,R_{\alpha_5\alpha_6 c_1c_2} \\
&-360\,(\Gamma_{a_1\ldots a_5})_{\alpha_1\alpha_2}\,R_{\alpha_3\alpha_4 c_1c_2}\,T_{c_2\alpha_5}{}^{\epsilon}\,T_{c_1\epsilon}{}^{\beta}\,(\Gamma_{a_6})_{\beta\alpha_6} \\
&+720\,(\Gamma_{a_1\ldots a_5})_{\alpha_1\alpha_2}\,R_{\alpha_3\alpha_4 c_1c_2}\,T_{c_1\alpha_5}{}^{\epsilon}\,T_{a_6\epsilon}{}^{\beta}\,(\Gamma_{c_2})_{\beta\alpha_6} \\
&-540\,i\,(\Gamma_{a_1\ldots a_5})_{(\alpha_1\epsilon|}\,T_{a_6\alpha_2}{}^{\epsilon}\,R_{|\alpha_3\alpha_4|}{}^{c_1c_2}\,R_{|\alpha_5\alpha_6)c_2c_1}\,.
\end{aligned}\tag{A.17}$$

The last term in (A.17) can be split in two parts,

$$-540\,i\left(\frac{2}{6}(\Gamma_{a_1\ldots a_5})_{\alpha_1\epsilon}\,T_{a_6\alpha_2}{}^{\epsilon}\,R_{\alpha_3\alpha_4}{}^{c_1c_2}\,R_{\alpha_5\alpha_6c_2c_1} + \frac{4}{6}(\Gamma_{a_1\ldots a_5})_{\alpha_1\alpha_2}\,T_{a_6\alpha_3}{}^{\epsilon}\,R_{\epsilon\alpha_4}{}^{c_1c_2}\,R_{\alpha_5\alpha_6c_2c_1}\right).$$

The first one leads to,

$$(\Gamma^g)_{\alpha_1\alpha_2}\left(\frac{5}{8}\,i\,\epsilon_{ga_1\ldots a_6}{}^{b_1\ldots b_4}\,G_{b_1\ldots b_4}\,R_{\alpha_3\alpha_4}{}^{c_1c_2}\,R_{\alpha_5\alpha_6c_2c_1}\right) + 225\,i\,G_{a_1\ldots a_4}\,(\Gamma_{a_5a_6})_{\alpha_1\alpha_2}\,R_{\alpha_3\alpha_4}{}^{c_1c_2}\,R_{\alpha_5\alpha_6c_2c_1}\,,$$

where the first term is τ-exact, and the second term cancels with the first one in (A.17). It can then be verified that the three remaining G^3 terms cancel out. Moreover there are three terms of the schematic form $G(DG)$,

$$\begin{aligned}
&- 360\, i\, (\Gamma^{a_1\dots a_5})_{\alpha_1\alpha_2}\, R_{\alpha_3\alpha_4}{}^{c_1c_2}\, \mathrm{d}_{c_2} T_{c_1\alpha_5}{}^{\beta}(\Gamma_{a_6})_{\beta\alpha_6} \\
&- 720\, i\, (\Gamma^{a_1\dots a_5})_{\alpha_1\alpha_2}\, R_{\alpha_3\alpha_4}{}^{c_1c_2}\, \mathrm{d}_{c_1} T_{a_6\alpha_5}{}^{\beta}(\Gamma_{c_2})_{\beta\alpha_6} \\
&- 180\, i\, (\Gamma^{a_1\dots a_5})_{\alpha_1\alpha_2}\, R_{\alpha_3\alpha_4}{}^{c_1c_2}\, \mathrm{d}_{a_6} R_{\alpha_5\alpha_6c_2c_1}\ ,
\end{aligned} \tag{A.18}$$

which cancel each other out. There are two RG terms which read,

$$\begin{aligned}
&- 1350\, (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\, (\Gamma_{a_3a_4})_{\alpha_3\alpha_4}\, R_{a_5a_6}{}^{c_1c_2}\, R_{\alpha_5\alpha_6c_2c_1} \\
&+ 45\, (\Gamma_{a_1\dots a_5})_{\alpha_1\alpha_2}\, R_{\alpha_3\alpha_4}{}^{c_1c_2} \Big((\Gamma^{e_1e_2}\, \Gamma_{a_6})_{\alpha_5\alpha_6}\, R_{c_2c_1e_1e_2} - 2\, (\Gamma^{e_1e_2}\, \Gamma_{c_2})_{\alpha_5\alpha_6}\, R_{c_1a_6e_1e_2} \Big)\ .
\end{aligned} \tag{A.19}$$

The first term of (A.20)can be put in a τ-exact form,

$$\begin{aligned}
&- 1350\, (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\, (\Gamma_{a_3a_4})_{\alpha_3\alpha_4}\, R_{a_5a_6}{}^{c_1c_2}\, R_{\alpha_5\alpha_6c_2c_1} = \\
&\quad (\Gamma^{g})_{\alpha_1\alpha_2} \Big(- 630\, i\, (\Gamma_{[ga_1\dots a_4|})_{\alpha_3\alpha_4}\, R_{|a_5a_6]}{}^{c_1c_2}\, R_{\alpha_5\alpha_6c_2c_1} + 180\, (\Gamma_{a_1\dots a_5})_{\alpha_1\alpha_2}\, R_{a_6g}{}^{c_1c_2}\, R_{\alpha_5\alpha_6c_2c_1} \Big)\ ,
\end{aligned}$$

while the remaining RG terms cancel out. There are two T^2 terms which read,

$$\begin{aligned}
&+ 2700\, (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\, (\Gamma_{a_3a_4})_{\alpha_3\alpha_4}\, R_{\alpha_5a_5}{}^{c_1c_2}\, R_{\alpha_6a_6c_2c_1} \\
&+ 1080\, i\, (\Gamma_{a_1\dots a_5})_{\alpha_1\alpha_2} \Big((\Gamma^{e_1e_2})_{\alpha_3\alpha_4} (\Gamma_{[c_1c_2}\, T_{e_1e_2]})_{\alpha_5} + \frac{1}{24} (\Gamma_{c_1c_2}{}^{e_1\dots e_4})_{\alpha_3\alpha_4} (\Gamma_{e_1e_2}\, T_{e_3e_4})_{\alpha_5} \Big) R_{\alpha_6a_6c_2c_1}\ .
\end{aligned}$$

The first term can be put in a τ-exact form,

$$\begin{aligned}
&2700\, (\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\, (\Gamma_{a_3a_4})_{\alpha_3\alpha_4}\, R_{\alpha_5a_5}{}^{c_1c_2}\, R_{\alpha_6a_6c_2c_1} = \\
&\quad \frac{1}{3}\, 2700\, (\Gamma^{g})_{\alpha_1\alpha_2} \Big(\frac{7}{5}\, (\Gamma_{ga_1\dots a_4})_{\alpha_3\alpha_4}\, R_{\alpha_5a_5}{}^{c_1c_2}\, R_{\alpha_6a_6c_2c_1} + \frac{2}{5}\, (\Gamma_{a_1\dots a_5})_{\alpha_3\alpha_4}\, R_{\alpha_5a_6}{}^{c_1c_2}\, R_{\alpha_6gc_2c_1} \Big)\ ,
\end{aligned}$$

while the remaining TT terms cancel out. Taking the triviality of $H_{\tau}^{7,4}$ into account, the non-vanishing terms extracted from the RG, T^2, and G^3 terms lead to the solution,

$$\begin{aligned}
K^{(3)}_{a_1\dots a_7\alpha_1\dots\alpha_4} &= 504\, i\, (\Gamma_{a_1\dots a_5})_{\alpha_1\alpha_2} \Big(- R_{a_6a_7}{}^{c_1c_2}\, R_{\alpha_5\alpha_6c_2c_1} + 2\, R_{\alpha_5a_6}{}^{c_1c_2}\, R_{\alpha_6a_7c_2c_1} \Big) \\
&\quad - \frac{1}{2}\, i\, \epsilon_{a_1\dots a_7}{}^{b_1\dots b_4}\, G_{b_1\dots b_4}\, R_{\alpha_1\alpha_2}{}^{c_1c_2}\, R_{\alpha_3\alpha_4c_2c_1}\ ,
\end{aligned} \tag{A.20}$$

up to τ-exact terms.

Dimension $7/2$ **-** ($A_1 \ldots A_5 \to \alpha_1 \ldots \alpha_5$**,** $A_6 \ldots A_{12} \to a_1 \ldots a_7$)

At dimension $7/2$, eq. (A.13) reads,

$$
\begin{aligned}
&\frac{5}{12}\, D_{\alpha_1} K_{\alpha_2 \ldots \alpha_5 a_1 \ldots a_7} \;-\; \frac{7}{12}\, D_{a_1} K_{a_2 \ldots a_7 \alpha_1 \ldots \alpha_5} \\
&-\frac{11}{2}\left(\frac{5}{33}\, T_{\alpha_1 \alpha_2}{}^{f}\, K_{f a_1 \ldots a_7 \alpha_3 \ldots \alpha_5} \;-\; \frac{7}{22}\, T_{a_1 a_2}{}^{\gamma}\, K_{\gamma \alpha_1 \ldots \alpha_5 a_3 \ldots a_7} \;+\; \frac{35}{66}\, T_{a_1 \alpha_1}{}^{\gamma}\, K_{\gamma \alpha_3 \ldots \alpha_5 a_2 \ldots a_7}\right) \\
&\quad = \frac{11!}{4(4!)^2}\Bigg(2\, \frac{1}{11}\, R_{\alpha_1 a_1}{}^{c_1 c_2}\, R_{a_2 a_3 c_2 c_1}\, G_{a_4 a_5 \alpha_3 \alpha_4}\, G_{a_6 a_7 \alpha_5 \alpha_6} \\
&\qquad\qquad\qquad\qquad + 4\, \frac{1}{66}\, R_{\alpha_1 a_1}{}^{c_1 c_2}\, R_{\alpha_2 \alpha_3 c_2 c_1}\, G_{a_2 a_3 \alpha_4 \alpha_5}\, G_{a_4 \ldots a_7}\Bigg)\,.
\end{aligned}
$$

The right-hand side of the equation above contains terms of the form $G(DT)$, $T(DG)$, TR, and TG^2. The first two groups of terms simply vanish (without the use of any equations of motion or BI). Two τ-exact terms can be extracted from RT and TG^2, and the remaining terms cancel out. This leads to the solution,

$$
\begin{aligned}
K^{(7/2)}_{a_1 \ldots a_8 \alpha_1 \ldots \alpha_3} =\,& 2016\, i\, (\Gamma_{a_1 \ldots a_5})_{\alpha_1 \alpha_2}\, R_{a_6 a_7}{}^{c_1 c_2} R_{\alpha_3 a_8 c_2 c_1} \\
&+\, 4\, \epsilon_{a_1 \ldots a_7}{}^{b_1 \ldots b_4}\, G_{b_1 \ldots b_4}\, R_{\alpha_1 \alpha_2}{}^{c_1 c_2}\, R_{\alpha_3 a_8 c_2 c_1}\;,
\end{aligned}
$$

up to τ-exact terms.

Dimension 4 **-** ($A_1 \ldots A_4 \to \alpha_1 \ldots \alpha_4$**,** $A_5 \ldots A_{12} \to a_1 \ldots a_8$)

At dimension 4, eq. (A.13) reads,

$$
\begin{aligned}
&\frac{4}{12}\, D_{\alpha_1} K_{\alpha_2 \ldots \alpha_4 a_1 \ldots a_8} \;+\; \frac{8}{12}\, D_{a_1} K_{a_2 \ldots a_8 \alpha_1 \ldots \alpha_4} \\
&+\frac{11}{2}\left(\frac{1}{11}\, T_{\alpha_1 \alpha_2}{}^{f}\, K_{f a_1 \ldots a_8 \alpha_3 \alpha_4} \;+\; \frac{14}{33}\, T_{a_1 a_2}{}^{\gamma}\, K_{\gamma \alpha_1 \ldots \alpha_4 a_3 \ldots a_8} \;+\; \frac{16}{33}\, T_{a_1 \alpha_1}{}^{\gamma}\, K_{\gamma \alpha_2 \ldots \alpha_4 a_2 \ldots a_8}\right) \\
&= \frac{11!}{4(4!)^2}\Bigg(1260\, R_{a_1 a_2}{}^{c_1 c_2}\, R_{a_3 a_4 c_2 c_1}\, G_{a_5 a_6 \alpha_3 \alpha_4}\, G_{a_7 a_8 \alpha_5 \alpha_6} \\
&\qquad\qquad + 35\, R_{\alpha_1 \alpha_2}{}^{c_1 c_2}\, R_{\alpha_3 \alpha_4 c_2 c_1}\, G_{a_1 a_2 a_3 a_4}\, G_{a_5 \ldots a_8} \\
&\qquad\qquad\qquad + 4 \cdot 210\, R_{\alpha_1 \alpha_2}{}^{c_1 c_2}\, R_{a_1 a_2 c_2 c_1}\, G_{a_3 a_4 a_5 a_6}\, G_{a_7 a_8 \alpha_3 \alpha_4} \\
&\qquad\qquad\qquad\qquad - 2 \cdot 840\, R_{\alpha_1 a_1}{}^{c_1 c_2}\, R_{\alpha_2 a_2 c_2 c_1}\, G_{a_3 a_4 \alpha_3 \alpha_4}\, G_{a_5 a_6 a_7 a_8}\Bigg)\,.
\end{aligned}
$$

The terms in the equation above can be cast in eight groups : R^2, RG^2, $R(DG)$, G^4, $G^2(DG)$, GT^2, $T(DT)$, and $G(DR)$. Parts of the terms of the form R^2, G^2R, and GT^2 can be put in a τ-exact form, while the remaining terms cancel out. Taking into account the BI,

$$
D_{a_1} R_{a_2 a_3 c_1 c_2} = -T_{a_1 a_2}{}^{\gamma} R_{\gamma a_3 c_1 c_2}\;, \tag{A.21}
$$

we see that the term $G(DR)$ cancel against a term from GT^2. Taking into account the equation of motion of G we see that a term from $G^2(DG)$ cancels against a term in G^4,

$$\epsilon_{a_1\ldots a_7}{}^{b_1\ldots b_4}\, D_{a_8} G_{b_1\ldots b_4} = \frac{1}{2}\,\epsilon_{a_1\ldots a_8}{}^{b_1\ldots b_3}\, D^c G_{cb_1\ldots b_3} = 105\, G_{a_1\ldots a_4}\, G_{a_5\ldots a_8}\ . \tag{A.22}$$

We are thus led to the solution,

$$\begin{aligned} K^{(4)}_{a_1\ldots a_9\alpha_1\ldots\alpha_2} = & -1512\, i\, (\Gamma_{a_1\ldots a_5})_{\alpha_1\alpha_2}\, R_{a_6a_7}{}^{c_1c_2}\, R_{a_8a_9c_2c_1} \\ & - 6\,\epsilon_{a_1\ldots a_7}{}^{b_1\ldots b_4}\, G_{b_1\ldots b_4}\, R_{\alpha_1\alpha_2}{}^{c_1c_2}\, R_{a_8a_9c_2c_1} \\ & + 12\,\epsilon_{a_1\ldots a_7}{}^{b_1\ldots b_4}\, G_{b_1\ldots b_4}\, R_{\alpha_1a_8}{}^{c_1c_2}\, R_{\alpha_2a_9c_2c_1}\ , \end{aligned}$$

up to τ-exact terms.

Dimension $9/2$ **- (**$A_1\ldots A_3 \to \alpha_1\ldots\alpha_3$**,** $A_4\ldots A_{12} \to a_1\ldots a_9$**)**

At dimension $9/2$, eq. (A.13) reads,

$$\begin{aligned} & \frac{3}{12}\, D_{\alpha_1} K_{\alpha_2\alpha_3a_1\ldots a_9} - \frac{9}{12}\, D_{a_1} K_{a_2\ldots a_9\alpha_1\ldots\alpha_3} \\ & -\frac{11}{2}\left(\frac{1}{22}\, T_{\alpha_1\alpha_2}{}^f\, K_{fa_1\ldots a_9\alpha_3} + \frac{6}{11}\, T_{a_1a_2}{}^\gamma\, K_{\gamma\alpha_1\ldots\alpha_3a_3\ldots a_9} + \frac{9}{22}\, T_{a_1\alpha_1}{}^\gamma\, K_{\gamma\alpha_2\alpha_3a_2\ldots a_9}\right) \\ & = \frac{11!}{4(4!)^2}\Bigg(2\,\frac{1}{110}\, R_{\alpha_1\alpha_2}{}^{c_1c_2}\, R_{\alpha_3a_1c_2c_1}\, G_{a_2\ldots a_5}\, G_{a_6\ldots a_9} \\ & \qquad\qquad + 4\,\frac{3}{55}\, R_{a_1a_2}{}^{c_1c_2}\, R_{\alpha_1a_3c_2c_1}\, G_{\alpha_2\alpha_3a_4a_5}\, G_{a_6\ldots a_9}\Bigg)\ . \end{aligned}$$

The terms in the equation above can be cast in seven groups : $R(DT)$, RTG, $G^2(DT)$, G^3T, T^3, $TG(DG)$ and $T(DR)$. One term of the form RTG is τ-exact, while all the remaining terms can be seen to cancel out, using (A.21) and (A.22) to convert a term of the form $T(DR)$ to the form T^3, and a term of the form $TG(DG)$ to the form G^3T. Up to τ-exact terms, the component of dimension $9/2$ then reads,

$$K^{(9/2)}_{a_1\ldots a_9\alpha_1\alpha_2} = 60\, i\,\epsilon_{a_1\ldots a_7}{}^{b_1\ldots b_4}\, G_{b_1\ldots b_4}\, R_{a_8a_9}{}^{c_1c_2}\, R_{\alpha_1a_{10}c_2c_1}\ .$$

Dimensions 5 - ($A_1A_2 \to \alpha_1\alpha_2$, $A_3 \ldots A_{12} \to a_1 \ldots a_{10}$)

At dimension 5, eq. (A.13) reads,

$$\begin{aligned}
&\frac{2}{12} D_{\alpha_1} K_{\alpha_2 a_1 \ldots a_9} + \frac{10}{12} D_{a_1} K_{a_2 \ldots a_{10} \alpha_1 \alpha_2} \\
&+ \frac{11}{2} \left(\frac{1}{66} T_{\alpha_1 \alpha_2}{}^{f} K_{f a_1 \ldots a_{10}} + \frac{15}{22} T_{a_1 a_2}{}^{\gamma} K_{\gamma \alpha_1 \alpha_2 a_3 \ldots a_{10}} + \frac{10}{33} T_{a_1 \alpha_1}{}^{\gamma} K_{\gamma \alpha_2 a_2 \ldots a_{10}} \right) \\
&= \frac{11!}{4(4!)^2} \Bigg(2\, \frac{1}{66} R_{\alpha_1 \alpha_2}{}^{c_1 c_2} R_{a_1 a_2 c_2 c_1} G_{a_3 \ldots a_6} G_{a_7 \ldots a_{10}} \\
&\qquad\qquad - 1\, \frac{2}{33} R_{\alpha_1 a_1}{}^{c_1 c_2} R_{\alpha_2 a_2 c_2 c_1} G_{a_3 a_4 a_5 a_6} G_{a_7 \ldots a_{10}} \\
&\qquad\qquad\qquad + 2\, \frac{1}{11} R_{a_1 a_2}{}^{c_1 c_2} R_{a_3 a_4 c_2 c_1} G_{a_5 a_6 \alpha_1 \alpha_2} G_{a_7 a_8 \alpha_3 \alpha_4} \Bigg) .
\end{aligned}$$

The terms in the equation above can be cast in nine groups : RT^2, $GT(DT)$, G^2T^2, GR^2, $GR(DG)$, RG^3, $R(DR)$, $G^2(DR)$, and $T^2(DG)$. One term in GR^2 is τ-exact, while all the remaining terms cancel out, as can be seen using eq. (A.21) and (A.22) to convert a term of the form $R(DR)$ to the form RT^2, a term of the form $G^2(DR)$ to the form G^2T^2, and a term of the form $T^2(DG)$ to the form G^2T^2. Up to τ-exact terms, the component of dimension 5 then reads,

$$K^{(5)}_{a_1 \ldots a_{11}} = -165\, \epsilon_{a_1 \ldots a_7}{}^{b_1 \ldots b_4} G_{b_1 \ldots b_4} R_{a_8 a_9}{}^{c_1 c_2} R_{a_{10} a_{11} c_2 c_1} . \tag{A.23}$$

Dimension $11/2$ - ($A_1 \to \alpha_1$, $A_2 \ldots A_{12} \to a_1 \ldots a_{11}$)

Since there is no new component of K appearing, this equation should be satisfied automatically,

$$\begin{aligned}
&\frac{1}{12} D_{\alpha_1} K_{a_1 \ldots a_{11}} - \frac{11}{12} D_{a_1} K_{a_2 \ldots a_{11} \alpha_1} - \frac{11}{2} \left(\frac{1}{6} T_{a_1 \alpha_2}{}^{f} K_{f a_2 \ldots a_{11}} - \frac{5}{6} T_{a_1 a_2}{}^{\gamma} K_{\gamma \alpha_1 a_3 \ldots a_{11}} \right) \\
&= \frac{11!}{4(4!)^2} \left(\frac{2}{6} R_{\alpha_1 a_1}{}^{c_1 c_2} R_{a_2 a_3 c_2 c_1} G_{a_4 \ldots a_7} G_{a_8 \ldots a_{11}} \right) .
\end{aligned}$$

The equation contains six types of terms : TR^2, $GR(DT)$, G^2TR, $GT(DR)$, $RT(DG)$, and T^3G. As expected all the terms cancel out, as can be seen using (A.21) and (A.22) to convert a term of the form $GT(DR)$ to the form T^3G, and a term of the form $RT(DG)$ to the form G^2TR.

A.3 Supersymmetrization at $\mathcal{O}(l^6)$, with half of the X_8 term

Instead of considering the full X_8 tensor required for quantum consistency, we can first attempt to symmetrize one of its two parts, as mentioned in (2.4.2). In this section, we consider the first member of X_8, namely $\mathrm{tr}(R^2)\wedge\mathrm{tr}(R^2)$, whose simple form allows for a particular treatment. At $\mathcal{O}(l^3)$, if one defines G_7 so that :

$$dG_7^{(1)} = G_4^{(0)} \wedge \mathrm{tr}(R^2)^2 , \tag{A.24}$$

where $\mathrm{tr}(R^2)$ is at $\mathcal{O}(l^3)$, then the twelve-form W_{12} associated to the Chern-Simons term verifies, at $\mathcal{O}(l^6)$:

$$\begin{aligned} W_{12} &= G_4^{(0)} \wedge \mathrm{tr}(R^2)^2 \\ &\overset{(A.24)}{=} \mathrm{d}G_7^{(1)} \wedge \mathrm{tr}(R^2) \;=\; \mathrm{d}K_{11} \,, \end{aligned}$$

where K_{11} can be simply expressed as $G_7^{(1)} \wedge \mathrm{tr}(R^2)$. In that case, the $\mathcal{O}(l^6)$ superinvariant corresponding to this supersymmetrization is,

$$\Delta S = l^6 \int \Big(\mathrm{tr}(R^2)^2 \wedge C_3^{(0)} - G_7^{(1)} \wedge \mathrm{tr} R^2 \Big) \,,$$

and the whole difficulty lies in finding the term $G_7^{(1)}$ defined by (A.24). This should be considerably simpler than dealing with the entire term X_8, since the equation for $G_7^{(1)}$ only contains eight indices (instead of 12). In superspace components, eq. (A.24) becomes :

$$\frac{1}{7!}\Big(D_{[A_1} G_{A_2 \ldots A_8)} + \frac{7}{2}\, T_{[A_1 A_2|}{}^{F}\, G_{F|A_3 \ldots A_8)} \Big) = \Big(\frac{1}{4!}\, G_{[A_1 \ldots A_4|} \Big)\Big(\frac{1}{2!}\, R_{|A_5 A_6|}{}^{c_1 c_2} \Big)\Big(\frac{1}{2!}\, R_{|A_7 A_8) c_2 c_1} \Big) \,,$$

or more explicitly :

$$2 D_{[A_1}\, G_{A_2 \ldots A_8)} + 7\, T_{[A_1 A_2|}{}^{F}\, G_{F|A_3 \ldots A_8)} = 105\, G_{[A_1 \ldots A_4|}\, R_{|A_5 A_6|}{}^{c_1 c_2}\, R_{|A_7 A_8) c_2 c_1}$$

with the dimensions ranging from $[G_{\alpha_1 \ldots \alpha_7}] = 1/2$ to $[G_{a_1 \ldots a_7}] = 4$.

Dimension 1 **—** Since $[G_{\alpha_1 \ldots \alpha_7}] = 0$, it can only be composed of gamma matrices, and must also be gauge invariant. This means it can only behave as a scalar, but $(00000) \not\subset (00001)^{\otimes s 7}$, so $G_{\alpha_1 \ldots \alpha_7} = 0$. The first equation (of dimension 1) then takes the form :

$$-i\, (\Gamma^f)_{\alpha_1 \alpha_2}\, G_{f \alpha_3 \ldots \alpha_8} = 0 \,. \tag{A.25}$$

According to (2.26), $G_{f\alpha_3 \ldots \alpha_8}$ admits solutions of the form $(\Gamma_{f e_1})_{\alpha_3 \alpha_4}\, S^{e_1}{}_{\alpha_5 \ldots \alpha_8}$, where S is in the representation $(10000) \otimes (00001)^{\otimes s 4} = 2(00010) \oplus \cdots$. There must exist two expressions of dimensions 1 involving the field G. The two following independent expressions (valid *a priori*),

$$\begin{aligned} &(\Gamma_f{}^{e_1})(\Gamma_{e_1 e_2})(\Gamma^{e_2 h_1 \ldots h_4})\, G_{h_1 \ldots h_4} \\ &(\Gamma_f{}^{g_1})(\Gamma_{e_1 e_2 \ldots e_5})(\Gamma^{e_2 \ldots e_5 h_1 \ldots h_4})\, G_{h_1 \ldots h_4} \end{aligned} \tag{A.26}$$

appear to be τ-exact. For example, the first one can be expanded using one of the gamma identities of (B.1) as

$$(\Gamma_f{}^{e_1})\big(12\, \delta_{e_1}^{h_1}\, (\Gamma^{h_2 h_3})(\Gamma^{h_4}) - (\Gamma_{g_1})(\Gamma_{e_1}{}^{g_1 h_1 \ldots h_4}) \big)\, G_{h_1 \ldots h_4} \tag{A.27}$$

Using the identity $(\Gamma^a)(\Gamma_{ab}) = 0$, the first term of (A.27) $(\Gamma_f{}^{h_1})(\Gamma^{h_2 h_3})(\Gamma^{h_4})\, G_{h_1 \ldots h_4}$ can be forced into the τ-exact form $(\Gamma^{h_4})(\Gamma_{[f|}{}^{h_1})(\Gamma^{h_2h_3})\,G_{|h_4]h_1\ldots h_3}$. The second one, $(\Gamma_f{}^{e_1})\ (\Gamma_{g_1})$ $(\Gamma_{e_1}{}^{g_1h_1\ldots h_4})\ G_{h_1\ldots h_4}$, can be put as well in the form $(\Gamma_{g_1})(\Gamma_{[f|}{}^{e_1})(\Gamma_{e_1|g_1]}{}^{h_1\ldots h_4})\,G_{h_1\ldots h_4}$.

The same can be done with the second term of (A.26), and every other expression which might be candidate for $G_{f\alpha_3\ldots\alpha_8}$. The only valid solution (up to τ-exact terms) is then $G_{f\alpha_3\ldots\alpha_8}=0$.

Dimension $3/2$ **—** The following equation is :

$$-i\,(\Gamma^f)_{\alpha_1\alpha_2}\,G_{fa_1\alpha_3\ldots\alpha_7}=0\,.$$

According to (2.26), the only possible solutions for $G_{fa_1\alpha_3\ldots\alpha_7}$ have the form $(\Gamma^{fa_1})\,S_{\alpha_5\alpha_6\alpha_7}$. The representation of $S_{\alpha_5\alpha_6\alpha_7}$ is $(00001)^{\otimes_S 3}$, which contains only 1 irrep (01001). There must be a single solution involving the torsion field T, which can be, for example $z\,(\Gamma_{fa_1})_{\alpha_1\alpha_2}\ (\Gamma^{e_1e_2})_{\alpha_3\alpha_4}$ $T_{e_1e_2\,\alpha_5}$, where z is a constant that cannot be fixed for the moment.

Dimension 2 **—** The equation of dimension 2 is the first which has a non-zero rhs,

$$(\Gamma^f)_{\alpha_1\alpha_2}\,G_{fa_1a_2\alpha_3\ldots\alpha_6}\;=\;6\,(\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\,R_{\alpha_3\alpha_4}{}^{c_1c_2}\,R_{\alpha_5\alpha_6c_2c_1}-\frac{2}{5}\,i\,z\,D_{\alpha_1}G_{a_1a_2\alpha_2\ldots\alpha_6}\,. \quad \text{(A.28)}$$

The spinorial derivative of $G_{a_1a_2\alpha_2\ldots\alpha_6}$, will contain a term $z(\Gamma_{a_1a_2})_{\alpha_1\alpha_2}\ (\Gamma^{e_1e_2})_{\alpha_3\alpha_4}\ (\Gamma^{e_3e_4})_{\alpha_5\alpha_6}$ $R_{e_1\ldots e_4}$. Although it is closed, this term happens to be non-τ-exact, which means eq. (A.28) cannot be verified unless z is set to 0. If eq. (A.28) with $z=0$ admits a solution for G, the rhs must be τ-exact. However, if some τ-exact terms can be extracted from the rhs, there inescapably remains,

$$(\Gamma_{a_1a_2})\left((\Gamma_{u_0u_1})(\Gamma_{u_2u_3})\,G^{u_0u_1}{}_{u_4u_5}G^{u_2u_3u_4u_5}+(\Gamma_{u_0u_1})(\Gamma_{u_2\ldots u_7})\,G^{u_2u_3u_4u_5}G^{u_6u_7u_0u_1}\right)$$

which cannot be cast in the right form. This conclusion is only valid given the assumption that the superinvariant must be quartic or higher in the fields (if this condition is dropped, then $G_{\alpha_1\ldots\alpha_7}$ need not be zero, eq. (A.25) changes, and all the following equations take a different form. This confirms what was said in section (2.4.2) : for U and V to be separately Weil-trivial, they must be cubic or lower in the fields.

B
Eleven dimensional CJS supergravity

B.1 Conventions & Fields in eleven-dimensional supergravity

In this section we give our conventions for the eleven-dimensional gamma matrices, and list the Fierz identities used in the analysis presented in section (2.4.1) of the main text.

B.1.1 Gamma matrices

Hodge duality for gamma matrices in 11 dimensions is defined as follows (with the Hodge star operator defined in 1.4),

$$\star\Gamma^{(n)} = -(-1)^{\frac{1}{2}n(n-1)}\Gamma^{(11-n)} \,, \tag{B.1}$$

The symmetry properties of the gamma matrices are given by,

$$(\Gamma^{a_1\ldots a_n})_{\alpha\beta} = (-1)^{\frac{1}{2}(n-1)(n-2)}(\Gamma^{a_1\ldots a_n})_{\beta\alpha} \,,$$

where $\Gamma^{(0)}$ is identified with the anti-symmetric charge conjugation matrix $C_{\alpha\beta}$, acting as a metric on spinors,

$$\psi_\alpha = C_{\alpha\beta}\,\psi^\beta \,, \qquad \begin{matrix} (\Gamma^m)_{\alpha\beta} = (\Gamma^m)_\alpha{}^\epsilon\, C_{\epsilon\beta} \\ (\Gamma^m)^{\alpha\beta} = C_{\alpha\epsilon}\,(\Gamma^m)_\epsilon{}^\beta \end{matrix} \,, \qquad C_{\alpha\beta}C^{\beta\delta} = \delta^\beta_\alpha$$

B.1.2 Eleven-dimensional superspace fields

In this section we review the properties of on-shell eleven-dimensional superspace at lowest order in the Planck length [29]. The expression of all the mixed components of tensors and their spinorial derivative below are found when solving the superspace Bianchi identities with the conventional constraint (cf. 2.1.2).

The non-zero superfield components are,

$$\begin{aligned}
G_{ab\alpha\beta} &= -i\,(\Gamma_{ab})_{\alpha\beta} \\
T_{\alpha\beta}{}^{f} &= -i\,(\Gamma^{f})_{\alpha\beta} \\
T_{a\alpha}{}^{\beta} &= -\frac{1}{36}\left((\Gamma^{bcd})_{\alpha}{}^{\beta}G_{abcd} + \frac{1}{8}(\Gamma_{a}{}^{bcde})_{\alpha}{}^{\beta}G_{bcde}\right) \\
R_{\alpha\beta ab} &= \frac{i}{6}\left((\Gamma^{gh})_{\alpha\beta}\,G_{ghab} + \frac{1}{24}\,(\Gamma_{ab}{}^{ghij})_{\alpha\beta}\,G_{ghij}\right) \\
R_{\alpha abc} &= \frac{i}{2}\left((\Gamma_{a}T_{bc})_{\alpha} - 2(\Gamma_{[b}T_{c]a})_{\alpha}\right)\ .
\end{aligned} \tag{B.2}$$

The action of the spinorial derivative on the superfields reads,

$$\begin{aligned}
D_{\alpha}G_{abcd} &= 6\,i\,(\Gamma_{[ab|})_{\alpha\epsilon}\,T_{|cd]}{}^{\epsilon} \\
D_{\alpha}R_{abcd} &= \mathrm{d}_{[a|}R_{\alpha|b]cd} - T_{ab}{}^{\epsilon}R_{\epsilon\alpha cd} + 2\,T_{[a|\alpha}{}^{\epsilon}R_{\epsilon|b]cd} \\
D_{\alpha}T_{ab}{}^{\beta} &= \frac{1}{4}\,R_{abcd}(\Gamma^{cd})_{\alpha}{}^{\beta} - 2\,D_{[a}T_{b]\alpha}{}^{\beta} - 2\,T_{[a|\alpha}{}^{\epsilon}T_{|b]\epsilon}{}^{\beta}\ .
\end{aligned} \tag{B.3}$$

The equations of motion for the field-strengths G,R and T are given by,

$$\begin{aligned}
D^{f}G_{fa_1a_2a_3} &= -\frac{1}{1152}\,\epsilon_{a_1a_2a_3b_1\ldots b_4c_1\ldots c_4}G^{b_1\ldots b_4}G^{c_1\ldots c_4} \\
(\Gamma^{a})_{\alpha\epsilon}\,T_{ab}{}^{\epsilon} &= 0 \\
R_{ab} - \frac{1}{2}\eta_{ab}R &= \frac{1}{12}\left(G_{afgh}G_{b}{}^{fgh} - \frac{1}{8}\eta_{ab}G_{fghi}G^{fghi}\right)\ .
\end{aligned} \tag{B.4}$$

B.1.3 Fierz identities

The following Fierz identities were used in the analysis. Anti-symmetrization over the a_i and b_j indices is always understood, as well as symmetrization over all fermionic indices of the gamma matrices (which are suppressed here to avoid cluttering the notation),

$$(\Gamma^{a_1\dots a_5 e_1})(\Gamma_{b_1\dots b_5 e_1}) =$$
$$+\ 120\,\delta^{a_1\dots a_5}_{b_1\dots b_5}\,(\Gamma^{e_1})(\Gamma_{e_1})$$
$$+\ 1\,(\Gamma^{a_1\dots a_5})(\Gamma_{b_1\dots b_5})$$
$$-\ 600\,\delta^{a_1\dots a_3}_{b_1\dots b_3}\,(\Gamma_{e_1})(\Gamma^{e_1 a_4 a_5}{}_{b_4 b_5})$$
$$+\ 25\,\delta^{a_1}_{b_1}\,(\Gamma_{e_1})(\Gamma^{e_1 a_1\dots a_4}{}_{b_1\dots b_4})$$
$$-\ 150\,\frac{1}{2}\left(\delta^{a_1}_{b_1}\,(\Gamma^{a_2 a_3})(\Gamma^{a_4 a_5}{}_{b_2\dots b_5}) + (a \leftrightarrow b)\right)$$
$$+\ 600\,\delta^{a_1\dots a_3}_{b_1\dots b_3}\,(\Gamma^{a_4 a_5})(\Gamma_{b_4 b_5})$$

$$(\Gamma^{a_1\dots a_4 e_1 e_2})(\Gamma_{b_1\dots b_4 e_1 e_2}) =$$
$$-\ 12\,\frac{1}{2}\left((\Gamma^{a_1 a_2})(\Gamma^{a_3 a_4}{}_{b_1\dots b_4}) + (a \leftrightarrow b)\right)$$
$$+\ 288\,\delta^{a_1 a_2}_{b_1 b_2}\,(\Gamma^{a_3 a_4})(\Gamma_{b_3 b_4})$$
$$-\ 96\,\frac{1}{2}\left(\delta^{a_1}_{b_1}(\Gamma^{a_2})(\Gamma^{a_3 a_4}{}_{b_2\dots b_4}) + (a \leftrightarrow b)\right)$$
$$+\ 192\,\delta^{a_1\dots a_3}_{b_1\dots b_3}\,(\Gamma^{a_4})(\Gamma_{b_4})$$
$$+\ 2\,(\Gamma_{e_1})(\Gamma^{e_1 a_1\dots a_4}{}_{b_1\dots b_4})$$
$$-\ 144\,\delta^{a_1 a_2}_{b_1 b_2}\,(\Gamma_{e_1})(\Gamma^{e_1 a_3 a_4}{}_{b_3 b_4})$$
$$+\ 48\,\delta^{a_1\dots a_4}_{b_1\dots b_4}\,(\Gamma^{e_1})(\Gamma_{e_1})$$

$$(\Gamma^{a_1\dots a_3 e_1\dots e_3})(\Gamma_{b_1\dots b_3 e_1\dots e_3}) =$$
$$+\ 36\,\delta^{a_1\dots a_3}_{b_1\dots b_3}\,(\Gamma_{e_1})(\Gamma^{e_1})$$
$$-\ 108\,\delta^{a_1}_{b_1}\,(\Gamma_{e_1})(\Gamma^{e_1 a_2 a_3}{}_{b_2 b_3})$$
$$+\ 216\,\delta^{a_1}_{b_1}\,(\Gamma^{a_2 a_3})(\Gamma_{b_2 b_3})$$
$$-\ 36\,\frac{1}{2}\left((\Gamma^{a_1})(\Gamma^{a_2 a_3}{}_{b_1\dots b_3}) + (a \leftrightarrow b)\right)$$
$$+\ 324\,\delta^{a_1 a_2}_{b_1 b_2}\,(\Gamma^{a_3})(\Gamma_{b_3})$$

$$(\Gamma^{a_1 a_2 e_1\dots e_4})(\Gamma_{b_1 b_2 e_1\dots e_4}) =$$
$$+\ 48\,\delta^{a_1 a_2}_{b_1 b_2}\,(\Gamma_{e_1})(\Gamma^{e_1})$$
$$-\ 96\,(\Gamma_{e_1})(\Gamma^{e_1 a_1 a_2}{}_{b_1 b_2})$$
$$+\ 168\,(\Gamma^{a_1 a_2})(\Gamma_{b_1 b_2})$$
$$+\ 672\,\delta^{a_1}_{b_1}\,(\Gamma^{a_1})(\Gamma^{b_1})$$

$$(\Gamma^{a_1 e_1\dots e_5})(\Gamma_{b_1 e_1\dots e_5}) =$$
$$+\ 240\,\delta^{a_1}_{b_1}\,(\Gamma_{e_1})(\Gamma^{e_1})$$
$$+\ 1680\,(\Gamma^{a_1})(\Gamma_{b_1})$$

$$(\Gamma^{e_1\dots e_6})(\Gamma_{e_1\dots e_6}) =$$
$$+\ 4320\,(\Gamma_{e_1})(\Gamma^{e_1})$$

$$(\Gamma^{a_1 a_2 e_1\dots e_3})(\Gamma_{b_1 b_2 e_1\dots e_3}) =$$
$$-\ 36\,\delta^{a_1 a_2}_{b_1 b_2}\,(\Gamma_{e_1})(\Gamma^{e_1})$$
$$+\ 24\,(\Gamma_{e_1})(\Gamma^{e_1 a_1 a_2}{}_{b_1 b_2})$$
$$-\ 42\,(\Gamma^{a_1 a_2})(\Gamma_{b_1 b_2})$$
$$+\ 168\,\delta^{a_1}_{b_1}\,(\Gamma^{a_2})(\Gamma_{b_2})$$

$$(\Gamma^{a_1 e_1\dots e_4})(\Gamma_{b_1 e_1\dots e_4}) =$$
$$-\ 96\,\delta^{a_1}_{b_1}\,(\Gamma_{e_1})(\Gamma^{e_1})$$
$$+\ 336\,(\Gamma^{a_1})(\Gamma_{b_1})$$

$$(\Gamma^{e_1\dots e_5})(\Gamma_{e_1\dots e_5}) =$$
$$-\ 720\,(\Gamma_{e_1})(\Gamma^{e_1})$$

$$
\begin{aligned}
&(\Gamma^{a_1\ldots a_4 e_1})(\Gamma_{b_1\ldots b_4 e_1}) = \\
&+\ 6\,\frac{1}{2}\Big((\Gamma^{a_1a_2})(\Gamma^{a_3a_4}{}_{b_1\ldots b_4}) + (a \leftrightarrow b) \Big) \\
&-\ 72\,\delta^{a_1a_2}_{b_1b_2}\,(\Gamma^{a_3a_4})(\Gamma_{b_3b_4}) \\
&-\ 48\,\frac{1}{2}\Big(\delta^{a_1}_{b_1}\,(\Gamma^{a_2})(\Gamma^{a_3a_4}{}_{b_2\ldots b_4}) + (a \leftrightarrow b) \Big) \\
&+\ 96\,\delta^{a_1\ldots a_3}_{b_1\ldots b_3}\,(\Gamma^{a_4})(\Gamma_{b_4}) \\
&-\ 1\,(\Gamma_{e_1})(\Gamma^{e_1a_1\ldots a_4}{}_{b_1\ldots b_4}) \\
&+\ 72\,\delta^{a_1a_2}_{b_1b_2}\,(\Gamma_{e_1})(\Gamma^{e_1a_3a_4}{}_{b_3b_4}) \\
&-\ 24\,\delta^{a_1\ldots a_4}_{b_1\ldots b_4}\,(\Gamma^{e_1})(\Gamma_{e_1})
\end{aligned}
$$

$$
\begin{aligned}
&(\Gamma^{a_1\ldots a_3 e_1 e_2})(\Gamma_{b_1\ldots b_3 e_1 e_2}) = \\
&-\ 24\,\delta^{a_1\ldots a_3}_{b_1\ldots b_3}\,(\Gamma_{e_1})(\Gamma^{e_1}) \\
&+\ 36\,\delta^{a_1}_{b_1}\,(\Gamma_{e_1})(\Gamma^{e_1a_2a_3}{}_{b_2b_3}) \\
&-\ 54\,\delta^{a_1}_{b_1}\,(\Gamma^{a_2a_3})(\Gamma_{b_2b_3}) \\
&-\ 12\,\frac{1}{2}\Big((\Gamma^{a_1})(\Gamma^{a_2a_3}{}_{b_1\ldots b_3}) + (a \leftrightarrow b) \Big) \\
&+\ 108\,\delta^{a_1a_2}_{b_1b_2}\,(\Gamma^{a_3})(\Gamma_{b_3})
\end{aligned}
$$

$$
\begin{aligned}
&(\Gamma^{a_1e_1})(\Gamma_{b_1\ldots b_4 e_1}) = \\
&\qquad +\ 1\,(\Gamma_{e_1})(\Gamma^{e_1a_1}{}_{b_1\ldots b_4}) \\
&\qquad +\ 12\,\delta^{a_1}_{b_1}\,(\Gamma_{b_2})(\Gamma_{b_3b_4})
\end{aligned}
$$

$$
\begin{aligned}
(\Gamma^{a_1e_1})(\Gamma_{b_1e_1}) = &+\ 1\,(\Gamma^{a_1})(\Gamma_{b_1}) \\
&-\ 1\,\delta^{a_1}_{b_1}\,(\Gamma_{e_1})(\Gamma^{e_1})
\end{aligned}
$$

$$
\begin{aligned}
&(\Gamma^{a_1\ldots a_4 e_1})(\Gamma^{b_1\ldots b_5}{}_{e_1}) = \\
&\qquad -\ 60\,\eta^{a_1b_1}\,(\Gamma^{a_2a_3})(\Gamma^{a_4b_2\ldots b_5}) \\
&\qquad -\ 60\,\eta_{a_1b_1}\,(\Gamma^{a_2})(\Gamma^{a_3a_4b_2\ldots b_5}) \\
&\qquad -\ 720\,\delta^{a_1\ldots a_3}_{c_1\ldots c_3}\,\eta^{c_1b_1}\eta^{c_2b_2}\eta^{c_3b_3}\,(\Gamma^{a_4})(\Gamma^{b_4b_5}) \\
&\qquad +\ 240\,\delta^{a_1\ldots a_3}_{c_1\ldots c_3}\,\eta^{c_1b_1}\eta^{c_2b_2}\eta^{c_3b_3}\,(\Gamma^{a_4})(\Gamma^{b_4b_5}) \\
&\qquad +\ 140\,\eta^{a_1b_1}\,(\Gamma^{[a_2})(\Gamma^{a_3a_4b_2\ldots b_5]}) \\
&\qquad -\ 120\,\delta^{a_1a_2}_{c_1c2}\,\eta^{c_1b_1}\eta^{c_2b_2}\,(\Gamma_{e_1})(\Gamma^{e_1a_3a_4b_3\ldots b_5})
\end{aligned}
$$

$$
\begin{aligned}
&(\Gamma^{a_1e_1})(\Gamma^{b_1\ldots b_5}{}_{e_1}) = \\
&\qquad -\ 6\,(\Gamma^{[a_1})(\Gamma^{b_1\ldots b_5]}) \\
&\qquad -\ 5\,\eta^{a_1b_1}\,(\Gamma_{e_1})(\Gamma^{e_1b_2\ldots b_5}) \\
&\qquad +\ 1\,(\Gamma^{a_1})(\Gamma^{b_1\ldots b_5})
\end{aligned}
$$

C

Group theory & Tensorial representations

C.1 Tensor representation of Young diagrams

A Young diagram with n boxes, see [87] for a review, represents an irreducible representation of the symmetric group S_n. It is possible to give explicit expressions for Young diagrams in the form of tensors. The method is more easily understood using a specific example. Consider a tensor $T_{a_1a_2a_3a_4}$ without any a priori symmetry properties, and let us construct its projection onto $\ydiagram{3,1}$. Several symmetry operations will have to be applied on the tensor, but the Young diagram does not state which indices correspond to its different boxes. First one must determine all the *standard tableaux*, i.e. all the Young diagrams with numbered boxes, with increasing numbers in all rows and columns. Different Young tableaux corresponding to the same Young diagram give equivalent but distinct representations of the symmetric group. The diagram $\ydiagram{3,1}$ has three standard tableaux, $\ytableaushort{123,4}$, $\ytableaushort{134,2}$, and $\ytableaushort{124,3}$, to which correspond three tensors, $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ respectively.

To obtain the tensor corresponding to a given standard tableau, one must first symmetrize over the indices indicated in each row, and then anti-symmetrize over the indices indicated in each column. For example, $(T^{(1)})_{a_1a_2a_3a_4}$ will be obtained by first symmetrizing over the indices a_1, a_2 and a_3,

$$\left(T_{a_1a_2a_3a_4} + T_{a_1a_3a_2a_4} + T_{a_2a_1a_3a_4} + T_{a_2a_3a_1a_4} + T_{a_3a_1a_2a_4} + T_{a_3a_2a_1a_4}\right) ,$$

and then anti-symmetrizing over a_1 and a_4,

$$
\begin{aligned}
(\Pi^{(1)}T)_{a_1a_2a_3a_4} = (T^{(1)})_{a_1a_2a_3a_4} = \\
&\frac{1}{8}\big(T_{a_1a_2a_3a_4} + T_{a_1a_3a_2a_4} + T_{a_2a_1a_3a_4} + T_{a_2a_3a_1a_4} + T_{a_2a_3a_4a_1} + T_{a_2a_4a_3a_1} + \\
&\quad T_{a_3a_1a_2a_4} + T_{a_3a_2a_1a_4} + T_{a_3a_2a_4a_1} + T_{a_3a_4a_2a_1} + T_{a_4a_2a_3a_1} + T_{a_4a_3a_2a_1}\big)\,.
\end{aligned}
$$

The overall normalization above can be straightforwardly determined by imposing $\Pi^{(1)}\Pi^{(1)}T = \Pi^{(1)}T$, where $\Pi^{(1)}T = T^{(1)}$ is the projection of the tensor T onto the Young tableau $\begin{smallmatrix}1&2&3\\4\end{smallmatrix}$.

For example the tensors $T^{(1)}$ and $T^{(2)}$, associated with $\begin{smallmatrix}1&2&3\\4\end{smallmatrix}$ and $\begin{smallmatrix}1&3&4\\2\end{smallmatrix}$ respectively, obey the following properties,

$$
\begin{aligned}
&(T^{(1)})_{[ab|c|d]} = 0 &\qquad& (T^{(2)})_{[abc]d} = 0 \\
&(T^{(1)})_{[a|bc|d]} = (T^{(1)})_{abcd} && (T^{(2)})_{[ab]cd} = (T^{(2)})_{abcd} \\
&(T^{(1)})_{a(bc)d} = (T^{(1)})_{abcd} && (T^{(2)})_{ab(cd)} = (T^{(2)})_{abcd}\,.
\end{aligned}
$$

More generally, each $T^{(i)}$ has exactly three independent orderings of indices, which can be taken to be $T^{(1)}_{a_1a_4a_3a_2}$, $T^{(1)}_{a_2a_1a_4a_3}$ and $T^{(1)}_{a_2a_3a_1a_4}$. Any symmetry operation on the indices of $T^{(i)}$ can be expressed as a linear combination of these three orderings, e.g.,

$$
\begin{aligned}
T^{(1)}_{a_1(a_2a_3a_4)} &= T^{(1)}_{a_1a_4a_3a_2} + \frac{1}{3}\,T^{(1)}_{a_2a_1a_4a_3} + \frac{1}{3}\,T^{(1)}_{a_2a_3a_1a_4} \\
T^{(1)}_{[a_1a_2]a_3a_4} &= \frac{1}{2}\,T^{(1)}_{a_1a_4a_3a_2} + \frac{1}{2}\,T^{(1)}_{a_2a_1a_4a_3} + 0\,T^{(1)}_{a_2a_3a_1a_4}\,.
\end{aligned}
$$

A tensor T projected onto a non-standard tableau can be expressed as a linear combination of the three standard ones. For example it is straightforward (but tedious) to check that the projection onto the non-standard tableau $\begin{smallmatrix}3&4&2\\1\end{smallmatrix}$ can be decomposed as,

$$
\begin{aligned}
(\Pi^{(4)}T)_{a_1a_2a_3a_4} &= T^{(1)}_{a_1a_4a_3a_2} + T^{(1)}_{a_2a_1a_4a_3} + 0\,T^{(1)}_{a_2a_3a_1a_4} && \text{(C.1)}\\
&\quad + 0\,T^{(2)}_{a_1a_4a_3a_2} - T^{(2)}_{a_2a_1a_4a_3} + T^{(2)}_{a_2a_3a_1a_4} && \text{(C.2)}\\
&\quad + T^{(3)}_{a_2a_1a_3a_4} + 0\,T^{(3)}_{a_2a_1a_4a_3} - T^{(3)}_{a_2a_3a_1a_4}\,. && \text{(C.3)}
\end{aligned}
$$

Every other tableau (corresponding to the same Young diagram $\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}$) and any symmetry operation on the indices can be expressed as a linear combination of those nine elements. The automatization of general decompositions onto Young tableaux, such as the one above, has been implemented in SSGamma.

More generally a tensor $T_{a_1a_2a_3a_4}$ without any a priori symmetry properties can be decomposed into ten Young tableaux,

$$\underbrace{\square^{\otimes 4}}_{T} = \underbrace{\ydiagram{4}}_{T^S} \oplus \underbrace{3\,\ydiagram{3,1}}_{T^{(1,2,3)}} \oplus \underbrace{2\,\ydiagram{2,2}}_{T'^{(1,2)}} \oplus \underbrace{3\,\ydiagram{2,1,1}}_{T''^{(1,2,3)}} \oplus \underbrace{\ydiagram{1,1,1,1}}_{T^A}\,, \tag{C.4}$$

where $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are the Young tableaux appearing on the right-hand side of (C.3) above, and correspond to the term $3\,\ydiagram{3,1}$. The remaining Young tableaux in the decomposition can be explicitly constructed using the same method.

Consider now a tensor T with a symmetry structure given by, e.g., $\ydiagram{2,1} \otimes \square$. The previous decomposition of $\square^{\otimes 4}$ can also be used to decompose T into its irreducible components. Indeed, a tensor with structure $\ydiagram{2,1} \otimes \square$ can be viewed as a particular set of symmetry operations performed on the indices of a tensor without any symmetry (i.e. with structure $\square^{\otimes 4}$). Therefore T can be expressed as a linear combination of the tensors already used in the decomposition (C.4).

The following example shows the decomposition of the symmetric product of two 3-forms H,

$$T_{a_1 \ldots a_6} := H_{a_1 a_2 a_3} H_{a_4 a_5 a_6} \quad \longrightarrow \quad \ydiagram{1,1,1}^{\otimes_S 2} = \ydiagram{2,2,2} \oplus \ydiagram{2,1,1,1,1}\,.$$

There are five standard tableaux corresponding to each of the Young diagrams $\ydiagram{2,2,2}$, $\ydiagram{2,1,1,1,1}$. The tensors corresponding to these Young tableaux can be denoted by $T^{(1)}, \ldots T^{(5)}$ and $T'^{(1)}, \ldots T'^{(5)}$, respectively. In the particular example above, it can be shown that,

$$H_{a_1 a_2 a_3} H_{a_4 a_5 a_6} = T^{(1)}_{a_1 a_2 a_3 a_4 a_5 a_6} + T'^{(1)}_{a_1 a_2 a_3 a_4 a_5 a_6} + T'^{(1)}_{a_1 a_2 a_3 a_4 a_6 a_5} - T'^{(1)}_{a_1 a_2 a_3 a_5 a_6 a_4}\,,$$

i.e. only the tensors $T^{(1)}$ and $T'^{(1)}$, corresponding to $\begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{ytableau}$ and $\begin{ytableau} 1 & 6 \\ 2 \\ 3 \\ 4 \\ 5 \end{ytableau}$ respectively, enter the decomposition.

D
Ten dimensional IIA supergravities

D.1 Dilaton and Einstein equations from Romans generalized action

The dilaton and Einstein equations following from the action (4.34) of chapter 4 read,

$$
\begin{aligned}
0 = &-\hat{\nabla}^2\phi + \frac{3}{8}e^{3\phi/2}F^2 - \frac{1}{12}e^{-\phi}H^2 + \frac{1}{96}e^{\phi/2}G^2 + \frac{4}{5}m^2e^{5\phi/2} \\
&- \frac{5}{4}(36\beta^2 - 10\beta + \frac{21}{20})e^{5\phi/4}m(\bar{\Lambda}\Lambda) - \frac{3}{8}(29\beta^2 - \frac{9}{2}\beta + \frac{5}{16})e^{3\phi/4}F_{mn}(\bar{\Lambda}\Gamma^{mn}\Gamma_{11}\Lambda) \\
&+ \frac{1}{2}(4\beta^2 + \frac{1}{3}\beta)e^{-\phi/2}H_{mnp}(\bar{\Lambda}\Gamma^{mnp}\Gamma_{11}\Lambda) - \frac{1}{96}(21\beta^2 - \frac{1}{2}\beta - \frac{3}{16})e^{\phi/4}G_{mnpq}(\bar{\Lambda}\Gamma^{mnpq}\Lambda)\,,
\end{aligned} \tag{D.1}
$$

and,

$$
\begin{aligned}
0 = &\hat{R}_{mn} + \frac{1}{2}\partial_m\phi\partial_n\phi + \frac{1}{25}m^2e^{5\phi/2}\hat{g}_{mn} + \frac{1}{4}e^{3\phi/2}\Big(2F^2_{mn} - \frac{1}{8}\hat{g}_{mn}F^2\Big) \\
&+ \frac{1}{12}e^{-\phi}\Big(3H^2_{mn} - \frac{1}{4}\hat{g}_{mn}H^2\Big) + \frac{1}{48}e^{\phi/2}\Big(4G^2_{mn} - \frac{3}{8}\hat{g}_{mn}G^2\Big) \\
&+ (1 - 144\beta^2)\Big(\frac{1}{2}(\bar{\Lambda}\Gamma_{(m}\nabla_{n)}\Lambda) + \frac{1}{16}g_{mn}(\bar{\Lambda}\Gamma^i\nabla_i\Lambda)\Big) \\
&- \frac{1}{8}\hat{g}_{mn}\Big((36\beta^2 - 10\beta + \frac{21}{20})e^{5\phi/4}m(\bar{\Lambda}\Lambda) + (8c^2 - 69\sqrt{2}c^3 + \frac{1773}{8}c^4(\bar{\Lambda}\Lambda)^2\Big) \\
&- \frac{1}{2}(29\beta^2 - \frac{9}{2}\beta + \frac{5}{16})e^{3\phi/4}F_{(m}{}^i(\bar{\Lambda}\Gamma_{n)i}\Gamma_{11}\Lambda) \\
&- (4\beta^2 + \frac{1}{3}\beta)e^{-\phi/2}\Big(\frac{3}{2}H_{(m}{}^{ij}(\bar{\Lambda}\Gamma_{n)ij}\Gamma_{11}\Lambda) - \frac{1}{16}\hat{g}_{mn}H_{(3)}(\bar{\Lambda}\Gamma^{(3)}\Gamma_{11}\Lambda)\Big) \\
&- \frac{1}{24}(21\beta^2 - \frac{1}{2}\beta - \frac{3}{16})e^{\phi/4}\Big(2G_{(m}{}^{ijk}(\bar{\Lambda}\Gamma_{n)ijk}\Lambda) - \frac{1}{8}\hat{g}_{mn}G_{(4)}(\bar{\Lambda}\Gamma^{(4)}\Lambda)\Big)\,.
\end{aligned} \tag{D.2}
$$

D.2 Conventions & Fields in ten-dimensional supergravity

D.2.1 Gamma matrices

In our conventions, the position of fermionic indices in ten dimensions is correlated with chirality. We manipulate Majorana-Weyl spinors μ^α and λ_α of opposite chiralities. Correspondingly, gamma matrices and the conjugation matrix can be chosen to split as,

$$C = \begin{pmatrix} 0 & \delta^\alpha{}_\beta \\ -\delta_\alpha{}^\beta & 0 \end{pmatrix} , \qquad \Gamma^m = \begin{pmatrix} 0 & (\gamma^m)_{\alpha\beta} \\ (\gamma^m)^{\alpha\beta} & 0 \end{pmatrix} , \qquad \Gamma^{mn} = \begin{pmatrix} (\gamma^{mn})^\alpha{}_\beta & 0 \\ 0 & (\gamma^{mn})_\alpha{}^\beta \end{pmatrix} , \quad \text{etc.}$$

where the position of fermi indices now determine which bilinear is allowed. The spin flip formula,

$$C^{-1}\Gamma^{a_1\ldots a_n} = (-1)^{\frac{1}{2}(n-1)(n-2)} C^{-1}\,\Gamma^{a_1\ldots a_n} \;,$$

is still valid, but has to be particularized according to the splitting defined above.

D.2.2 The supersymmetry transformations

Although we do not directly make use of this in this thesis, it is instructive to work out the explicit form of the supersymmetry transformations. A superdiffeomorphism generated by the supervector field ξ^A acts on the vielbein as follows,

$$\delta_\xi E_M^A \;=\; \xi^N \left(\partial_N E_M^A\right) + \left(\partial_M \xi^N\right) E_N^A \;=\; \nabla_M \xi^A + \xi^B T_{BM}{}^A \;, \tag{D.3}$$

up to a ξ-dependent Lorentz transformation. The supersymmetry transformation of the gravitini, $\psi_m^\alpha := E_m^\alpha|$, $\psi_{m\alpha} := E_{m\alpha}|$, with parameters $(\epsilon^\alpha, \zeta_\alpha)$, is obtained from the above by setting $\epsilon^\alpha := \xi^\alpha|$, $\zeta_\alpha := \xi_\alpha|$, where the vertical bar denotes the lowest-order term in the theta-expansion. We thus obtain,

$$\begin{aligned} \delta\psi_m^\alpha &= \nabla_m \epsilon^\alpha + e_m{}^c(\epsilon^\beta T_{\beta c}{}^\alpha + \zeta_\beta T^\beta{}_c{}^\alpha)| \\ \delta\psi_{m\alpha} &= \nabla_m \zeta_\alpha + e_m{}^c(\epsilon^\beta T_{\beta c\alpha} + \zeta_\beta T^\beta{}_{c\alpha})| \;, \end{aligned} \tag{D.4}$$

up to gravitino-dependent, cubic fermion terms which we do not need to consider here. Correspondingly the supersymmetry transformation of the dilatini reads,

$$\begin{aligned} \delta\mu^\alpha &= (\epsilon^\beta \nabla_\beta \mu^\alpha + \zeta_\beta \nabla^\beta \mu^\alpha)| \\ &= L\epsilon + K_m \gamma^m \zeta - L_{mn}\gamma^{mn}\epsilon + K_{mnp}\gamma^{mnp}\zeta + L_{mnpq}\gamma^{mnpq}\epsilon \\ \delta\lambda_\alpha &= (\epsilon^\beta \nabla_\beta \lambda_\alpha + \zeta_\beta \nabla^\beta \lambda_\alpha)| \\ &= -L\zeta + K_m\gamma^m\epsilon - L_{mn}\gamma^{mn}\zeta - K_{mnp}\gamma^{mnp}\epsilon - L_{mnpq}\gamma^{mnpq}\zeta \;, \end{aligned} \tag{D.5}$$

where we have taken (4.5),(4.6) of [93] into account. Together with (4.8),(4.11) above we obtain, suppressing spinor indices,

$$\begin{aligned}
e^{-3\phi/4}\delta\mu &= \frac{i}{2}\partial_m\phi\hat{\gamma}^m\zeta + \frac{1}{2}me^{5\phi/4}\epsilon \\
&\quad + \frac{3}{16}e^{3\phi/4}F_{mn}\hat{\gamma}^{mn}\epsilon - \frac{i}{24}e^{-\phi/2}H_{mnp}\hat{\gamma}^{mnp}\zeta + \frac{1}{192}e^{\phi/4}G_{mnpq}\hat{\gamma}^{mnpq}\epsilon \\
e^{-3\phi/4}\delta\lambda &= \frac{i}{2}\partial_m\phi\hat{\gamma}^m\epsilon - \frac{1}{2}me^{5\phi/4}\zeta \\
&\quad + \frac{3}{16}e^{3\phi/4}F_{mn}\hat{\gamma}^{mn}\zeta + \frac{i}{24}e^{-\phi/2}H_{mnp}\hat{\gamma}^{mnp}\epsilon - \frac{1}{192}e^{\phi/4}G_{mnpq}\hat{\gamma}^{mnpq}\zeta\,,
\end{aligned} \tag{D.6}$$

up to cubic fermion terms; the curved gamma matrices $\hat{\gamma}$ are defined with respect to the rescaled metric (4.9). Similarly for the gravitino transformations we obtain,

$$\begin{aligned}
\delta\psi_{m\alpha} &= \nabla_m\zeta - S\gamma_m\epsilon + F^1_{ef}\gamma_m{}^{ef}\epsilon - F^2_{me}\gamma^e\epsilon \\
&\quad - H'^1_{fgh}\gamma_m{}^{fgh}\zeta + H'^2_{mgh}\gamma^{gh}\zeta - G^1_{efgh}\gamma_m{}^{efgh}\epsilon + G^2_{mefg}\gamma^{efg}\epsilon \\
\delta\psi^\alpha_m &= \nabla_m\epsilon + S\gamma_m\zeta + F^1_{ef}\gamma_m{}^{ef}\zeta - F^2_{me}\gamma^e\zeta \\
&\quad - H^1_{fgh}\gamma_m{}^{fgh}\epsilon + H^2_{mgh}\gamma^{gh}\epsilon + G^1_{efgh}\gamma_m{}^{efgh}\zeta - G^2_{mefg}\gamma^{efg}\zeta\,,
\end{aligned} \tag{D.7}$$

where we used (4.3) of [93]. Furthermore using (4.6) of [93] and (4.8),(4.11) above we obtain,

$$\begin{aligned}
\delta\psi_{m\alpha} &= \hat{\nabla}_m\zeta + \frac{2i}{5}me^{5\phi/4}\hat{\gamma}_m\epsilon + \frac{3}{8}\partial_e\phi\hat{\gamma}^e{}_m\zeta + \frac{i}{8}e^{3\phi/4}F_{ef}\gamma_m{}^{ef}\epsilon + \frac{i}{2}e^{3\phi/4}F_{me}\gamma^e\epsilon \\
&\quad + \frac{1}{24}e^{-\phi/2}H_{fgh}\gamma_m{}^{fgh}\zeta + \frac{i}{24}e^{\phi/4}G_{mefg}\gamma^{efg}\epsilon \\
\delta\psi^\alpha_m &= \hat{\nabla}_m\epsilon - \frac{2i}{5}me^{5\phi/4}\hat{\gamma}_m\zeta + \frac{3}{8}\partial_e\phi\hat{\gamma}^e{}_m\epsilon + \frac{i}{8}e^{3\phi/4}F_{ef}\gamma_m{}^{ef}\zeta + \frac{i}{2}e^{3\phi/4}F_{me}\gamma^e\zeta \\
&\quad - \frac{1}{24}e^{-\phi/2}H_{fgh}\gamma_m{}^{fgh}\epsilon - \frac{i}{24}e^{\phi/4}G_{mefg}\gamma^{efg}\zeta\,,
\end{aligned} \tag{D.8}$$

up to cubic fermion terms; $\hat{\nabla}$ is the covariant derivative associated to the spin connection of the rescaled metric (4.9) so that,

$$e^{3\phi/2}\omega_{nkm} = \hat{\omega}_{nkm} + \frac{3}{4}\hat{g}_{nk}\partial_m\phi - \frac{3}{4}\hat{g}_{nm}\partial_k\phi\,, \qquad \nabla_m\chi = \hat{\nabla}_m\chi + \frac{3}{8}\partial_n\phi(\gamma^n{}_m\chi)\,, \tag{D.9}$$

where $\hat{\omega}$, ω are the spin connections of $\hat{g}$, g respectively, and χ is a fermion of either chirality.

To make contact with the supersymmetry transformations as given in e.g. [21] we use the following ten-dimensional Dirac-matrix notation :

$$\Gamma_m = \begin{pmatrix} 0 & -i(\hat{\gamma}_m)_{\alpha\beta} \\ i(\hat{\gamma}_m)^{\alpha\beta} & 0 \end{pmatrix}\,, \qquad \Gamma_{11} = \begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & -\delta^\beta_\alpha \end{pmatrix}\,, \qquad C^{-1} = \begin{pmatrix} 0 & \delta^\alpha_\beta \\ -\delta^\beta_\alpha & 0 \end{pmatrix}\,, \tag{D.10}$$

and define the Dirac-Majorana spinors,

$$\Psi_m = e^{3\phi/8}\begin{pmatrix}\psi_{m\alpha}\\ \psi_m^{\alpha}\end{pmatrix} - \frac{3}{4}\Gamma_m\Lambda\,,\qquad \Lambda = e^{-3\phi/8}\Gamma_{11}\begin{pmatrix}\lambda_\alpha\\ \mu^\alpha\end{pmatrix}\,,\qquad \Theta = e^{3\phi/8}\begin{pmatrix}\zeta_\alpha\\ \epsilon^\alpha\end{pmatrix}\,,\tag{D.11}$$

which obey the reality conditions $\overline{\Psi}_m = \Psi_m^{\rm Tr}C^{-1}$, etc. In terms of these, the supersymmetry transformations (D.6), (D.8) take the form,

$$\delta\Lambda = \left\{-\frac{1}{2}\Gamma^m\hat{\nabla}_m\phi - \frac{me^{5\phi/4}}{2} + \frac{3e^{3\phi/4}}{16}F_{(2)}\Gamma^{(2)}\Gamma_{11} + \frac{e^{-\phi/2}}{24}H_{(3)}\Gamma^{(3)}\Gamma_{11} - \frac{e^{\phi/4}}{192}G_{(4)}\Gamma^{(4)}\right\}\Theta\,,$$

and

$$\begin{aligned}\delta\Psi_m = \Big\{&\hat{\nabla}_m - \frac{me^{5\phi/4}}{40}\Gamma_m - \frac{e^{3\phi/4}}{64}F_{np}(\Gamma_m{}^{np} - 14\delta_m{}^n\Gamma^p)\Gamma_{11}\\ &+ \frac{e^{-\phi/2}}{96}H_{npq}(\Gamma_m{}^{npq} - 9\delta_m{}^n\Gamma^{pq})\Gamma_{11} + \frac{e^{\phi/4}}{256}G_{npqr}(\Gamma_m{}^{npqr} - \frac{20}{3}\delta_m{}^n\Gamma^{pqr})\Big\}\Theta\,,\end{aligned}\tag{D.12}$$

respectively, up to cubic fermion terms. These are precisely the supersymmetry transformations expressed in the conventions of [21].

D.2.3 A note on conventions

In this section we compare our conventions to those of [89, 91]. The translation between the conventions of the present paper and those of [21] was explained previously.
The fermionic fields in [91] are related to those in the present paper via,

$$\psi_m^R = \Psi_m\,,\qquad \lambda^R = \frac{1}{\sqrt{2}}\Lambda\,,$$

where the R superscript denotes the fields in that reference. Moreover the bosonic fields of [91] are related to those in the present paper via,

$$\begin{aligned}m^R B_{(2)} = \frac{1}{2}F_{(2)}\,,\qquad G^R_{(3)} = \frac{1}{2}H_{(3)}\,,\qquad F^R_{(4)} = \frac{1}{2}G_{(4)}\\ m^R = \frac{4}{5}m\,,\qquad \phi^R = -\frac{1}{2}\phi\,,\qquad R^R = -\hat{R}\,.\end{aligned}\tag{D.13}$$

With these field redefinitions it can be seen that at the fermionic vacuum (4.28) the action of [91] precisely reduces to that given in (4.26), (4.15) of the present paper, up to the quartic-fermion term which was not computed in [91].

On the other hand the quartic-fermion terms are identical in the massive and massless IIA theories. In order to compare with the quartic-fermion terms of massless IIA as given in [89] we note that, upon setting $k = 1$ therein, the fermionic ψ_m^{GP}, λ^{GP} of that reference are related to the ones in this thesis via,

$$\Psi_m = \frac{1}{\sqrt{2}}\psi_m^{GP}\,, \qquad \Lambda = -\Gamma_{11}\lambda^{GP}\,.$$

Thus the fermionic vacuum (4.28) corresponds to setting,

$$\psi_m^{GP} \;\equiv\; -\frac{3}{2\sqrt{2}}\Gamma_{11}\Gamma_m\lambda^{GP}\,, \qquad \widehat{\Psi}_m^{GP} \;\equiv\; -\frac{2\sqrt{2}}{3}\Gamma_{11}\Gamma_m\lambda^{GP}\,, \tag{D.14}$$

where $\widehat{\Psi}_m^{GP} := \psi_m^{GP} + (\sqrt{2}/12)\Gamma_{11}\Gamma_m\lambda^{GP}$.

E

Repertory of Gamma functions

This section is a succinct dictionary of functions available in SSGamma. Each function introduced in chapter 3 is described in a few words (some optional arguments and features are left aside, but can be found in the main text). Throughout this section, we adopt a convenient notation to represent the arguments of functions :

- Lists with unspecified elements are represented by `l`, e.g. {a_1, 4, β},
- Integers are represented by `i`,
- Symbols are represented by `s`, e.g. a_1 or T are symbols,
- The letters above can be combined. A list of integers is denoted `li`, e.g. `{1,0,7}`, and `ls` is a list of symbols, e.g. {a_0, T, β_{397}}. `lls` represents a list of list of symbols, e.g. {{a,b},{β}}
- An argument followed by `...` represents a unspecified number of repetitions of that same type of argument, e.g. `li...` can be `{1,2}`, or `{1,2},{4,3}`, etc.,
- The word `expr` denote an unspecified expression on which the function can be applied.

Symbols with underscores are not supported by Mathematica, and they are displayed here for aesthetic reasons (e.g. s_{14} have to be replaced by $s14$ to be a valid index). To sum up, the expression below,

```
Fun[s...,expr,lli,{{i,s},{i,s}}]
```

represents a function called `Fun`, that can be applied on an expression `expr`, where the first argument is a series of symbols, whose third argument has to be a list of list of integers, followed by something like {{1,a_1},{0,α}}.

`TG[s:`γ`,ls...,{{i,s},{i,s}}]`	Defines a single, or a contraction of several γ matrices of name γ by default, with bosonic indices `ls` and two fermionic indices specified in the last argument.
`TE[s:`ϵ`,ls]`	Defines a Levi-Civita symbol, with default name ϵ, with indices `ls` (as numerous as Dim).
`TD[s:`δ`,ls,ls]`	Defines a generalized Kronecker symbol, with default name δ, with two set of indices as two last arguments.
`GME[expr,lls]`	Simplifies all contracted γ matrices inside `expr`. Computation can be quickened by specifying lists of anti-symmetrized indices lls.
`GMC[expr,lls]`	Simplifies in expr all γ matrices with repeated bosonic indices in contracted. Computation can be quickened by specifying lists of anti-symmetrized indices lls.
`GMT[expr,lls]`	Simplifies γ matrices with contracted fermionic indices in `expr`. Computation can be quickened by specifying lists of anti-symmetrized indices lls.
`GMS[expr]`	Regroups γ matrices with contracted fermionic indices into a single series of γ matrices.
`GMET[expr,s]`	Simplifies the γ-traceless tensor named `s` wherever it is contracted with γ matrices in expr.
`DeltaSim[expr,lls]`	Applies all Kronecker deltas if some of their indices are contracted with other objects. Computation can be quickned by specifying lists of anti-symmetrized indices `lls`.
`TensDef[s,l,lls,lls]`	Adds in the dictionary the tensor named `s`, with indices `l` (specified by a symbol for even, and `{i,s}` for odd indices), with symmetric/anti-symmetric sets of indices specified as third/fourth argument.
`TensDef[]`	Just displays all tensors in the dictionary with their specificity.
`TensClear[s...]`	Removes tensors named `s`... from the dictionary. Without arguments, all tenors are removed.
`TS[s,ls]`	Writes a tensor named `s`, with indices `ls`. If it is in the dictionary, it inherits the properties attached to its name. If it is not, it is assumed totaly anti-symmetric.
`Der[expr,s]`	Applies an operator D with index `s`, verifying the Leibniz rule over tensors.
`NCDer[expr,{i,s}]`	Applies an operator D with index `{i,s}`, verifying the anti-commutative Leibniz rule over spinors.
`ExplicitDer[expr]`	Separates the derivative from the tensor it is applied on, for performing a replacement of the tensor (the derivative must be reactivated with the built-in function ACTIVATE.
`RD[expr,i]`	Renames all contracted indices in expr, with symbol taken from the list $u_0, u_1 \dots z_8, z_9$

`CanonicalRD[expr]`	Renames all contracted indices in expr in a canonical way, thus gathering all non-trivially isomorphic tensor contractions, with symbol taken from the list $u_0, u_1 \dots z_8, z_9$.
`ASym[expr,ls]`	Explicitly anti-symmetrizes the indices `ls` in `expr`.
`Sym[expr,ls]`	Explicitly symmetrizes the indices `ls` in `expr`.
`AC[expr,ls]`	Makes implicit the anti-symmetry of the indices `ls`.
`SC[expr,ls]`	Makes implicit the symmetry of the indices `ls`.
`SumGraph[expr]`	Gives a visual representation of tensor contractions in `expr`. Vertex are tensors, superscripts are the number of indices, subscripts are free indices.
`FactorTens[expr]`	Gathers all identical tensorial expressions in `expr`, factorizing all constants.
`IrrTens[expr,lls]`	Generates a tensorial representation of the Young diagram over the indices specified in `lls`.
`IrrYT[li...]`	Computes the product of Young diagrams specified by `li`.