Clifford index and gonality of curves on special K3 surfaces

Marco Ramponi

Université de Poitiers
Faculté des Sciences Fondamentales et Appliquées
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Marco Ramponi

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Après avis des rapporteurs :

- Daniele FAENZI (Professeur, Université de Bourgogne)
- Andreas KNUTSEN (Professeur, Université de Bergen, Norvége)

Composition du Jury :

Daniele FAENZI (Professeur, Université de Bourgogne)	Rapporteur
Andreas KNUTSEN (Professeur, Université de Bergen, Norvége)	Rapporteur
Vincent KOZIARZ (Professeur, Université de Bordeaux)	Examinateur
Samuel BOISSIÉRE (Professeur, Université de Poitiers)	Examinateur
Boris PASQUIER (Professeur, Université de Poitiers)	Examinateur
Alessandra SARTI (Professeure, Université de Poitiers)	Directrice de thèse

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[1]Qualité de ce qui est au service de l'être humain, adapté à la nature humaine, à dimension humaine, en particulier un outil, 12 lettres.

Résumé de la thèse

Nous allons étudier les propriétés des courbes algébriques sur des surfaces K3 spéciales, du point de vue de la théorie de Brill-Noether.

La démonstration de Lazarsfeld [Laz86] du théorème de Gieseker-Petri a mis en lumière l'importance de la théorie de Brill-Noether des courbes admettant un plongement dans une surface K3. Nous allons donner une démonstration détaillée de ce résultat classique, inspirée par les idées de Pareschi. En suite, nous allons décrire le théorème de Green et Lazarsfeld [GL87], fondamental pour tout notre travail, qui établit le comportement de l'indice de Clifford des courbes sur les surfaces K3.

Watanabe [Wat14] a montré que l'indice de Clifford de courbes sur certaines surfaces K3, admettant un recouvrement double des surfaces de del Pezzo, est calculé en utilisant les involutions non-symplectiques. Nous étudions une situation similaire pour des surfaces K3 avec un réseau de Picard isomorphe à $U(m)$, avec $m > 0$ un entier quelconque. Nous montrons que la gonalité et l'indice de Clifford de toute courbe lisse sur ces surfaces, avec une seule exception en genre $g = m + 1 \geq 4$ déterminée explicitement, sont obtenus par restriction des fibrations elliptiques de la surface. Ce travail est basé sur l'article suivant :

> M. Ramponi, *Gonality and Clifford index of curves on elliptic K3 surfaces with Picard number two*, Archiv der Mathematik, 106(4), p. 355–362, 2016.

Knutsen et Lopez [KL08, KL15] ont étudié en détail la théorie de Brill-Noether des courbes sur les surfaces d'Enriques. En appliquant leurs résultats, nous allons pouvoir calculer la gonalité et l'indice de Clifford de *toute courbe* lisse sur les surfaces K3 qui sont des recouvrements universels d'une surface d'Enriques. Ce travail est basé sur l'article suivant :

> M. Ramponi, *Special divisors on curves on K3 surfaces carrying an enriques involution*, Manuscripta Mathematica, 153(1), p. 315–322, 2017.

ABSTRACT

We study the properties of algebraic curves lying on special K3 surfaces, from the viewpoint of Brill-Noether theory.

Lazarsfeld's proof [Laz86] of the Gieseker-Petri theorem has revealed the importance of Brill-Noether theory of curves which admit an embedding in a K3 surface. We give a proof of this classical result, inspired by the ideas of Pareschi [Par95]. We then describe the theorem of Green and Lazarsfeld [GL87], a key result for our work, which establishes the behaviour of the Clifford index of curves on K3 surfaces.

Watanabe [Wat14] showed that the Clifford index of curves lying on certain special K3 surfaces realizable as a double covering of a smooth del Pezzo surface, can be determined by a direct use of the non-simplectic involution carried by these surfaces. We study a similar situation for some K3 surfaces having a Picard lattice isomorphic to $U(m)$, with $m > 0$ any integer. We show that the gonality and the Clifford index of all smooth curves on these surfaces, with a single, explicitly determined exception in genus $g = m + 1 \geq 4$, are obtained by restriction of the elliptic fibrations of the surface. This work is based on the following article:

> M. Ramponi, *Gonality and Clifford index of curves on elliptic K3 surfaces with Picard number two*, Archiv der Mathematik, 106(4), p. 355–362, 2016.

Knutsen and Lopez [KL08, KL15] have studied in detail the Brill-Noether theory of curves lying on Enriques surfaces. Applying their results, we are able to determine and compute the gonality and Clifford index of any smooth curve lying on the general K3 surface which is the universal covering of an Enriques surface. This work is based on the following article:

> M. Ramponi, *Special divisors on curves on K3 surfaces carrying an Enriques involution*, Manuscripta Mathematica, 153(1), p. 315–322, 2017.

Contents

INTRODUCTION

In the theory of algebraic curves, Brill-Noether theory, introduced by Alexander von Brill and Max Noether in 1874, is the study of *special* line bundles, certain line bundles on a curve C which carry unexpectedly many sections.

One of the numerous features of algebraic curves is that their moduli theory is particularly well-behaved. The set of all isomorphism classes of curves of genus g forms a quasi-projective algebraic variety $\mathcal{M}_g$. Quite amazingly, several properties of this moduli space were determined long before it was actually constructed — in 1940 as an analytic space by Teichmüller and in 1965 as a GIT quotient of the Hilbert scheme of curves by Mumford. For instance, Riemann computed its dimension in 1857 and Klein showed that $\mathcal{M}_g$ is irreducible in 1882. In particular, one can talk about a property enjoyed by the *general* curve of genus g, when this property holds for curves in a dense open subset of $\mathcal{M}_g$. Indeed, several amongst the early results in Brill-Noether theory, such as the Brill-Noether theorem, started with the words "Let C be a curve with general moduli…".

A notable example of this sort is the Petri conjecture. Roughly speaking, it states that for a curve with general moduli, the *Bril-Noether varieties*, parameter spaces for line bundles of given degree and number of sections, behave as nicely as one could possibly hope for (in terms of smoothness and dimension). The first rigorous proof of the Petri conjecture was given by Gieseker in 1982, by degeneration methods. By the irreducibility of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ (which allows curves in $\mathcal{M}_g$ to degenerate to *stable* ones, i.e. complete, connected curves, with finitely many automorphisms and at most nodes as singularities), Gieseker's strategy is to check the desired property on a single, suitable example of stable curve, hence proving that the property holds for the general curve in $\mathcal{M}_g$.

Degenerations techniques have proven to be extremely useful in several other circumstances, although the proofs are often lengthy and may require some rather sophisticated combinatorial analysis.

In 1986, Lazarsfeld [Laz86] gave a new proof of the Petri-Gieseker theorem by specialization to *smooth* curves lying on general K3 surfaces. His idea is that the Brill-Noether theory of curves on a K3 surface can be translated into properties of some vector bundles which can be naturally defined on the whole ambient surface.

Lazarsfeld's approach has been a breakthrough. Arguably, it has influenced much of the subsequent theory of algebraic curves in the following decades. Only two years before Lazarsfeld's paper appeared, Green had formulated his famous conjecture regarding the behaviour of syzygies of canonically embedded curves [Gre84]. Roughly speaking, Green's conjecture states that the shape of the minimal resolution of the homogeneous ideal of a canonical curve is determined by the Clifford index of the curve, a Brill-Noether type invariant which measures the complexity of the curve inside the moduli space. As of today, this remains an open problem, and although a very large number of theorems confirming the conjecture in several special cases have appeared in the literature during the years, this is still considered by many experts as one of the major unsolved problems in the theory of algebraic curves.

Following the philosophy of Lazarsfeld, between 2002 and 2005, Voisin proved Green's conjecture for the general curve of any genus, by specialization to smooth curves in the primitive linear system on a K3 surface with Picard number one [Voi02, Voi05]. In 2011, Aprodu and Farkas have showed that the conjecture is indeed true for curves on any K3 surface, including special ones [AF11]. A very remarkable consequence of this result is that the knowledge of the Clifford index of a given curve C lying on a K3 surface gives us informations about the geometry of C in its canonical embedding and also essentially determines the syzygies of the K3 surface with respect to the morphism induced by the line bundle $L = \mathcal{O}_X(C)$.

Other than Green's conjecture, there exist similar problems which relate the geometry of a curve in some types of embedding to Brill-Noether type invariants of the curve, such as its gonality or its Clifford index. Examples are the Secant conjecture and the Gonality conjecture, both formulated (and the latter proved in 2014) by Green and Lazarsfeld [GL86], or the more recent Prym-Green conjecture [CEFS13], a statement similar to Green's conjecture for general Prym-canonical curves, i.e. curves C embedded by a line bundle of the form $\omega_C \otimes \eta$, with η a 2-torsion line bundle.

In studying this kind of problems, it has often proved to be very useful to make computations for curves on K3 surfaces. For instance, Farkas and Kemeny have recently made important advancements for both the Secant conjecture and the Prym-Green conjecture, by lattice-theoretic computations on Nikulin surfaces, which are special K3 surfaces characterized by the existence of a symplectic involution [FK16].

It is in fact a consequence of a general, fundamental result by Green and Lazarsfeld [GL87] that the Clifford index of curves lying on a K3 surface is essentially determined by the lattice theory on the surface. In practice, however, when one wants to perform *explicit* computations of the Clifford index in some specific situation, *ad hoc* strategies are to be developed, depending on the particular example, as the geometry of K3 surfaces can be extremely rich.

DESCRIPTION OF RESEARCH CARRIED OUT

This work began with the study of a paper of Watanabe [Wat14], whose main result is a characterization of the Clifford index of curves on some special K3 surfaces X which are double covers of a smooth del Pezzo surface of degree $4 \leq d \leq 8$ such that the involution on X induced by the double cover acts trivially on the Picard group of X. Watanabe shows that, with a few exceptions, the Clifford index of these curves is induced by the elliptic pencils carried by the surface. My interest in this paper lied in Watanabe's original approach to the problem, especially in the author's systematic use of the non-symplectic involution on X in throughout all computations.

The initial goal of this thesis was an attempt to enquire about the possible connections between the theory of automorphisms on K3 surfaces and the Brill-Noether theory of the curves lying on them. Concretely, I began by analysing a situation similar to the one studied by Watanabe, considering a class of K3 surfaces carrying a non-symplectic automorphism of order 3. I initially carried out the computations in the same spirit of Watanabe, and this eventually led to the first main result (see Theorem 3.1), where I considered K3 surfaces with Picard lattice isomorphic to $U(m)$, for any $m > 0$, and obtained a similar outcome as in Watanabe's Theorem for the curves on these surfaces. However, in my treatment, I finally dropped any assumption on the existence of automorphism on the K3 surfaces (and obtained a slightly more general result), persuaded that there is no deeper connection between the theory of automorphisms and the Brill-Noether theory of curves on K3 surfaces, other than the lattice theory on the surface.

Next, I turned my attention to K3 surfaces which admit an involution without fixed points. Such a K3 surface is the universal covering of an Enriques surface, and the Brill-Noether theory of curves on Enriques surfaces has been established by Knutsen and Lopez [KL08, KL15], and it presents several aspects which are rather different than the case of K3 surfaces. For instance, the gonality and the Clifford index are often non-constant in a linear system. In view of their results, it was a very natural question to ask about the Brill-Noether theory of curves on the relative K3-covers of these surfaces. I have provided a complete answer to this question for all curves on the generic K3-cover, showing that their gonalities and Clifford indices are induced by the elliptic fibrations carried by the surface (see Theorem

4.4).

Amongst the various topics I had to learn about during these three years of research, I consider that of Lazarsfeld-Mukai bundles a specially important one, due to its central role in the Brill-Noether theory of curves on K3 surfaces. These vector bundles can be defined in fact on any surface (even though for K3 surfaces they are particularly well-behaved) and they represent a very powerful tool in many problems and appeared in distinct areas of current research. Even if I did not need to make a direct use of this tool for the proof of the original results in this thesis, I have spent some efforts in learning more about it, also in view of future research projects. As a testing for my understanding, I have reworked the aforementioned proof of the Gieseker-Petri theorem by Lazarsfeld (see Theorem 2.10), following the main argument of Pareschi [Par95], although my exposition is slightly different than Pareschi's and I give a proof of all non-trivial statements.

Brill-Noether theory has applications to several distinct areas of algebraic geometry, and I will mention a few problems which I am currently working on. In a joint work with Daniele Agostini (in preparation), we use Brill-Noether theory of curves on projective Kummer surfaces to compute the Koszul cohomology of various projective models of Kummer surfaces. I was unfortunately unable to add these topics in this manuscript, due to a lack of time. We are currently trying to apply these results to characterize some Noether-Lefschetz divisors in the moduli space of projective Kummer surfaces, of which we showed the existence.

A second topic, which is work in progress, joint with Gavril Farkas, is around some problems related to a conjecture of Mercat in higher rank Brill-Noether theory, and involves Nikulin K3 surfaces. This area of research presents several still unexplored directions, and it has already proven to give many interesting applications, in particular in connection to the birational geometry of moduli spaces.

Structure of the Thesis

- ⋄ In Chapter 1 we briefly recover some foundational material, such as the basics of lattice theory, algebraic surfaces, K3 surfaces and Brill-Noether theory of curves.

- ⋄ In Chapter 2 we recall Lazarsfeld-Mukai vector bundles on K3 surfaces and, as an application, we give a proof of Lazarsfeld's celebrated result on the Gieseker-Petri theorem. We then illustrate Green-Lazarsfeld's theorem on the constancy of the Clifford index in linear systems of curves on K3 surfaces. Finally, we recall Knutsen's classification of exceptional curves on K3 surfaces.

- ◇ In Chapter 3 we study curves on some elliptic K3 surfaces. We first recall Watanabe's result on double covers of del Pezzo surfaces and explain how it motivated our study of curves on K3 surfaces with Picard lattice isomorphic to $U(m)$, for any $m > 0$.
- ◇ In Chapter 4 we recall the work of Knutsen and Lopez on Brill-Noether theory for curves on Enriques surfaces and apply their results to study curves lying on universal covers of Enriques surfaces. We show that the gonalities and Clifford indices of these curves are induced by the elliptic fibrations arising from the Enriques quotient.

Chapter 1

General facts

§1. Lattices

The main purpose of this section is to fix notations and recall, for later reference, some standard definitions and a few results concerning the theory of integral quadratic forms.

By the word *lattice* we will understand a free $\mathbb{Z}$-module Λ of finite rank, equipped with a non-degenerate, symmetric, integral bilinear form

$$\Lambda \times \Lambda \longrightarrow \mathbb{Z}, \quad (x, y) \longmapsto x \cdot y.$$

We often refer to this bilinear form as the *intersection form* on Λ.

We shall say that

- ⋄ A lattice Λ is *even* if $x^2 = x \cdot x \in 2\mathbb{Z}$ for any $x \in \Lambda$.
- ⋄ The *signature* (n_+, n_-) of Λ is defined as the signature of the $\mathbb{R}$-linear extension of the bilinear form on the real vector space $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$.
- ⋄ A lattice is *definite* if either $n_+ = 0$ or $n_- = 0$, *indefinite* otherwise.
- ⋄ A *hyperbolic* lattice is one of signature $(1, r-1)$, where $r = \operatorname{rk} \Lambda \geq 2$.
- ⋄ The *dual lattice* Λ^* is defined abstractly as

  $$\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$$

  which allows to consider Λ as a subgroup of Λ^* via the canonical embedding $x \mapsto (x, -)$. One can then define the *discriminant group* of Λ as the quotient group

  $$A_{\Lambda} := \Lambda^*/\Lambda$$

- ⋄ In practice, it is often convenient to give the following alternative, yet equivalent, definition of the dual lattice as the subspace of the vector space $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$ given by

  $$\Lambda^* := \{x \in \Lambda_{\mathbb{Q}} : \ x \cdot y \in \mathbb{Z} \ \text{ for all } y \in \Lambda\}$$

where $\Lambda \subset \Lambda^*$ is given by the natural inclusion $\Lambda \subset \Lambda_{\mathbb{Q}}$. It becomes then easier to work with the discriminant group A_Λ.

- An *isometry* of lattices is a group isomorphism which preserves the intersection forms.

Observe that the bilinear form $(\ ,\ )$ on Λ does not descend to A_Λ, for we do not generally have equality between (x_1, x_2) and $(x_1 + u_1, x_2 + u_2)$, where $x_i \in \Lambda^*$ and $u_i \in \Lambda$. On the other hand,

$$\begin{aligned}(x_1 + u_1, x_2 + u_2) &= (x_1, x_2) + (x_1, u_2) + (x_2, u_1) + (u_1, u_2)\\ &\equiv (x_1, x_2) \mod \mathbb{Z}\end{aligned}$$

Therefore we can define on A_Λ the bilinear form $(\ ,\ )$ modulo $\mathbb{Z}$, i.e. with values in $\mathbb{Q}/\mathbb{Z}$. Similarly, under the further assumption that Λ is even, one obtains the quadratic form on A_Λ modulo $2\mathbb{Z}$, i.e. with values in $\mathbb{Q}/2\mathbb{Z}$.

- Note that A_Λ is a finite group,

$$A_\Lambda \simeq (\mathbb{Z}/m_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/m_k\mathbb{Z})$$

  and its order is $|\operatorname{discr}\Lambda|$, where $\operatorname{discr}\Lambda$ (also denoted by $\det\Lambda$ in the literature) denotes the determinant of the bilinear form with respect to some $\mathbb{Z}$-basis of Λ.

- The *length* $\ell = \ell(\Lambda)$ of Λ is by definition the minimal number of generators of the discriminant group A_Λ.

- An embedding of lattices $\Lambda_0 \hookrightarrow \Lambda$ is called *primitive* if Λ/Λ_0 is free.

To some extent, the simpler the discriminant group is, the simpler the lattice. In particular, a lattice Λ is called

- *unimodular:* if $A_\Lambda = 0$ (i.e. $|\operatorname{discr}\Lambda| = 1$)

- p-*elementary:* if $A_\Lambda = (\mathbb{Z}/p\mathbb{Z})^{\oplus\ell}$, for some $\ell \geq 0$.

Let us give a couple of fundamental examples of lattices:

- We denote by $\langle m \rangle$, with $m \in \mathbb{Z}$, the lattice of rank 1 such that $u^2 = m$ for a generator u of the lattice $\langle m \rangle$.

- There exists a unique even unimodular hyperbolic lattice of rank two (up to isometry). It is called the *hyperbolic plane* and we identify it with the matrix of its bilinear form,

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Given any lattice Λ, we denote by $\Lambda(m)$ the lattice obtained by keeping the same $\mathbb{Z}$-module and multiplying the intersection form by $m \in \mathbb{Z}$. In the case of $U(m)$, we have

$$A_{U(m)} \simeq (\mathbb{Z}/m\mathbb{Z}) \oplus (\mathbb{Z}/m\mathbb{Z}).$$

◇ There exists a unique even unimodular positive definite lattice of rank eight. It is denoted by E_8 and we identify it with its matrix

$$E_8 = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

By changing the sign of the matrix elements of E_8 above we get the negative definite lattice $E_8(-1)$.

We recall the following classical result of Milnor.

THEOREM 1.1 (Milnor). *Let Λ be an indefinite unimodular lattice. If Λ is odd, then*

$$\Lambda \simeq \langle 1 \rangle^{\oplus a} \oplus \langle -1 \rangle^{\oplus b}$$

for some $a \geq 0$ and $b \geq 0$. If Λ is even then

$$\Lambda \simeq U^{\oplus a} \oplus E_8(\pm 1)^{\oplus b}$$

for some a and b. In particular, the signature and parity of Λ determine Λ up to isometry.

Definition. The *K3 lattice* is by definition

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

By Milnor's theorem, any even unimodular lattice of signature $(3, 19)$ is isomorphic to Λ_{K3}.

We recall the following result from [Nik80, Thm. 1.14.4].

Theorem 1.2. *Let Λ be an even, unimodular lattice of signature (n_+, n_-) and Λ_0 any even lattice of signature (m_+, m_-). Under the assumptions that*

$$m_\pm < n_\pm \quad \text{and} \quad \ell(\Lambda_0) + 2 \leq \operatorname{rk}\Lambda - \operatorname{rk}\Lambda_0,$$

then there exists a primitive embedding $\Lambda_0 \hookrightarrow \Lambda$, which is unique up to $\operatorname{Aut}(\Lambda)$.

Notice that $\ell(\Lambda_0) + 2 \leq \operatorname{rk}\Lambda - \operatorname{rk}\Lambda_0$ clearly follows from the stronger condition $\operatorname{rk}\Lambda_0 + 2 \leq \operatorname{rk}\Lambda - \operatorname{rk}\Lambda_0$. Morrison [Mor84, Cor. 2.9] uses this to show:

Corollary 1.3. *Let N be an even lattice of signature $(1, \rho - 1)$ with $\rho \leq 10$. Then there exists a projective K3 surface X with $\mathrm{NS}(X) \simeq N$. Moreover, the primitive embedding $N \hookrightarrow H^2(X, \mathbb{Z})$ is unique up to the action of $\operatorname{Aut}(H^2(X, \mathbb{Z}))$.*

Proof. By Theorem 1.2 applied to the K3 lattice Λ_{K3}, one gets a primitive embedding $N \hookrightarrow \Lambda_{K3}$. Then, the existence of a K3 surface X such that $N \simeq \mathrm{NS}(X)$ follows by the surjectivity of the period map together with standard arguments of Hodge theory, cf. [Mor84, Cor. 1.9]. $\square$

§2. Algebraic surfaces

By *surface* we always mean a smooth algebraic surface. Let S be a surface. Recall that

$$\chi(\mathcal{O}_S) = 1 - q + p_g,$$

where $q = h^1(\mathcal{O}_S)$ is the irregularity of S and $p_g = h^2(\mathcal{O}_S)$ its geometric genus. More generally, for any sheaf $\mathcal{F}$ on S we set $\chi(\mathcal{F}) = \sum(-1)^i h^i(\mathcal{F})$. We denote by $\operatorname{Pic}(S)$ the group of line bundles on S modulo isomorphism and by $\operatorname{Num}(S) = \operatorname{Div}(S)/\equiv$ the group of divisors on S modulo numerical equivalence. Linear equivalence is denoted by $\sim$.

Let us recall a few standard facts about surfaces, cf. [BHPvdV14].

Theorem 1.4 (Riemann-Roch formula). *For any line bundle L on a surface S, we have the formula*

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2} L \cdot (L - K_S),$$

where K_S denotes a canonical divisor on S. For any vector bundle V on S,

$$\chi(V) = \operatorname{rk}V \cdot \chi(\mathcal{O}_S) + \frac{1}{2} c_1(V) \cdot (c_1(V) - K_S) - c_2(V),$$

where $c_i(V)$ denotes the i-th Chern class of V.

Recall that the topological-Euler characteristic of the surface S is given by $e(S) = 2 - 4q + b_2(S)$, where $b_2 = \operatorname{rk} H^2(S, \mathbb{Z})$ is the second Betti number of S. We have,

THEOREM 1.5 (Noether's formula). *For any surface S we have*

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + e(S)).$$

By *a curve*, we always mean a reduced curve. Let now C be an irreducible curve on a surface S. Its *arithmetic genus* is by definition the integer $p_a(C) := 1 - \chi(\mathcal{O}_C)$. We have,

THEOREM 1.6 (Genus formula). *For any irreducible curve C on a surface S, we have*

$$2p_a(C) - 2 = C^2 + C \cdot K_S.$$

We need now to recall the Hodge index theorem and some immediate consequences of it. Fix some ample divisor H on S. Let $q : \mathrm{Num}(S) \to \mathbb{Z}$, where $\mathrm{Num}(S)$ denotes the group of divisors modulo numerical equivalence, be the non-degenerate quadratic form induced by the self-intersection product and consider its linear extension, still denoted by q, to the real vector space

$$\mathrm{Num}(S)_{\mathbb{R}} = \mathrm{Num}(S) \otimes_{\mathbb{Z}} \mathbb{R}.$$

We denote by h the class of H in $\mathrm{Num}(S)_{\mathbb{R}}$. The form q is positive definite on $\langle h \rangle$. Let

$$h^{\perp} = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : \ x \cdot h = 0\}$$

and consider the orthogonal decomposition $\mathrm{Num}(S)_{\mathbb{R}} = \mathbb{R}h \oplus h^{\perp}$. Then we have

THEOREM 1.7 (Hodge index). *With the same notations as above, the form q is negative definite on the hyperplane $h^{\perp}$ inside $\mathrm{Num}(S)_{\mathbb{R}}$.*

Rephrasing this in terms of divisors, this immediately yields:

COROLLARY 1.8. *Let H be an ample divisor on a surface S. If D is a divisor such that $D \cdot H = 0$, then either $D^2 < 0$, or $D \equiv 0$.*

The intersection form q on $\mathrm{Num}(S)_{\mathbb{R}}$ is positive on all lines $\mathbb{R}h$ where h is any ample class and negative on the hyperplanes $h^{\perp}$. In other words, the Hodge index theorem says that the signature of q is $(1, \rho - 1)$, where

$$\rho(S) := \operatorname{rk} \mathrm{Num}(S)$$

is the Picard number of S.

We now want to recall and give a proof of a well-known strengthening of the Hodge index theorem. First, we need the following

LEMMA 1.9. *Let q be a quadratic form on $\mathbb{R}^m$ with signature $(1, m-1)$. Suppose $\pi \subset \mathbb{R}^m$ is a plane containing one positive direction, i.e. there is $x \in \pi$ with $q(x) > 0$. Then, the signature of $q|_{\pi}$, the restriction of q to the plane π, is $(1, 1)$.*

Proof. Since π contains a positive direction, the signature of $q|_{\pi}$ is either $(1, 1)$ or $(1, 0)$. Thus, we only need to show that $q|_{\pi}$ is still non-degenerate. In fact, since $\dim(\pi) = 2$, π intersects any hyperplane $\Pi \simeq \mathbb{R}^{m-1}$ where q is negative-definite. Hence, π also contains a negative direction. $\square$

Here we have the strong version of the Hodge index theorem, which is in fact the most useful one for all practical purposes:

THEOREM 1.10. *Let L, D be divisors on S, with $L^2 > 0$. Then*

(i) $(L \cdot D)^2 \geq L^2 \cdot D^2$. *Moreover, if D is not numerically trivial, then:*

(ii) Equality in (i) holds if and only if L and D are linearly dependent in $\mathrm{Num}(S)_{\mathbb{Q}}$, in which case one has precisely $(L \cdot D)L \equiv (L^2)D$.

Proof. If $D \equiv 0$ there is nothing to prove. Hence, we assume that the class d of D in $\mathrm{Num}(S)_{\mathbb{R}}$ is non-zero. Consider the subspace $\pi = \langle l, d \rangle_{\mathbb{R}}$ in $\mathrm{Num}(S)_{\mathbb{R}}$, spanned by the classes of L and D. Since $q(l) = L^2 > 0$ by assumption, π contains a positive direction.

Let $x \in \pi$. If we write $x = x_1 l + x_2 d$, we have

$$q(x) = (x_1 l + x_2 d)^2 = (L^2)x_1^2 + 2(L \cdot D)x_1 x_2 + (D^2)x_2^2$$

In other words, the matrix of the quadratic form $q|_{\pi}$ is

$$A = \begin{pmatrix} L^2 & L \cdot D \\ L \cdot D & D^2 \end{pmatrix}$$

Now, either $\dim(\pi) = 2$ or $\dim(\pi) = 1$. If $\dim(\pi) = 2$, then the signature of $q|_\pi$ is $(1, 1)$ by Hodge index theorem. Thus,

$$\det A = L^2 D^2 - (L \cdot D)^2 < 0,$$

which proves (i). Obviously, $\dim(\pi) = 1$ if and only if l and d are $\mathbb{R}$-linearly dependent (in which case $\det A = 0$). Since l and d are *integral* classes (i.e. they lie in $\mathrm{Num}(S) = \mathbb{Z}^\rho \subset \mathbb{R}^\rho$), linear dependence over $\mathbb{R}$ means linear dependence over $\mathbb{Q}$. So, we can write $rL \equiv sD$, for some integers r, s. Intersecting the expression $rL \equiv sD$ with L and D respectively, we get $r/s = L \cdot D/L^2$, which proves (ii). □

COROLLARY 1.11. *Let L be a divisor on S with $L^2 > 0$. If D is a divisor such that $D \cdot L = 0$, then either $D^2 < 0$, or $D \equiv 0$.*

Proof. By Theorem 1.10 (i) we have $D^2 \leq 0$. We assume by contradiction $D^2 = 0$ with D non-zero in $\mathrm{Num}(S)$. By Theorem 1.10 (ii) we get $rL \equiv sD$ for some $r, s \in \mathbb{Z}$. Then $0 = s^2 D^2 = r^2 L^2$, hence $r = 0$. Therefore, $sD = 0$ in $\mathrm{Num}(S)$, that is $sD \equiv 0$. Since $\mathrm{Num}(S)$ has no torsion, $D \equiv 0$. □

§3. NEF AND AMPLE DIVISORS

In this section we recall the basic properties of nef and ample divisors on an algebraic surface. The reference is [BHPvdV14].

Let S be a (smooth, algebraic) surface. Recall that if L is a line bundle on S, we say that L is *very ample* if, for some basis $s_0, \dots, s_r$ of $H^0(L)$, the rational map $\varphi_L \colon S \dashrightarrow \mathbb{P}^r = \mathbb{P}H^0(L)$ given by $x \mapsto (s_0(x), \dots, s_r(x))$ is an embedding, and we say that L is *ample* if some positive multiple mL is very ample. The property of being ample is then defined in the obvious way for divisors by looking at their respective line bundles. The question is then how to describe the set of ample divisors in $\mathrm{Num}(S)$.

If S has Picard number $\rho(S) = 1$ then $\mathrm{Num}(S)$ is generated by a single class, that one of some ample line bundle H. This situation is then quite simple. We therefore assume $\rho(S) \geq 2$ in what follows.

We have the quadratic form $q : \mathrm{Num}(S)_\mathbb{R} \to \mathbb{R}$, $q(x) = x^2$. By standard linear algebra, there is a basis of $\mathrm{Num}(S)_\mathbb{R}$ for which q is expressed in canonical form. Since the signature of q is $(1, \rho - 1)$, the matrix of q in such a basis is given by

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

If $x \in \mathrm{Num}(S)_{\mathbb{R}}$ has coordinates $x_1, \dots, x_\rho$ with respect to this basis, then

$$q(x) = x_1^2 - x_2^2 - \cdots - x_\rho^2.$$

We define the set

$$\Omega = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : \; q(x) > 0\}$$

which is the interior of the quadric cone $\{q = 0\}$. Notice that Ω has two connected components: the one with $x_1 > 0$ and that with $x_1 < 0$.

Definition. The connected component of Ω which contains an ample class is called the *positive cone*, and denoted by $\mathscr{C} = \mathscr{C}_S$.

Of course, in order for this definition to make sense we have to show that $\mathscr{C}$ contains indeed all ample classes, which is a matter of convexity:

Proof. Suppose L, H are two ample classes in $\mathrm{Num}(S)$. Since $L^2 > 0$ and $H^2 > 0$, then $L, H \in \Omega$. Consider a class α_λ on the segment connecting the classes H and L in $\mathrm{Num}(S)_{\mathbb{R}}$, i.e. $\alpha_\lambda = \lambda H + (1-\lambda)L$, with $0 \leq \lambda \leq 1$. Note that

$$q(\alpha_\lambda) = \lambda^2 H^2 + 2\lambda(1-\lambda) H \cdot L + (1-\lambda)^2 L^2$$

is positive, since H^2, L^2 and $H \cdot L$ are, and because the coefficients above are all non-negative (but never simultaneously zero). Therefore, q is positive on the whole segment, which means that L and H must lie in the same connected component of Ω. □

We recall the following numerical criterion for amplitude:

THEOREM 1.12 (Nakai-Moishezon). *A divisor D in S is ample if and only if it satisfies $D^2 > 0$ and $D \cdot C > 0$ for all irreducible curves C.*

Whence, ampleness is a well defined property in $\mathrm{Num}(S)$, in the sense that each representative of an ample class is itself an ample divisor: if H is ample and $H' \equiv H$, then H' is ample. We then have,

COROLLARY 1.13. *The sum of two ample divisors is ample.*

Proof. It follows from the criterion of Nakai-Moishezon. □

We observe that the set

$$D^\perp = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : \; x \cdot D = 0\},$$

is a hyperplane through the origin of $\mathrm{Num}(S)_{\mathbb{R}}$. Hence

$$D_{\geq 0} = \{x \in \mathrm{Num}(S)_{\mathbb{R}} : \ x \cdot D \geq 0\},$$

is a closed half-space of $\mathrm{Num}(S)_{\mathbb{R}}$. This is clear in the canonical basis of $\mathrm{Num}(S)_{\mathbb{R}}$, where, if D and x have coordinates m_i and x_i respectively, then $x \cdot D = m_1 x_1 - m_2 x_2 - \cdots - m_\rho x_\rho$. In particular, taking the intersection of spaces of the form $D_{\geq 0}$ yields a (convex) cone.

Definition. The *ample cone* is defined as the convex cone in $\mathrm{Num}(S)_{\mathbb{R}}$ generated by the ample classes,

$$\mathrm{Amp}(S) = \{\sum \lambda_i H_i : \ \lambda_i > 0, \ H_i \ \text{ ample}\}$$

(where the sums are finite). In particular, $\mathrm{Amp}(S) \subset \mathscr{C}_S$. The *nef cone* of S is defined as

$$\mathrm{Nef}(S) = \{x \in \overline{\mathscr{C}} : \ x \cdot C \geq 0 \text{ for all irreducible curves } C \subset S\}.$$

Definition. A *nef* divisor, is a divisor D for which $D \cdot C \geq 0$ for all irreducible curves C. A *big and nef* divisor is a nef divisor such that $D^2 > 0$.

The Ample cone is a cone and it is generated by ample classes, by definition. It is less obvious that the nef cone is a actually a cone. If D is a nef divisor then $D^2 \geq 0$ (see below). Therefore, the classes of nef divisors lie in the nef cone: they are precisely the points of $\mathrm{Nef}(S)$ with coordinates in $\mathbb{Z}$. However, in general they do not generate it as a cone (Mumford-Ramanujam counterexample).

PROPOSITION 1.14. *Let S be a surface and let $\mathscr{C}$ denote its positive cone. We have that* $\mathrm{Amp}(S)$ *is open in* $\mathscr{C}$ *and* $\mathrm{Nef}(S)$ *is closed in* $\overline{\mathscr{C}}$. *Moreover,*

$$\mathrm{Amp}(S) = \{x \in \mathscr{C} : \ x \cdot C > 0 \textit{ for all irreducible curves } C\} = \mathrm{Int}\,\mathrm{Nef}(S)$$
$$\overline{\mathrm{Amp}(S)} = \{x \in \overline{\mathscr{C}} : \ x \cdot C \geq 0 \textit{ for all irreducible curves } C\} = \mathrm{Nef}(S)$$

The second identity shows that nefness is the *limit condition* which defines amplitude. In other words, amplitude fails right at the boundary of $\mathrm{Nef}(S)$, i.e. we have,

COROLLARY 1.15. *For any class $x \in \partial\,\mathrm{Nef}(S)$, either $x^2 = 0$ or there exists an irreducible curve C on S such that $x \cdot C = 0$.*

Proof. If $x \in \partial \operatorname{Nef}(S)$ then clearly $x^2 \geq 0$. Suppose $x^2 > 0$. Then, if $x \cdot C > 0$ for all irreducible curves we have $x \in \operatorname{Amp}(S) = \operatorname{Int}\operatorname{Nef}(S)$, contradicting the assumption $x \in \partial \operatorname{Nef}(S)$. Thus $x \cdot C = 0$ for some C. □

We now give a short proof of the above Proposition.

Proof of the Proposition. We first show that $\operatorname{Amp}(S)$ is open.

We prove that any point $h \in \operatorname{Amp}(S)$ belongs to the interior of $\operatorname{Amp}(S)$, that is, we construct an open neighbourhood of h contained in $\operatorname{Amp}(S)$. First, let us do this when h is just the class of an ample divisor $H \in \operatorname{Div}(S)$. Choose divisors $D_1, \cdots, D_\rho$ whose classes give a basis of $\operatorname{Num}(S)$. We can choose some integers $m_1, \ldots, m_\rho$ such that both $D_i + m_i H$ and $-D_i + m_i H$ are very ample, for each i. Dividing out by m_i we have $H \pm \frac{1}{m_i} D_i \in \operatorname{Amp}(S)$. We take these divisors as the vertices of a polyhedron. By convexity, this polyhedron is entirely inside $\operatorname{Amp}(S)$. This shows that H lies in the interior of $\operatorname{Amp}(S)$, as claimed.

On the other hand, if we consider multiples $h = \lambda H$, with $\lambda > 0$ it is clear that the same argument still works (up to rescaling by a factor $1/\lambda$). Same if we consider $h = \lambda H + \mu H'$, with $\lambda, \mu > 0$. This shows that any point $h \in \operatorname{Amp}(S)$, which we can write as $h = \sum \lambda_i H_i$ with $\lambda_i > 0$ and H_i ample, lies in the interior of $\operatorname{Amp}(S)$, and therefore $\operatorname{Amp}(S)$ is open in $\operatorname{Num}(S)_{\mathbb{R}}$.

We now show that $\overline{\operatorname{Amp}(S)} = \operatorname{Nef}(S)$. We have $\operatorname{Amp}(S) \subset \operatorname{Nef}(S)$. Next, notice that $\operatorname{Nef}(S)$ is closed. Indeed, by definition it is the intersection of the closure of the positive cone $\overline{\mathscr{C}}$ with the intersection of all the (closed) half-spaces $\{x \cdot C \geq 0\}$, for all irreducible curves $C \subset S$. Therefore $\overline{\operatorname{Amp}(S)} \subset \operatorname{Nef}(S)$ and we need to show the other inclusion. We want to prove that each $h \in \operatorname{Nef}(S)$ is the limit of a sequence in $\operatorname{Amp}(S)$. Since $\operatorname{Amp}(S)$ is open we may choose a basis $h_1, \ldots, h_\rho$ of $\operatorname{Num}(S)_{\mathbb{R}}$ such that each h_i is the class of an ample divisor. For each $n \geq 1$ we construct a little box containing h,

$$B_n = \{h + \sum_{i=1}^{\rho} t_i h_i : \ 0 < t_i < \frac{1}{n}\}$$

We pick any point $\tilde{h}_n \in B_n$ having rational coordinates (it exists by density of $\mathbb{Q}$ in $\mathbb{R}$). Hence some integer multiple $m\tilde{h}_n$ is the class of a divisor D. Then,

$$D^2 = m^2(\tilde{h}_n)^2 = m^2(h^2 + 2\sum t_i h_i \cdot h + \sum t_i t_j h_i \cdot h_j) > 0$$

since $h^2 \geq 0$ and $h \cdot h_i \geq 0$ (because h is nef) and $h_i \cdot h_j > 0$ (ample). Also, for any irreducible curve $C \subset S$,

$$D \cdot C = m\tilde{h}_n \cdot C = m(h \cdot C + \sum t_i h_i \cdot C) > 0$$

since $h \cdot C \geq 0$ (nef) and $h_i \cdot C > 0$ (ample). By the criterion of Nakai-Moishezon, it follows that D is ample, so that

$$\tilde{h}_n = \frac{1}{m} D \in \mathrm{Amp}(S).$$

Finally, it is clear that $\tilde{h}_n$ converges to h as $n \to \infty$. This shows that $\overline{\mathrm{Amp}(S)} = \mathrm{Nef}(S)$ and completes the proof of the Proposition. $\square$

We have defined *nef divisors* by the condition $D \cdot C \geq 0$ for any irreducible curve C on S. Equivalently, $D \cdot E \geq 0$ for any effective divisor E.

PROPOSITION 1.16 (Kleiman). *If* D *is nef, then* $D^2 \geq 0$.

Proof. Pick an ample divisor H and define, for $t \in \mathbb{R}$, the function

$$p(t) = (D + tH)^2 = D^2 + 2tD \cdot H + t^2 H^2.$$

Since $H^2 > 0$, we have $p(0) = D^2$ and $p(t) > 0$ when $t > 0$ is big enough. Hence we only need to find a sequence $\{t_n\}$, converging to zero for $n \to \infty$ and such that $p(t_n) > 0$. The simplest choice will do the job: let $t_0 \in \mathbb{Q}$ be such that $p(t_0) > 0$. We claim that $p(t_0/2) > 0$. Let us prove this claim. Up to clearing the denominator of t_0 we can assume it is an integer, hence $D + t_0 H$ is a divisor. Then $(D + t_0 H)^2 > 0$ implies that for $n > 0$ big enough, either $n(D + t_0 H)$ or its opposite is effective. Indeed, the former is true, because $H \cdot n(D + t_0 H) > 0$. Since D is nef, we must then have $D \cdot n(D + t_0 H) \geq 0$. This gives $p(t_0/2) = D^2 + t_0 D \cdot H + \frac{t_0^2}{4} H^2 > 0$. $\square$

By the Nakai-Moishezon criterion and the above Proposition we may compare the numerical conditions which define amplitude and nefness of a given divisor D on a surface:

AMPLITUDE	NEFNESS
$D \cdot C > 0$, and	$D \cdot C \geq 0$, hence
$D^2 > 0$	$D^2 \geq 0$

where "for any smooth irreducible curve C" is understood. It might be worth to point out that the "strictly nef" condition $D \cdot C > 0$ alone does not imply $D^2 > 0$. There exist in fact some (rather pathological) examples of strictly nef divisors with $D^2 = 0$, thus non ample.

There is another important cone:

Definition. The *effective cone* Eff(S) is the cone in $\mathrm{Num}(S)_{\mathbb{R}}$ generated by the (classes of) curves of S (or, equivalently, by the classes of the effective divisors).

Concretely, its elements are finite linear combinations

$$x = \sum \lambda_i C_i$$

where C_i are irreducible curves on S and $\lambda_i \in \mathbb{R}_{>0}$.

LEMMA 1.17. *The effective cone* contains *the positive cone,*

$$\mathscr{C}_S \subset \mathrm{Eff}(S).$$

Proof. Since they are cones, it suffices to check on the generators. Let $D \in \mathscr{C}_S$. Then $D^2 > 0$, thus either nD or $-nD$ is effective, for n sufficiently large. Pick an ample class $H \in \mathscr{C}_S$ and consider the hyperplane $H^\perp$ in $\mathrm{Num}(S)$. By Hodge index theorem q is negative definite on $H^\perp$, thus $\mathscr{C}_S$ and $H^\perp$ do not intersect, i.e. $\mathscr{C}_S$ is contained in the half-space $H_{>0}$. Since clearly $H \in H_{>0}$ we get $D \cdot H > 0$, hence nD is effective and, therefore, D too. $\square$

Definition. The closure $\overline{\mathrm{Eff}(S)}$ in $\mathrm{Num}(S)_\mathbb{R}$ is called the *Kleiman-Mori cone*.

The general picture is the following:

$$\begin{array}{ccccc}
\mathrm{Amp}(S) & \subset & \mathscr{C}_S & \subset & \mathrm{Eff}(S) \\
\cap & & \cap & & \cap \\
\overline{\mathrm{Amp}(S)} & \subset & \overline{\mathscr{C}}_S & \subset & \overline{\mathrm{Eff}(S)}
\end{array}$$

Recall that the closure $\overline{\Omega} = \{x \in \mathrm{Num}(S)_\mathbb{R} : x^2 \geq 0\}$ is the union of two components $\overline{\mathscr{C}} \cup (-\overline{\mathscr{C}})$ intersecting at zero, where $\mathscr{C}$ is the positive cone.

REMARK 1.18. *The effective classes inside $\overline{\Omega}$ all lie in $\overline{\mathscr{C}}$, i.e.*

$$\overline{\Omega} \cap \mathrm{Eff}(S) \subset \overline{\mathscr{C}}.$$

Proof. Indeed, let $x \in \overline{\Omega} \cap \mathrm{Eff}(S)$ and assume by contradiction $x \in (-\overline{\mathscr{C}})$ (and x non-zero). For any ample class $h \in \mathscr{C}$, since x is effective, $x \cdot h > 0$. Since $x^2 \geq 0$, we have $q(\lambda x + (1-\lambda)h) > 0$ for any $0 < \lambda < 1$. Thus this segment is entirely contained in $\Omega = \mathscr{C} \sqcup (-\mathscr{C})$ and connects $x \in (-\overline{\mathscr{C}})$ and $h \in \mathscr{C}$; therefore it has to go through the origin, a contradiction. $\square$

COROLLARY 1.19. *For any two* effective *classes x and y* such that *$x^2 \geq 0$ and $y^2 \geq 0$ (hence $x, y \in \overline{\mathscr{C}}$), we have $x \cdot y \geq 0$.*

Proof. This was already clear when either $x^2 > 0$ or $y^2 > 0$, by the Hodge index theorem. When $x^2 = y^2 = 0$, it suffices to observe that q cannot drop negative along the segment from x to y. $\square$

We end this section by mentioning some description of the Kleiman-Mori cone. If D is a divisor, we denote by $\mathbb{R}_+[D]$ the *ray* generated by (the class of) D in $\mathrm{Num}(S)_{\mathbb{R}}$, that is, $\mathbb{R}_+[D] = \{\lambda[D] : \lambda > 0\}$.

Now suppose we have a curve $C \subset S$ with negative self-intersection $C^2 < 0$. Then obviously $\mathbb{R}_+[C]$ is not contained in $\mathscr{C}_S$, though $\mathbb{R}_+[C]$ is inside $\overline{\mathrm{Eff}(S)}$. We have,

THEOREM 1.20 (Mori). *The Kleiman-Mori cone is the smallest convex (closed) cone inside* $\mathrm{Num}(S)_{\mathbb{R}}$ *which contains both the closed positive cone and the rays generated by all curves* C *with negative self-intersection. In other words,*

$$\overline{\mathrm{Eff}(S)} = \overline{\mathscr{C}}_S + \sum_{C^2<0} \mathbb{R}_+[C]$$

(where the sum runs over all curves C *on* S *with* $C^2 < 0$*, if any).*

REMARK 1.21. *There exists a more precise description of the part of* $\overline{\mathrm{Eff}(S)}$ *which lies in the half-space* $\{x : x \cdot K_S < 0\}$*. The full result is known as Mori's cone theorem.*

Recall the Nakai-Moishezon criterion: a divisor D is ample if and only if $D^2 > 0$ *and* $D \cdot C > 0$ for any irreducible curve C. The latter condition can be re-expressed by saying that the functional $\phi_D \colon \mathrm{Num}(S)_R \to \mathbb{R}$ given by

$$\phi_D(x) = x \cdot D$$

is *positive* on the effective cone $\mathrm{Eff}(S)$. This latter condition alone does not guarantee D to be ample. On the other hand, if we extend this condition *up to the boundary* of $\mathrm{Eff}(S)$, then D is ample. More precisely, we have

THEOREM 1.22 (Kleiman). D *is ample if and only if its associated functional* ϕ_D *is positive on the Kleiman-Mori cone* $\overline{\mathrm{Eff}(S)}$*.*

§4. K3 SURFACES

In this section we recall the basic definitions and properties of K3 surface. We refer to [BHPvdV14] as a reference.

By definition, a K3 surface is a surface X such that

$$q(X) = 0 \text{ and } K_X = 0.$$

One has $p_g(X) = 1$, thus $\chi(X) = 2$. In particular,

$$h^0(X, \Omega^1) = h^1(\mathcal{O}_X) = 0,$$

i.e. there are no global holomorphic 1-forms, and

$$h^0(X, \Omega^2) = h^2(\mathcal{O}_X) = 1,$$

i.e. on X there exists a unique (non-zero) global holomorphic 2-form ω_X, up to scalar multiples:

$$H^{2,0}(X) = H^0(X, \Omega^2) \simeq \mathbb{C}\omega_X.$$

Being a generator of $H^{2,0}$, the 2-form ω_X is nowhere vanishing, hence it gives X a holomorphic symplectic structure. By the Hodge decomposition,

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus \overline{H^{2,0}(X)},$$

we get $b_2(X) = \dim H^2(X, \mathbb{C}) = 2 + h^{1,1}$. By Noether's formula we get

$$e(X) = 12\chi(\mathcal{O}_X) = 24,$$

hence $h^{1,1} = 20$. Finally, we recall that $H^2(X, \mathbb{Z})$, together with the cup product, is an even unimodular lattice of signature $(3, 19)$, hence isomorphic to the K3 lattice

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

by Milnor's theorem.

Without any further mention, we will consider only algebraic K3 surfaces in the following.

LEMMA 1.23. *Let C be an irreducible curve on a K3 surface X. Then $\mathcal{O}_C(C) = \omega_C$ and $C^2 \geq -2$, with equality if and only if C is a smooth rational curve, and $C^2 = 0$ if and only if $p_a(C) = 1$. In particular, if $p_a(C) \geq 2$ then C is a nef and big divisor.*

Proof. Adjunction formula $\mathcal{O}_X(C) \simeq \omega_C$, gives $C^2 = 2p_a(C) - 2$. Since C is irreducible, $p_a(C) \geq 0$ and so $C^2 \geq -2$, with $C^2 = -2$ if and only if $p_a(C) = 0$, i.e. $C \simeq \mathbb{P}^1$. Obviously $C^2 = 0$ if and only if $p_a(C) = 1$. Finally, if $p_a(C) \geq 2$, then $C^2 > 0$, and for any other irreducible curve C' (which cannot be a component of C by the irreducibility assumption) one has $C \cdot C' > 0$, i.e. C is big and nef. $\square$

It follows by the Lemma that the closure of the ample cone $\mathrm{Amp}(X)$ is

$$\mathrm{Nef}(X) = \{x \in \mathscr{C}_X : \ x \cdot C \geq 0 \text{ for all } (-2)\text{-curves } C\}.$$

LEMMA 1.24. *Let C be an irreducible curve on a K3 surface X, with $p_a(C) \geq 1$. Then, the linear system $|C|$ is basepoint free. Moreover, $h^1(\mathscr{O}_X(C)) = 0$.*

Proof. Since $h^1(\mathscr{O}_X) = 0$, the cohomology sequence of

$$0 \to \mathscr{O}_X \to \mathscr{O}_X(C) \to \omega_C \to 0,$$

shows that the map $H^0(\mathscr{O}_X(C)) \to H^0(\omega_C)$ is onto, since $H^1(\mathscr{O}_X) = 0$. Hence the basepoints of $\mathscr{O}_X(C)$ are basepoints of ω_C, which is basepoint free for $p_a(C) \geq 1$. Observing that the boundary map $H^1(\omega_C) \to H^2(\mathscr{O}_X)$ is Serre dual to the (bijective) restriction map $H^0(\mathscr{O}_X) \to H^0(\mathscr{O}_C)$, one immediately obtains $h^1(\mathscr{O}_X(C)) = 0$. □

More generally, just as in the proof of the Lemma, if D is an effective reduced divisor on a K3 surface X, then by the short exact sequence

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0,$$

using $H^1(\mathscr{O}_X) = 0$ and Serre duality $h^1(\mathscr{O}_X(-D)) = h^1(\mathscr{O}_X(D))$, we immediately get the geometric interpretation of $h^1(\mathscr{O}_X(D))$ as counting the connected components of D, precisely:

$$\begin{aligned} h^1(\mathscr{O}_X(D)) &= h^0(\mathscr{O}_D) - 1 \\ &= \#\{\text{connected components of } D\} - 1. \end{aligned}$$

Let L be a line bundle on X. The Riemann-Roch formula gives

$$h^0(L) + h^0(-L) = 2 + \frac{1}{2}L^2 + h^1(L).$$

As a straightforward consequence we get the following.

LEMMA 1.25. *Let L be a non-trivial line bundle on a K3 surface X. We have,*

(i) $L^2 = 2(\chi(L) - 2)$ *is even.*

(ii) If $L^2 \geq -2$ *then either* $h^0(L)$ *or* $h^0(-L)$ *is not zero.*

(iii) If $L^2 \geq 0$ *then either* $h^0(L)$ *or* $h^0(-L)$ *is at least 2.*

(iv) If $h^0(L) = 1$*, then the effective divisor defined by the unique section of* L *is a sum of smooth rational curves (with multiplicities).*

We shall only give a proof of (iv):

Proof. Let $D = \sum a_i\Gamma_i$ be the effective divisor defined by the only section of L. Since $h^0(D) = 1$, for any integral component Γ_i of D we have $h^0(\Gamma_i) = 1$, hence $\Gamma_i^2 = -2$ by Riemann-Roch, therefore $\Gamma_i \simeq \mathbb{P}^1$. □

COROLLARY 1.26. *The fixed part of a linear system* $|L|$ *on a K3 surface is a linear combination* $\sum a_i\Gamma_i$, $a_i \geq 0$, *of smooth rational curves* $\Gamma_i \simeq \mathbb{P}^1$.

Proof. Let $|L| = |M| + F$ be the mobile and fixed part decomposition. Then $h^0(F) = 1$, since F is fixed. Apply (iv) above. □

Let us now give two basic examples of the fixed part of a linear system on a K3 surface. Suppose L is an effective line bundle, with $L^2 \geq 0$ and let

$$|L| = |M| + \sum a_i\Gamma_i$$

be the mobile and fixed part decomposition of the linear system $|L|$, with $\Gamma_i \simeq \mathbb{P}^1$ the reduced components of the fixed part. By construction, we have $h^0(L) = h^0(M)$ and $|M|$, being mobile, has no fixed components and is nef. In particular, $M^2 \geq 0$ and for any component Γ_i we have $M \cdot \Gamma_i \geq 0$.

EXAMPLE 1.27. *Consider the case* $|kE| + F$, *with* E *an elliptic curve,* $k \geq 1$ *and* F *the fixed part. Suppose there exists a component* Γ_1 *of* F *such that* $E \cdot \Gamma_1 > 0$. *Up to linear equivalence* $kE + \Gamma_1 \sim E_1 + \cdots + E_k + \Gamma_1$ *(with* $E_i \in |E|$ *all distinct) as a reduced connected divisor (since* Γ_1 *intersects each* E_i*), whence* $h^1(kE + \Gamma_1) = 0$, *and by Riemann-Roch we get* $h^0(kE+\Gamma_1) = kE\cdot\Gamma_1 + 1$. *On the other hand, since* Γ_1 *is fixed in* $|kE + \Gamma_1|$, *we have* $h^0(kE + \Gamma_1) = h^0(kE) = k + 1$. *Equating the two yields* $E \cdot \Gamma_1 = 1$. *Now, if* Γ_2 *is another fixed component with* $E \cdot \Gamma_2 = 1$, *then by the same arguments we find* $k + 1 = h^0(kE + \Gamma_1 + \Gamma_2) = 2k + \Gamma_1 \cdot \Gamma_2$, *that is* $k = 1 - \Gamma_1 \cdot \Gamma_2$ *which forces* $k = 1$ *and* $\Gamma_1 \cdot \Gamma_2 = 0$. *In particular, we find that when* $k \geq 2$ *there can be only one component* Γ_1 *such that* $E \cdot \Gamma_1 = 1$ *and for all other components* $E \cdot \Gamma_i = 0$. *With the same arguments one shows that for* $k \geq 1$, *a smooth rational curve* Γ *on* X *is fixed in* $|kE+\Gamma|$ *if and only if* $E\cdot\Gamma = 0$ *or* $E\cdot\Gamma = 1$.

REMARK 1.28. *When* $L \sim kE + F$ *is as in the example above, if there exists a component* Γ *of the fixed part* F *such that* $E \cdot \Gamma = 0$, *then* $L \cdot \Gamma = F \cdot \Gamma < 0$, *so* L *is not nef. Thus, when* $k \geq 2$ *the only* nef *linear systems having* $|kE|$ *as the mobile part are* $|kE|$ *and* $|kE + \Gamma|$, *with* $\Gamma \simeq \mathbb{P}^1$ *and* $E \cdot \Gamma = 1$.

EXAMPLE 1.29. *Consider the case* $|L| = |C| + F$, *with* C *an irreducible curve. Arguing as in the previous example one finds that any component* Γ *of* F *satisfies*

$0 \leq C \cdot \Gamma_i \leq 1$. *Conversely, if some smooth rational curve* Γ *on* X *is such that* $C \cdot \Gamma = 0$ *or* 1, *then* $h^0(C) = h^0(C + \Gamma)$ *by Riemann-Roch, which implies that* Γ *is fixed in* $|C + \Gamma|$. *We conclude that for any smooth rational curve* Γ *on* X, *one has that* Γ *is fixed in* $|C + \Gamma|$ *if and only if* $C \cdot \Gamma = 0$ *or* $C \cdot \Gamma = 1$. *Note that* $L \cdot \Gamma < 0$, *so* L *is not nef.*

We recall some well-known results from Saint-Donat' paper [SD74]. First of all, to understand the base locus is enough to understand the fixed part. Precisely, one has the following fact, cf. [SD74, Cor. 3.2].

THEOREM 1.30. *A complete linear system* $|L|$ *on a K3 surface has no isolated basepoints: its base locus coincides with its (eventually empty) fixed part.*

Also, Saint-Donat characterizes complete basepoint free linear systems as follows, cf.[SD74, Prop. 2.6].

THEOREM 1.31. *Let* L *be a non-trivial effective line bundle on a K3 surface* X *such that the linear system* $|L|$ *is free of fixed components. Then, we have:*

(i) *If* $L^2 = 0$, *then* $L = \mathcal{O}_X(rE)$, *where* E *is an elliptic curve and* $r \geq 1$. *In this case* $h^1(L) = r - 1$.

(ii) *If* $L^2 > 0$, *then* $L = \mathcal{O}_X(C)$, *for some smooth irreducible curve* C *on* X.

Consequently, for big and nef line bundles one has the following result, cf. [SD74, §2.7]. See also [Huy16, Cor. 3.15].

COROLLARY 1.32. *Let* L *be a big and nef line bundle. Then, either*

(i) $|L|$ *is basepoint free and* $L = \mathcal{O}_X(C)$, *for some curve* C; *or*

(ii) $L = \mathcal{O}_X(rE + \Gamma)$ *for some* $r \geq 2$, *where* $|E|$ *is an elliptic pencil and* Γ *is a smooth rational curve such that* $E \cdot \Gamma = 1$.

Proof. Let $|L| = |M| + F$ be the mobile and fixed part decomposition. If $M^2 > 0$, by Theorem 1.31 we get $M \sim C$ for some irreducible curve C, hence $F = 0$ (for otherwise $L \cdot F < 0$ contradicting the assumption that L is nef). This proves (i). Hence, we can assume $F \neq 0$ and $M^2 = 0$. Then $M \sim rE$, for some $r \geq 1$ and an elliptic curve E, by Theorem 1.31. In fact, $r \geq 2$, for by Riemann-Roch $h^0(L) = h^0(rE) > 2$, while $h^0(E) = 2$. Since $0 < L^2 = 2M \cdot F + F^2$ and $F^2 < 0$, we have $M \cdot F > 0$, hence there exists at least one irreducible component $\Gamma \simeq \mathbb{P}^1$ of F such that $\Gamma \cdot M > 0$, and in fact $\Gamma \cdot E = 1$, as we observed above. Remark 1.28 concludes the proof. $\square$

Let $\mathscr{C}_X$ be the positive cone of a K3 surface X and consider a line bundle $L \in \mathscr{C}_X$. Then $L^2 > 0$ and we may assume L is effective, by Lemma 1.25. Let C be an irreducible curve on X with $C^2 \geq 0$. Then $L \cdot C > 0$, by the Hodge index theorem and the fact that L is effective. We then have,

PROPOSITION 1.33 (Amplitude and Nefness on a K3 surface). *Let L be a line bundle on a K3 surface X. Then,*

(i) L is ample if and only if $L \in \mathscr{C}_X$ and $L \cdot \Gamma > 0$ for all smooth rational curves $\Gamma \simeq \mathbb{P}^1$ on X.

(ii) If L is effective and $L^2 \geq 0$, then L is nef if and only if $L \cdot \Gamma \geq 0$ for all smooth rational curves $\Gamma \simeq \mathbb{P}^1$ on X.

Proof. The first assertion follows immediately by the Nakai-Moishezon criterion for amplitude and the observations above Proposition 1.33, and the second assertion by the fact that $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$. $\square$

If L is an effective line bundle with $L^2 = 0$ on a K3 surface X, then we cannot conclude that there exist elliptic curves in $|L|$, as it might be the case that L fails to be nef: if there is a smooth rational curve Γ with $L \cdot \Gamma < 0$, then Γ sits inside the base locus of $|L|$ and so Theorem 1.31 does not apply to L. Thus, it may seem that the existence of an elliptic curve depends on the actual position of the effective cone $\mathrm{Eff}(X)$ inside $\mathrm{Num}(X) \otimes \mathbb{R}$. Instead, as it turns out, this can just be read off the Picard lattice:

PROPOSITION 1.34. *A K3 surface X admits an elliptic fibration if and only if there exists a non-trivial line bundle $L \in \mathrm{Pic}(X)$ with $L^2 = 0$.*

Proof. Let x denote the class of L in $H^2(X, \mathbb{Z})$. Up to sign, we have $x \in \overline{\mathscr{C}}_X$. Clearly, x is effective, by Riemann-Roch. It may not be nef, but, if this is the case, then there is a (-2)-class δ such that $x \cdot \delta < 0$. Set $x' = s_\delta(x)$, where s_δ denotes the Picard-Lefschetz reflection with respect to δ, defined as

$$s_\delta(x) = x + (\delta \cdot x)\delta.$$

Thus, we have $0 < s_\delta(x) \cdot h < x \cdot h$, for some fixed ample class h. If still x' is not nef, we may repeat the process which must end (since the degree with respect to h has to be positive but decreases at each step). At the end, we find a non-trivial (effective) nef class, let us denote it again by x, with $x^2 = 0$. To conclude the proof one then only has to show that if $L \in \mathrm{Pic}(X)$ is a non-trivial nef line bundle satisfying $L^2 = 0$, then $L \sim mE$, for some elliptic curve E and some $m > 0$. This follows, for example, from Theorem 1.31 above (see also [PS71, §3 Thm. 1] for a direct, yet slightly lengthy proof of this fact, or [Huy16, Prop. 3.10] for a more sophisticated approach). $\square$

Let us recall the following notion.

Definition. Let m be some integer. A curve C on a surface S is m-*connected* if for any effective decomposition $C \sim D_1 + D_2$ one has $D_1 \cdot D_2 \geq m$. (Note that D_1 and D_2 are allowed to have common components).

Note that the definition only depends on the line bundle $L = \mathcal{O}_S(C)$.

LEMMA 1.35. *Let C be a big and nef curve on a projective surface S. Then C is 1-connected.*

Proof. We apply the Hodge index theorem: for any class $\alpha \in \mathrm{NS}(S)_{\mathbb{Q}}$ such that $\alpha \cdot C = 0$, then either $\alpha = 0$ or $\alpha^2 < 0$. Let $C \sim C_1 + C_2$ and set

$$0 \leq t := \frac{C \cdot C_1}{C^2} \leq 1.$$

If $t = 0$, then $C \cdot C_1 = 0$, and therefore we must have $\alpha^2 := (C_1)^2 < 0$. Then, $0 = C \cdot C_1 = (C_1)^2 + C_1 \cdot C_2$ gives $C_1 \cdot C_2 > 0$. Similarly, if $t = 1$, then $C \cdot C_2 = C^2 - C \cdot C_1 = 0$ and the same argument applies. Thus, we assume $0 < t < 1$, i.e. $C \cdot C_i > 0$. Then, $\alpha := tC - C_1 \in \mathrm{NS}(S)_{\mathbb{Q}}$ satisfies $\alpha \cdot C = 0$. Either $\alpha = 0$ or $\alpha^2 < 0$. In the first case, by $\alpha \cdot C_2 = 0$, we get $C_1 \cdot C_2 = tC \cdot C_2 > 0$. In the latter case, $C_1 \cdot C_2 = (tC - \alpha)(C + \alpha - tC) = t(1-t)C^2 - \alpha^2 > 0$. $\square$

On K3 surfaces we can prove the following, cf. [SD74, Lemma 3.7].

PROPOSITION 1.36. *Let C be an irreducible curve such that $C^2 > 0$ on a K3 surface X. Then, any divisor in $|C|$ is 2-connected.*

Proof. Since each curve in $|C|$ is 1-connected, we assume $C \sim C_1 + C_2$ such that $C_1 \cdot C_2 = 1$ and we will find a contradiction. We analyse each one of the following three cases separately:

(a) $C \cdot C_1 \geq 2$ and $C \cdot C_2 \geq 2$

(b) $C \cdot C_1 = 1$

(c) $C \cdot C_1 \leq 0$

Case (a) implies $C_i^2 > 0$, $i = 1, 2$. Then $C_i^2 \geq 2$ by evenness of the pairing, hence $(C_2)^2(C_2)^2 > (C_1 \cdot C_2)^2 = 1$ contradicts the Hodge index theorem 1.10.

Case (b) gives $C_1^2 = 0$. Then $h^0(C_1) \geq 2$ and by

$$0 \to \mathcal{O}_X(-C_2) \to \mathcal{O}_X(C_1) \to \mathcal{O}_C(C_1) \to 0$$

one finds $h^0(\mathcal{O}_C(C_1) \geq 2$, i.e. a line bundle on C of degree $C \cdot C_1 = 1$ with at least 2 sections, forcing $C \simeq \mathbb{P}^1$. This contradicts $C^2 > 0$.

In case (c) we may assume the inequality is strict, as $C \cdot C_1 = 0$ would imply $C_1^2 = -1$, which is impossible on a K3 surface, by the evenness of the K3 lattice. If $C \cdot C_1 < 0$, then C, being irreducible, would have to be an irreducible component of C_1, which is clearly impossible (or C would be linearly equivalent to a reducible curve, one of whose irreducible components is C itself). □

§5. Brill-Noether theory

For all the results discussed in this section we refer to [ACGH10].

Let C be a smooth algebraic curve of genus g. A linear system of type g^r_d on C is a pair (A, V) where A is a line bundle of degree d on C and V a $(r+1)$-dimensional subspace of $H^0(C, A)$. When $V = H^0(C, A)$ coincides with the whole space of global sections of A we say that the g^r_d is *complete*.

Definition. A line bundle A on C is called *special* if $h^1(A) > 0$.

For a line bundle A we denote by

$$r(A) := h^0(A) - 1,$$

the projective dimension of the linear system $|A|$. By the Riemann-Roch theorem,

$$r(A) = d - g + h^1(A).$$

Therefore, a line bundle is special if and only if

$$r(A) > d - g.$$

Special line bundles are thus those line bundles which carry an exceptional number of sections and Brill-Noether theory can be regarded as the study of special line bundles on algebraic curves.

We denote by $\mathcal{M}_g$ the moduli space of curves of genus g. It is well-known that $\mathcal{M}_g$ is an irreducible quasi-projective variety. We say that a given property of a curve of genus g is an *open property* if it holds inside a

(Zariski) open subset of $\mathcal{M}_g$. Equivalently, we say that the property holds for the *general curve* in $\mathcal{M}_g$.

Given g, r, d one defines the *Brill-Noether number*

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

The following result is known as the *Brill-Noether theorem*:

THEOREM 1.37. *A general curve of $\mathcal{M}_g$ carries a g^r_d if and only if $\rho(g, r, d) \geq 0$.*

Complete linear systems on C of fixed degree $d \geq 1$ and carrying at least $r + 1 > 0$ independent global sections are parametrized by a variety

$$W^r_d(C) = \{A \in \mathrm{Pic}^d(C) : r(A) \geq r\}.$$

This varieties are obtained as determinantal subvarieties of $\mathrm{Pic}^d(C)$ and the scheme structure on $W^r_d(C)$ is the one naturally inherited as a subscheme of $\mathrm{Pic}^d(C)$. We gather in the following theorem some well-known properties of these varieties, cf. [ACGH10, Chap.V].

THEOREM 1.38. *Let C be any curve in $\mathcal{M}_g$. Let $d \geq 1$ and $r \geq 0$ be integers. If $\rho(g, r, d) \geq 0$, then $W^r_d(C)$ is non-empty. Furthermore, if $r \geq d - g$, then each component of $W^r_d(C)$ has dimension at least $\rho(g, r, d)$.*

Therefore, for any $A \in W^r_d(C)$, we have

$$\dim_A W^r_d(C) \geq \rho(g, r, d),$$

although the dimension of $W^r_d(C)$ might be strictly bigger. The Brill-Noether number $\rho(g, r, d)$ is hence called the *expected dimension* of $W^r_d(C)$.

Of course, by the Brill-Noether theorem 1.37, if C is general and $\rho(g, r, d)$ is negative, then $W^r_d(C)$ is empty, though there can be special curves carrying a g^r_d with negative Brill-Noether number; an immediate example being smooth plane curves: the linear system giving the embedding in $\mathbb{P}^2$ is such that $\rho(g, 2, d) = 9d - 2d^2 - 10 < 0$ whenever $d > 2$.

For a line bundle A on C we denote by $\mu_0 = \mu_0(A)$ the cup-product map

$$\mu_0 \colon H^0(A) \otimes H^0(\omega_C \otimes A^{-1}) \to H^0(\omega_C).$$

Observe that the difference between the dimensions of codomain and domain of μ_0 is precisely $g - (r + 1)(g - d + r) = \rho(g, r, d)$, with $d = \deg A$ and $r = r(A)$. Therefore the image of μ_0 has codimension at least $\rho(g, r, d)$.

In particular, μ_0 is injective if and only the image $\mathrm{Im}(\mu_0)$ has codimension exactly $\rho(g, r, d)$, or, equivalently, $\dim \ker(\mu_0^\vee) = \rho(g, r, d)$, where

$$\mu_0^\vee \colon H^1(\mathscr{O}_C) \to H^0(A)^\vee \otimes H^1(A),$$

is the map dual to μ_0.

Let us recall [ACGH10, IV, Prop.(4.2)], which describes the infinitesimal properties of $W^r_d(C)$.

PROPOSITION 1.39. *Let A be a point of $W^r_d(C)$ not belonging to $W^{r+1}_d(C)$ and let $\mu_0 = \mu_0(A)$. The tangent space to $W^r_d(C)$ at A is identified with $\ker(\mu_0^\vee)$. The tangent space to $W^r_d(C)$ at a point $B \in W^{r+1}_d(C)$ is equal to the whole tangent space $T_B \operatorname{Pic}^d(C)$. In particular, if $W^r_d(C)$ has the expected dimension $\rho(g, r, d)$ and $r > d - g$, then B is a singular point of $W^r_d(C)$.*

We therefore have, for $A \in W^r_d(C) \setminus W^{r+1}_d(C)$, the inequalities

$$\rho(g, r, d) \leq \dim_A W^r_d(C) \leq \dim \ker(\mu_0^\vee),$$

which, together with the observations above, shows the following:

COROLLARY 1.40. *$W^r_d(C)$ is smooth of dimension exactly $\rho(g, r, d)$ at a point $A \in W^r_d(C) \setminus W^{r+1}_d(C)$ if and only if $\mu_0(A)$ is injective.*

It had been conjectured by Petri that for a *general* curve C in $\mathscr{M}_g$, all line bundles A on C have injective μ_0 map. Since this condition is open in $\mathscr{M}_g$, it would suffice to find a single example of a curve with this property (one for any genus) in order to prove Petri's conjecture. However, all the explicit known examples fail the task. This conjecture was nevertheless proved by Gieseker:

THEOREM 1.41 (Gieseker). *For a general curve C, all line bundles on C have injective μ_0 map.*

The proof is by degeneration to stable curves. A much simpler proof of this theorem was later found by Lazarsfeld. In particular, it uses curves on K3 surfaces to show the existence of *smooth* curves satisfying Petri's conjecture in any genus. Due to the importance this result has had to the subsequent development of the theory of curves, and in particular the Brill-Noether theory of curves on K3 surfaces, we will recall it in detail and give an outline of the proof using vector bundles in the next chapter.

We end this section by recalling the basic properties of two important invariants of an algebraic curve: the gonality and the Clifford index.

Let C be a smooth curve of genus g. Recall that C is called *hyperelliptic* if there exists a finite map $C \to \mathbb{P}^1$ of degree 2. Generalizing this situation, one gives the following definition.

Definition. The *gonality* of C is by definition the minimum integer $d > 0$ such that the curve admits a non-constant morphism $C \to \mathbb{P}^1$ of degree d. Equivalently,

$$\mathrm{gon}(C) = \min\{\deg(A) : A \in \mathrm{Pic}(C),\ h^0(A) = 2\}.$$

In particular, $\mathrm{gon}(C) = 2$ if and only if C is hyperelliptic. A line bundle with two independent global sections is usually called a *pencil*, and the line bundles A appearing in the definition of the gonality are called *gonality pencils*, or just pencils of minimal degree on C.

By the Brill-Noether theorem 1.37, the general curve of genus g will carry a pencil of degree d if and only if $\rho(g, 1, d) = 2d - g - 3 \geq 0$, i.e. whenever $d \geq \lfloor \frac{g+3}{2} \rfloor$ (the lower integral part of $\frac{g+3}{2}$). In particular, every curve of genus g has a pencil of degree $\lfloor \frac{g+3}{2} \rfloor$, that is,

$$\mathrm{gon}(C) \leq \lfloor \frac{g+3}{2} \rfloor,$$

with an equality for the general curve in $\mathscr{M}_g$. We refer to $\lfloor \frac{g+3}{2} \rfloor$ as the *maximal (or general) gonality* of a curve of genus g.

The gonality being a classical invariant of algebraic curves, its modern counterpart, in some sense to be considered a "refinement" of the gonality, is given by the Clifford index.

Definition. For a line bundle A on C, one defines

$$\mathrm{Cliff}(A) = \deg A - 2h^0(A) + 2.$$

The *Clifford index* of the curve C itself is then defined as

$$\mathrm{Cliff}(C) = \min\{\mathrm{Cliff}(A) : A \in \mathrm{Pic}(C),\ h^0(A) \geq 2,\ h^1(A) \geq 2\}.$$

The line bundles A appearing in the definition of $\mathrm{Cliff}(C)$ are said to *contribute to the Clifford index of* C. If, in addition, $\mathrm{Cliff}(C) = \mathrm{Cliff}(A)$ then we say that A *computes* $\mathrm{Cliff}(C)$.

Note that $\mathrm{Cliff}(A) = \mathrm{Cliff}(K_C - A)$, by the Riemann-Roch theorem, so the Clifford index is symmetric in A and $K_C - A$. If A contributes to the Clifford index of C then, since A is non-special, we have $\deg(A) \leq 2g - 2$.

Let us observe that if A is a line bundle on C of degree $d = \deg(A)$ and dimension $r = r(A) = h^0(A) - 1$, then

$$\rho(g, r, d) = g - h^0(A)h^1(A).$$

Hence, by the Brill-Noether theorem 1.37, line bundles which contribute to the Clifford index of a general curve C exist only for $g \geq 4$. When $g = 2$ or 3 we adopt the standard convention that $\mathrm{Cliff}(C) = 0$ when C is hyperelliptic and $\mathrm{Cliff}(C) = 1$ otherwise.

THEOREM 1.42 (Clifford). *If A is a line bundle on C such that $r(A) > 0$ and $r(K_C - A) > 0$, then*

$$\deg(A) - 2r(A) \geq 0.$$

If equality holds, then C is hyperelliptic.

By Clifford's theorem, we have $\mathrm{Cliff}(C) \geq 0$ with equality if and only if C is hyperelliptic. Since $\mathrm{Cliff}(C) \leq \mathrm{gon}(C) - 2$, we immediately have

$$\mathrm{Cliff}(C) \leq \lfloor \frac{g-1}{2} \rfloor,$$

and the equality holds for the general curve in $\mathscr{M}_g$, again by Theorem 1.37.

Gonality and Clifford index are, in fact, very much related: for any curve C of Clifford index c and gonality k, one has [CM91]

$$c + 2 \leq k \leq c + 3,$$

and curves for which $k = c + 3$ are called *exceptional*, and they are conjectured to be extremely rare [ELMS89].

Notice that $k = c + 2$ if and only if the Clifford index of C is computed by a pencil, i.e. if and only if C has *Clifford dimension* 1, where

$$\mathrm{Cliffdim}(C) = \min\{\dim|A| : A \text{ computes } \mathrm{Cliff}(C)\}.$$

Exceptional curves lying on K3 surfaces have been classified by Knutsen [Knu09]. We will recall this result in the next Chapter.

CHAPTER 2

CURVES ON K3 SURFACES

In recent years, many computations of gonality and Clifford index of curves lying on some special classes of surfaces, such as rational elliptic surfaces, abelian surfaces, K3 and Enriques surfaces, have appeared in the literature.

A great deal of attention has been concentrated on K3 surfaces, mainly, though not solely, for reasons arising from the Brill-Noether theory of curves.

All of this started with Lazarfseld's discovery that we can find Brill-Noether general curves on K3 surfaces. This paved the way for several subsequent results, culminating with the arguably most important theorem in the direction of Green's conjecture, namely Voisin's proof that the conjecture holds for generic curves in any genus, [Voi02], [Voi05].

In many of these works, a crucial technique has been the use of some particular vector bundles, called Lazarsfeld-Mukai bundles.

We sketch the definition and main properties of these objects in the following section and then, as an application, we give a self-contained proof of Lazarsfeld's theorem, following the ideas of Pareschi [Par95].

Amongst the other numerous applications of this vector bundle approach, one which is fundamental in our work is the theorem of Green and Lazarsfeld [GL87] on the behaviour of the Clifford index for K3 sections. We describe this result below.

We close the Chapter with some useful observations concerning the computation of the Clifford index of curves on K3 surfaces, and we also recall Knutsen's classification of exceptional curves [Knu09].

§1. VECTOR BUNDLES

In this section we recall some well-known definitions and constructions. We refer to [ACGH10] and [ACG11, Ch.XXI] as a general reference.

§1.1. THE MAP μ_1 AND KERNEL BUNDLES

We denote by C a smooth curve of genus g together with a globally generated line bundle A. In other words, the evaluation map

$$H^0(A) \otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}_A} A \tag{2.1}$$

is surjective, and A gives rise to a morphism $\varphi_A \colon C \to \mathbb{P}$, mapping C to the projective space

$$\mathbb{P} = \mathbb{P}H^0(A)^\vee.$$

REMARK 2.1. *With a little abuse of notation, given a sheaf $\mathcal{F}$ on $\mathbb{P}$ we will denote by $\mathcal{F}|_C$ the sheaf on C obtained by pulling back $\mathcal{F}$ by the morphism $C \to \mathbb{P}$. Moreover, for the sake of simplicity of notation, we sometimes omit the symbol for the tensor product of two line bundles and write AB instead of $A \otimes B$ and A^k in place of $A^{\otimes k}$. For example, we will always write $\omega_C A^{-1}$ in place of $\omega_C \otimes A^{-1}$.*

Recall that on the projective space $\mathbb{P}$, we have the Euler sequence

$$0 \to \mathcal{O}_\mathbb{P} \to H^0(\mathcal{O}_\mathbb{P}(1))^\vee \otimes \mathcal{O}_\mathbb{P}(1) \to T_\mathbb{P} \to 0, \tag{2.2}$$

where $T_\mathbb{P}$ denotes the tangent bundle of $\mathbb{P}$. Pulling back to C, we get

$$0 \to \mathcal{O}_C \to H^0(A)^\vee \otimes A \to T_\mathbb{P}|_C \to 0. \tag{2.3}$$

At the level of global sections, (2.3) induces an exact sequence

$$H^1(\mathcal{O}_C) \to H^0(A)^\vee \otimes H^1(A) \to H^1(T_\mathbb{P}|_C) \to 0, \tag{2.4}$$

and taking the dual of the first map appearing in (2.4), we get the map

$$\mu_0 \colon H^0(A) \otimes H^0(\omega_C A^{-1}) \to H^0(\omega_C),$$

given by multiplication of sections. In particular, $\ker(\mu_0) = H^1(T_\mathbb{P}|_C)^\vee$.

Thus, the first map appearing in (2.4) is just $\mu_0^\vee$ and we identify

$$H^1(T_\mathbb{P}|_C) = \ker(\mu_0)^\vee.$$

On C, we also have the normal bundle sequence

$$0 \to T_C \to T_\mathbb{P}|_C \to \mathcal{N}_{C/\mathbb{P}} \to 0, \tag{2.5}$$

which induces on cohomology an exact sequence

$$H^0(\mathcal{N}_{C/\mathbb{P}}) \xrightarrow{\kappa} H^1(T_C) \xrightarrow{\mu_1^\vee} H^1(T_\mathbb{P}|_C) \to H^1(\mathcal{N}_{C/\mathbb{P}}) \to 0, \tag{2.6}$$

where the first map $\kappa \colon H^0(\mathcal{N}_{C/\mathbb{P}}) \to H^1(T_C)$ is the usual Kodaira-Spencer map associated to $C \to \mathbb{P}$.

Definition. We define the map

$$\mu_1 \colon \ker(\mu_0) \to H^0(\omega_C^2)$$

as the dual of the map $\mu_1^\vee \colon H^1(T_C) \to H^1(T_{\mathbb{P}}|_C) = \ker(\mu_0)^\vee$ which is defined by the sequence (2.6) above. We write $\mu_1 = \mu_1(A)$.

The dual sequence to the Euler sequence (2.2) is

$$0 \to \Omega_{\mathbb{P}} \to H^0(\mathcal{O}_{\mathbb{P}}(1)) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}} \to 0 \tag{2.7}$$

where $\Omega_{\mathbb{P}}$ is the cotangent bundle of $\mathbb{P}$. Pulling back to C we get

$$0 \to \Omega_{\mathbb{P}}|_C \to H^0(A) \otimes A^{-1} \to \mathcal{O}_C \to 0. \tag{2.8}$$

Tensoring this sequence by A we get

$$0 \longrightarrow \Omega_{\mathbb{P}}|_C \otimes A \longrightarrow H^0(A) \otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}_A} A \longrightarrow 0. \tag{2.9}$$

Definition. We define the *kernel bundle* M_A on the curve C as the vector bundle defined as the kernel of the evaluation map (2.1). Thus, M_A is defined by the following short exact sequence

$$0 \longrightarrow M_A \longrightarrow H^0(A) \otimes \mathcal{O}_C \xrightarrow{\mathrm{ev}_A} A \longrightarrow 0 \tag{2.10}$$

LEMMA 2.2. *The kernel bundle M_A is a vector bundle of rank $r(A) = h^0(A) - 1$ on C and determinant $\det M_A \simeq A^{-1}$. Moreover we have an isomorphism*

$$M_A \simeq \Omega_{\mathbb{P}}|_C \otimes A. \tag{2.11}$$

Proof. By (2.10) we see that M_A has rank equal to $\dim H^0(A) - 1 = r(A)$ and $\mathcal{O}_C = \det(H^0(A) \otimes \mathcal{O}_C) \simeq \det(M_A) \otimes \det A$, i.e. $\det(M_A) \simeq A^{-1}$. Comparing (2.10) with (2.9) above we see that $M_A \simeq \Omega_{\mathbb{P}}|_C \otimes A$. $\square$

Let us now show the connection between the kernel bundle M_A and the homomorphisms $\mu_0 = \mu_0(A)$ and $\mu_1 = \mu_1(A)$.

LEMMA 2.3. *We have a natural map*

$$m \colon M_A \otimes \omega_C A^{-1} \to \omega_C^2$$

and an isomorphism

$$H^0(M_A \otimes \omega_C A^{-1}) = \ker(\mu_0)$$

which is such that $H^0(m) = \mu_1$, i.e. the map induced by m on global sections is identified with $\mu_1 \colon \ker(\mu_0) \to H^0(\omega_C^2)$.

Proof. Tensoring the differential map $d\colon \mathcal{O}_C \to \Omega^1_C = \omega_C$ with the evaluation map ev_A, we get $\mathrm{ev}_A \otimes d\colon H^0(A) \otimes \mathcal{O}_C \to A \otimes \omega_C$ and, by restriction to $M_A \subset H^0(A) \otimes \mathcal{O}_C$, a map $M_A \to A \otimes \omega_C$. Twisting by $\omega_C A^{-1}$ yields a map $m\colon M_A \otimes \omega_C A^{-1} \to \omega_C^2$. Now, twisting the sequence (2.10) defining M_A by $\omega_C A^{-1}$ we get the short exact sequence

$$0 \to M_A \otimes \omega_C A^{-1} \to H^0(A) \otimes \omega_C A^{-1} \to \omega_C \to 0$$

which induces on global sections the exact sequence

$$0 \to H^0(M_A \otimes \omega_C A^{-1}) \to H^0(A) \otimes H^0(\omega_C A^{-1}) \xrightarrow{\mu_0} H^0(\omega_C)$$

which shows that $H^0(M_A \otimes \omega_C A^{-1}) = \ker(\mu_0)$. Finally, we show $H^0(m) = \mu_1$ or, dually, $H^1(m^\vee) = \mu_1^\vee$. Indeed, $m^\vee\colon (\omega_C^2)^\vee \to M_A^\vee \otimes A$ and we have $(\omega_C^2)^\vee = \omega_C^{-1} = T_C$, while, by (2.11), $M_A^\vee \otimes A = (\Omega_{\mathbb{P}}|_C \otimes A)^\vee \otimes A = T_{\mathbb{P}}|_C$, and therefore $H^1(m^\vee) = \mu_1^\vee\colon H^1(T_C) \to H^1(T_{\mathbb{P}}|_C)$. □

§1.2. The map μ_1^S and Lazarsfeld-Mukai bundles

We keep the same notations as before: C is a smooth curve and A a globally generated line bundle on C, which determines the maps

$$\mu_1\colon \ker(\mu_0) \to H^0(\omega_C^2)$$

and

$$\mu_1^\vee\colon H^1(T_C) \to H^1(T_{\mathbb{P}}|_C)$$

defined above. We now further assume that C lies on a K3 surface S. We then have a normal bundle sequence

$$0 \to T_C \to T_S|_C \to \mathcal{N}_{C/S} \to 0, \tag{2.12}$$

whose first coboundary map on cohomology is the Kodaira-Spencer map associated to $C \subset S$, which we denote by

$$\kappa_S\colon H^0(\mathcal{N}_{C/S}) \to H^1(T_C). \tag{2.13}$$

Definition. We define μ_1^S as the composition $(\kappa_S)^\vee \circ \mu_1$,

$$\begin{tikzcd} \ker(\mu_0) \arrow[r, "\mu_1"] \arrow[rr, bend right, "\mu_1^S"'] & H^0(\omega_C^2) \arrow[r, "\kappa_S^\vee"] & H^1(\mathcal{O}_C) \end{tikzcd}$$

and thus its dual map is the composition

$$
\begin{tikzcd}
H^0(\mathcal{N}_{C/S}) \arrow[r, "\kappa_S"] \arrow[rr, bend right, "(\mu_1^S)^\vee"'] & H^1(T_C) \arrow[r, "\mu_1^\vee"] & H^1(T_{\mathbb{P}}|_C)
\end{tikzcd}
$$

Given the line bundle A on C, we can consider the sheaf j_*A on the K3 surface S, where j is the inclusion of C in S.

REMARK 2.4. *With the purpose of keeping the notation simple, we will just write A in place of j_*A. It will be clear from the context whether we regard A as a sheaf on C or on S.*

Since A is globally generated, we have a surjective evaluation map

$$H^0(A) \otimes \mathcal{O}_S \xrightarrow{\mathrm{ev}_A^S} A.$$

Definition. The *Lazarsfeld-Mukai bundle* F_A on S, is defined by

$$0 \to F_A \to H^0(A) \otimes \mathcal{O}_S \xrightarrow{\mathrm{ev}_A^S} A \to 0. \tag{2.14}$$

Its dual $F_A^\vee = \mathcal{H}om(F_A, \mathcal{O}_S)$, will also be called a Lazarsfeld-Mukai bundle.

LEMMA 2.5. *Let D be a Cartier divisor on a smooth variety X. Then,*

$$\mathcal{E}xt^1(\mathcal{O}_D, \mathcal{O}_X) \simeq \mathcal{O}_D(D).$$

Proof. We have $\mathcal{H}om(\mathcal{O}_D, \mathcal{O}_X) = 0$, since $\mathcal{O}_D$ is torsion as a sheaf on X. Dualizing the exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, we thus get $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{E}xt^1(\mathcal{O}_D, \mathcal{O}_X) \to 0$. Comparing this exact sequence with $0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathcal{O}_D(D) \to 0$, we get $\mathcal{E}xt^1(\mathcal{O}_D, \mathcal{O}_X) \simeq \mathcal{O}_D(D)$. $\square$

LEMMA 2.6. *The Lazarsfeld-Mukai bundle F_A is a vector bundle on S of rank equal to $\operatorname{rk} F_A = h^0(A)$ and its dual $E_A := F_A^\vee$, sits in a short exact sequence*

$$0 \to H^0(A)^\vee \otimes \mathcal{O}_S \to E_A \to \omega_C A^{-1} \to 0. \tag{2.15}$$

Proof. Since A is torsion (as a sheaf on S), by (2.14) we immediately get that $\operatorname{rk} F_A = \dim H^0(A)$ and, moreover, $\mathcal{H}om(A, \mathcal{O}_S) = 0$. Thus, applying the functor $\mathcal{H}om(-, \mathcal{O}_S)$ to the sequence (2.14), we get an exact sequence $0 \to H^0(A)^\vee \otimes \mathcal{O}_S \to E_A \to \mathcal{E}xt^1(A, \mathcal{O}_S) \to 0$. Finally, we have that $\mathscr{E}xt^1(A, \mathscr{O}_S) \simeq \mathscr{E}xt^1(\mathscr{O}_C, \mathscr{O}_S) \otimes A^{-1}$, cf. [Har77, Prop.6.7] and, by the Lemma above, $\mathscr{E}xt^1(\mathscr{O}_C, \mathscr{O}_S) \simeq \mathscr{O}_C(C)$. Since S is a K3 surface, by adjunction $\mathscr{O}_C(C) \simeq \omega_C$. □

The basic properties of Lazarsfeld-Mukai bundles on a K3 surface S are as follows.

LEMMA 2.7. *Let F_A and E_A be Lazarsfeld-Mukai bundles defined by a pair (C, A) by the exact sequences (2.14) and (2.15) respectively. We have*

(i) $c_1(E_A) = [C]$ *and* $c_2(E_A) = \deg A$.

(ii) E_A *is globally generated off the base locus of* $\omega_C A^{-1}$.

(iii) $h^0(E) = h^0(A) + h^0(\omega_C A^{-1})$.

(iv) $h^1(E_A) = h^1(F_A) = 0$ *and* $h^2(E_A) = h^0(F_A) = 0$.

(v) $h^0(F_A \otimes E_A) = h^0(F_A \otimes \omega_C A^{-1})$ *and* $\chi(F_A \otimes E_A) = 2 - 2\rho(g, r, d)$, *where* $g = g(C), d = \deg A, r = r(A)$ *and* $\rho(g, r, d)$ *is the Brill-Noether number.*

Proof. Considering $\omega_C A^{-1}$ as a sheaf on S, we have $c_1(\omega_C A^{-1}) = [C]$ and $c_2(\omega_C A^{-1}) = [C]^2 - \deg(\omega_C A^{-1}) = \deg A$, cf. [Fri98, p.30, Lem. 1]. By Whitney's formula applied to the short exact sequence (2.15) we get

$$\begin{aligned} 1 + c_1(E_A) + c_2(E_A) &= (1) \cdot (1 + c_1(\omega_C A^{-1}) + c_2(\omega_C A^{-1})) \\ &= 1 + [C] + \deg A. \end{aligned}$$

Equating the terms on both sides yields the statement (i). By (2.15) we get a surjection $H^0(E_A) \to H^0(\omega_C A^{-1})$. Thus, every section of $\omega_C A^{-1}$ lifts to a section of E_A, proving (ii). The statements in (iii) and (iv) are also immediate consequences of (2.15), using that $h^1(\mathscr{O}_S) = 0$ on a K3 surface. Setting $r := h^0(A) - 1 = \operatorname{rk} E_A - 1$, the Riemann-Roch formula gives

$$\begin{aligned} \chi(F_A \otimes E_A) &= rc_1(E)^2 - 2(r+1)c_2(E) + 2(r+1)^2 \\ &= r(2g-2) - 2(r+1)d + 2(r+1)^2 \\ &= 2 - 2g + 2(r+1)(r+g-d) \\ &= 2 - 2\rho(g, r, d). \end{aligned}$$

Finally, tensoring (2.15) by F_A and passing to the exact sequence in cohomology we get $h^0(F_A \otimes E_A) = h^0(F_A \otimes \omega_C A^{-1})$. □

Recall that a vector bundle E is called *simple* if the only endomorphisms of E are the constants, i.e. if $h^0(E^\vee \otimes E) = 1$.

REMARK 2.8. *Part (v) of Lemma (2.7) yields, more explicitly,*

$$2h^0(E_A^\vee \otimes E_A) - h^1(E_A^\vee \otimes E_A) = 2 - 2\rho(g, r, d).$$

Thus, if the Lazarsfeld-Mukai bundle E_A associated to a complete linear series A of type g^r_d on C is simple, then $\rho(g, r, d) \geq 0$. To put it the other way around, any g^r_d on a curve lying on a K3 surface, having negative Brill-Noether number, gives rise to a non-simple Lazarsfeld-Mukai bundle.

Given a curve C and a globally generated line bundle A on C, we have defined the kernel bundle M_A on C and explained how it encodes information about the homomorphisms $\mu_0 = \mu_0(A)$ and $\mu_1 = \mu_1(A)$. When C lies on a K3 surface S, we have the Lazarsfeld-Mukai bundle F_A on S and the homomorphism $\mu_1^S = \mu_1^S(A)$ defined above. We now come to explain the connection between F_A with M_A and the map μ_1^S.

LEMMA 2.9. *There is a natural short exact sequence*

$$0 \to \mathcal{O}_C \to F_A \otimes \omega_C A^{-1} \to M_A \otimes \omega_C A^{-1} \to 0 \tag{2.16}$$

whose first coboundary map $H^0(M_A \otimes \omega_C A^{-1}) = \ker(\mu_0) \to H^1(\mathcal{O}_C)$ is identified with the homomorphism μ_1^S. In particular, if F_A is simple, then μ_1^S is injective.

Proof. Restricting F_A to the curve C and comparing the short exact sequences (2.10) and (2.14) defining M_A and F_A respectively, we get a surjective map $F_A \otimes \mathcal{O}_C \to M_A$. Since $\operatorname{rk} F_A - \operatorname{rk} M_A = 1$, the kernel of this surjection is a line bundle $\mathscr{L}$ on C. Since $\det F_A = \mathcal{O}_S(-C)$ and $\det M_A = A^{-1}$, we get

$$\mathscr{L} \simeq \det(F_A|_C) \otimes \det(M_A)^{-1} \simeq \mathcal{O}_C(-C) \otimes A \simeq \omega_C^{-1} A$$

Tensoring $0 \to \mathscr{L} \to F_A \otimes \mathcal{O}_C \to M_A \to 0$ with $\mathscr{L}^{-1} = \omega_C A^{-1}$ yields (2.16).

Now, tensoring the differential map $d\colon \mathcal{O}_S \to \Omega^1_S$ with the evaluation map ev^S_A, we get $ev^S_A \otimes d\colon H^0(A) \otimes \mathcal{O}_S \to A \otimes \omega^1_S$ and, by restriction to $F_A \subset H^0(A) \otimes \mathcal{O}_S$, a map $F_A \to A \otimes \omega_C$. Twisting by $\omega_C A^{-1}$, we get a map $f\colon F_A \otimes \omega_C A^{-1} \to \Omega^1_S \otimes \omega_C$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, "\mathrm{id}"] & F_A \otimes \omega_C A^{-1} \arrow[r] \arrow[d, "f"] & M_A \otimes \omega_C A^{-1} \arrow[r] \arrow[d, "m"] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & \Omega^1_S \otimes \omega_C \arrow[r] & \omega_C^2 \arrow[r] & 0
\end{tikzcd}$$

where the second row is the short exact sequence obtained by the conormal sequence $0 \to \mathcal{N}^\vee_{C/S} \to \Omega^1_S|_C \to \omega_C \to 0$ twisted by $\omega_C \simeq \mathcal{N}_{C/S}$, and the map m, for which $H^0(m) = \mu_1$, was described in Lemma 2.3. Applying the functor H^0 of global sections to the right-hand side column of the diagram above, we get a commutative diagram

$$\begin{tikzcd}
\ker(\mu_0) \arrow[d, "\mu_1"'] \arrow[dr, "\mu_1^S"] & \\
& H^1(\mathcal{O}_C) \\
H^0(\omega_C^2) \arrow[ur, "\kappa_S^\vee"'] &
\end{tikzcd}$$

Therefore, passing to the cohomology of (2.16) yields the exact sequence

$$0 \to H^0(\mathcal{O}_C) \to H^0(F_A \otimes \omega_C A^{-1}) \to \ker(\mu_0) \xrightarrow{\mu_1^S} H^1(\mathcal{O}_C).$$

Then, $\dim \ker \mu_1^S = h^0(F_A \otimes \omega_C A^{-1}) - 1$ and so, since $h^0(F_A \otimes \omega_C A^{-1}) = h^0(F_A^\vee \otimes F_A)$ by Lemma 2.7(v), if F_A is simple, then μ_1^S is injective. □

§2. Curves on general K3 surfaces are Brill-Noether general

In this section we give a proof of Lazarsfeld's celebrated result [Laz86].

THEOREM 2.10 (Lazarsfeld). *Let C_0 be a smooth irreducible curve on a K3 surface S. If every member of the linear system $|C_0|$ is reduced and irreducible, then the generic member $C \in |C_0|$ is such that for any line bundle A on C the multiplication map $\mu_0\colon H^0(A) \otimes H^0(\omega_C A^{-1}) \to H^0(\omega_C)$ is injective.*

In particular, the hypothesis of the theorem are satisfied on a K3 surface S with Picard number one such that $\mathrm{Pic}(S) = \mathbb{Z} \cdot [C]$.

We follow Pareschi's argument [Par95]. Let us start with a very simple, yet useful, remark:

REMARK 2.11. *Let Δ be the divisor of base-points of A. If $\mu_0(A(-\Delta))$ is injective, then also $\mu_0(A)$ is injective.*

Proof. We assume $\mu_0(A)$ is not injective and show that $\mu_0(A(-\Delta))$ is not injective. Denote by $i : H^0(\omega_C(-\Delta)) \to H^0(\omega_C)$ the natural inclusion and by $\nu : H^0(A(-\Delta)) \otimes H^0(\omega_C A^{-1}) \to H^0(\omega_C(-\Delta))$ the multiplication map. Via the natural identification $H^0(A) = H^0(A(-\Delta))$, we have $\mu_0(A) = i \circ \nu$, not injective by assumption. Denoting by

$$j : H^0(A(-\Delta)) \otimes H^0(\omega_C A^{-1}) \to H^0(A(-\Delta)) \otimes H^0(\omega_C A^{-1}(\Delta))$$

the natural inclusion, we have $\mu_0(A(-\Delta)) \circ j = i \circ \nu$. Since $i \circ \nu$ is not injective and j is injective, we get that $\mu_0(A(-\Delta))$ is not injective. □

Therefore, in order to prove Theorem 2.10, we only have to check the injectivity of $\mu_0(A)$ with A globally generated. Given A, we denote its Brill-Noether number by $\rho(A) = \rho(g, r, d)$, where $g = g(C), d = \deg(A)$ and $r = r(A)$. Recall from the previous section that we defined a map $\mu_1^S(A)\colon \ker(\mu_0(A)) \to H^1(\mathcal{O}_C)$. The first crucial step is the following

PROPOSITION 2.12. *Let C be a smooth irreducible curve on a K3 surface S such that every member of the linear system $|C|$ is reduced and irreducible. Then, for any globally generated line bundle A on C, the Lazarsfeld-Mukai bundle F_A defined by (2.14) is simple. In particular, $\rho(A) \geq 0$ and the map $\mu_1^S(A)$ is injective.*

Proof. We prove that the dual $E_A = F_A^\vee$ is simple. By contradiction, assume there exists a non-constant endomorphism $\phi \in \mathrm{End}(E_A) = H^0(F_A \otimes E_A)$. We can assume that $\phi\colon E_A \to E_A$ drops rank everywhere on S, i.e. the constant function

$$\det(\phi) \in H^0(\det(F_A \otimes E_A)) = H^0(\mathcal{O}_S)$$

is identically zero: if not, just replace ϕ with the endomorphism $\phi - \lambda\,\mathrm{id}$, where λ is an eigenvalue of ϕ at some given point $x \in S$, so that the constant function $\det(\phi - \lambda\,\mathrm{id})$ must vanish at x, whence everywhere on S. Therefore, the image $\mathrm{Im}\,\phi$ is a proper subsheaf of E_A and we have a short exact sequence

$$0 \to M \to E_A \to N_0 \to 0,$$

where $M = \mathrm{Im}(\phi)$ and $N_0 = \mathrm{coker}(\phi)$. Let $N = N_0/T(N_0)$, where $T(N_0)$ is the torsion subsheaf of N_0. Then,

$$[C] = c_1(E_A) = c_1(M) + c_1(N) + c_1(T(N_0)),$$

inside $H^2(S, \mathbb{Z})$. Now, $c_1(T(N_0))$ is represented by the (possibly empty) support of $T(N_0)$ in codimension 1 (see e.g. [Fri98, p.29]). We want to prove that both $c_1(M)$ and $c_1(N)$ are represented by effective curves on S, thus giving a decomposition of [C] which contradicts the hypothesis that every member of $|C|$ is reduced and irreducible. We observe that M and N, being quotients of the globally generated, off a finite set, bundle E_A, are also globally generated off a finite set. Also, $M^\vee$ and $N^\vee$ are sub-sheaves of $F_A = E_A^\vee$. Thus, if $M = \mathscr{O}_S^{\oplus r}$ was a trivial bundle, then $0 \to \mathscr{O}_S^{\oplus r} \to F_A$ would provide F_A with global sections, but $h^0(F_A) = 0$ by Lemma (2.7). Hence M and N are both non-trivial and globally generated off a finite set. We conclude by applying the following lemma. $\square$

LEMMA 2.13. *Let M be a torsion-free sheaf of rank r on a smooth projective surface S. If M is globally generated off a finite set, then the class $c_1(M)$ is represented by an effective curve. Moreover, $c_1(M) = 0$ if and only if $M = \mathscr{O}_S^{\oplus r}$.*

Proof. Up to replacing M with its double dual, we may assume M is locally free. Indeed, $M^{\vee\vee}$ is locally free and the inclusion $M \to M^{\vee\vee}$ is an isomorphism off a finite set (cf. [OSS80, Ch.II, 1.1]), whence $c_1(M) = c_1(M^{\vee\vee})$. We then assume that M is a non-trivial vector bundle. Choose $r = \operatorname{rk} M$ sections $s_i \colon \mathscr{O}_S^{\oplus r} \to M$ in $H^0(M)$ which generate M off a finite set. By choosing the s_i generically in $H^0(M)$, their *degeneracy locus*, defined locally by $\{x \in S\colon s_1(x) \wedge \cdots \wedge s_r(x) = 0\}$, is of codimension one. Thus, $s = s_1 \wedge \cdots \wedge s_r \colon \mathscr{O}_S \to \bigwedge^r(M) = \det M$ is a global section of $\det M$, not identically zero (hence $h^0(\det M) > 0$, and so $c_1(M)$ is effective), and vanishing at some point, since the s_i vanish at some point. Therefore, $\det M$ is non-trivial, i.e. $c_1(M) \neq 0$. $\square$

Let us introduce some notation. Given a curve C of genus g, we let $V_d^r(C)$ denote the variety of complete g_d^r's on C, parametrizing line bundles of degree d carrying exactly $r+1$ independent sections:

$$V_d^r(C) = \{A \in \operatorname{Pic}^d(C) : r(A) = r\}.$$

If $|L|$ is a linear system of curves on a K3 surface S, and $U \subset |L|$ a subset containing smooth curves, we let

$$\mathscr{V}_d^r(U) = \{(C, A) : C \in U,\ A \in V_d^r(C)\}.$$

Each pair (C, A) in $\mathscr{V}_d^r(U)$ provides us with a map from C to some projective space of dimension r, but we want to be able to consider all curves in U mapping to a *fixed* $\mathbb{P}^r$. So, we set

$$\widetilde{\mathscr{V}}_d^r(U) = \{(C, A, \sigma) : C \in U,\ A \in V_d^r(C),\ \sigma \text{ is a basis of } H^0(A).\}$$

Letting $\mathscr{H}$ be the Hilbert scheme of curves of degree d and genus g in $\mathbb{P}^r$, we now have a natural map $\widetilde{\mathscr{V}}_d^r(U) \to \mathscr{H}$. It is well known that the tangent space of $\mathscr{H}$ at the point corresponding to C is canonically isomorphic to $H^0(\mathscr{N}_{C/\mathbb{P}^r})$. Thus, for any triple (C, A, σ) in $\widetilde{\mathscr{V}}^r_d(U)$, we have a natural map

$$T_{(C,A,\sigma)}\widetilde{\mathscr{V}}^r_d(U) \to H^0(\mathscr{N}_{C/\mathbb{P}^r}).$$

We can now prove Lazarsfeld's theorem.

Proof of Theorem 2.10. Let $U \subset |C_0|$ be the open subset of smooth curves in $|C_0|$ and denote by $\pi\colon \mathscr{V}^r_d(U) \to U$ the projection. Fix a pair (C, A) with $C \in U$ generic and A a globally generated line bundle on C. We let $\mu_0 = \mu_0(A)$ and $\mu_1 = \mu_1(A)$. With the notations introduced above, we have the commutative diagram

$$\begin{tikzcd}
T_{(C,A,\sigma)}\widetilde{\mathscr{V}}^r_d(U) \arrow[rr] \arrow[d] & & H^0(\mathscr{N}_{C/\mathbb{P}^r}) \arrow[d, "\kappa"] \\
T_{(C,A)}\mathscr{V}^r_d(U) \arrow[r, "d\pi"] & H^0(\mathscr{N}_{C/S}) \arrow[r, "\kappa_S"] \arrow[dr, "(\mu_1^S)^\vee"'] & H^1(T_C) \arrow[d, "\mu_1^\vee"] \\
& & H^1(T_{\mathbb{P}^r}|_C)
\end{tikzcd}$$

where the vertical column on the right-hand side is the exact sequence (2.6), which defines the map μ_1. In particular, $\mathrm{Im}(\kappa) = \ker(\mu_1^\vee)$ by exactness. Let us now follow the diagram from the top left corner: since the map $T_{(C,A,\sigma)}\widetilde{\mathscr{V}}^r_d(U) \to T_{(C,A)}\mathscr{V}^r_d(U)$ is surjective, we have $\mathrm{Im}(\kappa_S \circ d\pi) \subset \mathrm{Im}(\kappa) = \ker(\mu_1^\vee)$. In other words, $(\mu_1^S)^\vee \circ d\pi = 0$. But $d\pi$ is surjective by construction, hence $(\mu_1^S)^\vee = 0$. However, we know that $(\mu_1^S)^\vee$ is surjective by Proposition 2.12. We conclude that $H^1(T_{\mathbb{P}^r}|_C) = 0$, whence $\ker(\mu_0) = H^1(T_{\mathbb{P}^r}|_C)^\vee = 0$, which proves the statement of Theorem 2.10. □

§3. Constancy of the Clifford index in a linear system of curves on a K3 surface

In this section we illustrate the result of Green and Lazarsfeld [GL87].

THEOREM 2.14 (Green-Lazarsfeld). *Let S be a K3 surface and C a smooth irreducible curve on S, of genus $g \geq 2$. Then $\mathrm{Cliff}(C) = \mathrm{Cliff}(C')$ for any smooth curve C' in $|C|$. Moreover, if $\mathrm{Cliff}(C) < \lfloor \frac{g-1}{2} \rfloor$, then there exists a line bundle M on S such that $\mathrm{Cliff}(C') = \mathrm{Cliff}(M \otimes \mathscr{O}_C)$, for any smooth curve C' in $|C|$.*

We now want to give an idea of the main arguments used in the proof. The original proof in [GL87] being long and very technical, we work under some simplifying assumptions along the way.

LEMMA 2.15. *Let S be a K3 surface and E a vector bundle of rank r, globally generated off a finite set, with $h^2(E) = 0$. If $c_1(E)^2 > 0$, then $h^0(\det E) \geq h^0(E)$.*

Proof. We prove the Lemma under the simplifying assumption that E is globally generated. Choose a generic subspace $V \subset H^0(E)$ of sections of dimension $\dim V = \operatorname{rk} E$, so that the evaluation map $e \colon V \otimes \mathcal{O}_S \to E$ drops rank along a curve C and the cokernel of e is a line bundle A on C, yielding a short exact sequence

$$0 \to V \otimes \mathcal{O}_S \to E \to A \to 0. \tag{2.17}$$

By exactly the same arguments as in the proof of Lemma 2.7, we obtain $\det E = \mathcal{O}_S(C)$ and $h^0(E) = h^0(A) + \dim V$, and the dual sequence is

$$0 \to E^\vee \to V^\vee \otimes \mathcal{O}_S \to \omega_C A^{-1} \to 0. \tag{2.18}$$

Thus, $h^0(\omega_C A^{-1}) = \dim V + h^1(E^\vee)$ and $c_1(E)^2 = C^2 > 0$. Moreover, $\det E = \mathcal{O}_S(C)$ is globally generated, since E is. Hence, $\mathcal{O}_S(C)$ is basepoint free and, as V was chosen generically, C is a smooth irreducible curve of genus $g = 1 + C^2/2$, by Bertini's theorem. By (2.17) and (2.18) it follows that A and $\omega_C A^{-1}$ are both globally generated, and thus Clifford's theorem 1.42 gives $\deg A \geq 2h^0(A) - 2$, or, equivalently $h^0(A) + h^0(\omega_C A^{-1}) \leq 1 + g$, by the Riemann-Roch theorem. Since $h^0(\mathcal{O}_S(C)) = 1 + g$, we get

$$h^0(E) = h^0(A) + \dim V \leq h^0(A) + h^0(\omega_C A^{-1}) \leq h^0(\mathcal{O}_S(C)). \qquad \square$$

The main idea for the proof of Theorem 2.14 is a reduction argument related to the following

Definition. Let E_A be a Lazarsfeld-Mukai bundle on S. A vector bundle E, of rank ≥ 2, is called a *reduction* of E_A if

(i) There is a map $E_A \to E$ surjective off a finite set.

(ii) $h^0(E) \geq h^0(E_A)$ and $h^i(E) = 0$ for $i = 1, 2$.

(iii) $c_1(E) = c_1(E_A)$ and $2 \operatorname{rk} E - c_2(E) \geq 2 \operatorname{rk} E_A - c_2(E_A)$.

Sketch of the proof of Theorem 2.14. We explain how the arguments apply under the simplifying assumption that S contains no elliptic nor smooth rational curves. Among the smooth curves in the linear system $|C|$, we fix one, say C_0, of minimal Clifford index. We assume $\mathrm{Cliff}(C_0) < \lfloor \frac{g-1}{2} \rfloor$ and we show that there exists a line bundle M on S such that for any smooth curve C in $|C_0|$ the restriction $M \otimes \mathcal{O}_C$ contributes to the Clifford index of C and $\mathrm{Cliff}(M \otimes \mathcal{O}_C) \leq \mathrm{Cliff}(C_0)$. Let A be a line bundle which computes the Clifford index of C_0, and let E_A be the associated Lazarsfeld-Mukai vector bundle. The key step of the proof is given by the following

CLAIM. Let E be a reduction of E_A of *minimal rank*. Then there exists a line bundle M with $h^0(M) \geq 2$ and a non-zero map $M \to E$.

We prove the Claim in the case $\mathrm{rk}\, E = 2$ only. Let us show that in this case E is not simple. In fact, by Serre duality and Riemann-Roch

$$\begin{aligned}\chi(E^\vee \otimes E) &= 2h^0(E^\vee \otimes E) - h^1(E^\vee \otimes E)\\ &= c_1(E)^2 - 4c_2(E) + 8\end{aligned}$$

while, by (iii) and the assumption that $\mathrm{Cliff}(A) < \lfloor \frac{g-1}{2} \rfloor$, we have

$$\begin{aligned}c_1(E)^2 - 4c_2(E) + 8 &\geq c_1(E_A)^2 + 4(2\,\mathrm{rk}\, E_A - c_2(E_A) - 2)\\ &= 2g - 2 - 4\,\mathrm{Cliff}(A)\\ &\geq 4.\end{aligned}$$

Thus, $h^0(E^\vee \otimes E) \geq 2$, i.e. E is non-simple, and by the same arguments as in the proof of Proposition 2.12 we get a short exact sequence

$$0 \to M \to E \to N \to 0$$

(where we assume for simplicity that N is locally free) with M and $\det N$ effective line bundles, $h^2(N) = 0$ and $\mathcal{O}_S(C_0) = M \otimes \det N$. In particular, thanks to our assumptions on $\mathrm{Pic}(S)$, we have $M^2 > 0$ and $c_1(N)^2 > 0$, whence $h^0(M) \geq 2$, by Riemann-Roch. This proves the Claim.

By the same argument, $h^0(\det N) \geq 2$, and so $\det N = \mathcal{O}_S(C_0) \otimes M^{-1}$ is effective and non-trivial, which shows $h^0(M \otimes \mathcal{O}_S(-C_0)) = 0$. Therefore, for any smooth curve C in $|C_0|$, considering the long exact sequence in cohomology induced by

$$0 \to M \otimes \mathcal{O}_S(-C) \to M \to M \otimes \mathcal{O}_C \to 0$$

we get $h^0(M \otimes \mathcal{O}_C) \geq h^0(M) \geq 2$ and, since $h^2(M) = 0$,

$$h^1(M \otimes \mathcal{O}_C) \geq h^2(M \otimes \mathcal{O}_S(-C)) = h^2((\det N)^\vee) = h^0(\det N) \geq 2.$$

This shows that $M \otimes \mathcal{O}_C$ contributes to the Clifford index of C.

Next, we notice that N, being a quotient of E, is also globally generated away from finitely many points, whence we can apply Lemma 2.15, which yields $h^0(\det N) \geq h^0(N)$.

Recall by Lemma 2.7 that one has

$$h^0(E_A) = h^0(A) + h^0(\omega_{C_0} A^{-1}) = g(C_0) + 1 - \mathrm{Cliff}(A),$$

where the second equality follows by Riemann-Roch on C.

Putting everything together, we get

$$\begin{aligned} g(C_0) + 1 - \mathrm{Cliff}(A) = h^0(E_A) \le h^0(E) &\le h^0(M) + h^0(N) \\ &\le h^0(M) + h^0(\det N) \\ &\le h^0(M \otimes \mathcal{O}_C) + h^1(M \otimes \mathcal{O}_C) \\ &= h^0(M \otimes \mathcal{O}_C) + h^0(\omega_C \otimes M^{-1}) \\ &= g(C) + 1 - \mathrm{Cliff}(M \otimes \mathcal{O}_C), \end{aligned}$$

which shows $\mathrm{Cliff}(M \otimes \mathcal{O}_C) \le \mathrm{Cliff}(A)$, as required. □

§4. Computing the Clifford index

Suppose that C is a smooth irreducible curve of genus $g \ge 2$ lying on a K3 surface S and we want to compute its Clifford index $c = \mathrm{Cliff}(C)$. The main tool is of course Theorem 2.14: either we have

$$c = \lfloor \frac{g-1}{2} \rfloor = \lfloor C^2/4 \rfloor,$$

or there exists a line bundle $M = \mathcal{O}_S(D)$, where D is an effective divisor on S whose restriction to C computes its Clifford index,

$$c = \mathrm{Cliff}(D|_C) = D \cdot C - 2h^0(\mathcal{O}_C(D)) + 2. \tag{2.19}$$

Assume we are in the latter case. By the proof of Theorem 2.14, one has $h^0(D) \ge 2$ and $h^0(C - D) \ge 2$. Also, by definition of $\mathrm{Cliff}(C)$, we have

$$h^0(\mathcal{O}_C(D)) \ge 2 \ \text{ and } \ h^0(K_C - D|_C) = h^0(\mathcal{O}_C(C - D)) \ge 2$$

In particular, since $\mathcal{O}_C(D)$ and $\mathcal{O}_C(C-D)$ move at least in a pencil, we have $\deg_C(D) \ge 2$ and $\deg_C(C - D) \ge 2$. On S, these inequalities read

$$2 \le D \cdot C \le C^2 - 2. \tag{2.20}$$

Also, both D and $C - D$ being non-trivial effective divisors on S, we have $h^0(-D) = h^0(D - C) = 0$. Thus, by the short exact sequence

$$0 \to \mathcal{O}_S(D - C) \to \mathcal{O}_S(D) \to \mathcal{O}_C(D) \to 0,$$

we get the following exact sequences in cohomology:

$$0 \to H^0(D) \to H^0(D|_C) \to H^1(D-C)$$

and

$$H^1(D) \to H^1(D|_C) \to H^2(D-C) \to 0.$$

Since $h^1(D-C) = h^1(C-D)$ and $h^2(D-C) = h^0(C-D)$, by Serre duality, this yields the following:

REMARK 2.16. *With C and D as above, we have*

(i) $h^0(D|_C) \geq h^0(D)$, *with an equality if* $h^1(C-D) = 0$.

(ii) $h^1(D|_C) \geq h^0(C-D)$, *with an equality if* $h^1(D) = 0$.

When the geometry of S is particularly simple, these remarks can be already enough to compute the Clifford indices of the curves in S.

EXAMPLE 2.17. *Suppose* $\mathrm{Pic}(S) = \mathbb{Z} \cdot [C_0]$ *and we want to compute the Clifford index of the curves on* S. *First, we notice that for numerical reasons there cannot exist a divisor* D *satisfying the inequalities (2.20) for* $C = C_0$, *whence* C_0 *has maximal Clifford index (as predicted also by Lazarsfeld's Theorem 2.10, since the general member of* $|C_0|$ *is Brill-Noether general). Take now a smooth curve* C *in* $|mC_0|$, *for some* $m \geq 2$ *and let* $D \in |C_0|$. *Then,*

$$h^0(D|_C) \geq h^0(D) \geq 2$$

and

$$h^1(D|_C) \geq h^0((m-1)C_0) \geq 2.$$

Hence, $D|_C$ *contributes to the Clifford index of* C. *By direct computation, one checks* $\mathrm{Cliff}(D|_C) < \lfloor \frac{C^2}{4} \rfloor$, *i.e.* C *has non-maximal Clifford index. Again, by direct computation one can check that for any other effective divisor* D' *on* S *such that* $C - D'$ *is effective, we have* $\mathrm{Cliff}(D|_C) \leq \mathrm{Cliff}(D'|_C)$, *so we conclude that* $D|_C$ *computes the Clifford index of* C.

On the other hand, already when the K3 surface S has Picard number two, the computation of the Clifford index of $C \subset S$ can be more subtle. Neither the statement of Green-Lazarsfeld's Theorem 2.14, nor its proof, give any indication for how to find the predicted line bundle $M = \mathcal{O}_S(D)$ (where D is as in the above discussion). Indeed, it will often be the case that $\mathcal{O}_S(D)$ is not unique:

EXAMPLE 2.18. *Let S be a K3 surface with Picard group generated by two classes H and E such that*

$$H^2 = 2,\ E^2 = 0 \text{ and } H \cdot E = 1.$$

Let C be a smooth curve of genus 3 in the linear system $|2H|$. One can check that $h^0(H) = 3$ and $h^1(H) = 0$, while $h^0(C - E) \geq 2$ and $h^1(C - E) = 0$. One finds that both $H|_C$ and $E|_C$ contribute to the Clifford index of C and $\mathrm{Cliff}(H|_C) = \mathrm{Cliff}(E|_C) = 0$. Therefore, both line bundles $\mathcal{O}_S(H)$ and $\mathcal{O}_S(E)$ play the role of M in the statement of Theorem 2.14.

As it turns out, though, we can choose D to satisfy certain properties. In [Knu01, Lem. 8.3], building on [Mar89], Knutsen proves the following.

PROPOSITION 2.19. *Let C be a smooth irreducible curve of genus $g \geq 2$ and Clifford index $c < \lfloor \frac{g-1}{2} \rfloor$ on a K3 surface S. Then, there exists a smooth irreducible curve D, with $h^1(D) = h^1(C - D) = 0$, such that the restriction of D to C computes the Clifford index of C, and satisfying*

(i) $D^2 \leq c + 2$ *and* $2D^2 \leq D \cdot C$,

(ii) $\mathrm{Cliff}(C) = D \cdot C - D^2 - 2$,

with either two of the inequalities in (i) being an equality if and only if $C \sim 2D$.

Proof. It follows by the proof of Green and Lazarsfeld's Theorem [GL87], as worked out by Martens in [Mar89, Lem. 2.2] that we can choose a line bundle $\mathcal{O}_S(D)$, whose restriction to C compute its Clifford index, such that $h^1(D) = 0$ and $h^1(C - D) = 0$ and with either $|D|$ or $|C - D|$ basepoint free. Choosing $|D|$ to be basepoint free, we get that the general member of $|D|$ is a smooth irreducible curve, by Theorem 1.31. By (2.19) and Remark 2.16 we get (ii). Now we prove (i). Since $\mathrm{Cliff}(D|_C) = \mathrm{Cliff}((C - D)|_C)$, up to replacing D with $C - D$, we can assume $D \cdot C \leq (C - D) \cdot C$, or, equivalently, $2D \cdot C \leq C^2$. Since $c = D \cdot C - D^2 - 2$ by (ii), the latter inequality is equivalent to $D^2 \leq c + 2$. Finally, multiplying both sides of $2D \cdot C \leq C^2$ by D^2 and combining with the Hodge index theorem 1.10, we get $2D^2 \leq D \cdot C$, with equality if and only if $C \sim 2D$. □

Motivated by the discussion above, we adopt the following notations from [JK04]. Let

$$\mathcal{A}(C) := \{D \in \mathrm{Div}(S) :\ h^0(D) \geq 2,\ h^0(C - D) \geq 2\}.$$

Observe that $\mathcal{A}(C)$ is non-empty if and only if the curve C admits an effective decomposition $C \sim D + D'$ into two moving classes. In such a case, we have $D \cdot D' \geq 2$, since the curve C is 2-connected, cf. Proposition 1.36. We let

$$\mu_C := \min_{D \in \mathscr{A}(C)} \{D \cdot (C - D) - 2\} \geq 0$$

and set $\mu_C = +\infty$ if $\mathscr{A}(C) = \varnothing$. We denote by $\mathscr{A}^0(C)$ the set of divisors in $\mathscr{A}(C)$ achieving this minimum,

$$\mathscr{A}^0(C) := \{D \in \mathscr{A}(C) : \ D \cdot (C - D) - 2 = \mu_C\}.$$

By the preceding discussion and Proposition 2.19, we have

$$\mathrm{Cliff}(C) = \min\{\mu_C, \lfloor \frac{g-1}{2} \rfloor\}.$$

Thus, either C has maximal Clifford index, or $\mathscr{A}(C)$ is non-empty, the Clifford index of C is cut out by some divisor $D \in \mathscr{A}^0(C)$ and $\mathrm{Cliff}(C) = \mu_C$.

§5. Exceptional curves on K3 surfaces

Let C be a smooth irreducible curve of genus $g \geq 2$. As we recalled in the previous Chapter, one has $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$ if and only if C has Clifford dimension 1, i.e. the Clifford index of C is computed by a pencil A on C. Otherwise, $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3$ and C is called an *exceptional* curve.

EXAMPLE 2.20. *A smooth curve C_d of degree $d \geq 3$ in $\mathbb{P}^2$ is exceptional. In fact,* $\mathrm{gon}(C_d) = d - 1$ *(the pencils obtained by projecting from a point of C_d). While,* $\mathrm{Cliff}(C) = d - 4$*, (computed by the g^2_d which gives the embedding of C_d in $\mathbb{P}^2$).*

It had originally been conjectured that the gonality of curves on K3 surfaces should be constant in the linear system. However, in contrast with Theorem 2.14, this is not the case. In fact, Donagi and Morrison [DM89] have observed that we can find smooth plane sextics as special linear subsystems of a linear system on a K3 surface:

EXAMPLE 2.21 (Donagi-Morrison). *Let $\pi \colon X \to \mathbb{P}^2$ be a double cover ramified over a smooth sextic C_6. Let $L = \pi^* \mathscr{O}_{\mathbb{P}^2}(3)$. A general curve C in $|L|$ is isomorphic to a smooth plane sextic, whence* $\mathrm{gon}(C) = 5$*. On the other hand, $|L|$ contains a (codimension 1) linear subspace of bielliptic curves (which are pull-backs of the smooth plane cubics), whence of gonality four.*

Quite interestingly, this turns out to be an isolated example. In fact, by the results of Ciliberto and Pareschi [CP95] and Knutsen [Knu09], one has the following.

THEOREM 2.22. *Let* S *be a K3 surface and* L *a globally generated line bundle on* S*. If the gonality of the smooth curves in* $|L|$ *is not constant then there exists a double cover* $\pi\colon S \to \mathbb{P}^2$ *such that* $L \simeq \pi^*\mathcal{O}_{\mathbb{P}^2}(3)$ *as in Donagi-Morrison's example.*

The result was first proved by Ciliberto and Pareschi under the additional assumption that L is ample, and later refined by Knutsen. In the same paper, Knutsen also gives a complete classification of exceptional curves on K3 surfaces, which we recall:

THEOREM 2.23. *Let* C *be an exceptional curve on a K3 surfaces* S*. Then, for* $L = \mathcal{O}_S(C)$ *either one of the following cases occurs:*

(i) $L \simeq \pi^*\mathcal{O}_{\mathbb{P}^2}(3)$ *and* C *is a plane sextic as in Donagi-Morrison's example.*

(ii) $C \sim 2D + \Gamma$*, for some smooth curve* D *of genus* ≥ 2*, a smooth rational curve* Γ *such that* $D \cdot \Gamma = 1$ *and there exists no divisor* B *on* S *satisfying* $0 \leq B^2 \leq D^2 - 1$ *and* $0 < B \cdot (C - B) \leq D^2$*. Moreover, all the smooth curves in* $|L|$ *are exceptional, with Clifford index equal to* $D^2 - 1 = \frac{g-4}{2}$ *and Clifford dimension* $1 + D^2/2$*. Furthermore, for any smooth curve* $C \in |L|$*, the Clifford index of* C *is computed only by* $\mathcal{O}_C(D)$*.*

CHAPTER 3

GONALITY AND CLIFFORD INDEX OF CURVES ON K3 SURFACES WITH PICARD LATTICE $U(m)$

In this Chapter we describe the main result from the paper [Ram16]. We explain the statement in Section §1. below, and give some background and explain our original motivation in Section §2. We give the proof in Section §3.

§1. STATEMENT OF THE MAIN RESULT

THEOREM 3.1. *Let X be a K3 surface with Picard lattice isomorphic to $U(m)$, with $m \in \mathbb{Z}$, $m \geq 1$. Denote by E and F two effective generators of* $\mathrm{Pic}(X)$, *with $E^2 = F^2 = 0$ and $E \cdot F = m$. Let C be a curve on X of genus $g \geq 2$. Then, either*

(i) The Clifford index of C is cut out on C by an elliptic curve E_C (which is linearly equivalent to the one among E and F having minimal intersection with C). Then $\mathrm{Cliff}(C) = C \cdot E_C - 2$ *and* $\mathrm{gon}(C) = C \cdot E_C$,

or

(ii) $m > 2$ and $C \sim E + F$. In this case, C has maximal Clifford index $\mathrm{Cliff}(C) = \lfloor m/2 \rfloor$.

In the statement, $U(m)$ denotes the hyperbolic lattice with intersection form multiplied by an integer $m > 0$, i.e. the lattice $\mathbb{Z} \oplus \mathbb{Z}$ with intersection matrix

$$U(m) = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

Notice that there always exists a class of square zero in $\mathrm{Pic}(X) \simeq U(m)$. Thus, a K3 surface satisfying the assumptions of the theorem admits an elliptic fibration, by Proposition 1.34.

§2. Double covers of del Pezzo surfaces

Before discussing the proof of the theorem we want to explain why the hypothesis for the Picard lattice of X is interesting to us. The original motivation for studying this problem came from Watanabe's result [Wat14], which we recall below.

THEOREM 3.2. *Let X be a K3 surface such that its Picard group is a 2-elementary lattice with discriminant group of the form $(\mathbb{Z}/2\mathbb{Z})^{\oplus a}$ and X has Picard number $\rho(X) = a$, with $2 \leq a \leq 6$. Let C be a smooth curve of genus $g(C) \geq 3$ on X. Then, one of the following two cases occurs.*

(i) The Clifford index of C is computed by $\mathcal{O}_C(F)$, for an elliptic curve F on X.

(ii) There exists a smooth curve B of genus 2 such that $C \sim rB$ and $r \geq 2$.

A K3 surface X as in the statement arises as a double cover of a smooth del Pezzo surface of degree $4 \leq d \leq 8$ such that the involution on X induced by the double cover acts trivially on the Picard group. The main idea in the proof of Watanabe is to use directly the involution on X and the structure of its fixed locus in order to characterize the properties of the curves on X. By the classification of 2-elementary lattices [Nik83], one finds that

$$\mathrm{Pic}(X) \simeq U(2), \quad \text{or} \quad \mathrm{Pic}(X) \simeq \langle 2 \rangle \oplus \langle -2 \rangle^{\oplus a}, \quad 2 \leq a \leq 6.$$

Theorem 3.2 has served as the initial motivation for our work [Ram16]. We started to investigate the analogous situation to Watanabe's Theorem in the case where the K3 surface X carries a non-symplectic automorphism of order 3 which acts trivially on the Picard group. By the classification results of Artebani-Sarti [AS08] and Taki [Tak11], cf. [Tak11, Table 2], we always have an embedding of either U or $U(3)$ inside the Picard lattice $\mathrm{Pic}(X)$, such that

$$\mathrm{Pic}(X) = U(m) \oplus L,$$

with $m = 1$ or 3 and L some ADE lattice (or a sum of such). So, for example, Theorem 3.1 applies to K3 surfaces of Picard number two carrying an automorphism of order three. However, our techniques for the proof is essentially based on lattice theory, and in fact we do not require the existence of automorphisms at all in our hypothesis (indeed, this is not trivial, e.g. not all K3 surfaces with Picard lattice $U(m)$ admit a non-symplectic automorphism, cf. [AST11]).

§3. PROOF OF THEOREM 3.1

For the rest of this section we let X be a K3 surface as in Theorem 3.1 and denote by E and F two (effective) generators of the Picard lattice. Up to the action of the Weyl group of X we may assume that one of them, say E, is an elliptic curve (c.f. the discussion below Proposition 1.34).

Assume $m = 1$, and let $\Gamma = F - E$. Since $\Gamma \cdot E = 1$ and E is effective, Γ is effective. Notice that the class of Γ, up to sign, is the only (-2)-class in $\mathrm{Pic}(X)$. Therefore, $\Gamma \simeq \mathbb{P}^1$ is a smooth rational curve, and yields a section of the elliptic fibration given by $|E|$. Moreover, the general curve in $|F|$ is reducible of the form $E' + \Gamma$, with E' an elliptic curve in $|E|$. In particular, the class of F is not represented by an irreducible curve.

On the other hand, when $m > 1$, since $x^2 \in 2m\mathbb{Z}$ for $x \in U(m)$, that there are no rational curves on X. Any effective divisor on X is therefore nef and basepoint free, cf. Proposition 1.33. In particular, we may assume that F is also a smooth elliptic curve. Any elliptic curve on X belongs to either $|E|$ or $|F|$.

Let us now take a close look at the linear system given by

$$|E + F|.$$

When $m = 1$, as $|E + F| = |2E + \Gamma|$ contains the rational curve $\Gamma = F - E$ as base component, there are no irreducible curves in $|E + F|$.

When $m \geq 2$, due to the absence of (-2)-curves, $|E+F|$ is base-point free. Hence, the general element of $|E + F|$ is a smooth curve of genus $g = m + 1$, by Theorem 1.31.

For a given curve C on X, we recall the following notations from the previous Chapter:

$$\mathscr{A}(C) = \{D \in \mathrm{Div}(X) : \ h^0(D) \geq 2, \ h^0(C - D) \geq 2\},$$

$$\mu_C = \min_{D \in \mathscr{A}(C)} \{D \cdot (C - D) - 2\} \geq 0,$$

$$\mathscr{A}^0(C) = \{D \in \mathscr{A}(C) : \ D \cdot (C - D) - 2 = \mu_C\}.$$

In the following lemma we give a criterion to recognize numerically the curves in $|E+F|$ in terms of the elliptic curves on X and compute the Clifford index of these curves.

LEMMA 3.3. *Let $m \geq 2$ and C be a smooth curve on X with $C^2 > 0$. Let E_C be an elliptic curve such that its intersection number with C is minimal among all elliptic curves on X. If $(C - E_C)^2 = 0$, then $C \in |E + F|$. Moreover, for such a curve we have* *(i) If* $m = 2$ *then* C *is a hyperelliptic curve of genus* 3.

(ii) If $m > 2$ *then* C *has maximal Clifford index* $\lfloor m/2 \rfloor$.

Proof. If $(C - E_C)^2 = 0$ then $C - E_C$ is linearly equivalent to a multiple of an elliptic curve E', so that we can write $C = E_C + (C - E_C) \sim E_C + kE'$, some $k \geq 1$. Since $C^2 > 0$ we see that E' is not linearly equivalent to E_C. Since $E'.C = E_C.C$, we get $k = 1$ and, therefore, $C \sim E_C + E' \sim E + F$, as claimed.

Let us now prove the second part of the statement. Assume $D \in \mathscr{A}^0(C)$ and let $D \sim aE + bF$, with $a, b \geq 0$. Then by definition of $\mathscr{A}(C)$ we may assume $C-D$ effective, so that $0 \leq (C-D).E = m(1-b)$ and $0 \leq (C-D).F = m(1-a)$. Hence $a, b \in \{0, 1\}$. This shows that the only curves D in $\mathscr{A}^0(C)$ are the members of $|E|$ and $|F|$. Then C has non-maximal Clifford index whenever $\mu_C = D.(C - D) - 2 = m - 2 < \lfloor C^2/4 \rfloor = \lfloor m/2 \rfloor$, that is for $m \leq 2$. □

Motivated by the lemma, we set the following notation. For any effective divisor C on X let us define

$$
\begin{aligned}
d_C &= \min\{E' \cdot C \mid E' \text{ is an elliptic curve on } X\},\\
\mathscr{E}^0(C) &= \{\text{elliptic curves } E_C \text{ such that } E_C \cdot C = d_C\}.
\end{aligned}
$$

In the proof of Theorem 3.1 we first show that the elliptic curves in $\mathscr{E}^0(C)$ are good candidates amongst divisors which compute the Clifford index of C. In other words, that we have

$$\mathscr{E}^0(C) \subset \mathscr{A}^0(C).$$

At least, we would like to know that $\mathscr{E}^0(C) \subset \mathscr{A}(C)$. This follows by the next simple computation:

LEMMA 3.4. *Let* $C \subset X$ *be an effective divisor with* $C^2 > 0$. *For any elliptic curve* E' *on* X *we have*

$$
\begin{aligned}
(C - E')^2 &\geq 0,\\
h^0(C - E') &\geq 2.
\end{aligned}
$$

Moreover, $|C - E'|$ *is basepoint free for* $m > 1$.

Proof. Since E and F, up to linear equivalence, are the only effective reduced divisors on X with self-intersection zero, it is clear that in order to show the Lemma we may assume $E' \in |E|$, by the symmetry of the roles of E and F in $\mathrm{Pic}(X)$. Let $C \sim aE + bF$ for some positive integers a and b. Since

$(C-E)^2 \geq 0$ and $C.E > 0$, we have $E.(C-E) > 0$, which shows that $C-E$ is effective. Thus, $h^0(C-E) \geq 2$ by Riemann-Roch. If $m > 1$, then there are no rational curves on X, so $|C-E|$ is basepoint free. $\square$

Let C be a curve on X of genus $g > 2$. Lemma 3.4 implies $\mathscr{E}^0(C) \subset \mathscr{A}(C)$. In particular $\mu_C \leq d_C - 2$. Moreover, we have the straightforward equivalence:

$$\mathscr{E}^0(C) \subset \mathscr{A}^0(C) \iff \mu_C = d_C - 2.$$

Indeed, let $E_C \in \mathscr{E}^0(C) \subset \mathscr{A}(C)$. Assuming $\mu_C = d_C - 2$ implies that E_C computes μ_C. Hence $E_C \in \mathscr{A}^0(C)$. The other implication is obvious.

LEMMA 3.5. *Under the assumptions of Theorem 3.1, we have*

$$\mathscr{E}^0(C) \subset \mathscr{A}^0(C).$$

Proof. Let E_C be an elliptic curve in $\mathscr{E}^0(C)$. By Lemma 3.4, $E_C \in \mathscr{A}(C)$, so that $\mathscr{A}^0(C)$ is not empty. We show that $E_C \in \mathscr{A}^0(C)$.

If $\mathscr{A}^0(C)$ contains an elliptic curve F, then $C.E_C \leq C.F$ and so

$$C.E_C - 2 \leq F \cdot (C-F) = \mu_C.$$

Since $E_C \in \mathscr{A}(C)$, we have $C.E_C - 2 = \mu_C$. Therefore $E_C \in \mathscr{A}^0(C)$.

By contradiction, we assume that $\mathscr{A}^0(C)$ contains no elliptic curves. Let D be an effective divisor in $\mathscr{A}^0(C)$. Since $D \in \mathscr{A}(C)$, applying [JK04, Prop. 2.6], we have that $h^1(D) = 0$ and the (possibly zero) base-divisor of $|D|$ does not intersect C. Hence, $D^2 \geq 0$.

Claim. $D^2 \geq 2$.

Indeed, assume by contradiction $D^2 = 0$.

- If $m \geq 2$, since X contains no rational curves, D is basepoint free, and so it is linearly equivalent to a multiple rE' of an elliptic curve E', by Theorem 1.31. Since $D \in \mathscr{A}^0(C)$, it must be $r = 1$, by minimality. This contradicts the assumption that $\mathscr{A}^0(C)$ contains no elliptic curves.
- If $m = 1$, let E and F be generators of the Picard group of X, with $E^2 = F^2 = 0$ and $E.F = 1$. Then we may assume that E is an elliptic curve and there exists a rational curve $\Gamma = F - E$, which is the base divisor of $|F|$. Since $D^2 = 0$ and $D \in \mathscr{A}^0(C)$, we have $D \sim E$ or F. However, E is not in $\mathscr{A}^0(C)$ by assumption, thus $D \sim F$, whence Γ is the base divisor of $|D|$. Therefore, $C \cdot \Gamma = 0$ and we get

$$\mu_C = C \cdot D - 2 = C \cdot (D - \Gamma) - 2 = C \cdot E - 2$$

and, since $E \in \mathscr{A}(C)$, this yields $E \in \mathscr{A}^0(C)$, a contradiction.

Thus, $D^2 \geq 2$. Notice that $C - D \in \mathscr{A}^0(C)$ and then $(C-D)^2 \geq 2$ by the same reason. Observe that $E_C \in \mathscr{A}^0(C)$ if and only if the following inequality holds:

$$E_C \cdot C \leq D \cdot (C - D).$$

Set $D' := C - D$ and rewrite this inequality as

$$(D - E_C) \cdot (D' - E_C) \geq 0 \tag{3.1}$$

For $E_D \in \mathscr{E}^0(D)$ and $E_{D'} \in \mathscr{E}^0(D')$ we let

$$\begin{aligned} n_D &= (D - E_D) \cdot (D' - E_{D'}) \\ r_D &= D \cdot (E_{D'} - E_C) \\ r_{D'} &= D' \cdot (E_D - E_C) \end{aligned}$$

so that we may now rewrite (3.1) as follows:

$$n_D + r_D + r_{D'} \geq E_D \cdot E_{D'} \tag{3.2}$$

Claim. For *any choice* of $E_D \in \mathscr{E}^0(D)$ and $E_{D'} \in \mathscr{E}^0(D')$, we have

$$n_D = (D - E_D) \cdot (D' - E_{D'}) \geq 0.$$

Indeed, by Lemma 3.4 the classes of $(D - E_D)$ and $(D' - E_{D'})$ have non-negative self-intersection and are effective, thus they lie in the closure of the positive cone and intersect non-negatively, cf. Corollary 1.19. This proves our claim.

Now, consider the following straightforward inequalities:

$$\begin{aligned} r_D &\geq r_D + C \cdot (E_C - E_{D'}) = D' \cdot (E_C - E_{D'}) \geq 0 \\ r_{D'} &\geq r_{D'} + C \cdot (E_C - E_D) = D \cdot (E_C - E_D) \geq 0 \end{aligned} \tag{3.3}$$

If either $r_D > 0$ or $r_{D'} > 0$ then (3.2) holds, since $n_D \geq 0$ and

$$E_C \cdot E_{D'} \leq m \quad \text{and} \quad r_D \geq m \text{ or } r_{D'} \geq m.$$

(recall that $x.y \in m\mathbb{Z}$ for $x, y \in U(m)$). Hence, we assume $r_D = 0$ and $r_{D'} = 0$. Substituting this in (3.3) we get $D.E_C = D.E_D$ and $D'.E_C = D'E_{D'}$. Thus,

$$E_C \in \mathscr{E}^0(D) \cap \mathscr{E}^0(D').$$

Using the claim above, we can replace both E_D and $E_{D'}$ by E_C in the definition of n_D and this yields the desired inequality (3.1) and therefore

$$\mathscr{E}^0(C) \subset \mathscr{A}^0(C),$$

as claimed. □

We can now give a proof of Theorem 3.1.

Proof of Thm. 3.1. We first determine all curves C on X having maximal Clifford index. Let C be any curve on X and choose $E_C \in \mathscr{E}^0(C)$. By Lemma 3.4 we know $C - E_C \in \mathscr{A}(C)$. Moreover, we also have $C - E_C \in \mathscr{A}^0(C)$ since $E_C \in \mathscr{A}^0(C)$. We distinguish two cases:

- ◇ $(C - E_C)^2 = 0$. Then, by Lemma 3.3, C has maximal Clifford index if and only if $m > 2$ and $C \in |E + F|$.
- ◇ $(C - E_C)^2 > 0$. (in particular C is not linearly equivalent to $E + F$). We then show that C has non-maximal Clifford index. This amounts to show
$$2\mu_C \leq g - 3$$
which, by the definition of μ_C and the genus formula, is equivalent to
$$(C - 2E_C)^2 \geq -4.$$
We may write $C \sim aE_C + D$, with $a \geq 1$, D effective and $D^2 = 0$. If $a = 1$ we get $C \in |E + F|$ by Lemma 3.3, which is not the case. So $a \geq 2$ and
$$(C - 2E_C)^2 \geq 0.$$
Therefore, C has non-maximal Clifford index.

This proves that C has maximal Clifford index if and only if $m > 2$ and $C \in |E + F|$, as in part (ii) of the Theorem. To show part (i) of Theorem 3.1, we can therefore assume that C has non-maximal Clifford index. Then $\text{Cliff}(C) = \mu_C$ and since $\mathscr{E}^0(C) \subset \mathscr{A}^0(C)$ we have $\mu_C = d_C - 2$. Therefore, the Clifford index of C is cut out by some elliptic curve $E_C \in \mathscr{E}^0(C)$. In particular, $\text{Cliff}(C)$ is computed by a pencil: the restriction of $|E_C|$ to C. We conclude that
$$\text{gon}(C) = \text{Cliff}(C) + 2 = d_C.$$
Hence, the assertions of (i) in Theorem 3.1 follow, and the proof is complete. □

CHAPTER 4

SPECIAL DIVISORS ON CURVES ON K3 SURFACES CARRYING AN ENRIQUES INVOLUTION

In this chapter we study the Brill-Noether theory of curves lying on general K3-covers of Enriques surfaces. We find that the gonalities and Clifford indices of these curves are induced by the elliptic fibrations carried by the ambient K3 surface, an outcome similar to the statement of Theorem 3.1. The strategy of the proof, however, is different: here we take advantage of the geometry of the covering map between the K3 surface and its Enriques quotient, and we manage to reduce our analysis to curves on Enriques surfaces, for which we apply the results of Knutsen and Lopez [KL08].

We start by recalling their work in the following section.

§1. BRILL-NOETHER THEORY OF CURVES ON ENRIQUES SURFACES

An Enriques surface is by definition a regular surface Y with non-trivial 2-torsion canonical bundle K_Y. In particular, K_Y is numerically trivial.

Let Y be an Enriques surface and fix a curve C on Y with $C^2 > 0$. As explained in [KL08], it may well happen that the gonality may drop when C moves in its linear system. Thus, one defines the *generic gonality* of $|C|$,

$$\operatorname{gengon}|C| = \max\{\operatorname{gon}(C') : C' \in |C|\},$$

where C' runs over the smooth curves in the linear system $|C|$. In other words, $\operatorname{gengon}|C|$ is the gonality of the generic curve in $|C|$.

Define the following two functions

$$\phi(C) = \min\{F \cdot C : F \in \operatorname{Pic}(Y), F > 0, F^2 = 0, F \not\equiv 0\},$$

(where $\equiv$ denotes *numerical equivalence* on Y) and

$$\mu(C) = \min\{B \cdot C - 2 : F \in \operatorname{Pic}(Y), B > 0, B^2 = 4, B \not\equiv C\}.$$

Knutsen and Lopez [KL08, Thm 1.3] show that the generic gonality of the curves in Y is controlled by the functions μ and ϕ:

THEOREM 4.1. *Let $|L|$ be a base-component free complete linear system on an Enriques surface, with $L^2 > 0$. Then, for a curve C in $|L|$ we have*

$$\operatorname{gengon}|C| = \min\{2\phi(C), \mu(C), \lfloor \frac{C^2}{4} \rfloor + 2\}. \tag{4.1}$$

Moreover, a complete classification of the cases in which $\mu(C) < 2\phi(C)$ is given. This leads to the following, cf. [KL08, Cor. 1.5].

COROLLARY 4.2. *Let $|L|$ be a base-component free complete linear system on an Enriques surface, with $L^2 > 0$ and let C be a general curve in $|L|$. Set $\phi := \phi(C)$. Then $\operatorname{gon}(C) = 2\phi$, unless we have one of the following cases:*

(a) $C^2 = \phi^2$ *with* $\phi \geq 2$ *and even. Then* $\operatorname{gon}(C) = 2\phi - 2$.

(b) $C^2 = \phi^2+\phi-2$ *with* $\phi \geq 3$ *and* $L \not\equiv 2D$ *for* D *such that* $D^2 = 10$, $\phi(D) = 3$. *Then* $\operatorname{gon}(C) = 2\phi - 1$, *except for* $\phi = 3, 4$, *when* $\operatorname{gon}(C) = 2\phi - 2$.

(c) $(C^2, \phi) = (30, 5), (22, 4), (20, 4), (14, 3), (12, 3), (6, 2)$. *Then* $\operatorname{gon}(C) = 2\phi - 1$.

The cases (a), (b) and (c) above are also described explicitly [KL08, Pr. 1.4].

By the Corollary above, for any curve C on an Enriques surface, we get the following inequality:

$$2\phi(C) \leq \operatorname{gengon}|C| + 2 \tag{4.2}$$

In a second paper [KL15], the same authors have also classified exceptional curves on Enriques surfaces:

THEOREM 4.3. *On an Enriques surfaces there are no exceptional curves other than smooth plane quintics.*

This means in particular that for any curve C on an Enriques Y such that $C^2 \neq 10$, one has $\operatorname{gon}(C) = \operatorname{Cliff}(C) + 2$. Obviously, one can combine Theorems 4.1 and 4.3 to obtain a statement about the Clifford index of a general curve in a linear system on an Enriques surface.

§2. CURVES ON GENERAL K3-COVERS OF ENRIQUES SURFACES

It is well-known, cf. [BHPvdV14, Chap. VIII] that if a K3 surface X carries a fixed-point free involution ϑ, then the quotient surface $Y = X/\vartheta$ is an Enriques surface and, moreover, any Enriques surface arises this way. It is therefore natural to ask what are the consequences of Theorem 4.1 for the curves lying on the relative K3-cover of an Enriques surface.

We now describe the main result we obtained in [Ram17].

THEOREM 4.4. *Let (X, ϑ) be a generic K3 surface X with an Enriques involution ϑ. The gonality and the Clifford index of any smooth curve C on X, with $C^2 > 0$, are cut out by any elliptic curve E on X having minimal intersection with C, i.e.* $\mathrm{gon}(C) = E \cdot C$ *and* $\mathrm{Cliff}(C) = E \cdot C - 2$.

In light of the fact that the gonality might fail to be constant amongst curves moving in linear systems on an Enriques surface, and of the trichotomy expressed by (4.1), the content of Theorem 4.4 might be somewhat surprising at a first glance. As it turns out, however, we can show that the degree of the pencils induced on curves on the K3 surface by the elliptic fibrations of the Enriques quotient are low enough to get the statement, thanks to the properties of the functions φ and μ which we recalled above.

Regarding the statement about the Clifford index in Theorem 4.4, the fact that one has $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$ for the curves in question is a consequence of the classification of exceptional curves on K3 surfaces, which was recorded in Theorem 2.23, and some basic observations from lattice theory (cf. the beginning of the proof of Theorem 4.4 below), showing that we can exclude the presence of exceptional curves on X.

For the sake of the reader, let us first recall a few well-known facts, some of which have been already discussed in detail in the previous sections, which will be used along the proof. If a curve C lies on a K3 surface X and has non-maximal Clifford index, then by Theorem 2.14, there exists a line bundle M on the surface such that $\mathrm{Cliff}(C) = \mathrm{Cliff}(M \otimes \mathcal{O}_C)$ and, by Proposition 2.19, we can choose M such that

$$\mathrm{Cliff}(C) = C \cdot M - M^2 - 2. \tag{4.3}$$

Moreover,

$$M \text{ is represented by an irreducible curve,} \tag{4.4}$$

and

$$2M^2 \leq M \cdot C, \quad \text{with equality only if } C \sim 2M. \tag{4.5}$$

Let now (X, ϑ) be a pair consisting of a K3 surface X together with a fixed-point free involution ϑ of X. We denote by

$$Y = X / \langle \vartheta \rangle$$

the quotient Enriques surface.

We let $\pi \colon X \to Y$ denote the natural projection and by

$$\pi^* \colon H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z}); \qquad \pi_* \colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

the natural induced maps, satisfying

$$\pi_* \pi^*(y) = 2y; \quad \pi^* \pi_*(x) = x + \vartheta^*(x); \quad (\pi^* y_1, \pi^* y_2) = 2(y_1, y_2).$$

Denoting by $H^2(X, \mathbb{Z})^\vartheta$ the set of classes in $H^2(X, \mathbb{Z})$ which are fixed by ϑ, the above properties imply that the restriction of π_* to $H^2(X, \mathbb{Z})^\vartheta$ is an isomorphism onto its image which multiplies the intersection form by 2. That is,

$$\pi_*(H^2(X, \mathbb{Z})^\vartheta) \simeq H^2(X, \mathbb{Z})^\vartheta(2) \tag{4.6}$$

REMARK 4.5. *It is well-known that the fixed-point free involution ϑ is a* non-symplectic *involution, i.e.* $\vartheta^* \omega_X = -\omega_X$, *where* ω_X *is a generator of* $H^{2,0}(X)$. *If* $x \in H^2(X, \mathbb{Z})^\vartheta$, *since* ϑ *acts on* $H^2(X, \mathbb{Z})$ *preserving the intersection form, we then have* $x \cdot \omega_X = \vartheta^* x \cdot \vartheta^* \omega_X = -x \cdot \omega_X$, *whence* $x \cdot \omega_X = 0$. *This shows*

$$H^2(X, \mathbb{Z})^\vartheta \subset H^2(X, \mathbb{Z}) \cap H^{1,1}(X).$$

By the Lefschetz theorem on $(1,1)$*-classes, we identify the member on the right hand side of the above equation with* $\mathrm{Pic}(X)$. *As shown in [DK07], one can construct a 10-dimensional period domain* $\mathscr{D}$ *for the pairs* (X, ϑ) *and, for the generic pair in* $\mathscr{D}$, *one has the equality*

$$H^2(X, \mathbb{Z})^\vartheta = \mathrm{Pic}(X). \tag{4.7}$$

We can now give a proof of Theorem 4.4.

Proof of Theorem 4.4. By our genericity assumptions on (X, ϑ), we can assume that X satisfies condition (4.7) above. Let L be a big and nef line bundle on X. First of all, by (4.6) and (4.7), we get

$$L^2 \equiv 0 \pmod 4. \tag{4.8}$$

This implies that there are no exceptional curves on X. Indeed, by Theorem 2.23 we would either have a double cover $X \to \mathbb{P}^2$, hence curve B with

$B^2 = 2$ on X (the pull-back of the line of $\mathbb{P}^2$) or there would be a (-2)-curve Γ on X. Both are excluded by (4.8). Hence, the gonality of smooth curves Σ in $|L|$ is constant and

$$\mathrm{Cliff}(\Sigma) = \mathrm{gon}(\Sigma) - 2.$$

Thus, whenever $\mathrm{Cliff}(\Sigma)$ is computed by the restriction of a divisor D on the surface, since $\deg_\Sigma(D) = D \cdot \Sigma \in 2\mathbb{Z}$ (again by (4.6)), we see that both the Clifford index and the gonality of Σ must be even.

Secondly, for any $D \in |L|$ one has $\vartheta^* D \sim D$, i.e. ϑ acts as an involution on $|L| \simeq \mathbb{P}^g$. This lifts to an involution

$$\vartheta^* \colon H^0(X, L) \to H^0(X, L),$$

at the level of sections. Let us denote by $V_\pm \subset H^0(X, L)$ the eigenspaces relative to the eigenvalues ± 1 for this action. The sections in V_+ and V_- yield the effective divisors in $|L|$ which are mapped to themselves by ϑ. With respect to the covering $\pi \colon X \to Y$, these divisors map 2 to 1 onto divisors on the Enriques quotient. In other words, we may choose an effective divisor $C \subset Y$ such that $\pi^* C$ belongs to $|L|$ and $\mathbb{P}(V_+)$, as a linear subspace of $|L|$, corresponds to $\pi^*|C|$. (With respect to this choice, $\mathbb{P}(V_-)$ would then correspond to $\pi^*|C + K_Y|$).

As $L^2 > 0$ by assumption, we have $C^2 > 0$, hence the general member of $|C|$ is a smooth irreducible curve. In fact, if $|C|$ is hyperelliptic then its general member is a smooth (hyperelliptic) curve by [CD89, Corollary 4.5.1 p. 248]. Else, $|C|$ is basepoint free [CD89, Proposition 4.5.1] and we apply Bertini's theorem. We therefore assume C itself to be a smooth irreducible curve.

Moreover, we choose C to be generic in its linear system, so that the gonality of C is equal to the generic gonality

$$\mathrm{gon}(C) = \mathrm{gengon}\,|C|.$$

Let $\widetilde{C} := \pi^* C$. It is well-known that the restriction of the canonical bundle K_Y to C is non-trivial. It follows that $\widetilde{C}$ is a smooth irreducible curve of genus g in $|L|$ and the restriction of the covering map

$$\pi|_{\widetilde{C}} \colon \widetilde{C} \to C$$

exhibits $\widetilde{C}$ as an unramified double covering of C. In particular, by push-forward of a pencil of minimal degree on $\widetilde{C}$, or by pull-back of gonality pencils from C, one has

$$\mathrm{gon}(C) \leq \mathrm{gon}(\widetilde{C}) \leq 2\,\mathrm{gon}(C). \tag{4.9}$$

Let now $|2F|$ be a genus 1 pencil on the Enriques surface Y such that

$$\phi(C) = F \cdot C.$$

We set $\widetilde{F} = \pi^* F$ and by (4.2) we obtain the following inequality

$$\operatorname{gon}(\widetilde{C}) \leq \widetilde{F} \cdot \widetilde{C} = 2\phi(C) \leq \operatorname{gon}(C) + 2 \tag{4.10}$$

We claim that the first inequality is, in fact, always an equality.
Indeed, assume by contradiction $\operatorname{gon}(\widetilde{C}) < \widetilde{F} \cdot \widetilde{C}$. By (4.9),

$$\operatorname{gon}(C) < 2\phi(C).$$

Applying Corollary 4.2, we have

$$C^2 \geq 10 \ \text{ or } (C^2, \phi(C)) = (6, 2) \ \text{ or } (C^2, \phi(C)) = (4, 2). \tag{4.11}$$

Claim. $\operatorname{gon}(\widetilde{C}) < \lfloor \frac{g(\widetilde{C})+3}{2} \rfloor$ (in particular, $\operatorname{gon}(\widetilde{C})$ is even).

Proof of the Claim. If $C^2 = 4$, then $\widetilde{C}^2 = 8$ and $g(\widetilde{C}) = 5$, so if equality holds, then it must be $\operatorname{gon}(\widetilde{C}) = 4 = 2\phi(C) = \widetilde{F} \cdot \widetilde{C}$, a contradiction.
If $C^2 \geq 6$, then $\widetilde{C}^2 \geq 12$, so that $g(\widetilde{C}) \geq 7$. Hence

$$\begin{aligned}
\operatorname{gon}(\widetilde{C}) &\leq \widetilde{F} \cdot \widetilde{C} - 1 \\
&\leq \operatorname{gon}(C) + 1 \\
&\leq \lfloor \frac{g(C)+3}{2} \rfloor + 1 \\
&= \lfloor \frac{\frac{g(\widetilde{C})+1}{2} + 3}{2} \rfloor + 1 \\
&= \lfloor \frac{g(\widetilde{C})+11}{4} \rfloor \\
&< \lfloor \frac{g(\widetilde{C})+3}{2} \rfloor
\end{aligned}$$

where the last inequality uses $g(\widetilde{C}) \geq 7$. □

Since, by our assumptions, $\operatorname{gon}(C) < 2\phi(C)$, we proceed as follows.
If $\operatorname{gon}(C) = 2\phi(C) - 1$, then $\operatorname{gon}(C) \leq \operatorname{gon}(\widetilde{C}) < 2\phi(C)$ is incompatible with the parity of $\operatorname{gon}(\widetilde{C})$, whence yielding a contradiction.
By (4.2), we may therefore assume

$$\operatorname{gon}(C) = 2\phi(C) - 2.$$

Then, necessarily $\operatorname{gon}(C) = \operatorname{gon}(\widetilde{C})$.
We pick a line bundle M on the K3 surface X, as in (4.3), i.e. such that

$$\begin{aligned}\mathrm{Cliff}(\widetilde{C}) &= \mathrm{Cliff}(M \otimes \mathcal{O}_{\widetilde{C}}) \\ &= M \cdot \widetilde{C} - M^2 - 2.\end{aligned}$$

If $M^2 = 0$, then it follows by (4.4) that M is represented by an elliptic curve E. By construction, the elliptic curve $\widetilde{F}$ has minimal intersection with $\widetilde{C}$ among all elliptic curves on X, whence $\mathrm{gon}(\widetilde{C}) = \mathrm{Cliff}(\widetilde{C}) - 2 = E \cdot \widetilde{C} = \widetilde{F} \cdot \widetilde{C}$, a contradiction.

We may therefore assume $M^2 > 0$. Then $M^2 \geq 4$ by (4.8). By (4.11) and (4.3) we obtain

$$4 \leq M^2 \leq M \cdot \widetilde{C} - M^2 = \mathrm{Cliff}(\widetilde{C}) + 2 = \mathrm{gon}(\widetilde{C}).$$

Assume $\mathrm{gon}(\widetilde{C}) = 4$. Then $M^2 = 4$ and $M \cdot \widetilde{C} = 8$, so that (4.5) gives $\widetilde{C} \sim 2M$, whence $\widetilde{C}^2 = 16$. It follows that $C^2 = 8$. This contradicts (4.11) and we may therefore assume

$$\mathrm{gon}(\widetilde{C}) > 4.$$

Arguing as above, ϑ acts as an involution on $|M|$, and we get subsystems $\mathbb{P}_{\pm}$ of $|M|$, corresponding to $\pi^*|D|$ and $\pi^*|K_Y + D|$, where D is some effective divisor on Y, with $\pi^* D \sim M$. Since $M^2 > 0$, we have $D^2 > 0$, whence

$$h^0(D) \geq 2.$$

We have $\pi^*(C - D) \sim \widetilde{C} - M$, whence $(C - D)^2 > 0$. Also,

$$2(C - D) \cdot C = \pi^*(C - D) \cdot \pi^* C = N \cdot \widetilde{C} = M \cdot N + N^2 > 0,$$

so that by Riemann-Roch, $h^0(C - D) \geq 2$ and, similarly, $h^0(C - D + K_Y) \geq 2$. Therefore, $C \cdot (C - D) \geq 2$ by the Hodge index theorem, so that

$$\begin{aligned}C^2 &= (D + C - D)^2 \\ &= D^2 + (C - D)^2 + 2C \cdot (C - D) \\ &\geq 2 + 2 + 2 \\ &= 6\end{aligned}$$

We may now apply [KL15, Lemma 2.3] and obtain

$$\mathrm{Cliff}(C) \leq D \cdot C - D^2.$$

By Theorem 4.3, unless C is a smooth plane quintic (which has gonality 4), one has $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$, and so the above inequality yields

$$2(D \cdot C - D^2) \geq 2\,\mathrm{gon}(C) - 4. \tag{4.12}$$

On the other hand, $\operatorname{gon}(\widetilde{C}) = \operatorname{Cliff}(\widetilde{C}) + 2 = \widetilde{C} \cdot M - M^2$, thus

$$2(D \cdot C - D^2) = \widetilde{C} \cdot M - M^2 = \operatorname{gon}(\widetilde{C}). \tag{4.13}$$

Since $\operatorname{gon}(C) = \operatorname{gon}(\widetilde{C}) > 4$, the equations (4.12) and (4.13) are incompatible. Hence, our assumption that $\operatorname{gon}(\widetilde{C}) < \widetilde{F} \cdot \widetilde{C}$ has led to a contradiction and we conclude

$$\operatorname{gon}(\widetilde{C}) = \widetilde{F} \cdot \widetilde{C}.$$

As we have already observed above, thanks to our genericity assumption on X, this holds true for all smooth curves Σ in $|\widetilde{C}| = |L|$. This concludes the proof of Theorem 4.4. $\square$

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Résumé de la thèse

Nous allons étudier les propriétés des courbes algébriques sur des surfaces K3 spéciales, du point de vue de la théorie de Brill-Noether.

La démonstration de Lazarsfeld du théorème de Gieseker-Petri a mis en lumière l'importance de la théorie de Brill-Noether des courbes admettant un plongement dans une surface K3. Nous allons donner une démonstration détaillée de ce résultat classique, inspirée par les idées de Pareschi. En suite, nous allons décrire le théorème de Green et Lazarsfeld, fondamental pour tout notre travail, qui établit le comportement de l'indice de Clifford des courbes sur les surfaces K3.

Watanabe a montré que l'indice de Clifford de courbes sur certaines surfaces K3, admettant un recouvrement double des surfaces de del Pezzo, est calculé en utilisant les involutions non-symplectiques. Nous étudions une situation similaire pour des surfaces K3 avec un réseau de Picard isomorphe à $U(m)$, avec $m > 0$ un entier quelconque. Nous montrons que la gonalité et l'indice de Clifford de toute courbe lisse sur ces surfaces, avec une seule exception en genre $g = m + 1 \geq 4$ déterminée explicitement, sont obtenus par restriction des fibrations elliptiques de la surface.

Knutsen et Lopez ont étudié en détail la théorie de Brill-Noether des courbes sur les surfaces d'Enriques. En appliquant leurs résultats, nous allons pouvoir calculer la gonalité et l'indice de Clifford de *toute courbe* lisse sur les surfaces K3 qui sont des recouvrements universels d'une surface d'Enriques.

Abstract

We study the properties of algebraic curves lying on special K3 surfaces, from the viewpoint of Brill-Noether theory.

Lazarsfeld's proof of the Gieseker-Petri theorem has revealed the importance of Brill-Noether theory of curves which admit an embedding in a K3 surface. We give a proof of this classical result, inspired by the ideas of Pareschi. We then describe the theorem of Green and Lazarsfeld, a key result for our work, which establishes the behaviour of the Clifford index of curves on K3 surfaces.

Watanabe showed that the Clifford index of curves lying on certain special K3 surfaces realizable as a double covering of a smooth del Pezzo surface, can be determined by a direct use of the non-simplectic involution carried by these surfaces. We study a similar situation for some K3 surfaces having a Picard lattice isomorphic to $U(m)$, with $m > 0$ any integer. We show that the gonality and the Clifford index of all smooth curves on these surfaces, with a single, explicitly determined exception in genus $g = m + 1 \geq 4$, are obtained by restriction of the elliptic fibrations of the surface.

Knutsen and Lopez have studied in detail the Brill-Noether theory of curves lying on Enriques surfaces. Applying their results, we are able to determine and compute the gonality and Clifford index of any smooth curve lying on the general K3 surface which is the universal covering of an Enriques surface.