

Analysis and control of some fluid models with variable density

Sourav Mitra

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THÈSE

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Présentée et soutenue le *23/10/2018* **par :**

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Analysis and control of some fluid models with variable density

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To my parents

Preface

In this thesis we study mathematical models concerning some fluid flow problems with variable density. Our main concern is to prove well posedness, stabilizability and controllability results for some models arising in the study of fluid dynamics where the density of a fluid can change significantly along a streamline. We consider fluids both of non-homogeneous incompressible and compressible nature. From mathematical point of view such fluids follow a coupled dynamics where the fluid velocity solves an equation of parabolic nature and the fluid density satisfies a hyperbolic transport equation. The nature of this coupling differs from a non-homogeneous incompressible fluid and a compressible fluid.

We start our study by considering a non-homogeneous incompressible Navier-Stokes equation in a 2d channel in Chapter 2. It is well known that the Poiseuille flow is a particular stationary solution of this model. We have proved that the solution of the non-stationary system can be stabilized around the Poiseuille flow by controlling only the fluid velocity by acting on the inflow boundary of the domain under suitable assumptions on the support of the initial density.

Next in Chapter 3 and 4, we consider existence and control issues of some fluid structure interaction models. Unlike Chapter 2, the fluid flow here is modeled by compressible Navier-Stokes equations and moreover the fluids interact with an Euler-Bernoulli damped beam located at the boundary of the fluid domain. One of the fundamental differences between the incompressible and compressible Navier-Stokes equations is that the pressure of the fluid in incompressible Navier-Stokes equations is interpreted as the Lagrange multiplier whereas in the case of compressible Navier-Stokes equations the pressure is given as a function of the density. Compressible fluid structure interaction problems arise in the study of many engineering systems (e.g, aircraft, bridges etc.) where the fluid flow corresponds to a high Mach number.
Let us now go a little bit into the contents of Chapter 3 and 4. In Chapter 3, we analyze an interactive dynamical system governed by a compressible viscous fluid occupying a 2d channel and adhering to a one dimensional elastic structure which is located at the fluid boundary and is modeled by an Euler-Bernoulli damped beam equation. The fluid and the structure velocities coincide at the fluid solid interface. The motion of the beam results from the net surface force which is the resultant of the force exerted by the fluid and a constant external force. We prove the local in time existence of a strong solution for this fluid structure interaction problem.
We begin the Chapter 4 by considering a non-linear compressible fluid structure interaction system in a 2d channel where the boundary and the coupling conditions are different from the ones considered in Chapter 3. In fact we make a simplification in the expression of the stress tensor applied by the fluid on the structure. Such simplified model is also considered in the article [24]. Along with that we assume that fluid boundary as well as the fluid-solid interface are impermeable and further implement no vorticity condition (which closely relates to the Navier-slip boundary condition) for the fluid velocity at all

the boundaries of the fluid domain. We then suitably linearize this problem around a constant trajectory in a reference configuration. We have managed to prove an observability inequality for an adjoint problem corresponding to the linearized system with the observation localized at a part of the fluid boundary. This observability inequality in principle implies the null controllability of the fluid structure interaction problem under consideration.

The content of this thesis is the result of the following articles which are written during my PHD:

1. Sourav Mitra, Stabilization of the non-homogeneous Navier-Stokes equations in a 2d channel, submitted, https://arxiv.org/abs/1807.04043.
2. Sourav Mitra, Local existence of strong solutions for a fluid-structure interaction model, submitted, https://arxiv.org/abs/1808.06716.
3. Sourav Mitra, Observability of the adjoint of a linearized compressible fluid-structure model in a 2d channel, in preparation.

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Chapter 1

Introduction

1.1 Context

Let us have a quick outlook into the contents of this thesis.

In Chapter 2 we study the local stabilization of the non-homogeneous Navier- Stokes equations in a 2d channel around Poiseuille flow. We design a feedback control of the velocity which acts on the inflow boundary of the domain such that both the fluid velocity and density are stabilized around Poiseuille flow provided the initial density is given by a constant added with a perturbation, such that the perturbation is supported away from the lateral boundary of the channel. Moreover the feedback control operator we construct has finite dimensional range.

In Chapter 3 we are interested in studying a system coupling the compressible Navier-Stokes equations with an elastic structure located at the boundary of the fluid domain. Initially the fluid domain is rectangular and the beam is located on the upper side of the rectangle. The elastic structure is modeled by an Euler-Bernoulli damped beam equation. We prove the local in time existence of strong solutions for that coupled system.

In Chapter 4 our objective is to study the null controllability of a linear fluid structure interaction problem in a 2d channel. The fluid flow here is modeled by the compressible Navier-Stokes equations. Concerning the structure we will consider a damped Euler-Bernoulli beam located on a portion of the boundary. In this chapter we establish an observability inequality for the linearized fluid structure interaction problem under consideration which in principle is equivalent with the null controllability of the system.

In what follows we will discuss more elaborately the results we have obtained, their connection with the existing literature and our contributions.

1.2 Presentation of the results

1.2.1 Chapter 2: Stabilization of the non-homogeneous Navier-Stokes equations in a 2d channel

In this chapter we are interested in stabilizing the density dependent Navier-Stokes equations around some stationary state (ρ_s, v_s) (where (ρ_s, v_s, p_s) is a stationary solution) in a two dimensional channel Ω. For that we will use an appropriate boundary control u_c acting on the velocity in the inflow part of the boundary $\partial\Omega$.

1.2.1.1 Introduction to the system

Let d be a positive constant. We use the following notations corresponding to the domain Ω and its boundary Γ

$$\Omega = (0,d)\times(0,1),\ \Gamma = \partial\Omega,\ Q_T = \Omega\times(0,T),\ \Sigma_T = \Gamma\times(0,T) \quad \text{for } 0 < T \leqslant \infty. \tag{1.2.1}$$

The unit outward normal to the boundary Γ is denoted by n. The velocity, density and pressure of the fluid are denoted respectively by v, ρ and p. The viscosity $\nu > 0$ of the fluid is a positive constant. We consider the following control system

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \operatorname{div}(\rho v) = 0 & \text{in} \quad Q_\infty, \\ \rho = \rho_s & \text{on} \quad \{(x,t)\in\Sigma_\infty \mid (v(x,t)\cdot n(x)) < 0\}, \\ \rho(x,0) = \rho_s + \rho_0 & \text{in} \quad \Omega, \\ \rho\left(\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v\right) - \nu\Delta v + \nabla p = 0 & \text{in} \quad Q_\infty, \\ \operatorname{div}(v) = 0 & \text{in} \quad Q_\infty, \\ v = v_s + u_c\chi_{\Gamma_c} & \text{on} \quad \Sigma_\infty, \\ v(x,0) = v_s + v_0 & \text{in} \quad \Omega, \end{cases} \tag{1.2.2}$$

where $u_c\chi_{\Gamma_c}$ is a control function for the velocity v with χ_{Γ_c} denoting the characteristics function of a set Γ_c which is compactly supported on Γ. The set Γ_c is defined as follows

$$\Gamma_c = \{0\}\times(L,1-L) \subset \Gamma_{in} = \{x\in\Gamma \mid (v_s\cdot n)(x) < 0\} = \{0\}\times(0,1), \tag{1.2.3}$$

for some fixed $0 < L < \frac{1}{2}$. The equation $(1.2.2)_1$ is the mass balance equation which is a hyperbolic transport equation and $(1.2.2)_4$ is the momentum balance equation which is of parabolic nature. The triplet (ρ_s, v_s, p_s) is the Poiseuille profile defined as follows

$$\rho_s(x_1, x_2) = 1, \quad v_s(x_1, x_2) = \begin{bmatrix} x_2(1 - x_2) \\ 0 \end{bmatrix}, \quad p_s = -2\nu x_1, \quad \text{in} \quad \Omega. \tag{1.2.4}$$

The triplet (ρ_s, v_s, p_s) (given by (1.2.4)) is a stationary solution of the Navier-Stokes equations (1.2.2).

1.2.1.2 Existing results

For the underlying physics of the system (1.2.2), one can consult [15], [32] and [23]. The controllability and stabilizability issues of the constant density (or homogeneous) incompressible Navier-Stokes equation (with Dirichlet or mixed boundary condition) by a finite dimensional feedback Dirichlet boundary control has already been studied in the literature. For instance in [36] it is proved that in a C^4 domain the velocity profile v, solution to system $(1.2.2)_4$-$(1.2.2)_7$ with $\rho = 1$ is locally stabilizable around a steady state v_s ($v_s \in H^3(\Omega; \mathbb{R}^2)$) by a finite dimensional Dirichlet boundary control localized in a portion of the boundary and moreover the control u_c is given as a feedback of the velocity field.
Unlike the constant density incompressible Navier-Stokes equations (which is of parabolic nature), the system (1.2.2) obeys a coupled parabolic-hyperbolic dynamics. To the best of our knowledge the controllability results for the system (1.2.2) appear only in the article [2]. Local exact controllability to trajectories of the system (1.2.2) was studied in [2]. In the present article we answer the question posed in [2] on the stabilizability of the system (1.2.2) around the Poiseuille profile. To obtain this stabilization result one of our main standing assumption corresponds to the support of the initial density and is given as follows

$$\text{supp}(\rho_0) \subset [0, d] \times (A_1, 1 - A_1). \tag{1.2.5}$$

The assumption (1.2.5) is made in view of the geometry of the Poiseuille profile to make sure that all the fluid particles initially located in the domain are flushed out by the fluid flow (which is a perturbation of the Poiseuille profile) at some finite time.

1.2.1.3 Presentation of our result and contribution

In order to state our result precisely we define the following functional spaces:

$$\begin{aligned} &V^s(\Omega) = \{y \in H^s(\Omega;\mathbb{R}^2) \mid \mathrm{div} y = 0 \quad \text{in} \quad \Omega\} \quad \text{for} \quad s \in \{0,2\}, \\ &V_0^1(\Omega) = \{y \in H^1(\Omega;\mathbb{R}^2) \mid \mathrm{div} y = 0 \quad \text{in} \quad \Omega, \quad y = 0 \quad \text{on} \quad \Gamma\}, \\ &V^{2,1}(Q_T) = H^1(0,T;V^0(\Omega)) \cap L^2(0,T;V^2(\Omega)). \end{aligned} \tag{1.2.6}$$

The central result of this chapter is stated as follows:

Theorem 1.2.1. *Let $\beta > 0$, $A_1 \in (0, \frac{1}{2})$. There exist a constant $\delta > 0$ such that for all $(\rho_0, v_0) \in L^\infty(\Omega) \times V_0^1(\Omega)$ satisfying (1.2.5) and*

$$\|(\rho_0, v_0)\|_{L^\infty(\Omega)\times V_0^1(\Omega)} \leqslant \delta,$$

there exists a control $u_c \in H^1(0,\infty;C^\infty(\overline{\Gamma}_c))$, for which the system (2.1.2) admits a solution

$$(\rho, v) \in L^\infty(Q_\infty) \times V^{2,1}(Q_\infty),$$

satisfying the following stabilization requirement

$$\|e^{\beta t}(\rho - \rho_s, v - v_s)\|_{L^\infty(Q_\infty)\times V^{2,1}(Q_\infty)} \leqslant C\|(\rho_0, v_0)\|_{L^\infty(\Omega)\times V_0^1(\Omega)}, \tag{1.2.7}$$

for some constant $C > 0$. Moreover, $\rho(t) = \rho_s$ in Ω for t sufficiently large.

The control function u_c acting on the velocity is constructed in the following form

$$u_c(x,t) = e^{-\beta t} \sum_{j=1}^{N_c} w_j(t) g_j(x), \tag{1.2.8}$$

where N_c is a natural number,

$$\{g_j \mid 1 \leqslant j \leqslant N_c\},$$

is a set of suitably chosen smooth functions supported on Γ_c, and $w_c(t) = (w_1(t), ..., w_{N_c}(t))$ is the control variable which is given in terms of a feedback operator $\mathcal{K}$.

Comment on the support condition (1.2.5): In proving the controllability results for the non-homogeneous incompressible Navier-Stokes equations, one of the main geometric assumptions of [2] is that

$$\overline{\Omega} = \Omega_{out}^T, \quad \text{where} \quad \Omega_{out}^T = \{x \in \overline{\Omega} \mid \exists t \in (0,T), \text{ s.t. } \overline{X}(t,0,x) \in \mathbb{R}^d \setminus \overline{\Omega}\}, \tag{1.2.9}$$

where $\overline{X}$ is the flow corresponding to the target velocity trajectory $\overline{v}_s$ defined as

$$\forall (x,t,s) \in \mathbb{R}^d \times [0,T]^2, \quad \partial_t \overline{X}(x,t,s) = \overline{v}_s(\overline{X}(x,t,s),t), \quad \overline{X}(x,s,s) = x.$$

In the article [2] the assumption (1.2.9) plays the key role in controlling the density of the fluid. In our case since the target velocity trajectory is v_s (defined in (1.2.4)) the assumption (1.2.9) is not satisfied because v_s vanishes at the lateral boundary of the domain Ω. Hence to control the density we make the assumption (1.2.5). Indeed, the assumption (1.2.5) implies that $supp(\rho_0) \Subset \Omega^T_{out}$, for T large enough. The assumption (1.2.5) exploits the hyperbolic nature of the continuity equation $(1.2.2)_1$ in order to control the coupled system (1.2.2). The condition (1.2.5) in fact guarantees that the density exactly equals $\rho_s = 1$, after some time $T_1 = T_{A_1} > \frac{d}{\inf_{x_2 \in [A_1, 1-A_1]} v_s}$, so that the non-homogeneous Navier-Stokes equations become homogeneous after some finite time. Also note that in [2] the authors uses two control functions (one for the density and one for velocity) for the purpose of controlling the non-homogeneous fluid. Contrary to that we use only one control acting on the velocity to stabilize the coupled system (1.2.2).

1.2.1.4 Idea and strategy

To obtain a local stabilization result for a non-linear system the usual method is to first linearize the problem around the target trajectory, solve the stabilization problem for the linearized system and then use a fixed point method to conclude the local stabilization of the original nonlinear problem. In this article due to regularity issues of the transport equation we avoid linearizing the whole system and instead we only linearize the momentum equation. For detailed calculation we refer the readers to Chapter 2, Section 2.1.5. This partial linearization strategy leads us to consider the stabilizability problem for the following equation:

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + ((v_s + e^{-\beta t} y) \cdot \nabla)\sigma - \beta \sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \\ \dfrac{\partial y}{\partial t} - \beta y - \nu \Delta y + (v_s \cdot \nabla) y + (y \cdot \nabla) v_s + \nabla q = f & \text{in} \quad Q_\infty, \\ \operatorname{div} y = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0 \cup \Gamma_{out}) \times (0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ y(x,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{1.2.10}$$

where f is a non-homogeneous source term with some suitable Sobolev regularity, Γ_{out} and Γ_0 are defined as follows

$$\Gamma_{out} = \{x \in \Gamma \mid (v_s \cdot n)(x) > 0\} = \{d\} \times (0,1),$$
$$\Gamma_0 = \Gamma \setminus (\Gamma_{in} \cup \Gamma_{out}).$$

One might be a bit surprised by the line $(1.2.10)_2$ where we have prescribed the boundary condition for σ only on the inflow boundary for the velocity field v_s (see (1.2.3)) and not on the inflow boundary of the perturbed vector field $(v_s + e^{-\beta t}y)$. This is because we will show that y is small in a suitable norm so that the inflow boundaries of both the vector fields v_s and $(v_s + e^{-\beta t}y)$ coincide. For further details we refer the readers to Chapter 2, Corollary 2.2.17.

We first study the stabilizability of (σ, y), the solution to (1.2.10) around the null state $(0, 0)$. Due to the incompressibility condition the parabolic-hyperbolic coupling involved in (1.2.10) is of weak nature and this allows us to study separately the system $(1.2.10)_1$-$(1.2.10)_3$ and $(1.2.10)_4$-$(1.2.10)_8$. One observes that the system $(1.2.10)_4$-$(1.2.10)_8$ is completely independent of σ. The stabilization problem $(1.2.10)_4$-$(1.2.10)_8$ is already studied in the literature in several contexts. We adapt mainly the strategy used in [34]. Following [34] to prove the stabilizability of this system we look for a control of the form (1.2.8). We choose the functions $\{g_j \mid 1 \leqslant j \leqslant N_c\}$, supported on Γ_c, so that we can prove some unique continuation property equivalent to the stabilizability of the system under consideration. Using the fact that g_j (for all $1 \leqslant j \leqslant N_c$) is supported on a smooth subset of Γ we further show that g_j is in $C^\infty(\Gamma)$. This in particular implies that the control u_c, of the form (1.2.8), is smooth in the space variable.

Our aim is to find a boundary control, in feedback form, able to stabilize y. At the same time we design the control such that the velocity y belongs to the space $V^{2,1}(Q_\infty)$. The $H^2(\Omega)$ regularity of the velocity field is important in our analysis since it is used to prove the stabilization of the continuity equation. This creates a difficulty because to prove the $V^{2,1}(Q_\infty)$ regularity of y solution of $(1.2.10)_4$-$(1.2.10)_8$, one must have a compatibility condition between the initial velocity y_0, assumed to be in $V_0^1(\Omega)$, and the boundary condition (*i.e.* the control u). We deal with this issue by adding a system of ordinary differential equations satisfied by $w_c(t) = (w_1(t), ..., w_{N_c}(t))$. To be precise we extend the system $(1.2.10)_4$-$(1.2.10)_8$ by adding the following ODE

$$w_c' = -\gamma w_c + \mathcal{K}\begin{pmatrix} e^{-\beta t}Py \\ w_c \end{pmatrix} \quad \text{in} \quad (0,\infty), \quad w_c(0) = 0,$$

where γ is a positive constant, P is the Leray projector from $L^2(\Omega)$ to $V_n^0(\Omega)$ ([39, Section 1.4] for the definition of Leray projector) and $\mathcal{K} \in \mathcal{L}(V_n^0(\Omega) \times \mathbb{R}^{N_c}, \mathbb{R}^{N_c})$ is a feedback operator. Since $y_0 \,|_\Gamma = 0$, imposing $w_c(0) = 0$ furnishes the desired compatibility condition between the initial and boundary conditions of y which is necessary to obtain the $V^{2,1}(Q_\infty)$ regularity of y.

Next we consider the problem $(1.2.10)_1$-$(1.2.10)_3$, for a given function y suitably small. Assuming that $\|y\|_{V^{2,1}(Q_\infty)}$ is small enough, σ_0 belongs to $L^\infty(\Omega)$ and satisfies the support condition (1.2.5) with $\sigma_0 = \rho_0$, we prove that the solution σ to the problem $(1.2.10)_1$-$(1.2.10)_3$ vanishes after a sufficiently large time $T_1 > 0$. Let us give a formal sketch of this proof. We extend the velocity fields v_s and y from Ω to $\mathbb{R}^2$. The extensions

are denoted by v_s and $\mathbb{E}y$ respectively. One observes that characteristic curves corresponding to the perturbed velocity field $(v_s+e^{-\beta t}\mathbb{E}y)$ stay close to that of v_s in a suitable norm and this can be obtained from the smallness assumption over $\|y\|_{V^{2,1}(Q_\infty)}$. In fact using that the characteristics corresponding to the velocity fields v_s and $(v_s + e^{-\beta t}\mathbb{E}y)$ are close we show that the particles initially lying in the support of σ_0 are transported out of the domain in some finite time $T > T_{A_1} = \frac{d}{A_1(1-A_1)}$ along the flow corresponding to the perturbed velocity field. This flushing phenomenon is eventually used to obtain that the solution Ψ to the following auxiliary transport problem

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} + ((v_s + e^{-\beta t}\mathbb{E}y)\cdot\nabla)\Psi = 0 & \text{in} \quad \mathbb{R}^2 \times (0, T_1), \\ \Psi(\cdot, 0) = \vartheta & \text{in} \quad \mathbb{R}^2, \end{cases} \tag{1.2.11}$$

where $\vartheta \in C^\infty(\mathbb{R}^2)$, $\vartheta(x_1, x_2) \in [0, 1]$ is such that

$$\vartheta(x_1, x_2) = \begin{cases} 0 & \text{if} \quad (x_1, x_2) \in [0, d] \times [A_1, 1 - A_1], \\ 1 & \text{if} \quad (x_1, x_2) \in \mathbb{R}^2 \setminus [-\frac{\varepsilon}{2}, d + \frac{\varepsilon}{2}] \times [A_1 - \frac{\varepsilon}{2}, 1 - A_1 + \frac{\varepsilon}{2}], \end{cases} \tag{1.2.12}$$

for $\varepsilon > 0$ small enough, satisfies

$$\forall x \in \Omega, \quad \Psi(x, T_1) = 1, \tag{1.2.13}$$

for

$$T_1 = \frac{d + \varepsilon}{(A_1 - \varepsilon)(1 - A_1 + \varepsilon)} > T_{A_1}.$$

Then we introduce a Lyapunov functional

$$E_{loc}(t) = \frac{1}{2}\int_\Omega \Psi(x, t)|\sigma(x, t)|^2 dx \quad \text{for all} \quad t \in [0, T_1]. \tag{1.2.14}$$

The idea is that this quantity $E_{loc}(t)$ will measure the L^2 norm of $\sigma(\cdot, t)$ localized in the support of $\Psi(\cdot, t)$. Then we show that $E_{loc}(T_1)$ vanishes in Ω as a consequence of the support condition (1.2.5). This observation along with (1.2.13) is used to infer that $\sigma(\cdot, T)$ is identically zero in Ω for $T \geqslant T_1$.

Once the system (1.2.10) is stabilized, we use the Schauder fixed point Theorem to prove a local stabilization result corresponding to the original non linear system under consideration.

1.2.2 Chapter 3: Local existence of strong solutions for a fluid-structure interaction model

In this chapter we are interested in studying a fluid structure interaction problem in a 2d channel. The fluid flow here is modeled by the compressible Navier-Stokes equations.

Concerning the elastic structure we will consider an Euler-Bernoulli damped beam located on a portion of the boundary. We establish a local in time existence result of strong solutions of such a fluid structure interaction problem. Let us precisely describe the dynamical system we consider.

1.2.2.1 Introduction to the system

Let Ω be the domain $\mathbb{T}_L \times (0,1) \subset \mathbb{R}^2$, where $\mathbb{T}_L$ is the one dimensional torus identified with $(0,L)$ with periodic conditions. The boundary of Ω is denoted by Γ. We set

$$\Gamma_s = \mathbb{T}_L \times \{1\}, \quad \Gamma_\ell = \mathbb{T}_L \times \{0\}, \quad \Gamma = \Gamma_s \cup \Gamma_\ell.$$

Now for a given function

$$\eta : \Gamma_s \times (0,\infty) \to (-1,\infty),$$

which will correspond to the displacement of the one dimensional beam, let us denote by Ω_t and $\Gamma_{s,t}$ the following sets

$$\begin{aligned} \Omega_t &= \{(x,y) \mid x \in (0,L), \quad 0 < y < 1 + \eta(x,t)\} &&= \text{domain of the fluid at time } t, \\ \Gamma_{s,t} &= \{(x,y) \mid x \in (0,L), \quad y = 1 + \eta(x,t)\} &&= \text{the beam at time } t. \end{aligned}$$

The reference configuration of the beam is Γ_s, and we set

$$\begin{aligned} &\Sigma_T = \Gamma \times (0,T), && \Sigma_T^s = \Gamma_s \times (0,T), \\ &\widetilde{\Sigma_T^s} = \cup_{t\in(0,T)} \Gamma_{s,t} \times \{t\}, && \Sigma_T^\ell = \Gamma_\ell \times (0,T), \\ &Q_T = \Omega \times (0,T), && \widetilde{Q_T} = \cup_{t\in(0,T)} \Omega_t \times \{t\}. \end{aligned} \tag{1.2.15}$$

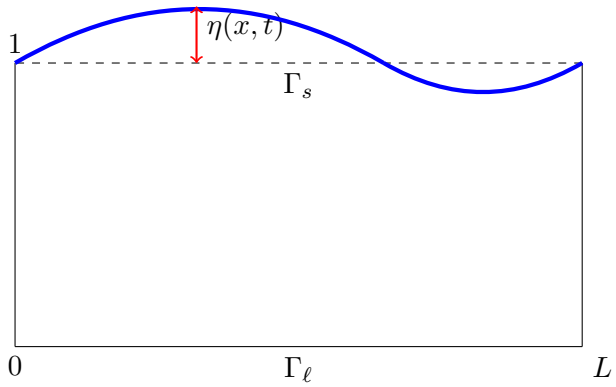


Figure 1.1: Domain Ω_t.

We consider a fluid with density ρ and velocity $\mathbf{u}$. The fluid structure interaction system

coupling the compressible Navier-Stokes and the Euler-Bernoulli damped beam equation is modeled by

$$\begin{cases} \rho_t + \mathrm{div}(\rho \mathbf{u}) = 0 & \text{in} \quad \widetilde{Q_T}, \\ (\rho \mathbf{u}_t + \rho(\mathbf{u}.\nabla)\mathbf{u}) - (2\mu \mathrm{div}(D(\mathbf{u})) + \mu' \nabla \mathrm{div}\mathbf{u}) + \nabla p(\rho) = 0 & \text{in} \quad \widetilde{Q_T}, \\ \mathbf{u}(\cdot, t) = (0, \eta_t) & \text{on} \quad \widetilde{\Sigma_T^s}, \\ \mathbf{u}(\cdot, t) = (0, 0) & \text{on} \quad \Sigma_T^\ell, \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 & \text{in} \quad \Omega, \\ \rho(\cdot, 0) = \rho_0 & \text{in} \quad \Omega, \\ \eta_{tt} - \beta \eta_{xx} - \delta \eta_{txx} + \alpha \eta_{xxxx} = (T_f)_2 & \text{on} \quad \Sigma_T^s, \\ \eta(\cdot, 0) = 0 \quad \text{and} \quad \eta_t(\cdot, 0) = \eta_1 & \text{in} \quad \Gamma_s. \end{cases} \tag{1.2.16}$$

The initial condition for the density is assumed to be positive and bounded. We fix the positive constants m and M such that

$$0 < m = \min_{\overline{\Omega}} \rho_0(x, y), \quad M = \max_{\overline{\Omega}} \rho_0(x, y). \tag{1.2.17}$$

In our model the fluid adheres to the plate and is viscous. This implies that the velocities corresponding to the fluid and the structure coincide at the interface and hence the condition $(1.2.16)_3$ holds. In the system (1.2.16), $D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ is the symmetric part of the gradient and the real constants μ, μ' are the Lamé coefficients which are supposed to satisfy

$$\mu > 0, \quad \mu' \geqslant 0.$$

In our case the fluid is isentropic i.e. the pressure $p(\rho)$ is only a function of the fluid density ρ and is given by

$$p(\rho) = a\rho^\gamma,$$

where $a > 0$ and $\gamma > 1$ are positive constants.

We assume that there exists a constant external force $p_{ext} > 0$ which acts on the beam. The external force p_{ext} can be written as follows

$$p_{ext} = a\overline{\rho}^\gamma,$$

for some positive constant $\overline{\rho}$.

To incorporate this external forcing term p_{ext} into the system of equations (1.2.16), we introduce the following

$$P(\rho) = p(\rho) - p_{ext} = a\rho^\gamma - a\overline{\rho}^\gamma. \tag{1.2.18}$$

Since $\nabla p(\rho) = \nabla P(\rho)$, from now onwards we will use $\nabla P(\rho)$ instead of $\nabla p(\rho)$ in the equation $(1.2.16)_2$.

In the beam equation the constants, $\alpha > 0$, $\beta \geqslant 0$ and $\delta > 0$ are respectively the adimensional rigidity, stretching and friction coefficients of the beam. The non-homogeneous source term of the beam equation $(T_f)_2$ is the net surface force on the structure which is the resultant of the force exerted by the fluid on the structure and the external force p_{ext} and it is given by

$$(T_f)_2 = ([-2\mu D(\mathbf{u}) - \mu'(\text{div}\mathbf{u})\mathbf{I}_d] \cdot \mathbf{n}_t + P\mathbf{n}_t) \mid_{\Gamma_{s,t}} \sqrt{1+\eta_x^2} \cdot \vec{e}_2 \quad \text{on} \quad \Sigma_T^s, \tag{1.2.19}$$

where $\mathbf{I}_d$ is the identity matrix, $\mathbf{n}_t$ is the outward unit normal to $\Gamma_{s,t}$ given by

$$\mathbf{n}_t = -\frac{\eta_x}{\sqrt{1+\eta_x^2}}\vec{e}_1 + \frac{1}{\sqrt{1+\eta_x^2}}\vec{e}_2$$

($\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$).

Our interest is to prove the local in time existence of a strong solution to system (1.2.16)-(1.2.18)-(1.2.19) i.e we prove that given a prescribed initial data $(\rho_0, \mathbf{u}_0, \eta_1)$, there exists a solution of system (1.2.16)-(1.2.18)-(1.2.19) with a certain Sobolev regularity in some time interval $(0,T)$, provided that the time T is small enough.

1.2.2.2 Existing results

In the last decades fluid-structure interaction problems have been an area of active research. There is a rich literature concerning the motion of a structure inside or at the boundary of a domain containing a viscous incompressible Newtonian fluid, whose behavior is described by Navier-Stokes equations. We will quote some articles dealing with incompressible fluid-structure interaction problems where the structure appears on the fluid boundary and is modeled by Euler-Bernoulli damped beam equations $(1.2.16)_7$-$(1.2.16)_8$. For example we refer the readers to [3] (local in time existence of strong solutions), [12] (existence of weak solutions), [35] (feedback stabilization), [26] (global in time existence) and the references therein for a very detailed discussion of such problems.

In the past few years there have been works exploring the fluid-structure interaction problems comprising the compressible Navier-Stokes equations with rigid/elastic body immersed in the fluid domain. For instance one can consult the articles [5], [4], [6], [30] and [8]. We will comment more on the contents of these articles in Chapter 3 Section 3.1.6.
On the other hand there is a very limited number of works on the compressible fluid-structure interaction problems with the structure appearing on the boundary of the fluid domain. The article [24] deals with a 1-D structure governed by plate equations coupled with a bi-dimensional compressible fluid where the structure is located at a part of the boundary. Here the authors consider the velocity field as a potential and in their

case the non linearity occurs only in the equation modeling the density. In a recent article [1] the authors prove the Hadamard well posedness of a linear compressible fluid structure interaction problem (three dimensional compressible fluid interacting with a bi-dimensional elastic structure) defined in a fixed domain and considering the Navier-slip boundary condition at the interactive boundary. In yet another recent article [10] the authors consider a three dimensional compressible fluid structure interaction model where the structure located at the boundary is a shell of Koiter-type with some prescribed thickness. In the spirit of [33] and [20] the authors prove the existence of a weak solution for their model with the adiabatic constant restricted to $\gamma > \frac{12}{7}$. They show that a weak solution exists until the structure touches the boundary of the fluid domain.

1.2.2.3 Presentation of our result and contribution

The notion of solution to the system (1.2.16)-(1.2.18)-(1.2.19) will be understood up to the use of a diffeomorphism which transfers the system (1.2.16)-(1.2.18)-(1.2.19) to a reference time independent domain. To transform the system (1.2.16)-(1.2.18)-(1.2.19) in the reference configuration, for η satisfying $1+\eta(x,t) > 0$ for all $(x,t) \in \Sigma_T^s$, we introduce the following change of variables

$$\begin{aligned}
&\Phi_{\eta(t)} : \Omega_t \longrightarrow \Omega \quad \text{defined by} \quad \Phi_{\eta(t)}(x,y) = (x,z) = \left(x, \frac{y}{1+\eta(x,t)}\right), \\
&\Phi_{\eta} : \widetilde{Q_T} \longrightarrow Q_T \quad \text{defined by} \quad \Phi_{\eta}(x,y,t) = (x,z,t) = \left(x, \frac{y}{1+\eta(x,t)}, t\right).
\end{aligned}$$

For each $t \in [0,T)$, the map $\Phi_{\eta(t)}$ is a C^1- diffeomophism from Ω_t onto Ω provided that $(1+\eta(x,t)) > 0$ for all $x \in \mathbb{T}_L$ and that $\eta(\cdot,t) \in C^1(\Gamma_s)$.
We set the following notations

$$\widehat{\rho}(x,z,t) = \rho(\Phi_\eta^{-1}(x,z,t)), \ \widehat{\mathbf{u}}(x,z,t) = (\widehat{u}_1, \widehat{u}_2) = \mathbf{u}(\Phi_\eta^{-1}(x,z,t)). \tag{1.2.20}$$

After transformation the nonlinear system (1.2.16)-(1.2.18)-(1.2.19) is rewritten in the following form

$$\begin{cases}
\widehat{\rho}_t + \begin{bmatrix} \widehat{u}_1 \\ \frac{1}{(1+\eta)}(\widehat{u}_2 - \eta_t z - \widehat{u}_1 z \eta_x) \end{bmatrix} \cdot \nabla \widehat{\rho} + \widehat{\rho} \mathrm{div} \widehat{\mathbf{u}} = F_1(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) & \text{in} \quad Q_T, \\
\widehat{\rho}\widehat{\mathbf{u}}_t - \mu \Delta \widehat{\mathbf{u}} - (\mu' + \mu) \nabla(\mathrm{div} \widehat{\mathbf{u}}) + \nabla P(\widehat{\rho}) = F_2(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) & \text{in} \quad Q_T, \\
\widehat{\mathbf{u}} = \eta_t \vec{e_2} & \text{on} \quad \Sigma_T^s, \\
\widehat{\mathbf{u}}(\cdot,t) = 0 & \text{on} \quad \Sigma_T^\ell, \\
\widehat{\mathbf{u}}(\cdot,0) = \mathbf{u}_0 & \text{in} \quad \Omega, \\
\widehat{\rho}(\cdot,0) = \rho_0 & \text{in} \quad \Omega, \\
\eta_{tt} - \beta \eta_{xx} - \delta \eta_{txx} + \alpha \eta_{xxxx} = F_3(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) & \text{on} \quad \Sigma_T^s, \\
\eta(0) = 0 \quad \text{and} \quad \eta_t(0) = \eta_1 & \text{in} \quad \Gamma_s.
\end{cases} \tag{1.2.21}$$

For the exact expressions of the non-homogeneous source terms (which are not necessarily non-linear) $F_1(\widehat{\rho},\widehat{\mathbf{u}},\eta)$, $F_2(\widehat{\rho},\widehat{\mathbf{u}},\eta)$ and $F_3(\widehat{\rho},\widehat{\mathbf{u}},\eta)$ one can look into the Chapter 3, Section 3.1.2.
Now we define the notion of strong solution of the system (1.2.16)-(1.2.18)-(1.2.19) in terms of the strong solution of the system (1.2.21).

Definition 1.2.2. *The triplet $(\rho,\mathbf{u},\eta)$ is a strong solution of the system (1.2.16)-(1.2.18)-(1.2.19) if*

$$\eta\in C^0([0,T];H^{9/2}(\Gamma_s)),\qquad \eta_t\in L^2(0,T;H^4(\Gamma_s))\cap C^0([0,T];H^3(\Gamma_s)),$$
$$\eta_{tt}\in L^2(0,T;H^2(\Gamma_s))\cap C^0([0,T];H^1(\Gamma_s)),\quad \eta_{ttt}\in L^2(0,T;L^2(\Gamma_s)),$$

the following holds

$$1+\eta(x,t)\geqslant\delta_0>0\quad on\quad \Sigma_T^s, \tag{1.2.22}$$

for some positive constant $\delta_0>0$ and the triplet $(\widehat{\rho},\widehat{\mathbf{u}},\eta)=(\rho\circ\Phi_\eta^{-1},\mathbf{u}\circ\Phi_\eta^{-1},\eta)$ solves (1.2.21) in the following Sobolev spaces

$$\begin{aligned}&\widehat{\rho}\in C^0([0,T];H^2(\Omega)),\ \widehat{\rho}_t\in C^0([0,T];H^1(\Omega)),\\ &\widehat{\mathbf{u}}\in L^2(0,T;\mathbf{H}^3(\Omega))\cap C^0([0,T];\mathbf{H}^{5/2}(\Omega)),\\ &\widehat{\mathbf{u}}_t\in L^2(0,T;\mathbf{H}^2(\Omega))\cap C^0([0,T];\mathbf{H}^1(\Omega)),\\ &\widehat{\mathbf{u}}_{tt}\in L^2(0,T;\mathbf{L}^2(\Omega)).\end{aligned} \tag{1.2.23}$$

Note that $(\rho,\mathbf{u})$ can then be obtained from $(\widehat{\rho},\widehat{\mathbf{u}})$ by $(\rho,\mathbf{u})=(\widehat{\rho}\circ\Phi_\eta,\widehat{\mathbf{u}}\circ\Phi_\eta)$.

Now we precisely state the main result of the article.

Theorem 1.2.3. *Assume that*

$$\begin{cases}(i)\quad (a)\ \textit{Regularity of initial conditions}:\rho_0\in H^2(\Omega),\ \eta_1\in H^3(\Gamma_s),\ \mathbf{u}_0\in\mathbf{H}^3(\Omega).\\ \qquad (b)\ \textit{Compatibility between initial and boundary conditions}:\\ \qquad (b)_1\ \left(\mathbf{u}_0-\begin{bmatrix}0\\ z\eta_1\end{bmatrix}\right)=0\quad on\quad\Gamma,\\ \qquad (b)_2\ -P'(\rho_0)\nabla\rho_0-(\delta\eta_{1,xx}-(\mu+2\mu')(u_0)_{2,z}+P(\rho_0))z\rho_0\vec{e}_2+z\rho_0(\mathbf{u}_0)_z\eta_1\\ \qquad\qquad -\rho_0(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0-(-\mu\Delta-(\mu+\mu')\nabla\mathrm{div})\mathbf{u}_0=0\quad on\quad\Gamma,\\ (ii)\quad (1.2.17)\ holds,\end{cases} \tag{1.2.24}$$

where we use the notations $P'(\rho_0)=\nabla P(\rho_0)$, $P(\rho_0)=(a\rho_0^\gamma-a\overline{\rho}^\gamma)$ and $\mathbf{u}_0=((u_0)_1,(u_0)_2)$. Then there exists $T>0$ such that the system (1.2.21) admits a solution $(\widehat{\rho},\widehat{\mathbf{u}},\eta)$ in the following functional framework

$$\widehat{\rho}\in\{\rho\in C^0([0,T];H^2(\Omega))\mid\rho_t\in C^0([0,T];H^1(\Omega))\},$$

$$\widehat{\mathbf{u}}\in\{\mathbf{u}\in L^2(0,T;\mathbf{H}^3(\Omega))\cap C^0([0,T];\mathbf{H}^{5/2}(\Omega))\mid\mathbf{u}_t\in L^2(0,T;\mathbf{H}^2(\Omega))\cap C^0([0,T];\mathbf{H}^1(\Omega)),$$
$$\mathbf{u}_{tt}\in L^2(0,T;\mathbf{L}^2(\Omega))\},$$

$$\eta\in\{\eta\in C^0([0,T];H^{9/2}(\Gamma_s)),\ \eta(x,0)=0\mid\eta_t\in L^2(0,T;H^4(\Gamma_s))\cap C^0([0,T];H^3(\Gamma_s)),$$
$$\eta_{tt}\in L^2(0,T;H^2(\Gamma_s))\cap C^0([0,T];H^1(\Gamma_s)),\eta_{ttt}\in L^2(0,T;L^2(\Gamma_s))\}.$$

Consequently in the sense of Definition 3.1.6 the system (1.2.16)-(1.2.18)-(1.2.19) *admits a strong solution* $(\rho, \mathbf{u}, \eta)$.

To the best of our knowledge there is no prior existing work (neither in 3 nor in 2 dimensions) proving the existence of strong solutions for the non-linear compressible fluid-structure interaction problems (defined in a time dependent domain) considering the structure at the boundary of the fluid domain. In the present article we address this problem in the case of a fluid contained in a 2d channel and interacting with a 1d structure at the boundary. The system we consider shares a close resemblance with the ones considered in [1] (existence of mild solution) and [10] (existence of weak solution) but our approach is different from that of [1] and [10]. In [1], since the problem itself is linearized in a fixed domain, the authors can directly use a semigroup formulation to study the existence of mild solution, whereas [10] considers weak solutions and a 4 level approximation process (using artificial pressure, artificial viscosity, regularization of the boundary and Galerkin approximation for the momentum equation). In the study of weak solutions (in [33], [20], [10]) one of the major difficulties is to pass to the limit in the non-linear pressure term which is handled by introducing a new unknown called the effective viscous flux. In our case of strong regularity framework we do not need to introduce the effective viscous flux and for small enough time T, the term $\nabla P(\sigma + \overline{\rho})$ can be treated as a non homogeneous source term. Our approach is based on studying the regularity properties of a decoupled parabolic equation, continuity equation and a beam equation. This is done by obtaining some a priori estimates and exploiting the analyticity of the semigroup corresponding to the beam equation. Then the existence result for the non-linear coupled problem is proved by using the Schauder fixed point Theorem. We prove the existence of the fixed point in a suitable convex set, which is constructed very carefully based on the estimates of the decoupled problems and the estimates of the non-homogeneous source terms. This led us to choose this convex set as a product of balls (in various functional spaces) of different radius. In the present article we prove a local in time existence result. For incompressible Navier-Stokes equation interacting with a damped beam at the fluid boundary, local in time existence result of strong solutions can be found in [3] and [31].

1.2.2.4 Idea and strategy

We prove the existence of local in time strong solution of the system (1.2.16)-(1.2.18)-(1.2.19) only when the beam displacement η is close to zero. Again observe that $(\widehat{\rho} = \overline{\rho}, \widehat{\mathbf{u}} = 0, \widehat{\eta} = 0)$ is a steady state solution of the system (1.2.16)-(1.2.18)-(1.2.19) and hence of the system (1.2.21). So to work in a neighborhood of $\eta = 0$, we make the following change of unknowns in (1.2.21),

$$\sigma = \widehat{\rho} - \overline{\rho}, \quad \mathbf{v} = (v_1, v_2) = \widehat{\mathbf{u}} - 0, \quad \eta = \eta - 0.$$

We transform the resulting system into a homogeneous Dirichlet boundary value problem by performing further the following change of unknown

$$\mathbf{w} = (w_1, w_2) = \mathbf{v} - z\eta_t \vec{e_2}.$$

With the new unknown $\mathbf{w}$, we write the transformed system in the following form

$$\begin{cases} \sigma_t + \begin{bmatrix} w_1 \\ \frac{1}{(1+\eta)}(w_2 - w_1 z\eta_x) \end{bmatrix} \cdot \nabla\sigma = G_1(\sigma, \mathbf{w}, \eta) & \text{in} \quad Q_T, \\ (\sigma + \overline{\rho})\mathbf{w}_t - \mu\Delta\mathbf{w} - (\mu + \mu')\nabla \text{div}\mathbf{w} = G_2(\sigma, \mathbf{w}, \eta) & \text{in} \quad Q_T, \\ \mathbf{w} = 0 & \text{on} \quad \Sigma_T, \\ \mathbf{w}(\cdot, 0) = \mathbf{w}_0 = \mathbf{u}_0 - z\eta_1 \vec{e}_2 & \text{in} \quad \Omega, \\ \sigma(\cdot, 0) = \sigma_0 = \rho_0 - \overline{\rho} & \text{in} \quad \Omega, \\ \eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = G_3(\sigma, \mathbf{w}, \eta) & \text{on} \quad \Sigma_T^s, \\ \eta(0) = 0 \quad \text{and} \quad \eta_t(0) = \eta_1 & \text{in} \quad \Gamma_s. \end{cases} \tag{1.2.25}$$

For the explicit expression of the non homogeneous source terms $G_1(\sigma, \mathbf{w}, \eta)$, $G_2(\sigma, \mathbf{w}, \eta)$ and $G_3(\sigma, \mathbf{w}, \eta)$ we refer the readers to Chapter 3 Section 3.1.4.

We observe that in the new system (1.2.25) the coupling between the velocity of the fluid and the elastic structure appears only through source terms. In order to solve the system (1.2.25) we first study some linear equations. In order to analyze the local in time existence of strong solution the difficulty is to track the dependence of the constants (appearing in the inequalities) with respect to the time parameter 'T'. In this direction we first obtain a priori estimates for the linear density and velocity equations with non homogeneous source terms in the spirit of [40]. Then we prove the existence of strong solutions for a linear beam equation. The proof strongly relies on the analyticity of the corresponding beam semigroup (see [14] for details). Also inspired by a technique from [37], we obtain estimates with the constants independent of 'T' for the beam equation by fixing a constant $\overline{T} > 0$ and restrict ourselves to work in the time interval $(0, T)$ where $T < \overline{T}$.

Finally we use the Schauder fixed point theorem to prove the local in time existence of strong solutions to the problem (1.2.25). This in turn proves the local in time existence of a strong solution to problem (1.2.21) and consequently to problem (1.2.16)-(1.2.18)-(1.2.19) in the sense of Definition 1.2.2.

1.2.3 Chapter 4: Observability of the adjoint of a linearized compressible fluid-structure model in a 2d channel

In this chapter we study the observability of the adjoint of a linearized compressible fluid-structure model in a 2d channel. The fluid flow here is modeled by the compressible Navier-Stokes equations. Concerning the structure we will consider a damped

Euler-Bernoulli beam located on a portion of the boundary. The fluid solid interactive dynamics considered in this chapter resembles with the one in Chapter 3 but the coupling conditions at the boundary are different. In the present article we establish an observability inequality for the adjoint linearized fluid structure interaction problem under consideration which in principle is equivalent to the null controllability of the system.

1.2.3.1 Introduction to the system

The result of this chapter corresponds to a linearized fluid structure interaction problem posed in a time independent domain. The linearized model is derived from a non linear problem by means of a suitable linearization process. Here we only present the linear model and refer to Chapter 4 for the details of the corresponding non linear system. Let us introduce a few notations:

$$\begin{aligned} \Omega = (0,d)\times(0,1) \quad &= \text{The fluid domain in the reference configuration,} \\ \Gamma_s = (0,d)\times\{1\} \quad &= \text{The structural domain in the reference configuration,} \end{aligned}$$

for some positive constant d. Let $\overline{\rho}$ and $\overline{u}_1$ are positive constants. We fix the final time as follows

$$T > \tfrac{d}{\overline{u}_1}. \tag{1.2.26}$$

The main motivation of the present chapter is to study the controllability of a fluid structure interaction problem with the controls acting locally at the lateral boundaries of the channel. In fact, for technical reasons, it is easier to deal with distributed controls. We therefore choose to extend the domain where the equation is set and consider distributed controls localized in the extension, so that one can recover boundary controllability results by a suitable restriction argument. To make our point precise we set

$$L = 3\overline{u}_1 T > 0,$$

embed Γ_s into $\mathbb{T}_L \times \{1\}$ and Ω into $\mathbb{T}_L \times (0,1)$, where $\mathbb{T}_L$ is the one dimensional torus identified with $(-L, d+L) = (-3\overline{u}_1 T, d+3\overline{u}_1 T)$ with periodic conditions. The control zones in the reference configuration are written as follows

$$\begin{aligned} \omega &= ((-L,0)\times(0,1)) \cup ((d,d+L]\times(0,1)), \\ \omega_1 &= ((-L,0)\times\{1\}) \cup ((d,d+L]\times\{1\}). \end{aligned} \tag{1.2.27}$$

The linearized model is given as follows

$$\begin{cases} \partial_t \widetilde{\sigma} + \overline{u}_1 \partial_x \widetilde{\sigma} + \overline{\rho} \mathrm{div} \widetilde{u} = v_{\widetilde{\sigma}} \chi_\omega & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \overline{\rho}(\partial_t \widetilde{u} + \overline{u}_1 \partial_x \widetilde{u}) - \mu \Delta \widetilde{u} - (\mu' + \mu) \nabla (\mathrm{div} \widetilde{u}) & \\ \qquad + P'(\overline{\rho}) \nabla \widetilde{\sigma} = v_{\widetilde{u}} \chi_\omega & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \widetilde{u} \cdot n = \widetilde{u}_2 = \partial_t \beta + \overline{u}_1 \partial_x \beta & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \widetilde{u}(\cdot, t) \cdot n = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T), \\ \mathrm{curl} \widetilde{u} = 0 & \text{on } ((\mathbb{T}_L \times \{0,1\}) \times (0,T), \\ \widetilde{u}(\cdot, 0) = \widehat{u}_0 - \overline{u} = \widetilde{u}_0 & \text{in } \mathbb{T}_L \times (0,1), \\ \widetilde{\sigma}(\cdot, 0) = \widehat{\rho}_0 - \overline{\rho} = \widetilde{\sigma}_0 & \text{in } \mathbb{T}_L \times (0,1), \\ \partial_{tt} \beta - \partial_{txx} \beta + \partial_{xxxx} \beta & \\ \qquad = \big(- (\mu' + 2\mu) \mathrm{div} \widetilde{u} + P'(\overline{\rho}) \widetilde{\sigma} \big) + v_\beta \chi_{\omega_1} & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \beta(0) = \beta_0 \quad \text{and} \quad \partial_t \beta(0) = \beta_1 & \text{in } \mathbb{T}_L \times \{1\}. \end{cases} \tag{1.2.28}$$

Here, $v_{\widetilde{\sigma}} \chi_\omega$ $v_{\widetilde{u}} \chi_\omega$ and $v_\beta \chi_{\omega_1}$ are the control functions corresponding to the fluid density, velocity and the beam displacement and velocity respectively and the goal is to study the null controllability of (1.2.28).

1.2.3.2 Existing results

We have already discussed the existence results for compressible fluid structure interaction problems in Section 1.2.2. Here we mainly concentrate on the literature concerning the controllability results related to some fluid models. Concerning the incompressible Navier-Stokes equations in a $2D$ domain one can find a result proving the local exact controllability to trajectories with localized boundary control in [25]. It is assumed in [25] that the fluid satisfies no vorticity boundary condition in the complement of the control part of the boundary. Local exact controllability to trajectories for incompressible Navier-Stokes equations in a $3D$ domain with distributed control and homogeneous Dirichlet boundary condition can be found in [29]. With less regularity assumption on the target trajectory the result in [29] was improved in [22]. We would also like to mention the article [27] for the local exact distributed controllability to trajectories for incompressible Navier-Stokes equations in a $3D$ domain with non linear Navier-slip boundary condition. In all of these articles the fluid is assumed to be homogeneous $i.e$ the fluid density is constant. In a very recent article [2] the authors prove the local exact boundary controllability to smooth trajectories for a non homogeneous incompressible Navier-Stokes equation in a three dimensional domain. For global controllability results for incompressible Navier-Stokes equations we refer the readers to [16], [13] and the references therein.
Now we quote a few articles dealing with the controllability issues of fluid structure interaction models. In fact to the best of our knowledge the only known results concerning the controllability issues of a fluid structure interaction problem in dimension greater

than one deals with the motion of a rigid body inside an incompressible fluid modeled by Navier-Stokes equations where the structural motion are given by the balance of linear and angular momentum. Local null controllability of such an interaction problem in dimension two can be found in [9] and [28]. In dimension three a local null controllability for such a system is proved in [7]. The article [35] deals with the problem of feedback stabilization (in infinite time) for an incompressible fluid structure interaction problem in a $2D$ channel where the structure appears at the fluid boundary and is modeled by an Euler-Bernoulli damped beam, the one we consider in $(1.2.28)_8$-$(1.2.28)_9$. To our knowledge so far there does not exist any article dealing with the finite time controllability of a fluid structure interaction problem (neither for incompressible nor compressible fluids) in dimension more than one where the structure appears at the fluid boundary.

We also mention the related literature on controllability issues for compressible Navier-Stokes equations. In fact our strategy to handle the coupling of the fluid velocity and density in the system (1.2.28) amounts in introducing a new unknown namely the effective viscous flux and this strategy is inspired from the article [18]. The articles [19] and [17] concern the motion of a fluid in dimension one whereas [18] deals with fluid flows in dimension two and three. In fact in the next section we will comment on the relation between our work and an open question posed in the article [18].

1.2.3.3 Presentation of our result and contribution

We are primarily interested in the null controllability of the system (1.2.28) at time T, where T is already fixed in (1.2.26). Hence as a first step of proving null controllability we will prove an observability inequality of the following adjoint system

$$\begin{cases} -\partial_t\sigma - \overline{u}_1\partial_x\sigma - P'(\overline{\rho})\mathrm{div} v = 0 & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T),\\ -\overline{\rho}(\partial_t v + \overline{u}_1\partial_x v) - \mu\Delta v - (\mu'+\mu)\nabla(\mathrm{div} v) & \\ \qquad -\overline{\rho}\nabla\sigma = 0 & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T),\\ v\cdot n = v_2 = \psi & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T),\\ v(\cdot,t)\cdot n = 0 & \text{on } (\mathbb{T}_L\times\{0\})\times(0,T),\\ \mathrm{curl} v = 0 & \text{on } (\mathbb{T}_L\times\{0,1\})\times(0,T),\\ v(\cdot,T) = v_T & \text{in } \mathbb{T}_L\times(0,1),\\ \sigma(\cdot,T) = \sigma_T & \text{in } \mathbb{T}_L\times(0,1),\\ \partial_{tt}\psi + \partial_{txx}\psi + \partial_{xxxx}\psi & \\ = (\partial_t + \overline{u}_1\partial_x)[(\mu'+2\mu)\mathrm{div}\, v + \overline{\rho}\sigma] & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T),\\ \psi(T) = \psi_T \quad \text{and} \quad \partial_t\psi(T) = \psi_T^1 & \text{in } \mathbb{T}_L\times\{1\}. \end{cases} \tag{1.2.29}$$

Both the systems (1.2.28) and (1.2.29) are well posed provided the initial datum are sufficiently regular and satisfy some compatibility conditions at the boundary. For details about the well posedness we refer the readers to Chapter 4. Let us now state the central theorem of this chapter as follows:

Theorem 1.2.4. *There exists a positive constant C such that for all*

$$(\sigma_T, v_T, \psi_T, \psi_T^1) \in H^2(\mathbb{T}_L \times (0,1)) \times H^3(\mathbb{T}_L \times (0,1)) \times H^3(\mathbb{T}_L \times \{1\}) \times H^1(\mathbb{T}_L \times \{1\}), \tag{1.2.30}$$

if the following compatibility conditions hold:

$$\begin{aligned}
&(i)\ (a) && v_T \cdot n = (v_T)_2 = \psi_T, && \text{on } \mathbb{T}_L \times \{1\},\\
&\quad (b) && v_T \cdot n = (v_T)_2 = 0, && \text{on } \mathbb{T}_L \times \{0\},\\
&(ii) && \operatorname{curl} v_T = 0, && \text{on } (\mathbb{T}_L \times \{0,1\})\\
&(iii)\ (a) && -\overline{u}_1 \partial_x (v_T)_2 + \tfrac{\mu}{\overline{\rho}} \Delta (v_T)_2 + \tfrac{(\mu+\mu')}{\overline{\rho}} \partial_z (\operatorname{div} v_T) && \\
& && -\partial_z \sigma_T = \psi_T^1 && \text{on } \mathbb{T}_L \times \{1\},\\
&\quad (b) && -\overline{u}_1 \partial_x (v_T)_2 + \tfrac{\mu}{\overline{\rho}} \Delta (v_T)_2 + \tfrac{(\mu+\mu')}{\overline{\rho}} \partial_z (\operatorname{div} v_T) && \\
& && -\partial_z \sigma_T = 0 && \text{on } \mathbb{T}_L \times \{0\},
\end{aligned} \tag{1.2.31}$$

then the solution (σ, v, ψ) of the problem (1.2.29) solves the following observability inequality

$$\begin{aligned}
&\int\limits_{\mathbb{T}_L \times (0,1)} |\sigma|^2(\cdot,0) + \int\limits_{\mathbb{T}_L \times (0,1)} |\nabla \sigma|^2(\cdot,0) + \int\limits_{\mathbb{T}_L \times (0,1)} |v|^2(\cdot,0) + \int\limits_{\mathbb{T}_L \times (0,1)} |\nabla v|^2(\cdot,0)\\
&+ \int\limits_{\mathbb{T}_L \times (0,1)} |\nabla^2 v|^2(\cdot,0) + \int\limits_{\mathbb{T}_L \times \{1\}} |\psi|^2(\cdot,0) + \int\limits_{\mathbb{T}_L \times \{1\}} |\partial_x \psi|^2(\cdot,0)\\
&+ \int\limits_{\mathbb{T}_L \times \{1\}} |\partial_{xx} \psi|^2(\cdot,0) + \int\limits_{\mathbb{T}_L \times \{1\}} |\partial_{xxx} \psi|^2(\cdot,0) + \int\limits_{\mathbb{T}_L \times \{1\}} |\partial_t \psi|^2(\cdot,0)\\
&+ \int\limits_{\mathbb{T}_L \times \{1\}} |\partial_{tx} \psi|^2(\cdot,0) \leqslant C\Big(\iint\limits_{\omega \times (0,T)} |\sigma|^2 + \iint\limits_{\omega \times (0,T)} |\nabla \sigma|^2 + \iint\limits_{\omega \times (0,T)} |\partial_t \sigma|^2\\
&+ \iint\limits_{\omega \times (0,T)} |v|^2 + \iint\limits_{\omega \times (0,T)} |\nabla v|^2 + \iint\limits_{\omega \times (0,T)} |\nabla^2 v|^2 + \iint\limits_{\omega \times (0,T)} |\nabla \partial_t v|^2 + \iint\limits_{\omega_1 \times (0,T)} |\psi|^2 \Big).
\end{aligned} \tag{1.2.32}$$

In principle the observability inequality (1.2.32) is equivalent to the null controllability of the primal problem (1.2.28) but unfortunately with this observability inequality it is not very clear in which functional settings one can prove the null controllability of the problem (1.2.28).

At this moment we recall that in the article [18] the local exact controllability to constant trajectories for compressible Navier-Stokes equations was proved by using control functions acting on the entire fluid boundary. One of the open questions posed in the article [18] concerns the use of a localized boundary control to prove such a result. In the direction of solving this open problem, as a particular implication of Theorem 1.2.4 one can infer an observability inequality for the adjoint of a linearized compressible fluid model in dimension two where in the complement of the control boundary we assume the Navier-Slip boundary condition without friction for the fluid velocity.

1.2.3.4 Idea and Strategy

The underlying idea behind the proof of (1.2.32) is the identification of the suitable unknowns to track down the dynamics of (σ, v, ψ). If for a moment we forget about the beam at the fluid boundary we are left with a system satisfied by (σ, v). The coupling of σ and v is quite strong. When considering the primal problem (1.2.28), the dynamics between $\widetilde{\sigma}$, and $\widetilde{u}$ can be made simpler by introducing the effective flux, [33] and [20]. For the adjoint problem, a similar quantity, already used in [18], also simplifies the description of the dynamics:

$$q = (\mu' + 2\mu)\mathrm{div}\, v + \overline{\rho}\sigma.$$

This can be termed as the dual version of the effective viscous flux.
Representing (1.2.29) *with* (σ, q, ψ): Suppose the regularity of v and σ are strong enough so that we are able to take the trace of $\nabla^2 v$ and $\nabla\sigma$ on $(\mathbb{T}_L \times \{0,1\}) \times (0,T)$. Hence we can consider the trace of the equation $(1.2.29)_2$ and use $(1.2.29)_3$-$(1.2.29)_5$ to have the following on $(\mathbb{T}_L \times \{1\}) \times (0,T)$:

$$\begin{aligned}
&-\overline{\rho}(\partial_t v_2 + \overline{u}_1\partial_x v_2) - \mu(\partial_{xx}v_2 + \partial_{zz}v_2) - (\mu+\mu')(\partial_{xz}v_1 + \partial_{zz}v_2) - \overline{\rho}\partial_z\sigma = 0\\
&\Rightarrow -\overline{\rho}(\partial_t\psi + \overline{u}_1\partial_x\psi) - (\mu' + 2\mu)(\partial_{xz}v_1 + \partial_{zz}v_2) - \overline{\rho}\partial_z\sigma = 0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{using } (1.2.29)_3, (1.2.29)_5)\\
&\Rightarrow \partial_z q = -\overline{\rho}(\partial_t\psi + \overline{u}_1\partial_x\psi).
\end{aligned} \tag{1.2.33}$$

Similarly one can obtain that on the boundary $(\mathbb{T}_L \times \{0\}) \times (0,T)$, q satisfies

$$\partial_z q = 0 \quad \text{on} \quad (\mathbb{T}_L \times \{0\}) \times (0,T).$$

Hence with the formal calculations above and using (1.2.29) we obtain the following system satisfied by the unknowns (σ, q, ψ):

$$\begin{cases}
-\partial_t\sigma - \overline{u}_1\partial_x\sigma + \dfrac{P'(\overline{\rho})\overline{\rho}}{\nu}\sigma = \dfrac{P'(\overline{\rho})}{\nu}q & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T),\\
-(\partial_t q + \overline{u}_1\partial_x q) - \dfrac{\nu}{\overline{\rho}}\Delta q - \dfrac{P'(\overline{\rho})\overline{\rho}}{\nu}q & \\
\qquad = -\dfrac{P'(\overline{\rho})\overline{\rho}^2}{\nu}\sigma & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T),\\
\partial_z q = -\overline{\rho}(\partial_t\psi + \overline{u}_1\partial_x\psi) & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T),\\
\partial_z q = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T),\\
q(\cdot,T) = q_T = \nu\mathrm{div} v_T + \overline{\rho}\sigma_T & \text{in } \mathbb{T}_L \times (0,1),\\
\sigma(\cdot,T) = \sigma_T & \text{in } \mathbb{T}_L \times (0,1),\\
\partial_{tt}\psi + \partial_{txx}\psi + \partial_{xxxx}\psi = (\partial_t + \overline{u}_1\partial_x)q & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T),\\
\psi(T) = \psi_T \quad \text{and} \quad \partial_t\psi(T) = \psi_T^1 & \text{in } \mathbb{T}_L \times \{1\},
\end{cases} \tag{1.2.34}$$

where $\nu = (\mu' + 2\mu)$. One can observe that the coupling between σ, q and ψ involved in the system (1.2.34) is weaker than the coupling between σ, v and ψ involved in the

system (1.2.29). Hence the system (1.2.34) will play the key role both in proving the well posedness and the observability of the system (1.2.29).
Indeed, well-posedness results for (1.2.29) will be obtained from the well-posedness of (1.2.34), see Chapter 4 Section 4.2, which are easier to obtain. In fact we observe that the system (1.2.34) has less degrees of freedom compared to the system (1.2.29) and hence it is easier to deal with (1.2.34) rather than (1.2.29). Similarly, to prove the observability estimate (1.2.32), for the system (1.2.29), we shall start by proving an observability estimate for the system (1.2.34).

Hence, we first show the following observability estimate

$$\begin{aligned}
&\int\limits_{\mathbb{T}_L\times(0,1)} |\sigma|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times(0,1)} |\nabla\sigma|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times(0,1)} |q|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times(0,1)} |\nabla q|^2(\cdot,0)\\
&+ \int\limits_{\mathbb{T}_L\times(0,1)} |\nabla^2 q|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times\{1\}} |\psi|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times\{1\}} |\partial_x\psi|^2(\cdot,0)\\
&+ \int\limits_{\mathbb{T}_L\times\{1\}} |\partial_{xx}\psi|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times\{1\}} |\partial_{xxx}\psi|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times\{1\}} |\partial_t\psi|^2(\cdot,0)\\
&+ \int\limits_{\mathbb{T}_L\times\{1\}} |\partial_{tx}\psi|^2(\cdot,0)\\
&\leqslant C\Big(\iint\limits_{\omega_T} |\sigma|^2 + \iint\limits_{\omega_T} |\nabla\sigma|^2 + \iint\limits_{\omega_T} |q|^2 + \iint\limits_{\omega_T} |\partial_x q|^2 + \iint\limits_{\omega_T} |\partial_t q|^2 + \iint\limits_{\omega_T^1} |\psi|^2\Big).
\end{aligned} \tag{1.2.35}$$

Let us discuss the strategy to prove the inequality (1.2.35). First we separately obtain observability estimates for an adjoint beam equation, heat equation and transport equation. For the parabolic beam and heat equations the observability estimates are obtained by Carleman estimates and for the transport equation we use a duality argument and a controllability result from [18] to obtain an observability estimate. The Carleman estimate for the damped beam proved in this chapter seems to be new and is achieved by using one dimensional weight functions which resemble with the ones introduced in [25] to study the observability of the adjoint heat equation. We also recover the Carleman estimate for the non homogeneous adjoint heat equation proved in [21] but by using weight functions different than the ones used in [21]. In fact our weight functions are compatible with the ones used to prove Carleman estimate for an adjoint beam equation. At this point we choose not to introduce the technical jargons used to write this weighted estimates and refer to Chapter 4 Section 4.3.2, for observability of an adjoint beam equation, Chapter 4 Section 4.3.3, for observability of an adjoint heat equation and Chapter 4 Section 4.3.4, for observability of an adjoint transport equation. Finally we suitably combine these observability estimates to prove the estimate (1.2.35) for the system (1.2.34). Here, the fact that the equations are only weakly coupled is of primary importance.

One needs to recover the estimates on v to show (1.2.32). In that direction, we note that curl v satisfies a homogeneous adjoint heat equation with Dirichlet boundary condition, so that classical observability estimates, see [25], yield:

$$\int\limits_{\mathbb{T}_L\times(0,1)} |\mathrm{curl} v|^2(\cdot,0) + \int\limits_{\mathbb{T}_L\times(0,1)} |\nabla \mathrm{curl} v|^2(\cdot,0) \leqslant C \iint\limits_{\omega_T} |\mathrm{curl} v|^2. \tag{1.2.36}$$

We suitably use the following decomposition:

$$\Delta v(\cdot,0) = \nabla \mathrm{div}\,(v(\cdot,0)) + \begin{pmatrix} \partial_z \\ -\partial_x \end{pmatrix} (\mathrm{curl}\, v(\cdot,0)) \quad \text{in} \quad \mathbb{T}_L \times (0,1),$$

and combine (1.2.35) and (1.2.36) to prove (1.2.32).

1.3 Perspectives

In this thesis we have studied some well posedness and controllability issues of fluid models with time varying density. There arises some open questions which are directly linked to the contents of this thesis and require additional works. Let us pose some of these questions in the following discussion.

1.3.0.5 Chapter 2. Stabilization of the non-homogeneous Navier-Stokes equations in a 2d channel

In Chapter 2 we present a local stabilization result of the non-homogeneous Navier-Stokes equations in a 2d channel around the Poiseuille profile. Poiseuille profile is a particular solution of the stationary solution to the non-homogeneous incompressible Navier-Stokes equations in a 2d channel.

It is natural to generalize the stabilization result around different stationary trajectories and in other geometries of the domain Ω both in 2D and 3D. For instance the article [2] deals with the local exact controllability to trajectories of the non homogeneous incompressible Navier-Stokes equation where the domain is just assumed to be open and bounded. In that case it is not very clear if one can provide an explicit stationary trajectory like the Poiseuille profile. As in [2] one can then assume the existence of a sufficiently smooth trajectory $(\overline{\rho}_s, \overline{v}_s)$ such that the following holds

$$\overline{\Omega} = \Omega^T_{out}, \quad \text{where} \quad \Omega^T_{out} = \{x \in \overline{\Omega} \mid \exists t \in (0,T), \text{ s.t. } \overline{X}(t,0,x) \in \mathbb{R}^d \setminus \overline{\Omega}\}, \tag{1.3.1}$$

where $\overline{X}$ is the flow corresponding to the target velocity trajectory $\overline{v}_s$ defined as

$$\forall (x,t,s) \in \mathbb{R}^d \times [0,T]^2, \quad \partial_t \overline{X}(x,t,s) = \overline{v}_s(\overline{X}(x,t,s)), \quad \overline{X}(x,s,s) = x.$$

Of course as we have explained in Section 1.2.1, the assumption (1.3.1) is a suitable replacement of our assumption (1.2.5) in order to control the transport equation modeling the density.

In our case to prove the stabilization result the main strategy is to first study the stabilization of the linearized system and then prove a local result by considering the non linear terms as perturbations. To prove a stabilization result around the Poiseuille profile, it might be interesting to proceed differently using the non linearity of the system with the goal of dropping the assumption (1.2.5). In that direction the return method (see [16] and [13]) may provide some leads.

1.3.0.6 Chapter 3: Local existence of strong solutions for a fluid-structure interaction model

In Chapter 3 we prove the local in time existence of strong solutions for a compressible fluid structure interaction problem, where the one dimensional structure appearing at the fluid boundary is modeled by an Euler-Bernoulli damped beam equation.

There are several interesting questions which arise in connection with our work:

- The uniqueness of the solution to the fluid structure interaction problem under consideration in the functional framework in which we only have shown the existence.
- We have assumed the initial displacement of the beam to be zero. Can we prove a similar existence result if the initial displacement is non zero and lies in some suitably regular Sobolev space? For an incompressible fluid structure interaction problem the case of non zero initial displacement is dealt with in the article [11]. The approach of [11] might provide us some leads.
- One might be interested in the long time dynamics of the system (1.2.16). In that direction we conjecture that there exists a unique strong solution to the problem (1.2.16) in a time interval $(0, T)$, for any positive constant T, when the initial datum $(\rho_0, \mathbf{u}_0, \eta_1)$ lies in a small neighborhood (in some suitable norm) of the steady state $(\overline{\rho}, 0, 0)$, where $\overline{\rho}$ is a positive constant, as long as no collision occur. To prove this local result one can first study the linearized system around the steady state $(\overline{\rho}, 0, 0)$. We strongly believe that the corresponding operator generates an analytic semi group of negative type. If so then we can first prove the local in time existence of strong solution with certain decay estimates. Then these estimates can be used to extend the solution to prove the existence of strong solution in any time interval $(0, T)$, as long as no collision occur. Similar analysis for a compressible fluid flow model (in the absence of a structure) can be found in [38].

- To prove the existence and uniqueness of a strong solution to the problem (1.2.16) in maximal $L_p - L_q$ framework. We refer again to the article [38] for the existence of strong solution for an compressible fluid flow model (in the absence of a structure) in maximal $L_p - L_q$ framework.
- It is interesting to consider a compressible fluid structure interaction problem in a 3D channel with an elastic structure appearing at the fluid boundary. One can also analyze compressible fluid structure interaction problems where the structure follows a Koiter shell model.

1.3.0.7 Chapter 4: Observability of the adjoint of a linearized compressible fluid-structure model in a 2d channel

In Chapter 4 we prove an observability inequality of the adjoint of a control linearized compressible fluid-structure model in a 2d channel. To the best of our knowledge this is the first observability result for an adjoint fluid structure interaction problem in dimension more than one where the structure lies on the fluid boundary. Our work rises plenty of unsolved questions and we only collect some of them in the following:

- In our model we consider a simplified expression of the coupling term appearing at the right hand side of the structure equation. More precisely we consider the following expression

$$(T_f)_2 = (-(\mu' + 2\mu)(\mathrm{div} u) I_d \cdot n_t + P n_t) \mid_{\Gamma_{s,t}} \sqrt{1 + (\partial_x \beta)^2} \cdot \vec{e}_2, \tag{1.3.2}$$

  as the net force acting on the beam for the non linear problem, where u is the fluid velocity, $P(\rho) = p(\rho) - P_{ext}$ where p is the fluid pressure and P_{ext} is the external forcing term, β is the beam displacement, I_d is the identity matrix, n_t is the unit outward normal at the moving interface and $\vec{e}_2 = (0, 1)$. For more details we refer the readers to Chapter 4 Section 4.1.1.1. It is explained in Chapter 4 Remark 4.1.1 that a more physical expression of the net force acting on the beam is given as follows

$$(T_f)_2^{ph} = ([-2\mu D(u) - \mu' \mathrm{div} u I_d] \cdot n_t + P n_t) \mid_{\Gamma_{s,t}} \sqrt{1 + (\partial_x \beta)^2} \cdot \vec{e}_2. \tag{1.3.3}$$

  One of the most interesting questions to ask is if it is possible to obtain an observability inequality for the adjoint of the linearized fluid structure interaction problem which is obtained when we consider the expression (1.3.3) as the net force acting on the beam. One of our main strategies in this chapter is to introduce the effective viscous flux. When we consider the expression (1.3.3) as the net force acting on the beam, the interface boundary conditions can not be expressed only in terms of the boundary conditions of the effective viscous flux and the beam making the whole dynamics more delicate to investigate.

- We have proven an observability inequality for an adjoint of a control linearized fluid structure interaction problem. Next one can try proving a controllability result for the linearized fluid structure interaction problem in some suitable functional framework by exploiting the observability inequality proved in this chapter. This would also be interesting to deal with the controllability of the non linear problem which we introduce in Chapter 4 Section 4.1.1.

- Here we prove a new Carleman estimate for the adjoint of an Euler-Bernoulli damped beam equation. It needs a little bit more work to conclude the null controllability of the linear Euler-Bernoulli damped beam equation by using this Carleman estimate.

- One can recall that in our case the fluid velocity satisfies the no vorticity boundary condition at the fluid boundary. It will also be interesting to consider the generalized Navier-slip boundary condition satisfied by the fluid velocity at the boundary.

- *A special case*: The observability inequality we prove can be directly used to infer an observability inequality when there is no beam at the fluid boundary and the fluid velocity solves the Navier slip boundary condition without friction. This gives a direction to solve an open question posed in [18] which corresponds to the local exact controllability to constant trajectories of a compressible fluid model with localized boundary controls.

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Chapter 2

Stabilization of the non-homogeneous Navier-Stokes equations in a 2d channel

2.1 Introduction

2.1.1 Settings of the problem

We are interested in stabilizing the density dependent Navier-Stokes equations around some stationary state (ρ_s, v_s) (where (ρ_s, v_s, p_s) is a stationary solution) in a two dimensional channel Ω. For that we will use an appropriate boundary control u_c acting on the velocity in the inflow part of the boundary $\partial\Omega$.
Let d be a positive constant. Throughout this article we will use the following notations (see Figure 1.)

$$\Omega = (0,d)\times(0,1),\ \Gamma = \partial\Omega,\ Q_T = \Omega\times(0,T),\ \Sigma_T = \Gamma\times(0,T) \quad \text{for } 0 < T \leqslant \infty. \tag{2.1.1}$$

The unit outward normal to the boundary Γ is denoted by n. The velocity, density and pressure of the fluid are denoted respectively by v, ρ and p. The viscosity $\nu > 0$ of the

fluid is a positive constant. We consider the following control system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 & \text{in} \quad Q_\infty, \\ \rho = \rho_s & \text{on} \quad \{(x,t) \in \Sigma_\infty \mid (v(x,t) \cdot n(x)) < 0\}, \\ \rho(x,0) = \rho_s + \rho_0 & \text{in} \quad \Omega, \\ \rho\left(\dfrac{\partial v}{\partial t} + (v \cdot \nabla)v\right) - \nu \Delta v + \nabla p = 0 & \text{in} \quad Q_\infty, \\ \operatorname{div}(v) = 0 & \text{in} \quad Q_\infty, \\ v = v_s + u_c \chi_{\Gamma_c} & \text{on} \quad \Sigma_\infty, \\ v(x,0) = v_s + v_0 & \text{in} \quad \Omega, \end{cases} \tag{2.1.2}$$

where $u_c \chi_{\Gamma_c}$ is a control function for the velocity v with χ_{Γ_c} denoting the characteristics function of a set Γ_c which is compactly supported on Γ. The set Γ_c will be precisely defined shortly afterwards. The equation $(2.1.2)_1$ is the mass balance equation and $(2.1.2)_4$ is the momentum balance equation. The triplet (ρ_s, v_s, p_s) is the Poiseuille profile defined as follows

$$\rho_s(x_1,x_2) = 1, \quad v_s(x_1,x_2) = \begin{bmatrix} x_2(1-x_2) \\ 0 \end{bmatrix}, \quad p_s = -2\nu x_1, \quad \text{in} \quad \Omega. \tag{2.1.3}$$

Observe that (ρ_s, v_s, p_s) (given by (2.1.3)) is a stationary solution of the Navier-Stokes equations (2.1.2). We remark that in the definition (2.1.3) of the Poiseuille profile we can choose ρ_s to be any positive constant in place of one up to modifying p_s accordingly. Also in the definition (2.1.3) one can consider $v_s = (\alpha x_2(1-x_2), 0)$, for a positive constant $\alpha > 0$. The strategy and results of our analysis apply for any constant $\rho_s > 0$ and $\alpha > 0$. The aim of this article is to determine feedback boundary control u_c (the control of the velocity) such that the solution (ρ, v) of the controlled system is exponentially stable around the stationary solution (ρ_s, v_s) provided the perturbation (ρ_0, v_0) of the steady state (ρ_s, v_s) is sufficiently small (in some suitable norm).
In view of the stationary profile (2.1.3), it is natural to control the inflow part of the boundary, *i.e.* we will consider the control function u_c supported on

$$\Gamma_{in} = \{x \in \Gamma \mid (v_s \cdot n)(x) < 0\} = \{0\} \times (0,1). \tag{2.1.4}$$

In fact we do slightly more and control on some open subset Γ_c of Γ_{in}. We consider Γ_c of the following form

$$\Gamma_c = \{0\} \times (L, 1-L) \subset \Gamma_{in}, \tag{2.1.5}$$

for some fixed $0 < L < \frac{1}{2}$.

Remark 2.1.1. *We consider the control zone of the form (2.1.5) to simplify the notations. In fact our analysis allows to consider any subset $\{0\} \times (A,B)$ $(0 < A < B < 1)$ of Γ_{in} as the control zone.*

To state our results precisely, we introduce some appropriate functional spaces.

2.1.2 Functional framework for the Navier-Stokes equation

Let $H^s(\Omega;\mathbb{R}^N)$ and $L^2(\Omega;\mathbb{R}^N)$ denote the vector valued Sobolev spaces. If it is clear from the context, we may simply denote these spaces by $H^s(\Omega)$ and $L^2(\Omega)$ both for scalar and vector valued functions. The same notational conventions will be used for the trace spaces. We now introduce different spaces of divergence free functions and some suitable spaces of boundary data:

$$
\begin{aligned}
V^s(\Omega) &= \{y \in H^s(\Omega;\mathbb{R}^2) \mid \mathrm{div} y = 0 \quad \text{in} \quad \Omega\} \quad \text{for} \quad s \geqslant 0,\\
V_n^s(\Omega) &= \{y \in H^s(\Omega;\mathbb{R}^2) \mid \mathrm{div} y = 0 \quad \text{in} \quad \Omega, \quad y\cdot n = 0 \quad \text{on} \quad \Gamma\} \quad \text{for} \quad s \geqslant 0,\\
V_0^s(\Omega) &= \{y \in H^s(\Omega;\mathbb{R}^2) \mid \mathrm{div} y = 0 \quad \text{in} \quad \Omega, \quad y = 0 \quad \text{on} \quad \Gamma\} \quad \text{for} \quad s \in (\tfrac{1}{2},\tfrac{3}{2}),\\
V^s(\Gamma) &= \{y \in H^s(\Gamma;\mathbb{R}^2) \mid \int_\Gamma y\cdot n\, dx = 0\} \quad \text{for} \quad s \geqslant 0.
\end{aligned}
$$

The spaces $V^s(\Omega)$ and $V^s(\Gamma)$ are respectively equipped with the usual norms of $H^s(\Omega)$ and $H^s(\Gamma)$, which will be denoted by $\|\cdot\|_{V^s(\Omega)}$ and $\|\cdot\|_{V^s(\Gamma)}$.
From now onwards we will identify the space $V_n^0(\Omega)$ with its dual.
For $0 < T \leqslant \infty$ let us introduce the following functional spaces adapted to deal with functions of the time and space variables.

$$
\begin{aligned}
V^{s,\tau}(Q_T) &= H^\tau(0,T;V^0(\Omega)) \cap L^2(0,T;V^s(\Omega)) \quad \text{for} \quad s,\tau \geqslant 0,\\
V^{s,\tau}(\Sigma_T) &= H^\tau(0,T;V^0(\Gamma)) \cap L^2(0,T;V^s(\Gamma)) \quad \text{for} \quad s,\tau \geqslant 0.
\end{aligned}
$$

We also fix the convention that for any two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, the product space $\mathcal{X}\times\mathcal{Y}$ is endowed with the norm

$$
\forall\ (x,y) \in \mathcal{X}\times\mathcal{Y},\ \|(x,y)\|_{\mathcal{X}\times\mathcal{Y}} = \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}},
$$

where $\|.\|_{\mathcal{X}}$ and $\|.\|_{\mathcal{Y}}$ denotes the norms in the corresponding spaces.

2.1.3 The main result

We now precisely state our main result in form of the following theorem.

Theorem 2.1.2. *Let $\beta > 0$, $A_1 \in (0,\frac{1}{2})$. There exist a constant $\delta > 0$ such that for all $(\rho_0, v_0) \in L^\infty(\Omega)\times V_0^1(\Omega)$ satisfying*

$$
supp(\rho_0) \subset [0,d]\times(A_1, 1-A_1), \tag{2.1.6}
$$

and

$$
\|(\rho_0,v_0)\|_{L^\infty(\Omega)\times V_0^1(\Omega)} \leqslant \delta,
$$

there exists a control $u_c \in H^1(0,\infty;C^\infty(\overline{\Gamma}_c))$, *for which the system* (2.1.2) *admits a solution*

$$(\rho, v) \in L^\infty(Q_\infty) \times V^{2,1}(Q_\infty),$$

satisfying the following stabilization requirement

$$\|e^{\beta t}(\rho - \rho_s, v - v_s)\|_{L^\infty(Q_\infty)\times V^{2,1}(Q_\infty)} \leqslant C\|(\rho_0, v_0)\|_{L^\infty(\Omega)\times V_0^1(\Omega)}, \tag{2.1.7}$$

for some constant $C > 0$. *Moreover,* $\rho = \rho_s$ *for* t *sufficiently large.*

We now make precise the structure of the control function u_c we are going to construct. We will show the existence of a natural number N_c, and a family

$$\{g_j \mid 1 \leqslant j \leqslant N_c\},$$

of smooth functions supported on Γ_c such that the control u_c acting on the velocity is given as follows

$$u_c(x,t) = e^{-\beta t} \sum_{j=1}^{N_c} w_j(t) g_j(x), \tag{2.1.8}$$

where $w_c(t) = (w_1(t), ..., w_{N_c}(t))$ is the control variable and is given in terms of a feedback operator $\mathcal{K}$. More precisely, $w_c = (w_1, ..., w_{N_c})$ satisfies the following ODE

$$w_c' = -\gamma w_c + \mathcal{K}\begin{pmatrix} P(v - v_s) \\ w_c \end{pmatrix} \quad \text{in} \quad (0,\infty), \quad w_c(0) = 0,$$

where γ is a positive constant, P is the Leray projector from $L^2(\Omega)$ to $V_n^0(\Omega)$ ([45, Section 1.4]) and $\mathcal{K} \in \mathcal{L}(V_n^0(\Omega) \times \mathbb{R}^{N_c}, \mathbb{R}^{N_c})$ (the feedback operator $\mathcal{K}$ is determined in Section 2.2.2.2).
The boundary control (2.1.8) we construct has a finite dimensional range and resembles with the control designed in [43]. The construction of our control basis $\{g_j \mid 1 \leqslant j \leqslant N_c\}$ is different from the one done in [43]. In [43] it is constructed using generalized eigenvectors of the adjoint of Oseen operator while we construct it only by using eigenvectors of adjoint of Oseen operator relying on the construction of [37]. We will not consider any control on the transport equation modeling the density and as for the homogeneous Navier-Stokes equations, we show that considering a control u_c of the velocity is enough to stabilize the whole system (2.1.2).
The stabilizability of the constant density (or homogeneous) incompressible Navier-Stokes equation (with Dirichlet or mixed boundary condition) by a finite dimensional feedback Dirichlet boundary control has already been studied in the literature. For instance in [43] it is proved that in a C^4 domain the velocity profile v, solution to system $(2.1.2)_4$-$(2.1.2)_7$ with $\rho = 1$ is locally stabilizable around a steady state v_s ($v_s \in H^3(\Omega;\mathbb{R}^2)$) by a finite dimensional Dirichlet boundary control localized in a portion of the boundary and moreover the control u_c is given as a feedback of the velocity field.

Unlike the constant density incompressible Navier-Stokes equations (which is of parabolic nature), the system (2.1.2) obeys a coupled parabolic-hyperbolic dynamics. Local exact controllability to trajectories of the system (2.1.2) was studied in [3]. In the present article we answer the question posed in [3] on the stabilizability of the system (2.1.2) around the Poiseuille profile. In proving the controllability results one of the main geometric assumptions of [3] is that

$$\overline{\Omega} = \Omega^T_{out} = \{x \in \overline{\Omega} \mid \exists t \in (0,T), \text{ s.t } \overline{X}(t,0,x) \in \mathbb{R}^d \setminus \overline{\Omega}\}, \tag{2.1.9}$$

where $\overline{X}$ is the flow corresponding to the target velocity trajectory $\overline{v}_s$ defined as

$$\forall (x,t,s) \in \mathbb{R}^d \times [0,T]^2, \quad \partial_t \overline{X}(x,t,s) = \overline{v}_s(\overline{X}(x,t,s),t), \quad \overline{X}(x,s,s) = x.$$

In the article [3] the assumption (2.1.9) plays the key role in controlling the density of the fluid. In our case since the target velocity trajectory is v_s (defined in (2.1.3)) the assumption (2.1.9) is not satisfied because v_s vanishes at the lateral boundary of the domain Ω. Hence to control the density we make a parallel assumption (2.1.6). Indeed, the assumption (2.1.6) implies that $supp(\rho_0) \Subset \Omega^T_{out}$. The assumption (2.1.6) exploits the hyperbolic nature of the continuity equation $(2.1.2)_1$ in order to control the coupled system (2.1.2). The condition (2.1.6) in fact guarantees that the density exactly equals $\rho_s = 1$, after some time $T_1 = T_{A_1} > \frac{d}{\inf\limits_{x_2 \in [A_1, 1-A_1]} v_s}$ (will be detailed in Section 2.3) so that the non-homogeneous Navier-Stokes equations become homogeneous after some finite time. In [3] the authors uses two control functions (one for the density and one for velocity) for the purpose of controlling the non-homogeneous fluid. Contrary to that we use only one control acting on the velocity to stabilize the coupled system (2.1.2).

2.1.4 Decomposition of the boundary Γ and comment on the support of control

Based on the velocity profile v_s (as defined in (2.1.3)) we can rewrite the boundary of Ω as follows

$$\Gamma = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_0,$$

where

$$\begin{aligned} &\Gamma_{in} \text{ is defined in } (2.1.4), \\ &\Gamma_{out} = \{x \in \Gamma \mid (v_s \cdot n)(x) > 0\} = \{d\} \times (0,1), \\ &\Gamma_0 = ((0,d) \times \{0\}) \cup ((0,d) \times \{1\}) = \Gamma_b \cup \Gamma_h \quad \text{(Figure 1)}. \end{aligned} \tag{2.1.10}$$

Remark 2.1.3. *From now onwards we will use the notation Γ_{in} to denote the inflow boundary of both the vector fields v_s and v. This is a slight abuse of notation but we will prove the existence of the controlled trajectory v in a small neighborhood (in a suitable*

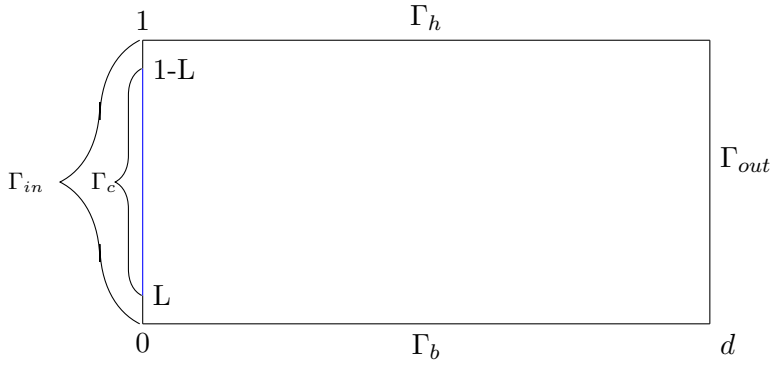


Figure 2.1: Picture of the domain Ω.

norm) of v_s provided the perturbation v_0 is small. This will guarantee that Γ_{in} and the inflow boundary of the vector field v_s are identical. For the details we refer the reader to the Corollary 2.2.17.

We will look for a control function u_c of the form (2.1.8) which is compactly supported in Γ_c. More particularly we will construct the finite dimensional basis $\{g_j \mid 1 \leqslant j \leqslant N_c\}$ of the control space in such a way that g_j $(\forall\, 1 \leqslant j \leqslant N_c)$ is smooth and supported in Γ_c.

2.1.5 Strategy

(i) As our goal is to stabilize the solution (ρ, v) of (2.1.2) around the stationary solution $(1, v_s)$ with a rate $e^{-\beta t}$ we introduce

$$y = e^{\beta t}(v - v_s), \quad \sigma = e^{\beta t}(\rho - 1), \quad q = e^{\beta t}(p - p_s), \quad u = e^{\beta t}u_c. \tag{2.1.11}$$

To be consistent with the notations y and σ, we further introduce the following

$$\sigma_0 = \rho_0, \quad y_0 = v_0. \tag{2.1.12}$$

As in our case the control (2.1.8) is supported in the inflow boundary, in view of the notations introduced in (2.1.10) and the Remark 2.1.3 we use (2.1.8) to rewrite the

system (2.1.2) in the following form

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + ((v_s + e^{-\beta t} y) \cdot \nabla)\sigma - \beta \sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \\ \dfrac{\partial y}{\partial t} - \beta y - \nu \Delta y + (v_s \cdot \nabla) y + (y \cdot \nabla) v_s + \nabla q = \mathcal{F}(y,\sigma) & \text{in} \quad Q_\infty, \\ \operatorname{div} y = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0 \cup \Gamma_{out}) \times (0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ y(x,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{2.1.13}$$

where

$$\mathcal{F}(y,\sigma) = -e^{-\beta t}\sigma \frac{\partial y}{\partial t} - e^{-\beta t}(y\cdot\nabla)y - e^{-\beta t}\sigma(v_s\cdot\nabla)y - e^{-\beta t}\sigma(y\cdot\nabla)v_s - e^{-2\beta t}\sigma(y\cdot\nabla)y + \beta e^{-\beta t}\sigma y.$$

To solve a nonlinear stabilization problem the usual method is to first solve the stabilization problem for the linearized system and then use a fixed point method to conclude the stabilizability of the original nonlinear problem (2.1.13). In this article due to regularity issues of the transport equation we avoid linearizing the whole system. Instead, we only linearize the equation $(2.1.13)_4$ satisfied by y *i.e.* we replace the nonlinear terms appearing in the equation $(2.1.13)_4$ by a non homogeneous source term f and we leave the equation of the density $(2.1.13)_1$ unchanged. Hence we start by analyzing the stabilizability of the system

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + ((v_s + e^{-\beta t} y) \cdot \nabla)\sigma - \beta \sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \\ \dfrac{\partial y}{\partial t} - \beta y - \nu \Delta y + (v_s \cdot \nabla) y + (y \cdot \nabla) v_s + \nabla q = f & \text{in} \quad Q_\infty, \\ \operatorname{div} y = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0 \cup \Gamma_{out}) \times (0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ y(x,0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{2.1.14}$$

(ii) Section 2.2 is devoted to study the stabilization of the linearized Oseen equations $(2.1.14)_4$-$(2.1.14)_8$. In that direction we first write $(2.1.14)_4$-$(2.1.14)_8$ using operator notations. This is done in the spirit of [41] but with suitable modifications which are

necessary since our domain is Lipschitz. To prove the stabilizability of this system we look for a control of the form (2.1.8). We will choose the functions $\{g_j \mid 1 \leqslant j \leqslant N_c\}$, supported on Γ_c, so that we can prove some unique continuation property equivalent to the stabilizability of the system under consideration. This is inspired from [37]. Using the fact that g_j (for all $1 \leqslant j \leqslant N_c$) is supported on a smooth subset of Γ we further show that g_j is in $C^\infty(\Gamma)$. This in particular implies that the control u_c, of the form (2.1.8), is smooth in the space variable.

(iii) Next our aim is to find a boundary control which is given in terms of a feedback law. At the same time we have to design the control such that the velocity y belongs to the space $V^{2,1}(Q_\infty)$. Indeed the $H^2(\Omega)$ regularity of the velocity field will be used later to prove the stabilization of the continuity equation. This creates another difficulty because to prove the $V^{2,1}(Q_\infty)$ regularity of y solution of $(2.1.14)_4$-$(2.1.14)_8$, one must have a compatibility between the initial velocity y_0, assumed to be in $V_0^1(\Omega)$ and the boundary condition (*i.e.* the control u). We deal with this issue by adding a system of ordinary differential equations satisfied by w_c. The corresponding extended system satisfied by (y, w_c) reads as follows

$$\begin{cases} \dfrac{\partial y}{\partial t} - \beta y - \nu\Delta y + (v_s \cdot \nabla)y + (y \cdot \nabla)v_s + \nabla q = f & \text{in} \quad Q_\infty, \\ \operatorname{div} y = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0 \cup \Gamma_{out}) \times (0, \infty), \\ y = \displaystyle\sum_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_{in} \times (0, \infty), \\ y(x, 0) = y_0 & \text{in} \quad \Omega, \\ w_c' + \gamma w_c = \varphi_c & \text{in} \quad (0, \infty), \\ w_c(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.1.15}$$

where $\gamma > 0$ is a positive constant and $\varphi_c (\in \mathbb{R}^{N_c})$ is a new control variable which will be determined later as a feedback of the pair (y, w_c). Since $y(., 0) = 0$, imposing $w_c(0) = 0$ furnishes the desired compatibility between the initial and boundary conditions of y which is necessary to obtain the $V^{2,1}(Q_\infty)$ regularity of y.

First we will construct the control φ_c given in terms of a feedback operator which is able to stabilize the homogeneous (*i.e.* when $f = 0$) extended system (2.1.15) by solving a Riccati equation. Then we show that the same control stabilizes the entire non-homogeneous (*i.e.* with the non-homogeneous source term f) extended system (2.1.15) by assuming that the non-homogeneous term f belongs to some appropriate space.

(iv) In Section 2.3, we study the stability of the continuity equation $(2.1.14)_1$-$(2.1.14)_3$. We assume the velocity field in $V^{2,1}(Q_\infty)$ and $\sigma_0 \in L^\infty(\Omega)$ such that (2.1.6) (recall from (2.1.12) that $\sigma_0 = \rho_0$) holds. Since $\sigma_0 \in L^\infty(\Omega)$ and the transport equation has no regularizing effect we expect that $\sigma \in L^\infty_{loc}(Q_\infty)$. The Cauchy problem for the continuity equation in the presence of an inflow boundary is rather delicate. In our case we use results from [9] for the existence of a unique renormalized weak solution of the problem $(2.1.14)_1$-$(2.1.14)_3$ in the space $L^\infty(Q_\infty)$. Our proof of the stabilization of the transport

equation satisfied by the density relies on the fact that the characteristics equation corresponding to the velocity field is well posed. As we are dealing with velocity fields in $L^2(0,\infty,H^2(\Omega))$, which is not embedded in $L^1_{loc}(0,\infty,W^{1,\infty}(\Omega))$ in dimension two, our analysis relies on [48] (see also [4, Theorem 3.7]), stating the well-posedness of the equation of the flow as a consequence of Osgood condition. Then considering the velocity field $(v_s + e^{-\beta t}y)$ as a small perturbation of v_s (see (2.1.3) for the definition) we prove that the characteristic curves corresponding to the perturbed velocity field stay close to that of v_s in a suitable norm. Using the fact that the characteristics corresponding to the velocity fields v_s and $(v_s+e^{-\beta t}y)$ are close we show that the particles initially lying in the support of σ_0 are transported out of the domain in some finite time $T > T_{A_1} = \frac{d}{A_1(1-A_1)}$ along the flow corresponding to the perturbed velocity field. Consequently, the solution ρ of the equation $(2.1.2)_1$-$(2.1.2)_3$ reaches exactly the target density $\rho_s = 1$ after the time T.
(v) Finally in Section 2.4, we will use Schauder's fixed point theorem to conclude that the control designed in step (iii) locally stabilizes the non linear coupled system (2.1.14) and consequently Theorem 2.1.2 follows.

2.1.6 Bibliographical comments

In the literature many works have been dedicated to the study of incompressible Navier-Stokes equations. For the classical results concerning the existence-uniqueness and regularity issues of the constant density incompressible Navier-Stokes equations we refer the reader to [45]. The reader can also look into [25] for a thorough analysis of the subject. Intricate situations may arise due to the lack of regularity when special geometric assumptions are imposed on the boundary $\partial\Omega$. For example, the domain can have corners or edges of prescribed geometric shape. For the analysis of these situations the interested reader may look into [34] and [14]. In the present article the functional settings for the incompressible Navier-Stokes equations is motivated from [41]. The results of [41] are stated in a domain with smooth boundary. Thus to adapt the functional framework from [41] in the case of a rectangular domain we have used some results from [26] and [28].
Regarding the Cauchy problem of the non-homogeneous Navier-Stokes equations, the existence of classical solution for the non-homogeneous Navier-Stokes equations with homogeneous Dirichlet boundary condition for velocity in space dimension three is studied in [1]. Results concerning the existence-uniqueness of global in time strong solution (with small initial data and small volume force) in space dimension three can be found in [30]. In dimension two the existence and uniqueness of global in time solution (without any smallness restriction on the data) is also proved in [30]. In both of these references the velocity field is Lipschitz and the initial condition of the density is smooth enough, hence the transport equation satisfied by the density can be classically solved using the method of characteristics. To deal with less regular velocity field the concept of renormalized solution was initially developed in [15] and later suitably adapted in

several contexts. For instance, one can find an application of a suitable variation of the Di-Perna-Lions theory to prove an existence and uniqueness result for the inhomogeneous Navier-Stokes equation in [13]. All of these articles assume that the velocity field satisfies $v \cdot n = 0$. In the present article we are dealing with the target velocity v_s, which is inflow on a part of the boundary $\partial\Omega$. For a velocity field with inflow, one must assume a suitable boundary condition for the density so that the transport equation satisfied by the density is well posed. This problem is analyzed in the articles [9, Chapter VI] and [7], where the authors suitably define the trace for the weak solution of the transport equation. They also prove that these traces enjoy the renormalization property. In the present article we use the existence, uniqueness and stability results for the transport equation from [9] and [7]. For a more intricate case involving nonlinear outflow boundary condition, similar results can be found in [8].

There is a rich literature where the question of the feedback boundary stabilization of the constant density incompressible Navier-Stokes equation is investigated. For the feedback boundary stabilization of a general semilinear parabolic equation one can look into the article [22]. The feedback stabilization of the 2D and 3D constant density Navier-Stokes equations can be found in the articles [23] and [24] respectively. Concerning the stabilization of homogeneous Navier-Stokes equations one can also consult [43] and [40] where the feedback boundary controls are achieved by solving optimal control problems. We would also like to mention the articles [36] and [5] where the authors prove the feedback stabilization of the same model around the Poiseuille profile by using normal velocity controllers. The idea of constructing a finite dimensional boundary feedback control to stabilize a linear parabolic equation dates back to the work [46]. In our case we adapt the ideas from the articles [37] and [43] in order to construct a feedback boundary control with finite dimensional range to stabilize the linear Oseen equations. Actually for constant density fluids, the article [37] deals with a more intricate case involving mixed boundary conditions. Control properties of the variable density Navier-Stokes equations have been studied in the article [21], which proves several optimal control results in the context of various cost functionals. We also refer to the article [3] where the authors prove the local exact controllability to a smooth trajectory of the non-homogeneous incompressible Navier-Stokes equation.

The study of the controllability and stabilizability issues of a system coupling equations of parabolic and hyperbolic nature is relatively new in the literature. We would like to quote a few articles in that direction. Null-controllability of a system of linear thermoelasticity (coupling wave and heat equations) in a $n-$ dimensional, compact, connected C^∞ Riemannain manifold is studied in [31]. Controllability and stabilizability issues of compressible Navier-Stokes equations are investigated in [11], [10], [18] (in dim 1) and [17] (in dim 2 and 3). The compressible Navier-Stokes equations are also modeled by a coupled system of momentum balance and mass balance equations but the coupling is different from the one we consider in system (2.1.2).

Let us emphasize that in the system (2.1.2) the control acts only on the velocity of the fluid and not on the density. In the literature there are articles dealing with controllability issues of a system of PDEs in which the controls act only on some components of the

system. We would like to quote a few of them. We refer to [12] where the authors prove local null-controllability of the three dimensional incompressible Navier-Stokes equations using distributed control with two vanishing components. A related result concerning the stabilizability of 2−d incompressible Navier-Stokes equations using a control acting on the normal component of the upper boundary is proved in [16]. In [31] to prove the null-controllability of a system of linear thermoelasticity the authors consider the control on the wave equation *i.e.* on the hyperbolic part and not on the parabolic equation modeling the temperature. On the other hand controllability and stabilizability issues of one dimensional compressible Navier-Stokes equations have been studied in [11] and [10] by using only a control acting on the velocity. In the present article we also consider the control on the velocity and not on the density but our approach exploits more directly and in a more intuitive manner the geometry of the flow of the target velocity in order to control the hyperbolic transport equation modeling the density.

2.1.7 Outline

In section 2.2 we study the feedback stabilization of the velocity. Section 2.3 is devoted to the stabilization of the density. In Section 2.4 we use a fixed point argument to prove the stabilizability of the coupled system (2.1.2). Finally in Section 2.5 we briefly comment on how to adapt our analysis if one wishes to control the outflow boundary Γ_{out} or the lateral boundary Γ_0 of the channel Ω.

2.2 Stabilization of the Oseen equations

The goal of this section is to discuss the stabilization of the Oseen equations $(2.1.14)_4$-$(2.1.14)_8$. We will first design a localized boundary control with finite dimensional range to stabilize the linear Oseen equation $(2.1.14)_4$-$(2.1.14)_8$. We will then construct the control as a feedback of (y, w_c), where the pair (y, w_c) solves the extended system (2.1.15). The plan of this section is as follows
(i) In Section 2.2.1, we study the stabilization of the homogeneous linear system (with $f = 0$) $(2.1.14)_4$-$(2.1.14)_8$, using a finite dimensional boundary control.
(ii) We will analyze the feedback stabilization of the extended system (2.1.15) in Section 2.2.2. Moreover with this feedback control we will prove the $V^{2,1}(Q_\infty)$ regularity of the solution of linear Oseen equations $(2.1.14)_4$-$(2.1.14)_8$. Using a further regularity regularity estimate (see (2.2.59)) of the control u we show that $(e^{-\beta t}y + v_s)$ has the same inflow and outflow as that of v_s, provided the initial condition y_0 and the non-homogeneous source term f (appearing in $(2.1.14)_4$-$(2.1.14)_8$) are suitably small (see Corollary 2.2.17).

2.2.1 Stabilization of the linear Oseen equations

In the following section we will define some operators and present some of their properties which helps in studying the linearized Oseen equations $(2.1.14)_4$-$(2.1.14)_8$.

2.2.1.1 Writing the equations with operators

The following results are taken from [41] where they are stated in a C^2 domain. It is necessary to make suitable changes to adapt those results in our case since the domain Ω in our case is Lipschitz. Without going into the details of the proofs we will just comment on how to adapt those results in our case.
Let P be the orthogonal projection operator from $L^2(\Omega)$ onto $V_n^0(\Omega)$ known as Helmholtz or Leray projector (see [45, Section 1.4]).
We denote by $(A, \mathcal{D}(A))$ (the Oseen operator) and $(A^*, \mathcal{D}(A^*))$ the unbounded operators in $V_n^0(\Omega)$, defined by

$$\begin{aligned} &\mathcal{D}(A) = V^2(\Omega) \cap V_0^1(\Omega), \qquad Ay = \nu P\Delta y + \beta y - P((v_s \cdot \nabla)y) - P((y \cdot \nabla)v_s), \\ &\mathcal{D}(A^*) = V^2(\Omega) \cap V_0^1(\Omega), \qquad A^*y = \nu P\Delta y + \beta y + P((v_s \cdot \nabla)y) - P((\nabla v_s)^T)y. \end{aligned} \tag{2.2.1}$$

For the $H^2(\Omega)$ regularity of the solutions of the homogeneous Dirichlet boundary value problems corresponding to the operators A and A^* in a rectangular domain Ω, one can apply [26, Theorem 3.2.1.3]. Since v_s is smooth with $\operatorname{div}(v_s) = 0$, we can prove the following lemma.

Lemma 2.2.1. *[39, Section 2.2] There exists $\lambda_0 > 0$ in the resolvent set of A such that the following hold*

$$\begin{aligned} &\langle(\lambda_0 I - A)y, y\rangle_{V_n^0(\Omega)} \geqslant \tfrac{1}{2}|y|^2_{V_0^1(\Omega)} \quad \textit{for all} \quad y \in \mathcal{D}(A), \\ &\textit{and} \\ &\langle(\lambda_0 I - A^*)y, y\rangle_{V_n^0}(\Omega) \geqslant \tfrac{1}{2}|y|^2_{V_0^1(\Omega)} \quad \textit{for all} \quad y \in \mathcal{D}(A^*). \end{aligned} \tag{2.2.2}$$

In Lemma 2.2.1 we can always choose $\lambda_0 > \beta$, taking λ_0 larger if necessary. Throughout this article we will stick to this assumption. Now, Lemma 2.2.1 can be used to prove the following.

Lemma 2.2.2. *The unbounded operator $(A, \mathcal{D}(A))$ (respectively $(A^*, \mathcal{D}(A^*))$) is the infinitesimal generator of an analytic semi group on $V_n^0(\Omega)$. Moreover the resolvent of A is compact.*

Proof. The proof of the fact that $(A, \mathcal{D}(A))$ (respectively $(A^*, \mathcal{D}(A^*))$) generates an analytic semigroup on $V_n^0(\Omega)$ uses the resolvent estimate (2.2.2) and can be found in [41,

Lemma 4.1]. One can mimic the arguments used in [23, Lemma 3.1] to show that the resolvent of A is compact. The reader can also look into [43, Section 3]. □

Now we want to find a suitable operator B to write down the Oseen equation as a boundary control system.
Consider the following system of equations

$$\begin{cases} \lambda_0 y - \nu\Delta y - \beta y + ((v_s\cdot\nabla)y) + ((y\cdot\nabla)v_s) + \nabla q = 0 & \text{in} \quad \Omega, \\ \operatorname{div}(y) = 0 & \text{in} \quad \Omega, \\ y = u & \text{on} \quad \Gamma. \end{cases} \tag{2.2.3}$$

Lemma 2.2.3. *Let λ_0 be as in Lemma 2.2.1. For $u \in V^{3/2}(\Gamma)$, the system (2.2.3) admits a unique solution $(y,q) \in V^2(\Omega)\times H^1(\Omega)/\mathbb{R}$ and moreover the following inequality holds*

$$\|y\|_{V^2(\Omega)} + \|q\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\|u\|_{H^{3/2}(\Gamma)}, \tag{2.2.4}$$

for some constant $C > 0$.

Remark 2.2.4. *The Lemma 2.2.3 is inspired from [41, Lemma B.1.] where it is proved in the case of a C^2 domain.*

Proof of Lemma 2.2.3. We write $(y,q) = (y_1,q_1) + (y_2,q_2)$, such that (y_1,q_1) satisfies

$$\begin{cases} \lambda_0 y_1 - \nu\Delta y_1 - \beta y_1 + \nabla q_1 = 0 & \text{in} \quad \Omega, \\ \operatorname{div}(y_1) = 0 & \text{in} \quad \Omega, \\ y_1 = u & \text{on} \quad \Gamma \end{cases} \tag{2.2.5}$$

and (y_2,q_2) satisfies

$$\begin{cases} \lambda_0 y_2 - \nu\Delta y_2 - \beta y_2 + ((v_s\cdot\nabla)y_2) + ((y_2\cdot\nabla)v_s) + \nabla q_2 \\ \qquad\qquad = -((v_s\cdot\nabla)y_1) - ((y_1\cdot\nabla)v_s) & \text{in} \quad \Omega, \\ \operatorname{div}(y_2) = 0 & \text{in} \quad \Omega, \\ y_2 = 0 & \text{on} \quad \Gamma. \end{cases} \tag{2.2.6}$$

As $u \in H^{3/2}(\Omega)$, the solution to (2.2.5) satisfies $(y_1,q_1) \in V^2(\Omega)\times H^1(\Omega)/\mathbb{R}$ (see [28]) and the following inequality is true

$$\|y_1\|_{V^2(\Omega)} + \|q_1\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\|u\|_{H^{3/2}(\Gamma)}, \tag{2.2.7}$$

where $C > 0$ is a constant. Using (2.2.7) we observe that the right hand side of $(2.2.6)_1$ is in $H^1(\Omega)$. Hence we get that $y_2 \in V^2(\Omega)\cap V_0^1(\Omega)$. Then the corresponding pressure $q_2 \in H^1(\Omega)/\mathbb{R}$ can be recovered using De Rham's theorem (see [45, Section 1.4]). Using (2.2.7) one also has the following inequality

$$\|y_2\|_{V^2(\Omega)} + \|q_2\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\|u\|_{H^{3/2}(\Gamma)}, \tag{2.2.8}$$

for some positive constant C. The inequalities (2.2.7) and (2.2.8) together yield (2.2.4). □

Now for $u \in V^{3/2}(\Gamma)$, we define the Dirichlet lifting operators $D_A u = y$ and $D_p u = q$, where (y, q) is the solution of (2.2.3) with Dirichlet data u.

Lemma 2.2.5. *(i) The operator D_A can be extended as a bounded linear map from $V^0(\Gamma)$ to $V^{1/2}(\Omega)$. Moreover $D_A \in \mathcal{L}(V^s(\Gamma), V^{s+1/2}(\Omega))$ for all $0 \leqslant s \leqslant 3/2$.*
(ii)The operator D_A^*, the adjoint of D_A computed as a bounded operator from $V^0(\Gamma)$ to $V^0(\Omega)$ is a bounded linear operator from $V^0(\Omega)$ to $V^0(\Gamma)$ and is given as follows

$$D_A^* g = -\nu \frac{\partial z}{\partial n} + \pi n - \frac{1}{|\Gamma|} \left(\int_\Gamma \pi \right) n, \tag{2.2.9}$$

where (z, π) is the solution of

$$\begin{cases} \lambda_0 z - \nu \Delta z - \beta z - (v_s \cdot \nabla) z + (\nabla v_s)^T z + \nabla \pi = g & \text{in} \quad \Omega, \\ \text{div} z = 0 & \text{in} \quad \Omega, \\ z = 0 & \text{on} \quad \Gamma, \end{cases} \tag{2.2.10}$$

Here $|\Gamma|$ is the one dimensional Lebesgue measure of Γ. Moreover $D_A^* \in \mathcal{L}(V^0(\Omega), V^{1/2}(\Gamma))$.
(iii) The operator D_A^* can be extended as a bounded linear operator from $H^{-\frac{1}{2}+\kappa}(\Omega)$ to $V^\kappa(\Gamma)$, for all $0 < \kappa < \frac{1}{2}$, i.e.

$$D_A^* \in \mathcal{L}(H^{-\frac{1}{2}+\kappa}(\Omega), V^\kappa(\Gamma)) \quad \text{for all} \quad 0 < \kappa < \tfrac{1}{2}. \tag{2.2.11}$$

Remark 2.2.6. *The first two parts of the Lemma 2.2.5 are inspired from [41, Lemma B.4.] where it is proved in the case of a C^2 domain.*

Proof of Lemma 2.2.5. (i) In Lemma 2.2.3 we have proved that D_A is a bounded operator from $V^{3/2}(\Gamma)$ into $V^2(\Omega)$. Following [41, Theorem B.1] one obtains that the operator D_A can be extended as a bounded linear map from $V^0(\Gamma)$ into $V^{1/2}(\Omega)$ (in the sense of variational formulation). Hence one can use interpolation to prove that D_A is bounded from $V^s(\Gamma)$ to $V^{s+1/2}(\Omega)$, for all $0 \leqslant s \leqslant 3/2$.
(ii) The second part can be done following the proof of [41, Lemma B.4]. It is therefore left to the reader.
(iii) For the final part, in view of [41, Appendix B, Lemma B.1.] one first observes that the map $\mathcal{M}_{g \to (z,\pi)}$, mapping g to (z, π) (where g, z and π are as in (2.2.10)) satisfies the following

$$\mathcal{M}_{g \to (z,\pi)} \in \mathcal{L}(L^2(\Omega), (V^2(\Omega) \cap V_0^1(\Omega)) \times H^1(\Omega)/\mathbb{R}). \tag{2.2.12}$$

Now following [41, Appendix B, Theorem B.1.] one can use method of transposition to define weak type solution of the problem (2.2.10) when $g \in (H^2(\Omega) \cap H_0^1(\Omega))'$, where $(H^2(\Omega) \cap H_0^1(\Omega))'$ is the dual of the space $H^2(\Omega) \cap H_0^1(\Omega)$ provided that $L^2(\Omega)$ is identified with its dual. In particular one has the following

$$\mathcal{M}_{g \to (z,\pi)} \in \mathcal{L}((H^2(\Omega) \cap H_0^1(\Omega))', V^0(\Omega) \times (H^1(\Omega)/\mathbb{R})'). \tag{2.2.13}$$

Now let us assume $g \in H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ denotes the dual of $H_0^1(\Omega)$ with $L^2(\Omega)$ as the pivot space. Using (2.2.13) and the fact that $H^{-1}(\Omega) \subset (H^2(\Omega) \cap H_0^1(\Omega))'$ (since $(H^2(\Omega) \cap H_0^1(\Omega))$ is dense in $H_0^1(\Omega)$) one can write (2.2.10) as follows

$$\begin{cases} -\nu\Delta z + \nabla\pi = g^* & \text{in} \quad \Omega, \\ \text{div} z = 0 & \text{in} \quad \Omega, \\ z = 0 & \text{on} \quad \Gamma, \end{cases} \tag{2.2.14}$$

where

$$g^* = g - \lambda_0 z + \beta z + (v_s \cdot \nabla)z - (\nabla v_s)^T z \in H^{-1}(\Omega).$$

Now [9, Theorem IV.5.2] furnishes the following regularity

$$\mathcal{M}_{g\to(z,\pi)} \in \mathcal{L}(H^{-1}(\Omega), V_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}). \tag{2.2.15}$$

Now from (2.2.12), (2.2.15) and using the interpolation result [33, Theorem 5.1.] one has

$$\mathcal{M}_{g\to(z,\pi)} \in \mathcal{L}(H^{-\frac{1}{2}+\kappa}(\Omega), (V^{\frac{3}{2}+\kappa}(\Omega) \cap V_0^1(\Omega)) \times H^{\frac{1}{2}+\kappa}(\Omega)/\mathbb{R}), \tag{2.2.16}$$

for $-\frac{1}{2} \leqslant \kappa \leqslant \frac{1}{2}$.
Finally the definition (2.2.9) of D_A^* and (2.2.16) in particular provide that

$$D_A^* \in \mathcal{L}(H^{-\frac{1}{2}+\kappa}(\Omega), V^\kappa(\Gamma)), \text{ for all } 0 < \kappa < \frac{1}{2}.$$

Hence we are done with the proof of Lemma 2.2.5. □

Remark 2.2.7. *In part (ii) of Lemma 2.2.5, the operator D_A^* is defined on the space of divergence free functions but in part (iii) we extended this definition by removing the divergence free constraint on the elements of the domain of D_A^*. This is possible since it is not necessary to have a divergence free function g in order to solve (2.2.10).*

In order to localize the control of the velocity on Γ_c (defined in (2.1.5)), we introduce the operator M, which is defined as follows

$$Mg(x) = m(x)g(x) - \frac{m}{\int_\Gamma m}\left(\int_\Gamma mg \cdot n\right) n(x) \quad \text{for all} \quad x \in \Gamma. \tag{2.2.17}$$

In the expression (2.2.17) the weight function $m \in C^\infty(\Gamma)$ takes values in $[0,1]$ and is supported in $\Gamma_c \subset \Gamma_{in}$. Moreover, m equals 1 in some open connected component

$$\Gamma_c^+ \Subset \Gamma_c. \tag{2.2.18}$$

So the operator M localizes the support of the control on Γ_{in} and also guarantees that $Mg \in V^0(\Gamma)$ for any $g \in L^2(\Gamma)$.

Lemma 2.2.8. *[43, Lemma 2.3] The operator $M \in \mathcal{L}(V^0(\Gamma))$ (defined in (2.2.17)) is symmetric.*

Sometimes we might use the notation

$$\mathbb{T}(v,p) = \nu(\nabla v + (\nabla v)^T) - pI, \tag{2.2.19}$$

to denote the Cauchy stress tensor corresponding to a vector field v and a pressure p. We now define the operator

$$B = (\lambda_0 I - A)PD_A M \in \mathcal{L}(V^0(\Gamma), (\mathcal{D}(A^*))'), \tag{2.2.20}$$

where $(\mathcal{D}(A^*))'$ denotes the dual of the space $\mathcal{D}(A^*)$ with $V_n^0(\Omega)$ as the pivot space.

Proposition 2.2.9. *(i) The adjoint of the operator B, computed for the duality structure $\langle \cdot, \cdot \rangle_{(\mathcal{D}(A^*)', \mathcal{D}(A^*))}$, that we will denote by B^* in the following, satisfies $B^* \in \mathcal{L}(\mathcal{D}(A^*), V^0(\Gamma))$ and for all $\Phi \in \mathcal{D}(A^*)$,*

$$B^*\Phi = M\left(-\nu\frac{\partial \Phi}{\partial n} + \left(\psi - \frac{1}{|\Gamma|}\left(\int_\Gamma \psi\right)\right) n\right) \tag{2.2.21}$$

$$= -M\mathbb{T}\left(\Phi, \left(\psi - \frac{1}{|\Gamma|}\left(\int_\Gamma \psi\right)\right)\right) n, \tag{2.2.22}$$

where

$$\nabla\psi = (I - P)[\nu\Delta\Phi + (v_s \cdot \nabla)\Phi - (\nabla v_s)^T\Phi], \tag{2.2.23}$$

and $\mathbb{T}$ denotes the stress tensor as defined in (2.2.19).
(ii) There exists a positive constant $\omega > 0$ such that the operator B^* can be extended as a bounded linear map from $\mathcal{D}((\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}})$ to $V^\kappa(\Gamma)$, for all $0 < \kappa < \frac{1}{2}$ i.e.

$$B^* \in \mathcal{L}(\mathcal{D}((\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}}), V^\kappa(\Gamma)) \quad \text{for all} \quad 0 < \kappa < \tfrac{1}{2}. \tag{2.2.24}$$

Proof. (i) From Lemma 2.2.5, we know that

$$B^*\Phi = MD_A^* P(\lambda_0 I - A^*)\Phi = M\left(-\nu\frac{\partial \hat{\Phi}}{\partial n} + \left(\psi - \frac{1}{|\Gamma|}\left(\int_\Gamma \psi\right)\right) n\right), \tag{2.2.25}$$

where

$$\begin{cases} \lambda_0\hat{\Phi} - \nu\Delta\hat{\Phi} - \beta\hat{\Phi} - ((v_s \cdot \nabla)\hat{\Phi}) + (\nabla v_s)^T\hat{\Phi} + \nabla\psi = P(\lambda_0 I - A^*)\Phi & \text{in} \quad \Omega, \\ \text{div}\hat{\Phi} = 0 & \text{in} \quad \Omega, \\ \hat{\Phi} = 0 & \text{on} \quad \Gamma. \end{cases} \tag{2.2.26}$$

This gives $\hat{\Phi} = \Phi$ and the expression (2.2.23). Hence the representation (2.2.21) directly follows from (2.2.25). Also (2.2.22) follows from (2.2.21) because $(\nabla\Phi)^T n = 0$ on Γ (this can be easily deduced from the fact that Φ on Γ is zero and $\mathrm{div}(\Phi) = 0$ on Ω).
(ii) Recall from Lemma 2.2.2 that $(A^*, \mathcal{D}(A^*))$ generates an analytic semigroup on $V_n^0(\Omega)$. Hence one can always choose a large enough positive constant ω from the resolvent set of A, such that the spectrum of $(A^* - \omega I)$ lies in the open left half-plane. Now following the definition [20, p. 329, Section 7.4, Eq. 7.4.3] one can define the operator $(\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}}$ where $0 < \kappa < \frac{1}{2}$. Let us consider $\Phi \in \mathcal{D}((\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}})$ where $0 < \kappa < \frac{1}{2}$. Since

$$\mathcal{D}((\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}}) = [V^2(\Omega) \cap V_0^1(\Omega), V_n^0(\Omega)]_{\frac{1}{4}-\frac{\kappa}{2}} = V^{\frac{3}{2}+\kappa}(\Omega) \cap V_0^1(\Omega)$$

(for details on the characterization of domains of fractional powers we refer to [32]), one observes the following

$$P(\lambda_0 I - A^*)\Phi \in H^{-\frac{1}{2}+\kappa}(\Omega). \tag{2.2.27}$$

Now one can use the expression of B^* as given by (2.2.25) and part (iii) of the Lemma 2.2.5 to prove (2.2.24). □

Now following [41] the Oseen equations

$$\begin{cases} \dfrac{\partial y}{\partial t} - \beta y - \nu\Delta y + (v_s \cdot \nabla)y + (y \cdot \nabla)v_s + \nabla q = 0 & \text{in} \quad Q_\infty, \\ \mathrm{div} y = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0 \cup \Gamma_{out}) \times (0, \infty), \\ y = Mu & \text{on} \quad \Gamma_{in} \times (0, \infty), \\ y(x, 0) = y_0 & \text{on} \quad \Omega, \end{cases} \tag{2.2.28}$$

can be written in the following evolution equation form

$$\begin{cases} Py' = APy + Bu & \text{in} \quad (0, \infty), \\ Py(0) = y_0, & \\ (I - P)y = (I - P)D_A Mu & \text{in} \quad (0, \infty). \end{cases} \tag{2.2.29}$$

In the following section we discuss some spectral properties of the Oseen operator A and then we define a suitable control space in order to construct a control function which stabilizes the Oseen equations.

2.2.1.2 Spectral properties of A and the stabilizability criterion

Since the resolvent of A is compact (see Lemma 2.2.2), the spectrum $spec(A)$ of the operator A is discrete. Moreover since A is the generator of an analytic semi group (see Lemma 2.2.2), $spec(A)$ is contained in a sector. Also the eigenvalues are of finite

multiplicity and appear in conjugate pairs when they are not real.
We denote by $(\lambda_k)_{k\in N}$ the eigenvalues of A. Without loss of generality we can always assume that there is no eigenvalue of A with zero real part by fixing a slightly larger β, if necessary. So we choose $N_u \in \mathbb{N}$ such that

$$...\mathrm{Re}\lambda_{N_u+1} < 0 < \mathrm{Re}\lambda_{N_u} \leqslant ... \leqslant \mathrm{Re}\lambda_1. \tag{2.2.30}$$

Following [37], we now choose the control space as follows

$$U_0 = \mathrm{vect}\, \oplus_{k=1}^{N_u} (\mathrm{Re}B^*\ker(A^* - \lambda_k I) \oplus \mathrm{Im}B^*\ker(A^* - \lambda_k I)). \tag{2.2.31}$$

The choice (2.2.31) of the control space plays an important role in proving a unique continuation property which implies the stabilizability of the pair (A, B). Let us choose the functions g_j in (2.1.8) such that

$$\{g_j \mid 1 \leqslant j \leqslant N_c\} \text{ is an orthonormal basis of } U_0. \tag{2.2.32}$$

For later use we now prove an additional regularity result for the elements of the control space U_0. The following regularity result is true only because the elements of U_0 are supported on a smooth subset of Γ.

Lemma 2.2.10. *The set U_0, defined in (2.2.31), is a subspace of $C^\infty(\Gamma)$.*

Proof. The function m is supported on Γ_c, which is C^∞. In view of the representation (2.2.21) of the operator B^*, we observe that to prove Lemma 2.2.10 it is enough to show that for each $1 \leqslant k \leqslant N_u$, any solution (ϕ, ψ) to the system (2.2.33) is C^∞ in some open set Ω_{Γ_c} $(\subset \Omega)$ such that $\partial\Omega_{\Gamma_c}$ contains Γ_c. Let us consider $k \in \{1, ..., N_u\}$ and (ϕ, ψ) solves the following

$$\begin{cases} \lambda_k\phi - \nu\Delta\phi - \beta\phi - ((v_s \cdot \nabla)\phi) + (\nabla v_s)^T\phi + \nabla\psi = 0 & \text{in} \quad \Omega, \\ \mathrm{div}\phi = 0 & \text{in} \quad \Omega, \\ \phi = 0 & \text{on} \quad \Gamma. \end{cases} \tag{2.2.33}$$

We thus apply the elliptic regularity result [26, Theorem 3.2.1.3] to show that

$$\phi \in \mathcal{D}(A^*) = V^2(\Omega) \cap V_0^1(\Omega) \quad \text{and} \quad \psi \in H^1(\Omega). \tag{2.2.34}$$

We will work in a neighborhood of Γ_c in order to avoid the singularities due to the presence of the corners $(0, 0)$ and $(0, 1)$. First consider a neighborhood $N^b_{\Gamma_c}$ of Γ_c such that neither of the points $(0, 0)$ and $(0, 1)$ belong to $N^b_{\Gamma_c}$. Now we consider an open set Ω_{Γ_c} such that $\Omega_{\Gamma_c} \subset \Omega$, $\partial\Omega_{\Gamma_c}$ (the boundary of Ω_{Γ_c}) is C^∞ and $\partial\Omega_{\Gamma_c} \cap \Gamma = N^b_{\Gamma_c}$. Let $\Theta \in C^\infty(\bar{\Omega}_{\Gamma_c})$ be such that $\Theta = 1$ on a subset of $\bar{\Omega}_{\Gamma_c}$ containing Γ_c and $\Theta = 0$ on $\partial\Omega_{\Gamma_c} \setminus N^b_{\Gamma_c}$. One can check that the function $(\Theta\phi, \Theta\psi)$ satisfies the following

$$-\nu\Delta(\Theta\phi) + \nabla(\Theta\psi) = F(\Theta, \phi, \psi) \quad \text{in} \quad \Omega_{\Gamma_c}, \tag{2.2.35}$$

where

$$F(\Theta,\phi,\psi) = -\nu\Delta\Theta\phi - 2\nu\nabla\Theta\nabla\phi - \phi(v_s\cdot\nabla)\Theta + \psi\nabla\Theta,$$

and also $\Theta\phi = 0$ on $\partial\Omega_{\Gamma_c}$, which implies $\int_{\Omega_{\Gamma_c}} \mathrm{div}(\Theta\phi) = \int_{\partial\Omega_{\Gamma_c}} (\Theta\phi)\cdot n = 0$. Using (2.2.34) one verifies that

$$F(\Theta,\phi,\psi) \in H^1(\Omega_{\Gamma_c}) \quad \text{and} \quad \mathrm{div}(\Theta\phi) = \phi\cdot\nabla\Theta \in H^2(\Omega_{\Gamma_c}).$$

Now we apply [9, Theorem IV.5.8] to obtain, $(\Theta\phi, \Theta\psi) \in H^3(\Omega_{\Gamma_c}) \times H^2(\Omega_{\Gamma_c})$. We can use a bootstrap argument to conclude that, $(\Theta\phi, \Theta\psi) \in C^\infty(\Omega_{\Gamma_c})$. Hence we finally have $g_j \in C^\infty(\Gamma)$, for all $1 \leqslant j \leqslant N_c$. □

We are looking for a control u taking values in U_0. We write

$$u(x,t) = \sum_{j=1}^{N_c} w_j(t) g_j(x), \tag{2.2.36}$$

where $w_c = (w_1, ..., w_{N_c}) \in L^2(0,\infty;\mathbb{R}^{N_c})$ is the control variable. Again in view of [37] we define a new control operator $\mathcal{B} \in \mathcal{L}(\mathbb{R}^{N_c}, (\mathcal{D}(A^*))')$ as

$$\mathcal{B}w_c = \sum_{j=1}^{N_c} w_j B g_j = \sum_{j=1}^{N_c} w_j(\lambda_0 I - A)PD_A g_j. \tag{2.2.37}$$

Observe that $\mathcal{B}$ is defined by restricting the action of the operator B to U_0.
Let us consider the controlled system

$$Py' = APy + Bu \text{ in } (0,\infty), \quad Py(0) = y_0, \tag{2.2.38}$$

which we obtain from $(2.2.29)_1$-$(2.2.29)_2$. With the definition (2.2.37) and a control of the form (2.2.36), the system (2.2.38) takes the form

$$Py' = APy + \mathcal{B}w_c \text{ in } (0,\infty), \quad Py(0) = y_0.$$

Theorem 2.2.11. *Assume that the spectrum of A obeys the condition (2.2.30), we choose $\{g_j \mid 1 \leqslant j \leqslant N_c\}$ as (2.2.32) and the operator $\mathcal{B}$ is as defined in (2.2.37). Then the pair $(A,\mathcal{B})$ is stabilizable in $V_n^0(\Omega)$.*

Before going into the proof of Theorem 2.2.11, let us recall that the pair $(A,\mathcal{B})$ is stabilizable in $V_n^0(\Omega)$ iff for all $y_0 \in V_n^0(\Omega)$, there exists a control $w_c \in L^2(0,\infty;\mathbb{R}^{N_c})$ such that the controlled system

$$Py' = APy + \mathcal{B}w_c \text{ in } (0,\infty), \quad Py(0) = y_0,$$

obeys

$$\int_0^\infty \|Py(t)\|^2_{V_n^0(\Omega)} dt < \infty.$$

The proof of Theorem 2.2.11 in a more intricate situation involving mixed boundary condition can be found in [37]. In [37] the localization operator M, localizing the control, is simply the cutoff function m whereas in our case M is as defined in (2.2.17). For the sake of completeness, we present the proof of Theorem 2.2.11 below, which follows step by step the one of [37] up to minor modifications.

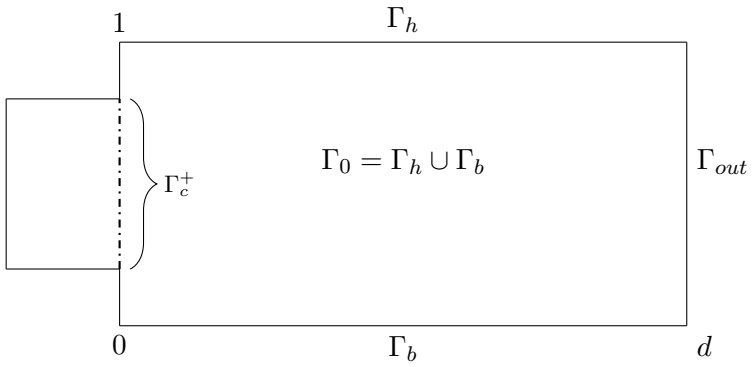


Figure 2.2: Domain Ω_{ex}.

Proof of Theorem 2.2.11. According to [42, Theorem 1.2] (one can also consult [6, Chapter V] for related results) proving the stabilizability of the pair $(A, \mathcal{B})$ is equivalent to verifying the Hautus criterion:

$$\ker(\lambda_k I - A^*) \cap \ker(\mathcal{B}^*) = \{0\}, \quad \text{for all} \quad 1 \leqslant k \leqslant N_u. \tag{2.2.39}$$

Let $\phi \in \ker(\lambda_k I - A^*)$. Also suppose that ψ is the pressure associated with ϕ, *i.e.* the pair (ϕ, ψ) solves (2.2.33). Now one can use (2.2.37) and Proposition 2.2.9 in order to verify that

$$\mathcal{B}^*\phi = -\left(\int_{\Gamma_c} g_j \overline{M\mathbb{T}(\phi,\psi)n}\, dx \right)_{1\leqslant j\leqslant N_c} \tag{2.2.40}$$

$$= -\left(\int_{\Gamma_c} g_j M\mathrm{Re}\mathbb{T}(\phi,\psi)n\, dx \right)_{1\leqslant j\leqslant N_c} + i\left(\int_{\Gamma_c} g_j M\mathrm{Im}\mathbb{T}(\phi,\psi)n\, dx \right)_{1\leqslant j\leqslant N_c}. \tag{2.2.41}$$

One can notice that $M\mathrm{Re}\mathbb{T}(\phi,\psi)n \in U_0$ and $M\mathrm{Im}\mathbb{T}(\phi,\psi)n \in U_0$. On the other hand we know that $\{g_j\}_{1\leqslant j\leqslant N_c}$ forms a basis of U_0. Hence $\mathcal{B}^*\phi = 0$ implies that

$$M(\mathbb{T}(\phi,\psi)n)\ |_{\Gamma_c} = 0.$$

This implies that

$$\mathbb{T}(\phi,\psi)n = C_0 n \quad \text{on} \quad supp\,(m), \tag{2.2.42}$$

where C_0 is a constant given by

$$C_0 = \frac{1}{\int\limits_\Gamma m} \left(\int\limits_\Gamma m\mathbb{T}(\phi,\psi)n \right).$$

Now recall that $\phi = 0$ on Γ and the unit outward normal on Γ_c^+ is $(-1, 0)$. Also since $\phi \in V^2(\Omega)$, one can consider the trace of divϕ on Γ to obtain that div$\phi = 0$ on Γ. Using these facts one can at once deduce from (2.2.42) that $\dfrac{\partial\phi}{\partial n} = 0$ and $\psi = C_0$ on Γ_c^+.
Now consider the domain Ω_{ex} which is an extension of the domain Ω (see Figure 2). Extend the function ϕ into Ω_{ex} by defining it zero outside Ω, denote the extension also by ϕ. Extend ψ into Ω_{ex} by the constant C_0 outside Ω. We denote the extension of ψ by ψ itself. It is not hard to verify that the extended pair $(\phi, \psi) \in V^2(\Omega_{ex}) \times H^1(\Omega_{ex})/\mathbb{R}$, solves the eigenvalue problem (2.2.33) in the extended domain Ω_{ex}. Finally the unique continuation property from [19] shows that $\phi = 0$ in Ω_{ex}, thus in particular on Ω. Hence we are done with the proof of the Hautus test (2.2.39). □

From Theorem 2.2.11 we know that the pair $(A, \mathcal{B})$ is stabilizable by a control $w_c \in L^2(0,\infty;\mathbb{R}^{N_c})$. Hence there exists a control u (of the form (2.2.36)) which belongs to the finite dimensional space U_0 (see (2.2.31)) and stabilizes the pair (A, B).
Now our aim is to construct w_c such that it is given in terms of a feedback control law. For that we will study the stabilization of the extended system (2.1.15) in the following section.

2.2.2 Stabilization of the extended system (2.1.15) by a feedback control

2.2.2.1 Evolution equation associated with the extended system (2.1.15)

We set

$$\widetilde{Z} = V_n^0(\Omega) \times \mathbb{R}^{N_c}. \tag{2.2.43}$$

Depending on the context the notation I denotes the identity operator for all of the spaces $V_n^0(\Omega)$, $\mathbb{R}^{N_c}$ and $\widetilde{Z}$. We equip the space $\widetilde{Z}$ with the inner product

$$(\widetilde{\zeta}_1, \widetilde{\zeta}_2)_{\widetilde{Z}} = (\zeta_1, \zeta_2)_{V_n^0(\Omega)} + (w_1, w_2)_{\mathbb{R}^{N_c}},$$

where $\widetilde{\zeta}_1 = (\zeta_1, w_1)$ and $\widetilde{\zeta}_2 = (\zeta_2, w_2)$.
We fix a positive constant γ (where γ is the constant appearing in the extended system (2.1.15)).
Now let us recall the representation (2.2.29) of the system (2.2.28). In the same note

it follows that $\widetilde{y} = (Py, w_c)$ is a solution to equation (2.1.15) iff (Py, w_c) solves the following set of equations

$$\begin{cases} \widetilde{y}' = \begin{pmatrix} Py \\ w_c \end{pmatrix}' = \begin{pmatrix} A & \mathcal{B} \\ 0 & -\gamma I \end{pmatrix} \begin{pmatrix} Py \\ w_c \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} \varphi_c + \widetilde{f} & \text{in} \quad (0,\infty), \\ \widetilde{y}(0) = \widetilde{y}_0 = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}, & \\ (I-P)y = \sum\limits_{j=1}^{N_c} w_j (I-P) D_A g_j & \text{in} \quad (0,\infty), \end{cases} \tag{2.2.44}$$

where $\widetilde{f} = (Pf, 0)$ and recall the definition of $\mathcal{B}$ from (2.2.37). Now we define the operator $(\widetilde{A}, \mathcal{D}(\widetilde{A}))$ in $\widetilde{Z}$ as follows

$$\mathcal{D}(\widetilde{A}) = \{(\zeta, w_c) \in \widetilde{Z} \mid A\zeta + \mathcal{B}w_c \in V_n^0(\Omega)\} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} A & \mathcal{B} \\ 0 & -\gamma I \end{pmatrix}. \tag{2.2.45}$$

As we have identified $V_n^0(\Omega)$ with its dual, the space $\widetilde{Z}$ and $\widetilde{Z}^*$ are also identified. We define the adjoint of $(\widetilde{A}, \mathcal{D}(\widetilde{A}))$ in $\widetilde{Z}$ as follows

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A^*) \times \mathbb{R}^{N_c} = \mathcal{D}(A) \times \mathbb{R}^{N_c} \quad \text{and} \quad \widetilde{A}^* = \begin{pmatrix} A^* & 0 \\ \mathcal{B}^* & -\gamma I \end{pmatrix}. \tag{2.2.46}$$

Remark 2.2.12. *We emphasize that due to the compatibility condition involved in the definition (2.2.45), $\mathcal{D}(\widetilde{A})$ can not be written as $\mathcal{D}(A) \times \mathbb{R}^{N_c}$. Contrary to that one does not require any compatibility condition in defining the domain of $\widetilde{A}^*$ which is given by (2.2.46).*

Theorem 2.2.13. *The operator $(\widetilde{A}, \mathcal{D}(\widetilde{A}))$ is the infinitesimal generator of an analytic semigroup on $\widetilde{Z}$.*

Proof. We will prove that $(\widetilde{A}, \mathcal{D}(\widetilde{A}))$ generates an analytic semigroup on $\widetilde{Z}$ by proving that $(\widetilde{A}^*, \mathcal{D}(\widetilde{A}^*))$ generates an analytic semigroup on $\widetilde{Z}$. This is enough since one has the following by using [27, Theorem 2.16.5, p. 56]

$$\begin{aligned} \|\mathcal{R}(\lambda, \widetilde{A})\|_{\mathcal{L}(\widetilde{Z})} = \|(\lambda I - \widetilde{A})^{-1}\|_{\mathcal{L}(\widetilde{Z})} = \|((\lambda I - \widetilde{A})^{-1})^*\|_{\mathcal{L}(\widetilde{Z})} &= \|(\overline{\lambda} I - \widetilde{A}^*)^{-1}\|_{\mathcal{L}(\widetilde{Z})} \\ &= \|\mathcal{R}(\overline{\lambda}, \widetilde{A}^*)\|_{\mathcal{L}(\widetilde{Z})}, \end{aligned}$$

where $\mathcal{R}(\lambda, \cdot)$ denotes the resolvent of the respective operator (see [27, Section 2.16] for details on resolvent) and hence $(\widetilde{A}, \mathcal{D}(\widetilde{A}))$ generates an analytic semigroup on $\widetilde{Z}$ follows from the fact that $(\widetilde{A}^*, \mathcal{D}(\widetilde{A}^*))$ generates an analytic semigroup on $\widetilde{Z}$ as a consequence of [47, p. 163, Def. 5.4.5].
Let us notice that the operator $\widetilde{A}^*$ can be decomposed as follows

$$\widetilde{A}^* = \widetilde{A}_1 + \widetilde{A}_2,$$

where

$$\widetilde{A}_1 = \begin{pmatrix} A^* & 0 \\ 0 & -\gamma I \end{pmatrix} \quad \text{and} \quad \widetilde{A}_2 = \begin{pmatrix} 0 & 0 \\ \mathcal{B}^* & 0 \end{pmatrix}.$$

Since $(A, \mathcal{D}(A))$ generates an analytic semigroup on $Z = V_n^0(\Omega)$ (see Lemma 2.2.2), $(A^*, \mathcal{D}(A^*))$ generates an analytic semigroup on Z (follows from the argument used in the beginning of the proof). Consequently the operator

$$\widetilde{A}_1^0 = \begin{pmatrix} A^* & 0 \\ 0 & 0 \end{pmatrix}$$

generates an analytic semigroup on $\widetilde{Z}$. Since $\widetilde{A}_1$ is a bounded perturbation of the operator $\widetilde{A}_1^0$, one uses [38, Corollary 2.2, Section 3.2] to conclude that $\widetilde{A}_1$ with domain $\mathcal{D}(A^*)\times\mathbb{R}^{N_c}$ generates an analytic semigroup on $\widetilde{Z}$. On the other hand the definition (2.2.37) of $\mathcal{B}$ and part (ii) of Proposition 2.2.9 furnish that

$$\mathcal{B}^* \in \mathcal{L}(\mathcal{D}((\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}}, \mathbb{R}^{N_c}) \quad \text{for all} \quad 0 < \kappa < \frac{1}{2}.$$

This implies the following

$$\widetilde{A}_2 \in \mathcal{L}(\mathcal{D}((\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}} \times \mathbb{R}^{N_c}, \widetilde{Z}) \quad \text{for all} \quad 0 < \kappa < \tfrac{1}{2}. \tag{2.2.47}$$

Now observe that $(\widetilde{A}_1 - \omega I)$ is a diagonal operator, hence the semigroup $e^{t(\widetilde{A}_1-\omega I)}$ on $\widetilde{Z}$ generated by $(\widetilde{A}_1 - \omega I)$ is of the form

$$e^{t(\widetilde{A}_1-\omega I)}(\zeta_1, w_1) = (e^{t(A^*-\omega I)}\zeta_1, e^{t(-\gamma-\omega)I}w_1), \quad \text{for all} \quad (\zeta_1, w_1) \in \widetilde{Z}.$$

Hence one can use the definition [20, p.329] of the domain of fractional power to have the following

$$\mathcal{D}((\omega I - \widetilde{A}_1)^{\frac{3}{4}+\frac{\kappa}{2}}) = \mathcal{D}((\omega I - A^*)^{\frac{3}{4}+\frac{\kappa}{2}}) \times \mathbb{R}^{N_c} \quad \text{for all} \quad 0 < \kappa < \tfrac{1}{2}. \tag{2.2.48}$$

Finally in view of (2.2.47) and (2.2.48), the result [44, p. 420, Lemma 12.38] furnish that $(\widetilde{A}^*, \mathcal{D}(\widetilde{A}^*))$ is the infinitesimal generator of an analytic semigroup on $\widetilde{Z}$. This in turn gives that $(\widetilde{A}, \mathcal{D}(\widetilde{A}))$ generates an analytic semigroup on $\widetilde{Z}$. □

From the definition (2.2.45) of the operator $\widetilde{A}$ one can easily observe that the spectrum of $\widetilde{A}$ is discrete and is explicitly given as follows

$$spec(\widetilde{A}) = spec(A) \cup \{-\gamma\}.$$

2.2.2.2 Existence of a feedback control law

We introduce the notation $\widetilde{J} = (0, I)$. Let us notice that $\widetilde{J}$ belongs to $\mathcal{L}(\mathbb{R}^{N_c}, \widetilde{Z})$. This section is devoted to the construction of a feedback control φ_c which is able to stabilize the linear equation

$$\widetilde{y}' = \widetilde{A}\widetilde{y} + \widetilde{J}\varphi_c \quad \text{in} \quad (0,\infty), \qquad \widetilde{y}(0) = \widetilde{y}_0, \tag{2.2.49}$$

which is obtained from $(2.2.44)_1$-$(2.2.44)_2$ after neglecting the non-homogeneous source term $\widetilde{f}$.

Proposition 2.2.14. *The pair $(\widetilde{A}, \widetilde{J})$ is stabilizable. More precisely there exists a feedback operator $\mathcal{K} \in \mathcal{L}(\widetilde{Z}, \mathbb{R}^{N_c})$ such that the operator $(\widetilde{A} + \widetilde{J}\mathcal{K})$ with domain $\mathcal{D}(\widetilde{A})$ generates an exponentially stable analytic semigroup on $\widetilde{Z}$.*

Before going into the proof of Proposition 2.2.14, let us recall that the pair $(\widetilde{A}, \widetilde{J})$ is stabilizable in $\widetilde{Z}$ iff for all $\widetilde{y}_0 \in \widetilde{Z}$, there exists a control $\varphi_c \in L^2(0,\infty;\mathbb{R}^{N_c})$ such that the controlled system

$$\widetilde{y}' = \widetilde{A}\widetilde{y} + \widetilde{J}\varphi_c \text{ in } (0,\infty), \qquad \widetilde{y} = \widetilde{y}_0,$$

satisfies

$$\|\widetilde{y}\|_{L^2(0,\infty;\widetilde{Z})} < \infty.$$

Proof of Proposition 2.2.14. We check the stabilizability of the pair $(\widetilde{A}, \widetilde{J})$ by verifying the Hautus criterion [42, Theorem 1.2] (one can also consult [6, Chapter V] for related results):

$$\ker(\widetilde{\lambda}_k I - \widetilde{A}^*) \cap \mathrm{Ker}(\widetilde{J}^*) = \{0\} \quad \text{for all} \quad \widetilde{\lambda}_k \in spec(\widetilde{A}) \quad \text{with} \quad \mathrm{Re}\widetilde{\lambda}_k > 0. \tag{2.2.50}$$

Let us prove (2.2.50). We consider

$$\begin{pmatrix} \phi \\ w \end{pmatrix} \in \ker(\widetilde{\lambda}_k I - \widetilde{A}^*) \cap \mathrm{Ker}(\widetilde{J}^*).$$

Recall that $\widetilde{J}^*(\phi, w) = w$. This gives $w = 0$.
Now use the relation $(\phi, 0) \in \ker(\widetilde{\lambda}_k I - \widetilde{A}^*)$, to obtain

$$(\widetilde{\lambda}_k I - A^*)\phi = \mathcal{B}^*\phi = 0.$$

Hence $\phi = 0$, since the pair $(A, \mathcal{B})$ is stabilizable (see Theorem 2.2.11). This furnishes the stabilizability of the pair $(\widetilde{A}, \widetilde{J})$.
We consider the following Riccati equation

$$\begin{cases} \widetilde{\mathcal{P}} \in \mathcal{L}(\widetilde{Z}, \widetilde{Z}), \quad \widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}^* > 0, \\ \widetilde{\mathcal{P}}\widetilde{A} + \widetilde{A}^*\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}\widetilde{J}\widetilde{J}^*\widetilde{P} = 0, \\ \widetilde{\mathcal{P}} \text{ is invertible.} \end{cases} \tag{2.2.51}$$

Using [29, Theorem 3], there exists a solution $\widetilde{\mathcal{P}}$ to the Riccati equation (2.2.51) and the operator $\mathcal{K} = -\widetilde{J}^*\widetilde{\mathcal{P}} \in \mathcal{L}(\widetilde{Z}, \mathbb{R}^{N_c})$, provides a stabilizing feedback for $(\widetilde{A}, \widetilde{J})$. The operator $(\widetilde{A} + \widetilde{J}\mathcal{K})$ with domain

$$\mathcal{D}(\widetilde{A} + \widetilde{J}\mathcal{K}) = \mathcal{D}(\widetilde{A})$$

is the generator of an exponentially stable analytic semigroup on $\widetilde{Z}$. □

From now onwards we will not use the explicit expression of the feedback controller $\mathcal{K}$ which was constructed in the proof of Proposition 2.2.14, in fact we will only use that $\mathcal{K} \in \mathcal{L}(\widetilde{Z}, \mathbb{R}^{N_c})$ and $\mathcal{D}(\widetilde{A} + \widetilde{J}\mathcal{K}) = \mathcal{D}(\widetilde{A})$.

2.2.2.3 Stabilization of the closed loop extended system with a non homogeneous source term

Using the feedback control $\mathcal{K}$, we write the equation $(2.2.44)_1$-$(2.2.44)_2$ as the following closed loop system

$$\begin{cases} \widetilde{y}' = \widetilde{A}\widetilde{y} + \widetilde{J}\mathcal{K}\widetilde{y} + \widetilde{f} \quad \text{in} \quad (0, \infty), \\ \widetilde{y}(0) = \widetilde{y}_0. \end{cases} \tag{2.2.52}$$

From now on the constant $K(> 0)$ appearing in the inequalities will denote a generic positive constant which may change from line to line. If we want to specify a constant (to use it for later purpose) we will denote it by K_i, for some natural number i.

Lemma 2.2.15. *Let the following hold*

$$\widetilde{f} \in L^2(0, \infty; V_n^0(\Omega) \times \mathbb{R}^{N_c}) \quad \textit{and} \quad \widetilde{y}_0 \in V_0^1(\Omega) \times \{0\}, \tag{2.2.53}$$

where $\{0\}$ *denotes the zero element of* $\mathbb{R}^{N_c}$. *Then the equation* (2.2.52) *admits a unique solution in*

$$\widetilde{y} \in H^1(0, \infty; \widetilde{Z}) \cap L^2(0, \infty; \mathcal{D}(\widetilde{A}))$$

which obeys

$$\|\widetilde{y}\|_{H^1(0,\infty;\widetilde{Z}) \cap L^2(0,\infty;\mathcal{D}(\widetilde{A}))} \leqslant K(\|\widetilde{y}_0\|_{V_0^1(\Omega)\times\mathbb{R}^{N_c}} + \|\widetilde{f}\|_{L^2(0,\infty;L^2(\Omega)\times R^{N_c})}), \tag{2.2.54}$$

for some positive constant K.

Proof. Observe that

$$\widetilde{y}_0 \in V_0^1(\Omega)\times\{0\} = [\mathcal{D}(A), V_n^0(\Omega)]_{1/2}\times\{0\} \underset{(i)}{=} [\mathcal{D}(A)\times\{0\}, V_n^0(\Omega)\times\{0\}]_{1/2} \underset{(ii)}{\subset} [\mathcal{D}(\widetilde{A}), \widetilde{Z}]_{1/2},$$

the steps (i) and (ii) in the calculation above directly follows by using the definition of interpolation spaces provided by using [33, p. 92, Theorem 14.1] and [33, Remark 14.1]. Since

$$\widetilde{f} \in L^2(0,\infty;\widetilde{Z}) \quad \text{and} \quad \widetilde{y}_0 \in [\mathcal{D}(\widetilde{A}),\widetilde{Z}]_{1/2},$$

one can use the isomorphism theorem [6, Part II, Section 3.6.3, Theorem 3.1] to conclude the proof of Lemma 2.2.15. □

Corollary 2.2.16. *Let the following hold*

$$f \in L^2(Q_\infty) \quad \textit{and} \quad y_0 \in V_0^1(\Omega). \tag{2.2.55}$$

Then the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \beta y - \nu\Delta y + (v_s\cdot\nabla)y + (y\cdot\nabla)v_s + \nabla q = f & \textit{in} \quad Q_\infty, \\ \mathrm{div} y = 0 & \textit{in} \quad Q_\infty, \\ y = 0 & \textit{on} \quad (\Gamma_0 \cup \Gamma_{out})\times(0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \textit{on} \quad \Gamma_{in}\times(0,\infty), \\ y(x,0) = y_0 & \textit{in} \quad \Omega, \\ w_c' + \gamma w_c - \mathcal{K}(Py, w_c) = 0 & \textit{in} \quad (0,\infty), \\ w_c(0) = 0 & \textit{in} \quad \Omega, \end{cases} \tag{2.2.56}$$

$w_c = (w_1,...,w_{N_c})$ and g_j, for all $1 \leqslant j \leqslant N_c$ are defined in (2.2.32), admits a unique solution (y, w_c) in $V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})$ and the pair (y, w_c) obeys the following estimate

$$\|(y,w_c)\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})} \leqslant K_1(\|y_0\|_{V_0^1(\Omega)} + \|f\|_{L^2(Q_\infty)}), \tag{2.2.57}$$

for some positive constant K_1.
In addition, there exists a constant $K_2 > 0$ such that the control

$$u(x,t) = \sum_{j=1}^{N_c} w_j(t) g_j(x), \tag{2.2.58}$$

satisfies the following estimate

$$\|u(x,t)\|_{L^\infty(\Sigma_\infty)} \leqslant K_2(\|y_0\|_{V_0^1(\Omega)} + \|f\|_{L^2(Q_\infty)}). \tag{2.2.59}$$

Proof. Using the notations used in (2.2.44), one observes that $\|\widetilde{y}_0\|_{V_0^1(\Omega)\times\mathbb{R}^{N_c}}$ $= \|(y_0,0)\|_{V_0^1(\Omega)\times\mathbb{R}^{N_c}} = \|y_0\|_{V_0^1(\Omega)}$ and $\|\widetilde{f}\|_{L^2(0,\infty;L^2(\Omega)\times\mathbb{R}^{N_c})} = \|(Pf,0)\|_{L^2(0,\infty;L^2(\Omega)\times\mathbb{R}^{N_c})} =$ $\|Pf\|_{L^2(Q_\infty)}$. Since the closed loop system (2.2.52) along with $(2.2.44)_3$ is the operator

representation of (2.2.56), one can use Lemma 2.2.15 (particularly the estimate (2.2.54)) to obtain the following

$$\|(Py, w_c)\|_{H^1(0,\infty;V_n^0(\Omega)\times\mathbb{R}^{N_c})\cap L^2(0,\infty;\mathcal{D}(\widetilde{A}))} \leqslant K(\|y_0\|_{V_0^1(\Omega)} + \|Pf\|_{L^2(Q_\infty)}). \tag{2.2.60}$$

Now we estimate

$$(I-P)y = \sum_{j=1}^{N_c} w_j (I-P)D_A g_j.$$

We know that there exists a positive constant K such that for all $1 \leqslant j \leqslant N_c$

$$\|D_A g_j\|_{V^2(\Omega)} \leqslant K\|g_j\|_{H^{3/2}(\Gamma)} \leqslant K. \tag{2.2.61}$$

Estimates (2.2.61) and (2.2.60) yield

$$\|(I-P)y\|_{H^1(0,\infty;H^2(\Omega))} \leqslant K(\|y_0\|_{V_0^1(\Omega)} + \|Pf\|_{L^2(Q_\infty)}). \tag{2.2.62}$$

Once again using (2.2.60) and (2.2.62) one has

$$\begin{aligned} \|y\|_{H^1(0,\infty;L^2(\Omega))} &\leqslant \|Py\|_{H^1(0,\infty;V_n^0(\Omega))} + \|(I-P)y\|_{H^1(0,\infty;H^2(\Omega))} \\ &\leqslant K(\|y_0\|_{V_0^1(\Omega)} + \|Pf\|_{L^2(Q_\infty)}). \end{aligned} \tag{2.2.63}$$

To prove higher regularity of y we will use a bootstrap argument. First we write $(2.2.56)_1$-$(2.2.56)_5$ as follows

$$\begin{cases} -\nu\Delta y + \nabla q = f^* & \text{in} \quad Q_\infty, \\ \text{div} y = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0 \cup \Gamma_{out}) \times (0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ y(x,0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{2.2.64}$$

where

$$f^* = f - \frac{\partial y}{\partial t} + \beta y - (v_s \cdot \nabla)y - (y \cdot \nabla)v_s.$$

Using (2.2.55) and (2.2.63) we obtain that $f^* \in L^2(0,\infty;H^{-1}(\Omega))$ and the following holds

$$\|f^*\|_{L^2(0,\infty;H^{-1}(\Omega))} \leqslant K(\|f\|_{L^2(Q_\infty)} + \|y_0\|_{V_0^1(\Omega)}). \tag{2.2.65}$$

Also

$$\sum_{j=1}^{N_c} w_j(t) g_j(x) \in H^1(0,\infty;V^0(\Gamma) \cap C^\infty(\Gamma)), \tag{2.2.66}$$

($g_j \in V^0(\Gamma)$ follows from the definition (2.2.32) and Proposition 2.2.9 whereas $C^\infty(\Gamma)$ regularity follows from Lemma 2.2.10). Hence one can use [9, Theorem IV.5.2], (2.2.65) and (2.2.60) to get $y \in L^2(0,\infty;V^1(\Omega))$ and the following inequality

$$\|y\|_{L^2(0,\infty;V^1(\Omega))} \leqslant K(\|f\|_{L^2(Q_\infty)} + \|y_0\|_{V_0^1(\Omega)}). \tag{2.2.67}$$

The regularity (2.2.63), (2.2.67) and (2.2.55) furnish $f^* \in L^2(Q_\infty)$ and

$$\|f^*\|_{L^2(Q_\infty)} \leqslant K(\|f\|_{L^2(Q_\infty)} + \|y_0\|_{V_0^1(\Omega)}). \tag{2.2.68}$$

In view of (2.2.66) and (2.2.68) one further obtains that $y \in L^2(0,\infty;V^2(\Omega))$ (using the regularity result from [28]) and the following

$$\|y\|_{L^2(0,\infty;V^2(\Omega))} \leqslant K(\|f\|_{L^2(Q_\infty)} + \|y_0\|_{V_0^1(\Omega)}). \tag{2.2.69}$$

Hence $y \in V^{2,1}(Q_\infty)$ and using (2.2.63) and (2.2.69) one has the following

$$\|y\|_{V^{2,1}(Q_\infty)} \leqslant K(\|y_0\|_{V_0^1(\Omega)} + \|f\|_{L^2(Q_\infty)}). \tag{2.2.70}$$

Finally, (2.2.60) and (2.2.70) provides the desired estimate (2.2.57).
Since $g_j \in L^\infty(\Gamma)$, the estimate (2.2.59) readily follows by using (2.2.57). □

The following result justifies our choice of denoting the inflow and outflow boundary of v_s and a perturbation of v_s using the same notation.

Corollary 2.2.17. *If we take*

$$(\|y_0\|_{V_0^1(\Omega)} + \|f\|_{L^2(Q_\infty)}) \leqslant \frac{L(1-L)}{2K_2}, \tag{2.2.71}$$

where K_2 is the constant in (2.2.59), then

$$\|y\,|_{\Sigma_\infty}\|_{L^\infty(\Sigma_\infty)} \leqslant \frac{L(1-L)}{2}$$

and hence in particular for all $t > 0$,

$$\begin{cases} (e^{-\beta t}y(\cdot,t) + v_s)\cdot n < 0 & \text{on } \Gamma_{in}, \\ (e^{-\beta t}y(\cdot,t) + v_s)\cdot n = 0 & \text{on } \Gamma_0, \\ (e^{-\beta t}y(\cdot,t) + v_s)\cdot n > 0 & \text{on } \Gamma_{out}, \end{cases} \tag{2.2.72}$$

where (y, w_c) is the solution to (2.2.56). This means that for all time $t > 0$, Γ_{in} and Γ_{out} are still the inflow and the outflow boundary for the perturbed vector field $(v_s + e^{-\beta t}y)$.

Proof. The proof is a direct consequence of Corollary 2.2.16, in particular the estimate (2.2.59). □

2.3 Stability of the continuity equation

This section is devoted to the study of the transport equation satisfied by density which is modeled by $(2.1.14)_1$ together with $(2.1.14)_2$ and $(2.1.14)_3$. This equation is linear in σ but nonlinear in (σ, y). First let us briefly discuss the stabilization of the linearized transport equation modeling the density with zero inflow boundary condition. This will give us an idea about how to obtain analogous results for its nonlinear counterpart.

2.3.1 Comments on the linear transport equation at velocity v_s

The linearized continuity equation with the zero inflow boundary condition is given by

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + (v_s \cdot \nabla)\sigma - \beta\sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega. \end{cases} \tag{2.3.1}$$

We can explicitly solve (2.3.1) to obtain

$$\sigma(x,t) = \begin{cases} e^{\beta t}\sigma_0(x_1 - (x_2(1-x_2))t, x_2) \quad \text{for} \quad t \leqslant \dfrac{1}{(x_2(1-x_2))}x_1, \\ \\ 0 \quad \text{for} \quad t > \dfrac{1}{(x_2(1-x_2))}x_1, \end{cases} \tag{2.3.2}$$

for all $(x_1, x_2) \in \Omega$. In particular if we assume that σ_0 satisfies the condition (2.1.6), the solution σ to (2.3.1) vanishes after some finite time $T_{A_1} = \frac{d}{A_1(1-A_1)}$. Hence we see that with zero inflow boundary condition the solution of the linearized transport equation is automatically stabilized (in fact controlled) after some finite time. The equation (2.3.1) is just a prototype of the transport equation $(2.1.14)_{1,2,3}$ exhibiting similar property and we will discuss this in the following section.

2.3.2 Stability of the transport equation (2.1.14) satisfied by density

We consider the transport equation satisfied by the density with the nonlinearity $(y\cdot\nabla)\sigma$. We assume that $\|y\|_{V^{2,1}(Q_\infty)}$ is small enough and the following holds

$$\begin{aligned} &(e^{-\beta t}y + v_s)\cdot n < 0 \text{ on } \Gamma_{in}, \ (e^{-\beta t}y + v_s)\cdot n = 0 \text{ on } \Gamma_0, \\ &\text{and } (e^{-\beta t}y + v_s)\cdot n > 0 \text{ on } \Gamma_{out}. \end{aligned} \tag{2.3.3}$$

Recalling Corollary 2.2.17, the condition (2.3.3) is automatically satisfied when y solves $(2.1.14)_4$-$(2.1.14)_8$ and (2.2.71) holds. Notice that the role of the condition (2.3.3) is only to guarantee that even if we perturb the vector field v_s by adding $e^{-\beta t}y$, the inflow boundary of the fluid remains unchanged.
Here the transport equation satisfied by the density is given by

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + ((v_s + e^{-\beta t}y)\cdot\nabla)\sigma - \beta\sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \end{cases} \tag{2.3.4}$$

where y is in $V^{2,1}(Q_\infty)$, (2.3.3) holds, $\sigma_0 \in L^\infty(\Omega)$ and satisfies the condition (2.1.6) (recall from (2.1.12) that $\sigma_0 = \rho_0$). Provided y is suitably small in the norm $V^{2,1}(Q_\infty)$,

(2.3.1) can be seen as an approximation of (2.3.4), and as we will see in Theorem 2.3.5, solutions of (2.3.1) and of (2.3.4) share some similar behavior.
We are in search of a unique solution of (2.3.4) in the space $L^\infty(Q_\infty)$. In the following discussion we will borrow several results from [9] on the existence, uniqueness and stability of the continuity equation. For later use, we shall consider a general transport equation of the form

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + (v \cdot \nabla)\sigma - \beta\sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Sigma_{in,v,\infty}, \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \end{cases} \tag{2.3.5}$$

where v is a divergence free vector field in $L^2(0,\infty;V^2(\Omega))$, and

$$\Sigma_{in,v,T} = \{(x,t) \in \Gamma \times (0,T) \mid v(x,t) \cdot n(x) < 0\}$$

First let us define the notion of weak solution for the transport equation $(2.3.5)_1$.

Definition 2.3.1. *Let $T > 0$ and v a divergence free vector field such that $v \in L^2((0,T);V^2(\Omega))$. A function $\sigma \in L^\infty(Q_T)$ is said to be a weak solution of $(2.3.5)_1$ if the following is true*

$$\int_0^T \int_\Omega \sigma(\partial_t \phi + v \cdot \nabla \phi + \beta\phi) dx dt = 0,$$

for any test function $\phi \in C^\infty(\bar{\Omega} \times [0,T])$ with $\phi(\cdot,T) = 0 = \phi(\cdot,0)$ in Ω and $\phi = 0$ on Σ_T.

One can interpret the boundary trace of a weak solution (as defined in Definition 2.3.1) of $(2.3.5)_1$ in a weak sense. Following [9] we introduce some notations which will be used to define the trace of a weak solution of $(2.3.5)_1$.
Let m denote the boundary Lebesgue measure on Γ. Now for any $T > 0$, associated to the vector field v, we introduce the measure

$$d\mu_v = (v \cdot n) dm dt \quad \text{on} \quad \Sigma_T$$

and denote by $d\mu_v^+$ (respectively $d\mu_v^-$) its positive (resp. negative) part in such a way that $|d\mu_v| = d\mu_v^+ + d\mu_v^-$. The support of $d\mu_v^+$ (resp. $d\mu_v^-$) is the outflow (resp. inflow) part of Σ_T corresponding to the vector field v.
The following two theorems, Theorem 2.3.2 and Theorem 2.3.3, are stated in [9] for a weaker assumption on the velocity field v. Here we state the results with $v \in L^2(0,T;V^2(\Omega))$ for the particular equation (2.3.5).

Theorem 2.3.2. *[9, Theorem VI.1.3] Let $T > 0$, $v \in L^2(0,T;V^2(\Omega))$ and $\sigma \in L^\infty(Q_T)$ be a weak solution of $(2.3.5)_1$ in the sense of the Definition 2.3.1. Then the following hold:*

(i) The function σ lies in $C^0([0,T],L^p(\Omega))$ for all $1 \leqslant p < +\infty$.
(ii) There exists a unique function $\gamma_\sigma \in L^\infty(\Sigma_T, |d\mu_v|)$ such that for any test function $\phi \in C^{0,1}(\bar{Q}_T)$ and for any $[t_0,t_1] \subset [0,T]$ we have

$$\int_{t_0}^{t_1}\int_\Omega \sigma\left(\frac{\partial\phi}{\partial t} + v\cdot\nabla\phi + \beta\phi\right)dxdt - \int_{t_0}^{t_1}\int_\Gamma \gamma_\sigma\phi d\mu_v + \int_\Omega \sigma(t_0)\phi(t_0)dx - \int_\Omega \sigma(t_1)\phi(t_1)dx = 0. \tag{2.3.6}$$

(iii) The renormalization property: For any function $\xi : \mathbb{R} \to \mathbb{R}$ of class C^1, for any $\phi \in C^{0,1}(\bar{Q}_T)$ and for any $[t_0,t_1] \subset [0,T]$ we have

$$\int_{t_0}^{t_1}\int_\Omega \xi(\sigma)\left(\frac{\partial\phi}{\partial t} + v\cdot\nabla\phi\right)dxdt + \int_{t_0}^{t_1}\int_\Omega \beta\sigma\xi'(\sigma)\phi - \int_{t_0}^{t_1}\int_\Gamma \xi(\gamma_\sigma)\phi d\mu_v + \int_\Omega \xi(\sigma(t_0))\phi(t_0)dx - \int_\Omega \xi(\sigma(t_1))\phi(t_1)dx = 0. \tag{2.3.7}$$

The following theorem states some results on the well posedness of the weak solution σ of the Cauchy-Dirichlet transport problem (2.3.5).

Theorem 2.3.3. *[9, Theorem VI.1.6] Let $T > 0$, $\sigma_0 \in L^\infty(\Omega)$ and $v \in L^2(0,T;V^2(\Omega))$. Then there exists a unique function $\sigma \in L^\infty(Q_T)$ such that*
(i) The function σ is a weak solution of the problem $(2.3.5)_1$ in Q_T in the sense of Definition 2.3.1.
(ii) The trace γ_σ of σ satisfies the inflow boundary condition, $\gamma_\sigma = 0$, $d\mu_v^-$ almost everywhere on $\Sigma_{in,v,T}$ and σ satisfies the initial condition $\sigma(x,0) = \sigma_0$ in Ω.
In the following, we call this function σ satisfying (i) and (ii), the solution of (2.3.5).
(iii) Moreover for $0 < t < T$, the solution σ of (2.3.5) satisfies

$$\|\sigma(\cdot,t)\|_{L^\infty(\Omega)} \leqslant \|\sigma_0\|_{L^\infty(\Omega)} e^{\beta t}. \tag{2.3.8}$$

Let us also recall, for later purpose, the following stability result for the transport equation with respect to its velocity field:

Lemma 2.3.4. *[9, Theorem VI.1.9] Let $T > 0$. Suppose that $\sigma_0 \in L^\infty(\Omega)$ and let $\{v_m\}_m$ be a sequence of functions in $L^2(0,T;V^2(\Omega))$ such that there exists $v \in L^2(0,T;V^2(\Omega))$ such that*

$$v_m \xrightarrow[m\to\infty]{} v \quad in \quad L^1(Q_T), \quad and \quad v_m\cdot n \xrightarrow[m\to\infty]{} v.n \quad in \quad L^1(\Sigma_T).$$

Now suppose that $\sigma_m \in L^\infty(Q_T)$ is the unique weak solution (in sense of Definition 2.3.1.) of the following initial and boundary value problem

$$\begin{cases} \dfrac{\partial \sigma_m}{\partial t} + (v_m \cdot \nabla)\sigma_m - \beta \sigma_m = 0 & \text{in} \quad Q_T, \\ \sigma_m(x,t) = 0 & \text{on} \quad \Sigma_{in,v_m,T}, \\ \sigma_m(x,0) = \sigma_0 & \text{in} \quad \Omega. \end{cases} \tag{2.3.9}$$

If we denote by σ the unique solution to the transport problem (2.3.5) in Q_T, then we have

$$\sigma_m \xrightarrow[m\to\infty]{} \sigma \quad \text{in} \quad C^0([0,T], L^p(\Omega)), \quad \text{for any} \quad 1 \leqslant p < +\infty. \tag{2.3.10}$$

Now we state the main theorem of this section:

Theorem 2.3.5. *Let $A_1 \in (0, \frac{1}{2})$ and $T_1 > T_{A_1} = \frac{d}{A_1(1-A_1)}$. There exists a constant $K_3 > 0$ such that if $y \in V^{2,1}(Q_\infty)$ satisfies*

$$\|y\|_{V^{2,1}(Q_\infty)} < K_3, \tag{2.3.11}$$

(2.3.3) holds, $\sigma_0 \in L^\infty(\Omega)$ and satisfies the condition (2.1.6), the solution σ of equation (2.3.4) satisfies the following

$$\begin{array}{l} (i)\, \forall t < T_1, \ \sigma(\cdot,t) \text{ satisfies the estimate (2.3.8)}, \\ (ii)\, \forall t \geqslant T_1, \ \|\sigma(\cdot,t)\|_{L^\infty(\Omega)} = 0. \end{array} \tag{2.3.12}$$

Proof of Theorem 2.3.5. Item (i) of (2.3.12) is automatically satisfied as a consequence of item (iii), Theorem 2.3.3.
We thus focus on the proof of item (ii) of Theorem 2.3.5. Let $T_1 > T_{A_1} = \frac{d}{A_1(1-A_1)}$ be fixed. Our approach will be based on the flow X corresponding to the vector field $v_s + e^{-\beta t} y$. In order to introduce it in a more convenient manner, we first extend the domain into $\mathbb{R}^2$. Observe that the definition of v_s can be naturally extended to $\mathbb{R}^2$ into a Lipschitz function by setting $v_s(x_1, x_2) = v_s(x_2)$ if $x_2 \in (0,1)$ and 0 if $x_2 \in \mathbb{R} \setminus (0,1)$. We denote this extension by v_s itself. For the following analysis we use the functional space

$$H^{2,1}(\mathbb{R}^2 \times (0,\infty)) = L^2(0,\infty; H^2(\mathbb{R}^2) \cap H^1(0,\infty; L^2(\mathbb{R}^2))$$

(this is consistent with the notations defined in Section 2.1.2). Now we introduce an extension operator $\mathbb{E}$ from Ω to $\mathbb{R}^2$.

$$\mathbb{E} : L^2(\Omega) \longrightarrow L^2(\mathbb{R}^2)$$

such that:

- for every $y \in L^2(\Omega)$, $\mathbb{E}y\,|_\Omega = y$,

- the restriction of $\mathbb{E}$ to $H^2(\Omega)$ defines a linear operator from $H^2(\Omega)$ to $H^2(\mathbb{R}^2)$,
- the restriction of $\mathbb{E}$ to $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ defines a linear operator from $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ to $H^2(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$,

The existence of such an extension operator is a direct consequence of [33, Theorem 2.2]. We now introduce the flow $X(x,t,s)$ defined for $x \in \mathbb{R}^2$ and $(t,s) \in [0,\infty)^2$, by the following differential equation:

$$\begin{cases} \dfrac{\partial X(x,t,s)}{\partial t} = (v_s + e^{-\beta t}\mathbb{E}y)(X(x,t,s),t), \\ X(x,t,s) \mid_{t=s} = x \in \mathbb{R}^2. \end{cases} \tag{2.3.13}$$

The integral formulation of (2.3.13) can be written as follows

$$\forall (x,t,s) \in \mathbb{R}^2 \times [0,\infty)^2, \quad X(x,t,s) = x + \int_s^t (v_s + e^{-\beta t}\mathbb{E}y)(X(x,\theta,s),\theta)d\theta. \tag{2.3.14}$$

As the vector field

$$(v_s + e^{-\beta t}\mathbb{E}y) \in L^2(0,\infty; W^{1,\infty}(\mathbb{R}^2)) + H^{2,1}(\mathbb{R}^2 \times (0,\infty)),$$

due to the Osgood condition (see [48] and [4, Theorem 3.7]) we know that equation (2.3.14) has a unique continuous solution. Similarly, we introduce the flow X_0 corresponding to the vector field v_s as the solution of the following differential equation:

$$\begin{cases} \dfrac{\partial X_0(x,t,s)}{\partial t} = v_s(X_0(x,t,s),t), \\ X_0(x,t,s) \mid_{t=s} = x \in \mathbb{R}^2. \end{cases} \tag{2.3.15}$$

As v_s is Lipschitz, the flow, which can also be seen as the solution of

$$X_0(x,t,s) = x + \int_s^t v_s(X_0(x,\theta,s),\theta)d\theta, \quad (x,t,s) \in \mathbb{R}^2 \times (0,\infty)^2, \tag{2.3.16}$$

is well defined in classical sense.

Lemma 2.3.6. *Let $T > 0$. There exists a constant $K_4 = K_4(T) > 0$ such that for all $y \in V^{2,1}(Q_\infty)$, $(t,s) \in [0,T]^2$ and $x \in \mathbb{R}^2$, the solutions of (2.3.13) and (2.3.15) satisfy the following*

$$\mid X(x,t,s) - X_0(x,t,s) \mid < K_4(T)\|y\|_{V^{2,1}(Q_\infty)}. \tag{2.3.17}$$

Proof. The proof of Lemma 2.3.6 can be performed by using arguments which are very standard in the literature. For the convenience of the reader we include the proof.

1. As $H^2(\mathbb{R}^2)$ is embedded in $L^\infty(\mathbb{R}^2)$, using Hölder's inequality we can at once obtain the following estimate for all $(t,s) \in [0,T]^2$ and $x \in \mathbb{R}^2$,

$$\left| \int_s^t e^{-\beta\theta} \mathbb{E}y(X(x,\theta,s),\theta) d\theta \right| \leqslant K \|\mathbb{E}y\|_{H^{2,1}(\mathbb{R}^2\times(0,\infty))},$$

for some constant $K > 0$.

2. Subtracting (2.3.14) from (2.3.16), we get, for all $(t,s) \in [0,\infty)^2$ and $x \in \mathbb{R}^2$,

$$\begin{aligned}
&|X(x,t,s) - X_0(x,t,s)| \\
&\leqslant \left| \int_s^t |v_s(X(x,\theta,s),\theta) - v_s(X_0(x,\theta,s),\theta)| d\theta \right| + \left| \int_s^t e^{-\beta\theta} |\mathbb{E}y(X(x,\theta,s),\theta)| d\theta \right| \\
&\leqslant \|\nabla v_s(.)\|_{L^\infty(\Omega)} \left| \int_s^t |X(x,\theta,s) - X_0(x,\theta,s)| d\theta \right| + K\|\mathbb{E}y\|_{H^{2,1}(\mathbb{R}^2\times(0,\infty))}.
\end{aligned}$$

Since $\mathbb{E}$ is a bounded operator from $L^2(\Omega)$ to $L^2(\mathbb{R}^2)$ and from $H^2(\Omega)$ to $H^2(\mathbb{R}^2)$, there exists a constant $K > 0$ such that

$$\begin{aligned}
|X(x,t,s) - X_0(x,t,s)| \leqslant \quad & \|\nabla v_s(.)\|_{L^\infty(\Omega)} \left| \int_s^t |X(x,\theta,s) - X_0(x,\theta,s)| d\theta \right| \\
& + K\|y\|_{V^{2,1}(Q_\infty)}.
\end{aligned} \tag{2.3.18}$$

Now we can use Grönwall's inequality to obtain (2.3.17). □

Recall that the solution of (2.3.1) vanishes after some finite time $T_{A_1} = \frac{d}{A_1(1-A_1)}$. At the same time Lemma 2.3.6 suggests that for any finite time $T > 0$, the flow $X_0(x,t,s)$ stays uniformly close to $X(x,t,s)$ in $\mathbb{R}^2 \times (0,T)$ provided $\|y\|_{V^{2,1}(Q_\infty)}$ is small enough. In view of these observations, in the following we design a Lyapunov functional corresponding to a localized energy, to prove that σ vanishes after the time $T_1 > T_{A_1}$ when $\|y\|_{V^{2,1}(Q_\infty)}$ is small enough, which will prove Theorem 2.3.5.
Let ε be a fixed positive constant in $(0, A_1)$ such that

$$T_1 = \frac{d+\varepsilon}{(A_1-\varepsilon)(1-A_1+\varepsilon)}. \tag{2.3.19}$$

Our primary goal is to prove that, for a velocity field y satisfying (2.3.3) and such that $\|y\|_{V^{2,1}(Q_{T_1})}$ is small enough and an initial condition $\sigma_0 \in L^\infty(\Omega)$ satisfying (2.1.6), the solution σ of (2.3.4) satisfies

$$\sigma(x,T_1) = 0 \quad \text{for all} \quad x \in \Omega. \tag{2.3.20}$$

In fact, the condition (2.3.3) does not play any role. We shall thus prove a slightly more general result: there exists $K_3 > 0$, such that for any velocity field y such that

$\|y\|_{V^{2,1}(Q_{T_1})} \leqslant K_3$ and any initial condition $\sigma_0 \in L^\infty(\Omega)$ satisfying (2.1.6), the solution σ of

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + ((v_s + e^{-\beta t} y) \cdot \nabla)\sigma - \beta\sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Sigma_{in,y,\infty}, \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \end{cases} \tag{2.3.21}$$

where

$$\Sigma_{in,y,\infty} = \{(x,t) \in \Gamma \times (0,T) \mid (v_s(x) + y(x,t)e^{-\beta t}) \cdot n(x) < 0\},$$

satisfies (2.3.20).

We will achieve this goal using two steps. In the first one, we shall consider smooth ($\in V^{2,1}(Q_{T_1}) \cap L^2(0,T_1; W^{1,\infty}(\Omega))$) vector field y. In the second one, we will explain how the same result can be obtained for all vector fields $y \in V^{2,1}(Q_{T_1})$.

Case $y \in V^{2,1}(Q_{T_1}) \cap L^2(0,T_1; W^{1,\infty}(\Omega))$. Here we assume that

$$y \in V^{2,1}(Q_{T_1}) \cap L^2(0,T_1; W^{1,\infty}(\Omega)). \tag{2.3.22}$$

With $\varepsilon > 0$ given by (2.3.19), we then define a function $\vartheta \in C^\infty(\mathbb{R}^2)$ and $\vartheta(x_1,x_2) \in [0,1]$ such that

$$\vartheta(x_1,x_2) = \begin{cases} 0 & \text{if} \quad (x_1,x_2) \in [0,d] \times [A_1, 1-A_1], \\ 1 & \text{if} \quad (x_1,x_2) \in \mathbb{R}^2 \setminus [-\frac{\varepsilon}{2}, d+\frac{\varepsilon}{2}] \times [A_1 - \frac{\varepsilon}{2}, 1 - A_1 + \frac{\varepsilon}{2}]. \end{cases} \tag{2.3.23}$$

We consider the following auxiliary transport problem

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} + ((v_s + e^{-\beta t}\mathbb{E}y) \cdot \nabla)\Psi = 0 & \text{in} \quad \mathbb{R}^2 \times (0,T_1), \\ \Psi(\cdot,0) = \vartheta & \text{in} \quad \mathbb{R}^2. \end{cases} \tag{2.3.24}$$

Since $v_s + e^{-\beta t}\mathbb{E}y$ belongs to $L^2(0,T_1; W^{1,\infty}(\mathbb{R}^2))$ the system (2.3.24) can be solved using the characteristics formula to obtain

$$\Psi(x,t) = \vartheta(X(x,0,t)) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2 \times [0,T_1], \tag{2.3.25}$$

where the flow $X(\cdot,\cdot,\cdot)$, defined by (2.3.13), is globally Lipschitz in $\mathbb{R}^2 \times [0,T_1]$. It follows that Ψ is also globally Lipschitz in $\mathbb{R}^2 \times [0,T_1]$. Besides, this formula immediately provides the non-negativity of Ψ in $\mathbb{R}^2 \times [0,T_1]$. We now introduce the following quantity:

$$E_{loc}(t) = \frac{1}{2}\int_\Omega \Psi(x,t)|\sigma(x,t)|^2 dx \quad \text{for all} \quad t \in [0,T_1]. \tag{2.3.26}$$

The idea is that this quantity will measure the L^2 norm of $\sigma(\cdot,t)$ localized in the support of $\Psi(\cdot,t)$.

In order to evaluate how the quantity E_{loc} evolves, we use the renormalization property (2.3.7) with $\xi(s) = s^2$ and we compute the time derivative of E_{loc} (in $\mathcal{D}'(0,T)$):

$$\begin{aligned}\frac{d}{dt}E_{loc}(t) &= \frac{1}{2}\int_\Omega (\frac{\partial \Psi}{\partial t} + (v_s + e^{-\beta t}\mathbb{E}y)\cdot\nabla\Psi)|\sigma|^2 dx + \beta\int_\Omega \Psi|\sigma|^2 dx \\ &\quad -\frac{1}{2}\int_\Gamma \Psi|\gamma_\sigma|^2((v_s + e^{-\beta t}\mathbb{E}y)\cdot n)dm \\ &\leqslant \beta\int_\Omega \Psi|\sigma|^2 dx = 2\beta E_{loc}(t).\end{aligned} \tag{2.3.27}$$

In the above calculation we have used that Ψ solves the equation $(2.3.24)_1$, γ_σ (the trace of σ, see Theorem 2.3.2, item (ii)) vanishes on Σ_{in,y,T_1}, and that Ψ stays non-negative in $(0,T_1)\times\mathbb{R}^2$. Now using Grönwall's inequality in (2.3.27), we get

$$\frac{1}{2}\int_\Omega \Psi(x,T_1)|\sigma(x,T_1)|^2 dx = E_{loc}(T_1) \leqslant e^{2\beta T_1}E_{loc}(0) = 0, \tag{2.3.28}$$

where the last identity comes from the fact that $\sigma_0 \in L^\infty(\Omega)$ satisfies the condition (2.1.6) and the choice of Ψ in (2.3.23), (2.3.24).
We now prove that

$$\forall x \in \Omega, \quad \Psi(x,T_1) = 1. \tag{2.3.29}$$

In order to prove (2.3.29), we will rely on the formula (2.3.25), and Lemma 2.3.6. Indeed, for $x = (x_1,x_2) \in \Omega$, we have

$$X_0(x,0,T_1) = \begin{pmatrix} x_1 - T_1(x_2(1-x_2)) \\ x_2 \end{pmatrix}.$$

Therefore, if $x = (x_1,x_2) \in \Omega$ satisfies $x_2 \in (A_1-\varepsilon, 1-A_1+\varepsilon)$, as one has $x_2(1-x_2) > (A_1-\varepsilon)(1-A_1+\varepsilon)$, $(X_0(x,0,T_1))_1 \leqslant d - T_1(A_1-\varepsilon)(1-A_1+\varepsilon) \leqslant -\varepsilon$. Similarly, if $x_2 \in [0,1]\setminus(A_1-\varepsilon, 1-A_1+\varepsilon)$, $(X_0(x,0,T_1))_2 \in [0,1]\setminus(A_1-\varepsilon,1-A_1+\varepsilon)$. In particular, one obtains that for all $x=(x_1,x_2)\in\Omega$

$$X_0(x,0,T_1) \in \mathbb{R}^2 \setminus (-\varepsilon, d+\varepsilon)\times(A_1-\varepsilon, 1-A_1+\varepsilon). \tag{2.3.30}$$

Now set $K_3 = K_3(T_1) = \dfrac{\varepsilon}{2K_4(T_1)} > 0$, where $K_4(T_1)$ is the constant appearing in Lemma 2.3.6, and assume that

$$\|y\|_{V^{2,1}(Q_{T_1})} < K_3. \tag{2.3.31}$$

The inequality (2.3.17), (2.3.30) and the assumption (2.3.31) furnish that for all $x \in \Omega$,

$$X(x,0,T_1) \in \mathbb{R}^2 \setminus [-\frac{\varepsilon}{2}, d+\frac{\varepsilon}{2}]\times[A_1-\frac{\varepsilon}{2}, 1-A_1+\frac{\varepsilon}{2}]. \tag{2.3.32}$$

Now using the representation (2.3.25) of Ψ, we immediately deduce (2.3.29). The estimate (2.3.28) then yields that σ vanishes at time T_1 in the whole set Ω, i.e. the identity

(2.3.20).

The general case $y \in V^{2,1}(Q_{T_1})$. We now discuss the case in which y does not satisfy the regularity (2.3.22) and y only belongs to $V^{2,1}(Q_\infty)$ as stated in Theorem 2.3.5. In order to deal with this case, we use the density of $V^{2,1}(Q_{T_1}) \cap L^2(0,T_1;W^{1,\infty}(\Omega))$ in $V^{2,1}(Q_{T_1})$. In particular, if y belongs to $V^{2,1}(Q_\infty)$ and satisfies (2.3.31), we can find a sequence y_n of functions of $V^{2,1}(Q_{T_1}) \cap L^2(0,T_1;W^{1,\infty}(\Omega))$ such that y_n strongly converges to y in $V^{2,1}(Q_T)$ and for all n, $\|y_n\|_{V^{2,1}(Q_{T_1})} < K_3$. Using then the previous arguments, we can show that for all n, $\sigma_n(x,T_1) = 0$ for all $x \in \Omega$, where σ_n denotes the solution of (2.3.9) on the time interval $(0,T_1)$. The strong convergence of (y_n) to y in $V^{2,1}(Q_{T_1})$, hence of y_n to y in $L^1(Q_{T_1})$ and of $y_n \cdot n$ to $y \cdot n$ in $L^1(\Sigma_{T_1})$, and Lemma 2.3.4 then imply (2.3.20).

End of the proof of Theorem 2.3.5. We shall then show that, when $y \in V^{2,1}(Q_\infty)$ satisfies the condition (2.3.31), the solution σ of (2.3.4) stays zero for times larger than T_1. This is obvious, as one can replace $(2.3.4)_3$ by $\sigma(x,T_1) = 0$ on Ω and solve the Cauchy problem (2.3.4) in the time interval $[T_1,\infty)$ to obtain that σ is the trivial solution

$$\sigma(x,s) = 0 \quad \text{for all} \quad (x,s) \in \Omega \times [T_1,\infty).$$

This concludes the proof of Theorem 2.3.5. □

Remark 2.3.7. *In the above proof, we have handled separately the case* $y \in V^{2,1}(Q_{T_1}) \cap L^2(0,T_1;W^{1,\infty}(\Omega))$ *from the case of a general vector field* $y \in V^{2,1}(Q_{T_1})$, *because the solution* Ψ *of* (2.3.24) *for a vector field* $y \in V^{2,1}(Q_{T_1})$ *has* a priori *only Hölder regularity (see in particular [4, Theorem 3.7]), and thus cannot be used directly as a test function in the weak formulation* (2.3.7) *to obtain* (2.3.27).

Remark 2.3.8. *In general to prove the stabilizability of a non-linear problem it is usual to first study the stabilizability of the corresponding linear problem and then consider the non-linear term as a source term to obtain analogous stabilizability result corresponding to the complete non-linear system. But the reader may notice that contrary to the usual method we did not consider the non-linear term* $(y \cdot \nabla)\sigma$ *(nonlinear in* (σ, y) *but linear in* σ*) as a source term while dealing with the system* (2.3.4). *This is because the transport equation has no regularizing effect on its solution, hence it is not possible to consider the non-linear term in* (2.3.4) *as a source term and to recover the solution in* $L^\infty(Q_\infty)$.

2.4 Stabilization of the two dimensional Navier-Stokes equations.

Proof of Theorem 2.1.2. We will prove Theorem 2.1.2 using the Schauder fixed point theorem. We now discuss the strategy of the proof.

(i) First we define an appropriate fixed point map. This will be done in Section 2.4.1.
(ii) Then we fix a suitable ball which is stable by the map defined in step (i). This is done in the Section 2.4.2.
(iii) In Section 2.4.3 we show that the ball defined in step (ii), is compact in some appropriate topology. We then prove that the fixed point map from step (i) in that topology is continuous.
(iv) At the end we draw the final conclusion to prove Theorem 2.1.2.

2.4.1 Definition of a fixed point map

Let us recall the fully non linear system (including the boundary controls) under consideration:

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + ((v_s + e^{-\beta t} y)\cdot\nabla)\sigma - \beta\sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \Gamma_{in}\times(0,\infty), \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \\ \dfrac{\partial y}{\partial t} - \beta y - \nu\Delta y + (v_s\cdot\nabla)y + (y\cdot\nabla)v_s + \nabla q = \mathcal{F}(y,\sigma) & \text{in} \quad Q_\infty, \\ \mathrm{div}(y) = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0\cup\Gamma_{out})\times(0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_{in}\times(0,\infty), \\ y(x,0) = y_0 & \text{in} \quad \Omega, \\ w_c^{'} + \gamma w_c - \mathcal{K}(Py, w_c) = 0 & \text{in} \quad (0,\infty), \\ w_c(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.4.1}$$

where

$$\mathcal{F}(y,\sigma) = -e^{-\beta t}\sigma\frac{\partial y}{\partial t} - e^{-\beta t}(y\cdot\nabla)y - e^{-\beta t}\sigma(v_s\cdot\nabla)y - e^{-\beta t}\sigma(y\cdot\nabla)v_s - e^{-2\beta t}\sigma(y\cdot\nabla)y + \beta e^{-\beta t}\sigma y,$$

and $w_c = (w_1, ..., w_{N_c})$. To prove the existence of a solution of the system (2.4.1) we are going to define a suitable fixed point map.
Now assume that $\sigma_0 \in L^\infty(\Omega)$ and satisfies (2.1.6). Recall the definition of g_j's from

(2.2.32). Let us suppose that $\widehat{y} \in V^{2,1}(Q_\infty)$ satisfies (2.3.11) and on the boundary it is given in the following form

$$\widehat{y} \mid_{\Sigma_\infty} = \sum_{j=1}^{N_c} \widehat{w}_j(t) g_j(x), \tag{2.4.2}$$

where $\widehat{w}_c = (\widehat{w}_1, ..., \widehat{w}_{N_c}) \in H^1(0, \infty; \mathbb{R}^{N_c})$. In addition the coefficients $\widehat{w}_c$ are assumed to be such that $\widehat{y}$ satisfies the following boundary condition

$$\| \widehat{y} \mid_{\Sigma_\infty} \|_{L^\infty(\Sigma_\infty)} \leqslant \frac{L(1-L)}{2}, \tag{2.4.3}$$

where the constant L was fixed in (2.1.5). We further assume that $y_0 \in V_0^1(\Omega)$.
We consider the following set of equations

$$\begin{cases} \dfrac{\partial \widehat{\sigma}}{\partial t} + ((v_s + e^{-\beta t}\widehat{y}) \cdot \nabla)\widehat{\sigma} - \beta \widehat{\sigma} = 0 & \text{in} \quad Q_\infty, \\ \widehat{\sigma}(x,t) = 0 & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ \widehat{\sigma}(x,0) = \sigma_0 & \text{in} \quad \Omega, \\ \dfrac{\partial y}{\partial t} - \beta y - \nu \Delta y + (v_s \cdot \nabla) y + (y \cdot \nabla) v_s + \nabla q = \mathcal{F}(\widehat{y}, \widehat{\sigma}) & \text{in} \quad Q_\infty, \\ \operatorname{div}(y) = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_0 \cup \Gamma_{out}) \times (0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_{in} \times (0,\infty), \\ y(x,0) = y_0 & \text{in} \quad \Omega, \\ w_c' + \gamma w_c - \mathcal{K}(Py, w_c) = 0 & \text{in} \quad (0,\infty), \\ w_c(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.4.4}$$

where

$$\mathcal{F}(\widehat{y}, \widehat{\sigma}) = -e^{-\beta t}\widehat{\sigma}\frac{\partial \widehat{y}}{\partial t} - e^{-\beta t}(\widehat{y}\cdot\nabla)\widehat{y} - e^{-\beta t}\widehat{\sigma}(v_s\cdot\nabla)\widehat{y} - e^{-\beta t}\widehat{\sigma}(\widehat{y}\cdot\nabla)v_s - e^{-2\beta t}\widehat{\sigma}(\widehat{y}\cdot\nabla)\widehat{y} + \beta e^{-\beta t}\widehat{\sigma}\widehat{y}$$

and $w_c = (w_1, ..., w_{N_c})$. Since (2.4.2) and (2.4.3) hold, one can verify that $\widehat{y}$ satisfies (2.3.3). Hence we can solve $(2.4.4)_1$-$(2.4.4)_3$ for $\widehat{\sigma}$ in $L^\infty(Q_\infty)$ (see Section 2.3). Now using this $\widehat{\sigma}$ and $\widehat{y}$ one can solve $(2.4.4)_4$-$(2.4.4)_{10}$ (see Section 2.2) for (y, w_c) provided $\mathcal{F}(\widehat{y}, \widehat{\sigma}) \in L^2(Q_\infty)$. This is indeed the case since we have $\widehat{y} \in V^{2,1}(Q_\infty)$ and $\widehat{\sigma} \in L^\infty(Q_\infty)$ and the detailed estimates are done in Lemma 2.4.1.
At this point we fix $T_1 > T_{A_1} = \frac{d}{A_1(1-A_1)}$ in Theorem 2.3.5. We also fix the constant K_3 appearing in Theorem 2.3.5. Let $0 < \mu < K_3$. We define a convex set D_μ as follows

$$D_\mu = \left\{ \begin{array}{l} \begin{pmatrix} \widehat{y} \\ \widehat{w}_c \end{pmatrix} \in V^{2,1}(Q_\infty) \times H^1(0,\infty;\mathbb{R}^{N_c}) \mid \\ \qquad \|(\widehat{y}, \widehat{w}_c)\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})} \leqslant \mu, \\ \text{and } \widehat{y} \mid_{\Sigma_\infty} \text{ is of the form (2.4.2) and satisfies the condition (2.4.3)} \end{array} \right\}. \tag{2.4.5}$$

Notice that $(0,0)$ belongs to D_μ, hence D_μ is non-empty.
Let $(\widehat{\sigma}, y, w_c) \in L^\infty(Q_\infty) \times V^{2,1}(Q_\infty) \times H^1(0,\infty;\mathbb{R}^{N_c})$ be the solution of system (2.4.4) corresponding to $(\widehat{y}, \widehat{w}_c) \in D_\mu$. We consider the following map

$$\begin{array}{rccc} \chi: & D_\mu & \longrightarrow & V^{2,1}(Q_\infty) \times H^1(0,\infty;\mathbb{R}^{N_c}) \\ & (\widehat{y}, \widehat{w}_c) & \mapsto & (y, w_c). \end{array} \tag{2.4.6}$$

In the sequel we will choose the constant $\mu \in (0, K_3)$, small enough such that χ maps D_μ into itself.
We will then look for a fixed point of the map χ. Indeed if $(y_f, w_{f,c})$ is a fixed point of the map χ, by construction, there exists a function σ_f such that the triplet $(\sigma_f, y_f, w_{f,c})$ solves (2.4.1). Hence in order to prove Theorem 2.1.2 it is enough to show that the map χ has a fixed point in D_μ.

2.4.2 χ maps D_μ into itself

In this section we will choose a suitable constant μ such that χ maps D_μ into itself, provided the initial data are small enough.
Now given $(\widehat{y}, \widehat{w}_c) \in D_\mu$, we can use (2.3.12) in order to show that $\widehat{\sigma}$, the solution of $(2.4.4)_1$-$(2.4.4)_3$ satisfies the following

$$\|\widehat{\sigma}\|_{L^\infty(Q_\infty)} \leqslant e^{\beta T_1} \|\sigma_0\|_{L^\infty(\Omega)}. \tag{2.4.7}$$

Lemma 2.4.1. *If $(\widehat{y}, \widehat{w}_c)$ belongs to D_μ (defined in (2.4.5)) and $\widehat{\sigma}$ is the solution of the problem $(2.4.4)_1$-$(2.4.4)_3$ then $\mathcal{F}(\widehat{y}, \widehat{\sigma}) \in L^2(Q_\infty)$. Besides there exist constants $K_5 > 0$, $K_6 > 0$ such that for all $(\widehat{y}, \widehat{w}_c) \in D_\mu$ and for all (σ_0, y_0) with σ_0 satisfying (2.1.6) and $e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)} < 1$, the following estimate is true:*

$$\|\mathcal{F}(\widehat{y}, \widehat{\sigma})\|_{L^2(Q_\infty)} \leqslant K_5 e^{\beta T_1} \|\sigma_0\|_{L^\infty(\Omega)} + K_6 \|\widehat{y}\|^2_{V^{2,1}(Q_\infty)}. \tag{2.4.8}$$

Proof. First use (2.4.7) to show

$$\|\widehat{\sigma} \frac{\partial \widehat{y}}{\partial t}\|_{L^2(Q_\infty)} \leqslant e^{\beta T_1} \|\sigma_0\|_{L^\infty(\Omega)} \|\widehat{y}\|_{V^{2,1}(Q_\infty)}. \tag{2.4.9}$$

Recall that $v_s \in C^\infty(\bar{\Omega})$. Hence we again apply (2.4.7) to get

$$\|\widehat{\sigma}(v_s \cdot \nabla)\widehat{y}\|_{L^2(Q_\infty)} \leqslant e^{\beta T_1} \|\sigma_0\|_{L^\infty(\Omega)} \|v_s\|_{W^{1,\infty}(\Omega)} \|\widehat{y}\|_{V^{2,1}(Q_\infty)}. \tag{2.4.10}$$

and

$$\|\widehat{\sigma}(\widehat{y} \cdot \nabla)v_s\|_{L^2(Q_\infty)} \leqslant e^{\beta T_1} \|\sigma_0\|_{L^\infty(\Omega)} \|v_s\|_{W^{1,\infty}(\Omega)} \|\widehat{y}\|_{V^{2,1}(Q_\infty)}. \tag{2.4.11}$$

Now we estimate $(\widehat{y} \cdot \nabla)\widehat{y}$ in $L^2(Q_\infty)$. We know that $V^{2,1}(Q_\infty)$ is continuously embedded in the space $L^\infty(0,\infty;H^1(\Omega))$. Hence $\widehat{y} \in L^\infty(0,\infty;H^1(\Omega))$, $\nabla\widehat{y} \in L^2(0,\infty;H^1(\Omega))$ and the following holds

$$\|(\widehat{y} \cdot \nabla)\widehat{y}\|_{L^2(Q_\infty)} \leqslant K \|\widehat{y}\|_{L^\infty(0,\infty;H^1(\Omega))} \|\nabla\widehat{y}\|_{L^2(0,\infty;H^1(\Omega))} \leqslant K \|\widehat{y}\|^2_{V^{2,1}(Q_\infty)}. \tag{2.4.12}$$

Similarly

$$\|\widehat{\sigma}(\widehat{y}\cdot\nabla)\widehat{y}\|_{L^2(Q_\infty)} \leqslant Ke^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}\|\widehat{y}\|^2_{V^{2,1}(Q_\infty)} \tag{2.4.13}$$

and

$$\|\beta\widehat{\sigma}\widehat{y}\|_{L^2(Q_\infty)} \leqslant |\beta|e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}\|\widehat{y}\|_{V^{2,1}(Q_\infty)}. \tag{2.4.14}$$

Now observe that

$$e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}\|\widehat{y}\|_{V^{2,1}(Q_\infty)} \leqslant \tfrac{1}{2}(e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)} + \|\widehat{y}\|^2_{V^{2,1}(Q_\infty)}). \tag{2.4.15}$$

Hence we use estimates (2.4.10)-(2.4.14) and (2.4.15) to prove Lemma 2.4.1 and the estimate (2.4.8). □

Lemma 2.4.2. *There exist constants $K_7 > \max\{1, K_5, K_6\} > 0$, $K_8 > \max\{K_5, K_6\} > 0$ such that for all $(\widehat{y}, \widehat{w}_c) \in D_\mu$ (defined in (2.4.5)), for all (σ_0, y_0) with σ_0 satisfying (2.1.6), $e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)} < 1$, and for $\widehat{\sigma}$ uniquely solving $(2.4.4)_1$-$(2.4.4)_3$, $(y, w_c) = \chi(\widehat{y}, \widehat{w}_c)$, solving $(2.4.4)_4$-$(2.4.4)_{10}$ is well defined and satisfies the following inequality*

$$\begin{aligned}&\|(y, w_c)\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})}\\ &\leqslant K_7\max\left\{e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}, \|y_0\|_{V_0^1(\Omega)}\right\} + K_8\|\widehat{y}\|^2_{V^{2,1}(Q_\infty)}.\end{aligned} \tag{2.4.16}$$

Proof. Corollary 2.2.16 shows that (y, w_c) satisfy the following estimate

$$\|(y, w_c)\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})} \leqslant K_1(\|y_0\|_{V_0^1(\Omega)} + \|\mathcal{F}(\widehat{y}, \widehat{\sigma})\|_{L^2(Q_\infty)}). \tag{2.4.17}$$

Now using (2.4.8) in (2.4.17), we get the desired result. □

From now on we will consider the initial data $\sigma_0 \in L^\infty(\Omega)$ and $y_0 \in V_0^1(\Omega)$ such that they satisfy

$$\begin{cases}\sigma_0 \text{ satisfies (2.1.6)},\\ \max\left\{e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}, \|y_0\|_{V_0^1(\Omega)}\right\} < \min\left\{\dfrac{L(1-L)}{8K_2K_7}, \dfrac{K_3}{2K_7}, \dfrac{1}{4K_7K_8}, 1\right\},\end{cases} \tag{2.4.18}$$

where K_2, K_7 and K_8 are the constants appearing respectively in (2.2.59) and (2.4.16).

Lemma 2.4.3. *For all (σ_0, y_0) satisfying (2.4.18), setting*

$$\mu = 2K_7 max\left\{e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}, \|y_0\|_{V_0^1(\Omega)}\right\}, \tag{2.4.19}$$

where K_7 is the constant in (2.4.16), the map χ (defined in (2.4.6)) maps D_μ (defined in (2.4.5)) into itself.

Proof. In view of $(2.4.18)_2$ and (2.4.19), one observes in particular that

$$0 < \mu < \min\left\{\frac{L(1-L)}{4K_2}, K_3, \frac{1}{2K_8}\right\}. \tag{2.4.20}$$

Now we will verify that with the choice (2.4.19) of μ, the map χ maps D_μ into itself. Let $(\widehat{y}, \widehat{w}_c) \in D_\mu$, for all (σ_0, y_0) obeying (2.4.18) and for $\widehat{\sigma}$ uniquely solving $(2.4.4)_1$-$(2.4.4)_3$, $(y, w_c) = \chi(\widehat{y}, \widehat{w}_c)$, solves $(2.4.4)_4$-$(2.4.4)_{10}$. We claim that $(y, w_c) \in D_\mu$.
First of all in view of (2.4.16), $(2.4.18)_2$, (2.4.19) and (2.4.20) we observe that

$$\|(y, w_c)\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})} \leqslant K_7 \max\left\{e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}, \|y_0\|_{V_0^1(\Omega)}\right\} + K_8\mu^2 \leqslant \mu.$$

Since $(y, w_c) = \chi(\widehat{y}, \widehat{w}_c)$ solves $(2.4.4)_4$-$(2.4.4)_{10}$, the function y on the boundary is given by $\sum_{j=1}^{N_c} w_j(t)g_j(x)$. This verifies (2.4.2).
Finally

$$\begin{aligned}
&\|y_0\|_{V_0^1(\Omega)} + \|\mathcal{F}(\widehat{y}, \widehat{\sigma})\|_{L^2(Q_\infty)} \\
&\leqslant (1 + K_5)\max\left\{e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}, \|y_0\|_{V_0^1(\Omega)}\right\} + K_6\mu^2 \\
&\leqslant \frac{3}{8}\frac{L(1-L)}{K_2} \leqslant \frac{L(1-L)}{2K_2},
\end{aligned} \tag{2.4.21}$$

where in $(2.4.21)_1$ we have used $(2.4.18)_2$, (2.4.8) and in $(2.4.21)_2$ we have used $(2.4.18)_2$, (2.4.20) and the fact that $K_7 > \max\{1, K_5, K_6\} > 0$, $K_8 > \max\{K_5, K_6\} > 0$ (which follows from the statement of Lemma 2.4.2). Now using Corollary 2.2.17 one verifies (2.4.3) for $y\,|_{\Sigma_\infty}$.
Hence we have verified that $(y, w_c) \in D_\mu$ and the proof of Lemma 2.4.3 is finished. □

At this point we fix μ as in Lemma 2.4.3.

2.4.3 Compactness and continuity

To start with, let us define the weighted space

$$\begin{aligned}
&L^2(0,\infty,(1+t)^{-1}dt; L^2(\Omega)\times\mathbb{R}^{N_c}) \\
&= \left\{\overline{z} = \begin{pmatrix} z(x,t) \\ w_c(t)\end{pmatrix} \in L^2(Q_\infty)\times L^2((0,\infty);\mathbb{R}^{N_c}) \;\middle|\; \int_0^\infty (1+t)^{-2}\|\overline{z}\|^2_{L^2(\Omega)\times\mathbb{R}^{N_c}}dt < \infty\right\}.
\end{aligned}$$

We endow the set D_μ, defined in (2.4.5), with the norm induced from $L^2(0,\infty,(1+t)^{-1}dt; L^2(\Omega)\times\mathbb{R}^{N_c})$.

Lemma 2.4.4. *The set D_μ is compact in $L^2(0,\infty,(1+t)^{-1}dt; L^2(\Omega)\times\mathbb{R}^{N_c})$.*

Proof. We divide the proof in two steps.
Step 1. We claim that D_μ is closed in the space $L^2(0,\infty,(1+t)^{-1}dt;L^2(\Omega)\times\mathbb{R}^{N_c})$. Consider a sequence $\{\overline{y}_n\}_n$ (where $\overline{y}_n=(y_n,w_{n,c})$) in D_μ such that $\{\overline{y}_n\}_n$ converges to some $\overline{y}$ (where $\overline{y}=(y,w_c)$) in the space $L^2(0,\infty,(1+t)^{-1}dt;L^2(\Omega)\times\mathbb{R}^{N_c})$. We will check that $\overline{y}\in D_\mu$. Since for all n, $\overline{y}_n\in D_\mu$, the definition of D_μ (see (2.4.5)) yields

$$\|\overline{y}_n\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})}\leqslant\mu. \tag{2.4.22}$$

Using the lower semi-continuity of the norms one obtains

$$\|\overline{y}\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})}\leqslant\mu. \tag{2.4.23}$$

Now we will verify that

$$y\mid_{\Sigma_\infty}=\sum_{j=1}^{N_c}w_j(t)g_j(x)\quad\text{for all}\quad(x,t)\in\Sigma_\infty, \tag{2.4.24}$$

where $w_c=(w_1,...,w_{N_c})$. From (2.4.22) one has the following weak convergence

$$y_n\rightharpoonup y\quad\text{in}\quad L^2(0,\infty;H^2(\Omega)),\quad\text{and}\quad w_{n,c}\rightharpoonup w_c\quad\text{in}\quad H^1(0,\infty;\mathbb{R}^{N_c}).$$

As the trace operator is linear and bounded from $H^2(\Omega)$ onto $H^{3/2}(\Gamma)$, $y_n\mid_{\Sigma_\infty}$ converges weakly to $y\mid_{\Sigma_\infty}$ in $L^2(0,\infty;H^{3/2}(\Gamma))$. On the other hand as $\overline{y}_n\in D_\mu$, for each n

$$y_n\mid_{\Sigma_\infty}=\sum_{j=1}^{N_c}w_{n,j}(t)g_j(x),$$

where $w_{n,c}=(w_{n,1},...,w_{n,N_c})$. Now since $w_{n,c}$ converges weakly to w_c in $H^1(0,\infty;\mathbb{R}^{N_c})$ we have the following convergence in the sense of distribution

$$y_n\mid_{\Sigma_\infty}\xrightarrow[n\to\infty]{}\sum_{j=1}^{N_c}w_j(t)g_j(x)\quad\text{in}\quad\mathcal{D}'(\Sigma_\infty).$$

Since the distributional limit and the weak limit (in the space $L^2(0,\infty;H^{3/2}(\Gamma))$) of $y_n\mid_{\Sigma_\infty}$ coincides, one at once obtains the expression (2.4.24) of $y\mid_{\Sigma_\infty}$. Also using the continuous embedding $H^1(0,\infty)\hookrightarrow L^\infty(0,\infty)$ one observes that

$$y_n\mid_{\Sigma_\infty}\overset{*}{\rightharpoonup}y\mid_{\Sigma_\infty}\quad\text{in}\quad L^\infty(\Sigma_\infty).$$

Hence one has the following by lower semi continuity of norm with respect to the above weak type convergence

$$\|y\mid_{\Sigma_\infty}\|_{L^\infty(\Sigma_\infty)}\leqslant\frac{L(1-L)}{2}.$$

Hence $y\mid_{\Sigma_\infty}$ satisfies (2.4.3). This finishes the proof of $\bar{y}\in D_\mu$.

Step 2. Now to prove Lemma 2.4.4, it is enough to show that $V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})$

is compactly embedded in $L^2(0,\infty,(1+t)^{-1}dt, L^2(\Omega)\times\mathbb{R}^{N_c})$. Let $\{\overline{z}_n\}_n$ be a sequence in $V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})$ such that

$$\|\overline{z}_n\|_{V^{2,1}(Q_\infty)\times H^1((0,\infty);\mathbb{R}^{N_c})} \leqslant 1.$$

This implies that for any $T>0$

$$\int_T^\infty (1+t)^{-2}\|\overline{z}_n\|^2_{L^2(\Omega)\times\mathbb{R}^{N_c}}dt \leqslant \frac{1}{(1+T)^2}, \tag{2.4.25}$$

for all $n\in\mathbb{N}$. Let $\epsilon>0$. Choose $T_\epsilon>0$ such that

$$\frac{1}{(1+T_\epsilon)^2}\leqslant\epsilon.$$

So using (2.4.25) we have

$$\|\overline{z}_n-\overline{z}_m\|^2_{L^2(T_\epsilon,\infty,(1+t)^{-1}dt;L^2(\Omega)\times\mathbb{R}^{N_c})}\leqslant 4\epsilon, \tag{2.4.26}$$

for all $m,n\in\mathbb{N}$.
We know from Rellich's compactness theorem and Aubin-Lions lemma ([2]) that the embedding of $V^{2,1}(Q_{T_\epsilon})\times H^1(0,T_\epsilon;\mathbb{R}^{N_c})$ into $L^2(0,T_\epsilon,L^2(\Omega)\times\mathbb{R}^{N_c})$ is compact. Hence up to a subsequence (denoted by the same notation) $\{\overline{z}_n\}_n$ is Cauchy in $L^2(0,T_\epsilon,L^2(\Omega)\times\mathbb{R}^{N_c})$.
So it follows that there exists $N_0\in\mathbb{N}$ such that for all natural numbers $m,n\geqslant N_0$,

$$\|\overline{z}_n-\overline{z}_m\|^2_{L^2(0,T_\epsilon,(1+t)^{-1}dt,L^2(\Omega)\times\mathbb{R}^{N_c})}\leqslant\epsilon. \tag{2.4.27}$$

Now combining (2.4.26), (2.4.27) and a diagonal extraction argument, we can construct a subsequence $\{\overline{z}_n\}_n$ which is a Cauchy sequence in the Banach space $L^2(0,\infty,(1+t)^{-1}dt;L^2(\Omega)\times\mathbb{R}^{N_c})$.
The proof is complete. □

Lemma 2.4.5. *If a sequence $\{\overline{z}_n\}$ in D_μ converges weakly to some $\overline{z}$ in $V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})$ then up to a subsequence*

$$e^{-\beta t}\overline{z}_n\xrightarrow[n\to\infty]{} e^{-\beta t}\overline{z}\quad \textit{strongly in}\quad L^2(0,\infty;L^2(\Omega)\times\mathbb{R}^{N_c}). \tag{2.4.28}$$

Proof. The proof follows from the arguments used in proving Lemma 2.4.4 and is left to the reader. □

Lemma 2.4.6. *The map χ is continuous in D_μ, endowed with the norm $L^2(0,\infty,(1+t)^{-1}dt;L^2(\Omega))$.*

Proof. Let $\{\overline{\widehat{y}}_n\}_n$ $\Big($where $\overline{\widehat{y}}_n = (\widehat{y}_n, \widehat{w}_{n,c})\Big)$ be a sequence in D_μ and assume that this sequence $\{\overline{\widehat{y}}_n\}_n$ strongly converges to $\overline{\widehat{y}}$ $\Big($where $\overline{\widehat{y}} = (\widehat{y}, \widehat{w}_c)\Big)$ in the norm $L^2(0,\infty,(1+t)^{-1}dt; L^2(\Omega)\times\mathbb{R}^{N_c})$.
As for all $n\in\mathbb{N}$, $\|\overline{\widehat{y}}_n\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})} \leqslant \mu$, up to a subsequence we have the following weak convergence

$$\{\overline{\widehat{y}}_n\}_n \rightharpoonup \overline{\widehat{y}} \quad \text{in} \quad V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c}) \quad \text{as} \quad n\to\infty. \tag{2.4.29}$$

Now corresponding to the vector field $\widehat{y}_n$, let us denote by $\widehat{\sigma}_n$ the solutions to $(2.4.4)_1$-$(2.4.4)_3$. Similarly $\widehat{\sigma}$ is the solution to $(2.4.4)_1$-$(2.4.4)_3$ which corresponds to the vector field $\widehat{y}$. As $\widehat{y}_n$ converges strongly to $\widehat{y}$ in the norm $L^2(0,\infty,(1+t)^{-1}dt, L^2(\Omega))$, for any $T>0$, $\widehat{y}_n$ converges to $\widehat{y}$ in particular in the norm $L^1(Q_T)$. Besides, the strong $L^1(\Sigma_T)$ convergence of $\widehat{y}_n\cdot\vec{n}$ towards $\widehat{y}\cdot\vec{n}$ is obvious in view of the identities (2.4.2) and the strong convergence of $\widehat{w}_n$ to $\widehat{w}$ in $L^1(0,T)$, which immediately follows from the weak convergence of $\widehat{w}_n$ to $\widehat{w}$ in $H^1(0,\infty)$. Hence from Lemma 2.3.4, we obtain that $\widehat{\sigma}_n$ strongly converges to $\widehat{\sigma}$ in $C^0([0,T],L^q(\Omega))$ for all $1\leqslant q<+\infty$. Due to the suitable choice of μ in Lemma 2.4.3, we can conclude from Theorem 2.3.5 (in particular from (2.3.12)) that each of $\widehat{\sigma}_n$ and $\widehat{\sigma}$ vanishes for $t\geqslant T_1$. So

$$\begin{aligned}&\widehat{\sigma}_n \xrightarrow[n\to\infty]{} \widehat{\sigma} \quad \text{strongly in} \quad L^\infty(0,\infty;L^q(\Omega))\ \forall\ 1\leqslant q<+\infty,\\ &\forall n\in\mathbb{N},\ \widehat{\sigma}_n(t)=\widehat{\sigma}(t)=0 \quad \text{for all} \quad t\geqslant T_1.\end{aligned} \tag{2.4.30}$$

Also from (2.4.7) and (2.4.18) we know that the $L^\infty(Q_\infty)$ norm of the sequence $\widehat{\sigma}_n$ is uniformly bounded.
We will now check that $\mathcal{F}(\widehat{y}_n,\widehat{\sigma}_n)$ converges weakly in $L^2(Q_\infty)$ to $\mathcal{F}(\widehat{y},\widehat{\sigma})$. As $(\widehat{y}_n,\widehat{w}_{n,c})\in D_\mu$, from the estimate (2.4.8) we obtain a uniform bound for $\|\mathcal{F}(\widehat{y}_n,\widehat{\sigma}_n)\|_{L^2(Q_\infty)}$. So there exists a subsequence of $\mathcal{F}(\widehat{y}_n,\widehat{\sigma}_n)$ which weakly converges in $L^2(0,\infty;L^2(\Omega))$. This is therefore enough to show that the sequence $\mathcal{F}(\widehat{y}_n,\widehat{\sigma}_n)$ converges to $\mathcal{F}(\widehat{y},\widehat{\sigma})$ weakly in $\mathcal{D}'(Q_\infty)$ (*i.e.* in the sense of distribution).
Let us first check the weak convergence of the term $-e^{-\beta t}\widehat{\sigma}_n\frac{\partial\widehat{y}_n}{\partial t}$. From (2.4.30) we know that $\widehat{\sigma}_n$ strongly converges to $\widehat{\sigma}$ in $L^2(Q_\infty)$ and each of $\widehat{\sigma}_n$ and $\widehat{\sigma}$ vanishes for all $t\geqslant T_1$ (see (2.4.30)). Also from (2.4.29) we have that $\frac{\partial\widehat{y}_n}{\partial t}$ converges weakly to $\frac{\partial\widehat{y}_n}{\partial t}$ in $L^2(Q_\infty)$. Hence their product $\widehat{\sigma}_n\frac{\partial\widehat{y}_n}{\partial t}$ converges weakly to $\widehat{\sigma}\frac{\partial\widehat{y}}{\partial t}$ in $L^1(Q_\infty)$. So it is now easy to verify that $e^{-\beta t}\widehat{\sigma}_n\frac{\partial\widehat{y}_n}{\partial t}$ converges to $e^{-\beta t}\widehat{\sigma}\frac{\partial\widehat{y}}{\partial t}$ weakly in $L^1(Q_\infty)$.
Now we consider $e^{-2\beta t}(\widehat{y}_n\cdot\nabla)\widehat{y}_n$. As $\widehat{y}_n$ is bounded and weakly convergent to $\widehat{y}$ in $V^{2,1}(Q_\infty)$, using Lemma 2.4.5, we have

$$e^{-2\beta t}\widehat{y}_n \xrightarrow[n\to\infty]{} e^{-2\beta t}\widehat{y} \quad \text{strongly in} \quad L^2(Q_\infty), \tag{2.4.31}$$

and

$$\nabla\widehat{y}_n \rightharpoonup \nabla\widehat{y} \quad \text{in} \quad L^2(Q_\infty) \quad \text{as} \quad n\to\infty. \tag{2.4.32}$$

Therefore $e^{-2\beta t}(\widehat{y}_n\cdot\nabla)\widehat{y}$ converges to $e^{-2\beta t}(\widehat{y}\cdot\nabla)\widehat{y}$ weakly in $L^1(Q_\infty)$.
Since $\widehat{y}_n$ converges weakly to y in $V^{2,1}(Q_\infty)$, one has the following

$$\nabla\widehat{y}_n \rightharpoonup \nabla\widehat{y} \quad \text{in} \quad L^\infty(0,\infty;L^2(\Omega))\cap L^2(0,\infty;H^1(\Omega)) \quad \text{as} \quad n\to\infty.$$

We use the interpolation result [9, Theorem II.5.5] to obtain the following in particular

$$\nabla \widehat{y}_n \rightharpoonup \nabla \widehat{y} \quad \text{in} \quad L^3(0,\infty;L^3(\Omega)) \quad \text{as} \quad n \to \infty. \tag{2.4.33}$$

Using (2.4.30), (2.4.31) and (2.4.33) one has the following weak convergence

$$e^{-2\beta t}\widehat{\sigma}_n(\widehat{y}_n \cdot \nabla)\widehat{y}_n \rightharpoonup e^{-2\beta t}\widehat{\sigma}(\widehat{y} \cdot \nabla)\widehat{y} \quad \text{in} \quad L^1(Q_\infty) \quad \text{as} \quad n \to \infty.$$

The convergences of the remaining terms $e^{-\beta t}\widehat{\sigma}_n(v_s\cdot\nabla)\widehat{y}_n$, $e^{-\beta t}\widehat{\sigma}_n(\widehat{y}_n\cdot\nabla)v_s$ and $\beta e^{-\beta t}\widehat{\sigma}_n\widehat{y}_n$ can be analyzed similarly using the convergences (2.4.29) and $(2.4.30)_1$. We thus conclude that $\mathcal{F}(\widehat{y}_n,\widehat{\sigma}_n)$ converges weakly to $\mathcal{F}(\widehat{y},\widehat{\sigma})$ in the space $\mathcal{D}'(Q_\infty)$. Hence this is also the $L^2(Q_\infty)$ weak limit.
From Corollary 2.2.16, we know that for the closed loop system (2.2.56), the map

$$\begin{array}{ccc} L^2(0,\infty;L^2(\Omega)) \times V_0^1(\Omega) & \mapsto & V^{2,1}(Q_\infty) \times H^1(0,\infty;\mathbb{R}^{N_c}) \\ (f,y_0) & \mapsto & \overline{y} \end{array}$$

is linear and bounded. Hence we obtain that $\overline{y}_n = \chi(\overline{\widehat{y}}_n)$ $(\overline{y}_n = (y_n, w_{n,c}))$ weakly converges to $\overline{y} = \chi(\overline{\widehat{y}})$ $(\overline{y} = (y, w_c))$ in $(D_\mu, \|.\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})})$. Finally as D_μ is compact in $L^2(0,\infty;(1+t)^{-1}dt$,
$L^2(\Omega)\times\mathbb{R}^{N_c})$ (see Lemma 2.4.4), $\overline{y}_n$ strongly converges to $\overline{y}$ in $L^2(0,\infty;(1+t)^{-1}dt, L^2(\Omega)\times\mathbb{R}^{N_c})$. The proof of Lemma 2.4.6 is complete. □

2.4.4 Conclusion

Let μ is as in Lemma 2.4.3. Then
(i) For an initial datum (σ_0, y_0) satisfying (2.4.18), the map χ defined in (2.4.6) maps D_μ defined in (2.4.5) into itself.
(ii) The non-empty convex set D_μ is compact in the topology of $L^2(0,\infty,(1+t)^{-1}dt; L^2(\Omega)\times\mathbb{R}^{N_c})$ (see Lemma 2.4.4).
(iii) The map χ is continuous on D_μ, endowed with the norm $L^2(0,\infty,(1+t)^{-1}dt; L^2(\Omega))$ (Lemma 2.4.6).
One observes that all the assumptions of Schauder fixed point theorem are satisfied by the map χ on D_μ, endowed with the norm $L^2(0,\infty,(1+t)^{-1}dt; L^2(\Omega)\times\mathbb{R}^{N_c})$. Therefore, Schauder fixed point theorem yields a fixed point $(y_f, w_{f,c})$ of the map χ in D_μ. Hence the trajectory $(\sigma_f, y_f, w_{f,c})$ solves the non linear problem (2.4.1). Moreover, as a consequence of Theorem 2.3.5 the following holds

$$\sigma_f(.,t) = 0 \quad \text{in} \quad \Omega \quad \text{for} \quad t \geqslant T_1. \tag{2.4.34}$$

Using (2.4.19) in (2.4.16) and (2.4.20), one further obtains

$$\|(y_f, w_{f,c})\|_{V^{2,1}(Q_\infty)\times H^1(0,\infty;\mathbb{R}^{N_c})} \leqslant C\max\{e^{\beta T_1}\|\sigma_0\|_{L^\infty(\Omega)}, \|y_0\|_{V_0^1(\Omega)}\}, \tag{2.4.35}$$

for some positive constant C. Once again using Theorem 2.3.5, (2.4.35) furnish the following continuous dependence on initial data

$$\|(\sigma_f, y_f)\|_{L^\infty(Q_\infty)\times V^{2,1}(Q_\infty)} \leqslant C\|(\sigma_0, y_0)\|_{L^\infty(\Omega)\times V_0^1(\Omega)}, \tag{2.4.36}$$

for some positive constant C. Now in view of the change of unknowns (2.1.11), we obtain the existence of a trajectory $(\rho, v) \in L^\infty(Q_\infty) \times V^{2,1}(Q_\infty)$ which solves (2.1.2) and satisfies the decay estimate (2.1.7). The proof of Theorem 2.1.2 is complete. □

2.5 Further comments

Our result considers that the control u_c is supported on Γ_c, which is an open subset of the inflow part Γ_{in} (see (2.1.5)) of the boundary. This is in fact natural to control the inflow boundary of the channel. At the same time we remark that our analysis applies if one wants to control the outflow boundary Γ_{out} or the lateral boundary Γ_0 of the channel Ω. In what follows we briefly discuss these cases.

(i) **Controlling the outflow boundary.** In this case the control zone Γ_c is an open subset of Γ_{out}. After the change of unknowns (2.1.11), one can imitate the linearization procedure (as done while transforming (2.1.13) into (2.1.14)). In this linearized system the transport equation modeling the density $(2.1.14)_1$-$(2.1.14)_3$ will remain unchanged but the boundary conditions on the velocity equations $(2.1.14)_4$-$(2.1.14)_8$ should be replaced by $y = 0$ on $(\Gamma_0 \cup \Gamma_{in}) \times (0, \infty)$ and $y = \sum_{j=1}^{N_c} w_j(t) g_j(x)$ on $\Gamma_{out} \times (0, \infty)$. Still the proof of the boundary controllability of the Oseen equations can be carried in a similar way as done in Section 2.2 and in the same spirit of Corollary 2.2.17, one can prove that if the initial condition y_0 and the non-homogeneous term f are suitably small then the inflow and the outflow boundaries of the perturbed vector field $(v_s + e^{-\beta t} y)$ coincide with that of v_s. Since the transport equation $(2.1.14)_1$-$(2.1.14)_3$ remains unchanged in this case, the analysis done in Section 2.3 applies without any change. The fixed point argument done in Section 2.4 to prove the stabilization of the coupled system (2.1.2) also applies without change.

(ii) **Controlling the lateral boundary.** In this case the control zone Γ_c is an open subset of Γ_0. In particular we assume that $\Gamma_c \subset \Gamma_b$ (where $\Gamma_b = (0, d) \times \{0\} \subset \Gamma_0$). Now the inflow and outflow boundaries of the velocity vector $(e^{-\beta t} y + v_s)$ cannot be characterized by using the notations Γ_{in} and Γ_{out} (as defined in (2.1.10)), since Γ_c can contain an inflow part and an outflow part and one can not prove a result similar to Corollary 2.2.17. More precisely here we can use the following notations for time $t > 0$,

$$\begin{cases} \Gamma^*_{in,y}(t) = \Gamma_{in} \cup \{x \in \Gamma_c \mid (v_s(x) + e^{-\beta t} y(x,t) \cdot n(x)) < 0\} \subset \Gamma_{in} \cup \Gamma_b, \\ \Gamma_h = (0, d) \times \{1\}. \end{cases} \tag{2.5.1}$$

In a similar way as we have obtained (2.1.14) from (2.1.2), one gets the following system

$$\begin{cases} \dfrac{\partial \sigma}{\partial t} + ((v_s + e^{-\beta t} y) \cdot \nabla)\sigma - \beta\sigma = 0 & \text{in} \quad Q_\infty, \\ \sigma(x,t) = 0 & \text{on} \quad \bigcup_{t\in(0,\infty)} (\Gamma^*_{in,y}(t) \times \{t\}), \\ \sigma(x,0) = \sigma_0 & \text{in} \quad \Omega, \\ \dfrac{\partial y}{\partial t} - \beta y - \nu\Delta y + (v_s \cdot \nabla)y + (y \cdot \nabla)v_s + \nabla q = f & \text{in} \quad Q_\infty, \\ \text{div} y = 0 & \text{in} \quad Q_\infty, \\ y = 0 & \text{on} \quad (\Gamma_{in} \cup \Gamma_h \cup \Gamma_{out}) \times (0,\infty), \\ y = \sum\limits_{j=1}^{N_c} w_j(t) g_j(x) & \text{on} \quad \Gamma_b \times (0,\infty), \\ y(x,0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{2.5.2}$$

One can use arguments similar to the ones in Section 2.2 in order to stabilize y solving $(2.5.2)_4$-$(2.5.2)_8$. The functions g_j can be constructed with compact support in Γ_b (imitating the construction (2.2.32)), and we can recover the C^∞ regularity of the boundary control and $V^{2,1}(Q_\infty)$ regularity of y. Hence the flow corresponding to the vector field $(e^{-\beta t} y + v_s)$ is well defined in classical sense, consequently one can adapt the arguments used in Section 2.3 to prove that σ, the solution of $(2.5.2)_1$-$(2.5.2)_3$ belongs to $L^\infty(Q_\infty)$ and vanishes after some finite time provided the initial condition σ_0 is supported away from the lateral boundaries and y is small enough. The use of a fixed point argument to prove the stabilizability of the solution of (2.1.2) is again a straightforward adaptation of the arguments used in Section 2.4.

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Chapter 3

Local existence of strong solutions for a fluid-structure interaction model

3.1 Introduction

3.1.1 Statement of the problem

Our objective is to study a fluid structure interaction problem in a 2d channel. The fluid flow here is modeled by the compressible Navier-Stokes equations. Concerning the structure we will consider an Euler-Bernoulli damped beam located on a portion of the boundary. As remarked in [2], such dynamical models arise in the study of many engineering systems (e.g., aircraft, bridges etc). In the present article we establish a result on the local in time existence of strong solutions of such a fluid structure interaction problem. To the best of our knowledge, this is the first article dealing with the existence of local in time strong solutions for the complete non-linear model considered here.
We consider data and solutions which are periodic in the 'channel direction' (with period L, where $L > 0$ is a constant). Here L-periodicity of a function f (defined on $\mathbb{R}$) means that $f(x + L) = f(x)$ for all $x \in \mathbb{R}$.
We now define a few notations. Let Ω be the domain $\mathbb{T}_L \times (0, 1) \subset \mathbb{R}^2$, where $\mathbb{T}_L$ is the one dimensional torus identified with $(0, L)$ with periodic conditions. The boundary of Ω is denoted by Γ. We set

$$\Gamma_s = \mathbb{T}_L \times \{1\}, \quad \Gamma_\ell = \mathbb{T}_L \times \{0\}, \quad \Gamma = \Gamma_s \cup \Gamma_\ell.$$

Now for a given function

$$\eta : \Gamma_s \times (0,\infty) \to (-1,\infty),$$

which will correspond to the displacement of the one dimensional beam, let us denote by Ω_t and $\Gamma_{s,t}$ the following sets

$$\begin{aligned}
\Omega_t &= \{(x,y) \mid x \in (0,L), \quad 0 < y < 1+\eta(x,t)\} &&= \text{domain of the fluid at time } t,\\
\Gamma_{s,t} &= \{(x,y) \mid x \in (0,L), \quad y = 1+\eta(x,t)\} &&= \text{the beam at time } t.
\end{aligned}$$

The reference configuration of the beam is Γ_s, and we set

$$\begin{aligned}
&\Sigma_T = \Gamma \times (0,T), && \Sigma_T^s = \Gamma_s \times (0,T),\\
&\widetilde{\Sigma_T^s} = \cup_{t\in(0,T)} \Gamma_{s,t} \times \{t\}, && \Sigma_T^\ell = \Gamma_\ell \times (0,T),\\
&Q_T = \Omega \times (0,T), && \widetilde{Q_T} = \cup_{t\in(0,T)} \Omega_t \times \{t\}.
\end{aligned} \tag{3.1.1}$$

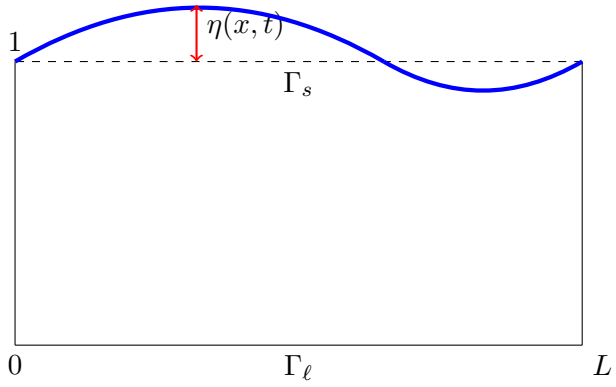


Figure 3.1: Domain Ω_t.

We consider a fluid with density ρ and velocity $\mathbf{u}$. The fluid structure interaction system coupling the compressible Navier-Stokes and the Euler-Bernoulli damped beam equation is modeled by

$$\begin{cases}
\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0 & \text{in} \quad \widetilde{Q_T},\\
(\rho\mathbf{u}_t + \rho(\mathbf{u}.\nabla)\mathbf{u}) - (2\mu \mathrm{div}(D(\mathbf{u})) + \mu'\nabla \mathrm{div}\mathbf{u}) + \nabla p(\rho) = 0 & \text{in} \quad \widetilde{Q_T},\\
\mathbf{u}(\cdot,t) = (0,\eta_t) & \text{on} \quad \widetilde{\Sigma_T^s},\\
\mathbf{u}(\cdot,t) = (0,0) & \text{on} \quad \Sigma_T^\ell,\\
\mathbf{u}(\cdot,0) = \mathbf{u}_0 & \text{in} \quad \Omega_{\eta(0)} = \Omega,\\
\rho(\cdot,0) = \rho_0 & \text{in} \quad \Omega_{\eta(0)} = \Omega,\\
\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = (T_f)_2 & \text{on} \quad \Sigma_T^s,\\
\eta(\cdot,0) = 0 \quad \text{and} \quad \eta_t(\cdot,0) = \eta_1 & \text{in} \quad \Gamma_s.
\end{cases} \tag{3.1.2}$$

The initial condition for the density is assumed to be positive and bounded. We fix the positive constants m and M such that

$$0 < m = \min_{\overline{\Omega}} \rho_0(x,y), \quad M = \max_{\overline{\Omega}} \rho_0(x,y). \tag{3.1.3}$$

In our model the fluid adheres to the plate and is viscous. This implies that the velocities corresponding to the fluid and the structure coincide at the interface and hence the condition $(3.1.2)_3$ holds. In the system (3.1.2), $D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$ is the symmetric part of the gradient and the real constants μ, μ' are the Lamé coefficients which are supposed to satisfy

$$\mu > 0, \quad \mu' \geqslant 0.$$

In our case the fluid is isentropic i.e. the pressure $p(\rho)$ is only a function of the fluid density ρ and is given by

$$p(\rho) = a\rho^\gamma,$$

where $a > 0$ and $\gamma > 1$ are positive constants.
We assume that there exists a constant external force $p_{ext} > 0$ which acts on the beam. The external force p_{ext} can be written as follows

$$p_{ext} = a\overline{\rho}^\gamma,$$

for some positive constant $\overline{\rho}$.
To incorporate this external forcing term p_{ext} into the system of equations (3.1.2), we introduce the following

$$P(\rho) = p(\rho) - p_{ext} = a\rho^\gamma - a\overline{\rho}^\gamma. \tag{3.1.4}$$

Since $\nabla p(\rho) = \nabla P(\rho)$, from now onwards we will use $\nabla P(\rho)$ instead of $\nabla p(\rho)$ in the equation $(3.1.2)_2$.
In the beam equation the constants, $\alpha > 0$, $\beta \geqslant 0$ and $\delta > 0$ are respectively the adimensional rigidity, stretching and friction coefficients of the beam. The non-homogeneous source term of the beam equation $(T_f)_2$ is the net surface force on the structure which is the resultant of force exerted by the fluid on the structure and the external force p_{ext} and it is assumed to be of the following form

$$(T_f)_2 = ([-2\mu D(\mathbf{u}) - \mu'(\mathrm{div}\mathbf{u})\mathbf{I}_d] \cdot \mathbf{n}_t + P\mathbf{n}_t) \mid_{\Gamma_{s,t}} \sqrt{1+\eta_x^2} \cdot \vec{e}_2 \quad \text{on} \quad \Sigma_T^s, \tag{3.1.5}$$

where $\mathbf{I}_d$ is the identity matrix, $\mathbf{n}_t$ is the outward unit normal to $\Gamma_{s,t}$ given by

$$\mathbf{n}_t = -\frac{\eta_x}{\sqrt{1+\eta_x^2}}\vec{e}_1 + \frac{1}{\sqrt{1+\eta_x^2}}\vec{e}_2$$

($\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$).
Observe that $(\rho, \mathbf{u}, \eta) = (\overline{\rho}, 0, 0)$ is a stationary solution to (3.1.2)-(3.1.4)-(3.1.5).

Remark 3.1.1. *Now we can formally derive a priori estimates for the system* (3.1.2)-(3.1.4)-(3.1.5) *and show the following energy equality*

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_t}\rho|\mathbf{u}|^2\,dx\right)+\frac{d}{dt}\left(\int_{\Omega_t}\frac{a}{(\gamma-1)}\rho^\gamma\,dx\right)+\frac{1}{2}\frac{d}{dt}\left(\int_0^L|\eta_t|^2\,dx\right)+\frac{\beta}{2}\frac{d}{dt}\left(\int_0^L|\eta_x|^2\,dx\right)$$
$$+\frac{\alpha}{2}\frac{d}{dt}\left(\int_0^L|\eta_{xx}|^2\,dx\right)+2\mu\int_{\Omega_t}|D\mathbf{u}|^2\,dx+\mu'\int_{\Omega_t}|\mathrm{div}\mathbf{u}|^2\,dx+\delta\int_0^L|\eta_{tx}|^2\,dx \tag{3.1.6}$$
$$=-p_{ext}\int_{\Gamma_s}\eta_t.$$

The equality (3.1.6) *underlines the physical interpretation of each coefficient and in particular of the viscosity coefficients,* μ, μ' *and* δ.

Remark 3.1.2. *Observe that in* (3.1.2) *we have considered the initial displacement* $\eta(0)$ *of the beam to be zero. This is because we prove the local existence of strong solution of the system* (3.1.2) *with the beam displacement* η *close to the steady state zero. There are several examples in the literature where the authors consider the initial displacement of the structure (in a fluid-structure interaction problem) to be equal to zero. For instance the readers can look into the articles [24] and [8]. We also refer to the article [3] where the initial displacement of the structure is non zero but is considered to be suitably small. The issues involving the existence of strong solution for the model* (3.1.2) *but with a non zero initial displacement* $\eta(0)$ *of the beam is open. The case of a system coupling the incompressible Navier-Stokes equations and an Euler-Bernoulli damped beam with a non zero initial beam displacement is addressed in [11].*

Our interest is to prove the local in time existence of a strong solution to system (3.1.2)-(3.1.4)-(3.1.5) i.e we prove that given a prescribed initial datum $(\rho_0, \mathbf{u}_0, \eta_1)$, there exists a solution of system (3.1.2)-(3.1.4)-(3.1.5) with a certain Sobolev regularity in some time interval $(0, T)$, provided that the time T is small enough.
We study the system (3.1.2)-(3.1.4)-(3.1.5) by transforming it into the reference cylindrical domain Q_T. This is done by defining a diffeomorphism from Ω_t onto Ω. We adapt the diffeomorphism used in [3] in the study of an incompressible fluid-structure interaction model. The reader can also look at [33], [23] where the authors use a similar map in the context of a coupled fluid-structure model comprising an incompressible fluid.

3.1.2 Transformation of the problem to a fixed domain

To transform the system (3.1.2)-(3.1.4)-(3.1.5) in the reference configuration, for η satisfying $1+\eta(x,t)>0$ for all $(x,t)\in\Sigma_T^s$, we introduce the following change of variables

$$\begin{aligned}
&\Phi_{\eta(t)} : \Omega_t \longrightarrow \Omega \quad \text{defined by} \quad \Phi_{\eta(t)}(x,y) = (x,z) = \left(x, \frac{y}{1+\eta(x,t)}\right), \\
&\Phi_{\eta} : \widetilde{Q_T} \longrightarrow Q_T \quad \text{defined by} \quad \Phi_{\eta}(x,y,t) = (x,z,t) = \left(x, \frac{y}{1+\eta(x,t)}, t\right).
\end{aligned} \tag{3.1.7}$$

Remark 3.1.3. *It is easy to prove that for each $t \in [0,T)$, the map $\Phi_{\eta(t)}$ is a C^1- diffeomophism from Ω_t onto Ω provided that $(1+\eta(x,t)) > 0$ for all $x \in \mathbb{T}_L$ and that $\eta(\cdot,t) \in C^1(\Gamma_s)$.*

Notice that since $\eta(\cdot,0) = 0$, $\Phi_{\eta(0)}$ is just the identity map. We set the following notations

$$\widehat{\rho}(x,z,t) = \rho(\Phi_\eta^{-1}(x,z,t)),\ \widehat{\mathbf{u}}(x,z,t) = (\widehat{u}_1, \widehat{u}_2) = \mathbf{u}(\Phi_\eta^{-1}(x,z,t)). \tag{3.1.8}$$

After transformation and using the fact that $\widehat{u}_{1,x} = 0$ on Σ_T^s (since $\widehat{\mathbf{u}} = \eta_t \vec{e_2}$ on Σ_T^s) the nonlinear system (3.1.2)-(3.1.4)-(3.1.5) is rewritten in the following form

$$\begin{cases}
\widehat{\rho}_t + \begin{bmatrix} \widehat{u}_1 \\ \frac{1}{(1+\eta)}(\widehat{u}_2 - \eta_t z - \widehat{u}_1 z \eta_x) \end{bmatrix} \cdot \nabla \widehat{\rho} + \widehat{\rho} \mathrm{div} \widehat{\mathbf{u}} = F_1(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) & \text{in} \quad Q_T, \\
\widehat{\rho}\widehat{\mathbf{u}}_t - \mu \Delta \widehat{\mathbf{u}} - (\mu' + \mu) \nabla(\mathrm{div} \widehat{\mathbf{u}}) + \nabla P(\widehat{\rho}) = F_2(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) & \text{in} \quad Q_T, \\
\widehat{\mathbf{u}} = \eta_t \vec{e_2} & \text{on} \quad \Sigma_T^s, \\
\widehat{\mathbf{u}}(\cdot,t) = 0 & \text{on} \quad \Sigma_T^\ell, \\
\widehat{\mathbf{u}}(\cdot,0) = \mathbf{u}_0 & \text{in} \quad \Omega, \\
\widehat{\rho}(\cdot,0) = \rho_0 & \text{in} \quad \Omega, \\
\eta_{tt} - \beta \eta_{xx} - \delta \eta_{txx} + \alpha \eta_{xxxx} = F_3(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) & \text{on} \quad \Sigma_T^s, \\
\eta(0) = 0 \quad \text{and} \quad \eta_t(0) = \eta_1 & \text{in} \quad \Gamma_s,
\end{cases} \tag{3.1.9}$$

where

$$
\begin{aligned}
F_1(\widehat{\rho},\widehat{\mathbf{u}},\eta) =& \frac{1}{(1+\eta)}(\widehat{u}_{1,z} z\eta_x\widehat{\rho} + \eta\widehat{\rho}\widehat{u}_{2,z}),\\
F_2(\widehat{\rho},\widehat{\mathbf{u}},\eta) =& -\eta\widehat{\rho}\widehat{\mathbf{u}}_t + z\widehat{\rho}\widehat{\mathbf{u}}_z\eta_t - \eta\widehat{\rho}\widehat{u}_1\widehat{\mathbf{u}}_x + \widehat{u}_1\widehat{\mathbf{u}}_z\eta_x\widehat{\rho}z + \mu(\eta\widehat{\mathbf{u}}_{xx} - \frac{\eta\widehat{\mathbf{u}}_{zz}}{(1+\eta)}\\
&- 2\eta_x z\widehat{\mathbf{u}}_{zx} + \frac{\widehat{\mathbf{u}}_{zz}z^2\eta_x^2}{(1+\eta)} - \widehat{\mathbf{u}}_z(\frac{(1+\eta)z\eta_{xx} - 2\eta_x^2 z}{(1+\eta)})) - \widehat{\rho}(\widehat{\mathbf{u}}.\nabla)\widehat{\mathbf{u}}\\
&+ (\mu+\mu')\cdot\begin{bmatrix} \eta\widehat{u}_{1,xx} - \widehat{u}_{1,xz}z\eta_x - \eta_x z(\widehat{u}_{1,zx} - \frac{\widehat{u}_{1,zz}z\eta_x}{(1+\eta)})\\ +\widehat{u}_{1,z}(\frac{(1+\eta)z\eta_{xx} - 2\eta_x^2 z}{(1+\eta)}) - \frac{\eta_x\widehat{u}_{2,z}}{(1+\eta)} - \frac{\eta_x z\widehat{u}_{2,zz}}{(1+\eta)}\\ \\ -\frac{\eta_x\widehat{u}_{1,z}}{(1+\eta)} - \frac{\eta_x z\widehat{u}_{1,zz}}{(1+\eta)} - \frac{\eta\widehat{u}_{2,zz}}{(1+\eta)}\end{bmatrix}\\
&- (\eta P_x(\widehat{\rho}) - P_z(\widehat{\rho})z\eta_x)\vec{e_1},\\
F_3(\widehat{\rho},\widehat{\mathbf{u}},\eta) =& -\mu(-\widehat{u}_{2,z} + \eta_x\widehat{u}_{2,x} + \frac{\widehat{u}_{2,z}}{(1+\eta)}\eta_x^2 z - \frac{2\eta\widehat{u}_{2,z}}{(1+\eta)} - \frac{\eta_x\widehat{u}_{1,z}}{(1+\eta)})\\
&- \mu'(-2\widehat{u}_{2,z} + \frac{\widehat{u}_{1,z}}{(1+\eta)}\eta_x z - \frac{\eta\widehat{u}_{2,z}}{(1+\eta)}) + P(\widehat{\rho}).
\end{aligned}
\tag{3.1.10}
$$

The transport equation for density $(3.1.9)_1$-$(3.1.9)_6$ is of the form

$$
\begin{cases}
\widehat{\rho}_t + \begin{bmatrix} \widehat{u}_1 \\ \frac{1}{(1+\eta)}(\widehat{u}_2 - \eta_t z - \widehat{u}_1 z\eta_x)\end{bmatrix}\cdot\nabla\widehat{\rho} + \widehat{\rho}\mathrm{div}\widehat{\mathbf{u}} = F_1 & \text{in} \quad Q_T,\\
\widehat{\rho}(\cdot,0) = \rho_0 & \text{in} \quad \Omega.
\end{cases}
\tag{3.1.11}
$$

Due to the interface condition, $\widehat{\mathbf{u}} = \eta_t\vec{e_2}$ on Σ_T^s, we get that the velocity field $(\widehat{u}_1, \frac{1}{(1+\eta)}(\widehat{u}_2 - \eta_t z - \widehat{u}_1 z\eta_x))$ satisfies

$$
\begin{bmatrix} \widehat{u}_1 \\ \frac{1}{(1+\eta)}(\widehat{u}_2 - \eta_t z - \widehat{u}_1 z\eta_x)\end{bmatrix}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma_T^s,
$$

where $\mathbf{n}$ is the unit outward normal to Ω. Hence we shall not prescribe any boundary condition on the density for the system (3.1.11) to be well posed.
To avoid working in domains which deform when time evolves, the meaning of solutions for (3.1.2)-(3.1.4)-(3.1.5) will be understood as follows: The triplet $(\rho,\mathbf{u},\eta)$ solves (3.1.2)-(3.1.4)-(3.1.5) if and only if $(\widehat{\rho},\widehat{\mathbf{u}},\eta)$ solves (3.1.9). This notion will be detailed in the next section.

3.1.3 Functional settings and the main result

In the fixed domain Ω we have the following spaces of functions with values in $\mathbb{R}^2$,

$$\mathbf{H}^s(\Omega) = H^s(\Omega;\mathbb{R}^2) \quad \text{for all} \quad s \geqslant 0.$$

We also introduce the following spaces of vector valued functions

$$\begin{aligned} &\mathbf{H}^1_0(\Omega) = \{\mathbf{z} \in \mathbf{H}^1(\Omega) \mid \mathbf{z} = 0 \text{ on } \Gamma\},\\ &\mathbf{H}^{2,1}(Q_T) = L^2(0,T;\mathbf{H}^2(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega)),\\ &\mathbf{H}^{2,1}_{\Sigma_T}(Q_T) = \{\mathbf{z} \in \mathbf{H}^{2,1}(Q_T) \mid \mathbf{z} = 0 \text{ on } \Sigma_T\}. \end{aligned} \tag{3.1.12}$$

Similarly for $s \geqslant 0$, we can define $H^s(\Omega)$, the Sobolev space for the scalar valued functions defined on Ω. Now for $\theta, \tau \geqslant 0$, we introduce the following spaces which we use to analyze the beam equation

$$H^{\theta,\tau}(\Sigma_T^s) = L^2(0,T;H^\theta(\Gamma_s)) \cap H^\tau(0,T;L^2(\Gamma_s)).$$

Remark 3.1.4. *Since $\Omega = \mathbb{T}_L \times (0,1)$ and $\Gamma_s = \mathbb{T}_L \times \{1\}$, the above definitions of the functional spaces implicitly assert that the functions are $L-$ periodic in the x variable.*

Proposition 3.1.5. *Let $T > 0$. If η is regular enough in the space variable, say $\eta(\cdot,t) \in H^m(\Gamma_s)$ for $m \geqslant 2$ and the following holds*

$$1 + \eta(x,t) \geqslant \delta_0 > 0 \quad \text{on} \quad \Sigma_T^s, \tag{3.1.13}$$

for some constant δ_0, the map $g \mapsto \widehat{g} = g(\Phi^{-1}_{\eta(t)}(x,z))$ is a homeomorphism from $H^s(\Omega_t)$ to $H^s(\Omega)$ for any $s \leqslant m$.

The proposition stated above can be proved in the same spirit of [23, Proposition 2, Section 3].
Now in view of Proposition 3.1.5, we define the notion of strong solution of the system (3.1.2)-(3.1.4)-(3.1.5) in terms of the strong solution of the system (3.1.9).

Definition 3.1.6. *The triplet $(\rho, \mathbf{u}, \eta)$ is a strong solution of the system (3.1.2)-(3.1.4)-(3.1.5) if*

$$\begin{aligned} &\eta \in C^0([0,T];H^{9/2}(\Gamma_s)), \qquad \eta_t \in L^2(0,T;H^4(\Gamma_s)) \cap C^0([0,T];H^3(\Gamma_s)),\\ &\eta_{tt} \in L^2(0,T;H^2(\Gamma_s)) \cap C^0([0,T];H^1(\Gamma_s)), \qquad \eta_{ttt} \in L^2(0,T;L^2(\Gamma_s)), \end{aligned} \tag{3.1.14}$$

(3.1.13) holds for every $(x,t) \in \Sigma_T^s$ and the triplet $(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) = (\rho \circ \Phi_\eta^{-1}, \mathbf{u} \circ \Phi_\eta^{-1}, \eta)$ solves (3.1.9) in the following Sobolev spaces

$$\begin{aligned} &\widehat{\rho} \in C^0([0,T];H^2(\Omega)),\ \widehat{\rho}_t \in C^0([0,T];H^1(\Omega)),\\ &\widehat{\mathbf{u}} \in L^2(0,T;\mathbf{H}^3(\Omega)) \cap C^0([0,T];\mathbf{H}^{5/2}(\Omega)),\\ &\widehat{\mathbf{u}}_t \in L^2(0,T;\mathbf{H}^2(\Omega)) \cap C^0([0,T];\mathbf{H}^1(\Omega)),\\ &\widehat{\mathbf{u}}_{tt} \in L^2(0,T;\mathbf{L}^2(\Omega)). \end{aligned} \tag{3.1.15}$$

(η is in the space mentioned in (3.1.14)). Note that $(\rho, \mathbf{u})$ can then be obtained from $(\widehat{\rho}, \widehat{\mathbf{u}})$ by $(\rho, \mathbf{u}) = (\widehat{\rho} \circ \Phi_\eta, \widehat{\mathbf{u}} \circ \Phi_\eta)$.

In relation with Definition 3.1.6, we introduce the following functional spaces

$$
\begin{aligned}
Y_1^T =& \{\rho \in C^0([0,T]; H^2(\Omega)) \mid \rho_t \in C^0([0,T]; H^1(\Omega))\},\\
Y_2^T =& \{\mathbf{u} \in L^2(0,T; \mathbf{H}^3(\Omega)) \cap C^0([0,T]; \mathbf{H}^{5/2}(\Omega)) \mid \mathbf{u}_t \in L^2(0,T; \mathbf{H}^2(\Omega)) \cap C^0([0,T]; \mathbf{H}^1(\Omega)),\\
& \qquad \mathbf{u}_{tt} \in L^2(0,T; \mathbf{L}^2(\Omega))\},\\
Y_3^T =& \{\eta \in C^0([0,T]; H^{9/2}(\Gamma_s)),\ \eta(x,0) = 0 \mid \eta_t \in L^2(0,T; H^4(\Gamma_s)) \cap C^0([0,T]; H^3(\Gamma_s)),\\
& \qquad \eta_{tt} \in L^2(0,T; H^2(\Gamma_s)) \cap C^0([0,T]; H^1(\Gamma_s)), \eta_{ttt} \in L^2(0,T; L^2(\Gamma_s))\}. \quad (3.1.16)
\end{aligned}
$$

The spaces Y_1^T, Y_2^T and Y_3^T correspond to the spaces in which the unknowns $\widehat{\rho}$, $\widehat{\mathbf{u}}$ and η respectively belong.
Now we precisely state the main result of the article.

Theorem 3.1.7. *Assume that*

$$
\begin{cases}
(i) & (a)\ \textit{Regularity of initial conditions}: \ \rho_0 \in H^2(\Omega),\ \eta_1 \in H^3(\Gamma_s),\\
& \qquad\qquad \mathbf{u}_0 \in \mathbf{H}^3(\Omega).\\
& (b)\ \textit{Compatibility between initial and boundary conditions}:\\
& \quad (b)_1 \ \left(\mathbf{u}_0 - \begin{bmatrix} 0 \\ z\eta_1 \end{bmatrix}\right) = 0 \ \textit{on}\, \Gamma,\\
& \quad (b)_2 \ -P'(\rho_0)\nabla\rho_0 - (\delta\eta_{1,xx} - (\mu+2\mu')(u_0)_{2,z} + P(\rho_0))z\rho_0\vec{e}_2\\
& \qquad +z\rho_0(\mathbf{u}_0)_z\eta_1 - \rho_0(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 - (-\mu\Delta - (\mu+\mu')\nabla\mathrm{div})\mathbf{u}_0 = 0 \ \textit{on}\, \Gamma,\\
(ii) & (3.1.3)\ \textit{holds},
\end{cases}
\quad (3.1.17)
$$

where we use the notations $P'(\rho_0) = \nabla P(\rho_0)$, $P(\rho_0) = (a\rho_0^\gamma - a\overline{\rho}^\gamma)$ and $\mathbf{u}_0 = ((u_0)_1, (u_0)_2)$. Then there exists $T > 0$ such that the system (3.1.9) admits a solution $(\widehat{\rho}, \widehat{\mathbf{u}}, \eta) \in Y_1^T \times Y_2^T \times Y_3^T$. Consequently in the sense of Definition 3.1.6 the system (3.1.2)-(3.1.4)-(3.1.5) admits a strong solution $(\rho, \mathbf{u}, \eta)$.

Remark 3.1.8. *Our analysis throughout the article can be suitably adapted to consider any pressure law $p(\cdot) \in C^2(\mathbb{R}^+)$ (in this article we present the proofs with the pressure law given by $p(\rho) = a\rho^\gamma$, with $\gamma > 1$) such that there exists a positive constant $\overline{\rho}$ satisfying $p(\overline{\rho}) = p_{ext}$, where $p_{ext}(> 0)$ is the external force acting on the beam. The adaptation is possible since we only consider the case where the fluid density ρ has a positive lower and upper bound.*

Now let us sketch the strategy towards the proof of Theorem 3.1.7.

3.1.4 Strategy

(i) *Changing* (3.1.9) *to a homogeneous boundary value problem*: Recall that (see Remark 3.1.2) we will prove the existence of local in time strong solution of the system (3.1.2)-(3.1.4)-(3.1.5) only when the beam displacement η is close to zero. Again observe that $(\widehat{\rho} = \overline{\rho}, \widehat{\mathbf{u}} = 0, \widehat{\eta} = 0)$ is a steady state solution of the system (3.1.2)-(3.1.4)-(3.1.5) and hence of the system (3.1.9). So to work in a neighborhood of $\eta = 0$, we make the following change of unknowns in (3.1.9),

$$\sigma = \widehat{\rho} - \overline{\rho}, \quad \mathbf{v} = (v_1, v_2) = \widehat{\mathbf{u}} - 0, \quad \eta = \eta - 0. \tag{3.1.18}$$

In view of the change of unknowns (3.1.18) one obtains

$$\begin{cases} \sigma_t + \begin{bmatrix} v_1 \\ \frac{1}{(1+\eta)}(v_2 - \eta_t z - v_1 z \eta_x) \end{bmatrix} \cdot \nabla\sigma + (\sigma + \overline{\rho})\mathrm{div}(\mathbf{v}) = F_1(\sigma + \overline{\rho}, \mathbf{v}, \eta) & \text{in} \quad Q_T, \\ (\sigma + \overline{\rho})\mathbf{v}_t - \mu\Delta\mathbf{v} - (\mu + \mu')\nabla\mathrm{div}\mathbf{v} = -P'(\sigma + \overline{\rho})\nabla\sigma + F_2(\sigma + \overline{\rho}, \mathbf{v}, \eta) & \text{in} \quad Q_T, \\ \mathbf{v} = \eta_t \vec{e_2} & \text{on} \quad \Sigma_T^s, \\ \mathbf{v} = 0 & \text{on} \quad \Sigma_T^\ell, \\ \mathbf{v}(\cdot, 0) = \mathbf{u}_0 & \text{in} \quad \Omega, \\ \sigma(\cdot, 0) = \sigma_0 = \rho_0 - \overline{\rho} & \text{in} \quad \Omega, \\ \eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = F_3(\sigma + \overline{\rho}, \mathbf{v}, \eta) & \text{on} \quad \Sigma_T^s, \\ \eta(0) = 0 \quad \text{and} \quad \eta_t(0) = \eta_1 & \text{in} \quad \Gamma_s. \end{cases} \tag{3.1.19}$$

We transform the system (3.1.19) into a homogeneous Dirichlet boundary value problem by performing further the following change of unknown

$$\mathbf{w} = (w_1, w_2) = \mathbf{v} - z\eta_t \vec{e_2}. \tag{3.1.20}$$

Since $\mathbf{v}$ and η_t both are L-periodic in the $x-$direction, the new unknown $\mathbf{w}$ is also L-periodic in the $x-$direction. With the new unknown $\mathbf{w}$, we write the transformed system in the following form

$$\begin{cases} \sigma_t + \begin{bmatrix} w_1 \\ \frac{1}{(1+\eta)}(w_2 - w_1 z \eta_x) \end{bmatrix} \cdot \nabla\sigma = G_1(\sigma, \mathbf{w}, \eta) & \text{in} \quad Q_T, \\ (\sigma + \overline{\rho})\mathbf{w}_t - \mu\Delta\mathbf{w} - (\mu + \mu')\nabla\mathrm{div}\mathbf{w} = G_2(\sigma, \mathbf{w}, \eta) & \text{in} \quad Q_T, \\ \mathbf{w} = 0 & \text{on} \quad \Sigma_T, \\ \mathbf{w}(\cdot, 0) = \mathbf{w}_0 = \mathbf{u}_0 - z\eta_1 \vec{e_2} & \text{in} \quad \Omega, \\ \sigma(\cdot, 0) = \sigma_0 = \rho_0 - \overline{\rho} & \text{in} \quad \Omega, \\ \eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = G_3(\sigma, \mathbf{w}, \eta) & \text{on} \quad \Sigma_T^s, \\ \eta(0) = 0 \quad \text{and} \quad \eta_t(0) = \eta_1 & \text{in} \quad \Gamma_s, \end{cases} \tag{3.1.21}$$

where

$$\begin{aligned} G_1(\sigma,\mathbf{w},\eta) &= -(\sigma+\overline{\rho})\mathrm{div}(\mathbf{w}+z\eta_t\vec{e_2})+F_1(\sigma+\overline{\rho},\mathbf{w}+z\eta_t\vec{e_2},\eta),\\ G_2(\sigma,\mathbf{w},\eta) &= -P'(\sigma+\overline{\rho})\nabla\sigma - z\eta_{tt}(\sigma+\overline{\rho})\vec{e_2}-(-\mu\Delta-(\mu+\mu')\nabla\mathrm{div})(z\eta_t\vec{e_2})\\ &\quad + F_2(\sigma+\overline{\rho},\mathbf{w}+z\eta_t\vec{e_2},\eta),\\ G_3(\sigma,\mathbf{w},\eta) &= F_3(\sigma+\overline{\rho},\mathbf{w}+\eta_t\vec{e_2},\eta). \end{aligned} \tag{3.1.22}$$

(ii) *Study of some decoupled linear problems*: Observe that in the new system (3.1.21) the coupling between the velocity of the fluid and the elastic structure appears only as source terms. In order to solve the system (3.1.21) we first study some linear equations in Section 3.2. In order to analyze the local in time existence of strong solution the difficulty is to track the dependence of the constants (appearing in the inequalities) with respect to the time parameter 'T'. In this direction we first obtain a priori estimates for the linear density and velocity equations with non homogeneous source terms in the spirit of [38]. Then we prove the existence of strong solutions for a linear beam equation. The proof strongly relies on the analyticity of the corresponding beam semigroup (see [13] for details). At this point we refer the readers to the articles [18] (maximal L^p-L^q regularity of structurally damped beam equation), [19] (analyticity and exponential stability of beam semigroup), [33] (study of beam equation in the context of an incompressible fluid structure interaction problem) and the references therein for the existence and regularity issues of the damped beam equation. In our case to obtain estimates with the constants independent of 'T' for the beam equation we first fix a constant $\overline{T}>0$ and restrict ourselves to work in the time interval $(0,T)$ where

$$T<\overline{T}. \tag{3.1.23}$$

This technique is inspired from [34].
(iii) *Fixed point argument*: In Section 3.3 we prove the existence of a strong solution of (3.1.21) by using the Schauder's fixed point theorem based on (3.1.21)-(3.1.22).

Remark 3.1.9. *Since $\eta(0)=0$ the regularity (3.1.14) of η guarantees that*

$$\|\eta\|_{L^\infty(\Sigma_T^s)}\leqslant CT\|\eta_t\|_{L^\infty(0,T;H^3(\Gamma_s))}, \tag{3.1.24}$$

for a constant C independent of T. For small enough time T, (3.1.24) furnishes $\eta\approx 0$ and hence during small times, the beam stays close to the steady state zero.

3.1.5 Comments on initial and compatibility conditions

(i) Recall from (3.1.17)$(i)(a)$ that we assume $\mathbf{u}_0\in\mathbf{H}^3(\Omega)$. Also observe that in our solution (see (3.1.15)) the vector field $\widehat{\mathbf{u}}\in C^0([0,T];\mathbf{H}^{5/2}(\Omega))$ i.e for the velocity field there is a loss of $\frac{1}{2}$ space regularity as the time evolves. One can find such instances of a loss of space regularity in many other articles in the literature, for instance we refer the

readers to [7], [24] (for the coupling of fluid-elastic structure comprising a compressible fluid) and [14], [15], [34] (for incompressible fluid structure interaction models).

(ii) We use $(3.1.22)_3$ to obtain the following expression of $G_3 \mid_{t=0}$ (the value of $G_3(\sigma, \mathbf{w}, \eta)$ at time $t = 0$)

$$G_3 \mid_{t=0} = -(\mu + 2\mu')(u_0)_{2,z} + P(\rho_0). \tag{3.1.25}$$

Using $\rho_0 \in H^2(\Omega)$, $\mathbf{u}_0 \in \mathbf{H}^3(\Omega)$ (see $(3.1.17)(i)(a)$) and standard trace theorems one easily checks that

$$G_3 \mid_{t=0} \in H^{3/2}(\Gamma_s). \tag{3.1.26}$$

We will use the regularity of $G_3 \mid_{t=0}$ (in fact we will only use $G_3 \mid_{t=0} \in H^1(\Gamma_s)$) to prove the regularity of η. This will be detailed in Theorem 3.2.7.

(iii) We use (3.1.25) and the equation $(3.1.21)_6$ to check that

$$\eta_{tt}(\cdot, 0) = \delta\eta_{1,xx} - (\mu + 2\mu')(u_0)_{2,z} + P(\rho_0).$$

Hence using $(3.1.22)_2$ one obtains the following expression of $G_2 \mid_{t=0}$ (the value of $G_2(\sigma, \mathbf{w}, \eta)$ at time $t = 0$)

$$\begin{aligned} G_2 \mid_{t=0} = &-P'(\rho_0)\nabla\rho_0 - (\delta\eta_{1,xx} - (\mu + 2\mu')(u_0)_{2,z} + P(\rho_0))z\rho_0\vec{e}_2 + z\rho_0(\mathbf{u}_0)_z\eta_1 \\ &- \rho_0(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - (-\mu\Delta - (\mu + \mu')\nabla\mathrm{div})(z\eta_1\vec{e}_2). \end{aligned} \tag{3.1.27}$$

This gives

$$\begin{aligned} G_2 \mid_{t=0} - (-\mu\Delta - (\mu + \mu')\nabla\mathrm{div})\left(\mathbf{u}_0 - \begin{bmatrix} 0 \\ z\eta_1 \end{bmatrix}\right) = &-P'(\rho_0)\nabla\rho_0 - (\delta\eta_{1,xx} \\ &- (\mu + 2\mu')(u_0)_{2,z} + P(\rho_0))z\rho_0\vec{e}_2 + z\rho_0(\mathbf{u}_0)_z\eta_1 - \rho_0(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 - (-\mu\Delta \\ &- (\mu + \mu')\nabla\mathrm{div})\mathbf{u}_0. \end{aligned} \tag{3.1.28}$$

The regularity assumptions $(3.1.17)(i)(a)$ and (3.1.28) furnish the following

$$G_2 \mid_{t=0} -(-\mu\Delta - (\mu + \mu')\nabla\mathrm{div})\left(\mathbf{u}_0 - \begin{bmatrix} 0 \\ z\eta_1 \end{bmatrix}\right) \in \mathbf{H}^1(\Omega). \tag{3.1.29}$$

Hence one obtains (recalling that $\mathbf{w}_0 = \mathbf{u}_0 - z\eta_1\vec{e}_2$)

$$\begin{aligned} &\text{the assumption } (3.1.17)(i)(a) \text{ and } (3.1.17)(i)(b)_2 \\ &\Longrightarrow G_2 \mid_{t=0} -(-\mu\Delta\mathbf{w}_0 - (\mu + \mu')\nabla\mathrm{div}\mathbf{w}_0) \in \mathbf{H}_0^1(\Omega). \end{aligned} \tag{3.1.30}$$

We need this to prove some regularity of $\mathbf{w}$ and hence of $\widehat{\mathbf{u}}$. This will be detailed in Theorem 3.2.1.

3.1.6 Bibliographical comments

Here we mainly focus on the existing literature devoted to the study of fluid structure interaction problems.
To begin with we quote a few articles dedicated to the mathematical study of compressible Navier-Stokes equations. The existence of local in time classical solutions for the compressible Navier-Stokes equations in a time independent domain was first proved in [31] and the uniqueness was established in [35]. The global existence of strong solutions for a small perturbation of a stable constant state was established in the celebrated work [28]. In the article [38] the authors established the local in time existence of strong solutions in the presence of inflow and outflow of the fluid through the boundary. In the same article they also present the proof of global in time existence for small data in the absence of the inflow. P.-L. Lions proved (in [27]) the global existence of renormalized weak solution with bounded energy for an isentropic fluid (i.e $p(\rho) = \rho^\gamma$) with the adiabatic constant $\gamma > 3d/(d+2)$, where d is the space dimension. E. Feireisl *et al.* generalized the approach to cover the range $\gamma > 3/2$ in dimension 3 and $\gamma > 1$, in dimension 2 in [20]. The well-posedness issues of the compressible Navier-Stokes equations for critical regularity data can be found in [16], [17]. For further references and a very detailed development of the mathematical theory of compressible flow we refer the reader into the books [32] and [10].
In the last decades the fluid-structure interaction problems have been an area of active research. There is a rich literature concerning the motion of a structure inside or at the boundary of a domain containing a viscous incompressible Newtonian fluid, whose behavior is described by Navier-Stokes equations. For instance local existence and uniqueness of strong solutions of incompressible fluid-structure models with the structure immersed inside the fluid are studied in [14] (the elastic structure is modeled by linear Kirchhoff equations) and [15] (the elastic structure is governed by quasilinear elastodynamics). There also exist articles dealing with incompressible fluid-structure interaction problems where the structure appears on the fluid boundary and is modeled by Euler-Bernoulli damped beam equations $(3.1.2)_7$-$(3.1.2)_8$. For example we refer the readers to [3] (local in time existence of strong solutions), [12] (existence of weak solutions), [33] (feedback stabilization), [23] (global in time existence) and the references therein for a very detailed discussion of such problems.
Despite of the growing literature on incompressible fluids the number of articles addressing the compressible fluid-structure interaction problems is relatively limited and the literature has been rather recently developed. One of the fundamental differences between the incompressible and compressible Navier-Stokes equations is that the pressure of the fluid in incompressible Navier-Stokes equations is interpreted as the Lagrange multiplier whereas in the case of compressible Navier-Stokes equations the pressure is given as a function of density with the density modeled by a transport equation of hyperbolic nature. The strong coupling between the parabolic and hyperbolic dynamics is one of the intricacies in dealing with the compressible Navier-Stokes equations and this results in the regularity incompatibilities between the fluid and the solid structure.

However in the past few years there have been works exploring the fluid-structure interaction problems comprising the compressible Navier-Stokes equations with an elastic body immersed in the fluid domain. For instance in the article [6] the authors prove the existence and uniqueness of strong solutions of a fluid structure interaction problem for a compressible fluid and a rigid structure immersed in a regular bounded domain in dimension 3. The result is proved in any time interval $(0, T)$, where $T > 0$ and for a small perturbation of a stable constant state provided there is no collision between the rigid body and the boundary $\partial\Omega$ of the fluid domain. In [5] the existence of weak solution is obtained in three dimension for an elastic structure immersed in a compressible fluid. The structure equation considered in [5] is strongly regularized in order to obtain suitable estimates on the elastic deformations. A result concerning the local in time existence and uniqueness of strong solutions for a problem coupling compressible fluid and an elastic structure (immersed inside the fluid) can be found in [7]. In the article [7] the equation of the structure does not contain any extra regularizing term. The flow corresponding to a Lagrangian velocity is used in [7] in order to transform the fluid structure interaction problem in a reference fluid domain $\Omega_F(0)$, whereas in the present article we use the non physical change of variables (3.1.7) for the similar purpose of writing the entire system in a reference configuration. A similar Navier-Stokes-Lamé system as that of [7] is analyzed in [24] to prove the existence of local in time strong solutions but in a different Sobolev regularity framework. In the article [24] the authors deal with less regular initial data. We also quote a very recent work [8] where the authors prove the local in time existence of a unique strong solution of a compressible fluid structure interaction model where the structure immersed inside the fluid is governed by the Saint-Venant Kirchhoff equations.
On the other hand there is a very limited number of works on the compressible fluid-structure interaction problems with the structure appearing on the boundary of the fluid domain. The article [21] deals with a 1-D structure governed by plate equations coupled with a bi-dimensional compressible fluid where the structure is located at a part of the boundary. Here the authors consider the velocity field as a potential and in their case the non linearity occurs only in the equation modeling the density. Instead of writing the system in a reference configuration in [21] the authors proved the existence and uniqueness of solution in Sobolev-like spaces defined on time dependent domains. The existence of weak solution for a different compressible fluid structure interaction model (with the structure appearing on the boundary) is studied in dimension three by the same authors in [22]. In the model considered in [22], the fluid velocity v satisfies $\mathrm{curl} v \wedge n = 0$ on the entire fluid boundary and the plate is clamped everywhere on the structural boundary. In a recent article [2] the authors prove the Hadamard well posedness of a linear compressible fluid structure interaction problem (three dimensional compressible fluid interacting with a bi-dimensional elastic structure) defined in a fixed domain and considering the Navier-slip boundary condition at the interactive boundary.

They write the coupled system in the form

$$\frac{d}{dt}\begin{pmatrix}\rho\\ \mathbf{u}\\ \eta\\ \eta_t\end{pmatrix} = A\begin{pmatrix}\rho\\ \mathbf{u}\\ \eta\\ \eta_t\end{pmatrix} \text{ in } (0,T), \quad \text{and} \quad \begin{pmatrix}\rho(0)\\ \mathbf{u}(0)\\ \eta(0)\\ \eta_t(0)\end{pmatrix} = \begin{pmatrix}\rho_0\\ \mathbf{u}_0\\ \eta_1\\ \eta_2\end{pmatrix},$$

and prove the existence of mild solution $(\rho, \mathbf{u}, \eta, \eta_t)$ in the space $C^0([0,T];\mathcal{D}(A))$ where $\mathcal{D}(A)$ is the domain of the operator A. Their approach is based on using the Lumer-Phillips theorem to prove that A generates a strongly continuous semigroup. In yet another recent article [9] the authors consider a three dimensional compressible fluid structure interaction model where the structure located at the boundary is a shell of Koiter-type with some prescribed thickness. In the spirit of [27] and [20] the authors prove the existence of a weak solution for their model with the adiabatic constant restricted to $\gamma > \frac{12}{7}$. They show that a weak solution exists until the structure touches the boundary of the fluid domain.

To the best of our knowledge there is no existing work (neither in dimension 2 nor in 3) proving the existence of strong solutions for the non-linear compressible fluid-structure interaction problems (defined in a time dependent domain) considering the structure at the boundary of the fluid domain. In the present article we address this problem in the case of a fluid contained in a 2d channel and interacting with a 1d structure at the boundary. Our approach is different from that of [2] and [9]. In [2], since the problem itself is linearized in a fixed domain, the authors can directly use a semigroup formulation to study the existence of mild solution, whereas [9] considers weak solutions and a 4 level approximation process (using artificial pressure, artificial viscosity, regularization of the boundary and Galerkin approximation for the momentum equation). In the study of weak solutions (in [27], [20], [9]) one of the major difficulties is to pass to the limit in the non-linear pressure term which is handled by introducing a new unknown called the effective viscous flux. In our case of strong regularity framework we do not need to introduce the effective viscous flux and for small enough time T, the term $\nabla P(\sigma + \overline{\rho})$ can be treated as a non homogeneous source term. Our approach is based on studying the regularity properties of a decoupled parabolic equation, continuity equation and a beam equation. This is done by obtaining some apriori estimates and exploiting the analyticity of the semigroup corresponding to the beam equation. Then the existence result for the non-linear coupled problem is proved by using the Schauder's fixed point argument. We prove the existence of the fixed point in a suitable convex set, which is constructed very carefully based on the estimates of the decoupled problems and the estimates of the non-homogeneous source terms. This led us to choose this convex set as a product of balls (in various functional spaces) of different radius. In the present article we prove a local in time existence result of strong solutions whose incompressible counterpart was proved in [3].

Let us also mention the very recent article [36] where the global existence for the compressible viscous fluids (without any structure on the boundary) in a bounded domain is proved in the maximal $L^p - L^q$ regularity class. In this article the authors consider a

slip type boundary condition. More precisely the fluid velocity $\mathbf{u}$ satisfies the following on the boundary

$$D(\mathbf{u})\mathbf{n} - \langle D(\mathbf{u})\mathbf{n}, \mathbf{n}\rangle \mathbf{n} = 0, \quad \text{and} \quad \mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \times (0,T).$$

In a similar note one can consider a fluid structure interaction problem with slip type boundary condition. In that case the velocity field $\mathbf{u}$ solves the following

$$D(\mathbf{u})\mathbf{n} - \langle D(\mathbf{u})\mathbf{n}, \mathbf{n}\rangle \mathbf{n} = 0, \quad \text{and} \quad \mathbf{u}\cdot\mathbf{n} = \eta_t \quad \text{on} \quad \Gamma_s \times (0,T), \tag{3.1.31}$$

where η_t is the structural velocity at the interactive boundary $\Gamma_s \times (0,T)$. To the best of our knowledge for a compressible fluid structure interaction problem the condition (3.1.31) is treated only in [2], proving the existence of mild solution. Of course the boundary condition (3.1.31) is different from the one we consider in the present article since in our case we do not allow the fluid to slip tangentially through the fluid structure interface (i.e recall in our case $u_1 = 0$ on Σ_T^s).
A more generalized slip boundary condition is considered in [30] in the context of an incompressible fluid structure interaction problem. In the model examined in [30] the structural displacement has both tangential and normal components with respect to the reference configuration. At the interface the fluid and the structural velocities are coupled via a kinematic coupling condition and a dynamic coupling condition (stating that the structural dynamics is governed by the jump of the normal stress at the interface). The kinematic coupling conditions at the interface treated in [30] consists of continuity of the normal velocities and a second condition stating that the slip between the tangential components of the fluid and structural velocities is proportional to the fluid normal stress. The authors in [30] prove the existence of a weak solution for their model.

3.1.7 Outline

Section 3.2 contains results involving the existence and uniqueness of some decoupled linear equations. We state the existence and uniqueness result for a parabolic equation in Section 3.2.1, continuity equation in Section 3.2.2, linear beam equation in Section 3.2.3. In Section 3.3 we prove Theorem 3.1.7 by using the Schauder fixed point theorem.

3.2 Analysis of some linear equations

We will prove the existence and uniqueness of strong solutions of a parabolic equation, a continuity equation and a damped beam equation with prescribed initial datum and source terms in appropriate Sobolev spaces.
From now onwards all the constants appearing in the inequalities will be independent

of the final time T, unless specified. We also comment that we will denote many of the constants in the inequalities using the same notation although they might vary from line to line.

3.2.1 Study of a parabolic equation

At first we consider the following linear problem

$$\begin{cases} \overline{\sigma}\mathbf{w}_t - \mu\Delta\mathbf{w} - (\mu+\mu')\nabla\mathrm{div}\mathbf{w} = G_2 & \text{in} \quad Q_T, \\ \mathbf{w} = 0 & \text{on} \quad \Sigma_T, \\ \mathbf{w}(0) = \mathbf{w}_0 & \text{in} \quad \Omega, \end{cases} \tag{3.2.1}$$

where $\overline{\sigma}$, $\mathbf{w}_0$ and G_2 are known functions which are L-periodic in the x direction.
Let m and M be positive constants such that $m < M$. We are going to study (3.2.1) where $\overline{\sigma}$, $\mathbf{w}_0$ and G_2 satisfy the following

$$\begin{cases} \overline{\sigma} \in L^\infty(0,T;\mathbf{L}^\infty(\Omega)),\ 0 < m/2 \leqslant \overline{\sigma} \leqslant 2M \text{ in } Q_T,\ 0 < m \leqslant \overline{\sigma}(\cdot,0) \leqslant M \text{ in } \Omega, \\ \nabla\overline{\sigma} \in L^2(0,T;\mathbf{L}^3(\Omega)),\ \overline{\sigma}_t \in L^2(0,T;L^3(\Omega)), \end{cases} \tag{3.2.2}$$

and

$$\begin{cases} G_2 \in L^2(0,T;\mathbf{H}^1(\Omega)),\ G_{2,t} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ \mathbf{w}_0 \in \mathbf{H}_0^1(\Omega), \\ (G_2 \mid_{t=0} -(-\mu\Delta\mathbf{w}_0 - (\mu+\mu')\nabla\mathrm{div}\mathbf{w}_0)) \in \mathbf{H}_0^1(\Omega). \end{cases} \tag{3.2.3}$$

The following theorem corresponds to the existence and the regularity properties of the solution $\mathbf{w}$ of the system (3.2.1).

Theorem 3.2.1. *Let m, M be positive constants such that $m < M$. Then for all $\overline{\sigma}$, G_2 and $\mathbf{w}_0$ satisfying (3.2.2) and (3.2.3), there exists a unique solution $\mathbf{w}$ of (3.2.1) which satisfies the following*

$$\begin{aligned} &\mathbf{w} \in L^2(0,T;\mathbf{H}^3(\Omega)) \cap C^0([0,T];\mathbf{H}^{5/2}(\Omega)),\ \mathbf{w}_t \in L^2(0,T;\mathbf{H}^2(\Omega)) \\ &\cap C^0([0,T];\mathbf{H}^1(\Omega)), \mathbf{w}_{tt} \in L^2(0,T;\mathbf{L}^2(\Omega)). \end{aligned} \tag{3.2.4}$$

Besides, there exists a constant c_1 (depending on m and M but independent of T, $\overline{\sigma}$, G_2 and $\mathbf{w}_0$) such that $\mathbf{w}$ satisfies the following inequality

$$\begin{aligned} &\|\mathbf{w}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} + \|\mathbf{w}\|_{L^2(0,T;\mathbf{H}^3(\Omega))} + \|\mathbf{w}_t\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{w}_t\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \\ &+ \|\mathbf{w}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant c_1\{\|G_2\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \|G_2\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \\ &+ \left(\|G_{2,t}\|_{L^2(0,T;\mathbf{L}^2(\Omega)} + \left\|\frac{G_2 \mid_{t=0} -(-\mu\Delta\mathbf{w}_0 - (\mu+\mu')\nabla\mathrm{div}\mathbf{w}_0)}{\overline{\sigma}(0)}\right\|_{\mathbf{H}^1(\Omega)}\right) \\ &\cdot (1 + \|\overline{\sigma}_t\|_{L^2(0,T;L^3(\Omega))} + \|\nabla\overline{\sigma}\|_{L^2(0,T;\mathbf{L}^3(\Omega))}) \cdot \exp(c_1\|\overline{\sigma}_t\|^2_{L^2(0,T;L^3(\Omega))})\}. \end{aligned} \tag{3.2.5}$$

Remark 3.2.2. *Observe from (3.2.4) that* $\mathbf{w} \in C^0([0,T];\mathbf{H}^{5/2}(\Omega))$ *but in (3.2.5) we only include the estimate of* $\|\mathbf{w}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))}$ *and not of* $\|\mathbf{w}\|_{L^\infty(0,T;\mathbf{H}^{5/2}(\Omega))}$. *Using interpolation one can recover an estimate of* $\|\mathbf{w}\|_{L^\infty(0,T;\mathbf{H}^{5/2}(\Omega))}$ *from the estimates of* $\|\mathbf{w}\|_{L^2(0,T;\mathbf{H}^3(\Omega))}$ *and* $\|\mathbf{w}_t\|_{L^2(0,T;\mathbf{H}^2(\Omega))}$ *where the constant of interpolation may depend on the final time* T.

Remark 3.2.3. *Using (3.2.3) let us observe that* $G_2 \in L^2(0,T;\mathbf{H}^1(\Omega)) \cap H^1(0,T;\mathbf{L}^2(\Omega))$ *and hence by interpolation* $G_2 \mid_{t=0} \in \mathbf{H}^{1/2}(\Omega)$. *Now from* $(3.2.3)_3$ *one gets that*

$$(-\mu\Delta\mathbf{w}_0 - (\mu+\mu')\nabla \mathrm{div}\mathbf{w}_0) \in \mathbf{H}^{1/2}(\Omega).$$

The elliptic regularity result furnishes that $\mathbf{w}_0 \in \mathbf{H}^{5/2}(\Omega)$. *Since* $\mathbf{w} \in C^0([0,T];\mathbf{H}^{5/2}(\Omega))$, *for the linear equation (3.2.1) we do not loose any regularity as time evolves.*

Proof of Theorem 3.2.1. In the context of a smooth domain and with homogeneous Dirichlet boundary condition Theorem 3.2.1 is proved in the article [38]. There is no particular difficulty to adapt the same proof in Ω with L-periodic (in the x direction) boundary condition. Hence we refer the readers to the proofs of [38, Lemma 2.1]. For a related result we also refer the reader to [37, Lemma 2.2]. □

3.2.2 Study of a continuity equation

In this section we consider the following linear problem

$$\begin{cases} \sigma_t + \overline{\mathbf{w}} \cdot \nabla\sigma = G_1 & \text{in} \quad Q_T, \\ \sigma(0) = \sigma_0 & \text{in} \quad \Omega, \end{cases} \tag{3.2.6}$$

where the functions $\overline{\mathbf{w}}$, G_1 and σ_0 are L-periodic (in the x direction) functions. The following theorem asserts the existence and regularity of the solution σ of the density equation (3.2.6).

Theorem 3.2.4. *Let* $\overline{\mathbf{w}} \in L^1(0,T;\mathbf{H}^3(\Omega))$, $\overline{\mathbf{w}} \cdot \mathbf{n} = 0$ *on* Σ_T, $\sigma_0 \in H^2(\Omega)$ *and* $G_1 \in L^1(0,T;H^2(\Omega))$. *Then there exists a unique solution* σ *of (3.2.6) such that* $\sigma \in C^0([0,T];H^2(\Omega))$ *and*

$$\|\sigma\|_{L^\infty(0,T;H^2(\Omega))} \leqslant (\|\sigma_0\|_{H^2(\Omega)} + c_2\|G_1\|_{L^1(0,T;H^2(\Omega))})\exp(c_2\|\overline{\mathbf{w}}\|_{L^1(0,T;\mathbf{H}^3(\Omega))}). \tag{3.2.7}$$

If in addition $G_1 \in L^\infty(0,T;H^1(\Omega))$ *and* $\overline{\mathbf{w}} \in L^\infty(0,T;\mathbf{H}^2(\Omega))$ *then* $\sigma_t \in L^\infty(0,T;H^1(\Omega))$ *and*

$$\begin{aligned} \|\sigma_t\|_{L^\infty(0,T;H^1(\Omega))} \leqslant & c_3\|\overline{\mathbf{w}}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))}(\|\sigma_0\|_{H^2(\Omega)} + c_2\|G_1\|_{L^1(0,T;H^2(\Omega))}) \\ & \cdot \exp(c_2\|\overline{\mathbf{w}}\|_{L^1(0,T;\mathbf{H}^3(\Omega))}) + \|G_1\|_{L^\infty(0,T;H^1(\Omega))}. \end{aligned} \tag{3.2.8}$$

The constants c_2 *and* c_3 *appearing respectively in (3.2.7) and (3.2.8) are independent of* T, $\overline{\mathbf{w}}$, σ_0 *and* G_1.

Proof. The theorem is proved in [38, Lemma 2.4] with a particular expression of the function G_1. In our case we adapt the same proof with minor changes.
The existence of solution of (3.2.6) follows from the method of characteristics. The representation formula for the solution σ is

$$\sigma(x,t) = \sigma_0(U(x,0,t)) + \int_0^t G_1(U(x,s,t),s)ds, \tag{3.2.9}$$

where $U(x,t,s)$ solves the following ODE

$$\begin{cases} \dfrac{d}{dt}U(x,t,s) = \overline{\mathbf{w}}(U(x,t,s),t) & \text{in} \quad Q_T, \\ U(x,s,s) = x & \text{in} \quad \Omega. \end{cases} \tag{3.2.10}$$

Observe

$$U(\cdot,\cdot,\cdot) \in C^0([0,T]\times[0,T];\mathbf{H}^3(\Omega))$$

and consequently

$$\sigma(\cdot,\cdot) \in C^0([0,T];H^2(\Omega)).$$

Now to prove the estimate (3.2.7), we multiply $(3.2.6)_1$ by σ and integrate in Ω. Integrating by parts the term $\displaystyle\int_\Omega \overline{\mathbf{w}}\cdot\nabla\sigma\sigma$ and using the fact that $\overline{\mathbf{w}}\cdot n = 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|^2_{L^2(\Omega)} \leqslant \frac{1}{2}\int_\Omega \mathrm{div}\overline{\mathbf{w}}\sigma^2 + \|G_1\|_{L^2(\Omega)}\|\sigma\|_{L^2(\Omega)}.$$

Due to the embedding $\mathbf{H}^3(\Omega)\hookrightarrow C^1(\overline{\Omega})$ one has

$$\frac{d}{dt}\|\sigma\|^2_{L^2(\Omega)} \leqslant c(\|\overline{\mathbf{w}}\|_{\mathbf{H}^3(\Omega)}\|\sigma\|^2_{L^2(\Omega)} + \|G_1\|_{L^2(\Omega)}\|\sigma\|_{L^2(\Omega)}). \tag{3.2.11}$$

Before going into the next estimate let us observe that

$$\int_\Omega [(\overline{\mathbf{w}}\cdot\nabla)\nabla\sigma]\cdot\nabla\sigma = -\frac{1}{2}\int_\Omega (\mathrm{div}\overline{\mathbf{w}})|\nabla\sigma|^2. \tag{3.2.12}$$

Now take the gradient of $(3.2.6)_1$, multiply by $\nabla\sigma$ and integrate in Ω. Using (3.2.12) one obtains

$$\frac{d}{dt}\int_\Omega |\nabla\sigma|^2 \leqslant c(\|\overline{\mathbf{w}}\|_{\mathbf{H}^3(\Omega)}\|\nabla\sigma\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla G_1\|_{\mathbf{L}^2(\Omega)}\|\nabla\sigma\|_{\mathbf{L}^2(\Omega)}). \tag{3.2.13}$$

In a similar way for the second derivative we have

$$\frac{d}{dt}\int_\Omega |D^2\sigma|^2 \leqslant c(\|\overline{\mathbf{w}}\|_{\mathbf{H}^3(\Omega)}\|D^2\sigma\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega |D^2\overline{\mathbf{w}}||\nabla\sigma||D^2\sigma| + \|D^2G_1\|_{\mathbf{L}^2(\Omega)}\|D^2\sigma\|_{\mathbf{L}^2(\Omega)}). \tag{3.2.14}$$

One has the following estimate

$$\begin{aligned}\int_\Omega |D^2\overline{\mathbf{w}}||\nabla\sigma||D^2\sigma| &\leqslant \|D^2\overline{\mathbf{w}}\|_{\mathbf{L}^3(\Omega)}\|\nabla\sigma\|_{\mathbf{L}^6(\Omega)}\|D^2\sigma\|_{\mathbf{L}^2(\Omega)}\\ &\leqslant c\|D^2\overline{\mathbf{w}}\|_{\mathbf{L}^3(\Omega)}\|\nabla\sigma\|_{\mathbf{H}^1(\Omega)}\|D^2\sigma\|_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{3.2.15}$$

The estimates (3.2.11) and (3.2.13)-(3.2.14)-(3.2.15) furnish the following

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|^2_{H^2(\Omega)} \leqslant c(\|\overline{\mathbf{w}}\|_{\mathbf{H}^3(\Omega)}\|\sigma\|^2_{H^2(\Omega)} + \|G_1\|_{H^2(\Omega)}\|\sigma\|_{H^2(\Omega)}). \tag{3.2.16}$$

Now (3.2.7) is a consequence of (3.2.16) and Gronwall lemma. Finally the estimate (3.2.8) is a direct consequence of $(3.2.6)_1$ and (3.2.7). □

The following corollary directly follows from $(3.2.6)_1$ and the regularity $\sigma \in C^0([0,T];H^2(\Omega))$ which we have obtained in Theorem 3.2.4.

Corollary 3.2.5. *In addition to the assumptions of Theorem 3.2.4 if* $G_1 \in C^0([0,T];H^1(\Omega))$ *and* $\overline{\mathbf{w}} \in C^0([0,T];\mathbf{H}^2(\Omega))$ *then* $\sigma_t \in C^0([0,T];H^1(\Omega))$.

3.2.3 Study of a linear beam equation

The linearized beam equation with a non homogeneous source term is the following

$$\begin{cases} \eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = G_3 & \text{in} \quad \Sigma_T^s,\\ \eta(0) = 0 \quad \text{and} \quad \eta_t(0) = \eta_1 & \text{in} \quad \Gamma_s, \end{cases} \tag{3.2.17}$$

where G_2 and η_1 are known L-periodic (in the x direction) functions. Let us denote

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -\alpha\Delta^2 + \beta\Delta & \delta\Delta \end{bmatrix}. \tag{3.2.18}$$

The unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ is defined in

$$H_s = H^2(\Gamma_s) \times L^2(\Gamma_s), \tag{3.2.19}$$

with domain

$$D(\mathcal{A}) = H^4(\Gamma_s) \times H^2(\Gamma_s).$$

Hence with the notations

$$\mathbf{Y}(t) = \begin{bmatrix} \eta(t) \\ \eta_t(t) \end{bmatrix}, \quad \mathbf{Y}_0 = \begin{bmatrix} 0 \\ \eta_1 \end{bmatrix} \quad \text{and} \quad \widetilde{G}_3 = \begin{bmatrix} 0 \\ G_3 \end{bmatrix}, \tag{3.2.20}$$

we can equivalently write (3.2.17) as

$$\begin{cases} \mathbf{Y}_t(t) = \mathcal{A}\mathbf{Y}(t) + \widetilde{G}_3 \quad \text{on} \quad (0,T),\\ \mathbf{Y}(0) = \mathbf{Y}_0. \end{cases} \tag{3.2.21}$$

Lemma 3.2.6. *Let*

$$\widetilde{G}_3 \in L^2(0,T;H^2(\Gamma_s)\times L^2(\Gamma_s)) \quad \textit{and} \quad \mathbf{Y}_0 \in H^3(\Gamma_s)\times H^1(\Gamma_s). \tag{3.2.22}$$

Then the equation (3.2.21) *admits a unique solution* $\mathbf{Y}$ *which satisfies*

$$\begin{aligned}\mathbf{Y} \in L^2(0,T;H^4(\Gamma_s)\times H^2(\Gamma_s)) \cap H^1(0,T;H^2(\Gamma_s)\times L^2(\Gamma_s))\\ \cap C^0([0,T];H^3(\Gamma_s)\times H^1(\Gamma_s)).\end{aligned} \tag{3.2.23}$$

In addition if

$$\widetilde{G}_{3,t} \in L^2(0,T;H^2(\Gamma_s)\times L^2(\Gamma_s)) \quad \textit{and} \quad \mathcal{A}\mathbf{Y}_0 + \widetilde{G}_3\mid_{t=0} \in H^3(\Gamma_s)\times H^1(\Gamma_s), \tag{3.2.24}$$

the solution $\mathbf{Y}$ *of the problem* (3.2.21) *has the following additional regularities*

$$\begin{aligned}&\mathbf{Y}_t \in L^2(0,T;H^4(\Gamma_s)\times H^2(\Gamma_s)) \cap C^0([0,T];H^3(\Gamma_s)\times H^1(\Gamma_s)),\\ &\mathbf{Y}_{tt} \in L^2(0,T;H^2(\Gamma_s)\times L^2(\Gamma_s)).\end{aligned} \tag{3.2.25}$$

Proof. To prove this result we will use the maximal parabolic regularity results from [4]. Recall the definition of H_s in (3.2.19). The unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on H_s (for the proof see [13]). Hence using the isomorphism theorem [4, Theorem 3.1, p. 143] and the assumption (3.2.22), which can be read as $\widetilde{G}_3 \in L^2(0,T;H_s)$ and $\mathbf{Y}_0 \in D(\mathcal{A}^{1/2})$, we get that the equation (3.2.21) admits a unique solution $\mathbf{Y}$ satisfying the following:

$$\mathbf{Y} \in L^2(0,T;H^4(\Gamma_s)\times H^2(\Gamma_s)) \cap H^1(0,T;H^2(\Gamma_s)\times L^2(\Gamma_S)).$$

Using interpolation (see [26]) one also obtains that

$$\mathbf{Y} \in C^0([0,T];H^3(\Gamma_s)\times H^1(\Gamma_s)).$$

This proves (3.2.23).
Now we assume that (3.2.24) holds. In order to obtain the time regularity of $\mathbf{Y}$ let us differentiate (3.2.21) with respect to t and write $\mathbf{Z} = \mathbf{Y}_t$,

$$\begin{cases} \mathbf{Z}_t(t) = \mathcal{A}\mathbf{Z}(t) + \widetilde{G}_{3,t} \quad \text{on} \quad (0,T),\\ \mathbf{Z}(0) = \mathbf{Z}^0 = \mathcal{A}\mathbf{Y}_0 + \widetilde{G}_3\mid_{t=0}. \end{cases} \tag{3.2.26}$$

Due to the assumptions (3.2.24), $\widetilde{G}_{3,t} \in L^2(0,T;H_s)$ and $\mathcal{A}\mathbf{Y}_0 + \widetilde{G}_3\mid_{t=0} \in D(\mathcal{A}^{1/2})$ $(= H^3(\Gamma_s)\times H^1(\Gamma_s))$. We can use the isomorphism theorem [4, Theorem 3.1, p. 143] again to conclude

$$\mathbf{Z} = \mathbf{Y}_t \in L^2(0,T;H^4(\Gamma_s)\times H^2(\Gamma_s)) \cap H^1(0,T;H^2(\Gamma_s)\times L^2(\Gamma_s)).$$

Once again using interpolation we verify that

$$\mathbf{Y}_t \in C^0([0,T];H^3(\Gamma_s)\times H^1(\Gamma_s)).$$

This completes the proof of Lemma 3.2.6. □

We are going to use the representation (3.2.21) of (3.2.17) to state the existence and regularity result for the problem (3.2.17).

Theorem 3.2.7. *Assume that $T < \overline{T}$ (recall that $\overline{T}$ was fixed in (3.1.23)), $G_3 \in L^\infty(0,T;H^{1/2}(\Gamma_s))$ and $G_{3,t} \in L^2(0,T;L^2(\Gamma_s))$. Also suppose that $\eta_1 \in H^3(\Gamma_s)$ and $G_3 \mid_{t=0} \in H^1(\Gamma_s)$. Then the equation (3.2.17) admits a unique solution η which satisfies*

$$\begin{aligned}
&\eta \in L^\infty(0,T;H^{9/2}(\Gamma_s)),\\
&\eta_t \in L^2(0,T;H^4(\Gamma_s)) \cap C^0([0,T];H^3(\Gamma_s)),\\
&\eta_{tt} \in L^2(0,T;H^2(\Gamma_s)) \cap C^0([0,T];H^1(\Gamma_s)),\\
&\eta_{ttt} \in L^2(0,T;L^2(\Gamma_s)),
\end{aligned} \tag{3.2.27}$$

and for some positive constant c_4 independent of T, G_3 and η_1 we have the following estimate

$$\begin{aligned}
&\|\eta\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} + \|\eta_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_t\|_{L^\infty(0,T;H^3(\Gamma_s))} + \|\eta_{tt}\|_{L^2(0,T;H^2(\Gamma_s))}\\
&+ \|\eta_{tt}\|_{L^\infty(0,T;H^1(\Gamma_s))} + \|\eta_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \leqslant c_4\big(\|\eta_1\|_{H^3(\Gamma_s)}\\
&+ \|G_3 \mid_{t=0}\|_{H^1(\Gamma_s)} + \|G_3\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))} + \|G_{3,t}\|_{L^2(0,T;L^2(\Gamma_s))}\big).
\end{aligned} \tag{3.2.28}$$

Proof. We first consider

$$G_3 \in L^2(0,T;L^2(\Gamma_s)) \quad \text{and} \quad \eta_1 \in H^1(\Gamma_s). \tag{3.2.29}$$

In view of the notations (3.2.20), (3.2.29) corresponds to the case (3.2.22) of Lemma 3.2.6. Hence we can use (3.2.23) to obtain

$$\|\eta\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_t\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{tt}\|_{L^2(0,T;L^2(\Gamma_s))} \leqslant c\big(\|\eta_1\|_{H^1(\Gamma_s)} + \|G_3\|_{L^2(0,T;L^2(\Gamma_s))}\big), \tag{3.2.30}$$

where the constant c might depend on the final time T. We want to show that there exists a constant c independent of T such that the inequality (3.2.30) is true. For that we extend G_3 by defining it zero in $(T,\overline{T})$ and denote the extended function also by G_3. Observe that $G_3 \in L^2(0,\overline{T};L^2(\Gamma_s))$. We can solve (3.2.17) in the time interval $(0,\overline{T})$ and consequently

$$\begin{aligned}
&\|\eta\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_t\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{tt}\|_{L^2(0,T;L^2(\Gamma_s))}\\
&\leqslant \|\eta\|_{L^2(0,\overline{T};H^4(\Gamma_s))} + \|\eta_t\|_{L^2(0,\overline{T};H^2(\Gamma_s))} + \|\eta_{tt}\|_{L^2(0,\overline{T};L^2(\Gamma_s))}\\
&\leqslant c(\overline{T})\big(\|\eta_1\|_{H^1(\Gamma_s)} + \|G_3\|_{L^2(0,\overline{T};L^2(\Gamma_s))}\big)\\
&= c(\overline{T})\big(\|\eta_1\|_{H^1(\Gamma_s)} + \|G_3\|_{L^2(0,T;L^2(\Gamma_s))}\big).
\end{aligned} \tag{3.2.31}$$

So we are able to get a constant $c(\overline{T})$ which is independent of T.
To prove the regularity estimates of η_t, we will use

$$G_{3,t} \in L^2(0,T;L^2(\Gamma_s)), \quad \eta_1 \in H^3(\Gamma_s) \quad \text{and} \quad G_3 \mid_{t=0} \in H^1(\Gamma_s). \tag{3.2.32}$$

Indeed, observe that (3.2.32) implies $\delta\Delta\eta_1 + G_3 \mid_{t=0} \in H^1(\Gamma_s)$. Now differentiate the equation (3.2.17) with respect to t,

$$\begin{cases} (\eta_t)_{tt} - \beta(\eta_t)_{xx} - \delta(\eta_t)_{txx} + \alpha(\eta_t)_{xxxx} = G_{3,t} & \text{on} \quad \Sigma_T^s, \\ \eta_t(0) = \eta_1 \quad \text{and} \quad \eta_{tt}(0) = \delta\Delta\eta_1 + G_3 \mid_{t=0} & \text{in} \quad \Gamma_s. \end{cases} \tag{3.2.33}$$

In view of the notations (3.2.20), (3.2.32) and (3.2.33) correspond respectively to (3.2.24) and (3.2.26) in Lemma 3.2.6. Hence we can use (3.2.25) to furnish the following

$$\begin{aligned} &\|\eta_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \\ &\leqslant c(\|\eta_1\|_{H^3(\Gamma_s)} + \|G_3 \mid_{t=0}\|_{H^1(\Gamma_s)} + \|G_{3,t}\|_{L^2(0,T;L^2(\Gamma_s))}), \end{aligned} \tag{3.2.34}$$

where the constant c might depend on the final time T. Since we are interested in proving (3.2.34) with a constant c independent of T, we extend the function $G_{3,t}$ by defining it zero in the interval $(T, \overline{T})$ and denote the extended function also by $G_{3,t}$. In a similar spirit of the computation (3.2.31) one can prove

$$\begin{aligned} &\|\eta_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \\ &\leqslant c(\overline{T})(\|\eta_1\|_{H^3(\Gamma_s)} + \|G_3 \mid_{t=0}\|_{H^1(\Gamma_s)} + \|G_{3,t}\|_{L^2(0,T;L^2(\Gamma_s))}) \end{aligned} \tag{3.2.35}$$

for some constant $c(\overline{T})$ independent on T. In order to get explicit bounds on the $L^\infty(0,T)$ norms of η, η_t and η_{tt} we first multiply $(3.2.17)_1$ by η_{txx} and integrate over Γ_s. We use the L-periodicity (in the x direction) of η and integrate the terms by parts to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Gamma_s} \eta_{tx}^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_{\Gamma_s} \eta_{xx}^2 dx + \delta\int_{\Gamma_s} \eta_{txx}^2 dx + \frac{\alpha}{2}\frac{d}{dt}\int_{\Gamma_s} \eta_{xxx}^2 dx \\ &\leqslant \frac{\delta}{8}\|\eta_{txx}(t)\|_{L^2(\Gamma_s)}^2 + \frac{2}{\delta}\|G_3\|_{L^2(\Gamma_s)}^2. \end{aligned} \tag{3.2.36}$$

Now integrating (3.2.36) with respect to t,

$$\begin{aligned} &\|\eta_{tx}\|_{L^\infty(0,T;L^2(\Gamma_s))}^2 + \|\eta_{xx}\|_{L^\infty(0,T;L^2(\Gamma_s))}^2 + \|\eta_{txx}\|_{L^2(0,T;L^2(\Gamma_s))}^2 \\ &+ \|\eta_{xxx}\|_{L^\infty(0,T;L^2(\Gamma_s))}^2 \leqslant c(\|\eta_{tx}(0)\|_{L^2(\Gamma_s)}^2 + \|G_3\|_{L^2(0,T;L^2(\Gamma_s))}^2). \end{aligned} \tag{3.2.37}$$

From (3.2.37) we get in particular

$$\|\eta\|_{L^\infty(0,T;H^3(\Gamma_s))}^2 \leqslant c(\|\eta_1\|_{H^1(\Gamma_s)}^2 + \|G_3\|_{L^2(0,T;L^2(\Gamma_s))}^2). \tag{3.2.38}$$

Now consider the equations (3.2.33). One imitates the analysis used to obtain (3.2.37) to find

$$\begin{aligned} &\|\eta_{ttx}\|_{L^\infty(0,T;L^2(\Gamma_s))}^2 + \|\eta_{txx}\|_{L^\infty(0,T;L^2(\Gamma_s))}^2 + \|\eta_{ttxx}\|_{L^2(0,T;L^2(\Gamma_s))}^2 \\ &+ \|\eta_{txxx}\|_{L^\infty(0,T;L^2(\Gamma_s))}^2 \leqslant c(\|\eta_1\|_{H^3(\Gamma_s)}^2 + \|G_3 \mid_{t=0}\|_{H^1(\Gamma_s)}^2 + \|G_{3,t}\|_{L^2(0,T;L^2(\Gamma_s))}^2). \end{aligned} \tag{3.2.39}$$

Hence in particular

$$\|\eta_t\|^2_{L^\infty(0,T;H^3(\Gamma_s))} + \|\eta_{tt}\|^2_{L^\infty(0,T;H^1(\Gamma_s))} \leqslant c(\|\eta_1\|^2_{H^3(\Gamma_s)} + \|G_3\mid_{t=0}\|^2_{H^1(\Gamma_s)} + \|G_{3,t}\|^2_{L^2(0,T;L^2(\Gamma_s))}). \tag{3.2.40}$$

Now we will use that

$$G_3 \in L^\infty(0,T;H^{1/2}(\Gamma_s)). \tag{3.2.41}$$

Write $(3.2.17)_1$ as

$$\eta_{xxxx} = \tfrac{1}{\alpha}(G_3 + \delta\eta_{txx} + \beta\eta_{xx} - \eta_{tt}). \tag{3.2.42}$$

In view of (3.2.41) one observes that all the terms appearing in the right hand side of (3.2.42) belongs to $L^\infty(0,T;H^{1/2}(\Gamma_s))$. As the beam in our problem is one dimensional, $\eta \in L^\infty(0,T;H^{9/2}(\Gamma_s))$ and the estimates (3.2.38) and (3.2.40) furnish the following

$$\|\eta\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} \leqslant c(\|\eta_1\|_{H^3(\Gamma_s)} + \|G_3\mid_{t=0}\|_{H^1(\Gamma_s)} + \|G_{3,t}\|_{L^2(0,T;L^2(\Gamma_s))} + \|G_3\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))}). \tag{3.2.43}$$

Hence combining all the above estimates we here conclude the proof of Theorem 3.2.7. □

The following corollary follows directly by using the regularities (3.2.27) and the expression (3.2.42) of η_{xxxx}.

Corollary 3.2.8. *In addition to the assumptions of Theorem 3.2.7 if G_3 further satisfies the regularity assumption $G_3 \in C^0([0,T];H^{1/2}(\Gamma_s))$ then $\eta \in C^0([0,T];H^{9/2}(\Gamma_s))$.*

3.3 Local existence of the non linear coupled system

From now on up to the end of this article, we fix the initial data $(\rho_0, \mathbf{u}_0, \eta_1)$ such that they satisfy the assumptions stated in (3.1.17). We also fix the constant

$$\delta_0 \in (0,1). \tag{3.3.1}$$

The constant δ_0 will be used to keep a positive distance between the beam and the bottom Γ_ℓ of the domain Ω. Also recall that the positive constants m and M were fixed in (3.1.3) and $\overline{T}$ was fixed in (3.1.23).

Proof of Theorem 3.1.7. This section is devoted to the study of the non linear system (3.1.21). We will prove here that the system (3.1.21) admits a strong solution in a time interval $(0,T)$, for some $T > 0$ small enough and hence we will conclude Theorem 3.1.7. Now we sketch the steps towards the proof of Theorem 3.1.7:

(i) First in Section 3.3.1 we define a suitable map for which a fixed point gives a solution of the system (3.1.21).
(ii) Next we design a suitable convex set such that the map defined in step (i) maps this set into itself. This is done in Section 3.3.2.
(iii) In Section 3.3.3 we show that the convex set defined in step (ii) is compact in some appropriate topology. We further prove that the fixed point map from step (i), is continuous in that topology.
(iv) At the end in Section 3.3.4 we draw the final conclusion to prove Theorem 3.1.7.
In what follows all the constants appearing in the inequalities may vary from line to line but will never depend on T.

3.3.1 Definition of the fixed point map

For $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ satisfying

$$\begin{cases} \widetilde{\sigma} \in L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega)), \\ \widetilde{\mathbf{w}} \in L^\infty(0,T;\mathbf{H}^{5/2}(\Omega)) \cap L^2(0,T;\mathbf{H}^3(\Omega)) \cap W^{1,\infty}(0,T;\mathbf{H}^1(\Omega)) \\ \qquad \cap H^1(0,T;\mathbf{H}^2(\Omega)) \cap H^2(0,T;\mathbf{L}^2(\Omega)), \\ \widetilde{\eta} \in L^\infty(0,T;H^{9/2}(\Gamma_s)) \cap W^{1,\infty}(0,T;H^3(\Gamma_s)) \cap H^1(0,T;H^4(\Gamma_s)) \\ \qquad \cap W^{2,\infty}(0,T;H^1(\Gamma_s)) \cap H^2(0,T;H^2(\Gamma_s)) \cap H^3(0,T;L^2(\Gamma_s)), \end{cases} \tag{3.3.2}$$

we consider the following problem:

$$\begin{cases} \sigma_t + \widetilde{W}(\widetilde{\mathbf{w}},\widetilde{\eta}) \cdot \nabla \sigma = G_1(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) & \text{in} \quad Q_T, \\ (\widetilde{\sigma}+\overline{\rho})\mathbf{w}_t - \mu\Delta\mathbf{w} - (\mu+\mu')\nabla \mathrm{div}\mathbf{w} = G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) & \text{in} \quad Q_T, \\ \mathbf{w} = 0 & \text{on} \quad \Sigma_T, \\ \mathbf{w}(\cdot,0) = \mathbf{w}_0 = \mathbf{u}_0 - z\eta_1\vec{e}_2 & \text{in} \quad \Omega, \\ \sigma(\cdot,0) = \sigma_0 = \rho_0 - \overline{\rho} & \text{in} \quad \Omega, \\ \eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = G_3(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) & \text{on} \quad \Sigma_T^s, \\ \eta(0) = 0 \quad \text{and} \quad \eta_t(0) = \eta_1 & \text{in} \quad \Gamma_s, \end{cases} \tag{3.3.3}$$

where G_1, G_2, G_3 are as defined in (3.1.22) and $\widetilde{W}(\widetilde{\mathbf{w}},\widetilde{\eta})$ is defined as follows

$$\widetilde{W}(\widetilde{\mathbf{w}},\widetilde{\eta}) = \begin{bmatrix} \widetilde{w}_1 \\ \frac{1}{(1+\widetilde{\eta})}(\widetilde{w}_2 - \widetilde{w}_1 z\widetilde{\eta}_x) \end{bmatrix}, \qquad \left(\widetilde{\mathbf{w}} = \begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \end{pmatrix}\right). \tag{3.3.4}$$

It turns out that it will be important for us to check that $G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ and $G_3(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ respectively coincide at time $t=0$ with the values G_2^0 and G_3^0 computed in (3.1.27) and (3.1.25), and given as follows:

$$\begin{aligned} G_2^0 = &-P'(\rho_0)\nabla\rho_0 - (\delta\eta_{1,xx} - (\mu+2\mu')(u_0)_{2,z} + P(\rho_0))z\rho_0\vec{e}_2 + z\rho_0(\mathbf{u}_0)_z\eta_1 \\ &- \rho_0(\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 - (-\mu\Delta - (\mu+\mu')\nabla\mathrm{div})(z\eta_1\vec{e}_2), \end{aligned} \tag{3.3.5}$$

$$G_3^0 = -(\mu + 2\mu')(u_0)_{2,z} + P(\rho_0). \tag{3.3.6}$$

This will be imposed by assuming $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}, \widetilde{\eta}_t)(\cdot, 0) = (\sigma_0, \mathbf{w}_0, 0, \eta_1)$ and

$$\begin{aligned} \widetilde{\eta}_{tt}(\cdot, 0) &= \delta\eta_{1,xx} - (\mu + 2\mu')(u_0)_{2,z} + P(\rho_0) \text{ in } \Omega, \\ \widetilde{\mathbf{w}}_t(\cdot, 0) &= \frac{1}{\rho_0}(G_2^0 - (-\mu\Delta - (\mu + \mu')\nabla\mathrm{div})(\mathbf{u}_0 - z\eta_1\vec{e_2})) \text{ in } \Omega, \end{aligned} \tag{3.3.7}$$

Indeed, under the above conditions, one can check from the expressions of $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ and $G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ that $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \mid_{t=0} = G_2^0$ and $G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \mid_{t=0} = G_3^0$.

Lemma 3.3.1. *Let the constant δ_0 be fixed by (3.3.1). For $T < \overline{T}$, let us assume the following*

$$(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \text{ satisfies (3.3.2)}, \tag{3.3.8}$$

$$\widetilde{\mathbf{w}} = 0 \text{ on } \Sigma_T, \tag{3.3.9}$$

$$(\widetilde{\sigma}(\cdot, 0), \widetilde{\mathbf{w}}(\cdot, 0), \widetilde{\eta}(\cdot, 0), \widetilde{\eta}_t(\cdot, 0)) = (\rho_0 - \overline{\rho}, \mathbf{u}_0 - z\eta_1\vec{e_2}, 0, \eta_1) \text{ in } \Omega, \tag{3.3.10}$$

$$(3.3.7) \text{ holds}, \tag{3.3.11}$$

$$1 + \widetilde{\eta}(x, t) \geqslant \delta_0 > 0 \text{ on } \Sigma_T^s, \tag{3.3.12}$$

$$0 < \frac{m}{2} \leqslant \widetilde{\sigma} + \overline{\rho} \leqslant 2M \text{ in } Q_T, \tag{3.3.13}$$

where m and M were fixed in (3.1.3).
Then $G_1(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$, $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ and $G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ satisfy the following

$$\begin{aligned} &G_1(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \in L^1(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \\ &G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \in L^2(0, T; \mathbf{H}^1(\Omega)) \cap H^1(0, T; \mathbf{L}^2(\Omega)) \cap L^\infty(0, T; \mathbf{L}^2(\Omega)), \\ &G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \in L^\infty(0, T; H^{1/2}(\Gamma_s)) \cap H^1(0, T; L^2(\Gamma_s)), \\ &\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta}) \in L^1(0, T; \mathbf{H}^3(\Omega)) \cap L^\infty(0, T; \mathbf{H}^2(\Omega)), \\ &G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \mid_{t=0} = G_2^0 \text{ and } G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \mid_{t=0} = G_3^0. \end{aligned} \tag{3.3.14}$$

Proof. The detailed computations to verify (3.3.14) follows from Lemma 3.3.8 (for estimates of G_1), Lemma 3.3.10 (for estimates of G_2), Lemma 3.3.12 (for estimates of G_3) and Lemma 3.3.14 (for estimates of $\widetilde{W}$) in the Section 3.3.2.2. □

Observe that the condition (3.3.9) implies that $\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta}) \cdot \mathbf{n} = 0$ (where $\widetilde{W}$ is as defined in (3.3.4)) on Σ_T. Hence in view of Lemma 3.3.1, for all $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ satisfying the conditions (3.3.8)-(3.3.9)-(3.3.10)-(3.3.11)-(3.3.12) -(3.3.13), the system (3.3.3) admits a

unique solution as a consequence of Theorem 3.2.1, Theorem 3.2.4 and Theorem 3.2.7 in the space $Z_1^T \times Y_2^T \times Z_3^T$, where Y_2^T is defined in (3.1.16), Z_1^T and Z_3^T are defined as follows

$$\begin{aligned} Z_1^T &= \{\rho \in C^0([0,T];H^2(\Omega)) \mid \rho_t \in L^\infty(0,T;H^1(\Omega))\},\\ Z_3^T &= \{\eta \in L^\infty(0,T;H^{9/2}(\Gamma_s)),\ \eta(x,0)=0 \mid \eta_t \in L^2(0,T;H^4(\Gamma_s)) \cap C^0([0,T];H^3(\Gamma_s)),\\ &\quad \eta_{tt} \in L^2(0,T;H^2(\Gamma_s)) \cap C^0([0,T];H^1(\Gamma_s)), \eta_{ttt} \in L^2(0,T;L^2(\Gamma_s))\}. \end{aligned} \tag{3.3.15}$$

Observe that the only difference between Y_1^T (defined in (3.1.16)) and Z_1^T is that the elements of Y_1^T belongs to $C^1([0,T];H^1(\Omega))$ while the elements of Z_1^T are in $W^{1,\infty}(0,T;H^1(\Omega))$. Also one observes that the elements of Y_3^T (defined in (3.1.16)) are in $C^0([0,T];H^{9/2}(\Gamma_s))$ while Z_3^T is only a subset of $L^\infty(0,T;H^{9/2}(\Gamma_s))$.
Before defining a suitable fixed point map (in order to solve the non-linear problem (3.1.21)), we will introduce a convex set $\mathcal{C}_T$ (where we will show the existence of a fixed point). The set $\mathcal{C}_T$ will be defined as a subset of $L^2(0,T;L^2(\Omega)) \times L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;L^2(\Gamma_s))$ such that the elements of $\mathcal{C}_T$ satisfy some norm bounds and some conditions at initial time $t=0$.
Let us make precise the assumptions which will be used to define the set $\mathcal{C}_T$.
Regularity assumptions and norm bounds of $(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$:

$$\|\widetilde{\sigma}\|_{L^\infty(0,T;H^2(\Omega))} \leqslant B_1, \quad \|\widetilde{\sigma}_t\|_{L^\infty(0,T;H^1(\Omega))} \leqslant B_2, \tag{3.3.16a}$$

$$\begin{aligned} \|\widetilde{\mathbf{w}}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} + \|\widetilde{\mathbf{w}}\|_{L^2(0,T;\mathbf{H}^3(\Omega))} + \|\widetilde{\mathbf{w}}_t\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\widetilde{\mathbf{w}}_t\|_{L^2(0,T;\mathbf{H}^2(\Omega))}\\ + \|\widetilde{\mathbf{w}}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant B_3, \end{aligned} \tag{3.3.16b}$$

$$\begin{aligned} \|\widetilde{\eta}\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} + \|\widetilde{\eta}_t\|_{L^\infty(0,T;H^3(\Gamma_s))} + \|\widetilde{\eta}_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\widetilde{\eta}_{tt}\|_{L^\infty(0,T;H^1(\Gamma_s))}\\ + \|\widetilde{\eta}_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} + \|\widetilde{\eta}_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \leqslant B_4, \end{aligned} \tag{3.3.16c}$$

$$1 + \widetilde{\eta}(x,t) \geqslant \delta_0 > 0 \text{ on } \Sigma_T^s, \tag{3.3.16d}$$

$$0 < \frac{m}{2} \leqslant \widetilde{\sigma} + \overline{\rho} \leqslant 2M \text{ in } Q_T, \tag{3.3.16e}$$

where B_i's $(1 \leqslant i \leqslant 4)$ are positive constants and will be chosen in the sequel. The norm bound (3.3.16b) implicitly asserts (by interpolation) that $\widetilde{\mathbf{w}}$ is in $C^0([0,T];\mathbf{H}^{5/2}(\Omega))$.
Assumptions on initial and boundary conditions:

$$\widetilde{\mathbf{w}} = 0 \text{ on } \Sigma_T, \tag{3.3.17a}$$

$$(\widetilde{\sigma}(\cdot,0),\widetilde{\mathbf{w}}(\cdot,0),\widetilde{\eta}(\cdot,0),\widetilde{\eta}_t(\cdot,0)) = (\rho_0-\overline{\rho},\mathbf{u}_0 - z\eta_1\vec{e}_2,0,\eta_1) \text{ in } \Omega, \tag{3.3.17b}$$

$$\widetilde{\eta}_{tt}(\cdot,0) = \delta\eta_{1,xx} - (\mu+2\mu')(u_0)_{2,z} + P(\rho_0) \text{ in } \Omega, \tag{3.3.17c}$$

$$\widetilde{\mathbf{w}}_t(\cdot,0) = \frac{1}{\rho_0}(G_2^0 - (-\mu\Delta - (\mu+\mu')\nabla\text{div})(\mathbf{u}_0 - z\eta_1\vec{e_2})) \text{ in } \Omega. \tag{3.3.17d}$$

For $T < \overline{T}$, let us define the following set

$$\begin{aligned}\mathcal{C}_T(B_1,B_2,B_3,B_4) = \{(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) \in L^2(0,T;L^2(\Omega)) \times L^2(0,T;\mathbf{L}^2(\Omega)) \times L^2(0,T;L^2(\Gamma_s)) \mid \\ \text{the relations (3.3.16)} - \text{(3.3.17) are true}\}.\end{aligned} \tag{3.3.18}$$

Now for $(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) \in \mathcal{C}_T(B_1,B_2,B_3,B_4)$, let $(\sigma,\mathbf{w},\eta) \in Z_1^T \times Y_2^T \times Z_3^T$ (recall the definition of Y_2^T, from (3.1.16) and Z_1^T, Z_3^T are defined in (3.3.15)) be the solution of the problem (3.3.3) corresponding to $(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$. This defines the map

$$\begin{array}{ccc} L : \mathcal{C}_T(B_1,B_2,B_3,B_4) & \longrightarrow & Z_1^T \times Y_2^T \times Z_3^T \\ (\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) & \mapsto & (\sigma,\mathbf{w},\eta). \end{array} \tag{3.3.19}$$

Now observe that if the map L admits a fixed point $(\sigma_f,\mathbf{w}_f,\eta_f)$ on the set $\mathcal{C}_T(B_1,B_2,B_3,B_4)$, then the triplet $(\sigma_f,\mathbf{w}_f,\eta_f)$ is a solution to the system (3.1.21). Thus, our goal from now is to prove the existence of a fixed point to the map L. In that direction we first show that for suitable parameters B_i $(1 \leqslant i \leqslant 4)$ and T, the set $\mathcal{C}_T(B_1,B_2,B_3,B_4)$ is non-empty.

Lemma 3.3.2. *There exists a constant $B_0^* > 0$ such that for all $B_i \geqslant B_0^*$ $(1 \leqslant i \leqslant 4)$ there exists $T_0^*(B_1,B_2,B_3,B_4) \in (0,\min\{1,\overline{T}\})$ such that for all $0 < T \leqslant T_0^*(B_1,B_2,B_3,B_4)$ the set $\mathcal{C}_T(B_1,B_2,B_3,B_4)$ is non empty.*

Proof. The choice of the constant B_0^* will be done based on the calculations performed in the following steps.

Step 1. In this step we will prove the existence of a function $\mathbf{w}^*$ which satisfies the norm bound (3.3.16b) and the condition (3.3.17d) at time $t = 0$. We begin by recalling that $(\rho_0,\mathbf{u}_0,\eta_1)$ satisfies (3.1.17) and hence one observes that $(\mathbf{u}_0 - z\eta_1\vec{e}_2) \in \mathbf{H}^3(\Omega)$. As G_2^0 is given by the expression (3.3.5), using (3.1.29) one has $G_2^0 \in H^1(\Omega)$. We can thus find a lifting $\mathbf{h} \in L^2(\mathbb{R}^+;\mathbf{H}^1(\Omega))$ and $\mathbf{h}_t \in L^2(\mathbb{R}^+;\mathbf{L}^2(\Omega))$ (see e.g. [25, Theorem 3.2, p. 21]) such that $\mathbf{h}(0) = G_2^0$ in Ω. (In fact, we only need $G_2^0 \in H^{1/2}(\Omega)$ in this step.)
Let $\mathbf{w}^*$ be the solution of the following system

$$\begin{cases} \rho_0\mathbf{w}_t^* - \mu\Delta\mathbf{w}^* - (\mu+\mu')\nabla\text{div}\mathbf{w}^* = \mathbf{h} & \text{in} \quad Q_\infty, \\ \mathbf{w}^* = 0 & \text{on} \quad \Sigma_\infty, \\ \mathbf{w}^*(0) = \mathbf{w}_0 = (\mathbf{u}_0 - z\eta_1\vec{e_2}) & \text{in} \quad \Omega. \end{cases} \tag{3.3.20}$$

In view of (3.1.30) one can uniquely solve (3.3.20) such that the function $\mathbf{w}^*$ satisfies the following estimate

$$\begin{aligned}
&\|\mathbf{w}^*\|_{L^\infty(0,\infty;\mathbf{H}^2(\Omega))} + \|\mathbf{w}^*\|_{L^2(0,\infty;\mathbf{H}^3(\Omega))} + \|\mathbf{w}^*_t\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))} \\
&+ \|\mathbf{w}^*_t\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))} + \|\mathbf{w}^*_{tt}\|_{L^2(0,\infty;\mathbf{L}^2(\Omega))} \\
&\leqslant c(\|\mathbf{h}\|_{L^2(0,\infty;H^1(\Omega))} + \|\mathbf{h}_t\|_{L^2(0,\infty;L^2(\Omega))} + \|G_2\mid_{t=0}\|_{H^1(\Omega)} + \|\mathbf{u}_0 - z\eta_1\vec{e_2}\|_{H^3(\Omega)}) \\
&\leqslant c_5(\|G_2\mid_{t=0}\|_{H^1(\Omega)} + \|\mathbf{u}_0 - z\eta_1\vec{e_2}\|_{H^3(\Omega)}).
\end{aligned} \tag{3.3.21}$$

Using (3.3.20) one also observes the following

$$\mathbf{w}^*_t(\cdot,0) = \frac{1}{\rho_0}(G_2^0 - (-\mu\Delta - (\mu+\mu')\nabla\mathrm{div})(\mathbf{u}_0 - z\eta_1\vec{e_2})). \tag{3.3.22}$$

In view of (3.3.21) and (3.3.22) one observes that $\mathbf{w}^*$ satisfies (3.3.16b) and (3.3.17d) respectively.

Step 2. In this step we will prove the existence of a function η^* which satisfies the norm bound (3.3.16c), (3.3.16d) and the condition (3.3.17c) at time $t = 0$. In that direction first recall that $G_3^0 \in H^1(\Gamma_s)$. We use in particular the regularity $G_3^0 \in H^{1/2}(\Gamma_s)$ to obtain a lifting h_1 of G_3^0 such that $h_1 \in L^2(\mathbb{R}^+; H^1(\Gamma_s)) \cap H^1(\mathbb{R}^+; L^2(\Gamma_s)) \cap L^\infty(\mathbb{R}^+; H^{1/2}(\Gamma_s))$ and $h_1(0) = G_3^0$ in Γ_s. Let η^* be the solution of equation (3.2.17) with G_3 replaced by h_1. From Theorem 3.2.7 and inequality (3.2.28) one obtains

$$\begin{aligned}
&\|\eta^*\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} + \|\eta^*_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta^*_t\|_{L^\infty(0,T;H^3(\Gamma_s))} + \|\eta^*_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} \\
&+ \|\eta^*_{tt}\|_{L^\infty([0,T];H^1(\Gamma_s))} + \|\eta^*_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \\
&\leqslant c(\|(h_1)_t\|_{L^2(0,\infty;L^2(\Gamma_s))} + \|h_1\|_{L^\infty(0,\infty;H^{1/2}(\Gamma_s))} + \|G_3^0\|_{H^1(\Gamma_s)} + \|\eta_1\|_{H^3(\Gamma_s)}) \\
&\leqslant c_4(\|G_3^0\|_{H^1(\Gamma_s)} + \|\eta_1\|_{H^3(\Gamma_s)})
\end{aligned} \tag{3.3.23}$$

where the constant c_4 is independent of T. One further uses (3.3.6) to check that

$$\eta^*_{tt}(\cdot,0) = \delta\eta_{1,xx} - (\mu+2\mu')(u_0)_{2,z} + P(\rho_0). \tag{3.3.24}$$

In view of (3.3.23) and (3.3.24) we get that η^* satisfies (3.3.16c) and (3.3.17c). Since $\eta^*(.,0) = 0$, we observe the following by interpolation

$$\begin{aligned}
\|\eta^*\|_{C^0(\overline{\Sigma}^s_T)} &\leqslant c\|\eta^*\|^{1/3}_{L^\infty(0,T;H^1(\Gamma_s))}\|\eta^*\|^{2/3}_{L^\infty(0,T;H^2(\Gamma_s))} \\
&\leqslant cT^{1/3}(\|\eta^*_t\|_{L^\infty(0,T;H^1(\Gamma_s))})^{1/3}\cdot(\|\eta^*\|_{L^\infty(0,T;H^2(\Gamma_s))})^{2/3}.
\end{aligned} \tag{3.3.25}$$

At this point we set

$$\begin{aligned}
B_0^* = \max\{&c_5(\|G_2^0\|_{H^1(\Omega)} + \|\mathbf{u}_0 - z\eta_1\vec{e_2}\|_{H^3(\Omega)}),\ \|\rho_0 - \overline{\rho}\|_{H^2(\Omega)}, \\
&c_4(\|G_3^0\|_{H^1(\Gamma_s)} + \|\eta_1\|_{H^3(\Gamma_s)})\}
\end{aligned} \tag{3.3.26}$$

and for all $1 \leqslant i \leqslant 4$, $B_i \geqslant B_0^*$.
Hence in view of (3.3.25), there exists $T_0^*(B_1, B_2, B_3, B_4) \in (0, \min\{1, \overline{T}\})$ such that for all $0 < T \leqslant T_0^*(B_1, B_2, B_3, B_4)$ we verify that

$$1 + \eta^* \geqslant \delta_0 > 0 \quad \text{on} \quad \Sigma_T^s,$$

i.e η^* satisfies (3.3.16d).
Step 3. One easily checks that $\sigma^* = \rho_0 - \overline{\rho}$ verifies (3.3.16a) and (3.3.16e).
We further observe that $(\sigma^*, \mathbf{w}^*, \eta^*)$ satisfies (3.3.17a) and (3.3.17b) automatically by construction.
So we have shown that if we choose B_0^* (and hence $B_i \geqslant B_0^*$, for all $1 \leqslant i \leqslant 4$) as in (3.3.26) and $0 < T \leqslant T_0^*(B_1, B_2, B_3, B_4)$ then $(\sigma^* = \rho_0 - \overline{\rho}, \mathbf{w}^*, \eta^*) \in \mathcal{C}_T(B_1, B_2, B_3, B_4)$, *i.e.*

$$\mathcal{C}_T(B_1, B_2, B_3, B_4) \neq \varnothing.$$

□

Remark 3.3.3. *Observe from the proof of Lemma 3.3.2, the constant $T_0^*(B_1, B_2, B_3, B_4)$ depends on $\delta_0 \in (0, 1)$. Since δ_0 is fixed (see (3.3.1)) we do not write explicitly the dependence of $T_0^*(B_1, B_2, B_3, B_4)$ on δ_0.*

3.3.2 For small enough T, L maps $\mathcal{C}_T(B_1, B_2, B_3, B_4)$ into itself

To prove that the map L admits a fixed point we first show that for T small enough and a suitable choice of parameters (B_1, B_2, B_3, B_4), the set $\mathcal{C}_T(B_1, B_2, B_3, B_4)$ is mapped into itself by L.
Provided $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \in \mathcal{C}_T(B_1, B_2, B_3, B_4)$, we have to estimate the terms $G_1(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$, $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$, $G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ and $\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta})$ (recall the definition of G_1, G_2, G_3 and $\widetilde{W}$ from (3.1.22) and (3.3.4) respectively). For this purpose we will require some results which we collect in the following section.

3.3.2.1 Useful lemmas

The following lemma concerning the Sobolev regularity of the product of two functions is standard in the literature.

Lemma 3.3.4. *Consider a bounded domain Ω_0 in $\mathbb{R}^d$ (for $d = 1, 2$). Let $r > \frac{d}{2}$, $0 \leqslant s \leqslant r$. If $v \in H^r(\Omega_0)$ and $w \in H^s(\Omega_0)$ then $vw \in H^s(\Omega_0)$ with*

$$\|vw\|_{H^s(\Omega_0)} \leqslant K(\Omega_0)\|v\|_{H^r(\Omega_0)}\|w\|_{H^s(\Omega_0)}.$$

Similar estimates hold when v and w are vector valued functions i.e for $\mathbf{v} \in \mathbf{H}^r(\Omega_0)$ and $\mathbf{w} \in \mathbf{H}^s(\Omega_0)$.

Lemma 3.3.5. *Let $T < \overline{T}$ (recall that we have fixed $\overline{T}$ in (3.1.23)). We assume that $\mathbf{f} \in \mathbf{H}^{2,1}_{\Sigma_T}(Q_T)$. As usual we use the notation $\mathbf{f}_z$ to denote the partial derivative $\partial_z \mathbf{f}$ of $\mathbf{f}$ with respect to z. Also suppose that Γ_s is a smooth subset of Γ. Then the trace $\mathbf{f}_z \mid_{\Sigma_T}$ on Γ_s (i.e the normal derivative of $\mathbf{f}$ on Γ_s) belongs to $H^{1/6}(0,T;\mathbf{L}^2(\Gamma_s))$. In particular there exists a constant $K > 0$ such that for all $\mathbf{f} \in \mathbf{H}^{2,1}_{\Sigma_T}(Q_T)$ we have the following*

$$\|\mathbf{f}_z \mid_{\Sigma_T} \|_{L^2(0,T;\mathbf{L}^2(\Gamma_s))} \leqslant T^{1/6} K(\|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)}), \tag{3.3.27}$$

where $\mathbf{f}(0)$ denotes the function $\mathbf{f}$ at time $t = 0$. We specify that in our case the space $\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)$ is endowed with the following norm

$$\|\mathbf{f}\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)} = \|\mathbf{f}\|_{L^2(0,T;\mathbf{H}^2(\Omega))} + \|\mathbf{f}_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))}.$$

Remark 3.3.6. *The appearance of $\mathbf{f}(0)$ in the inequality (3.3.27) might seem redundant since for all, $\mathbf{f} \in \mathbf{H}^{2,1}_{\Sigma_T}(Q_T)$*

$$\|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)} \leqslant K_T \|\mathbf{f}\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)}.$$

But the constant K_T there may depend on T while the constant K in (3.3.27) is independent of T. This is the reason why we prefer working with (3.3.27).

Proof of Lemma 3.3.5. We have to estimate $\|\mathbf{f}_z \mid_{\Sigma_T} \|_{L^2(0,T;L^2(\Gamma_s))}$. Using Hölder's inequality we get the following

$$\begin{aligned}\left(\int_0^T \|\mathbf{f}_z \mid_{\Sigma_T} \|^2_{L^2(\Gamma_s)}\right)^{1/2} &\leqslant \left(\int_0^T \|\mathbf{f}_z \mid_{\Sigma_T} \|^3_{L^2(\Gamma_s)}\right)^{1/3} T^{1/6} \\ &\leqslant K(\Omega) \left(\int_0^T \|\mathbf{f}\|^3_{\mathbf{H}^{5/3}(\Omega)}\right)^{1/3} T^{1/6}.\end{aligned} \tag{3.3.28}$$

To prove (3.3.27), in view of (3.3.28) it is enough to show the following inequality

$$\|\mathbf{f}\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))} \leqslant K(\Omega,\overline{T})(\|\mathbf{f}\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)} + \|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)}). \tag{3.3.29}$$

In order to prove (3.3.29), first let us consider the solution $\mathbf{f}^*$ of

$$\begin{cases} \mathbf{f}^*_t - \Delta \mathbf{f}^* = 0 & \text{in} \quad Q_T, \\ \mathbf{f}^* = 0 & \text{on} \quad \Sigma_T, \\ \mathbf{f}^*(.,0) = \mathbf{f}(0) & \text{in} \quad \Omega. \end{cases} \tag{3.3.30}$$

As $\mathbf{f}(0) \in \mathbf{H}^1_0(\Omega)$, $\mathbf{f}^* \in \mathbf{H}^{2,1}_{\Sigma_T}(Q_T)$. It is also well known that there exists a constant $K(\Omega)$ such that $\mathbf{f}^*$ satisfies the following inequalities

$$\begin{aligned} &(i) \quad \|\mathbf{f}^*\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)} \leqslant K(\Omega)\|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)}, \\ &(ii) \quad \|\mathbf{f}^*\|_{L^\infty(0,T;\mathbf{H}^1_0(\Omega))} + \|\mathbf{f}^*\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \leqslant K(\Omega)\|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)}. \end{aligned} \tag{3.3.31}$$

Now we will estimate the norm of $\mathbf{f}^*$ in $L^3(0,T;\mathbf{H}^{5/3}(\Omega))$. Using interpolation we have for $a.e\ t$

$$\|\mathbf{f}^*(t)\|_{\mathbf{H}^{5/3}(\Omega)} \leqslant K(\Omega)\|\mathbf{f}^*(t)\|^{2/3}_{\mathbf{H}^2(\Omega)}\|\mathbf{f}^*(t)\|^{1/3}_{\mathbf{H}^1_0(\Omega)}.$$

From the last inequality one obtains the following

$$\begin{aligned}\|\mathbf{f}^*\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))} &= \left(\int_0^T \|\mathbf{f}^*(t)\|^3_{\mathbf{H}^{5/3}(\Omega)}\right)^{1/3} \\ &\leqslant K(\Omega)\|\mathbf{f}^*\|^{1/3}_{L^\infty(0,T;\mathbf{H}^1_0(\Omega))}\|\mathbf{f}^*\|^{2/3}_{L^2(0,T;\mathbf{H}^2(\Omega))}.\end{aligned} \tag{3.3.32}$$

Hence using inequality (ii) of (3.3.31) in (3.3.32) we obtain

$$\|\mathbf{f}^*\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))} \leqslant K(\Omega)\|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)}. \tag{3.3.33}$$

Now let us observe that $(\mathbf{f}-\mathbf{f}^*)(0)=0$. Extend the function $(\mathbf{f}-\mathbf{f}^*)$ by defining it zero in the time interval $(T-\overline{T},0)$ (the extended function is also denoted by $(\mathbf{f}-\mathbf{f}^*)$). In what follows we will use the notation

$$Q_{T-\overline{T},T} = \Omega\times(T-\overline{T},T).$$

We also introduce the space $\mathbf{H}^{2,1}_{\Sigma_T}(Q_{T-\overline{T},T})$ which is defined as in (3.1.12) with Q_T replaced by $Q_{T-\overline{T},T}$.
One can check that the extended function $(\mathbf{f}-\mathbf{f}^*)\in\mathbf{H}^{2,1}_{\Sigma_T}(Q_{T-\overline{T},T})$ and

$$\|(\mathbf{f}-\mathbf{f}^*)\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_{T-\overline{T},T})} = \|(\mathbf{f}-\mathbf{f}^*)\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)}. \tag{3.3.34}$$

Again due to the embedding $\mathbf{H}^{2,1}_{\Sigma_T}(Q_{T-\overline{T},T})\hookrightarrow H^{1/6}(T-\overline{T},T;\mathbf{H}^{5/3}(\Omega))$ we have the following

$$\|\mathbf{f}-\mathbf{f}^*\|_{H^{1/6}(T-\overline{T},T;\mathbf{H}^{5/3}(\Omega))} \leqslant K(\overline{T},\Omega)\|\mathbf{f}-\mathbf{f}^*\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_{T-\overline{T},T})}. \tag{3.3.35}$$

Since $H^{1/6}(T-\overline{T},T)$ is continuously embedded into $L^3(T-\overline{T},T)$, hence from (3.3.35)

$$\|\mathbf{f}-\mathbf{f}^*\|_{L^3(T-\overline{T},T;\mathbf{H}^{5/3}(\Omega))} \leqslant K(\overline{T},\Omega)\|\mathbf{f}-\mathbf{f}^*\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_{T-\overline{T},T})}. \tag{3.3.36}$$

Use of triangle inequality furnishes the following

$$\|\mathbf{f}\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))} \leqslant K(\|\mathbf{f}-\mathbf{f}^*\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))} + \|\mathbf{f}^*\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))}). \tag{3.3.37}$$

Incorporate inequalities (3.3.33) and (3.3.36) in (3.3.37) in order to obtain

$$\|\mathbf{f}\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))} \leqslant K(\Omega,\overline{T})(\|\mathbf{f}-\mathbf{f}^*\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_{T-\overline{T},T})} + \|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)}). \tag{3.3.38}$$

In view of the equality (3.3.34) we can obtain the following from (3.3.38),

$$\|\mathbf{f}\|_{L^3(0,T;\mathbf{H}^{5/3}(\Omega))} \leqslant K(\Omega,\overline{T})(\|\mathbf{f}-\mathbf{f}^*\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)} + \|\mathbf{f}(0)\|_{\mathbf{H}^1_0(\Omega)}). \tag{3.3.39}$$

Once again use triangle inequality and (3.3.31) (i), in order to prove (3.3.29).
Finally use (3.3.29) in (3.3.28) to show (3.3.27). This completes the proof. □

The following lemma is a simple consequence of the fundamental theorem of calculus, whose proof is left to the reader.

Lemma 3.3.7. *Fix $i \geqslant 0$ and a domain Ω_0 in $\mathbb{R}^d$ (d is either 1 or 2). Then there exists a constant $K > 0$ such that for all $\psi \in H^1(0,T;H^i(\Omega_0))$, the following holds*

$$\|\psi\|_{L^\infty(0,T;H^i(\Omega_0))} \leqslant K(\|\psi(0)\|_{H^i(\Omega_0)} + T^{1/2}\|\psi_t\|_{L^2(0,T;H^i(\Omega_0))}), \tag{3.3.40}$$

where $\psi(0)$ denotes ψ at time $t = 0$. The inequality (3.3.40) is true even for a vector valued function $\Psi \in H^1(0,T;\mathbf{H}^i(\Omega_0))$.

3.3.2.2 Estimates of G_1, G_2, G_3 and $\widetilde{W}$

Lemma 3.3.8. *Let B_0^* and T_0^* are as in Lemma 3.3.2 and $B_i \geqslant B_0^*$ ($\forall\, 1 \leqslant i \leqslant 4$). Then there exist $K_1 = K_1(B_1,B_2,B_3,B_4) > 0$ and $K_2 > 0$ such that for all $0 < T \leqslant T_0^*(B_1,B_2,B_3,B_4)$ and $(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) \in \mathcal{C}_T(B_1,B_2,B_3,B_4)$, $G_1(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ (defined in (3.1.22)) satisfies the following estimates*

$$\begin{aligned}
&(i) \quad \|G_1(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})\|_{L^1(0,T;H^2(\Omega))} \leqslant K_1(B_1,B_2,B_3,B_4)T^{1/2},\\
&(ii) \quad \|G_1(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})\|_{L^\infty(0,T;H^1(\Omega))} \leqslant K_2\|\rho_0 \mathrm{div}(\mathbf{u}_0)\|_{H^1(\Omega)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + K_1(B_1,B_2,B_3,B_4)T^{1/2}.
\end{aligned} \tag{3.3.41}$$

Remark 3.3.9. *In (3.3.41), the constant K_2 does not depend on any of the B_i ($1 \leqslant i \leqslant 4$).*

Proof of Lemma 3.3.8. (i) We will first prove (3.3.41)(i).
Estimate of $(\widetilde{\sigma}+\overline{\rho})\mathrm{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)$ in $L^1(0,T;H^2(\Omega))$: From (3.3.8), we get that $(\widetilde{\sigma}+\overline{\rho}) \in L^\infty(0,T;H^2(\Omega))$ and $(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2) \in L^2(0,T;\mathbf{H}^2(\Omega))$. Hence we have the following inequality

$$\begin{aligned}
&\|(\widetilde{\sigma}+\overline{\rho})\mathrm{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)\|_{L^1(0,T;H^2(\Omega))}\\
&\leqslant K(\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\|(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)\|_{L^1(0,T;\mathbf{H}^3(\Omega))}) \quad \text{(using Lemma 3.3.4)}\\
&\leqslant KT^{1/2}(\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\|(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)\|_{L^2(0,T;\mathbf{H}^3(\Omega))}) \quad \text{(using Hölder's inequality)}\\
&\leqslant K(B_1,B_3,B_4)T^{1/2}, \quad \text{(using (3.3.16a), (3.3.16b), (3.3.16c)).}
\end{aligned} \tag{3.3.42}$$

Estimate of $F_1(\widetilde{\sigma}+\overline{\rho},\widetilde{w}+z\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta})$ (defined in (3.1.10)) in $L^1(0,T;H^2(\Omega))$: First observe that, as $\widetilde{\eta} \in L^\infty(0,T;H^{9/2}(\Gamma_s))$ and (3.3.16d) holds, one can verify the following

$$\frac{1}{(1+\widetilde{\eta})} \in L^\infty(0,T;H^{9/2}(\Gamma_s)) \tag{3.3.43}$$

and

$$\left\|\frac{1}{(1+\widetilde{\eta})}\right\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} \leqslant K\|\widetilde{\eta}\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} \leqslant K(B_4), \quad \text{(using (3.3.16c))}. \tag{3.3.44}$$

Hence we get the following estimate of $\dfrac{z\widetilde{\eta}_x(\widetilde{\sigma}+\overline{\rho})(\widetilde{w}_1+z\widetilde{\eta}_t\vec{e_2})_z}{(1+\widetilde{\eta})}$,

$$\begin{aligned}
&\left\|\frac{\widetilde{\eta}_x(\widetilde{\sigma}+\overline{\rho})(\widetilde{w}_1+z\widetilde{\eta}_t\vec{e_2})_z}{(1+\widetilde{\eta})}\right\|_{L^1(0,T;H^2(\Omega))}\\
&\leqslant K(\|\widetilde{\eta}_x\|_{L^\infty(0,T;H^{7/2}(\Gamma_s))}\left\|\frac{1}{(1+\widetilde{\eta})}\right\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))} \qquad (3.3.45)\\
&\quad \|(\widetilde{w}_1+z\widetilde{\eta}_t\vec{e_2})_z\|_{L^1(0,T;H^2(\Omega))}) \qquad \text{(using Lemma 3.3.4)}\\
&\leqslant K(B_1,B_3,B_4)T^{1/2}, \quad \text{(using Hölder's inequality and (3.3.16a), (3.3.16b), (3.3.16c)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and (3.3.44))}.
\end{aligned}$$

Rest of the terms in the expression of $F_1(\widetilde{\sigma}+\overline{\rho},\widetilde{w}+z\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta})$ can be estimated in a similar way. Hence we can show the following

$$\|F_1(\widetilde{\sigma}+\overline{\rho},\widetilde{w}+z\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta})\|_{L^1(0,T;H^2(\Omega))} \leqslant K(B_1,B_3,B_4)T^{1/2}. \tag{3.3.46}$$

We combine (3.3.42) and (3.3.46) to prove (3.3.41)(i).

(ii) We will now prove (3.3.41)(ii).
Estimate of $(\widetilde{\sigma}+\overline{\rho})\text{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)$ in $L^\infty(0,T;H^1(\Omega))$: We observe the following

$$\begin{aligned}
&\|((\widetilde{\sigma}+\overline{\rho})\text{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2))_t\|_{L^2(0,T;H^1(\Omega))}\\
&\leqslant K(\|\widetilde{\sigma}_t\text{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)\|_{L^2(0,T;H^1(\Omega))}+\|(\widetilde{\sigma}+\overline{\rho})\text{div}(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e}_2)\|_{L^2(0,T;H^1(\Omega))})\\
&\leqslant K(\|\widetilde{\sigma}_t\|_{L^\infty(0,T;H^1(\Omega))}\|\text{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)\|_{L^2(0,T;H^2(\Omega))}\\
&+\|\widetilde{\sigma}+\overline{\rho}\|_{L^\infty(0,T;H^2(\Omega))}\|\text{div}(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e}_2)\|_{L^2(0,T;H^1(\Omega))}) \quad \text{(using Lemma 3.3.4)}\\
&\leqslant K(B_1,B_2,B_3,B_4), \quad \text{(using (3.3.16a), (3.3.16b), (3.3.16c))}.
\end{aligned} \tag{3.3.47}$$

Now apply the inequality (3.3.40) with $\psi=(\widetilde{\sigma}+\overline{\rho})\text{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)$. We obtain

$$\begin{aligned}
&\|(\widetilde{\sigma}+\overline{\rho})\text{div}(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2)\|_{L^\infty(0,T;H^1(\Omega))}\\
&\leqslant K\|\rho_0\text{div}(\mathbf{u}_0)\|_{H^1(\Omega)}+T^{1/2}K(B_1,B_2,B_3,B_4), \quad \text{(using (3.3.47))}.
\end{aligned} \tag{3.3.48}$$

Estimate of $F_1(\widetilde{\sigma}+\overline{\rho},\widetilde{w}+z\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta})$ in $L^\infty(0,T;H^1(\Omega))$: We can have the following

estimate

$$
\begin{aligned}
&\left\|\frac{\widetilde{\eta}_x(\widetilde{\sigma}+\overline{\rho})(\widetilde{w}_1+z\widetilde{\eta}_t\vec{e_2})_z}{(1+\widetilde{\eta})}\right\|_{L^\infty(0,T;H^1(\Omega))}\\
&\leqslant K(\|\widetilde{\eta}_x\|_{L^\infty(0,T;H^2(\Gamma_s))}\left\|\frac{1}{(1+\widetilde{\eta})}\right\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\\
&\qquad\|(\widetilde{w}_1+z\widetilde{\eta}_t\vec{e_2})_z\|_{L^\infty(0,T;H^1(\Omega))}) \qquad \text{(using Lemma 3.3.4)}\\
&\leqslant KT^{1/2}(\|\widetilde{\eta}_{xt}\|_{L^2(0,T;H^2(\Gamma_s))}\left\|\frac{1}{(1+\widetilde{\eta})}\right\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\\
&\qquad\|(\widetilde{w}_1+z\widetilde{\eta}_t\vec{e_2})_z\|_{L^\infty(0,T;H^1(\Omega))})\\
&\text{(using (3.3.40) with } \psi=\widetilde{\eta}_x \text{ and the fact that } \widetilde{\eta}_x(0)=0)\\
&\leqslant K(B_1,B_3,B_4)T^{1/2}, \quad \text{(using (3.3.16a),(3.3.16b),(3.3.16c) and (3.3.44)).}
\end{aligned}
\tag{3.3.49}
$$

A similar analysis can be applied to estimate other summands of $F_1(\widetilde{\sigma}+\overline{\rho},\widetilde{w}+z\widetilde{\eta}_t\vec{e_2},\widetilde{\eta})$. Hence we can now show that

$$
\|F_1(\widetilde{\sigma}+\overline{\rho},\widetilde{w}+z\widetilde{\eta}_t\vec{e_2},\widetilde{\eta})\|_{L^\infty(0,T;H^1(\Omega))}\leqslant K(B_1,B_2,B_4)T^{1/2}. \tag{3.3.50}
$$

Combine (3.3.48) with (3.3.50) to show (3.3.41)(ii). □

Lemma 3.3.10. *Let B_0^* and T_0^* are as in Lemma 3.3.2 and $B_i\geqslant B_0^*$ $(\forall\, 1\leqslant i\leqslant 4)$. Then there exist $K_3=K_3(B_1,B_2,B_3,B_4)>0$, $K_4=K_4(B_1,B_4)>0$ and $K_5>0$ such that for all $0<T\leqslant T_0^*(B_1,B_2,B_3,B_4)$ and $(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})\in\mathcal{C}_T(B_1,B_2,B_3,B_4)$, $G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ (defined in (3.1.22)) satisfies the following estimates*

$$
\begin{aligned}
&(i) \quad \|G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\leqslant K_3(B_1,B_2,B_3,B_4)T^{1/2},\\
&(ii) \quad \|(G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}))_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\leqslant K_3(B_1,B_2,B_3,B_4)T^{1/2}+K_4(B_1,B_4),\\
&(iii) \quad \|G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}\leqslant K_5\|G_2^0\|_{\mathbf{L}^2(\Omega)}+K_3(B_1,B_2,B_3,B_4)T^{1/2}.
\end{aligned}
\tag{3.3.51}
$$

Remark 3.3.11. *The estimates in (3.3.51) are inspired from the results stated in [38, p. 269] which is done in absence of the beam unknown η but includes the evolution of the temperature of the fluid.*
We further emphasize that the constant K_4 does not depend on (B_2,B_3) and K_5 does not depend on any of the B_i $(1\leqslant i\leqslant 4)$.

Proof of Lemma 3.3.10. One can use (3.3.16e) to show that for $\gamma>1$,

$$
(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}\in C^0([0,T];H^2(\Omega)) \text{ and } (\widetilde{\sigma}+\overline{\rho})^{\gamma-2}\in C^0([0,T];H^2(\Omega))
$$

and

$$
\begin{cases}
\|(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}\|_{L^\infty(0,T;H^2(\Omega))}\leqslant K\|\widetilde{\sigma}\|_{L^\infty(0,T;H^2(\Omega))}\leqslant K(B_1),\\
\|(\widetilde{\sigma}+\overline{\rho})^{\gamma-2}\|_{L^\infty(0,T;H^2(\Omega))}\leqslant K\|\widetilde{\sigma}\|_{L^\infty(0,T;H^2(\Omega))}\leqslant K(B_1).
\end{cases}
\tag{3.3.52}
$$

(i) We first estimate $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ in $L^2(0,T;\mathbf{H}^1(\Omega))$.
Estimate of $P'(\widetilde{\sigma}+\overline{\rho})\nabla\widetilde{\sigma}$ in $L^2(0,T;\mathbf{H}^1(\Omega))$:

$$
\begin{aligned}
&\|P'(\widetilde{\sigma}+\overline{\rho})\nabla\widetilde{\sigma}\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\\
&\leqslant T^{1/2}\|P'(\widetilde{\sigma}+\overline{\rho})\nabla\widetilde{\sigma}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}\\
&\leqslant T^{1/2}K(\|(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}\|_{L^\infty(0,T;H^2(\Omega))}\|\nabla\widetilde{\sigma}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}) \qquad (3.3.53)\\
&\qquad\qquad \text{(using the definition of } P \text{ and Lemma 3.3.4)}\\
&\leqslant K(B_1)T^{1/2}, \quad \text{(using (3.3.16a))}.
\end{aligned}
$$

Estimate of $z\widetilde{\eta}_{tt}(\widetilde{\sigma}+\overline{\rho})\vec{e}_2 - (\mu\Delta+(\mu+\mu')\nabla\mathrm{div})(z\widetilde{\eta}_t\vec{e}_2)$ in $L^2(0,T;\mathbf{H}^1(\Omega))$:

$$
\begin{aligned}
&\|z\widetilde{\eta}_{tt}(\widetilde{\sigma}+\overline{\rho})\vec{e}_2 - (\mu\Delta+(\mu+\mu')\nabla\mathrm{div})(z\widetilde{\eta}_t\vec{e}_2)\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\\
&\leqslant T^{1/2}\|z\widetilde{\eta}_{tt}(\widetilde{\sigma}+\overline{\rho})\vec{e}_2 - (\mu\Delta+(\mu+\mu')\nabla\mathrm{div})(z\widetilde{\eta}_t\vec{e}_2)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}\\
&\leqslant T^{1/2}K(\|\widetilde{\eta}_{tt}\|_{L^\infty(0,T;H^1(\Omega))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))} + \|\widetilde{\eta}_t\|_{L^\infty(0,T;H^3(\Omega))}) \qquad (3.3.54)\\
&\qquad\qquad \text{(using Lemma 3.3.4)}\\
&\leqslant K(B_1,B_4)T^{1/2}, \quad \text{(using (3.3.16a), (3.3.16c))}.
\end{aligned}
$$

Estimate of $F_2(\widetilde{\sigma}+\overline{\rho}, \widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2, \widetilde{\eta})$ (defined in (3.1.10)) in $L^2(0,T;\mathbf{H}^1(\Omega))$: We will only estimate the terms of $F_2(\widetilde{\sigma}+\overline{\rho}, \widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e}_2, \widetilde{\eta})$ which are the most intricate to deal with. The others are left to the reader.

(a)
$$
\begin{aligned}
&\|\widetilde{\eta}(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e}_2)\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\\
&\leqslant T^{1/2}\|\widetilde{\eta}(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e}_2)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \qquad (3.3.55)\\
&\leqslant T^{1/2}K(\|\widetilde{\eta}\|_{L^\infty(0,T;H^2(\Gamma_s))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\|(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e}_2)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))})\\
&\qquad\qquad \text{(using Lemma 3.3.4)}\\
&\leqslant K(B_1,B_3,B_4)T^{1/2}, \quad \text{(using (3.3.16a), (3.3.16b), (3.3.16c))}.
\end{aligned}
$$

(b)
$$
\begin{aligned}
&\|z(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_z+\widetilde{\eta}_t\vec{e}_2)\widetilde{\eta}_t\|_{L^2(0,T;\mathbf{H}^1(\Omega))}\\
&\leqslant T^{1/2}\|z(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_z+\widetilde{\eta}_t\vec{e}_2)\widetilde{\eta}_t\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}\\
&\leqslant T^{1/2}K(\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\|(\widetilde{\mathbf{w}}_z+\widetilde{\eta}_t\vec{e}_2)\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}\|\widetilde{\eta}_t\|_{L^\infty(0,T;H^2(\Gamma_s))})\\
&\qquad\qquad \text{(using Lemma 3.3.4)}\\
&\leqslant K(B_1,B_3,B_4)T^{1/2}, \quad \text{(using (3.3.16a), (3.3.16b), (3.3.16c))}. \qquad (3.3.56)
\end{aligned}
$$

$$
\begin{aligned}
(c)\quad & \left\| \frac{\widetilde{\mathbf{w}}_{zz} z^2 \eta_x^2}{(1+\widetilde{\eta})} \right\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \\
&\leqslant K(\|\widetilde{\mathbf{w}}_{zz}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \|\widetilde{\eta}_x^2\|_{L^\infty(0,T;H^2(\Gamma_s))} \left\| \frac{1}{(1+\widetilde{\eta})} \right\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))}) \\
&\qquad\qquad \text{(using Lemma 3.3.4)} \\
&\leqslant T^{1/2} K(\|\widetilde{\mathbf{w}}_{zz}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \|\widetilde{\eta}_{xt}^2\|_{L^2(0,T;H^2(\Gamma_s))} \left\| \frac{1}{(1+\widetilde{\eta})} \right\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))}) \\
&\qquad\qquad \text{(using (3.3.40) with } \psi = \widetilde{\eta}_x^2 \text{ and the fact } \widetilde{\eta}_x(,0) = 0) \\
&\leqslant K(B_3, B_4) T^{1/2}, \quad \text{(using (3.3.16a), (3.3.16b), (3.3.16c) and (3.3.44)).}
\end{aligned} \tag{3.3.57}
$$

(d) Using arguments similar to that in the computation (3.3.53) we show the following

$$
\|(\widetilde{\eta} P' \widetilde{\sigma}_x - P' \widetilde{\sigma}_z z \widetilde{\eta}_x)\vec{e_1}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leqslant K(B_1, B_4) T^{1/2}. \tag{3.3.58}
$$

Now the reader can deal with the other terms using similar arguments in order to prove

$$
\|F_2(\widetilde{\sigma} + \overline{\rho}, (\widetilde{\mathbf{w}} + z\widetilde{\eta}_t \vec{e_2}), \widetilde{\eta})\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leqslant K(B_1, B_3, B_4) T^{1/2}. \tag{3.3.59}
$$

Combining the estimates (3.3.53), (3.3.54) and (3.3.59) we conclude the proof of the inequality (3.3.51)(i).

(ii) We now estimate $\|(G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}))_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$.
Estimate of $(P'(\widetilde{\sigma} + \overline{\rho})\nabla\widetilde{\sigma})_t$ in $L^2(0,T;\mathbf{L}^2(\Omega))$:

$$
\begin{aligned}
&\|(P'(\widetilde{\sigma} + \overline{\rho})\nabla\widetilde{\sigma})_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \\
&\leqslant T^{1/2} \|(P'(\widetilde{\sigma} + \overline{\rho})\nabla\widetilde{\sigma})_t\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \\
&\leqslant T^{1/2} K(\|(\widetilde{\sigma} + \overline{\rho})^{(\gamma-2)} \widetilde{\sigma}_t \nabla\widetilde{\sigma}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} + \|(\widetilde{\sigma} + \overline{\rho})^{\gamma-1} \nabla\widetilde{\sigma}_t\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}) \\
&\leqslant T^{1/2} K(\|(\widetilde{\sigma} + \overline{\rho})^{\gamma-2}\|_{L^\infty(0,T;H^2(\Omega))} \|\widetilde{\sigma}_t\|_{L^\infty(0,T;H^1(\Omega))} \|\nabla\widetilde{\sigma}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \\
&\qquad + \|(\widetilde{\sigma} + \overline{\rho})^{\gamma-1}\|_{L^\infty(0,T;H^2(\Omega))} \|\nabla\widetilde{\sigma}_t\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}) \qquad \text{(using Lemma 3.3.4)} \\
&\leqslant K(B_1, B_2) T^{1/2}, \quad \text{(using (3.3.16a) and (3.3.52)).}
\end{aligned} \tag{3.3.60}
$$

Estimate of $(z\widetilde{\eta}_{tt}(\widetilde{\sigma} + \overline{\rho})\vec{e_2} - (\mu\Delta + (\mu + \mu')\nabla\mathrm{div})(z\widetilde{\eta}_t\vec{e_2}))_t$ in $L^2(0,T;\mathbf{L}^2(\Omega))$:

$$
\begin{aligned}
&\|(z\widetilde{\eta}_{tt}(\widetilde{\sigma} + \overline{\rho})\vec{e_2} - (\mu\Delta + (\mu + \mu')\nabla\mathrm{div})(z\widetilde{\eta}_t\vec{e_2}))_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \\
&\leqslant T^{1/2} K(\|\widetilde{\eta}_{tt}\|_{L^\infty(0,T;H^1(\Gamma_s))} \|\widetilde{\sigma}_t\|_{L^\infty(0,T;H^1(\Omega))}) + K(\|\widetilde{\eta}_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \\
&\qquad \cdot \|(\widetilde{\sigma} + \overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))} + \|\widetilde{\eta}_{tt}\|_{L^2(0,T;H^2(\Gamma_s))}) \qquad \text{(using Lemma 3.3.4)} \\
&\leqslant K(B_2, B_4) T^{1/2} + K(B_1, B_4).
\end{aligned} \tag{3.3.61}
$$

Estimate of $(F_2(\widetilde{\sigma}+\overline{\rho},(\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e_2}),\widetilde{\eta}))_t$ in $L^2(0,T;\mathbf{L}^2(\Omega))$:

(a) $\|(\widetilde{\eta}(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e_2}))_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$

$$\begin{aligned}
&\leqslant K(\|(\widetilde{\eta}_t(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e_2}))\|_{L^2(0,T;\mathbf{L}^2(\Omega))}+\|\widetilde{\eta}\widetilde{\sigma}_t(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e_2})\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\\
&\quad+\|(\widetilde{\eta}(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_{tt}+z\widetilde{\eta}_{ttt}\vec{e_2}))\|_{L^2(0,T;\mathbf{L}^2(\Omega))})\\
&\leqslant T^{1/2}K(\|\widetilde{\eta}_t\|_{L^\infty(0,T;H^2(\Gamma_s))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\|(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e_2})\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}\\
&\quad+\|\widetilde{\eta}\|_{L^\infty(0,T;H^2(\Gamma_s))}\|\widetilde{\sigma}_t\|_{L^\infty(0,T;H^1(\Omega))}\|(\widetilde{\mathbf{w}}_t+z\widetilde{\eta}_{tt}\vec{e_2})\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))})\qquad(3.3.62)\\
&\quad+\|\widetilde{\eta}\|_{L^\infty(0,T;H^2(\Gamma_s))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\|(\widetilde{\mathbf{w}}_{tt}+z\widetilde{\eta}_{ttt}\vec{e_2})\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\\
&\leqslant T^{1/2}K(B_1,B_2,B_3,B_4)+T^{1/2}\|\widetilde{\eta}_t\|_{L^2(0,T;H^2(\Gamma_s))}\|(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^2(\Omega))}\\
&\quad\|(\widetilde{\mathbf{w}}_{tt}+z\widetilde{\eta}_{ttt}\vec{e_2})\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\\
&\quad(\text{using (3.3.40) with } \psi=\widetilde{\eta} \text{ and the fact } \widetilde{\eta}(,0)=0)\\
&\leqslant K(B_1,B_2,B_3,B_4)T^{1/2}.
\end{aligned}$$

(b) Using similar estimates we can have the following

$$\|(z(\widetilde{\sigma}+\overline{\rho})(\widetilde{\mathbf{w}}_z+\widetilde{\eta}_t\vec{e_2})\eta_t)_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\leqslant K(B_1,B_2,B_3,B_4)T^{1/2}.\qquad(3.3.63)$$

(c) Now we estimate

$$\left\|\left(\frac{\widetilde{\mathbf{w}}_{zz}z^2\widetilde{\eta}_x^2}{(1+\widetilde{\eta})}\right)_t\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))}.$$

To start with, we have the following identity of distributional derivatives

$$\left(\frac{\widetilde{\mathbf{w}}_{zz}z^2\widetilde{\eta}_x^2}{(1+\widetilde{\eta})}\right)_t=\frac{z^2\widetilde{\mathbf{w}}_{tzz}\widetilde{\eta}_x^2}{(1+\widetilde{\eta})}+\frac{2\widetilde{\eta}_x\widetilde{\eta}_{xt}\widetilde{\mathbf{w}}_{zz}}{(1+\widetilde{\eta})}-\frac{\widetilde{\mathbf{w}}_{zz}z^2\widetilde{\eta}_x^2\widetilde{\eta}_t}{(1+\widetilde{\eta})^2}.\qquad(3.3.64)$$

We now estimate the first term of the summands. Using (3.3.44) one obtains

$$\left\|\frac{z^2\widetilde{\mathbf{w}}_{tzz}\widetilde{\eta}_x^2}{(1+\widetilde{\eta})}\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\leqslant K(B_4)(\|\widetilde{\mathbf{w}}_{tzz}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\|\widetilde{\eta}_x\|^2_{L^\infty(0,T;L^\infty(\Gamma_s))}).\qquad(3.3.65)$$

Now we use inequality (3.3.40) and $\widetilde{\eta}_x(\cdot,0)=0$ to get

$$\|\widetilde{\eta}_x\|_{L^\infty(0,T;L^\infty(\Gamma_s))}\leqslant C\|\widetilde{\eta}_x\|_{L^\infty(0,T;H^2(\Gamma_s))}\leqslant K(B_3)T^{1/2}.\qquad(3.3.66)$$

Hence we use (3.3.66) in (3.3.65) to obtain

$$\left\|\frac{z^2\widetilde{\mathbf{w}}_{tzz}\widetilde{\eta}_x^2}{(1+\widetilde{\eta})}\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))}\leqslant K(B_3,B_4)T.\qquad(3.3.67)$$

For the second and third summands of (3.3.64), we similarly obtain:

$$\begin{aligned}\left\|\frac{2\widetilde{\eta}_x\widetilde{\eta}_{xt}\widetilde{\mathbf{w}}_{zz}}{(1+\widetilde{\eta})}\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))} &\leqslant K(B_3,B_4)T^{1/2},\\ \left\|-\frac{\widetilde{\mathbf{w}}_{zz}z^2\widetilde{\eta}_x^2\widetilde{\eta}_t}{(1+\widetilde{\eta})^2}\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))} &\leqslant K(B_3,B_4)T.\end{aligned} \tag{3.3.68}$$

So altogether we get

$$\left\|\left(\frac{\widetilde{\mathbf{w}}_{zz}z^2\widetilde{\eta}_x^2}{(1+\widetilde{\eta})}\right)_t\right\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant K(B_3,B_4)T^{1/2}. \tag{3.3.69}$$

The remaining terms in the expression of F_2 are relatively easier to deal with and hence we leave the details to the reader to show

$$\|(F_2(\widetilde{\sigma}+\overline{\rho},\widetilde{\mathbf{w}}+z\widetilde{\eta}_t\vec{e_2},\widetilde{\eta}))_t\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant K(B_1,B_2,B_3,B_4)T^{1/2}. \tag{3.3.70}$$

Hence combining the estimates (3.3.60), (3.3.61) and (3.3.70) one gets (3.3.51)(ii).

(iii) In (3.3.40) replace ψ by $G_2(\widetilde{\mathbf{w}},\widetilde{\sigma},\widetilde{\eta})$ and use the estimate (3.3.51)(ii) to prove (3.3.51)(iii). □

Lemma 3.3.12. *Let B_0^* and T_0^* are as in Lemma 3.3.2 and $B_i \geqslant B_0^*$ $(\forall\, 1 \leqslant i \leqslant 4)$. Then there exist $K_6 > 0$ and $K_7 = K_7(B_1,B_2,B_3,B_4) > 0$ such that for all $0 < T \leqslant T_0^*(B_1,B_2,B_3,B_4)$ and $(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}) \in \mathcal{C}_T(B_1,B_2,B_3,B_4)$, $G_3(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ (defined in (3.1.22)) satisfies the following estimates*

$$\begin{aligned}&(i)\|G_3(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))} \leqslant K_6\|(\rho_0,\mathbf{u}_0)\|_{H^2(\Omega)\times\mathbf{H}^2(\Omega)} + K_7(B_1,B_2,B_3,B_4)T^{1/2},\\ &(ii)\|(G_3(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta}))_t\|_{L^2(0,T;L^2(\Gamma_s))} \leqslant T^{1/6}K_7(B_1,B_2,B_3,B_4).\end{aligned} \tag{3.3.71}$$

Remark 3.3.13. *We emphasize that K_6 does not depend on any of the B_i $(1 \leqslant i \leqslant 4)$.*

Proof. In this proof we will consider the function $\widetilde{\mathbf{w}}$ and $(\widetilde{\sigma}+\overline{\rho})$ on Γ_s, i.e we take the trace of these functions and make use of well known trace theorem without mentioning it explicitly.

(i) Estimate of $F_3(\widetilde{\sigma}+\overline{\rho},\widetilde{\mathbf{w}}+\widetilde{\eta}_t\vec{e_2},\widetilde{\eta})$ (defined in (3.1.10)) in $L^\infty(0,T;H^{1/2}(\Gamma_s))$:

(a) First let us estimate $(\widetilde{w}_{2,z}+\widetilde{\eta}_t)$ in $L^\infty(0,T;H^{1/2}(\Gamma_s))$:

$$\begin{aligned}\|(\widetilde{w}_{2,z}+\widetilde{\eta}_t)\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))} &\leqslant K(\|\mathbf{u}_0\|_{H^2(\Omega)} + T^{1/2}\|(\widetilde{w}_{2,z}+\widetilde{\eta}_t)_t\|_{L^2(0,T;H^{1/2}(\Gamma_s))})\\ &\qquad\qquad\qquad

(b) Let us estimate $P(\widetilde{\sigma}+\overline{\rho})$ in $L^\infty(0,T;H^{1/2}(\Gamma_s))$.

$$\begin{aligned}\|P(\widetilde{\sigma}+\overline{\rho})\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))} &\leqslant K(\|\rho_0\|_{H^2(\Omega)}+T^{1/2}\|(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}\widetilde{\sigma}_t\|_{L^2(0,T;H^{1/2}(\Gamma_s))}ds)\\ &\qquad\qquad\text{(using (3.3.40))}\\ &\leqslant K(\|\rho_0\|_{H^2(\Omega)}+T^{1/2}K(B_1,B_2)),\qquad\text{(using (3.3.16a)).}\end{aligned}\tag{3.3.73}$$

(c) Now we estimate $\dfrac{\widetilde{\eta}_x\widetilde{w}_{1,z}}{(1+\widetilde{\eta})}$ in $L^\infty(0,T;H^{1/2}(\Gamma_s))$,

$$\begin{aligned}\|\frac{\widetilde{\eta}_x\widetilde{w}_{1,z}}{(1+\widetilde{\eta})}\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))} &\leqslant K(B_4)(\|\widetilde{\eta}_x\|_{L^\infty(0,T;H^2(\Gamma_s))}\|\widetilde{w}_{1,z}\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))})\\ &\qquad\text{(using (3.3.44))}\\ &\leqslant T^{1/2}K(B_3,B_4),\qquad\text{(using (3.3.40) with } \psi=\widetilde{\eta}_x \text{ and } \widetilde{\eta}_x(.,0)=0).\end{aligned}\tag{3.3.74}$$

We use similar sort of arguments to show that

$$\|F_3(\widetilde{\sigma}+\overline{\rho},\widetilde{\mathbf{w}}+\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta})\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))}\leqslant K\|\rho_0\|_{H^2(\Gamma_s)}+K(B_1,B_2,B_3,B_4)T^{1/2}.\tag{3.3.75}$$

Combine (3.3.72) with (3.3.75) to prove (3.3.71)(i).

(ii) Estimate of $(F_3(\widetilde{\sigma}+\overline{\rho},\widetilde{\mathbf{w}}+\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta}))_t$ (defined in (3.1.10)) in $L^2(0,T;L^2(\Gamma_s))$:
First let us estimate $(\widetilde{w}_{2,z}+\widetilde{\eta}_t\vec{e}_2)_t$ in $L^2(0,T;L^2(\Gamma_s))$:

$$\begin{aligned}&\|(\widetilde{w}_{2,z}+\widetilde{\eta}_t)_t\|_{L^2(0,T;L^2(\Gamma_s))}\\ &\leqslant\|\widetilde{w}_{2,tz}\|_{L^2(0,T;L^2(\Gamma_s))}+\|\widetilde{\eta}_{tt}\|_{L^2(0,T;L^2(\Gamma_s))}\\ &\leqslant T^{1/6}K(\|\widetilde{\mathbf{w}}_t(\cdot,0)\|_{\mathbf{H}^1_0(\Omega)}+\|\widetilde{\mathbf{w}}_t\|_{\mathbf{H}^{2,1}_{\Sigma_T}(Q_T)})+T^{1/2}K\|\widetilde{\eta}_{tt}\|_{L^\infty(0,T;L^2(\Gamma_s))}\\ &\qquad\qquad\text{(using Lemma 3.3.5 with } \mathbf{f} \text{ replaced by } \widetilde{\mathbf{w}}_t)\\ &\leqslant T^{1/6}K(\Big\|\frac{1}{\rho_0}(G_2\mid_{t=0}-(-\mu\Delta-(\mu+\mu')\nabla\mathrm{div})(\mathbf{u}_0-z\eta_1\vec{e_2}))\Big\|_{\mathbf{H}^1_0(\Omega)}\\ &\qquad+K(B_3))+T^{1/2}K(B_4).\qquad\text{(using (3.3.17d) and the inequalities}\\ &\qquad\qquad\text{(3.3.16b) and (3.3.16c))}\end{aligned}\tag{3.3.76}$$

Using similar line of arguments one can prove that the trace of $(F_3(\widetilde{\sigma}+\overline{\rho},\widetilde{\mathbf{w}}+\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta}))_t$ on Γ_s belongs to $L^2(0,T;L^2(\Gamma_s))$ and the following inequality is true for $T<1$,

$$\|(F_3(\widetilde{\sigma}+\overline{\rho},\widetilde{\mathbf{w}}+\widetilde{\eta}_t\vec{e}_2,\widetilde{\eta}))_t\|_{L^2(0,T;L^2(\Gamma_s))}\leqslant T^{1/6}K(B_1,B_2,B_3,B_4).\tag{3.3.77}$$

Combining (3.3.76) and (3.3.77), we conclude (3.3.71)(ii). □

Lemma 3.3.14. *Let B_0^* and T_0^* are as in Lemma 3.3.2 and $B_i \geqslant B_0^*$ $(\forall\, 1 \leqslant i \leqslant 4)$. Then there exist $K_8 = K_8(B_1, B_2, B_3, B_4) > 0$, and $K_9 = K_9(B_3, B_4) > 0$ such that for all $0 < T \leqslant T_0^*(B_1, B_2, B_3, B_4)$ and $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \in \mathcal{C}_T(B_1, B_2, B_3, B_4)$, we have the following estimates (recall the notation $\widetilde{W}$ from (3.3.4))*

$$\begin{aligned}
&(i) \quad \|\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta})\|_{L^1(0,T;\mathbf{H}^3(\Omega))} \leqslant K_8(B_1, B_2, B_3, B_4)T^{1/2},\\
&(ii) \quad \|\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta})\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} \leqslant K_9(B_3, B_4) + K_8(B_1, B_2, B_3, B_4)T^{1/2},\\
&(iii) \quad \|\widetilde{\sigma}_t\|_{L^2(0,T;L^3(\Omega))} + \|\nabla\widetilde{\sigma}\|_{L^2(0,T;L^3(\Omega))} \leqslant K_8(B_1, B_2, B_3, B_4)T^{1/2}.
\end{aligned} \tag{3.3.78}$$

Remark 3.3.15. *We emphasize that K_9 does not depend on B_1 and B_2.*

Proof. (i) One can use Lemma 3.3.4 to check that $\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta}) \in L^2(0, T; \mathbf{H}^3(\Omega))$. As a consequence, (3.3.78)(i) follows.

(ii) The following estimates follow from the regularity of $\widetilde{\mathbf{w}}$.

$$\|\widetilde{w}_1\|_{L^\infty(0,T;H^2(\Omega))} \leqslant K(B_3), \tag{3.3.79}$$

$$\left\|\frac{1}{(1+\widetilde{\eta})}\widetilde{w}_2\right\|_{L^\infty(0,T;H^2(\Omega))} \leqslant K(B_3, B_4), \qquad (\text{using } (3.3.44)) \tag{3.3.80}$$

and

$$\begin{aligned}
&\left\|\frac{1}{(1+\widetilde{\eta})}\widetilde{w}_1 z\widetilde{\eta}_x\right\|_{L^\infty(0,T;H^2(\Omega))}\\
&\leqslant K(B_4)(\|\widetilde{\mathbf{w}}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))}\|\widetilde{\eta}_x\|_{L^\infty(0,T;H^2(\Gamma_s))})\\
&\qquad\qquad (\text{using } (3.3.44) \text{ and Lemma } 3.3.4)\\
&\leqslant T^{1/2}K(B_4)(\|\widetilde{\mathbf{w}}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))}\|\widetilde{\eta}_{xt}\|_{L^2(0,T;H^2(\Gamma_s))})\\
&\qquad\qquad (\text{using } (3.3.40) \text{ with } \psi = \widetilde{\eta}_x \text{ and } \widetilde{\eta}_x(.,0) = 0)\\
&\leqslant T^{1/2}K(B_3, B_4).
\end{aligned} \tag{3.3.81}$$

Combine (3.3.79), (3.3.80) and (3.3.81) to prove (3.3.78)(ii).

(iii) From the definition of $\mathcal{C}_T(B_1, B_2, B_3, B_4)$ we know that $\widetilde{\sigma}_t$ is in $L^\infty(0, T; H^1(\Omega))$ and $\nabla\widetilde{\sigma}$ belongs to $L^\infty(0, T; \mathbf{H}^1(\Omega))$. Hence one uses the continuous embedding $H^1(\Omega) \hookrightarrow L^3(\Omega)$ to obtain that the embedding from $L^\infty(0, T; H^1(\Omega)) \hookrightarrow L^2(0, T; L^3(\Omega))$ has a norm of size $\sqrt{T}$. We then easily derive (3.3.78)(iii). □

3.3.2.3 Choices of B_1, B_2, B_3 and B_4

Now we will choose the constants $B_i \geqslant B_0^*$ $(\leqslant i \leqslant 4)$ such that for a small enough time $0 < T \leqslant T_0^*(B_1, B_2, B_3, B_4)$, L maps $\mathcal{C}_T(B_1, B_2, B_3, B_4)$ into itself.

Lemma 3.3.16. *Let B_0^* and T_0^* are as in Lemma 3.3.2. There exist constants $B_i \geqslant B_0^*$ $(1 \leqslant i \leqslant 4)$ and a time $T^*(B_1, B_2, B_3, B_4)$ satisfying $0 < T^*(B_1, B_2, B_3, B_4) \leqslant T_0^*(B_1, B_2, B_3, B_4)$ such that for all $0 < T \leqslant T^*(B_1, B_2, B_3, B_4)$, L maps $\mathcal{C}_T(B_1, B_2, B_3, B_4)$ into itself.*

Proof. In the following we will fix B_i $(1 \leqslant i \leqslant 4)$ in a hierarchical order. We use the constants B_0^* (Lemma 3.3.2), c_4 (Theorem 3.2.7), K_6 (Lemma 3.3.12), c_1 (Theorem 3.2.1), K_4 (Lemma 3.3.10), K_5 (Lemma 3.3.10), c_3 (Theorem 3.2.4), K_9 (Lemma 3.3.14) and K_2 (Lemma 3.3.8). First we set B_1 and B_4 as follows

$$\begin{cases} B_1 = \max\{2(\|\sigma_0\|_{H^2(\Omega)} + 1), B_0^*\}, \\ B_4 = \max\{c_4(\|\eta_1\|_{H^3(\Gamma_s)} + \|G_3 \mid_{t=0}\|_{H^1(\Gamma_s)} \\ \qquad\qquad + K_6\|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)\times\mathbf{H}^2(\Omega)} + 1), B_0^*\}. \end{cases} \tag{3.3.82}$$

Now using B_1 and B_4 we choose B_2 and B_3 in the following order.

$$\begin{aligned} B_3 = \max\{c_1(2 + K_5\|G_2 \mid_{t=0}\|_{\mathbf{L}^2(\Omega)} + 4(1 + K_4(B_1, B_4) \\ + \left\| \frac{G_2 \mid_{t=0} -(-\mu\Delta - (\mu+\mu')\nabla\mathrm{div})(\mathbf{u}_0 - z\eta_1\vec{e_2})}{\rho_0} \right\|_{\mathbf{H}^1(\Omega)}\!\!)), B_0^*\}, \end{aligned} \tag{3.3.83}$$

and

$$B_2 = \max\{c_3K_9(B_3, B_4)\|\sigma_0\|_{H^2(\Omega)} + K_2\|\rho_0\mathrm{div}(\mathbf{u}_0)\|_{H^1(\Omega)} + 1, B_0^*\}. \tag{3.3.84}$$

In the rest of the proof we verify that with the choices (3.3.82), (3.3.83) and (3.3.84) of B_i $(\forall\, 1 \leqslant i \leqslant 4)$, there exists a time $T^*(B_1, B_2, B_3, B_4)$ such that for all $0 < T \leqslant T^*(B_1, B_2, B_3, B_4)$, L maps $\mathcal{C}_T(B_1, B_2, B_3, B_4)$ into itself.
Let $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \in \mathcal{C}_T(B_1, B_2, B_3, B_4)$ and $L(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) = (\sigma, \mathbf{w}, \eta)$. From Theorem 3.2.1, Theorem 3.2.4 and Theorem 3.2.7 we know that $(\sigma, \mathbf{w}, \eta)$ satisfies the following inequalities with

$$(G_1, G_2, G_3) = (G_1(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}), G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}), G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})).$$

$$\begin{cases} \|\mathbf{w}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} + \|\mathbf{w}\|_{L^2(0,T;\mathbf{H}^3(\Omega))} + \|\mathbf{w}_t\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{w}_t\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \\ + \|\mathbf{w}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant c_1\{\|G_2\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \|G_2\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))} \\ + (\|G_{2,t}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} + \left\| \frac{G_2 \mid_{t=0} -(-\mu\Delta - (\mu+\mu')\nabla\mathrm{div})(\mathbf{u}_0 - z\eta_1\vec{e_2})}{\rho_0} \right\|_{\mathbf{H}^1(\Omega)}\!\!) \\ \cdot (1 + \|\widetilde{\sigma}_t\|_{L^2(0,T;L^3(\Omega))} + \|\nabla\widetilde{\sigma}\|_{L^2(0,T;L^3(\Omega))})\exp(c_1\|\widetilde{\sigma}_t\|^2_{L^2(0,T;L^3(\Omega))})\}, \end{cases} \tag{3.3.85}$$

$$\begin{cases} \|\sigma\|_{L^\infty(0,T;H^2(\Omega))} \leqslant (\|\sigma_0\|_{H^2(\Omega)} + c_2\|G_1\|_{L^1(0,T;H^2(\Omega))}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \cdot \exp(c_2\|\widetilde{W}\|_{L^1(0,T;\mathbf{H}^3(\Omega))}), \\ \|\sigma_t\|_{L^\infty(0,T;H^1(\Omega))} \leqslant c_3\|\widetilde{W}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))}[(\|\sigma_0\|_{H^2(\Omega)} + c_2\|G_1\|_{L^1(0,T;H^2(\Omega))}) \\ \qquad\qquad\qquad \cdot \exp(c_2\|\widetilde{W}\|_{L^1(0,T;\mathbf{H}^3(\Omega))})] + \|G_1\|_{L^\infty(0,T;H^1(\Omega))}, \end{cases} \tag{3.3.86}$$

and

$$\begin{cases} \|\eta\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} + \|\eta_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_t\|_{L^\infty(0,T;H^3(\Gamma_s))} \\ + \|\eta_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{tt}\|_{L^\infty([0,T];H^1(\Gamma_s))} + \|\eta_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \\ \leqslant c_4(\|\eta_1\|_{H^3(\Gamma_s)} + \|G_3\ |_{t=0}\ \|_{H^1(\Gamma_s)} + \|G_3\|_{L^\infty(0,T;H^{1/2}(\Gamma_s))} \\ \qquad + \|G_{3,t}\|_{L^2(0,T;L^2(\Gamma_s))}). \end{cases} \tag{3.3.87}$$

(i) Using the estimate (3.3.71) on $G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ in (3.3.87) we obtain:

$$\begin{cases} \|\eta\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} + \|\eta_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_t\|_{L^\infty(0,T;H^3(\Gamma_s))} \\ + \|\eta_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{tt}\|_{L^\infty([0,T];H^1(\Gamma_s))} + \|\eta_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \\ \leqslant c_4(\|\eta_1\|_{H^3(\Gamma_s)} + \|G_3\ |_{t=0}\ \|_{H^1(\Gamma_s)} + K_6\|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)\times\mathbf{H}^2(\Omega)} \\ \qquad + K_7(B_1, B_2, B_3, B_4)T^{1/2} + T^{1/6}K_7(B_1, B_2, B_3, B_4). \end{cases} \tag{3.3.88}$$

Now choose $T_1^* = T_1^*(B_1, B_2, B_3, B_4)(\leqslant T_0^*(B_1, B_2, B_3, B_4))$, positive and small enough such that

$$K_7(B_1, B_2, B_3, B_4)(T_1^*)^{1/2} + K_7(B_1, B_2, B_3, B_4)(T_1^*)^{1/6} < 1. \tag{3.3.89}$$

In view of the choice of B_4 (see (3.3.82)) and (3.3.89), for all $0 < T \leqslant T_1^*$ one verifies that

$$\begin{cases} \|\eta\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} + \|\eta_t\|_{L^2(0,T;H^4(\Gamma_s))} + \|\eta_t\|_{L^\infty(0,T;H^3(\Gamma_s))} \\ + \|\eta_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} + \|\eta_{tt}\|_{L^\infty([0,T];H^1(\Gamma_s))} + \|\eta_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \leqslant B_4. \end{cases} \tag{3.3.90}$$

(ii) Using the estimates (3.3.41)(i) on $G_1(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ and (3.3.78)(i) on $\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta})$ in $(3.3.86)_1$ furnish

$$\begin{aligned} \|\sigma\|_{L^\infty(0,T;H^2(\Omega))} \leqslant\ & (\|\sigma_0\|_{H^2(\Omega)} + c_2K_1(B_1, B_2, B_3, B_4)T^{1/2}) \\ & \exp(c_2K_8(B_1, B_2, B_3, B_4)T^{1/2}). \end{aligned} \tag{3.3.91}$$

Choose $T_2^* = T_2^*(B_1, B_2, B_3, B_4)(\leqslant T_1^*)$, positive and small enough such that

$$\begin{aligned} & c_2K_1(B_1, B_2, B_3, B_4)(T_2^*)^{1/2} < 1, \\ & \exp(c_2K_8(B_1, B_2, B_3, B_4)(T_2^*)^{1/2}) < 2. \end{aligned} \tag{3.3.92}$$

In view of the choice of B_1 (see (3.3.82)) and (3.3.92), for all $0 < T \leqslant T_2^*$ the following holds

$$\|\sigma\|_{L^\infty(0,T;H^2(\Omega))} \leqslant B_1. \tag{3.3.93}$$

(iii) Using the estimates (3.3.51) on $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ and (3.3.78)(iii) on $\widetilde{\sigma}$ in (3.3.85) to obtain

$$\begin{cases} \|\mathbf{w}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} + \|\mathbf{w}\|_{L^2(0,T;\mathbf{H}^3(\Omega))} + \|\mathbf{w}_t\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\mathbf{w}_t\|_{L^2(0,T;\mathbf{H}^2(\Omega))} \\ + \|\mathbf{w}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant c_1\{K_5\|G_2\mid_{t=0}\|_{\mathbf{L}^2(\Omega)} + 2K_3(B_1,B_2,B_3,B_4)T^{1/2} \\ + (K_3(B_1,B_2,B_3,B_4)T^{1/2} + K_4(B_1,B_4) \\ + \left\|\frac{G_2\mid_{t=0} -(-\mu\Delta - (\mu+\mu')\nabla\mathrm{div})(\mathbf{u}_0 - z\eta_1\vec{e_2})}{\rho_0}\right\|_{\mathbf{H}^1(\Omega)}) \cdot (1 \\ + K_8(B_1,B_2,B_3,B_4)T^{1/2}) \cdot \exp(c_1K_8^2(B_1,B_2,B_3,B_4)T)\}. \end{cases} \tag{3.3.94}$$

Choose $T_3^* = T_3^*(B_1,B_2,B_3,B_4)(\leqslant T_2^*(B_1,B_2,B_3,B_4))$, positive and small enough such that

$$\begin{aligned} &K_3(B_1,B_2,B_3,B_4)(T_3^*)^{1/2} < 1, \\ &\text{and } (1 + K_8(B_1,B_2,B_3,B_4)(T_3^*)^{1/2})\exp(c_1K_8^2(B_1,B_2,B_3,B_4)T_3^*) < 4. \end{aligned} \tag{3.3.95}$$

In view of the choice of B_3 (see (3.3.83)) and (3.3.95), for all $0 < T \leqslant T_3^*$ we have

$$\begin{aligned} \|\mathbf{w}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} + \|\mathbf{w}\|_{L^2(0,T;\mathbf{H}^3(\Omega))} + \|\mathbf{w}_t\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} \\ + \|\mathbf{w}_t\|_{L^2(0,T;\mathbf{H}^2(\Omega))} + \|\mathbf{w}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} < B_3. \end{aligned} \tag{3.3.96}$$

(iv) Using the estimates (3.3.41) on $G_1(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ and (3.3.78)(i)-(3.3.78)(ii) on $\widetilde{W}(\widetilde{\mathbf{w}}, \widetilde{\eta})$ in $(3.3.86)_2$ furnish

$$\begin{aligned} \|\sigma_t\|_{L^\infty(0,T;H^1(\Omega))} &\leqslant c_3(K_9(B_3,B_4) + K_8(B_1,B_2,B_3,B_4)T^{1/2})[(\|\sigma_0\|_{H^2(\Omega)} \\ &+ c_2K_1(B_1,B_2,B_3,B_4)T^{1/2}) \cdot \exp(c_2K_8(B_1,B_2,B_3,B_4)T^{1/2})] + K_2\|\rho_0\mathrm{div}(\mathbf{u}_0)\|_{H^1(\Omega)} \\ &+ K_1(B_1,B_2,B_3,B_4)T^{1/2}. \end{aligned} \tag{3.3.97}$$

Choose $T_4^* = T_4^*(B_1,B_2,B_3,B_4)(\leqslant T_3^*)$, positive and small enough such that

$$\begin{aligned} &c_2c_3K_9(B_3,B_4)K_1(B_1,B_2,B_3,B_4)(T_4^*)^{1/2} \cdot \exp(c_2K_8(B_1,B_2,B_3,B_4)(T_4^*)^{1/2}) \\ &+ K_8(B_1,B_2,B_3,B_4)(T_4^*)^{1/2}[(\|\sigma_0\|_{H^2(\Omega)} + c_2K_1(B_1,B_2,B_3,B_4)(T_4^*)^{1/2}) \\ &\cdot \exp(c_2K_8(B_1,B_2,B_3,B_4)(T_4^*)^{1/2})] + K_1(B_1,B_2,B_3,B_4)(T_4^*)^{1/2} < 1. \end{aligned} \tag{3.3.98}$$

In view of the choice of B_2 (see (3.3.84)) and (3.3.98), we check that for all $0 < T \leqslant T_4^*$ the following holds

$$\|\sigma_t\|_{L^\infty(0,T;H^1(\Omega))} < B_2. \tag{3.3.99}$$

Hence with the choices (3.3.82), (3.3.83) and (3.3.84) of the constants B_i $(1 \leqslant i \leqslant 4)$, $(\sigma, \mathbf{w}, \eta)$ satisfies the estimates (3.3.93), (3.3.99), (3.3.96) and (3.3.90) respectively for all $0 < T \leqslant T_4^*$. We can also use similar kind of interpolation arguments as used in (3.3.25) to show that there exists a $T_5^* = T_5^*(B_1, B_2, B_3, B_4)$ $(\leqslant T_4^*)$ such that for all $0 < T \leqslant T_5^*$,

$$\begin{aligned} &1 + \eta(x,t) \geqslant \delta_0 > 0, && \text{on} \quad \Sigma_T^s \\ &\frac{m}{2} \leqslant \sigma(x,z,t) + \overline{\rho} \leqslant 2M, && \text{in} \quad Q_T. \end{aligned} \tag{3.3.100}$$

Again it follows from the equation $(3.3.3)_2$ that $\mathbf{w}_t(0)$ satisfies the condition (3.3.17d). Similarly one uses $(3.3.3)_6$ to show that $\eta_{tt}(\cdot, 0)$ satisfies (3.3.17c). Now we set

$$T^* = T^*(B_1, B_2, B_3, B_4) = T_5^*.$$

Hence if B_i $(\forall\, 1 \leqslant i \leqslant 4)$ is chosen as in (3.3.82), (3.3.83) and (3.3.84) and $0 < T \leqslant T^*$, $(\sigma, \mathbf{w}, \eta)$ satisfies all the conditions (3.3.16)-(3.3.17), guaranteeing $(\sigma, \mathbf{w}, \eta) \in \mathcal{C}_T(B_1, B_2, B_3, B_4)$.
This concludes the proof of Lemma 3.3.16. □

We fix the choice of B_i $(\forall\, 1 \leqslant i \leqslant 4)$ and $T = T^*(B_1, B_2, B_3, B_4)$ as in Lemma 3.3.16. Hence in the following we will simply use the notations

$$T = T^* \text{ and } \mathcal{C}_T = \mathcal{C}_T(B_1, B_2, B_3, B_4). \tag{3.3.101}$$

3.3.3 Compactness and continuity

Let us observe that $\mathcal{C}_T$ is a convex, bounded subset of the space

$$\mathcal{X} = \{(\sigma, \mathbf{w}, \eta) \in C^0([0,T], H^1(\Omega)) \times C^0([0,T]; \mathbf{H}^1(\Omega)) \times C^1([0,T]; H^1(\Gamma_s)) \cap C^0([0,T]; H^2(\Gamma_s))\},$$

endowed with the topology induced by the norm

$$\|(\sigma, \mathbf{w}, \eta)\|_{\mathcal{X}} = \sup_{t \in [0,T]} (\|\sigma(t)\|_{H^1(\Omega)} + \|\mathbf{w}(t)\|_{\mathbf{H}^1(\Omega)} + \|\eta(t)\|_{H^2(\Gamma_s)} + \|\eta_t(t)\|_{H^1(\Gamma_s)}).$$

Lemma 3.3.17. *Let $\mathcal{C}_T$ be the set as introduced in (3.3.101). The set $\mathcal{C}_T$, when endowed with the topology of $\mathcal{X}$, is compact in $\mathcal{X}$.*

Proof. We claim that the set $\mathcal{C}_T$ is closed in $\mathcal{X}$.
Assume that a sequence $(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n) \in \mathcal{C}_T$ and that $(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n) \to (\overline{\sigma}, \overline{\mathbf{w}}, \overline{\eta})$ in $\mathcal{X}$. Now $\widetilde{\eta}_n \to \overline{\eta}$ in $C^1([0,T]; H^1(\Gamma_s))$ implies that $\widetilde{\eta}_{n,t} \to \overline{\eta}_t$ in $C^0([0,T]; H^1(\Gamma_s))$, $\widetilde{\eta}_{n,tt} \to \overline{\eta}_{tt}$ and $\widetilde{\eta}_{n,ttt} \to \overline{\eta}_{ttt}$ in $\mathcal{D}'(0,T; L^2(\Gamma_s))$ in particular, where $\mathcal{D}'(0,T; L^2(\Gamma_s))$ denotes the space of distributions on $(0,T)$ with values in $L^2(\Gamma_s)$.

We recall the norm bounds over η in the set $\mathcal{C}_T$. Hence we have up to a subsequence (still denoted by $\widetilde{\eta}_n$) that $\widetilde{\eta}_n \to \overline{\eta}$ weak* in $L^\infty(0,T;H^{9/2}(\Gamma_s))$, $\widetilde{\eta}_{n,t} \to \overline{\eta}_t$ weakly in $L^2(0,T;H^4(\Gamma_s))$ and weak* in $L^\infty(0,T;H^3(\Gamma_s))$, $\widetilde{\eta}_{n,tt} \to \overline{\eta}_{tt}$ weakly in $L^2(0,T;H^2(\Gamma_s))$ and weak* in $L^\infty(0,T;H^1(\Gamma_s))$, $\widetilde{\eta}_{n,ttt} \to \overline{\eta}_{ttt}$ weakly in $L^2(0,T;L^2(\Gamma_s))$. Also by the lower semi-continuity of the norms with respect to the above weak type convergences we get

$$\begin{aligned}&\|\overline{\eta}\|_{L^\infty(0,T;H^{9/2}(\Gamma_s))} + \|\overline{\eta}_t\|_{L^\infty(0,T;H^3(\Gamma_s))} + \|\overline{\eta}_t\|_{L^2(0,T;H^4(\Gamma_s))}\\ &+ \|\overline{\eta}_{tt}\|_{L^\infty(0,T;H^1(\Gamma_s))} + \|\overline{\eta}_{tt}\|_{L^2(0,T;H^2(\Gamma_s))} + \|\overline{\eta}_{ttt}\|_{L^2(0,T;L^2(\Gamma_s))} \leqslant B_4.\end{aligned} \tag{3.3.102}$$

As $\widetilde{\eta}_n \to \overline{\eta}$ in $C^1([0,T];H^1(\Gamma_s))$ and $\widetilde{\eta}_{n,t} \to \overline{\eta}_t$ in $C^0([0,T];H^1(\Gamma_s))$, hence

$$\overline{\eta}(\cdot,0) = 0 \quad \text{and} \quad \overline{\eta}_t(0) = \eta_1. \tag{3.3.103}$$

The uniform bounds of $\|\widetilde{\eta}_{n,tt}\|_{L^\infty(0,T;H^1(\Gamma_s))}$ and $\|\widetilde{\eta}_{n,ttt}\|_{L^2(0,T;L^2(\Gamma_s))}$ and Aubin Lions lemma ([1]) furnish that up to a subsequence (still denoted by $\widetilde{\eta}_n$), $\widetilde{\eta}_{n,tt}$ strongly converges to $\overline{\eta}_{tt}$ in $C^0([0,T];L^2(\Gamma_s))$. Hence

$$\overline{\eta}_{tt}(\cdot,0) = \delta\eta_{1,xx} - (\mu+2\mu')(u_0)_{2,z} + P(\rho_0). \tag{3.3.104}$$

Similar arguments (used to show (3.3.102)) can be used to show that

$$\begin{aligned}\|\overline{\mathbf{w}}\|_{L^\infty(0,T;\mathbf{H}^2(\Omega))} + \|\overline{\mathbf{w}}\|_{L^2(0,T;\mathbf{H}^3(\Omega))} \quad &+\|\overline{\mathbf{w}}_t\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))} + \|\overline{\mathbf{w}}_t\|_{L^2(0,T;\mathbf{H}^2(\Omega))}\\ &+\|\overline{\mathbf{w}}_{tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega))} \leqslant B_3,\end{aligned} \tag{3.3.105}$$

$$\|\overline{\sigma}\|_{L^\infty(0,T;H^2(\Omega))} \leqslant B_1, \quad \|\overline{\sigma}_t\|_{L^\infty(0,T;H^1(\Omega))} \leqslant B_2. \tag{3.3.106}$$

Since $\widetilde{\eta}_n$ converges to $\overline{\eta}$ in $L^\infty(\Sigma_T^s)$ (follows from the continuous embedding $H^2(\Gamma_s) \hookrightarrow L^\infty(\Gamma_s)$), one has the following (as $\widetilde{\eta}_n$ satisfies (3.3.16d))

$$1 + \overline{\eta}(x,t) \geqslant \delta_0 > 0 \quad \text{on} \quad \Sigma_T^s. \tag{3.3.107}$$

Observe that the weak* convergence of $\widetilde{\sigma}_n$ to $\overline{\sigma}$ in $L^\infty(0,T;L^2(\Omega))$ is enough to conclude that (since $\widetilde{\sigma}_n$ satisfies (3.3.16e))

$$\frac{m}{2} \leqslant \overline{\sigma}(x,z,t) + \overline{\rho} \leqslant 2M \quad \text{in} \quad Q_T. \tag{3.3.108}$$

Using the strong convergence of $(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)$ to $(\overline{\sigma}, \overline{\mathbf{w}}, \overline{\eta})$ in $\mathcal{X}$ furnishes

$$\begin{aligned}\overline{\mathbf{w}}(\cdot,0) &= (\mathbf{u}_0 - z\eta_1\vec{e_2}) \quad &&\text{in} \quad \Omega,\\ \overline{\sigma}(\cdot,0) &= \sigma_0 &&\text{in} \quad \Omega.\end{aligned} \tag{3.3.109}$$

Now we can use the uniform bounds of $\|\widetilde{\mathbf{w}}_{n,t}\|_{L^\infty(0,T;\mathbf{H}^1(\Omega))}$ and $\|\widetilde{\mathbf{w}}_{n,tt}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$ and the Aubin Lions lemma to have the convergence $\widetilde{\mathbf{w}}_{n,t} \to \overline{\mathbf{w}}_t$ in $C^0([0,T];\mathbf{L}^2(\Omega))$. Consequently

$$\overline{\mathbf{w}}_t(.,0) = \frac{1}{\rho_0}(G_2^0 - (-\mu\Delta - (\mu+\mu')\nabla\mathrm{div})(\mathbf{u}_0 - z\eta_1\vec{e}_2)). \tag{3.3.110}$$

So combining (3.3.102)-(3.3.103)-(3.3.104)-(3.3.105)-(3.3.106)-(3.3.107)-(3.3.108)-(3.3.109)-(3.3.110) we conclude that the limit point $(\overline{\sigma}, \overline{\mathbf{w}}, \overline{\eta}) \in \mathcal{C}_T$ and hence $\mathcal{C}_T$ is closed in $\mathcal{X}$. Once again using Aubin Lions lemma we get that $\mathcal{C}_T$ is a compact subset of $\mathcal{X}$. □

Now to apply Schauder's fixed point theorem one only needs to prove that L is continuous on $\mathcal{C}_T$.

Lemma 3.3.18. *Let $\mathcal{C}_T$ be the set in (3.3.101). The map L is continuous from $\mathcal{C}_T$ into itself for the topology of $\mathcal{X}$.*

Proof. Suppose that $(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n) \in \mathcal{C}_T$, converges to $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ strongly in $\mathcal{X}$. Then, according to Lemma 3.3.17, $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta}) \in \mathcal{C}_T$. We thus set $(\widehat{\sigma}_n, \widehat{\mathbf{w}}_n, \widehat{\eta}_n) = L(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)$, $(\widehat{\sigma}, \widehat{\mathbf{w}}, \widehat{\eta}) = L(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$. Our goal is to show that $(\widehat{\sigma}_n, \widehat{\mathbf{w}}_n, \widehat{\eta}_n)$ strongly converges to $(\widehat{\sigma}, \widehat{\mathbf{w}}, \widehat{\eta})$ in $\mathcal{X}$. Using that $(\widehat{\sigma}_n, \widehat{\mathbf{w}}_n, \widehat{\eta}_n)$ belongs to $\mathcal{C}_T$ (see Lemma 3.3.16) we get that there exists a triplet $(\overline{\sigma}, \overline{\mathbf{w}}, \overline{\eta})$ such that up to a subsequence

$$\begin{aligned}
&\widehat{\sigma}_n \overset{*}{\rightharpoonup} \overline{\sigma} && \text{in } L^\infty(0,T;H^2(\Omega)) \cap W^{1,\infty}(0,T;H^1(\Omega)),\\
&\widehat{\mathbf{w}}_n \rightharpoonup \overline{\mathbf{w}} && \text{in } L^2(0,T;\mathbf{H}^3(\Omega)) \cap H^1(0,T;\mathbf{H}^2(\Omega)) \cap H^2(0,T;\mathbf{L}^2(\Omega)),\\
&\widehat{\mathbf{w}}_n \overset{*}{\rightharpoonup} \overline{\mathbf{w}} && \text{in } L^\infty(0,T;\mathbf{H}^2(\Omega)) \cap W^{1,\infty}(0,T;\mathbf{H}^1(\Omega)),\\
&\widehat{\eta}_n \rightharpoonup \overline{\eta} && \text{in } H^1(0,T;H^4(\Gamma_s)) \cap H^2(0,T;H^2(\Gamma_s)) \cap H^3(0,T;L^2(\Gamma_s)),\\
&\widehat{\eta}_n \overset{*}{\rightharpoonup} \overline{\eta} && \text{in } L^\infty(0,T;H^{9/2}(\Gamma_s)) \cap W^{1,\infty}(0,T;H^3(\Gamma_s)) \cap W^{2,\infty}(0,T;H^1(\Gamma_s)).
\end{aligned} \tag{3.3.111}$$

The compactness result proved in Lemma 3.3.17 provides the strong convergence in $\mathcal{X}$ i.e, up to a subsequence, $(\widehat{\sigma}_n, \widehat{\mathbf{w}}_n, \widehat{\eta}_n)$ converges strongly in $\mathcal{X}$ to $(\overline{\sigma}, \overline{\mathbf{w}}, \overline{\eta})$. It is clear that in order to prove that the map L is continuous it is enough to show that $(\overline{\sigma}, \overline{\mathbf{w}}, \overline{\eta}) = (\widehat{\sigma}, \widehat{\mathbf{w}}, \widehat{\eta})$. This will be verified in the following steps.
(i) We first claim that $G_2(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)$ converges weakly to $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ in $L^2(0,T;\mathbf{L}^2(\Omega))$. Since $(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)$ belongs to $\mathcal{C}_T$ and we have fixed B_i (for all $1 \leqslant i \leqslant 4$) and T, one can use Lemma 3.3.10 to show that $\|G_2(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)\|_{L^2(0,T;L^2(\Omega))}$ is uniformly bounded. Hence, to prove our claim it is enough to show that $G_2(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)$ converges to $G_2(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ in $\mathcal{D}'(Q_T)$ ($\mathcal{D}'(Q_T)$ is the space of distributions on Q_T).
Let us consider the term $\dfrac{\widetilde{\mathbf{w}}_{n,zz} z^2 \widetilde{\eta}_{n,x}^2}{(1+\widetilde{\eta}_n)}$. From the uniform norm bound over $\|\widetilde{\mathbf{w}}_{n,zz}\|_{L^2(0,T;\mathbf{H}^1(\Omega))}$ we get that $\widetilde{\mathbf{w}}_{n,zz}$ converges weakly in $L^2(0,T;\mathbf{H}^1(\Omega))$ to $\widetilde{\mathbf{w}}_{zz}$. Since $\widetilde{\eta}_n$ strongly converges to $\widetilde{\eta}$ in $C^0([0,T];H^2(\Gamma_s))$ and both $\widetilde{\eta}_n$ and $\widetilde{\eta}$ satisfy (3.3.16d), $\dfrac{1}{(1+\widetilde{\eta}_n)}$ and $\widetilde{\eta}_{n,x}$ converge strongly to $\dfrac{1}{(1+\widetilde{\eta})}$ and $\widetilde{\eta}_x$ respectively in the spaces $C^0([0,T];H^2(\Gamma_s))$ and $C^0([0,T];H^1(\Gamma_s))$. Hence one gets in particular the strong convergence of $\widetilde{\eta}_{n,x}^2$ to $\widetilde{\eta}_x^2$ in the space $C^0([0,T];L^2(\Gamma_s))$. This implies that $\dfrac{\widetilde{\mathbf{w}}_{n,zz} z^2 \widetilde{\eta}_{n,x}^2}{(1+\widetilde{\eta}_n)}$ converges to $\dfrac{\widetilde{\mathbf{w}}_{zz} z^2 \widetilde{\eta}_x^2}{(1+\widetilde{\eta})}$ weakly in $L^2(0,T;L^1(\Omega))$ and hence particularly in the space $\mathcal{D}'(Q_T)$.
Now we consider the term $P'\widetilde{\sigma}_{n,z} z \widetilde{\eta}_{n,x} \vec{e}_1 = (\widetilde{\sigma}_n + \overline{\rho})^{\gamma-1} \widetilde{\sigma}_{n,z} z \widetilde{\eta}_{n,x} \vec{e}_1$. Since $\|(\widetilde{\sigma}_n + \overline{\rho})\|_{C^0(0,T;H^2(\Omega))}$

is uniformly bounded so is $\|(\widetilde{\sigma}_n+\overline{\rho})^{\gamma-1}\|_{C^0(0,T;H^2(\Omega))}$ and hence $(\widetilde{\sigma}_n+\overline{\rho})^{\gamma-1}$ converges weakly to $(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}$ in $L^2(0,T;H^2(\Omega))$. We also have that $\widetilde{\sigma}_{n,z}$ converges strongly to $\widetilde{\sigma}_z$ in $C^0([0,T];L^2(\Omega))$. Hence $(\widetilde{\sigma}_n+\overline{\rho})^{\gamma-1}\widetilde{\sigma}_{n,z}$ converges weakly to $(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}\widetilde{\sigma}_z$ in $L^2(0,T;L^2(\Omega))$. Now the strong convergence of $\widetilde{\eta}_{n,x}$ to $\widetilde{\eta}_x$ in $C^0([0,T];H^1(\Gamma_s))$ furnish that $(\widetilde{\sigma}_n+\overline{\rho})^{\gamma-1}\widetilde{\sigma}_{n,z}z\widetilde{\eta}_{n,x}$ weakly converges to $(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}\widetilde{\sigma}_z z\widetilde{\eta}_x$ in $L^2(0,T;L^1(\Omega))$. Hence $(\widetilde{\sigma}_n+\overline{\rho})^{\gamma-1}\widetilde{\sigma}_{n,z}z\widetilde{\eta}_{n,x}\vec{e}_1$ converges to $(\widetilde{\sigma}+\overline{\rho})^{\gamma-1}\widetilde{\sigma}_z z\widetilde{\eta}_x\vec{e}_1$ in the space $\mathcal{D}'(Q_T)$.
We can apply similar line of arguments to prove that $G_2(\widetilde{\sigma}_n,\widetilde{\mathbf{w}}_n,\widetilde{\eta}_n)$ converges to $G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ in $\mathcal{D}'(Q_T)$. Hence we have proved that $G_2(\widetilde{\sigma}_n,\widetilde{\mathbf{w}}_n,\widetilde{\eta}_n)$ converges to $G_2(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ weakly in $L^2(0,T;\mathbf{L}^2(\Omega))$.
Also observe that $(\widetilde{\sigma}_n+\overline{\rho})$ converges strongly to $(\widetilde{\sigma}+\overline{\rho})$ in $C^0([0,T];H^1(\Omega))$ and $\widehat{\mathbf{w}}_{n,t}$, $(-\mu\Delta-(\mu'+\mu)\nabla(\mathrm{div}))\widehat{\mathbf{w}}_n$ converge up to a subsequence weakly to $\overline{\mathbf{w}}_t$ and $(-\mu\Delta-(\mu'+\mu)\nabla(\mathrm{div}))\overline{\mathbf{w}}$ respectively in the spaces $L^2(0,T;\mathbf{H}^2(\Omega))$ and $L^2(0,T;\mathbf{H}^1(\Omega))$. Hence up to a subsequence one obtains in particular the following convergence

$$(\widetilde{\sigma}_n+\overline{\rho})\widehat{\mathbf{w}}_{n,t}-\mu\Delta\widehat{\mathbf{w}}_n-(\mu'+\mu)\nabla(\mathrm{div}\widehat{\mathbf{w}}_n)\rightharpoonup(\widetilde{\sigma}+\overline{\rho})\overline{\mathbf{w}}_t-\mu\Delta\overline{\mathbf{w}}-(\mu'+\mu)\nabla(\mathrm{div}\overline{\mathbf{w}})$$
$$\text{in } L^2(0,T;\mathbf{L}^2(\Omega)).$$

Now consider $(3.3.3)_2$ with $(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ and $\mathbf{w}$ replaced respectively by $(\widetilde{\sigma}_n,\widetilde{\mathbf{w}}_n,\widetilde{\eta}_n)$ and $\widehat{\mathbf{w}}_n$. The weak convergences discussed so far allow to pass to the limits in both sides of this equation. So using the uniqueness of weak solution for the linear problem (3.2.1) we conclude that $\overline{\mathbf{w}}=\widehat{\mathbf{w}}$.
(ii) Now we claim that $G_1(\widetilde{\sigma}_n,\widetilde{\mathbf{w}}_n,\widetilde{\eta}_n)$ converges weakly to $G_1(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ in $L^2(0,T;L^2(\Omega))$. Let us consider the term $\dfrac{1}{(1+\widetilde{\eta}_n)}(\widetilde{w}_n)_{1,z}z\widetilde{\eta}_{n,x}(\widetilde{\sigma}_n+\overline{\rho})$. We already know that $\dfrac{1}{(1+\widetilde{\eta}_n)}$ and $\widetilde{\eta}_{n,x}$ converge strongly to $\dfrac{1}{(1+\widetilde{\eta})}$ and $\widetilde{\eta}_x$ respectively in the spaces $C^0([0,T];H^2(\Gamma_s)$ and $C^0([0,T];H^1(\Gamma_s))$. One also observes that $(\widetilde{w}_n)_{1,z}$ weakly converges to $\widetilde{w}_{1,z}$ in $L^2(0,T;H^2(\Omega))$ (since $\widetilde{\mathbf{w}}_n\rightharpoonup\widetilde{\mathbf{w}}$ in $L^2(0,T;\mathbf{H}^3(\Omega))$). Finally the strong convergence of $(\widetilde{\sigma}_n+\overline{\rho})$ to $(\widetilde{\sigma}+\overline{\rho})$ in $C^0([0,T];H^1(\Omega))$ furnish the weak convergence of $\dfrac{1}{(1+\widetilde{\eta}_n)}(\widetilde{w}_n)_{1,z}z\widetilde{\eta}_{n,x}(\widetilde{\sigma}_n+\overline{\rho})$ to $\dfrac{1}{(1+\widetilde{\eta})}(\widetilde{w})_{1,z}z\widetilde{\eta}_x(\widetilde{\sigma}+\overline{\rho})$ in $L^2(0,T;L^2(\Omega))$. We can apply similar arguments for other terms in the expression of $G_1(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ in order to prove the weak convergence of $G_1(\widetilde{\sigma}_n,\widetilde{\mathbf{w}}_n,\widetilde{\eta}_n)$ to $G_1(\widetilde{\sigma},\widetilde{\mathbf{w}},\widetilde{\eta})$ in $L^2(0,T;L^2(\Omega))$.
We further observe that $\nabla\widehat{\sigma}_n$ strongly converges to $\nabla\overline{\sigma}$ in $C^0([0,T];L^2(\Omega))$. Since $(\widetilde{w}_n)_1$ weakly converges to $\widetilde{w}_1$ in $L^2(0,T;H^3(\Omega))$, $(\widetilde{\eta}_n)_x$ strongly converges to $\widetilde{\eta}_x$ in $L^\infty(\Sigma_T^s)$ (because $(\widetilde{\eta}_n)_x$ strongly converges to $\widetilde{\eta}_x$ in $C^0([0,T];H^1(\Gamma_s))$ and the embedding $H^1(\Gamma_s)\hookrightarrow L^\infty(\Gamma_s)$ is continuous) and $\frac{1}{(1+\widetilde{\eta}_n)}$ strongly converges to $\frac{1}{(1+\widetilde{\eta})}$ in $C^0([0,T];H^2(\Gamma_s))$, $\frac{1}{(1+\widetilde{\eta}_n)}(\widetilde{w}_n)_1z(\widetilde{\eta}_n)_x(\widehat{\sigma}_n)_z$ weakly converges to $\frac{1}{(1+\widetilde{\eta})}\widetilde{w}_1z\widetilde{\eta}_x\widehat{\sigma}_z$ in $L^2(0,T;L^2(\Omega))$. Besides, up to a subsequence $(\widehat{\sigma}_n)_t$ weakly converges to $\overline{\sigma}_t$ in $L^2(0,T;L^2(\Omega))$. Hence up to a subsequence we have

$$(\widehat{\sigma}_n)_t+\begin{bmatrix}(\widetilde{w}_n)_1\\ \frac{1}{(1+\widetilde{\eta}_n)}((\widetilde{w}_n)_2-(\widetilde{w}_n)_1z(\widetilde{\eta}_n)_x)\end{bmatrix}\cdot\nabla\widehat{\sigma}_n\rightharpoonup\overline{\sigma}_t+\begin{bmatrix}\widetilde{w}_1\\ \frac{1}{(1+\widetilde{\eta})}(\widetilde{w}_2-\widetilde{w}_1z\widetilde{\eta}_x)\end{bmatrix}\cdot\nabla\overline{\sigma}$$
$$\text{in}\quad L^2(0,T;L^2(\Omega)).$$

Now consider $(3.3.3)_1$ with $(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ and σ replaced respectively by $(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)$ and $\widehat{\sigma}_n$. The weak type convergences discussed so far allow to pass to the limits in both sides of this equation. Hence from uniqueness of weak solution of the linear problem (3.2.6) we conclude that $\overline{\sigma} = \widehat{\sigma}$.
(iii) One can use similar line of arguments as used so far to show that $G_3(\widetilde{\sigma}_n, \widetilde{\mathbf{w}}_n, \widetilde{\eta}_n)$ converges weakly to $G_3(\widetilde{\sigma}, \widetilde{\mathbf{w}}, \widetilde{\eta})$ in $L^2(0,T;L^2(\Gamma_s))$. Using the norm bounds of $\widehat{\eta}_n$ (since $(\widehat{\sigma}_n, \widehat{\mathbf{w}}_n, \widehat{\eta}_n) \in \mathcal{C}_T$) we can prove that up to a subsequence the left hand side of $(3.3.3)_6$ with η replaced by $\widehat{\eta}_n$ converges weakly to

$$\overline{\eta}_{tt} - \beta\overline{\eta}_{xx} - \delta\overline{\eta}_{txx} + \alpha\overline{\eta}_{xxxx}$$

in $L^2(0,T;L^2(\Gamma_s))$. Now the uniqueness of weak solution to the problem (3.2.17) furnishes $\overline{\eta} = \widehat{\eta}$. Hence the proof of Lemma 3.3.18 is complete. □

3.3.4 Conclusion

The following properties hold
(i) The convex set $\mathcal{C}_T$ is non-empty (Lemma 3.3.2) and is a compact subset of $\mathcal{X}$ (Lemma 3.3.17).
(ii) The map L, defined in (3.3.19), is continuous on $\mathcal{C}_T$ in the topology of $\mathcal{X}$ (Lemma 3.3.18).
(iii) The map L maps $\mathcal{C}_T$ to itself (Lemma 3.3.16).
Thus, all the assumptions of Schauder fixed point theorem are satisfied by the map L on $\mathcal{C}_T$, endowed with the topology of $\mathcal{X}$. Therefore, Schauder fixed point theorem yields a fixed point $(\sigma_f, \mathbf{w}_f, \eta_f)$ of the map L in $\mathcal{C}_T$. From the definition of the map L, one has $(\sigma_f, \mathbf{w}_f, \eta_f) \in Z_1^T \times Y_2^T \times Z_3^T$. Hence we have the following time continuities (since still now one only has the regularities (3.3.41) of $G_1(\sigma_f, \mathbf{w}_f, \eta_f)$, (3.3.51) of $G_2(\sigma_f, \mathbf{w}_f, \eta_f)$ and (3.3.71) of $G_3(\sigma_f, \mathbf{w}_f, \eta_f)$)

$$\begin{aligned} &\sigma_f \in C^0([0,T];H^2(\Omega)),\\ &\mathbf{w}_f \in C^0([0,T];\mathbf{H}^{5/2}(\Omega)) \cap C^1([0,T];\mathbf{H}^1(\Omega)),\\ &\eta_f \in C^0([0,T];H^4(\Gamma_s)) \cap C^1([0,T];H^3(\Gamma_s)) \cap C^2([0,T];H^1(\Gamma_s)). \end{aligned} \tag{3.3.112}$$

The regularities (3.3.112) can be used to further check that $G_1(\sigma_f, \mathbf{w}_f, \eta_f) \in C^0([0,T];H^1(\Omega))$ and $G_3(\sigma_f, \mathbf{w}_f, \eta_f) \in C^0([0,T];H^{1/2}(\Gamma_s))$. Hence we use Corollary 3.2.5 and the Corollary 3.2.8 to obtain the following

$$(\sigma_f)_t \in C^0([0,T];H^1(\Omega)) \text{ and } \eta_f \in C^0([0,T];H^{9/2}(\Gamma_s)).$$

Hence, $(\sigma_f, \mathbf{w}_f, \eta_f) \in Y_1^T \times Y_2^T \times Y_3^T$. The trajectory $(\sigma_f, \mathbf{w}_f, \eta_f)$ solves the nonlinear problem (3.1.21) in $Y_1^T \times Y_2^T \times Y_3^T$. Consequently the system (3.1.19) admits a solution. This further implies that the original system (3.1.2)-(3.1.4)-(3.1.5) admits a strong solution in sense of the Definition 3.1.6. Finally the proof of Theorem 3.1.7 is complete. □

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Chapter 4

Observability of the adjoint of a linearized compressible fluid-structure model in a 2d channel

4.1 Introduction

Our objective is to study the null controllability of a linearized fluid structure interaction problem in a 2d channel. The fluid flow here is modeled by the compressible Navier-Stokes equations. Concerning the structure we will consider a damped Euler-Bernoulli beam located on a portion of the boundary. In the present article we establish an observability inequality for the adjoint of the linearized fluid structure interaction problem under consideration which in principle is equivalent with the null controllability of the system.
In order to introduce our model in a fixed domain it is first important to present the non linear fluid structure interaction dynamics and obtain the linear model via a suitable linearization procedure. We remark that this linearization process is not unique and depend on the structure of the map which we will use to bring the time dependent domain to a fixed reference configuration.

4.1.1 Motivation

In this section, we introduce the full non-linear compressible fluid structure interaction model which we aim at studying from the controllability point of view, even though our work is only a preliminary work in this direction.
Our goal here is to explain how, starting from a control problem for a compressible fluid-structure interaction model, we derive a linear model (see Section 4.1.1.4) which should, in principle, contain some of the main difficulties related to the non-linear model.

4.1.1.1 The non-linear model

We first define a few notations corresponding to the fluid and the structural domain. Let $d > 0$ be a constant and $\Omega = (0, d) \times (0, 1)$. We set

$$\Gamma_s = (0, d) \times \{1\}, \quad \Gamma_\ell = (0, d) \times \{0\}, \quad \Gamma = \Gamma_s \cup \Gamma_\ell.$$

For a given function

$$\beta : \Gamma_s \times (0, \infty) \to (-1, \infty),$$

which will correspond to the displacement of the one dimensional beam, let us denote by Ω_t and $\Gamma_{s,t}$ the following sets

$$\begin{aligned} \Omega_t &= \{(x, y) \mid x \in (0, L), \quad 0 < y < 1 + \beta(x, t)\} &&= \text{domain of the fluid at time } t, \\ \Gamma_{s,t} &= \{(x, y) \mid x \in (0, L), \quad y = 1 + \beta(x, t)\} &&= \text{the beam at time } t. \end{aligned}$$

The reference configuration of the beam is Γ_s and we set

$$\begin{aligned} &\Sigma_T = \Gamma \times (0, T), && \Sigma_T^s = \Gamma_s \times (0, T), \\ &\widetilde{\Sigma_T^s} = \cup_{t\in(0,T)} \Gamma_{s,t} \times \{t\}, && \Sigma_T^\ell = \Gamma_\ell \times (0, T), \\ &Q_T = \Omega \times (0, T), && \widetilde{Q_T} = \cup_{t\in(0,T)} \Omega_t \times \{t\}. \end{aligned} \tag{4.1.1}$$

We consider a fluid with density ρ and velocity u. The fluid structure interaction system coupling the compressible Navier-Stokes and the damped Euler-Bernoulli beam equation is modeled by the following equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 & \text{in} \quad \widetilde{Q_T}, \\ \rho(\partial_t u + (u.\nabla)u) - \mu\Delta u - (\mu + \mu')\nabla \operatorname{div} u + \nabla p(\rho) = 0 & \text{in} \quad \widetilde{Q_T}, \\ \partial_{tt}\beta - \partial_{txx}\beta + \partial_{xxxx}\beta = (T_f)_2 & \text{on} \quad \Sigma_T^s. \end{cases} \tag{4.1.2}$$

We assume that at the fluid structure interface the following impermeability condition holds

$$u(\cdot, t) \cdot n_t = (0, \partial_t \beta) \cdot n_t \quad \text{on} \quad \widetilde{\Sigma_T^s}, \tag{4.1.3}$$

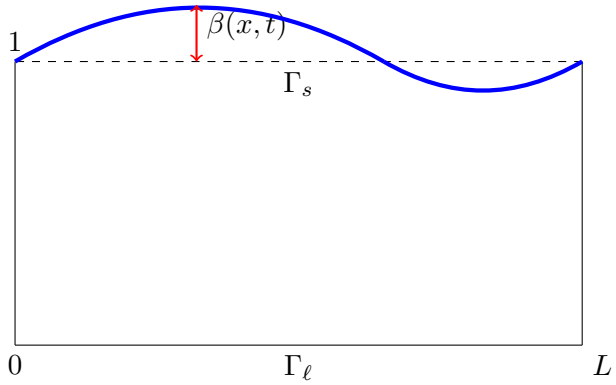


Figure 4.1: Domain Ω_t.

where n_t is the outward unit normal to $\Gamma_{s,t}$ given by

$$n_t = -\frac{\partial_x \beta}{\sqrt{1+(\partial_x\beta)^2}}\vec{e}_1 + \frac{1}{\sqrt{1+(\partial_x\beta)^2}}\vec{e}_2, \qquad (\vec{e}_1 = (1,0) \text{ and } \vec{e}_2 = (0,1)).$$

The fixed boundary Σ_T^ℓ is assumed to be impermeable and here the impermeability condition is given as follows

$$u(\cdot,t)\cdot n = 0 \quad \text{on} \quad \Sigma_T^\ell, \tag{4.1.4}$$

where n is the unit outward normal to Γ_ℓ. The fluid boundary is supplemented with the following slip condition

$$\text{curl}(u) = 0 \quad \text{on} \quad \widetilde{\Sigma_T^s} \cup \Sigma_T^l, \tag{4.1.5}$$

where $\text{curl}u = (\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x})$, denotes the vorticity of the vector field u. In the system (4.1.2), the real constants μ, μ' are the Lamé coefficients which are supposed to satisfy

$$\mu > 0, \quad (\mu' + 2\mu) > 0.$$

In our case the fluid is isentropic i.e. the pressure $p(\rho)$ is only a function of the fluid density ρ and is given by

$$p(\rho) = a\rho^\gamma,$$

where $a > 0$ and $\gamma > 1$ are positive constants.
We assume that there exists a constant external force $P_{ext} > 0$ which acts on the beam. We then introduce the positive constant $\overline{\rho}$ defined by the relation

$$P_{ext} = a\overline{\rho}^\gamma.$$

To incorporate this external forcing term P_{ext} into the system of equations (4.1.2), we introduce the following

$$P(\rho) = p(\rho) - P_{ext} = a\rho^\gamma - a\overline{\rho}^\gamma. \tag{4.1.6}$$

The non-homogeneous source term of the beam equation $(T_f)_2$ is the net surface force on the structure which is the resultant of force exerted by the fluid on the structure and the external force P_{ext} and it is assumed to be of the following form

$$(T_f)_2 = (-(\mu' + 2\mu)(\mathrm{div} u) I_d \cdot n_t + P n_t) \mid_{\Gamma_{s,t}} \sqrt{1 + (\partial_x \beta)^2} \cdot \vec{e}_2 \quad \text{on} \quad \Sigma_T^s, \tag{4.1.7}$$

where I_d is the identity matrix.

Remark 4.1.1 (The physical model and simplification.)**.** *The stress tensor corresponding to a Newtonian fluid with velocity u and pressure p is of the following form:*

$$\mathbb{S}(u,p) = (2\mu D(u) + \mu' div\, u I_d) - pI_d, \tag{4.1.8}$$

where $D(u)$ is the symmetric gradient given by

$$D(u) = \frac{1}{2}(\nabla u + \nabla^T u).$$

In view of the expression (4.1.8) of the stress tensor, the net force acting on the beam should be given as follows:

$$(T_f)_2^{ph} = ([-2\mu D(u) - \mu' div\, u I_d] \cdot n_t + P n_t) \mid_{\Gamma_{s,t}} \sqrt{1 + (\partial_x \beta)^2} \cdot \vec{e}_2 \quad on \quad \Sigma_T^s. \tag{4.1.9}$$

Instead of using the force (4.1.9), we assume, for technical reasons (see Remark 4.2.1), that the net force acting on the beam is given by (4.1.7).
Although this might seem physically irrelevant, let us point out that the resulting simplified model (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7) admits an energy equality which is explained in the following.
Assuming the data and the unknowns ρ, u and β are periodic in the x direction, we can formally derive the following energy dissipation law for the system (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7) (the detailed computation is included in Section 4.5)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_{\beta(t)}} \rho |u|^2 \, dx\right) + \frac{d}{dt}\left(\int_{\Omega_{\beta(t)}} \frac{a}{(\gamma-1)}\rho^\gamma \, dx\right) + \frac{1}{2}\frac{d}{dt}\left(\int_0^L |\partial_t \beta|^2 \, dx\right) \\ &+ \frac{1}{2}\frac{d}{dt}\left(\int_0^L |\partial_{xx}\beta|^2 \, dx\right) + \mu \int_{\Omega_{\beta(t)}} |\mathrm{curl} u|^2 \, dx + (\mu' + 2\mu) \int_{\Omega_{\beta(t)}} |\mathrm{div} u|^2 \, dx \\ &+ \int_0^L |\partial_{tx}\beta|^2 \, dx = -P_{ext} \int_{\Gamma_s} \partial_t \beta. \end{aligned} \tag{4.1.10}$$

Let us also point out that a similar model was considered in [18] and [17]. We would like to refer the readers to the Remark 4.2.5 for the technical details behind considering the simplified model (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7).

4.1.1.2 Control problem and extension arguments

Our goal will be to discuss a control problem with controls acting from the boundary in the x-variable. So far, we did not make precise the boundary conditions in the x-variable, as the controls we shall consider will precisely act on these boundaries. But in fact, the boundary control functions will never appear explicitly, as we will first do an extension argument in the direction of the channel and then study the distributed controllability for (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7) in the extended domain with controls localized in the extension of the domain. We thus take $L > 0$ and embed Γ_s into $\mathbb{T}_L \times \{1\}$ and Ω into $\mathbb{T}_L \times (0,1)$ where $\mathbb{T}_L$ is the one dimensional torus identified with $(-L, d+L)$ with periodic conditions. Then we consider the controls $v_\rho \chi_{\widetilde{\omega}_t}$ (for the density), $v_u \chi_{\widetilde{\omega}_t}$ (for the velocity) and $v_\beta \chi_{\widetilde{\omega}_{1,t}}$ (for the beam), where $\chi_{\widetilde{\omega}_t}$ and $\chi_{\widetilde{\omega}_{1,t}}$ are the characteristics functions of the sets $\widetilde{\omega}_t$ and $\widetilde{\omega}_{1,t}$ which are defined as follows

$$\begin{aligned}
\widetilde{\omega}_t &= \{(x,y) \mid x \in [-L,0),\ 0 < y < 1+\beta(x,t)\} \cup \{(x,y) \mid x \in (d,d+L], \\
&\qquad\qquad 0 < y < 1+\beta(x,t)\}, \\
\widetilde{\omega}_{1,t} &= \{(x,y) \mid x \in [-L,0),\ y = 1+\beta(x,t)\} \cup \{(x,y) \mid x \in (d,d+L], \\
&\qquad\qquad y = 1+\beta(x,t)\}.
\end{aligned} \tag{4.1.11}$$

To write the control system we further introduce the following notations

$$\begin{aligned}
\Omega_t^{ex} &= \{(x,y) \mid x \in \mathbb{T}_L, \quad 0 < y < 1+\beta(x,t)\} &&= \text{extended domain of the fluid at time } t, \\
\Gamma_{s,t}^{ex} &= \{(x,y) \mid x \in \mathbb{T}_L, \quad y = 1+\beta(x,t)\} &&= \text{the extended beam at time } t.
\end{aligned}$$

Our control system then reads as follows

$$\left\{\begin{array}{ll}
\partial_t \rho + \mathrm{div}(\rho u) = v_\rho \chi_{\widetilde{\omega}_t} & \text{in } \cup_{t\in(0,T)} \Omega_t^{ex} \times \{t\}, \\
\rho(\partial_t u + (u.\nabla)u) - \mu\Delta u - (\mu+\mu')\nabla \mathrm{div} u + \nabla P(\rho) = v_u \chi_{\widetilde{\omega}_t} & \text{in } \cup_{t\in(0,T)} \Omega_t^{ex} \times \{t\}, \\
u_2 = \partial_t \beta + \partial_x \beta u_1 & \text{on } \cup_{t\in(0,T)} \Gamma_{s,t}^{ex} \times \{t\}, \\
u_2 = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T), \\
\mathrm{curl}(u) = 0 & \text{on } ((\mathbb{T}_L \times \{0,1\}) \times (0,T), \\
u(\cdot,0) = u_0 & \text{in } \mathbb{T}_L \times (0,1), \\
\rho(\cdot,0) = \rho_0 & \text{in } \mathbb{T}_L \times (0,1), \\
\partial_{tt}\beta - \partial_{txx}\beta + \partial_{xxxx}\beta = (T_f)_2 + v_\beta \chi_{\widetilde{\omega}_{1,t}} & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\
\beta(\cdot,0) = \beta_0 \quad \text{and} \quad \partial_t \beta(\cdot,0) = \beta_1 & \text{in } \mathbb{T}_L \times \{1\}.
\end{array}\right. \tag{4.1.12}$$

It is standard to deduce a boundary controllability result for the system (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7) from a controllability result of the system (4.1.12) by restricting the data at the boundaries in the x-variable.

4.1.1.3 Transformation of the problem to a fixed domain

To transform the system (4.1.12) in the reference configuration, for β satisfying $1+\beta(x,t) > 0$ for all $(x,t) \in (\mathbb{T}_L \times \{1\}) \times (0,T)$, we introduce the following changes of variables

$$\begin{aligned}
&\Phi_{\beta(t)} : \Omega_t^{ex} \longrightarrow \mathbb{T}_L \times (0,1) \quad \text{defined by} \\
&\qquad\qquad \Phi_{\beta(t)}(x,y) = (x,z) = \left(x, \frac{y}{1+\beta(x,t)}\right), \\
&\Phi_{\beta} : \cup_{t\in(0,T)} \Omega_t^{ex} \times \{t\} \longrightarrow (\mathbb{T}_L \times (0,1)) \times (0,T) \quad \text{defined by} \\
&\qquad\qquad \Phi_{\beta}(x,y,t) = (x,z,t) = \left(x, \frac{y}{1+\beta(x,t)}, t\right).
\end{aligned} \tag{4.1.13}$$

Remark 4.1.2. *It is easy to prove that for each $t \in [0,T)$, the map $\Phi_{\beta(t)}$ is a C^1- diffeomorphism from Ω_t^{ex} onto $\mathbb{T}_L \times (0,1)$ provided that $(1+\beta(x,t)) > 0$, for all $x \in \mathbb{T}_L$ and that $\beta(\cdot,t) \in C^1(\Gamma_s)$.*

Observe that the map $\Phi_{\beta(t)}$ can be uniquely extended to the boundary $\Gamma_{s,t}^{ex}$ with values in $\mathbb{T}_L \times \{1\}$, by using the same formula $(4.1.13)_1$. With the change of variable $\Phi_{\beta(t)}$ (introduced in (4.1.13)), the control zones in the reference configuration are written as follows

$$\begin{aligned}
&\omega = ((-L,0)\times(0,1)) \cup ((d,d+L]\times(0,1)) = \{\Phi_{\beta(t)}(x,y) \mid (x,y) \in \widetilde{\omega}_t\}, \\
&\omega_1 = ((-L,0)\times\{1\}) \cup ((d,d+L]\times\{1\}) = \{\Phi_{\beta(t)}(x,y) \mid (x,y) \in \widetilde{\omega}_{1,t}\}.
\end{aligned} \tag{4.1.14}$$

We set the following notations

$$\begin{aligned}
&\widehat{\rho}(x,z,t) = \rho(\Phi_{\beta}^{-1}(x,z,t)),\ \widehat{u}(x,z,t) = (\widehat{u}_1,\widehat{u}_2) = u(\Phi_{\beta}^{-1}(x,z,t)), \\
&\qquad\qquad \forall (x,z,t) \in (\mathbb{T}_L \times (0,1)) \times (0,T), \\
&\widehat{\rho}_0(x,z) = \rho_0(\Phi_{\beta(0)}^{-1}(x,z)),\ \widehat{u}_0(x,z) = u_0(\Phi_{\beta(0)}^{-1}(x,z)), \quad \forall (x,z) \in \mathbb{T}_L \times (0,1), \\
&v_{\widehat{\rho}}\chi_{\omega}(x,z,t) = v_{\rho}\chi_{\widetilde{\omega}}(\Phi_{\beta}^{-1}(x,z,t)),\ v_{\widehat{u}}\chi_{\omega}(x,z,t) = v_u\chi_{\widetilde{\omega}}(\Phi_{\beta}^{-1}(x,z,t)), \\
&\qquad\qquad \forall (x,z,t) \in (\mathbb{T}_L \times (0,1)) \times (0,T), \\
&v_{\beta}\chi_{\omega_1}(x,1,t) = v_{\beta}\chi_{\widetilde{\omega}_1}(\Phi_{\beta}^{-1}(x,1,t)), \quad \forall x \in \mathbb{T}_L \text{ and } \forall t \in (0,T).
\end{aligned} \tag{4.1.15}$$

After transformation the nonlinear control problem (4.1.12) is rewritten as following

$$
\begin{cases}
\partial_t\widehat{\rho}+\begin{bmatrix}\widehat{u}_1\\ \frac{1}{(1+\beta)}(\widehat{u}_2-z\partial_t\beta-z\widehat{u}_1\partial_x\beta)\end{bmatrix}\cdot\nabla\widehat{\rho}+\widehat{\rho}\mathrm{div}\widehat{u} & \\
\qquad\qquad =F_1(\widehat{\rho},\widehat{u},\beta)+v_{\widehat{\rho}}\chi_\omega & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T),\\
\widehat{\rho}(\partial_t\widehat{u}+(\widehat{u}\cdot\nabla)\widehat{u})-\mu\Delta\widehat{u}-(\mu'+\mu)\nabla(\mathrm{div}\widehat{u})+\nabla P(\widehat{\rho}) & \\
\qquad\qquad =F_2(\widehat{\rho},\widehat{u},\beta)+v_{\widehat{u}}\chi_\omega & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T),\\
\widehat{u}\cdot n=\widehat{u}_2=\partial_t\beta+\partial_x\beta\widehat{u}_1 & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T),\\
\widehat{u}(\cdot,t)\cdot n=0 & \text{on } (\mathbb{T}_L\times\{0\})\times(0,T),\\
\mathrm{curl}\widehat{u}=\dfrac{\beta\partial_z\widehat{u}_1}{(1+\beta)}-\dfrac{z\partial_x\beta\partial_z\widehat{u}_2}{(1+\beta)} & \text{on } ((\mathbb{T}_L\times\{0,1\})\times(0,T),\\
\widehat{u}(\cdot,0)=\widehat{u}_0 & \text{in } \mathbb{T}_L\times(0,1),\\
\widehat{\rho}(\cdot,0)=\widehat{\rho}_0 & \text{in } \mathbb{T}_L\times(0,1),\\
\partial_{tt}\beta-\partial_{txx}\beta+\partial_{xxxx}\beta & \\
\qquad =(-(\mu'+2\mu)\mathrm{div}u+P(\widehat{\rho}))+F_3(\widehat{\rho},\widehat{u},\beta)+v_\beta\chi_{\omega_1} & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T),\\
\beta(0)=\beta_0\quad\text{and}\quad\partial_t\beta(0)=\beta_1 & \text{in } \mathbb{T}_L\times\{1\},
\end{cases}
\tag{4.1.16}
$$

where

$$
\begin{aligned}
&F_1(\widehat{\rho},\widehat{u},\beta)=\frac{1}{(1+\beta)}(z\partial_z\widehat{u}_1\partial_x\beta\widehat{\rho}+\beta\widehat{\rho}\partial_z\widehat{u}_2),\\
&F_2(\widehat{\rho},\widehat{u},\beta)=-\beta\widehat{\rho}\partial_t\widehat{u}+z\widehat{\rho}\partial_z\widehat{u}\partial_t\beta-\beta\widehat{\rho}\widehat{u}_1\partial_x\widehat{u}+\widehat{u}_1\partial_z\widehat{u}\partial_x\beta\widehat{\rho}z+\mu(\beta\partial_{xx}\widehat{u}-\frac{\beta\partial_{zz}\widehat{u}}{(1+\beta)}\\
&-2\partial_x\beta z\partial_{xz}\widehat{u}+\frac{\partial_{zz}\widehat{u}z^2(\partial_x\beta)^2}{(1+\beta)}+\partial_z\widehat{u}(\frac{(1+\beta)z\partial_{xx}\beta-2(\partial_x\beta)^2z}{(1+\beta)}))+(\mu+\mu')\cdot\\
&\begin{bmatrix}\beta\widehat{\partial}_{xx}u_1-\partial_{xz}\widehat{u}_1z\beta_x-\partial_x\beta z(\partial_{xz}\widehat{u}_1-\dfrac{\partial_{zz}\widehat{u}_1z\partial_x\beta}{(1+\beta)})+\partial_z\widehat{u}_1(\dfrac{(1+\beta)z\partial_{xx}\beta-2(\partial_x\beta)^2z}{(1+\beta)})\\ -\dfrac{\partial_x\beta\partial_z\widehat{u}_2}{(1+\beta)}-\dfrac{\partial_x\beta z\partial_{zz}\widehat{u}_2}{(1+\beta)}\\ \\ -\dfrac{\partial_x\beta\partial_z\widehat{u}_1}{(1+\beta)}-\dfrac{\partial_x\beta z\partial_{zz}\widehat{u}_1}{(1+\beta)}-\dfrac{\beta\partial_{zz}\widehat{u}_2}{(1+\beta)}\end{bmatrix}\\
&-(\beta P_x(\widehat{\rho})-P_z(\widehat{\rho})z\beta_x)\vec{e_1},\\
&F_3(\widehat{\rho},\widehat{u},\beta)=(\mu'+2\mu)\left(\frac{z\partial_z\widehat{u}_1\partial_x\beta}{(1+\beta)}-\frac{\beta\partial_z\widehat{u}_2}{(1+\beta)}\right),
\end{aligned}
\tag{4.1.17}
$$

and n denotes the unit normal to the boundary $(\mathbb{T}_L\times\{0,1\})$ of the extended reference domain $(\mathbb{T}_L\times(0,1))$, *i.e.*

$$
n=\begin{cases}(0,1) & \text{on}\quad\Gamma_s,\\ (0,-1) & \text{on}\quad\Gamma_l.\end{cases}
$$

The transport equation satisfied by the density $(4.1.16)_1$, $(4.1.16)_7$ is of the form

$$\begin{cases} \partial_t \widehat{\rho} + \begin{bmatrix} \widehat{u}_1 \\ \frac{1}{(1+\beta)}(\widehat{u}_2 - \partial_t\beta z - \widehat{u}_1 z\partial_x\beta) \end{bmatrix} \cdot \nabla\widehat{\rho} + \widehat{\rho}\mathrm{div}\widehat{u} = F_1 & \text{in} \quad (\mathbb{T}_L \times (0,1)) \\ & \qquad \times(0,T), \\ \widehat{\rho}(\cdot,0) = \rho_0 & \text{in} \quad \mathbb{T}_L \times (0,1). \end{cases} \tag{4.1.18}$$

Due to the interface condition, $\widehat{u}_2 = \partial_t\beta + \partial_x\beta\widehat{u}_1$ on $(\mathbb{T}_L \times \{1\}) \times (0,T)$ and $(4.1.16)_4$ we get that the velocity field $(\widehat{u}_1, \frac{1}{(1+\beta)}(\widehat{u}_2 - \partial_t\beta z - \widehat{u}_1 z\partial_x\beta))$ satisfies

$$\begin{bmatrix} \widehat{u}_1 \\ \dfrac{1}{(1+\beta)}(\widehat{u}_2 - \partial_t\beta z - \widehat{u}_1 z\partial_x\beta) \end{bmatrix} \cdot n = 0 \quad \text{on} \quad (\mathbb{T}_L \times \{0,1\}) \times (0,T).$$

Hence one shall not prescribe any further boundary condition on the density for the above problem (4.1.18) to be well-posed. This will be made more precise in Section 4.2 on the linearized model.

4.1.1.4 A control problem for the linearized model around the stationary state $(\overline{\rho}, \overline{u}, 0)$

Hereafter, we focus on the local exact controllability to the steady state $(\overline{\rho}, \overline{u}, 0)$, where

$$\overline{\rho} > 0 \text{ is such that } \; P_{ext} = a\overline{\rho}^{\gamma} \text{ and } \overline{u} = \begin{pmatrix} \overline{u}_1 \\ 0 \end{pmatrix}, \; \overline{u}_1 > 0 \text{ is a constant,} \tag{4.1.19}$$

which is obviously a stationary solution of the system (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7) or (4.1.12).
To be more precise, we would like to prove that, if $(\rho_0, u_0, \beta_0, \beta_1)$ is close in a suitable topology to $(\overline{\rho}, \overline{u}, 0, 0)$, then one can find control functions (v_ρ, v_u, v_β) such that the solution of (4.1.12) satisfies $(\rho(T), u(T), \beta(T), \partial_t\beta(T)) = (\overline{\rho}, \overline{u}, 0, 0)$.
To study this local exact controllability problem, we do the following change of unknowns

$$\widetilde{\sigma} = \widehat{\rho} - \overline{\rho}, \quad \widetilde{u} = \widehat{u} - \overline{u}, \quad \beta = \beta - 0, \tag{4.1.20}$$

so that the controllability problem to the state $(\overline{\rho}, \overline{u}, 0, 0)$ for $(\rho, u, \beta, \partial_t\beta)$ is now reduced to a local null controllability problem for $(\widetilde{\sigma}, \widetilde{u}, \beta, \partial_t\beta)$.
We further introduce the following notations corresponding to the control functions which are consistent with the new unknowns defined in (4.1.20).

$$v_{\widetilde{\sigma}}\chi_\omega = v_{\widehat{\rho}}\chi_\omega, \quad v_{\widetilde{u}}\chi_\omega = v_{\widehat{u}}\chi_\omega. \tag{4.1.21}$$

In fact, as we would like to obtain a local null-controllability problem for $(\widetilde{\sigma}, \widetilde{u}, \beta, \partial_t\beta)$, it seems reasonable to start by considering the linearized problem around the state

$(0,0,0,0)$. Starting from system (4.1.16) and the non-linear terms (4.1.17) and dropping all the non-linear terms in $(\widetilde{\sigma},\widetilde{u},\beta,\partial_t\beta)$, we obtain

$$\begin{cases} \partial_t\widetilde{\sigma}+\overline{u}_1\partial_x\widetilde{\sigma}+\overline{\rho}\mathrm{div}\widetilde{u}=v_{\widetilde{\sigma}}\chi_\omega & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T),\\ \overline{\rho}(\partial_t\widetilde{u}+\overline{u}_1\partial_x\widetilde{u})-\mu\Delta\widetilde{u}-(\mu'+\mu)\nabla(\mathrm{div}\widetilde{u}) & \\ \qquad +P'(\overline{\rho})\nabla\widetilde{\sigma}=v_{\widetilde{u}}\chi_\omega & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T),\\ \widetilde{u}\cdot n=\widetilde{u}_2=\partial_t\beta+\overline{u}_1\partial_x\beta & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T),\\ \widetilde{u}(\cdot,t)\cdot n=0 & \text{on } (\mathbb{T}_L\times\{0\})\times(0,T),\\ \mathrm{curl}\widetilde{u}=0 & \text{on } ((\mathbb{T}_L\times\{0,1\})\times(0,T),\\ \widetilde{u}(\cdot,0)=\widehat{u}_0-\overline{u}=\widetilde{u}_0 & \text{in } \mathbb{T}_L\times(0,1),\\ \widetilde{\sigma}(\cdot,0)=\widehat{\rho}_0-\overline{\rho}=\widetilde{\sigma}_0 & \text{in } \mathbb{T}_L\times(0,1),\\ \partial_{tt}\beta-\partial_{txx}\beta+\partial_{xxxx}\beta & \\ \qquad =\big(-(\mu'+2\mu)\mathrm{div}\widetilde{u}+P'(\overline{\rho})\widetilde{\sigma}\big)+v_\beta\chi_{\omega_1} & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T),\\ \beta(0)=\beta_0\quad\text{and}\quad\partial_t\beta(0)=\beta_1 & \text{in } \mathbb{T}_L\times\{1\}. \end{cases} \tag{4.1.22}$$

Before going further, let us give the following well-posedness result for (4.1.22), whose proof is postponed to Section 4.2:

Theorem 4.1.3. *Let*

$$\begin{aligned}(\widetilde{\sigma}_0,\widetilde{u}_0,\beta_0,\beta_1)\in H^2(\mathbb{T}_L\times(0,1))\times H^3(\mathbb{T}_L\times(0,1))\times H^{9/2}(\mathbb{T}_L\times\{1\})\\ \times H^{5/2}(\mathbb{T}_L\times\{1\}),\end{aligned} \tag{4.1.23}$$

the following compatibility conditions hold

$$\begin{array}{lll} (i)\ (a) & \widetilde{u}_0\cdot n=(\widetilde{u}_0)_2=\beta_1+\overline{u}_1\partial_x\beta_0, & \text{on } \mathbb{T}_L\times\{1\},\\ \quad (b) & \widetilde{u}_0\cdot n=(\widetilde{u}_0)_2=0, & \text{on } \mathbb{T}_L\times\{0\},\\ (ii) & \mathrm{curl}\,\widetilde{u}_0=0, & \text{on } \mathbb{T}_L\times\{0,1\},\\ (iii)\ (a) & -\overline{u}_1\partial_x(\widetilde{u}_0)_2+\frac{\mu}{\overline{\rho}}\Delta(\widetilde{u}_0)_2+\frac{(\mu+\mu')}{\overline{\rho}}\partial_z(\mathrm{div}\,\widetilde{u}_0) & \\ & -P'(\overline{\rho})\partial_z\widetilde{\sigma}_0=(v_\beta\chi_{\omega_1}(\cdot,0)-(v_{\widetilde{u}}\chi_\omega)_2(\cdot,0)) & \\ & -\nu\mathrm{div}\,\widetilde{u}_0+P'(\overline{\rho})\widetilde{\sigma}_0+\partial_{xx}\beta_1-\partial_{xxxx}\beta_0 & \\ & +\overline{u}_1\partial_x\beta_1, & \text{on } \mathbb{T}_L\times\{1\},\\ \quad (b) & -\overline{u}_1\partial_x(\widetilde{u}_0)_2+\frac{\mu}{\overline{\rho}}\Delta(\widetilde{u}_0)_2+\frac{(\mu+\mu')}{\overline{\rho}}\partial_z(\mathrm{div}\,\widetilde{u}_0) & \\ & \qquad =-P'(\overline{\rho})\partial_z\widetilde{\sigma}_0-(v_{\widetilde{u}}\chi_\omega)_2(\cdot,0) & \text{on } \mathbb{T}_L\times\{0\} \end{array} \tag{4.1.24}$$

and $(v_{\widetilde{\sigma}}\chi_\omega, v_{\widetilde{u}}\chi_\omega, v_\beta\chi_{\omega_1})$ satisfy the following regularity assumptions

$$\begin{aligned} &(a)\ v_{\widetilde{\sigma}}\chi_\omega\in C^0([0,T];H^2(\mathbb{T}_L\times(0,1))),\\ &(b)\ v_{\widetilde{u}}\chi_\omega\in L^2(0,T;H^2(\mathbb{T}_L\times(0,1)))\cup H^1(0,T;L^2(\mathbb{T}_L\times(0,1))),\\ &(c)\ v_\beta\chi_{\omega_1}\in L^2(0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cup H^{3/4}(0,T;L^2(\mathbb{T}_L\times\{1\})), \end{aligned} \tag{4.1.25}$$

then the system (4.1.22) admits a unique solution $(\widetilde{\sigma},\widetilde{u},\beta)$ in the following functional

framework

$$\begin{cases} \widetilde{\sigma} \in C^0([0,T];H^2(\mathbb{T}_L\times(0,1)))\cap C^1([0,T];H^1(\mathbb{T}_L\times(0,1))), \\ \widetilde{u} \in L^2(0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^1(0,T;H^2(\mathbb{T}_L\times(0,1))) \\ \qquad \cap H^2(0,T;L^2(\mathbb{T}_L\times(0,1))), \\ \beta \in L^2(0,T;H^{11/2}(\mathbb{T}_L\times\{1\}))\cap H^1(0,T;H^{7/2}(\mathbb{T}_L\times\{1\})) \\ \qquad \cap H^2(0,T;H^{3/2}(\mathbb{T}_L\times\{1\})). \end{cases} \tag{4.1.26}$$

Unfortunately, as we will see, we did not manage yet to get a control result for (4.1.22) in the above functional setting. Instead, as will be presented next, we will obtain an observability result for the adjoint system of (4.1.22), which is a first step to study the controllability problem for the linearized model (4.1.22) and then for the fully non-linear model (4.1.12) (or equivalently for (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7)).

4.1.2 Main result: Observability of the adjoint of (4.1.22)

In order to study the null controllability of the linearized problem (4.1.22), the classical strategy is to prove the observability of the adjoint system of (4.1.22). Here, the adjoint system of (4.1.22), computed with respect to the scalar product $L^2(\mathbb{T}_L\times(0,1))\times L^2(\mathbb{T}_L\times(0,1))\times L^2(\mathbb{T}_L)\times L^2(\mathbb{T}_L)$, reads as follows:

$$\begin{cases} -\partial_t\sigma-\overline{u}_1\partial_x\sigma-P'(\overline{\rho})\mathrm{div}v=0 & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T), \\ -\overline{\rho}(\partial_t v+\overline{u}_1\partial_x v)-\mu\Delta v-(\mu'+\mu)\nabla(\mathrm{div}v)-\overline{\rho}\nabla\sigma=0 & \text{in } (\mathbb{T}_L\times(0,1))\times(0,T), \\ v\cdot n=v_2=\psi & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T), \\ v(\cdot,t)\cdot n=0 & \text{on } (\mathbb{T}_L\times\{0\})\times(0,T), \\ \mathrm{curl}v=0 & \text{on } ((\mathbb{T}_L\times\{0,1\})\times(0,T), \\ v(\cdot,T)=v_T & \text{in } \mathbb{T}_L\times(0,1), \\ \sigma(\cdot,T)=\sigma_T & \text{in } \mathbb{T}_L\times(0,1), \\ \partial_{tt}\psi+\partial_{txx}\psi+\partial_{xxxx}\psi & \\ \qquad =(\partial_t+\overline{u}_1\partial_x)[(\mu'+2\mu)\mathrm{div}\,v+\overline{\rho}\sigma] & \text{on } (\mathbb{T}_L\times\{1\})\times(0,T), \\ \psi(T)=\psi_T \quad\text{and}\quad \partial_t\psi(T)=\psi_T^1 & \text{in } \mathbb{T}_L\times\{1\}. \end{cases} \tag{4.1.27}$$

The well-posedness of system (4.1.27) is then obtained similarly as the one of system (4.1.22), and we can derive the following result, which is proved in Section 4.2.1:

Theorem 4.1.4. *Let*

$$\begin{aligned}(\sigma_T,v_T,\psi_T,\psi_T^1)\in H^2(\mathbb{T}_L\times(0,1))\times H^3(\mathbb{T}_L\times(0,1))\times H^3(\mathbb{T}_L\times\{1\}) \\ \times H^1(\mathbb{T}_L\times\{1\})\end{aligned} \tag{4.1.28}$$

and the following compatibility relations hold

$$\begin{array}{lll} (i)\ (a) & v_T \cdot n = (v_T)_2 = \psi_T, & \text{on } \mathbb{T}_L \times \{1\}, \\ \quad (b) & v_T \cdot n = (v_T)_2 = 0, & \text{on } \mathbb{T}_L \times \{0\}, \\ (ii) & \operatorname{curl} v_T = 0, & \text{on } \mathbb{T}_L \times \{0,1\} \\ (iii)\ (a) & -\overline{u}_1 \partial_x (v_T)_2 + \frac{\mu}{\overline{\rho}} \Delta (v_T)_2 + \frac{(\mu+\mu')}{\overline{\rho}} \partial_z (\operatorname{div} v_T) & \\ & -\partial_z \sigma_T = \psi_T^1 & \text{on } \mathbb{T}_L \times \{1\}, \\ \quad (b) & -\overline{u}_1 \partial_x (v_T)_2 + \frac{\mu}{\overline{\rho}} \Delta (v_T)_2 + \frac{(\mu+\mu')}{\overline{\rho}} \partial_z (\operatorname{div} v_T) & \\ & -\partial_z \sigma_T = 0 & \text{on } \mathbb{T}_L \times \{0\}. \end{array} \tag{4.1.29}$$

Then the system (4.1.27) *admits a unique solution* (σ, v, ψ) *which satisfies the following regularity*

$$\begin{cases} \sigma \in C^0([0,T]; H^2(\mathbb{T}_L \times (0,1))) \cap C^1([0,T]; H^1(\mathbb{T}_L \times (0,1))), \\ v \in L^2(0,T; H^3(\mathbb{T}_L \times (0,1))) \cap H^1(0,T; H^2(\mathbb{T}_L \times (0,1))) \\ \qquad \cap H^2(0,T; L^2(\mathbb{T}_L \times (0,1))), \\ \psi \in L^2(0,T; H^4(\mathbb{T}_L \times \{1\})) \cap H^1(0,T; H^2(\mathbb{T}_L \times \{1\})) \\ \qquad \cap H^2(0,T; L^2(\mathbb{T}_L \times \{1\})). \end{cases} \tag{4.1.30}$$

The central result of the present article is the observability inequality of the adjoint system (4.1.27):

Theorem 4.1.5. *Let* $(\overline{\rho}, \overline{u}, 0)$ *be as in* (4.1.19), $T > 0$ *be such that*

$$T > \frac{d}{\overline{u}_1}, \tag{4.1.31}$$

and

$$L = 3\overline{u}_1 T > 0. \tag{4.1.32}$$

There exists a positive constant C *such that for all*

$$\begin{aligned} (\sigma_T, v_T, \psi_T, \psi_T^1) \in H^2(\mathbb{T}_L \times (0,1)) \times H^3(\mathbb{T}_L \times (0,1)) \times H^3(\mathbb{T}_L \times \{1\}) \\ \times H^1(\mathbb{T}_L \times \{1\}), \end{aligned} \tag{4.1.33}$$

satisfying the compatibility conditions (4.1.29), *then the solution* (σ, v, ψ) *of the problem* (4.1.27) *(in the sense of Theorem 4.1.4) satisfies the following observability inequality:*

$$\begin{aligned} \|\sigma(\cdot,0)\|_{H^1(\mathbb{T}_L \times (0,1))} + \|v(\cdot,0)\|_{H^2(\mathbb{T}_L \times (0,1))} + \|(\psi(\cdot,0), \partial_t \psi(\cdot,0))\|_{H^3(\mathbb{T}_L \times (0,1))) \times H^1(\mathbb{T}_L \times (0,1))} \\ \leqslant C \|\psi\|_{L^2(\omega_1^T)} + C \|v\|_{L^2(0,T; H^2(\omega)) \cap H^1(0,T; H^1(\omega))} + C \|\sigma\|_{H^1(\omega_T)}. \end{aligned} \tag{4.1.34}$$

where the notations ω *and* ω_1 *for the observation sets were introduced in* (4.1.14) *and* $\omega_T = \omega \times (0,T)$, $\omega_1^T = \omega_1 \times (0,T)$.

Comments on the choice of T and L in Theorem 4.1.5: We recall (4.1.31) and (4.1.32). The condition (4.1.31) means that the time of observability should be greater than the time taken to cross the channel length d by a particle moving with a velocity $(\overline{u}_1, 0)$. This condition is imposed due to the hyperbolic nature of the transport equation satisfied by σ in the system (4.1.27). In fact this condition plays a key role in obtaining the observability estimate for a hyperbolic transport equation in Section 4.3.4. Our proof in Section 4.3.4 depends on a duality argument and a controllability result for the dual to the problem considered in Section 4.3.4 which is obtained in [12]. Hence one can consult [12] for a more explicit construction of the controlled trajectory where the restriction of the final time (4.1.31) comes into play.
The role of (4.1.32) can be better explained after we define some Carleman weights in Section 4.3. Hence we refer the readers to the Remark 4.3.1 for the explanation behind choosing L as in (4.1.32). Still, let us mention that from the control point of view, the value of $L > 0$ does not play any role in the restriction argument presented in Section 4.1.1.2.

4.1.3 Ideas and Strategy

Now we will briefly discuss our ideas and strategy to prove Theorem 4.1.5.
The underlying idea behind the proof of Theorem (4.1.5) is the identification of the suitable unknowns to track down the dynamics of (σ, v, ψ). It is well known that the coupling of σ and v is strong. When considering the primal problem (4.1.22), the dynamics between $\widetilde{\sigma}$, $\widetilde{u}$ can be made simpler by introducing the effective viscous flux, see [25] and [14]. For the adjoint problem, a similar quantity, already used in [12], also simplifies the description of the dynamics:

$$q = (\mu' + 2\mu)\mathrm{div}\, v + \overline{\rho}\sigma. \tag{4.1.35}$$

This can be termed as the dual version of the effective viscous flux. Now in our case it is important to identify the behavior of q at the boundaries and specially at the fluid solid interface. This way we obtain a closed loop system solved by (σ, q, ψ). For details we refer the readers to Section 4.2. One can in particular look into the system (4.2.3) to observe that unlike the coupling between σ and v in system (4.1.27), the coupling between σ and q is of lower order. Also it is easier to deal with (σ, q, ψ), since it has less degrees of freedom in comparison with (σ, v, ψ).

We use this new set of unknowns (σ, q, ψ) both to prove the well posedness result stated in Theorem 4.1.4 and the observability Theorem 4.1.5. In Section 4.2 we first prove Theorem 4.1.4 and next we sketch the proof of Theorem 4.1.3 which corresponds to the well posedness of the primal problem (4.1.22). We do not present a detailed proof of Theorem 4.1.3 since it shares the same strategy as the one of Theorem 4.1.4. In fact for the primal problem we use the effective viscous flux for a similar purpose of introducing the dual version (4.1.35), see Section 4.2.2 for details.

Next we focus in proving an observability inequality for the system satisfied by (σ, q, ψ). In that direction we first separately study the observability inequalities of some scalar equations *i.e* an adjoint damped beam equation, an adjoint heat equation and an adjoint transport equation. The observability estimates for the adjoint damped beam and the adjoint heat equations rely on Carleman estimates while for the adjoint transport equation we use a duality argument and a controllability result from [12]. The main difficulty here is to obtain these separate observability estimates with a single goal of combining them suitably to obtain an observability for the coupled system solved by (σ, q, ψ). For the parabolic hyperbolic couplings this question is handled in the articles [1], [12] (for compressible Navier-Stokes equations) and [8] (for damped viscoelasticity equations). The idea is to use compatible weight functions for the parabolic and hyperbolic equations so that the resulting observability estimates can be suitably combined. In our case along with a parabolic hyperbolic coupling there is a direct coupling between q and ψ at the fluid boundary (see the system (4.2.3) for details). Hence to use the ideas from [1], [8] and [12] in our case there is not many options other than considering a one dimensional weight function, since the beam is one dimensional. We introduce such weight function in Section 4.3.1. Then using this weight function we derive a Carleman estimate for the adjoint damped beam equation, see Section 4.3.2 for details. To the best of our knowledge the Carleman estimate for the adjoint beam proved in this article is completely new. Then using the same weight function for an adjoint heat equation with Neumann boundary condition we recover a Carleman estimate proved in [15] but with a different weight function. This Carleman estimate is included in Section 4.3.3. In the beginning of Section 4.3.3, we also point out the difference between our weight function used in proving the Carleman estimate for an adjoint heat equation with the one standard in the literature. Then the same weight function is used to obtain an observability estimate for an adjoint transport equation in Section 4.3.4. This is done by using a controllability result for a primal transport problem proved in [12] and a duality argument.

Next in Section 4.4 we combine the Carleman estimates obtained in Section 4.3. First using suitably large values of the Carleman parameters we are able to prove an inequality corresponding to the unique continuation property for the system satisfied by (σ, q, ψ). This inequality is explicitly given by (4.4.8). It is not surprising to obtain an estimate of the form (4.4.8) by combining three different observability estimates. In fact as it is well known that the Carleman parameters quantify the compactness of a system hence our strategy to obtain the inequality (4.4.8) strongly relates on the proof of Step 1 in Theorem 4.1.4, where we prove the well posedness of (4.2.3) by gaining a time integrability of a suitable fixed point map. After this unique continuation estimate is used to show an observability estimate of (σ, q, ψ) at some intermediate time, see (4.4.9) for details. We further use a well posedness result for the system satisfied by (σ, q, ψ) to obtain an observability estimate over (σ, q, ψ) at initial time $t = 0$. Finally using this observability estimate over $(\sigma, q, \psi)(\cdot, 0)$ we recover an observability estimate of $(\sigma, \text{div } v, \psi)(\cdot, 0)$, which combined with an observability inequality of curl $v(\cdot, 0)$ furnishes

the desired inequality (4.1.34).

4.1.4 Related bibliography

Concerning the incompressible Navier-Stokes equations in a $2D$ domain one can find a result proving the local exact controllability to trajectories with localized boundary control in [20]. It is assumed in [20] that the fluid satisfies no vorticity boundary condition in the complement of the control part of the boundary. Local exact controllability to trajectories for incompressible Navier-Stokes equations in a $3D$ domain with distributed control and homogeneous Dirichlet boundary condition can be found in [23]. With less regularity assumption on the target trajectory the result in [23] was improved in [16]. We would also like to mention the article [21] for the local exact distributed controllability to trajectories for incompressible Navier-Stokes equations in a $3D$ domain with non linear Navier-slip boundary condition. In all of these articles the fluid is assumed to be homogeneous *i.e* the fluid density is constant. In a very recent article [2] the authors prove the local exact boundary controllability to smooth trajectories for a non homogeneous incompressible Navier-Stokes equation in a three dimensional domain. For global controllability results for incompressible Navier-Stokes equations we refer the readers to [10], [7] and the references therein.
Now we quote a few articles dealing with the controllability issues of fluid structure interaction models. In fact to the best of our knowledge the only known results concerning the controllability issues of a fluid structure interaction problem in dimension greater than one deals with the motion of a rigid body inside a incompressible fluid modeled by Navier-Stokes equations where the structural motion are given by the balance of linear and angular momentum. Local null controllability of such an interaction problem in dimension two can be found in [5] and [22]. In dimension three a local null controllability for such a system is proved in [4]. The article [28] deals with the problem of feedback stabilization (in infinite time) for an incompressible fluid structure interaction problem in a $2D$ channel where the structure appears at the fluid boundary and is modeled by an Euler-Bernoulli damped beam, the one we consider in $(4.1.22)_8$-$(4.1.22)_9$. To our knowledge so far there does not exist any article dealing with the finite time controllability of a fluid structure interaction problem (neither for incompressible nor compressible fluids) in dimension more than one where the structure appears at the fluid boundary. We would also like to refer the readers to [19], [32] and [33] for observability estimates individually for the Euler-Bernoulli plate equations and Kirchoff plate systems without damping.

We also like to quote a few articles from the literature dealing with the controllability issues of compressible Navier-Stokes equation. In fact our strategy to handle the coupling of the fluid velocity and density in the system (4.1.22) amounts in introducing a new unknown namely the effective viscous flux and this strategy is inspired from the article [12]. The article [13] concerns the motion of a fluid in dimension one whereas [12] deals with fluid flows in dimension two and three.

4.2 Well posedness results for the primal problem (4.1.22) and the adjoint problem (4.1.27)

Of course, the proofs of Theorem 4.1.3 and of Theorem 4.1.4 are very similar and we shall thus develop one more extensively and give the main steps of the proof for the other one.
As we will in fact work only on the adjoint system (4.1.27) to establish Theorem 4.1.5, we do the choice to focus on the proof of Theorem 4.1.4. As it turns out, this proof will also give some insights of the strength of the various coupling between the equations in (4.1.27), which will also help in the proof of Theorem 4.1.5.
Section 4.2.1 provides the proof of Theorem 4.1.4, while Section 4.2.2 sketches the main steps of the proof of Theorem 4.1.3.

4.2.1 Proof of Theorem 4.1.4

The proof is divided into two main steps. The first one consists in introducing the new unknown

$$q = (\mu' + 2\mu)\mathrm{div}\, v + \overline{\rho}\sigma, \tag{4.2.1}$$

and looking at the system satisfied by (σ, q, β), which turns out to be easier to analyze than the full system (4.1.27). The second step will then consist in deducing from the regularity on (σ, q, β) suitable estimates for the function v in (4.1.27).

Remark 4.2.1. *The unknown q in (4.2.1) can be interpreted as the dual version of the effective viscous flux introduced for instance in [25], see also [14]. In fact, this quantity already appeared in [12] when studying the controllability properties of a compressible fluid (without structure and controls acting on the whole boundary), where it helps to weaken the coupling of the parabolic and hyperbolic effects of the system.*
Here, the interesting point is that this quantity is also suitable to deal with the coupling with the structure lying on the boundary. There, we strongly use that the force acting on the beam is given by (4.1.7) instead of the more natural one (4.1.9), which would not yield such a clean understanding of the coupling between the fluid and the structure.

4.2.1.1 Step 1. The system satisfied by (σ, q, β)

We first derive the system that (σ, q, β) should satisfy provided (σ, v, β) satisfies (4.1.27) and has the regularity given by (4.1.30). Indeed, these regularities allow to take the trace of $\nabla^2 v$ and $\nabla\sigma$ on $(\mathbb{T}_L \times \{0, 1\} \times (0, T)$. Hence we can consider the trace of the equation

$(4.1.27)_2$ and use $(4.1.27)_3$-$(4.1.27)_5$ to have the following on $((\mathbb{T}_L \times \{1\})) \times (0,T)$:

$$\begin{aligned} 0 &= -\overline{\rho}(\partial_t v_2 + \overline{u}_1 \partial_x v_2) - \mu(\partial_{xx} v_2 + \partial_{zz} v_2) - (\mu + \mu')(\partial_{xz} v_1 + \partial_{zz} v_2) - \overline{\rho}\partial_z \sigma \\ &= -\overline{\rho}(\partial_t \psi + \overline{u}_1 \partial_x \psi) - (\mu' + 2\mu)(\partial_{xz} v_1 + \partial_{zz} v_2) - \overline{\rho}\partial_z \sigma \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{using } (4.1.27)_3, (4.1.27)_5) \\ &= -\overline{\rho}(\partial_t \psi + \overline{u}_1 \partial_x \psi) - \partial_z q. \end{aligned} \tag{4.2.2}$$

Similarly one can obtain that on the boundary $(\mathbb{T}_L \times \{0\}) \times (0,T)$, q satisfies

$$\partial_z q = 0 \quad \text{on} \quad (\mathbb{T}_L \times \{0\}) \times (0,T).$$

Hence with the formal calculations above and using (4.1.27) we obtain the following system satisfied by the unknowns (σ, q, ψ) :

$$\begin{cases} -\partial_t \sigma - \overline{u}_1 \partial_x \sigma + \dfrac{P'(\overline{\rho})\overline{\rho}}{\nu}\sigma = \dfrac{P'(\overline{\rho})}{\nu} q & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ -(\partial_t q + \overline{u}_1 \partial_x q) - \dfrac{\nu}{\overline{\rho}}\Delta q - \dfrac{P'(\overline{\rho})\overline{\rho}}{\nu} q = -\dfrac{P'(\overline{\rho})\overline{\rho}^2}{\nu}\sigma & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \partial_z q = -\overline{\rho}(\partial_t \psi + \overline{u}_1 \partial_x \psi) & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \partial_z q = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T), \\ q(\cdot, T) = q_T & \text{in } \mathbb{T}_L \times (0,1), \\ \sigma(\cdot, T) = \sigma_T & \text{in } \mathbb{T}_L \times (0,1), \\ \partial_{tt}\psi + \partial_{txx}\psi + \partial_{xxxx}\psi = (\partial_t + \overline{u}_1 \partial_x) q & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \psi(T) = \psi_T \quad \text{and} \quad \partial_t \psi(T) = \psi_T^1 & \text{in } \mathbb{T}_L \times \{1\}, \end{cases} \tag{4.2.3}$$

where $\nu = (\mu' + 2\mu)$, and $q_T = \nu \mathrm{div} v_T + \overline{\rho}\sigma_T$.

We then deal with the well-posedness of (4.2.3):

Lemma 4.2.2. *There exists a constant $C > 0$ such that for any*

$$(\sigma_T, q_T, \psi_T, \psi_T^1) \in H^1(\mathbb{T}_L \times (0,1)) \times H^2(\mathbb{T}_L \times (0,1)) \times H^3(\mathbb{T}_L \times \{1\}) \times H^1(\mathbb{T}_L \times \{1\}) \tag{4.2.4}$$

satisfying the following compatibility conditions

$$\begin{aligned} &(i)\, \partial_z q_T = -\overline{\rho}(\psi_T^1 + \overline{u}_1 \partial_x \psi_T) \text{ on } \mathbb{T}_L \times \{1\}, \\ &(ii)\, \partial_z q_T = 0 \text{ on } \mathbb{T}_L \times \{0\}, \end{aligned} \tag{4.2.5}$$

the system (4.2.3) admits a unique solution (σ, q, ψ) which satisfies the following estimate

$$\begin{aligned} &\|\sigma\|_{C^0([0,T];H^1(\mathbb{T}_L\times(0,1)))\cap C^1([0,T];L^2(\mathbb{T}_L\times(0,1)))} + \|q\|_{C^0([0,T];H^2(\mathbb{T}_L\times(0,1)))} \\ &+ \|\psi\|_{C^0([0,T];H^3(\mathbb{T}_L\times\{1\}))\cap C^1([0,T];H^1(\mathbb{T}_L\times\{1\}))} \\ &\leqslant C(\|q_T\|_{H^2(\mathbb{T}_L\times(0,1))} + \|\sigma_T\|_{H^1(\mathbb{T}_L\times(0,1))} + \|(\psi_T, \psi_T^1)\|_{H^3(\mathbb{T}_L\times\{1\})\times H^1(\mathbb{T}_L\times\{1\})}). \end{aligned} \tag{4.2.6}$$

Proof of Lemma 4.2.2. In this step we will prove the existence and regularity result for (σ, q, ψ) solving the system (4.2.3). In fact, we will prove that under the assumptions (4.2.4) and (4.2.5), the system (4.2.3) admits a unique solution in the following functional framework:

$$\begin{aligned}(\sigma, q, \psi) \in &(C^0([0,T]; H^1(\mathbb{T}_L \times (0,1))) \cap C^1([0,T]; L^2(\mathbb{T}_L \times (0,1))))\\ &\times (L^2(0,T; H^3(\mathbb{T}_L \times (0,1))) \cap H^{3/2}([0,T]; L^2(\mathbb{T}_L \times (0,1))))\\ &\times (L^2(0,T; H^4(\mathbb{T}_L \times \{1\})) \cap H^1(0,T; H^2(\mathbb{T}_L \times \{1\}))\\ &\qquad \cap H^2(0,T; L^2(\mathbb{T}_L \times \{1\})))\end{aligned} \tag{4.2.7}$$

We will first prove a local in time existence result for the problem (4.2.3). Then using the linearity of (4.2.3) we iterate the time steps in order to show (4.2.7). Since the problem (4.2.3) is posed backward in time by local in time existence, we first work in some time interval of the form $(T - T_0, T)$ for T_0 sufficiently small.
Let $0 < T_0 < T$. We consider the system (4.2.3). We are going to define a suitable map whose fixed point gives a solution to the system (4.2.3) in the time interval $(T - T_0, T)$. We define

$$\begin{aligned}\mathbb{H}_1^{T_0} = &(L^2(T - T_0, T; H^1(\mathbb{T}_L \times (0,1))) \cap H^1(T - T_0, T; L^2(\mathbb{T}_L \times (0,1))))\\ &\times (H^1(T - T_0, T; H^{3/2}(\mathbb{T}_L \times \{1\})) \cap H^{7/4}(T - T_0, T; L^2(\mathbb{T}_L \times \{1\}))\\ &\cap L^2(T - T_0, T; H^{5/2}(\mathbb{T}_L \times \{1\})) \cap H^{3/4}(T - T_0, T; H^1(\mathbb{T}_L \times \{1\}))),\end{aligned} \tag{4.2.8}$$

and for $(\widehat{\sigma}, \widehat{\psi}) \in \mathbb{H}_1^{T_0}$ satisfying the condition

$$(\widehat{\psi}, \partial_t \widehat{\psi})(\cdot, T) = (\psi_T, \psi_T^1), \tag{4.2.9}$$

we solve the system

$$\begin{cases}
-\partial_t \sigma - \overline{u}_1 \partial_x \sigma + \dfrac{P'(\overline{\rho})\overline{\rho}}{\nu}\sigma = \dfrac{P'(\overline{\rho})}{\nu} q & \text{in } (\mathbb{T}_L \times (0,1)) \times (T - T_0, T),\\
-(\partial_t q + \overline{u}_1 \partial_x q) - \dfrac{\nu}{\overline{\rho}}\Delta q - \dfrac{P'(\overline{\rho})\overline{\rho}}{\nu} q = -\dfrac{P'(\overline{\rho})\overline{\rho}^2}{\nu}\widehat{\sigma} & \text{in } (\mathbb{T}_L \times (0,1)) \times (T - T_0, T),\\
\partial_z q = -\overline{\rho}(\partial_t \widehat{\psi} + \overline{u}_1 \partial_x \widehat{\psi}) & \text{on } (\mathbb{T}_L \times \{1\}) \times (T - T_0, T),\\
\partial_z q = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (T - T_0, T),\\
q(\cdot, T) = q_T & \text{in } \mathbb{T}_L \times (0,1),\\
\sigma(\cdot, T) = \sigma_T & \text{in } \mathbb{T}_L \times (0,1),\\
\partial_{tt}\psi + \partial_{txx}\psi + \partial_{xxxx}\psi = (\partial_t + \overline{u}_1 \partial_x) q & \text{on } (\mathbb{T}_L \times \{1\}) \times (T - T_0, T),\\
\psi(T) = \psi_T \quad \text{and} \quad \partial_t \psi(T) = \psi_T^1 & \text{in } \mathbb{T}_L \times \{1\},
\end{cases} \tag{4.2.10}$$

This defines the following map:

$$\mathcal{L}_{T_0} : (\widehat{\sigma}, \widehat{\psi}) \in \mathbb{H}_1^{T_0} \longmapsto (\sigma, \psi). \tag{4.2.11}$$

We will show that $\mathcal{L}_{T_0}$ maps $\mathbb{H}_1^{T_0}$ into itself and is a contraction there. Observe that a fixed point of the map infers a solution to the system (4.2.3) in the time interval $(T-T_0,T)$. In the sequel we will show that the map $\mathcal{L}_{T_0}$ admits a fixed point in $\mathbb{H}_{T_0}^1$, for T_0 sufficiently small.
In that direction we first claim that there exists a positive constant C, independent of T_0 such that

$$\begin{aligned}&\|q\|_{L^2(T-T_0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\&\leqslant C(\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}\\&+\|\widehat{\sigma}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}+\|q_T)\|_{H^2(\mathbb{T}_L\times(0,1))}))\end{aligned}\tag{4.2.12}$$

To begin with, we will just use the following regularities for the non-homogeneous source term, boundary data and the initial condition

$$\begin{cases}\widehat{\sigma}\in L^2(T-T_0,T;L^2(\mathbb{T}_L\times(0,1))),\\-\overline{\rho}(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\in L^2(T-T_0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\\\qquad\qquad\qquad\cap H^{1/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\})),\\q_T\in H^1(\mathbb{T}_L\times(0,1)).\end{cases}\tag{4.2.13}$$

Hence using the regularities (4.2.13) one can apply [24, Theorem 5.3, p. 32] to solve $(4.2.10)_2$-$(4.2.10)_5$ in the following functional framework

$$q\in L^2(T-T_0,T;H^2(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1))).\tag{4.2.14}$$

Moreover there exists a positive constant C independent of T_0 such that

$$\begin{aligned}&\|q\|_{L^2(T-T_0,T;H^2(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\&\leqslant C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}+\|(\nu\mathrm{div}\,v_T+\overline{\rho}\sigma_T)\|_{H^1(\mathbb{T}_L\times(0,1))}\\&\quad+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\cap H^{1/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}).\end{aligned}\tag{4.2.15}$$

Let us explain how we obtain a constant C independent of T_0 in the inequality (4.2.15). The technique is inspired from [29]. We extend $\widehat{\sigma}$ and $(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})$ in $(0,T)$ by defining them zero in the time interval $(0,T-T_0)$. The extended functions are also denoted by the same notations $\widehat{\sigma}$ and $(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})$. It is easy to verify that

$$\begin{aligned}\widehat{\sigma}\in L^2(0,T;L^2(\mathbb{T}_L\times(0,1)))\text{ and }(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\in&L^2(0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\\&\cap H^{1/4}(0,T;L^2(\mathbb{T}_L\times\{1\})).\end{aligned}$$

One then has the following

$$\begin{aligned}&\|q\|_{L^2(T-T_0,T;H^2(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\&\leqslant\|q\|_{L^2(0,T;H^2(\mathbb{T}_L\times(0,1)))\cap H^1(0,T;L^2(\mathbb{T}_L\times(0,1)))}\\&\leqslant C(\|\widehat{\sigma}\|_{L^2(0,T;L^2(\mathbb{T}_L\times(0,1)))}+\|q_T\|_{H^1(\mathbb{T}_L\times(0,1))}\\&\qquad+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\cap H^{1/4}(0,T;L^2(\mathbb{T}_L\times\{1\}))})\\&=C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}+\|q_T\|_{H^1(\mathbb{T}_L\times(0,1))}\\&\qquad+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\cap H^{1/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}),\end{aligned}\tag{4.2.16}$$

where the constant C is independent of T_0.
Now to prove (4.2.12) we write the equations $(4.2.10)_2$-$(4.2.10)_5$ as follows:

$$\begin{cases} -\partial_t q - \dfrac{\nu}{\overline{\rho}}\Delta q = -\dfrac{P'(\overline{\rho})\overline{\rho}^2}{\nu}\widehat{\sigma} + \overline{u}_1\partial_x q + \dfrac{P'(\overline{\rho})\overline{\rho}}{\nu}q := \mathcal{G}_{\widehat{\sigma},q} & \text{in } (\mathbb{T}_L\times(0,1))\times(T-T_0,T),\\ \partial_z q = -\overline{\rho}(\partial_t\widehat{\psi} + \overline{u}_1\partial_x\widehat{\psi}) & \text{on } (\mathbb{T}_L\times\{1\})\times(T-T_0,T),\\ \partial_z q = 0 & \text{on } (\mathbb{T}_L\times\{0\})\times(T-T_0,T),\\ q(\cdot,T) = q_T = \nu\mathrm{div}v_T + \overline{\rho}\sigma_T & \text{in } \mathbb{T}_L\times(0,1), \end{cases} \tag{4.2.17}$$

In view of (4.2.16) one has the following regularity estimate of $\partial_x q$ by interpolation

$$\begin{aligned} &\|\partial_x q\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^{1/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\ &\leqslant C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))} + \|q_T\|_{H^1(\mathbb{T}_L\times(0,1))}\\ &+ \|(\partial_t\widehat{\psi} + \overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\cap H^{1/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}), \end{aligned} \tag{4.2.18}$$

for some positive constant C independent of T_0. Indeed, this can be obtained by performing the interpolation process in the time interval $(0,T)$ instead of $(T-T_0,T)$. Hence the assumption (4.2.8), the obtained regularity (4.2.16) and (4.2.18) implies that

$$\mathcal{G}_{\widehat{\sigma},q}\in L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^{1/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))$$

and

$$\begin{aligned} &\|\mathcal{G}_{\widehat{\sigma},q}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^{1/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\ &\leqslant C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\ &\quad + \|(\partial_t\widehat{\psi} + \overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\cap H^{1/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}\\ &\quad + \|q_T\|_{H^1(\mathbb{T}_L\times(0,1))}). \end{aligned} \tag{4.2.19}$$

At this stage we will use the following regularities of the boundary and initial datum which follows from (4.2.8) and (4.1.28):

$$\begin{cases} -\overline{\rho}(\partial_t\widehat{\psi} + \overline{u}_1\partial_x\widehat{\psi}) \in L^2(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\\ \qquad\qquad\qquad\qquad \cap H^{3/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\})),\\ q_T\in H^2(\mathbb{T}_L\times(0,1)). \end{cases} \tag{4.2.20}$$

Furthermore, in view of (4.2.5), (4.2.19) and (4.2.20), we can apply [24, Theorem 5.3, p. 32] to solve (4.2.17) in the functional framework

$$q\in L^2(T-T_0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1))). \tag{4.2.21}$$

It is well known that one can obtain a positive constant independent of T_0 such that the

solution q of the heat equation (4.2.17) satisfies the following estimate

$$\begin{aligned}
&\|q\|_{L^2(T-T_0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\
&\leqslant C(\|\mathcal{G}_{\widehat{\sigma},q}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^{1/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\
&+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}\\
&+\|q_T\|_{H^2(\mathbb{T}_L\times(0,1))})\\
&\leqslant C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\
&+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}\\
&+\|q_T\|_{H^2(\mathbb{T}_L\times(0,1))}).
\end{aligned}\tag{4.2.22}$$

In the final step of the estimate (4.2.22) we have used (4.2.19).
The regularity (4.2.21) implies that $q\in C^0([T-T_0,T];H^1(\mathbb{T}_L\times(0,1)))$. Hence using the regularity assumption $\sigma_T\in H^1(\mathbb{T}_L\times(0,1))$, we obtain the following by solving $(4.2.10)_1$ and $(4.2.10)_6$,

$$\begin{aligned}
&\sigma\in\mathbb{H}_2^{T_0},\\
&\text{where we have set } H_2^{T_0}=C^0([T-T_0,T];H^1(\mathbb{T}_L\times(0,1)))\cap C^1([T-T_0,T];L^2(\mathbb{T}_L\times(0,1))).
\end{aligned}\tag{4.2.23}$$

Moreover, we have the estimate

$$\|\sigma\|_{\mathbb{H}_2^{T_0}}\leqslant C(\|q\|_{L^\infty(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))}+\|\sigma_T\|_{H^1(\mathbb{T}_L\times(0,1))}),\tag{4.2.24}$$

for some positive constant C independent of T_0. For the proofs of (4.2.23) and (4.2.24) one can follow the arguments used in proving [30, Lemma 2.4]. In fact the only difference with [30, Lemma 2.4] is that we solve the transport equation with a non homogeneous source term but this adaptation is straightforward.
Now one considers the equations $(4.2.10)_7$-$(4.2.10)_8$. Using standard trace theorem and (4.2.21) one in particular has

$$(\partial_t+\overline{u}_1\partial_x)q\mid_{\mathbb{T}_L\times\{1\}}\in L^2(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))\tag{4.2.25}$$

and the following holds in the view of (4.2.22)

$$\begin{aligned}
&\|(\partial_t+\overline{u}_1\partial_x)q\|_{L^2(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}\\
&\leqslant C\|q\|_{L^2(T-T_0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\
&\leqslant C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\
&\quad+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}\\
&\quad+\|q_T\|_{H^2(\mathbb{T}_L\times(0,1))}).
\end{aligned}\tag{4.2.26}$$

Now the regularity (4.2.25) and the assumption (4.1.28) on (ψ_T,ψ_T^1) furnish the following regularities for ψ :

$$\begin{aligned}
\psi\in&L^2(T-T_0,T;H^4(\mathbb{T}_L\times\{1\}))\cap H^1(T-T_0,T;H^2(\mathbb{T}_L\times\{1\}))\\
&\cap H^2(T-T_0,T;L^2(\mathbb{T}_L\times\{1\})).
\end{aligned}\tag{4.2.27}$$

The above regularity result for ψ is a consequence of the fact that the corresponding damped beam operator is the generator of an analytic semigroup. This result can be found in [9] (see also [28]). Using standard interpolation results it is not hard to observe from (4.2.27) that

$$\begin{aligned}&\psi \in \mathbb{H}_3^{T_0},\\ &\text{where we have set } \mathbb{H}_3^{T_0} = H^{5/4}(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^1(T-T_0,T;H^2((\mathbb{T}_L\times\{1\})))\\ &\qquad\cap H^2(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(T-T_0,T;H^{5/2}((\mathbb{T}_L\times\{1\}))).\end{aligned} \tag{4.2.28}$$

Furthermore, one has the following in view of the estimate (4.2.26)

$$\begin{aligned}&\|\psi\|_{\mathbb{H}_3^{T_0}}\\ &\leqslant C(\|(\partial_t+\overline{u}_1\partial_x)q\|_{L^2(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}+\|(\psi_T,\psi_T^1)\|_{H^3(\mathbb{T}_L\times\{1\})\times H^1(\mathbb{T}_L\times\{1\})})\\ &\leqslant C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\ &+\|(\psi_T,\psi_T^1)\|_{H^3(\mathbb{T}_L\times\{1\})\times H^1(\mathbb{T}_L\times\{1\})}+\|q_T\|_{H^2(\mathbb{T}_L\times(0,1))}\\ &+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}),\end{aligned} \tag{4.2.29}$$

for some positive constant C independent of T_0. One can obtain a constant C independent of T_0 in the inequality (4.2.29) by defining $(\partial_t+\overline{u}_1\partial_x)q$ equal zero in the times interval $(0,T-T_0)$ and following the line of arguments already used in (4.2.16).
We further use (4.2.22) in (4.2.24) and combine with the estimate (4.2.29) to infer the following

$$\begin{aligned}&\|(\sigma,\psi)\|_{\mathbb{H}_2^{T_0}\times\mathbb{H}_3^{T_0}}\\ &\leqslant C(\|\widehat{\sigma}\|_{L^2(T-T_0,T;H^1(\mathbb{T}_L\times(0,1)))\cap H^1(T-T_0,T;L^2(\mathbb{T}_L\times(0,1)))}\\ &+\|(\partial_t\widehat{\psi}+\overline{u}_1\partial_x\widehat{\psi})\|_{L^2(T-T_0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(T-T_0,T;L^2(\mathbb{T}_L\times\{1\}))}\\ &+\|(\sigma_T,q_T,\psi_T,\psi_T^1)\|_{H^1(\mathbb{T}_L\times(0,1))\times H^2(\mathbb{T}_L\times(0,1))\times H^3(\mathbb{T}_L\times\{1\})\times H^1(\mathbb{T}_L\times\{1\})})\\ &\leqslant C(\|(\widehat{\sigma},\widehat{\psi})\|_{\mathbb{H}_1^{T_0}}\\ &\qquad+\|(\sigma_T,q_T,\psi_T,\psi_T^1)\|_{H^1(\mathbb{T}_L\times(0,1))\times H^2(\mathbb{T}_L\times(0,1))\times H^3(\mathbb{T}_L\times\{1\})\times H^1(\mathbb{T}_L\times\{1\})}),\end{aligned} \tag{4.2.30}$$

for some positive constant C independent of T_0.
So far we have observed that $\mathcal{L}_{T_0}$ (defined in (4.2.11)) maps $\mathbb{H}_1^{T_0}$ to $\mathbb{H}_2^{T_0}\times\mathbb{H}_3^{T_0}$, which is obviously a subset of $\mathbb{H}_1^{T_0}$.
Now, we will choose T_0 small enough such that the map $\mathcal{L}_{T_0}$ admits a fixed point in the space $\mathbb{H}_1^{T_0}$.
Let us compare the space $\mathbb{H}_1^{T_0}$ with $\mathbb{H}_2^{T_0}\times\mathbb{H}_3^{T_0}$ to observe that there exists a constant $s>0$ such that

$$\|(\cdot,\cdot)\|_{\mathbb{H}_1^{T_0}}\leqslant CT_0^s\|(\cdot,\cdot)\|_{\mathbb{H}_2^{T_0}\times\mathbb{H}_3^{T_0}}, \tag{4.2.31}$$

for some positive constant C independent of T_0. Now let $(\widehat{\sigma}_1,\widehat{\psi}_1)\in\mathbb{H}_1^{T_0}$ and $(\widehat{\sigma}_2,\widehat{\psi}_2)\in\mathbb{H}_1^{T_0}$ and $(\sigma_1,\psi_1)=\mathcal{L}_{T_0}(\widehat{\sigma}_1,\widehat{\psi}_1)$ and $(\sigma_2,\psi_2)=\mathcal{L}_{T_0}(\widehat{\sigma}_2,\widehat{\psi}_2)$.

Using (4.2.31) and the estimate (4.2.30), one has

$$\begin{aligned}\|(\sigma_1-\sigma_2,\psi_1-\psi_2)\|_{\mathbb{H}_1^{T_0}} &\leqslant CT_0^s\|(\sigma_1-\sigma_2,\psi_1-\psi_2)\|_{\mathbb{H}_2^{T_0}\times\mathbb{H}_3^{T_0}}\\ &\leqslant CT_0^s\|(\widehat{\sigma}_1-\widehat{\sigma}_2,\widehat{\psi}-\widehat{\psi}_2)\|_{\mathbb{H}_1^{T_0}},\end{aligned} \tag{4.2.32}$$

for some positive constant C independent of T_0.
In view of these arguments for $T-T_0$ close to T, *i.e* for T_0 small enough, the map $\mathcal{L}_{T_0}$ is a contraction from $\mathbb{H}_1^{T_0}$ to itself. Hence applying the Banach fixed point theorem we obtain that for T_0 small enough, the map $\mathcal{L}_{T_0}$ admits a unique fixed point $(\widehat{\sigma},\widehat{\psi})$ in $\mathbb{H}_1^{T_0}$. As $(\widehat{\sigma},\widehat{\psi})=\mathcal{L}_{T_0}(\widehat{\sigma},\widehat{\psi})$, we also have that $(\widehat{\sigma},\widehat{\psi})$ belongs to $\mathbb{H}_2^{T_0}\times\mathbb{H}_3^{T_0}$ and the following regularities coming from (4.2.21) and (4.2.27):

$$\left\{\begin{array}{l}\sigma\in C^0([T-T_0,T];H^1(\mathbb{T}_L\times(0,1)))\cap C^1([T-T_0,T];L^2(\mathbb{T}_L\times(0,1))),\\ q\in L^2(T-T_0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(T-T_0,T;L^2(\mathbb{T}_L\times(0,1))),\\ \psi\in L^2(T-T_0,T;H^4(\mathbb{T}_L\times\{1\}))\cap H^1(T-T_0,T;H^2(\mathbb{T}_L\times\{1\}))\\ \qquad\cap H^2(T-T_0,T;L^2(\mathbb{T}_L\times\{1\})),\end{array}\right. \tag{4.2.33}$$

provided T_0 is sufficiently small. Further (4.2.33) infers that

$$\left\{\begin{array}{l}\sigma\in C^0([T-T_0,T;H^1(\mathbb{T}_L\times(0,1))]),\ q\in C^0([T-T_0,T];H^2(\mathbb{T}_L\times(0,1)))\\ \psi\in C^0([T-T_0,T];H^3(\mathbb{T}_L\times\{1\}))\cap C^1([T-T_0,T];H^1(\mathbb{T}_L\times(0,1))).\end{array}\right. \tag{4.2.34}$$

The continuities (4.2.34) in time and the system (4.2.3) can be used to check the following compatibilities at time $t=T-T_0$:

$$\begin{array}{l}(i)\,\partial_z q(\cdot,T-T_0)=-\overline{\rho}(\partial_t\psi+\overline{u}_1\partial_x\psi)(\cdot,T-T_0)\text{ on }\mathbb{T}_L\times\{1\},\\ (ii)\,\partial_z q(\cdot,T-T_0)=0\text{ on }\mathbb{T}_L\times\{0\}.\end{array} \tag{4.2.35}$$

Further one recalls that in proving (4.2.33) we did no assumption on the size of the initial datum. In view of (4.2.34), (4.2.35) and using that the linearity of the system (4.2.3) the solution (σ,q,ψ) can be extended to the time interval $(0,T)$ by iteration in order to prove:

$$\left\{\begin{array}{l}\sigma\in C^0([0,T];H^1(\mathbb{T}_L\times(0,1)))\cap C^1([0,T];L^2(\mathbb{T}_L\times(0,1))),\\ q\in L^2(0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(0,T;L^2(\mathbb{T}_L\times(0,1))),\\ \psi\in L^2(0,T;H^4(\mathbb{T}_L\times\{1\}))\cap H^1(0,T;H^2(\mathbb{T}_L\times\{1\}))\\ \qquad\cap H^2(0,T;L^2(\mathbb{T}_L\times\{1\})).\end{array}\right. \tag{4.2.36}$$

This finishes the proof of (4.2.7) and thus of Lemma 4.2.2. □

4.2.1.2 Step 2: Constructing v.

In order to make complete the proof of the existence of a solution (σ,v,ψ) of (4.1.27), we first set $q_T=\nu\mathrm{div}v_T+\overline{\rho}\sigma_T$ and solve the system (4.2.3) with initial data $(\sigma_T,q_T,\psi_T,\psi_T^1)$.

Note that the regularity and compatibility conditions of the initial data $(\sigma_T, v_T, \psi_T, \psi_T^1)$ in Theorem 4.1.4 precisely imply the regularity and compatibility conditions of $(\sigma_T, q_T, \psi_T, \psi_T^1)$ required by Lemma 4.2.2, so that Lemma 4.2.2 applies, yielding functions (σ, q, ψ) solving (4.2.3). Besides, if $\sigma_T \in H^2(\mathbb{T}_L \times \{1\})$, it is clear that the solution σ of $(4.2.3)_{(1)}$ in fact satisfies

$$\sigma \in C^0([0,T]; H^2(\mathbb{T}_L \times (0,1))) \cap C^1([0,T]; H^1(\mathbb{T}_L \times (0,1))),$$

since the source term of the transport equation belongs to $C^0([0,T]; H^2(\mathbb{T}_L \times (0,1)))$. We then construct the function v by solving the equation

$$\begin{cases} -\overline{\rho}(\partial_t v + \overline{u}_1 \partial_x v) - \mu \Delta v - (\mu' + \mu)\nabla(\mathrm{div} v) - \overline{\rho}\nabla\sigma = 0 & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ v \cdot n = v_2 = \psi & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ v(\cdot,t) \cdot n = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T), \\ \mathrm{curl} v = 0 & \text{on } (\mathbb{T}_L \times \{0,1\}) \times (0,T), \\ v(\cdot,T) = v_T & \text{in } \mathbb{T}_L \times (0,1). \end{cases} \tag{4.2.37}$$

Note that we already know the regularity of the source terms $\overline{\rho}\nabla\sigma$ and ψ from (4.2.36), so this step only consists in constructing a solution to a parabolic equation with source terms having given regularities. In order to analyze it properly, we will first do a lifting of the boundary condition to recast it into a distributed source term.

Step 2.a. In this step we will only use the following regularity assumption on the initial datum (which follows from (4.1.28))

$$v_T \in H^1(\mathbb{T}_L \times (0,1)), \tag{4.2.38}$$

and the following compatibility assumption (which follows from $(4.1.29)(i)$),

$$\begin{aligned} &(i)\ v_T \cdot n = (v_T)_2 = \psi_T, \text{ on } \mathbb{T}_L \times \{1\}, \\ &(ii)\ v_T \cdot n = (v_T)_2 = 0, \text{ on } \mathbb{T}_L \times \{0\}, \end{aligned} \tag{4.2.39}$$

and the regularities (4.2.36) obtained in *Step 1* to prove that the solution v of (4.2.37) has the following regularity

$$v \in L^2(0,T; H^2(\mathbb{T}_L \times (0,1))) \cap H^1(0,T; L^2(\mathbb{T}_L \times (0,1))).$$

We consider the system (4.2.37). We will consider a lifting of the boundary term ψ to work with a homogeneous boundary value problem. We define the lifting function ψ_{lift} which satisfies, for all $t \in (0,T)$,

$$\begin{cases} \Delta \psi_{lift}(\cdot,t) = 0 \text{ in } \mathbb{T}_L \times (0,1), \\ \partial_z \psi_{lift}(\cdot,t) \,|_{\mathbb{T}_L \times \{1\}} = \psi(\cdot,t), \\ \partial_z \psi_{lift}(\cdot,t) \,|_{\mathbb{T}_L \times \{0\}} = 0, \end{cases} \tag{4.2.40}$$

for all $t \in (0,T]$.
The regularities of ψ from (4.2.36) infer the following regularity of ψ_{lift} :

$$\begin{aligned}\psi_{lift} \in & L^2(0,T;H^{11/2}(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;H^{7/2}(\mathbb{T}_L \times (0,1))) \\ & \cap H^2(0,T;H^{3/2}(\mathbb{T}_L \times (0,1))).\end{aligned} \tag{4.2.41}$$

We define a new unknown as follows

$$V = (v - \nabla\psi_{lift}).$$

By using (4.2.37), we obtain the following equation satisfied by V:

$$\begin{cases} -\partial_t V - \overline{u}_1 \partial_x V - \frac{\mu}{\overline{\rho}}\Delta V - \frac{(\mu+\mu')}{\overline{\rho}}\nabla \mathrm{div} V = \mathcal{G} & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ V(\cdot,t)\cdot n = 0 & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ V(\cdot,t)\cdot n = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T), \\ \mathrm{curl} V = 0 & \text{on } (\mathbb{T}_L \times \{0,1\}) \times (0,T), \\ V(\cdot,T) = V_T = v_T - \nabla\psi_{lift}(\cdot,T) & \text{in } \mathbb{T}_L \times (0,1). \end{cases} \tag{4.2.42}$$

where

$$\mathcal{G} = \nabla\partial_t\psi_{lift} + \overline{u}_1\nabla\partial_x\psi_{lift} + \frac{\mu}{\overline{\rho}}\Delta\nabla\psi_{lift} + \frac{(\mu+\mu')}{\overline{\rho}}\nabla\mathrm{div}\nabla\psi_{lift} + \nabla\sigma. \tag{4.2.43}$$

In order to prove the existence of a solution to the system (4.2.42), we first define the following space

$$H_n = \{V \in H^1(\mathbb{T}_L \times (0,1)) \mid V \cdot n = 0 \text{ on } (\mathbb{T}_L \times \{0,1\}).$$

Now we introduce the following unbounded operator on $L^2(\mathbb{T}_L \times (0,1))$ by

$$\mathcal{A} = -\overline{u}_1\partial_x - \frac{\mu}{\overline{\rho}}\Delta - \frac{(\mu+\mu')}{\overline{\rho}}\nabla\mathrm{div}, \tag{4.2.44}$$

with domain

$$\begin{aligned}\mathcal{D}(\mathcal{A}) &= \{V \in H_n \mid \mathcal{A}V \in L^2(\mathbb{T}_L \times (0,1)),\ \mathrm{curl} V = 0 \text{ and } V\cdot n = 0 \text{ on } \\ &\qquad\qquad (\mathbb{T}_L \times \{0,1\})\} \\ &= \{V \in H^2(\mathbb{T}_L \times (0,1)) \mid \mathrm{curl} V = 0 \text{ and } V\cdot n = 0 \text{ on } \\ &\qquad\qquad (\mathbb{T}_L \times \{0,1\})\}.\end{aligned} \tag{4.2.45}$$

In $(4.2.45)_2$ we have used that an element V of H_n satisfying

$$\begin{cases} -\overline{u}_1\partial_x V - \frac{\mu}{\overline{\rho}}\Delta V - \frac{(\mu+\mu')}{\overline{\rho}}\nabla\mathrm{div} V \in L^2(\mathbb{T}_L \times (0,1)), \\ \mathrm{curl} V = 0 \text{ on } (\mathbb{T}_L \times \{0,1\}), \\ V\cdot n = 0 \text{ on } (\mathbb{T}_L \times \{0,1\}), \end{cases} \tag{4.2.46}$$

admits $H^2(\mathbb{T}_L \times (0,1))$ regularity and this can be proved by following the line of arguments used to prove [27, Theorem 3.2] (one can also consult [6, p. 332, Section 9.2] and [31, p. 173, Lemma 2.4]). One can notice that in [27] and [31] the authors consider Navier slip boundary condition whereas we consider no vorticity condition in (4.2.46). Actually since our boundaries are flat, one can easily verify that

$$\begin{aligned} &V \cdot n = 0 \text{ and } \mathrm{curl} V = 0 \text{ on } (\mathbb{T}_L \times \{0,1\}) \\ &\Leftrightarrow \text{Navier slip boundary condition } i.e.\, V \cdot n = 0 \\ &\quad \text{and } (2\mu D(V) + \mu' \mathrm{div}\, V I_d) n \cdot \vec{\tau} = 0 \text{ on } (\mathbb{T}_L \times \{0,1\}) \text{ as considered in} \\ &\quad [27] \text{ and } [31], \end{aligned} \tag{4.2.47}$$

where $\vec{\tau}$ denotes the unit tangent vector to the boundary.
Corresponding to the operator $\mathcal{A}$ we define the following bilinear form

$$a(V_1, V_2) = -\overline{u}_1 \int\limits_{\mathbb{T}_L \times (0,1)} \partial_x V_1 V_2 + \frac{\mu}{\overline{\rho}} \int\limits_{\mathbb{T}_L \times (0,1)} \mathrm{curl} V_1 \mathrm{curl} V_2 + \frac{(2\mu + \mu')}{\overline{\rho}} \int\limits_{\mathbb{T}_L \times (0,1)} \mathrm{div} V_1 \mathrm{div} V_2$$

on the space H_n.
Using the x-periodicity of the functions in H_n one readily checks that $\forall\, V \in H_n$,

$$a(V, V) = \frac{\mu}{\overline{\rho}} \int\limits_{\mathbb{T}_L \times (0,1)} |\mathrm{curl} V|^2 + \frac{(2\mu + \mu')}{\overline{\rho}} \int\limits_{\mathbb{T}_L \times (0,1)} |\mathrm{div} V|^2.$$

Hence one verifies that $a(\cdot, \cdot)$ is a continuous bilinear form on H_n and satisfies the following coercivity

$$a(V, V) + \|V\|^2_{L^2(\mathbb{T}_L \times (0,1))} \geqslant c \|V\|^2_{H_n}, \tag{4.2.48}$$

for some small positive constant c.
At this point one can apply [3, p. 115, Theorem 3.12] to deduce that $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ generates an analytic semigroup on $L^2(\mathbb{T}_L \times (0,1))$.
We will use this analyticity and apply the isomorphism theorem [3, p. 143, Theorem 3.1] to prove the existence of a unique solution to the system (4.2.42). In that direction in view of the compatibility condition (4.2.39) one observes that

$$V_T \cdot n = 0 \text{ on } \ (\mathbb{T}_L \times \{0,1\}).$$

This in particular implies that

$$V_T \in \mathcal{D}(\mathcal{A}^{1/2}) = \{V \in H^1(\mathbb{T}_L \times (0,1)) \mid V \cdot n = 0 \text{ on } (\mathbb{T}_L \times \{0,1\})\}. \tag{4.2.49}$$

Also using the regularities of ψ_{lift} from (4.2.41) and σ from (4.2.36) we can verify that $\mathcal{G}$ (defined in (4.2.43)) has the following regularity

$$\mathcal{G} \in L^2(\mathbb{T}_L \times (0,1)).$$

At this point we apply the isomorphism theorem [3, p. 143, Theorem 3.1] to infer the existence of a unique solution of the problem (4.2.42) in the following functional framework

$$V \in L^2(0,T;\mathcal{D}(\mathcal{A})) \cap H^1(0,T;L^2(\mathbb{T}_L \times (0,1))). \tag{4.2.50}$$

One further uses (4.2.41) to furnish that

$$v \in L^2(0,T;H^2(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;L^2(\mathbb{T}_L \times (0,1))). \tag{4.2.51}$$

Step 2.b. In this step we will use the full regularity assumption (4.1.28), i.e.

$$(\sigma_T, v_T) \in H^2(\mathbb{T}_L \times (0,1)) \times H^3(\mathbb{T}_L \times (0,1)), \tag{4.2.52}$$

the compatibility conditions (consequences of (4.1.29)(ii)-(4.1.29)(iii)),

$$\begin{aligned}
&(i)\ \operatorname{curl} v_T = 0, \text{ on } (\mathbb{T}_L \times \{0,1\}),\\
&(ii)\ -\overline{u}_1 \partial_x (v_T)_2 + \frac{\mu}{\overline{\rho}} \Delta (v_T)_2 + \frac{(\mu+\mu')}{\overline{\rho}} \partial_z (\operatorname{div} v_T) \partial_z \sigma_T = \psi_T^1 \text{ on } \mathbb{T}_L \times \{1\},\\
&\quad -\overline{u}_1 \partial_x (v_T)_2 + \frac{\mu}{\overline{\rho}} \Delta (v_T)_2 + \frac{(\mu+\mu')}{\overline{\rho}} \partial_z (\operatorname{div} v_T) \partial_z \sigma_T = 0 \text{ on } \mathbb{T}_L \times \{0\},
\end{aligned} \tag{4.2.53}$$

and the regularities obtained in (4.2.36) to prove that the solution v of (4.2.37) has the following improved regularity

$$v \in L^2(0,T;H^3(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;H^2(\mathbb{T}_L \times (0,1))) \cap H^2(0,T;L^2(\mathbb{T}_L \times (0,1))).$$

We introduce the new unknowns

$$V^* = \partial_t V, \quad \text{and} \quad \psi^*_{lift} = \partial_t \psi_{lift}.$$

One obtains the following system solved by V^*:

$$\begin{cases}
-\partial_t V^* - \overline{u}_1 \partial_x V^* - \frac{\mu}{\overline{\rho}} \Delta V^* - \frac{(\mu+\mu')}{\overline{\rho}} \nabla \operatorname{div} V^* = \mathcal{G}^* & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T),\\
V^*(\cdot,t) \cdot n = 0 & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T),\\
V^*(\cdot,t) \cdot n = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T),\\
\operatorname{curl} V^* = 0 & \text{on } (\mathbb{T}_L \times \{0,1\}) \times (0,T),\\
V^*(\cdot,T) = V_T^* = \mathcal{V}_T - \nabla \psi^*_{lift}(\cdot,T) & \text{in } \mathbb{T}_L \times (0,1),
\end{cases} \tag{4.2.54}$$

where

$$\mathcal{G}^* = \nabla \partial_t \psi^*_{lift} + \overline{u}_1 \nabla \partial_x \psi^*_{lift} + \frac{\mu}{\overline{\rho}} \Delta \nabla \psi^*_{lift} + \frac{(\mu+\mu')}{\overline{\rho}} \nabla \operatorname{div} \nabla \psi^*_{lift} + \nabla \partial_t \sigma. \tag{4.2.55}$$

Now, we recall from (4.2.36) that

$$\partial_t \psi \in L^2(0,T;H^2(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;L^2(\mathbb{T}_L \times (0,1))).$$

Hence ψ^*_{lift} satisfies the following regularities

$$\begin{aligned} \psi^*_{lift} \in L^2(0,T;H^{7/2}(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;H^{3/2}(\mathbb{T}_L \times (0,1))) \\ \cap C^0([0,T];H^{5/2}(\mathbb{T}_L \times (0,1))). \end{aligned} \tag{4.2.56}$$

Applying (4.2.36) and (4.2.56) we infer

$$\mathcal{G}^* \in L^2(0,T;L^2(\mathbb{T}_L \times (0,1))). \tag{4.2.57}$$

Furthermore the regularity assumptions (4.2.52) and (4.2.36) furnish

$$V^*_T \in H^1(\mathbb{T}_L \times (0,1)). \tag{4.2.58}$$

One now uses (4.2.57), (4.2.58), the compatibility assumption (4.2.53)(ii) and imitates the analysis used to show (4.2.50) to infer

$$V^* \in L^2(0,T;H^2(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;L^2(\mathbb{T}_L \times (0,1))).$$

Hence in view of (4.2.56) we conclude that

$$\partial_t v \text{ and } \partial_t V \text{ belong to } L^2(0,T;H^2(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;L^2(\mathbb{T}_L \times (0,1))). \tag{4.2.59}$$

Now we will improve on the space regularity of v. For that once again we get back to the system (4.2.42) solved by V $(= v - \nabla\psi_{lift})$. Using (4.2.59) and the regularity (4.2.36) for σ we observe that $(\mathcal{G} + \partial_t V)$ (where $\mathcal{G}$ is defined in (4.2.43)) satisfies

$$(\mathcal{G} + \partial_t V) \in L^2(0,T;H^1(\mathbb{T}_L \times (0,1))). \tag{4.2.60}$$

We write the equation satisfied by V in the following form

$$\begin{cases} -\overline{u}_1 \partial_x V - \frac{\mu}{\overline{\rho}} \Delta V - \frac{(\mu+\mu')}{\overline{\rho}} \nabla \mathrm{div} V = \mathcal{G} + \partial_t V & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ V \cdot n = 0 & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ V \cdot n = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T), \\ \mathrm{curl} V = 0 & \text{on } ((\mathbb{T}_L \times \{0,1\}) \times (0,T). \end{cases} \tag{4.2.61}$$

Once again we recall the remark (4.2.47), imitate the proof of boundary regularity done in [27, Theorem 3.2] along with the fact that the boundary of $\mathbb{T}_L \times (0,1)$ is smooth (since $\mathbb{T}_L$ is a torus) in order to prove

$$V \in L^2(0,T;H^3(\mathbb{T}_L \times (0,1))).$$

This regularity of V with (4.2.41) implies

$$v \in L^2(0,T;H^3(\mathbb{T}_L \times (0,1))). \tag{4.2.62}$$

Conclusion. Next (4.2.51), (4.2.59), (4.2.62) combined together and the compatibilities (4.1.29) allow us to conclude that v enjoys the regularity

$$\begin{aligned} v \in L^2(0,T;H^3(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;H^2(\mathbb{T}_L \times (0,1))) \\ \cap H^2(0,T;L^2(\mathbb{T}_L \times (0,1))) \end{aligned} \tag{4.2.63}$$

and solves the system (4.2.37). With the strong regularity framework (4.2.63) and with σ in the space (4.2.36) one can verify the formal argument (4.2.2) and conclude that the triplet (σ, v, ψ) solves the system (4.1.27) in the functional framework (4.1.30).
Now in order to prove the uniqueness of the solution (σ, v, ψ) of (4.1.27) in the spaces (4.1.30), we proceed as follows. Let us assume that there exists two solutions (σ_1, v_1, ψ_1) and (σ_2, v_2, ψ_2) to the problem (4.1.27) in the functional spaces (4.1.30) with the same initial datum. The strong regularities of v_1 (as well as v_2) and σ_1 (as well as σ_2) allows us to verify that (σ_1, q_1, ψ_1) and (σ_2, q_2, ψ_2) solves (4.2.3), where

$$q_1 = (\nu \mathrm{div}\, v_1 + \overline{\rho}\sigma_1) \text{ and } q_2 = (\nu \mathrm{div}\, v_2 + \overline{\rho}\sigma_2).$$

But the solution of the system (4.2.3) is unique (thanks to the Banach fixed point argument used in *Step* 2) in the framework (4.2.36). Hence $\sigma_1 = \sigma_2$ and $\psi_1 = \psi_2$. Now one observes that v_1 (respectively v_2) solves (4.2.37) with (σ_1, ψ_1) (respectively (σ_2, ψ_2)). Since $(\sigma_1, \psi_1) = (\sigma_2, \psi_2)$, from the uniqueness of the solution to the linear problem (4.2.37) we infer that $v_1 = v_2$. Hence the solution to the problem (4.1.27) is unique in the functional framework (4.1.30).
This concludes the proof of Theorem 4.1.4.

4.2.2 Sketch of the proof of Theorem 4.1.3

The strategy used to prove Theorem 4.1.3 is very similar to the one used to prove Theorem 4.1.4.
In a similar spirit of the *Step 1* of the proof of Theorem 4.1.4 we introduce the new unknown

$$\widetilde{q} = -\nu \mathrm{div}\widetilde{u} + P'(\overline{\rho})\widetilde{\sigma},$$

where $\nu = (\mu' + 2\mu)$. Again, the system satisfied by $(\widetilde{\sigma}, \widetilde{q}, \beta)$ will be easier to analyze than the full system (4.1.22). Hence, once we will have constructed $(\widetilde{\sigma}, \widetilde{q}, \beta)$, we will construct $\widetilde{u}$ from $(4.1.22)_{(2,3,4)}$ and the knowledge of $\widetilde{\sigma}$ and β.

4.2.2.1 Step 1. The system satisfied by $(\widetilde{\sigma}, \widetilde{q}, \beta)$

One can follow the line of arguments used in the calculation (4.2.2) to formally verify that the triplet $(\widetilde{\sigma}, \widetilde{q}, \beta)$ solves the following system

$$\begin{cases} \partial_t\widetilde{\sigma} + \overline{u}_1\partial_x\widetilde{\sigma} + \dfrac{\overline{\rho}P'(\overline{\rho})}{\nu}\widetilde{\sigma} = \dfrac{\overline{\rho}}{\nu}\widetilde{q} + v_{\widetilde{\sigma}}\chi_\omega & \text{in} \quad (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \partial_t\widetilde{q} + \overline{u}_1\partial_x\widetilde{q} - \dfrac{\nu}{\overline{\rho}}\Delta\widetilde{q} - \dfrac{\overline{\rho}P'(\overline{\rho})}{\nu}\widetilde{q} = -\dfrac{\overline{\rho}(P'(\overline{\rho}))^2}{\nu}\widetilde{\sigma} & \\ \qquad\qquad + (P'(\overline{\rho})v_{\widetilde{\sigma}}\chi_\omega - \dfrac{\nu}{\overline{\rho}}\mathrm{div}(v_{\widetilde{u}}\chi_\omega)) & \text{in} \quad (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \partial_z\widetilde{q} = -\overline{\rho}(\partial_t + \overline{u}_1\partial_x)^2\beta + (v_{\widetilde{u}}\chi_\omega)_2 & \text{on} \quad (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \partial_z\widetilde{q} = 0 & \text{on} \quad (\mathbb{T}_L \times \{0\}) \times (0,T), \\ \widetilde{q}(\cdot,0) = \widetilde{q}_0 = -\nu\mathrm{div}\widetilde{u}_0 + P'(\overline{\rho})\widetilde{\sigma}_0 & \text{in} \quad \mathbb{T}_L \times (0,1), \\ \widetilde{\sigma}(\cdot,0) = \widehat{\rho}_0 - \overline{\rho} = \widetilde{\sigma}_0 & \text{in} \quad \mathbb{T}_L \times (0,1), \\ \partial_{tt}\beta - \partial_{txx}\beta + \partial_{xxxx}\beta = \widetilde{q} + v_\beta\chi_{\omega_1} & \text{on} \quad (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \beta(0) = \beta_0 \quad \text{and} \quad \partial_t\beta(0) = \beta_1 & \text{in} \quad \mathbb{T}_L \times \{1\}. \end{cases} \tag{4.2.64}$$

We will follow the similar strategy as used in the proof of Theorem 4.1.4 to prove the existence of a unique solution to the problem (4.1.22):

Lemma 4.2.3. *Let*

$$\begin{aligned} (\widetilde{\sigma}_0, \widetilde{q}_0, \beta_0, \beta_1) \in H^2(\mathbb{T}_L \times (0,1)) \times H^2(\mathbb{T}_L \times (0,1)) \times H^{9/2}(\mathbb{T}_L \times \{1\}) \\ \times H^{5/2}(\mathbb{T}_L \times \{1\}), \end{aligned} \tag{4.2.65}$$

the following compatibility conditions hold

$$\begin{aligned} &(i)\ \partial_z\widetilde{q}(\cdot,0) = -\overline{\rho}(\partial_t + \overline{u}_1\partial_x)^2\beta(\cdot,0) + (v_{\widetilde{u}}\chi_\omega)_2(\cdot,0) \quad \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ &(ii)\ \partial_z\widetilde{q}(\cdot,0) = (v_{\widetilde{u}}\chi_\omega)_2(\cdot,0) \quad \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T). \end{aligned} \tag{4.2.66}$$

and $(v_{\widetilde{\sigma}}\chi_\omega, v_{\widetilde{u}}\chi_\omega, v_\beta\chi_{\omega_1})$ satisfy the regularity assumptions (4.1.25), then the system (4.2.64) admits a unique solution $(\widetilde{\sigma}, \widetilde{q}, \beta)$ in the following functional framework

$$\begin{cases} \widetilde{\sigma} \in C^0([0,T]; H^2(\mathbb{T}_L \times (0,1))) \cap C^1([0,T]; H^1(\mathbb{T}_L \times (0,1))), \\ \widetilde{q} \in L^2(0,T; H^3(\mathbb{T}_L \times (0,1))) \cap H^{3/2}(0,T; L^2(\mathbb{T}_L \times (0,1))), \\ \beta \in L^2(0,T; H^{11/2}(\mathbb{T}_L \times \{1\})) \cap H^1(0,T; H^{7/2}(\mathbb{T}_L \times \{1\})) \\ \qquad\qquad \cap H^2(0,T; H^{3/2}(\mathbb{T}_L \times \{1\})). \end{cases} \tag{4.2.67}$$

Sketch of the proof of Lemma 4.2.3. As for the proof of Lemma 4.2.2, we construct solutions of (4.2.64) using a fixed point argument.
Namely, for

$$\begin{aligned} (\widehat{\widetilde{\sigma}}, \widehat{\beta}) \in & L^2(0,T; L^2(\mathbb{T}_L \times (0,1))) \times (L^2(0,T; H^4(\mathbb{T}_L \times \{1\})) \\ & \cap H^1(0,T; H^2(\mathbb{T}_L \times \{1\})) \cap H^2(0,T; L^2(\mathbb{T}_L \times \{1\}))), \end{aligned} \tag{4.2.68}$$

we solve the following problem

$$\begin{cases} \partial_t\widetilde{\sigma}+\overline{u}_1\partial_x\widetilde{\sigma}+\dfrac{\overline{\rho}P'(\overline{\rho})}{\nu}\widetilde{\sigma}=\dfrac{\overline{\rho}}{\nu}\widetilde{q}+v_{\widetilde{\sigma}}\chi_\omega & \text{in} \quad (\mathbb{T}_L\times(0,1))\times(0,T),\\ \partial_t\widetilde{q}+\overline{u}_1\partial_x\widetilde{q}-\dfrac{\nu}{\overline{\rho}}\Delta\widetilde{q}-\dfrac{\overline{\rho}P'(\overline{\rho})}{\nu}\widetilde{q}=-\dfrac{\overline{\rho}(P'(\overline{\rho}))^2}{\nu}\widehat{\widetilde{\sigma}} & \\ \qquad\qquad +(P'(\overline{\rho})v_{\widetilde{\sigma}}\chi_\omega-\dfrac{\nu}{\overline{\rho}}\mathrm{div}(v_{\widetilde{u}}\chi_\omega)) & \text{in} \quad (\mathbb{T}_L\times(0,1))\times(0,T),\\ \partial_z\widetilde{q}=-\overline{\rho}(\partial_t+\overline{u}_1\partial_x)^2\widehat{\beta}+(v_{\widetilde{u}}\chi_\omega)_2 & \text{on} \quad (\mathbb{T}_L\times\{1\})\times(0,T),\\ \partial_z\widetilde{q}=0 & \text{on} \quad (\mathbb{T}_L\times\{0\})\times(0,T),\\ \widetilde{q}(\cdot,0)=\widetilde{q}_0 & \text{in} \quad \mathbb{T}_L\times(0,1),\\ \widetilde{\sigma}(\cdot,0)=\widetilde{\sigma}_0 & \text{in} \quad \mathbb{T}_L\times(0,1),\\ \partial_{tt}\beta-\partial_{txx}\beta+\partial_{xxxx}\beta=\widetilde{q}+v_\beta\chi_{\omega_1} & \text{on} \quad (\mathbb{T}_L\times\{1\})\times(0,T),\\ \beta(0)=\beta_0 \quad \text{and} \quad \partial_t\beta(0)=\beta_1 & \text{in} \quad \mathbb{T}_L\times\{1\}. \end{cases} \tag{4.2.69}$$

In view of (4.1.23), (4.1.25)(b) and (4.2.68) one solves $(4.2.69)_2$-$(4.2.69)_5$ in the framework

$$\widetilde{q}\in L^2(0,T;H^{3/2}(\mathbb{T}_L\times(0,1)))\cap H^{3/4}(0,T;L^2(\mathbb{T}_L\times(0,1))), \tag{4.2.70}$$

and introduce the map $\mathcal{L}_T:(\widehat{\widetilde{\sigma}},\widehat{\beta})\mapsto(\widetilde{\sigma},\beta)$.
With the regularity (4.2.70) we in particular have the following for the trace of $\widetilde{q}$ (in fact one can have a better trace regularity of $\widetilde{q}$ but we just use the following in our proof):

$$\widetilde{q}\,|_{\mathbb{T}_L\times\{1\}}\in L^2(0,T;H^{1/2}(\mathbb{T}_L\times(0,1)))\cap H^{1/4}(0,T;L^2(\mathbb{T}_L\times(0,1))). \tag{4.2.71}$$

Now, to recover regularity estimates on β solving $(4.2.69)_{(7,8)}$, we use the following lemma, whose proof can be found in Section 4.6.

Lemma 4.2.4. (i) *Let*

$$(\beta_0,\beta_1)\in H^{7/2}(\mathbb{T}_L\times\{1\})\times H^{3/2}(\mathbb{T}_L\times\{1\}), \tag{4.2.72}$$

and

$$\widetilde{\mathcal{G}}\in L^2(0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\cap H^{1/4}(0,T;L^2(\mathbb{T}_L\times\{1\})), \tag{4.2.73}$$

then the equations

$$\begin{cases} \partial_{tt}\beta-\partial_{txx}\beta+\partial_{xxxx}\beta=\widetilde{\mathcal{G}} & \text{on}\,(\mathbb{T}_L\times\{1\})\times(0,T),\\ \beta(0)=\beta_0 \quad \text{and} \quad \partial_t\beta(0)=\beta_1 & \text{in}\,\mathbb{T}_L\times\{1\}. \end{cases} \tag{4.2.74}$$

admits a unique solution in the following functional framework

$$\begin{aligned} (\beta,\partial_t\beta)\in&L^2(0,T;H^{9/2}(\mathbb{T}_L\times\{1\})\times H^{5/2}(\mathbb{T}_L\times\{1\}))\\ &\cap H^1(0,T;H^{5/2}(\mathbb{T}_L\times\{1\})\times H^{1/2}(\mathbb{T}_L\times\{1\}))\\ &\cap H^{5/4}(0,T;H^2(\mathbb{T}_L\times\{1\})\times L^2(\mathbb{T}_L\times\{1\})). \end{aligned} \tag{4.2.75}$$

(ii) Let

$$(\beta_0, \beta_1) \in H^{9/2}(\mathbb{T}_L \times \{1\}) \times H^{5/2}(\mathbb{T}_L \times \{1\}), \tag{4.2.76}$$

and

$$\widetilde{\mathcal{G}} \in L^2(0,T; H^{3/2}(\mathbb{T}_L \times \{1\})) \cap H^{3/4}(0,T; L^2(\mathbb{T}_L \times \{1\})), \tag{4.2.77}$$

then the equations (4.2.74) *admits a unique solution in the following functional framework*

$$\begin{aligned}(\beta, \partial_t \beta) \in & L^2(0,T; H^{11/2}(\mathbb{T}_L \times \{1\}) \times H^{7/2}(\mathbb{T}_L \times \{1\})) \\ & \cap H^1(0,T; H^{7/2}(\mathbb{T}_L \times \{1\}) \times H^{3/2}(\mathbb{T}_L \times \{1\})) \\ & \cap H^{3/2}(0,T; H^2(\mathbb{T}_L \times \{1\}) \times L^2(\mathbb{T}_L \times \{1\})).\end{aligned} \tag{4.2.78}$$

Using item (i) of Lemma 4.2.4, the regularity (4.2.71) along with (4.1.23) we show that

$$\begin{aligned}(\beta, \partial_t \beta) \in & L^2(0,T; H^{9/2}(\mathbb{T}_L \times \{1\}) \times H^{5/2}(\mathbb{T}_L \times \{1\})) \\ & \cap H^1(0,T; H^{5/2}(\mathbb{T}_L \times \{1\}) \times H^{1/2}(\mathbb{T}_L \times \{1\})) \\ & \cap H^{5/4}(0,T; H^2(\mathbb{T}_L \times \{1\}) \times L^2(\mathbb{T}_L \times \{1\})).\end{aligned} \tag{4.2.79}$$

Furthermore, one uses (4.2.70) and (4.1.23) to solve $(4.2.69)_1$ and $(4.2.69)_6$ with

$$\widetilde{\sigma} \in C^0([0,T]; L^2(\mathbb{T}_L \times (0,1))). \tag{4.2.80}$$

Using interpolation the regularities (4.2.79) and (4.2.80) yield

$$\begin{aligned}&\widetilde{\sigma} \in C^0([0,T]; L^2(\mathbb{T}_L \times (0,1))), \\ &\beta \in H^{1/4}(0,T; H^4(\mathbb{T}_L \times \{1\})) \cap H^{5/4}(0,T; H^2(\mathbb{T}_L \times \{1\})) \\ &\quad \cap H^{9/4}(0,T; L^2(\mathbb{T}_L \times \{1\})).\end{aligned} \tag{4.2.81}$$

In view of (4.2.68) and (4.2.81) we observe that the couple $(\widetilde{\sigma}, \beta)$ gains time integrability in comparison with $(\widehat{\widetilde{\sigma}}, \widehat{\beta})$. One can make this point precise by applying a similar line of arguments used in the proof of Lemma 4.2.2 and prove that the map $\mathcal{L}_T$ is a contraction for T small enough. Using then Banach fixed point theorem, we show that, for T small enough, the auxiliary system (4.2.64) admits a unique solution in the following functional framework

$$\begin{cases} \widetilde{q} \in L^2(0,T; H^{3/2}(\mathbb{T}_L \times (0,1))) \cap H^{3/4}(0,T; L^2(\mathbb{T}_L \times (0,1))), \\ \widetilde{\sigma} \in C^0([0,T]; L^2(\mathbb{T}_L \times (0,1))), \\ \beta \in L^2(0,T; H^{9/2}(\mathbb{T}_L \times \{1\})) \cap H^1(0,T; H^{5/2}(\mathbb{T}_L \times \{1\})) \\ \quad \cap H^2(0,T; H^{1/2}(\mathbb{T}_L \times \{1\})) \cap H^{9/4}(0,T; L^2(\mathbb{T}_L \times \{1\})). \end{cases} \tag{4.2.82}$$

One can iterate this argument to deduce that for any $T > 0$, the auxiliary system (4.2.64) admits a unique solution in the framework (4.2.82).

Now we use a bootstrap argument to improve the regularities of $(\widetilde{\sigma}, \widetilde{q}, \beta)$. With the regularity of β from (4.2.82) we obtain that

$$(\partial_t + \overline{u}_1\partial_x)^2\beta\,|_{\mathbb{T}_L\times\{1\}} \in L^2(0,T;H^{1/2}(\mathbb{T}_L\times\{1\}))\cap H^{1/4}(0,T;L^2(\mathbb{T}_L\times\{1\})). \quad (4.2.83)$$

The assumption (4.1.23), (4.1.25), the regularities (4.2.82) of σ and (4.2.83) can be used to show that $\widetilde{q}$ solving $(4.2.64)_2$-$(4.2.64)_5$ has the following improved regularity

$$\widetilde{q} \in L^2(0,T;H^2(\mathbb{T}_L\times(0,1)))\cap H^1(0,T;L^2(\mathbb{T}_L\times(0,1))). \quad (4.2.84)$$

This in particular implies that

$$\begin{cases} \widetilde{q}\,|_{\mathbb{T}_L\times\{1\}} \in L^2(0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(0,T;L^2(\mathbb{T}_L\times\{1\})), \\ \widetilde{q} \in C^0([0,T];H^1(\mathbb{T}_L\times(0,1))) \end{cases} \quad (4.2.85)$$

The regularities (4.2.85) along with the regularities of the initial datum (4.1.23) can be used to solve consecutively the beam equation $(4.2.64)_7$-$(4.2.64)_8$ and the transport equation $(4.2.64)_1$-$(4.2.64)_6$ in the following settings

$$\begin{cases} \beta \in L^2(0,T;H^{11/2}(\mathbb{T}_L\times\{1\}))\cap H^1(0,T;H^{7/2}(\mathbb{T}_L\times\{1\})) \\ \qquad \cap H^2(0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{11/4}(0,T;L^2(\mathbb{T}_L\times\{1\})), \\ \widetilde{\sigma} \in C^0([0,T];H^1(\mathbb{T}_L\times(0,1)))\cap C^1([0,T];L^2(\mathbb{T}_L\times(0,1))). \end{cases} \quad (4.2.86)$$

In obtaining $(4.2.86)_1$ we have used the item (ii) of Lemma 4.2.4. From (4.2.86) one in particular obtains that

$$(\partial_t + \overline{u}_1\partial_x)^2\beta\,|_{\mathbb{T}_L\times\{1\}} \in L^2(0,T;H^{3/2}(\mathbb{T}_L\times\{1\}))\cap H^{3/4}(0,T;L^2(\mathbb{T}_L\times\{1\})). \quad (4.2.87)$$

Once again in view of the regularities (4.1.23) of the initial datum, (4.1.25) of the source terms, (4.2.87) and the compatibility condition (4.2.66) are used to infer the following additional regularity for $\widetilde{q}$

$$\widetilde{q} \in L^2(0,T;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(0,T;L^2(\mathbb{T}_L\times(0,1))). \quad (4.2.88)$$

This by interpolation in particular implies that

$$\widetilde{q} \in C^0([0,T];H^2(\mathbb{T}_L\times(0,1))). \quad (4.2.89)$$

One can use the assumption on $\widetilde{\sigma}_0$ from (4.1.23), on $v_{\widetilde{\sigma}}\chi_\omega$ from (4.1.25) and (4.2.89) to bootstrap and improve the regularity of $\widetilde{\sigma}$

$$\widetilde{\sigma} \in C^0([0,T];H^2(\mathbb{T}_L\times(0,1)))\cap C^1([0,T];H^1(\mathbb{T}_L\times(0,1))). \quad (4.2.90)$$

This concludes the proof of Lemma 4.2.3. □

4.2.2.2 Step 2. Constructing $\widetilde{u}$

Similarly as in the proof of Theorem 4.1.4, we start by solving (4.2.64) with initial datum $(\widetilde{\sigma}_0, \widetilde{q}_0, \beta_0, \beta_1)$, where $\widetilde{q}_0 = -\nu \mathrm{div}\widetilde{u} + P'(\overline{\rho})\widetilde{\sigma}$. Note that with this choice, the assumptions (4.1.23), (4.1.24), (4.1.25) on $(\widetilde{\sigma}_0, \widetilde{u}_0, \beta_0, \beta_1)$ allow to derive (4.2.65) and (4.2.66) for $(\widetilde{\sigma}_0, \widetilde{q}_0, \beta_0, \beta_1)$, so that Lemma 4.2.3 applies and provides functions $(\widetilde{\sigma}, \widetilde{q}, \widetilde{\beta})$ solving (4.2.64).
We then use the regularities obtained in Lemma 4.2.3 to verify that the solution $\widetilde{u}$ of

$$\begin{cases} \overline{\rho}(\partial_t \widetilde{u} + \overline{u}_1 \partial_x \widetilde{u}) - \mu \Delta \widetilde{u} - (\mu' + \mu)\nabla(\mathrm{div}\widetilde{u}) \\ \qquad\qquad + P'(\overline{\rho})\nabla\widetilde{\sigma} = v_{\widetilde{u}}\chi_\omega & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \widetilde{u}\cdot n = \widetilde{u}_2 = \partial_t \beta + \overline{u}_1 \partial_x \beta & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \widetilde{u}(\cdot,t)\cdot n = 0 & \text{on } (\mathbb{T}_L \times \{0\}) \times (0,T), \\ \mathrm{curl}\widetilde{u} = 0 & \text{on } (\mathbb{T}_L \times \{0,1\}) \times (0,T), \\ \widetilde{u}(\cdot,0) = \widetilde{u}_0 & \text{in } \mathbb{T}_L \times (0,1). \end{cases} \tag{4.2.91}$$

satisfies

$$\begin{aligned} \widetilde{u} \in L^2(0,T;H^3(\mathbb{T}_L \times (0,1))) \quad & \cap H^1(0,T;H^2(\mathbb{T}_L \times (0,1))) \\ & \cap H^2(0,T;L^2(\mathbb{T}_L \times (0,1))). \end{aligned} \tag{4.2.92}$$

This can be proved by following the line of arguments used in the *Step 2* of the proof of Theorem 4.1.4.
Finally one can conclude the proof of Theorem 4.1.3 by imitating the arguments in the conclusion of the proof of Theorem 4.1.4.

Remark 4.2.5. *At this point the role of the effective viscous flux $\widetilde{q}$ and the dual version of effective viscous flux q is clear in order to prove the existence results Theorem 4.1.3 and Theorem 4.1.4. We now explain more clearly the reason behind considering the simplified model (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7) instead of using the more physical expression (4.1.9) (recall Remark 4.1.1). In our analysis we use the effective viscous flux (the dual version of that) to reduce the strength of the parabolic hyperbolic coupling in the system (4.1.2) (respectively (4.1.27)). With the simplified expression (4.1.7) of the net surface force on the beam we are able to write the linearized version of the system (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.7) as a closed loop system in terms of the fluid density, effective viscous flux and the beam displacement. On the other hand it does not seem to be possible if one considers the expression (4.1.9) for the net force acting on the beam. In some sense the effective viscous flux does not prove to be a good unknown to weaken the parabolic hyperbolic coupling near the boundary for the system (4.1.2)-(4.1.3)-(4.1.4)-(4.1.5)-(4.1.6)-(4.1.9).*

4.3 Carleman estimates for scalar equations

From now onwards we fix our final time horizon T and the length L of our torus such that they satisfy (4.1.31) and (4.1.32) respectively.
The goal of this section is to provide Carleman estimates for the various equations involved in the system (4.1.27), in particular:

- a Carleman estimate for the beam equation set in the torus;
- a Carleman estimate for the heat equation with non-homogeneous Neumann boundary conditions;
- a Carleman estimate for the transport equation.

One of the difficulties of our work is that, in order to prove Theorem 4.1.5, one should be able to couple all these Carleman estimates in a suitable way. In order to do that, we will consider one weight function which allows to derive Carleman estimates for the beam equation, the heat equation and the transport equation simultaneously.

4.3.1 Construction of the weight function

1. We first introduce a function η on $\mathbb{T}_L$ such that

$$\begin{aligned} &\eta \in C^6(\mathbb{T}_L),\ \eta(x) > 0 \text{ in } \mathbb{T}_L, \\ &\inf\{|\nabla\eta(x)| \mid x \in \mathbb{T}_L \setminus \{(-3\overline{u}_1 T, -2\overline{u}_1 T) \cup (d + \overline{u}_1 T, d + 3\overline{u}_1 T)\}\} > 0, \end{aligned} \tag{4.3.1}$$

2. Now we define $\eta^0 \in C^6(\mathbb{T}_L \times [0,T])$ as follows

$$\eta^0(x,t) = \eta(x - \overline{u}_1 t) \quad \text{for all} \quad (x,t) \in \mathbb{T}_L \times [0,T]. \tag{4.3.2}$$

In view of (4.3.1), one can easily verify the following

$$\inf\{|\nabla\eta^0(x,t)| \mid (x,t) \in [-\overline{u}_1 T, d + \overline{u}_1 T] \times [0,T]\} > 0, \tag{4.3.3}$$

3. Next we will define a weight function in the time variable. Let $T_0 > 0$, $T_1 > 0$, small enough such that

$$2T_0 + 2T_1 < T - \frac{d}{\overline{u_1}}. \tag{4.3.4}$$

Now we choose a weight function $\theta(t) \in C^4(0,T)$ such that

$$\theta(t) = \begin{cases} \dfrac{1}{t^2}, & \forall\ t \in [0, T_0], \\ \theta \text{ is strictly decreasing} & \forall\ t \in [T_0, 2T_0], \\ 1 & \forall\ t \in [2T_0, T - 2T_1], \\ \theta \text{ is strictly increasing} & \forall\ t \in [T - 2T_1, T - T_1], \\ \dfrac{1}{(T-t)^2}, & \forall\ t \in [T - T_1, T]. \end{cases} \tag{4.3.5}$$

Observe that $\theta(t)$ blows up at the terminal points $\{0\}$ and $\{T\}$ of the interval $(0,T)$.
4. In view of $\eta^0(x,t)$ and $\theta(t)$ we finally introduce the following weight functions in $\mathbb{T}_L \times [0,T]$,

$$\begin{cases} \phi(x,t) = \theta(t)(e^{6\lambda\|\eta^0\|_\infty} - e^{\lambda(\eta^0(x,t)+4\|\eta^0\|_\infty)}), \\ \xi(x,t) = \theta(t) e^{\lambda(\eta^0(x,t)+4\|\eta^0\|_\infty)}, \end{cases} \tag{4.3.6}$$

where $\lambda \geqslant 1$ is a positive parameter.

Remark 4.3.1. *Recall that we have fixed $L = 3\overline{u}_1 T$. The reason lies in the choice (4.3.2) of η^0 which travels along $\mathbb{T}_L$ with a velocity $\overline{u}_1$. Of course this choice plays a very important role in Section 4.3.4 while obtaining an observability estimate for a hyperbolic transport equation. In that case the domain $(0,d)\times(0,1)$ needs to be embedded in $\mathbb{T}_L \times (0,1)$, for L large enough such that $\inf\{|\nabla\eta^0(x,t)|\}$ is positive in a neighborhood of $(0,d)\times(0,1)$, since this is crucial to obtain parabolic Carleman estimates. Now the choice $L = 3\overline{u}_1 T$ serves this purpose and provides enough room so that (4.3.3) holds.*

From now on we will denote by c, a generic strictly positive small constant and by C, a large constant, where both of them are independent of the parameters s ($\geqslant 1$) and λ ($\geqslant 1$).
In our computations afterwards we will frequently use the following estimates, valid on $\mathbb{T}_L \times (0,T)$:

$$\begin{array}{l} |\partial_x^{(i)}\phi| \leqslant C\lambda^i\xi \quad \text{for all } i \in \{1,2,3,4\}, \\ |\partial_t\phi| \leqslant C\lambda\xi^{3/2}, \quad |\partial_{tt}\phi| \leqslant C\lambda^2\xi^2, \quad |\partial_{tx}\phi| \leqslant C\lambda^2\xi^{3/2}, \quad |\partial_{txx}\phi| \leqslant C\lambda^3\xi^{3/2}, \\ |\partial_{txxx}\phi| \leqslant C\lambda^4\xi^{3/2}, \quad |\partial_{ttx}\phi| \leqslant C\lambda^3\xi^2 \quad \text{and} \quad |\partial_{ttxx}\phi| \leqslant C\lambda^4\xi^2, \end{array} \tag{4.3.7}$$

and

$$\begin{array}{l} |\partial_x^{(i)}\xi| \leqslant C\lambda^i\xi \quad \text{for all } i \in \{1,2,3,4\}, \\ |\partial_t\xi| \leqslant C\lambda\xi^{3/2}, \quad |\partial_{tt}\xi| \leqslant C\lambda^2\xi^2, \quad |\partial_{tx}\xi| \leqslant C\lambda^2\xi^{3/2}, \quad |\partial_{txx}\xi| \leqslant C\lambda^3\xi^{3/2} \\ |\partial_{txxx}\xi| \leqslant C\lambda^4\xi^{3/2}, \quad |\partial_{ttx}\xi| \leqslant C\lambda^3\xi^2 \quad \text{and} \quad |\partial_{ttxx}\xi| \leqslant C\lambda^4\xi^2, \end{array} \tag{4.3.8}$$

and, for λ large enough, for all $(x,t) \in [-\overline{u}_1 T, d + \overline{u}_1 T] \times (0,T)$ and $i \in \{1,2,3,4\}$,

$$-\partial_x^{(i)}\phi = \partial_x^{(i)}\xi \geqslant c\lambda^i\xi. \tag{4.3.9}$$

4.3.2 Carleman estimate for an adjoint damped beam equation

In the following section we derive a Carleman estimate for the adjoint of the damped

$$\begin{cases} \partial_{tt}\psi + \partial_{txx}\psi + \partial_{xxxx}\psi = f_\psi & \text{in } \mathbb{T}_L \times (0,T), \\ \psi(.,T) = \psi_T, \quad \partial_t\psi(.,T) = \psi_T^1 & \text{in } \mathbb{T}_L. \end{cases} \tag{4.3.10}$$

The main theorem of this section is stated as follows

Theorem 4.3.2. *There exist constants $C > 0$, $s_0 \geqslant 1$, $\lambda_0 \geqslant 1$ such that for all ψ solving (4.3.10) with initial datum $\psi_T \in H^3(\mathbb{T}_L)$ and $\psi_T^1 \in H^1(\mathbb{T}_L)$ and source term $f_\psi \in L^2(\mathbb{T}_L \times (0,T))$, for all $s \geqslant s_0$, and $\lambda \geqslant \lambda_0$,*

$$\begin{aligned} &s^7\lambda^8 \int_0^T \int_{\mathbb{T}_L} \xi^7 |\psi|^2 e^{-2s\phi} + s^5\lambda^6 \int_0^T \int_{\mathbb{T}_L} \xi^5 |\partial_x\psi|^2 e^{-2s\phi} \\ &+ s^3\lambda^4 \int_0^T \int_{\mathbb{T}_L} \xi^3 (|\partial_{xx}\psi|^2 + |\partial_t\psi|^2) e^{-2s\phi} + s\lambda^2 \int_0^T \int_{\mathbb{T}_L} \xi (|\partial_{tx}\psi|^2 + |\partial_{xxx}\psi|^2) e^{-2s\phi} \\ &+ \frac{1}{s} \int_0^T \int_{\mathbb{T}_L} \frac{1}{\xi} (|\partial_{tt}\psi|^2 + |\partial_{txx}\psi|^2 + |\partial_{xxxx}\psi|^2) e^{-2s\phi} \\ &\leqslant C \int_0^T \int_{\mathbb{T}_L} |f_\psi|^2 e^{-2s\phi} + Cs^7\lambda^8 \int_0^T \int_{\omega_1} \xi^7 |\psi|^2 e^{-2s\phi}, \end{aligned} \tag{4.3.11}$$

where the notation ω_1 was introduced in (4.1.14).

Proof. We introduce the change of unknown

$$w = e^{-s\phi}\psi.$$

In view of (4.3.10), w satisfies:

$$\begin{aligned} e^{-s\phi} f_\psi &= e^{-s\phi}(\partial_{tt}\psi + \partial_{txx}\psi + \partial_{xxxx}\psi) \\ &= e^{-s\phi}(\partial_{tt}(e^{s\phi}w) + \partial_{txx}(e^{s\phi}w) + \partial_{xxxx}(e^{s\phi}w)) = P_\phi w. \end{aligned} \tag{4.3.12}$$

We write $P_\phi w$ in the form:

$$P_\phi w = P_1 w + P_2 w + Rw, \tag{4.3.13}$$

where

$$\begin{cases} P_1 w = & s^4(\partial_x\phi)^4 w + 6s^2(\partial_x\phi)^2\partial_{xx}w + \partial_{xxxx}w + 2s\partial_x\phi\partial_{xt}w + \partial_{tt}w, \\ P_2 w = & 4s^3(\partial_x\phi)^3\partial_x w + 4s\partial_x\phi\partial_{xxx}w + \partial_{xxt}w + s^2(\partial_x\phi)^2\partial_t w \\ & +6(1+\zeta)s^3(\partial_x\phi)^2\partial_{xx}\phi w, \\ Rw = & s^2(\partial_t\phi)^2 w + s\partial_t\phi\partial_{xx}w + s^3\partial_t\phi(\partial_x\phi)^2 w + s\partial_t\phi\partial_t w + 2s^2\partial_t\phi\partial_x\phi\partial_x w \\ & +\frac{s}{2}\partial_{tt}\phi w - s\partial_{xxt}\phi w + 2s\partial_{xt}\phi\partial_x w + 4s^2\partial_x\phi\partial_{xxx}\phi w + s\partial_{xx}\phi\partial_t w \\ & +12s^2\partial_x\phi\partial_{xx}\phi\partial_x w + 3s^2(\partial_{xx}\phi)^2 w + \frac{s}{2}\partial_{tt}\phi w + 2s^2\partial_{xt}\phi\partial_x\phi w \\ & +s^2\partial_t\phi\partial_{xx}\phi w + 6s\partial_{xx}\phi\partial_{xx}w + s\partial_{xxxx}\phi w + 4s\partial_{xxx}\phi\partial_x w \\ & -6\zeta s^3(\partial_x\phi)^2\partial_{xx}\phi w, \end{cases} \tag{4.3.14}$$

where ζ is a free parameter which will be fixed later.
Based on the identity

$$P_1 w + P_2 w = f_\psi e^{-s\phi} - Rw,$$

we obtain

$$\begin{aligned}\int_0^T \int_{\mathbb{T}_L} |P_1 w|^2 + \int_0^T \int_{\mathbb{T}_L} |P_2 w|^2 + 2\int_0^T \int_{\mathbb{T}_L} P_1 w P_2 w = \int_0^T \int_{\mathbb{T}_L} |f_\psi e^{-s\phi} - Rw|^2 \\ \leqslant 2\int_0^T \int_{\mathbb{T}_L} |f_\psi|^2 e^{-2s\phi} + 2\int_0^T \int_{\mathbb{T}_L} |Rw|^2.\end{aligned} \tag{4.3.15}$$

The crucial point is to obtain suitable estimates for the product term $\int_0^T \int_{\mathbb{T}_L} P_1 w P_2 w$. We will denote by $I_{i,j}$ the cross product of the i-th term of $P_1 w$ and of the j-th term of $P_2 w$, so that

$$\int_0^T \int_{\mathbb{T}_L} P_1 w P_2 w = \sum_{i,j=1}^{i=5,j=5} I_{ij}.$$

In the following estimates to make the presentation simpler we will write L.O.T (lower order terms) for the terms which are small (for large values of the parameters s and λ) with respect to the left hand side of (4.3.11), i.e. for which there exists a constant C independent of s and λ such that

$$\begin{aligned}|L.O.T| \leqslant C\left(\frac{1}{s} + \frac{1}{\lambda}\right)\Bigg(s^7\lambda^8 \int_0^T \int_{\mathbb{T}_L} \xi^7 |\psi|^2 e^{-2s\phi} + s^5\lambda^6 \int_0^T \int_{\mathbb{T}_L} \xi^5 |\partial_x \psi|^2 e^{-2s\phi} \\ + s^3\lambda^4 \int_0^T \int_{\mathbb{T}_L} \xi^3 (|\partial_{xx}\psi|^2 + |\partial_t \psi|^2) e^{-2s\phi} + s\lambda^2 \int_0^T \int_{\mathbb{T}_L} \xi (|\partial_{tx}\psi|^2 + |\partial_{xxx}\psi|^2) e^{-2s\phi}\Bigg).\end{aligned}$$

In particular, note that we immediately get that

$$\int_0^T \int_{\mathbb{T}_L} |Rw|^2 = L.O.T. \tag{4.3.16}$$

We list below the computations of each I_{ij}.

$$I_{11} = 4s^7 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^7 w \partial_x w = -14 s^7 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^6 \partial_{xx}\phi w^2. \tag{4.3.17}$$

$$
\begin{aligned}
I_{12} &= 4s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^5 w\partial_{xxx}w = -120s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2(\partial_{xx}\phi)^3w^2\\
&\quad - 80s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_{xxx}\phi w^2 - 40s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_{xxx}\phi w^2\\
&\quad - 10s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xxxx}\phi w^2 + 30s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}\phi(\partial_x w)^2\\
&= L.O.T + 30s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}\phi(\partial_x w)^2.
\end{aligned}
\tag{4.3.18}
$$

$$
\begin{aligned}
I_{13} &= s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4 w\partial_{xxt}w\\
&= -12s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)\partial_{xt}\phi(\partial_{xx}\phi)^2w^2 - 12s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi\partial_{xxt}\phi w^2\\
&\quad - 6s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xt}\phi\partial_{xxx}\phi w^2 \quad - 2s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{txxx}\phi w^2\\
&\quad + 2s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{tx}\phi(\partial_x w)^2 + 4s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_x w\partial_t w\\
&= L.O.T + 4s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_x w\partial_t w.
\end{aligned}
\tag{4.3.19}
$$

$$
I_{14} = 3s^6\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^5\partial_{tx}\phi w^2 = L.O.T. \tag{4.3.20}
$$

$$
I_{15} = 6(1+\zeta)s^7\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^6\partial_{xx}\phi w^2. \tag{4.3.21}
$$

$$
I_{21} = -60s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}\phi(\partial_x w)^2. \tag{4.3.22}
$$

$$
I_{22} = -36s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi(\partial_{xx}w)^2. \tag{4.3.23}
$$

$$
I_{23} = -12s^2\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)\partial_{xt}\phi(\partial_{xx}w)^2 = L.O.T. \tag{4.3.24}
$$

$$
\begin{aligned}
I_{24} &= 6s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}w\partial_t w\\
&= -24s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_x w\partial_t w + 12s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xt}\phi(\partial_x w)^2\\
&= L.O.T - 24s^4\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_x w\partial_t w.
\end{aligned}
\tag{4.3.25}
$$

$$
\begin{aligned}
I_{25} &= 36(1+\zeta)s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}\phi w\partial_{xx}w = (1+\zeta)\left(216s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2(\partial_{xx}\phi)^3w^2\right.\\
&+144s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_{xxx}\phi w^2 \ +72s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_{xxx}\phi w^2 \qquad (4.3.26)\\
&\left.+18s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xxxx}\phi w^2 - 36s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}\phi(\partial_x w)^2\right)\\
&= L.O.T - 36(1+\zeta)s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}\phi(\partial_x w)^2.
\end{aligned}
$$

$$
\begin{aligned}
I_{31} &= 4s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_x w\partial_{xxxx}w\\
&= -12s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi\partial_x w\partial_{xxx}w - 4s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}w\partial_{xxx}w \qquad (4.3.27)\\
&= L.O.T + 18s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi(\partial_{xx}w)^2.
\end{aligned}
$$

$$
I_{32} = -2s\int_0^T\int_{\mathbb{T}_L}\partial_{xx}\phi(\partial_{xxx}w)^2. \qquad (4.3.28)
$$

$$
I_{33} = \int_0^T\int_{\mathbb{T}_L}\partial_{xxxx}w\partial_{xxt}w = 0. \qquad (4.3.29)
$$

$$
\begin{aligned}
I_{34} &= s^2\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xxxx}w\partial_t w\\
&= -2s^2\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)\partial_{xx}\phi\partial_{xxx}w\partial_t w - s^2\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xxx}w\partial_{tx}w \qquad (4.3.30)\\
&= L.O.T + 4s^2\int_0^T\int_{\mathbb{T}_L}\partial_x\phi\partial_{xx}\phi\partial_{xx}w\partial_{tx}w.
\end{aligned}
$$

$$
\begin{aligned}
I_{35} &= 6(1+\zeta)s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi\partial_{xxxx}ww\\
&= -12(1+\zeta)s^3\int_0^T\int_{\mathbb{T}_L}\partial_x\phi(\partial_{xx}\phi)^2\partial_{xxx}ww - 6(1+\zeta)s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xxx}\phi\partial_{xxx}ww\\
&- 6(1+\zeta)s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi\partial_{xxx}w\partial_x w \qquad (4.3.31)\\
&= L.O.T + 6(1+\zeta)s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi(\partial_{xx}w)^2.
\end{aligned}
$$

$$I_{41} = -16s^4 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_{tx} \phi (\partial_x w)^2 = L.O.T. \tag{4.3.32}$$

$$\begin{aligned} I_{42} &= 8s^2 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xt} w \partial_{xxx} w \\ &= -16s^2 \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{xx} \phi \partial_{xt} w \partial_{xx} w + 8s^2 \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{xt} \phi (\partial_{xx} w)^2 \\ &= L.O.T - 16s^2 \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{xx} \phi \partial_{xt} w \partial_{xx} w. \end{aligned} \tag{4.3.33}$$

$$I_{43} = -s \int_0^T \int_{\mathbb{T}_L} \partial_{xx} \phi (\partial_{xt} w)^2. \tag{4.3.34}$$

$$I_{44} = -3s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xx} \phi (\partial_t w)^2. \tag{4.3.35}$$

$$\begin{aligned} I_{45} &= 12(1+\zeta)s^4 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_{xx} \phi \partial_{xt} w w \\ &= (1+\zeta) \Bigg(-36s^4 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 (\partial_{xx} \phi)^2 \partial_t w w - 12s^4 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_{xxx} \phi \partial_t w w \\ &\quad -12s^4 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_{xx} \phi \partial_t w \partial_x w \Bigg) \\ &= L.O.T - 12(1+\zeta)s^4 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_{xx} \phi \partial_t w \partial_x w. \end{aligned} \tag{4.3.36}$$

$$\begin{aligned} I_{51} &= 4s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_x w \partial_{tt} w \\ &= -12s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xt} \phi \partial_x w \partial_t w - 4s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_{xt} w \partial_t w \\ &= L.O.T + 6s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xx} \phi (\partial_t w)^2. \end{aligned} \tag{4.3.37}$$

$$\begin{aligned} I_{52} &= 4s \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{xxx} w \partial_{tt} w \\ &= -4s \int_0^T \int_{\mathbb{T}_L} \partial_{xt} \phi \partial_{xxx} w \partial_t w - 4s \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{xxxt} w \partial_t w \\ &= L.O.T - 6s \int_0^T \int_{\mathbb{T}_L} \partial_{xx} \phi (\partial_{xt} w)^2. \end{aligned} \tag{4.3.38}$$

$$I_{53} = \int_0^T \int_{\mathbb{T}_L} \partial_{tt} w \partial_{xxt} w = \int_0^T \int_{\mathbb{T}_L} \partial_t (\partial_{tx} w)^2 = 0. \tag{4.3.39}$$

$$\begin{aligned} I_{54} &= s^2 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_t w \partial_{tt} w \\ &= -s^2 \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{tx} \phi (\partial_t w)^2 = L.O.T. \end{aligned} \tag{4.3.40}$$

$$\begin{aligned} I_{55} &= 6(1+\zeta) s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xx} \phi \partial_{tt} w w = (1+\zeta) \left(-6s^3 \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{tx} \phi \partial_{xx} \phi \partial_t (w^2) \right. \\ &\left. -3s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{txx} \phi \partial_t (w^2) - 6s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xx} \phi (\partial_t w)^2 \right) \\ &= L.O.T - 6(1+\zeta) s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xx} \phi (\partial_t w)^2. \end{aligned} \tag{4.3.41}$$

Hence we find that

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}_L} P_1 w P_2 w = \sum_{i,j=1}^{i=5,j=5} I_{ij} \\ &= (-8+6\zeta) s^7 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^6 \partial_{xx} \phi w^2 + (-66-36\zeta) s^5 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^4 \partial_{xx} \phi (\partial_x w)^2 \\ &+ (-12+6\zeta) s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xx} \phi (\partial_{xx} w)^2 + (-3-6\zeta) s^3 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^2 \partial_{xx} \phi (\partial_t w)^2 \\ &- 2s \int_0^T \int_{\mathbb{T}_L} \partial_{xx} \phi (\partial_{xxx} w)^2 - 7s \int_0^T \int_{\mathbb{T}_L} \partial_{xx} \phi (\partial_{xt} w)^2 \\ &+ (-32-12\zeta) s^4 \int_0^T \int_{\mathbb{T}_L} (\partial_x \phi)^3 \partial_{xx} \phi \partial_x w \partial_t w - 12 s^2 \int_0^T \int_{\mathbb{T}_L} \partial_x \phi \partial_{xx} \phi \partial_{xt} w \partial_{xx} w + L.O.T \\ &= \sum_{n=1}^{8} E_n + L.O.T. \end{aligned} \tag{4.3.42}$$

Now, we adjust the parameter ζ such that all the coefficients of E_n for $n \in \{1, \cdots, 6\}$ are negative and for some $\epsilon > 0$ independent of s and λ,

$$|E_7| + |E_8| \leqslant (1-\epsilon)(|E_2| + |E_3| + |E_4| + |E_6|). \tag{4.3.43}$$

In that direction we observe that, according to Young's inequality, for α_1 and α_2 positive,

$$\begin{aligned}|E_7| &= (32+12\zeta)s^4\Big|\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^3\partial_{xx}\phi\partial_x w\partial_t w\Big| \\ &\leqslant \frac{(32+12\zeta)}{2\alpha_1}s^5\int_0^T\int_{\mathbb{T}_L}|(\partial_x\phi)^4\partial_{xx}\phi(\partial_x w)^2| + \frac{(32+12\zeta)\alpha_1}{2}s^3\int_0^T\int_{\mathbb{T}_L}|(\partial_x\phi)^2\partial_{xx}\phi(\partial_t w)^2| \\ &\leqslant \frac{(32+12\zeta)}{2\alpha_1|66+36\zeta|}|E_2| + \frac{(32+12\zeta)\alpha_1}{2|3+6\zeta|}|E_4|, \end{aligned} \tag{4.3.44}$$

and

$$\begin{aligned}|E_8| &= 12s^2\int_0^T\int_{\mathbb{T}_L}|\partial_x\phi\partial_{xx}\phi\partial_{xt}w\partial_{xx}w| \\ &\leqslant \frac{12\alpha_2}{2}s\int_0^T\int_{\mathbb{T}_L}|\partial_{xx}\phi(\partial_{xt}w)^2| + \frac{12}{2\alpha_2}s^3\int_0^T\int_{\mathbb{T}_L}|(\partial_x\phi)^2\partial_{xx}\phi(\partial_{xx}w)^2| \\ &\leqslant \frac{12\alpha_2}{14}|E_6| + \frac{12}{2\alpha_2|12-6\zeta|}|E_3|, \end{aligned} \tag{4.3.45}$$

We then choose ζ, such that

$$\max\{-8+6\zeta, -66-36\zeta, -12+6\zeta, -3-6\zeta\} < 0, \tag{4.3.46}$$

which imposes $\zeta \in (-1/2, 4/3)$, and such that there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\max\left\{\frac{(32+12\zeta)}{2\alpha_1|66+36\zeta|}, \frac{(32+12\zeta)\alpha_1}{2|3+6\zeta|}, \frac{12\alpha_2}{14}, \frac{12}{2\alpha_2|12-6\zeta|}\right\} < 1. \tag{4.3.47}$$

This can be done provided $\zeta \in (-1/2, 4/3)$ satisfies

$$\frac{8+3\zeta}{33+18\zeta} < \frac{3+6\zeta}{16+6\zeta}, \quad \text{and} \quad \frac{1}{2-\zeta} < \frac{7}{6}.$$

These conditions can be easily satisfied by taking

$$\zeta = 1. \tag{4.3.48}$$

At this point in view of the choice (4.3.48), we fix α_1 and α_2 such that they satisfy (4.3.47).
Hence from (4.3.42) we get that there exist positive constants K_1, K_2, K_3, K_4, K_5 and K_6 such that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}_L} P_1 w P_2 w \geqslant -K_1 s^7\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^6\partial_{xx}\phi w^2 - K_2 s^5\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^4\partial_{xx}\phi(\partial_x w)^2 \\ - K_3 s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi(\partial_{xx}w)^2 - K_4 s^3\int_0^T\int_{\mathbb{T}_L}(\partial_x\phi)^2\partial_{xx}\phi(\partial_t w)^2 \\ - K_5 s\int_0^T\int_{\mathbb{T}_L}\partial_{xx}\phi(\partial_{xxx}w)^2 - K_6 s\int_0^T\int_{\mathbb{T}_L}\partial_{xx}\phi(\partial_{xt}w)^2 + L.O.T. \end{aligned} \tag{4.3.49}$$

Hence in view of (4.3.7) and (4.3.9) one obtains that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}_L} P_1wP_2w \geqslant c\left(s^7\lambda^8\int_0^T\int_{\mathbb{T}_L}\xi^7w^2+s^5\lambda^6\int_0^T\int_{\mathbb{T}_L}\xi^5(\partial_x w)^2\right.\\
&+s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_{xx}w)^2+s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_t w)^2+s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xxx}w)^2\\
&\left.+s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xt}w)^2\right)-C\left(s^7\lambda^8\iint\limits_{\omega_T^2}\xi^7w^2+s^5\lambda^6\iint\limits_{\omega_T^2}\xi^5(\partial_x w)^2\right.\\
&+s^3\lambda^4\iint\limits_{\omega_T^2}\xi^3(\partial_{xx}w)^2+s\lambda^2\iint\limits_{\omega_T^2}\xi(\partial_{xxx}w)^2+s^3\lambda^4\iint\limits_{\omega_T^2}\xi^3(\partial_t w)^2\\
&\left.+s\lambda^2\iint\limits_{\omega_T^2}\xi(\partial_{xt}w)^2\right),
\end{aligned}\tag{4.3.50}$$

where

$$\omega_T^2=(\mathbb{T}_L\setminus(-\overline{u}_1T,d+\overline{u}_1T))\times(0,T).\tag{4.3.51}$$

Now in view of (4.3.15) and (4.3.16), (4.3.50) furnishes that for large enough values of the parameter s and λ the following holds

$$\begin{aligned}
&s^7\lambda^8\int_0^T\int_{\mathbb{T}_L}\xi^7w^2+s^5\lambda^6\int_0^T\int_{\mathbb{T}_L}\xi^5(\partial_x w)^2+s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_{xx}w)^2\\
&\quad+s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_t w)^2+s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xxx}w)^2+s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xt}w)^2\\
&\leqslant C\left(\int_0^T\int_{\mathbb{T}_L}|f_\psi|^2e^{-2s\phi}+s^7\lambda^8\iint\limits_{\omega_T^2}\xi^7w^2+s^5\lambda^6\iint\limits_{\omega_T^2}\xi^5(\partial_x w)^2+s^3\lambda^4\iint\limits_{\omega_T^2}\xi^3(\partial_{xx}w)^2\right.\\
&\qquad\left.+s\lambda^2\iint\limits_{\omega_T^2}\xi(\partial_{xxx}w)^2+s^3\lambda^4\iint\limits_{\omega_T^2}\xi^3(\partial_t w)^2+s\lambda^2\iint\limits_{\omega_T^2}\xi(\partial_{xt}w)^2\right).
\end{aligned}\tag{4.3.52}$$

Now, our goal is to estimate $\partial_{tt}w$, $\partial_{txx}w$ and $\partial_{xxxx}w$. In order to do that, we set

$$\tau=\frac{1}{\sqrt{s\xi}}w.\tag{4.3.53}$$

Using (4.3.12), let us observe that (since $e^{-s\phi}$ vanishes at time T) the new unknown τ

solves the following set of equations

$$\begin{cases} \partial_{tt}\tau + \partial_{txx}\tau + \partial_{xxxx}\tau = \dfrac{1}{\sqrt{s\xi}} f_\psi e^{-s\phi} - (\mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3) - \mathcal{F}_4 & \text{in } \mathbb{T}_L \times (0,T), \\ \tau(.,T) = 0, \quad \partial_t \tau(.,T) = 0 & \text{in } \mathbb{T}_L, \end{cases} \tag{4.3.54}$$

where

$$\begin{aligned} \mathcal{F}_1 &= \partial_{tt}\tau - \tfrac{1}{\sqrt{s\xi}}\partial_{tt} w, \\ &= \left[\partial_{tt}, \tfrac{1}{\sqrt{s\xi}}\right] w, \end{aligned} \qquad \begin{aligned} \mathcal{F}_2 &= \partial_{txx}\tau - \tfrac{1}{\sqrt{s\xi}}\partial_{txx} w, \\ &= \left[\partial_{txx}, \tfrac{1}{\sqrt{s\xi}}\right] w, \end{aligned} \qquad \begin{aligned} \mathcal{F}_3 &= \partial_{xxxx}\tau - \tfrac{1}{\sqrt{s\xi}}\partial_{xxxx} w, \\ &= \left[\partial_{xxxx}, \tfrac{1}{\sqrt{s\xi}}\right] w. \end{aligned}$$

and $\mathcal{F}_4$ is given by

$$\begin{aligned} \sqrt{s\xi}\mathcal{F}_4 = Rw + s^4(\partial_x\phi)^4 w + 6s^2(\partial_x\phi)^2\partial_{xx} w + 2s\partial_x\phi\partial_{xt} w + 4s^3(\partial_x\phi)^3\partial_x w \\ + 4s\partial_x\phi\partial_{xxx} w + s^2(\partial_x\phi)^2\partial_t w + 6(1+\zeta)s^3(\partial_x\phi)^2\partial_{xx}\phi w. \end{aligned}$$

It is then easy to check that

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}_L} \left(|\mathcal{F}_1|^2 + |\mathcal{F}_2|^2 + |\mathcal{F}_3|^2 + |\mathcal{F}_4|^2\right) \\ &\quad \leqslant C\left(s^7\lambda^8 \int_0^T \int_{\mathbb{T}_L} \xi^7 w^2 + s^5\lambda^6 \int_0^T \int_{\mathbb{T}_L} \xi^5 (\partial_x w)^2 + s^3\lambda^4 \int_0^T \int_{\mathbb{T}_L} \xi^3 (\partial_{xx} w)^2 \right. \\ &\qquad \left. + s^3\lambda^4 \int_0^T \int_{\mathbb{T}_L} \xi^3 (\partial_t w)^2 + s\lambda^2 \int_0^T \int_{\mathbb{T}_L} \xi (\partial_{xxx} w)^2 + s\lambda^2 \int_0^T \int_{\mathbb{T}_L} \xi (\partial_{xt} w)^2 \right). \end{aligned}$$

Hence the maximal parabolic regularity result for the system (4.3.54) furnishes the following

$$\tau \in L^2(0,T; H^4(\mathbb{T}_L)) \cap H^2(0,T; L^2(\mathbb{T}_L)). \tag{4.3.55}$$

Besides one has the following inequality

$$\begin{aligned} \|\tau\|^2_{L^2(0,T;H^4(\mathbb{T}_L))\cap H^2(0,T;L^2(\mathbb{T}_L))} \leqslant C(\|f_\psi e^{-s\phi}\|^2_{L^2(\mathbb{T}_L\times(0,T))} + \|\mathcal{F}_1\|^2_{L^2(\mathbb{T}_L\times(0,T))} \\ + \|\mathcal{F}_2\|^2_{L^2(\mathbb{T}_L\times(0,T))} + \|\mathcal{F}_3\|^2_{L^2(\mathbb{T}_L\times(0,T))} + \|\mathcal{F}_4\|^2_{L^2(\mathbb{T}_L\times(0,T))}). \end{aligned} \tag{4.3.56}$$

This then yields the following estimate:

$$\begin{aligned} \frac{1}{s}\int_0^T \int_{\mathbb{T}_L} \frac{1}{\xi}(|\partial_{tt} w|^2 + |\partial_{txx} w|^2 + |\partial_{xxxx} w|^2) e^{-2s\phi} \leqslant C(\|f_\psi e^{-s\phi}\|^2_{L^2(\mathbb{T}_L\times(0,T))} + \|\mathcal{F}_1\|^2_{L^2(\mathbb{T}_L\times(0,T))} \\ + \|\mathcal{F}_2\|^2_{L^2(\mathbb{T}_L\times(0,T))} + \|\mathcal{F}_3\|^2_{L^2(\mathbb{T}_L\times(0,T))} + \|\mathcal{F}_4\|^2_{L^2(\mathbb{T}_L\times(0,T))}). \end{aligned} \tag{4.3.57}$$

Combining the inequalities (4.3.52) and (4.3.57) on obtains the following

$$
\begin{aligned}
&s^7\lambda^8\int_0^T\int_{\mathbb{T}_L}\xi^7w^2+s^5\lambda^6\int_0^T\int_{\mathbb{T}_L}\xi^5(\partial_x w)^2+s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_{xx}w)^2\\
&\quad+s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xxx}w)^2+s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_t w)^2+s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xt}w)^2\\
&\qquad+\frac{1}{s}\int_0^T\int_{\mathbb{T}_L}\frac{1}{\xi}(|\partial_{tt}w|^2+|\partial_{txx}w|^2+|\partial_{xxxx}w|^2)e^{-2s\phi}\\
&\leqslant C\left(\int_0^T\int_{\mathbb{T}_L}|f_\psi|^2e^{-2s\phi}+s^7\lambda^8\iint\limits_{\omega_T^2}\xi^7w^2+s^5\lambda^6\iint\limits_{\omega_T^2}\xi^5(\partial_x w)^2+s^3\lambda^4\iint\limits_{\omega_T^2}\xi^3(\partial_{xx}w)^2\right.\\
&\qquad\left.+s^3\lambda^4\iint\limits_{\omega_T^2}\xi^3(\partial_t w)^2+s\lambda^2\iint\limits_{\omega_T^2}\xi(\partial_{xxx}w)^2+s\lambda^2\iint\limits_{\omega_T^2}\xi(\partial_{xt}w)^2\right).
\end{aligned}
\tag{4.3.58}
$$

Now, we need to suitably absorb the third to seventh observability terms appearing in the R.H.S of (4.3.58). This is rather standard and such arguments can be found for instance in [15, p. 461] and [2, p. 565]. We absorb it in a reverse way, starting from the last terms.

We introduce a smooth cut-off function Υ_2 such that

$$
\begin{aligned}
&\Upsilon_2\in C_c^\infty(\mathbb{T}_L;[0,1]),\quad \Upsilon_2(x)=1\,\text{in}\,\omega^2,\quad \Upsilon_2(x)=0\,\text{for}\,x\notin\omega^3,\\
&\quad\text{where }\omega^2=\mathbb{T}_L\setminus[-\overline{u}_1T,d+\overline{u}_1T],\ \text{and }\omega^3=\mathbb{T}_L\setminus[-\overline{u}_1T/2,d+\overline{u}_1T/2].
\end{aligned}
\tag{4.3.59}
$$

In the following, we shall also use the notation $\omega_T^3=\omega^3\times(0,T)$.

Using Young's inequality, we have, for all $\varepsilon>0$,

$$
\begin{aligned}
&s\lambda^2\iint\limits_{\omega_T^2}\xi(\partial_{xt}w)^2\leqslant s\lambda^2\iint\limits_{\omega_T^3}\Upsilon_2\xi(\partial_{xt}w)^2=s\lambda^2\iint\limits_{\omega_T^3}\Upsilon_2\xi\partial_{xx}w\partial_{tt}w+L.O.T.\\
&\leqslant\frac{\varepsilon}{2s}\iint\limits_{\omega_T^3}\Upsilon_2\xi^{-1}(\partial_{tt}w)^2+\frac{s^3\lambda^4}{2\varepsilon}\iint\limits_{\omega_T^3}\Upsilon_2\xi^3(\partial_{xx}w)^2+L.O.T.\\
&\leqslant\frac{\varepsilon}{2s}\iint\limits_{\omega_T^3}\xi^{-1}(\partial_{tt}w)^2+\frac{Cs^3\lambda^4}{2\varepsilon}\iint\limits_{\omega_T^3}\xi^3(\partial_{xx}w)^2+L.O.T.
\end{aligned}
$$

Similarly, we get

$$s\lambda^2 \iint_{\omega_T^2} \xi(\partial_{xxx}w)^2 \leqslant s\lambda^2 \iint_{\omega_T^3} \Upsilon_2\xi(\partial_{xxx}w)^2 = -s\lambda^2 \iint_{\omega_T^3} \Upsilon_2\xi\partial_{xxxx}w\partial_{xx}w + L.O.T.$$
$$\leqslant \frac{\varepsilon}{2s}\iint_{\omega_T^3} \xi^{-1}(\partial_{xxxx}w)^2 + \frac{s^3\lambda^4}{2\varepsilon}\iint_{\omega_T^3} \xi^3(\partial_{xx}w)^2 + L.O.T.$$

We then choose $\varepsilon > 0$ small enough so that $C\varepsilon < 0$, where C is the constant in (4.3.58), and we plug these two estimates in (4.3.58). We obtain that there exists a constant $C > 0$ such that for all s and λ large enough,

$$\begin{aligned} &s^7\lambda^8\int_0^T\int_{\mathbb{T}_L}\xi^7w^2 + s^5\lambda^6\int_0^T\int_{\mathbb{T}_L}\xi^5(\partial_x w)^2 + s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_{xx}w)^2 \\ &\quad + s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xxx}w)^2 + s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_t w)^2 + s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xt}w)^2 \\ &\quad + \frac{1}{s}\int_0^T\int_{\mathbb{T}_L}\frac{1}{\xi}(|\partial_{tt}w|^2 + |\partial_{txx}w|^2 + |\partial_{xxxx}w|^2)e^{-2s\phi} \leqslant C\left(\int_0^T\int_{\mathbb{T}_L}|f_\psi|^2e^{-2s\phi}\right. \\ &\left. + s^7\lambda^8\iint_{\omega_T^3}\xi^7w^2 + s^5\lambda^6\iint_{\omega_T^3}\xi^5(\partial_x w)^2 + s^3\lambda^4\iint_{\omega_T^3}\xi^3(\partial_{xx}w)^2 + s^3\lambda^4\iint_{\omega_T^3}\xi^3(\partial_t w)^2 + L.O.T.\right). \end{aligned} \tag{4.3.60}$$

We now introduce a smooth cut-off function Υ_3 such that

$$\Upsilon_3 \in C_c^\infty(\mathbb{T}_L;[0,1]), \quad \Upsilon_3(x) = 1 \text{ in } \omega^3, \quad \Upsilon_3(x) = 0 \text{ in } [0,d],$$

and we use the notation $\omega_1 = \mathbb{T}_L \setminus [0,d]$, and $\omega_1^T = \omega_1 \times (0,T)$.
Now, as before we can write, for $\varepsilon_1 > 0$ to be fixed later,

$$s^3\lambda^4\iint_{\omega_T^3}\xi^3(\partial_t w)^2 \leqslant s^3\lambda^4\iint_{\omega_T^1}\Upsilon_3\xi^3(\partial_t w)^2 = -s^3\lambda^4\iint_{\omega_T^1}\Upsilon_3\xi^3\partial_{tt}ww + L.O.T$$
$$\leqslant \frac{\varepsilon_1}{2s}\iint_{\omega_T^1}\xi^{-1}(\partial_{tt}w)^2 + \frac{s^7\lambda^8}{2\varepsilon_1}\iint_{\omega_T^1}\xi^7w^2 + L.O.T,$$

and

$$s^3\lambda^4\iint_{\omega_T^3}\xi^3(\partial_{xx}w)^2 \leqslant s^3\lambda^4\iint_{\omega_T^1}\Upsilon_3\xi^3(\partial_{xx}w)^2 = s^3\lambda^4\iint_{\omega_T^1}\Upsilon_3\xi^3\partial_{xxxx}ww + L.O.T$$
$$\leqslant \frac{\varepsilon_1}{2s}\iint_{\omega_T^1}\xi^{-1}(\partial_{xxxx}w)^2 + \frac{s^7\lambda^8}{2\varepsilon_1}\iint_{\omega_T^1}\xi^7w^2 + L.O.T.$$

Similarly,

$$s^5\lambda^6\iint\limits_{\omega_T^3}\xi^5(\partial_x w)^2 \leqslant s^5\lambda^6\iint\limits_{\omega_T^1}\Upsilon_3\xi^5(\partial_x w)^2 = -s^5\lambda^6\iint\limits_{\omega_T^1}\Upsilon_3\xi^5\partial_{xx}ww + L.O.T.$$
$$\leqslant \frac{\varepsilon_1 s^3\lambda^4}{2}\iint\limits_{\omega_T^1}\xi^3(\partial_{xx}w)^2 + \frac{s^7\lambda^8}{2\varepsilon_1}\iint\limits_{\omega_T^1}\xi^7 w^2 + L.O.T..$$

Choosing now $\varepsilon_1 > 0$ small enough so that $C\varepsilon_1 \leqslant 1$ where C is the constant in (4.3.60), we deduce the following inequality from (4.3.60): for all s and λ large enough,

$$\begin{aligned}
&s^7\lambda^8\int_0^T\int_{\mathbb{T}_L}\xi^7 w^2 + s^5\lambda^6\int_0^T\int_{\mathbb{T}_L}\xi^5(\partial_x w)^2 + s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_{xx}w)^2\\
&\quad + s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xxx}w)^2 + s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_t w)^2 + s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xt}w)^2\\
&\quad + \frac{1}{s}\int_0^T\int_{\mathbb{T}_L}\frac{1}{\xi}(|\partial_{tt}w|^2 + |\partial_{txx}w|^2 + |\partial_{xxxx}w|^2)e^{-2s\phi}\\
&\quad \leqslant C\left(\int_0^T\int_{\mathbb{T}_L}|f_\psi|^2 e^{-2s\phi} + s^7\lambda^8\iint\limits_{\omega_T^1}\xi^7 w^2 + L.O.T.\right). \qquad (4.3.61)
\end{aligned}$$

Now, the lower order terms $L.O.T$ can be absorbed by taking s and λ large enough, so that from (4.3.61), we obtain that for all s and λ large enough,

$$\begin{aligned}
&s^7\lambda^8\int_0^T\int_{\mathbb{T}_L}\xi^7 w^2 + s^5\lambda^6\int_0^T\int_{\mathbb{T}_L}\xi^5(\partial_x w)^2 + s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_{xx}w)^2\\
&\quad + s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xxx}w)^2 + s^3\lambda^4\int_0^T\int_{\mathbb{T}_L}\xi^3(\partial_t w)^2 + s\lambda^2\int_0^T\int_{\mathbb{T}_L}\xi(\partial_{xt}w)^2\\
&\quad + \frac{1}{s}\int_0^T\int_{\mathbb{T}_L}\frac{1}{\xi}(|\partial_{tt}w|^2 + |\partial_{txx}w|^2 + |\partial_{xxxx}w|^2)e^{-2s\phi}\\
&\quad \leqslant C\left(\int_0^T\int_{\mathbb{T}_L}|f_\psi|^2 e^{-2s\phi} + s^7\lambda^8\iint\limits_{\omega_T^1}\xi^7 w^2\right). \qquad (4.3.62)
\end{aligned}$$

Finally to obtain (4.3.11) from (4.3.62) we just need to recall that $w = e^{-s\phi}\psi$, or equivalently that $\psi = we^{s\phi}$. This argument is very standard and is left to the reader. □

4.3.3 Carleman estimate for an adjoint heat equation

In this section we consider the following adjoint heat equation:

$$\begin{cases} -\dfrac{\overline{\rho}}{\nu}\partial_t q - \Delta q = f_1 & \text{in} \quad (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \partial_z q = f_2 & \text{on} \quad (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \partial_z q = 0 & \text{on} \quad (\mathbb{T}_L \times \{0\}) \times (0,T), \\ q(.,T) = q_T & \text{in} \quad (\mathbb{T}_L \times (0,1)), \end{cases} \tag{4.3.63}$$

where $f_1 \in L^2((\mathbb{T}_L \times (0,1)) \times (0,T))$, $f_2 \in L^2((\mathbb{T}_L \times \{1\}) \times (0,T))$.
For convenience, we further introduce the following shorthand notations which will mainly be used in writing the domain of integrals.

$$\begin{aligned} &Q_T^{ex} = (\mathbb{T}_L \times (0,1)) \times (0,T), \quad \omega_T = \omega \times (0,T), \\ &\mathbb{T}_T^1 = (\mathbb{T}_L \times \{1\}) \times (0,T), \quad \mathbb{T}_T^0 = (\mathbb{T}_L \times \{0\}) \times (0,T). \end{aligned} \tag{4.3.64}$$

Theorem 4.3.3. *There exist positive constants C, $s_2 \geqslant 1$ and $\lambda_2 \geqslant 1$ such that for all*

$$f_1 \in L^2(Q_T^{ex}), \quad \text{and} \quad f_2 \in L^2(\mathbb{T}_T^1), \tag{4.3.65}$$

and for all $q_T \in L^2(\mathbb{T}_L \times (0,1))$, for all $s \geqslant s_1$ and $\lambda \geqslant \lambda_1$, then the weak solution q of (4.3.63) satisfies the following inequality

$$\begin{aligned} &\iint\limits_{Q_T^{ex}} e^{-2s\phi}(s\lambda^2\xi|\nabla q|^2 + s^3\lambda^4\xi^3|q|^2) + s^2\lambda^3 \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi^2|q|^2 \\ &\leqslant C\left(\iint\limits_{Q_T^{ex}} e^{-2s\phi}|f_1|^2 + s\lambda \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi|f_2|^2 + s^3\lambda^4 \iint\limits_{\omega_T} e^{-2s\phi}\xi^3|q|^2\right), \end{aligned} \tag{4.3.66}$$

where the notations Q_T^{ex}, $\mathbb{T}_T^1$ and ω_T are introduced in (4.3.64).

Theorem 4.3.3 will be proved mainly by using the similar line of arguments as used to prove [15, Theorem 1]. The difference with [15] occurs in the construction of the weight functions. To be precise, the weight $\theta(t)$ in time is defined in [15] as $\theta(t) = \frac{1}{t(T-t)}$, whereas in our case $\theta(t)$ is as defined in (4.3.5). Further unlike [15], the function η^0 (defined in (4.3.2)) travels in time with a constant velocity $\overline{u}_1$. Both of these differences can be classically handled just by using the estimates (4.3.7), (4.3.8) and (4.3.9) of ξ and ϕ. One can also consult [2] for similar issues.
Above all in [15] and in most other articles in the literature it is assumed that η^0 vanishes at the boundary of the domain. Of course this assumption does not serve our purpose since we are working with a beam at the boundary. In our case, η^0 just depends on (x,t) implying in particular

$$\partial_z\phi = \partial_z\xi = 0 \quad \text{on} \quad ((\mathbb{T}_L \times \{1\}) \cup (\mathbb{T}_L \times \{0\})) \times (0,T). \tag{4.3.67}$$

Using this property, we still recover the same Carleman estimate for the heat equation with non homogeneous boundary condition obtained in [15].
Hence without going into the details we will just sketch the main steps of the proof of Theorem 4.3.3 and comment with references for their proofs.

For the proof of Theorem 4.3.3 we will need an auxiliary result: a Carleman inequality for heat equation with homogeneous Neumann boundary conditions, stated below:

Lemma 4.3.4. *There exist positive constants C, $s_1 \geqslant 1$ and $\lambda_1 \geqslant 1$ such that for all $s \geqslant s_1$, $\lambda \geqslant \lambda_1$, for all $\vartheta_0 \in H^1(\mathbb{T}_L \times (0,1))$, and for all $f_3 \in L^2(Q_T^{ex})$, the solution ϑ of the following problem*

$$\begin{cases} -\dfrac{\overline{\rho}}{\nu}\partial_t \vartheta - \Delta \vartheta = f_3 & \text{in} \quad Q_T^{ex}, \\ \partial_z \vartheta = 0 & \text{on} \quad \mathbb{T}_T^1 \cup \mathbb{T}_T^0, \\ \vartheta(.,T) = \vartheta_0 & \text{in} \quad \mathbb{T}_L \times (0,1), \end{cases} \tag{4.3.68}$$

satisfies

$$\begin{aligned} &\iint\limits_{Q_T^{ex}} e^{-2s\phi} \left(s\lambda^2 \xi |\nabla \vartheta|^2 + s^3\lambda^4 \xi^3 |\vartheta|^2 \right) \\ &\leqslant C \left(\iint\limits_{Q_T^{ex}} e^{-2s\phi} |f_3|^2 + s^3\lambda^4 \iint\limits_{\omega_T} e^{-2s\phi} \xi^3 |\vartheta|^2 \right). \end{aligned} \tag{4.3.69}$$

Lemma 4.3.4 will allow to construct by duality suitable solutions to a control problem:

Lemma 4.3.5. *There exist positive constants $C > 0$, $s_1 \geqslant 1$ and $\lambda_1 \geqslant 1$, such that for all $s \geqslant s_1$, $\lambda \geqslant \lambda_1$ and for all G satisfying*

$$\iint\limits_{Q_T^{ex}} \xi^{-3} |G|^2 e^{2s\phi} < \infty, \tag{4.3.70}$$

there exists a solution (Y, H) of the following control problem

$$\begin{cases} \dfrac{\overline{\rho}}{\nu}\partial_t Y - \Delta Y = G + H\chi_\omega & \text{in} \quad Q_T^{ex}, \\ \partial_z Y = 0 & \text{on} \quad \mathbb{T}_T^1 \cup \mathbb{T}_T^0, \\ Y(\cdot,0) = 0 & \text{in} \quad \mathbb{T}_L \times (0,1), \\ Y(\cdot,T) = 0 & \text{in} \quad \mathbb{T}_L \times (0,1), \end{cases} \tag{4.3.71}$$

which satisfies

$$Y \in L^2(0,T; H^2(\mathbb{T}_L \times (0,1))) \cap H^1(0,T; L^2(\mathbb{T}_L \times (0,1))) \cap C^0([0,T]; H^1(\mathbb{T}_L \times (0,1))).$$

and the following estimate:

$$\begin{aligned} &s^3\lambda^4 \iint\limits_{Q_T^{ex}} |Y|^2 e^{2s\phi} + s\lambda^2 \iint\limits_{Q_T^{ex}} \xi^{-2}|\nabla Y|^2 e^{2s\phi} \\ &+s^2\lambda^3 \iint\limits_{\mathbb{T}_T^1} \xi^{-1}|Y|^2 e^{2s\phi} + \iint\limits_{\omega_T} \xi^{-3}|H|^2 e^{2s\phi} \leqslant C \iint\limits_{Q_T^{ex}} \xi^{-3}|G|^2 e^{2s\phi}. \end{aligned} \tag{4.3.72}$$

The proof of Theorem 4.3.3 then follows from Lemma 4.3.5 by duality.

Details of the proof are given hereafter. Section 4.3.3.1 is dedicated to the proof of Lemma 4.3.4, Section 4.3.3.2 to the proof of Lemma 4.3.5, and Section 4.3.3.3 to the proof of Theorem 4.3.3.

In Section 4.3.3.4, we prove an additional result, which is a corollary of Theorem 4.3.3:

Corollary 4.3.6. *There exist positive constants C, $s_2(> s_1)$ and $\lambda_2(> \lambda_1)$ (where s_1 and λ_1 are the constants as in Theorem 4.3.3) such that if all the assumptions of Theorem 4.3.3 satisfied and if $s \geqslant s_2$ and $\lambda \geqslant \lambda_2$, then the solution q of (4.3.63) satisfies the following inequality*

$$\begin{aligned} &\iint\limits_{Q_T^{ex}} e^{-2s\phi}\Big(\frac{1}{s\xi}|\nabla q|^2 + s\lambda^2\xi|q|^2\Big) + \lambda \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}|q|^2 \\ &\leqslant C\left(\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|f_1|^2 + \frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\frac{1}{\xi}|f_2|^2 + s\lambda^2\iint\limits_{\omega_T} e^{-2s\phi}\xi|q|^2\right), \end{aligned} \tag{4.3.73}$$

where the notations Q_T^{ex}, $\mathbb{T}_T^1$ are introduced in (4.3.64).

4.3.3.1 Proof of Lemma 4.3.4

We set

$$\vartheta_1 = e^{-s\phi}\vartheta. \tag{4.3.74}$$

In view of (4.3.67) one observes that

$$\partial_z\vartheta_1 = 0 \quad \text{on} \quad \mathbb{T}_T^1 \cup \mathbb{T}_T^0. \tag{4.3.75}$$

Besides, with f_3 as in (4.3.68), ϑ_1 satisfies

$$e^{-s\phi}f_3 = e^{-s\phi}\Big(-\frac{\bar\rho}{\nu}\partial_t\vartheta - \Delta\vartheta\Big) = e^{-s\phi}\Big(-\frac{\bar\rho}{\nu}\partial_t(e^{s\phi}\vartheta_1) - \Delta(e^{s\phi}\vartheta_1)\Big) = \mathcal{P}_\phi\vartheta_1,$$

where the operator $\mathcal{P}$ is given by

$$\mathcal{P}_\phi \vartheta_1 = -\frac{\bar\rho}{\nu}\partial_t\vartheta_1 - s\frac{\bar\rho}{\nu}\partial_t\phi\vartheta_1 - \Delta\vartheta_1 - 2s\nabla\phi\cdot\nabla\vartheta_1 - s^2|\nabla\phi|^2\vartheta_1 - s\Delta\phi\vartheta_1.$$

Now we use

$$\nabla\phi = -\xi\lambda\nabla\eta^0,$$

and rewrite $\mathcal{P}_\phi$ as

$$\mathcal{P}_\phi = \mathcal{P}_1\vartheta_1 + \mathcal{P}_2\vartheta_1 + \mathcal{R}\vartheta_1,$$

where

$$\begin{array}{l}\mathcal{P}_1\vartheta_1 = -\frac{\bar\rho}{\nu}\partial_t\vartheta_1 + 2s\lambda\xi\nabla\eta^0\cdot\nabla\vartheta_1 + 2s\lambda^2|\nabla\eta^0|^2\xi\vartheta_1,\\ \mathcal{P}_2\vartheta_1 = -\Delta\vartheta_1 - \frac{\bar\rho}{\nu}s\partial_t\phi\vartheta_1 - s^2\lambda^2\xi^2|\nabla\eta^0|^2\vartheta_1,\\ \mathcal{R}\vartheta_1 = -s\lambda\Delta\eta^0\xi\vartheta_1 - s\lambda^2|\nabla\eta^0|^2\xi\vartheta_1.\end{array} \tag{4.3.76}$$

We then use that $\mathcal{P}_1\vartheta_1 + \mathcal{P}_2\vartheta_1 = f_3e^{-s\phi} - \mathcal{R}\vartheta_1$ and then

$$\begin{aligned}\iint\limits_{Q_T^{ex}} |\mathcal{P}_1\vartheta_1|^2 + \iint\limits_{Q_T^{ex}} |\mathcal{P}_2\vartheta_1|^2 + 2\iint\limits_{Q_T^{ex}} \mathcal{P}_1\vartheta_1\mathcal{P}_2\vartheta_1 &= \iint\limits_{Q_T^{ex}} |f_3e^{-s\phi} - \mathcal{R}\vartheta_1|^2\\ &\leqslant 2\iint\limits_{Q_T^{ex}} |f_3|^2e^{-2s\phi} + 2\iint\limits_{Q_T^{ex}} |\mathcal{R}\vartheta_1|^2.\end{aligned} \tag{4.3.77}$$

We now compute the scalar product of $\mathcal{P}_1\vartheta_1$ with $\mathcal{P}_2\vartheta_1$. In fact, these computations are very similar to the classical ones, and one should only remark that the integrations by parts do not yield any bad terms on the boundaries of the domain.
Again, we shall write $LO.T.$ to design lower order terms, that is terms which can be bounded as follows:

$$|L.O.T| \leqslant C\left(\frac{1}{s} + \frac{1}{\lambda}\right)\iint\limits_{Q_T^{ex}} e^{-2s\phi}\left(s\lambda^2\xi|\nabla\vartheta|^2 + s^3\lambda^4\xi^3|\vartheta|^2\right).$$

Note in particular that we have

$$\iint\limits_{Q_T^{ex}} |\mathcal{R}\vartheta_1|^2 = L.O.T.$$

Computations. We write

$$\iint\limits_{Q_T^{ex}} \mathcal{P}_1\vartheta_1\mathcal{P}_2\vartheta_1 = \sum_{i,j=1}^{3} J_{ij},$$

where $J_{i,j}$ is the scalar product of the i-th term of $\mathcal{P}_1\vartheta_1$ with the j-th term of $\mathcal{P}_2\vartheta_1$.
Computation of J_{11} :

$$J_{11} = \frac{\bar\rho}{\nu}\iint\limits_{Q_T^{ex}} \partial_t\vartheta_1\Delta\vartheta_1 = -\frac{\bar\rho}{\nu}\iint\limits_{Q_T^{ex}} \partial_t\left(\frac{|\nabla\vartheta_1|^2}{2}\right) = 0. \tag{4.3.78}$$

Computation of J_{12} :

$$\begin{aligned} J_{12} &= \left(\frac{\bar{\rho}}{\nu}\right)^2 s \iint\limits_{Q_T^{ex}} (\partial_t \vartheta_1 \cdot \vartheta_1)\, \partial_t \phi = \frac{1}{2}\left(\frac{\bar{\rho}}{\nu}\right)^2 s \iint\limits_{Q_T^{ex}} \partial_t |\vartheta_1|^2 \partial_t \phi \\ &= -\frac{1}{2}\left(\frac{\bar{\rho}}{\nu}\right)^2 s \iint\limits_{Q_T^{ex}} |\vartheta_1|^2 \partial_{tt} \phi = L.O.T. \end{aligned} \tag{4.3.79}$$

Computation of J_{13} :

$$\begin{aligned} J_{13} &= \frac{\bar{\rho}}{\nu} s^2\lambda^2 \iint\limits_{Q_T^{ex}} \xi^2 |\nabla \eta^0|^2 \partial_t \vartheta_1 \cdot \vartheta_1 = \frac{\bar{\rho}}{\nu}\frac{s^2\lambda^2}{2} \iint\limits_{Q_T^{ex}} \partial_t |\vartheta_1|^2 \xi^2 |\nabla \eta^0|^2 \\ &= -\frac{\bar{\rho}}{\nu}\frac{s^2\lambda^2}{2} \iint\limits_{Q_T^{ex}} |\vartheta_1|^2 \partial_t (\xi^2 |\nabla \eta^0|^2) = L.O.T. \end{aligned} \tag{4.3.80}$$

Computation of J_{12} :

$$\begin{aligned} J_{21} &= -2s\lambda \iint\limits_{Q_T^{ex}} \xi (\nabla \eta^0 \cdot \nabla \vartheta_1) \Delta \vartheta_1 = 2s\lambda \iint\limits_{Q_T^{ex}} \nabla(\xi(\nabla \eta^0 \cdot \nabla \vartheta_1)) \cdot \nabla \vartheta_1 \\ &= 2s\lambda^2 \iint\limits_{Q_T^{ex}} \xi |\nabla \eta^0 \cdot \nabla \vartheta_1|^2 + 2s\lambda \iint\limits_{Q_T^{ex}} \xi \nabla^2 \eta^0 (\nabla \vartheta_1, \nabla \vartheta_1) + s\lambda \iint\limits_{Q_T^{ex}} \xi \nabla \eta^0 \cdot \nabla |\nabla \vartheta_1|^2 \\ &= 2s\lambda^2 \iint\limits_{Q_T^{ex}} \xi |\nabla \eta^0 \cdot \nabla \vartheta_1|^2 + 2s\lambda \iint\limits_{Q_T^{ex}} \xi \nabla^2 \eta^0 (\nabla \vartheta_1, \nabla \vartheta_1) - s\lambda^2 \iint\limits_{Q_T^{ex}} \xi |\nabla \eta^0|^2 |\nabla \vartheta_1|^2 \\ &\quad - s\lambda \iint\limits_{Q_T^{ex}} \xi \Delta \eta^0 |\nabla \vartheta_1|^2 \\ &= 2s\lambda^2 \iint\limits_{Q_T^{ex}} \xi |\nabla \eta^0 \cdot \nabla \vartheta_1|^2 - s\lambda^2 \iint\limits_{Q_T^{ex}} \xi |\nabla \eta^0|^2 |\nabla \vartheta_1|^2 + L.O.T. \end{aligned} \tag{4.3.81}$$

Computation of J_{22} :

$$J_{22} = -2\frac{\bar{\rho}}{\nu} s^2\lambda \iint\limits_{Q_T^{ex}} \xi (\nabla \eta^0 \cdot \nabla \vartheta_1) \partial_t \phi \vartheta_1 = \frac{\bar{\rho}}{\nu} s^2\lambda \iint\limits_{Q_T^{ex}} \operatorname{div}\left(\partial_t \phi \nabla \eta^0 \xi\right) |\vartheta_1|^2 = L.O.T. \tag{4.3.82}$$

Computation of J_{23} :

$$J_{23} = -2s^3\lambda^3 \iint\limits_{Q_T^{ex}} \xi^3 (\nabla \eta^0 \cdot \nabla \vartheta_1) |\nabla \eta^0|^2 \vartheta_1 = s^3\lambda^3 \iint\limits_{Q_T^{ex}} \operatorname{div}(|\nabla \eta^0|^2 \nabla \eta^0 \xi^3) |\vartheta_1|^2. \tag{4.3.83}$$

Computation of J_{31} :

$$\begin{aligned} J_{31} &= -2s\lambda^2 \iint\limits_{Q_T^{ex}} |\nabla\eta^0|^2\xi\vartheta_1\Delta\vartheta_1 \\ &= 2s\lambda^2 \iint\limits_{Q_T^{ex}} |\nabla\eta^0|^2\xi|\nabla\vartheta_1|^2 + 2s\lambda^2 \iint\limits_{Q_T^{ex}} \nabla(|\nabla\eta^0|^2\xi)\vartheta_1\cdot\nabla\vartheta_1 \\ &= 2s\lambda^2 \iint\limits_{Q_T^{ex}} |\nabla\eta^0|^2\xi|\nabla\vartheta_1|^2 - s\lambda^2 \iint\limits_{Q_T^{ex}} \Delta(|\nabla\eta^0|^2\xi)|\vartheta_1|^2 \\ &= 2s\lambda^2 \iint\limits_{Q_T^{ex}} |\nabla\eta^0|^2\xi|\nabla\vartheta_1|^2 + L.O.T. \end{aligned} \tag{4.3.84}$$

Computation of J_{32} :

$$J_{32} = -2s^2\lambda^2\frac{\bar\rho}{\nu}\iint\limits_{Q_T^{ex}} |\nabla\eta^0|^2\xi|\vartheta_1|^2\partial_t\phi = L.O.T. \tag{4.3.85}$$

Computation of J_{33} :

$$J_{33} = -2s^3\lambda^4 \iint\limits_{Q_T^{ex}} \xi^3|\nabla\eta^0|^4|\vartheta_1|^2. \tag{4.3.86}$$

Combining the above computations (4.3.78)-(4.3.86), we obtain the following:

$$\begin{aligned} \iint\limits_{Q_T^{ex}} \mathcal{P}_1\vartheta_1\mathcal{P}_2\vartheta_1 &= +2s\lambda^2 \iint\limits_{Q_T^{ex}} \xi|\nabla\eta^0\cdot\nabla\vartheta_1|^2 + s\lambda^2 \iint\limits_{Q_T^{ex}} \xi|\nabla\eta^0|^2|\nabla\vartheta_1|^2 \\ &+ s^3\lambda^3 \iint\limits_{Q_T^{ex}} |\vartheta_1|^2\Big(\mathrm{div}(|\nabla\eta^0|^2\nabla\eta^0\xi^3) - 2\lambda\xi^3|\nabla\eta^0|^4\Big) + L.O.T. \end{aligned} \tag{4.3.87}$$

Now, it is clear that

$$\Big|\Big(\mathrm{div}(|\nabla\eta^0|^2\nabla\eta^0\xi^3) - 2\lambda\xi^3|\nabla\eta^0\Big) - \lambda\xi^3|\nabla\eta^0|^4\Big| \leqslant C\xi^3.$$

We thus immediately deduce from (4.3.3) and (4.3.77) that there exists $C>0$ such that for all s and λ large enough,

$$\begin{aligned} &\iint\limits_{Q_T^{ex}} s\lambda^2\xi|\nabla\vartheta_1|^2 + \iint\limits_{Q_T^{ex}} s^3\lambda^4\xi^3|\vartheta_1|^2 \\ &\leqslant C\left(\iint\limits_{Q_T^{ex}} e^{-2s\phi}|f_3|^2 + s^3\lambda^4\iint\limits_{\tilde\omega_T}\xi^3|\vartheta_1|^2 + s\lambda^2\iint\limits_{\tilde\omega_T}\xi|\nabla\vartheta_1|^2\right), \end{aligned} \tag{4.3.88}$$

where $\tilde\omega_T = (\mathbb{T}_L\setminus[-\overline{u}_1T, d+\overline{u}_1T]\times(0,1))\times(0,T)$. Now there are two steps to obtain (4.3.69) from (4.3.88):

- Absorbing the observation in (4.3.88) involving $\nabla\vartheta_1$ on $\tilde{\omega}_T$: This can be done similarly as in the proof of Theorem 4.3.2 when passing from (4.3.58) to (4.3.62). In fact, here we can avoid estimating the highest order term $\partial_t\vartheta_1$, $D^2\vartheta_1$ by simply multiplying $\mathcal{P}\vartheta_1$ by $\Upsilon\vartheta_1\xi$, for a suitable cut off function Υ, similar to the one defined in (4.3.59).
- One should then come back to the original unknown ϑ from ϑ_1. This can be done as it is done classically by recalling that $\vartheta = e^{s\phi}\vartheta_1$.

This concludes the proof of Lemma 4.3.4.

4.3.3.2 Proof of Lemma 4.3.5

The proof of this lemma follows the one of [2, Theorem 2.6], and we recall it only for the convenience of the reader.
In order to solve (4.3.71) we will introduce a functional J whose Euler Lagrange equation provides a solution of (4.3.71). For smooth functions ϑ on $\overline{Q_T^{ex}}$ with $\partial_z\vartheta = 0$ on $\overline{\mathbb{T}_T^1}\cup\overline{\mathbb{T}_T^0}$, let us define

$$J(\vartheta) = \frac{1}{2}\iint\limits_{Q_T^{ex}} |(-\frac{\overline{\rho}}{\nu}\partial_t - \Delta)\vartheta|^2 e^{-2s\phi} + \frac{s^3\lambda^4}{2}\iint\limits_{\omega_T}\xi^3|\vartheta|^2e^{-2s\phi} - \iint\limits_{Q_T^{ex}} G\vartheta. \tag{4.3.89}$$

We introduce the following space

$$X_{obs} = \overline{\{\vartheta \in C^\infty(\overline{Q_T^{ex}}) \text{ such that } \partial_z\vartheta = 0 \text{ on } \overline{\mathbb{T}_T^1}\cup\overline{\mathbb{T}_T^0}\}}^{\|\cdot\|_{obs}}, \tag{4.3.90}$$

where $\|\cdot\|_{obs}$ is the Hilbert norm defined by

$$\|\vartheta\|_{obs}^2 = \iint\limits_{Q_T^{ex}} |(-\frac{\overline{\rho}}{\nu}\partial_t - \Delta)\vartheta|^2 e^{-2s\phi} + s^3\lambda^4\iint\limits_{\omega_T}\xi^3|\vartheta|^2e^{-2s\phi}. \tag{4.3.91}$$

We endow the space X_{obs} with the Hilbert structure given by $\|\cdot\|_{obs}$. Of course, the fact that $\|\cdot\|_{obs}$ is a norm follows from the Carleman estimate (4.3.69).
Observe that, from the assumption (4.3.70) and the Carleman estimate (4.3.69), one has

$$\left|\iint\limits_{Q_T^{ex}} G\vartheta\right| \leqslant C\|\vartheta\|_{obs}\left(\frac{1}{s^3\lambda^4}\iint\limits_{Q_T^{ex}}\xi^{-3}|G|^2e^{2s\phi}\right)^{1/2}, \tag{4.3.92}$$

for some constant $C > 0$. Hence in view of (4.3.92), one observes that the functional J can be uniquely extended as a continuous functional on X_{obs}. We denote this extension

by the notation J itself. The inequality (4.3.92) also infers the coercivity of J on X_{obs}. It is easy to check that J is strictly convex on X_{obs}. Hence J admits a unique minimizer $\vartheta_{\min}$ on X_{obs}.
We further set

$$Y = (-\frac{\overline{\rho}}{\nu}\partial_t - \Delta)\vartheta_{\min} e^{-2s\phi} \quad \text{and} \quad H = -s^3\lambda^4\xi^3\vartheta_{\min}e^{-2s\phi}\chi_{\omega_T}. \tag{4.3.93}$$

From the Euler Lagrange equation of J at $\vartheta_{\min}$, for all smooth functions ϑ on $\overline{Q_T^{ex}}$ such that $\partial_z\vartheta = 0$ on $\overline{T}_T^1 \cup \overline{\mathbb{T}_T^0}$,

$$\iint\limits_{Q_T^{ex}} Y(-\frac{\overline{\rho}}{\nu}\partial_t\vartheta - \Delta\vartheta) - \iint\limits_{Q_T^{ex}} G\vartheta - \iint\limits_{\omega_T} H\vartheta = 0, \tag{4.3.94}$$

This equation easily implies that the solution Y of $(4.3.71)_{(1,2,3)}$ in the sense of transposition with source term $G + H\chi_\omega$ with H given by (4.3.93) coincides with the function Y given by (4.3.93) and satisfies the null controllability requirement $Y(\cdot, T) = 0$ in $\mathbb{T}_L \times (0,1)$.
Note that, since $G \in L^2(Q_T^{ex})$ and $H \in L^2(Q_T^{ex})$,

$$Y \in L^2(0,T;H^2(\mathbb{T}_L\times(0,1))) \cap H^1(0,T;L^2(\mathbb{T}_L\times(0,1))) \cap C^0([0,T];H^1(\mathbb{T}_L\times(0,1))).$$

Moreover since $\vartheta_{\min}$ is the minimizer of J on X_{obs}, using (4.3.92) one gets

$$s^3\lambda^4 \iint\limits_{Q_T^{ex}} |Y|^2 e^{2s\phi} + \iint\limits_{\omega_T} \xi^{-3}|H|^2 e^{2s\phi} \leqslant C \iint\limits_{Q_T^{ex}} \xi^{-3}|G|^2 e^{2s\phi}, \tag{4.3.95}$$

for large enough values of the parameters s and λ.
One then follows the arguments used in proving [2, Theorem 2.6], which mainly consists in a suitable energy estimate, i.e. a multiplication of (4.3.71) by $\xi^{-2}e^{2s\phi}Y$, to show the following

$$s\lambda^2 \iint\limits_{Q_T^{ex}} \xi^{-2}|\nabla Y|^2 e^{2s\phi} \leqslant C \iint\limits_{Q_T^{ex}} \xi^{-3}|G|^2 e^{2s\phi}, \tag{4.3.96}$$

for large enough values of the parameters s and λ.
In order to finish proving (4.3.72) one only needs to show

$$s^2\lambda^3 \iint\limits_{\mathbb{T}_T^1} \xi^{-1}|Y|^2 e^{2s\phi} \leqslant C \iint\limits_{Q_T^{ex}} \xi^{-3}|G|^2 e^{2s\phi}, \tag{4.3.97}$$

for large enough values of the parameters s and λ.
This can be achieved thanks to the estimates (4.3.95) and (4.3.96). In this direction we introduce a function

$$\kappa = \kappa(x,z) = \begin{pmatrix} 0 \\ z \end{pmatrix}$$

on $\mathbb{T}_L \times (0,1)$. The function κ verifies

$$\kappa \cdot n = \begin{cases} 1 \text{ on } \mathbb{T}_L \times \{1\}, \\ 0 \text{ on } \mathbb{T}_L \times \{0\}, \end{cases} \tag{4.3.98}$$

where n denotes the unit outward normal to $(\mathbb{T}_L \times \{1\}) \cup (\mathbb{T}_L \times \{0\})$.
Now we perform the following calculations, using that $\partial_z(\xi^{-1}e^{2s\phi}) = 0$:

$$\begin{aligned} s^2\lambda^3 \iint_{\mathbb{T}_T^1} \xi^{-1}|Y|^2 e^{2s\phi} &= s^2\lambda^3 \iint_{Q_T^{ex}} \operatorname{div}(\kappa\xi^{-1}|Y|^2 e^{2s\phi}) \\ &= s^2\lambda^3 \iint_{Q_T^{ex}} \operatorname{div}(\kappa\xi^{-1}e^{2s\phi})|Y|^2 + 2s^2\lambda^3 \iint_{Q_T^{ex}} \kappa\xi^{-1}e^{2s\phi} Y \cdot \nabla Y \\ &= s^2\lambda^3 \iint_{Q_T^{ex}} \xi^{-1}e^{2s\phi}|Y|^2 + 2s^2\lambda^3 \iint_{Q_T^{ex}} \kappa\xi^{-1}e^{2s\phi} Y \cdot \nabla Y \\ &\leqslant C\left(s^3\lambda^4 \iint_{Q_T^{ex}} |Y|^2 e^{2s\phi} + s\lambda^2 \iint_{Q_T^{ex}} \xi^{-2}|\nabla Y|^2 e^{2s\phi} \right). \end{aligned} \tag{4.3.99}$$

Finally combining the estimates (4.3.95), (4.3.96) and (4.3.97), we conclude the proof of the estimate (4.3.72).

4.3.3.3 Proof of Theorem 4.3.3

Since we only assume that $f_1 \in Q_T^{ex}$, $f_2 \in \mathbb{T}_T^1$, the solution of the problem (4.3.63) has to be considered in the sense of transposition. In particular, for

$$Y \in L^2(0,T;H^2(\mathbb{T}_L \times (0,1))) \cap H^1(0,T;L^2(\mathbb{T}_L \times (0,1))) \cap C^0([0,T];H^1(\mathbb{T}_L \times (0,1))),$$

satisfying $\partial_n Y = 0$ on $\mathbb{T}_T^1 \cup \mathbb{T}_T^0$ with $Y(\cdot,0) = Y(\cdot,T) = 0$ in $\mathbb{T}_L \times (0,1)$, we have

$$\iint_{Q_T^{ex}} f_1 Y = \iint_{Q_T^{ex}} (\frac{\overline{\rho}}{\nu}\partial_t Y - \Delta Y)q + \iint_{\mathbb{T}_T^1} Y f_2 \tag{4.3.100}$$

We thus choose a particular function G which satisfies (4.3.70), namely

$$G = \xi^3 q e^{-2s\phi}, \tag{4.3.101}$$

and Y the function given by Lemma 4.3.5, so that (4.3.100) yields:

$$\iint_{Q_T^{ex}} \xi^3|q|^2 e^{-2s\phi} = \iint_{Q_T^{ex}} f_1 Y - \iint_{\mathbb{T}_T^1} Y f_2 - \iint_{Q_T^{ex}} qG\chi_\omega. \tag{4.3.102}$$

With the choice (4.3.101) of G, observe in particular that

$$\iint\limits_{Q_T^{ex}} \xi^{-3}|G|^2 e^{2s\phi} = \iint\limits_{Q_T^{ex}} \xi^3 |q|^2 e^{-2s\phi}.$$

Besides (4.3.102) furnishes, for all $\epsilon > 0$,

$$\iint\limits_{Q_T^{ex}} \xi^3|q|^2e^{-2s\phi} \leqslant C\left(\epsilon s^3\lambda^4 \iint\limits_{Q_T^{ex}} |Y|^2e^{2s\phi} + \frac{1}{\epsilon s^3\lambda^4}\iint\limits_{Q_T^{ex}} |f_1|^2e^{-2s\phi} + \frac{1}{\epsilon}\iint\limits_{\omega_T} \xi^3|q|^2e^{-2s\phi}\right.$$
$$\left. +\epsilon \iint\limits_{\omega_T} \xi^{-3}|H|^2e^{2s\phi} + \frac{1}{\epsilon s^2\lambda^3}\iint\limits_{\mathbb{T}_T^1} \xi|f_2|^2e^{-2s\phi} + \epsilon s^2\lambda^3 \iint\limits_{\mathbb{T}_T^1} \xi^{-1}|Y|^2e^{2s\phi}\right). \quad (4.3.103)$$

On the other hand with the particular choice (4.3.101) of G, the inequality (4.3.72) in particular takes the form

$$s^3\lambda^4 \iint\limits_{Q_T^{ex}} |Y|^2e^{2s\phi} + s^2\lambda^3 \iint\limits_{\mathbb{T}_T^1} \xi^{-1}|Y|^2e^{2s\phi} + \iint\limits_{\omega_T} \xi^{-3}|H|^2e^{2s\phi} \leqslant C\iint\limits_{Q_T^{ex}} \xi^3|q|^2e^{-2s\phi}. \quad (4.3.104)$$

Incorporating (4.3.104) in (4.3.103) and choosing small enough value for the parameter ϵ, we prove that

$$s^3\lambda^4 \iint\limits_{Q_T^{ex}} \xi^3|q|^2e^{-2s\phi}$$
$$\leqslant C\left(\iint\limits_{Q_T^{ex}} e^{-2s\phi}|f_1|^2 + s\lambda \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi|f_2|^2 + s^3\lambda^4 \iint\limits_{\omega_T} e^{-2s\phi}\xi^3|q|^2\right). \quad (4.3.105)$$

Now one needs to estimate the integral of ∇q on Q_T^{ex} and the integral of q on $\mathbb{T}_T^1$ to finish proving the inequality (4.3.66).
In order to do that, we perform a weighted energy estimate on q by multiplying (4.3.63) by $\xi q e^{-2s\phi}$. This will lead to

$$\iint\limits_{Q_T^{ex}} \xi|\nabla q|^2e^{-2s\phi} \leqslant Cs^2\lambda^2 \iint\limits_{Q_T^{ex}} \xi^3|q|^2e^{-2s\phi} + C\iint\limits_{Q_T^{ex}} \xi|q||f_1|e^{-2s\phi} + C\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi f_2 q$$
$$\leqslant Cs^2\lambda^2 \iint\limits_{Q_T^{ex}} \xi^3|q|^2e^{-2s\phi} + \frac{C}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} |f_1|^2e^{-2s\phi} + Cs\lambda \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi|q|^2$$
$$+ \frac{C}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi|f_2|^2. \quad (4.3.106)$$

Now, similarly as in (4.3.99), we can obtain

$$s\lambda \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi|q|^2 \leqslant s\lambda \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi^2|q|^2 \tag{4.3.107}$$
$$\leqslant C\left(s^2\lambda^2(1+\frac{1}{\varepsilon})\iint\limits_{Q_T^{ex}} \xi^3|q|^2e^{-2s\phi} + \varepsilon \iint\limits_{Q_T^{ex}} \xi|\nabla q|^2e^{-2s\phi}\right).$$

Therefore, choosing $\varepsilon > 0$ small enough, we deduce from (4.3.106) and (4.3.107) that there exists a constant $C > 0$ such that for all s and λ large enough,

$$\iint\limits_{Q_T^{ex}} \xi|\nabla q|^2e^{-2s\phi} + s\lambda \iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi^2|q|^2$$
$$\leqslant Cs^2\lambda^2 \iint\limits_{Q_T^{ex}} \xi^3|q|^2e^{-2s\phi} + \frac{C}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} |f_1|^2e^{-2s\phi} + \frac{C}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi|f_2|^2.$$

With the estimate (4.3.105), this finishes the proof of Theorem 4.3.3.

4.3.3.4 Proof of Corollary 4.3.6

Let us introduce the new unknown:

$$q_1 = \frac{1}{s\lambda\xi}q.$$

From (4.3.63) we obtain the following system satisfied by q_1:

$$\begin{cases} -\dfrac{\overline{\rho}}{\nu}\partial_t q_1 - \Delta q_1 = \dfrac{1}{s\lambda\xi}f_1 + \mathcal{F}_5 & \text{in} \quad (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \partial_z q_1 = \dfrac{1}{s\lambda\xi}f_2 & \text{on} \quad (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \partial_z q_1 = 0 & \text{on} \quad (\mathbb{T}_L \times \{0\}) \times (0,T), \\ q_1(.,T) = \dfrac{1}{s\lambda\xi}q_T & \text{in} \quad (\mathbb{T}_L \times (0,1)), \end{cases} \tag{4.3.108}$$

where

$$\mathcal{F}_5 = \frac{\overline{\rho}}{\nu}\frac{1}{s\lambda\xi^2}\partial_t\xi q + \frac{1}{s\lambda\xi^2}\nabla q\cdot\nabla\xi + \frac{1}{s\lambda\xi^2}q\Delta\xi + \frac{2}{s\lambda\xi^3}|\nabla\xi|^2q - \frac{1}{s\lambda\xi^2}\nabla\xi\cdot\nabla q. \tag{4.3.109}$$

In view of the estimates (4.3.8), $\mathcal{F}_5$ satisfies

$$\iint\limits_{Q_T^{ex}} e^{-2s\phi}|\mathcal{F}_5|^2 \leqslant C\left(\frac{\lambda^2}{s^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi}|q|^2 + \frac{1}{s^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|\nabla q|^2\right). \tag{4.3.110}$$

Now we apply Theorem 4.3.3 for the system (4.3.108) to have the following

$$\begin{aligned}
&\iint\limits_{Q_T^{ex}} e^{-2s\phi}(s\lambda^2\xi|\nabla q_1|^2+s^3\lambda^4\xi^3|q_1|^2)+s^2\lambda^3\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi^2|q_1|^2\\
&\leqslant C\left(\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|f_1|^2+\iint\limits_{Q_T^{ex}} e^{-2s\phi}|\mathcal{F}_5|^2+\frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi\frac{1}{\xi^2}|f_2|^2\right.\\
&\qquad\left.+s^3\lambda^4\iint\limits_{\omega_T} e^{-2s\phi}\xi^3|q_1|^2\right)\\
&\leqslant C\left(\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|f_1|^2+\frac{\lambda^2}{s^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi}|q|^2+\frac{1}{s^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|\nabla q|^2\right.\\
&\qquad\left.+\frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\frac{1}{\xi}|f_2|^2+s\lambda^2\iint\limits_{\omega_T} e^{-2s\phi}\xi|q|^2\right),
\end{aligned}\tag{4.3.111}$$

where the last step of (4.3.111) from the penultimate step follows by using (4.3.110) and the definition of q_1. Now, using $q=s\lambda\xi q_1$, one further checks the following

$$\begin{aligned}
&\iint\limits_{Q_T^{ex}} e^{-2s\phi}(\frac{1}{s\xi}|\nabla q|^2+s\lambda^2\xi|q|^2)+\lambda\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}|q|^2\\
&\qquad\leqslant C\left(\iint\limits_{Q_T^{ex}} e^{-2s\phi}(s\lambda^2\xi|\nabla q_1|^2+s^3\lambda^4\xi^3|q_1|^2)+s^2\lambda^3\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\xi^2|q_1|^2\right).
\end{aligned}\tag{4.3.112}$$

Combining (4.3.111) and (4.3.112), we get

$$\begin{aligned}
&\iint\limits_{Q_T^{ex}} e^{-2s\phi}(\frac{1}{s\xi}|\nabla q|^2+s\lambda^2\xi|q|^2)+\lambda\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}|q|^2\\
&\qquad\leqslant C\left(\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|f_1|^2+\frac{\lambda^2}{s^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi}|q|^2+\frac{1}{s^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|\nabla q|^2\right.\\
&\qquad\qquad\left.+\frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\frac{1}{\xi}|f_2|^2+s\lambda^2\iint\limits_{\omega_T} e^{-2s\phi}\xi|q|^2\right).
\end{aligned}$$

Taking s and λ large enough, we conclude Corollary 4.3.6.

4.3.4 Observability of an adjoint transport equation

In this section we derive an observability inequality for the adjoint transport equation

$$\begin{cases} -\partial_t\sigma - \overline{u}_1\partial_x\sigma + \frac{P'(\overline{\rho})}{\nu}\sigma = f_4 & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \sigma(\cdot,T) = \sigma_T & \text{in } \mathbb{T}_L \times (0,1). \end{cases} \tag{4.3.113}$$

Theorem 4.3.7. *Let us recall the notations Q_T^{ex} and ω_T introduced in (4.3.64). There exists a positive constant C such that for all $\sigma_T \in L^2(\mathbb{T}_L \times (0,1))$, $f_4 \in L^2(Q_T^{ex})$ and for all values of the parameters $s \geqslant 1$ and $\lambda \geqslant 1$, the solution σ of (4.3.113) satisfies the following inequality*

$$\|\sigma e^{-s\phi}\|^2_{L^2(Q_T^{ex})} \leqslant C(\|f_4 e^{-s\phi}\|^2_{L^2(Q_T^{ex})} + \|\sigma e^{-s\phi}\|^2_{L^2(\omega_T)}). \tag{4.3.114}$$

There exists a positive constant C such that for all $\sigma_T \in H^1(\mathbb{T}_L \times (0,1))$, $f_4 \in L^2(0,T;H^1(\mathbb{T}_L \times (0,1)))$, and for all values of the parameter $s \geqslant 1$, the solution σ of (4.3.113) satisfies the following inequality

$$\|\partial_x\sigma e^{-s\phi}\|^2_{L^2(Q_T^{ex})} \leqslant C(\|\partial_x f_4 e^{-s\phi}\|^2_{L^2(Q_T^{ex})} + \|\partial_x\sigma e^{-s\phi}\|^2_{L^2(\omega_T)}). \tag{4.3.115}$$

Consequently there exists a positive constant C such that for all $f_4 \in L^2(0,T;H^1(\mathbb{T}_L \times (0,1)))$, and for all values of the parameter $s \geqslant 1$, the solution σ of (4.3.113) satisfies the following inequality

$$\begin{aligned} \|\partial_t\sigma e^{-s\phi}\|^2_{L^2(Q_T^{ex})} \leqslant & C(\|\partial_x f_4 e^{-s\phi}\|^2_{L^2(Q_T^{ex})} + \|f_4 e^{-s\phi}\|^2_{L^2(Q_T^{ex})} \\ & + \|\sigma e^{-s\phi}\|^2_{L^2(\omega_T)} + \|\partial_x\sigma e^{-s\phi}\|^2_{L^2(\omega_T)}). \end{aligned} \tag{4.3.116}$$

Proof. We first prove the inequality (4.3.114). The proof depends on a controllability estimate of the following problem

$$\begin{cases} \partial_t\widetilde{\sigma} + \overline{u}_1\partial_x\widetilde{\sigma} + \dfrac{P'(\overline{\rho})}{\nu}\widetilde{\sigma} = \widetilde{f}_4 + v_{\widetilde{\sigma}}\chi_\omega & \text{in } (\mathbb{T}_L \times (0,1)) \times (0,T), \\ \widetilde{\sigma}(\cdot,0) = 0 & \text{in } \mathbb{T}_L \times (0,1), \\ \widetilde{\sigma}(\cdot,T) = 0 & \text{in } \mathbb{T}_L \times (0,1). \end{cases} \tag{4.3.117}$$

and a duality argument. The controllability estimate for the problem (4.3.117) will be adapted from [12]. In fact following [12, Theorem 3.5], we have the following result: If

$$\|\widetilde{f}_4 e^{s\phi}\|_{L^2(Q_T^{ex})} < \infty,$$

there exists a control function $v_{\widetilde{\sigma}}\chi_\omega \in L^2(Q_T^{ex})$, such that $\widetilde{\sigma}$ solving $(4.3.117)_{(1,2)}$ satisfies the control requirement $(4.3.117)_3$. Besides, $\widetilde{\sigma}$ solves

$$\|\widetilde{\sigma} e^{s\phi}\|_{L^2(Q_T^{ex})} + \|v_{\widetilde{\sigma}} e^{s\phi}\|_{L^2(\omega_T)} \leqslant C\|\widetilde{f}_4 e^{s\phi}\|_{L^2(Q_T^{ex})}. \tag{4.3.118}$$

In fact, the result stated in [12, Theorem 3.5] involves an additional weight in time $\theta^{-3/2}$, but the proof of the above result truly follows line by line the one of [12, Theorem 3.5]. The only difference is that one should use the multiplier $\widetilde{\sigma}e^{2s\phi}$ instead of $\theta^{-3}\widetilde{\sigma}e^{2s\phi}$ in [12, Theorem 3.5].
Now, for $\widetilde{f}_4$ satisfying $\|\widetilde{f}_4 e^{s\phi}\|_{L^2(Q_T^{ex})} < \infty$, we associate $(\widetilde{\sigma}, v_{\widetilde{\sigma}})$ given by the above process.
We then do the following computations

$$\begin{aligned}\|\sigma e^{-s\phi}\|_{L^2(Q_T^{ex})} &= \sup_{\|\widetilde{f}_4 e^{s\phi}\|\leqslant 1} |\langle \widetilde{f}_4, \sigma\rangle_{L^2(Q_T^{ex})}| \\ &\leqslant \sup_{\|\widetilde{f}_4 e^{s\phi}\|\leqslant 1} (|\langle \widetilde{\sigma}, f_4\rangle_{L^2(Q_T^{ex})}| + |\langle v_{\widetilde{\sigma}}\chi_\omega, \sigma\rangle_{L^2(Q_T^{ex})}|) \\ &\leqslant C(\|f_4 e^{-s\phi}\|_{L^2(Q_T^{ex})} + \|\sigma e^{-s\phi}\|_{L^2(\omega_T)}).\end{aligned}$$

This provides the inequality (4.3.114).
The inequality (4.3.115) can be obtained by applying (4.3.114) to the system satisfied by $\partial_x\sigma$. Once we have (4.3.115), the estimate (4.3.116) can be obtained directly by using the equation $(4.3.113)_1$. □

4.4 Observability of the system (4.1.27)

The goal of this section is to prove Theorem 4.1.5. In order to do that, as for the proof of Theorem 4.1.4, the key step is first to prove an observability result for the system (4.2.3). In fact, we will start by showing the following result:

Lemma 4.4.1. *There exists a constant $C > 0$ such that for all (σ, q, ψ) solving (4.2.3) with initial datum $(\sigma_T, q_T, \psi_T, \psi_T^1)$ having the regularity (4.2.4) and satisfying the compatibility conditions (4.2.5),*

$$\begin{aligned}\|(\psi(\cdot,0), \partial_t\psi(\cdot,0))\|_{H^3(\mathbb{T}_L\times(0,1)))\times H^1(\mathbb{T}_L\times(0,1))} + \|q(\cdot,0)\|_{H^2(\mathbb{T}_L\times(0,1))} + \|\sigma(\cdot,0)\|_{H^1(\mathbb{T}_L\times(0,1))} \\ \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}, \quad (4.4.1)\end{aligned}$$

where ω_T is defined in (4.3.64).

Lemma 4.4.1 is proved in Section 4.4.1 below, and follows from a suitable use of the various Carleman estimates proved in the previous section.

Based on Lemma 4.4.1, it will be rather easy to derive an estimate on $v(\cdot, 0)$ in $L^2(\mathbb{T}_L \times (0,1))$ and conclude the proof of Theorem 4.1.5, which will be done in Section 4.4.2.

4.4.1 Proof of Lemma 4.4.1

Note that, from Lemma 4.2.2, for $(\sigma_T, q_T, \psi_T, \psi_T^1)$ having the regularity (4.2.4) and satisfying the compatibility conditions (4.2.5), we have that $(\partial_t + \overline{u}_1\partial_x)q_{|\mathbb{T}_L\times\{1\}}$ belongs to $L^2(\mathbb{T}_L \times \{1\} \times (0,T))$. We thus apply Theorem 4.3.2 with $f_\psi = (\partial_t + \overline{u}_1\partial_x)q_{|\mathbb{T}_L\times\{1\}}$:

$$
\begin{aligned}
&s^7\lambda^8 \iint\limits_{\mathbb{T}_T^1} \xi^7|\psi|^2 e^{-2s\phi} + s^5\lambda^6 \iint\limits_{\mathbb{T}_T^1} \xi^5|\partial_x\psi|^2 e^{-2s\phi} + s^3\lambda^4 \iint\limits_{\mathbb{T}_T^1} \xi^3(|\partial_{xx}\psi|^2 + |\partial_t\psi|^2)e^{-2s\phi} \\
&+ s\lambda^2 \iint\limits_{\mathbb{T}_T^1} \xi(|\partial_{tx}\psi|^2 + |\partial_{xxx}\psi|^2)e^{-2s\phi} + \frac{1}{s}\iint\limits_{\mathbb{T}_T^1} \frac{1}{\xi}(|\partial_{tt}\psi|^2 + |\partial_{txx}\psi|^2 + |\partial_{xxxx}\psi|^2)e^{-2s\phi} \\
&\leqslant C \iint\limits_{\mathbb{T}_T^1} |\partial_t q + \overline{u}_1\partial_x q|^2 e^{-2s\phi} + Cs^7\lambda^8 \iint\limits_{\omega_1^T} \xi^7|\psi|^2 e^{-2s\phi}. \qquad (4.4.2)
\end{aligned}
$$

From this estimate we deduce that $\partial_t\psi + \overline{u}_1\partial_x\psi \in H^1(0,T;L^2(\mathbb{T}_L)) \cap L^2(0,T;H^2(\mathbb{T}_L))$. We also know from Lemma 4.2.2 that $\sigma \in C^0([0,T];H^1(\mathbb{T}_L \times (0,1))$, so we can apply Corollary 4.3.6 to q, $\partial_t q$ and $\partial_x q$:

$$
\begin{aligned}
&\iint\limits_{Q_T^{ex}} e^{-2s\phi}\Big(\frac{1}{s\xi}|\nabla q|^2 + s\lambda^2\xi|q|^2\Big) \qquad (4.4.3) \\
&\leqslant C\left(\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|\sigma|^2 + \frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\frac{1}{\xi}|\partial_t\psi + \overline{u}_1\partial_x\psi|^2 + s\lambda^2\iint\limits_{\omega_T} e^{-2s\phi}\xi|q|^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&\iint\limits_{Q_T^{ex}} e^{-2s\phi}\Big(\frac{1}{s\xi}|\nabla\partial_t q|^2 + s\lambda^2\xi|\partial_t q|^2\Big) + \lambda\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}|\partial_t q|^2 \qquad (4.4.4) \\
&\leqslant C\left(\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|\partial_t\sigma|^2 + \frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\frac{1}{\xi}|\partial_{tt}\psi + \overline{u}_1\partial_{xt}\psi|^2 + s\lambda^2\iint\limits_{\omega_T} e^{-2s\phi}\xi|\partial_t q|^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&\iint\limits_{Q_T^{ex}} e^{-2s\phi}\Big(\frac{1}{s\xi}|\nabla\partial_x q|^2 + s\lambda^2\xi|\partial_x q|^2\Big) + \lambda\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}|\partial_x q|^2 \qquad (4.4.5) \\
&\leqslant C\left(\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}|\partial_x\sigma|^2 + \frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\frac{1}{\xi}|\partial_{tx}\psi + \overline{u}_1\partial_{xx}\psi|^2 + s\lambda^2\iint\limits_{\omega_T} e^{-2s\phi}\xi|\partial_x q|^2\right).
\end{aligned}
$$

Now, we apply the observability estimates of Theorem 4.3.7:

$$\iint\limits_{Q_T^{ex}} e^{-2s\phi}(|\sigma|^2+|\partial_x\sigma|^2+|\partial_t\sigma|^2)$$
$$\leqslant C\iint\limits_{\omega_T} e^{-2s\phi}(|\sigma|^2+|\partial_x\sigma|^2+|\partial_t\sigma|^2)+C\iint\limits_{Q_T^{ex}} e^{-2s\phi}(|q|^2+|\partial_t q|^2+|\partial_x q|^2). \quad (4.4.6)$$

Therefore, summing up (4.4.2)–(4.4.3)–(4.4.4)–(4.4.5)–(4.4.6), we obtain:

$$s^7\lambda^8\iint\limits_{\mathbb{T}_T^1}\xi^7|\psi|^2e^{-2s\phi}+s^5\lambda^6\iint\limits_{\mathbb{T}_T^1}\xi^5|\partial_x\psi|^2e^{-2s\phi}+s^3\lambda^4\iint\limits_{\mathbb{T}_T^1}\xi^3(|\partial_{xx}\psi|^2+|\partial_t\psi|^2)e^{-2s\phi}$$
$$+s\lambda^2\iint\limits_{\mathbb{T}_T^1}\xi(|\partial_{tx}\psi|^2+|\partial_{xxx}\psi|^2)e^{-2s\phi}+\frac{1}{s}\iint\limits_{\mathbb{T}_T^1}\frac{1}{\xi}(|\partial_{tt}\psi|^2+|\partial_{txx}\psi|^2+|\partial_{xxxx}\psi|^2)e^{-2s\phi}$$
$$+\iint\limits_{Q_T^{ex}} e^{-2s\phi}(\frac{1}{s\xi}(|\nabla q|^2+|\nabla\partial_t q|^2+|\nabla\partial_x q|^2)+s\lambda^2\xi(|q|^2+|\partial_t q|^2+|\partial_x q|^2))$$
$$+\lambda\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}(|\partial_t q|^2+|\partial_x q|^2)+\iint\limits_{Q_T^{ex}} e^{-2s\phi}(|\sigma|^2+|\partial_x\sigma|^2+|\partial_t\sigma|^2)$$
$$\leqslant C\iint\limits_{\mathbb{T}_T^1}|\partial_t q+\overline{u}_1\partial_x q|^2e^{-2s\phi}+Cs^7\lambda^8\iint\limits_{\omega_1^T}\xi^7|\psi|^2e^{-2s\phi}+\frac{1}{s^2\lambda^2}\iint\limits_{Q_T^{ex}} e^{-2s\phi}\frac{1}{\xi^2}(|\sigma|^2$$
$$+|\partial_x\sigma|^2+|\partial_t\sigma|^2)+\frac{1}{s\lambda}\iint\limits_{\mathbb{T}_T^1} e^{-2s\phi}\frac{1}{\xi}(|\partial_{tt}\psi|^2+|\partial_{xt}\psi|^2+|\partial_{xx}\psi|^2+|\partial_t\psi|^2+|\partial_x\psi|^2)$$
$$+s\lambda^2\iint\limits_{\omega_T} e^{-2s\phi}\xi(|q|^2+|\partial_x q|^2+|\partial_t q|^2)$$
$$+C\iint\limits_{\omega_T} e^{-2s\phi}(|\sigma|^2+|\partial_x\sigma|^2+|\partial_t\sigma|^2)+C\iint\limits_{Q_T^{ex}} e^{-2s\phi}(|q|^2+|\partial_t q|^2+|\partial_x q|^2).$$

Using that $\xi \geqslant \theta \geqslant 1$, taking s and λ large enough, we can absorb all the terms in the right hand side which are not localized in the observation set, so that we deduce

$$
\begin{aligned}
&s^7\lambda^8 \iint\limits_{\mathbb{T}_T^1} \xi^7|\psi|^2 e^{-2s\phi} + \frac{1}{s}\iint\limits_{\mathbb{T}_T^1} \frac{1}{\xi}(|\partial_{tt}\psi|^2 + |\partial_{xxxx}\psi|^2)e^{-2s\phi} \\
&+ \iint\limits_{Q_T^{ex}} e^{-2s\phi}\Big(\frac{1}{s\xi}(|\nabla\partial_t q|^2 + |\nabla\partial_x q|^2) + s\lambda^2\xi|q|^2\Big) + \iint\limits_{Q_T^{ex}} e^{-2s\phi}(|\sigma|^2 + |\partial_x\sigma|^2 + |\partial_t\sigma|^2) \\
&\leqslant Cs^7\lambda^8 \iint\limits_{\omega_1^T} \xi^7|\psi|^2 e^{-2s\phi} + Cs\lambda^2 \iint\limits_{\omega_T} e^{-2s\phi}\xi(|q|^2 + |\partial_x q|^2 + |\partial_t q|^2) \qquad (4.4.7) \\
&\quad + C\iint\limits_{\omega_T} e^{-2s\phi}(|\sigma|^2 + |\partial_x\sigma|^2 + |\partial_t\sigma|^2).
\end{aligned}
$$

We now fix s and λ such that (4.4.7) holds. We then use the fact that

$$
\begin{aligned}
&e^{-2s\phi} \leqslant C \quad \text{on}\,\omega_T, \qquad \xi^i e^{-2s\phi} \leqslant C \quad \text{on}\ \omega_T \quad \text{for}\ i \in \{1,7\}, \\
&\exists c > 0,\ \text{s.t.}\ e^{-2s\phi} \geqslant c \quad \text{on}\,(\mathbb{T}_L \times (0,1)) \times (2T_0, T-2T_1) \\
&\qquad \text{and}\ \ \xi^i e^{-2s\phi} \geqslant c \quad \text{on}\,(\mathbb{T}_L \times \{1\}) \times (2T_0, T-2T_1) \quad \text{for}\ i \in \{1,7\}.
\end{aligned}
$$

to deduce from (4.4.7) that

$$
\begin{aligned}
&\|\psi\|_{L^2(2T_0,T-2T_1;H^4(\mathbb{T}_L\times(0,1)))\cap H^2(2T_0,T-2T_1;L^2(\mathbb{T}_L\times(0,1)))} + \|q\|_{H^1(2T_0,T-2T_1;H^1(\mathbb{T}_L\times(0,1)))} \\
&\qquad + \|\sigma\|_{L^2(2T_0,T-2T_1;H^1(\mathbb{T}_L\times(0,1)))\cap H^1(2T_0,T-2T_1;L^2(\mathbb{T}_L\times(0,1)))} \\
&\qquad\qquad \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}. \qquad (4.4.8)
\end{aligned}
$$

The next step consists in proving that there exists a constant C such that

$$
\begin{aligned}
&\|(\psi(\cdot,2T_0), \partial_t\psi(\cdot,2T_0))\|_{H^3(\mathbb{T}_L\times(0,1)))\times H^1(\mathbb{T}_L\times(0,1))} + \|q(\cdot,2T_0)\|_{H^2(\mathbb{T}_L\times(0,1))} \\
&\qquad + \|\sigma(\cdot,2T_0)\|_{H^1(\mathbb{T}_L\times(0,1))} \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}. \qquad (4.4.9)
\end{aligned}
$$

From (4.4.8), we have an estimate on ψ in $L^2(2T_0, T-2T_1; H^4(\mathbb{T}_L\times(0,1)))\cap H^2(2T_0, T-2T_1; L^2(\mathbb{T}_L\times(0,1)))$ and thus by interpolation on $\psi \in C^0([2T_0, T-2T_1]; H^3(\mathbb{T}_L\times(0,1)))\cap C^1([2T_0, T-2T_1]; H^1(\mathbb{T}_L\times(0,1)))$, so that we obtain

$$
\|(\psi(\cdot,2T_0), \partial_t\psi(\cdot,2T_0))\|_{H^3(\mathbb{T}_L\times(0,1)))\times H^1(\mathbb{T}_L\times(0,1))} \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}. \qquad (4.4.10)
$$

Then, from (4.4.8), we also have an estimate on q in $C^0([2T_0, T-2T_1]; H^1(\mathbb{T}_L\times(0,1)))$, therefore

$$
\|q(\cdot, T-2T_1)\|_{H^1(\mathbb{T}_L\times(0,1))} \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}. \qquad (4.4.11)
$$

Now, q satisfies the heat equation $(4.2.3)_{(2,3,4)}$ so that, taking $T^* \in (2T_0, T-2T_1)$,

$$\begin{aligned}
&\|q\|_{L^2(2T_0,T^*;H^3(\mathbb{T}_L\times(0,1)))\cap H^{3/2}(2T_0,T^*;L^2(\mathbb{T}_L\times(0,1)))}\\
&\leqslant C\|q(\cdot,T-2T_1)\|_{H^1(\mathbb{T}_L\times(0,1))} + C\|\sigma\|_{H^1(\mathbb{T}_L\times(0,1)\times(2T_0,T-2T_1))}\\
&\qquad + C\|\partial_t\psi + \overline{u}_1\partial_x\psi\|_{L^2(2T_0,T-2T_1;H^2(\mathbb{T}_L\times(0,1)))\cap H^1(2T_0,T-2T_1;L^2(\mathbb{T}_L\times(0,1)))}\\
&\leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)},
\end{aligned}$$

where the last estimate is a consequence of (4.4.8) and (4.4.11). We thus immediately deduce that

$$\|q(\cdot,2T_0)\|_{H^2(\mathbb{T}_L\times(0,1))} \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}. \tag{4.4.12}$$

Finally, to get an estimate on σ at time $2T_0$, we first remark that (4.4.8) gives an estimate on σ in $C^0([2T_0,T-2T_1];L^2(\mathbb{T}_L\times(0,1)))$, so that we already have

$$\|\sigma(\cdot,2T_0)\|_{L^2(\mathbb{T}_L\times(0,1))} \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}. \tag{4.4.13}$$

Then, we take the gradient of $(4.2.3)_1$ and multiply by $\nabla\sigma$ for $t \in [2T_0, T-2T_1]$:

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{T}_L\times(0,1)}|\nabla\sigma(\cdot,t)|^2\right) - \frac{\overline{\rho}P'(\overline{\rho})}{\nu}\int_{\mathbb{T}_L\times(0,1)}|\nabla\sigma(\cdot,t)|^2 = -\frac{P'(\overline{\rho})}{\nu}\int_{\mathbb{T}_L\times(0,1)}\nabla q(\cdot,t)\cdot\nabla\sigma(\cdot,t).$$

Setting $\alpha = -\overline{\rho}P'(\overline{\rho})/\nu$, it follows that

$$\left|\frac{d}{dt}\left(e^{\alpha t}\|\nabla\sigma(\cdot,t)\|_{L^2(\mathbb{T}_L\times(0,1))}\right)\right| \leqslant Ce^{\alpha t}\|\nabla q(\cdot,t)\|_{L^2(\mathbb{T}_L\times(0,1))}.$$

We then easily deduce that

$$\|\nabla\sigma(\cdot,2T_0)\|_{L^2(\mathbb{T}_L\times(0,1))} \leqslant C\|\nabla\sigma\|_{L^1(2T_0,T-2T_1;L^2(\mathbb{T}_L\times(0,1)))} + C\|\nabla q\|_{L^1(2T_0,T-2T_1;L^2(\mathbb{T}_L\times(0,1)))}$$

In view of (4.4.8), we deduce that

$$\|\nabla\sigma(\cdot,2T_0)\|_{L^2(\mathbb{T}_L\times(0,1))} \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}. \tag{4.4.14}$$

The combination of (4.4.10)–(4.4.12)–(4.4.13)–(4.4.14) then concludes the proof of (4.4.9). Then, to deduce (4.4.1), we use Lemma 4.2.2 to solve (4.2.3) starting from the time $2T_0$ and the estimate (4.4.9) (note that $(\sigma(\cdot,2T_0), q(\cdot,2T_0), \psi(\cdot,2T_0), \partial_t\psi(\cdot,2T_0))$ satisfies the regularity assumption (4.2.4) and the compatibility conditions (4.2.5)).

4.4.2 Proof of Theorem 4.1.5

For $(\sigma_T, v_T, \psi_T, \psi_T^1)$ as in Theorem 4.1.5, we start by solving (4.2.3) with initial datum $(\sigma_T, q_T, \psi_T, \psi_T^1)$, where $q_T = \nu\operatorname{div} v_T + \overline{\rho}\sigma_T$. Using Lemma 4.4.1 and recalling that

$q = \nu \mathrm{div}\, v + \overline{\rho}\sigma$, we deduce that

$$\begin{aligned}\|\sigma(\cdot,0)\|_{H^1(\mathbb{T}_L\times(0,1))} + \|\mathrm{div}\, v(\cdot,0)\|_{H^1(\mathbb{T}_L\times(0,1))} + \|(\psi(\cdot,0),\partial_t\psi(\cdot,0))\|_{H^3(\mathbb{T}_L\times(0,1)))\times H^1(\mathbb{T}_L\times(0,1))}& \\ \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|q\|_{H^1(\omega_T)} + C\|\sigma\|_{H^1(\omega_T)}& \\ \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|v\|_{L^2(0,T;H^2(\omega))\cap H^1(0,T;H^1(\omega))} + C\|\sigma\|_{H^1(\omega_T)}.&\end{aligned} \tag{4.4.15}$$

Thus, to obtain the result of Theorem 4.1.5, it only remains to estimate $v(\cdot,0)$. As we already have an estimate on $\mathrm{div}\, v(\cdot,0)$, we first focus on getting an estimate on $\mathrm{curl}\, v$. One now uses the system (4.2.37) to obtain the following set of equations solved by $\mathrm{curl} v$:

$$\begin{cases} -\overline{\rho}(\partial_t(\mathrm{curl} v) + \overline{u}_1\partial_x(\mathrm{curl} v)) - \mu\Delta(\mathrm{curl} v) = 0 & \text{in} \quad (\mathbb{T}_L\times(0,1))\times(0,T), \\ \mathrm{curl} v = 0 & \text{on} \quad (\mathbb{T}_L\times\{0,1\})\times(0,T), \\ \mathrm{curl} v(\cdot,T) = \mathrm{curl} v_T & \text{in} \quad \mathbb{T}_L\times(0,1). \end{cases} \tag{4.4.16}$$

Thus, $\mathrm{curl}\, v$ satisfies a parabolic heat type equation with homogeneous Dirichlet boundary condition. Classical observability estimates for the heat equation (see e.g. [20] or [15]) immediately yields

$$\|\mathrm{curl}\, v(\cdot,0)\|_{H^1(\mathbb{T}_L\times(0,1))} \leqslant C\|\mathrm{curl}\, v\|_{L^2(\omega_T)} \leqslant C\|v\|_{L^2(0,T;H^1(\omega))}. \tag{4.4.17}$$

Now we recover $v(\cdot,0)$ by solving the following elliptic problem at times $t = 0$:

$$\begin{cases} \Delta v(\cdot,0) = \nabla \mathrm{div}\,(v(\cdot,0)) + \begin{pmatrix} \partial_z \\ -\partial_x \end{pmatrix}(\mathrm{curl}\, v(\cdot,0)) & \text{in} \quad \mathbb{T}_L\times(0,1), \\ v(\cdot,0)\cdot n = \psi(\cdot,0) & \text{on}\, \mathbb{T}_L\times\{1\}, \\ v(\cdot,0)\cdot n = 0 & \text{on}\, \mathbb{T}_L\times\{0\}, \\ \mathrm{curl}\, v(\cdot,0) = 0 & \text{on}\, \mathbb{T}_L\times\{0,1\}. \end{cases} \tag{4.4.18}$$

One can then do a lifting of the non-homogeneous boundary condition and use [27, Theorem 3.2] to deduce that

$$\begin{aligned}\|v(\cdot,0)\|_{H^2(\mathbb{T}_L\times(0,1))} \leqslant C\|\mathrm{curl}\, v(\cdot,0)\|_{H^1(\mathbb{T}_L\times(0,1))} \quad & +C\|\mathrm{div}\, v(\cdot,0)\|_{H^1(\mathbb{T}_L\times(0,1))} \\ & +C\|\psi(\cdot,0)\|_{H^3(\mathbb{T}_L\times\{1\})}.\end{aligned}$$

With the estimates (4.4.15)–(4.4.17) and this last estimate, we then deduce that

$$\|v(\cdot,0)\|_{H^1(\mathbb{T}_L\times(0,1))} \leqslant C\|\psi\|_{L^2(\omega_1^T)} + C\|v\|_{L^2(0,T;H^2(\omega))\cap H^1(0,T;H^1(\omega))} + C\|\sigma\|_{H^1(\omega_T)}. \tag{4.4.19}$$

Together with (4.4.15), this concludes the proof of Theorem 4.1.5.

Appendix

4.5 Derivation of the energy dissipation law (4.1.10)

First write Δu in the following form

$$\Delta u = \nabla \mathrm{div}(u) + \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} (\mathrm{curl} u).$$

Multiply $(4.1.2)_2$ by u and integrate over $\Omega_{\eta(t)}$. After integration by parts and using the periodicity of u in the $x-$direction and the boundary conditions (4.1.4)-(4.1.5) we get

$$\begin{aligned} \int\limits_{\Omega_{\eta(t)}} \rho \partial_t u \cdot u + \int\limits_{\Omega_{\eta(t)}} \rho (u \cdot \nabla) u \cdot u + \mu \int\limits_{\Omega_{\eta(t)}} |\mathrm{curl} u|^2 + (\mu' + 2\mu) \int\limits_{\Omega_{\eta(t)}} |\mathrm{div} u|^2 \\ + \int\limits_{\Omega_{\eta(t)}} \nabla P(\rho) \cdot u - \int\limits_{\Gamma_{s,t}} (((\mu' + 2\mu)(\mathrm{div} u) I_d) \cdot n_t) \cdot u = 0. \end{aligned} \tag{4.5.1}$$

We know that for any function g which is integrable on $\Omega_{\eta(t)}$

$$\frac{d}{dt} \left(\int\limits_{\Omega_{\eta(t)}} g \right) = \int\limits_{\Omega_{\eta(t)}} \partial_t g + \int\limits_{\partial \Omega_{\eta(t)}} g u \cdot n_t. \tag{4.5.2}$$

Now let us simplify some terms from (4.5.1).

$$\begin{aligned} \int\limits_{\Omega_{\eta(t)}} \rho (u \cdot \nabla) u \cdot u &= \int\limits_{\Omega_{\eta(t)}} \rho (\sum_{i=1}^{2} u_i \partial_{x_i} u) \cdot u \\ &= \frac{1}{2} \sum_{\{i,j\} \in \{1,2\}} \int\limits_{\Omega_{\eta(t)}} \rho u_i \partial_{x_i} |u_j|^2 \\ &= -\frac{1}{2} \sum_{\{i,j\} \in \{1,2\}} \Big(\int\limits_{\Omega_{\eta(t)}} \partial_{x_i} (\rho u_i) |u_j|^2 \Big) + \frac{1}{2} \int\limits_{\Gamma_{s,t}} (\rho |u|^2) u \cdot n_t. \end{aligned} \tag{4.5.3}$$

Using (4.5.2) from $(4.1.2)_1$ and (4.5.3) we get

$$\begin{aligned} \frac{d}{dt} \left(\int\limits_{\Omega_{\eta(t)}} \rho |u|^2 \right) &= \int\limits_{\Omega_{\eta(t)}} \partial_t (\rho |u|^2) + \int\limits_{\Gamma_{s,t}} (\rho |u|^2) u \cdot n_t \\ &= 2 \int\limits_{\Omega_{\eta(t)}} (\rho u \cdot \partial_t u + \rho ((u \cdot \nabla) u) \cdot u). \end{aligned} \tag{4.5.4}$$

As $P(\rho) = a\rho^\gamma - a\overline{\rho}^\gamma$ for $\gamma > 1$, we have:

$$\begin{aligned} u \cdot \nabla P(\rho) = u \cdot \nabla(a\rho^\gamma) &= a\gamma\rho^{\gamma-1}\nabla\rho \cdot u \\ &= \frac{a\gamma}{(\gamma-1)}\rho^{\gamma-1}\left((\partial_t\rho + \mathrm{div}(\rho u)) + (\gamma-1)u\cdot\nabla\rho\right) \\ &= \frac{a\gamma}{(\gamma-1)}\rho^{\gamma-1}\left(\partial_t\rho + \rho\mathrm{div}u + \gamma u\cdot\nabla\rho\right) \\ &= \partial_t\left(\frac{a}{(\gamma-1)}\rho^\gamma\right) + \mathrm{div}\left(u\frac{a\gamma}{(\gamma-1)}\rho^\gamma\right). \end{aligned} \tag{4.5.5}$$

Hence using (4.5.2) and (4.5.5) we obtain

$$\begin{aligned} \int\limits_{\Omega_{\eta(t)}} (\nabla(a\rho^\gamma))\cdot u &= \int\limits_{\Omega_{\eta(t)}} \left(\partial_t(\frac{a}{(\gamma-1)}\rho^\gamma) + \mathrm{div}(u\frac{a\gamma}{(\gamma-1)}\rho^\gamma)\right) \\ &= \int\limits_{\Omega_{\eta(t)}} \partial_t\left(\frac{a}{(\gamma-1)}\rho^\gamma\right) + \int\limits_{\Gamma_{s,t}} \frac{a\gamma}{(\gamma-1)}\rho^\gamma u\cdot n_t \\ &= \frac{d}{dt}\left(\int\limits_{\Omega_{\eta(t)}} \frac{a}{(\gamma-1)}\rho^\gamma\right) + \int\limits_{\Gamma_{s,t}} a\rho^\gamma u\cdot n_t. \end{aligned} \tag{4.5.6}$$

We combine the identities (4.5.1)-(4.5.6) and the definition (4.1.6)of $P(\rho)$ to deduce

$$\begin{aligned} &\frac{d}{dt}\left(\int\limits_{\Omega_{\eta(t)}} (\frac{1}{2}\rho \mid u \mid^2 + \frac{a}{(\gamma-1)}\rho^\gamma)\right) + \mu\int\limits_{\Omega_{\eta(t)}} |\mathrm{curl}u|^2 + (\mu^{'}+2\mu)\int\limits_{\Omega_{\eta(t)}} |\mathrm{div}u|^2 \\ &- \int\limits_{\Gamma_{s,t}} (((\mu'+2\mu)(\mathrm{div}u)I_d)\cdot n_t - Pn_t)\cdot u = -P_{ext}\int\limits_{\Gamma_s}\partial_t\beta. \end{aligned} \tag{4.5.7}$$

Multiplying $(4.1.2)_3$ by $\partial_t\beta$ and integrate over $(0, L)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\int\limits_0^L |\partial_t\beta|^2\right) + \int\limits_0^L |\partial_{tx}\beta|^2 + \frac{1}{2}\frac{d}{dt}\left(\int\limits_0^L |\partial_{xx}\beta|^2\right) = \int\limits_0^L (T_f)_2\beta_t. \tag{4.5.8}$$

Recalling the definition (4.1.7) of T_f, using the interface condition (4.1.3) and adding (4.5.7) with (4.5.8) the energy equality (4.1.10) follows.

4.6 Proof of Lemma 4.2.4

Proof. It is well known (see for instance [28]) that provided

$$(\beta_0, \beta_1) \in H^3(\mathbb{T}_L \times \{1\}) \times H^1(\mathbb{T}_L \times \{1\}), \tag{4.6.1}$$

and

$$\widetilde{\mathcal{G}} \in L^2(0,T;L^2(\mathbb{T}_L \times \{1\})), \tag{4.6.2}$$

the equations (4.2.74) admits a unique solution in the space

$$\beta \in L^2(0,T;H^4(\mathbb{T}_L \times \{1\})) \cap H^2(0,T;L^2(\mathbb{T}_L \times \{1\})). \tag{4.6.3}$$

Now we will improve on this regularity by assuming that

$$(\beta_0, \beta_1) \in H^5(\mathbb{T}_L \times \{1\}) \times H^3(\mathbb{T}_L \times \{1\}), \tag{4.6.4}$$

and

$$\widetilde{\mathcal{G}} \in L^2(0,T;H^2(\mathbb{T}_L \times \{1\})) \cap H^1(0,T;L^2(\mathbb{T}_L \times \{1\})). \tag{4.6.5}$$

We write the system (4.2.74) in the following form

$$\begin{cases} \partial_t \begin{pmatrix} \beta \\ \partial_t \beta \end{pmatrix} - \begin{pmatrix} 0 & I \\ -\partial_{xxxx} & \partial_{xx} \end{pmatrix} \begin{pmatrix} \beta \\ \partial_t \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\mathcal{G}} \end{pmatrix} & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \beta(0) = \beta_0 \quad \text{and} \quad \partial_t \beta(0) = \beta_1 & \text{in } \mathbb{T}_L \times \{1\}. \end{cases} \tag{4.6.6}$$

Let us operate $(4.6.6)_1$ by ∂_t and write the resulting equation in the following form

$$\begin{cases} \partial_t \begin{pmatrix} \partial_t \beta \\ \partial_{tt} \beta \end{pmatrix} - \begin{pmatrix} 0 & I \\ -\partial_{xxxx} & \partial_{xx} \end{pmatrix} \begin{pmatrix} \partial_t \beta \\ \partial_{tt} \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_t \widetilde{\mathcal{G}} \end{pmatrix} & \text{on } (\mathbb{T}_L \times \{1\}) \times (0,T), \\ \partial_t \beta(0) = \beta_1 \quad \partial_{tt} \beta(0) = \widetilde{\mathcal{G}}(0) + \partial_{xx} \beta_1 - \partial_{xxxx} \beta_0 & \text{in } \mathbb{T}_L \times \{1\}. \end{cases} \tag{4.6.7}$$

In view of (4.6.4) and (4.6.5) one has the following

$$\begin{cases} \begin{pmatrix} 0 \\ \partial_t \widetilde{\mathcal{G}} \end{pmatrix} \in L^2(0,T;L^2(\mathbb{T}_L \times \{1\})), \\ (\partial_t \beta(0), \partial_{tt} \beta(0)) \in H^3(\mathbb{T}_L \times \{1\}) \times H^1(\mathbb{T}_L \times \{1\}). \end{cases} \tag{4.6.8}$$

Hence in view of (4.6.8) one can solve (4.6.7) in the following settings

$$\partial_t \beta \in L^2(0,T;H^4(\mathbb{T}_L \times \{1\})) \cap H^2(0,T;L^2(\mathbb{T}_L \times \{1\})). \tag{4.6.9}$$

Now we write $(4.6.6)_1$ in the form

$$-\begin{pmatrix} 0 & I \\ -\partial_{xxxx} & \partial_{xx} \end{pmatrix} \begin{pmatrix} \beta \\ \partial_t \beta \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\mathcal{G}} \end{pmatrix} - \partial_t \begin{pmatrix} \beta \\ \partial_t \beta \end{pmatrix} \text{ on } (\mathbb{T}_L \times \{1\}) \times (0,T). \tag{4.6.10}$$

In view of (4.6.5) and (4.6.9) we infer that the right hand side of the equation (4.6.10) belongs to $L^2(0,T;H^4(\mathbb{T}_L \times \{1\}) \times H^2(\mathbb{T}_L \times \{1\}))$.
Since the one dimensional beam is defined on a torus $\mathbb{T}_L$, one can use the Fourier transform to prove that

$$\beta \in L^2(0,T;H^6(\mathbb{T}_L \times \{1\})) \cap H^3(0,T;L^2(\mathbb{T}_L \times \{1\})). \tag{4.6.11}$$

Hence in view of (4.6.9) and (4.6.11), we show that the assumptions (4.6.4) and (4.6.5) imply the existence of a unique solution β of the problem (4.2.74) in the following settings

$$\beta \in L^2(0,T;H^6(\mathbb{T}_L \times \{1\})) \cap H^3(0,T;L^2(\mathbb{T}_L \times \{1\})). \tag{4.6.12}$$

Now we will use interpolation results for the linear problem (4.2.74) stating the following map

$$\begin{array}{ll} H_{\mathcal{G},\ell} \times H_{\beta_0,\beta_1,\ell} & \longrightarrow H_{\beta,\ell} \\ (\widetilde{\mathcal{G}}, \beta(0), \partial_t\beta(0)) & \longmapsto \beta \end{array} \tag{4.6.13}$$

is an isomorphism for all $\ell \in [0,1]$, where

$$\begin{aligned} H_{\mathcal{G},\ell} &= [L^2(0,T;L^2(\mathbb{T}_L \times \{1\})), L^2(0,T;H^2(\mathbb{T}_L \times \{1\})) \cap H^1(0,T;L^2(\mathbb{T}_L \times \{1\}))]_\ell \\ &= L^2(0,T;H^{2\ell}(\mathbb{T}_L \times \{1\})) \cap H^\ell(0,T;L^2(\mathbb{T}_L \times \{1\})), \end{aligned}$$

$$\begin{aligned} H_{\beta_0,\beta_1,\ell} &= [H^3(\mathbb{T}_L \times \{1\}) \times H^1(\mathbb{T}_L \times \{1\}), H^5(\mathbb{T}_L \times \{1\}) \times H^3(\mathbb{T}_L \times \{1\})]_\ell \\ &= H^{2\ell+3}(\mathbb{T}_L \times \{1\}) \times H^{2\ell+1}(\mathbb{T}_L \times \{1\}), \end{aligned}$$

$$\begin{aligned} &H_{\beta,\ell} = [L^2(0,T;H^4(\mathbb{T}_L \times \{1\})) \cap H^2(0,T;L^2(\mathbb{T}_L \times \{1\})), L^2(0,T;H^6(\mathbb{T}_L \times \{1\})) \\ &\cap H^3(0,T;L^2(\mathbb{T}_L \times \{1\}))]_\ell = L^2(0,T;H^{2\ell+4}(\mathbb{T}_L \times \{1\}) \cap H^{\ell+2}(0,T;L^2(\mathbb{T}_L \times \{1\})). \end{aligned}$$

Hence using the isomorphism result (4.6.13) one proves the item (i) of Lemma 4.2.4 by taking $\ell = \frac{1}{4}$ and the item (ii) of Lemma 4.2.4 by taking $\ell = \frac{3}{4}$. □

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Analysis and control of some fluid models with variable density

Titre de la thèse: Analyse et contrôle de certains modèles de fluide à densité variable.

Résumés: Dans cette thèse, nous étudions des modèles mathématiques concernant certains problèmes d'écoulement de fluide à densité variable. Le premier chapitre résume l'ensemble de la thèse et se concentre sur les résultats obtenus, la nouveauté et la comparaison avec la littérature existante. Dans le deuxième chapitre, nous étudions la stabilisation locale des équations non homogènes de Navier-Stokes dans un canal 2d autour du flot de Poiseuille. Nous concevons un contrôle feedback de la vitesse qui agit sur l'entrée du domaine de sorte que la vitesse et la densité du fluide soient stabilisées autour du flot de Poiseuille, à condition que la densité initiale soit donnée par une constante additionnée d'une perturbation dont le support se situe loin du bord latéral du canal. Dans le troisième chapitre, nous étudions un système couplant les équations de Navier-Stokes compressibles à une structure élastique située à la frontière du domaine fluide. Nous prouvons l'existence locale de solutions solides pour ce système couplé. Dans le quatrième chapitre, notre objectif est d'étudier la nulle-contrôlabilité d'un problemè d'interaction fluide-structure linéarisé dans un canal bi dimensional. L'écoulement du fluide est ici modélisé par les équations de Navier-Stokes compressibles. En ce qui concerne la structure, nous considérons une poutre de type Euler-Bernoulli amortie située sur une partie du bord. Dans ce chapitre, nous établissons une inégalité d'observabilité pour le problème considéré d'interaction fluid-structure linéarisé qui constitue le premier pas vers la preuve de la nulle contrôlabilité du système.

Mots-clés: Equations de Navier-Stokes incompressibles, Equations de Navier-Stokes compressibles, interaction fluide-structure, stabilisation, existence locale en temps, nulle contrôlabilité, inégalité d'observabilité.

Abstract: In this thesis we study mathematical models concerning some fluid flow problems with variable density. The first chapter is a summary of the entire thesis and focuses on the results obtained, novelty and comparison with the existing literature. In the second chapter we study the local stabilization of the non-homogeneous Navier-Stokes equations in a 2d channel around Poiseuille flow. We design a feedback control of the velocity which acts on the inflow boundary of the domain such that both the fluid velocity and density are stabilized around Poiseuille flow provided the initial density is given by a constant added with a perturbation, such that the perturbation is supported away from the lateral boundary of the channel. In the third chapter we prove the local in time existence of strong solutions for a system coupling the compressible Navier-Stokes equations with an elastic structure located at the boundary of the fluid domain. In the fourth chapter our objective is to study the null controllability of a linearized compressible fluid structure interaction problem in a 2d channel where the structure is elastic and located at the fluid boundary. In this chapter we establish an observability inequality for the linearized fluid structure interaction problem under consideration which is the first step towards the direction of proving the null controllability of the system.

Keywords: Incompressible Navier-Stokes equations, Compressible Navier-Stokes equations, fluid-structure interaction, stabilization, local in time existence, null controllability, observability inequality.