Optimal investment in friction markets and equilibrium theory with unbounded attainable sets

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Résumé

Cette thèse traite des phénomènes liés aux mathématiques financières et économiques. Elle est composée de deux sujets de recherche indépendants. La première partie est consacrée à deux contributions au problème de Merton.

Pour commencer, nous étudions le problème de l'investissement optimal et de la consommation de Merton dans le cas de marchés discrets dans un horizon infini. Nous supposons qu'il y a des frictions sur les marchés en raison de la perte due aux échanges financières. Ces frictions sont modélisées par des fonctions de pénalités non linéaires où les modèles classiques de coût de transactions étudiés par Magill et Constantinides [31] et les marchés illiquides étudiés par Cetin, Jarrow et Protter dans [6] sont inclus dans cette formulation. Dans ce contexte, la région de solvabilité est définie en tenant compte de cette fonction de pénalité et chaque investisseur doit maximiser son utilité, dérivée de la consommation. Nous donnons la programmation dynamique du modèle et nous prouvons l'existence et l'unicité de la fonction valeur. Des stratégies optimales d'investissement et de consommation sont également construites. Ensuite, nous étendons le modèle de Merton à un problème à plusieurs investisseurs. Notre approche consiste à construire un modèle d'équilibre général déterministe dynamique. Nous prouvons ensuite l'existence d'un équilibre du problème qui est un ensemble de contrôles composés de processus de consommation et de portefeuille, ainsi que les processus de prix qui en découlent afin que la politique de consommation de chaque investisseur maximise son profil. Les résultats obtenus dans cette partie étendent principalement les résultats récemment obtenus par Chebbi et Soner [10] ainsi qu'aux d'autres résultats obtenus dans ce cadre dans la littérature.

Dans la deuxième partie, nous traitons le problème de l'existence d'un équilibre d'une économie de production avec des ensembles d'allocations réalisables non-bornés où les consommateurs peuvent avoir des préférences non-transitives non-complètes. Nous introduisons une propriété asymptotique sur les préférences pour les consommations réalisables afin de prouver l'existence d'un équilibre. Nous montrons que cette condition est vraie lorsque l'ensemble des allocations réalisables est compact ou aussi lorsque les préférences sont représentées par des fonctions d'utilité dans le cas où l'ensemble des niveaux d'utilité rationnels individuels réalisables est compact. Cette hypothèse généralise la condition de CPP de Allouch [1] et couvre l'exemple de Page et al. [40] lorsque les niveaux d'utilité disponibles définis ne sont pas compacts. Nous étendons donc les résultats existants dans la littérature avec des ensembles réalisables non bornés de deux façons en ajoutant la production et en prenant en compte des préférences générales.

Mots-clés: problème de Merton, marché discret, horizon infini, marché à friction, programmation dynamique, équilibre, économie de production, quasi-équilibre, préférences non transitives non complètes.

Abstract

This PhD dissertation studies two independent research topics dealing with phenomena issues from financial and economic mathematics.

This thesis is organized in two parts. The first part is devoted to two contributions to the Merton problem.

First, we investigate the problem of optimal investment and consumption of Merton in the case of discrete markets in an infinite horizon. We suppose that there is frictions in the markets due to loss in trading. These frictions are modeled through nonlinear penalty functions and the classical transaction cost studied by Magill and Constantinides in [31] and illiquidity models studied by Cetin, Jarrow and Protter in [6] are included in this formulation. In this context, the solvency region is defined taking into account this penalty function and every investigator have to maximize his utility, that is derived from consumption, in this region. We give the dynamic programming of the model and we prove the existence and uniqueness of the value function. Optimal investment and consumption strategies are constructed as well. We second extend the Merton model to a multi-investors problem. Our approach is to construct a dynamic deterministic general equilibrium model. We then provide the existence of equilibrium of the problem which is a set of controls that is composed of consumption and portfolio processes, as well as the resulting price processes so that each investor's consumption policy maximizes his lifetime expected. The results obtained in this part extends mainly the results recently obtained by Chebbi and Soner [10] and other corresponding results in the litterature.

The second part of this thesis deals with the problem of the existence of an equilibrium of a production economy with unbounded attainable allocations sets where the consumers may have non-complete non-transitive preferences. We introduce an asymptotic property on preferences for the attainable consumptions in order to prove the existence of an equilibrium. We show that this condition holds true if the set of attainable allocations is compact or, when preferences are representable by utility functions, if the set of attainable individually rational utility levels is compact. This assumption generalizes the CPP condition of Allouch [1] and covers the example of Page et al. [40] when the attainable utility levels set is not compact. So we extend the previous existence results with unbounded attainable sets in two ways by adding a production sector and considering general preferences.

Keywords: Merton problem, discrete market, infinite horizon, market friction, after-liquidation value, dynamic programming, equilibrium, production economy, unbounded attainable allocations, quasi-equilibrium, non complete non transitive preferences.

Contents

Chapter 1

Introduction

In this thesis, we first study the optimal investment and consumption problem of Merton in discrete time when there are frictions and second, we address the existence of equilibrium for a production economy with unbounded attainable allocations sets.
In chapter 2, we study the Merton problem in an infinite horizon and discrete time with frictions. We suppose that friction in the market is due to loss in trading. In this context, we prove the dynamic programming of the model and by using a fixed point approach, we deduce the existence and uniqueness of the value function.
In chapter 3, we extend the Merton model to a multi-investors problem. We model the agent's optimization problem in a framework of a dynamic deterministic general equilibrium model. We use an intermediary model of truncated economy to prove the existence of equilibrium of the problem which represents a set of controls that is composed of consumption and portfolio processes, as well as the resulting price processes so that each investor's consumption policy maximizes his lifetime expected.
Chapter 4 deals with the problem of the existence of an equilibrium of a production economy with unbounded attainable allocations sets. We introduce a sufficient asymptotic condition on preferences that allows to prove the existence of equilibrium with unbounded consumption sets and preference orderings need not be transitive. Our approach is based on the use of a fixed point like argument and an asymptotic argument.

1.1 Merton Problem in discrete time with frictions

1.1.1 Overview on the Merton Problem

The portfolio optimization problem in the continuous time diffusion model was first introduced by Merton in the two landmark papers [32,33], where he examined the combined problem of optimal portfolio selection and consumption rules for an individual when the individual's income is generated by returns on assets and these returns are stochastic. By assuming a model with constant coefficients and solving the relevant Hamilton-Jacobi-Bellman equation, Merton [32] derived explicit solutions and optimal strategies of the value function for both finite- and infinite-horizon models with special types of utility functions for a multi-asset problem when the rate of returns are generated by the Wiener Brownian-motion.
Formally, an investor must choose how much to consume and must allocate his wealth between stocks and a risk-free asset so as to maximize expected utility. This investment strategy π, coupled with its c consumption, leads to a portfolio value $X_T^{c,\pi}$ at time T.
The investor seeks to solve the following problem

$$\sup_{(c,\pi)\in\mathbb{A}} \mathbb{E}[\int_0^T \exp^{-\beta s} U_1(c_s)ds + \exp^{-\beta T} U_2(X_T^{c,\pi})]$$

where U_1, U_2 are two classical utility functions, i.e. concave and increasing functions. U_1 and U_2 represent respectively the utility of the agent relating to his instantaneous consumption and to his final wealth. The growth property of these utility functions reflects the fact that happiness increases with consumption, as to the concavity, it represents the decrease property of the marginal interest to get (or to consume) a little more money. The parameter β should not be confused with the rate of interest without risk; in this model it represents a preference for the present. The set $\mathbb{A}$ is the set of admissible strategies.

This problem has been extended and extensively used in financial models. One direction of extension has been to include market frictions and to study their impact on the optimal decisions. This part of this thesis is a study in this direction in a case of discrete time formulation.

1.1.2 Market Friction

A friction market is one that has transaction costs (including taxes) and restrictions on trade (e.g. short sale constraints). Indeed, in financial markets expenses incurred in the purchase or sale of a security are generally defined as transaction costs and usually include commission, bid-ask spread and market impact. Commission is the amount of a broker fee to effect a transaction, which can be a set price or can depend on the size of the trade. The bid-ask spread is the difference between prices at which one can buy a share of stock and then immediately sell it. The impact on the market is the cost associated to the effect that a market participant has when he buys or sells an asset, i.e. changing the price of the stock. Transaction costs can be expressed in terms of trading vector $\triangle x$. Typically, for small trades, meaning transaction costs that come from the bid-ask spread and other brokerage fees, are simply modeled as a function that is proportional to the amount traded. For large trades, there is market price impact, which can be temporary when it affects a single transaction or permanent when it affects every future transaction. Transaction costs are important to investors as they are one of the key determinants of their net return. In addition, transaction costs may represent capital gains taxes and are therefore to be included in the portfolio rebalancing. Incorporating transaction costs into the portfolio optimization model allows us to determine the optimal portfolio in the most cost-effective way.
In our context, market friction will be modeled by a convex penalty function which englobes both transaction costs and illiquid market.

1.1.3 Optimal investment in friction market

As it is mentioned previously in 1971, Merton developed a mathematical model of the optimal investment and consumption problem in continuous time. Ever since then, there has been many attempts to generalize and develop the results of Merton in different ways. Dealing with the complete market in a direction to which a large portion of the works has been devoted. One of the directions considered was the problem with stochastic drift returns. In this sense, one could refer to Campbell and Viceira [5] and Wachter [50]. In the meantime, other topics have received much attention and popularity among which we can mention stochastic volatility and transaction cost problems. We refer to Chacko and Viceira [9] for some explicit results for the cases with stochastic volatility in incomplete markets.

In recent papers, the impact of transaction costs on the trading decisions of investors has been studied intensively. The proportional transaction costs is a consequence of the bid-ask spread and was first studied in the context of the Merton problem by Magill and Constantinides [31] and later by Constantinides [12]. They showed that when investors pay proportional transaction costs to their investments, there is an area of asset price variation within which no portfolio redesign will be considered (contrary to the traditional portfolio theory results that show an optimal portfolio corresponding to each new market state) and it is only when the change in the price of the securities brings out the optimal portfolio of this zone that the investor proceeds a redevelopment. This analysis justifies that interventions of the savers are sporadic, especially when the variations of stock prices are of small amplitude. It should be noted that they should involve large volumes and may occasionally strengthen market volatility. In [12], Constantinides derives the optimal investment policy of an infinitely lived agent who can trade a riskless and a risky asset. The return of the riskless asset is constant over time and that of the risky asset is Independently and identically distributed. The risky asset carries transaction costs which are proportional to the dollar value traded. Because the agent has CRRA preferences, the optimal policy in the absence of transaction costs is to maintain a constant fraction of wealth invested in the risky asset as in Merton [33]. In the presence of transaction costs, the agent prevents this fraction from exiting an interval. When the fraction is strictly inside the interval, the agent does not trade. The agent incurs a small utility loss from transaction costs, even though he trades infinitely often in their absence. Intuitively, the derivative of his utility at the optimal policy is zero, and hence derivation from that policy results in the second-order loss. This model was later put in a modern mathematical framework in continuous time by Davis and Norman [15] who show that the optimal no-trade interval policy exists, and propose a numerical method to compute it.

In discrete theory, the model is fully developed by Jouini and Kallal [27]. They showed that a bid-ask spread price process is arbitrage free if and only if there exists an equivalent probability measure that transforms some process between the bid and the ask price processes into a martingale. A similar concept of friction is due to liquidity. Indeed, recently Cetin, Jarrow and Protter [6] developed a mathematical model for an illiquid market in which the notion of the stochastic supply curve which gives the price of stock as a function of the trade size.

In Chebbi and Soner [10], the approach is similar to that of [7] and they provide several extensions by studying the problem in multi-dimensions in a generality that covers both transaction cost and illiquidity. Indeed, the model they consider proposes a penalty function for trading. The penalty function considered represents trading results in a loss which is a certain small percentage of the traded dollar amount. To simplify the discussion, let us assume in this introduction that there is only one risky asset. If at any time the investor decides to make a portfolio adjustment of α dollars in his stock account, then he loses $g(\alpha)$ dollars to market friction. This function is assumed to be a general non-negative convex with $g(0) = 0$. In the case of proportional transaction costs $g(\alpha) = \lambda|\alpha|$ and in one particular example of an illiquid market with no bid-ask spread $g(\alpha) = \lambda\alpha^2$, where $\lambda > 0$ is a (small) market parameter. The discrete time formulation has the advantage of studying several different types of market frictions together through a general penalty function g. In a continuous time only the structure of g near the origin is relevant. Hence, one has to distinguish the cases when g is differentiable at the origin and when not. In contrast to continuous time, a unifying approach is possible in discrete time.

Our first contribution in the first part of this thesis is to extend the model considered in [10] to the case of an infinite horizon. Using the penalty function, we give the dynamics of the cash and stock position. We then study the optimal investment and the consumption problem of Merton. This problem is formulated as an optimization problem in which every investor has to maximize his expected utility function under a constraint condition defined by a solvency region. The utility function is derived from consumptions and the solvency region is defined through a natural condition concerning the non negativeness of what we call the after liquidation value, when an investor is forced to liquidate all stock positions. Consequently, we prove the dynamic programming of the model and by using a fixed point approach, we deduce the existence and uniqueness of the value function.

1.1.4 Extensions of Merton problem to mutil-investors case

Our interest in the second contribution of the first part is to extend the Merton Problem studied in the previous section to a multi-agent problem in market with frictions based on the equilibrium general theory.

In the litterature, the articles Heaton and Lucas [26], Vayanos [48] , Vayanos and Vila [49] and Lo, Mamaysky and Wang [30] study the behavior of the equilibrium resulting from transmission costs. Heaton and Lucas [26] provide a stationary equilibrium under transactions cost in which investors trade all the time in small quantities. In [48, 49], the investor has a finite lifetime, transaction costs induce him to buy securities when he is young that he can resell in order to secure his life during his old age.

Inspired by the recent work of Le Van and Pham in [29], we derive an equilibrium of the Merton problem in a financial market with multi-agent in infinite horizon. We study, in terms of price, consumption and portfolio, the dynamics of a financial-market equilibrium that we can expect to observe when there are frictions. We assume a riskless, perfectly liquid bond with a constant rate of return, and many risky stocks that carry frictions. The prices that are accepted by agents when determining their optimal consumption and portfolio policies requiring the fact that for all commodity to be exactly owned, actually represent the prices at equilibrium. Equilibrium in this market is defined as a set of controls which is composed of consumption and portfolio processes, as well as the resulting price processes for financial assets, so that each agent's consumption policy maximizes his lifetime expected; that is this consumption process is financed by the optimal portfolio process, financial markets clear and the market for consumption good clears. Then, we determine a simple set of conditions that are sufficient for equilibrium and we construct the equilibrium in the case where there are transaction costs.

1.2 Equilibrium theory with unbounded allocation sets

1.2.1 Finite-dimensional economy

The economy is a human activity that involves production, distribution, exchange and consumption of goods and services. Its complexity contrasts sharply with the simplicity of a question that must be raised about its functioning. Since the seventies, with the exception of the seminal paper of Mas-Colell [20] and a first paper of Shafer-Sonnenschein [46], equilibrium for a finite dimensional standard economy has attracted the attention of several researchers in different directions.

In the private ownership economy, a finite number of commodities are exchanged, produced or consumed. a finite number of consumers who are endowed with initial holdings of different commodities and consume the goods available in the market in such a way that optimizes their preferences and satisfies their consumption plans and budget constraints. For given prices, producers choose their production in the production plan so as to maximize their profits.
The consumption of each consumer must be feasible. The production of each producer must be possible and the market must be at a state of equilibrium.
A classical private ownership economy is completely characterized by

$$\mathcal{E} = (\mathbb{R}^l, (X_i, P_i, \omega_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{i,j})_{i\in I, j\in J})$$

where L is a finite set of goods, so that $\mathbb{R}^L$ is the commodity space and the price space. I is a finite set of consumers, each consumer i has a consumption set $X_i \subset \mathbb{R}^L$ and an initial endowment $\omega_i \in \mathbb{R}^L$. The tastes of this consumer are described by a preference correspondence $P_i : \prod_{k\in I} X_k \to X_i$. $P_i(x)$ represents the set of strictly preferred consumption to $x_i \in X_i$ given the consumption $(x_k)_{k\neq i}$ of the other consumers. J is a finite set of producers and $Y_j \subset \mathbb{R}^L$ is the set of possible productions of firm $j \in J$. For each i and j, θ_{ij} is the portfolio of shares of the consumer i on the profit of the producer j. The θ_{ij} are nonnegative and for every $j \in J$, $\sum_{i\in I} \theta_{ij} = 1$. These shares together with their initial endowment determine the wealth of each consumer.

We recall that an allocation $(x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ is called attainable if and only if

$$\sum_{i\in I} x_i = \sum_{j\in J} y_j + \sum_{i\in I} \omega_i.$$

An equilibrium of a private ownership economy is a t-uple $(\bar{x}, \bar{y}, \bar{p})$ consisting of an attainable allocation $(\bar{x}, \bar{y})$ and a nonzero price vector $\bar{p}$ such that:

(a) For each $i \in I$, $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i + \bar{p} \cdot (\sum_{j\in J} \theta_{ij}\bar{y}_j)$ and
$x_i \in P_i(\bar{x}, \bar{y}, \bar{p}) \Longrightarrow \bar{p} \cdot x_i > \bar{p} \cdot \bar{x}_i$,

(b) For every $j \in J$, for every $y_j \in Y_j$, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$,

(c) $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} \omega_i + \sum_{j\in J} \bar{y}_j$.

Condition (a) states that every consumer has chosen a consumption vector which satisfies his preferences in X_i under his budget constraint. Condition (b) states that each producer has maximized his profit in the production set. Condition (c) expresses that $(\bar{x}, \bar{y})$ is in a state of equilibrium.

One basic assumption of the consumer theory is that a consumer can always rank consistently different commodity bundles to determine which one he prefers. This is corresponds to what we call complete preferences. Second, transitivity of preferences implies their strong coherence of their choices.

In the non-transitive equilibrium, consumers are not supposed to have a complete preference preorder on their consumption set. This is the case where potential consumer choices have a strong coherence associated with transitivity, but where consumers can not compare all pairs of commodity bundles in their consumption sets. The non-transitive equilibrium also encompasses the case where consumers' strict binary preferences on consumption are the only data on consumer preferences and where these are non-transitive relations. Finally, considering preference correspondences instead of complete preference preorders is tantamount to introducing into a non-transitive equilibrium the case where consumers have preferences over consumption vectors established by the actions of other agents and relative prices.

Several works including ([3], [19], [18], [24], [44], [45]) have studied the existence of equilibrium in economies with interdependent preferences and price-dependent preferences that may be non-transitive and non-complete.

1.2.2 Non compact attainable allocation

The problem of existence of an Arrow-Debreu equilibrium in economies with consumption sets unbounded from below has appeared in several economic contexts. Among those we can cite Equilibrium Models of asset Markets (Hart [25], Page [38], Milne [35]), General Equilibrium more widely (Werner [51], Nelson [36], Page and Wooders [41], [39]) that established various conditions for existence of equilibria. Some of these conditions imply the compactness of "the individually rational attainable allocations set".

Earlier papers in the literature on existence of equilibrium in models with consumption sets unbounded from below focused on establishing sufficient and necessary conditions for existence of equilibrium. This is due to the fact that the consumption set need not be bounded from below in an asset market economy where unlimited short sales are allowed. Hart [25], first introduced the condition on preferences eventually known as the no arbitrage condition which allows to prove the existence of a Walrasian equilibrium with consumption sets unbounded from below. Since then, several works have dealt with this question which has allowed to have a wide literature with different arbitrage notions which generalizes and develops the condition of Hart on preferences.

One extension of the no-arbitrage condition (see Hammond, Page, Nielson, and Page and Wooders) was to consider a weaker hypothesis, "the compactness of the individually rational utility set" that represents the set of utility vectors in which each agent receives no less than the utility of its initial endowment and no more than the utility of his consumption in a feasible allocation. Furthermore, it has been shown that the arbitrage conditions are not only sufficient but also necessary for the existence of equilibrium in certain cases.

However, these conditions present some limitations. Firstly, they are not always necessary to prove the existence of equilibrium. Indeed, Page et al. [40] provide an example of an economy in which an equilibrium exists but neither it satisfies any known no-arbitrage condition nor it has a compact utility set for individually rational allocations. Secondly, preferences may not satisfy transitivity.

In the same context, Allouch [1] provide a new type of no-arbitrage condition, called CPP condition, to prove the existence of a quasi-equilibrium in an exchange economy with short selling when the preference relations of the investors are represented by a partial preorder. This setting encompasses the case of preference relations derived from a utility function.

It is natural to consider an extension of the notions of arbitrage to the case with non-transitive preferences. A new condition was introduced by Won and Yannelis [52] where preferences need not be transitive or complete and the consumption set need not be bounded from below. Within the framework of a general economy with non-transitive and lower semi-continuous preferences, they provide a sufficient condition for proving the existence of equilibrium, through which they emphasize non-symmetric treatment of consumers where one of the consumers plays a particular role.

Our work is in this direction, we provide a sufficient condition (H3) to replace the standard compactness of the attainable allocation set, which is suitably written to deal with general preferences and we give a simple numerical example where the set of the attainable allocations is not bounded and preferences are not representable by utility functions but the preferences satisfy our asymptotic condition. More precisely, we assume that for each sequence of attainable consumptions, there exists an attainable consumption where the preferred sets are asymptotically close to the preferred sets of the elements of the sequence. We also restrict our attention to the attainable allocation, which are individually rational, in a sense adapted to the fact that preferences may not be transitive.

We prove that our condition is satisfied when the attainable set is compact and when preferences are represented by utility functions and the set of attainable individually rational utility levels is compact. So, our result extends the previous ones in the literature. Our asymptotic assumption is weaker than the CPP condition within the framework considered by Allouch where preferences are transitive and have open lower sections.

As for the contribution of Won and Yannelis, we provide an asymmetric assumption (EWH3) for exchange economies. Won and Yannelis condition and the (EWH3) are not comparable and both of them cover the example of Page et al [40].

Our assumption can be stated for an exchange economy or for a production economy since it deals only with feasible consumption vectors and not with the associated production vectors.

1.2.3 Existence of equilibrium with unbounded allocation sets

Since the seventies, with the exception of the seminal paper of Mas-Colell [20] and a first paper of Shafer-Sonnenschein [46], equilibrium for a finite dimensional standard economy is commonly proved using explicitly or implicitly equilibrium existence for the associated abstract economy first introduced by Debreu and then developed in many directions by several papers (see for example [3], [19], [18], [24], [44], [45]).

The methodology of applying this result to the economic model consists in considering the equilibrium functioning of an economy as the equilibrium of a generalized game in which agents are the consumers, the producers and an hypothetic additional agent, the Walrasian auctioneer that corrects the eventual disequilibrium by making the excess of demand over supply as expensive as possible.

Based on this approach and by using the CPP condition, Allouch [1] prove the existence of a quasi-equilibrium in an exchange economy when the preference relations of the investors are represented by a partial preorder and when the consumption sets are unbounded from below. Won and Yannelis [52] extend Allouch [1] to the case of non-transitive preferences and they show that each economy with the truncated consumption sets has an equilibrium under their sequential new assumption. Their argument is based on the existence of a bounded sequence of allocations which are asymptotically supported by the sequence of equilibrium prices for the truncated economies. The limit of those prices and bounded allocations gives the equilibrium of the economy.

In our work, we consider a production economy with an unbounded attainable set where the consumers may have non-complete non-transitive preferences. We use the asymptotic property on preferences for the attainable consumptions to get the equilibrium. We posit classical hypothesis such as closedness and convexity on consumption and production sets. The definition of the "augmented preferences" due to Gale and Mas-Collel (see [21], [22]) is slightly modified by using the convex hull of preferences correspondences since, in our setting, we use non-convex preferences. This definition allows to have the local insatiability of consumers at any point of their attainable consumption set. We show that some properties are transmitted from equivalent properties of the preferences correspondences like convexity, lower-semi-continuity and irreflexivity. We define an abstract economy and a quasi-equilibrium of this abstract economy that is equivalent to the quasi-equilibrium of the private ownership economy initially introduced. We truncate consumption and production sets with a closed ball with a radius large enough. Following an idea of Bergstrom [3], we modify the budget sets in such a way that it will coincides with the original ones when the price belongs to the unit sphere; in order to apply a fixed point like argument to the artificial truncated economy. By applying our asymptotic assumption to the sequence of allocations in growing associated compact economies, we prove that the attainable consumption having preferred sets close to the ones of the consumption vectors of the compact economies is a quasi-equilibrium of the original economy. Note that the originality of the proof comes from the fact that the attainable allocation is not necessarily the limit of the sequence of allocations considered.

Hence, our result on the existence of equilibrium extends the previous existence results with unbounded attainable sets in two ways by adding a production sector and considering general preferences.

Chapter 2

Merton Problem in an Infinite Horizon and discrete time with Frictions

ABSTRACT. We investigate the problem of optimal investment and consumption of Merton in the case of discrete markets in an infinite horizon. We suppose that there is frictions in the markets due to loss in trading. These frictions are modeled through nonlinear penalty functions and the classical transaction cost and liquidity models are included in this formulation. In this context, the solvency region is defined taking into account this penalty function and every investigator have to maximize his utility, that is derived from consumption, in this region. We give the dynamic programming of the model and we prove the existence and uniqueness of the value function.

Keywords: Merton problem, discrete market, infinite horizon, market frictions, after liquidation value, dynamic programming, value function.

2.1 Introduction

In a very known paper appeared in 1971, Merton developed and modeled the problem of optimal investment and consumption in continuous time. Since it appeared, this problem has been extensively investigated in the literature and extended in many directions, we refer to the book of Karatzas and Shreve [28] for some extensions in this way. Recently, Chebbi and Soner in [10] consider the model of Merton when there is frictions in the market due to loss in trading. This paper is a study in this direction and the markets considered are discrete in infinite horizon.

In the literature, we can find several types of market friction. The first one that receive the most attention is the proportional transaction costs, first introduced and studied in the context of Merton problem by Magill and Constantinides [31] and later by Constantinides [12]. Recently, another concept of friction has been introduced by Cetin, Jarrow and Protter [6] for an illiquid market and a related super-replication problem studied by Cetin, Soner and Touzi [8]. Our concept of friction in this paper will be formulated through a convex penalty function g in a discrete market considered in an infinite horizon. This formulation will included both the function of proportional costs considered in [31] and the one considered for an illiquid market with no bid and ask spread [6] and it was also considered by Dolinsky and Soner [16]. The discrete time formulation of Merton problem was firstly developed by Jouini and Kallal [27] and in our context, the advantage of this type of formulation is that we can give a uniform approach that covers both the two principal types of frictions, i.e. proportional costs and illiquid markets, while in continuous time one have to distinguish the case when g is differentiable at the origin or not.

In section 2, we extend the model of Merton with friction studied in [10] to the case of an infinite horizon. Using the penalty function, we give the dynamics of the cash and stock position.

In section 3, we study the optimal investment and the consumption problem of Merton. This problem is formulated as an optimization problem in which every investor has to maximize his expected utility function under a constraint condition defined by a solvency region. The utility function is derived from consumptions and the solvency region is defined through a natural condition concerning the non negativeness of what we call the after liquidation value, when an investor is forced to liquidate all stock positions. Then, we prove the dynamic programming of the model and by using a fixed point approach, we deduce the existence and uniqueness of the value function.

2.2 The Model

We consider a discrete market model in an infinite horizon. We suppose that the market is with a money market account and N risky assets and we assume that the money market account pays a return of fraction $r > 0$ of the invested amount. The risky assets, called the stocks, provide a random return of $R = (R_k)_{k\geq 1}$ with values in $[-1, \infty)^N$. The returns are supposed to be identically and independently distributed over time. We let μ be the common probability measure of $R_k's$, which is supposed to be finite on $\mathbb{R}^N$. We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = (\mathbb{R}^N)^\infty$ denotes the space of events $(\omega_k)_{k\geq 1}$ such that for all $k \in \mathbb{N}^*$, $\omega_k \in \mathbb{R}^N$. For $k \in \mathbb{N}^*$, we define the canonical mapping process $B_k(\omega) = \omega_k,\ k \geq 1,\ \omega \in \Omega$. We denote by $\mathcal{F}_k = \sigma(B_s; s \in \{1, 2, ..., k\})$ the σ-field generated by the canonical map, which represents the information that the investor has at any time k. We set $\mathcal{F}_\infty = \sigma(\bigcup_{k\in\mathbb{N}} \mathcal{F}_k)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the trivial σ-algebra.
Let $\mathbb{P}$ the product probability measure given by

$$\mathbb{P}(\{\omega \in \Omega, \omega_k \in A_k, k \geq 1\}) = \prod_{k\geq 1} \mu(A_k).$$

Now, we let the return vector at time k be given by $R_k(\omega) = B_k(\omega) = \omega_k$, $k \in \mathbb{N}^*$. Then $R_k's$ are $\mathcal{F}_k$-measurable, hence $R = (R_k)_{k\geq 1}$ is an $(\mathbb{R})^N$-valued, $\mathcal{F}$-adapted process. The connection between the stock process $S = (S_k)_{k\geq 1}$, where S_k^i is the ith stock at time k, and the return process R is simply given by

$$S_k^i = S_0^i \prod_{j\geq 1}[1 + R_j^i] \Longleftrightarrow R_k^i = \frac{S_k^i - S_{k-1}^i}{S_{k-1}^i}, \quad i = 1, \cdots, N.$$

where S_0^i is the initial stock value. Since $R_j^i \geq -1$, S is an $(\mathbb{R}^+)^N$-valued $\mathcal{F}$-adapted process.

The portfolio position of the investor is an $\mathcal{F}$-adapted, $\mathbb{R} \times (\mathbb{R}^+)^N$-valued process (x, y) and it has the following interpretation,

$x = (x_k)_{k\geq 1}$ = process of money invested in the money market account at any time k.
$y = (y_k^i)_{k\geq 1}$ = process of money invested in the i-th stock at any time k prior to the

portfolio adjustment.

For $k \geq 1$, let $z = (z_k)_{k\geq 1}$ be the process of the number of shares of stock held by the investor at time k prior to the portfolio adjustment. Hence, z_k is $\mathcal{F}_{k-1}$-measurable and z in an $\mathcal{F}$-predictable process with values in $\mathbb{R}^N$. Moreover,

$$y_k^i = z_k^i S_k^i, \quad i = 1, \cdots, N; \quad k \in \mathbb{N}^*.$$

In our model, we assume that the market is with friction since trading results in a loss is a certain small percentage of the traded dollar amount:

$$\alpha_k^i := S_k^i \Delta_k z^i = S_k^i \ (z_{k+1}^i - z_k^i), \quad i = 1, \ldots, N, \quad k \geq 1. \tag{2.1}$$

We thus suppose that there is a penalty function $g : \mathbb{R}^N \to [0, \infty)$, in the market which is assumed to be convex with $g(0) = 0$ and $g \geq 0$.

In this context, the dynamics for the cash position will be the following:

$$x_{k+1} = (x_k - \langle \alpha_k, 1 \rangle - g(\alpha_k) - c_k)(1 + r), \qquad k \geq 1, \tag{2.2}$$

where the non-negative, $\mathcal{F}$-adapted process c is the *consumption* of the investor, $\langle \cdot, \cdot \rangle$ denotes the usually inner product in $\mathbb{R}^N$.

Specific examples of a loss function in the literature are

$$g(\alpha) = \sum_{i=1}^{N} \lambda_i |\alpha^i|, \quad \text{or} \quad g(\alpha) = \sum_{i=1}^{N} \lambda_i (\alpha^i)^2,$$

where λ^i's are given non-negative (small) constants. The first of the above example corresponds to the classical example of the proportional costs [15, 17, 27, 31, 47]. The second, however, is a model of illiquidity [6,7,23]. origin. The main difference between the two examples is the differentiability at the origin. Indeed, a non-differentiability of g at the origin corresponds to a proportional transaction costs, or equivalently the existence of a bid-ask spread in the market.

The dynamics of the y process is the classical one defined for $k \geq 1$ by:

$$
\begin{aligned}
y^i_{k+1} &= y^i_k + \left[z^i_{k+1} S^i_{k+1} - z^i_k S^i_k\right] \\
&= y^i_k + S^i_k \left[z^i_{k+1} - z^i_k\right] + z^i_{k+1} \left[S^i_{k+1} - S^i_k\right] \\
&= y^i_k + \alpha^i_k + S^i_k z^i_{k+1} \left(\frac{S^i_{k+1} - S^i_k}{S^i_k} \right) \\
&= y^i_k + \alpha^i_k + \left[S^i_k (z^i_{k+1} - z^i_k) + z^i_k S^i_k\right] \; R^i_{k+1} \\
&= y^i_k + \alpha^i_k + \left(\alpha^i_k + y^i_k\right) \; R^i_{k+1} \\
&= \left(y^i_k + \alpha^i_k\right) \; (1 + R^i_{k+1}).
\end{aligned} \tag{2.3}
$$

Notice that the dynamics of the state variables (x, y) in (3.2)-(3.4) are given only through the process α and not z. Hence, in whatever follows, we use the $\mathcal{F}$-adapted process α instead of z.

We also note that the mark-to-market value

$$
\omega_k := x_k + \langle y_k, 1 \rangle = x_k + \sum_{i=1}^{N} y^i_k
$$

satisfies the equation

$$
\begin{aligned}
\omega_{k+1} &= \omega_k + r x_k + [\alpha_k + y_k] \cdot R_k - \alpha_k \cdot \vec{1} r - c_k (1 + r) - g(\alpha_k)(1 + r) \\
&= \omega_k \left[1 + r + \pi_k \cdot (R_{k+1} - r)\right] - c_k (1 + r) - g(\alpha_k)(1 + r),
\end{aligned}
$$

where $\pi^i_k := [\alpha^i_k + y^i_k]/w_k$ is the fraction of the mark-to-market value invested in the stock after the portfolio adjustment. Indeed, this is the classical wealth equation when there is no friction, i.e, when $g \equiv 0$.

2.3 Solvency Region

It is well known that the optimal investment and consumptions type problem of Merton require a lower bound on the wealth like variables, see [28]. Otherwise, one may easily obtain non intuitive trivial results as consumption with no bound would be admissible. In this context, an appropriate notion is to require the mark-to-market value of the portfolio to be non-negative. In our model of markets with frictions, an admissibility type condition can be defined by taking into account the penalty function.

For a portfolio position $(x, y) \in \mathbb{R} \times (\mathbb{R}^+)^N$, we define the *cash value* or the *after liquidation value* simply as the cash value of the position after the investor is forced to liquidate (i.e., sell or close) all stock positions. Due to the loss function postulated in (3.2) this value differs from the mark-to-market value defined in the previous subsection. Indeed, using the idea behind (3.2), with $z_0 = y/S_0$, $z_1 = 0$, we obtain $\alpha_0 = -y$ and define,

$$L(x, y) := x + \langle y, 1 \rangle - g(-y). \tag{2.4}$$

The solvency region is then defined by,

$$\mathbb{L} := \{(x, y) \in \mathbb{R} \times \mathbb{R}^N : L(x, y) > 0\}.$$

Using these we define the admissible controls as the ones which keep all the future portfolio values solvent with probability one.

Definition 2.1. *A* control process $\nu := \{(c_k, \alpha_k)\}_{k=0,1,...}$ *consists of a non-negative, $\mathcal{F}$-adapted consumption process c and an $\mathbb{R}^N$-valued, $\mathcal{F}$-adapted portfolio adjustment process α. We say that a control process ν is* admissible *with initial position $(x, y) \in \overline{\mathbb{L}}$, if the solution $(x_k, y_k)_{k\geq 1}$ corresponding to 3.2-3.4 with initial data $x_0 = x$, $y_0 = y$ and controls (c_k, α_k) satisfies*

$$L(x_k, y_k) = x_k + \langle y_k, 1 \rangle - g(-y_k) \geq 0, \quad \Longleftrightarrow \quad (x_k, y_k) \in \overline{\mathbb{L}}, \quad \forall k \geq 1,$$

$\mathbb{P}$-almost surely. We denote by $\mathbb{A}(x, y)$ the set of all admissible controls.

□

In the general context, we simply define

$$\mathbb{U}(x, y) := \left\{(c, \alpha) \in \mathbb{R}^+ \times \mathbb{R}^N \ : \ L(x_1((x, y), (c, \alpha)), y_1((x, y), (c, \alpha))) \geq 0, \ \mathbb{P} - a.s.\right\}, \tag{2.5}$$

where $(x_1((x,y),(c,\alpha)), y_1((x,y),(c,\alpha)))$ is the solution of the (3.2)-(3.4) with initial data (x,y) and control process with $(c_0,\alpha_0)=(c,\alpha)$. We may rewrite the admissibility criterion using the sets $\mathbb{U}(x,y)$ as well. For future reference, we record this simple connection,

$$(c,\alpha)\in\mathbb{A}(x,y) \qquad \Longleftrightarrow \qquad (c_k,\alpha_k)\in\mathbb{U}(x_k,y_k), \quad \forall\ k\geq 0, \tag{2.6}$$

where (x_k,y_k) is the solution of (3.2)-(3.4).

Lemma 2.3.1. *For any $(x,y)\in\overline{\mathbb{L}}$, the admissible class of controls $\mathbb{A}(x,y)$) (and also $\mathbb{U}(x,y)$) is nonempty and convex.*

Proof.

To prove that $\mathbb{A}(x,y)\neq\emptyset$, take as a control process: $c\equiv 0$, $\alpha_0=-y$ and $\alpha_k=0$ for all $k\geq 1$. Then, the solution of (3.2)-(3.4) at time $k\geq 1$ is given by $y_k=0$ and

$$x_k=(x+\langle y,1\rangle-g(-y))(1+r)^k.$$

Then,

$$L(x_k,y_k)=x_k=(x+\langle y,1\rangle-g(-y))(1+r)^k\geq 0,$$

since $(x,y)\in\overline{\mathbb{L}}$ is equivalent to $x+\langle y,1\rangle-g(-y))\geq 0$. So $\mathbb{U}(x,y)$ (resp. $\mathbb{A}(x,y)$) is nonempty.

Now we want to show that $\mathbb{A}(x,y)$ is convex. Take $(c^i,\alpha^i)\in\mathbb{A}(x^i,y^i)$, for $i=1,2$, i.e. $(c^i_k,\alpha^i_k)\in\mathbb{U}(x^i_k,y^i_k)$ for $i=1,2$ and $k\geq 1$. For $\lambda\in[0,1]$, we note by $\bar{c}_k=\lambda c^1_k+(1-\lambda)c^2_k$ and similarly $\bar{\alpha}_k$, $\bar{x}_k$, $\bar{y}_k$. We have:

$$\begin{aligned} L(\bar{x}_k,\bar{y}_k) &= \bar{x}_k+\bar{y}_k-g(\bar{\alpha}_k)\\ &= \lambda(x^1_k+y^1_k)+(1-\lambda)(x^2_k+y^2_k)-g(\bar{\alpha}_k)\\ &\geq \lambda g(\alpha^1_k)+(1-\lambda)g(\alpha^2_k)-g(\bar{\alpha}_k)\\ &\geq 0 \end{aligned}$$

since g is convex and $(x^i_k, y^i_k) \in \overline{\mathbb{L}}$ for $i = 1, 2$ and $k \geq 1$.

□

Now for $\delta > 0$ and $I \subset \{1, \ldots, N\}$ define the set

$$\Omega^{\delta,I} := \{R^i_1 \leq r - \delta, \text{ for } i \in I, \text{ and } R^j_1 \geq r + \delta, \text{ for } j \notin I\}.$$

We provide a natural sufficient condition for $\mathbb{U}$ to be bounded.

Lemma 2.3.2. *Suppose that for some $\delta > 0$:*

$$\mu\left(\Omega^{\delta,I}\right) > 0, \tag{2.7}$$

for every subset $I \subset \{1, \ldots, N\}$. Then $\mathbb{U}(x, y)$ is a bounded subset of $\mathbb{R}^+ \times \mathbb{R}^N$ for all $(x, y) \in \overline{\mathbb{L}}$. In fact, it is locally uniformly bounded in (x, y).

Proof. It is clear that if $(c, \alpha) \in \mathbb{U}(x, y)$, then c must be bounded by above. Now suppose that there are $(c^m, \alpha^m) \in \mathbb{U}(x, y)$ so that $|\alpha^m|$ tends to infinity. Considering a subsequence, we may assume that all components of α^m converge (including the limit points $\pm\infty$). First assume that $(\alpha^m)^i$ converges to plus infinity for some i. Set I to be set of indices for which the limit point is plus infinity. Then, one can argue that on the set $\Omega^{\delta,I}$,

$$L\left((x - \alpha \cdot \vec{1} - g(\alpha) - c)(1 + r), (y + \alpha)(1 + R_1)\right)$$

converges to minus infinity. Hence a contradiction to the fact that $(c^m, \alpha^m) \in \mathbb{U}(x, y)$ and thus the above expression is non-negative with probability one.

Now, if $(\alpha^m)^i$ converges to minus infinity for some i. We set I to be the complement of the set on which the limit point is minus infinity and argue similarly.

□

2.4 Investment-consumption problem

In this model, we consider the classical problem of optimal investment and consumption of Merton [28, 34]. In our context of an infinite horizon, we assume that the investor derives utility from consumption. For a given initial position (x, y) and an admissible process $\nu = (c, \alpha) \in \mathbb{A}(x, y)$, the utility is given by:

$$\mathbb{J}(x, y, c, \alpha) := \mathbb{E}\left[\sum_{k=0}^{\infty} \rho^k U(c_k)\right], \tag{2.8}$$

where $U : \mathbb{R}^+ \to \mathbb{R}$, is a classical *utility function*, i.e., a concave, non-decreasing function satisfying the *Inada condition* and the given constant $\rho \in (0, 1)$ is the *impatience parameter*. Then, the problem is to maximize the total expected utility function $\mathbb{J}$ over all admissible controls.

In what follows, the resulting optimal value is called the *value function* and is given by:

$$V(x, y) = \sup_{(c,\alpha)\in\mathbb{A}(x,y)} \mathbb{J}(x, y, c, \alpha).$$

To simplify the presentation, we make the following assumption. However, most of the result hold without this condition as well.

$$U \text{ is bounded.} \tag{2.9}$$

We set U_{max} be the upper bound of $|U|$.

Then, clearly,

$$|J(x, y, c, \alpha)| \leq \sum_{k=0}^{\infty} \rho^k U_{max} = \frac{U_{max}}{1-\rho}.$$

In view of this

$$|V(x, y)| \leq \frac{U_{max}}{1-\rho} \quad \forall (x, y) \in \overline{\mathbb{L}}, \tag{2.10}$$

recall that $\mathbb{L}$ is defined in (3.5).

Remark 2.4.1. *We recall that in the finite horizon case, the utility considered in [10] for a given initial position (x, y), an horizon t and an admissible process $\nu = (c, \alpha) \in \mathbb{A}(x, y)$ is the following:*

$$\mathbb{J}(x, y, c, \alpha) := \mathbb{E}\left[\sum_{k=0}^{t-1} \rho^k U(c_k) + \rho^t \hat{U}(L(x_t, y_t))\right],$$

where $\hat{U}$ is as U. It is important to notice that when t is large, the second member of this utility function formally goes to 0. This provides a connection between the two problems. □

2.5 Dynamic Programming

In this section, we prove that the value function is the unique solution of the dynamic programming equation.

Theorem 2.1. (Dynamic Programming) *Assume* (2.9) *and* (2.7)*. Then, the value function is the unique continuous, concave, bounded solution of the following equation:*

$$V(x, y) = \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}\left[U(c) + \rho\, V(x_1, y_1)\right], \quad \forall\ (x, y) \in \overline{\mathbb{L}}. \tag{2.11}$$

where

$$(x_1, y_1) = \Big((x - \alpha \cdot \vec{1} - g(\alpha) - c)(1 + r), (y + \alpha)(1 + R_1)\Big).$$

Proof. Let $\mathcal{C}$ be the set of all continuous, concave, bounded function on $\overline{\mathbb{L}}$ with the supremum norm. Define a nonlinear operator T on $\mathcal{C}$ by,

$$T(h)(x, y) := \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \mathbb{E}\left[U(c) + \rho\, h(x_1, y_1)\right], \quad \forall\ (x, y) \in \overline{\mathbb{L}}.$$

Since $\mathbb{U}$ is convex, U and h are concave, it is easy to show that $T(h)$ is a concave function on $\overline{\mathbb{L}}$. Moreover, for every $(x, y) \in \overline{\mathbb{L}}$,

$$|T(h)(x, y)| \leq U_{max} + \|h\|_\infty.$$

Hence, $T(h)$ is concave and bounded on $\overline{\mathbb{L}}$. Since h is continuous on $\overline{\mathbb{L}}$ and $\mathbb{U}(x,y)$ is locally uniformly bounded ly Lemma (2.3.2), it directly follows that Th is also continuous. Hence, $T(h) \in \mathcal{C}$.

Moreover, T is monotone, i.e.,

$$T(h) \leq T(g),$$

whenever $g \leq h$. Finally, for any $g, h \in \mathcal{C}$,

$$|g(x,y) - h(x,y)| \leq \sup_{(c,\alpha)\in\mathbb{U}(x,y)} \rho\mathbb{E}\left[|\ h(x_1,y_1) - g(x_1,y_1)|\right] \leq \rho\|g-h\|_\infty \quad \forall\ (x,y) \in \overline{\mathbb{L}}.$$

Hence, T is a monotone, contraction on $\mathcal{C}$. Therefore, it has a unique fixed point in $\mathcal{C}$.

Let $v \in \mathcal{C}$ be the unique fixed point of T. We claim that $v = V$. Indeed, let

$$(c^*, \alpha^*) : \overline{\mathbb{L}} \to \mathbb{R}^+ \times \mathbb{R}^d$$

be a measurable function such that

$$(c^*, \alpha^*)(x,y) \in \mathbb{U}(x,y), \tag{2.12}$$

and

$$v(x,y) = Tv(x,y) = U(c^*(x,y)) + \rho\mathbb{E}[v(x_1^*, y_1^*)], \quad \forall\ (x,y) \in \overline{\mathbb{L}}, \tag{2.13}$$

where

$$(x_1^*, y_1^*) = \Big((x - \alpha^*(x,y)\cdot\vec{1} - g(\alpha^*(x,y)) - c)(1+r), (y + \alpha^*(x,y))(1+R_1)\Big).$$

Such a measurable function exists because $\mathbb{U}$ is compact, U and v are concave and continuous.

Now, fix $(x_0, y_0) \in \overline{\mathbb{L}}$ and define (x_k^*, y_k^*) be the solution of (3.2)-(3.4) with feedback control (c^*, α^*). Set

$$(c_k^*, \alpha_k^*) := (c^*(x_k^*, y_k^*), \alpha(x_k^*, y_k^*)), \quad k = 0, 1, \ldots.$$

Then, in view of (2.12) the resulting strategy $\nu^* := \{(c_k^*, \alpha^*)\}_{k=0,1,\ldots} \in \mathbb{A}(x_0, y_0)$. Moreover, by (2.13),

$$v(x_k^*, y_k^*) = U(c_k^*) + \rho\mathbb{E}[v(x_{k+1}^*, y_{k+1}^*)|\mathcal{F}_k], \quad k = 0, 1, \ldots.$$

Therefore,

$$\begin{aligned} v(x_0, y_0) &= \mathbb{E}[U(c_0^*) + \rho v(x_1^*, y_1^*)] \\ &= \mathbb{E}[U(c_0^*) + \rho U(c_1^*) + \rho^2 v(x_1^*, y_1^*)] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} \rho^k U(c_k^*)\right] + \rho^{N+1}\mathbb{E}[v(x_{N+1}^*, y_{N+1}^*)]. \end{aligned}$$

Since v is bounded, the last term converges to zero as N tends to infinity. Hence,

$$v(x_0, y_0) = J(x_0, y_0, \nu^*).$$

Now, let $\nu \in \mathbb{A}(x_0, y_0)$ be arbitrary and let (x_k, y_k) be the solution of (3.2)-(3.4) with control ν and initial condition (x_0, y_0). Since $v = T(v)$, for any $k = 0, 1, \ldots$,

$$v(x_k, y_k) \geq U(c_k) + \rho\mathbb{E}\left[v(x_{k+1}, y_{k+1})|\mathcal{F}_k\right].$$

By iterating the above inequality as done in the previous argument, we conclude that

$$v(x, y) \geq J(x_0, y_0, \nu), \quad \forall\ \nu \in \mathbb{A}(x_0, y_0).$$

Therefore, $v = V$. In particular, $V \in \mathcal{C}$. □

Remark 2.5.1. *Assume that the utility function U takes non-negative values. Then we can show that the value function of (2.11) satisfies the transversality condition:*

$$\forall (c, \alpha) \in \mathbb{U}(x, y), \lim_{T \to +\infty} \rho^T V(x_T, y_T) = 0.$$

Indeed, let (c, α) *be in* $\mathbb{A}(x, y)$, *since* $\mathbb{U}$ *is a bounded subset of* $\mathbb{R}^+ \times \mathbb{R}^N$ *and the utility function* U *is non-negative, we have:*

$$\sup_{(c,\alpha)\in\mathbb{A}(x,y)} \mathbb{J}(x, y, c, \alpha) \leq \frac{U(M)}{1-\rho}.$$

Hence,

$$\forall T, \qquad \lim_{T\to\infty} \rho^T V(x_T, y_T) = 0.$$

Chapter 3

Multi-Agent equilibrium of Merton problem with frictions

ABSTRACT. This paper considers the problem of optimal investment and consumption of Merton in market frictions with many investors. We build an infinite-horizon dynamic deterministic general equilibrium model in which each investor's objective is to choose a commodity consumption process and to manage his portfolio so as to maximize the expected utility of his consumption over all controls, subject to having nonnegative after liquidation value. The main result of this paper extends the corresponding results obtained recently by Ounaies, Bonnisseau, Chebbi and Soner in [37] and by Chebbi and Soner in [10], our approach is very different and is based on the general equilbrium theory.

Keywords: Merton problem, infinite horizon, market frictions, dynamic programming, T-truncated economy, equilibrium.

3.1 Introduction

The problem of optimal investment was first introduced by Merton in the two landmark papers [32,34]. Since it appears, this problem has been widely studied and generalized in different contexts as for example by including viscosity theory in Shreve and Soner [47] and for illiquid markets by Cetin, Yarrow and Protter in [6].

In this paper, we focus our works on the Merton problem in the case of market frictions and our objective is to study their effects on the asset prices and their impact on the optimal decisions. In the literature, Magill and Constantinides [31], first studied the proportional transaction costs in the context of the Merton problem in a continuous time. In discrete time, the study of models with proportional transaction costs was developed by Jouini and Kallal in [27] who considered a financial market with one non risky asset taken as a numerary and normalized to 1, and one risky asset. They showed that the absence of arbitrage is equivalent to the existence of at least an equivalent probability measure that transforms some process between the bid and the ask price processes of traded securities into a martingale.

Recently, Chebbi and Soner in [10] studied the problem of Merton with frictions in discrete time and finite horizon for one investor. Their argument to prove the existence of an optimal strategy is based on solving a dynamic optimization problem and then constructing the solution. This paper was extended to the infinite horizon by Ounaies, Bonnisseau, Chebbi and Soner in [37] and the optimal strategy is obtained by an argument of fixed points. Our work is in this direction, we consider the problem of Merton in market frictions when there are many investors and our approach is very different. We develop a general equilibrium model with multiple agents, we assume a riskless, perfectly liquid bond with a constant rate of return and many risky stocks that carry frictions in trading. There is a single infinitely-divisible commodity, and each agent wishes to maximize his expected total utility from consumption of this commodity over time. The prices that are accepted by agents when determining their optimal consumption and portfolio policies requiring the fact that for all commodity to be exactly owned, actually represent the prices at equilibrium.

Our paper is structured as follows: In section 2, We describe the model of Merton problem and the dynamic programming of this model. As in [10] and [37], frictions in markets are modelled by a non-linear (convex) penalty functions and a constraint condition about liquidation value is defined.

In section 3, we study the agent's optimization problem and market clearing. The model of general equilibrium theory corresponding to Merton problem is constructed and the notion of equilibrium of this economy is then defined.

In Section 4, we prove the main result of the paper about the existence of an optimal strategy for the optimal investment and consumption problem of Merton. As an intermediary step of the proof, we define the corresponding truncated economy, we compactify this econmy and by an argument of limit, the equilibrium of this economy will give us the equilibrium of the economy initially defined.

3.2 The Model

Let $(\Omega, \mathcal{F}, P)$ be the probability space where $\Omega = (\mathbb{R}^N)^\infty$ denotes the space of events $(\omega_t)_{t\geq 1}$ such that for all $t \in \mathbb{N}^*$, $\omega_t \in \mathbb{R}^N$. For $t \in \mathbb{N}^*$, we define the canonical mapping process $B_t(\omega) = \omega_t,\ t \geq 1,\ \omega \in \Omega$. We denote by $\mathcal{F}_t = \sigma(B_s; s \in \{1, 2, ..., t\})$ the σ-field generated by the canonical map, which represents the information available to the investors at time t. We set $\mathcal{F}_\infty = \sigma(\bigcup_{t\in\mathbb{N}} \mathcal{F}_t)$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the trivial σ-algebra. The probability measure on this space is represented by $P : \mathcal{F} \to [0, 1]$ with the usual properties that $P(\emptyset) = 0, P(\Omega) = 1$ and for a set of disjoint events $A_i \in \mathcal{F}$ we have that $P(\cup_i A_i) = \sum_i P(A_i)$. All the random variables and stochastic processes in this and subsequent sections will be defined on this base.

In a discrete time setting, we consider a market with a money market account that pays a return of fraction $r > 0$ of the invested amount and N risky assets that provide a random return of $R = (R_t)_{t\geq 1}$ with values in $[-1, \infty)^N$. The returns are supposed to be identically and independently distributed over time. Let $(p^j)_{1\leq j\leq N}$ denote the strictly positive asset price process, which we shall suppose has the property

$$p_t^j = p_0^j \prod_{k\geq 1} [1 + R_k^j] \Longleftrightarrow R_t^j = \frac{p_t^j - p_{t-1}^j}{p_{t-1}^j}, \quad j = 1, \cdots, N. \tag{3.1}$$

where p_0^j is the initial stock value. We let the return vector at time t given by $R_t(\omega) = B_t(\omega) = \omega_t,\ t \in \mathbb{N}^*,\ j = 1, \cdots, N$. Then R_t's are $\mathcal{F}_t$-measurable. Hence $R = (R_t)_{t\geq 1}$ is an $(\mathbb{R})^N$-valued, $\mathcal{F}$-adapted process. The process p is an $(\mathbb{R}^+)^N$-valued $\mathcal{F}$-adapted process.

We shall assume that there is a finite number m of individuals labeled i, $(i = 1, 2, \cdots, m)$. Let us consider the behavior of one individual. He has to select a portfolio of assets, and there are $N+1$ different assets to choose from, labeled j, $(j = 0, 1, 2, \cdots, N)$. The yield on any asset is assumed to be a random variable whose distribution is known to the individual.
We shall use $y = (y_{i,t}^j)_{t\geq 1}$ to denote individual i's process of money invested in the j-th stock at any time t prior to the portfolio adjustment. We shall take the riskless asset $x = (x_{i,t})_{t\geq 1}$ to be the initial one which denote the process of money invested in the money market account at any time t. Shares are traded, after payment of real dividends, at a competitively determined price vector $p_t = (p_t^1, \cdots, p_t^N)$.
For $t \geq 1$, the process $z_{i,t}$ the number of shares held by the i-th invetor at time t with values in $\mathbb{R}^N$. Moreover,

$$y_{i,t}^j = z_{i,t}^j p_t^j, \qquad j = 1, \cdots, N, \quad i = 1, \cdots, m, \quad t \geq 1.$$

In our model, we suppose that the market is with frictions. For that, we thus assume that there is transaction costs involved in buying or selling these financial assets which are represented by a penalty function $g_i : \mathbb{R}^N \to \mathbb{R}_+^N$, in the market for an individual i.

In this context, the dynamics of the riskless asset will be the following:

$$x_{i,t+1} = \big(x_{i,t} - \alpha_{i,t} \cdot 1 - p_t g_i\big((z_{i,t+1} - z_{i,t})\big) \cdot 1 - c_{i,t}\big)\,(1 + r), \qquad t \geq 1, \tag{3.2}$$

where the non-negative, $\mathcal{F}$-adapted process c_i is the *consumption* of the i-th investor, and α_i is the portfolio adjustment process which is expressed as follows

$$\alpha_{i,t}^j := p_t^j \Delta_t z_i^j = p_t^j\ (z_{i,t+1}^j - z_{i,t}^j), \quad j = 1, \ldots, N, \quad t \geq 1. \tag{3.3}$$

Now, suppose that portfolio rebalancing occurs between two time points; between time t and time $t+1$, we change the number of shares held from $z_{i,t}$ to $z_{i,t+1}$ of the j-th risky asset, and the cash held changes from $y_{i,t}$ to $y_{i,t+1}$ such that

$$\begin{aligned}
y^j_{i,t+1} &= y^j_{i,t} + \left[z^j_{i,t+1}p^j_{t+1} - z^j_{i,t}p^i_t\right] \\
&= y^j_{i,t} + p^j_t\left[z^j_{t+1} - z^j_t\right] + z^j_{t+1}\left[p^j_{t+1} - p^j_t\right] \\
&= y^j_t + \alpha^j_{i,t} + p^j_t z^j_{i,t+1}\left(\frac{p^j_{t+1} - p^j_t}{p^j_t}\right) \\
&= y^j_{i,t} + \alpha^j_{i,t} + \left[p^j_t(z^j_{i,t+1} - z^j_{i,t}) + z^j_{i,t}p^j_t\right]\ R^j_{t+1} \\
&= y^j_{i,t} + \alpha^j_{i,t} + \left(\alpha^j_{i,t} + y^j_{i,t}\right)\ R^j_{t+1} \\
&= \left(y^j_{i,t} + \alpha^j_{i,t}\right)\ (1 + R^j_{t+1}).
\end{aligned} \tag{3.4}$$

One can point out that the process $z = (z^j_{i,t})_{t\geq 1}$ is $\mathcal{F}_{t-1}$-measurable, which in turn implies the processes x and y are previsibles.
We also note that the mark-to-market value

$$\omega_{i,t} := x_{i,t} + y_{i,t}\cdot \mathbf{1} = x_{i,t} + \sum_{j=1}^{N} y^j_{i,t}$$

Remark 3.2.1. *In what follow, we use the dynamics of the state variables (x, y) $\mathcal{F}$-adapted process with terms of the $\mathcal{F}$-adapted z_i instead of α_i since it is more adapted to our approach for equilibrium.*

3.3 The general equilibrium model of Merton problem

For a portfolio position $(x, y) \in \mathbb{R}\times(\mathbb{R}^+)^N$, we define the after-liquidation value as the cash value of the position after the investor is forced to liquidate (i.e., sell or close) all stock positions. Due to the loss function postulated in (3.2) this value differs from the mark-to-market value defined above and thus is written as follows

$$\begin{aligned}
L(x_{i,t}, y_{i,t}) &= x_{i,t} + y_{i,t}\cdot\mathbf{1} - p_t g_i\big((z_{i,t+1} - z_{i,t})\big)\cdot\mathbf{1} \\
&= x_{i,t} + p_t z_{i,t}\cdot\mathbf{1} - p_t g_i\big((z_{i,t+1} - z_{i,t})\big)\cdot\mathbf{1}
\end{aligned} \tag{3.5}$$

Then, the solvency condition is simply given by the requirement that $L(x_{i,t}, y_{i,t}) \geq 0$ for all $t \geq 1$, P-almost surely.

Now, each investor seeks to maximize his or her life time expected utility and solves the following problem

$$
\begin{aligned}
Q_i(x,y) \quad &: \quad \sup_{(c_{i,t}, z_{i,t})} E\left[\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t})\right] \\
\text{subject to (solvency constraint)} \quad &: \quad x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i\big((z_{i,t+1} - z_{i,t})\big) \cdot 1 \geq 0 \quad a.e.
\end{aligned}
$$

where for each investor i, u_i is the classical utility function and ρ_i^t is the impatience parameter.

Remark 3.3.1. *One may use a model in which the loss function $p_t g_i\big((z_{i,t+1} - z_{i,t})\big)$ is replaced by $G_i(p_t(z_{i,t+1} - z_{i,t}))$, with some function G_i.*

We define an infinite-horizon sequence of prices and quantities by:

$$(p, (c_i, z_i)_{i=1}^m)$$

where, for each $i = 1, \cdots, m$,

$$(p, c_i, z_i) = \big((p_t)_{t=0}^{+\infty}, (c_{i,t})_{t=0}^{+\infty}, (z_{i,t})_{t=0}^{+\infty}\big) \in (\mathbb{R}_+^{+\infty})^N \times \mathbb{R}_+^{+\infty} \times (\mathbb{R}_+^{+\infty})^N,$$

We Denote by $\mathcal{E}$ the economy which is characterized by a list

$$\mathcal{E} = (\mathbb{R}^N, (u_i, \rho_i, z_{i,-1})_{i=1}^m)$$

where $z_{i,-1}$ is the initial number of stocks held.

Equilibrium in this economy is defined as a set of consumption policies and portfolio policies along with the resulting price processes for the financial assets, such that the consumption policy of each agent maximizes her lifetime expected utility, that this consumption policy is financed by the optimal portfolio policy, financial markets clear so that and the market for consumption good clears. More precisely:

Definition 3.1. *The stochastic process* $\left(\bar{p}_t, (\bar{c}_{i,t}, \bar{z}_{i,t})_{i=1}^m\right)_{t=0}^\infty$ *is an equilibrium of the economy* $\mathcal{E}$ *if it satisfies the following conditions*

1. *Price positivity:* $\bar{p}_t > 0$ *for* $t \geq 0$.
2. *Market clearing: at each* $t \geq 0$,

$$\begin{array}{ll} \sum_{i=1}^m & \bar{c}_{i,t} + p_t g_i\big((z_{i,t+1} - z_{i,t}) \cdot 1 = \omega_t, \qquad a.e. \\ \sum_{j=1}^N & z_{i,t}^j = 1 \quad a.e., \quad \forall i \in \{1, \cdots, m\}, \\ \sum_{i=1}^m & z_{i,t}^0 = 0 \qquad a.e. \end{array}$$

3. *Optimal consumption plans: for each* i, $\left((\bar{c}_{i,t}, \bar{z}_{i,t})_{i=1}^m\right)_{t=0}^\infty$ *is a solution of the problem* $Q_i(x, y)$.

Here, 1_m is the m-dimensional column vector with all components equal to 1 and *a.e.* means almost everywhere.

3.4 Existence of equilibrium

To prove the main result of this paper on the existence of equilibrium, some standard assumptions are required.

Assumption(H1): u_i is in C^1, $u_i(0) = 0$, $u_i'(0) = \infty$ and u_i is strictly increasing, concave, continuously differentiable.

Assumption(H2): At initial period 0, $z_{i,-1} \geq 0$, and $z_{i,-1} \neq 0$ for $i = 1, \cdots, m$. Moreover, we assume that $\sum_{i=1}^m z_{i,-1} = 1_m$.

Assumption(H3): The penalty function $g_i : \mathbb{R}^N \to \mathbb{R}_+^N$, in the market is assumed to be convex with $g_i(0) = 0$ and $g_i \geq 0$ for $i = 1, \cdots, m$.

Assumption(H4): For each i, utility of agent i is finite

$$\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t}) < \infty.$$

We first define the T-truncated economy $\mathcal{E}^T$as $\mathcal{E}$ in which there are no activities from period $T+1$ to the infinity, i.e., $c_{i,t} = z_{i,t} = 0$ for every $i = 1, \cdots, m$ and $t \geq T+1$ and this economy will be compactified in the following way:

we define the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ but all the random variables are bounded. Consider a finite-horizon bounded economy which goes on for $T+1$ periods: $t = 0, \cdots, T$. We fix sufficiently large quantity bounds B_c, B_z and so on, with:

$$\begin{aligned} C_i &:= \left\{(c_{i,0}, \cdots, c_{i,T}) : 0 \leq c_{i,t} \leq B_c, \quad \forall t \in \{1, \cdots, T\}\right\} = [0, B_c]^{T+1}; \\ Z_i &:= \left\{(z_{i,1}^j, \cdots, z_{i,T}^j) : 0 \leq z_{i,t}^j \leq B_z, \quad \forall t \in \{1, \cdots, T\}\right\} = [0, B_z]^T. \end{aligned}$$

Now, we focus on the solvency constraint with $z_{i,T+1} = 0$. Consider the solvency set:

$$\mathbb{U}_i^T(x, y) := \left\{(c_i, z_i) \in C_i \times Z_i \ : \ x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 \geq 0, \ P - a.s\right\}.$$

and its interior

$$\mathbb{L}_i^T(x, y) := \left\{(c_i, z_i) \in C_i \times Z_i \ : \ x_{i,t} + p_t z_{i,t} \cdot 1 - p_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 > 0, \ P - a.s\right\}.$$

Once the T-truncated economy is well defined, we introduce the economy $\mathcal{E}_b^{T,\epsilon}$ which is defined as follows: For each $\epsilon > 0$ such that $m\epsilon < 1$, we define ϵ-economy $\mathcal{E}_b^{T,\epsilon}$ by adding ϵ units for each agent at date 0. Note that this trick is used to ensure that the solvency set is non-empty. More precisely, the feasible set of agent i is given by

$$\begin{aligned}
\mathbb{U}_i^{T,\epsilon}(x,y) \ := \ & \Big\{(c_i,z_i) \in \mathbb{R}_+^{T+1} \times (\mathbb{R}_+^{T+1})^N \ : \\
& (x_{i,0} + p_0(z_{i,0}+\epsilon)\cdot 1 - p_0 g_i(-(z_{i,0}+\epsilon))\cdot 1 - c_{i,0})(1+r) \geq 0, \\
& \text{ for each } 1 \leq t \leq T : x_{i,t} + p_t z_{i,t}\cdot 1 - p_t g_i(z_{i,t+1} - z_{i,t})\cdot 1 \geq 0, \ P-a.s.\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}_i^{T,\epsilon}(x,y) \ := \ & \Big\{(c_i,z_i) \in \mathbb{R}_+^{T+1} \times (\mathbb{R}_+^{T+1})^N \ : \\
& (x_{i,0} + p_0(z_{i,0}+\epsilon)\cdot 1 - p_0 g_i(-(z_{i,0}+\epsilon))\cdot 1 - c_{i,0})(1+r) > 0, \\
& \text{for each } 1 \leq t \leq T : x_{i,t} + p_t z_{i,t}\cdot 1 - p_t g_i(z_{i,t+1} - z_{i,t})\cdot 1 > 0, \ P-a.s.\Big\}
\end{aligned}$$

Lemma 3.4.1. *The set* $\mathbb{L}_i^{T,\epsilon}(x,y)$ *is non empty, for* $t = 0, \cdots, T$.

Proof. Indeed,

$$\begin{aligned}
& L \quad (x_{i,1}, y_{i,1}) \\
= \ & L\left((x_{i,0} + p_0(z_{i,0}+\epsilon)\cdot 1 - p_0 g_i(-(z_{i,0}+\epsilon))\cdot 1 - c_{i,0})(1+r), (y_{i,0}^{j\epsilon} + \alpha_{i,0}^{j\epsilon})(1+R_1^j)\right) \\
= \ & L\big((x_{i,0} + p_0(z_{i,0}+\epsilon)\cdot 1 - p_0 g_i(-(z_{i,0}+\epsilon))\cdot 1 - c_{i,0})(1+r), 0\big) \\
= \ & (x_{i,0} + p_0(z_{i,0}+\epsilon)\cdot 1 - p_0 g_i(-(z_{i,0}+\epsilon))\cdot 1 - c_{i,0})\,(1+r) \geq 0
\end{aligned}$$

Now, since $\epsilon, (z_{i,0}+\epsilon) > 0$, we can choose $c_{i,0} \in (0, B_c)$ and $z_{i,0} \in (0, B_z)$ such that

$$(x_{i,0} + p_0(z_{i,0}+\epsilon)\cdot 1 - p_0 g_i(-(z_{i,0}+\epsilon))\cdot 1 - c_{i,0})\,(1+r) > 0$$

□

Lemma 3.4.2. *The set* $\mathbb{U}_i^T(x,y)$ *has convex values.*

Proof. Now we want to show that $\mathbb{U}(x,y)$ is convex. Take $(c_{i,t}^k, \alpha_{i,t}^k) \in \mathbb{U}(x^k, y^k)$, for $k = 1, 2$ and $t \geq 1$. For $\lambda \in [0,1]$, we note by $\bar{c}_{i,t} = \lambda c_{i,t}^1 + (1-\lambda)c_{i,t}^2$ and similarly $\bar{x}_{i,t}$, $\bar{z}_{i,t}$. We have:

$$
\begin{aligned}
L(\bar{x}_{i,t}, \bar{y}_{i,t}) &= \bar{x}_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1 - \bar{p}_t g(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \\
&= \lambda(x^1_{i,t} + \bar{p}_t z^1_{i,t} \cdot 1) + (1-\lambda)(x^2_{i,t} + \bar{p}_t z^2_{i,t} \cdot 1) - \bar{p}_t g(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \\
&\geq \bar{p}_t[\lambda g(z^1_{i,t+1} - z^1_{i,t}) \cdot 1 + (1-\lambda) g(z^2_{i,t+1} - z^2_{i,t}) \cdot 1 - g(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1] \\
&\geq 0
\end{aligned}
$$

since g is convex and $(x^k_{i,t}, y^k_{i,t}) \in \bar{\mathbb{L}}$ for $k = 1, 2$ and $t \geq 1$.

□

For simplicity, we denote $U_i = C_i \times Z_i$.

Lemma 3.4.3. *$\mathbb{L}^{T,\epsilon}_i(x, y)$ is lower semi-continuous correspondence on U_i and $\mathbb{U}^{T,\epsilon}_i(x, y)$ is upper semi-continuous with compact convex values.*

Proof. Since $\mathbb{L}^{T,\epsilon}_i(x, y)$ is non-empty and has open graph, then it is lower semi-continuous correspondence. Since U_i is compact and the correspondence $\mathbb{U}^{T,\epsilon}_i(x, y)$ has a closed graph, then $\mathbb{U}^{T,\epsilon}_i(x, y)$ is upper semi-continuous with compact values.

□

Definition 3.2. *The stochastic process $\left(\bar{p}_t, (\bar{c}_{i,t}, \bar{z}_{i,t})_{i=1}^m\right)_{t=0}^T$ is an equilibrium of the economy $\mathcal{E}^T_b$ if it satisfies the following conditions:*

1. *Price positivity: $\bar{p}_t > 0$ for $t = 0, 1, \cdots, T$*

2. *Market clearing:*

$$
\begin{aligned}
\sum_{i=1}^m \bar{c}_{i,0} + p_0 g_i(-(\bar{z}_{i,0} + \epsilon)) &= \sum_{i=1}^m x_{i,0} + p_0(\bar{z}^j_{i,0} + \epsilon) \cdot \vec{1}, \qquad a.e. \\
\sum_{i=1}^m \bar{c}_{i,t} + p_t g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) &= \sum_{i=1}^m x_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1, \qquad a.e.
\end{aligned}
$$

3. *Optimal consumption plans: for each i, $\left(\bar{c}_{i,t}, \bar{z}_{i,t}\right)_{t=1}^{T}$ is a solution of the maximization problem of agent i with the feasible set $\mathbb{U}_i^{T,\epsilon}(x,y)$ such that*

$$Q_i^{T,\epsilon}(x,y) : \sup_{(c_{i,t},z_{i,t})} \mathbb{E}\left[\sum_{t=0}^{T} \rho^t u_i(c_{i,t})\right].$$

Now, for $i = 0, \cdots, m$, we consider an element $h = (h_i)$ defined on $X := B \times \prod_{i=1}^{m} U_i$ by

$$h_i = \begin{cases} p & \text{for } i = 0 \\ (c_i, z_i) & \text{for } i = 1, \cdots, m \end{cases}$$

where $B = \{p \in \mathbb{R}^N | \|p\| \leq 1\}$.

We now define some correspondences. First, let φ_0 (for additional agent 0) be a correspondence defined as follows:

$$\varphi_0 : \prod_{i=1}^{m} U_i \to 2^B$$

$$\begin{aligned} \varphi_0((h_i)_{i=0}^m) \quad &:= \quad \arg\max_{p \in B} \Big\{ (\sum_{i=1}^{m} c_{i,0} + p_0 g_i(-(z_{i,0}+\epsilon)) \cdot 1 - x_{i,0} - p_0(z_{i,0}+\epsilon) \cdot 1 \\ &+ \quad \sum_{t=1}^{T} \sum_{i=1}^{m} c_{i,t} + p_t g_i(z_{i,t+1} - z_{i,t}) \cdot 1 - x_{i,t} - p_t z_{i,t}^j \cdot 1 \Big\}. \end{aligned}$$

For each $i = 1, \cdots, m$, we define

$$\varphi_i : B \to 2^{U_i}$$

$$\varphi_i(p) := \arg\max_{(c_i,z_i) \in \mathbb{U}(x,y)} \mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t})\right].$$

Lemma 3.4.4. *The correspondence φ_i is upper semi-continuous and non-empty, convex, compact valued for each $i = 1, \cdots, m$.*

Proof. This is a direct consequence of the Maximum Theorem. □

According to the Kakutani Theorem, there exists $\big(\bar{p},(\bar{c}_{i,t},\bar{z}_{i,t})\big)$ such that

$$\begin{aligned}\bar{p} &\in \varphi_0\big((\bar{c}_i,\bar{z}_i)_{i=1}^m\big) &(3.6)\\ (\bar{c}_i,\bar{z}_i) &\in \varphi_i(\bar{p}). &(3.7)\end{aligned}$$

For simplicity, we denote by

$$\begin{aligned}\bar{E}_t &= \sum_{i=1}^m \bar{c}_{i,t} - x_{i,t}, \quad t \geq 0\\ \bar{F}_0 &= \sum_{i=1}^m g_i(-(\bar{z}^j_{i,0}+\epsilon)) - (\bar{z}^j_{i,0}+\epsilon)\cdot 1\\ \bar{F}_t &= \sum_{i=1}^m g_i(\bar{z}_{i,t+1}-\bar{z}_{i,t}) - \bar{z}^j_{i,t}\cdot 1, \quad t \geq 1\end{aligned}$$

Lemma 3.4.5. *Under Assumptions (H1), (H2) and (H3) there exists an equilibrium for the finite-horizon bounded ϵ-economy $\mathcal{E}_b^{T,\epsilon}$.*

Proof. We first start by proving that $\bar{E}_t + \bar{p}_t\bar{F}_t = 0$ and $\bar{p}_t > 0$ for $t = 0,\cdots,T$. Indeed, From (3.6), one can easily check that for every $p \in B$, we have:

$$\sum_{t=0}^T (p_t - \bar{p}_t)\bar{F}_t \leq 0. \tag{3.8}$$

We recall the solvency constraint,

$$x_{i,t} + \bar{p}_t z_{i,t}\cdot 1 - \bar{p}_t g_i(z_{i,t+1} - z_{i,t})\cdot 1 \geq 0$$

Moreover, in any market satisfying the dynamic portfolio {(3.2),(3.4)} and for any consumption process such that $(c,z) \in \mathbb{U}^{T,\epsilon}(x,y)$, the value of an agent's consumption cannot exceed the value of his wealth. This leads to the following inequality:

$$\begin{aligned} x_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1 - \bar{p}_t g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 &\geq \bar{c}_{i,t} \\ x_{i,t} - \bar{c}_{i,t} + \bar{p}_t \bar{z}_{i,t} \cdot 1 - \bar{p}_t g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 &\geq 0 \end{aligned} \tag{3.9}$$

By summing the inequality (3.9) over i, we get that, for each t:

$$\begin{aligned} \sum_{i=1}^{m} x_{i,t} - \bar{c}_{i,t} + \bar{p}_t \left[\sum_{i=1}^{m} \bar{z}_{i,t}^{j} \cdot 1 - g_i(\bar{z}_{i,t+1} - \bar{z}_{i,t}) \cdot 1 \right] &\geq 0 \\ \bar{E}_t + \bar{p}_t \bar{F}_t &\leq 0 \end{aligned} \tag{3.10}$$

If $\bar{p}_t = 0$, we obtain that $\bar{c}_{i,t} = B_c > \omega_{i,t}$. Therefore for all t, we get $\sum_{i=1}^{m} \bar{c}_{i,t} > \sum_{i=1}^{m} x_{i,t}$ which contradicts (3.10). Hence, we obtain as a result, $\bar{p}_t > 0$.

Now, since prices are strictly positive and the utility functions are strictly increasing, all budget constraints are binding. By summing budget constraints (over i) at date t we have.

$$\bar{E}_t + \bar{p}_t \bar{F}_t = 0.$$

Finally, the optimality of $(\bar{c}_i, \bar{z}_i)$ is from (3.7). □

Note that if wealth becomes zero before time T, it stays there, and no further consumption or investment takes place and this is due to the fact bankruptcy is an absorbing state for the wealth process when $(c,z) \in \mathbb{U}(x,y)$.

Lemma 3.4.6. *Under Assumptions (H1), (H2) and (H3) there exists an equilibrium for the finite-horizon bounded economy $\mathcal{E}_b^T$.*

Proof. We have proved that for each $\epsilon = \frac{1}{n} > 0$, where n is an integer and large enough, there exists an equilibrium called

$$equi(n) := \big(\bar{p}(n), (\bar{c}_{i,t}(n), \bar{z}_{i,t}(n))_{i=1}^{m}\big)_{t=0}^{T};$$

for the economy, $\mathcal{E}_b^{T,\epsilon_n}$. Now, since prices and allocations are bounded, there exists a subsequence $(n_1, n_2, \cdots,)$ such that $equi(n_s)$ converges. However, without loss of generality, we can assume that

$$\big(\bar{p}(n), (\bar{c}_i(n), \bar{z}_i(n))_{i=1}^{m}\big) \rightarrow \big(\bar{p}, (\bar{c}_i, \bar{z}_i)_{i=1}^{m}\big)$$

when n goes to infinity.

Furthermore, by taking limit of market clearing conditions of the economy $\mathcal{E}_b^{T,\epsilon_n}$, we obtain market clearing conditions of the bounded truncated economy $\mathcal{E}_b^{T}$. □

Remark 3.4.1. *we will go back to two points that will be useful later. First, one can remark from (3.1) that at equilibrium $\bar{p}_0 > 0$. Indeed, if $\bar{p}_0 = 0$, then $\bar{p}_t = 0$, for all $t = 1, \cdots, T$, according to the optimality of $(\bar{c}_i, \bar{z}_i)$. Second, if $\bar{p}_0 \neq 0$ and $z_{i,0}^j \neq 0$, then the feasible set $\mathbb{L}_i^T(\bar{x}, \bar{y}) \neq \emptyset$. We can use the same argument in Lemma 4.3.1 to prove that $\mathbb{L}_i^T(\bar{x}, \bar{y})$ is non empty.*

Lemma 3.4.7. *For each i, $(\bar{c}_i, \bar{z}_i)$ is optimal.*

Proof. Since $\sum_{i=1}^{m} z_{i,-1}^j = 1$, for all $j \in \{1, \cdots, N\}$, there exists an agent i such that $z_{i,-1} > 0$. According to Remark 3.4.1, we have $\mathbb{L}_i^T(x, y) \neq \emptyset$. We are going to prove the optimality of allocation $(\bar{c}_i, \bar{z}_i)$.
Let (c_i, z_i) be a feasible allocation of the maximization problem of agent i with the feasible set $\mathbb{U}_i^T(\bar{x}, \bar{y})$. We have to prove that $\mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^{T} \rho_i^t u_i(\bar{c}_{i,t})\right]$.

Since $\mathbb{L}_i^T(\bar{x}, \bar{y}) \neq \emptyset$, there exists $(h)_{h \geq 0}$ and $(c_i^h, z_i^h) \in \mathbb{L}_i^T(\bar{x}, \bar{y})$ such that (c_i^h, z_i^h) converges to (c_i, z_i). Then, for each i, we have

$$x_{i,t} + \bar{p}_t z_{i,t}^h \cdot \vec{1} - \bar{p}_t g_i(z_{i,t+1}^h - z_{i,t}^h) > 0.$$

Fixe h. Let n_0 (n_0 depends on h) be high enough such that for every $n \geq n_0$, $(c_i^h, z_i^h) \in \mathbb{U}_i^{T,\frac{1}{n}}(\bar{x}(n), \bar{y}(n))$. Therefore, we have $\mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(c_{i,t}^h)\right] \leq \mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(\bar{c}_{i,t}(n))\right]$.

Let n tend to infinity, we obtain $\mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(c_{i,t}^h)\right] \leq \mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(\bar{c}_{i,t})\right]$.
Let h tend to infinity, we obtain $\mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(\bar{c}_{i,t})\right]$.

We have just proved the optimality of $(\bar{c}_i, \bar{z}_i)$.
We now prove that $\bar{p}_t > 0$ for every t. Indeed, if $\bar{p}_t = 0$, the optimality of $(\bar{c}_i, \bar{z}_i)$ implies that $\bar{c}_{i,t} = B_c > x_{i,t}$, contradiction.

□

Once the existence of the equilibrium has been proved when ϵ tends to 0, one proves that this equilibrium holds for the truncated unbounded economy.

Lemma 3.4.8. *An equilibrium for $\mathcal{E}_b^T$ is an equilibrium for $\mathcal{E}^T$.*

Proof. Let $\left(\bar{p}_t, (\bar{c}_{i,t}, \bar{z}_{i,t})_{i=1}^m\right)_{t=0}^T$ be an equilibrium of $\mathcal{E}_b^T$. Note that $z_{i,T+1} = 0$ for every $i = 1, \cdots T$. We can see that conditions (i) and (ii) in Definition (3.2) are hold. We will show that condition (iii) are hold too.

For condition (iii), let $a_i := (\bar{c}_{i,t}, \bar{z}_{i,t})_{t=0}^T$ be a feasible plan of agent i.
Assume that $\sum_{t=0}^T \rho_i^t u_i(c_{i,t}) > \sum_{t=0}^T \rho_i^t u_i(\bar{c}_{i,t})$. For each $\gamma \in (0,1)$, we define $a_i(\gamma) := \gamma a_i + (1-\gamma)\bar{a}_i$. By definition of $\mathcal{E}_b^T$,we can choose γ sufficiently close to 0 such that $a_i(\gamma) \in C_i \times Z_i$. It is clear that $a_i(\gamma)$ is a feasible allocation. By the concavity of the utility function, we have

$$\begin{aligned}\sum_{t=0}^T \rho_i^t u_i(c_{i,t}(\gamma)) &\geq \gamma \sum_{t=0}^T \rho_i^t u_i(c_{i,t}) + (1-\gamma) \sum_{t=0}^\infty \rho_i^t u_i(\bar{c}_{i,t}) \\ &> \sum_{t=0}^T \rho_i^t u_i(\bar{c}_{i,t})\end{aligned}$$

The linearity of mathematical expectation allows us to deduce

$$\mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(c_{i,t}(\gamma))\right] > \mathbb{E}\left[\sum_{t=0}^T \rho_i^t u_i(\bar{c}_{i,t})\right]$$

Contradiction to the optimality of $\bar{a}_i$. So, we have shown that condition (iii) in definition is hold.

□

We consider the limit of sequences of equilibria in $\mathcal{E}^T$, when $T \to \infty$. We use convergence for the product topology.

We denote by $\left(\bar{p}^T, (\bar{c}_i^T, \bar{z}_i^T)_{i=1}^m\right)$ an equilibrium of the T-truncated economy $\mathcal{E}^T$. Since $\|\bar{p}_t\| \leq 1$, for every $t \leq T$, $\bar{c}_i^T \leq B_c$ and$\sum_{i=1}^m \bar{z}_i^T = 1$ .Thus, we can assume that

$$\left(\bar{p}^T, (\bar{c}_i^T, \bar{z}_i^T)_{i=1}^m\right) \longrightarrow \left(\bar{p}, (\bar{c}_i, \bar{z}_i)_{i=1}^m\right)$$

when T goes to infinity.
One can easily check that all markets clear.

Now we can give the main result of this paper:

Theorem 3.1. *Under Assumptions (H1), (H2),(H3) and (H4) there exists an equilibrium in the infinite horizon economy* $\mathcal{E}$.

Proof. We have shown that for each $T \geq 1$, there exists an equilibrium for the economy $\mathcal{E}^T$.

Now, we consider a feasible allocation (c_i, z_i) of the problem $Q_i(\bar{p}, \bar{z})$. We have to prove that $\mathbb{E}\left[\sum_{t=0}^\infty \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^\infty \rho_i^t u_i(\bar{c}_{i,t})\right]$.

We define $(c_i', z_i')_{t=0}^T$ as follows:

$$\begin{aligned} z'_{i,t} &= z_{i,t} \quad \text{if } t \leq T-1, \\ c'_{i,t} &= c_{i,t} \quad \text{if } t \leq T-1, \\ c_{i,t} &= z_{i,t} = 0 \quad \text{if } t > T \\ x_{i,T} + \bar{p}_T z'_{i,T} - \bar{p}_T g_i(-z'_{i,T}) &= x_{i,T} + \bar{p}_T z_{i,T} - \bar{p}_T g_i(-z_{i,T}) \end{aligned}$$

We see that $(c'_i, z'_i)_{t=0}^T \in \mathbb{U}_i^T(\bar{x}, \bar{y})$.
Since $\mathbb{L}_i^T(\bar{x}, \bar{y}) \neq \emptyset$, there exists a sequence $\big((c_i^n, z_i^n)_{t=0}^T\big)_{n=0}^\infty \in \mathbb{L}_i^T(\bar{x}, \bar{y})$ with $z_{i,T+1}^n = 0$ and this sequence converges to $(c'_i, z'_i)_{t=0}^T$ when n tends to infinity . We have

$$x_{i,t}^n + \bar{p}_t z_{i,t}^n - \bar{p}_t g_i(z_{i,t+1}^n - z_{i,t}^n) > 0.$$

We can choose s_0 high enough such that $s_0 > T$ and for every $s \geq s_0$, we have

$$x_{i,t}^n + \bar{p}_t^s z_{i,t}^n - \bar{p}_t^s g_i(z_{i,t+1}^n - z_{i,t}^n) > 0.$$

It means that $(c_i^n, z_i^n)_{t=0}^T \in \mathbb{U}_i^T(\bar{x}^s, \bar{y}^s)$. Therefore, we get $\sum_{t=0}^T \rho_i^t u_i(c_{i,t}^n) \leq \sum_{t=0}^s \rho_i^t u_i(\bar{c}_{i,t}^s)$.
Let s tend to infinity, we obtain $\sum_{t=0}^T \rho_i^t u_i(c_{i,t}^n) \leq \sum_{t=0}^\infty \rho_i^t u_i(\bar{c}_{i,t})$.
Let n tend to infinity, we have $\sum_{t=0}^T \rho_i^t u_i(c_{i,t}) \leq \sum_{t=0}^\infty \rho_i^t u_i(\bar{c}_{i,t})$ for every T. As a consequence, we have for every T

$$\sum_{t=0}^{T-1} \rho_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t}).$$

Let T tend to infinity, we obtain

$$\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t}) \leq \sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t}).$$

Then,

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \rho_i^t u_i(c_{i,t})\right] \leq \mathbb{E}\left[\sum_{t=0}^{\infty} \rho_i^t u_i(\bar{c}_{i,t})\right].$$

Therefore, we have proved the optimality of $(\bar{c}_i, \bar{z}_i)$.
Prices $\bar{p}_t$ are strictly positive since the utility function of agent i is strictly increasing.

□

Chapter 4

Equilibrium of a production economy with unbounded attainable allocations set

ABSTRACT. In this paper, we consider a production economy with an unbounded attainable set where the consumers may have non-complete non-transitive preferences. To get the existence of an equilibrium, we provide an asymptotic property on preferences for the attainable consumptions. We show that this condition holds true if the set of attainable allocations is compact or, when preferences are representable by utility functions, if the set of attainable individually rational utility levels is compact. This assumption generalizes the CPP condition of Allouch (2002) and covers the example of Page et al. (2000) when the attainable utility levels set is not compact. So we extend the previous existence results with unbounded attainable sets in two ways by adding a production sector and considering general preferences.

Keywords: production economy, unbounded attainable allocations, quasi-equilibrium, non complete non transitive preferences.

4.1 Introduction

Since the seventies, with the exception of the seminal paper of Mas-Colell [20] and a first paper of Shafer-Sonnenschein [46], equilibrium for a finite dimensional standard economy is commonly proved using explicitly or implicitly equilibrium existence for the associated abstract economy (see [3], [19], [18], [24], [44], [45]) in which agents are the consumers, the producers and an hypothetic additional agent, the Walrasian auctioneer. Moreover, in exchange economies, it is well-known that the existence of equilibrium with consumption sets that are unbounded from below requires some non-arbitrage conditions (see [25], [51], [11], [4], [13], [14], [2]). In [14], it is shown that these conditions imply the compactness of the individually rational utility level set, which is clearly weaker than assuming the compactness of the attainable allocation, and the authors prove an existence result of an equilibrium under this last condition.

The purpose of our paper is to extend this result to finite dimensional production economies with non-complete, non-transitive preferences, which may not be representable by a utility function. Furthermore, we also allow preferences to be other regarding in the sense that the preferred set of an agent depends on the consumption of the other consumers. We posit the standard assumptions about the closedness, the convexity and the continuity on the consumption side as well as on the production side of the economy like in Florenzano [19] and a survival assumption. We only consider quasi-equilibrium and we refer to the usual interiority of initial endowments or irreducibility condition to get an equilibrium from a quasi-equilibrium (see for example Florenzano [19] section 3.2).

The unboundedness of the attainable sets appears naturally in an economy with financial markets and short-selling. Using the Hart's trick [25], we can reduce the problem to a standard exchange economy when the financial markets are frictionless. But, if there are some transaction costs, intermediaries like clearing house mechanisms or other kind of frictions, this method is no more working and we then need to introduce a production sector to encompass these frictions. That is why we add in this paper a production sector, which is also justified if we want to analyze a stock market where the payments of an asset depend on the production plan of a firm.

Considering non-complete, non-transitive preferences allows us to deal with Bewley preferences where the agents have several criterions and a consumption is preferred to another one only if all criterions are improved. Such preferences are not representable by utility functions. They appear naturally in financial models where the objectives is to minimize the risk according to some consistent measures.

Our main contribution is to provide a sufficient condition (H3) to replace the standard compactness of the attainable allocation set, which is suitably written to deal with general preferences. More precisely, we assume that for each sequence of attainable consumptions, there exists an attainable consumption where the preferred consumptions can be approximated by preferred consumptions of the elements of the sequence. Actually, we also restrict our attention to the attainable allocation, which are individually rational, in a sense adapted to the fact that preferences may not be transitive. The formulation of our assumption is in the same spirit as the CPP condition of Allouch [1].

We prove that our condition is satisfied when the attainable set is compact and when preferences are represented by utility functions and the set of attainable individually rational utility levels is compact. So, our result extends the previous ones in the literature. Our asymptotic assumption is weaker than the CPP condition within the framework considered by Allouch where preferences are transitive and have open lower sections.

To compare our work with the contribution of Won and Yannelis [52], we provide an asymmetric assumption (EWH3) for exchange economies which is less demanding for one particular consumer. We are not please with this assumption since the fundamentals of the economy are symmetric and there is no reason to treat a consumer differently from the others. Won and Yannelis condition and the (EWH3) are not comparable and both of them cover the example of Page et al [40]. Nevertheless, neither of these conditions covers Example 3.1.2 of Won and Yannelis. So, there is room for further works to provide a symmetric assumption covering both examples.

We also remark that our condition deals only with feasible consumptions and not with the associated productions. So, our condition can be identically stated for an exchange economy or for a production economy. This means that even, if there exists unbounded feasible productions, an equilibrium still exists if the attainable consumption set remains compact. In other words, the key problem comes from the behavior of the preferences for large consumptions and not from the geometry of the productions sets at infinity.

To prove the existence of a quasi-equilibrium, we use several tricks borrowed from various authors. Using a truncated economy in order to apply a fixed point theorem to an artificial compact economy is an old trick as in the first equilibrium proofs. The definition of the "augmented preferences" due to Gale and Mas-Collel (see [21], [22]) is slightly modified by using the convex hull of preferences correspondences since, in our setting, we used non-convex preferences. This definition allows to have the local insatiability of consumers at any point of their attainable consumption set. Further, we restrict prices to be in the closed unit-ball of $\mathbb{R}^L$, the commodity space, and we used modified budget sets, which are reduced to the original ones when prices belong to the unit-sphere using Bergstrom's trick (see [3]). In this way, it is possible to avoid the problem of discontinuity at the origin of some correspondences. Finally, we are also considering a weakening of Assumption (H3) to prepare the discussion about Won-Yannelis work. We apply our assumption on the asymptotic behavior of preferences to a sequence of quasi-equilibrium allocations in growing associated truncated economies. We prove that the attainable consumption given by Assumption (H3) is a quasi-equilibrium consumption of the original economy. The originality of the proof is mainly contained in this last section.

The reminder of this paper is organized as follows. Section 2 describes the model, gives the definition of a quasi-equilibrium and provide a simple numerical example where the set of the attainable allocations is not bounded and preferences are not representable by utility functions but satisfy our asymptotic condition. This example is extended to a class of production economies illustrating the fact the key issue lies in the consumption sector. In Section 3, we first introduce the definition of modified "augmented preferences" and we prove some properties that are transmitted from equivalent properties of the preferences correspondences like convexity, lower semicontinuity and irreflexivity. Then, we define a new economy $\mathcal{E}'$ where the production sets are the closed convex hull of the initial production sets. We show that the hypothesis made on the original economy $\mathcal{E}$ still hold for $\mathcal{E}'$. Last, we prove that one easily deduces a quasi-equilibrium of $\mathcal{E}$ from a quasi-equilibrium of $\mathcal{E}'$. Section 4 concludes the present work by giving a proof of the existence of the quasi-equilibrium of the production economy $\mathcal{E}'$ in two steps, a fixed point like argument and an asymptotic argument. The last section is devoted to compare Assumption (H3) with other conditions in the literature on the existence of equilibrium with unbounded consumption sets, in particular with the CPP condition of Allouch and to discuss the relationships with the condition of Won and Yannelis.

4.2 The Model

In this paper, we consider the private ownership economy:

$$\mathcal{E} = (\mathbb{R}^L, (X_i, P_i, \omega_i)_{i\in I}, (Y_j)_{j\in J}, (\theta_{ij})_{(i,j)})$$

where L is a finite set of goods, so that $\mathbb{R}^L$ is the commodity space and the price space. I is a finite set of consumers, each consumer i has a consumption set $X_i \subset \mathbb{R}^L$ and an initial endowment $\omega_i \in \mathbb{R}^L$. The tastes of this consumer are described by a preference correspondence $P_i : \prod_{k\in I} X_k \to X_i$. $P_i(x)$ represents the set of strictly preferred consumption to $x_i \in X_i$ given the consumption $(x_k)_{k\neq i}$ of the other consumers. J is a finite set of producers and $Y_j \subset \mathbb{R}^L$ is the set of possible productions of firm $j \in J$. For each i and j, θ_{ij} is the portfolio of shares of the consumer i on the profit of the producer j. The θ_{ij} are nonnegative and for every $j \in J$, $\sum_{i\in I} \theta_{ij} = 1$. These shares together with their initial endowment determine the wealth of each consumer.

Definition 4.2.1. *An allocation* $(x, y) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ *is called attainable if:*

$$\sum_{i\in I} x_i = \sum_{j\in J} y_j + \sum_{i\in I} \omega_i.$$

We denote by $\mathcal{A}(\mathcal{E})$ *the set of attainable allocations.*

In this paper, we are only dealing with the existence of quasi-equilibrium. We refer to the large literature on irreducibility, which provides sufficient conditions for a quasi-equilibrium to be an equilibrium. The simplest one is the interiority of the initial endowments linked with the possibility of inaction for the producers.

Definition 4.2.2. *A quasi-equilibrium of the private ownership economy is a pair of an allocation* $((\bar{x}_i)_{i\in I}, (\bar{y}_j)_{j\in J}) \in \prod_{i\in I} X_i \times \prod_{j\in J} Y_j$ *and a non-zero price vector* $\bar{p} \neq 0$*, such that:*

(a) (Profit maximization): for every $j \in J$*, for every* $y_j \in Y_j$*,* $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$*,*

(b) (Quasi-demand): for each $i \in I$*,* $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i + \bar{p} \cdot (\sum_{j\in J} \theta_{ij}\bar{y}_j)$ *and*
$x_i \in P_i(\bar{x}) \Rightarrow \bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$

(c) (Attainability): $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} \omega_i + \sum_{j\in J} \bar{y}_j$*.*

Notice that, in view of Condition (c), Condition (b) can be rephrased as

$$\text{for every } i \in I, \bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i + \bar{p} \cdot (\sum_{j\in J} \theta_{ij}\bar{y}_j) \text{ and } [x_i \in P_i(\bar{x}) \Rightarrow \bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i]$$

Before stating the assumptions considered on $\mathcal{E}$, let us introduce some notations:

- $\omega = \sum_{i\in I} \omega_i$ is the total initial endowment;
- $Y = \sum_{j\in J} Y_j$ is the total production set;
- $\hat{X} = \{x \in \prod_{i\in I} X_i : \exists y \in Y : \sum_{i\in I} x_i = \omega + y\}$ is the set of all attainable consumption allocations;
- $\hat{Y} = \{y \in Y : \exists x \in \prod_{i\in I} X_i : \sum_{i\in I} x_i = \omega + y\}$ is the attainable total production set.

In this paper, we consider the following hypothesis:

Assumption (H1) For every $i \in I$

(a) X_i is a non-empty, closed, convex subset of $\mathbb{R}^L$;

(b) [irreflexivity]$\forall x \in \prod_{i\in I} X_i$, $x_i \notin \mathrm{co}P_i(x)$ (the convex hull of $P_i(x)$);

(c) [lower semicontinuous] $P_i : \prod_{k\in I} X_k \to X_i$ is lower semicontinuous;

(d) $\omega_i \in X_i - \sum_{j\in J} \theta_{i,j} Y_j$, i.e. there exists $(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j\in J} Y_j$ such that $\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j} \underline{y}_{i,j}$;

(e) For each $x \in \hat{X}$, one has $P_i(x) \neq \emptyset$.

Assumption (H2) Y is a non-empty, closed and convex subset of $\mathbb{R}^L$.

To overcome the fact that we do not assume local non-satiation but only non-satiation, we introduce the definition of "augmented preferences" as in Gale and Mas-Collel ([21], [22]). We can avoid the use of augmented preferences if Assumption (H1)(e) is replaced by x_i belongs to the closure of $P_i(x)$.

$$\hat{P}_i(x) = \{x'_i \in X_i | x'_i = \lambda x_i + (1-\lambda)x''_i, 0 \leq \lambda < 1, x''_i \in \text{co}P_i(x)\},$$

Assumption (H3) For all sequence $((x_i^\nu))$ of $\hat{X}$ such that for all i, $\underline{x}_i \in \overline{\hat{P}_i(x^\nu)^c}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i, for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to (ξ_i) such that for all $\nu \geq \nu_1$, for all $i \in I$, $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$.

Closedness and convexity are standard assumptions on consumptions and productions sets. They imply in particular that commodities are perfectly divisible. Assumption (H1)(c) is a weak continuity assumption on preferences. Assumption (H1)(b), i.e. the irreflexivity, is made on the sets $\text{co}P_i(x)$ to avoid to assume the convexity of the preference correspondences P_i. Assumption (H1)(d) implies that using his own shares in the productive system, consumer i can survive without participating in any exchange. This implies no trader will be allowed to starve no matter what the prices are. It also insures that the set $\mathcal{A}(\mathcal{E})$ is nonempty. Usually, in exchange economy, this assumption is merely written as $\omega_i \in X_i$, which corresponds to $\omega_i = \underline{x}_i$ and $\underline{y}_{i,j} = 0$ for all j. Assumption (H1)(e) assumes, for every i, the insatiability of the ith consumer at any point of his attainable consumption set.

Assumption (H3) is an attempt to weaken the compactness assumption on the global attainable set $\mathcal{A}(\mathcal{E})$. A large literature tackles this question by considering what is called a non-arbitrage condition (see for example [2], [4], [13], [14]). Our work is much in the spirit of Dana et al. [13, 14] considering a compact set of attainable utility levels as generalized by Allouch [1]. But, we remove the transitivity assumption on preferences like in Won and Yannelis [52]. We discuss into details the relationships with these contributions in the last section of the paper.

We assume that for each sequence of attainable consumptions, there exists an attainable consumption where the preferred consumptions can be approximated by preferred consumptions of the elements of the sequence. Indeed, the element $\bar{x}$ of $\hat{X}$ is not necessarily a cluster point of the sequence (x^ν) but any element strictly preferred to $\bar{x}$ by any agent is approachable by a sequence of elements strictly preferred to $(x^{\varphi(\nu)})$. This condition imposes some restriction on the asymptotic behaviour of the preferences for attainable allocations in the sense that some preferred elements remain at a finite distance of the origin even if the allocation is very far.

Note that the productions are not considered in Assumption (H3). So, only the total production set matters since it determines the attainable consumptions. The fact that some unbounded sequences of individual productions can be attainable does not prevent the existence of an equilibrium as long as the total production set is not modified.

Example 4.2.1. *We present an example of an exchange economy where Assumption (H3) is satisfied while the attainable set is not bounded and the preference correspondences are not representable by utility functions. Then we extend it to a production economy with a class of productions sets. Let us consider an exchange economy with two commodities A and B and two consumers.*

The consumption sets are given by

$$X_1 = X_2 = \{(a, b) \in \mathbb{R}^2 | a + b \geq 0\}$$

The attainable allocations set $\mathcal{A}(\mathcal{E})$ of the economy is then

$$\mathcal{A}(\mathcal{E}) = \{((a, b), (\omega_A - a, \omega_B - b)) | 0 \leq a + b \leq \omega_A + \omega_B\}$$

where (ω_A, ω_B) with $\omega_A + \omega_B > 0$ denotes the global endowment. The set $\mathcal{A}(\mathcal{E})$ is clearly unbounded.

We consider the following continuous function $\Pi : X_i \to \mathbb{R}^2$ defined by:

$$\Pi(a,b) = (\frac{1}{2} + \frac{a-b}{(|a-b|+1)(a^2+b^2+2)}, \frac{1}{2} + \frac{b-a}{(|a-b|+1)(a^2+b^2+2)}).$$

The preference correspondence is the same for the two consumers and it is defined by $P_i : X_1 \times X_2 \to X_i$

$$P_i((a_1,b_1),(a_2,b_2)) = \{(\alpha,\beta) \in X_i | \Pi(a_i,b_i)\cdot(\alpha,\beta) > \Pi(a_i,b_i)\cdot(a_i,b_i)\}$$

One easily checks that Assumption (H1) is satisfied by the preference relations since Π is continuous so P_i has an open graph and $\Pi(a,b) \gg (0,0)$ so that the local non-satiation holds true everywhere.

We remark that if (a_i^ν, b_i^ν) is a sequence of X_i such that $\|(a_i^\nu, b_i^\nu)\|$ converges to $+\infty$ and $a_i^\nu + b_i^\nu$ converges to a finite limit c, then $\Pi(a_i^\nu, b_i^\nu)$ converges to $(1/2, 1/2)$ and $\Pi(a_i^\nu, b_i^\nu)\cdot(a_i^\nu, b_i^\nu)$ converges to $\lim_\nu (1/2)(a_i^\nu + b_i^\nu) = \frac{c}{2}$.

Let $((a_1^\nu, b_1^\nu),(a_2^\nu, b_2^\nu))$ be a sequence of $\mathcal{A}(\mathcal{E})$. If it has a bounded subsequence, then this subsequence has a cluster point $((\bar{a}_1, \bar{b}_1),(\bar{a}_2, \bar{b}_2))$. Then the desired property of Assumption (H3) holds true thanks to the fact that the preference correspondences have an open graph. See the proof of Proposition 4.5.1 (i).

If the sequence is unbounded, we remark that the sequences $(a_1^\nu + b_1^\nu)$ and $(a_2^\nu + b_2^\nu)$ belongs to $[0, \omega_A + \omega_B]$ and for all ν, $a_1^\nu + b_1^\nu + a_2^\nu + b_2^\nu = \omega_A + \omega_B$. So, there exists a subsequence $((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}),(a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)}))$ such that the sequences $(a_1^{\varphi(\nu)} + b_1^{\varphi(\nu)})$ and $(a_2^{\varphi(\nu)} + b_2^{\varphi(\nu)})$ converges respectively to $c \in [0, \omega_A + \omega_B]$ and to $\omega_A + \omega_B - c$. Let us consider the attainable allocation $((\bar{a}_1 = c/2, \bar{b}_1 = c/2),(\bar{a}_2 = (\omega_A + \omega_B - c)/2, \bar{b}_2 = (\omega_A + \omega_B - c)/2))$. We remark that $\Pi(\bar{a}_1, \bar{b}_1) = \Pi(\bar{a}_2, \bar{b}_2) = (1/2, 1/2)$ and $\Pi(\bar{a}_1, \bar{b}_1)\cdot(\bar{a}_1, \bar{b}_1) = (1/2)(\bar{a}_1 + \bar{b}_1) = c/2$ and $\Pi(\bar{a}_2, \bar{b}_2)\cdot(\bar{a}_2, \bar{b}_2) = (1/2)(\bar{a}_2 + \bar{b}_2) = (\omega_A + \omega_B - c)/2$. Let $i = 1, 2$ and $(a_i, b_i) \in X_i$ such that $(a_i, b_i) \in P_i((\bar{a}_1, \bar{b}_1),(\bar{a}_2, \bar{b}_2))$. From the definition of P_i, one deduces that $(1/2)(a_i + b_i) > (1/2)(\bar{a}_i + \bar{b}_i) = (1/2)\lim_{\nu\to\infty}(a_i^{\varphi(\nu)} + b_i^{\varphi(\nu)}) = \lim_{\nu\to\infty}\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})\cdot(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$. Furthermore, since $\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$ converges to $(1/2, 1/2)$, $(1/2)(a_i + b_i) = \lim_{\nu\to\infty}\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})\cdot(a_i, b_i)$. Consequently, for ν large enough, $\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})\cdot(a_i, b_i) > \lim_{\nu\to\infty}\Pi(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})\cdot(a_i^{\varphi(\nu)}, b_i^{\varphi(\nu)})$, which means that

$(a_i, b_i) \in P_i((a_1^{\varphi(\nu)}, b_1^{\varphi(\nu)}), (a_2^{\varphi(\nu)}, b_2^{\varphi(\nu)}))$, *so the desired property in Assumption (H3) holds true.* □

We now consider a finite collection of production sets $(Y_j)_{j\in J}$ *of* $\mathbb{R}^2$ *such that* $Y = \sum_{j\in J} Y_j$ *is closed, convex, contains* 0 *and* $y_A + y_B \leq 0$ *for all* $(y_A, y_B) \in Y$. *Let us consider the production economy where the consumption sector is as above, the production sector is described by* $(Y_j)_{j\in J}$ *and the portfolio shares* (θ_{ij}) *are any ones satisfying the standard conditions. One easily checks that Assumption (H3) is satisfied by this production economy since the attainable consumption set is smaller or equal to the one of the exchange economy.*

The main result of this paper is the following existence theorem of a quasi-equilibrium for a production economy.

Theorem 4.1. *Under Assumptions* $(H1), (H2)$ *and* $(H3)$, *there exists a quasi-equilibrium of the economy* $\mathcal{E}$.

4.3 Preliminary results

First, we show that some properties of the preference correspondences P_i are still true for $\hat{P}_i$.

Proposition 4.3.1. *Assume that for all i,* X_i *is convex*

(i) If P_i *is lower semicontinuous on* $\prod_{i\in I} X_i$, *then the same is true for* $\hat{P}_i$.

(ii) $\hat{P}_i(x)$ *has convex values. Furthermore, if for all* $x_i \in X_i$, $x_i \notin coP_i(x)$, *then* $x_i \notin \hat{P}_i(x)$.

Proof.

(i) Let $x \in \prod_{i\in I} X_i$ and V an open subset of X_i such that

$$V \cap \hat{P}_i(x) \neq \emptyset.$$

Then, there exists $\xi_i \in \hat{P}_i(x) \cap V$, which means that $\xi_i = \lambda x_i + (1-\lambda)\zeta_i$ for some $\lambda \in [0,1[$, $\zeta_i \in \mathrm{co}P_i(x)$. Let $\epsilon > 0$ such that $B(\xi_i, \epsilon) \subset V$. Since the correspondence P_i is lower semicontinuous, then $\mathrm{co}P_i$ is lower semicontinuous (see [19], page 154). Consequently, there exists a neighborhood W of x in $\prod_{i\in I} X_i$ such that

$$x' \in W \Rightarrow \mathrm{co}P_i(x') \cap B(\zeta_i, \epsilon) \neq \emptyset.$$

Thus, for all $x' \in W$, there exists $\zeta'_i \in \mathrm{co}P_i(x') \cap B(\zeta_i, \epsilon)$. Let W' such that

$$W' = \{x' \in W | \|x'_i - x_i\| < \epsilon\}.$$

Let $x' \in W'$ and $\xi'_i = \lambda x'_i + (1-\lambda)\zeta'_i$, then $\xi'_i \in \hat{P}_i(x')$

$$\|\xi'_i - \xi_i\| \leq \lambda \|x'_i - x_i\| + (1-\lambda)\|\zeta'_i - \zeta_i\| < \epsilon.$$

Then, one gets $\xi'_i \in B(\xi_i, \epsilon) \subset V$. Hence, $\xi'_i \in \hat{P}_i(x') \cap V$, which proves the lower semi-continuity of $\hat{P}_i$.

(ii) Let $x \in \prod_{i\in I} X_i$ and $z_i, z'_i \in \hat{P}_i(x)$ such that $z_i = x_i + \lambda(\xi_i - x_i)$ and $z'_i = x_i + \beta(\xi'_i - x_i)$ for some $\lambda, \beta \in]0,1]$ and $\xi_i, \xi'_i \in \mathrm{co}P_i(x)$. For $\alpha \in]0,1[$, we have:

$$\begin{aligned}
\alpha z_i + (1-\alpha)z'_i &= x_i + \alpha\lambda\xi_i + (1-\alpha)\beta\xi'_i - [\alpha\lambda x_i + (1-\alpha)\beta x_i] \\
&= x_i + \alpha\lambda\xi_i + (1-\alpha)\beta\xi'_i - [\alpha\lambda + (1-\alpha)\beta]x_i \\
&= x_i + \gamma(\xi''_i - x_i).
\end{aligned}$$

where $\gamma = \alpha\lambda + (1-\alpha)\beta$ and $\xi''_i = \frac{\alpha\lambda}{\gamma}\xi_i + \frac{(1-\alpha)\beta}{\gamma}\xi'_i$. One easily checks that $\gamma \in]0,1]$ since $\lambda, \beta \in]0,1]$ and $\xi''_i \in \mathrm{co}P_i(x)$. Then, $\alpha z_i + (1-\alpha)z'_i \in \hat{P}_i(x)$ which means that $\hat{P}_i$ has convex values.

We prove by contraposition the irreflexivity. Let us suppose that $x_i \in \hat{P}_i(x)$ for some i, then $x_i = \lambda x_i + (1-\lambda)x'_i$ with $\lambda \in [0,1[$ and $x'_i \in \mathrm{co}P_i(x)$. Hence, we have $x_i = x'_i \in \mathrm{co}P_i(x)$ which contradicts Assumption (H1)(b). □

Now, we consider the following economy

$$\mathcal{E}' = (\mathbb{R}^L, (X_i, \hat{P}_i, \omega_i)_{i\in I}, (Y'_j)_{j\in J}, (\theta_{ij})_{(i,j)})$$

where the preference correspondences are replaced by the augmented preference correspondences and the production sets are replaced by their closed convex hull, that is for each j, $Y'_j = \overline{\text{co}}Y_j$.

Lemma 4.3.1. *Under assumption (H2), the economies $\mathcal{E}$ and $\mathcal{E}'$ have the same total production set so the same attainable consumption set $\hat{X}$.*

Proof. Let $Y' = \sum_{j\in J} Y'_j$.

It is clear that $Y \subset Y'$. Conversely, $Y' = \sum_{j\in J} \overline{\text{co}}Y_j \subset cl(\sum_{j\in J} \text{co}Y_j)$, see [43] (Corollary 6.6, page 48). Since the convex hull of a sum is the sum of the convex hulls, one gets

$$Y' = \sum_{j\in J} \bar{\text{co}}Y_j \subset \text{cl}(\sum_{j\in J} \text{co}Y_j) = \text{cl}(\text{co}(\sum_{j\in J} Y_j)) = \overline{\text{co}}Y.$$

Since Y is a non-empty closed, convex subset of $\mathbb{R}^L$, then $\overline{\text{co}}Y = Y$. Hence $Y = Y'$.
□

Proposition 4.3.2. *If $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ is a quasi-equilibrium of $\mathcal{E}'$, then there exists $\bar{y} \in \prod_{j\in J} Y_j$ such that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is a quasi-equilibrium of $\mathcal{E}$.*

Proof. Let $((\bar{x}_i), (\bar{\zeta}_j), \bar{p})$ be a quasi-equilibrium of $\mathcal{E}'$. So, $\sum_{j\in J} \bar{\zeta}_j \in \sum_{j\in J} Y'_j$. By Lemma 4.3.1, $\sum_{j\in J} Y'_j = Y$. Consequently, there exists $\bar{y} \in \prod_{j\in J} Y_j$ such that $\sum_{j\in J} \bar{\zeta}_j = \sum_{j\in J} \bar{y}_j$. Hence $\sum_{i\in I} \bar{x}_i = \omega + \sum_{j\in J} \bar{y}_j$. In other words, Condition (c) of Definition 4.2.2 is satisfied.

Moreover, one can remark that $\bar{y}_j \in Y'_j$ for every j. Consequently, $\bar{p} \cdot \bar{y}_j \leq \bar{p} \cdot \bar{\zeta}_j$. But since $\sum_{j\in J} \bar{\zeta}_j = \sum_{j\in J} \bar{y}_j$, one gets $\bar{p} \cdot \bar{y}_j = \bar{p} \cdot \bar{\zeta}_j$.

We now show that condition (a) is satisfied. Let $j \in J$ and $y_j \in Y_j$. Then, $y_j \in Y'_j$, so, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$. Hence, $\bar{p} \cdot y_j \leq \bar{p} \cdot \bar{y}_j$ and Condition (a) of Definition 4.2.2 is satisfied.

Last, we show that condition (b) is satisfied. Since $\bar{p} \cdot \bar{\zeta}_j = \bar{p} \cdot \bar{y}_j$ for all $j \in J$, we have, $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j$ for all i. Now, let $i \in I$ and $x_i \in X_i$ such that $x_i \in P_i(\bar{x})$. Since $P_i(\bar{x}) \subset \hat{P}_i(\bar{x})$, $\bar{p} \cdot x_i \geq \bar{p} \cdot \bar{x}_i$. □

4.4 Existence of quasi-equilibria

In this section we consider the economy $\mathcal{E}'$ as defined above. We have seen in the previous section that we can deduce the existence of a quasi-equilibrium of $\mathcal{E}$ from a quasi-equilibrium of $\mathcal{E}'$.

In what follow, we will consider Assumptions (H1'), (H2') whose correspond to (H1), (H2) but adapted to $\mathcal{E}'$ and the asymptotic assumption (WH3). In the previous section, we have shown that (H1') and (H2') are satisfied by $\mathcal{E}'$ if Assumptions (H1), (H2) are satisfied by $\mathcal{E}$ and (WH3) is weaker than (H3).

Assumption (H1') For every $i \in I$

(a) X_i is a non-empty closed, convex subset of $\mathbb{R}^L$;

(b) [irreflexivity] $\forall x \in \prod_{i\in I} X_i$, $x_i \notin \hat{P}_i(x)$;

(c) [lower semicontinuous] $\hat{P}_i : \prod_{k\in I} X_k \to X_i$ is lower semicontinuous and convex valued;

(d) $\omega_i \in X_i - \sum_{j\in J} \theta_{i,j} Y'_j$, i.e. there exists $(\underline{x}_i, (\underline{y}_{i,j})) \in X_i \times \prod_{j\in J} Y'_j$ such that $\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j} \underline{y}_{i,j}$;

(e) For each $x \in \hat{X}$, one has $\hat{P}_i(x) \neq \emptyset$ and for all $\xi_i \in \hat{P}_i(x)$, for all $t \in]0,1]$, $t\xi_i + (1-t)x_i \in \hat{P}_i(x)$.

Assumption (H2') For each $j \in J$, Y'_j is a closed, convex subset of $\mathbb{R}^L$.

To prepare the discussion on the relationships with the paper of Won and Yannelis, we consider the following weakening of Assumption (H3). If A is a subset of $\mathbb{R}^L$, cone$A$ is the cone spanned by A.

Assumption (WH3) There exists a consumer i_0 such that, for all sequence $((x_i^\nu))$ of $\hat{X}$ such that for all i, $\underline{x}_i \in \overline{\hat{P}_i(x^\nu)^c}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i, for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to (ξ_i) such that for all $\nu \geq \nu_1$,

$\xi_{i_0}^{\varphi(\nu)} \in \text{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}] + \bar{x}_{i_0}^{\varphi(\nu)}$
for all $i \neq i_0$,

$\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$.

Assumption (WH3) is clearly weaker than (H3) since

$$\hat{P}_{i_0}(x^{\varphi(\nu)}) \subset \text{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}] + \bar{x}_{i_0}^{\varphi(\nu)}$$

But this assumption exhibits the drawback of being asymmetric. That is why we did not emphasise it before since we think that further works should provide an even weaker but symmetric assumption. We provide more comments in the last section when we discuss the link with the work of Won and Yannelis.

We now state the existence result of a quasi-equilibrium for a finite private ownership economy satisfying Assumptions (H1'), (H2') and (WH3).

Theorem 4.2. *If Assumptions (H1'), (H2') and (WH3) are satisfied, then there exists a quasi-equilibrium of the economy $\mathcal{E}'$.*

The idea of the proof is as follows: we first truncate consumption and production sets with a closed ball with a radius large enough; following an idea of Bergstrom [3], we modify the budget sets in such a way that it will coincides with the original ones when the price belongs to the unit sphere; then, by applying the well known result of Gale and Mas-Colell - Bergstrom about the existence of maximal elements to a suitable family of lower semi-continuous correspondences, we obtain a sequence $((x^\nu), (y^\nu), p^\nu)$ such that $((x^\nu), (y^\nu))$ is an attainable allocation of the economy $\mathcal{A}(\mathcal{E}')$, p^ν belongs to the unit ball of $\mathbb{R}^L$, the domain of admissible prices, the producers maximize the profit over the truncated production sets and the consumptions are maximal elements of the preferences on the truncated consumption sets but with a relaxed budget constraint; from Assumption (WH3) and the compactness of the price set, we obtain a subsequence $(x^{\varphi(\nu)}, y^{\varphi(\nu)}, p^{\varphi(\nu)})$ and an element $(\bar{x}, \bar{y}, \bar{p})$ such that the preferences at this point are close to the preferences at $x^{\varphi(\nu)}$ for ν large enough and $p^{\varphi(\nu)}$ converges to $\bar{p}$; finally, we prove that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of $\mathcal{E}'$. Note that the difficulty of the limit argument comes from the fact that $(\bar{x}, \bar{y})$ is not necessarily the limit of $(x^{\varphi(\nu)}, y^{\varphi(\nu)})$.

4.4.1 The fixed-point argument

From Assumption (H1')(d), let us fix $\underline{x}_i \in X_i$ and $\underline{y}_{i,j} \in Y'_j$ such that $\underline{x}_i = \omega_i + \sum_{j\in J}\theta_{i,j}\underline{y}_{i,j}$ for every $i \in I$. Let $\bar{B}^\nu$ be the closed ball with center 0 and radius ν with ν large enough so that ω, $\underline{x}_i$, $\underline{y}_{i,j}$ and ω_i belong to B^ν, the interior of $\bar{B}^\nu$, for all i, j. We consider the truncated economy obtained by replacing agent's consumption set by $X_i^\nu = X_i \cap \bar{B}^\nu$ for all $i_0 \neq i$, and $X_{i_0}^\nu = X_i \cap \bar{B}^{(\sharp I+\sharp J)\nu}$. The production set becomes $Y_j^\nu = Y'_j \cap \bar{B}^\nu$ and the augmented preference correspondences are $\hat{P}_i^\nu = \hat{P}_i \cap B^\nu$ and for $i_0 \neq i$, $\hat{P}_{i_0}^\nu = \hat{P}_{i_0} \cap B^{(\sharp I+\sharp J)\nu}$. The closed unit ball $\bar{B} = \{x \in \mathbb{R}^L : \|x\| \leq 1\}$ will be the price set. The truncation of X_{i_0} is chosen in such a way that if $(x, y) \in \prod_{i\in I} X_i^\nu \times \prod_{j\in J} Y_j^\nu$ is feasible, that is, $\sum_{i\in I} x_i = \omega + \sum_{j\in J} y_j$, then x_{i_0} belongs to the open ball $B^{(\sharp I+\sharp J)\nu}$.

We now consider the economy

$$\mathcal{E}^\nu = \left(\mathbb{R}^L, (X_i^\nu, \hat{P}_i^\nu, \omega_i)_{i\in I}, (Y_j^\nu)_{j\in J}, (\theta_{i,j})_{(i\in I, j\in J)}\right)$$

where the consumption and production sets are compact.

Remark 4.4.1. *For all i, the correspondence $\hat{P}_i^\nu$ is lower semi-continuous. Indeed, $\hat{P}_i^\nu$ is the intersection of the lower semi-continuous correspondence $\hat{P}_i$ and the constant correspondence B^ν (or $B^{(\sharp I+\sharp J)\nu}$), which has on open graph.*

Remark 4.4.2. *With the above remark and since $\bar{B}^\nu$ is convex and closed, note that the compact economy $\mathcal{E}^\nu$ satisfies Assumption (H1') but the non satiation of preferences at attainable allocations and Assumption (H2'). Furthermore, Y_j^ν is now compact.*

Since each Y_j^ν is compact, we can define for every $p \in \bar{B}$ the profit function

$$\pi_j^\nu(p) = \sup p.Y_j^\nu = \sup\{p.y_j : y_j \in Y_j^\nu\}$$

and the wealth of consumer i is defined by:

$$\gamma_i^\nu(p) = p.\omega_i + \sum_{j\in J}\theta_{ij}\pi_j^\nu(p).$$

Note that the function $\pi_j^\nu : \bar{B} \to \mathbb{R}$ is continuous since it is finite and convex.

In what follows, we will use the following notations for simplicity

- $Z^\nu = \prod_{i\in I} X_i^\nu \times \prod_{j\in J} Y_j^\nu \times \bar{B}$ and $z = (x, y, p)$ denotes a typical element of Z^ν
- $\hat{\gamma}_i^\nu(z) = \gamma_i^\nu(p) + \frac{1-\|p\|}{\sharp I}$
- $\widetilde{\gamma}_i^\nu(z) = \max\{\hat{\gamma}_i^\nu(z), \frac{1}{2}[\hat{\gamma}_i^\nu(p) + p \cdot x_i]\}$

Remark 4.4.3. *Note that* $p \cdot x_i > \widetilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$ *when* $p \cdot x_i > \hat{\gamma}_i^\nu(z)$ *and* $\widetilde{\gamma}_i^\nu(z) = \hat{\gamma}_i^\nu(z)$ *when* $p \cdot x_i \leq \hat{\gamma}_i^\nu(z)$.

Let now $N = I \cup J \cup \{0\}$ be the union of the set of consumers I indexed by i, the set of producers J indexed by j and an additional agent 0 whose function is to react with prices to a given total excess demand.

For all $i \in I$, we define the correspondences $\alpha_i^\nu : Z^\nu \to X_i^\nu$ and $\widetilde{\beta}_i^\nu : Z^\nu \to X_i^\nu$ as follows.

$$\alpha_i^\nu(z) = \{\xi_i \in X_i^\nu : p \cdot \xi_i \leq \hat{\gamma}_i^\nu(z)\}$$
$$\widetilde{\beta_i}^\nu(z) = \{\xi_i \in X_i^\nu : p \cdot \xi_i < \widetilde{\gamma}_i^\nu(z)\}$$

From the construction of the extended budget set, one checks that for all i, the consumption $\underline{x}_i$ belongs to $\widetilde{\beta_i}^\nu(z)$ if $x_i \notin \alpha_i^\nu(z)$. Indeed, from (H1')(d),

$$\underline{x}_i = \omega_i + \sum_{j\in J} \theta_{i,j} \underline{y}_{i,j}$$

since $x_i \notin {\alpha_i}^\nu(z)$, $p \cdot x_i > \hat{\gamma}_i^\nu(z)$ and $\widetilde{\gamma}_i^\nu(z) > \hat{\gamma}_i^\nu(z)$. Furthermore

$$\begin{aligned} p \cdot \underline{x}_i &= p \cdot \omega_i + \sum_{j\in J} \theta_{i,j} p \cdot \underline{y}_{i,j} \\ &\leq p \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \pi_j^\nu(p) \\ &\leq \hat{\gamma}_i^\nu(z) \\ &< \widetilde{\gamma}_i^\nu(z) \end{aligned}$$

which means that $\underline{x}_i$ belongs to $\widetilde{\beta_i}^\nu(z)$. Furthermore, since $\widetilde{\gamma}_i^\nu$ is continuous, the correspondence $\widetilde{\beta_i}^\nu$ has an open graph in $Z^\nu \times X_i^\nu$.

Now, for $i \in I$, we consider the mapping ϕ_i^ν defined from Z^ν to X_i^ν by:

$$\phi_i^\nu(z) = \begin{cases} \widetilde{\beta}_i^\nu(z) & \text{if } x_i \notin \alpha_i^\nu(z) \\ \widetilde{\beta}_i^\nu(z) \cap \hat{P}_i^\nu(x) & \text{if } x_i \in \alpha_i^\nu(z) \end{cases}$$

For $j \in J$, we define ϕ_j^ν from Z^ν to Y_j^ν by:

$$\phi_j^\nu(z) = \{y_j' \in Y_j^\nu \mid p \cdot y_j < p \cdot y_j'\},$$

and the mapping ϕ_0^ν from Z^ν to $\bar{B}$ is defined by:

$$\phi_0^\nu(z) = \{q \in \bar{B} \mid (q - p) \cdot (\sum_{i \in I} x_i - \omega - \sum_{j \in J} y_j) > 0\}$$

Now we will apply to Z^ν and the correspondences $(\phi_i)_{i \in I}^\nu, (\phi_j)_{j \in J}^\nu, \phi_0^\nu$ the well known theorem of Gale and Mas-Colell [22]. We will actually use the Bergstrom version of this theorem in [3], which is more adapted to our setting.

Theorem 4.3. *(Gale and Mas-Colell - Bergstrom) For each $k = 1, \cdots, \bar{k}$, let Z_k be a nonempty, compact, convex subset of some finite dimensional Euclidean space. Given $Z = \prod_{k=1}^{\bar{k}} Z_k$, let for each k, $\phi_k : Z \to Z_k$ be a lower semicontinuous correspondences satisfying for all $z \in Z$, $z_k \notin co\phi_k(z)$. Then there exists $\bar{z} \in Z$ such that for each $k = 1, \cdots, \bar{k}$:*

$$\phi_k(\bar{z}) = \emptyset \tag{4.1}$$

For the correspondences $(\phi_j^\nu)_{j \in J}$ and ϕ_0^ν, one easily checks that they are convex valued, irreflexive and lower semi-continuous since they have an open graph.

We now check that for all $i \in I$, the correspondence ϕ_i^ν satisfies the assumption of Theorem 4.3. We first remark that ϕ_i^ν has convex valued since $\widetilde{\beta}_i^\nu$ and $\hat{P}_i$ are so. We now check the irreflexivity. If $x_i \in \alpha_i^\nu(z)$, then, from Assumption (H1')(b), $x_i \notin \hat{P}_i(x)$, so $x_i \notin \phi_i^\nu(x)$ since $\phi_i^\nu(x) \subset \hat{P}_i(x)$. If $x_i \notin \alpha_i^\nu(z)$, then from Remark 4.4.3, $p \cdot x_i > \widetilde{\gamma}_i^\nu(z)$, so $x_i \notin \widetilde{\beta}_i^\nu(z) = \phi_i^\nu(z)$.

For the lower semi-continuity, let V be an open set and z such that $\phi_i^\nu(z) \cap V \neq \emptyset$. If $x_i \notin \alpha_i^\nu(z)$, then $p \cdot x_i > \hat{\gamma}_i^\nu(z)$. Since $\hat{\gamma}_i^\nu$ is continuous, there exists a neighborhood W of z such that for all $z' \in W$, $p' \cdot x_i' > \hat{\gamma}_i^\nu(z')$. Since $\widetilde{\beta}_i^\nu$ has an open graph, there existe a neighborhood W' of z such that for all $z' \in W'$, $\widetilde{\beta}_i^\nu(z') \cap V \neq \emptyset$. So, for all $z' \in W \cap W'$, $\phi_i^\nu(z') \cap V \neq \emptyset$ and consequently, ϕ_i^ν is lower semi-continuous at z. If $x_i \in \alpha_i^\nu(z)$, we first remark that $\widetilde{\beta_i}^\nu \cap \hat{P}_i^\nu$ is lower semicontinuous as an intersection of a lower semicontinuous correspondence with an open graph correspondence. So, there exists a neighborhood W of z such that for all $z' \in W$, $\widetilde{\beta_i}^\nu(z') \cap \hat{P}_i^\nu(x') \cap V \neq \emptyset$. This implies that $\widetilde{\beta_i}^\nu(z') \cap V \neq \emptyset$. Hence, in both cases, $x_i' \in \alpha_i^\nu(z')$ or $x_i' \notin \alpha_i^\nu(z')$, $\phi_i^\nu(z') \cap V \neq \emptyset$ from the definition of ϕ_i^ν. Thus ϕ_i^ν is also lower semi-continuous at z in this case.

From Theorem 4.3 , there exists $\bar{z}^\nu = (\bar{x}^\nu, \bar{y}^\nu, \bar{p}^\nu) \in Z^\nu$ such that, for all $k \in N$

$$\phi_k^\nu(\bar{z}^\nu) = \emptyset \tag{4.2}$$

As already noticed, since for all $i \in I$, $\underline{x}_i \in \widetilde{\beta}_i^\nu(\bar{z}^\nu)$ and $\phi_i^\nu(\bar{z}^\nu) = \emptyset$, we conclude from the definition of ϕ_i^ν that

$$\begin{cases} \bar{p}^\nu \cdot \bar{x}_i^\nu \leq \hat{\gamma}_i^\nu(\bar{z}^\nu) \\ \widetilde{\beta}_i^\nu(\bar{z}^\nu) \cap \hat{P}_i^\nu(\bar{x}^\nu) = \emptyset \end{cases} \tag{4.3}$$

Furthermore, from Remark 4.4.3, one deduces that $\widetilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$.

In addition, for all $j \in J$, since $\phi_j^\nu(\bar{z}^\nu) = \emptyset$, we deduce that:

$$\forall y_j \in Y_j^\nu, \bar{p}^\nu \cdot y_j \leq \bar{p}^\nu \cdot \bar{y}_j^\nu = \pi_j^\nu(\bar{p}^\nu), \tag{4.4}$$

and since $\phi_0^\nu(\bar{z}^\nu) = \emptyset$,

$$\forall p \in \bar{B}, p \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) \leq \bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) \tag{4.5}$$

We now prove that $(\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) = 0$. Indeed, if not, it follows from (4.5) that $\bar{p}^\nu$ belongs to the boundary of $\bar{B}$, that is $\|\bar{p}^\nu\| = 1$ and $\bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) > 0$. Now, by (4.3) and (4.4), for all i, $\bar{p}^\nu \cdot \bar{x}_i^\nu \leq \hat{\gamma}_i^\nu(\bar{z}^\nu) = \gamma_i^\nu(\bar{z}^\nu) = \bar{p}^\nu \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^\nu \cdot \bar{y}_j^\nu$. Summing up over $i \in I$ these inequalities, one gets, $\bar{p}^\nu \cdot (\sum_{i \in I} \bar{x}_i^\nu - \omega - \sum_{j \in J} \bar{y}_j^\nu) \leq 0$, which yields a contradiction. We thus have proved that $(\bar{x}^\nu, \bar{y}^\nu) \in \mathcal{A}(\mathcal{E}^\nu)$.

Remark 4.4.4. *Since $(\bar{x}^\nu, \bar{y}^\nu)$ is feasible, we deduce that $\bar{x}_{i_0}^\nu$ belongs to the open ball $B^{(\sharp I + \sharp J)\nu}$. From Assumption (H1')(e), $\hat{P}_{i_0}(\bar{x}^\nu)$ is nonempty and for all $\xi_{i_0} \in \hat{P}_{i_0}(\bar{x}^\nu)$*

and for all $t \in]0,1]$, $t\xi_{i_0} + (1-t)\bar{x}^\nu_{i_0} \in \hat{P}_{i_0}(\bar{x}^\nu)$. For t small enough, $t\xi_{i_0} + (1-t)\bar{x}^\nu_{i_0}$ belongs to $B^{(\sharp I+\sharp J)\nu}$, so to $\hat{P}^\nu_{i_0}(\bar{x}^\nu)$. From (4.3), $\bar{p}^\nu \cdot (t\xi_{i_0} + (1-t)\bar{x}^\nu_{i_0}) \geq \hat{\gamma}^\nu_{i_0}(\bar{z}^\nu)$. At the limit when t tends to 1, knowing from (4.3) that $\bar{p}^\nu \cdot \bar{x}^\nu_{i_0} \leq \hat{\gamma}^\nu_{i_0}(\bar{z}^\nu)$, one gets

$$\bar{p}^\nu \cdot \bar{x}^\nu_{i_0} = \hat{\gamma}^\nu_{i_0}(\bar{z}^\nu) \tag{4.6}$$

from which one deduces that

$$\forall \xi_{i_0} \in \hat{P}_{i_0}(\bar{x}^\nu), \bar{p}^\nu \cdot \xi_{i_0} \geq \hat{\gamma}^\nu_{i_0}(\bar{z}^\nu) \tag{4.7}$$

4.4.2 The limit argument

We first show that we can apply Assumption (WH3) to the sequence $((\bar{x}^\nu_i))$ built in the previous sub-section. We have already proved that $\bar{x}^\nu$ is attainable in the truncated economy $\mathcal{E}^\nu$, so it is also attainable in the economy $\mathcal{E}'$. It remains to show that $\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}$ for all i.

There are two cases. First, if $\bar{p}^\nu \cdot \underline{x}_i < \hat{\gamma}^\nu_i(\bar{z}^\nu)$, which means that $\underline{x}_i \in \widetilde{\beta}^\nu_i(\bar{z}^\nu)$, then, from (4.3), $\underline{x}_i \notin \hat{P}^\nu_i(\bar{x}^\nu) = \hat{P}_i(\bar{x}^\nu) \cap B^\nu$. Since $\underline{x}_i \in B^\nu$ as ν has been chosen large enough, one deduces that $\underline{x}_i \notin \hat{P}_i(\bar{x}^\nu)$ and therefore $\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}$.

If $\bar{p}^\nu \cdot \underline{x}_i \geq \hat{\gamma}^\nu_i(\bar{z}^\nu)$, as $\underline{x}_i \in \widetilde{\beta}^\nu_i(\bar{z}^\nu)$ and $\hat{\gamma}^\nu_i(\bar{z}^\nu) = \widetilde{\gamma}^\nu_i(\bar{z}^\nu)$, we actually have the equality $\bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}^\nu_i(\bar{z}^\nu)$. We remark that $\hat{\gamma}^\nu_i(\bar{z}^\nu) = \gamma^\nu_i(\bar{z}^\nu) + \frac{1-\|\bar{p}^\nu\|}{\sharp I} = \bar{p}^\nu \cdot \underline{x}_i = \bar{p}^\nu \cdot (\omega_i + \sum_{j\in J} \theta_{i,j}\underline{y}_{i,j}) \leq \gamma^\nu_i(\bar{z}^\nu)$. So, $\|\bar{p}^\nu\| = 1$. By contradiction, we prove that $\underline{x}_i \in \overline{\hat{P}_i(\bar{x}^\nu)^c}$. Indeed, if not, $\underline{x}_i \in \text{int } \hat{P}_i(\bar{x}^\nu)$ and there exists $\rho > 0$ such that $B(\underline{x}_i, \rho) \subset \hat{P}_i(\bar{x}^\nu)$ and $B(\underline{x}_i, \rho) \subset B^\nu$. Since $\bar{p}^\nu \neq 0$, there exists $\xi^\nu_i \in B(\underline{x}_i, \rho)$ such that $\bar{p}^\nu \cdot \xi^\nu_i < \bar{p}^\nu \cdot \underline{x}_i = \hat{\gamma}^\nu_i(\bar{z}^\nu)$ and this contradicts (4.3) since $\xi^\nu_i \in B(\underline{x}_i, \rho) \subset \hat{P}^\nu_i(\bar{x}^\nu)$.

We now consider the subsequence $((\bar{x}^{\varphi(\nu)}_i))$ of $\hat{X}$ and $((\bar{x}_i)) \in \hat{X}$ as given by Assumption (WH3). From the definition of $\hat{X}$, there exists $(\bar{y}_j) \in \prod_{j\in J} Y'_j$ such that $\sum_{i\in I} \bar{x}_i = \sum_{i\in I} \omega_i + \sum_{j\in J} \bar{y}_j$. Since $\bar{B}$ is compact, we can assume without any loss of generality that the sequence $(\bar{p}^{\varphi(\nu)})$ converges to $\bar{p} \in \bar{B}$.

Now let $(y_j) \in \prod_{j\in J} Y'_j$, $(\xi_i) \in \prod_{i\in I} \hat{P}_i(\bar{x})$ and $\lambda \in [0,1[$. Such (ξ_i) exists from Assumption (H1')(e). Furthermore, from the definition of the extended preferences, note that $\xi^\lambda_i = \lambda\bar{x}_i + (1-\lambda)\xi_i \in \hat{P}_i(\bar{x})$.

By (WH3), there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i^λ such that for all $\nu \geq \nu_1$, $\xi_{i_0}^{\varphi(\nu)} \in \mathrm{cone}\{\hat{P}_{i_0}(\bar{x}^{\varphi(\nu)}) - \bar{x}_{i_0}^{\varphi(\nu)}\} + \bar{x}_{i_0}^{\varphi(\nu)}$ and for all $i \neq i_0$, $\xi_i^{\varphi(\nu)} \in \hat{P}_i(\bar{x}^{\varphi(\nu)})$. Since the sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ is convergent, it is bounded and for ν large enough, for all $i \neq i_0$, $\xi_i^{\varphi(\nu)}$ belong to B^ν, so $\xi_i^{\varphi(\nu)} \in \hat{P}_i^\nu(\bar{x}^{\varphi(\nu)})$, $\forall \nu \geq \nu_1$. We deduce from (4.3) that $\xi_i^{\varphi(\nu)} \notin \widetilde{\beta}_i^\nu(\bar{z}^{\varphi(\nu)})$, that is, $\bar{p}^\nu \cdot \xi_i^{\varphi(\nu)} \geq \widetilde{\gamma}_i^\nu(\bar{z}^\nu) = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. Using the same argument as in Remark 4.4.4, one deduces that $\bar{p}^\nu \cdot \bar{x}_i^{\varphi(\nu)} = \hat{\gamma}_i^\nu(\bar{z}^\nu)$. So, from Remark 4.4.4, for all $i \in I$,

$$\bar{p}^\nu \cdot \bar{x}_i^{\varphi(\nu)} = \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)} + \frac{1 - \|\bar{p}^{\varphi(\nu)}\|}{\sharp I}$$

Summing over i these inequalities and knowing that $(\bar{x}^{\varphi(\nu)}, \bar{y}^{\varphi(\nu)})$ is a feasible allocation, we conclude that $\|\bar{p}^{\varphi(\nu)}\| = 1$ and at the limit, $\|\bar{p}\| = 1$.

For all $i \neq i_0$,

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \gamma_i^\nu(\bar{z}^\nu) = \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)}$$

and for i_0, there exists $\alpha \geq 0$ and $\zeta_{i_0}^{\varphi(\nu)} \in \hat{P}_{i_0}(\bar{x}^{\varphi(\nu)})$ such that,

$$\bar{p}^{\varphi(\nu)} \cdot \xi_{i_0}^{\varphi(\nu)} = \bar{p}^{\varphi(\nu)} \cdot [\bar{x}_{i_0}^{\varphi(\nu)} + \alpha(\zeta_{i_0}^{\varphi(\nu)} - \bar{x}_{i_0}^{\varphi(\nu)})]$$

From (4.6) and (4.7), $\bar{p}^{\varphi(\nu)} \cdot \bar{x}_{i_0}^{\varphi(\nu)} = \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) = \gamma_{i_0}^\nu(\bar{z}^\nu)$ and $\bar{p}^{\varphi(\nu)} \cdot \zeta_{i_0}^{\varphi(\nu)} \geq \hat{\gamma}_{i_0}^\nu(\bar{z}^\nu) = \gamma_{i_0}^\nu(\bar{z}^\nu)$, so, since $\alpha \geq 0$, one concludes that

$$\bar{p}^{\varphi(\nu)} \cdot \xi_{i_0}^{\varphi(\nu)} \geq \gamma_{i_0}^\nu(\bar{z}^\nu) = \bar{p}^{\varphi(\nu)} \cdot \omega_{i_0} + \sum_{j \in J} \theta_{i_0,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j^{\varphi(\nu)}$$

For ν large enough, for all $j \in J$, $y_j \in \bar{B}^\nu$. So, $(y_j) \in \prod_{j \in J} Y_j^\nu$, and from (4.4), one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j \in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot y_j \tag{4.8}$$

In particular, for $(\bar{y}_j) \in \prod_{j\in J} Y'_j$, one gets

$$\bar{p}^{\varphi(\nu)} \cdot \xi_i^{\varphi(\nu)} \geq \bar{p}^{\varphi(\nu)} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p}^{\varphi(\nu)} \cdot \bar{y}_j \tag{4.9}$$

Passing to the limit in (4.8) and (4.9), we obtain:

$$\bar{p} \cdot \xi_i^{\lambda} \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot y_j \tag{4.10}$$

and

$$\bar{p} \cdot \xi_i^{\lambda} \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j \tag{4.11}$$

The two above inequalities hold true for any $i \in I$, $\xi_i \in \hat{P}_i(\bar{x})$, $\lambda \in [0, 1[$ and $(y_j) \in \prod_{j\in J} Y'_j$. Knowing that $(\bar{x}, \bar{y})$ is an attainable allocation, we will show that $(\bar{x}, \bar{y}, \bar{p})$ is a quasi-equilibrium of the economy $\mathcal{E}'$, which completes the proof.

When λ goes to 1 in (4.10) and (4.11), one gets

$$\bar{p} \cdot \bar{x}_i \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot y_j \tag{4.12}$$

and

$$\bar{p} \cdot \bar{x}_i \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j \tag{4.13}$$

Since $(\bar{x}, \bar{y})$ is a feasible allocation, summing over i the inequalities in (4.13), one deduces that

$$\bar{p} \cdot \bar{x}_i = \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j \tag{4.14}$$

Taken $\lambda = 0$ in (4.11), we obtain for all $i \in I$, for all $\xi_i \in \hat{P}_i(\bar{x})$,

$$\bar{p} \cdot \xi_i \geq \bar{p} \cdot \omega_i + \sum_{j\in J} \theta_{i,j} \bar{p} \cdot \bar{y}_j \tag{4.15}$$

So, the quasi-demand condition (b) of Definition 4.2.2 is satisfied.

Finally, from (4.12) and (4.13), for all $(y_j) \in \prod_{j\in J} Y'_j$, one gets

$$\bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot y_j \le \bar{p}\cdot\omega_i + \sum_{j\in J}\theta_{i,j}\bar{p}\cdot\bar{y}_j \tag{4.16}$$

Summing over i, we get

$$\sum_{j\in J}\bar{p}\cdot y_j \le \sum_{j\in J}\bar{p}\cdot\bar{y}_j$$

For any $j \in J$, applying this inequality to $y' \in \prod_{j\in J} Y'_j$ defined by $y'_{j'} = \begin{cases} y_j & \text{if } j' = j \\ \bar{y}'_j & \text{if } j' \neq j \end{cases}$, it readily follows that

$$\bar{p}\cdot y_j \le \bar{p}\cdot\bar{y}_j \tag{4.17}$$

which means that the profit maximization condition (a) of Definition 4.2.2 is also satisfied. □

4.5 Relationship with the literature

In this section, we compare Assumption (H3) with other conditions in the literature on the existence of equilibrium with unbounded consumption sets. We show that Assumption (H3) is weaker than the compactness of the set of individually rational and attainable allocations or utility levels and the CPP condition of Allouch. We also explain the relationships with the condition of Won and Yannelis.

4.5.1 Compactness of the attainable utility set

The following proposition shows that Assumption (H3) is weaker than the compactness of $\mathcal{A}(\mathcal{E})$ or U the attainable utility set. We use the following assumption on preferences as in Allouch.

Assumption (H4) The utility function u_i is lower semi-continuous and strictly quasi-concave, that is, for all $(x_i, z_i) \in X_i \times X_i$ with $u_i(z_i) > u_i(x_i)$ then $u_i(\lambda x_i + (1-\lambda)z_i) > u_i(x_i)$ for all $\lambda \in [0, 1[$.

If P_i is represented by a utility function u_i satisfying Assumption (H4), i.e.:

$$P_i(x) = \{x'_i \in X_i : u_i(x'_i) > u_i(x_i)\}$$

then, $P_i(x) = \hat{P}_i(x)$, for all $x \in \prod_{i \in I} X_i$. If the preferences of all consumers are represented by a utility function, the set of attainable utility level U is defined as:

$$U = \{(\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_m) \in \mathbb{R}^I_+ : \exists x \in \hat{X} \text{ s.t. } u_i(\underline{x}_i) \leq \mathrm{v}_i \leq u_i(x_i)\},$$

In an exchange economy with the survival assumption $\omega_i \in X_i$ for all i, the set U is just the set of individually rational attainable consumptions.

Proposition 4.5.1. *Under Assumption (H1),*

(i) If $\mathcal{A}(\mathcal{E})$ is compact, then (H3) is satisfied.

(ii) If the preferences of all consumers are represented by a utility function satisfying Assumption (H4) and if U is compact, then Assumption (H3) is satisfied.

Proof.

(i) Let $((x_i^\nu))$ be a sequence in $\hat{X}$. From the definition of $\hat{X}$, there exists a sequence $((y_j^\nu))$ of $\prod_{j \in J} Y_j$ such that $((x_i^\nu), (y_j^\nu)) \in \mathcal{A}(\mathcal{E})$. Since $\mathcal{A}(\mathcal{E})$ is compact, there exists a subsequence $((x_i^{\varphi(\nu)}), (y_j^{\varphi(\nu)}))$ convergent to $((\bar{x}_i), (\bar{y}_j)) \in \mathcal{A}(\mathcal{E})$. Let $i \in I$ and $\xi_i \in \hat{P}_i(\bar{x})$. For all integer $k \geq 1$, we set $V_k = \{x \in \prod_{i \in I} X_i | B(\xi_i, \frac{1}{k}) \cap \hat{P}_i(x) \neq \emptyset\}$. Since P_i is lower semi-continuous, so is $\hat{P}_i$. Hence, V_k is an open neighborhood containing $\bar{x}$. Since $(x^{\varphi(\nu)})$ converges to $\bar{x}$, there exists an integer $\bar{\nu}(k)$ such that for all $\nu \geq \bar{\nu}(k)$, $x^{\varphi(\nu)} \in V_k$. We can assume without loss of generality that for all $k \geq 1$, the sequence $(\bar{\nu}(k))$ is strictly increasing, which implies that for all $\nu \geq \bar{\nu}(1)$, there exists a unique integer $\kappa(\nu)$ such that $\bar{\nu}(\kappa(\nu)) \leq \varphi(\nu) < \bar{\nu}(\kappa(\nu) + 1)$. Hence, we have $x^{\varphi(\nu)} \in V_{\kappa(\nu)}$ and there exists $(\xi_i^{\varphi(\nu)})$ such that $\xi_i^{\varphi(\nu)} \in B(\xi_i, \frac{1}{\kappa(\nu)}) \cap \hat{P}_i(x^{\varphi(\nu)})$. Now, since $\varphi(\nu)$ goes to infinity and for all $k \geq 1$, $(\bar{\nu}(k))$ is strictly increasing, then $\kappa(\nu)$ goes to infinity. Hence, the sequence $(\xi_i^{\varphi(\nu)})$ converges to ξ_i and for all $\varphi(\nu) \geq \bar{\nu}(1)$, $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$. So, Assumption (H3) holds true.

(ii) Let $((x_i^\nu))$ be a sequence in $\hat{X}$ such that for all i, $\underline{x}_i \in \overline{\hat{P}_i(x^\nu)^c}$. Since $P_i(x) = \{x'_i \in X_i : u_i(x'_i) > u_i(x_i)\}$ and $\underline{x}_i \in \hat{P}_i(x^\nu)^c \subset \overline{P_i(x^\nu)^c}$, for all $i \in I$, the lower semicontinuity of u_i implies that $u_i(\underline{x}_i) \leq u_i(x_i^\nu)$. Let now consider the sequence (v_i^ν) defined by

$$(v_i^\nu) = (u_i(x_i^\nu))$$

For all $i \in I$, we have $u_i(\underline{x}_i) \leq v_i^\nu = u_i(x_i^\nu)$ so $(v_i^\nu) \in U$. Since U is compact, there exists a subsequence $(v_i^{\varphi(\nu)}) = (u_i(x_i^{\varphi(\nu)}))$ convergent to $\bar{v}_i \in U$. By definition of U, there exists $(\bar{x}_i) \in \hat{X}$ such that $\bar{v}_i \leq u_i(\bar{x}_i)$ for all $i \in I$. Let $i \in I$ and $\xi_i \in \hat{P}_i(\bar{x})$. Since under Assumption (H4), $P_i(\bar{x}) = \hat{P}_i(\bar{x})$, then $u_i(\bar{x}_i) < u_i(\xi_i)$. Consequently $\bar{v}_i \leq u_i(\bar{x}_i) < u_i(\xi_i)$ for all $i \in I$. Since $(u_i(x_i^{\varphi(\nu)}))$ converges to $\bar{v}_i$, there exists ν_1 such that for all $\nu \geq \nu_1$, $u_i(x_i^{\varphi(\nu)}) < u_i(\xi_i)$, hence $\xi_i \in P_i(x^{\varphi(\nu)}) \subset \hat{P}_i(x^{\varphi(\nu)})$.Then, the constant sequence (ξ_i) satisfies $\xi_i \in \hat{P}_i(x^{\varphi(\nu)})$ for all $\nu \geq \nu_1$ and converges to ξ_i. □

4.5.2 Comparison with the CPP condition of Allouch

We recall the following definition of the CPP condition considered by Allouch [1].

Definition 4.5.1. *The economy $\mathcal{E}$ satisfies the CPP condition if for every sequence $((x_i^\nu))$ of $\hat{X}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$, an element $(\xi_i) \in \hat{X}$ and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)})$, for all ν.*

Beside this assumption, Allouch also assumes that the preference relations are transitive, have open lower-section and that the augmented preferences are equal to the preferences. Assumption (H3) and the CPP condition have the same flavour, but the transitivity allows to consider a unique sequence $(\xi_i^{\varphi(\nu)})$ whereas Assumption (H3) needs a sequence for each preferred element.

Proposition 4.5.2. *Let us assume that the preference relations are transitive, have open lower-section and are equal to the augmented preferences. Then if the CPP condition is satisfied, Assumption (H3) holds true.*

Proof. Let a sequence $((x_i^\nu))$ of $\hat{X}$. From the CPP condition, there exists a subsequence $((x_i^{\varphi(\nu)}))$ and $(\xi_i) \in \hat{X}$, there exists a sequence $(\xi_i^{\varphi(\nu)})$ convergent to ξ_i with $\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)}) = P_i(x^{\varphi(\nu)})$, for all ν. Let $i \in I$ and $\zeta_i \in P_i(\xi)$. Since P_i has open lower sections, there exists a neighborhood V of ξ such that for all $\xi' \in V$, $z_i \in P_i(\xi')$.

Since the sequence $(\xi^{\varphi(\nu)})$ converges to ξ, we have $\xi^{\varphi(\nu)} \in V$, for ν large enough. Consequently, $z_i \in P_i(\xi^{\varphi(\nu)})$. Since P_i is transitive and since $\xi_i^{\varphi(\nu)} \in P_i(x^{\varphi(\nu)})$ and $z_i \in P_i(\xi^{\varphi(\nu)})$, one gets $z_i \in P_i(x^{\varphi(\nu)})$. Therefore, the constant sequence (z_i) satisfies $z_i \in \hat{P}_i(x^{\varphi(\nu)})$ for all ν large enough and converges to z_i. □

4.5.3 Comparison with Won and Yannelis work

To compare our contribution to the one of Won and Yannelis [52], we restrict our attention to an exchange economy. Indeed, the initial endowments ω_i are used as a reference point on the budget line and there is no equivalent consumption in a production economy. The frameworks and the basic assumptions are quite similar and we focused our attention on the asymptotic condition corresponding to our Assumption (H3). To state it, we borrow the following notations from [52]. For $x \in \prod_{i\in I} X_i$, for all $i \in I$, $r_i(x) = \max\{\|x_k\| \mid k \neq i\}$ and $\bar{B}(0, r)$ denotes the closed ball of center 0 and radius r. We now state Assumption (B7a) of Won and Yannelis.

Assumption (B7a) There exists a consumer $i_0 \in I$ such that for all sequence $((x_i^\nu))$ of $\hat{X}$ with $\omega_i \in \overline{\hat{P}_i(x^\nu)}^c$ for all i and for all ν, there exists a subsequence $((x_i^{\varphi(\nu)}))$ and a sequence $(y^{\varphi(\nu)})$ convergent to a point $y \in \hat{X}$, such that, for all ν,

$$P_{i_0}(y^{\varphi(\nu)}) \subset \text{cone}[P_{i_0}(x^{\varphi(\nu)}) - \{\omega_{i_0}\}] + \{\omega_{i_0}\}$$

and for all $i \neq i_0$,

$$P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) \subset \text{cone}[P_i(x^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) - \{\omega_i\}] + \{\omega_i\}.$$

We first remark that Assumption (H3) does not require the sequence $(y^{\varphi(\nu)})$ and the inclusion of the associated preferred set, or a truncation of it, in a set generated by the preferred set of $x^{\varphi(\nu)}$. Indeed, our assumption has the flavour of the CPP condition of Allouch.

Note that the use of the cone operator enlarges the set $[P_{i_0}(x^{\varphi(\nu)})]-\{\omega_{i_0}\}]$ or $[P_i(x\{\omega_{i_0}\})\cap \bar{B}(0, r_{i_0}(x\{\omega_{i_0}\}))-\{\omega_i\}]$, so the condition is weaker than assuming $P_{i_0}(y^{\varphi(\nu)}) \subset P_{i_0}(x^{\varphi(\nu)})$ and $P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)})) \subset P_i(y^{\varphi(\nu)}) \cap \bar{B}(0, r_i(x^{\varphi(\nu)}))$ for all $i \neq i_0$. Note that, thanks to the lower semi-continuity of the preferences, Assumption (H3) is weaker than assuming the existence of the convergent sequence $(y^{\varphi(\nu)})$ and the inclusion $P_i(y^{\varphi(\nu)}) \subset P_i(x^{\varphi(\nu)})$. So, at this stage, the two assumptions are not comparable.

But Assumption (B7a) exhibits a non symmetric treatment of the consumers with Consumer i_0 playing a particular role with no truncation by the ball $B(0, r_{i_0}(x))$. Furthermore, the radius $r_{i_0}(x)$ is suitably chosen to be the smallest radius such that all consumptions $x_i^{\varphi(\nu)}$ belongs to the ball for all $i \neq i_0$. We have no hint about the possibility to choose a larger radius. So, the choice of this particular radius seems to be ad hoc. In view of our argument above, we can understand why we can have a weaker condition on one consumer. In the truncated economy before the limit argument, we can choose a large enough radius to truncate the consumption set of one consumer i_0 such that all feasible allocation (x), the consumption x_{i_0} belongs to the interior of the ball. This allows us to prove that the whole preferred set is above the budget line. Whereas for the other consumers, we have only the truncated preferred set above the budget line. That is why, in Assumption (B7a), the condition for all $i \neq i_0$ involves the truncated preferred set $P_i(x^{\varphi(\nu)}) \cap \bar{B}(0, r_{i_0}(x^{\varphi(\nu)}))$.

The major advantage of Assumption (B7a) comes from the fact that it is satisfied by the example Page et al [40] where an equilibrium exists with an unbounded set of attainable individually rational utility level. We easily check that this example satisfies the following asymmetric weakening of Assumption (H3) in the framework of an exchange economy:

Assumption (EWH3) There exists a consumer $i_0 \in I$, such that for all sequence $((x_i^\nu))$ of $\hat{X}$ such that for all i, $\omega_i \in \overline{\hat{P}_i(x^\nu)^c}$, there exists a subsequence $((x_i^{\varphi(\nu)})) \in \hat{X}$ and $(\bar{x}_i) \in \hat{X}$ such that for all i, for all $\xi_i \in \hat{P}_i(\bar{x})$, there exists an integer ν_1 and a sequence $(\xi_i^{\varphi(\nu)})_{\nu \geq \nu_1}$ convergent to ξ_i with, for all $\nu \geq \nu_1$,

$$\xi_{i_0}^{\varphi(\nu)} - \omega_{i_0} \in \text{cone}[\hat{P}_{i_0}(x^{\varphi(\nu)}) - \omega_{i_0}]$$

and for all $i \neq i_0$,

$$\xi_i^{\varphi(\nu)} \in \hat{P}_i(x^{\varphi(\nu)}).$$

We did not consider and emphasise this assumption previously since its asymmetry is an hint that there is still room for improvements to get a still weaker and symmetric assumption. We can easily adapt the proof of Section 4 to check that assumption (EWH3) is sufficient for the existence of quasi-equilibrium in exchange economies.

Finally, we discuss Example 3.1.2 of Won and Yannelis. Clearly, Assumption (EWH3) does not cover this example.The authors claim that this example satisfies their weaker assumption (B7). The argument is based on the fact that there is no equilibrium in the truncated economy except the no-trade one with the two consumptions equal to 0 and any positive price. Actually, it seems to us that the price $p = (0, 1)$ associated to the consumptions $x_1 = (r, 0)$ and $x_2 = (-r, 0)$ is an equilibrium when the first agent has a truncated budget set $\bar{B}(0, r)$. In that case, the set $P_1(x) \cap \bar{B}(0, r_2(x))$ is empty, so is the set $G_2(x)$ with the notation of the paper. Consequently, finding an assumption covering Example 3.1.2 of [52] is still an open challenge.

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