Détection de la non-réalisabilité et stratégies de régularisation en optimisation non linéaire

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À ma famille

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Chapter 1

Introduction

A constrained optimization problem involves minimizing an objective function over a set of constraints. If the objective function and constraints are linear, this problem is called a linear optimization problem, otherwise, it is a nonlinear optimization problem. Optimization problems can be found in many areas such as science and technology, economics, industrial production and so on. Since the first appearance of the simplex algorithm for solving a linear optimization problem by G. Dantzig in 1947, developing algorithms for solving optimization problems has become an active area in optimization research. Moreover, the development of computers led to the birth of many linear and nonlinear optimization solvers. This thesis is focused on the numerical solution of the nonlinear optimization problems.

In this work, we consider nonlinear optimization problems under this form:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & c(x) = 0, \\ & x \geq 0, \end{aligned} \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions. Any differentiable optimization problem with equality and inequality constraints can be reformulated under this standard form by possibly adding nonnegative slack variables and by splitting free variables into positive and negative parts.

A constrained optimization algorithm focuses on two tasks: minimizing the objective function and satisfying the constraints. When the set of the optimal solutions is nonempty, many efficient algorithms in the literature are proposed to find an optimal solution of the problem. But in practice, infeasible instances also appear quite a lot. They can arise, for example, in mathematical modeling, from varying the parameters of a model to study the system response. Infeasibility may also appear when solving a sequence of subproblems in an algorithm like a branch-and-bound method. Even if the problem is feasible, algorithm may encounter difficulties in finding a feasible point. In all these cases, an efficient solver should quickly return an infeasible stationary point, with the goal to avoid a long sequence of iterations or a convergence to a spurious solution. In this context, a rapid infeasibility detection is an important issue in nonlinear optimization since many contemporary methods either fail or take an excessive number of computational time to notify that a problem is infeasible, see, e.g., [118]. The first part of this thesis focuses on developing a numerical method to rapidly detect the infeasibility in the framework of a primal-dual method.

In the literature, the local convergence analysis of nonlinear optimization algorithms is handled under the following usual assumptions: linear independence constraint qualification (LICQ), second order sufficient conditions (SOSCs) and strict complementarity. These assumptions imply that the Jacobian of the optimality system to solve is locally nonsingular, an essential property of Newtonian methods to get a superlinear or a quadratic rate of convergence. However, in practice, these assumptions are not always satisfied, for example, in mathematical programming with equilibrium constraints (MPECs) or when the optimal solution is not isolated. In these degenerate cases, a regularization technique must be applied to recover a rapid rate of convergence. In this thesis, we analyze the local behaviors of two regularization techniques incorporated in two primal-dual algorithms to deal with the lack of SOSCs or of any constraint qualification.

In the following section, we summarize the state of the art of optimization methods used to handle problem (1.1).

1.1 The sequential quadratic programming method

The sequential quadratic programming (SQP) is one of the most effective methods for nonlinear constrained optimization. This method, which was first proposed by Wilson in his PhD [152], generates iterates by solving a sequence of quadratic subproblems. For solving (1.1), at an iterate x_k, a basic SQP algorithm defines a search direction d_k as a solution to the quadratic minimization problem

$$\begin{aligned}
\min_{d} \quad & \nabla f(x_k)^\top d + \tfrac{1}{2} d^\top \nabla^2_{xx}\mathcal{L}(x_k, y_k, z_k) d \\
\text{s.t.} \quad & c(x_k) + \nabla c(x_k)^\top d = 0, \\
& x_k + d \geq 0,
\end{aligned}$$

where $\mathcal{L}(x, y, z) = f(x) + y^\top c(x) - z^\top x$, for $(x, y, z) \in \mathbb{R}^{n+m+n}$, is the Lagrangian function associated to (1.1).

The solvers FILTER [67] and SNOPT [79] are two of the most popular implementations of SQP methods. Both of them contain two phases: a main phase and a feasibility restoration phase which is devoted to minimize the constraint violations. There is a switching technique between these two phases. FILTER uses the feasibility of a trust region subproblem to activate the feasibility restoration phase. Whereas, SNOPT transforms the standard SQP algorithm into a Sl_1QP [66] when the problem seems to be infeasible. Gould and Robison [87, 88], Byrd *et al.* [40] and Burke *et al.* [30] proposed exact penalty SQP methods with only one single optimization phase. A steering rule [39] is used to update the penalty parameter in these methods. In particular, an additional subproblem is solved to choose a penalty parameter that ensures balanced progress toward feasibility and optimality. However, since the above SQP methods require the solution of a general quadratic problem with large cost, the size of problems that can be solved in practice is limited. Moreover, the indefinite Hessian matrix used in these methods may cause difficulties (see, e.g. [84]). To overcome these obstacles, Fletcher and Saínz de la Maza [68] proposed the sequential linear-quadratic programming (SLQP) method which was further developed by Chin and Fletcher [43] and Byrd *et al.* [37]. The former possesses a filter method and a feasibility restoration phase to deal with infeasibility. By contrast, the latter is a penalty function method and is implemented in the KNITRO/ACTIVE code [38].

The local convergence analysis of SQP methods has been studied widely, see, e.g., [66, 88, 130, 137, 145]. However, these analyses focus mostly on the feasible instance under some usual assumptions. Byrd *et al.* [40] and Burke *et al.* [30] proposed first local convergence analysis for infeasible problems. The rapid convergence is mostly based on the rules to update the penalty parameters. More specifically, the spirit of the steering rules is used to update the penalty parameter in Byrd *et al.* [40]. The update of penalty parameter in the algorithm of Burke *et al.* [30] is considered twice at each iteration after the solutions of quadratic optimization subproblems.

On the other hand, the local convergence analysis of optimization algorithms without linear independence constraint qualification (LICQ) and constraint qualifications in general was investigated by stabilized SQP. Wright [153] and Fisher [65] demonstrated the superlinear convergence of their algorithms under the Mangasarian-Fromovitz constraint qualification (MFCQ) and strict complementarity. Qi and Wei [126] introduced a weaker condition than LICQ and used it together with strict complementarity. The later works of Anitescu [6] and of Wright [155, 156] dropped the strict complementarity. Both MFCQ and strict complementarity were removed in the studies of Hager [93], Wright [157], Izmailov and Solodov [101], Gill *et al.* [80], Arreckx and Orban [17].

1.2 The quadratic penalty method and the augmented Lagrangian method

The quadratic penalty method is proposed firstly by Courant [48] in 1943. This approach is to solve a sequence of unconstrained problem created by a combination of the objective function f and the l_2 feasibility measure of the constraints. For example, with the following equality constrained minimization

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & c(x) = 0, \end{aligned} \tag{1.2}$$

the quadratic penalty method solves a sequence of subproblems

$$\min_{x \in \mathbb{R}^n} \quad \phi_\sigma(x) := f(x) + \frac{1}{2\sigma}\|c(x)\|^2, \tag{1.3}$$

where the penalty parameter $\sigma > 0$ is forced to decrease to zero. Lootsma [110] and Murray [114] demonstrated that the small value of σ can raise numerical difficulty. This one comes from the third term in the Hessian matrix of penalty function $\phi_\sigma(\cdot)$

$$\nabla^2 \phi_\sigma(x) = \nabla^2 f(x) + \frac{1}{\sigma}\sum_{i=1}^{m} c_i(x)\nabla^2 c_i(x) + \frac{1}{\sigma}\nabla c(x)\nabla c(x)^\top.$$

(This term becomes increasingly ill-conditioned as σ tends to zero.) This inconvenience of the quadratic penalty method caused it to be shunned by practitioners for a long time. To overcome this difficulty, some kinds of auxiliary variable is introduced to obtain well-conditioned systems in framework of both primal algorithm (Broyden and Attia [27, 28], Gould [85, 86]) and primal-dual algorithm (Armand *et al.* [16]).

Augmented Lagrangian method were proposed independently by Hestenes [96] and Powell [125] in 1969 as another alternative to deal with the ill-conditioning associated with the quadratic penalty method. At each iteration, with a fixed penalty parameter σ and an estimate of Lagrange multiplier λ, the augmented Lagrangian method performs minimization with respect to x of the following unconstrained problem

$$\min_{x\in\mathbb{R}^n} \quad \mathcal{L}_{\lambda,\sigma}(x) := f(x) + \lambda^\top c(x) + \frac{1}{\sigma}\|c(x)\|^2. \tag{1.4}$$

After that, the penalty parameter and the Lagrange multiplier will be updated based on the satisfaction of constraints. If the constraints have decreased sufficiently, the penalty parameter σ is not changed in the next iteration and the Lagrange multiplier λ^+ is updated by the formula

$$\lambda^+ = \lambda + \frac{c(x)}{\sigma},$$

where x is an approximate solution of (1.4). Otherwise, we decrease the penalty parameter to ensure that the next iterate takes more emphasis on decreasing the constraint violations. Simultaneously, the new Lagrange multiplier λ^+ will be kept as the previous one λ. The augmented Lagrangian method of Powell and Hestenes was proposed only for the equality constrained minimization (1.2). In the early 1970s, Rockafellar [131, 132] and Buys [31] extended this method to inequality constrained optimization. Di Pillo and Grippo [54, 55] developed algorithms based on this method for problems in both framework of equality and inequality constrained optimizations.

The global convergence of the augmented Lagrangian method has been discussed in both the convex case (see, e.g., [99, 104, 105]) and the general non-convex case (see, e.g., [21, 44, 122, 131, 133]) under assumptions on the boundedness of the sequence of the Lagrange multiplier. The merit functions to globalize the augmented Lagrangian method have been used in two forms: the primal merit function (see, e.g., Byrd *et al.* [32], Gill *el al.* [78], Schittkowski [134, 135], Tapia [138]) and the primal-dual merit function (see, e.g. Gill and Robinson [76, 77], Gill *et al.* [81], Armand and Omheni [12]). Some softwares have been implemented based on the augmented Lagrangian method to solve problem (1.1). MINOS [115] used the projected augmented Lagrangian method which involves a sequence of sparse, linearly constrained subproblems. The objective functions of these subproblems include a modified Lagrangian term and a modified quadratic penalty function. It is most successful in problems with nonlinear objective and linear or near-linear constraints. On the other hand, LANCELOT [45] is more effective on problems with relatively few constraints. It uses a gradient projection method with trust region to solve bound-constrained nonlinear subproblems

$$\min_{x \in \mathbb{R}^n} \mathcal{L}_{\lambda,\sigma}(x) \quad \text{subject to} \quad l \leq x \leq u.$$

This type of subproblems is also addressed by ALGENCAN [3, 5]. However, it uses the line search method for the globalization. PENNON [106] is based on an augmented Lagrangian approach and can solve nonlinear optimization problems with the semi-definite matrix constraints. In this software, a combination of the line search and trust region method is used to solve a sequence of unconstrained optimization problem in which the inequality constraints is treated by the barrier function. SPDOPT-AL [12] combined the Newton-type method and a line search strategy.

We note that these augmented Lagrangian methods were studied mainly for feasible optimization problems. Recently, Martínez and Prudente [112] modified the algorithm in ALGENCAN [3] by changing the convergence tolerances for subproblems. They defined an adaptive stopping criterion for subproblems depending on the constraints, the penalty parameter and the multipliers. The algorithm will return a notification of infeasibility when the penalty parameter becomes very small. The numerical examples show that the new modification is better than the original ALGENCAN. Birgin *et al.* [25] give a necessary and sufficient condition to characterize the infeasibility of a problem. The tolerance sequence is also chosen adaptively. In particular, this sequence depends on the constraints, the penalty parameter and the multipliers. The numerical experiments demonstrate the advantages of this new algorithm compared to the original augmented Lagrangian method [23] in the infeasible case. Gonçalves *el al.* [83] introduced an algorithm based on the general class of Kort-Bertsekas Lagrangian function [21, 105]. Similarly to [25], a necessary and sufficient condition to indicate the infeasibility of problem is also given. From some numerical comparisons on feasible and infeasible problems, three types of augmented Lagrangian function

$$L(x, \lambda, \sigma) = f(x) + \frac{1}{\sigma} \sum_{i=1}^{m} W(\sigma c_i(x), \lambda_i),$$

where

$$W(s,t) = \begin{cases} t(e^s - 1) + \frac{1}{3}[\max\{0, s\}]^3, \\ \begin{cases} ts + \cosh(s) - 1 & \text{if } t + \sinh(s) \geq 0, \\ t\sinh^{-1}(-t) + \cosh(\sinh^{-1}(-t)) - 1 & \text{otherwise,} \end{cases} \\ \begin{cases} ts + ts^2 + s^3 & \text{if } s \geq 0, \\ ts/(1-s) & \text{otherwise.} \end{cases} \end{cases},$$

are as efficients as the Powell-Hestenes-Rockafellar (PHR) augmented Lagrangian function (1.4) for feasible instances. Moreover, they detect infeasibility in less iterations than PHR. Birgin *et al.* [26] modified ALGENCAN in the way to update the penalty parameter. Instead of using the minimum of the inequality constraints and its multipliers, this new algorithm uses their products to measure complementarity. It converges to the minimizers of a feasibility measure. The numerical results show that the performances of ALGENCAN and this new algorithm are nearly the same. The augmented Lagrangian method of Armand and Omheni [12] may detect infeasibility when the sequence of dual variables becomes unbounded and the penalty parameter is forced to zero. The main drawback is that the infeasibility can take a long time to be detected.

Under some classical assumptions, the local convergence results of augmented Lagrangian method in the feasible problems have been demonstrated, see e.g. [12, 20, 21, 24, 31, 82, 123, 124]. The local behavior of augmented Lagrangian algorithms in infeasible problems has never been mentioned in literature. On the other hands, Hager [93], Wright [153, 157], Gill *el al.* [80] investigated the local behavior of augmented Lagrangian method without the conventional assumption related to the linearly independence of vectors of gradient constraints.

1.3 The interior point method

The (primal) interior point method was proposed by Frisch [73] in 1955 to solve the optimization problems with inequality constraints

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & c(x) \geq 0. \end{array} \tag{1.5}$$

The idea of this method is to solve a sequence of unconstrained problems

$$\min_{x \in \mathbb{R}^n} \quad P_\mu(x) := f(x) - \mu \sum_{i=1}^{m} \log c_i(x), \tag{1.6}$$

where the barrier parameter $\mu > 0$ is driven to zero. Fiacco and McCormick [64] showed that under some standard assumptions, a solution of problem (1.6) $x^*(\mu)$ converges to x^* and $\frac{\mu}{c(x^*(\mu))}$ converges to z^*, where x^* and z^* are optimal solution and optimal Lagrange multiplier associated to the constraints of the problem (1.5), respectively. This primal interior point method has fallen into disuse after the appearance of SQP methods and has not been recovered because it suffers of several drawbacks. One of them is due to the increasing of the ill conditioning associated with the minimization P_μ when μ approaches zero (see, e.g., [110, 114]). The success of the Karmarkar's algorithm [102] in solving the linear optimization problems [103] and the impressive computational performance of primal-dual interior point methods for linear programming [111] motivated the concentration of the optimization community in extending these methods to the general nonlinear problem (1.1) (see, e.g., [71, 84, 116]). In particular, by defining $z_i = \mu / c_i(x)$, the optimality conditions of problem (1.6) can be rewritten as

$$\begin{aligned} \nabla f(x) - \nabla c(x) z &= 0 \\ C(x) z - \mu e &= 0, \end{aligned} \tag{1.7}$$

where $C(x) = \mathrm{diag}(c(x))$. We note that this system is equivalent to the perturbed KKT conditions for problem (1.5) if we introduce nonnegative slack variables $s \in \mathbb{R}^m_{++}$ to get the following problem

$$\begin{array}{ll} \min\limits_{x,s} & f(x) - \mu \sum\limits_{i=1}^{m} \log s_i \\ \text{s.t.} & c(x) - s = 0. \end{array} \tag{1.8}$$

One of the most visible advantages of this formulation is that the starting point can be chosen more easily than in the original formulation (1.6).

The interior point methods were implemented in some softwares. LOQO [146] uses a line search strategy to obtain the global convergence. KNITRO [38] processes two strategies: a line search in which a linear system is solved by direct factorization and a trust region in which a linear system is solved by conjugate gradient method. IPOPT [149], on the other hand, invokes the filter technique for globalization. SPDOPT [13] is a combination of interior point method and augmented Lagrangian method with a line search strategy.

In literature, there are two approaches of interior point methods. The first approach called barrier-SQP method computes a search direction d_x by applying Newton's method directly to the KKT conditions of (1.8). After that, a step length α_k is determined to maintain the positivity of slack variables s. The second one called barrier-penalty method uses a classical way to treat equality constraints (the constraint-removal spirit of the 1960s). In particular, the equality constraints of (1.8) are eliminated and introduced as a penalty term of a composite function. Consequently, this approach tends to solve unconstrained minimization problem

$$\min_{x,s} \quad f(x) - \mu \sum_{i=1}^{p} \log s_i + \frac{1}{2\mu}\|c(x) - s\|^2.$$

Regardless of which approaches are used, a globalization strategy such as line search, trust region and filter (or a combination of these strategies) must be applied to ensure the global convergence. Wächter and Biegler [148] gave an example to show that barrier-SQP methods using line search (see, e.g., [1, 2, 7, 15, 60, 74, 108, 109, 113, 146, 159]) may converge to false solutions. In contrast, line search strategy applied to barrier-penalty methods is not affected by this failure, see, e.g., [8, 13, 18, 42, 70, 76, 77, 82, 139, 161]. Further discussions on interior point methods using trust region and filter strategies can be found in the literature, see, e.g., [34–36, 46, 47, 52, 75, 141, 150, 163] (trust region methods) and [19, 67, 69, 144, 149] (filter methods). Castro and Cuesta [41], Friedlander and Orban [72] considered the regularization techniques for interior point method in the framework of convex quadratic programs.

Byrd *et al.* [35], Chen and Goldfarb [42] mentioned about the infeasibility in their researches. The filter line search method of Wächter and Biegler [149] possesses a feasibility restoration phase. Besides finding a new acceptable iterate to the filter, the significant role of this phase is to detect (local) infeasibility. Recently, Curtis [49] proposed a method which is a combination of penalty method and interior point method to solve problem (1.5). Numerical results demonstrate that this approach maybe efficient to detect infeasibility. Nocedal *et al.* [118] presented a trust region interior point method with two phases: a main phase (solve the barrier problem (1.8) with $\mu \to 0$) and a feasibility phase (minimization the feasibility violation measure). Numerical experiments showed that the new method is better than the older one KNITRO/CG [38] in detecting infeasible problems and there is no loss robustness in solving feasible problems. The capability to detect infeasibility of SPDOPT [13] is closely related to the behaviors of the penalty parameter and of dual variables. In particular, an infeasible stationary point is declared by this algorithm if the sequence of dual variables tends to infinity and the penalty parameter converges to zero. Nevertheless, the fast infeasibility detection has not been studied in both of these algorithms. Very recently, Dai *et al.* [50] introduce a primal-dual interior point method with the fast convergence to a KKT point or to an infeasible stationary point.

Local convergence analyses of the primal-dual interior point methods were demonstrated under some standard assumptions in papers [9, 14, 59, 89, 90, 139, 160, 163]. These researches concentrated on demonstrating the superlinear and quadratic convergence or on studying the behavior of the primal-dual interior point methods in the neighborhood of the central path defined by (1.7). On the other hands, Ralph and Wright [127] showed the superlinear convergence of an algorithm for solving a monotone variational inequality under a constant-rank constraint qualification without any assumption about the uniqueness of the multipliers. In [128], they argued that the result in the previous paper still hold without the constant-rank condition. The research of Wright [154] can be seen as an extension of [127, 128] to general nonconvex nonlinear problems. In his work, the analysis was done under the MFCQ assumption. We note that all of these algorithms impose a centrality condition on the iterates. Vicente and Wright [147] proposed an algorithm with quadratic rate of convergence under a weaker constraint qualification (MFCQ) and the strict complementarity. The primal variables and their multipliers are modified as some components approach zero. On the other hand, the full step is permitted even if the non-negativity of constraints are not valid. Wright and Orban [158] showed the uniqueness of the local minimizer of the barrier problem (1.6) under assumptions MFCQ and strict complementarity. Without the latter, the distances between the minimizers and the solution of the problem (1.6) are estimated in terms of the barrier parameter μ. Another interior point method was introduced by Yamashita and Yabe [162] where the quadratic convergence was obtained by assuming the linear independence of gradients of equality constraints and the strict complementarity.

The whole content of this thesis is organized as follows. The current chapter is a general introduction of this dissertation. The next chapter recalls some basic backgrounds and notations which will be used throughout the thesis. Especially, an original proof of the first and the second order necessary optimality conditions based on penalty function will be given. Chapter 3 introduces a new augmented Lagrangian method for solving problem (1.2). In chapter 4, we develop the result in previous chapter to the general optimization problem (1.1) by taking into account the interior point method and the augmented Lagrangian method. Two next chapters are devoted to study the local properties of regularized methods in the absence of classical assumptions.

In addition to theoretical researches, we implemented algorithms in this dissertation in language C. In particular, the algorithms in Chapter 3 and Chapter 4 can be seen as the developments of SPDOPT [12, 13] with the aim of detecting infeasibility.

Chapter 2

Preliminaries

In this chapter, we introduced notations and concepts which will be used frequently in the next chapters. Some backgrounds related to real analysis, linear algebra and optimization problem are also recalled without specific demonstration, unless the proof is original. Finally, we mention a useful tool to compare the performances of several algorithms which will be helpful in Chapters 3 and 4.

2.1 Elementary notations and concepts

Throughout this dissertation, scalars and vectors are denoted by lowercase letters, matrices are denoted by capital letters and the capitalization of vector name indicates the diagonal matrix formed by placing elements of that vector on the diagonal, e.g., $X = \mathrm{diag}(x)$. For notational convenience, we often omit transpose notation and write $(x, y, z) = (x^\top, y^\top, z^\top)^\top$. The identity matrix is denoted by I and e stands for the vector of all ones with an arbitrary size. The ith component of a vector $x \in \mathbb{R}^n$ will be denoted by $[x]_i$ or x_i if there is no ambiguity. Vector inequalities are understood componentwise, for example $x \geq 0$ means that $x_i \geq 0$, for all $i = 1, \ldots, n$. The notations $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ respectively stand for the nonnegative and the positive orthants, i.e. $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x > 0\}$. The minimum and the maximum of vectors which have the same size are understood componentwise, i.e., $\min\{x, y\} = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})^\top$. Given two vectors $x, y \in \mathbb{R}^n$, their Euclidean scalar product is denoted by $x^\top y$ and is defined by

$$x^\top y = \sum_{i=1}^{n} x_i y_i.$$

We denote $x \perp y$, if $x^\top y = 0$. The associated l_2 norm is $\|x\| = (x^\top x)^{\frac{1}{2}}$. A property that holds for the Euclidean norm l_2 is the Cauchy-Schwarz inequality, which states that

$$|x^\top y| \leq \|x\| \|y\|.$$

The open Euclidean ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ is denoted by $B(x, r)$, that is

$$B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| < r\}.$$

The Hadamard product of two vectors x and y, denoted by $x \circ y$, is defined by

$$[x \circ y]_i = x_i y_i, \qquad \text{for all } i = 1, \ldots, n.$$

The positive part of a real vector x is a vector in $\mathbb{R}^n$ defined by $x_+ = \max\{x, 0\}$, where the maximum is understood componentwise.

For two nonnegative scalar sequences $\{a_k\}$ and $\{b_k\}$, we use the Landau symbols $a_k = \mathrm{O}(b_k)$ if there exists a constant $C > 0$ such that $a_k \leq C b_k$, for all $k \in \mathbb{N}$ and $a_k = \mathrm{o}(b_k)$ if there exists a sequence $\{\epsilon_k\}$ such that $\lim\limits_{k \to \infty} \epsilon_k = 0$ and $a_k \leq \epsilon_k b_k$ for all $k \in \mathbb{N}$. If $a_k = \mathrm{O}(b_k)$, we also write that $b_k = \Omega(a_k)$. Finally, we use the notation $a_k = \Theta(b_k)$ to indicate that $a_k = \mathrm{O}(b_k)$ and $b_k = \mathrm{O}(a_k)$.

For a function f and an iterate x_k, to simplify the notation we denote $f_k = f(x_k)$. Likewise, f^* stands for $f(x^*)$, and so on. The notation $g(x)$ stands for the gradient of f at x, while the transpose of the Jacobian of c at x is denoted by $A(x) = \nabla c(x)$. The Lagrangian associated to the problem (1.1) is defined by $\mathcal{L}(w) = f(x) + y^\top c(x) - z^\top x$, where $w = (x, y, z)$ is the vector of primal-dual variables and $(y, z) \in \mathbb{R}^{m+n}$ is the vector of Lagrange multipliers corresponding to the constraints. The notations $\nabla_x \mathcal{L}(\cdot)$ and $\nabla^2_{xx} \mathcal{L}(\cdot)$ are used to stand for the gradient and the Hessian matrix of $\mathcal{L}(\cdot)$ with respect to the primal variable x.

For a rectangular matrix $M \in \mathbb{R}^{m \times n}$, the induced matrix norm is defined by

$$\|M\| = \max\{\|Mx\| : \|x\| \leq 1\}.$$

Let M be a symmetric matrix. The smallest eigenvalue is denoted by $\lambda_{\min}(M)$. The notation $M \succ 0$ (resp., $\succeq 0$) stands for M positive definite (resp., M positive semidefinite). For two symmetric matrices A and B with the same size, we use the notation $A \succ B$ (resp., $A \succeq B$) to mean that $A - B \succ 0$ (resp., $A - B \succeq 0$). The inertia of a symmetric matrix M is the integer triple that indicates the numbers n_+, n_- and n_0 of positive, negative and zero eigenvalues of this matrix, respectively, that is

$$\mathrm{In}(M) = (n_+, n_-, n_0).$$

For a closed set $C \subset \mathbb{R}^n$, the distance from $x \in \mathbb{R}^n$ to C is defined by

$$d(x, C) = \min_{\xi \in C} \|x - \xi\|.$$

In some certain situations, the set is clearly defined by the context, so we frequently omit this set in the notation of distance function, i.e., $d(x) = d(x, C)$. In chapters 5 and 6, for a vector x, the notation $\bar{x}$ denotes any element of C such that $\|x - \bar{x}\| = d(x, C)$.

2.2 Linear Algebra

We recall a property related to the minimum eigenvalue of symmetric matrices.

Proposition 2.1. *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be two real symmetric matrices. Then,*

$$\lambda_{\min}(A) + \lambda_{\min}(B) \le \lambda_{\min}(A + B).$$

Proof. This result can be seen as a consequence of the interlacing property of eigenvalues, see, e.g. Horn [98, Theorem 4.3.1]. For the sake of completeness, we give a simple proof here. Let $u \in \mathbb{R}^n$ with $\|u\| = 1$ be an eigenvector of $A + B$ corresponding to the eigenvalue $\lambda := \lambda_{\min}(A + B)$, i.e., $(A + B)u = \lambda u$. It implies that $\lambda = u^\top (A + B)u$. By noting that for all $x \in \mathbb{R}^n$, one has

$$\lambda_{\min}(A)\|x\|^2 \le x^\top Ax \quad \text{and} \quad \lambda_{\min}(B)\|x\|^2 \le x^\top Bx.$$

These above facts imply that

$$\lambda_{\min}(A) + \lambda_{\min}(B) = (\lambda_{\min}(A) + \lambda_{\min}(B))\|u\|^2 \le u^\top (A + B)u = \lambda = \lambda_{\min}(A + B).$$

□

We have the following properties about the relation between the positive definiteness and inertia. The proof can be found in [70, Lemma 4.1], [51, Theorem 3].

Lemma 2.2. *Let $H \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $A \in \mathbb{R}^{n\times m}$ and $\delta > 0$. Define the matrices*

$$K = H + \frac{1}{\delta}AA^\top \quad \textit{and} \quad M = \begin{pmatrix} H & A \\ A^\top & -\delta I \end{pmatrix}.$$

(i) The matrix K is positive definite if and only if $In(M) = (n, m, 0)$.

(ii) For all $u \in \mathbb{R}^n \backslash \{0\}$ such that $A^\top u = 0$, $u^\top H u > 0$ if and only if there exists a number $\delta > 0$ such that the matrix K is positive definite.

A square real matrix Q is said to be orthogonal if

$$Q^\top Q = QQ^\top = I.$$

We recall spectral decomposition theorem of a symmetric matrix.

Proposition 2.3 (Spectral Decomposition). *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. There exist n eigenvalues $\lambda_1, \dots, \lambda_n$ in $\mathbb{R}$ and n eigenvectors $u_1, \dots, u_n$ in $\mathbb{R}^n$, such that*

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^\top = Q\Lambda Q^\top,$$

where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n\times n}$ and $Q = (u_1, \dots, u_n)$ is orthogonal.

We recall Singular Value Decomposition (SVD) theorem which will be useful.

Proposition 2.4 ([98],Theorem 7.3.2). *Let A be an $n \times m$ real matrix and let $r = rank(A)$. Assume that $A^\top A = V\Lambda V^\top$, in which V is an $m \times m$ orthogonal matrix, $\Lambda = \Sigma^2$, where $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r, 0_{m-r})$, 0_{m-r} is the zero vector of size $m - r$ and $\sigma_1 \geq \dots \geq \sigma_r > 0$. Let define $\Sigma_r = \operatorname{diag}(\sigma_1, \dots, \sigma_r) \in \mathbb{R}^{n\times n}$ and partition $V = \begin{pmatrix} V_1 & V_2 \end{pmatrix}$, in which $V_1 \in \mathbb{R}^{m\times r}$. Then there exists a partitioned orthogonal matrix $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$, where $U_1 \in \mathbb{R}^{n\times r}$ such that*

$$A = U\Sigma V^\top = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^\top \\ V_2^\top \end{pmatrix}.$$

We recall some results about the norm of the inverse matrix.

Proposition 2.5 ([53], Theorem 3.1.4). *Let A and B be two $n \times n$ real matrices. If A is nonsingular and $\|A^{-1}(B - A)\| < 1$, then B is nonsingular and*

$$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}(B - A)\|}$$

Proposition 2.6 ([98], Corollary 5.6.16)**.** *Let $A \in \mathbb{R}^{n\times n}$. If there exists a matrix norm $\|\cdot\|$ such that $\|I - A\| < 1$, then the matrix A is nonsingular and*

$$A^{-1} = \sum_{k=0}^{\infty} (I - A)^k.$$

The determinant of block matrices has the following property.

Proposition 2.7. *Let A, B, C, D be real matrices. If D is nonsingular, we then have*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(S) \det(D),$$

where $S = A - BD^{-1}C$ is the Schur complement of D.

Proof. The result follows from the following formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} S & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ C & D \end{pmatrix}.$$

□

The next important result is about the nonsingularity of a Jacobian matrix related to a bound constrained optimization problem and it will be useful in the local analysis of our algorithms.

Proposition 2.8. *Let $H \in \mathbb{R}^{n\times n}$ be symmetric, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ and*

$$M = \begin{pmatrix} H & -I \\ Z & X \end{pmatrix},$$

where $X = \operatorname{diag}(x)$ and $Z = \operatorname{diag}(z)$.

If $0 \leq x \perp z \geq 0$, $\min\{x_i + z_i : i = 1 \ldots, n\} > 0$ and for all $u \in \ker(Z)$, $u^\top Hu > 0$, then the matrix M is nonsingular.

Proof. A usual proof, see, e.g., [64], is to show that M is injective. We propose here another quite simple proof.

Without loss of generality, we may assume that $z = (z_1, 0)$, $x = (0, x_2)$ where $(z_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ and $(z_1, x_2) > 0$, for some $0 \leq p \leq n$. Using this partition, let us write the symmetric matrix H under the form

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where H_{11}, H_{22} are symmetric matrices and $H_{12} = H_{21}^\top$. Since $z_1 > 0$, it follows from the assumption that for all nonzero vector $u \in \mathbb{R}^{n-p}$,

$$u^\top H_{22} u = \begin{pmatrix} 0 & u^\top \end{pmatrix} H \begin{pmatrix} 0 \\ u \end{pmatrix} > 0,$$

which implies that $H_{22} \succ 0$. By making a permutation of the first and the third column blocks, we have

$$\begin{aligned}
|\det M| &= \left| \det \begin{pmatrix} H_{11} & H_{12} & -I & 0 \\ H_{21} & H_{22} & 0 & -I \\ Z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_2 \end{pmatrix} \right| \\
&= \left| \det \begin{pmatrix} -I & H_{12} & H_{11} & 0 \\ 0 & H_{22} & H_{21} & -I \\ 0 & 0 & Z_1 & 0 \\ 0 & 0 & 0 & X_2 \end{pmatrix} \right|, \\
&= \det(H_{22}) \det(Z_1) \det(X_2) \\
&> 0,
\end{aligned}$$

which implies the nonsingularity of the matrix M. $\square$

2.3 Real analysis

This section is devoted to some basic backgrounds of real analysis which will be used in studying the local behavior of algorithms. Firstly, we recall the notion of Lipschitz continuity.

Definition 2.9. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on a set $D \subset \mathbb{R}^n$ if there exists $L > 0$ such that for every $x, y \in D$,

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

The next Lemma is a direct consequence of the Lipschitz continuity.

Proposition 2.10 ([53], Lemma 4.1.12)**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function in the open convex set $D \subset \mathbb{R}^n$. Assume that the Jacobian* *matrix of F at w is Lipschitz continuous on D. Then, for any $w, w' \in D$,*

$$\|F(w') - F(w) - F'(w)(w' - w)\| \leq \frac{L}{2}\|w - w'\|^2.$$

The next result give us upper and lower bounds of $\|F(w) - F(w')\|$.

Proposition 2.11 ([53], Lemma 4.1.16)**.** *Under the assumptions of Proposition 2.10, assume that $F'(\bar{w})^{-1}$ exists for some $\bar{w} \in D$. Then, there exists $\varepsilon > 0$ and $0 < a_1 \leq a_2$ such that for all $w, w' \in B(\bar{w}, \varepsilon) \cap D$,*

$$a_1\|w - w'\| \leq \|F(w) - F(w')\| \leq a_2\|w - w'\|.$$

We now present a well-known result in multivariable calculus which permits us to represent some variables via some other ones.

Theorem 2.12 (Implicit Function Theorem)**.** *Let $F : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be a continuously differentiable function and a point $(x^*, y^*) \in \mathbb{R}^{n+m}$ satisfying $F(x^*, y^*) = 0$. If the Jacobian matrix $F'_x(x^*, y^*) = \left[\frac{\partial F_i}{\partial x_j}(x^*, y^*)\right]_{i,j}$ is nonsingular, then there exist positive constants δ, ε and a unique continuously differentiable function $\varphi : B(y^*, \varepsilon) \to \mathbb{R}^n$ such that for all $(x, y) \in B(x^*, \delta) \times B(y^*, \varepsilon)$,*

$$F(x, y) = 0 \quad \textit{if and only if} \quad y = \varphi(x).$$

Finally, we end this section with some definitions about the rate of convergence.

Definition 2.13 ([21] Rate of convergence)**.** Let $\{x_n\}$ be a sequence in $\mathbb{R}^n$ which converges to $x^* \in \mathbb{R}^n$. If there exist constants $p > 1$ such that $\|x_{k+1} - x^*\| = \mathrm{O}(\|x_k - x^*\|^p)$, then $\{x_k\}$ is said to converge to x^* at least superlinearly with order (or rate of) p. If $\|x_{k+1} - x^*\| = \Omega(\|x_k - x^*\|^p)$, then $\{x_k\}$ is said to converge to x^* at most superlinearly with order p. This sequence is said to converge superlinearly with order p, where $p > 1$, if it converges both at most and at least superlinearly with order p.

If $\|x_{k+1} - x^*\| = \mathrm{O}(\|x_k - x^*\|^2)$, then the convergence is said to be quadratic.

2.4 Optimization problem

Let us consider two kinds of optimization problem: the optimization problem with only equality constraints (1.2) and the general optimization problem (1.1). To simplify, we introduce notions related to the general optimization problem (1.1).

Similar concepts and definitions can be easily extended to (1.2).
The feasible set $\mathcal{F}$ of (1.1) is a set of points x that satisfy the constraints, i.e.,

$$\mathcal{F} = \{x \in \mathbb{R}^n : c(x) = 0 \text{ and } x \geq 0\}.$$

At a feasible point $x \in \mathcal{F}$, we define the set of active bound as

$$\mathcal{A}(x) = \{i : x_i = 0\}.$$

The linearized cone at a feasible point $x \in \mathcal{F}$ is given by

$$L(x) = \{d \in \mathbb{R}^n : \nabla c(x)^\top d = 0 \quad \text{and} \quad d_j \geq 0, \text{ for all } j \in \mathcal{A}(x)\}.$$

We recall the notion of constraint qualifications. These are sufficient conditions under which the linearized cone at a feasible point equals to the tangent cone at this point. We introduce here the two most famous constraint qualifications which will be used throughout this thesis.

Definition 2.14 (LICQ)**.** The linear independence constraint qualification is satisfied at a feasible point $x \in \mathbb{R}^n$ if the set of the gradients of active constraints $\{\nabla c_i(x) : i = 1, \ldots, m\} \cup \{e_i : i \in \mathcal{A}(x)\}$ is linearly independent.

Definition 2.15 (MFCQ)**.** The Mangasarian-Fromovitz constraint qualification is satisfied at a feasible point $x \in \mathbb{R}^n$ if the set of equality constraint gradients $\{\nabla c_i(x) : i = 1, \ldots, m\}$ is linearly independent and there exists a vector $d \in \mathbb{R}^n$ such that $\nabla c(x)^\top d = 0$ and $d_i > 0$ for all $i \in \mathcal{A}(x)$.

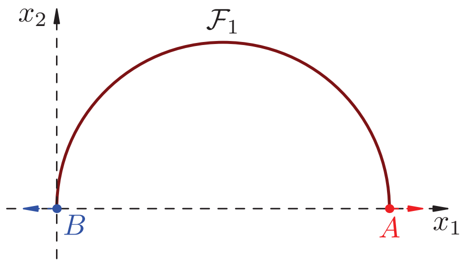
It is worth noting that the MFCQ holds at x^* if and only if there does not exist $(y, z) \neq 0$ such that $z_i \geq 0$, for all $i \in \mathcal{A}$ and

$$\nabla c(x)y + \sum_{j \in \mathcal{A}} z_j = 0.$$

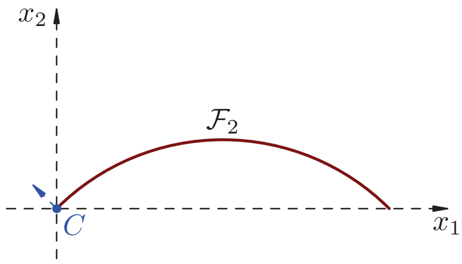
The LICQ is stronger than the MFCQ in the sense that the MFCQ can be implied by the LICQ. In the framework of equality constrained problem (1.2), these two constraint qualifications are equivalent and can be interpreted as full rank of the Jacobian matrix $\nabla c(x)$.

Example 2.16. We consider two feasible sets

$$\mathcal{F}_1 = \{x \in \mathbb{R}^2 | (x_1 - 1)^2 + x_2^2 - 1 = 0, x_1 \geq 0, x_2 \geq 0\}$$



(a) LICQ is satisfied at A. MFCQ is not valid at B.



(b) MFCQ holds at C, but LICQ does not hold at this point

Fig. 2.1: Illustration of LICQ and MFCQ

and

$$\mathcal{F}_2 = \{x \in \mathbb{R}^2 | (x_1 - 1)^2 + (x_2 + 1)^2 - 2 = 0, x_1 \geq 0, x_2 \geq 0\}.$$

The LICQ is valid at $A = (2, 0) \in \mathcal{F}_1$ and does not hold at $C = (0, 0) \in \mathcal{F}_2$. The MFCQ does not hold at $B = (0, 0) \in \mathcal{F}_1$, but it holds at $C = (0, 0) \in \mathcal{F}_2$.

We now give some definitions about local and global minimizers.

Definition 2.17. A point x^* is a *global minimizer* of the problem (1.1) if $x^* \in \mathcal{F}$ and $f(x^*) \leq f(x)$ for all $x \in \mathcal{F}$.
A point x^* is a *local minimizer* of the problem (1.1) if $x^* \in \mathcal{F}$ and there exists a positive number ε such that $f(x^*) \leq f(x)$ for all $x \in \mathcal{B}(x^*, \varepsilon) \cap \mathcal{F}$.

In practice, it is difficult to find global solutions. Hence, optimization algorithms aim to seek local ones. Throughout this thesis, the term "optimal solution" is meant to be a local minimizer. In numerical optimization, the aim is to solve the first order optimality conditions. A solution of these conditions is called a stationary point. We recall here some definitions of stationary points of the optimization problem (1.1).

Definition 2.18 (Fritz-John point)**.** A point $x \in \mathbb{R}^n$ is called a Fritz-John (FJ) point of problem (1.1) if there exists a nonzero vector $(u, y, z) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^n$ such that

$$ug(x) + A(x)y - z = 0, \quad c(x) = 0 \quad \text{and} \quad 0 \leq x \perp z \geq 0.$$

When we consider a FJ point in a relation with constraint qualifications, we have the following definitions

Definition 2.19 (KKT point)**.** A point $x \in \mathbb{R}^n$ is called a Karush-Kuhn-Tucker (KKT) point of problem (1.1) if there exists $(y, z) \in \mathbb{R}^{m+n}$ such that

$$g(x) + A(x)y - z = 0, \quad c(x) = 0 \quad \text{and} \quad 0 \leq x \perp z \geq 0. \tag{2.1}$$

Definition 2.20 (Singular stationary point)**.** A point $x \in \mathbb{R}^n$ is called a singular stationary point of problem (1.1) if there exists a nonzero vector $(y, z) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$A(x)y - z = 0, \quad c(x) = 0 \quad \text{and} \quad 0 \leq x \perp z \geq 0.$$

In other word, a singular stationary point of problem (1.1) is a feasible point at which the MFCQdoes not hold.

Definition 2.21 (Infeasible stationary point)**.** A point $x \in \mathbb{R}^n$ is called an infeasible stationary point of problem (1.1) if there exists $z \in \mathbb{R}^n$ such that

$$c(x) \neq 0, \quad A(x)c(x) - z = 0 \quad \text{and} \quad 0 \leq x \perp z \geq 0.$$

An infeasible stationary point is not feasible for problem (1.1) and is a stationary point of the feasibility problem

$$\begin{array}{ll} \min & \frac{1}{2}\|c(x)\|^2 \\ \text{s.t.} & x \geq 0. \end{array}$$

We introduce a condition related to the constant rank in a neighborhood of a minimizer which will be used to state the second order necessary conditions.

Definition 2.22 (WCR, [4])**.** The weak constant-rank (WCR) property holds at $x^* \in \mathcal{F}$, if the rank of the matrix $[\nabla c_1(x), \ldots, \nabla c_m(x), e_{i_1}, \ldots, e_{i_p}]$ is constant for all x in a neighborhood of x^*, where $\mathcal{A}(x^*) = \{i_1, \ldots, i_p\}$.

The next theorem states necessary conditions for x^* to be a local minimizer. The proof of the first order conditions is based on the use of a mixed-penalty function and was proposed by P. Armand in his Master's course. We propose here to complete the proof for the second order necessary conditions. Other proofs which used different kinds of penalty function without the log barrier term can be found in (see, e.g., of Bertsekas [22], Güler [92], Andreani *et al.* [4]).

Theorem 2.23 (Necessary optimality conditions)**.** *Suppose that x^* is a local minimizer of (1.1) and that the function f and c are twice continuously differentiable in a neighborhood of x^*. Then x^* is a Fritz-John point of this problem.*

Moreover, if MFCQ and WCR hold at x^*, *then there exists* $(y^*, z^*) \in \mathbb{R}^{m+n}$ *such that* $w^* = (x^*, y^*, z^*)$ *satisfies* (2.1) *and* $d^\top \nabla^2_{xx}\mathcal{L}(w^*)d \geq 0$, *for all* $d \in \mathbb{R}^n$ *satisfying* $\nabla c(x^*)^\top d = 0$ *and* $d_i = 0$ *for all* $i \in \mathcal{A}(x^*)$.

Proof. Let $x^* \in \mathcal{F}$ be a local minimum of (1.1). To simplify the notation, let us denote $\mathcal{A} = \mathcal{A}(x^*) = \{i_1, \ldots, i_p\}$ and $\mathcal{A}^c = \{1, \ldots, n\}\backslash\mathcal{A}$. Let us choose $\varepsilon \in (0, \min_{i\in\mathcal{A}^c} x_i^*)$, such that for all $x \in \mathcal{B} \cap \mathcal{F}$, $f(x^*) \leq f(x)$, where $\mathcal{B}$ is the closed ball in $\mathbb{R}^n$ of radius ε and center x^*.

Let us define $\mathcal{B}_+ = \{x \in \mathcal{B} : x > 0\}$. Let $k \in \mathbb{N}^*$ and define the quadratic barrier penalty function on $\mathcal{B}_+$ by

$$\varphi_k(x) = f(x) + \frac{k}{2}\|c(x)\|^2 - \frac{1}{k}\sum_{i=1}^{n}\log x_i + \frac{1}{4}\|x - x^*\|^4.$$

Let us show that φ_k has a global minimum over $\mathcal{B}_+$. Select an index $i \in \{1, \ldots, n\}$. For all $x \in \mathcal{B}_+$ we have

$$\varphi_k(x) \geq \bar{f} - (n-1)\log\gamma - \frac{1}{k}\log x_i,$$

where $\bar{f}$ is the minimum of f over $\mathcal{B}$ and γ is an upper bound on all the components of x over $\mathcal{B}_+$. It follows that $\varphi_k(x)$ tends to infinity when x approaches the boundary of the nonnegative orthant. Therefore, for all $r \in \mathbb{R}$ the sublevel set $\{x \in \mathcal{B}_+ : \varphi_k(x) \leq r\}$ is compact and thus φ_k has at least one global minimum $x_k \in \mathcal{B}_+$.

Let $\mathcal{K} \subset \mathbb{N}^*$ be such that the subsequence $\{x_k\}_\mathcal{K}$ tends to $\bar{x} \in \mathcal{B}$. Let us show that $\bar{x} = x^*$. For $k \in \mathbb{N}^*$, define $\tilde{x}_k$ by

$$[\tilde{x}_k]_i = \begin{cases} x_i^* & \text{if } i \in \mathcal{A}^c, \\ \frac{1}{k}[x_k]_i & \text{if } i \in \mathcal{A}. \end{cases}$$

The sequence $\{\tilde{x}_k\}$ converges to x^* and $\|\tilde{x}_k - x^*\| \leq \frac{1}{k}\|x_k - x^*\| \leq \frac{\varepsilon}{k}$ for all $k \in \mathbb{N}^*$, which implies that $\{\tilde{x}_k\} \subset \mathcal{B}_+$. Because x_k is a global minimum of φ_k over $\mathcal{B}_+$, we have $\varphi_k(x_k) \leq \varphi_k(\tilde{x}_k)$ for all $k \in \mathbb{N}^*$. The continuous differentiability of c implies that there exists $L > 0$ such that $\|c(x)\| = \|c(x) - c(x^*)\| \leq L\|x - x^*\|$, for all $x \in \mathcal{B}$. Using the two previous facts, the choice of ε and the upper bound γ, then defining the constants $p = |\mathcal{A}|$ and $M = 2\max\{\frac{L^2\varepsilon^2}{2} + \frac{\varepsilon^4}{4} + (n-p)\log\frac{\gamma}{\varepsilon}, p\}$, for all $k \geq 3$

we then have

$$\begin{aligned}
& f(x_k)+\frac{k}{2}\|c(x_k)\|^2+\frac{1}{4}\|x_k-x^*\|^4 \\
\leq\ & f(\tilde{x}_k)+\frac{k}{2}\|c(\tilde{x}_k)\|^2+\frac{1}{4}\|\tilde{x}_k-x^*\|^4-\frac{1}{k}\sum_{i\in\mathcal{A}^c}\log\frac{[\tilde{x}_k]_i}{[x_k]_i}-\frac{1}{k}\sum_{i\in\mathcal{A}}\log\frac{[\tilde{x}_k]_i}{[x_k]_i} \\
\leq\ & f(\tilde{x}_k)+\frac{L^2\varepsilon^2}{2k}+\frac{\varepsilon^4}{4k^4}+\frac{n-p}{k}\log\frac{\gamma}{\varepsilon}+\frac{p}{k}\log k \\
\leq\ & f(\tilde{x}_k)+\frac{M}{k}\log k,
\end{aligned}$$

where we use the inequality $1 \leq \log k$, for $k \geq 3$.

On one hand, we deduce that for all $k \geq 3$,

$$\|c(x_k)\|^2 \leq \frac{2}{k}(f(\tilde{x}_k)-f(x_k))+\frac{2M}{k^2}\log k.$$

By taking the limit $k \to \infty$ in $\mathcal{K}$, we deduce that $c(\bar{x})=0$. We also have $\bar{x} \geq 0$, because $x_k > 0$ for all $k \in \mathbb{N}^*$. We have proved that $\bar{x} \in \mathcal{B} \cap \mathcal{F}$, therefore $f(x^*) \leq f(\bar{x})$. On the other hand, we have

$$f(x_k)+\frac{1}{4}\|x_k-x^*\|^4 \leq f(\tilde{x}_k)+\frac{M}{k}\log k.$$

By taking the limit $k \to \infty$ in $\mathcal{K}$, we obtain $f(\bar{x})+\frac{1}{4}\|\bar{x}-x^*\|^4 \leq f(x^*) \leq f(\bar{x})$, and thus $\bar{x}=x^*$. Since this property holds for any limit point of the sequence $\{x_k\}$, the whole sequence converges to x^*.

Having proved that $\{x_k\} \to x^*$, the minimization of φ_k over $\mathcal{B}_+$ is an unconstrained minimization problem for k large enough. Then, there exists $k_0 \in \mathbb{N}^*$ such that for all $k \geq k_0$, $\nabla\varphi_k(x_k)=0$ and the matrix $\nabla^2\varphi_k(x_k)$ is positive semidefinite. The first order optimality conditions $\nabla\varphi_k(x_k)=0$ can be rewritten under the form

$$\begin{aligned}
\nabla f(x_k)+\nabla c(x_k)y_k-z_k+\|x_k-x^*\|^2(x_k-x^*) &= 0, && (2.2)\\
c(x_k) &= \frac{1}{k}y_k, && (2.3)\\
X_k z_k &= \frac{1}{k}e. && (2.4)
\end{aligned}$$

where we introduced the vectors $y_k = kc(x_k)$ and $z_k = \frac{1}{k}X_k^{-1}e$. Let us rewrite the equation (2.2) under the form $A_k v_k = 0$ where $A_k = \begin{pmatrix}\nabla f(x_k) & \nabla c(x_k) & -I & \|x_k-x^*\|^2(x_k-x^*)\end{pmatrix}$ and $v_k = \begin{pmatrix}1 & y_k^\top & z_k^\top & 1\end{pmatrix}^\top$. By dividing by $\|v_k\|$ and by taking the limit for some subsequence we obtain

$$u^*\nabla f(x^*) + \nabla c(x^*)y^* - z^* = 0$$

for some $(u^*, y^*, z^*) \in \mathbb{R}^{1+m+n}$. The equation (2.4) implies that $z_i^* = 0$ for all $i \in \mathcal{A}^c$. Therefore, x^* is a Fritz John point of the problem.

We now assume that MFCQ and WCR are satisfied at x^*. Let us show that the sequence $\{(y_k, z_k)\}$ is bounded. Indeed, if this sequence is unbounded, by dividing the both sides of equations (2.2) and (2.4) by $\|(y_k, z_k)\|$ and by taking the limit for some subsequence, we obtain

$$\nabla c(x^*)\bar{y} - \sum_{j\in\mathcal{A}} \bar{z}_j = 0 \quad \text{and} \quad \bar{z} \geq 0,$$

for some nonzero vector $(\bar{y}, \bar{z}) \in \mathbb{R}^{m+n}$. This implies that the MFCQ is not valid at x^*.

Because $\{(y_k, z_k)\}$ is bounded, there exists a subset $\mathcal{K} \subset \mathbb{N}$ such that $\lim_{k\in\mathcal{K}}(y_k, z_k) = (y^*, z^*)$ and (x^*, y^*, z^*) satisfies (2.1).

Let us now show that the second order necessary optimality conditions are satisfied at (x^*, y^*, z^*). We use the same kind of proof as in the paper of Andreani *et al.* [4]. Let $d \in \mathbb{R}^n$ such that

$$\nabla c_i(x^*)^\top d = 0, i = 1, \dots, m \quad \text{and} \quad d_j = 0, \text{ for all } j \in \mathcal{A}. \tag{2.5}$$

Since the MFCQ is valid at x^*, without loss of generality, we can assume that there exist vectors $e_{i_1}, \dots, e_{i_q}$ such that the set of vectors $\{\nabla c_1(x^*), \dots, \nabla c_m(x^*), e_{i_1}, \dots, e_{i_q}\}$ is linearly independent and $\text{rank}([\nabla c_1(x^*), \dots, \nabla c_m(x^*), e_{i_1}, \dots, e_{i_p}]) = m + q$, where $0 \leq q \leq p$. This fact and the continuity of the gradients implies that for all x in a neighborhood of x^* the set of vector $\{\nabla c_1(x), \dots, \nabla c_m(x), e_{i_1}, \dots, e_{i_q}\}$ is linearly independent. For all $x \in \mathbb{R}^n$, we define the matrix

$$M(x) = [\nabla c_1(x), \dots, \nabla c_m(x), e_{i_1}, \dots, e_{i_q}].$$

The WCR implies that for all $i = 1, \dots, m, j = 1, \dots, p$, the vectors $\nabla c_i(x)$ and e_{i_j} are linear combinations of the columns of $M(x)$ for all x in a neighborhood of

x^*. In particular, since $\{x_k\}$ converges to x^*, for all $j = q+1, \ldots, p$,

$$e_j = \sum_{i=1}^{m} [\alpha_k]_i \nabla c_i(x_k) + \sum_{i \in \mathcal{J}} [\alpha_k]_i e_i \tag{2.6}$$

for some $\alpha_k \in \mathbb{R}^{m+|\mathcal{J}|}$.

For each $k \in \mathbb{N}$, let d_k be the orthogonal projection of d onto the nullspace of $M(x_k)^\top$, i.e.,

$$d_k = (I - M(x_k)(M(x_k)^\top M(x_k))^{-1} M(x_k)^\top) d.$$

From the property of the orthogonal projection and (2.6), we get

$$\nabla c_i(x_k)^\top d_k = 0, i = 1, \ldots, m \quad \text{and} \quad [d_k]_j = 0, j \in \mathcal{A} \tag{2.7}$$

for every $k \in \mathbb{N}$. The convergence of $\{(x_k, z_k)\}_{k \in \mathcal{K}}$ to (x^*, z^*) gives us

$$\lim_{k \in \mathcal{K}} d_k = [I - M(x^*)(M(x^*)^\top M(x^*))^{-1} M(x^*)^\top] d = d,$$

where, the last equality comes from (2.5).

Reminding that x_k is an unconstrained minimum of the function φ_k for all $k \in \mathbb{N}$, we then have

$$\begin{aligned}
0 &\le d_k^\top \nabla^2 \varphi_k(x_k) d_k \\
&= d_k^\top \left(\nabla^2 f(x_k) + \sum_{i=1}^{m} k c_i(x_k) \nabla^2 c_i(x_k) \right) d_k \\
&\quad + k \|\nabla c(x_k)^\top d_k\|^2 + \frac{1}{k} \sum_{j=1}^{n} \frac{1}{[x_k]_j^2} [d_k]_j^2 + 2((x_k - x^*)^\top d_k)^2 + \|x_k - x^*\|^2 \|d_k\|^2 \\
&\le d_k^\top \left(\nabla^2 f(x_k) + \sum_{i=1}^{m} k c_i(x_k) \nabla^2 c_i(x_k) \right) d_k + \frac{1}{k} \sum_{j \notin \mathcal{A}} \frac{[d_k]_j^2}{[x_k]_j^2} + 3\|x_k - x^*\|^2 \|d_k\|^2,
\end{aligned}$$

where the last inequality comes from (2.7) and the Cauchy-Schwarz inequality. By taking the limit with noting that $\{(x_k, y_k, z_k)\}_{k \in \mathcal{K}'} \to (x^*, y^*, z^*)$, $\{d_k\}_{k \in \mathcal{K}'} \to d$ and $[x^*]_j > 0$ for $j \notin \mathcal{A}$, we obtain $d^\top \nabla^2_{xx} \mathcal{L}(x^*, y^*, z^*) d \ge 0$ which completes the proof. $\square$

Conversely, second order sufficient conditions (SOSCs) permits us to conclude that KKT point is a (strict) minimizer.

Definition 2.24 (SOSCs)**.** Let w^* satisfy KKT condition (2.1). The second order sufficient conditions hold at w^* if $u^\top \nabla^2_{xx} \mathcal{L}(w^*) u > 0$ for all $u \neq 0$ satisfying

$\nabla c(x^*)^\top u = 0$ and $u_i = 0$ for all $i \in \mathcal{A}$.

Let $\mathcal{N}_\mathcal{A}$ be a matrix whose columns created by a basis for the null space of $\nabla c(x^*)^\top$. The second-order necessary (sufficient) conditions can be expressed under the condition that the reduced Hessian matrix of Lagrangian $\mathcal{N}_\mathcal{A}^\top \nabla^2_{xx}\mathcal{L}(w^*)\mathcal{N}_\mathcal{A}$ is positive semidefinite (positive definite).

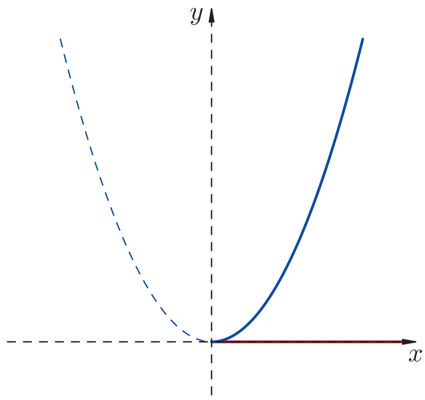


Fig. 2.2: Validity of SOSCs

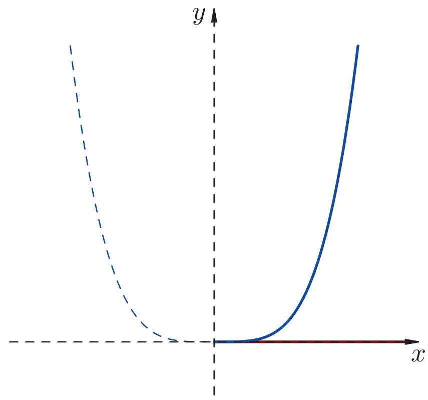


Fig. 2.3: Failure of SOSCs

Sometimes, the SOSCs are too restricted and hard to satisfy in a large number of optimization problems. In this case, the local convergence analysis can be performed under a local error bound condition. More details on this can found in Pang [120] which is a survey of the broad theory and rich applications of error bounds for inequality and optimization problems.

Definition 2.25. Let C be a nonempty closed subset in $\mathbb{R}^n$ and $\Phi : \mathbb{R}^n \to \mathbb{R}_+$ be a function satisfying $\Phi(x) = 0$ if and only if $x \in C$. The function Φ provides a local error bound of C at $x^* \in C$ if there exist $\kappa > 0, r > 0$ such that

$$\forall x \in B(x^*, r), \qquad d(x, C) \leq \kappa\Phi(x).$$

In local convergence analysis of algorithms, we also assume the satisfaction of the strict complementarity (SC) at a primal-dual solution w^*, i.e. $z_i^* > 0$ for all $i \in \mathcal{A}$. Formally, it is stated in the following definition.

Definition 2.26 (Strict Complementarity (SC))**.** Let w^* satisfy KKT conditions (2.1). The strict complementarity (SC) is satisfied at w^* if

$$\min\{x_i^* + z_i^* : i = 1, \ldots, n\} > 0.$$

2.5 Performance profiles

For the purpose of objectively comparing algorithms, we often use the performance profiles which proposed by Dolan and Moré [56]. Suppose that we have a set of solvers S and we want to compare their performances on a set of problem P in term of computing time (we can apply this comparision to other measures, e.g., the number of function evaluations, number of factorizations). For each problem p and solver s, we define $t_{p,s}$ as computing time required to solve problem p by solver s.

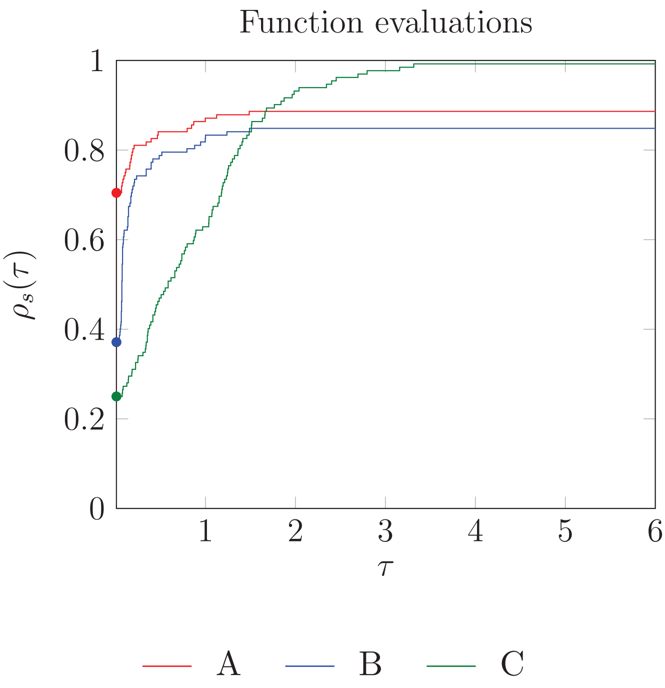


Fig. 2.4: Performance profile comparing three solvers on a set of 132 problems

The *performance ratio* for a problem p and solver s is defined by

$$r_{p,s} = \begin{cases} \dfrac{t_{p,s}}{\min\{t_{p,s} : s \in S\}}, & \text{if } p \text{ is solved by } s \\ +\infty & \text{otherwise.} \end{cases}$$

In other words, this is the ratio between the performance on problem p by solver s and the best performance by all solvers on this problem. For each solver s, we define

$$\rho_s(\tau) = \frac{1}{n_p}|\{p \in P : r_{p,s} \leq \tau\}|,$$

then $\rho_s(\tau)$ is the fraction of the test problems which are solved by the solver s

within a factor $\tau \in \mathbb{R}$ of the performance of the best solver. In this manuscript, we use the logarithmic scale for the $\tau-$axis to present ρ_s as a function of τ for each solver s. The leftmost and the rightmost values of the graph plotting the performance profiles of solvers give us the efficiency and the robustness of solvers, respectively. We say that a solver is effective if it takes less time to solve a given problem. If this solver succeeds in finding an optimal solution, then it is a robust solver for solving this problem. Figure 2.4 gives us the performance profile comparing the number of function evaluations of three solvers A, B and C on a set of 132 problems. We can see that A is the most efficient solver and C is the most robust solver.

Chapter 3

An Augmented Lagrangian method for equality constrained optimization with rapid infeasibility detection capabilities

In this chapter we concentrate on the rapid detection of the infeasibility in the framework of a solution of an equality constrained optimization problem (1.2) by means of a primal-dual augmented Lagrangian method. The new algorithm can be seen as an improvement of Armand and Omheni [12]. We propose to introduce a new parameter, called feasibility parameter, whose role is to control the progress of the iterates to the feasible set. This parameter scales the objective function relatively to the constraints until a nearly feasible point is detected. From a formal point of view, the algorithm can be interpreted as the numerical solution of the Fritz-John optimality conditions, but with a perturbation of the constraints due to the augmented Lagrangian parameters (Lagrange multiplier and quadratic penalty term). The feasibility parameter is updated dynamically. In particular, its value depends on the norm of the residual of a primal-dual system related to the minimization of the feasibility measure. This leads to a superlinear or quadratic convergence of the sequence of iterates to an infeasible stationary point. To our knowledge, this is the first local convergence result in the infeasible case of an augmented Lagrangian method.

3.1 Algorithm

We consider the equality constrained optimization problem

$$\text{minimize} \quad \rho f(x) \quad \text{subject to} \quad c(x) = 0, \tag{P$_\rho$}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$ are smooth, and where $\rho \geq 0$. For the value $\rho = 1$, the problem (P_1) is referred as the original problem (1.2). For the value $\rho = 0$, any feasible solution is optimal for (P_0). The parameter ρ is then called as the feasibility parameter.

The augmented Lagrangian associated with (P_ρ) is defined as

$$\mathcal{L}_{\rho,\sigma}(x, \lambda) := \rho f(x) + \lambda^\top c(x) + \tfrac{1}{2\sigma}\|c(x)\|^2, \tag{3.1}$$

where $\lambda \in \mathbb{R}^m$ is an estimate of the vector of Lagrange multipliers associated with the equality constraints and $\sigma > 0$ is a quadratic penalty parameter. Recall that when x^* is a KKT point for (P_ρ), with an associated vector of Lagrange multipliers λ^*, if the sufficient second order optimality conditions hold at x^*, then x^* is a strict local minimum of $\mathcal{L}_{\rho,\sigma}(\cdot, \lambda^*)$ provided that σ is small enough, see, e.g., [21, Proposition 1.26]. This result serves as a basis of augmented Lagrangian methods, in which the augmented Lagrangian is minimized while the parameters λ and σ are updated in an appropriate manner, see, e.g., [117, Chapter 17].

The first order optimality conditions for minimizing $\mathcal{L}_{\rho,\sigma}(\cdot, \lambda)$ are

$$\rho g(x) + A(x)\left(\lambda + \tfrac{1}{\sigma}c(x)\right) = 0.$$

By introducing the dual variable $y \in \mathbb{R}^m$ and the notation $w := (x, y)$, these optimality conditions can be reformulated as

$$\Phi(w, \lambda, \rho, \sigma) := \begin{pmatrix} \rho g(x) + A(x)y \\ c(x) + \sigma(\lambda - y) \end{pmatrix} = 0.$$

This formulation of the optimality conditions for minimizing (3.1) serves as a basis of our algorithm. Note that by setting $\lambda = y$, we retrieve the optimality conditions of problem (P_ρ).

Let us define the regularized Jacobian matrix of the function Φ with respect to w by

$$J_{\rho,\sigma,\theta}(w) = \begin{pmatrix} H_{\rho,\theta}(w) & A(x) \\ A(x)^\top & -\sigma I \end{pmatrix},$$

where $\theta \geq 0$ is a regularization parameter and where

$$H_{\rho,\theta}(w) = \rho\nabla^2 f(x) + \sum_{i=1}^{m} y_i \nabla^2 c_i(x) + \theta I$$

is the regularized Hessian of the Lagrangian associated with (P_ρ). During the iterations, the regularization parameter is chosen to control the inertia of regularized Jacobian matrix of Φ. It is well known that $\mathrm{In}(J_{\rho,\sigma,\theta}(w)) = (n, m, 0)$ if and only if the matrix

$$K_{\rho,\sigma,\theta}(w) := H_{\rho,\theta}(w) + \tfrac{1}{\sigma}A(x)A(x)^\top$$

is positive definite (see, e.g., Lemma 2.2). A link with the augmented Lagrangian is given by the following formula:

$$K_{\rho,\sigma,\theta}(w) = \nabla^2_{xx}\mathcal{L}_{\rho,\sigma}(x, y - \tfrac{1}{\sigma}c(x)) + \theta I.$$

The algorithm is a Newton-type method for the solution of the optimality system $\Phi = 0$ and it follows the one proposed in [12]. The globalization scheme of the algorithm uses two kinds of iteration. At a main iteration, called outer iteration, all the parameters λ, ρ and σ are updated and a full Newton step for the solution of $\Phi = 0$ is performed. If the norm of the residual $\|\Phi\|$ at the trial iterate is deemed sufficiently small, then the new iterate is updated and a new outer iteration is called, otherwise the parameters are fixed to their current values and a sequence of inner iterations is applied in order to reduce sufficiently $\|\Phi\|$. The inner iteration algorithm is a backtracking line search applied to a primal-dual merit function, whose first order optimality conditions correspond to $\Phi = 0$.

We now describe the outer iteration algorithm in detail. Initially, a starting point $w_0 = (x_0, y_0) \in \mathbb{R}^{n+m}$ is chosen, then we set $\lambda_0 = y_0$, choose $\rho_0 > 0$, $\sigma_0 > 0$ and three constants $a \in (0,1)$, $\ell \in \mathbb{N}$ and $\tau \in (0,1)$. The iteration counter is set to $k = 0$ and an additional index is set to $i_0 = 0$. Let $\mathtt{F}$ be a flag to indicate if the algorithm is in the feasibility detection phase or not. Initially the flag is set to $\mathtt{F} = 1$. A feasibility tolerance $\epsilon > 0$ is chosen.

The algorithm is quite similar to [12, Algorithm 1], except for the first four steps which are related to the updating of the parameters.

Initially, a primal-dual starting point is defined and the values of the parameters are choosen. A flag is used to indicate if the algorithm is in the feasibility detection phase ($\mathtt{F} = \mathtt{true}$) or not ($\mathtt{F} = \mathtt{false}$).

Algorithm 1 (Outer iteration)

0. Initialize $w_0 = (x_0, y_0) \in \mathbb{R}^{n+m}$ and set $\lambda_0 = y_0$. Choose parameters $\rho_0 > 0$, $\sigma_0 > 0$, $\epsilon > 0$, $a \in]0,1[$, $\ell \in \mathbb{N}$ and $\tau \in]0,1[$. Set $k = 0$ and $i_0 = 0$. Set $\mathtt{F} = \mathtt{1}$.

1. If $\|c_k\| \leq \epsilon$, then set $\mathtt{F} = 0$.

2. Choose $\zeta_k > 0$ such that $\{\zeta_k\} \to 0$. If $k = 0$ or
$$\|c_k\| \leq a \max\{\|c_{i_j}\| \colon (k-\ell)_+ \leq j \leq k\} + \zeta_k \tag{3.2}$$
then set $i_{k+1} = k$ and go to Step 4, otherwise set $i_{k+1} = i_k$.

3. If $\mathtt{F} = 1$, then choose $0 < \rho_{k+1} \leq \tau\rho_k$ and set $\sigma_{k+1} = \sigma_k$, else choose $0 < \sigma_{k+1} \leq \tau\sigma_k$ and set $\rho_{k+1} = \rho_k$. Set $\lambda_{k+1} = \frac{\rho_{k+1}}{\rho_k}\lambda_k$ and go to Step 5.

4. Choose $0 < \sigma_{k+1} \leq \sigma_k$. Set $\rho_{k+1} = \rho_k$ and $\lambda_{k+1} = y_k$.

5. Choose the regularization parameter $\theta_k \geq 0$ such that $\mathrm{In}(J_k) = (n, m, 0)$, where $J_k = J_{\rho_{k+1},\sigma_{k+1},\theta_k}(w_k)$. Compute w_k^+ by solving the linear system
$$J_k(w_k^+ - w_k) = -\Phi(w_k, \lambda_{k+1}, \rho_{k+1}, \sigma_{k+1}).$$

6. Choose $\varepsilon_k > 0$ such that $\{\varepsilon_k\} \to 0$. If
$$\|\Phi(w_k^+, \lambda_{k+1}, \rho_{k+1}, \sigma_{k+1})\| \leq \varepsilon_k, \tag{3.3}$$
then set $w_{k+1} = w_k^+$. Otherwise, apply a sequence of inner iterations to find w_{k+1} such that
$$\|\Phi(w_{k+1}, \lambda_{k+1}, \rho_{k+1}, \sigma_{k+1})\| \leq \varepsilon_k. \tag{3.4}$$

7. If termination criteria hold for (P_ρ) then stop, else increment k by 1 and go to Step 1.

At the first step, the algorithm tests if a nearly feasible point, with regards to a feasibility tolerance $\epsilon > 0$, has been detected. If it is the case, the algorithm switches into the normal operating mode of [12, Algorithm 1]. This means in particular that the feasibility parameter ρ_k will remain constant for all further iterations.

This switching mechanism is necessary to avoid the undesirable situation where the feasibility measure goes to zero very slowly, while the condition (3.2) is alternatively satisfied and not satisfied an infinite number of times, leading to decreasing the feasibility parameter to zero. Moreover, in this situation, it would be impossible to make the distinction between the satisfaction of the KKT conditions and the regularity of the constraints.

At the second step, the algorithm tests if a sufficient reduction of the feasibility measure has been obtained. If it is the case, the feasibility parameter is kept constant, the Lagrange multiplier estimate is set to the current value of the dual variable and a new value of the quadratic penalty parameter is chosen. For $k \geq 1$, the index i_k is the number of the last iteration prior to k at which inequality (3.2) holds. Note that, at Step 4, the quadratic penalty parameter is chosen in such a way that it could remain constant all along the iterations. But in that case, the convergence to a KKT point is only linear and the numerical experiments in [12] have shown that, in practice, it is better to force the convergence of σ_k to zero.

If the algorithm detects that the constraints have not decreased sufficiently, because condition (3.2) is not satisfied, then there are two situations. If $\mathtt{F} = 1$, then the algorithm is still in the feasibility detection phase. In that case, the feasibility parameter is sufficiently decreased, the quadratic penalty parameter is kept constant and the Lagrange multiplier estimate is rescaled. This scaling is important to force the convergence to zero of $\{\lambda_k\}$ when this step is always executed from some iteration (see Lemma 3.1-(ii)), ensuring that the sequence of iterates approaches stationarity of the feasibility problem (see Theorem 3.3-(ii)). The second situation is when $\mathtt{F} = 0$. In that case the algorithm has left the feasibility detection phase. Then the feasibility parameter is kept constant, but the quadratic penalty parameter is decreased to penalize the constraints and the Lagrange multiplier estimate is kept constant as in a classical augmented Lagrangian algorithm.

The following lemma summarizes the behavior of the algorithm regarding the feasibility detection strategy and the update rules of the parameters.

Lemma 3.1. *Assume that Algorithm 1 generates an infinite sequence $\{w_k\}$. Let $\mathcal{K} \subset \mathbb{N}$ be the set of iteration indices for which the condition (3.2) is satisfied.*

(i) If $\mathcal{K}$ is infinite, then the subsequence $\{c_k\}_{k\in\mathcal{K}}$ converges to zero and $\{\rho_k\}$ is eventually constant.

(ii) If $\mathcal{K}$ is finite, then $\liminf \|c_k\| > 0$ and both sequences $\{\sigma_k\rho_k\}$ and $\{\sigma_k\lambda_k\}$ converge to zero.

Proof. For $k \in \mathbb{N}$, set $\beta_k = \|c_{i_k}\|$. We then have for all $k \in \mathcal{K}$,

$$\beta_{k+1} \leq a \max\{\beta_j : (k-\ell)_+ \leq j \leq k\} + \zeta_k$$

and for all $k \notin \mathcal{K}$, $\beta_{k+1} = \beta_k$. It has been shown in [12, Lemma 3.1] that such a sequence converges to zero. This proves the first conclusion of assertion (i). Since $\{c_k\}_{k\in\mathcal{K}}$ converges to zero, then there exists $k_0 \in \mathcal{K}$ such that $\|c_{k_0}\| \leq \epsilon$ and thus $\mathtt{F} = 0$ for all further iterations. The update rules of the feasibility parameter at Step 3 and Step 4 imply that $\rho_k = \rho_{k_0}$ for all $k \geq k_0$, which proves the second conclusion of assertion (i).

To prove conclusion (ii), suppose that $\mathcal{K}$ is finite and let $k_0 = \max \mathcal{K}$. For all $k \geq k_0+1$, $i_k = k_0$ and Step 3 is executed. It follows that for all $k \geq k_0+\ell$, we have $\|c_k\| > a\|c_{k_0}\|$, therefore $\liminf \|c_k\| > 0$. We consider two cases. If at some iteration k, $\|c_k\| \leq \epsilon$, then $\mathtt{F} = 0$ for all further iterations. The update of the parameters at Step 3 implies that both sequences $\{\rho_k\}$ and $\{\lambda_k\}$ are eventually constant and $\{\sigma_k\}$ tends to zero. It follows that $\{\sigma_k\rho_k\}$ and $\{\sigma_k\lambda_k\}$ tend to zero. The second case is when $\|c_k\| > \epsilon$ for all $k \in \mathbb{N}$, which implies that $\mathtt{F} = 1$ at each iteration. In that case, for all $k \geq k_0 + 1$, $\rho_{k+1} \leq \tau\rho_k$, $\sigma_{k+1} = \sigma_k$ and $\lambda_{k+1} = \frac{\rho_{k+1}}{\rho_{k_0}} y_{k_0}$. We deduce that $\{\rho_k\}$ goes to zero, $\{\sigma_k\}$ is eventually constant and $\{\lambda_k\}$ goes to zero, which implies that both sequences $\{\sigma_k\rho_k\}$ and $\{\sigma_k\lambda_k\}$ tend to zero. $\square$

At Step 5 of Algorithm 1, the parameter θ_k is selected to control the inertia of the matrix J_k. This issue is important to avoid that the outer iterates converge to a stationary point which is not a local minimum, see [15].

At Step 6, a tolerance $\varepsilon_k > 0$ is chosen to check if a sufficient reduction of the norm of the optimality conditions at the candidate iterate w_k^+ has been obtained. An example of choice of ε_k is detailed in Section 3.4. If the residual norm is not smaller than this tolerance, then a sequence of inner iterations is called to compute the new iterate.

The inner iteration algorithm consists of a minimization procedure of the primal-dual merit function defined by

$$\varphi_{\lambda,\rho,\sigma,\nu}(w) = \mathcal{L}_{\rho,\sigma}(x,\lambda) + \frac{\nu}{2\sigma}\|c(x) + \sigma(\lambda - y)\|^2,$$

where $\nu > 0$ is a scaling parameter. It is easy to see that w is a stationary point of this function if and only if $\Phi(w, \lambda, \rho, \sigma) = 0$. This primal-dual merit function was first introduced by Robinson [129] and Gill and Robinson [76] in framework of SQP methods. It also used successfully in framework of the quadratic penalty method [16] and the augmented Lagrangian method [12]. The minimization procedure is a backtracking line search algorithm quite similar to [12, Algorithm 2]. The only difference is that in our description of Algorithm 1, the quadratic parameter

σ_{k+1} is kept constant during the inner iterations, while in [12] it can be increased. This choice has no impact from a theoretical point of view and simplifies the presentation of the algorithm. In our numerical experiments, the value of the quadratic penalty parameter is also kept constant during the inner iterations. We now describe the detail of the inner algorithm. We set the initial values $w^0 = w_k^+$ and choose constants $\nu > 0, \omega \in (0,1)$. We fix $\lambda = \lambda_{k+1}, \rho = \rho_{k+1}, \sigma = \sigma_{k+1}$ during the inner iterations.

Algorithm 2 Inner iteration

1. If $\|\Phi(w^i, \lambda, \rho, \sigma)\| \leq \varepsilon_k$, then set $w_{k+1} = w^i$ and return to Algorithm 1.
2. Choose $\theta^i \geq 0$ such that $\mathrm{In}(J^i) = (n, m, 0)$, where $J^i = J_{\rho,\sigma,\theta^i}(w^i)$, and solve the system $J^i d^i = -\Phi(w^i, \lambda, \rho, \sigma)$ to compute the direction d^i.
3. Starting from $\alpha^i = 1$, perform a backtracking line-search to find $\alpha \in (0,1]$ such that
$$\varphi_{\lambda,\rho,\sigma,\nu}(w^i + \alpha^i d^i) \leq \varphi_{\lambda,\rho,\sigma,\nu}(w^i) + \alpha^i \omega \nabla \varphi_{\lambda,\rho,\sigma,\nu}(w^i)^\top d^i.$$
Set $w^{i+1} = w^i + \alpha^i d^i$.

3.2 Global convergence analysis

Firstly, we have the following theorem which is similar to [12, Theorem 2.3]. It shows that if the function f is bounded from below and if the gradient of the constraints and the regularized Hessian of the Lagrangian stay bounded during the inner iterations, then the iterate w_{k+1} can be computed in a finite number of inner iterations.

Theorem 3.2. *Suppose that an infinite sequence $\{w^i\}$ is generated by Algorithm 2. Assume also that the sequences $\{A(w^i)\}$ and $\{H_{\rho,\theta^i}(w^i)\}$ are bounded and that the matrices $K_{\rho,\sigma,\theta^i}(w^i)$ are uniformly positive definite for $i \in \mathbb{N}$. Then, either the function value f^i goes to $-\infty$ or a subsequence of $\{\Phi(w^i, \lambda, \rho, \sigma)\}$ goes to zero.*

In view of this result, it will be assumed that the inner iteration algorithm succeeds in a finite number of iterations in finding w_{k+1} each time it is called at Step 6 of Algorithm 1.

Theorem 3.3. *Assume that Algorithm 1 generates an infinite sequence $\{w_k\}$. Assume also that the sequence $\{(g_k, A_k)\}$ is bounded. In any case, the iterates* *approach feasible or infeasible stationarity of problem* (P_1). *More precisely, let* $\mathcal{K} \subset \mathbb{N}$ *be the set of iteration indices for which the condition* (3.2) *is satisfied. Then, at least one of the following situations occurs.*

(i) If $\mathcal{K}$ *is infinite, then the subsequence* $\{c_k\}_{\mathcal{K}}$ *tends to zero. In addition, if* $\{y_k\}_{\mathcal{K}}$ *is bounded, then the sequence* $\{(g_k, A_k)\}$ *has a limit point* $(\bar{g}, \bar{A})$ *such that* $\bar{g} + \bar{A}\bar{y} = 0$ *for some* $\bar{y} \in \mathbb{R}^m$. *If* $\{y_k\}_{\mathcal{K}}$ *is unbounded, then* $\{A_k\}$ *has a limit point* $\bar{A}$ *which is rank deficient.*

(ii) If $\mathcal{K}$ *is finite, then* $\{\|c_k\|\}$ *is bounded away from zero and* $\{A_k c_k\}$ *tends to zero.*

Proof. First note that the convergence to zero of the sequence $\{\rho_k g_k + A_k y_k\}$ is a direct consequence of Step 6 of Algorithm 1.

Let us prove outcome (i). Lemma 3.1-(i) implies that $\lim_{\mathcal{K}} c_k = 0$ and $\{\rho_k\}$ is eventually constant. If $\{y_k\}_{\mathcal{K}}$ is bounded, then the assumptions imply that the whole sequence $\{(g_k, A_k, y_k/\rho_k)\}_{\mathcal{K}}$ is bounded and so has a limit point $(\bar{g}, \bar{A}, \bar{y})$ such that $\bar{g} + \bar{A}\bar{y} = 0$, which proves the first part of outcome (i). Suppose now that $\{y_k\}_{\mathcal{K}}$ is unbounded.There exists $\mathcal{K}' \subset \mathcal{K}$ such that $y_k \neq 0$ for all $k \in \mathcal{K}'$ and $\lim_{\mathcal{K}'} \|y_k\| = \infty$. For $k \in \mathcal{K}'$, we have

$$\|A_k u_k\| \leq \tfrac{1}{\|y_k\|} \|\rho_k g_k + A_k y_k\| + \tfrac{\rho_k}{\|y_k\|} \|g_k\|,$$

where $u_k = y_k/\|y_k\|$. Because $\{(A_k, u_k)\}_{\mathcal{K}'}$ is bounded, this sequence has a limit point $(\bar{A}, \bar{u})$, with $\bar{u} \neq 0$. By taking the limit in the previous inequality, knowing that the two terms of the right-hand side tend to zero, we deduce that $\bar{A}\bar{u} = 0$, which proves the second part of outcome (i).

For outcome (ii), suppose that $\mathcal{K}$ is finite. Lemma 3.1-(ii) implies that $\{\|c_k\|\}$ is bounded away from zero and $\{\sigma_k \rho_k, \sigma_k \lambda_k\}$ tends to zero. For all $k \in \mathbb{N}$, we have

$$A_k c_k = A_k(c_k + \sigma_k(\lambda_k - y_k)) - \sigma_k A_k \lambda_k + \sigma_k(\rho_k g_k + A_k y_k) - \sigma_k \rho_k g_k.$$

By taking the norm on both sides, for all k we have

$$\begin{aligned} \|A_k c_k\| &\leq \|A_k\| \|c_k + \sigma_k(\lambda_k - y_k)\| + \sigma_k \|A_k\| \|\lambda_k\| + \sigma_k \|\rho_k g_k + A_k y_k\| + \sigma_k \rho_k \|g_k\| \\ &\leq \max\{\|A_k\|, \sigma_k, \|g_k\|\}(2\|\Phi(w_k, \lambda_k, y_k, \sigma_k)\| + \sigma_k \|\lambda_k\| + \sigma_k \rho_k). \end{aligned}$$

Because the first term of the right-hand side of this inequality is bounded above and all the terms in the parenthesis tend to zero, we have $\lim A_k c_k = 0$, which concludes the proof. □

To sum up, the next result shows the behavior of the algorithm when the sequence of primal iterates remains bounded, a usual and mild assumption in a global convergence analysis.

Theorem 3.4. *Assume that Algorithm 1 generates an infinite sequence $\{w_k\}$ such that the sequence $\{x_k\}$ lies in a compact set.*

(i) Any feasible limit point of the sequence $\{x_k\}$ is a Fritz-John point of problem (P_1).

(ii) If the sequence $\{x_k\}$ has no feasible limit point, then any limit point is an infeasible stationary point of problem (P_1).

Proof. The compactness assumption implies that the sequences $\{g_k\}$ and $\{A_k\}$ are bounded and so Theorem 3.3 applies.

Let $\bar{x}$ be a limit point of $\{x_k\}$ such that $\bar{c} = 0$. From Lemma 3.1-(ii) we have that the condition (3.2) is satisfied an infinite number of times. It follows from Lemma 3.1-(i) that there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\rho_k = \rho_{k_0}$. Let $\mathcal{J} \subset \mathbb{N}$ such that the subsequence $\{x_k\}_{\mathcal{J}}$ tends to $\bar{x}$. Step 6 of Algorithm 1 implies that the sequence $\{\rho_{k_0} g_k + A_k y_k\}$ tends to zero. Dividing by $\|(\rho_{k_0}, y_k)\|$ and because $\rho_{k_0} \neq 0$, we have

$$\lim_{\substack{k \to \infty \\ k \in \mathcal{J}}} \frac{\rho_{k_0} g_k + A_k y_k}{\|(\rho_{k_0}, y_k)\|} = 0.$$

By compactness, the sequence $\{(\rho_{k_0}, y_k)/\|(\rho_{k_0}, y_k)\|\}_{\mathcal{J}}$ has a limit point $(\bar{\rho}, \bar{y})$, such that $\|(\bar{\rho}, \bar{y})\| = 1$ and $\bar{\rho}\bar{g} + \bar{A}\bar{y} = 0$, which proves assertion (i).

Suppose now that $\{x_k\}$ has no feasible limit point. From Lemma 3.1-(i) we have that the condition (3.2) is only satisfied a finite number of times. Theorem 3.3-(ii) implies that $\bar{A}\bar{c} = 0$ for any limit point $\bar{x}$ of $\{x_k\}$, which proves assertion (ii). □

3.3 Asymptotic analysis

3.3.1 Asymptotic behavior near the KKT point

In this section, it is assumed that Algorithm 1 generates a convergent sequence $\{w_k\}$ to a primal-dual solution $w^* := (x^*, y^*) \in \mathbb{R}^{n+m}$ of the problem (1.2). In this case, because $\{c_k\}$ converges to zero, the feasibility parameter becomes constant after a finite number of iterations and the algorithm is reduced to Algorithm 1 in

[12] applied to the solution of problem (P_ρ) with a fixed value of ρ. For a fixed parameter $\rho \geq 0$ and $w := (x, y) \in \mathbb{R}^n$, we define

$$\bar{F}(w) = \begin{pmatrix} \rho g(x) + A(x)y \\ g(x) \end{pmatrix}.$$

We need some assumptions below in this section.

Assumption 3.1. The sequence $\{w_k\}$ converges to w^* and $\{\sigma_k\} \to 0$.

Assumption 3.2. The functions f and c are twice continuously differentiable and their second derivatives are Lipschitz continuous over an open neighborhood of x^*.

Assumption 3.3. The Jacobian matrix $\nabla c(x^*)$ is of full row rank.

Assumption 3.4. The second order sufficient conditions hold at w^*, i.e. for all $u \in \mathbb{R}^n$, if $u \neq 0$ and $A(x^*)^\top u = 0$, then $u^\top \nabla^2_{xx}\mathcal{L}_\rho(w^*)u > 0$, where $\nabla^2_{xx}\mathcal{L}_\rho(w) = \rho\nabla^2 f(x) + \sum_{i=1}^m y\nabla^2 c_i(x)$.

We now state the main result of this section which gives the conditions on the choice of parameters in Algorithm 1 to get a quadratic convergence rate.

Theorem 3.5. *Under Assumption 3.1- 3.4, if the parameters of Algorithm 1 are chosen such that* $\zeta_k = \Omega(\sigma_k), \sigma_{k+1} = \Theta(\|\bar{F}(w_k)\|)$ *and* $\varepsilon_k = \Omega(\sigma_{k+1})$*, then for* $k \in \mathbb{N}$ *large enough,* $w_{k+1} = w_k^+, \lambda_{k+1} = y_k$ *and* $\|w_{k+1} - w^*\| = O(\|w_k - w^*\|^2)$.

The proof of this theorem is similar to the one in [12, Theorem 4.5] with noting that the feasibility parameter becomes constant after a finite number of iterations and the algorithm is reduced to Algorithm 1 in [12] applied to the solution of problem (P_ρ) with a fixed value of ρ.

3.3.2 Asymptotic behavior near the infeasible stationary point

In this section, we assume that Algorithm 1 generates a convergent sequence $\{x_k\}$ to an infeasible stationary point.

Assumption 3.5. Algorithm 1 generates an infinite sequence $\{w_k\}$ which converges to $w^* = (x^*, y^*) \in \mathbb{R}^{n+m}$, where x^* is an infeasible stationary point of problem (P_1)).

This assumption is very usual for the analysis of the rate of convergence of a numerical optimization algorithm. Note that it is equivalent to assume that $\{x_k\}$ converges to an infeasible stationary point x^* and the algorithm always stays in the feasibility detection phase, i.e., $\mathtt{F} = 1$ for all iteration. Indeed, by Lemma 3.1-(i), the convergence of $\{x_k\}$ to an infeasible stationary point x^* implies that the condition (3.2) is satisfied for a finite number of iterations. In that case, by Lemma 3.1-(ii), the sequence $\{\sigma_k\lambda_k\}$ tends to zero. Step 3 of Algorithm 1 implies that if $\mathtt{F} = 1$ at any iteration, then $\{\sigma_k\}$ is eventually constant and on the contrary, if $\mathtt{F} = 1$ at some iteration, then $\{\sigma_k\} \to 0$. Since $\{c_k + \sigma_k(\lambda_k - y_k)\}$ tends to zero, the sequence $\{\sigma_k y_k\}$ tends to $c^* \neq 0$ and thus $\{y_k\}$ has a finite limit if and only if $\mathtt{F}$ keeps the value 1 at all the iterations. This indicates that the choice of the value of the feasibility tolerance ϵ is an important issue related to the behavior of the algorithm. In practice, ϵ is chosen equal to, or smaller than, the stopping tolerance of the overall algorithm.

Lemma 3.6. *Under Assumption 3.5, the inequality (3.2) is satisfied a finite number of times, the sequence $\{\rho_k\}$ converges to zero, $\{\sigma_k\}$ is eventually constant and $\|\lambda_k\| = \mathrm{O}(\rho_k)$.*

Proof. Assumption 3.5 implies that $\{c_k\}$ converges to a non-zero value. Therefore, by virtue of Lemma 3.1-(i), the inequality (3.2) is satisfied only a finite number of times. It follows that Step 3 of Algorithm 1 is always executed for k sufficiently large and that $\mathtt{F} = 1$ for all iteration. Indeed, for all $k \in \mathbb{N}$ we have

$$\|c_k\| \leq \|c_k + \sigma_k(\lambda_k - y_k)\| + \sigma_k\|\lambda_k\| + \sigma_k\|y_k\|.$$

Step 6 and Lemma 3.1-(ii) imply that the first two terms of the right-hand side of the inequality tend to zero. Because $\{y_k\}$ is supposed to be convergent, we deduce that the sequence $\{\sigma_k\}$ does not converge to zero, which implies that $\mathtt{F} = 1$ for all iteration. Therefore, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\rho_k \leq \tau^{k-k_0}\rho_{k_0}$, $\sigma_k = \sigma_{k_0}$ and $\lambda_k/\rho_k = \lambda_{k_0}/\rho_{k_0}$, the conclusion follows. □

Let $\sigma > 0$ be the limit value of $\{\sigma_k\}$. For $w = (x, y) \in \mathbb{R}^{n+m}$, let us define

$$F(w) = \begin{pmatrix} A(x)y \\ c(x) - \sigma y \end{pmatrix}. \tag{3.5}$$

We have $\lim \Phi(w_k, \lambda_k, \rho_k, \sigma_k) = \Phi(w^*, 0, 0, \sigma) = F(w^*)$, therefore $y^* = \frac{1}{\sigma}c^*$.

Assumption 3.6. The function f and c are twice continuously differentiable and their second derivatives are Lipschitz continuous over an open neighborhood of x^*.

The Hessian matrix of the function $\frac{1}{2}\|c\|^2$ is defined as

$$C := \textstyle\sum_i c_i \nabla^2 c_i + AA^\top.$$

Assumption 3.7. The sufficient second order optimality conditions hold at x^* for the feasibility problem $\min_{x\in\mathbb{R}^n} \frac{1}{2}\|c(x)\|^2$, i.e., the matrix C^* is positive definite.

The following lemma is a direct consequence of these assumptions.

Lemma 3.7. *Under Assumptions 3.6 and 3.7, there exist a neighborhood W of w^*, $M > 0$, $L > 0$ and $0 < a_1 \le a_2$ such that for all $w, w' \in W$ we have*

(i) $||F'(w)^{-1}|| \le M$,

(ii) $||F(w') - F(w) - F'(w)(w' - w)|| \le \frac{L}{2}\|w - w'\|^2$,

(iii) $a_1\|w - w'\| \le \|F(w) - F(w')\| \le a_2\|w - w'\|$.

Proof. To prove (i) it suffices to show that $F'(w^*)$ is nonsingular. By using the fact that $y^* = \frac{1}{\sigma}c^*$, we have

$$F'(w^*) = \begin{pmatrix} \frac{1}{\sigma}\sum_i c_i^* \nabla^2 c_i^* & A^* \\ A^{*\top} & -\sigma I \end{pmatrix}.$$

By virtue of Proposition 2.7, the matrix $F'(w^*)$ is nonsingular if and only if the matrix $\frac{1}{\sigma}C^*$, the Schur complement of $-\sigma I$ of the matrix $F'(w^*)$, is positive definite. Thus, item (i) follows from Assumption 3.7. Assumption 3.6 implies that F' is Lipschitz continuous on W with the Lipschitz constant L. Property (ii) then follows from the Lipschitz continuity of F' and from Proposition 2.10. The assertion (iii) follows from Proposition 2.11. □

The next lemma shows that the matrix J_k used at Step 5 of Algorithm 1 is a good approximation of the Jacobian matrix of F at w_k when the feasibility parameter goes to zero.

Lemma 3.8. *Under Assumptions 3.5-3.7, there exists $\beta > 0$ such that for all $k \in \mathbb{N}$ large enough,*

$$\|J_k - F_k'\| \le \beta\rho_{k+1} \quad \textit{and} \quad \|J_k^{-1}\| \le 2M,$$

where M is defined by Lemma 3.7.

Proof. From the definition of J_k, for all $k \in \mathbb{N}$ we have

$$\|J_k - F'_k\| = \|\rho_{k+1}\nabla^2 f_k + \theta_k I\|.$$

Since $\{x_k\}$ converges to x^* and f is assumed to be twice continuously differentiable in a neighborhood of x^*, the first inequality will be proved if we show that $\theta_k = 0$ for k large enough. This is the case if $\mathrm{In}(J_k) = (n, m, 0)$ or, equivalently, if $K_{\rho_{k+1},\sigma,0}(w_k)$ is positive definite for k large enough. For all $k \in \mathbb{N}$ we have

$$\begin{aligned} K_{\rho_{k+1},\sigma,0}(w_k) &= H_{\rho_{k+1},0}(w_k) + \tfrac{1}{\sigma}A_k A_k^\top \\ &= \tfrac{1}{\sigma}C^* + \tfrac{1}{\sigma}(C_k - C^*) + H_{\rho_{k+1},0}(x_k, y_k - \tfrac{1}{\sigma}c_k) \end{aligned}$$

By assumption C^* is positive definite and the two other matrices tend to zero when k tends to infinity. It follows that $K_{\rho_{k+1},\sigma,0}(w_k)$ is positive definite for k large enough, which proves the first inequality.

Using Lemma 3.7-(i), the inequality just proved and the fact that $\{\rho_k\}$ tends to zero, for k large enough we have

$$\begin{aligned} \|{F'}_k^{-1}(J_k - F'_k)\| &\le \|{F'}_k^{-1}\|\|J_k - F'_k\| \\ &\le M\beta\rho_{k+1} \\ &\le \tfrac{1}{2}. \end{aligned}$$

By applying Proposition 2.5 with $A = F'_k$ and $B = J_k$, we then obtain the second inequality. □

The last lemma gives an estimate of the distance of the Newton iterate w_k^+ to the solution w^*.

Lemma 3.9. *Assume that Assumptions 3.5-3.7 hold. The sequence of iterates generated by Algorithm 1 satisfies*

$$\|w_k^+ - w^*\| = \mathrm{O}(\|w_k - w^*\|^2) + \mathrm{O}(\rho_{k+1}).$$

Proof. Let $k \in \mathbb{N}$. From the definition of the trial iterate w_k^+ at Step 5 of Algorithm 1, we have

$$\begin{aligned} w_k^+ - w^* &= w_k - w^* - J_k^{-1}\Phi(w_k, \lambda_{k+1}, \rho_{k+1}, \sigma) \\ &= J_k^{-1}((J_k - F'_k)(w_k - w^*) + F'_k(w_k - w^*) - F_k \\ &\quad + F_k - \Phi(w_k, \lambda_{k+1}, \rho_{k+1}, \sigma)). \end{aligned}$$

By using $F^* = 0$, by taking the norm on both sides, then by applying Lemma 3.8, Lemma 3.7-(ii), finally by using the convergence of $\{w_k\}$ to w^*, the boundedness of $\{g_k\}$ and $\|\lambda_k\| = \mathrm{O}(\rho_k)$ from Lemma 3.6, we obtain

$$\begin{aligned}
&\|w_k^+ - w^*\| \\
&\quad \le \ \|J_k^{-1}\|(\|J_k - F_k'\|\|w_k - w^*\| + \|F^* - F_k - F_k'(w^* - w_k)\| \\
&\qquad\quad + \|F_k - \Phi(w_k, \lambda_{k+1}, \rho_{k+1}, \sigma)\|) \\
&\quad \le \ 2M(\beta\rho_{k+1}\|w_k - w^*\| + \tfrac{L}{2}\|w_k - w^*\|^2 + \rho_{k+1}\|g_k\| + \sigma\|\lambda_{k+1}\|) \\
&\quad = \ \mathrm{O}(\rho_{k+1}) + \mathrm{O}(\|w_k - w^*\|^2),
\end{aligned}$$

which concludes the proof. □

We now state the main result of this section. This theorem shows the rapid rate of convergence of Algorithm 1 in the infeasible case. In addition, under a suitable choice of the parameters, there is no need of inner iterations for k large enough. In this case, the cost of the one iteration of the algorithm is reduced to the solution of the linear system at Step 5, which can be done with $\mathrm{O}((n+m)^3)$ arithmetic operations.

Theorem 3.10. *Assume that Assumptions 3.5-3.7 hold. Let $t \in (0,2]$. If the feasibility parameter of Algorithm 1 is chosen so that $\rho_{k+1} = \mathrm{O}(\|F_k\|^t)$, then*

$$\|w_{k+1} - w^*\| = \mathrm{O}(\|w_k - w^*\|^t). \tag{3.6}$$

In addition, if $\rho_{k+1} = \Theta(\|F_k\|^t)$ and if $\varepsilon_k = \Omega(\rho_k^{t'})$ for $0 < t' < t$, then for k large enough there is no inner iterations, i.e., $w_{k+1} = w_k^+$.

Proof. The assumption on the value of ρ_{k+1} and the Lipschitz property of F from Lemma 3.7-(iii) imply that

$$\rho_{k+1} = \mathrm{O}(\|w_k - w^*\|^t). \tag{3.7}$$

Using this estimate in Lemma 3.9, we deduce that

$$\|w_k^+ - w^*\| = \mathrm{O}(\|w_k - w^*\|^t). \tag{3.8}$$

At Step 6 of Algorithm 1, we have either $w_{k+1} = w_k^+$ or w_{k+1} is computed by means of the inner iterations. In the first case, it is clear that (3.6) follows from (3.8). Suppose now that the second case holds, i.e., the inequality (3.3) is not satisfied at iteration k. We then have

$$\|\Phi(w_{k+1}, \lambda_{k+1}, \rho_{k+1}, \sigma)\| \leq \varepsilon_k < \|\Phi(w_k^+, \lambda_{k+1}, \rho_{k+1}, \sigma)\|. \tag{3.9}$$

From (3.8), the sequence $\{w_k^+\}$ tends to w^*, therefore $\{g_k^+\}$ is bounded. Using the second inequality of Lemma 3.7-(iii) and Lemma 3.6, then (3.7) and (3.8), we deduce that

$$\begin{aligned}
\|\Phi(w_k^+, \lambda_{k+1}, \rho_{k+1}, \sigma)\| &\leq \|F_k^+ - F^*\| + \rho_{k+1}\|g_k^+\| + \sigma\|\lambda_{k+1}\| \\
&= \mathrm{O}(\|w_k^+ - w^*\|) + \mathrm{O}(\rho_{k+1}) \\
&= \mathrm{O}(\|w_k - w^*\|^t).
\end{aligned} \tag{3.10}$$

Combining (3.9) and(3.10) we obtain

$$\|\Phi(w_{k+1}, \lambda_{k+1}, \rho_{k+1}, \sigma)\| = \mathrm{O}(\|w_k - w^*\|^t).$$

Finally, from the first inequality of Lemma 3.7-(iii), the last estimate, the boundedness of $\{g_k\}$, Lemma 3.6 and the estimate (3.7), we have

$$\begin{aligned}
a_1\|w_{k+1} - w^*\| &\leq \|F_{k+1} - F^*\| \\
&= \|F_{k+1}\| \\
&\leq \|\Phi(w_{k+1}, \lambda_{k+1}, \rho_{k+1}, \sigma)\| + \rho_{k+1}\|g_{k+1}\| + \sigma\|\lambda_{k+1}\| \\
&= \mathrm{O}(\|w_k - w^*\|^t) + \mathrm{O}(\rho_{k+1}) \\
&= \mathrm{O}(\|w_k - w^*\|^t),
\end{aligned}$$

which proves (3.6).

Let us now prove the second assertion of the theorem. On one hand, Lemma 3.7-(iii) and (3.6) imply that $\|F_{k+1}\| = \mathrm{O}(\|F_k\|^t)$. By assumption, we have $\rho_{k+1} = \Theta(\|F_k\|^t)$, thus $\rho_{k+1} = \mathrm{O}(\rho_k^t)$. Since $t' < t$, we then have $\rho_{k+1} = \mathrm{o}(\rho_k^{t'})$. On the other hand, the estimate (3.10) implies that

$$\|\Phi(w_k^+, \lambda_{k+1}, \rho_{k+1}, \sigma)\| = \mathrm{O}(\|F_k\|^t) = \mathrm{O}(\rho_{k+1}).$$

By assumption, $\varepsilon_k = \Omega(\rho_k^{t'})$, therefore for k large enough, the inequality (3.3) is satisfied. □

3.4 Numerical experiments

Our algorithm is called SPDOPT-ID (Strongly Primal-Dual Optimization with Infeasibility Detection) and has been implemented in C. The performances of SPDOPT-ID are compared with those of SPDOPT-AL [12] on a set of 130 *standard* problems from the CUTEr collection [91]. The selected problems are those with equality constraints and a solution time less than 300 seconds. To create a second set of 130 *infeasible* problems, the constraint $c_1^2 + 1 = 0$, where c_1 is the first component of c, has been added to each problem. Note that the addition of this new constraint leads to a twofold difficulty. Indeed, not only the constraints are infeasible, but their gradients are linearly dependent.

We also compare the condition used to update the parameters in Step 2 of Algorithm 1, with the one used in [12, Algorithm 1]. The algorithm called SPDOPT-IDOld, is Algorithm 1, but with the inequality (3.2) which is replaced by

$$\|c_k\| \leq a \max\{\|c_{i_j}\| + \zeta_{i_j} : (k - \ell)_+ \leq j \leq k\}. \tag{3.11}$$

We will show that this modification is of importance when solving an infeasible problem and that the use of (3.2) in place of (3.11), leads to better numerical performances.

The feasibility parameter is initially set to $\rho_0 = 1$. When $\mathtt{F} = 1$, the feasibility parameter in Step 3 is updated by the formula

$$\rho_{k+1} = \min\{0.2\rho_k, 0.2\|F_k\|^2, 1/(k+1)\}.$$

The assumption on ρ_{k+1} in the statements of Theorem 3.10 are satisfied with $t = 2$. The rate of convergence of $\{w_k\}$ to w^* is then quadratic. A lower bound of 10^{-16} is applied on this parameter.

The parameters σ_k and θ_k are updated at Step 3, Step 4 and Step 5 as in [12, Algorithm 1]. In particular, for each $k \in \mathbb{N}$ we set

$$\sigma_{k+1} = \min\{\tau\sigma_k, \tau\|\widehat{F}_k\|, 1/(k+1)\},$$

where $\tau = 0.1$ at Step 3 and $\tau = 0.2$ at Step 4 and $\widehat{F}_k = \begin{pmatrix} g_k + A_k y_k \\ c_k \end{pmatrix}$.

To be able to solve a quadratic problem in only one iteration, we adopt the same procedure as in [16] for the choice of the starting point. Let $\bar{w} = (\bar{x}, \bar{y})$, where $\bar{x}$ is the default starting point and $\bar{y} = (1, \ldots, 1)^\top$. Initially, the following linear system $J_{1,0,0}(\bar{w})d = -\Phi(\bar{w},\bar{y},1,0)$ is solved. If the inequality $\|\Phi(\bar{w}+d,0,1,0)\|_\infty \leq \|\Phi(\bar{w},0,1,0)\|_\infty$ is satisfied, then $w_0 = \bar{w}+d$, otherwise $w_0 = \bar{w}$.

The algorithm is terminated and an optimal solution is declared to be found if $\|(g_k + A_k y_k/\rho_k, c_k)\|_\infty \leq \varepsilon_{\text{tol}}$ with $\varepsilon_{\text{tol}} = 10^{-8}$. Otherwise, if $\rho_k \leq \varepsilon_{\text{tol}}$, $\|c_k\| \geq \varepsilon_{\text{tol}}$ and $\|\Phi(w_k,0,0,\sigma_k)\|_\infty \leq \varepsilon_{\text{tol}}$, the algorithm returns a notification that an infeasible stationary point has been found. For SPDOPT-AL, we also add the stopping condition $\|c_k\| \geq \varepsilon_{\text{tol}}$, $\|A_k c_k\| \leq \varepsilon_{\text{tol}}$ and $\sigma_k \leq \varepsilon_{\text{tol}}$ to terminate this algorithm at an infeasible stationary point.

As mentioned in Section 3.3.2, the feasibility tolerance at Step 1 is set to $\epsilon = \varepsilon_{\text{tol}}$, to get a fast local convergence when the algorithm converges to an infeasible stationary point.

At Step 2 of Algorithm 1, we choose $a = 0.9, \ell = 2$ and $\zeta_k = 10\sigma_k\rho_k$ for all iteration k.

The sequence of tolerance $\{\varepsilon_k\}$ in Step 6 is defined by the following formula

$$\varepsilon_k = 0.9\max\{\|\Phi(w_i,\lambda_i,\rho_i,\sigma_i)\| \colon (k-4)_+ \leq i \leq k\} + \zeta_k.$$

The convergence to zero of the sequence $\{\varepsilon_k\}$ is a consequence of [16, Proposition 1]. This choice meets the requirements to get a fast convergence in both feasible case, i.e., $\varepsilon_k = \Omega(\sigma_{k+1})$, and in the infeasible case, i.e., $\varepsilon_k = \Omega(\rho_k^{t'})$, with $t' = 1$.

The symmetric indefinite factorization code MA57 [58] is used to factorize and regularize the matrix J_k. Since this factorization reveals the inertia of the matrix, the correction parameter θ_k, initially set to zero, is increased and a new factorization is performed until the inertia of J_k has the correct value.

The maximum number of iterations, counting both the inner and the outer iterations, is limited to 3000.

For the standard problems, only 129 problems solved by at least one of three algorithms are selected for the comparison purpose (problem `dixchlng` has not been solved). Figure 3.1 shows the performance profiles of Dolan and Moré [56] on the numbers of function and gradient evaluations. For $\tau \geq 0$, $\rho_s(\tau)$ is the fraction of test problems for which the performance of the solver s is within a factor 2^τ of the best one. These profiles show that the performances of the three algorithms are very similar, the difference is not significant. In term of robustness, the three algorithms solve successfully the same number of problems (128 problems). We can conclude that the infeasibility detection does not reduce the performances of the algorithm for solving standard problems.

Figure 3.2 shows the performances of these algorithms in terms of numbers

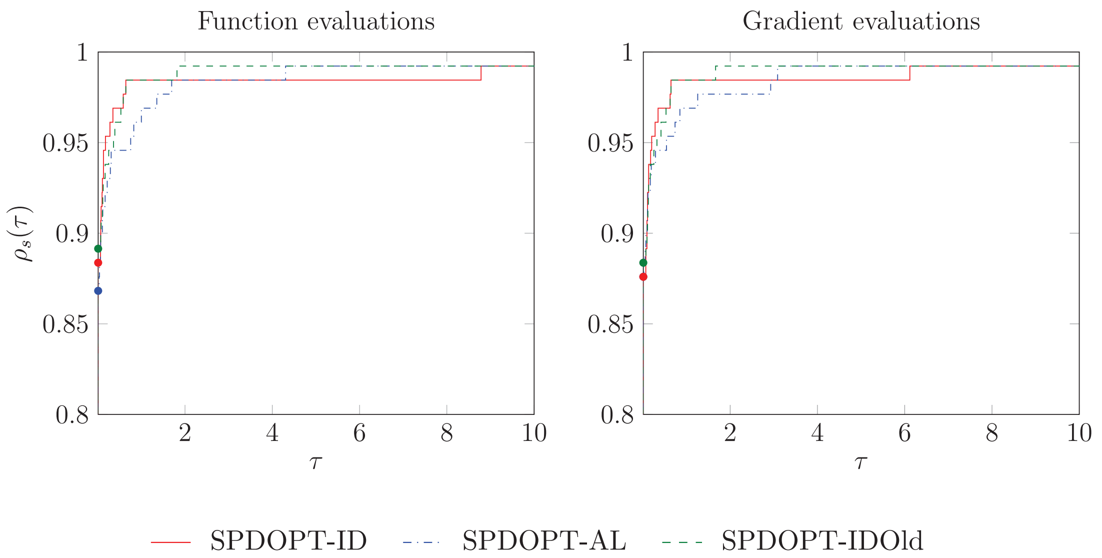


Fig. 3.1: Performance profiles comparing the three algorithms on the set of standard problems

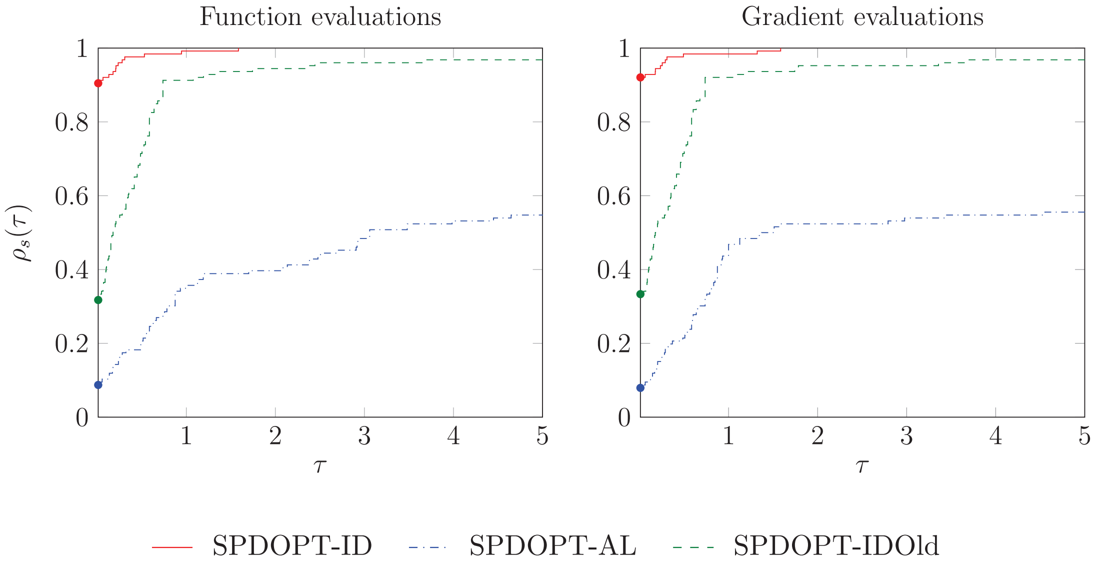


Fig. 3.2: Performance profiles comparing the three algorithms on the set of infeasible problems

of function and gradient evaluations on a set of 126 infeasible problems (the problems `gilbert, hager3, porous1, porous2` have been eliminated since three algorithms cannot detect the infeasibility). We observe that SPDOPT-ID is the most efficient algorithm for detecting infeasible problems, with an efficiency rate of approximately 90%. In any case, the efficiency of SPDOPT-ID and SPDOPT-IDOld is very significant comparing to SPDOPT-AL. In term of robustness, our two algorithms are more robust than SPDOPT-AL since they can detect more than 95% of problems, whereas SPDOPT-AL only detects less than 60%. This figure also shows that SPDOPT-ID is better comparing to SPDOPT-IDOld, justifying the choice of new criterion for updating parameters.

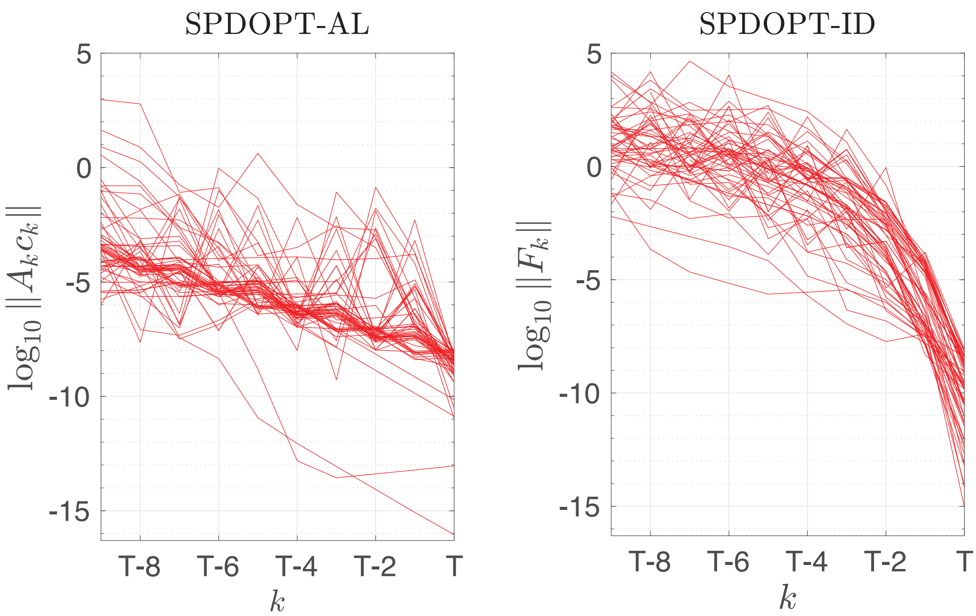


Fig. 3.3: Values of $\log_{10}\|A_k c_k\|$ and $\log_{10}\|F_k\|$ for the last ten iterations of SPDOPT-AL and SPDOPT-ID. T represents the index of the stopping iteration for each run.

We conclude this section by a comparison of a numerical estimate of the rate of convergence of the new algorithm SPDOPT-ID and of the original one SPDOPT-AL, when the sequence of iterates converges to an infeasible stationary point. We used a graphical representation inspired by [30]. We selected a set of 58 problems among the collection of infeasible problems, for which both algorithms generate a sequence converging to an infeasible stationary point. Figure 3.3 shows the last ten values of $\|A_k c_k\|$ for SPDOPT-AL and of $\|F_k\|$ for SPDOPT-ID. We cannot plot the values $\|F_k\|$ for SPDOPT-AL, because when the sequence of iterates converges to an infeasible stationary point, $\{\sigma_k\}$ goes to zero and $\{y_k\}$ becomes unbounded. Under some regularity assumptions, we obviously have $\|A_k c_k\| = \Theta(\|x_k - x^*\|)$

and $\|F_k\| = \Theta(\|w_k - w^*\|)$. These curves empirically show that there is a true improvement of the rate of convergence of the algorithm, from linear to quadratic.

For the solution of these infeasible problems, we observed that for the last four outer iterations, there is no inner iterations (i.e., $w_{k+1} = w_k^+$) for 90% of the problems. This percentage is more than 93% if one considers the last three outer iterations. For the infeasible problems for which SPDOPT-ID uses inner iterations at the last outer iterations, either the algorithm does not terminate successfully or the quadratic convergence is not observed. These observations confirm the asymptotic property of our algorithm.

Chapter 4

Rapid infeasibility detection in a mixed logarithmic barrier-augmented Lagrangian method for nonlinear optimization

This section is devoted to extend the approach in the previous chapter to the solution of general optimization problem with equality and inequality constraints. One possibility is to introduce slack variables to inequality constraints and apply augmented Lagrangian method in the case of simple bounds as in [3]. Armand and Omheni [13] proposed a nonlinear optimization algorithm, called SPDOPT, which is a mix of an interior point method and of an augmented Lagrangian method. The capability of this algorithm to detect infeasibility is closely related to the behaviors of the penalty parameter and of the dual variables. Nevertheless, a fast infeasibility detection has not been observed in practice or theoretically proved for this algorithm. In order to accelerate the infeasibility detection, the idea of Chapter 3 will be used to modify SPDOPT. More specifically, a new parameter, called the feasibility parameter, is introduced to balance the minimization of the barrier function and the realization of the equality constraints. If a nearly feasible point is detected, the feasibility parameter remains constant and our algorithm acts as the original algorithm. When the algorithm tends to an infeasible stationary point, the feasibility parameter acts as a barrier parameter. In this case, the exact solution of the perturbed system parametrized by the feasibility parameter defines a smooth trajectory. With a suitable rule of updating the feasibility parameter, the iterates tangentially follow this trajectory. Consequently, when the sequence of iterates converges to an infeasible stationary point, the algorithm can achieve a superlinear rate of convergence. To the best of our knowledge, this is the first local convergence analysis in the infeasible case related to interior point methods.

4.1 Algorithm

In this chapter, we consider the following nonlinear optimization problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f(x) \ \text{ subject to } c(x)=0 \text{ and } x \geq 0, \tag{4.1}$$

where $f:\mathbb{R}^n \to \mathbb{R}$ and $c:\mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable functions.

To the problem (4.1), we associate the mixed penalty function

$$\varphi_{\rho,\lambda,\sigma,\mu}(x) = \rho f(x) + \lambda^\top c(x) + \frac{1}{2\sigma}\|c(x)\|^2 - \rho\mu\sum_{i=1}^{n}\log x_i, 6 \tag{4.2}$$

where $\rho > 0$ is the *feasibility parameter*, $\lambda \in \mathbb{R}^m$ is an estimate of the vector of Lagrange multipliers associated with the equality constraints, $\sigma > 0$ is the *quadratic penalty parameter* and $\mu > 0$ is the *barrier parameter*. This penalty function is a mix of the augmented Lagrangian and of the logarithmic barrier function. It can be interpreted as the augmented Lagrangian associated with the log-barrier problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \rho\left(f(x) - \mu\sum_{i=1}^{n}\log x_i\right), \quad \text{subject to } c(x)=0. \tag{4.3}$$

The first order optimality conditions for the minimization of (4.2) are

$$\rho g(x) + A(x)(\lambda + \tfrac{1}{\sigma}c(x)) - \rho\mu X^{-1}e = 0. \tag{4.4}$$

By introducing the dual variables $y = \lambda + \frac{1}{\sigma}c(x)$ and $z = \rho\mu X^{-1}e$, the equation (4.4) is equivalently formulated as

$$\Phi(w,\lambda,\rho,\sigma,\mu) := \begin{pmatrix} \rho g(x) + A(x)y - z \\ c(x) + \sigma(\lambda - y) \\ XZe - \rho\mu e \end{pmatrix} = 0, \tag{4.5}$$

where $w := (x, y, z) \in \mathbb{R}^N$, with $N = n + m + n$. This primal-dual system of equations can be seen as a perturbation of the Fritz-John (FJ) optimality conditions of problem (4.1) (Definition 2.18). The equality constraints are perturbed thanks to the term $\sigma(\lambda - y)$. Note that when $\lambda = y$, this system is the primal-dual optimality system associated with the problem (4.3). Each complementarity product is perturbed thanks to the term $\rho\mu$. An important feature of our approach is that each time the parameters are updated, either ρ is reduced, or μ is reduced, but not both. We will see that, under some usual regularity assumptions, this allows to guarantee the superlinear convergence in case of convergence to a local minimum or to an infeasible stationary point.

The algorithm involves applying a Newton-type method for the solution of the system $\Phi = 0$, while updating the parameters along the iterations to control the convergence to a FJ point or to an infeasible stationary point. There are two kinds of iteration: outer and inner. At an outer iteration, the parameters are updated, while at an inner iteration the parameters are kept constant. At each iteration, outer or inner, a candidate iterate w^+ is computed as a solution of the following linear system:

$$J_{\rho,\theta,\delta}(w)(w^+ - w) = -\Phi(w, \lambda, \rho, \sigma, \mu)$$

where the matrix is defined as

$$J_{\rho,\theta,\delta}(w) := \begin{pmatrix} H_{\rho,\theta}(x, y) & A(x) & -I \\ A(x)^\top & -\delta I & 0 \\ Z & 0 & X \end{pmatrix},$$

with

$$H_{\rho,\theta}(x, y) := \rho \nabla^2 f(x) + \sum_{i=1}^{m} y_i \nabla^2 c_i(x) + \theta I.$$

The regularization parameters $\theta \geq 0$ and $\delta > 0$ are chosen such that the matrix

$$K_{\rho,\theta,\delta}(w) := H_{\rho,\theta}(x, y) + X^{-1}Z + \frac{1}{\delta} A(x)A(x)^\top$$

is positive definite. This property implies that not only the matrix $J_{\rho,\theta,\delta}(w)$ is nonsingular, but also that the solution of the linear system solved during the inner iterations is a descent direction for some merit function.

We now describe in detail the outer iteration algorithm. At the beginning, a starting point $w_0 = (x_0, y_0, z_0) \in \mathbb{R}^{2n+m}$ satisfying $v_0 = (x_0, z_0) > 0$ is chosen, then we set $\lambda_0 = y_0$. Besides, we choose $\sigma_0 > 0$, $\mu_0 > 0$, $\rho_0 = 1$ and four constants κ,

β, $\bar{\tau} \in (0,1)$ and $l \in \mathbb{N}$. The outer iteration counter is set to $k = 0$. A feasibility tolerance $\epsilon > 0$ must be chosen. A flag is used to indicate if the algorithm is in the feasibility detection phase ($\mathtt{F} = 1$) or not ($\mathtt{F} = 0$). Initially $\mathtt{F} = 1$, then this value is kept until a feasible, or nearly feasible, point has been detected. The index i_k stores the last iteration number prior to k at which inequality (4.6) is satisfied. It is initialized to $i_0 = 0$.

The algorithm can be seen as an extension of [13, Algorithm 1]. The first four steps are related to the updating of the parameters and are drawn from the feasibility detection algorithm for the equality constrained case proposed in Chapter 3.

The first step aims to detect a feasible or nearly feasible point. Whenever this is the case, the flag is set to $\mathtt{F} = 0$ and this value is kept for all further iterations. In this case, the algorithm is exactly the same as [13, Algorithm 1]. In particular, the sequence $\{\rho_k\}$ is eventually constant. It is worth noting that this switching mechanism is necessary to avoid the undesirable situation in which the condition (4.6) is alternatively satisfied and not satisfied an infinite number of times, for example when the feasibility measure decreases very slowly. In this case, it would be difficult to distinguish between a convergence to a KKT point or to a singular stationary point. Moreover, in practice, a point is deemed to be feasible if the norm of the constraints is smaller than some predefined tolerance. Suppose, for example, that we want to minimize x in $\mathbb{R}$, subject to the constraint $e^x \leq 0$. Most of well established softwares for nonlinear numerical optimization return a message like "optimal solution found" when solving this problem. Hence, it seems natural to state that the problem is feasible, whenever the feasibility tolerance is small enough.

Depending on the reduction of the feasibility measure, the trial values ρ_k^+ and μ_k^+ for the feasibility and barrier parameters, as well as the new values σ_{k+1} and λ_{k+1} for the augmented Lagrangian parameters, are chosen. If the inequality (4.6) is satisfied, the feasibility parameter is kept constant, new values of the barrier and quadratic penalty parameters are chosen, the Lagrange multiplier estimate is set to the current value of the dual variable. As shown in [12], from a theoretical point of view and in practice, it is better to force the convergence of $\{\sigma_k\}$ to zero to get a rapid rate of convergence. The primal-dual nature of our algorithm allows to avoid the difficulties related to tiny values of the quadratic penalty parameter, as in a classical penalty or augmented Lagrangian approach. On the other hand, if the condition (4.6) is not satisfied, then there are two situations. The first situation is when the algorithm is still in the feasibility detection phase ($\mathtt{F} = 1$). In that case,

Algorithm 3 (Outer iteration)

Initialize $w_0 = (x_0, y_0, z_0) \in \mathbb{R}^{2n+m}$ such that $v_0 = (x_0, z_0) > 0$. Choose parameters $\epsilon > 0$, $\kappa \in (0,1)$, $\beta \in (0,1)$, $\bar{\tau} \in (0,1)$, $l \in \mathbb{N}$, $\sigma_0 > 0$ and $\mu_0 > 0$. Set $\lambda_0 = y_0$, $\rho_0 = 1$, $k = 0$, $i_0 = 0$ and $\mathtt{F} = 1$.

1. If $\|c_k\| \leq \epsilon$, then set $\mathtt{F} = 0$.
2. Choose $\zeta_k > 0$ such that $\{\zeta_k\} \to 0$. If $k = 0$ or
$$\|c_k\| \leq \kappa \max\{\|c_{i_j}\| \colon (k-l)_+ \leq j \leq k\} + \zeta_k \tag{4.6}$$
then set $i_{k+1} = k$ and go to Step 4, otherwise set $i_{k+1} = i_k$.
3. If $\mathtt{F} = \mathtt{1}$, then choose $0 < \rho_k^+ \leq \beta\rho_k$ and set $\sigma_{k+1} = \sigma_k, \mu_k^+ = \mu_k, \lambda_{k+1} = \rho_k^+\lambda_k$, else set $\rho_k^+ = \rho_k$, choose $0 < \sigma_{k+1} \leq \beta\sigma_k, 0 < \mu_k^+ \leq \beta\mu_k$ and set $\lambda_{k+1} = \lambda_k$. Go to Step 5.
4. Set $\rho_k^+ = \rho_k$, choose $0 < \sigma_{k+1} \leq \sigma_k$ and $0 < \mu_k^+ \leq \beta\mu_k$. Set $\lambda_{k+1} = y_k$.
5. If $\rho_k^+ = \rho_k$, then choose $\delta_k = \Omega(\mu_k)$, else set $\delta_k = \sigma_k$. Choose regularization parameter $\theta_k \geq 0$ such that $K_{\rho_k^+,\theta_k,\delta_k}(w_k) \succ 0$. Set $J_k = J_{\rho_k^+,\theta_k,\delta_k}(w_k)$. Compute w_k^+ by solving the linear system
$$J_k(w_k^+ - w_k) = -\Phi(w_k, \lambda_{k+1}, \rho_k^+, \sigma_{k+1}, \mu_k^+). \tag{4.7}$$
6. Choose $\tau_k \in [\bar{\tau}, 1[$. Compute α_k as the largest $\alpha \in (0,1]$ such that
$$v_k + \alpha(v_k^+ - v_k) \geq (1 - \tau_k)v_k, \tag{4.8}$$
where $v_k = (x_k, z_k)$. Choose a vector $a_k = (a_k^x, a_k^y, z_k^z) \in [\alpha_k, 1]^N$ such that $v_k + a_k^v \circ (v_k^+ - v_k) > 0$, where $a_k^v = (a_k^x, a_k^z)$.
7. Set $\widehat{w}_k = w_k + a_k \circ (w_k^+ - w_k)$, $(\rho_{k+1}, \mu_{k+1}) = (\rho_k, \mu_k) + \alpha_k(\rho_k^+ - \rho_k, \mu_k^+ - \mu_k)$.
8. Choose $\varepsilon_k > 0$ such that $\{\varepsilon_k\} \to 0$. If $\|\Phi(\widehat{w}_k, \lambda_{k+1}, \rho_{k+1}, \sigma_{k+1}, \mu_{k+1})\| \leq \varepsilon_k$, then set $w_{k+1} = \widehat{w}_k$. Otherwise, apply the inner iteration algorithm to find w_{k+1} such that
$$\|\Phi(w_{k+1}, \lambda_{k+1}, \rho_{k+1}, \sigma_{k+1}, \mu_{k+1})\| \leq \varepsilon_k. \tag{4.9}$$
9. If termination criteria hold for (4.1) then stop, else $k \leftarrow k+1$ and go to Step 1.

the feasibility parameter is sufficiently reduced, while the barrier and quadratic penalty parameters are kept constant, and then the Lagrange multiplier estimate is rescaled. When these updates are always done from some iteration, this scaling of the Lagrange multiplier estimate is useful for the convergence of the iterates to an infeasible stationary point (see Theorem 4.3-ii). The second situation is when the algorithm has left the feasibility detection phase ($\mathtt{F} = 0$). In that case, the feasibility parameter is kept constant, the barrier parameter is reduced as in a standard interior point method, the quadratic penalty parameter is also reduced to penalize the constraints violation and the Lagrange multiplier estimate is kept constant as in a classical augmented Lagrangian algorithm.

At Step 5, the choice of the regularization parameter δ_k of the regularized Jacobian matrix is done as follows. When the feasibility parameter is unchanged, because (4.6) is satisfied or $\mathtt{F} = 0$, we set $\delta_k = \Omega(\mu_k)$. This choice is imposed by the global convergence theory of the algorithm in the feasible case, see [15, Theorem 3.3] and [13, Theorem 4.2]. In case the feasibility parameter is reduced, we set $\delta_k = \sigma_k$. This choice is motivated to get a rapid convergence when the sequence of iterates converges to an infeasible stationary point, see Lemma 4.8 below.

Once the Newton iterate w_k^+ is computed at Step 5, the fraction to the boundary rule is applied to ensure the positivity of the primal and dual variables. As in [15], the step length can be selected componentwise to calculate the trial iterate $\widehat{w}_k$. The values of barrier and feasibility parameters are then updated according to formulae at Step 7. These formulae avoid too large discrepancies between the magnitude of these parameters and the one of $\|\Phi\|$ and increase robustness [9].

Finally, at Step 8, if the trial iterate $\widehat{w}_k$ satisfies a sufficient reduction of the residual norm of the perturbed optimality conditions, then $w_{k+1} = \widehat{w}_k$. If this is not the case, a sequence of inner iterations with all the parameters fixed to their current values will be carried out to find the new iterate w_{k+1}.

The inner iteration algorithm (Algorithm 4) is a backtracking line search applied to the primal-dual merit function

$$\mathcal{M}_{\rho,\lambda,\sigma,\mu}(w) = \varphi_{\rho,\lambda,\sigma,\mu}(x) + \nu_1\psi_{\sigma,\lambda}(x,y) + \nu_2\pi_{\rho,\mu}(x,z),$$

where $\varphi_{\rho,\lambda,\sigma,\mu}(x)$ is defined by (4.2),

$$\psi_{\sigma,\lambda}(x,y) = \frac{1}{2\sigma}\|c(x) + \sigma(\lambda - y)\|^2 \quad \text{and} \quad \pi_{\rho,\mu}(x,z) = x^\top z - \rho\mu\sum_{i=1}^{p}\log(x_i z_i),$$

where $\nu_1 > 0, \nu_2 > 0$ are scaling parameters. This is motivated by the fact that the

Algorithm 4 Inner iteration

1. If $\|\Phi(w^i,\lambda,\rho,\sigma,\mu)\| \leq \varepsilon_k$, then set $w_{k+1} = w^i$ and return to Algorithm 3.
2. Choose $\theta^i \geq 0$ such that $\mathrm{In}(J_{\rho,\theta^i,\sigma}(w^i)) = (n,m,0)$ and compute the direction $d^i = (d^i_x, d^i_y, d^i_z)$ by solving the system $J_{\rho,\theta^i,\sigma}(w^i)d = -\Phi(w^i,\lambda,\rho,\sigma,\mu)$.
3. Compute $\hat{\alpha}$ as the largest $\alpha \in (0,1]$ such that
$$v^i + \alpha d^i_v \geq (1-\tau)v^i.$$
where $d^i_v = (d^i_x, d^i_z)$ and $\tau \in (0,1)$.
4. Start from $\alpha^i = \widehat{\alpha}$, apply a backtracking line search to find $\alpha^i \in (0,\widehat{\alpha}]$ such that
$$\mathcal{M}_{\rho,\lambda,\sigma,\mu}(w^i + \alpha^i d^i) \leq \mathcal{M}_{\rho,\lambda,\sigma,\mu}(w^i) + \omega\alpha^i \nabla\mathcal{M}_{\rho,\lambda,\sigma,\mu}(w^i)^\top d^i.$$
5. Set $w^{i+1} = w^i + \alpha^i d^i$.

first order optimality conditions for minimizing this merit function correspond to (4.5). To simplify the presentation, we consider that the quadratic parameter σ_{k+1} remains constant all along the inner iterations, while in [13, Algorithm 2] it can be increased. This choice has no impact from the theoretical point of view and in our numerical experiments, the value of this parameter is allowed to increase during the inner iterations. Once the inner algorithm is triggered we set $w^0 = \widehat{w}_k$ and choose constants $\nu_1, \nu_2 > 0, \omega \in (0,1)$. We fix the values $\rho = \rho_{k+1}, \sigma = \sigma_{k+1}, \mu = \mu_{k+1}$, and $\lambda = \lambda_{k+1}$.

We end this section by showing some properties related to the behavior of the parameters with respect to the feasibility detection by means of the criterion (4.6).

Lemma 4.1. *Assume that Algorithm 3 generates an infinite sequence of iterates $\{w_k\}$. Let $\mathcal{K} \subset \mathbb{N}$ be the set of iteration indices at which the inequality (4.6) is satisfied.*

(i) If $\mathcal{K}$ is infinite, then $\{c_k\}_{k\in\mathcal{K}}$ converges to zero and $\{\rho_k\}$ is eventually constant.

(ii) If $\mathcal{K}$ is finite, then $\liminf\|c_k\| > 0$.

In addition, suppose that the sequence $\{H_{\rho_k^+,\theta_k}(x_k,y_k), g_k, A_k\}$ is bounded and that the matrices $K_{\rho_k^+,\theta_k,\delta_k}(w_k)$ are uniformly positive definite for $k \in \mathbb{N}$.

(iii) If $\mathcal{K}$ is infinite, then the sequence $\{\mu_k\}$ converges to zero.

(iv) If $\mathcal{K}$ is finite, then the sequences $\{\sigma_k\rho_k\}$ and $\{\sigma_k\lambda_k\}$ converge to zero.

Proof. The outcomes (i) and (ii) are proved in Lemma 3.1, while (iii) is a direct consequence of [13, Theorem 4.2 (iv)].

To prove (iv), suppose that $\mathcal{K}$ is finite and let us define $k_0 := \max \mathcal{K}$. Two cases are considered. The first case is when $\|c_k\| \leq \epsilon$ at some iteration k. We then have $\mathtt{F} = 0$ for all further iterations. The update of the parameters at Step 3 implies that both sequences $\{\rho_k\}$ and $\{\lambda_k\}$ are eventually constant and $\{\sigma_k\}$ tends to zero. It follows that the two sequences $\{\sigma_k\rho_k\}$ and $\{\sigma_k\lambda_k\}$ converge to zero. The second case is when $\|c_k\| > \epsilon$ for all $k \in \mathbb{N}$, which implies that $\mathtt{F} = 1$ at each iteration. In that case, for all $k \geq k_0 + 1$, $\rho_k^+ \leq \beta\rho_k$, $\sigma_{k+1} = \sigma_{k_0}$, $\mu_{k+1} = \mu_{k_0}$ and $\|\lambda_{k+1}\| \leq \beta\rho_k\|\lambda_k\| \leq \beta\rho_k\|y_{k_0}\|$. By using similar arguments as in the proof of [13, Theorem 4.2 (iv)] and [15, Theorem 3.3], we will show that $\{\rho_k\}$ converges to zero, which will imply that the two sequences $\{\sigma_k\rho_k\}$ and $\{\sigma_k\lambda_k\}$ converge to zero. The proof is by contradiction by supposing that $\{\rho_k\}$ is bounded away from zero by some constant $\bar{\rho} > 0$. For all $k \geq k_0$, we have $\rho_k^+ \leq \beta\rho_k$ with $\beta \in (0, 1)$. From Step 7 of Algorithm 3, for all $k \geq k_0$ we have

$$\begin{aligned} \rho_{k+1} &= \rho_k + \alpha_k(\rho_k^+ - \rho_k) \\ &\leq (1 - (1-\beta)\alpha_k)\rho_k. \end{aligned}$$

Because $\{\rho_k\}$ is supposed to be bounded away from zero, this inequality implies that $\{\alpha_k\}$ converges to zero. We will get a contradiction by showing that $\{\alpha_k\}$ is bounded away from zero. The inequality (33) in [13, Theorem 4.2] shows that for all k large enough, we have

$$\frac{\bar{\tau}}{1 + \|w_k^+ - w_k\|/\sqrt{\bar{\rho}}} \leq \alpha_k. \tag{4.10}$$

Recall that the inequality (4.9) is satisfied at each iteration with a sequence $\{\varepsilon_k\}$ going to zero. Therefore, the sequence $\{\Phi(w_k, \lambda_k, \rho_k, \sigma_k, \mu_k)\}$ converges to zero. In particular, $\{X_k Z_k e - \rho_k\mu_k e\}$ tends to zero. Consequently, for k large enough

$$[x_k]_i[z_k]_i \geq \frac{\bar{\rho}\mu_{k_0}}{2}, \quad \text{for all } i = 1, \ldots, n.$$

Keeping in mind all the assumptions and the previous lower bound on $\{x_k \circ z_k\}$, [10, Theorem 1] shows that there exists a constant $K_1 > 0$, such that for all $k \geq 0$

$$\|J_{\rho_k^+,\theta_k,\delta_k}(w_k)^{-1}\| \leq K_1. \tag{4.11}$$

From the definition of $\Phi(\cdot)$, for all $k \geq 0$ we have

$$\begin{aligned}\Phi(w_k, \lambda_{k+1}, \rho_k^+, \sigma_{k+1}, \mu_k^+) &= \Phi(w_k, \lambda_k, \rho_k, \sigma_{k+1}, \mu_k^+)\\ &\quad + ((\rho_k^+ - \rho_k)g_k^\top, \sigma_{k+1}(\lambda_{k+1} - \lambda_k)^\top, (\rho_k - \rho_k^+)\mu_k^+ e^\top)^\top.\end{aligned}$$

By using (4.7), (4.9), (4.11) and noting that $\sigma_{k+1} = \sigma_k = \sigma_{k_0}$, $\mu_k^+ = \mu_k = \mu_{k_0}$, $\lambda_{k+1} = \rho_k^+ \lambda_k$ and $\rho_k^+ \leq \rho_k \leq 1$, for k large enough we then get

$$\begin{aligned}\|w_k^+ - w_k\| &= \|J_{\rho_k^+,\theta_k,\delta_k}(w_k)^{-1}\Phi(w_k, \lambda_{k+1}, \rho_k^+, \sigma_{k+1}, \mu_k^+)\|\\ &\leq K_1(\|\Phi(w_k, \lambda_k, \rho_k, \sigma_k, \mu_k)\| + |\rho_k - \rho_k^+|(\|g_k\| + \mu_{k_0}\|e\|) + \sigma_{k_0}|\rho_k^+ - 1|\|\lambda_k\|)\\ &\leq K_1(\varepsilon_k + \|g_k\| + \mu_{k_0}\|e\| + \sigma_{k_0}\|\lambda_k\|).\end{aligned}$$

Because $\{\varepsilon_k\}$ tends to zero, $\{g_k\}$ is bounded and $\|\lambda_k\| \leq \beta\|y_{k_0}\|$, we deduce that $\|w_k^+ - w_k\|$ is bounded from above, which contradicts inequality (4.10) and the fact that $\{\alpha_k\}$ is supposed to converge to zero. We then deduce that $\rho_k \to 0$, which completes the proof of (iv). □

4.2 Global convergence analysis

For the global convergence of the inner iterations, we have the following theorem.

Theorem 4.2. *[13, Theorem4.1] Suppose that an infinite sequence $\{w^i\}$ is generated by Algorithm 4.Assume also that the sequence $\{x^i\}$ lies in a compact set of $\mathbb{R}^n$ and that the matrices $K_{\rho,\theta^i,\sigma}(w^i)$ are uniformly positive definite for $i \in \mathbb{N}$. Then, the sequence $\{\Phi(w^i, \lambda, \rho, \sigma, \mu)\}$ converges to zero.*

Roughly speaking, this theorem shows that under some standard assumptions, the inner iteration algorithm is able to find a new iterate w_{k+1} satisfying the stopping test (4.9) after a finite number of iterations. Hence, we can assume that the inner iteration algorithm successfully terminates each time it is called at Step 8.

We then have the following result about the global convergence of Algorithm 3.

Theorem 4.3. *Assume that all the assumptions of Lemma 4.1 are satisfied and that Algorithm 3 generates an infinite sequence $\{w_k\}$. Let $\mathcal{K} \subset \mathbb{N}$ be the set of* *iteration indices for which the condition (4.6) is satisfied.*

(i) If $\mathcal{K}$ is infinite, then $\rho_k = \bar{\rho} > 0$ for k large enough and the iterates approach stationarity of the problem (4.1), i.e., the sequences $\{\bar{\rho} g_k + A_k y_k - z_k\}$, $\{c_k\}_\mathcal{K}$ and $\{X_k z_k\}$ converge to zero.

(ii) If $\mathcal{K}$ is finite, then $\{\|c_k\|\}$ is bounded away from zero and the sequence of primal iterates approaches stationarity of the feasibility problem, i.e., there exists a sequence $\{u_k\} \subset \mathbb{R}^n_+$ such that $\lim \|(A_k c_k - u_k, X_k u_k)\| = 0$.

Proof. Let us denote $\Phi_k := \Phi(w_k, \lambda_k, \rho_k, \sigma_k, \mu_k)$ for $k \in \mathbb{N}$. Step 8 of Algorithm 3 implies that $\{\Phi_k\}$ converges to zero.

Suppose that $\mathcal{K}$ is infinite. Lemma 4.1-(i) shows that $\{c_k\}_\mathcal{K}$ tends to zero and that $\rho_k = \bar{\rho}$ for sufficiently large k. The first block of Φ_k is $\rho_k g_k + A_k y_k - z_k$, therefore $\lim \bar{\rho} g_k + A_k y_k - z_k = 0$. The third block of Φ_k is $X_k Z_k e - \rho_k \mu_k e$. Lemma 4.1-(iii) implies that $\{\mu_k\}$ tends to zero. Therefore $\{X_k z_k\}$ tends to zero, which concludes the proof of outcome (i).

Suppose now that $\mathcal{K}$ is finite. Lemma 4.1-(ii) implies that the sequence $\{\|c_k\|\}$ is bounded away from zero. Let us define $u_k := \sigma_k z_k > 0$ for $k \in \mathbb{N}$. For all $k \in \mathbb{N}$, we have

$$A_k c_k - u_k = \sigma_k(\rho_k g_k + A_k y_k - z_k) - \sigma_k \rho_k g_k + A_k(c_k + \sigma_k(\lambda_k - y_k)) - \sigma_k A_k \lambda_k$$

and

$$X_k u_k = \sigma_k(X_k Z_k e - \rho_k \mu_k e) + \sigma_k \rho_k \mu_k e.$$

By taking the norm on both sides, for all k we then get

$$\|A_k c_k - u_k\| + \|X_k u_k\| \leq (2\sigma_k + \|g_k\| + 2\|A_k\| + \mu_k \|e\|) \max\{\|\Phi_k\|, \sigma_k \rho_k, \sigma_k \|\lambda_k\|\}.$$

By assumptions, the sequences $\{g_k\}$ and $\{A_k\}$ are bounded. The first assertion of this proof and Lemma 4.1-(iv) imply that the second term of this inequality tends to zero. We then deduce that $\{(A_k c_k - u_k, X_k u_k)\}$ converges to zero, which ends the proof. □

The next result summarizes the behavior of the Algorithm 3 under a mild and usual assumption about the boundedness of the sequence of primal iterates. It is shown that, under the boundedness of the primal sequence and for an appropriate choice of the stopping criteria, the algorithm stops after a finite number of iterations, either on a FJ point or on an infeasible stationary point.

Theorem 4.4. *Assume that Algorithm 3 generates an infinite sequence of iterates $\{w_k\}$ such that $\{x_k\}$ lies in a compact set. Assume also that the regularization parameters chosen at Step 5 are such that the matrices $K_{\rho_k^+,\theta_k,\delta_k}(w_k)$ are uniformly positive definite for $k \in \mathbb{N}$.*

(i) Any feasible limit point of $\{x_k\}$ is a Fritz-John point of problem (4.1).

(ii) If $\{x_k\}$ has no feasible limit point, then any limit point is an infeasible stationary point of problem (4.1).

Proof. The compactness assumption implies that the sequence $\{(g_k, A_k)\}$ is bounded.

Assume that $\{x_k\}$ has a feasible limit point $\bar{x}$, i.e., $\bar{c} = 0$. Lemma 4.1-(ii) implies that the updating condition (4.6) is satisfied an infinite number of times. Let $\mathcal{K} \subset \mathbb{N}$ such that $\lim_{\mathcal{K}} x_k = \bar{x}$. From Lemma 4.1-(i), we have $\rho_k = \bar{\rho}$ for k large enough. We consider two situations regarding the boundedness of the sequence $\{y_k\}_{\mathcal{K}}$. Suppose that $\{y_k\}_{\mathcal{K}}$ is bounded. With the compactness of $\{x_k\}$ we have that the sequence $\{H_{\rho_k^+,\theta_k}(x_k, y_k)\}$ is bounded and so Theorem 4.3 applies. The sequence $\{z_k\}$ is also bounded. Indeed, for all $k \in \mathbb{N}$, we have

$$\|z_k\| \leq \|\rho_k g_k + A_k y_k - z_k\| + \|\rho_k g_k + A_k y_k\|.$$

The first term on the right hand side tends to zero and the second one is bounded. Consequently, by virtue of Theorem (4.3)-(i), any limit point of the sequence $\{x_k, y_k, z_k\}_{\mathcal{K}}$ satisfies the FJ conditions. Because $\bar{\rho} > 0$, $\bar{x}$ is a KKT point of problem (4.1). The second situation is when $\{y_k\}_{\mathcal{K}}$ is unbounded. For all $k \in \mathcal{K}$, let $(r_k, s_k) := \frac{(y_k, z_k)}{\|(y_k, z_k)\|}$. Note that $s_k > 0$, because of Step 6 of Algorithm 3. By taking a subsequence if necessary, we can assume that $\lim_{\mathcal{K}}(r_k, s_k) = (\bar{a}, \bar{b}) \neq 0$, with $\bar{b} \geq 0$. For all $k \in \mathbb{N}$ we have

$$\|A_k r_k - s_k\| \leq \frac{1}{\|(y_k, z_k)\|} (\|\rho_k g_k + A_k y_k - z_k\| + \|\rho_k g_k\|)$$

Because $\{\rho_k g_k + A_k y_k - z_k\}$ converges to zero, $\{g_k\}$ is bounded, $\{\rho_k\}$ is eventually constant and $\{y_k\}_{\mathcal{K}}$ is unbounded, we get $\bar{A}\bar{a} - \bar{b} = 0$. For all $k \in \mathbb{N}$ we have

$$\|X_k s_k\| = \frac{1}{\|(y_k, z_k)\|} (\|X_k Z_k e - \rho_k \mu_k e\| + \rho_k \mu_k \|e\|)$$

Because $\{X_k Z_k e - \rho_k \mu_k e\}$ tends to zero and $\{(y_k, z_k)\}_{\mathcal{K}}$ is unbounded, we also get $\bar{X}\bar{b} = 0$. We can conclude that $\bar{x}$ is a singular stationary point of problem (4.1).

Let us now consider the second outcome for which any limit point of $\{x_k\}$ is infeasible. It follows from Lemma 4.1-(i) that the Step 3 of Algorithm 3 is executed at all iteration $k \geq k_0$ for some $k_0 \in \mathbb{N}$. There are two cases depending on the boundedness of $\{y_k\}$. If $\{y_k\}$ is bounded, then the sequence $\{H_{\rho_k^+,\theta_k}(x_k, y_k)\}$ is bounded. Therefore, Theorem 4.3-(ii) shows that any limit point $\bar{x}$ of $\{x_k\}$ is an infeasible stationary point of (4.1). In the second case, $\{y_k\}$ is unbounded. From the convergence to zero of the sequence $\{c_k + \sigma_k(\lambda_k - y_k)\}$ and the boundedness of $\{\lambda_k\}$ (since $\|\lambda_k\| \leq \|\lambda_{k_0}\|$ for all $k \geq k_0$), we deduce that the sequence $\{\sigma_k\}$ tends to zero. As a consequence, the sequences $\{\sigma_k \rho_k\}$ and $\{\sigma_k \lambda_k\}$ converge to zero. We then obtain the same conclusion as Lemma 4.1-(iv). It suffices to repeat the proof of Theorem (4.3)-(ii) to show that any limit point $\{x_k\}$ is an infeasible stationary point of (4.1). □

4.3 Asymptotic behavior

There are two cases to analyse. The first one is when $\{w_k\}$ converges to a primal-dual solution of problem (4.1) $w^* = (x^*, y^*, z^*)$. Because the flag `F` is switched to zero at some iteration, the feasibility parameter becomes constant after a finite number of iterations. Consequently, the analysis in [119, Section 1.4] can be directly applied to demonstrate that under some classical assumptions and a suitable choice of the parameters, the rate of convergence of the sequence $\{w_k\}$ is superlinear. In particular, we have the following results.

Theorem 4.5. *Assume that the following assumptions are fulfilled:*

(i) The functions f and c are twice continuously differentiable and their second derivatives are Lipschitz continuous over an open neighborhood of x^*.

(ii) The LICQ (Definition 2.14) is satisfied at x^*.

(iii) The SOSCs (Definition 2.24) hold at w^*.

(iv) The strict complementarity (Definition 2.26) holds at w^*.

(v) The sequences $\{\mu_k^+\}$, $\{\sigma_k\}$, $\{\delta_k\}$, $\{\tau_k\}$, and $\{\varepsilon_k\}$ are chosen such that

$$\theta_1 \mu_k^{t+1} \leq \mu_k^+ \leq \theta_2 \mu_k, \sigma_k = \mathrm{O}(\mu_k), \delta_k = \mathrm{O}(\mu_k), \lim_{k\to\infty} (1-\tau_k)\frac{\mu_k}{\mu_k^+} = 0 \quad \textit{and} \quad \varepsilon = \Omega(\mu_k).$$

for some $\theta_1 > 0$ and $t, \theta_2 \in (0, 1)$.

Then, for k sufficiently large, we have the following results

1. *Algorithm 3 does not need the inner iterations from which implies that $w_{k+1} = \widehat{w}_k$.*

2. *The iterates are asymptotically tangent to the trajectory in the sense that $w_k = \mathbf{w}(\mu_k) + \mathrm{o}(\mu_k)$, where the function $\mathbf{w}(\cdot)$ is implied by applying the Implicit Function Theorem (Theorem 2.12) for the function $F : \mathbb{R}^{N+1} \to \mathbb{R}^N$*

$$F(w, \mu) = \begin{pmatrix} g + Ay - z \\ c(x) \\ XZe \end{pmatrix}.$$

3. *The unit step is accepted by the fraction to the boundary rule, i.e. $\mu_{k+1} = \mu^+, \widehat{w}_k = w_k^+$.*

   Moreover, if the sequence $\{\zeta_k\}$ is chosen such that $\zeta_k = \Omega(\mu_k)$, then the inequality (4.6) is satisfied at each iteration.

The second case is when Algorithm 3 generates a convergent sequence $\{x_k\}$ to an infeasible stationary point $x^* \in \mathbb{R}^n$ of the problem (4.1). This section concentrates on this case.

4.3.1 Assumptions and basic results

The first assumption is about the regularity of the problem data. This assumption is standard in our framework.

Assumption 4.1. The function f and c are twice continuously differentiable and their second derivatives are Lipschitz continuous over an open neighborhood of x^*.

To analyse the rate of convergence of the sequence of iterates to an infeasible stationary point, a natural assumption is that the whole sequence of iterates converges to such a point.

Assumption 4.2. Algorithm 3 generates an infinite sequence $\{w_k\}$ which converges to $w^* = (x^*, y^*, z^*) \in \mathbb{R}^{2n+m}$, where x^* is an infeasible stationary point of problem (4.1).

This assumption implies that the algorithm always stays in feasibility detection phase, i.e., the feasibility flag keeps the value $\mathtt{F} = 1$ all along the iterations. More precisely, some direct consequences are the following.

Lemma 4.6. *Under Assumption 4.2,* $\mathtt{F} = 1$ *at each iteration, there exists* $k_0 \in \mathbb{N}$ *such that for all* $k \geq k_0$, $\sigma_k = \sigma := \sigma_{k_0} > 0$ *and* $\mu_k = \mu := \mu_{k_0} > 0$, $\{\rho_k\}$ *converges to zero and* $\|\lambda_{k+1}\| = \mathrm{o}(\rho_k^+)$.

Proof. Assumption 4.2 implies that $\{c_k\}$ is bounded away from zero. By virtue of Lemma 4.1-(i), the inequality (4.6) is satisfied only a finite number of times. It follows that Step 3 of Algorithm 3 is always executed for k sufficiently large. For all $k \in \mathbb{N}$, we have

$$\|c_k\| \leq \|c_k + \sigma_k(\lambda_k - y_k)\| + \sigma_k\|\lambda_k\| + \sigma_k\|y_k\|.$$

Inequality (4.9) implies that the first term of the right-hand side tends to zero. The update rule of λ_k at Step 3 implies that $\{\lambda_k\}$ remains bounded. Since $\{y_k\}$ is supposed to be convergent, if $\{\sigma_k\}$ goes to zero, then the previous inequality shows that $\{c_k\}$ goes to zero, which would contradict the initial assumption. This means that σ_k is constant for k large enough and thus $\mathtt{F} = 1$ at each iteration. Hence, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $\sigma_k = \sigma_{k_0}, \mu_k = \mu_{k_0}$ and $\lambda_{k+1}/\rho_k^+ = \rho_{k-1}^+ \dots \rho_{k_0}^+ \lambda_{k_0}$. By using the same arguments as in the proof of Lemma 4.1-(iv), we deduce that the sequence $\{\rho_k\}$ converges to zero. Thanks to the fact that $\rho_{k-1}^+ \leq \rho_k$, we get $\|\lambda_{k+1}\| = \mathrm{o}(\rho_k^+)$. □

For $w = (x, y, z) \in \mathbb{R}^{2n+m}$ and $\rho > 0$, let us define

$$F(w, \rho) = \begin{pmatrix} \rho g(x) + A(x)y - z \\ c(x) - \sigma y \\ XZe - \rho\mu e \end{pmatrix},$$

where σ and μ are the values defined by Lemma 4.6. From the fact that $\lim \Phi(w_k, \lambda_k, \rho_k, \sigma_k, \mu_k) = \Phi(w^*, 0, 0, \sigma, \mu) = F(w^*, 0)$, one has $y^* = \frac{1}{\sigma}c^*$, $z^* = \frac{1}{\sigma}A^*c^*$ and $0 \leq x^* \perp z^* \geq 0$.

Let us denote the Hessian matrix of the function $\frac{1}{2}\|c\|^2$ at $x \in \mathbb{R}^n$ by

$$S(x) := \sum_{i=1}^{m} c_i(x)\nabla^2 c_i(x) + A(x)A(x)^\top.$$

The next assumption related to the second order sufficient optimality conditions of the feasibility problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \tfrac{1}{2}\|c(x)\|^2 \ \text{ subject to } x \geq 0. \tag{4.12}$$

Assumption 4.3. The second order sufficient optimality conditions (SOSCs) for the feasibility problem (4.12) hold at x^*, i.e., $u^\top S^* u > 0$ for all $u \neq 0$ satisfying $u_i = 0$ for all $i \in \mathcal{A}$.

Assumption 4.4. Strict complementarity holds at w^*, that is

$$a := \min\{x_i^* + z_i^* : i = 1, ..., n\} > 0.$$

The next lemma summarizes some basic results which are direct consequences of these assumptions.

Lemma 4.7. *Under Assumptions 4.1-4.4, there exist positive constants r^*, ρ^*, K, C, L, L_1, L_2 and a continuously differentiable function $\mathbf{w} : (-\rho^*, \rho^*) \to \mathbb{R}^N$, such that for all $w, w' \in B(w^*, r^*)$ and for all $\rho, \rho' \in (-\rho^*, \rho^*)$, we have*

(i) $\|F'_w(w, \rho)^{-1}\| \leq K$,

(ii) $F(w, \rho) = 0$ *if and only if* $\mathbf{w}(\rho) = w$,

(iii) $\|\mathbf{w}(\rho) - \mathbf{w}(\rho')\| \leq C|\rho - \rho'|$,

(iv) $\|F'_w(w, \rho) - F'_w(w', \rho)\| \leq L(\|w - w'\|)$,

(v) $L_1\|w - w'\| \leq \|F(w, \rho) - F(w', \rho)\| \leq L_2\|w - w'\|$.

Proof. In order to prove (i), we only need to show that the matrix $F'_w(w^*, 0)$ is nonsingular. Let $u \in \mathbb{R}^N$ such that $F'_w(w^*, 0)u = 0$. By writing $u := (u_1, u_2, u_3)$ and by using the fact that $y^* = \frac{1}{\sigma}c^*$, we have

$$\begin{pmatrix} \frac{1}{\sigma}\sum_i c_i^* \nabla^2 c_i^* & A^* & -I \\ A^{*\top} & -\sigma I & 0 \\ Z^* & 0 & X^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The third equation of this linear system and Assumption 4.4 imply that $[u_1]_i = 0$ for all $i \in \mathcal{A}$ and $[u_3]_i = 0$ for all $i \notin \mathcal{A}$. Consequently, one has $u_1^\top u_3 = 0$. The second equation of the linear system gives us $u_2 = \frac{1}{\sigma}A^{*\top}u_1$. Substituting this equality into the first equation and premultiplying by $u_1^\top$, we get $\frac{1}{\sigma}u_1^\top S^* u_1 = 0$. By Assumption 4.3 we deduce that $u_1 = 0$, from which we deduce that $u_2 = 0$ and $u_3 = 0$.

The properties (ii) and (iii) are direct consequences of the Implicit Function Theorem (Theorem 2.12).

The Lipschitz continuity of F'_w, property (iv), follows from Assumption 4.1. The last assertion (v) is a consequence of [53, Lemma 15]. □

Our asymptotic analysis also requires some specifications on the choice of the feasibility parameter. For all $k \in \mathbb{N}$, the trial value of feasibility parameter ρ_k^+ is chosen such that

$$\theta_1 \rho_k^{1+t} \leq \rho_k^+ \leq \theta_2 \rho_k \tag{4.13}$$

for some constants $\theta_1 > 0$, $\theta_2 \in (0,1)$ and $t \in (0,1)$. From the fact that $\{\rho_k\}$ goes to zero, the left inequality of (4.13) implies that

$$\rho_k^2 = o(\rho_k^+). \tag{4.14}$$

At last, the parameter τ_k used in (4.8) must satisfy the following condition:

$$\lim_{k\to\infty} (1-\tau_k)\frac{\rho_k}{\rho_k^+} = 0. \tag{4.15}$$

The conditions (4.13) and (4.15) are standard in the asymptotic analysis of this family of primal-dual methods, see e.g., [14, 15]. The left inequality of (4.13) means that the parameter of the path $\mathbf{w}$ must converge to zero with a subquadratic rate of convergence (converge at most superlinearly with order $1+t$) in order that the Newton iterates, which converge naturally with a quadratic rate, catch the path prior the parameter is nearly zero.

For k large enough, we have $\rho_k^+ < \rho_k$ and $\sigma_k = \sigma$. Therefore, at Step 5 of Algorithm 3, the regularization parameter of the matrix J_k is set to $\delta_k = \sigma$. The next lemma shows that the matrix J_k coincides with the Jacobian of $F(\cdot, \rho_k^+)$ at w_k when the feasibility parameter goes to zero.

Lemma 4.8. *Under Assumptions 4.1-4.4, for all $k \in \mathbb{N}$ large enough, one has*

$$J_k = F'_w(w_k, \rho_k^+) \quad \textit{and} \quad \|J_k^{-1}\| \leq K,$$

where K is defined by Lemma 4.7.

Proof. For the first claim, it suffices to show that $\theta_k = 0$ for k large enough. This is true whenever $K_{\rho_k^+,0,\sigma}(w_k)$ is positive definite. For all $k \in \mathbb{N}$ we have

$$\begin{aligned} K_{\rho_k^+,0,\sigma}(w_k) &= H_{\rho_k^+,0}(x_k,y_k) + \tfrac{1}{\sigma}A_kA_k^\top + X_k^{-1}Z_k \\ &= \frac{1}{\sigma}(S_k - S^*) + H_{\rho_k^+,0}(x_k, y_k - \frac{1}{\sigma}c_k) + \tfrac{1}{\sigma}S^* + X_k^{-1}Z_k. \end{aligned}$$

The first two matrices tend to zero when k tends to infinity because $\{(x_k, y_k)\}$ and $\{\rho_k^+\}$ respectively converge to $(x^*, \frac{1}{\sigma}c^*)$ and zero. It remains to show that the matrices $\frac{1}{\sigma}S^* + X_k^{-1}Z_k$ are uniformly positive definite for k large enough. Let us define the $n \times n$ diagonal matrix E, whose ith diagonal element is equal to one if $i \in \mathcal{A}$ and zero otherwise. Assumption 4.3 means that S^* is positive definite on the null space of E. Therefore, from Lemma 2.2-(ii), there exists $\gamma > 0$ such that the matrix $\frac{1}{\sigma}S^* + \gamma E$ is positive definite. For $k \in \mathbb{N}$, let us define the diagonal matrix Ξ_k whose ith diagonal element is $[z_k]_i/[x_k]_i$ if $i \in \mathcal{A}$ and zero otherwise. Because of the strict complementarity assumption, the matrices $X_k^{-1}Z_k - \Xi_k$ tend to zero and each nonzero component of Ξ_k goes to infinity. It follows that $\Xi_k - \gamma E$ is positive semidefinite for all k large enough. By writing

$$\frac{1}{\sigma}S^* + X_k^{-1}Z_k = \frac{1}{\sigma}S^* + \gamma E + \Xi_k - \gamma E + X_k^{-1}Z_k - \Xi_k,$$

we deduce that the matrices $\frac{1}{\sigma}S^* + X_k^{-1}Z_k$ are uniformly positive definite for all k large enough.

The second assertion follows directly from the first claim and Lemma 4.7-(i). $\square$

Throughout this section, we assume that Assumptions 4.1-4.4 are satisfied. The following result gives an estimate of the distance of the Newton iterate to the trajectory $\mathbf{w}$.

Lemma 4.9. *There exists $M > 0$, such that for all sufficiently large k*

$$\|w_k^+ - \mathbf{w}(\rho_k^+)\| \leq M(\|w_k - \mathbf{w}(\rho_k)\|^2 + \rho_k^2 + \|\lambda_{k+1}\|).$$

Proof. Let $k \in \mathbb{N}$ be large enough such that $\rho_k \leq \rho^*$ and $w_k \in B(w^*, r^*)$. Define $e_k := \mathbf{w}(\rho_k^+) - w_k$. From the linear system (4.7), then using $\Phi(w, \lambda, \rho, \sigma, \mu) = F(w, \rho) + (0, \sigma\lambda^\top, 0)^\top$ and $F(\mathbf{w}(\rho_k^+), \rho_k^+) = 0$, we have

$$\begin{aligned}
w_k^+ - \mathbf{w}(\rho_k^+) &= -J_k^{-1}\Phi(w_k, \lambda_{k+1}, \rho_k^+, \sigma, \mu) - e_k \\
&= J_k^{-1}\left(F(\mathbf{w}(\rho_k^+), \rho_k^+) - F(w_k, \rho_k^+) - (0, \sigma\lambda_{k+1}^\top, 0)^\top - J_k e_k\right) \\
&= J_k^{-1}\int_0^1 (F'_w(w_k + te_k, \rho_k^+) - F'_w(w_k, \rho_k^+))e_k dt \\
&\quad + J_k^{-1}(F'_w(w_k, \rho_k^+) - J_k)e_k - J_k^{-1}(0, \sigma\lambda_{k+1}^\top, 0)^\top.
\end{aligned}$$

By taking norm on both sides and next applying Lemma 4.7-(iv) and Lemma 4.8,

we then have

$$\|w_k^+ - \mathbf{w}(\rho_k^+)\| \leq \frac{1}{2} KL \|e_k\|^2 + K\sigma\|\lambda_{k+1}\|.$$

By applying Lemma 4.7-(iii) and by using the inequality $\rho_k^+ \leq \rho_k$, we also get

$$\begin{aligned} \|e_k\| &\leq \|w_k - \mathbf{w}(\rho_k)\| + \|\mathbf{w}(\rho_k) - \mathbf{w}(\rho_k^+)\| \\ &\leq \|w_k - \mathbf{w}(\rho_k)\| + C\rho_k. \end{aligned}$$

Finally, by using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for two real numbers a and b, we deduce that

$$\|w_k^+ - \mathbf{w}(\rho_k^+)\| \leq KL\|w_k - \mathbf{w}(\rho_k)\|^2 + KLC^2\rho_k^2 + K\sigma\|\lambda_{k+1}\|.$$

Set $M = K \max\{L, LC^2, \sigma\}$ to complete the proof. □

4.3.2 Asymptotic result

According to Assumptions 4.1-4.4 and Lemma 4.9, the convergence to w^* of w_k implies that there exists $\bar{r} \in (0, r^*]$ such that for all k large enough, w_k, w_{k+1}, $\mathbf{w}(\rho_k)$, $\mathbf{w}(\rho_k^+)$, and $\mathbf{w}(\rho_{k+1})$ belong to $B(w^*, \bar{r})$ and $\rho_k \in (0, \rho^*)$. Without loss of generality, we can assume that these properties are true for all $k \in \mathbb{N}$. Note that, by definition of the constant a in Assumption 4.4, we implicitly have $r^* < a$, see [14, Section 4]. We can then define the positive constant $b = \frac{1}{a-\bar{r}}$, which will be used in Lemma 4.11 below.

The following lemma gives an evaluation of the distance between the next iterate w_{k+1} and the path $\mathbf{w}$.

Lemma 4.10. *There exists a constant $C_1 > 0$ such that for all $k \in \mathbb{N}$,*

$$\|w_{k+1} - \mathbf{w}(\rho_{k+1})\| \leq C_1(\|\widehat{w}_k - \mathbf{w}(\rho_{k+1})\| + \|\lambda_{k+1}\|). \tag{4.16}$$

Proof. If there is no inner iteration, we then have $w_{k+1} = \widehat{w}_k$, therefore the inequality (4.16) holds trivially with $C_1 = 1$. Suppose now that w_{k+1} is obtained by applying a sequence of inner iterations. By virtue of Lemma 4.7-(v), Lemma 4.7-(ii) and Step 8 of Algorithm 3, there exists an index $k_0 \in \mathbb{N}$ such that for all

$k \geq k_0$,

$$\begin{aligned}
L_1 \| w_{k+1} - \mathbf{w}(\rho_{k+1}) \| &\leq \| F(w_{k+1}, \rho_{k+1}) - F(\mathbf{w}(\rho_{k+1}), \rho_{k+1}) \| \\
&= \| \Phi(w_{k+1}, \lambda_{k+1}, \rho_{k+1}, \sigma, \mu) - (0, \sigma \lambda_{k+1}^\top, 0)^\top \| \\
&\leq \varepsilon_k + \sigma \| \lambda_{k+1} \| \\
&< \| \Phi(\widehat{w}_k, \lambda_{k+1}, \rho_{k+1}, \sigma, \mu) \| + \sigma \| \lambda_{k+1} \| \\
&\leq \| F(\widehat{w}_k, \rho_{k+1}) - F(\mathbf{w}(\rho_{k+1}), \rho_{k+1}) \| + 2\sigma \| \lambda_{k+1} \| \\
&\leq L_2 \| \widehat{w}_k - \mathbf{w}(\rho_{k+1}) \| + 2\sigma \| \lambda_{k+1} \|.
\end{aligned}$$

By defining $C_1 := \frac{1}{L_1} \max\{L_2, 2\sigma\}$, the inequality (4.16) is true for all $k \geq k_0$. This inequality also holds for all $k < k_0$ by increasing the constant C_1 if necessary. $\square$

The next lemma gives a lower bound on the step length computed by applying the fraction to the boundary rule (4.8). The proof is given in [14, Corollary 1].

Lemma 4.11. *For all $k \in \mathbb{N}$, the step length α_k computed by (4.8) satisfies*

$$1 - \alpha_k \leq 1 - \tau_k + b \| w_k - w_k^+ \|, \tag{4.17}$$

where $b = \dfrac{1}{a - \bar{r}} > 0.$

The first step of the asymptotic analysis is to show that the distance of the iterates to the trajectory $\mathbf{w}$ is upper bounded by a constant times the feasibility parameter.

Lemma 4.12. *The iterates w_k generated by Algorithm 3 satisfy*

$$\| w_k - \mathbf{w}(\rho_k) \| = \mathrm{O}(\rho_k).$$

Proof. First of all, let us prove that there exist constants $D_1 \in (0, 1)$, $D_2 > 0$ and $D_3 > 0$ such that for all $k \in \mathbb{N}$,

$$d_{k+1} \leq d_k^2 + D_1 \frac{\rho_{k+1}}{\rho_k} d_k + D_2 \rho_{k+1}, \tag{4.18}$$

where $d_k := D_3 \| w_k - \mathbf{w}(\rho_k) \|$. Indeed, if a sequence $\{d_k\}$ satisfies this inequality, it has been proved in [14, Lemma 7] that $d_k = \mathrm{O}(\rho_k)$ and thus the conclusion of the lemma will follow.

Let us choose $k \geq 0$. Invoking inequality (4.16) and Lemma 4.7-(iii), we get

$$\begin{aligned} C_1^{-1}\|w_{k+1} - \mathbf{w}(\rho_{k+1})\| &\leq \|\widehat{w}_k - \mathbf{w}(\rho_k^+)\| + \|\mathbf{w}(\rho_k^+) - \mathbf{w}(\rho_{k+1})\| + \|\lambda_{k+1}\| \\ &\leq \|\widehat{w}_k - \mathbf{w}(\rho_k^+)\| + C|\rho_k^+ - \rho_{k+1}| + \|\lambda_{k+1}\|. \end{aligned} \tag{4.19}$$

By using the definition of ρ_{k+1} at Step 7 of Algorithm 3, one has

$$\begin{aligned} |\rho_k^+ - \rho_{k+1}| &= (1-\alpha_k)(\rho_k - \rho_k^+) \\ &\leq (1-\alpha_k)\rho_k \\ &\leq (1 - \tau_k + b\|w_k - w_k^+\|)\rho_k, \end{aligned} \tag{4.20}$$

where the last inequality is due to (4.17). In the same manner, we also have

$$\widehat{w}_k - \mathbf{w}(\rho_k^+) = a_k \circ (w_k^+ - \mathbf{w}(\rho_k^+)) + (e - a_k) \circ (w_k - \mathbf{w}(\rho_k^+)).$$

Taking the norm on both sides and noting that $\|a_k\|_\infty \leq 1$ and $\|e - a_k\|_\infty \leq 1 - \alpha_k$, we obtain

$$\begin{aligned} \|\widehat{w}_k - \mathbf{w}(\rho_k^+)\| &\leq \|a_k\|_\infty \|w_k^+ - \mathbf{w}(\rho_k^+)\| + \|e - a_k\|_\infty \|w_k - \mathbf{w}(\rho_k^+)\| \\ &\leq \|w_k^+ - \mathbf{w}(\rho_k^+)\| + (1-\alpha_k)\|w_k - \mathbf{w}(\rho_k^+)\|. \end{aligned}$$

Applying Lemma 4.7-(iii), Lemma 4.9 and inequality (4.17) to the previous inequality, we deduce

$$\begin{aligned} \|\widehat{w}_k - \mathbf{w}(\rho_k^+)\| &\leq \|w_k^+ - \mathbf{w}(\rho_k^+)\| + (1-\alpha_k)(\|w_k - \mathbf{w}(\rho_k)\| + C\rho_k) \\ &\leq M(\|w_k - \mathbf{w}(\rho_k)\|^2 + \rho_k^2 + \|\lambda_{k+1}\|) \\ &\quad + (1 - \tau_k + b\|w_k - w_k^+\|)(\|w_k - \mathbf{w}(\rho_k)\| + C\rho_k). \end{aligned} \tag{4.21}$$

By using Lemma 4.7-(iii), Lemma 4.9, $\|w_k - \mathbf{w}(\rho_k)\| \leq 2\bar{r}$ and $\rho_k \leq \rho^*$, we obtain

$$\begin{aligned} \|w_k - w_k^+\| &\leq \|w_k - \mathbf{w}(\rho_k)\| + \|\mathbf{w}(\rho_k) - \mathbf{w}(\rho_k^+)\| + \|\mathbf{w}(\rho_k^+) - w_k^+\| \\ &\leq \|w_k - \mathbf{w}(\rho_k)\| + C\rho_k + M(\|w_k - \mathbf{w}(\rho_k)\|^2 + \rho_k^2 + \|\lambda_{k+1}\|) \\ &\leq K_1\|w_k - \mathbf{w}(\rho_k)\| + K_2\rho_k + M\|\lambda_{k+1}\|, \end{aligned} \tag{4.22}$$

where $K_1 = 1 + 2M\bar{r}$ and $K_2 = C + M\rho^*$. Combining (4.20)-(4.22) into (4.19), using again $\|w_k - \mathbf{w}(\rho_k)\| \leq 2\bar{r}$ and also the inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ for two real numbers a and b, we obtain

$$\begin{aligned} C_1^{-1}\|w_{k+1}-\mathbf{w}(\rho_{k+1})\| &\leq C_2\|w_k-\mathbf{w}(\rho_k)\|^2+(1-\tau_k)\|w_k-\mathbf{w}(\rho_k)\| \\ &\quad + C_3\rho_k^2+2C(1-\tau_k)\rho_k+C_4\|\lambda_{k+1}\|, \end{aligned}$$

where $C_2 = M + bK_1 + bCK_1 + \frac{1}{2}bK_2$, $C_3 = M + 2bCK_2 + bCK_1 + \frac{1}{2}bK_2$ and $C_4 = M + 2bCM + 2bM\bar{r} + 1$. By using the properties (4.14) and (4.15), we get

$$C_1^{-1}\|w_{k+1}-\mathbf{w}(\rho_{k+1})\|\leq C_2\|w_k-\mathbf{w}(\rho_k)\|^2+\mathrm{o}\left(\frac{\rho_k^+}{\rho_k}\right)\|w_k-\mathbf{w}(\rho_k)\|+\mathrm{o}(\rho_k^+)+C_4\|\lambda_{k+1}\|.$$

Now, we use $\|\lambda_{k+1}\|= \mathrm{o}(\rho_k^+)$ given in Lemma 4.6 and $\rho_k^+ \leq \rho_{k+1}$ to get

$$\|w_{k+1}-\mathbf{w}(\rho_{k+1})\|\leq C_1C_2\|w_k-\mathbf{w}(\rho_k)\|^2+\mathrm{o}\left(\frac{\rho_{k+1}}{\rho_k}\right)\|w_k-\mathbf{w}(\rho_k)\|+\mathrm{o}(\rho_{k+1}).$$

Multiplying both sides by $D_3 := C_1C_2$, for k large enough, we get

$$d_{k+1} \leq d_k^2 + \mathrm{o}\left(\frac{\rho_{k+1}}{\rho_k}\right) d_k + \mathrm{o}(\rho_{k+1}).$$

By increasing the constant $D_2 > 0$ if necessary, inequality (4.18) is satisfied for all $k \in \mathbb{N}$. □

Lemma 4.13. *The Newton-like iterate w_k^+ generated at Step 5 of Algorithm 3 satisfies*

$$\|w_k - w_k^+\|= \mathrm{O}(\rho_k).$$

Proof. Let $k \in \mathbb{N}$. In view of Lemmas 4.9, 4.12, 4.6 and the relation (4.14), we get

$$\|w_k^+ - \mathbf{w}(\rho_k^+)\|\leq M(\|w_k-\mathbf{w}(\rho_k)\|^2+\rho_k^2+\|\lambda_{k+1}\|) = \mathrm{o}(\rho_k^+). \tag{4.23}$$

By virtue of Lemmas 4.12, 4.7-(iii), (4.23) and (4.13), we then have

$$\begin{aligned} \|w_k - w_k^+\| &\leq \|w_k-\mathbf{w}(\rho_k)\|+\|\mathbf{w}(\rho_k)-\mathbf{w}(\rho_k^+)\|+\|\mathbf{w}(\rho_k^+)-w_k^+\| \\ &\leq \mathrm{O}(\rho_k)+C|\rho_k-\rho_k^+|+\mathrm{o}(\rho_k^+) \\ &= \mathrm{O}(\rho_k). \end{aligned}$$

□

The following lemma shows that iterate $\widehat{w}_k$ is asymptotically tangent to the trajectory $\mathbf{w}$.

Lemma 4.14. *The candidate iterate $\widehat{w}_k$ computed at Step 7 of Algorithm 3 satisfies*

$$\|\widehat{w}_k - \mathbf{w}(\rho_{k+1})\| = \mathrm{o}(\rho_{k+1}).$$

Proof. Let $k \in \mathbb{N}$. A first order Taylor expansion of $\mathbf{w}$ at $\rho = 0$ and the definition of ρ_{k+1} at Step 7 of Algorithm 3 yield

$$\begin{aligned}
\mathbf{w}(\rho_{k+1}) &= w^* + \mathbf{w}'(0)\rho_{k+1} + \mathrm{o}(\rho_{k+1}) \\
&= \alpha_k(w^* + \mathbf{w}'(0)\rho_k^+) + (1-\alpha_k)(w^* + \mathbf{w}'(0)\rho_k) + \mathrm{o}(\rho_{k+1}) \\
&= \alpha_k(\mathbf{w}(\rho_k^+) + \mathrm{o}(\rho_k^+)) + (1-\alpha_k)(\mathbf{w}(\rho_k) + \mathrm{o}(\rho_k)) + \mathrm{o}(\rho_{k+1}) \\
&= \alpha_k \mathbf{w}(\rho_k^+) + (1-\alpha_k)\mathbf{w}(\rho_k) + \mathrm{o}(\rho_{k+1}).
\end{aligned}$$

By using the definition of $\widehat{w}_k$ at Step 7 of Algorithm 3, we get

$$\begin{aligned}
\widehat{w}_k - \mathbf{w}(\rho_{k+1}) &= a_k \circ w_k^+ + (e - a_k) \circ w_k - \mathbf{w}(\rho_{k+1}) \\
&= a_k \circ (w_k^+ - \mathbf{w}(\rho_k^+)) + (e - a_k) \circ (w_k - \mathbf{w}(\rho_k)) \\
&\quad + (a_k - \alpha_k e) \circ (\mathbf{w}(\rho_k^+) - \mathbf{w}(\rho_k)) + \mathrm{o}(\rho_{k+1}).
\end{aligned}$$

Taking the norm on both sides, using $\|a_k\|_\infty \leq 1$, $\|e - a_k\|_\infty \leq 1 - \alpha_k$ and $\|a_k - \alpha_k e\|_\infty \leq 1 - \alpha_k$ and then invoking Lemma 4.7-(iii), Lemma 4.12 and inequality (4.23), we deduce that

$$\begin{aligned}
\|\widehat{w}_k - \mathbf{w}(\rho_{k+1})\| &\leq \|a_k\|_\infty \|w_k^+ - \mathbf{w}(\rho_k^+)\| + \|e - a_k\|_\infty \|w_k - \mathbf{w}(\rho_k)\| \\
&\quad + \|a_k - \alpha_k e\|_\infty \|\mathbf{w}(\rho_k^+) - \mathbf{w}(\rho_k)\| + \mathrm{o}(\rho_{k+1}) \\
&\leq \|w_k^+ - \mathbf{w}(\rho_k^+)\| + (1-\alpha_k)\|w_k - \mathbf{w}(\rho_k)\| \\
&\quad + C(1-\alpha_k)\rho_k + \mathrm{o}(\rho_{k+1}) \\
&= \mathrm{o}(\rho_k^+) + \mathrm{O}((1-\alpha_k)\rho_k) + \mathrm{o}(\rho_{k+1}).
\end{aligned} \tag{4.24}$$

The bound (4.17) on the step length gives

$$(1-\alpha_k)\rho_k \leq (1 - \tau_k + b\|w_k - w_k^+\|)\rho_k.$$

Using Lemma 4.13, properties (4.15) and (4.14), we deduce that

$$(1-\alpha_k)\rho_k = \mathrm{o}(\rho_k^+) + \mathrm{O}(\rho_k^2) = \mathrm{o}(\rho_k^+). \tag{4.25}$$

We conclude by substituting (4.25) into (4.24) and by using the fact that $\rho_k^+ \leq$

ρ_{k+1}. □

We now state the main result of this section, which states that the sequence of iterates becomes asymptotically tangent to the trajectory $\mathbf{w}$.

Theorem 4.15. *Under Assumptions 4.1-4.4, if the feasibility parameter satisfies (4.13), if the fraction to the boundary parameter satisfies (4.15), if the tolerance is such that $\varepsilon_k = \Omega(\rho_{k+1})$, then we have the following results*

(i) Algorithm 3 asymptotically needs no inner iterations, i.e., $w_{k+1} = \widehat{w}_k$ for k large enough.

(ii) The iterates generated by Algorithm 3 satisfy

$$\|w_k - \mathbf{w}(\rho_k)\| = \mathrm{o}(\rho_k).$$

(iii) The unit step is asymptotically accepted by the fraction to the boundary rule, i.e., $\alpha_k = 1$ for k large enough.

In particular, $\{w_k\}$ and $\{\rho_k\}$ have the same rate of convergence, i.e.,

$$\|w_k - w^*\| = \Theta(\rho_k). \tag{4.26}$$

Proof. To prove the result (i), it suffices to show that the stopping condition in Step 8 of Algorithm 3 is satisfied for k large enough. According to Lemmas 4.7-(ii), 4.7-(v), 4.14 and 4.6 with noting that $\rho_k^+ \leq \rho_{k+1}$, we have

$$\begin{aligned}
\|\Phi(\widehat{w}_k, \lambda_{k+1}, \rho_{k+1}, \sigma, \mu)\| &\leq \|F(\widehat{w}_k, \rho_{k+1}) - F(\mathbf{w}(\rho_{k+1}), \rho_{k+1}) - (0, \sigma\lambda_{k+1}^\top, 0)^\top\| \\
&\leq L_2\|\widehat{w}_k - \mathbf{w}(\rho_{k+1})\| + \sigma\|\lambda_{k+1}\| \\
&= \mathrm{o}(\rho_{k+1}) \\
&< \varepsilon_k.
\end{aligned}$$

The second result (ii) follows directly from (i) and Lemma 4.14.

The proof of the acceptation of the unit step obtained by the fraction to the boundary rule (4.8) is similar to the one in [15, Lemma 4.20].

From (ii) we have $w_k - w^* = \mathbf{w}'(0)\rho_k + \mathrm{o}(\rho_k)$. The strict complementarity assumption implies that $\mathbf{w}'(0) \neq 0$, from which we deduce (4.26). □

4.4 Numerical experiments

We refer to our algorithm as SPDOPT-ID (Strongly Primal-Dual Optimization with Infeasibility Detection) which has been implemented in C. We compared it with SPDOPT [13] on two sets of problems. The *standard* set consists of 186 problems in the Hock and Schittkowski collection [97] with at least one inequality. The *infeasible* set is created from the *standard* set by adding the constraint $c_1^2+1 = 0$ or $(x_1 - u_1)^2 + 1 = 0$, where c_1 is the first component of c and u_1 is a bound of the first variable ($x_1 \le u_1$ or $x_1 \ge u_1$). It is clear that all problems of the latter set are infeasible.

With a default starting point x_0 and $z_0 = (1, \dots, 1)^\top$, y_0 is defined as the least squares solution of $g_0 + A_0 y - z_0 = 0$. The barrier parameter is initialized by $\mu_0 = 0.1$. If this parameter is updated with a trial value $\mu_k^+ < \mu_k$, we adopt the rule described in [15, Algorithm 2].

The feasibility parameter is initially set to $\rho_0 = 1$. When $\mathtt{F} = 1$, a trial value of the feasibility parameter in Step 3 is updated as follows

$$\rho_k^+ = \min\{0.2\rho_k, \rho_k^{1.4}\}.$$

This choice of ρ_k^+ satisfies the requirement (4.13) with $t = 0.4, \theta_1 = 1, \theta_2 = 0.2$. A lower bound of 10^{-16} is also imposed on this parameter.

For the fraction to the boundary rule, the choice $\tau_k = \max\{0.99, 1 - \rho_k\mu_k\}$ verifies condition (4.15). The regularization parameter δ_k is updated by the following rule: if $\mathtt{F} = 1$ and the condition (4.6) is not satisfied, then $\delta_k = \sigma_k$; otherwise, $\delta_k = \max\{10^{-2}\mu_k, 10^{-8}\}$. It is easy to verify that all assumptions of δ_k in the global and the local analysis are fulfilled. The parameters σ_k and θ_k are updated as in [13, Algorithm 1].

If $\|(g_k + A_k y_k/\rho_k - z_k/\rho_k, c_k, x_k \circ z_k/\rho_k)\|_\infty \le \varepsilon_{\text{tol}}$ with $\varepsilon_{\text{tol}} = 10^{-8}$, the algorithm is terminated and an optimal solution is declared to be found. Otherwise, if $\|c_k\| > \varepsilon_{\text{tol}}, \|\Phi(w_k, 0, 0, \sigma_k, \mu_k)\|_\infty \le \varepsilon_{\text{tol}}$ and $\rho_k \le \varepsilon_{\text{tol}}$, the algorithm is stopped at an infeasible stationary point. For SPDOPT, the stopping conditions $\|c_k\| > \varepsilon_{\text{tol}}, \|A_k c_k \circ x_k\|_\infty \le \varepsilon_{\text{tol}}$ and $\sigma_k \le \varepsilon_{\text{tol}}$ are added to terminate this algorithm at an infeasible stationary point.

For the aim of getting a fast local convergence when the algorithm converges to an infeasible stationary point, the feasibility tolerance at Step 1 is set to $\epsilon = \varepsilon_{\text{tol}}$. At Step 2 of Algorithm 3, we choose $\kappa = 0.9$, $l = 2$ and $\zeta_k = 10\sigma_k\rho_k$ for all iteration

k. The sequence of tolerance $\{\varepsilon_k\}$ in Step 8 is defined by the following formula

$$\varepsilon_k = 0.9 \max\{\|\Phi(w_i, \lambda_i, \rho_i, \sigma_i, \mu_i)\| \colon (k-4)_+ \leq i \leq k\} + 10 \min\{\alpha_k^x, \alpha_k^z\}^{0.2} \mu_{k+1}\rho_{k+1}.$$

By applying [16, Proposition 1], it is easy to see that $\{\varepsilon_k\}$ converges to zero. This choice meets the requirements to get a fast convergence in the feasible case, for which we must have $\varepsilon_k = \Omega(\mu_{k+1})$ (see, [119, Theorem 4.4.1]) and in the infeasible case, for which we must have $\varepsilon_k = \Omega(\rho_{k+1})$ (see, Theorem 4.15).

The linear solver MA57 [58] is used for all the algorithms. The maximum number of iterations, counting both the inner and the outer iterations, is limited to 3000.

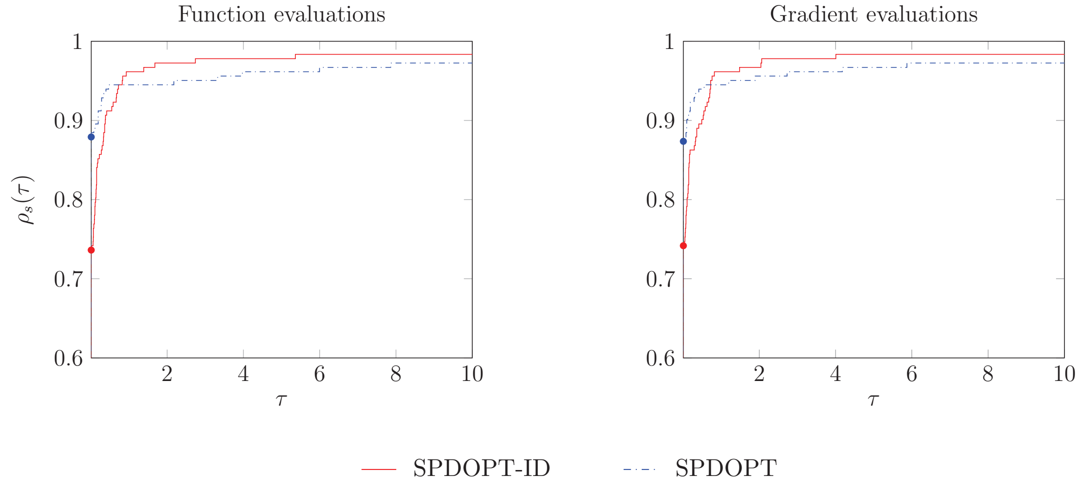


Fig. 4.1: Performance profiles comparing the two algorithms on the set of standard problems

For the standard problems, only 182 problems solved by at least one of two algorithms are selected for the comparison purpose (problems `hs099`, `hs102`, `hs103` and `s332` have not been solved). Figure 4.1 gives us the performance profiles of Dolan and Moré [56] on the numbers of function and gradient evaluations. These profiles show that SPDOPT-ID and its predecessor SPDOPT have their own strengths in terms of efficiency and robustness. In particular, SPDOPT is slightly more efficient than SPDOPT-ID (14%). SPDOPT-ID is slightly more robust than SPDOPT, because it solves 180 problems, while SPDOPT solves 178 problems. We can conclude that the infeasibility detection does not impact the performances of the original algorithm (SPDOPT) for solving standard problems.

Figure 4.2 shows the performances of these algorithms in terms of numbers of function and gradient evaluations on a set of 174 infeasible problems

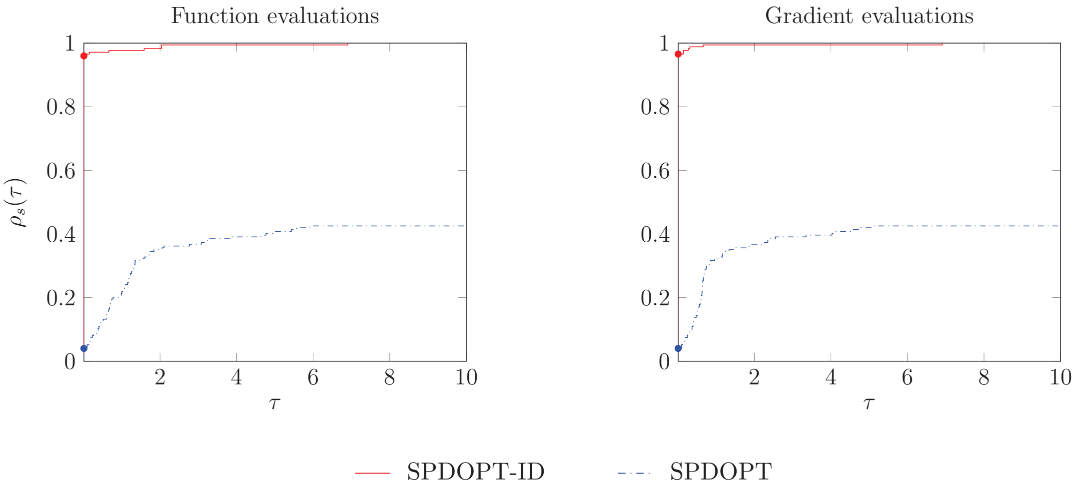


Fig. 4.2: Performance profiles comparing the two algorithms on the set of infeasible problems

(the problems `hs044`, `hs057`, `hs064`, `hs083`, `hs084`, `hs099`, `hs105`, `s220`, `s331`, `s332`, `s357`, `s376` have been eliminated since two algorithms cannot detect the infeasibility). We observe that SPDOPT-ID is the most efficient algorithm for detecting infeasible problems. In particular, the efficiency rate of SPDOPT-ID is over 95%. In terms of robustness, SPDOPT-ID is also more robust than SPDOPT since it can detect more than 93% of problems (174 problems), while the rate of SPDOPT is only 40%.

Finally, we observed that amongst these 174 problems, there are 139 (respectively 137 and 133) for which there is no inner iteration during the last three (respectively four and five) last outer iterations. These observations confirm the asymptotic property (i) of Theorem 4.15.

Chapter 5

Local convergence of a primal-dual method for bound constrained optimization without SOSCs

The current chapter deals with the convergence of an interior point algorithm for solving a bound constrained optimization problem of the form:

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f(x)\ \text{ subject to } x \geq 0, \tag{5.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function. In the literature, the local convergence analysis is usually done under the second order sufficient conditions (SOSCs) and the strict complementarity (SC) assumptions. However, in practice, the SOSCs are not satisfied. For example, suppose that we minimize the function $f(x) = \frac{1}{2}\|c(x)\|^2$, where $c : \mathbb{R}^n \to \mathbb{R}^m$ is smooth with $m < n$. For any solution $x \in \mathbb{R}^n$ such that $x \geq 0$ and $c(x) = 0$, the matrix $\nabla^2 f(x) = \nabla c(x)\nabla c(x)^\top$ is singular, i.e., the SOSCs does not holds at this solution.

We propose to replace this assumption by a weaker one based on a local error bound condition. This condition can be seen as a natural extension of the local error bound condition in unconstrained optimization. In addition, because the Jacobian matrix can be locally singular or nearly singular, we need to introduce a regularization technique to our algorithm in order to get a superlinear or quadratic rate of convergence. These regularization techniques are usual for solving nonlinear equations [61, 63, 164], for unconstrained optimization [11, 95, 107, 143] and for constrained optimization [41, 72, 142]. Our algorithm belongs to the class of interior point methods. The local convergence analysis for this algorithm differs from previous ones in the literature on some following points. *Firstly*, there is no requirement on the proximity to the central trajectory of iterates in our algorithm as in analyses of Armand and Benoist [9], Byrd *et al.* [33], Gould *et al.* [89, 90]. In fact, the central path does not exist because of the absence of the conventional assumption - the second order sufficient conditions. Our analysis is proceeded under a weaker assumption concerning the gradient of the objective function. *Secondly*, we demonstrate that the regularization technique will not affect the rapid convergence of the interior point method. More specifically, with a suitable choice of parameters, we will show that our method converges superlinearly to the solution under milder conditions than classical conditions. In the literature, Tseng and Yun [142] demonstrated the linear convergence of their regularized algorithm. Our research can be seen as a continuation of Armand and Lankoandé [11] for bound constrained optimization. We will apply their idea of regularization parameter update and follow a similar process to demonstrate the fast convergence. Besides, the appearance of constraints makes some differences between our analysis and the one of [11]. In particular, we introduce some upper bounds on the inverse of a regularized Jacobian matrix. One of these bounds is a generalization of [10, Theorem 1].

5.1 Local algorithm

Let $w = (x, z) \in \mathbb{R}^{2n}$ be a vector of primal variables and Lagrange multipliers associated to the constraints of problem (5.1). The first order optimality conditions of this problem can be written as

$$F(w) = 0 \quad \text{and} \quad w \geq 0,$$

where $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by

$$F(w) = \begin{pmatrix} \nabla f(x) - z \\ XZe \end{pmatrix},$$

In order to solve the problem (5.1), we introduce a regularized primal-dual interior point algorithm, by considering a sequence of problems of the form

$$\min_{\substack{x\in\mathbb{R}^n\\ x>0}} f(x) + \frac{\theta_k}{2}\|x - x_k\|^2 - \mu_k \sum_{i=1}^{n} \log x_i,$$

where x_k is the iterate at iteration k, μ_k is the barrier parameter and θ_k is the regularization parameter. These problems belong to the class of proximal algorithms for solving the nonlinear optimization problems, see e.g., Parikh and Boyd [121].

From an initial point $w_0 = (x_0, z_0) > 0$ and $\mu_{-1} > 0$, the regularization and the barrier parameters are chosen such that for all $k \in \mathbb{N}$,

$$\theta_k = \gamma_1 \|F(w_k)\|^\sigma, \tag{5.2}$$

$$\mu_k = \gamma_2 \min\{\|F(w_k)\|^{1+\sigma}, \mu_{k-1}\}, \tag{5.3}$$

where $\gamma_1 > 0$, γ_2, $\sigma \in (0, 1)$. The choice of the regularization parameter is related to the one in [11, 143]. The sequence $\{\delta_k\}$ is chosen such that

$$\max\left\{0, -\lambda_{\min}\left(\nabla^2 f(x_k) + X_k^{-1} Z_k\right)\right\} \le \delta_k \le \beta \max\left\{0, -\lambda_{\min}\left(\nabla^2 f(x_k) + X_k^{-1} Z_k\right)\right\}, \tag{5.4}$$

where $\beta \ge 1$ is a given constant. We will propose a simple algorithm to satisfy condition (5.4), see Algorithm 6 in Section 5.4. A Newton iterate $w_k^+ = (x_k^+, z_k^+)$ is then computed by solving the linear system

$$\begin{pmatrix} H_k + \theta_k I & -I \\ Z_k & X_k \end{pmatrix} \begin{pmatrix} x_k^+ - x_k \\ z_k^+ - z_k \end{pmatrix} = - \begin{pmatrix} \nabla f(x_k) - z_k \\ X_k Z_k e - \mu_k e \end{pmatrix}. \tag{5.5}$$

where $H_k = \nabla^2 f(x_k) + \delta_k I$. We note that (5.4) and $\theta_k > 0$ imply that the matrix $H_k + X_k^{-1} Z_k + \theta_k I$ is positive definite, which will be useful to guarantee that the Newton direction is a descent direction of some merit function. To maintain the strict feasibility of the iterates, the *fraction to the boundary rule* is applied: compute a step-length α_k as the greatest $\alpha \in (0, 1]$ such that

$$w_k + \alpha(w_k^+ - w_k) \ge (1 - \tau_k) w_k, \tag{5.6}$$

where $\tau_k \in (0, 1)$ is chosen such that

$$1 - \tau_k = \mathrm{O}(\|F(w_k)\|). \tag{5.7}$$

The new iterate is then set according to

$$w_{k+1} = w_k + \alpha_k(w_k^+ - w_k). \tag{5.8}$$

In the sequel, the algorithm defined by (5.2)-(5.8) will be called as the *local algorithm*.

5.2 Local convergence analysis

5.2.1 Assumptions and preliminary results

Let $\mathcal{X} = \{x \in \mathbb{R}^n : x \text{ is a solution of } (5.1)\}$ be the set of primal solution and $\mathcal{S} = \{(x, \nabla f(x)) : x \in \mathcal{X}\}$ be the set of primal-dual solution of (5.1). We assume that the original minimization problem has a local solution $x^* \in \mathcal{X}$. We remind that $\mathcal{A} := \{i : x_i^* = 0\}$ is the index set of active bounds. Let us denote by $\mathcal{A}^c$ the index set of inactive bounds. The partial derivatives are denoted by $\partial_i f = \dfrac{\partial f}{\partial x_i}$, for $i = 1, \dots, n$.

The local analysis of the *local algorithm* will be done under the following assumptions.

Assumption 5.1. The function f is twice continuously differentiable and their second derivatives are Lipschitz continuous over an open neighborhood of $\mathcal{X}$.

Let $z^* = \nabla f(x^*)$ be a vector of Lagrange multipliers associated to the bound constraints at x^*.

Assumption 5.2. Strict complementarity holds at $w^* = (x^*, z^*)$ (Definition 2.26), i.e., $a := \min\{x_i^* + z_i^* : i = 1, \dots, n\} > 0$.

We note that if SC holds at w^*, then it also holds at another solution belonging to some neighborhoods of this solution. Indeed, for all $i \in \mathcal{A}$, we have $z_i^* = \partial_i f(x^*) > 0$. From Assumption 5.1 related to the continuity of ∇f, we deduce that there exists a neighborhood of x^* such that for all $i \in \mathcal{A}$, $\partial_i f(x)$ is positive for all x belonging to this neighborhood. Hence, if $w \in \mathcal{S}$ is near w^*, SC is satisfied at w.

We now introduce our new assumption related to the local error bound condition.

Assumption 5.3. The Hadamard product $x \circ \nabla f(x)$ provides a local error bound at $x^* \in \mathcal{X}$, i.e., there exist $\kappa > 0$ and $r > 0$ such that

$$d(x, \mathcal{X}) \leq \kappa \|x \circ \nabla f(x)\| \quad \text{for all } x \in B(x^*, r). \tag{5.9}$$

The local error bound condition (5.9) is a natural extension of the one in unconstrained optimization. Indeed, in unconstrained optimization, one has $\mathcal{A} = \emptyset$. By virtue of Assumption 5.2, there exists $\varepsilon > 0$ such that for all $x \in B(x^*, \varepsilon)$, $x \geq \frac{a}{2}$. Hence, the condition (5.9) can be restated under the form: there exist $\kappa > 0$ and $r \in (0, \varepsilon)$ such that for all $x \in B(w^*, r)$, one has $d(x, \mathcal{X}) \leq \kappa \|\nabla f(x)\|$. This is known as local error bound condition for the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} f(x).$$

The next result gives us a necessary and sufficient condition for this local error bound condition.

Proposition 5.1. *The Hadamard product $x \circ \nabla f(x)$ provides a local error bound at $x^* \in \mathcal{X}$ as in (5.9) if and only if the function F provides a local error bound at some $w^* \in \mathcal{S}$ in the sense that there exist constants $b > 0$ and $\eta > 0$ such that $d(w, \mathcal{S}) \leq b\|F(w)\|$, for all $w \in B(w^*, \eta)$.*

Proof. Assume that the Hadamard product $x \circ \nabla f(x)$ provides a local error bound at $x^* \in \mathcal{X}$, i.e., there exist constants $r > 0$ and $\kappa > 0$ such that (5.9) holds. Recall that for $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, we have

$$u \circ v = UVe \quad \text{and} \quad \|u \circ v\| = \|Uv\| \leq \|U\|\|v\| \leq \|u\|\|v\|,$$

where $U = \operatorname{diag}(u)$ and $V = \operatorname{diag}(v)$. For all $x \in B(x^*, r)$ and $z \in \mathbb{R}^n$, we then have

$$\begin{aligned} \|x \circ \nabla f(x)\| &= \|x \circ (\nabla f(x) - z) + XZe\| \\ &\leq \|x \circ (\nabla f(x) - z)\| + \|XZe\| \\ &\leq \|x\|\|\nabla f(x) - z\| + \|XZe\| \\ &\leq (\|x^*\| + r + 1)\|F(w)\|. \end{aligned}$$

Let $M := \kappa(\|x^*\| + r + 1) > 0$ and $w = (x, z) \in B(w^*, r)$. By combining the last inequality and (5.9), we then get

$$\|x - \bar{x}\| = d(x, \mathcal{X}) \leq \kappa \|x \circ \nabla f(x)\| \leq M\|F(w)\|. \tag{5.10}$$

From Assumption 5.1, there exists $l_1 > 0$ such that $\|\nabla f(x) - \nabla f(y)\| \leq l_1 \|x - y\|$, for all $x, y \in B(x^*, r)$. We then get

$$\begin{aligned}\|z - \nabla f(\bar{x})\| &\leq \|z - \nabla f(x)\| + \|\nabla f(x) - \nabla f(\bar{x})\| \\ &\leq \|F(w)\| + l_1\|x - \bar{x}\| \\ &\leq (1 + l_1 M)\|F(w)\|. \end{aligned} \tag{5.11}$$

Let $b = M + 1 + l_1 M$. By definition of the distance function and noting that $(\bar{x}, \nabla f(\bar{x})) \in \mathcal{S}$, then using the two inequalities (5.10) and (5.11), we have

$$d(w, \mathcal{S}) \leq \|w - (\bar{x}, \nabla f(\bar{x})) \leq |x - \bar{x}\| + \|z - \nabla f(\bar{x})\| \leq b\|F(w)\|,$$

which implies that F provides the local error bound at w^*.

Conversely, we suppose that there exist $b > 0$ and $\eta > 0$ such that $d(w, \mathcal{S}) \leq b\|F(w)\|$ for all $w \in B(w^*, \eta)$. Let us define $r = \frac{\eta}{\sqrt{1+l_1^2}}$, where l_1 is the Lipschitz constant related to ∇f in the ball $B(x^*, \eta)$. Let $x \in B(x^*, r)$ and set $w = (x, \nabla f(x))$. By virtue of Assumption 5.1, we get

$$\|w - w^*\|^2 = \|x - x^*\|^2 + \|\nabla f(x) - \nabla f(x^*)\|^2 \leq (1 + l_1^2)\|x - x^*\|^2 \leq \eta^2,$$

which implies that $w \in B(w^*, \eta)$. Consequently, we obtain

$$d(x, \mathcal{X}) \leq d(w, \mathcal{S}) \leq b\|F(w)\| = b\|x \circ \nabla f(x)\|,$$

meaning that the Hadamard product $x \circ \nabla f(x)$ provides a local error bound. □

We recall the second order sufficient conditions (SOSCs) for problem (5.1) which are used very often in local convergence analysis of optimization algorithms: the SOSCs are satisfied at w^* if $u^\top \nabla^2 f(x^*) u > 0$ for all $u \neq 0$ satisfying $u_i = 0$ for all $i \in \mathcal{A}$. The next result shows us that the SOSCs are sufficient conditions for our local error bound condition.

Proposition 5.2. *Under Assumptions 5.1, 5.2, the SOSCs imply that the local error bound condition defined in Assumption 5.3 holds.*

Proof. The SOSCs and Assumptions 5.1, 5.2 imply that $\mathcal{S} = \{x^*\}$. By virtue of 2.8, the matrix $F'(w)$ is nonsingular in a neighborhood of w^*. The Taylor's expansion of F at w^* gives us

$$F(w) = F(w^*) + F'(w^*)(w - w^*) + \mathrm{o}(\|w - w^*\|).$$

We then imply that

$$d(w,\mathcal{S}) = \|w - w^*\| \leq \frac{\|F'(w^*)^{-1}\|}{1+\mathrm{o}(1)}\|F(w)\|,$$

from which F provides a local error bound at w^*. It implies from Proposition 5.1 that Assumption 5.3 is also satisfied. □

Conversely, the following example shows us that Assumption 5.3 is milder than the SOSCs.

Example 5.3. Let consider optimization problem in $\mathbb{R}^2$

$$\begin{aligned} \min \quad & f(x) := \frac{1}{2}(x_1 - x_2)^2 \\ \text{s.t.} \quad & x \geq 0. \end{aligned} \tag{5.12}$$

For all $x \in \mathbb{R}^n$, we have

$$\nabla f(x) = \begin{pmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The set of primal solutions of (5.12) satisfying the strict complementarity condition is

$$\mathcal{X} = \{(\lambda,\lambda) : \lambda > 0\}.$$

We note that the matrix $\nabla^2 f(x)$ is indefinite for all $x \in \mathbb{R}^2$. It follows that the SOSCs are not satisfied at any $x^* \in \mathcal{X}$. We now show that $x \circ \nabla f(x)$ provides a local error bound at $x^* = (\lambda,\lambda) \in \mathcal{X}$, for $\lambda > 0$ from which we imply the existence of local error bound condition (5.9). Indeed, let $r = \lambda > 0$, for all $x \in B(x^*, r)$, we note that $x > 0$ and

$$x_1 + x_2 \geq \frac{1}{2\lambda}(x_1^2 + x_2^2 + \lambda^2) \geq \frac{\lambda}{2}.$$

Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ for positive real numbers a, b and the above one, for all $x \in B(x^*, r)$, we obtain

$$\begin{aligned} d(x, \mathcal{X})^2 &= \|x - \bar{x}\|^2 \\ &= \frac{1}{2}(x_1 - x_2)^2 \qquad \text{(see Figure 5.1)} \\ &\leq \frac{4}{\lambda^2}\frac{1}{2}(x_1 + x_2)^2 (x_1 - x_2)^2 \\ &\leq \frac{4}{\lambda^2}(x_1^2 + x_2^2)(x_1 - x_2)^2 \\ &\leq \left(\frac{2}{\lambda}\right)^2 \|x \circ \nabla f(x)\|^2 \end{aligned}$$

which implies that the local error bound (5.9) holds at x^* with $\kappa = \frac{2}{\lambda}$ and $r = \lambda$.

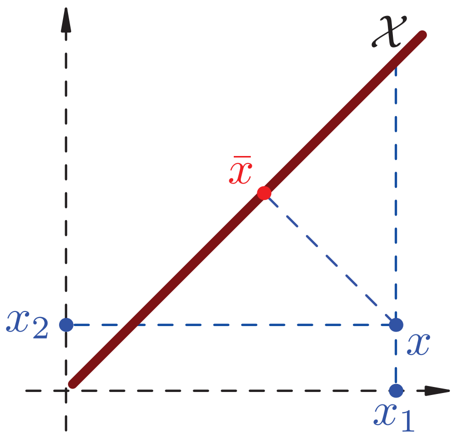


Fig. 5.1: The projection onto the solution set $\mathcal{X} = \{(\lambda, \lambda) : \lambda > 0\}$.

Some local error bound conditions are introduced in the literature. For example, Wang *et al.* [151] proposed a local error bound condition used in the framework of a trust region method for minimizing a nonlinear function over a convex set by means of inequalities. This local error bound condition is deduced from the notion of weak sharp minima [29] and is used for analyzing the convergence and the finite termination of their algorithm. Tseng [140] defined a local error bound condition for the complementarity problem: there exist constants $r_1 > 0$ and $\kappa_1 > 0$ such that

$$d(x, \mathcal{X}) \leq \kappa_1 \|\min\{x, \nabla f(x)\}\| \quad \text{whenever} \quad \|\min\{x, \nabla f(x)\}\| \leq r_1. \tag{5.13}$$

This error bound condition differs from ours, in the sense that it is not related to the choice of an optimal solution. Despite this difference, we now show that if the error bound condition of Tseng is satisfied, then at any optimal solution

$x^* \in \mathcal{X}$ at which Assumptions 5.1 and 5.2 hold, the local error bound condition of Assumption 5.3 is satisfied, but the converse is not true.

Proposition 5.4. *Under Assumptions 5.1-5.2, the local error bound condition of Tseng (5.13) implies that the local error bound condition of Assumption 5.3 is satisfied, but the converse is not true.*

Proof. Suppose that there exist $\kappa_1 > 0$ and $r_1 > 0$ such that (5.13) holds. Recall that from the SC Assumption 5.2, we have

$$a = \min\left\{\min_{i\in\mathcal{A}^c} x_i^*, \min_{x\in\mathcal{A}} \partial_i f(x^*)\right\} > 0.$$

By continuity, there exists $0 < r \le r_1$ such that for all $x \in B(x^*, r)$

$$\min\{x_i, \partial_i f(x)\} = x_i \quad \text{and} \quad \partial_i f(x) > \frac{a}{2} \quad \text{for all } i \in \mathcal{A}$$

and

$$\min\{x_i, \partial_i f(x)\} = \partial_i f(x) \quad \text{and} \quad x_i > \frac{a}{2} \quad \text{for all } i \notin \mathcal{A}.$$

From these facts, for all $x \in B(x^*, r)$, we get

$$\begin{aligned} d(x,\mathcal{X})^2 &\le \kappa_1^2 \|\min\{x, \nabla f(x)\}\|^2 \\ &= \kappa_1^2 \left(\sum_{i\in\mathcal{A}} x_i^2 + \sum_{i\notin\mathcal{A}} \partial_i f(x)^2\right) \\ &< \kappa_1^2 \left(\sum_{i\in\mathcal{A}} x_i^2 \left(\frac{2}{a}\partial_i f(x)\right)^2 + \sum_{i\notin\mathcal{A}} \left(\frac{2}{a}x_i\right)^2 \partial_i f(x)^2\right) \\ &= \left(\frac{2\kappa_1}{a}\right)^2 \|x \circ \nabla f(x)\|^2, \end{aligned}$$

which implies that the condition (5.9) is satisfied with the constant $\kappa = \frac{2\kappa_1}{a}$.

We now show that the converse is not true. Let us consider the problem (5.1) in $\mathbb{R}^2$ where $f(x) = -e^{-x_1} + \frac{1}{2}(x_2 - 1)^2$. The unique solution of this problem is $x^* = (0, 1)$ with the dual solution $z^* = (1, 0)$. The SOSCs are satisfied at x^*, which imply that (5.9) holds. The gradient of f is given by $\nabla f(x) = (e^{-x_1}, x_2 - 1)$. Assume that the condition (5.13) is true. Then there exist positive constants κ_1, r_1 such that $d(x, \mathcal{X}) \le \kappa_1 \|\min\{x, \nabla f(x)\}\|$ for all x satisfying $\|\min\{x, \nabla f(x)\}\| \le r_1$. Let us take $\bar{x}_1 > 0$ sufficiently large such that $\bar{x}_1 > \max\{1, \kappa_1\} e^{-\bar{x}_1}$, $e^{-\bar{x}_1} \le r_1$. By setting $\bar{x} = (\bar{x}_1, 1)$, we then get

$$\|\min\{\bar{x}, \nabla f(\bar{x})\}\| = \left\| \left(\min\{\bar{x}_1, e^{-\bar{x}_1}\}, \quad \min\{1, 0\} \right)^\top \right\| = e^{-\bar{x}_1} \leq r_1.$$

However, one notes that $d(\bar{x}, \mathcal{X}) = \bar{x}_1 > \kappa_1 e^{-\bar{x}_1} = \kappa_1 \|\min\{\bar{x}, \nabla f(\bar{x})\}\|$ which is a contradiction. This means that (5.13) is not valid. □

5.2.2 Regularized Jacobian matrix

In this section, we show that the inverse of a regularized Jacobian matrix is uniformly bounded. The next lemma is a generalization, in the case of a bound constrained optimization problem, of [10, Theorem 1]. The statement of this lemma can be understood in a general context.

Lemma 5.5. *Let $\{H_k\}$ be a sequence of $n \times n$ real symmetric matrices and $\{\rho_k\}$ be a sequence of positive scalars. Let $\{x_k\}$ and $\{z_k\}$ be two positive sequences in $\mathbb{R}^n$. Let us define for all $k \in \mathbb{N}$, $X_k = \operatorname{diag}(x_k)$, $Z_k = \operatorname{diag}(z_k)$ and*

$$J_k := \begin{pmatrix} H_k + \rho_k I & -I \\ Z_k & X_k \end{pmatrix}.$$

Assume that the following properties are satisfied.

(i) The sequences $\{H_k\}$ and $\{\rho_k\}$ are bounded.

(ii) There exists $\nu > 0$ such that for all $k \in \mathbb{N}$, $\max\{x_k, z_k\} \geq \nu$.

(iii) For all $k \in \mathbb{N}$, $H_k + X_k^{-1} Z_k \succeq 0$.

(iv) If $\liminf \rho_k = 0$, then for any subsequence $\{\rho_k\}_\mathcal{K}$ converging to zero, there exist $r > 0$ and $t \in (0,1)$ such that for all $k \in \mathcal{K}$, $\|x_k \circ z_k\|^t \leq r\rho_k$.

Then there exists $C > 0$ such that for all $k \in \mathbb{N}$ the matrix J_k is nonsingular and

$$\|J_k^{-1}\| \leq \frac{C}{\rho_k}.$$

Proof. The idea of the proof is based on [10]. Let us show that for all $k \in \mathbb{N}$ the matrix J_k is nonsingular. By Proposition 2.7, for all $k \in \mathbb{N}$ we have $\det J_k = \det S_k \det X_k$, where $S_k = H_k + \rho_k I + X_k^{-1} Z_k$. Assumption (iii) and $\rho_k > 0$ imply that $S_k \succ 0$. We then deduce that $\det J_k > 0$ and the sequence $\{J_k^{-1}\}$ is well defined.

We now prove the boundedness of the sequence $\{\rho_k \| J_k^{-1} \|\}$ by a contradiction reasoning. Taking a subsequence if necessary, we can suppose that the sequence $\{\rho_k \| J_k^{-1} \|\}$ tends to infinity. From the definition of a matrix norm, there exists a sequence of unit vectors $\{v_k\} \subset \mathbb{R}^{2n}$ such that $\| J_k^{-1} \| = \| J_k^{-1} v_k \|$. Define for all $k \in \mathbb{N}$, $u_k := J_k^{-1} v_k / \| J_k^{-1} \|$. It follows that $\{u_k\}$ is a sequence of unit vectors with $\lim \| \frac{1}{\rho_k} J_k u_k \| = 0$. By introducing the notation

$$u_k = \begin{pmatrix} p_k \\ q_k \end{pmatrix} \in \mathbb{R}^{2n} \quad \text{and} \quad \frac{1}{\rho_k} J_k u_k = \begin{pmatrix} \lambda_k \\ \zeta_k \end{pmatrix},$$

we have

$$\begin{aligned} (H_k + \rho_k I) p_k - q_k &= \rho_k \lambda_k, \\ Z_k p_k + X_k q_k &= \rho_k \zeta_k, \end{aligned} \tag{5.14}$$

where the sequences $\{\lambda_k\}$ and $\{\zeta_k\}$ converge to zero.

Let us show the existence of a subset $\mathcal{I}$ of $\{1, \ldots, n\}$ and of an infinite subset $\mathcal{K}$ of $\mathbb{N}$ such that

$$\liminf_{k \in \mathcal{K}} [x_k]_i > 0 \quad \text{for all } i \in \mathcal{I},$$

and

$$\lim_{k \in \mathcal{K}} [x_k]_i = 0 \quad \text{for all } i \in \mathcal{J} := \{1, \ldots, n\} \backslash \mathcal{I}.$$

This is performed by the following process. At first, define $\mathcal{I}_0 = \emptyset$ and $\mathcal{K}_0 = \mathbb{N}$. Then, for i from 1 to n we do as follows. If

$$\liminf_{k \in \mathcal{K}_{i-1}} [x_k]_i > 0,$$

then set $\mathcal{I}_i = \mathcal{I}_{i-1} \cup \{i\}$ and $\mathcal{K}_i = \mathcal{K}_{i-1}$, otherwise set $\mathcal{I}_i = \mathcal{I}_{i-1}$ and choose $\mathcal{K}_i \subset \mathcal{K}_{i-1}$ such that

$$\lim_{k \in \mathcal{K}_i} [x_k]_i = 0.$$

Finally, let us define $\mathcal{I} := \mathcal{I}_n$, $\mathcal{J} := \{1, \ldots, n\} \backslash \mathcal{I}$ and $\mathcal{K} := \mathcal{K}_n$. The construction of $\mathcal{I}$ and $\mathcal{J}$ implies that

$$\lim_{k \in \mathcal{K}} x_k^{\mathcal{J}} = 0 \tag{5.15}$$

and there exists $\varepsilon > 0$ such that for all $k \in \mathcal{K}$ large enough

$$x_k^{\mathcal{I}} \geq \varepsilon. \tag{5.16}$$

Assumption (ii) and (5.15) imply that for $k \in \mathcal{K}$ large enough

$$\max\{x_k^{\mathcal{J}}, z_k^{\mathcal{J}}\} = z_k^{\mathcal{J}} \geq \nu. \tag{5.17}$$

Let $k \in \mathcal{K}$. By reordering the indices, we rewrite $x_k = (x_k^{\mathcal{I}}, x_k^{\mathcal{J}}) \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$, $z_k = (z_k^{\mathcal{I}}, z_k^{\mathcal{J}}) \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$, and the matrix J_k under the form

$$J_k = \begin{pmatrix} H_k^{\mathcal{I}\mathcal{I}} + \rho_k I & H_k^{\mathcal{I}\mathcal{J}} & -I & 0 \\ H_k^{\mathcal{J}\mathcal{I}} & H_k^{\mathcal{J}\mathcal{J}} + \rho_k I & 0 & -I \\ Z_k^{\mathcal{I}} & 0 & X_k^{\mathcal{I}} & 0 \\ 0 & Z_k^{\mathcal{J}} & 0 & X_k^{\mathcal{J}} \end{pmatrix}.$$

To simply the notation, let us denote

$$\begin{aligned} a_k &:= p_k^{\mathcal{I}}, \quad b_k := p_k^{\mathcal{J}}, \quad c_k := q_k^{\mathcal{I}}, \quad d_k := q_k^{\mathcal{J}}, \\ \alpha_k &:= \lambda_k^{\mathcal{I}}, \quad \beta_k := \lambda_k^{\mathcal{J}}, \quad \gamma_k := \zeta_k^{\mathcal{I}}, \quad \delta_k := \zeta_k^{\mathcal{J}}, \end{aligned}$$

where the sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{\gamma_k\}$ and $\{\delta_k\}$ converge to zero and $\{(a_k, b_k, c_k, d_k)\}$ is a sequence of unit vectors. By using these notations, the system of equations (5.14) becomes

$$\begin{aligned} &(H_k^{\mathcal{I}\mathcal{I}} + \rho_k I)a_k + H_k^{\mathcal{I}\mathcal{J}} b_k - c_k = \rho_k \alpha_k, \\ &H_k^{\mathcal{J}\mathcal{I}} a_k + (H_k^{\mathcal{J}\mathcal{J}} + \rho_k I)b_k - d_k = \rho_k \beta_k, \\ &Z_k^{\mathcal{I}} a_k + X_k^{\mathcal{I}} c_k = \rho_k \gamma_k, \\ &Z_k^{\mathcal{J}} b_k + X_k^{\mathcal{J}} d_k = \rho_k \delta_k. \end{aligned}$$

This implies that

$$\begin{aligned} &\left(H_k^{\mathcal{I}\mathcal{I}} + \rho_k I\right) a_k + H_k^{\mathcal{I}\mathcal{J}} b_k - c_k = \rho_k \alpha_k, \\ &d_k = H_k^{\mathcal{J}\mathcal{I}} a_k + (H_k^{\mathcal{J}\mathcal{J}} + \rho_k I)b_k - \rho_k \beta_k, \\ &c_k = \left(X_k^{\mathcal{I}}\right)^{-1} \left(-Z_k^{\mathcal{I}} a_k + \rho_k \gamma_k\right), \\ &b_k = \left(Z_k^{\mathcal{J}}\right)^{-1} \left(-X_k^{\mathcal{J}} d_k + \rho_k \delta_k\right). \end{aligned} \tag{5.18}$$

The fourth equation in (5.18) and (5.17) give us $\|b_k\| \leq \frac{1}{\nu}(\|x_k^{\mathcal{J}}\| \|d_k\| + \rho_k \|\delta_k\|)$ for all $k \in \mathcal{K}$ large enough. The convergence to zero of $\{\delta_k\}$, the boundedness of $\{d_k\}$

and $\{\rho_k\}$, and (5.15) imply that

$$\lim_{k\in\mathcal{K}} b_k = 0. \tag{5.19}$$

Let us show that

$$\mathcal{I} \neq \emptyset \quad \text{and} \quad \liminf_{k\in\mathcal{K}} \|a_k\| > 0. \tag{5.20}$$

Indeed, if this is not the case, the first two equations of (5.18) imply that for all $k \in \mathbb{N}$,

$$\begin{aligned} \|c_k\| &\leq \|H_k^{\mathcal{I}\mathcal{I}} + \rho_k I\|\|a_k\| + \|H_k^{\mathcal{I}\mathcal{J}}\|\|b_k\| + \rho_k\|\alpha_k\|, \\ \|d_k\| &\leq \|H_k^{\mathcal{J}\mathcal{I}}\|\|a_k\| + \|H_k^{\mathcal{J}\mathcal{J}} + \rho_k I\|\|b_k\| + \rho_k\|\beta_k\|. \end{aligned}$$

According to Assumption (i), (5.19) and the negation of (5.20), we obtain

$$\lim_{k\in\mathcal{K}'} c_k = 0 \quad \text{and} \quad \lim_{k\in\mathcal{K}'} d_k = 0,$$

where $\mathcal{K}' \subset \mathcal{K}$ such that (5.20) is not true, which is in contradiction with the fact that $\{u_k\}$ is the sequence of unit vectors, meaning that (5.20) is true.

By substituting the expression of c_k from the third equation of (5.18) to the first one, then by premultiplying by $a_k^\top$, we get

$$\frac{1}{\rho_k} a_k^\top \left(H_k^{\mathcal{I}\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}} + \rho_k I \right) a_k = a_k^\top \left(-\frac{1}{\rho_k} H_k^{\mathcal{I}\mathcal{J}} b_k + \alpha_k + \left(X_k^{\mathcal{I}}\right)^{-1} \gamma_k \right).$$

By using the inequality (5.16), the above equality implies that for all $k \in \mathcal{K}$ sufficiently large, we have

$$\frac{1}{\rho_k} \left| a_k^\top \left(H_k^{\mathcal{I}\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}} + \rho_k I \right) a_k \right| \leq \|a_k\| \left(\frac{1}{\rho_k} \|H_k^{\mathcal{I}\mathcal{J}}\|\|b_k\| + \|\alpha_k\| + \frac{1}{\varepsilon}\|\gamma_k\| \right). \tag{5.21}$$

According to Assumption (iii), we then have

$$\|a_k\|^2 = \frac{\rho_k}{\rho_k}\|a_k\|^2 \leq \frac{1}{\rho_k} a_k^\top \left(H_k^{\mathcal{I}\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}} + \rho_k I \right) a_k.$$

Combining this inequality and (5.21), and using the boundedness of $\{H_k\}$, for all $k \in \mathcal{K}$ sufficiently large, we have

$$\|a_k\| \leq \frac{1}{\rho_k} \left(\sup_{k\in\mathcal{K}} \|H_k\| \right) \|b_k\| + \|\alpha_k\| + \frac{1}{\varepsilon}\|\gamma_k\|. \tag{5.22}$$

We consider the two following cases. The first case is when $\{\rho_k\}_{\mathcal{K}}$ is bounded away from zero. This implies that the sequence $\left\{\frac{1}{\rho_k}\right\}_{\mathcal{K}}$ is bounded. By using this fact, (5.19) and taking the limit for $k \in \mathcal{K}$ in (5.22), we obtain

$$\lim_{k\in\mathcal{K}} \|a_k\| \leq 0,$$

which is in contradiction with (5.20). Hence, this case cannot happen.

The second case is when there exists an infinite subset $\mathcal{K}'' \subset \mathcal{K}$ such that $\lim_{k\in\mathcal{K}''} \rho_k = 0$. We rewrite the fourth equation of (5.18) under the form

$$b_k = \left(Z_k^{\mathcal{J}}\right)^{-1}\left(-\left(Z_k^{\mathcal{J}}\right)^{-1}\left(Z_k^{\mathcal{J}} X_k^{\mathcal{J}}\right) d_k + \rho_k \delta_k\right).$$

According to Assumption (iv) and (5.17), the above equation implies that there exists $r > 0$ such that for all $k \in \mathcal{K}''$ large enough, we have

$$\begin{aligned}
\|b_k\| &\leq \frac{1}{\nu^2}\|x_k \circ z_k\| \|d_k\| + \frac{1}{\nu}\rho_k\|\delta_k\| \\
&\leq \frac{r^{1/t}}{\nu^2}\rho_k^{1/t}\|d_k\| + \frac{1}{\nu}\rho_k\|\delta_k\|.
\end{aligned}$$

This implies that for all $k \in \mathcal{K}''$ large enough,

$$\frac{1}{\rho_k}\|b_k\| \leq \frac{r^{1/t}}{\nu^2}\rho_k^{1/t-1}\|d_k\| + \frac{1}{\nu}\|\delta_k\|.$$

By using the fact that the sequence $\{d_k\}$ is bounded, the sequences $\{(\alpha_k, \gamma_k, \delta_k)\}$ and $\{\rho_k^{1/t-1}\}_{\mathcal{K}''}$ tend to zero (since $t \in (0,1)$), substituting the last inequality to (5.22) and taking the limit for $k \in \mathcal{K}''$, we obtain

$$\liminf_{k\in\mathcal{K}} \|a_k\| \leq \lim_{k\in\mathcal{K}''} \|a_k\| \leq 0,$$

which is again in contradiction with (5.20).

In sum, we proved that the sequence $\{\rho_k \|J_k^{-1}\|\}$ is bounded from which the proof is completed. □

The next result is a generalization of the previous Lemma for a general nonlinear optimization problem.

Corollary 5.6. *Let $\{H_k\}$ be a sequence of $n \times n$ real symmetric matrices, $\{A_k\}$ be a sequence of $n \times m$ real matrices, and $\{\delta_k\}$ and $\{\rho_k\}$ be two sequences of positive scalars. Let $\{x_k\}$ and $\{z_k\}$ be two positive sequences in $\mathbb{R}^n$. Let us define for all*

$k \in \mathbb{N}$, $X_k = \operatorname{diag}(x_k)$, $Z_k = \operatorname{diag}(z_k)$ and

$$J_k := \begin{pmatrix} H_k + \rho_k I & -I & A_k \\ Z_k & X_k & 0 \\ A_k^\top & 0 & -\delta_k I \end{pmatrix}.$$

Assume that the following properties are satisfied.

(i) The sequences $\{H_k\}$, $\{A_k\}$ and $\{\rho_k\}$ are bounded.

(ii) The sequence $\{\delta_k\}$ is bounded away from zero.

(iii) There exists $\nu > 0$ such that for all $k \in \mathbb{N}$, $\max\{x_k, z_k\} \geq \nu$.

(iv) For all $k \in \mathbb{N}$, $H_k + X_k^{-1}Z_k + \frac{1}{\delta_k}A_kA_k^\top \succeq 0$.

(v) If $\liminf \rho_k = 0$, then for all $\mathcal{K} \subset \mathbb{N}$ such that the subsequence $\{\rho_k\}_{\mathcal{K}}$ goes to zero, there exist $r > 0$ and $t \in (0, 1)$ such that for all $k \in \mathcal{K}$, $\|x_k \circ z_k\|^t \leq r\rho_k$.

Then, there exists $C > 0$ such that for all $k \in \mathbb{N}$ the matrix J_k is nonsingular and

$$\|J_k^{-1}\| \leq \frac{C}{\rho_k}.$$

Proof. Let $k \in \mathbb{N}$. Let us show that for all $k \in \mathbb{N}$ the matrix J_k is nonsingular. By Proposition 2.7, we get $\det J_k = \det S_k \det D_k$, where

$$S_k = H_k + \rho_k I + X_k^{-1}Z_k + \frac{1}{\delta_k}A_kA_k^\top \quad \text{and} \quad D_k = \begin{pmatrix} X_k & 0 \\ 0 & -\delta_k I \end{pmatrix}.$$

From Assumption (iv) and $\rho_k > 0$, one has $S_k \succ 0$. Assumption (ii) and the positivity of the sequence $\{x_k\}$ imply that $\det D_k \neq 0$. We then deduce that $\det J_k > 0$, therefore the sequence $\{J_k^{-1}\}$ is well defined. We now prove that $\|J_k^{-1}\| = \mathrm{O}\left(\frac{1}{\rho_k}\right)$. The definition of a matrix norm implies that there exists a sequence of unit vectors $\{v_k\} \subset \mathbb{R}^{2n+m}$ such that $\|J_k^{-1}\| = \|J_k^{-1}v_k\|$. Define $u_k := J_k^{-1}v_k$. Introduce the notation

$$u_k = \begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix} \in \mathbb{R}^{2n+m} \quad \text{and} \quad v_k = \begin{pmatrix} \alpha_k \\ \beta_k \\ \gamma_k \end{pmatrix}.$$

We then have

$$\begin{aligned} H_k a_k - b_k + A_k c_k &= \alpha_k, \\ Z_k a_k + X_k b_k &= \beta_k, \\ A_k^\top a_k - \delta_k c_k &= \gamma_k. \end{aligned}$$

By eliminating c_k from the first equation, we have

$$\widetilde{J}_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} = r_k, \tag{5.23}$$

where

$$\widetilde{J}_k = \begin{pmatrix} H_k + \rho_k I + \frac{1}{\delta_k} A_k A_k^\top & -I \\ Z_k & X_k \end{pmatrix} \quad \text{and} \quad r_k = \begin{pmatrix} \alpha_k + \frac{1}{\delta_k} A_k \gamma_k \\ \beta_k \end{pmatrix}.$$

Lemma 5.5 implies that $\widetilde{J}_k^{-1} = \mathrm{O}\left(\frac{1}{\rho_k}\right)$. Assumption (i) and the boundedness of $\{v_k\}$ imply that the sequence $\{r_k\}$ is bounded. Two above facts and (5.23) give us

$$\|(a_k, b_k)\| = \|\widetilde{J}_k^{-1} r_k\| = \|\widetilde{J}_k^{-1}\| \|r_k\| = \mathrm{O}\left(\frac{1}{\rho_k}\right). \tag{5.24}$$

By using Assumptions (i) and (ii), (5.24) and $\|c_k\| \leq \frac{1}{\delta_k}(\|A_k^\top\| \|a_k\| + \|\gamma_k\|)$, we deduce that

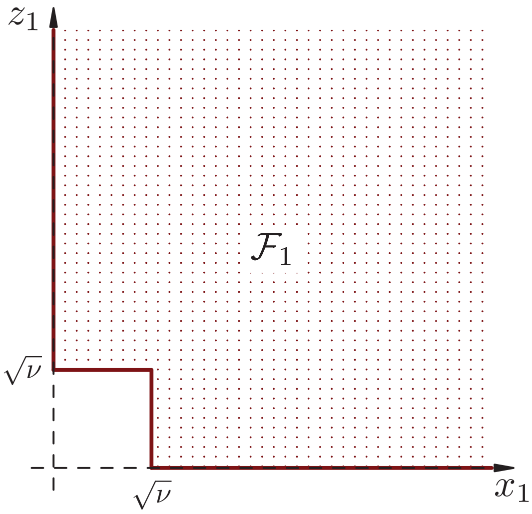
$$\|J_k^{-1}\| = \|J_k^{-1} v_k\| = \|u_k\| = \mathrm{O}\left(\frac{1}{\rho_k}\right).$$

□

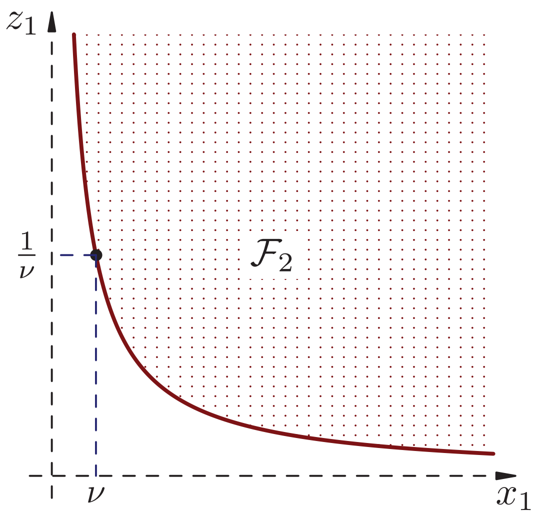
Remark 5.7. Let $\{M_k\}$ be a sequence of $n \times n$ real symmetric matrices, $\{A_k\}$ be a sequence of $n \times m$ real matrices, and $\{\delta_k\}$ be a sequence of positive scalars. Let $\{x_k\}$ and $\{z_k\}$ be two positive sequences in $\mathbb{R}^n$. Let us define for all $k \in \mathbb{N}$, $X_k = \operatorname{diag}(x_k)$, $Z_k = \operatorname{diag}(z_k)$ and

$$\widehat{J}_k := \begin{pmatrix} M_k & -I & A_k \\ Z_k & X_k & 0 \\ A_k^\top & 0 & -\delta_k I \end{pmatrix}.$$

In [10, Theorem 1], it is shown that the sequence $\{\widehat{J}_k^{-1}\}$ is bounded under the following assumptions:



(a) The set $\mathcal{F}_1 = \{(x_1, z_1) : \max\{x_1, z_1\} \geq \sqrt{\nu}\}$.



(b) The set $\mathcal{F}_2 = \{(x_1, z_1) : x_1 z_1 \geq \nu\}$.

Fig. 5.2: The sets $\mathcal{F}_1 = \{(x_1, z_1) : \max\{x_1, z_1\} \geq \nu\}$ and $\mathcal{F}_2 = \{(x_1, z_1) : x_1 z_1 \geq \nu\}$ for $\nu > 0$.

1. The sequences $\{M_k\}$ and $\{A_k\}$ are bounded,
2. The sequence $\{\delta_k\}$ is bounded away from zero,
3. There exists $\nu > 0$ such that for all $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$

$$[x_k]_i [z_k]_i \geq \nu,$$

4. There exists $\lambda > 0$ such that for all $k \in \mathbb{N}$ and all $d \in \mathbb{R}^n$

$$d^\top \left(M_k + X_k^{-1} Z_k + \frac{1}{\delta_k} A_k A_k^\top \right) d \geq \lambda \|d\|.$$

This theorem is a direct consequence of Corollary 5.6. Indeed, by assuming that all above assumptions hold, we will apply this Corollary to deduce the boundedness of $\{\widehat{J}_k^{-1}\}$. For all k, let define $H_k = M_k - \lambda I$ and $\rho_k = \lambda$. By noting that for all k, $\max\{[x_k]_i, [z_k]_i\} \geq \sqrt{[x_k]_i [z_k]_i} \geq \sqrt{\nu}$ for all $i = 1, \ldots, n$ (see Figure 5.2), then Corollary 5.6 implies that there exists $C > 0$ such that $\|\widehat{J}_k^{-1}\| \leq \frac{C}{\lambda}$. This implies that the sequence $\{\widehat{J}_k^{-1}\}$ is bounded.

We now give a result about the uniform boundedness of the inverse of the regularized Jacobian matrices near a point w^*. This result will be useful to analyze the local behavior of the Newton method.

Lemma 5.8. *Let* $w^* = (x^*, z^*) \in \mathbb{R}^{2n}$ *such that*

$$0 \leq x^* \perp z^* \geq 0 \quad \text{and} \quad a := \min\{x_i^* + z_i^* | i = 1, \ldots n\} > 0.$$

Let $H : \mathbb{R}^{2n} \to \mathbb{R}^{n \times n}$ *be a continuous function such that for all* $w \in \mathbb{R}^{2n}$, $H(w) = H(w)^\top$. *Let* $\theta : \mathbb{R}^{2n} \to \mathbb{R}_{++}$ *be a continuous function such that* $\theta(w) \geq \gamma \|x \circ z\|^t$, *for all* $w \in \mathbb{R}^{2n}$, *for some* $\gamma > 0$ *and* $t \in (0, 1)$. *Assume that for all* $w \in \mathbb{R}^{2n}_{++}$, $H(w) + X^{-1}Z \succeq 0$. *For all* $w = (x, z) \in \mathbb{R}^{2n}$, *let us define the matrix*

$$J(w) = \begin{pmatrix} H(w) + \theta(w)I & -I \\ Z & X \end{pmatrix}.$$

For all $w \in \mathbb{R}^{2n}_{++}$, *the matrix* $J(w)$ *is nonsingular and for all* $\varepsilon \in (0, a)$, *there exists* $C > 0$ *such that for all* $w \in B(w^*, \varepsilon) \cap \mathbb{R}^{2n}_{++}$,

$$\|J(w)^{-1}\| \leq \frac{C}{\theta(w)}.$$

Proof. Let us show that for all $w \in \mathbb{R}^{2n}_{++}$, the matrix $J(w)$ is nonsingular. By Proposition 2.7, we have $\det J(w) = \det S(w) \det X$, where $S(w) = H(w) + \theta(w)I + X^{-1}Z$, for all $w \in \mathbb{R}^{2n}_{++}$. Assumptions $x > 0$, $H(w) + X^{-1}Z \succeq 0$ and $\theta(w) > 0$ imply that $X \succ 0$ and $S(w) \succ 0$. We then deduce that $\det J(w) > 0$, therefore the matrix $J(w)^{-1}$ is well defined for all $w \in \mathbb{R}^{2n}_{++}$.

The second part of the lemma is proved by a contradiction reasoning. Suppose that there exist $\varepsilon \in (0, a)$ and a sequence $\{w_k\} \subset B(w^*, \varepsilon) \cap \mathbb{R}^{2n}_{++}$ such that for all $k \in \mathbb{N}$, $H(w_k) + X_k^{-1} Z_k \succeq 0$, $\theta(w_k) \geq \gamma \|x_k \circ z_k\|^t$, but the sequence of matrices $\{\theta(w_k) J(w_k)^{-1}\}$ is unbounded. The continuity of H and θ implies that the sequences $\{H(w_k)\}$ and $\{\theta(w_k)\}$ are bounded. Let us define the set $\mathcal{A} = \{i \in \{1, \ldots, n\} : x_i^* = 0\}$. Definitions of a and of $\mathcal{A}$ imply that $x_i^* \geq a$ for all $i \notin \mathcal{A}$ and $z_i^* \geq a$, for all $i \in \mathcal{A}$. Let $a_1 = a - \varepsilon > 0$. For all $k \in \mathbb{N}$, we then have

$$[x_k]_i \geq x_i^* - \|w_k - w^*\| > a_1, \quad \text{for all } i \notin \mathcal{A}$$

and

$$[z_k]_i \geq z_i^* - \|w_k - w^*\| > a_1, \quad \text{for all } i \in \mathcal{A}.$$

From which we get $0 < a_1 < \max\{x_k, z_k\}$ for all $k \in \mathbb{N}$. By virtue of Lemma 5.5, there exists $C > 0$ such that for all $k \in \mathbb{N}$,

$$\|J(w_k)^{-1}\| \le C\frac{1}{\theta(w_k)},$$

which is in contradiction with the assumption that the sequence $\{\theta(w_k)J(w_k)^{-1}\}$ is unbounded. $\square$

We now give another result related to the upper bound of the inverse of regularized Jacobian matrix at a solution $w \in \mathcal{S}$. From this lemma, we can prove the boundedness of the sequence $\{\delta_k\}$ given in (5.4), see Lemma 5.10 below.

Lemma 5.9. *Let us define*

$$\mathcal{C} = \{w = (x, z) \in \mathbb{R}^{2n} : 0 \le x \perp z \ge 0\}.$$

Let $w^* \in \mathcal{C}$ such that $a := \min\{x_i^* + z_i^* : i = 1, \dots, n\} > 0$. Let $H : \mathbb{R}^{2n} \to \mathbb{R}^{n\times n}$ be a continuous function such that $H(w) = H(w)^\top$ for all $w \in \mathbb{R}^{2n}$. Assume that $u^\top H(w^*)u \ge 0$, for all $u \in \ker(Z^*)$. For all $w = (x, z) \in \mathbb{R}^{2n}$ and $\rho > 0$, let us define the matrix

$$G_\rho(w) = \begin{pmatrix} H(w) + \rho I & -I \\ Z & X \end{pmatrix}.$$

Then, for all $\varepsilon \in (0, a)$, there exists $C > 0$ such that for all $w \in B(w^*, \varepsilon) \cap \mathcal{C}$ and $\rho > 0$, the matrix $G_\rho(w)$ is nonsingular and

$$\|G_\rho(w)^{-1}\| \le C \max\left\{\rho, \frac{1}{\rho}\right\}.$$

Proof. Let us define $\mathcal{A} = \{i \in \{1, \dots, n\} : x_i^* = 0\}$. It implies that $x_i^* \ge a$ for all $i \notin \mathcal{A}$ and $z_i^* \ge a$, for all $i \in \mathcal{A}$. Let $\varepsilon \in (0, a)$ and let us define $a_1 = a - \varepsilon > 0$. For all $w = (x, z) \in B(w^*, \varepsilon)$, we then have

$$x_i \ge x_i^* - \|w - w^*\| > a_1, \quad \text{for all } i \notin \mathcal{A}$$

and

$$z_i \ge z_i^* - \|w - w^*\| > a_1, \quad \text{for all } i \in \mathcal{A}.$$

Let $w = (x, z) \in B(w^*, \varepsilon) \cap \mathcal{C}$ and $\rho > 0$. From the definition of the set $\mathcal{C}$ and the two above inequalities, without loss of generality, we may assume that $z = (z_1, 0)$, $x = (0, x_2)$ with $(z_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$ and $(z_1, x_2) > a_1$, where $p = |\mathcal{A}|$. Using this partition, let us write the symmetric matrix $H(w)$ under the form

$$H(w) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where H_{11}, H_{22} are symmetric matrices and $H_{12} = H_{21}^\top$. The assumption of H implies that $H_{22} \succeq 0$. Then, we have $H_{22}+\rho I \succ 0$. In other word, for all $u \in \ker(Z)$, $u \neq 0$, one has $u^\top(H(w)+\rho I)u > 0$. By virtue of Proposition 2.8, the matrix $G_\rho(w)$ is nonsingular.

By definition of a matrix norm, there exists a unit vector $v \in \mathbb{R}^{2n}$ such that $\|G_\rho(w)^{-1}\| = \|G_\rho(w)^{-1}v\|$. Define $u := G_\rho(w)^{-1}v$. By introducing the notation

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

we have

$$\begin{aligned}
&(H_{11} + \rho I)u_1 + H_{12}u_2 - u_3 = v_1, \\
&H_{21}u_1 + (H_{22} + \rho I)u_2 - u_4 = v_2, \\
&Z_1 u_1 = v_3, \\
&X_2 u_4 = v_4.
\end{aligned}$$

This implies that

$$\begin{aligned}
u_1 &= Z_1^{-1} v_3, \\
u_2 &= (H_{22} + \rho I)^{-1}\zeta, \\
u_3 &= (H_{11} + \rho I)Z_1^{-1} v_3 + H_{12}(H_{22} + \rho I)^{-1}\zeta - v_1, \\
u_4 &= X_2^{-1} v_4,
\end{aligned}$$

where $\zeta = v_2 - H_{21}Z_1^{-1}v_3 + X_2^{-1}v_4$. We then deduce that

$$\begin{aligned}
\|u\| &\leq \|u_1\| + \|u_2\| + \|u_3\| + \|u_4\| \\
&\leq C_1 + C_2\|H_{11} + \rho I\| + C_4\|(H_{22} + \rho I)^{-1}\|,
\end{aligned}$$

where $C_1 = \|Z_1^{-1}v_3\| + \|X_2^{-1}v_4\| + \|v_1\|$, $C_2 = \|Z_1^{-1}v_3\|$ and $C_4 = (1 + \|H_{12}\|)\|\zeta\|$.

From the assumptions, we obtain $C_1 \leq 2/a_1 + 1$, $C_2 \leq 1/a_1$ and $C_4 \leq (1+M)(1+(M+1)/a_1)$, where $M = \sup_{w \in \bar{B}} \|H(w)\|$, where $\bar{B} = B(w^*, \varepsilon) \cap \mathcal{C}$. Because $H_{22} \succeq 0$, we then have $\|(H_{22} + \rho I)^{-1}\| \leq \frac{1}{\rho}$. By using $\|H_{11} + \rho I\| \leq \|H_{11}\| + \rho \leq M + \rho$ and the fact that $1 \leq \max\{1/t, t\}$, for all positive number t, we conclude that

$$\begin{aligned} \|G_\rho(w)^{-1}\| = \|u\| &\leq \frac{2}{a_1} + 1 + \frac{1}{a_1}(M+\rho) + (1+M)\left(1 + \frac{1}{a_1}(M+1)\right)\frac{1}{\rho} \\ &\leq C \max\{\frac{1}{\rho}, \rho\}, \end{aligned}$$

where $C = (3 + M + (1+M)^2)\,\frac{1}{a_1} + 2 + M$. □

5.2.3 Properties of the Newton iterate w^+

Example 5.3 shows that the SOSCs may not be satisfied at any solution at which the strict complementarity holds. As a consequence, the standard local convergence analysis of the Newton's method in the literature cannot be directly applied. In this section, we will show the rapid convergence of our algorithm under the milder condition (Assumption 5.3). In order to show the superlinear convergence of our algorithm, at first we need to analyze the properties of the Newton step $w_k^+ - w_k$ solution of the system (5.5).

To simplify the notation, the iteration index k will be removed in this subsection. Hereafter, Assumptions 5.1-5.3 will be assumed to be satisfied at a solution point $w^* \in \mathcal{S}$. The distance of a point $w \in \mathbb{R}^n$ to the set $\mathcal{S}$ is shortly denoted by $d(w)$. Under Assumptions 5.1-5.3, there exist positive numbers l, L, b, and $\eta < a$ such that for all $w, w' \in B(w^*, \eta)$

$$\|F(w) - F(w')\| \leq l\|w - w'\|, \tag{5.25}$$

$$\|F'(w) - F'(w')\| \leq L\|w - w'\|, \tag{5.26}$$

$$d(w) \leq b\|F(w)\|. \tag{5.27}$$

For $w \in B(w^*, \eta) \cap \mathbb{R}^{2n}_{++}$ let us denote the matrix

$$J_\theta(w) := \begin{pmatrix} \nabla^2 f(x) + (\delta+\theta)I & -I \\ Z & X \end{pmatrix} = F'(w) + (\delta+\theta)\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where the parameters δ and θ are chosen as follows. Let $\beta \geq 1$, choose

$$0 \leq \delta \leq \beta \max\left\{0, -\lambda_{\min}\left(\nabla^2 f(x) + X^{-1}Z\right)\right\}, \tag{5.28}$$

such that $\nabla^2 f(x) + X^{-1}Z + \delta I \succeq 0$. Let $\gamma_1 > 0$ and $\sigma \in (0,1)$, set

$$\theta = \gamma_1 \|F(w)\|^{\sigma} > 0. \tag{5.29}$$

We will consider the behavior of the Newton iterate w^+ solved by the linear system

$$J_\theta(w)(w^+ - w) = -(F(w) - \mu\tilde{e}), \tag{5.30}$$

where $\tilde{e} = \begin{pmatrix} 0 \\ e \end{pmatrix} \in \mathbb{R}^{2n}$ and the parameter μ is set by the following formula

$$0 < \mu \le \gamma_2 \|F(w)\|^{1+\sigma}, \tag{5.31}$$

for $\gamma_2 \in (0,1)$.

Firstly, we will show that the sequence $\{\delta_k\}$ defined in (5.4) is bounded by the distance function.

Lemma 5.10. *There exists $M_1 > 0$ such that for all $w \in B(w^*, \eta/2) \cap \mathbb{R}^{2n}_{++}$ and δ satisfies (5.28), for some $\beta \ge 1$, we have $\delta \le M_1 d(w)$.*

Proof. For all $w = (x,z) \in \mathbb{R}^{2n}$ and $\rho > 0$, let us define the matrix

$$G_\rho(w) = \begin{pmatrix} \nabla^2 f(x) + \rho I & -I \\ Z & X \end{pmatrix}.$$

For any $w = (x,z) \in \mathbb{R}^{2n}_{++}$, let us denote $\lambda = -\lambda_{\min}(\nabla^2 f(x) + X^{-1}Z)$.

Let $w \in B(w^*, \eta/2) \cap \mathbb{R}^{2n}_{++}$. If $\lambda \le 0$, then $\delta = 0$ and the result clearly holds. Suppose now that $\lambda > 0$. Recall that $\bar{w}$ is defined by $\|w - \bar{w}\| = d(w)$. We then have

$$\begin{aligned} \|\bar{w} - w^*\| &\le \|\bar{w} - w\| + \|w - w^*\| \\ &\le 2\|w - w^*\| \\ &< \eta, \end{aligned}$$

meaning that $\bar{w} \in B(w^*, \eta) \cap \mathcal{S}$. By virtue of Lemma 5.9, the matrix $G_\lambda(\bar{w})$ is nonsingular and there exists a constant $C > 0$ such that

$$\|G_\lambda(\bar{w})^{-1}\| \le C \max\left\{\frac{1}{\lambda}, \lambda\right\}. \tag{5.32}$$

Because $-\lambda$ is an eigenvalue of the matrix $\nabla^2 f(x) + X^{-1}Z$, the matrix

$\nabla^2 f(x) + X^{-1}Z + \lambda I$ is singular . Proposition 2.7 implies that $\det G_\lambda(w) = \det X \det(\nabla^2 f(x) + X^{-1}Z + \lambda I) = 0$. Therefore, the matrix $G_\lambda(w)$ is singular. By applying Proposition 2.6 for the matrix $G_\lambda(w)$ and using (5.26), (5.32), it follows that

$$\begin{aligned} 1 &\leq \|G_\lambda(\bar{w})^{-1} G_\lambda(w) - I\| \\ &= \|G_\lambda(\bar{w})^{-1} \left(G_\lambda(w) - G_\lambda(\bar{w})\right)\| \\ &\leq \|G_\lambda(\bar{w})^{-1}\| \|F'(w) - F'(\bar{w})\| \\ &\leq C \max\left\{\frac{1}{\lambda}, \lambda\right\} L\|w - \bar{w}\|. \end{aligned}$$

This implies that

$$\min\left\{\frac{1}{\lambda}, \lambda\right\} \leq CL\|w - \bar{w}\|.$$

If $\lambda \in (0, 1)$, then the definition of δ implies that

$$\delta \leq \beta\lambda = \beta \min\left\{\frac{1}{\lambda}, \lambda\right\} \leq \beta CL\|w - \bar{w}\|$$

and the result follows with $M_1 = \beta CL$. Otherwise, if $\lambda \geq 1$, then

$$\begin{aligned} \lambda &= -\lambda_{\min}(\nabla^2 f(x) + X^{-1}Z) \\ &\leq -\lambda_{\min}(\nabla^2 f(x)) \\ &\leq \|\nabla^2 f(x)\| \\ &\leq m_1, \end{aligned}$$

where $m_1 = \sup_{x \in B(w^*, \frac{\eta}{2})} \|\nabla^2 f(x)\|$ and the first inequality is obtained by applying Proposition 2.1. By setting $M_1 = \beta m_1^2 CL$, one has

$$\delta \leq \beta\lambda = \beta\lambda^2 \frac{1}{\lambda} = \beta\lambda^2 \min\left\{\frac{1}{\lambda}, \lambda\right\} \leq \beta m_1^2 CL\|w - \bar{w}\| \leq M_1\|w - \bar{w}\|.$$

□

The next lemma provides an upper bound on the length of the solution of the linear system (5.30) through the distance from the current iterate to the solution set.

Lemma 5.11. *There exists $C_1 > 0$ such that for all $w \in B(w^*, \eta/2) \cap \mathbb{R}^{2n}_{++}$, we have*

$$\|w^+ - w\| \leq C_1 d(w).$$

Proof. By the choice of δ, for all $w \in B(w^*, \eta/2) \cap \mathbb{R}^{2n}_{++}$, one has $\nabla^2 f(x) + X^{-1}Z + \delta I \succeq 0$. The definition of θ implies that $\theta \geq \gamma_1 \|x \circ z\|^\sigma$. By virtue of Lemma 5.8, there exists $C \geq 1$ such that for all $w \in B(w^*, \eta/2) \cap \mathbb{R}^{2n}_{++}$,

$$\|J_\theta(w)^{-1}\| \leq \frac{C}{\theta}. \tag{5.33}$$

Let $w \in B(w^*, \eta/2) \cap \mathbb{R}^{2n}_{++}$. From the linear system (5.30), we have

$$\begin{aligned}
w^+ - w &= -J_\theta(w)^{-1}(F(w) - \mu\tilde{e}) \\
&= J_\theta(w)^{-1}(F(\bar{w}) - (F(w) - \mu\tilde{e})) \\
&= J_\theta(w)^{-1} \int_0^1 [F'(w + t(\bar{w} - w))(\bar{w} - w) + \mu\tilde{e}]dt \\
&= J_\theta(w)^{-1} \int_0^1 \left(F'(w + t(\bar{w} - w)) - F'(w)\right)(\bar{w} - w)dt + \mu J_\theta(w)^{-1}\tilde{e} \\
&\quad + J_\theta(w)^{-1}F'(w)(\bar{w} - w) \\
&= J_\theta(w)^{-1} \int_0^1 \left(F'(w + t(\bar{w} - w)) - F'(w)\right)(\bar{w} - w)dt + \mu J_\theta(w)^{-1}\tilde{e} \\
&\quad + \bar{w} - w - (\delta + \theta)J_\theta(w)^{-1}\begin{pmatrix} \bar{x} - x \\ 0 \end{pmatrix}.
\end{aligned}$$

By taking the norm on both sides and using (5.26), we get

$$\begin{aligned}
\|w^+ - w\| \leq \|J_\theta(w)^{-1}\| \frac{L}{2}\|\bar{w} - w\|^2 + \|\bar{w} - w\| + (\delta + \theta)\|J_\theta(w)^{-1}\|\|\bar{w} - w\| \\
+ \mu\|J_\theta(w)^{-1}\|\sqrt{n}.
\end{aligned}$$

Using (5.33) and reminding that $\|\bar{w} - w\| = d(w)$ and $C \geq 1$, the last inequality implies that

$$\|w^+ - w\| \leq C\left(\left(\frac{L}{2\theta}d(w) + 2 + \frac{\delta}{\theta}\right)d(w) + \sqrt{n}\frac{\mu}{\theta}\right). \tag{5.34}$$

From the local error bound condition (5.27), the formula (5.29) of θ gives us

$$\frac{1}{\theta} \leq \frac{b^\sigma}{\gamma_1}\frac{1}{d(w)^\sigma}. \tag{5.35}$$

The formula of μ given by (5.31), $F(\bar{w}) = 0$ and (5.25) imply that

$$\begin{aligned}\mu &\leq \gamma_2 \|F(w) - F(\bar{w})\|^{1+\sigma} \\ &\leq \gamma_2 l^{1+\sigma} d(w)^{1+\sigma}.\end{aligned} \tag{5.36}$$

By using (5.34)-(5.36) and Lemma 5.10, we deduce that

$$\|w^+ - w\| \leq C \left(\frac{Lb^\sigma}{2\gamma_1} d(w)^{1-\sigma} + 2 + \frac{M_1 b^\sigma}{\gamma_1} d(w)^{1-\sigma} + \frac{\gamma_2 b^\sigma l^{1+\sigma}\sqrt{n}}{\gamma_1} \right) d(w).$$

Finally, by using $d(w) \leq \|w - w^*\| < \eta$, we obtain

$$\|w^+ - w\| \leq C \left(\frac{Lb^\sigma \eta^{1-\sigma}}{2\gamma_1} (1 + 2\beta M_1) + 2 + \frac{\gamma_2 b^\sigma l^{1+\sigma}\sqrt{n}}{\gamma_1} \right) d(w),$$

which completes the proof. □

The next result gives us a relation between two distance functions evaluated at w and w^+.

Lemma 5.12. *Let $C_1 > 0$ be defined as in Lemma 5.11. There exists $C_2 > 0$ such that for all $w \in B\left(w^*, \frac{\eta}{2(1+C_1)}\right) \cap \mathbb{R}^{2n}_{++}$, we have*

$$d(w^+) \leq C_2 d(w)^{1+\sigma}.$$

Proof. Let $w \in B\left(w^*, \frac{\eta}{2(1+C_1)}\right) \cap \mathbb{R}^{2n}_{++}$. By applying Lemma 5.11, we get

$$\begin{aligned}\|w^+ - w^*\| &\leq \|w^+ - w\| + \|w - w^*\| \\ &\leq C_1 d(w) + \|w - w^*\| \\ &\leq (C_1 + 1)\|w - w^*\| \\ &< \frac{\eta}{2}.\end{aligned}$$

This means that $w^+ \in B(w^*, \frac{\eta}{2})$ and the inequality (5.27) can be applied at w^+ to get

$$d(w^+) \leq b\|F(w^+)\|. \tag{5.37}$$

From the linear system (5.30), the fundamental theorem of calculus permits us to write

$$
\begin{aligned}
F(w^+) &= F(w) + \int_0^1 F'(w + t(w^+ - w))(w^+ - w)dt \\
&= -J_\theta(w)(w^+ - w) + \mu\tilde{e} + \int_0^1 F'(w + t(w^+ - w))(w^+ - w)dt \\
&= -(\delta + \theta)(x^+ - x) + \mu\tilde{e} + \int_0^1 [F'(w + t(w^+ - w)) - F'(w)](w^+ - w)dt.
\end{aligned}
$$

By taking the norm on both sides, using (5.26) and applying Lemma 5.11, we get

$$
\begin{aligned}
\|F(w^+)\| &\leq (\delta + \theta)\|w^+ - w\| + \frac{L}{2}\|w^+ - w\|^2 + \mu\sqrt{n} \\
&\leq (\delta + \theta)C_1 d(w) + \frac{L}{2}C_1^2 d(w)^2 + \mu\sqrt{n}.
\end{aligned}
\tag{5.38}
$$

The Lipschitz property (5.25) of F, the choice of θ given by (5.29) and $F(\bar{w}) = 0$ imply that

$$
\begin{aligned}
\theta &= \gamma_1\|F(w) - F(\bar{w})\|^\sigma \\
&\leq \gamma_1 l^\sigma d(w)^\sigma.
\end{aligned}
\tag{5.39}
$$

Starting from (5.37), then using (5.38), Lemma 5.10, (5.39) and (5.36), and reminding that $d(w) < \eta$, we deduce that

$$
\begin{aligned}
d(w^+) &\leq b\left((M_1 d(w) + \gamma_1 l^\sigma d(w)^\sigma) C_1 d(w) + \frac{L}{2}C_1^2 d(w)^2 + \sqrt{n}\gamma_2 l^{1+\sigma} d(w)^{1+\sigma}\right) \\
&\leq b\left(\left(M_1\eta^{1-\sigma} + \gamma_1 l^\sigma\right) C_1 + \frac{L}{2}C_1^2\eta^{1-\sigma} + \sqrt{n}\gamma_2 l^{1+\sigma}\right) d(w)^{1+\sigma},
\end{aligned}
$$

from which the result follows. □

5.2.4 Convergence of $\{w_k\}$

In this section, we will demonstrate the convergence of the sequence $\{w_k\}$ to a solution $\widetilde{w} \in \mathcal{S}$. Let $\{w_k\}$ be a sequence created by the local algorithm (5.2)-(5.8). Firstly, we recall a result about the bound on step length α_k, see, e.g., [9, Corollary 1].

Lemma 5.13. *For all $k \in \mathbb{N}$ such that $w_k \in B(w^*, \eta) \cap \mathbb{R}^{2n}_{++}$, the following inequality holds*

$$
1 - \alpha_k \leq 1 - \tau_k + \frac{1}{a - \eta}\|w_k^+ - w_k\|. \tag{5.40}
$$

The next lemma shows that if a starting point w_0 belongs to some neighborhood of w^*, then the whole sequence $\{w_k\}$ converges to a solution of (5.1).

Lemma 5.14. *There exist constants $0 < \bar{\varepsilon} < \varepsilon$ such that if $w_0 \in B(w^*, \bar{\varepsilon}) \cap \mathbb{R}^{2n}_{++}$, then for all $k \geq 1, w_k \in B(w^*, \varepsilon) \cap \mathbb{R}^{2n}_{++}$. In addition, the sequence $\{w_k\}$ converges to $\widetilde{w} \in \mathcal{S}$.*

Proof. From the choice of $\{\tau_k\}$ given by (5.7), $F(\bar{w}) = 0$ for all $\bar{w} \in \mathcal{S}$, and (5.25), there exists $C_3 > 0$ such that for all $k \in \mathbb{N}$ satisfying $w_k \in B(w^*, \eta) \cap \mathbb{R}^{2n}_{++}$, one has

$$\begin{aligned} 1 - \tau_k &\leq C_3 \|F(w_k) - F(\bar{w}_k)\| \\ &\leq C_3 l d(w_k). \end{aligned}$$

By using this inequality and Lemma 5.11, we deduce from (5.40) that for all $k \in \mathbb{N}$ satisfying $w_k \in B(w^*, \eta/2) \cap \mathbb{R}^{2n}_{++}$,

$$\begin{aligned} 1 - \alpha_k &\leq C_3 l d(w_k) + \frac{C_1}{a - \eta} d(w_k) \\ &= C_4 d(w_k), \end{aligned} \tag{5.41}$$

where $C_4 := C_3 l + \frac{C_1}{a-\eta}$. Let us define $C_5 = C_1 C_4 \eta^{1-\sigma} + C_2$,

$$\varepsilon = \min\left\{\frac{\eta}{2(1 + C_1)}, \frac{1}{(2C_5)^{1/\sigma}}\right\} \quad \text{and} \quad \bar{\varepsilon} = \frac{\varepsilon}{2C_1 + 1}. \tag{5.42}$$

For each k, let $\bar{w}_k^+ \in \mathcal{S}$ such that $d(w_k^+) = \|w_k^+ - \bar{w}_k^+\|$. By applying Lemmas 5.11 and 5.12, inequality (5.41) and noting that $w_{k+1} - w_k^+ = (\alpha_k - 1)(w_k^+ - w_k)$ and $d(w_k) = \|w_k - \bar{w}_k\| < \eta$, for all $k \in \mathbb{N}$ such that $w_k \in B(w^*, \varepsilon) \cap \mathbb{R}^{2n}_{++}$, we have

$$\begin{aligned} d(w_{k+1}) &\leq \|w_{k+1} - \bar{w}_k^+\| \\ &\leq \|w_{k+1} - w_k^+\| + \|w_k^+ - \bar{w}_k^+\| \\ &\leq C_4 d(w_k) \|w_k^+ - w_k\| + d(w_k^+) \\ &\leq C_1 C_4 d(w_k)^2 + C_2 d(w_k)^{1+\sigma} \\ &\leq C_5 d(w_k)^{1+\sigma}. \end{aligned} \tag{5.43}$$

The proof of the first assertion is performed by induction on $k \in \mathbb{N}^*$. Since at each $k \in \mathbb{N}^*$, the *fraction to the boundary rule* (5.6) is applied and $w_0 > 0$, the sequence $\{w_k\}$ is positive. For the base case $k = 1$, by virtue of Lemma 5.11 and reminding that $d(w_0) \leq \|w_0 - w^*\| < \bar{\varepsilon}$, we have

$$\begin{aligned}\|w_1 - w^*\| &\leq \|w_1 - w_0^+\| + \|w_0^+ - w_0\| + \|w_0 - w^*\| \\ &\leq (1-\alpha_0)\|w_0^+ - w_0\| + \|w_0^+ - w_0\| + \|w_0 - w^*\| \\ &< 2C_1 d(w_0) + \bar{\varepsilon} \\ &< (2C_1 + 1)\bar{\varepsilon} \\ &= \varepsilon.\end{aligned}$$

Suppose now that for $k \geq 1$, $w_j \in B(w^*, \varepsilon) \cap \mathbb{R}^{2n}_{++}$ for all $j \in \{1, \ldots, k\}$. Let $j \in \{1, \ldots, k\}$. By noting that $\alpha_j \in (0, 1]$, Lemma 5.11 and (5.8) imply that

$$\|w_{j+1} - w_j\| = \alpha_j \|w_j^+ - w_j\| \leq C_1 d(w_j). \tag{5.44}$$

By using (5.43), the inequalities $d(w_0) < \bar{\varepsilon}$ and $(1+\sigma)^j \geq 1 + j\sigma$, we get

$$\begin{aligned}d(w_j) &\leq C_5 d(w_{j-1})^{1+\sigma} \\ &\leq C_5^{\frac{(1+\sigma)^j - 1}{\sigma}} d(w_0)^{(1+\sigma)^j} \\ &\leq (C_5 \bar{\varepsilon}^\sigma)^{\frac{(1+\sigma)^j - 1}{\sigma}} \bar{\varepsilon} \\ &\leq \frac{1}{2^j}\bar{\varepsilon},\end{aligned} \tag{5.45}$$

where the last inequality is deduced from the definition of ε and $\bar{\varepsilon}$ which implies that $C_5\bar{\varepsilon}^\sigma < C_5\varepsilon^\sigma \leq \frac{1}{2}$. By using (5.44) and (5.45), the triangle inequality gives us

$$\begin{aligned}\|w_{k+1} - w^*\| &\leq \|w_0 - w^*\| + \sum_{j=0}^{k} \|w_{j+1} - w_j\| \\ &< \bar{\varepsilon} + C_1 \sum_{j=0}^{k} \frac{1}{2^j}\bar{\varepsilon} \\ &\leq (1 + 2C_1)\bar{\varepsilon} \\ &= \varepsilon.\end{aligned}$$

To prove the last part of this theorem, let us take two nonnegative integers p and

q. From (5.44) and (5.45) we have

$$\begin{aligned}\|w_{p+q} - w_p\| &\le \sum_{k=p}^{p+q-1} \|w_{k+1} - w_k\| \\ &\le C_1 \bar{\varepsilon} \sum_{k=p}^{p+q-1} \frac{1}{2^k} \\ &\le \frac{C_1 \bar{\varepsilon}}{2^{p-1}}.\end{aligned}$$

This means that $\{w_k\}$ is a Cauchy sequence and therefore converges to $\widetilde{w} \in \mathbb{R}^{2n}_+$. By virtue of (5.45), we get $d(\widetilde{w}) = \lim d(w_k) = 0$, meaning that $\widetilde{w} \in \mathcal{S}$. □

5.2.5 Rate of convergence of $\{w_k\}$

This section is devoted to evaluate the rate of convergence of the sequence $\{w_k\}$. The remainder of this section relies on the proof of [107, Theorem 3.2]. Firstly, we introduce the lemma below which will be used to demonstrate the superlinear convergence of $\{w_k\}$.

Lemma 5.15. *If a positive sequence $\{a_k\}$ converges to zero at least superlinearly with a rate of $1 + \sigma$, where $\sigma \in (0, 1]$, then there exists a constant $C \in (0, 1)$ and a positive integer $\bar{k}$ such that, for all $k \ge \bar{k}$*

$$\sum_{i=1}^{\infty} a_{k+i} \le C a_k.$$

Proof. Since $\{a_k\}$ converges to zero at least superlinearly with a rate of $1+\sigma$, then Definition 2.13 implies that there exists $M > 0$ such that for all k sufficiently large

$$a_{k+1} \le M a_k^{1+\sigma}.$$

Besides, from the convergence to zero of $\{a_k\}$, there exists $\bar{k} \in \mathbb{N}$ such that for all $k \ge \bar{k}$

$$a_k < \frac{1}{(3M)^{\frac{1}{\sigma}}}.$$

For all $k \geq \bar{k}$ and $i \in \mathbb{N}$, let $\lambda_k = Ma_k^\sigma \in \left(0, \frac{1}{3}\right)$ and we have

$$\begin{aligned} a_{k+i} &\leq Ma_{k+i-1}^{1+\sigma} \\ &\leq M^{\frac{(1+\sigma)^i - 1}{\sigma}} a_k^{(1+\sigma)^i} \\ &\leq (Ma_k^\sigma)^{\frac{(1+\sigma)^i - 1}{\sigma}} a_k \\ &\leq \lambda_k^i a_k, \end{aligned}$$

where the last inequality is obtained by using the inequality $(1+\sigma)^i \geq 1 + i\sigma$. By taking the sum on both sides, we get

$$\begin{aligned} \sum_{i=1}^{\infty} a_{k+i} &\leq \sum_{i=1}^{\infty} \lambda_k^i a_k \\ &= \frac{\lambda_k}{1-\lambda_k} a_k. \end{aligned}$$

Since $\lambda_k \in (0, \frac{1}{3})$, we then have $\frac{\lambda_k}{1-\lambda_k} \in (0, \frac{1}{2})$. And the proof is completed with $C = \sup\limits_{k \geq \bar{k}} \frac{\lambda_k}{1-\lambda_k} \in (0, 1)$. □

We have the following result about the superlinear convergence of the sequence $\{w_k\}$.

Theorem 5.16. *Let Assumptions 5.1–5.3 hold. There exists $\bar{\varepsilon} > 0$ such that if $w_0 \in B(w^*, \bar{\varepsilon}) \cap \mathbb{R}^{2n}_{++}$, then the sequence $\{w_k\}$ converges at least superlinearly with a rate of $1 + \sigma$ to a solution $\widetilde{w} \in \mathcal{S}$*

Proof. Let $\varepsilon, \bar{\varepsilon}$ be defined as in Lemma 5.14. By virtue of Lemma 5.14, we deduce that $w_k \in B(w^*, \varepsilon) \cap \mathbb{R}^{2n}_{++}$ for all $k \in \mathbb{N}$ and the sequence $\{w_k\}$ converges to a solution $\widetilde{w} \in \mathcal{S}$. We now analyze the rate of convergence of the sequence $\{w_k\}$. Let $k \in \mathbb{N}$ and let us define $d_k := w_{k+1} - w_k$. From (5.44), we then have

$$\|d_k\| \leq C_1 d(w_k). \tag{5.46}$$

By combining this fact with (5.45), we imply that the series $\sum\limits_{i=k_0}^{\infty} d_i$ is absolutely convergent. Let $\bar{w}_{k+1} \in \mathcal{S}$ such that $\|w_{k+1} - \bar{w}_{k+1}\| = d(w_{k+1})$. As $w_{k+1} \in B(w^*, \varepsilon) \cap$

$\mathbb{R}^{2n}_{++}$, the property of the distance function and the inequality (5.43) give us

$$\begin{aligned} d(w_k) &\leq \|w_k - \bar{w}_{k+1}\| \\ &\leq \|w_k - w_{k+1}\| + \|w_{k+1} - \bar{w}_{k+1}\| \\ &\leq \|d_k\| + C_5 d(w_k)^{1+\sigma} \\ &\leq \|d_k\| + C_5 \varepsilon^\sigma d(w_k). \end{aligned}$$

From the definition (5.42) of ε, we then have $C_5\varepsilon^\sigma \leq \frac{1}{2}$. The above inequality implies that

$$\|d_k\| \geq C_6 d(w_k),$$

where $C_6 := 1 - C_5\varepsilon^\sigma > 0$. Combining this inequality with (5.46) and (5.43), we deduce that the two sequences $\{\|d_k\|\}$ and $\{d(w_k)\}$ converges to zero at least superlinearly with the same rate of $1+\sigma$. By applying Lemma 5.15 for the sequence $\{\|d_k\|\}$, there exists $C \in (0,1)$ and $\bar{k} \geq 0$ such that for all $k \geq \bar{k}$

$$\left\| \sum_{i=1}^{\infty} d_{k+i} \right\| \leq \sum_{i=1}^{\infty} \|d_{k+i}\| \leq C\|d_k\|.$$

It follows from the triangle inequality and the above one that for $k \geq \bar{k}$

$$\left\| \sum_{i=0}^{\infty} d_{k+1+i} \right\| \leq \|d_{k+1}\| + \left\| \sum_{i=1}^{\infty} d_{k+1+i} \right\| \leq (1+C)\|d_{k+1}\|$$

and

$$\left\| \sum_{i=0}^{\infty} d_{k+i} \right\| \geq \|d_k\| - \left\| \sum_{i=1}^{\infty} d_{k+i} \right\| \geq (1-C)\|d_k\|.$$

For all $k \geq \bar{k}$, by noting that the limit point $\widetilde{w}$ can be expressed as

$$\widetilde{w} = w_k + \sum_{i=0}^{\infty} d_{k+i},$$

from the two above inequalities, we obtain

$$\frac{\|w_{k+1} - \widetilde{w}\|}{\|w_k - \widetilde{w}\|^{1+\sigma}} = \frac{\|\sum_{i=0}^{\infty} d_{k+1+i}\|}{\|\sum_{i=0}^{\infty} d_{k+i}\|^{1+\sigma}} \leq \frac{1+C}{(1-C)^{1+\sigma}} \frac{\|d_{k+1}\|}{\|d_k\|^{1+\sigma}}.$$

Because $\{\|d_k\|\}$ converges to zero at least superlinearly, we then obtain from the above inequality that

$$\|w_{k+1} - \widetilde{w}\| = \mathrm{O}\left(\|w_k - \widetilde{w}\|^{1+\sigma}\right),$$

which completes the proof. □

5.3 Globalization of the local algorithm

In this section, we introduce a simple way to globalize the *local algorithm* (5.2)-(5.8). After computing w_k^+ from the linear system (5.5) and performing the *fraction to the boundary rule* (5.6), we consider the possibility to use an inner algorithm. At an iteration k, if the iterate $\widehat{w}_k := w_k + \alpha(w_k^+ - w_k)$ satisfies the condition $\|F(\widehat{w}_k) - \mu_k\tilde{e}\| \leq c\|F(w_k) - \mu_{k-1}\tilde{e}\| + \zeta_k$, for some $c \in (0, 1)$ where $\{\zeta_k\}$ converges to zero, then we set $w_{k+1} = \widehat{w}_k$. Otherwise, we apply a sequence of inner iterations in order to find a point w_{k+1} such that (5.48) is satisfied. The inner iterations can be obtained by applying some kinds of globalization techniques used in interior point methods, see, e.g., Armand and Omheni [13], Wachter and Biegler [149]. More specifically, the inner algorithm aims to minimize the function $\varphi(x) = f(x) - \mu\sum_{i=1}^{n}\log x_i$ for some fixed parameter μ. Because our main purpose is to study the fast convergence of our algorithm, we will not mention the inner iteration scheme in more detail. We will show that if the sequence $\{w_k\}$ converges to some neighborhood of the solution set, then the rate of convergence is superlinear. In particular, Algorithm 5 does not need the inner algorithm.

At the beginning, we choose a starting point $w_0 = (x_0, z_0) \in \mathbb{R}^{2n}_{++}$ and some constants $\mu_{-1} > 0$, $\beta \geq 1$, $\gamma_1 > 0$, $\gamma_2 \in (0, 1)$, $c, \sigma \in (0, 1)$. The outer iteration counter is set by $k = 0$. The outer algorithm is described in Algorithm 5.

Algorithm 5 (kth outer iteration)

---

1. Update $\theta_k > 0$, $\delta_k > 0$ by formulas (5.2) and (5.4), respectively. The barrier parameter μ_k is set by
$$\mu_k = \gamma_2 \min\{\|F(w_k)\|^{1+\sigma}, \mu_{k-1}\}. \tag{5.47}$$
2. Compute $w_k^+ = (x_k^+, z_k^+)$ by solving the linear system (5.5).
3. Compute a step-length $\alpha_k \in (0, 1]$ by using (5.6). Set $\widehat{w}_k = w_k + \alpha_k(w_k^+ - w_k)$.
4. Choose $\zeta_k > 0$ such that $\{\zeta_k\} \to 0$. If $\|F(\widehat{w}_k) - \mu_k\tilde{e}\| \leq c\|F(w_k) - \mu_{k-1}\tilde{e}\| + \zeta_k$, then set $w_{k+1} = \widehat{w}_k$. Otherwise, apply a sequence of inner iterations to find $w_{k+1} \in \mathbb{R}^{2n}_{++}$ such that
$$\|F(w_{k+1}) - \mu_k\tilde{e}\| \leq c\|F(w_k) - \mu_{k-1}\tilde{e}\| + \zeta_k. \tag{5.48}$$

5.3.1 Global convergence

From now on, we assume that Algorithm 5 generates an infinite sequence of iterates $\{w_k\}$. This implies that under some standard assumptions, the inner algorithm terminates after a finite number of iterations and returns a point w_{k+1} satisfying the condition (5.48), see, e.g., [13, Section 4.1]. The next theorem gives us the global behavior of Algorithm 5.

Theorem 5.17. *Assume that $\{\zeta_k\}$ tends to zero and Algorithm 5 generates an infinite sequence of iterates $\{w_k\}$. Then, $\{F(w_k)\}$ converges to zero.*

Proof. Let $M = \max\{\|F(w_0) - \mu_{-1}\tilde{e}\|, \bar{\zeta}\}$, where $\bar{\zeta} = \sup_{k\in\mathbb{N}} \zeta_k$. Since $c \in (0,1)$, the inequality (5.48) implies that for all $k \in \mathbb{N}$

$$\begin{aligned}\|F(w_{k+1}) - \mu_k\tilde{e}\| \leq c\|F(w_k) - \mu_{k-1}\tilde{e}\| + \bar{\zeta} \\ &\leq c^{k+1}\|F(w_0) - \mu_{-1}\tilde{e}\| + \sum_{i=0}^{k} c^i\bar{\zeta} \\ &\leq \frac{M}{1-c}.\end{aligned}$$

Hence, the limit superior of the sequence $\{\|F(w_{k+1}) - \mu_k\tilde{e}\|\}$ is finite. By taking the limit superior in (5.48) and reminding that $\lim \zeta_k = 0$, we have

$$(1-c)\limsup_{k\to\infty}\|F(w_k) - \mu_{k-1}\tilde{e}\| \leq 0,$$

and thus $\limsup_{k\to\infty}\|F(w_k) - \mu_{k-1}\tilde{e}\| = 0$. In addition, the triangle inequality gives us

$$\|F(w_k)\| \leq \|F(w_k) - \mu_{k-1}\tilde{e}\| + \mu_{k-1}\|\tilde{e}\|.$$

By taking the limit superior on both sides and using the convergence to zero of the sequence $\{\mu_k\}$ from (5.47), we obtain $\limsup_{k\to\infty}\|F(w_k)\| = 0$ which implies that $\{F(w_k)\}$ converges to zero. □

5.3.2 Asymptotic analysis

In Section 5.2, we demonstrated that the sequence $\{w_k\}$ created by the local algorithm (5.2)-(5.8) converges superlinearly to a solution $\tilde{w} \in \mathcal{S}$. We will show that the global scheme (Algorithm 5) does not affect the fast convergence of the sequence of iterates in some neighborhoods of the point w^*. In particular, we prove that for k large enough, the iterate $\widehat{w}_k$ will be accepted by the acceptance criterion at Step 4. This implies that there is no need to call the inner iteration algorithm.

Lemma 5.18. *Let Assumptions 5.1–5.3 hold and let us choose* $\zeta_k = \Omega(\mu_k^{\frac{1}{1+\sigma}})$. *There exists* $\varepsilon_g > 0$ *such that if at an iteration* k, $w_k \in B(w^*, \varepsilon_g) \cap \mathbb{R}^{2n}_{++}$, *then* $w_{k+1} = \widehat{w}_k$.

Proof. The assumption on $\{\zeta_k\}$ implies that there exists $C_7 > 0$ such that for all $k \in \mathbb{N}$

$$\zeta_k \geq C_7 \mu_k^{\frac{1}{1+\sigma}}. \tag{5.49}$$

Let us define

$$C_8 = bl(C_5 + \gamma_2 l^\sigma \sqrt{n}) \quad \text{and} \quad \varepsilon_g = \min\left\{\varepsilon, \left(\frac{C_7}{C_8}\gamma_2^{\frac{1}{1+\sigma}}\right)^{\frac{1}{\sigma}}, \left(\frac{c}{C_8 + cl^\sigma\sqrt{n}}\right)^{\frac{1}{\sigma}}\right\},$$

where ε is defined in (5.42).

Assume that $w_k \in B(w^*, \varepsilon_g) \cap \mathbb{R}^{2n}_{++}$ at some iteration k, we will show that $w_{k+1} = \widehat{w}_k$. Indeed, by applying the same argument to get (5.43), we then obtain

$$d(\widehat{w}_k) \leq C_5 d(w_k)^{1+\sigma}. \tag{5.50}$$

Let $\bar{\widehat{w}}_k \in \mathcal{S}$ such that $d(\widehat{w}_k) = \|\widehat{w}_k - \bar{\widehat{w}}_k\|$. For all $w_k \in B(w^*, \varepsilon_g)$, by applying inequalities (5.25), (5.36), (5.50), and (5.27), we then get

$$\begin{aligned}
\|F(\widehat{w}_k) - \mu_k \tilde{e}\| &\leq \|F(\widehat{w}_k) - F(\bar{\widehat{w}}_k)\| + \mu_k\sqrt{n} \\
&\leq l\|\widehat{w}_k - \bar{\widehat{w}}_k\| + \gamma_2 l^{1+\sigma}\sqrt{n} d(w_k)^{1+\sigma} \\
&\leq l(C_5 + \gamma_2 l^\sigma \sqrt{n}) d(w_k)^{1+\sigma} \\
&< bl(C_5 + \gamma_2 l^\sigma \sqrt{n}) \varepsilon_g^\sigma \|F(w_k)\| \\
&\leq C_8 \varepsilon_g^\sigma \|F(w_k)\|.
\end{aligned} \tag{5.51}$$

From the definition (5.47) of $\{\mu_k\}$, we consider the two following cases. The first case is when $\mu_k = \gamma_2\|F(w_k)\|^{1+\sigma}$, the inequality (5.49) and the definition of ε_g imply that

$$\zeta_k \geq C_7 \gamma_2^{\frac{1}{1+\sigma}} \|F(w_k)\| \geq C_8 \varepsilon_g^\sigma \|F(w_k)\|.$$

By substituting this inequality to (5.51), we then get

$$\|F(\widehat{w}_k) - \mu_k \tilde{e}\| \leq \zeta_k,$$

which implies that $w_{k+1} = \widehat{w}_k$.

The second case is when $\mu_k = \gamma_2 \mu_{k-1}$. This means that $\mu_{k-1} \leq \|F(w_k)\|^{1+\sigma}$. By using inequalities (5.51) and (5.25) and reminding that $F(\bar{w}_k) = 0$, it follows from the previous inequality that

$$\begin{aligned}\|F(\widehat{w}_k) - \mu_k \tilde{e}\| + c\mu_{k-1}\|\tilde{e}\| &\leq (C_8 \varepsilon_g^\sigma + c\sqrt{n}\|F(w_k) - F(\bar{w}_k)\|^\sigma)\|F(w_k)\| \\ &\leq (C_8 \varepsilon_g^\sigma + cl^\sigma \sqrt{n} d(w_k)^\sigma)\|F(w_k)\|.\end{aligned}$$

By using $d(w_k) < \varepsilon_g$, the definition of ε_g implies that $C_8 \varepsilon_g^\sigma + cl^\sigma \sqrt{n} d(w_k)^\sigma < c$. The triangle inequality then gives us

$$\begin{aligned}\|F(\widehat{w}_k) - \mu_k \tilde{e}\| &\leq c(\|F(w_k)\| - \mu_{k-1}\|\tilde{e}\|) \\ &\leq c\|F(w_k) - \mu_{k-1}\tilde{e}\| \\ &< c\|F(w_k) - \mu_{k-1}\tilde{e}\| + \zeta_k,\end{aligned}$$

from which the proof is completed. □

We now state the main result of this section which is a direct consequence of Theorem 5.16 and Lemma 5.18.

Theorem 5.19. *Let Assumptions 5.1–5.3 hold. There exists $r > 0$ such that if at an iteration k_0, $w_{k_0} \in B(w^*, r) \cap \mathbb{R}^{2n}_{++}$, then for all $k \geq k_0$, $w_{k+1} = \widehat{w}_k$ and the sequence $\{w_k\}$ converges at least superlinearly with a rate of $1 + \sigma$ to a solution $\widetilde{w} \in \mathcal{S}$.*

Proof. Let us define

$$r = \min\{\bar{\varepsilon}, \varepsilon_g\},$$

where $\bar{\varepsilon}$ and ε_g are respectively defined in Theorem 5.16 and Lemma 5.18. Let $k_0 \in \mathbb{N}$ be such that $w_{k_0} \in B(w^*, r) \cap \mathbb{R}^{2n}_{++}$. From the above definition of r and Lemma 5.18, we have $w_{k+1} = \widehat{w}_k$ for all $k \geq k_0$. This means that Algorithm 5 is reduced to the local algorithm (5.2)-(5.8). We now apply Theorem 5.16 to conclude the result. □

5.4 Examples and numerical results

In this section, we give some examples and numerical results to demonstrate the effectiveness of our method. Our algorithm called SPDOPT-R (Strongly Primal-Dual OTImization-Regularization) is implemented in C. Initially, a starting point

$w_0 = (x_0, z_0)$ is defined, where x_0 is given by user and $z_0 = e$. For each k, the parameters θ_k, μ_k, τ_k are set by

$$\begin{aligned}
\theta_k &= \min\{\gamma_1 \|F(w_k)\|^\sigma, \bar{\theta}\},\\
\mu_k &= \gamma_2 \min\{\|F(w_k)\|^{1+\sigma}, \mu_{k-1}\},\\
\tau_k &= \max\{0.99, 1-\mu_k\},
\end{aligned}$$

where $\gamma_1 = 0.1, \sigma = 0.5, \bar{\theta} = 10^{-3}, \gamma_2 = 0.1, \mu_{-1} = 0.1$. We note that these choices of parameters fulfill all assumptions in both global and local convergence analysis for all k such that $\|F(w_k)\| \le 10^{-3}$. At each k, the parameter δ_k is chosen by the following Algorithm 6, where $\beta = 100, \delta_0 = 0.01$.

Algorithm 6 Computation of δ_k

1. Let $M_k = \nabla^2 f(x_k) + X_k^{-1} Z_k$. If $M_k \succeq 0$, then return $\delta_k = 0$, else set $\delta^0 = \delta_0$.
2. Set $i = 0$. If $M_k + \delta^i I \succeq 0$, then go to Step 4.
3. Set $i = i+1$ and $\delta^i = \beta\delta^{i-1}$ until $M_k + \delta^i I \succeq 0$. Return $\delta_k = \delta^i$.
4. Set $i = i+1$ and $\delta^i = \delta^{i-1}/\beta$ until $M_k + \delta^i I \not\succeq 0$. Return $\delta_k = \beta\delta^{i-1}$.

We note that the sequence $\{\delta_k\}$ generated by Algorithm 6 satisfies (5.4). Indeed, let $k \in \mathbb{N}$ and let us denote $\lambda_{\min}^k := \lambda_{\min}(\nabla^2 f(x_k) + X_k^{-1} Z_k)$. If $M_k \succeq 0$, then $\lambda_{\min}^k \ge 0$ and $\delta_k = 0$ will satisfy the condition (5.4). Otherwise, if $\lambda_{\min}^k < 0$, let us consider two following cases. The first case is when δ_k is obtained from Step 3. We then have $M_k + \delta_k I \succeq 0$ and $M_k + \frac{\delta_k}{\beta} I \not\succeq 0$. This implies that

$$\lambda_{\min}^k + \delta_k \ge 0 \quad \text{and} \quad \lambda_{\min}^k + \frac{\delta_k}{\beta} < 0,$$

We then have

$$\max\{0, -\lambda_{\min}^k\} = -\lambda_{\min}^k \le \delta_k < -\beta\lambda_{\min}^k = \beta \max\{0, -\lambda_{\min}^k\}.$$

By using the same argument, we can demonstrate that if δ_k is returned by Step 4, then it also satisfies (5.4).

In Step 4 of Algorithm 5, we set $\zeta_k = 10\mu_k^{1/(1+\sigma)}$ and $c = 0.9$. We use the same kind of inner iterations as in [13, Algorithm 2]. The algorithm terminates if $\|F(w_k)\|_\infty \le 10^{-8}$. We will compare our SPDOPT-R with SPDOPT [13] in next examples.

5.4.1 Example 1: empty active set

We consider the problem (5.1) in $\mathbb{R}^2$, where

$$f(x) = \begin{cases} \frac{1}{2}(x_2 - 1)^2 & \text{if } x_1 \in [1, 3] \\ \frac{1}{8}(x_1 - 1)^4 (x_1 - 3)^4 + \frac{1}{2}(x_2 - 1)^2 & \text{otherwise.} \end{cases}$$

The first and second derivatives of f are

$$\nabla f(x) = \begin{cases} \begin{pmatrix} 0 \\ x_2 - 1 \end{pmatrix} & \text{if } x_1 \in [1, 3] \\ \begin{pmatrix} (x_1 - 1)^3 (x_1 - 3)^3 (x_1 - 2) \\ x_2 - 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

and

$$\nabla^2 f(x) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x_1 \in [1, 3] \\ \begin{pmatrix} (x_1 - 1)^2 (x_1 - 3)^2 (7x_1^2 - 28x_1 + 27) & 0 \\ 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

The function f is twice continuously differentiable and the second derivative $\nabla^2 f$ is Lipschitz continuous on $\mathbb{R}^2$. The solution set of this problem is $\mathcal{X} = [1, 3] \times \{1\}$ and the active set is empty. Let us choose a constant $\eta \in (0, 1)$. For any $x_1^* \in [1 + \eta, 3 - \eta]$, the local error bound condition (5.9) is provided at $x^* = (x_1^*, 1) \in \mathcal{X}$. Indeed, let $x = (x_1, x_2) \in B(x^*, \eta)$. We have

$$x_1^* - \eta < x_1 < x_1^* + \eta \quad \text{and} \quad 1 - \eta < x_2.$$

By noting that $1 + \eta \leq x_1^* \leq 3 - \eta$, this implies that

$$1 < x_1 < 3 \quad \text{and} \quad \frac{x_2}{1 - \eta} > 1.$$

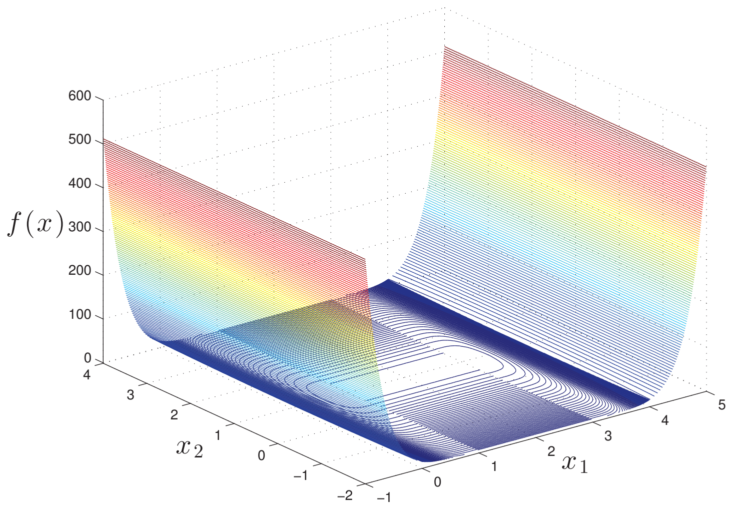


Fig. 5.3: SOSCs are not satisfied at any solution

Hence, we deduce that $\bar{x} = (x_1, 1)$ and

$$\begin{aligned} d(x, \mathcal{X}) = \|x - \bar{x}\| &= |x_2 - 1| \\ &< \frac{1}{1-\eta} x_2 |x_2 - 1| \\ &= \frac{1}{1-\eta} \|x \circ \nabla f(x)\|, \end{aligned}$$

which implies that the local error bound condition (5.9) is validated at x^*. We note, however, that the SOSCs do not hold at any $x^* \in \mathcal{X}$, since matrix $\nabla^2 f(x)$ is singular for all $x \in \mathbb{R}^2$.

From the starting point $x_0 = (1.5, 1.5)$, our algorithm converges to the solution $\widehat{x} = (1.5023, 1)$ after 5 iterations. SPDOPT takes 21 iteration to converge linearly to the solution $(3, 1)$. Figure 5.4 shows us the behaviors of these algorithms in 5 last iterations. We can see that SPDOPT-R converges superlinearly to $\widehat{x}$.

5.4.2 Example 2: nonempty active set

Let us consider the problem (5.1) in $\mathbb{R}^3$, where

$$f(x) = \begin{cases} \frac{1}{2}(x_2 - 1)^2 + \frac{1}{2}x_3^2 + x_3 & \text{if } x_1 \in [1, 3] \\ \frac{1}{8}(x_1 - 1)^4 (x_1 - 3)^4 + \frac{1}{2}(x_2 - 1)^2 + \frac{1}{2}x_3^2 + x_3 & \text{otherwise.} \end{cases}$$

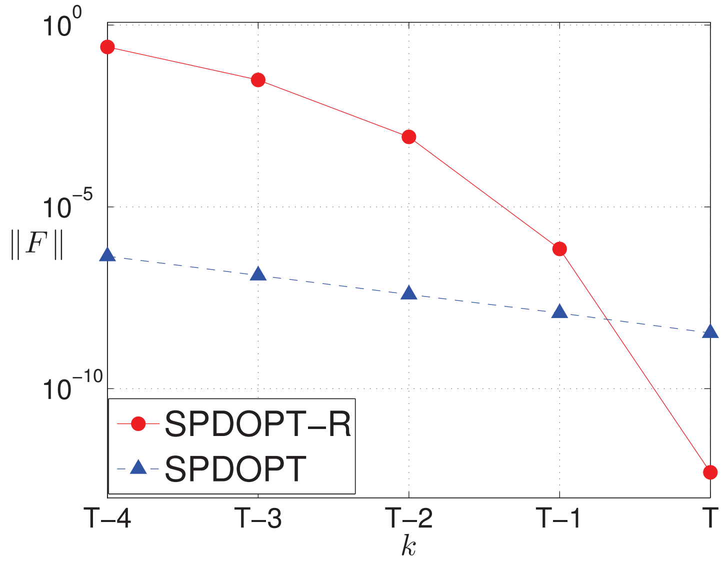


Fig. 5.4: Illustration of the rate of convergence (T= last iteration)

The first and second derivatives of f are given by

$$\nabla f(x) = \begin{cases} \begin{pmatrix} 0 \\ x_2 - 1 \\ x_3 + 1 \end{pmatrix} & \text{if } x_1 \in [1,3] \\ \begin{pmatrix} (x_1-1)^3(x_1-3)^3(x_1-2) \\ x_2 - 1 \\ x_3 + 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

and

$$\nabla^2 f(x) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{if } x_1 \in [1,3] \\ \begin{pmatrix} (x_1-1)^2(x_1-3)^2(7x_1^2 - 28x_1 + 27) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

The function f is twice continuously differentiable and the second derivative $\nabla^2 f$ is Lipschitz continuous on $\mathbb{R}^2$. The solution set is $\mathcal{X} = [1,3] \times \{1\} \times \{0\}$. Let choose a positive radius $\eta \in (0,1]$. For any $x_1^* \in [1+\eta, 3-\eta]$, the local error bound condition (5.9) is provided at $x^* = (x_1^*, 1, 0) \in \mathcal{X}$. Indeed, for all $x \in B(x^*, \eta)$, we have

$$x_1^* - \eta < x_1 < x_1^* + \eta, \quad 1 - \eta < x_2, \quad \text{and} \quad -\eta < x_3$$

By noting that $1 + \eta \le x_1^* \le 3 - \eta$, this implies that

$$1 < x_1 < 3, \quad \frac{x_2}{1-\eta} > 1, \quad \text{and} \quad \frac{x_3+1}{1-\eta} > 1.$$

Hence, we deduce that $\bar{x} = (x_1, 1, 0)$ and then

$$\begin{aligned} d(x, \mathcal{X})^2 = \|x - \bar{x}\|^2 &= (x_2 - 1)^2 + x_3^2 \\ &\le \frac{1}{(1-\eta)^2} x_2^2 (x_2 - 1)^2 + \frac{1}{(1-\eta)^2} (x_3 + 1)^2 x_3^2 \\ &= \frac{1}{(1-\eta)^2} \|x \circ \nabla f(x)\|^2, \end{aligned}$$

from which the Hadamard product $x \circ \nabla f(x)$ provides a local error bound at x^*.

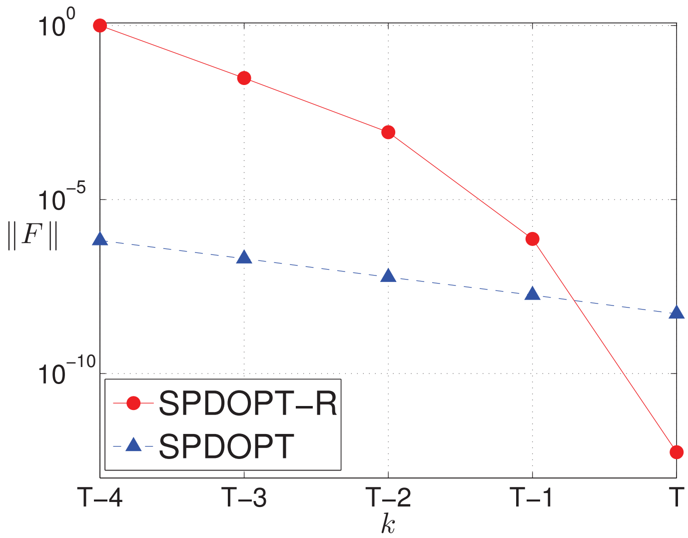


Fig. 5.5: Illustration of the rate of convergence (T= last iteration)

By choosing the starting point $x_0 = (2, 1.5, 1)$, SPDOPT-R converges to the solution $\hat{x} = (2.0021, 1, 0)$ after 5 iterations. SPDOPT takes 23 iterations to return the solution $(3, 1, 0)$. Figure 5.5 shows us that SPDOPT-R and SPDOPT converges to solutions superlinearly and linearly, respectively.

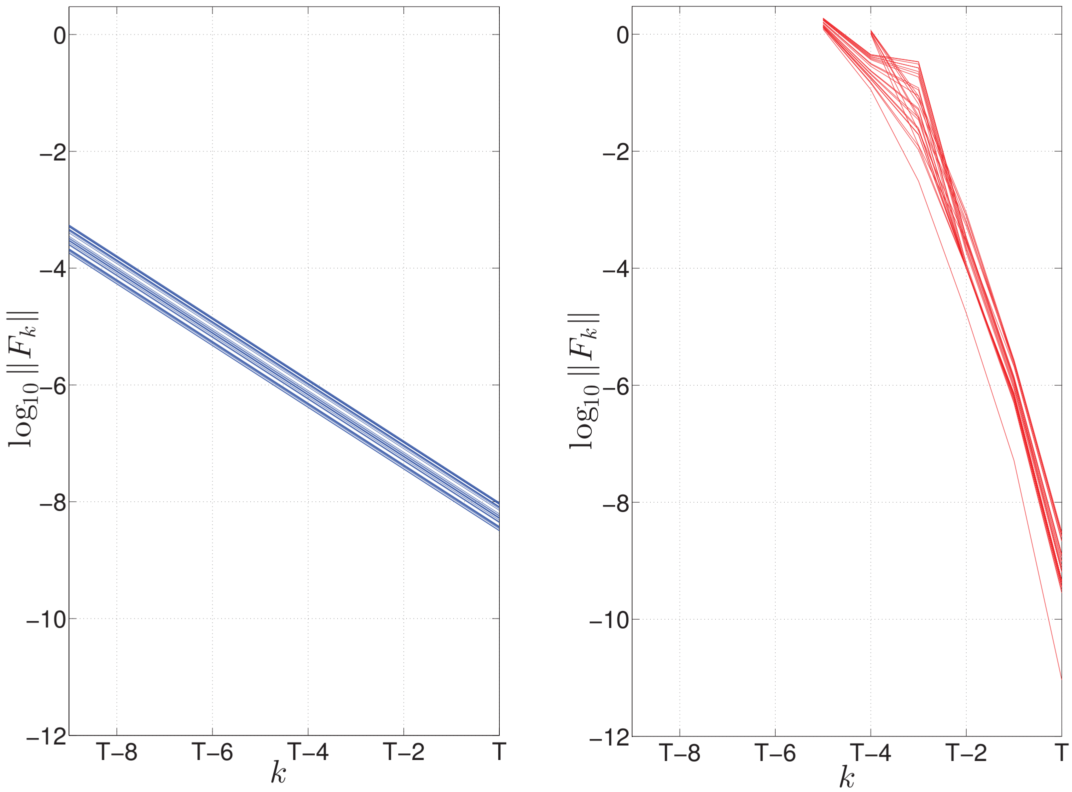
5.4.3 Example 3: large scale problem

Let us consider the bound constrained optimization problem (5.1) in $\mathbb{R}^n$, where

$$f(x) = \begin{cases} \frac{1}{2}(x_n - 1)^2 & \text{if } \sum_{i=1}^{n-1} x_i^2 \le n \\ \frac{1}{8}\left(\sum_{i=1}^{n-1} x_i^2 - n\right)^4 + \frac{1}{2}(x_n - 1)^2 & \text{otherwise,} \end{cases}$$

The set of primal solutions at which the SC holds is

$$\mathcal{X}_1 = \{x \in \mathbb{R}^n_{++} : \sum_{i=1}^{n-1} x_i^2 \le n, x_n = 1\}.$$



(a) SPDOPT (b) SPDOPT-R

Fig. 5.6: Values of $\log_{10}\|F_k\|$ for the last ten iterations of SPDOPT and SPDOPT-R when solving Example 3. T represents the index of the stopping iteration for each run.

We note that the SOSCs are not satisfied at any solution x^* of $\mathcal{X}_1$. However, the local error bound condition (5.9) holds at every relative interior point of $\mathcal{X}_1$. For each $n = 10$, 100, 1000, SPDOPT and SPDOPT-R solve 10 problems which use random starting points in $[0, 1]^n$. Figure 5.6 shows us the logarithms of $\|F(w_k)\|$ for the last ten iterations of SPDOPT and SPDOPT-R when solving these problems. These figures show us that the new algorithm SPDOPT-R converges superlinealy to solutions x^* belonging to $\mathcal{X}_1$, whereas the rate of convergence of SPDOPT-R is just linear.

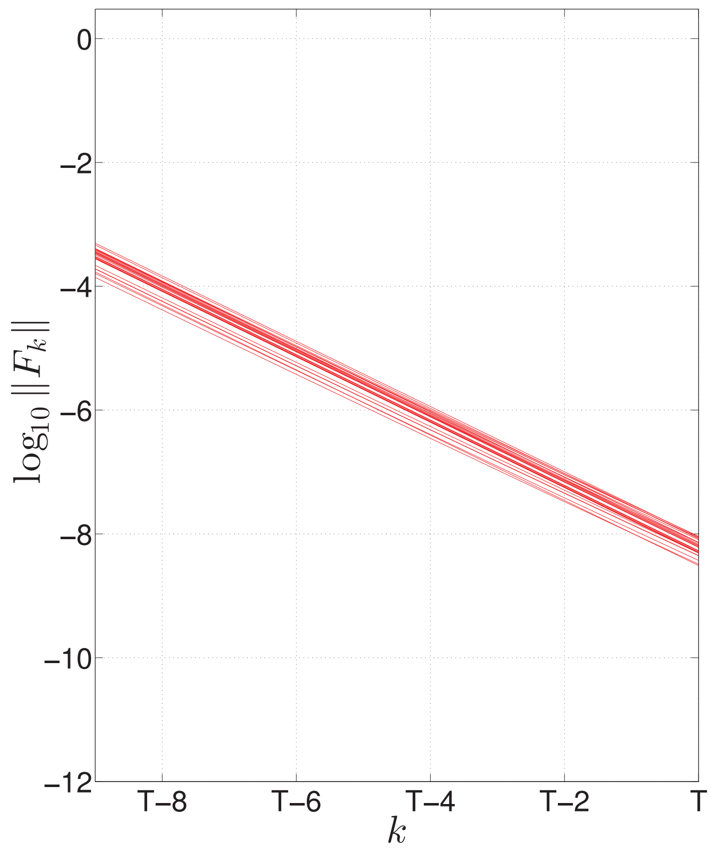


Fig. 5.7: Values of $\log_{10}\|F_k\|$ for the last ten iterations of SPDOPT-R when solving 30 problems in the form of Example 3 with $x_0 \in [1.1, 2]^n$. T represents the index of the stopping iteration for each run.

We note that for each $n = 10,\ 100,\ 1000$, if we choose random starting points $x_0 \in [1.1, 2]^n$, only a linear rate of convergence of SPDOPT-R is observed, see Figure 5.7. In this case, the sequence of iterates converges to a relative boundary point of $\mathcal{X}_1$ at which the local error bound (5.9) is not satisfied. From this observation, an open question is how to drive the iterates to a "good" neighborhood on which the local error bound condition (5.7) holds.

Chapter 6

Local convergence analysis of regularized primal-dual algorithms without constraint qualification

In this chapter, we study the local convergence properties of two regularized algorithms without any constraint qualification. This study was motivated by practical observations. Indeed, when solving degenerate problems for which the linear independence constraint qualification (LICQ) is not satisfied, the algorithm SPDOPT-AL of Armand and Omheni [12] has good performances and we can observe superlinear or quadratic convergence. The asymptotic analysis of this algorithm was made under the classical assumptions of second order sufficient conditions (SOSCs) and LICQ, but the quadratic convergence without constraint qualification has not been proved. In this work, we will answer the question raised by the authors of [12] on the rate of convergence without constraint qualification. We propose a method to update the parameters of SPDOPT [13] to achieve a fast rate of convergence of this algorithm without any constraint qualification. The main characteristics of our study are summarized as follows:

- For equality constrained optimization (1.2), the algorithm of Izmailov and Solodov [100] used the singular value decomposition (SVD) to identify locally the rank of the constraints degeneracy. This information is used to create a system of modified primal-dual optimality conditions with a unique solution. This algorithm is not really applicable for large scale problems because of the high computational cost of an SVD factorization. The linear system to solve at each iteration in the equality constrained phase of Wright [157] or in the outer algorithm of Arreckx and Orban [17] is quite similar to the one of SPDOPT-AL. In addition, since SPDOPT-AL possesses a rule to update parameters and a condition to call inner algorithm, they will be considered in our local convergence analysis.

- For the solution of the general nonlinear optimization problem (1.1), by means of a mixed interior point-augmented Lagrangian method [13], the quadratic penalty parameter introduces a natural regularization for the linear system to solve at each iteration. One advantage of this regularization parameter is that the superlinear convergence will be done without any additional conditions related to constraint qualifications required in other studies in the literature, for example, the constant-rank condition (Ralph and Wright [127]), MFCQ (Wright [154], Vicente and Wright [147]) or the linear independence of gradients of equality constraints (Yamashita and Yabe [162]). The centrality conditions will not be enforced as in other algorithms, see, e.g. [127, 128, 162]. Instead, the *fraction to the boundary rule* is applied to maintain the strict feasibility of the iterates for the bound constraints. It is worth noting that this rule is easier to implement and is commonly used in interior point algorithms.

- In the local convergence analysis of an optimization algorithm, the uniform boundedness of the inverse of the Jacobian matrix is a key fact, see, e.g., Byrd *et al.* [33], Armand and Benoist [9]. However, the lack of constraint qualification may lead to the unboundedness of the inverse of this matrix. In this chapter, we will investigate relations between regularized Jacobian matrices raised by linear systems of primal-dual algorithms and their regularization parameters.

6.1 An augmented Lagrangian algorithm for equality constrained optimization

In the two Sections 6.1 and 6.2, we consider the following equality constrained optimization problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ f(x) \quad \text{subject to } c(x)=0, \tag{EP}$$

where $f : \mathbb{R}^n \to \mathbb{R}, c : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions. The first order optimality conditions can be written as

$$F(w) = 0, \tag{6.1}$$

where $F : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ is defined by

$$F(w) = \begin{pmatrix} \nabla f(x) + A(x)y \\ c(x) \end{pmatrix},$$

where $w = (x, y) \in \mathbb{R}^{n+m}$. Armand and Omheni [12] introduced a primal-dual algorithm based on the augmented Lagrangian method to solve (EP). Let us define the augmented Lagrangian function by

$$\mathcal{L}_{\sigma,\lambda}(x) = f(x) + \lambda^\top x + \frac{1}{2\sigma}\|c(x)\|^2,$$

where $\lambda \in \mathbb{R}^m$ is a vector of Lagrange multipliers associated with the constraints of (EP) and $\sigma > 0$ is a quadratic penalty parameter. The main idea of SPDOPT-AL is to apply a Newton type method to the system $g(x)+A(x)\left(\lambda + \frac{1}{\sigma}c(x)\right) = 0$ which are the first order optimality conditions for minimizing $\mathcal{L}_{\sigma,\lambda}(x)$. By introducing the variable $y = \lambda + \frac{1}{\sigma}c(x)$, these conditions can be rewritten under the form $\Phi(w, \lambda, \sigma) = 0$, where $w = (x, y) \in \mathbb{R}^{n+m}$ and

$$\Phi(w, \lambda, \sigma) = \begin{pmatrix} \nabla f(x) + A(x)y \\ c(x) + \sigma(\lambda - y) \end{pmatrix}.$$

There are two kinds of iterations in SPDOPT-AL. The outers are devoted to update parameters and to compute a candidate iterate. If the residual $\|\Phi\|$ at the candidate iterate is sufficiently decreased, then the new iterate is set to the candidate and a new outer iteration is performed. Otherwise, a sequence of inner iterations will be applied to reduce this residual.

The outer iteration of SPDOPT-AL [12, Algorithm 1] is recalled in Algorithm 7. The algorithm is initialized with a starting point $w_0 := (x_0, y_0) \in \mathbb{R}^{n+m}$, a penalty parameter $\sigma_0 > 0$ and a Lagrange multiplier estimate $\lambda_0 = y_0$. Some constants $\kappa \in (0, 1), l \in \mathbb{N}$ and $\rho \in (0, 1)$ are chosen. The iteration counter is set to $k = 0$ and an index i_k is initially set to $i_0 = 0$.

The inner iteration algorithm is a backtracking line search applied to a merit function (see, [12, Algorithm 2]). In our analysis, we will prove that this algorithm is not called when the iterates belong to some neighborhood of an optimal solution.

Algorithm 7 (kth outer iteration of SPDOPT-AL)

1. Choose $\zeta_k \geq 0$ such that $\{\zeta_k\} \to 0$ and set $\eta_k = \|c_k\| + \zeta_k$. If $k = 0$ or
$$\|c(x_k)\| \leq \kappa \max\{\eta_{i_j} : (k-l)^+ \leq j \leq k\}, \tag{6.2}$$
then go to Step 3.
2. Choose $\sigma_{k+1} \leq \rho\sigma_k$. Set $\lambda_{k+1} = \lambda_k$, $i_{k+1} = i_k$ and go to Step 4.
3. Choose $\sigma_{k+1} \leq \sigma_k$. Set $\lambda_{k+1} = y_k$, $i_{k+1} = k$.
4. Choose a symmetric matrix H_k such that $\mathrm{In}(J_k) = (n, m, 0)$ and compute w_k^+ by solving the linear system
$$J_k(w_k^+ - w_k) = -\Phi(w_k, \lambda_{k+1}, \sigma_{k+1}), \tag{6.3}$$
where $J_k = \begin{pmatrix} H_k & A_k \\ A_k^\top & -\sigma_{k+1} I \end{pmatrix}$.
5. Choose $\varepsilon_k > 0$ such that $\{\varepsilon_k\} \to 0$. If
$$\|\Phi(w_k^+, \lambda_{k+1}, \sigma_{k+1})\| \leq \varepsilon_k, \tag{6.4}$$
then set $w_{k+1} = w_k^+$. Otherwise, apply a sequence of inner iterations to find w_{k+1} such that
$$\|\Phi(w_{k+1}, \lambda_{k+1}, \sigma_{k+1})\| \leq \varepsilon_k.$$

6.2 Asymptotic analysis for SPDOPT-AL without constraint qualification

6.2.1 Uniform boundedness of the inverse of a regularized Jacobian matrix

At each iteration k, the algorithm SPDOPT-AL solves a linear system with a regularized Jacobian matrix J_k. In SPDOPT-AL, the regularization parameter σ is updated dynamically, i.e., $\sigma_{k+1} = \Theta(\|F(w_k)\|)$. This implies that when the sequence of iterates $\{w_k\}$ converges to an optimal solution of (EP), the sequence $\{\sigma_k\}$ will tend to zero. Therefore, we cannot apply Corollary 5.6 of the previous chapter to deduce the uniform boundedness of the sequence $\{J_k^{-1}\}$. Nevertheless, we still have a similar result for the sequence $\{J_k^{-1}\}$ as follows.

Lemma 6.1. *Let $w^* = (x^*, y^*)$ be a vector in $\mathbb{R}^N$, where $N = n + m$ and n, m are natural numbers. Let $H : \mathbb{R}^N \to \mathbb{R}^{n\times n}$ be a bounded function such that for all $w \in \mathbb{R}^N$, $H(w) = H(w)^\top$ and let $A : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be a Lipschitz continuous* *function. Let* $\rho : \mathbb{R}^N \to \mathbb{R}_{++}$ *be a bounded function such that if* ρ *is not bounded away from zero, then* $\rho(w) = \Omega(\|x - x^*\|^t)$*, for all* $w = (x, y) \in \mathbb{R}^N$*, for some* $t \in (0, 1]$*. For all* $w \in \mathbb{R}^N$*, let us define the matrix*

$$J(w) := \begin{pmatrix} H(w) & A(x) \\ A(x)^\top & -\rho(w)I \end{pmatrix}.$$

Assume that there exist $r > 0$ *and* $\nu > 0$ *such that for all* $w \in B(w^*, r)$

$$H(w) + \frac{1}{\rho(w)} A(x)A(x)^\top \succeq \nu I. \tag{6.5}$$

Then, there exists $C > 0$ *such that for all* $w \in B(w^*, r)$*, the matrix* $J(w)$ *is nonsingular and* $\|J(w)^{-1}\| \leq \dfrac{C}{\rho(w)}$*.*

Proof. Let $w \in B(w^*, r)$ and let us define $\rho := \rho(w)$. Let us show that the matrix $J(w)$ is nonsingular. By Proposition 2.7, we have $\det J(w) = \det\left(H(w) + \frac{1}{\rho}A(x)A(x)^\top\right)\det(-\rho I)$. The assumption (6.5) and $\rho > 0$ imply that $\det\left(H(w) + \frac{1}{\rho}A(x)A(x)^\top\right) > 0$ and $\det(-\rho I) \neq 0$. It follows that the matrix $J(w)$ is nonsingular.

To prove the second assertion, let us consider the two following cases. The first case is when the function ρ is bounded away from zero. The conclusion will follow if the inverse of $J(w)$ is bounded. For all $w \in B(w^*, r)$, let us define $M(w) = H(w) + \dfrac{1}{\rho(w)}A(x)A(x)^\top$. We deduce from (6.5) that for all $w \in B(w^*, r)$,

$$\|M(w)^{-1}\| \leq \frac{1}{\nu}.$$

By noting that

$$J(w)^{-1} = \begin{pmatrix} M(w)^{-1} & \dfrac{1}{\rho(w)} M(w)^{-1}A(x) \\ \dfrac{1}{\rho(w)} A(x)^\top M(w)^{-1} & \dfrac{1}{\rho(w)^2} A(x)^\top M(w)^{-1}A(x) - \dfrac{1}{\rho(w)} I \end{pmatrix},$$

the boundedness of $\left(\|A(x)\|, \rho(w), \dfrac{1}{\rho(w)}\right)$ and the above inequality imply that $\|J(w)^{-1}\| = O(1)$.

We now consider the second case in which there exists a sequence $\{w_k\}$ in $B(w^*, r)$ such that $\lim \rho(w_k) = 0$. To simplify the notation, let us denote $J_k := J(w_k)$, $H_k := H(w_k)$, $A_k := A(x_k)$ and $\rho_k := \rho(w_k)$. The proof is based on a contradiction reasoning. We assume that $\lim \rho_k \|J_k^{-1}\| = \infty$. Let us define $r = \operatorname{rank}(A^*) \leq \min\{m, n\}$, where $A^* := A(x^*)$. By Proposition 2.4, the matrix A^* can be expressed under the form

$$A^* = U\Sigma V^\top = \begin{pmatrix} U_{\mathcal{I}} & U_{\mathcal{J}} \end{pmatrix} \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_{\mathcal{I}}^\top \\ V_{\mathcal{J}}^\top \end{pmatrix}, \tag{6.6}$$

where $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_1 \geq \ldots \geq \sigma_r > 0$ are the singular values of A^*, $U = \begin{pmatrix} U_{\mathcal{I}} & U_{\mathcal{J}} \end{pmatrix}$ and $V = \begin{pmatrix} V_{\mathcal{I}} & V_{\mathcal{J}} \end{pmatrix}$ are orthogonal matrices, $U_{\mathcal{I}} \in \mathbb{R}^{n \times r}, U_{\mathcal{J}} \in \mathbb{R}^{n \times (n-r)}, V_{\mathcal{I}} \in \mathbb{R}^{m \times r}, V_{\mathcal{J}} \in \mathbb{R}^{m \times (m-r)}$. For $k \in \mathbb{N}$, let us define

$$G_k := U^\top (A_k - A^*) V.$$

For $k \in \mathbb{N}$, we then have

$$\begin{aligned} A_k = A^* + U G_k V^\top &= U(\Sigma + G_k) V^\top \\ &= \begin{pmatrix} U_{\mathcal{I}} & U_{\mathcal{J}} \end{pmatrix} \begin{pmatrix} \Sigma_r + G_k^{11} & G_k^{12} \\ G_k^{21} & G_k^{22} \end{pmatrix} \begin{pmatrix} V_{\mathcal{I}}^\top \\ V_{\mathcal{J}}^\top \end{pmatrix}, \end{aligned} \tag{6.7}$$

where $G_k^{11} \in \mathbb{R}^{r \times r}$, $G_k^{12} \in \mathbb{R}^{r \times (m-r)}$, $G_k^{21} \in \mathbb{R}^{(n-r) \times r}$, $G_k^{22} \in \mathbb{R}^{(n-r) \times (m-r)}$. From the Lipschitz continuity of A and $\|x_k - x^*\| = \mathrm{O}(\rho_k^{1/t}) = \mathrm{O}(\rho_k)$, we get

$$\|G_k^{ij}\| = \mathrm{O}(\rho_k) \qquad \text{for all } i, j = 1, 2. \tag{6.8}$$

Since the l_2 norm is invariant under multiplication with orthogonal matrices, for all k, one has

$$\|J_k^{-1}\| = \|Q^\top J_k^{-1} Q\|,$$

where $Q = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ and $Q^\top Q = Q^\top Q = I$. From the definition of a matrix norm, there exists a sequence of unit vectors $\{v_k\} \subset \mathbb{R}^{2n}$ such that $\|J_k^{-1}\| = \|Q^\top J_k^{-1} Q\| = \|Q^\top J_k^{-1} Q v_k\|$. Define for all $k \in \mathbb{N}$, $u_k := Q^\top J_k^{-1} Q v_k / \|J_k^{-1}\|$. It follows that $\{u_k\}$ is a sequence of unit vectors with $\lim \frac{1}{\rho_k} \|Q^\top J_k Q u_k\| = 0$. Let $k \in \mathbb{N}$. By introducing the notation

$$u_k = \begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \end{pmatrix} \in \mathbb{R}^{2n} \quad \text{and} \quad \frac{1}{\rho_k} Q^\top J_k Q u_k = \begin{pmatrix} \alpha_k \\ \beta_k \\ \gamma_k \\ \delta_k \end{pmatrix}$$

and using (6.7), we have

$$\begin{aligned}
U_{\mathcal{I}}^{\top} H_k U_{\mathcal{I}} a_k + U_{\mathcal{I}}^{\top} H_k U_{\mathcal{J}} b_k + \left(\Sigma_r + G_k^{11}\right) c_k + G_k^{12} d_k &= \rho_k \alpha_k, \\
U_{\mathcal{J}}^{\top} H_k U_{\mathcal{I}} a_k + U_{\mathcal{J}}^{\top} H_k U_{\mathcal{J}} b_k + G_k^{21} c_k + G_k^{22} d_k &= \rho_k \beta_k, \\
\left(\Sigma_r + (G_k^{11})^{\top}\right) a_k + (G_k^{21})^{\top} b_k - \rho_k c_k &= \rho_k \gamma_k, \\
(G_k^{12})^{\top} a_k + (G_k^{22})^{\top} b_k - \rho_k d_k &= \rho_k \delta_k,
\end{aligned} \tag{6.9}$$

where the sequence $\{(\alpha_k, \beta_k, \gamma_k, \delta_k)\}$ converges to zero. From the assumption (6.5) and the orthogonality of the matrix U, we deduce that

$$\begin{aligned}
b_k^{\top} U_{\mathcal{J}}^{\top} \left(H_k + \frac{1}{\rho_k} A_k A_k^{\top}\right) U_{\mathcal{J}} b_k &\geq \nu \|U_{\mathcal{J}} b_k\|^2 \\
&= \nu \|b_k\|^2.
\end{aligned} \tag{6.10}$$

From (6.7) and $U_{\mathcal{J}}^{\top} U = \begin{pmatrix} 0 & I \end{pmatrix}$, we get

$$\begin{aligned}
U_{\mathcal{J}}^{\top} A_k &= \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_r + G_k^{11} & G_k^{12} \\ G_k^{21} & G_k^{22} \end{pmatrix} V^{\top} \\
&= \begin{pmatrix} G_k^{21} & G_k^{22} \end{pmatrix} V^{\top},
\end{aligned}$$

which implies that

$$U_{\mathcal{J}}^{\top} A_k A_k^{\top} U_{\mathcal{J}} = G_k^{21} (G_k^{21})^{\top} + G_k^{22} (G_k^{22})^{\top}.$$

By substituting this equality to (6.10), we obtain

$$\nu \|b_k\|^2 \leq b_k^{\top} \left(U_{\mathcal{J}}^{\top} H_k U_{\mathcal{J}} + \frac{1}{\rho_k} \left(G_k^{21} (G_k^{21})^{\top} + G_k^{22} (G_k^{22})^{\top}\right)\right) b_k. \tag{6.11}$$

We deduce from the third equation of (6.9) that

$$\|a_k\| \leq \left\|\Sigma_r^{-1}\right\| \left(\|(G_k^{11})^{\top}\| \|a_k\| + \|(G_k^{21})^{\top}\| \|b_k\| + \rho_k \left(\|c_k\| + \|\gamma_k\|\right)\right).$$

From $\|\Sigma_r^{-1}\| = \frac{1}{\sigma_r}$, the boundedness of $\{(a_k, b_k, c_k, \gamma_k)\}$, (6.8) and the convergence to zero of $\{\rho_k\}$, taking the limit for $k \in \mathbb{N}$, we then get

$$\lim a_k = 0. \tag{6.12}$$

Let us show that

$$\mathcal{J} \neq \emptyset \quad \text{and} \quad \liminf b_k > 0. \tag{6.13}$$

Indeed, if this is not the case, we consider the two following possibilities. The first case is when $\mathcal{J} = \emptyset$. By reminding that $\|\Sigma_r^{-1}\| = \frac{1}{\sigma_r}$, $\|U_{\mathcal{I}}^\top\| = \|U_{\mathcal{I}}\| \leq 1$ and the sequence $\{(\|H_k\|, \|c_k\|, \|\alpha_k\|)\}$ is bounded, the first equation of (6.9) and (6.8) gives us

$$\begin{aligned}\|c_k\| &\leq \left\|\Sigma_r^{-1}\right\| \left(\|U_{\mathcal{I}}^\top\|\|H_k\|\|U_{\mathcal{I}}\|\|a_k\| + \|G_k^{11}\|\|c_k\| + \rho_k\|\alpha_k\|\right) \\ &= \mathrm{O}(\|a_k\|) + \mathrm{O}(\rho_k).\end{aligned}$$

Taking the limit in both sides of the above inequality, using (6.12) and the convergence to zero of $\{\rho_k\}$, we then get

$$\lim c_k = 0,$$

which is in contradiction with the fact that that $\{(a_k, c_k)\}$ is a sequence of unit vectors. The second case is when $\mathcal{J} \neq \emptyset$ and there exists an infinite subset $\mathcal{K} \subset \mathbb{N}$ such that $\lim_{k\in\mathcal{K}} b_k = 0$. By using the boundedness of $\{(\|H_k\|, \|c_k\|, \|d_k\|, \|\alpha_k\|)\}$, (6.8) and the fact that $\|U_{\mathcal{I}}^\top\| = \|U_{\mathcal{I}}\| \leq 1$, $\|U_{\mathcal{J}}\| \leq 1$ and $\|\Sigma_r^{-1}\| = \frac{1}{\sigma_r}$, we deduce from the first and the fourth equations of (6.9) that

$$\begin{aligned}\|c_k\| &\leq \left\|\Sigma_r^{-1}\right\| \Big(\|U_{\mathcal{I}}^\top\|\|H_k\|(\|U_{\mathcal{I}}\|\|a_k\| + \|U_{\mathcal{J}}\|\|b_k\|) \\ &\qquad + \|G_k^{11}\|\|c_k\| + \|G_k^{12}\|\|d_k\| + \rho_k\|\alpha_k\|\Big) \\ &= \mathrm{O}(\|a_k\|) + \mathrm{O}(\|b_k\|) + \mathrm{O}(\rho_k), \\ \|d_k\| &\leq \frac{1}{\rho_k}\left(\|(G_k^{12})^\top\|\|a_k\| + \|(G_k^{22})^\top\|\|b_k\|\right) + \|\delta_k\| \\ &= \mathrm{O}(\|a_k\|) + \mathrm{O}(\|b_k\|) + \mathrm{O}(\|\delta_k\|).\end{aligned}$$

From (6.12) and the convergence to zero of $\{(b_k, \rho_k, \delta_k)\}_{\mathcal{K}}$, taking the limit for $k \in \mathcal{K}$, we then get

$$\lim_{k\in\mathcal{K}} c_k = 0 \quad \text{and} \quad \lim_{k\in\mathcal{K}} d_k = 0,$$

which is again in contradiction with the fact that $\{(a_k, b_k, c_k, d_k)\}$ is a sequence of unit vectors. Hence, (6.13) must be true.

Premultiplying the second equation of (6.9) by $b_k^\top$, using (6.11) and the Cauchy-

Schwarz inequality, we get

$$
\begin{aligned}
\nu\|b_k\|^2 &\le b_k^\top \left(U_{\mathcal{J}}^\top H_k U_{\mathcal{J}} + \frac{1}{\rho_k}(G_k^{21}(G_k^{21})^\top + G_k^{22}(G_k^{22})^\top) \right) b_k \\
&= \|b_k\| \big(\|U_{\mathcal{J}}^\top\| \|H_k\| \|U_{\mathcal{I}}\| \|a_k\| + \|G_k^{21}\| \|c_k\| + \|G_k^{22}\| \|d_k\| + \rho_k \|\beta_k\| \big) \\
&\quad + \frac{1}{\rho_k} \left(\|G_k^{21}\|^2 + \|G_k^{22}\|^2 \right) \|b_k\|^2.
\end{aligned}
$$

By reminding that $\|U_{\mathcal{J}}^\top\| \le 1$, $\|U_{\mathcal{I}}\| \le 1$ and the sequences $\{(b_k, c_k, d_k, \beta_k)\}$ and $\{H_k\}$ are bounded, the above inequality and (6.8) gives us

$$\|b_k\| = \mathrm{O}(\|a_k\|) + \mathrm{O}(\rho_k).$$

Taking the limit for $k \in \mathbb{N}$ in the above inequality, using (6.12) and the convergence to zero of $\{\rho_k\}$, we obtain

$$\lim b_k = 0,$$

which is in contradiction with (6.13).

Hence, there exists $C > 0$ such that for all $w \in B(w^*, r)$, $\|J(w)^{-1}\| \le \dfrac{C}{\rho(w)}$. $\square$

6.2.2 Assumptions and preliminary results

Let $x^* \in \mathbb{R}^n$ be an optimal solution of (EP). Let us assume that the following assumptions are satisfied.

Assumption 6.1. The functions f and c are twice continuously differentiable and their second derivatives are Lipschitz continuous over an open neighborhood of x^*.

Assumption 6.2. There exists $y^* \in \mathbb{R}^m$ such that the KKT conditions (6.1) are satisfied at $w^* = (x^*, y^*)$.

Assumption 6.3. The second order sufficient conditions (SOSCs) hold at w^*, i.e., for all $u \in \mathbb{R}^n$, if $u \neq 0$ and $A(x^*)^\top u = 0$, then $u^\top \nabla^2_{xx} \mathcal{L}(w^*) u > 0$.

Under these above assumptions, x^* is a strict local solution of (EP), see, e.g., [22, Proposition 3.2.1]. Let us define the set of dual solutions

$$\mathcal{S}_D = \{y^* \in \mathbb{R}^m | (x^*, y^*) \text{ satisfies the KKT condtidions (6.1)}\},$$

and the set of primal-dual solutions

$$\mathcal{S} = \{x^*\} \times \mathcal{S}_D.$$

Assumption 6.2 implies that $\mathcal{S} \neq \emptyset$. Since $\mathcal{S}$ is closed and the norm is coercive, for all $w = (x, y) \in \mathbb{R}^{n+m}$, there exists $\bar{w} = (x^*, \bar{y}) \in \mathcal{S}$ such that

$$\|w - \bar{w}\| = \min_{\xi \in \mathcal{S}} \|w - \xi\| =: d(w).$$

We now introduce a property of the distance from a point to the solution set of problem (EP). In the literature, some similar results have been demonstrated, see, e.g., Wright [157, Theorem 3.4], Arreckx and Orban [17, Lemmas 7 and 8].

Lemma 6.2. *Let us consider the problem* (EP) *and assume that Assumptions 6.1-6.3 hold at w^*. Then, there exist constants $\varepsilon > 0$ and $\beta > 0$ such that for all $w \in B(w^*, \varepsilon)$, we have*

$$\frac{1}{\beta} d(w) \leq \|F(w)\| \leq \beta d(w).$$

Proof. Under Assumption 6.1 and reminding that $F(\bar{w}) = 0$, there exists a positive constant r such that for all $w \in B(w^*, r)$,

$$\|F(w)\| = \|F(w) - F(\bar{w})\| = \mathrm{O}(\|w - \bar{w}\|) = \mathrm{O}(d(w)).$$

Conversely, for the sake of convenience, let us recall [94, Lemma 2] in the framework of problem (EP) which will be used to prove $d(w) = \mathrm{O}(\|F(w)\|)$: assume that Assumptions 6.1-6.3 hold at w^*. Then, there exist constants $\eta > 0$, $\kappa > 0$ and $\gamma > 0$ such that for all $w = (x, y) \in B(w^*, \eta)$, for each $t \in \mathbb{R}^{n+m}$ with $\|t\| \leq \kappa$ and

$$F(w) + t = 0,$$

we then have

$$\|x - x^*\| + \|y - \bar{y}\| \leq \gamma \|t\|,$$

where $\bar{y} \in \mathcal{S}_D$ such that $\|(x, y) - (x^*, \bar{y})\| = d(w)$.

By virtue of $\|F(w)\| = \mathrm{O}(d(w)) = \mathrm{O}(r)$, we can choose $\varepsilon \in (0, \min\{r, \eta\}]$, such that $\|F(w)\| \leq \kappa$. By applying the above result for $t = -F(w)$, we deduce that for all $w \in B(w^*, \varepsilon)$,

$$d(w) \leq \|x - x^*\| + \|y - \bar{y}\| \leq \gamma \|F(w)\|,$$

from which the result is concluded. □

The next result shows us the uniformly positive definiteness of the augmented matrix $\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\sigma}A(x)A(x)^\top$ in some neighborhood of the solution w^*, with σ sufficiently small.

Lemma 6.3. *Let us consider the problem (EP) and assume that Assumptions 6.1-6.3 hold at w^*. Then, there exist $r > 0$, $\bar{\sigma} > 0$ and $\nu > 0$ such that for all $w \in B(w^*, r)$ and $\sigma \in (0, \bar{\sigma}]$, we have*

$$\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\sigma}A(x)A(x)^\top \succeq \nu I. \tag{6.14}$$

In particular, we have $In(J_\sigma(w)) = (n, m, 0)$, where

$$J_\sigma(w) := \Phi'_w(w, \lambda, \sigma) = \begin{pmatrix} \nabla^2_{xx}\mathcal{L}(w) & A(x) \\ A(x)^\top & -\sigma I \end{pmatrix}.$$

Proof. By virtue of Lemma 2.2, Assumption 6.3 implies that there exists a number $\bar{\sigma} > 0$ such that

$$\nabla^2_{xx}\mathcal{L}(w^*) + \frac{1}{\bar{\sigma}}A(x^*)A(x^*)^\top \succ 0.$$

Let us denote $\lambda = \lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w^*) + \frac{1}{\bar{\sigma}}A(x^*)A(x^*)^\top\right) > 0$. By noting that the eigenvalues of a matrix are continuous functions of its entries, see, e.g., [136, Theorem 5.2], we deduce that there exists $r > 0$ such that for all $w \in B(w^*, r)$,

$$\lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\bar{\sigma}}A(x)A(x)^\top\right) \geq \frac{\lambda}{2}.$$

From the above inequality and Proposition 2.1, it follows that for all $w \in B(w^*, r)$ and $\sigma \in (0, \bar{\sigma}]$,

$$\begin{aligned}\lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\sigma}A(x)A(x)^\top\right) &\geq \lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\bar{\sigma}}A(x)A(x)^\top\right) \\ &\quad + \left(\frac{1}{\sigma} - \frac{1}{\bar{\sigma}}\right)\lambda_{\min}\left(A(x)A(x)^\top\right) \\ &\geq \nu := \frac{\lambda}{2},\end{aligned}$$

which concludes the first assertion.

The second assertion is implied from the first one and Lemma 2.2. □

6.2.3 Properties of the Newton iterates w^+

Under Assumptions 6.1-6.3 and by virtue of Lemmas 6.2 and 6.3, there exists positive numbers η, l, L, b and $\bar{\sigma}$ such that for all w, $w' \in B(w^*, \eta)$ and $\sigma^+ \in (0, \bar{\sigma}]$, (6.14) holds, $\mathrm{In}(J_{\sigma^+}(w)) = (n, m, 0)$ and

$$\|F(w) - F(w')\| \leq l\|w - w'\|, \tag{6.15}$$

$$\|F'(w) - F'(w')\| \leq L\|w - w'\|, \tag{6.16}$$

$$\frac{1}{b}\|F(w)\| \leq d(w) \leq b\|F(w)\|, \tag{6.17}$$

where

$$J_\sigma(w) = \begin{pmatrix} \nabla^2_{xx}\mathcal{L}(w) & A(x) \\ A(x)^\top & -\sigma I \end{pmatrix} = F'(w) - \sigma \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

In this section, for $w = (x, y) \in B(w^*, \eta)$, $\lambda^+ = y$ and a regularization parameter σ^+ satisfying

$$\sigma^+ \in (0, \bar{\sigma}] \quad \text{and} \quad \frac{1}{a}\|F(w)\| \leq \sigma^+ \leq a\|F(w)\|, \tag{6.18}$$

where $a > 0$, we will consider the behavior of the Newton iterate w^+ solved by the linear system

$$J_{\sigma^+}(w)(w^+ - w) = -\Phi(w, \lambda^+, \sigma^+)(w). \tag{6.19}$$

Firstly, we estimate an upper bound on the length of a solution of the linear system (6.19).

Lemma 6.4. *There exists $C_1 > 0$ such that for all $w \in B(w^*, \eta)$, $\lambda^+ = y$ and σ^+ satisfying (6.18), one has*

$$\|w^+ - w\| \leq C_1 d(w).$$

Proof. For all $w \in B(w^*, \eta)$, the choice (6.18) of σ^+ and (6.17) give us

$$\sigma^+ = \Theta(\|F(w)\|) = \Theta(d(w)). \tag{6.20}$$

For all $w \in B(w^*, \eta)$, by noting that $\|x - x^*\| \leq \|w - \bar{w}\| = d(w) < \eta$, it follows from the above that

$$\sigma^+ = \mathrm{O}(d(w)) < \eta \quad \text{and} \quad \sigma^+ = \Omega(\|x - x^*\|). \tag{6.21}$$

From Assumption 6.1, Lemma 6.3 and (6.21), by virtue of Lemma 6.1, there exists a constant $C > 0$ such that for all $w \in B(w^*, \eta)$, we then have

$$\|J_{\sigma^+}(w)^{-1}\| \leq \frac{C}{\sigma^+}. \tag{6.22}$$

Let $w \in B(w^*, \eta)$. By noting that $\Phi(w, \lambda^+, \sigma^+) = F(w)$ and $F(\bar{w}) = 0$, the system (6.19) gives us

$$\begin{aligned}
w^+ - w &= -J_{\sigma^+}(w)^{-1}\Phi(w, \lambda^+, \sigma^+) \\
&= J_{\sigma^+}(w)^{-1}(F(\bar{w}) - F(w)) \\
&= J_{\sigma^+}(w)^{-1} \int_0^1 F'(w + t(\bar{w} - w))(\bar{w} - w)dt \\
&= J_{\sigma^+}(w)^{-1} \int_0^1 [F'(w + t(\bar{w} - w)) - F'(w)](\bar{w} - w)dt \\
&\quad + J_{\sigma^+}(w)^{-1}F'(w)(\bar{w} - w) \\
&= J_{\sigma^+}(w)^{-1} \int_0^1 [F'(w + t(\bar{w} - w)) - F'(w)](\bar{w} - w)dt \\
&\quad + \sigma^+ J_{\sigma^+}(w)^{-1} \begin{pmatrix} 0 \\ \bar{y} - y \end{pmatrix} + \bar{w} - w.
\end{aligned}$$

Taking the norm on both sides and using (6.16), we get

$$\|w^+ - w\| \leq \|J_{\sigma^+}(w)^{-1}\| \left(\frac{L}{2} \|\bar{w} - w\|^2 + \sigma^+ \|\bar{w} - w\| \right) + \|\bar{w} - w\|. \tag{6.23}$$

By substituting (6.20) and (6.22) to (6.23), and reminding that $\|\bar{w} - w\| = d(w)$, we obtain

$$\|w^+ - w\| = \mathrm{O}(\|\bar{w} - w\|) = \mathrm{O}(d(w)).$$

□

The next lemma gives us a relation between distance functions evaluated at the Newton iterate w^+ and at the current point w.

Lemma 6.5. *Let $C_1 > 0$ be in Lemma 6.4. There exists $C_2 > 0$ such that for all $w \in B\left(w^*, \frac{\eta}{1+C_1}\right)$, $\lambda^+ = y$ and σ^+ satisfying (6.18), one has*

$$d(w^+) \leq C_2 d(w)^2.$$

Proof. Let $w \in B\left(w^*, \frac{\eta}{1+C_1}\right)$. By noting that $d(w) = \|w - \bar{w}\| \leq \|w - w^*\|$, we deduce from Lemma 6.4 that

$$\begin{aligned}\|w^+ - w^*\| &\leq \|w^+ - w\| + \|w - w^*\| \\ &\leq C_1 d(w) + \|w - w^*\| \\ &\leq (C_1 + 1)\|w - w^*\| \\ &< \eta,\end{aligned} \tag{6.24}$$

meaning that $w^+ \in B(w^*, \eta)$ and the inequality (6.17) can be applied at w^+ to get

$$d(w^+) \leq b\|F(w^+)\|. \tag{6.25}$$

From the linear system (6.19) and noting that $F(w) = \Phi(w, \lambda^+, \sigma^+)$, we have

$$\begin{aligned}F(w^+) &= F(w) + \int_0^1 F'(w + t(w^+ - w))(w^+ - w)dt \\ &= -J_{\sigma^+}(w)(w^+ - w) + \int_0^1 F'(w + t(w^+ - w))(w^+ - w)dt \\ &= -\left(F'(w) - \sigma^+ \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right)(w^+ - w) + \int_0^1 F'(w + t(w^+ - w))(w^+ - w)dt \\ &= \sigma^+ \begin{pmatrix} 0 \\ y^+ - y \end{pmatrix} + \int_0^1 [F'(w + t(w^+ - w)) - F'(w)](w^+ - w)dt.\end{aligned}$$

By taking the norm on both sides, using (6.16) and applying Lemma 6.4, we get

$$\begin{aligned}\|F(w^+)\| &\leq \sigma^+\|w^+ - w\| + \frac{L}{2}\|w^+ - w\|^2 \\ &\leq \sigma^+ C_1 d(w) + \frac{L}{2} C_1^2 d(w)^2.\end{aligned}$$

Finally, the result follows by combing the above inequality, (6.25) and (6.20). □

The next lemma shows that there is no need to call the inner iteration scheme if the current iterate w belongs to some neighborhood of w^*. Moreover, we will show that the condition (6.2) is also satisfied by the Newton iterate x^+. In particular, we will prove that $\|c(x^+)\| \leq \kappa(\|c(x)\| + \zeta^+)$, for some $\zeta^+ > 0$.

Lemma 6.6. *There exists $r > 0$ such that if $w \in B(w^*, r)$, $\lambda^+ = y$, σ^+ satisfying (6.18), $\zeta^+ \geq \theta_1 \sigma^+$ and $\varepsilon \geq \theta_2 \sigma^+$ for some θ_1, $\theta_2 > 0$, then*

$$\begin{aligned}&\|\Phi(w^+, \lambda^+, \sigma^+)\| \leq \varepsilon, \\ &\|c(x^+)\| \leq \kappa(\|c(x)\| + \zeta^+).\end{aligned}$$

Proof. Let us define

$$r = \min\left\{\frac{\eta}{1+C_1}, \frac{\kappa\theta_1}{ab^2C_2}, \frac{\theta_2}{ab^2C_2+C_1}\right\}, \tag{6.26}$$

where a and b are respectively given in (6.18) and (6.17).
Let $w \in B(w^*, r)$. By apply the same argument to get (6.24) and noting that $r \leq \frac{\eta}{1+C_1}$, we deduce that $w^+ \in B(w^*, \eta)$. By using (6.17), Lemma 6.5 and (6.18) we then get

$$\begin{aligned}\|F(w^+)\| &\leq bd(w^+) \leq bC_2 d(w)^2 \leq b^2C_2 d(w)\|F(w)\| \\ &\leq ab^2C_2 d(w)\sigma^+. \end{aligned} \tag{6.27}$$

By using the definition of $\Phi(\cdot)$, this evaluation, Lemmas 6.4 and the definition (6.26) of r, and noting that $\lambda^+ = y$ and $d(w) \leq \|w - w^*\| < r$, we then get

$$\begin{aligned}\|\Phi(w^+, \lambda^+, \sigma^+)\| &\leq \|F(w^+)\| + \sigma^+\|w^+ - w\| \\ &\leq ab^2C_2 d(w)\sigma^+ + C_1\sigma^+ d(w) \\ &< \theta_2\sigma^+ \\ &\leq \varepsilon.\end{aligned}$$

We now prove the second assertion. From the inequality (6.27), the definition (6.26) of r, the choice of ζ^+ and reminding that $d(w) \leq \|w - w^*\| < r$, we then have

$$\begin{aligned}\|c(x^+)\| &\leq \|F(w^+)\| \\ &\leq ab^2C_2 d(w)\sigma^+ \\ &< \kappa\theta_1\sigma^+ \\ &\leq \kappa(\|c(x)\| + \zeta^+),\end{aligned}$$

which completes the proof. □

6.2.4 Convergence of the sequence $\{w_k\}$

In this section, we will consider the behavior of the sequence $\{w_k\}$ generated by Algorithm 7. For each $k \in \mathbb{N}$, the parameter σ_{k+1} is chosen such that

$$\frac{1}{a}\|F(w_k)\| \leq \sigma_{k+1} \leq a\|F(w_k)\|, \tag{6.28}$$

for some $a > 0$. We now show that if at an iteration k_0, the iterate w_{k_0} is sufficiently close to w^*, then the algorithm is reduced to a sequence of outer iterations and the whole sequence converges quadratically to an optimal solution of (EP).

Theorem 6.7. *Assume that Assumptions 6.1–6.3 hold. Let the sequence $\{\sigma_k\}$ be chosen to satisfy (6.28) and $\sigma_0 \leq \bar{\sigma}$. Let the sequences $\{\zeta_k\}$ and $\{\varepsilon_k\}$ be chosen so that for all $k \in \mathbb{N}$, $\zeta_k \geq \theta_1 \sigma_k$ and $\varepsilon_k \geq \theta_2 \sigma_{k+1}$, where $\theta_1, \theta_2 > 0$. Then, there exist $0 < \bar{r} < r$ such that if at an iteration k_0, $w_{k_0} \in B(w^*, \bar{r})$ and the condition (6.2) is satisfied, then for all $k \in \mathcal{K} := \{k \in \mathbb{N} : k \geq k_0\}$, $w_{k+1} = w_k^+ \in B(w^*, r)$ and the condition (6.2) holds at $k+1$. In addition, the sequence $\{w_k\}$ converges quadratically to $\widehat{w} \in \mathcal{S}$.*

Proof. Let us define r as in (6.26) and

$$\bar{r} = \min \left\{ \frac{r}{1 + 2C_1}, \frac{1}{2C_2} \right\}.$$

For all $k \in \mathbb{N}$, we note that $\sigma_{k+1} \leq \sigma_k$. Since $\sigma_0 \leq \bar{\sigma}$, this implies that $\sigma_{k+1} \leq \bar{\sigma}$ for all $k \in \mathbb{N}$. The first part is proved by induction on $\mathcal{K}$. For the base case $k = k_0$, (6.4), (6.2) and Lemma 6.6 implies that $w_{k_0+1} = w_{k_0}^+$ and the condition (6.2) holds at $k_0 + 1$. It follows from this fact, Lemma 6.4 and $d(w_{k_0}) \leq \|w_{k_0} - w^*\| < \bar{r}$ that

$$\begin{aligned} \|w_{k_0+1} - w^*\| &= \|w_{k_0}^+ - w^*\| \\ &\leq \|w_{k_0}^+ - w_{k_0}\| + \|w_{k_0} - w^*\| \\ &\leq C_1 d(w_{k_0}) + \|w_{k_0} - w^*\| \\ &< (C_1 + 1)\bar{r} \\ &\leq r, \end{aligned}$$

which implies that $w_{k_0+1} \in B(w^*, r)$.

Suppose now that for an index $k \geq k_0 + 1$, $w_{j+1} = w_j^+ \in B(w^*, r)$ and the condition (6.2) holds at $j+1$, for all $j = k_0, \ldots, k-1$. Let $j \in \{k_0, \ldots, k\}$. Lemma 6.6, (6.2) and (6.4) imply that the condition (6.2) holds at $j+1$ and $w_{j+1} = w_j^+$. By applying Lemmas 6.4 and 6.5, one has

$$\|w_{j+1} - w_j\| = \|w_j^+ - w_j\| \leq C_1 d(w_j) \tag{6.29}$$

and

$$\begin{aligned} d(w_{j+1}) &= d(w_j^+) \\ &\leq C_2 d(w_j)^2. \end{aligned} \tag{6.30}$$

By using (6.30), the inequality $2^{j-k_0}-1 \geq j-k_0$ and noting that $d(w_{k_0}) < \bar{r} \leq \frac{1}{2C_2}$, we get

$$\begin{aligned} d(w_j) &\leq C_2 d(w_{j-1})^2 \\ &\leq C_2^{2^{j-k_0}-1} d(w_{k_0})^{2^{j-k_0}} \\ &\leq (C_2\bar{r})^{2^{j-k_0}-1}\,\bar{r} \\ &\leq \frac{1}{2^{j-k_0}}\bar{r}. \end{aligned} \tag{6.31}$$

By combining (6.29) and (6.31), we deduce that

$$\begin{aligned} \|w_{k+1}-w^*\| \leq \|w_{k_0}-w^*\| + \sum_{j=k_0}^{k} \|w_{j+1}-w_j\| &< \bar{r} + C_1 \sum_{j=0}^{k-k_0} \frac{1}{2^j}\bar{r} \\ &\leq (1+2C_1)\bar{r} \\ &< r. \end{aligned}$$

To prove the last part of this theorem, let us take two nonnegative integers p and q such that $p \geq k_0$. From (6.29) and (6.31), we have

$$\begin{aligned} \|w_{p+q}-w_p\| &\leq \sum_{k=p}^{p+q-1} \|w_{k+1}-w_k\| \\ &\leq C_1\bar{r} \sum_{k=p}^{p+q-1} \frac{1}{2^{k-k_0}} \\ &\leq \frac{C_1\bar{r}}{2^{p-k_0-1}}. \end{aligned}$$

It means that $\{w_k\}$ is a Cauchy sequence and converges to a point $\widehat{w} \in \mathbb{R}^{n+m}$. From (6.31), we deduce that $d(\widehat{w}) = \lim d(w_k) = 0$, which means that $\widehat{w} \in \mathcal{S}$.

The proof of the quadratic convergence can be performed similarly as the one of Theorem 5.16. □

Remark 6.8. In Theorem 6.7, there is no assumption about the convergence of the sequence $\{w_k\}$ to an optimal solution. Instead, we assume the existence of a neighborhood of an optimal solution w^* and of an iteration k_0 at which

w_{k_0} belongs to this neighborhood and the condition (6.2) is satisfied. We then show that the sequence $\{w_k\}$ converges quadratically to an optimal solution $\widehat{w} \in \mathcal{S}$. We note that Algorithm 7 is a Newton-type method applied to the system $\Phi(w, \lambda, \sigma) = 0$–the first order optimality conditions for minimizing augmented Lagrangian. Hence, when we consider the asymptotic behavior of this algorithm, we need an assumption on the Lagrange multiplier estimate $\lambda_{k_0+1} = y_0$ at some iteration k_0. If we only consider Algorithm 7 in some neighborhood of w^*, there is no need to take into account the condition (6.2) and we can set $\lambda_{k+1} = y_k$ for all k. In this case, this algorithm is reduced to a Newton-type method applied to the KKT conditions $F(w) = 0$ with a regularized Jacobian matrix.

Instead of assuming that (6.2) is satisfied at some k_0 where $w_{k_0} \in B(w^*, \bar{r})$ as in Theorem 6.7, we can use the assumption that the sequence $\{w_k\}$ converges to an optimal solution $w^* = (x^*, y^*) \in \mathcal{S}$, see, e.g. [12, Assumption A4], [17, Assumption 5.1]. Indeed, in this case, we will show that (6.2) occurs infinitely often by a contradiction reasoning. If this is not the case, there exists $k_0 \in \mathbb{N}$ such that (6.2) is satisfied at $k = k_0$ and never more satisfied for $k > k_0$. Let $k > k_0$. By noting from Step 2 of Algorithm 7 that $\eta_{i_k} = \eta_{k_0}$, we then deduce that $\|c(x_k)\| > \kappa\eta_{k_0} > 0$, which is in a contradiction with the assumption that $\{c(x_k)\}$ converges to zero. Hence, (6.2) must be satisfied infinitely often. This fact and the assumption that $\{w_k\}$ converges to w^* imply that there exists $\bar{r} > 0$ and $k_0 \in \mathbb{N}$ such that $w_{k_0} \in B(w^*, \bar{r})$. Now, by repeating the arguments in this section, we can obtain the same results as in Theorem 6.7 with $\widehat{w} = w^*$.

6.3 A mixed interior point-augmented Lagrangian algorithm for nonlinear optimization

In the next two sections, let us consider the following nonlinear optimization problem

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad c(x) = 0, \quad x \geq 0, \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable functions. We use the notation $w = (x, y, z) \in \mathbb{R}^N$ and $v = (x, z) \in \mathbb{R}^{2n}$, where $N = n+m+n$. We will analyze the asymptotic behavior of SPDOPT [13, Algorithm 1] with a new update strategy for parameters. At each outer iteration k, regularization parameters $\theta_k \geq 0$, $\delta_k > 0$ are chosen such that $\mathrm{In}(\widehat{J}_{\theta_k,\delta_k}(w_k)) = (n, m, 0)$, where the reduced matrix $\widehat{J}_{\theta,\delta}(w)$ is defined by

$$\widehat{J}_{\theta,\delta}(w) := \begin{pmatrix} \nabla^2_{xx}\mathcal{L}(w) + \theta I + X^{-1}Z & A(x) \\ A(x)^\top & -\delta I \end{pmatrix}. \tag{6.32}$$

Next, SPDOPT solves the linear system

$$J_{\theta_k,\delta_k}(w_k)(w_k^+ - w_k) = -\Phi(w_k, \lambda_{k+1}, \sigma_{k+1}, \mu_{k+1}), \tag{6.33}$$

where

$$J_{\theta,\delta}(w) = \begin{pmatrix} \nabla^2_{xx}\mathcal{L}(w) + \theta I & A(x) & -I \\ A(x)^\top & -\delta I & 0 \\ Z & 0 & X \end{pmatrix}, \quad \Phi(w,\lambda,\sigma,\mu) = \begin{pmatrix} \nabla_x \mathcal{L}(w) \\ c(x) + \sigma(\lambda - y) \\ XZe - \mu e \end{pmatrix}, \tag{6.34}$$

λ_k is the vector of Lagrange multipliers associated with the constraint of (P), $\sigma_k > 0$ and $\mu_k > 0$ are quadratic penalty and barrier parameters, respectively. To maintain the positivity of v_k, we apply the *fraction to the boundary rule*: let α_k be the largest $\alpha \in (0,1]$ such that

$$v_k + \alpha_k(v_k^+ - v_k) \geq (1 - \tau_k)v_k, \tag{6.35}$$

for $\tau_k \in (0,1]$. For a positive sequence $\{\varepsilon_k\}$ converging to zero, if the candidate iterate $\widehat{w}_k = w_k + \alpha_k(w_k^+ - w_k)$ satisfies $\|\Phi(\widehat{w}_k, \lambda_{k+1}, \sigma_{k+1}, \mu_{k+1})\| \leq \varepsilon_k$, we then set $w_{k+1} = \widehat{w}_k$. Otherwise, a sequence of inner iterations is applied to find w_{k+1} satisfying $v_{k+1} = (x_{k+1}, z_{k+1}) > 0$ and $\|\Phi(w_{k+1}, \lambda_{k+1}, \sigma_{k+1}, \mu_{k+1})\| \leq \varepsilon_k$.

We now recall the algorithm SPDOPT. Initially, we select parameters $\sigma_0 > 0$, $\mu_0 > 0, \lambda_0 \in \mathbb{R}^m$ and a starting point $w_0 \in \mathbb{R}^N$ satisfying $v_0 > 0$. Three constants $\kappa, \rho \in (0,1)$ and $l \in \mathbb{N}$ are also chosen. The iteration index k and an index i_k are set to $k = 0$ and $i_0 = 0$, respectively. The detail of this algorithm is given in Algorithm 8.

Remark 6.9. In the literature, to force the positivity of iterates, instead of the *fraction to the boundary rule*, interior point algorithms for solving degenerate problem (without LICQ or MFCQ) usually use the lower bound of type

$$(x_k + \alpha_k(x_k^+ - x_k)) \circ (z_k + \alpha_k(z_k^+ - z_k)) \geq \gamma_1 \mu_{k+1},$$

for some constant $\gamma > 0$, see, e.g., [128, 154, 162]. However, two matters should

Algorithm 8 (Outer iteration)

1. Choose $\mu_{k+1} > 0, \tau_k > 0$ and $\zeta_k > 0$ such that $\{\zeta_k\} \to 0$. If $k = 0$ or
$$\|c(x_k)\| \leq \kappa \max\{\|c(x_{i_j})\| + \zeta_{i_j} : (k-l)^+ \leq j \leq k\}, \tag{6.36}$$
then go to Step 3.
2. Choose $\sigma_{k+1} \leq \rho\sigma_k$. Set $\lambda_{k+1} = \lambda_k, i_{k+1} = i_k$ and go to Step 4.
3. Choose $\sigma_{k+1} \leq \sigma_k$. Set $\lambda_{k+1} = y_k, i_{k+1} = k$.
4. Choose $\theta_k \geq 0, \delta_k > 0$ such that $\mathrm{In}(\widehat{J}_{\theta_k,\delta_k}(w_k)) = (n, m, 0)$. Compute w_k^+ by solving the linear system (6.33).
5. Compute the step length α_k as the largest $\alpha \in (0, 1]$ satisfying (6.35) and set $\widehat{w}_k = w_k + \alpha_k(w_k^+ - w_k)$.
6. Choose $\varepsilon_k > 0$ such that $\{\varepsilon_k\} \to 0$. If
$$\|\Phi(\widehat{w}_k, \lambda_{k+1}, \sigma_{k+1}, \mu_{k+1})\| \leq \varepsilon_k, \tag{6.37}$$
then set $w_{k+1} = \widehat{w}_k$. Otherwise, apply a sequence of inner iterations to find w_{k+1} such that $v_{k+1} = (x_{k+1}, z_{k+1}) > 0$ and $\|\Phi(w_{k+1}, \lambda_{k+1}, \sigma_{k+1}, \mu_{k+1})\| \leq \varepsilon_k$.

be dealt with this kind of condition. Firstly, it is more difficult to implement this condition than the *fraction to the boundary rule*. Secondly, we cannot show that the sequence of step lengths $\{\alpha_k\}$ eventually tends to one which is a usual requirement to get a fast convergence. To overcome this difficulty, Yamashita and Yabe [162] added an upper bound of type

$$(x_k + \alpha_k(x_k^+ - x_k)) \circ (z_k + \alpha_k(z_k^+ - z_k)) \leq \gamma_2\mu_{k+1},$$

for some constant $\gamma_2 > \gamma_1$. Once again, the question related to the implementation of this condition in practice has not been addressed by the authors.

6.4 Asymptotic analysis for SPDOPT without constraint qualification

6.4.1 Uniform boundedness of the inverse of a regularized Jacobian matrix

In this section, we introduce a result related to the boundedness of a regularized Jacobian matrix which is a key property to demonstrate the superlinear convergence of SPDOPT. It can be seen as a generalization of Lemma 6.1 in the framework of a general nonlinear optimization problem.

Lemma 6.10. *Let $w^* = (x^*, y^*, z^*)$ be a vector in $\mathbb{R}^N$ such that*

$$0 \leq x^* \perp z^* \geq 0 \quad \textit{and} \quad a := \min\{x_i^* + z_i^* | i = 1, \dots n\} > 0,$$

where $N = n + m + n$ and n, m are natural numbers. Let $A : \mathbb{R}^N \to \mathbb{R}^{n \times m}$ be a Lipschitz continuous function and $H : \mathbb{R}^N \to \mathbb{R}^{n \times n}$ be a bounded function such that for all $w \in \mathbb{R}^N$, $H(w) = H(w)^\top$. Let $\delta : \mathbb{R}^N \to \mathbb{R}_{++}$ be a bounded function such that if δ is not bounded away from zero, then for all $w = (x, y, z) \in \mathbb{R}^N$, $\delta(w) = \Omega\left(\|x - x^*\|^t\right)$ for some $t \in (0, 1)$. For all $w = (x, y, z) \in \mathbb{R}^N$, let us define the matrices $X = \operatorname{diag}(x)$, $Z = \operatorname{diag}(z)$ and

$$J(w) := \begin{pmatrix} H(w) & A(x) & -I \\ A(x)^\top & -\delta(w) I & 0 \\ Z & 0 & X \end{pmatrix}.$$

Assume that there exist $r \in (0, a)$ and $\lambda > 0$ such that for all $w = (x, y, z) \in B(w^*, r)$ satisfying $v = (x, z) > 0$,

$$H(w) + X^{-1}Z + \frac{1}{\delta(w)} A(x) A(x)^\top \succeq \lambda I. \tag{6.38}$$

Then, there exists $C > 0$ such that for all $w \in B(w^*, r)$ satisfying $v > 0$, the matrix $J(w)$ is nonsingular and $\|J(w)^{-1}\| \leq \dfrac{C}{\delta(w)}$.

Proof. Let $w \in B(w^*, r)$ such that $v > 0$ and let us denote $\delta = \delta(w)$. Let us show that the matrix $J(w)$ is nonsingular. By Proposition 2.7, we have

$$\begin{aligned} \det J(w) &= \det X \det \begin{pmatrix} H(w) + X^{-1}Z & A(x) \\ A(x)^\top & -\delta I \end{pmatrix} \\ &= \det X \det(-\delta I) \det\left(H(w) + X^{-1}Z + \frac{1}{\delta} A(x) A(x)^\top\right). \end{aligned}$$

From $x > 0$, $\delta > 0$ and the assumption (6.38), we have $\det X > 0$, $\det(-\delta I) \neq 0$ and $\det\left(H(w) + X^{-1}Z + \frac{1}{\delta} A(x) A(x)^\top\right) > 0$. It follows that the matrix $J(w)$ is nonsingular.

We now prove the uniform boundedness of the matrix $\delta(w) J(w)^{-1}$, for all $w \in B(w^*, r)$ such that $v > 0$, by a contradiction reasoning. Suppose that there exists a sequence $\{w_k\} \subset B(w^*, r)$ such that $v_k = (x_k, z_k) > 0$ and a sequence $\{\delta_k\}$

such that for all $k \in \mathbb{N}$,

$$H_k + X_k^{-1} Z_k + \frac{1}{\delta_k} A_k A_k^\top \succeq \lambda I, \tag{6.39}$$

but the sequence $\{\delta_k \| J_k^{-1} \|\}$ tends to infinity, where we use the notation $H_k := H(w_k)$, $A_k := A(x_k)$ $\delta_k := \delta(w_k)$ and $J_k := J(w_k)$. The boundedness of the functions H and δ imply that the sequences $\{H_k\}$ and $\{\delta_k\}$ are bounded. Let us define the set $\mathcal{J} = \{i \in \{1, \ldots, n\} : x_i^* = 0\}$ and $\mathcal{I} = \{1, \ldots, n\} \backslash \mathcal{J}$. The definition of a implies that $x_i^* \geq a$ for all $i \in \mathcal{I}$ and $z_i^* \geq a$, for all $i \in \mathcal{J}$. Let $\nu = a - r > 0$. For all $k \in \mathbb{N}$, we then have

$$[x_k]_i \geq x_i^* - \| w_k - w^* \| > \nu, \quad \text{for all } i \in \mathcal{I}$$

and

$$[z_k]_i \geq z_i^* - \| w_k - w^* \| > \nu, \quad \text{for all } i \in \mathcal{J},$$

which imply that $0 < \nu < \max\{x_k, z_k\}$ for all $k \in \mathbb{N}$. We consider the two following cases. The first case is when the sequence $\{\delta_k\}$ is bounded away from zero. For each $k \in \mathbb{N}$, let us define the matrix

$$\widetilde{J}_k := \begin{pmatrix} (H_k - \lambda I) + \lambda I & -I & A_k \\ Z_k & X_k & 0 \\ A_k^\top & 0 & -\delta_k I \end{pmatrix}.$$

By applying Corollary 5.6 for the sequences $\{\widetilde{J}_k\}$ and $\{\rho_k\}$, where $\rho_k = \lambda$ for all $k \in \mathbb{N}$, then there exists $C > 0$ such that for all $k \in \mathbb{N}$, $\| \widetilde{J}_k^{-1} \| \leq \frac{C}{\lambda}$. We note that for all $k \in \mathbb{N}$, $\| J_k^{-1} \| = \| \widetilde{J}_k^{-1} \| \leq \frac{C}{\lambda}$. Therefore, the sequence $\{\delta_k \| J_k^{-1} \|\}$ is bounded which is in contradiction with the assumption that this sequence tends to infinity. Hence, this case cannot happen.

Let us consider the second case in which there exists an infinite subset $\mathcal{K} \subset \mathbb{N}$ such that $\lim_{k \in \mathcal{K}} \delta_k = 0$. The assumption of δ implies that $\delta_k = \Omega(\| x_k - x^* \|^t)$, for some $t \in (0, 1)$. By reordering the indices, we rewrite $x_k = (x_k^{\mathcal{I}}, x_k^{\mathcal{J}}) \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$, $z_k = (z_k^{\mathcal{I}}, z_k^{\mathcal{J}}) \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$, the matrices H_k and A_k under the form

$$H_k = \begin{pmatrix} H_k^{\mathcal{I}\mathcal{I}} & H_k^{\mathcal{I}\mathcal{J}} \\ H_k^{\mathcal{J}\mathcal{I}} & H_k^{\mathcal{J}\mathcal{J}} \end{pmatrix} \quad \text{and} \quad A_k = \begin{pmatrix} A_k^{\mathcal{I}} \\ A_k^{\mathcal{J}} \end{pmatrix}.$$

From the definition of the sets $\mathcal{I}$ and $\mathcal{J}$, for all $k \in \mathbb{N}$, we have

$$x_k^{\mathcal{I}} \geq \nu, \tag{6.40}$$

and

$$z_k^{\mathcal{J}} \geq \nu. \tag{6.41}$$

The definition of a matrix norm implies that there exists a sequence of unit vectors $\{v_k\} \subset \mathbb{R}^n$ such that $\|J_k^{-1}\| = \|J_k^{-1} v_k\|$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let us define $u_k := J_k^{-1} v_k / \|J_k^{-1}\|$. It follows that $\{u_k\}$ is a sequence of unit vectors and $\lim \frac{1}{\delta_k} \|J_k u_k\| = 0$. Let $k \in \mathcal{K}$. Introduce the notation

$$u_k = \begin{pmatrix} a_k^{\mathcal{I}} \\ a_k^{\mathcal{J}} \\ b_k \\ c_k^{\mathcal{I}} \\ c_k^{\mathcal{J}} \end{pmatrix} \quad \text{and} \quad \frac{1}{\delta_k} J_k u_k = \begin{pmatrix} \alpha_k^{\mathcal{I}} \\ \alpha_k^{\mathcal{J}} \\ \beta_k \\ \gamma_k^{\mathcal{I}} \\ \gamma_k^{\mathcal{J}} \end{pmatrix},$$

we then have

$$\begin{aligned}
&H_k^{\mathcal{I}\mathcal{I}} a_k^{\mathcal{I}} + H_k^{\mathcal{I}\mathcal{J}} a_k^{\mathcal{J}} + A_k^{\mathcal{I}} b_k - c_k^{\mathcal{I}} = \delta_k \alpha_k^{\mathcal{I}}, \\
&H_k^{\mathcal{J}\mathcal{I}} a_k^{\mathcal{I}} + H_k^{\mathcal{J}\mathcal{J}} a_k^{\mathcal{J}} + A_k^{\mathcal{J}} b_k - c_k^{\mathcal{J}} = \delta_k \alpha_k^{\mathcal{J}}, \\
&\left(A_k^{\mathcal{I}}\right)^\top a_k^{\mathcal{I}} + \left(A_k^{\mathcal{J}}\right)^\top a_k^{\mathcal{J}} - \delta_k b_k = \delta_k \beta_k, \\
&Z_k^{\mathcal{I}} a_k^{\mathcal{I}} + X_k^{\mathcal{I}} c_k^{\mathcal{I}} = \delta_k \gamma_k^{\mathcal{I}}, \\
&Z_k^{\mathcal{J}} a_k^{\mathcal{J}} + X_k^{\mathcal{J}} c_k^{\mathcal{J}} = \delta_k \gamma_k^{\mathcal{J}},
\end{aligned} \tag{6.42}$$

where $\left\{\left(a_k^{\mathcal{I}}, a_k^{\mathcal{J}}, b_k, c_k^{\mathcal{I}}, c_k^{\mathcal{J}}\right)\right\}$ is a sequence of unit vectors and the sequence $\left\{\left(\alpha_k^{\mathcal{I}}, \alpha_k^{\mathcal{J}}, \beta_k, \gamma_k^{\mathcal{I}}, \gamma_k^{\mathcal{J}}\right)\right\}$ converges to zero. From the fifth equation of (6.42), (6.41) and the fact that $\|c_k^{\mathcal{J}}\| \leq 1$, we then get

$$\begin{aligned}
\|a_k^{\mathcal{J}}\| &\leq \left\|\left(Z_k^{\mathcal{J}}\right)^{-1}\right\| \left(\|X_k^{\mathcal{J}}\| \|c_k^{\mathcal{J}}\| + \delta_k \|\gamma_k^{\mathcal{J}}\|\right) \\
&\leq \frac{1}{\nu} \delta_k \left(\delta_k^{-1} \|x_k^{\mathcal{J}}\| + \|\gamma_k^{\mathcal{J}}\|\right).
\end{aligned} \tag{6.43}$$

Since $\delta_k = \Omega(\|x_k - x^*\|^t)$ and $(x^*)^{\mathcal{J}} = 0$, one has

$$\delta_k^{-1} \|x_k^{\mathcal{J}}\| = \mathrm{O}(\delta_k^{-1} \|x_k - x^*\|) = \mathrm{O}\left(\delta_k^{1/t-1}\right).$$

Substituting the previous equality to (6.43) and reminding that the sequences $\{\delta_k\}_{\mathcal{K}}$ and $\{\gamma_k^{\mathcal{J}}\}$ converge to zero, $t \in (0,1)$, we deduce that

$$\lim_{k\in\mathcal{K}} \frac{1}{\delta_k} \|a_k^{\mathcal{J}}\| = 0. \tag{6.44}$$

Let us show that

$$\mathcal{I} \neq \emptyset \quad \text{and} \quad \liminf_{k\in\mathcal{K}} \left\| \left(a_k^{\mathcal{I}}, b_k\right) \right\| > 0. \tag{6.45}$$

Indeed, if this is not the case, there are two possibilities. The first one is when $\mathcal{I} = \emptyset$. By noting that the sequence $\{(\|H_k^{\mathcal{J}\mathcal{J}}\|, \|A_k^{\mathcal{J}}\|, \delta_k)\}$ is bounded, the second and the third equations of (6.42) give us

$$\|c_k^{\mathcal{J}}\| = \mathrm{O}(\|a_k^{\mathcal{J}}\|) + \mathrm{O}(\|b_k\|) + \mathrm{O}(\|\alpha_k^{\mathcal{J}}\|) \quad \text{and} \quad \|b_k\| = \frac{1}{\delta_k}\mathrm{O}(\|a_k^{\mathcal{J}}\|) + \|\beta_k\|.$$

By taking the limit for $k \in \mathcal{K}$, using (6.44) and the convergence to zero of $\{(\alpha_k^{\mathcal{J}}, \beta_k)\}$, we obtain

$$\lim_{k\in\mathcal{K}} b_k = \lim_{k\in\mathcal{K}} c_k^{\mathcal{J}} = 0,$$

which is in contradiction with the fact that $\{(a_k^{\mathcal{J}}, b_k, c_k^{\mathcal{J}})\}$ is a sequence of unit vectors. The second possibility is when $\mathcal{I} \neq \emptyset$ and there exists an infinite subset $\mathcal{K}' \subset \mathcal{K}$ such that $\lim_{k\in\mathcal{K}'} \left\| \left(a_k^{\mathcal{I}}, b_k\right) \right\| = 0$. The first two equations of (6.42) and the boundedness of $\{(\|H_k\|, \|A_k\|, \delta_k)\}$ imply that

$$\begin{aligned}\left\| \left(c_k^{\mathcal{I}}, c_k^{\mathcal{J}}\right) \right\| &\leq \|H_k\| \left\| \left(a_k^{\mathcal{I}}, a_k^{\mathcal{J}}\right) \right\| + \|A_k\| \|b_k\| + \delta_k \left\| \left(\alpha_k^{\mathcal{I}}, \alpha_k^{\mathcal{J}}\right) \right\| \\ &= \mathrm{O}\left(\left\| \left(\frac{1}{\delta_k} a_k^{\mathcal{J}}, a_k^{\mathcal{I}}, b_k \right) \right\| \right) + \mathrm{O}\left(\left\| \left(\alpha_k^{\mathcal{I}}, \alpha_k^{\mathcal{J}}\right) \right\| \right).\end{aligned}$$

From (6.44) and the convergence to zero of $\left\{\left(\alpha_k^{\mathcal{I}}, \alpha_k^{\mathcal{J}}\right)\right\}$, the above inequality implies that

$$\lim_{k\in\mathcal{K}'} \left\| \left(c_k^{\mathcal{I}}, c_k^{\mathcal{J}}\right) \right\| = 0,$$

which is again in contradiction with the fact that $\{u_k\}$ is a sequence of unit vectors. Hence, (6.45) is true.

By eliminating $c_k^{\mathcal{I}}$ in the first equation of (6.42), the first and the third equations of this system can be rewritten under the form

$$\begin{pmatrix} H_k^{\mathcal{I}\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}} & A_k^{\mathcal{I}} \\ \left(A_k^{\mathcal{I}}\right)^{\top} & -\delta_k \end{pmatrix} \begin{pmatrix} a_k^{\mathcal{I}} \\ b_k \end{pmatrix} = \begin{pmatrix} -H_k^{\mathcal{I}\mathcal{J}} a_k^{\mathcal{J}} + \delta_k \left(\alpha_k^{\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} \gamma_k^{\mathcal{I}} \right) \\ -\left(A_k^{\mathcal{J}}\right)^{\top} a_k^{\mathcal{J}} + \delta_k \beta_k \end{pmatrix}. \tag{6.46}$$

Let us denote for each $k \in \mathcal{K}$,

$$\widehat{J}_k := \begin{pmatrix} H_k^{\mathcal{I}\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}} & A_k^{\mathcal{I}} \\ \left(A_k^{\mathcal{I}}\right)^\top & -\delta_k \end{pmatrix}, \quad \hat{v}_k := \begin{pmatrix} -H_k^{\mathcal{I}\mathcal{J}} a_k^{\mathcal{J}} + \delta_k \left(\alpha_k^{\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} \gamma_k^{\mathcal{I}}\right) \\ -\left(A_k^{\mathcal{J}}\right)^\top a_k^{\mathcal{J}} + \delta_k \beta_k \end{pmatrix}.$$

On the one hand, by virtue of (6.40), the sequence $\left\{\left(X_k^{\mathcal{I}}\right)^{-1}\right\}$ is bounded. Combining this fact with the boundedness of the sequence $\left\{\left(\|H_k^{\mathcal{I}\mathcal{J}}\|, \|A_k^{\mathcal{J}}\|\right)\right\}$, the convergence to zero of the sequence $\left\{(\alpha_k^{\mathcal{I}}, \beta_k, \gamma_k^{\mathcal{I}})\right\}$ and (6.44), we deduce that

$$\lim_{k \in \mathcal{K}} \frac{1}{\delta_k} \|\hat{v}_k\| = 0. \tag{6.47}$$

On the other hand, from (6.40) and the boundedness of $\{z_k\}$, the sequence $\left\{\left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}}\right\}$ is bounded. Besides, (6.39) implies that

$$H_k^{\mathcal{I}\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}} + \frac{1}{\delta_k} A_k^{\mathcal{I}} \left(A_k^{\mathcal{I}}\right)^\top \succeq \lambda I.$$

By virtue of Lemma 6.1, there exists $C > 0$ such that for all $k \in \mathcal{K}$, we then have $\|\widehat{J}_k^{-1}\| \leq \frac{C}{\delta_k}$. It follows from this fact and (6.46) that

$$\left\|\left(a_k^{\mathcal{I}}, b_k\right)\right\| = \|\widehat{J}_k^{-1} \hat{v}_k\| \leq \frac{C}{\delta_k} \|\hat{v}_k\|.$$

Taking the limit for $k \in \mathcal{K}$ and using (6.47), we then get

$$\lim_{k \in \mathcal{K}} \left\|\left(a_k^{\mathcal{I}}, b_k\right)\right\| = 0,$$

which is in contradiction with (6.45).

In sum, for all $r \in (0, a)$ such that (6.38) holds, there exists $C > 0$ such that for all $w \in B(w^*, r)$ satisfying $v > 0$, $\delta(w)\|J(w)^{-1}\| \leq C$, which concludes the proof. □

6.4.2 Assumptions and preliminary results

Let $x^* \in \mathbb{R}^n$ be an optimal solution of (P). We recall the index set of active bounds $\mathcal{A} := \{i \in \{1, \ldots, n\} : x_i^* = 0\}$. The Karush-Kuhn-Tucker (KKT) conditions (2.1) are satisfied at a point $w = (x, y, z) \in \mathbb{R}^N$ if

$$F(w) = 0 \quad \text{and} \quad v = (x, z) \geq 0, \tag{6.48}$$

where $F : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $F(w) = \begin{pmatrix} \nabla_x \mathcal{L}(w) \\ c(x) \\ XZe \end{pmatrix}$. Let us assume that the following assumptions are satisfied.

Assumption 6.4. The functions f and c are twice continuously differentiable and their second derivatives are Lipschitz continuous over an open neighborhood of x^*.

Assumption 6.5. There exists $(y^*, z^*) \in \mathbb{R}^{m+n}$ such that the KKT conditions (6.48) are satisfied at $w^* = (x^*, y^*, z^*)$.

Assumption 6.6. The second order sufficient conditions (SOSCs) (Definition 2.24) hold at w^*, i.e. $u^\top \nabla^2_{xx} \mathcal{L}(w^*) u > 0$, for all $u \in \mathbb{R}^n$ such that $u \neq 0$, $A(x^*)^\top u = 0$ and $u_i = 0$ for all $i \in \mathcal{A}$.

Assumption 6.7. The strict complementarity (Definition 2.26) holds at w^*, i.e.,

$$a := \min\{x_i^* + z_i^* : i = 1, \ldots, n\} > 0.$$

Under Assumptions 6.4–6.7, we note that x^* is a strict local solution of (P), see, e.g. [22, Proposition 3.3.2]. Let us define the set of dual solutions

$$\mathcal{S}_D = \{(y^*, z^*) | w^* = (x^*, y^*, z^*) \text{ satisfies the KKT conditions (6.48)}\}$$

and the set of primal-dual solutions

$$\mathcal{S} = \{x^*\} \times \mathcal{S}_D.$$

Assumption 6.5 implies that $\mathcal{S} \neq \emptyset$. Since $\mathcal{S}$ is closed and the norm is coercive, for all $w = (x, y, z) \in \mathbb{R}^N$, there exists $\bar{w} = (x^*, \bar{y}, \bar{z}) \in \mathcal{S}$ such that

$$\|w - \bar{w}\| = \min_{\xi \in \mathcal{S}} \|w - \xi\| =: d(w).$$

The next lemma gives us a relation between the distance function d and F. In particular, it shows that the function F provides a local error bound condition at w^*.

Lemma 6.11. *Assume that Assumptions 6.4–6.7 hold at w^*. Then, there exists $\varepsilon > 0$ such that for all $w \in B(w^*, \varepsilon)$ and $v = (x, z) \geq 0$, we have*

$$d(w) = \Theta(\|F(w)\|).$$

Proof. Let $\varepsilon_1 > 0$ such that the functions f and c are twice continuously differentiable on $B(w^*, \varepsilon_1)$. Let $w \in B(w^*, \varepsilon_1)$. By reminding that $F(\bar{w}) = 0$ and $\bar{w} = (x^*, \bar{y}, \bar{z})$, Assumption 6.4 implies that

$$\begin{aligned} \|\nabla_x \mathcal{L}(w)\| &= \|\nabla_x \mathcal{L}(w) - \nabla_x \mathcal{L}(\bar{w})\| = \mathrm{O}(\|w - \bar{w}\|) = \mathrm{O}(d(w)), \\ \|c(x)\| &= \|c(x) - c(x^*)\| = \mathrm{O}(\|w - \bar{w}\|) = \mathrm{O}(d(w)). \end{aligned} \tag{6.49}$$

By noting that $\|x - x^*\| \leq \|w - w^*\| \leq \varepsilon_1$ and $\|z - \bar{z}\| \leq \|w - w^*\| \leq \varepsilon_1$, the triangle inequality gives us

$$\begin{aligned} &\|x\| \leq \|x - x^*\| + \|x^*\| \leq \varepsilon_1 + \|w^*\|, \\ &\|\bar{z}\| \leq \|\bar{z} - z\| + \|z - z^*\| + \|z^*\| \leq 2\varepsilon_1 + \|w^*\|. \end{aligned}$$

Hence, we then get

$$\|XZe\| = \|X(z - \bar{z}) + \bar{Z}(x - x^*)\| \leq (3\varepsilon_1 + 2\|w^*\|)\|w - \bar{w}\| = \mathrm{O}(d(w)).$$

From this fact and (6.49), we deduce that $\|F(w)\| = \mathrm{O}(d(w))$.

Conversely, by virtue of [94, Theorem 1], there exist constants $\eta > 0$, $\kappa > 0$ and $\gamma > 0$ such that for all $w = (x, y, z) \in \mathbb{R}^N$, for each $\widehat{w} = (\hat{x}, \hat{y}, \hat{z}) \in B(w^*, \eta)$, and for each $t = (t_1, t_2, t_3) \in \mathbb{R}^N$ with $\|t\| \leq \kappa$ and

$$\nabla_x \mathcal{L}(\widehat{w}) + t_1 = 0, \quad c(\hat{x}) + t_2 = 0 \quad \text{and} \quad -\hat{x} + t_3 \in N(\hat{z}), \tag{6.50}$$

where the cone $N(\cdot)$ is defined by

$$N(\lambda) = \begin{cases} \{\mu : \mu \leq 0, \quad \mu^\top \lambda = 0\}, & \text{if } \lambda \geq 0 \\ \emptyset & \text{otherwise}, \end{cases}$$

we then have

$$\|x - x^*\| + \|(y, z) - (\bar{y}, \bar{z})\| \leq \|x - \hat{x}\| + \|(y, z) - (\hat{y}, \hat{z})\| + \gamma\|t\|, \tag{6.51}$$

where $(\bar{y}, \bar{z}) \in \mathcal{S}_D$ such that $\|(x, y, z) - (x^*, \bar{y}, \bar{z})\| = d(w)$. We note that $x_i^* \geq a$ for all $i \in \mathcal{A}^c := \{1, \dots, n\} \backslash \mathcal{A}$ and $z_i^* \geq a$, for all $i \in \mathcal{A}$, where a is defined in Assumption 6.7. Let us define

$$\bar{\eta} := \frac{1}{2} \min\{a, \eta\} \quad \text{and} \quad \bar{a} := a - \bar{\eta} \geq \bar{\eta}. \tag{6.52}$$

For all $w \in B(w^*, \bar{\eta})$, by using the definition of $\bar{a}$ and noting that for all $i \in \mathcal{A}$ and $j \in \mathcal{A}^c$, $x_i^* = z_j^* = 0$, we then have

$$x_i \geq x_i^* - \|w - w^*\| > \bar{a} > z_i - z_i^* = z_i, \quad \text{for all } i \in \mathcal{A}^c \tag{6.53}$$

and

$$z_i \geq z_i^* - \|w - w^*\| > \bar{a} > x_i - x_i^* = x_i, \quad \text{for all } i \in \mathcal{A}. \tag{6.54}$$

For all $w \in B(w^*, \bar{\eta})$ such that $v = (x, z) \geq 0$, let us define the vectors t_3 and $\hat{z}$ in $\mathbb{R}^n$ as follows

$$[t_3]_i = \begin{cases} 0 & \text{if } i \in \mathcal{A}^c \\ x_i & \text{if } i \in \mathcal{A} \end{cases} \quad \text{and} \quad \hat{z}_i = \begin{cases} 0 & \text{if } i \in \mathcal{A}^c \\ z_i & \text{if } i \in \mathcal{A}. \end{cases}$$

For a vector $x \in \mathbb{R}^n$ and a subset $\mathcal{B} \subset \{1, \ldots, n\}$, let us denote $x_{\mathcal{B}} = (x_i)_{i \in \mathcal{B}} \in \mathbb{R}^{|\mathcal{B}|}$. From (6.53) and (6.54), the definitions of t_3 and $\hat{z}$ imply that for all $w \in B(w^*, \bar{\eta})$ satisfying $v = (x, z) \geq 0$ one has

$$-x + t_3 \in N(\hat{z}) \tag{6.55}$$

and

$$\begin{aligned} \|t_3\| &= \|x_{\mathcal{A}}\| < \frac{1}{\bar{a}} \|x_{\mathcal{A}} z_{\mathcal{A}}\| \leq \frac{1}{\bar{a}} \|XZe\|, \\ \|z - \hat{z}\| &= \|z_{\mathcal{A}^c}\| < \frac{1}{\bar{a}} \|x_{\mathcal{A}^c} z_{\mathcal{A}^c}\| \leq \frac{1}{\bar{a}} \|XZe\|. \end{aligned} \tag{6.56}$$

By virtue of (6.49), we can choose

$$0 < \varepsilon \leq \min\left\{\varepsilon_1, \bar{\eta}, \frac{\kappa}{2}\right\} \tag{6.57}$$

such that $\|(\nabla \mathcal{L}(w), c(x))\| \leq \frac{\kappa}{2}$, for all $w \in B(w^*, \varepsilon)$. Let $w = (x, y, z) \in B(w^*, \varepsilon)$ such that $(x, z) \geq 0$ and let us define the vectors $\hat{w} = (x, y, \hat{z})$ and $t = (-\nabla_x \mathcal{L}(\hat{w}), -c(x), t_3)$ in $\mathbb{R}^N$. The definitions of t and of $\hat{w}$, and (6.55) imply that the condition (6.50) holds at t and $\hat{w}$. We now show that $\hat{w} \in B(w^*, \eta)$ and $\|t\| \leq \kappa$. By reminding that $z_{\mathcal{A}^c}^* = 0$ and $x_{\mathcal{A}}^* = 0$, the definition of $\hat{z}$ gives us

$$\begin{aligned} \|z - \hat{z}\| &= \|z_{\mathcal{A}^c}\| = \|z_{\mathcal{A}^c} - z_{\mathcal{A}^c}^*\| \leq \|w - w^*\| < \varepsilon, \\ \|t_3\| &= \|x_{\mathcal{A}}\| = \|x_{\mathcal{A}} - x_{\mathcal{A}}^*\| < \varepsilon \leq \frac{\kappa}{2}. \end{aligned}$$

Using these above inequalities, the definitions of ε given by (6.57), of $\bar{\eta}$ given by (6.52), of $\hat{w}$ and of t, we then get

$$\begin{aligned} \|\hat{w} - w^*\| &\leq \|w - w^*\| + \|z - \hat{z}\| < 2\varepsilon \leq 2\bar{\eta} \leq \eta, \\ \|t\| &\leq \|(\nabla_x \mathcal{L}(w), c(x))\| + \|t_3\| < \frac{\kappa}{2} + \frac{\kappa}{2} = \kappa. \end{aligned}$$

By reminding that $\nabla_x \mathcal{L}(\hat{w}) = \nabla_x \mathcal{L}(w) + (z - \hat{z})$, we then deduce from (6.51) and (6.56) that

$$\begin{aligned} \|x - x^*\| + \|(y, z) - (\bar{y}, \bar{z})\| &\leq \|z - \hat{z}\| + \gamma(\|\nabla_x \mathcal{L}(\hat{w})\| + \|c(x)\| + \|t_3\|) \\ &= \mathrm{O}(\|\nabla_x \mathcal{L}(x, y, z)\| + \|c(x)\| + \|z - \hat{z}\| + \|t_3\|) \\ &= \mathrm{O}(\|\nabla_x \mathcal{L}(w)\| + \|c(x)\| + \|XZe\|) \end{aligned} \tag{6.58}$$

Since all Euclidean norms are equivalent, without loss of generality, we may use l_2 norm here. Using this fact and the inequalities $\|(a, b)\| \leq \|a\| + \|b\| \leq \sqrt{2}\|(a, b)\|$, for any vectors $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, we then deduce from (6.58) that

$$\begin{aligned} d(w) &= \|(x - x^*, y - \bar{y}, z - \bar{z})\| \\ &\leq \|x - x^*\| + \|(y, z) - (\bar{y}, \bar{z})\| \\ &= \mathrm{O}(\|\nabla_x \mathcal{L}(w)\| + \|c(x)\| + \|XZe\|\|) \\ &= \mathrm{O}(\|F(w)\|). \end{aligned}$$

□

Remark 6.12. Lemma 6.11 is related to some other results of Fachinei *et al.* [62], Wright [155, Theorem A.1], Wright [157, Theorem 3.1], Yamashita and Yabe [162, Theorem 1]. We note that in these works, authors usually show that

$$d(w) = \|(\nabla_x \mathcal{L}(w), c(x), \min\{x, z\})\|.$$

To demonstrate that $d(w) = \mathrm{O}(x^\top z)$ or $d(w) = \mathrm{O}(\|F(w)\|)$, the centrality condition $x \circ z \geq \mu$ is needed and some conditions of μ are assumed, see, e.g. Wright [154, Theorem 3.3], Yamashita and Yabe [162, Lemma 7].

In Lemma 6.11, we show directly the property $d(w) = \mathrm{O}(\|F(w)\|)$ without any additional condition related to a specific algorithm.

The next lemma shows us that the matrix $\widehat{J}_{0,\delta}(w)$ defined in (6.32) has the correct inertia $(n, m, 0)$ in some neighborhood of w^*. Moreover, the minimum eigenvalue of the matrix $\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\delta}A(x)A(x)^\top + X^{-1}Z$ is uniformly bounded away from zero in some neighborhood of this point.

Lemma 6.13. *Assume that Assumptions 6.4–6.7 hold at w^*. Then there exist $\kappa > 0$, $\bar{\delta} > 0$ and $\nu > 0$ such that for all $w = (x, y, z) \in B(w^*, \kappa)$ satisfying $v = (x, z) > 0$ and for each $\delta \in (0, \bar{\delta}]$,*

$$\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\delta}A(x)A(x)^\top + X^{-1}Z \succeq \nu I. \tag{6.59}$$

In particular, for all $w \in B(w^*, \kappa)$ such that $v > 0$ and $\delta \in (0, \bar{\delta}]$, we have $In(\widehat{J}_\delta(w)) = (n, m, 0)$ where

$$\widehat{J}_\delta(w) := \widehat{J}_{0,\delta}(w) = \begin{pmatrix} \nabla^2_{xx}\mathcal{L}(w) + X^{-1}Z & A(x) \\ A(x)^\top & -\delta I \end{pmatrix}.$$

Proof. Let us define the diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$, where

$$d_i = \begin{cases} 1 & \text{if } i \in \mathcal{A} \\ 0 & \text{if } i \notin \mathcal{A}. \end{cases}$$

Assumption 6.6 can be restated under the form: $u^\top \nabla^2_{xx}\mathcal{L}(w^*)u > 0$, for all $u \in \mathbb{R}^N \backslash \{0\}$ such that $\begin{pmatrix} A(x^*)^\top \\ D \end{pmatrix} u = 0$. By virtue of Lemma 2.2, this implies that there exists $\bar{\delta} > 0$ such that

$$\nabla^2_{xx}\mathcal{L}(w^*) + \frac{1}{\bar{\delta}}A(x^*)A(x^*)^\top + \frac{1}{\bar{\delta}}D^2 \succ 0.$$

Let us denote $\lambda = \lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w^*) + \frac{1}{\bar{\delta}}A(x^*)A(x^*)^\top + \frac{1}{\bar{\delta}}D^2\right) > 0$. Since the eigenvalues of a matrix are continuous functions of its entries, see, e.g., [136, Theorem 5.2], there exists $\kappa \in \left(0, \frac{\bar{\delta}a}{1+\bar{\delta}}\right]$, where a is defined in Assumption 6.7, such that for all $w \in B(w^*, \kappa)$, we then have

$$\lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\bar{\delta}}A(x)A(x)^\top + \frac{1}{\bar{\delta}}D^2\right) \geq \frac{\lambda}{2}. \tag{6.60}$$

Let $w \in B(w^*, \kappa)$ such that $v > 0$. For all $i \in \mathcal{A}$, we note that

$$z_i \geq z_i^* - \|w - w^*\| > a - \kappa > 0 \quad \text{and} \quad x_i = x_i - x_i^* \leq \|w - w^*\| < \kappa,$$

which implies that for all $i \in \mathcal{A}$,

$$\frac{z_i}{x_i} > \frac{a-\kappa}{\kappa} \geq \frac{1}{\bar{\delta}}.$$

From the choice of v and the definition of the diagonal matrix D in which $(x_i, z_i) > 0$ and $d_i = 0$ for all $i \notin \mathcal{A}$, the above inequality implies that

$$\lambda_{\min}\left(X^{-1}Z - \frac{1}{\bar{\delta}}D^2\right) > 0. \tag{6.61}$$

By using (6.60) and (6.61), Proposition 2.1 gives us

$$\begin{aligned}
&\lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\bar{\delta}}A(x)A(x)^\top + X^{-1}Z\right) \\
&\geq \;\lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\bar{\delta}}A(x)A(x)^\top + \frac{1}{\bar{\delta}}D^2\right) + \lambda_{\min}\left(X^{-1}Z - \frac{1}{\bar{\delta}}D^2\right), \\
&\geq \;\frac{\lambda}{2}.
\end{aligned}$$

Therefore, for all $w \in B(w^*, \kappa)$ and $\delta \in (0, \bar{\delta}]$, by invoking again Proposition 2.1, we obtain

$$\begin{aligned}
&\lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\delta}A(x)A(x)^\top + X^{-1}Z\right) \\
&\geq \;\lambda_{\min}\left(\nabla^2_{xx}\mathcal{L}(w) + \frac{1}{\bar{\delta}}A(x)A(x)^\top + X^{-1}Z\right) + \left(\frac{1}{\delta} - \frac{1}{\bar{\delta}}\right)\lambda_{\min}\left(A(x)A(x)^\top\right) \\
&\geq \;\nu := \frac{\lambda}{2},
\end{aligned}$$

from which the first conclusion follows.

The second assertion related to the inertia of the matrix $\widehat{J}_\delta(w)$ is a direct consequence of the first one and Lemma 2.2. $\square$

6.4.3 Properties of the Newton iterates w^+

Under Assumptions 6.4-6.7 and by virtue of Lemmas 6.11 and 6.13, there exist positive numbers $\eta < a$, $\bar{\delta}$, l, L, b such that for all w, $w' \in B(w^*, \eta)$ satisfying $v > 0$ and $\delta \in (0, \bar{\delta}]$, the condition (6.59) holds and one has

$$\|F(w) - F(w')\| \leq l\|w - w'\| \tag{6.62}$$

$$\|F'(w) - F'(w')\| \leq L\|w - w'\| \tag{6.63}$$

$$\frac{1}{b}\|F(w)\| \leq d(w) \leq b\|F(w)\|. \tag{6.64}$$

In this section, we will analyze the properties of the Newton iterates w_k^+ solved by the linear system (6.33) under the assumption that the condition (6.36) holds. To simplify the notation, the iteration index k will be eliminated. For a point $w = (x, y, z) \in B(w^*, \eta)$ such that $v = (x, z) > 0$, let us denote

$$J_\delta(w) = \begin{pmatrix} \nabla^2_{xx}\mathcal{L}(w) & A(x) & -I \\ A(x)^\top & -\delta I & 0 \\ Z & 0 & X \end{pmatrix} = F'(w) - \delta\tilde{I},$$

where, $\tilde{I} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$. A regularization, a logarithm barrier and a quadratic penalty parameters are chosen such that

$$\delta \in (0, \bar{\delta}] \quad \text{and} \quad \delta = \gamma_1 \|F(w)\|^t, \tag{6.65}$$

$$\mu^+ \leq \gamma_2 \|F(w)\|^{1+t}, \tag{6.66}$$

$$\sigma^+ = \gamma_3 \|F(w)\|, \tag{6.67}$$

where $\gamma_1 > 0$ and γ_2, γ_3, $t \in (0, 1)$. Let $\lambda^+ = y$ and let us consider the Newton iterate w^+ generated by the system

$$J_\delta(w)(w^+ - w) = -\Phi(w, \lambda^+, \sigma^+, \mu^+). \tag{6.68}$$

Let us denote $\tilde{e} = \begin{pmatrix} 0 \\ 0 \\ e \end{pmatrix} \in \mathbb{R}^N$. From the choice $\lambda^+ = y$, we note that $\Phi(w, \lambda^+, \sigma^+, \mu^+) = F(w) - \mu^+\tilde{e}$.

We now estimate an upper bound on the length of the solution of the linear system (6.68).

Lemma 6.14. *There exists $C_1 > 0$ such that for all $w \in B(w^*, \eta)$ satisfying $v > 0$, $\lambda^+ = y$, parameters δ and μ^+ chosen as in (6.65), (6.66), we have*

$$\|w^+ - w\| \leq C_1 d(w).$$

Proof. For all $w \in B(w^*, \eta)$ such that $v > 0$, the choice (6.65) of δ and (6.64) imply that

$$\delta = \gamma_1 \|F(w)\|^t = \Theta(d(w)^t),$$

By reminding that $d(w) = \|(x, y, z) - (x^*, \bar{y}, \bar{z})\| \geq \|x - x^*\|$, it follows from the above fact that for all $w \in B(w^*, \eta)$ satisfying $v > 0$,

$$\delta = \mathrm{O}(d(w)^t), \tag{6.69}$$

$$\begin{aligned} \delta &= \Omega(d(w)^t) \\ &= \Omega(\|x - x^*\|^t). \end{aligned} \tag{6.70}$$

From the previous equality and Lemma 6.13, all assumptions of Lemma 6.10 are satisfied at all $w \in B(w^*, \eta)$ such that $v > 0$. Hence, there exists $C > 0$ such that for all $w \in B(w^*, \eta)$ satisfying $v > 0$,

$$\|J_\delta(w)^{-1}\| \leq \frac{C}{\delta}. \tag{6.71}$$

Let $w \in B(w^*, \eta)$ such that $v > 0$. From (6.68) and reminding that $F(\bar{w}) = 0$, we have

$$\begin{aligned} w^+ - w &= -J_\delta(w)^{-1}\Phi(w, \lambda^+, \sigma_{k+1}, \mu^+) \\ &= -J_\delta(w)^{-1}(F(w) - \mu^+\tilde{e}) \\ &= J_\delta(w)^{-1}(F(\bar{w}) - F(w)) + \mu^+ J_\delta(w)^{-1}\tilde{e} \\ &= J_\delta(w)^{-1}\int_0^1 F'(w + s(\bar{w} - w))(\bar{w} - w)\mathrm{d}s + \mu^+ J_\delta(w)^{-1}\tilde{e} \\ &= J_\delta(w)^{-1}\int_0^1 [F'(w + s(\bar{w} - w)) - F'(w)](\bar{w} - w)\mathrm{d}s \\ &\quad + J_\delta(w)^{-1}F'(w)(\bar{w} - w) + \mu^+ J_\delta(w)^{-1}\tilde{e} \\ &= J_\delta(w)^{-1}\int_0^1 [F'(w + s(\bar{w} - w)) - F'(w)](\bar{w} - w)\mathrm{d}s \\ &\quad + \bar{w} - w + \delta J_\delta(w)^{-1}\begin{pmatrix} 0 \\ \bar{y} - y \\ 0 \end{pmatrix} + \mu^+ J_\delta(w)^{-1}\tilde{e}. \end{aligned}$$

By taking the norm on both sides and using (6.63), we get

$$\|w^+ - w\| \leq \|J_\delta(w)^{-1}\|\left(\frac{L}{2}\|\bar{w} - w\|^2 + \delta\|\bar{w} - w\| + \mu^+\sqrt{n}\right) + \|\bar{w} - w\|. \tag{6.72}$$

It follows from the choice (6.66) of μ^+ and (6.64) that

$$\mu^+ \leq \gamma_2\|F(w)\|^{1+t} = \gamma_2 b^{1+t} d(w)^{1+t}. \tag{6.73}$$

By substituting (6.71), (6.70) and (6.73) to (6.72), and using $d(w) = \|w - \bar{w}\| < \eta$, we obtain

$$\|w^+ - w\| = \mathrm{O}(d(w)),$$

which concludes the result. □

6.4.4 Properties of the candidate iterate $\widehat{w}$

Let $w \in B(w^*, \eta)$ such that $v > 0$. Let w^+ be a solution of the linear system (6.68). Let α_v be the largest value in $(0, 1]$ such that

$$v + \alpha_v(v^+ - v) \geq (1 - \tau_v)v, \tag{6.74}$$

where $\tau_v \in (0, 1)$ satisfies

$$1 - \tau_v = \mathrm{O}(\|F(w)\|). \tag{6.75}$$

At first, we have the following bound on step length α_v.

Lemma 6.15. *Let $w \in B(w^*, \eta)$ such that $v > 0$, $\lambda^+ = y$, and parameters δ and μ^+ satisfying (6.65) and (6.66). Let w^+ be a solution of the linear system (6.68) and α_v be calculated by the formula (6.74). Then, there exists $C_2 > 0$ such that the following inequality holds*

$$1 - \alpha_v \leq C_2 d(w). \tag{6.76}$$

Proof. From the choice (6.75) of τ_v and (6.64), there exists $M > 0$ such that for all $w \in B(w^*, \eta)$ satisfying $v > 0$,

$$\begin{aligned} 1 - \tau_v &\leq M\|F(w)\| \\ &\leq Mbd(w). \end{aligned} \tag{6.77}$$

Let $w \in B(w^*, \eta)$ such that $v > 0$. By virtue of [14, Corollary 1], we have

$$0 \leq 1 - \alpha_v \leq 1 - \tau_v + \frac{1}{a - \eta}\|w^+ - w\|.$$

By using (6.77) and Lemma 6.14, we conclude that (6.76) holds with $C_2 := Mb + \frac{1}{a-\eta}C_1 > 0$. □

The next lemma gives us a relation between the distance functions evaluated at the candidate iterate $\widehat{w} := w + \alpha_v(w^+ - w)$ and at w.

Lemma 6.16. *Let $C_1 > 0$ be defined in Lemma 6.14. Let $w \in B(w^*, \frac{\eta}{1+C_1})$ such that $v > 0$, $\lambda^+ = y$, and parameters δ and μ^+ satisfying (6.65) and (6.66). Let w^+ be a solution of the linear system (6.68) and α_v be calculated by (6.74). Let us define $\widehat{w} := w + \alpha_v(w^+ - w)$. Then, there exists $C_3 > 0$ such that*

$$d(\widehat{w}) \le C_3 d(w)^{1+t}. \tag{6.78}$$

Proof. Let $w \in B(w^*, \frac{\eta}{1+C_1})$ such that $v > 0$. By applying Lemma 6.14, using the definition of $\widehat{w}$ and noting that $\alpha_v \in (0, 1]$ and $d(w) = \|w - \bar{w}\| \le \|w - w^*\|$, we get

$$\begin{aligned}
\|\widehat{w} - w^*\| &= \|(1 - \alpha_v)(w - w^*) + \alpha_v(w^+ - w^*)\| \\
&\le \|w - w^*\| + \|w^+ - w^*\| \\
&\le (1 + C_1)\|w - w^*\| \\
&< \eta,
\end{aligned}$$

from which (6.64) implies that

$$\frac{1}{b}\|F(\widehat{w})\| \le d(\widehat{w}) \le b\|F(\widehat{w})\|. \tag{6.79}$$

By virtue of the linear system (6.68) and noting that $F(w) - \mu^+\tilde{e} = \Phi(w, \lambda^+, \sigma^+, \mu^+)$, we have

$$\begin{aligned}
F(w^+) &= F(w) + \int_0^1 F'(w + s(w^+ - w))(w^+ - w)\mathrm{ds} \\
&= -J_\delta(w)(w^+ - w) + \mu^+\tilde{e} + \int_0^1 F'(w + s(w^+ - w))(w^+ - w)\mathrm{ds} \\
&= -\left(F'(w) - \delta\tilde{I}\right)(w^+ - w) + \mu^+\tilde{e} + \int_0^1 F'(w + s(w^+ - w))(w^+ - w)\mathrm{ds} \\
&= \delta\begin{pmatrix} 0 \\ y^+ - y \\ 0 \end{pmatrix} + \mu^+\tilde{e} + \int_0^1 [F'(w + s(w^+ - w)) - F'(w)](w^+ - w)\mathrm{ds}.
\end{aligned}$$

By taking the norm on both sides, using (6.63) and Lemma 6.14, we get

$$\begin{aligned}
\|F(w^+)\| &\le \delta\|w^+ - w\| + \mu^+\sqrt{n} + \frac{L}{2}\|w^+ - w\|^2 \\
&\le \delta C_1 d(w) + \mu^+\sqrt{n} + \frac{L}{2}C_1^2 d(w)^2. \qquad (6.80)
\end{aligned}$$

By combining (6.80), (6.69) and (6.73), and noting that $d(w) = \|w - \bar{w}\| < \eta$, we obtain

$$\|F(w^+)\| = \mathrm{O}(d(w)^{1+t}). \tag{6.81}$$

Besides, by reminding that $\widehat{w} - w^+ = (\alpha_v - 1)(w^+ - w)$ and $d(w) = \|w - \bar{w}\| < \eta$, we deduce from (6.62), (6.76) and Lemma 6.14 that

$$\begin{aligned}\|F(\widehat{w}) - F(w^+)\| &\leq l\|\widehat{w} - w^+\| \\ &\leq lC_2 d(w)\|w^+ - w\| \\ &= \mathrm{O}(d(w)^{1+t}).\end{aligned} \tag{6.82}$$

By combining (6.79), (6.81) and (6.82),we have

$$\begin{aligned}d(\widehat{w}) \leq b\|F(\widehat{w})\| &\leq b(\|F(\widehat{w}) - F(w^+)\| + \|F(w^+)\|) \\ &= \mathrm{O}(d(w)^{1+t}),\end{aligned}$$

from which the result follows. $\square$

The next lemma shows that inner iterates will not be needed when w is sufficiently close to w^*. In addition, the condition (6.2) is also satisfied by the candidate iterate $\widehat{x} := x + \alpha_v(x^+ - x)$.

Lemma 6.17. *For each $w \in B(w^*, \eta)$ such that $v > 0$, let $\lambda^+ = y$ and parameters be chosen as follows. The parameters δ, μ^+ and σ^+ satisfy (6.65), (6.66) and (6.67). Let us choose $\zeta^+ \geq \theta_1\sigma^+$ and $\varepsilon \geq \theta_2\sigma^+$, for some θ_1, $\theta_2 > 0$. Then, there exists $r > 0$ such that for all $w \in B(w^*, r)$,*

$$\begin{aligned}&\|\Phi(\widehat{w}, \lambda^+, \sigma^+, \mu^+)\| \leq \varepsilon, \\ &\|c(\widehat{x})\| \leq \kappa(\|c(x)\| + \zeta^+),\end{aligned}$$

where $\widehat{w} := w + \alpha_v(w^+ - w)$, w^+ be the solution of the linear system (6.68), α_v is given by (6.74) and a given constant $\kappa \in (0, 1)$.

Proof. Let us define $C_4 := b(C_3 + \gamma_3\eta^{1-t}C_1 + \gamma_2 b^t\sqrt{n})$ and

$$r := \min\left\{\frac{\eta}{1 + C_1}, \left(\frac{\theta_2\gamma_3}{C_4 b}\right)^{\frac{1}{t}}, \left(\frac{\kappa\theta_1\gamma_3}{b^2 C_3}\right)^{\frac{1}{t}}\right\}. \tag{6.83}$$

Let $w \in B(w^*, r)$ such that $v > 0$. The choice of σ^+ given by (6.67) and (6.64) imply that

$$\frac{\gamma_3}{b}d(w) \le \sigma^+ \le \gamma_3 b d(w). \tag{6.84}$$

From inequalities (6.78) and (6.79), we deduce that

$$\begin{aligned}\|F(\widehat{w})\| &\le bd(\widehat{w}) \\ &\le bC_3 d(w)^{1+t}. \end{aligned} \tag{6.85}$$

By virtue of inequalities (6.85), (6.84) and (6.73), of Lemma 6.14 and reminding that $\lambda^+ = y$, $\|\widehat{w} - w\| \le \|w^+ - w\|$ and $d(w) < r < \eta$, we then get

$$\begin{aligned}\|\Phi(\widehat{w}, \lambda^+, \sigma^+, \mu^+)\| &\le \|F(\widehat{w})\| + \sigma^+\|\widehat{w} - w\| + \mu^+\sqrt{n} \\ &\le bC_3 d(w)^{1+t} + \gamma_3 b C_1 d(w)^2 + \gamma_2 b^{1+t}\sqrt{n} d(w)^{1+t} \\ &\le b\left(C_3 + \gamma_3 \eta^{1-t} C_1 + \gamma_2 b^t \sqrt{n}\right) d(w)^{1+t} \\ &< C_4 r^t d(w).\end{aligned}$$

From the choice of ε, the leftmost inequality in (6.84) and the definition of r, we deduce that

$$\varepsilon \ge \theta_2 \sigma^+ \ge \frac{\theta_2 \gamma_3}{b} d(w) \ge C_4 r^t d(w) > \|\Phi(\widehat{w}, \lambda^+, \sigma^+, \mu^+)\|,$$

from which the first assertion follows.

We are now in a position to demonstrate the inequality $\|c(\widehat{x})\| \le \kappa(\|c(x)\| + \zeta^+)$. From (6.85), the leftmost inequality of (6.84) and the choice of ζ^+, and noting that $d(w) \le \|w - w^*\| < r$, we then get

$$\|c(\widehat{x})\| \le \|F(\widehat{w})\| \le bC_3 d(w)^{1+t} \le \frac{b^2 C_3}{\gamma_3} r^t \sigma^+ \le \kappa\theta_1 \sigma^+ \le \kappa(\|c(x)\| + \zeta^+),$$

which completes the proof. □

6.4.5 Superlinear convergence of the sequence $\{w_k\}$

This section is devoted to analyze the behavior of the sequence $\{w_k\}$ generated by Algorithm 8. For each $k \in \mathbb{N}$, from a point $w_0 \in \mathbb{R}^N$ such that $v_0 > 0$ and an initial value $\mu_0 > 0$, the regularization, the logarithm barrier and the quadratic penalty parameters are chosen such that

$$\delta_k \in (0, \bar{\delta}] \quad \text{and} \quad \delta_k = \gamma_1 \|F(w_k)\|^t, \tag{6.86}$$

$$\mu_{k+1} = \gamma_2 \min\{\|F(w_k)\|^{1+t}, \mu_k\}, \tag{6.87}$$

$$\sigma_{k+1} = \gamma_3 \|F(w_k)\|, \tag{6.88}$$

where $\gamma_1 > 0$ and $\gamma_2, \gamma_3, t \in (0, 1)$.

We now show that if at an iteration k_0, w_{k_0} is sufficiently near w^* and the condition (6.36) satisfies, then the algorithm is reduced to a sequence of outer iterations and the whole sequence converges superlinearly to an optimal solution of (P).

Theorem 6.18. *Let Assumptions 6.4–6.7 hold at w^*. Assume that sequences $\{\delta_k\}$, $\{\mu_k\}$, $\{\sigma_k\}$, $\{\zeta_k\}$ and $\{\varepsilon_k\}$ of Algorithm 8 are chose by formulas (6.86)–(6.88) and for all $k \in \mathbb{N}$, $\zeta_k \geq \theta_1 \sigma_k$, $\varepsilon_k \geq \theta_2 \sigma_{k+1}$ for some given positive constants θ_1, θ_2. There exist $0 < \bar{r} < r$ such that if at an iteration k_0, $w_{k_0} \in B(w^*, \bar{r})$, $v_{k_0} > 0$ and if the condition (6.36) is satisfied, then for all $k \in \mathcal{K} := \{k \in \mathbb{N} : k \geq k_0\}$, $w_{k+1} = \widehat{w}_k \in B(w^*, r)$, $v_k > 0$ and the condition (6.36) is satisfied at $k+1$. In addition, the sequence $\{w_k\}$ converges at least superlinearly with a rate of $1+t$ to a solution $\widetilde{w} \in \mathcal{S}$.*

Proof. Let $r > 0$ be defined by (6.83) and let us define

$$\bar{r} = \min\left\{\frac{r}{1 + 2C_1}, \frac{1}{(2C_3)^{1/t}}\right\}.$$

We note that at each iteration $k \in \mathbb{N}$, either the *fraction to the boundary rule* (6.35) or the inner algorithm is applied to create an iterate w_{k+1} such that $v_{k+1} > 0$, therefore the sequence $\{v_k\}$ is positive. We use an inductive argument to prove the first part. For the base case $k = k_0$, Lemma 6.17, (6.37) and (6.36) imply that $w_{k_0+1} = \widehat{w}_{k_0}$ and the condition (6.36) holds at $k_0 + 1$. By reminding that $\widehat{w}_{k_0} - w_{k_0}^+ = (1 - \alpha_{k_0})(w_{k_0} - w_{k_0}^+)$ and $d(w_{k_0}) \leq \|w_{k_0} - w^*\| < \bar{r}$, then by applying

Lemma 6.14, one has

$$\begin{aligned}\|w_{k_0+1} - w^*\| &\leq \|\widehat{w}_{k_0} - w_{k_0}^+\| + \|w_{k_0}^+ - w_{k_0}\| + \|w_{k_0} - w^*\| \\ &\leq (1-\alpha_{k_0})\|w_{k_0}^+ - w_{k_0}\| + \|w_{k_0}^+ - w_{k_0}\| + \|w_{k_0} - w^*\| \\ &< 2C_1 d(w_{k_0}) + \bar{r} \\ &< (2C_1 + 1)\bar{r} \\ &\leq r,\end{aligned}$$

which means that $w_{k_0+1} \in B(w^*, r)$.

Suppose now that for an index $k \geq k_0 + 1$, we have $w_{j+1} = \widehat{w}_j \in B(w^*, r)$ and (6.36) satisfies at $j+1$, for all $j \in \{k_0, \ldots, k-1\}$. Let $j \in \{k_0, \ldots, k\}$. By virtue of Lemma 6.17, (6.37) and (6.36), we imply that $w_{j+1} = \widehat{w}_j$ and (6.36) hold at the iteration $j+1$. Since $\alpha_j \in (0, 1]$, Lemma 6.14, (6.78) and the definition of $\widehat{w}_j$ give us

$$\|w_{j+1} - w_j\| = \|\widehat{w}_j - w_j\| = \alpha_j \|w_j^+ - w_j\| \leq C_1 d(w_j)$$

and

$$d(w_{j+1}) = d(\widehat{w}_j) \leq C_3 d(w_j)^{1+t}.$$

By using these facts and the same argument used in the proof of Theorem 6.7, we can deduce that $w_{k+1} \in B(w^*, r)$ and $\{w_k\}$ is a Cauchy sequence. Consequently, this sequence converges to a point $\tilde{w} \in \mathcal{S}$.

Finally, the rate of convergence of the sequence $\{w_k\}$ can be demonstrated similarly as Theorem 5.16. □

Similar to Remark 6.8, instead of assuming that the condition (6.36) is satisfied at an iteration k_0 such that $w_{k_0} \in B(w^*, \bar{r})$ and $v_{k_0} > 0$, we can prove the superlinear convergence of the sequence $\{w_k\}$ to $\tilde{w} \in \mathcal{S}$ under the assumption that this sequence converges to $\tilde{w}$.

6.5 Numerical illustration

In this section, we summarize behaviors of Algorithms 7 and 8 on two sets of degenerate problems which are created from 108 equality constrained problems of CUTEr [91] and COPS [57] collections. In particular, the first set called the *equality set* includes problems in which each problem is added a constraint of type $c_1(x)^2 = c_1(x)$, where $c_1(x)$ is the first constraint of this one. Each problem of the second set called the *inequality set* is generated by splitting the first equality constraint $c_1(x) = 0$ to two inequality constraints $c_1(x) \le 0$ and $c_1(x) \ge 0$. We note that the MFCQ fails at any point of each problem in the two above sets.

We adopt the rule to update the penalty parameters of SPDOPT-AL for Algorithm 7. More specifically, we set $\sigma_{k+1} = \rho \min\{\sigma_k, \|F(w_k)\|\}$, where $\rho = 0.1$ at Step 2 and $\rho = 0.2$ at Step 3 of Algorithm 7. This formula implies that $\sigma_{k+1} = \mathrm{O}(\|F(w_k)\|)$. If at some k, $w_k \in B\left(w^*, \min\left\{\bar{r}, \frac{\rho}{b^2 C_3}\right\}\right)$ and $\sigma_{k+1} = \rho\|F(w_k)\|$, then by virtue of Theorem 6.7, (6.17) and (6.30), we deduce that

$$\|F(w_{k+1})\| \le bd(w_{k+1}) \le bC_3 d(w_k)^2 \le b^2 C_3 \|w_k - w^*\| \|F(w_k)\| \le \rho\|F(w_k)\| = \sigma_{k+1},$$

meaning that $\sigma_{k+2} = \rho\|F(w_{k+1})\|$. In summary, the condition (6.28) eventually holds, for k large enough. This rule is also use to update the sequence $\{\sigma_k\}$ in Algorithm 8. At each iteration k, the regularization parameter δ_k and the barrier parameter μ_k of Algorithm 8 are updated by formulas

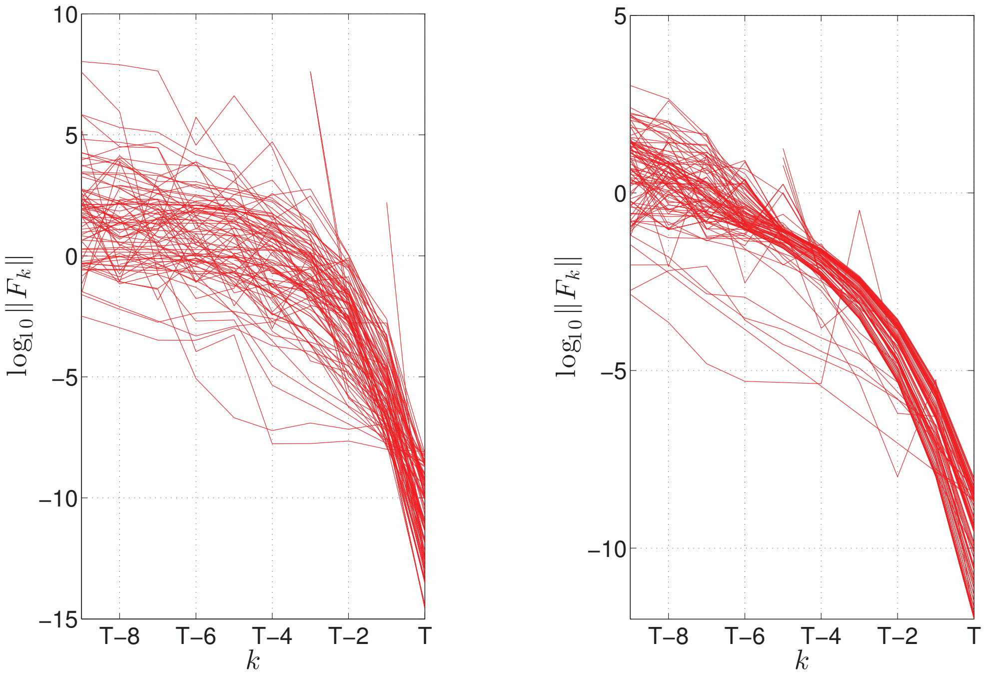
$$\delta_k = \min\{\gamma_1 \|F(w_k)\|^t, \bar{\delta}\} \quad \text{and} \quad \mu_{k+1} = \gamma_2 \min\{\|F(w_k)\|^{1+t}, \mu_k\},$$

where $\gamma_1 = \bar{\delta} = 0.01$, $\gamma_2 = 0.99$, $\mu_0 = 0.1$ and $t = 0.5$. We can see that these choices of parameters satisfy (6.86) and (6.87) for all k such that $\|F(w_k)\| \le 1$. The sequence $\{\tau_k\}$ is set by $\tau_k = \max\{0.99, 1 - \mu_{k+1}\}$ for all k. We note that for all k, $1 - \tau_k \le \mu_{k+1} \le \gamma_2 \|F(w_k)\|^{1+t}$, which verifies the fulfillment of the condition $1 - \tau_k = \mathrm{O}(\|F(w_k)\|)$.

At the first step of Algorithms 7 and 8, we chose $\kappa = 0.9$, $l = 2$, $\zeta_k = (10/\kappa)\sigma_k$. The sequence $\{\varepsilon_k\}$ in Algorithm 7 and Algorithm 8 are respectively chosen by the formulas $\varepsilon_k = 0.9 \max\{\|\Phi(w_i, \lambda_i, \sigma_i)\|\colon (k-l)^+ \le i \le k\} + 10\sigma_{k+1}$ and $\varepsilon_k = 0.9 \max\{\|\Phi(w_i, \lambda_i, \sigma_i, \mu_i)\|\colon (k-l)^+ \le i \le k\} + 10\sigma_{k+1}$. The assumptions of the sequences $\{\zeta_k\}$ and $\{\varepsilon_k\}$ in Theorems 6.7 and 6.18 are satisfied with the above choices.

The information about the initialization and the factorization can be found in [12, 13]. Both algorithms are terminated if $\|F(w_k)\| \le 10^{-8}$. To solve each problem, the number of (outer and inner) iterations is limited by 3000.

For the *equality set*, we eliminated problems `bt08`, `lukvle09`, `lukvle10`, `orthrdm2`, and `s335` since Algorithm 7 cannot find an optimal solution of problem (EP). For the same reason, problems `bt08`, `chain2`, and `lukvle09` of the *inequality set* are not taken into account in our illustration. Figure 6.1 shows us the logarithms of $\|F(w_k)\|$ for the last ten iterations of Algorithms 7 and 8 when solving optimization problems in two aforementioned sets. Through this figure,



(a) The set of equality problems (b) The set of inequality problems

Fig. 6.1: Values of $\log_{10}\|F_k\|$ for the last ten iterations of Algorithms 7 and 8 on the sets of degenerate problems. T represents the index of the stopping iteration for each run.

we can see that the two algorithms obtain the superlinear (quadratic) convergence on most of problems.

Chapter 7

Conclusions and perspectives

In Chapters 3 and 4, the algorithm SPDOPT-ID based on modifications of SPDOPT-AL [12] and SPDOPT [13] has been introduced to quickly detect infeasibility. This algorithm has a good global behavior in the sense that under some usual assumptions, this algorithm can end with three kinds of stationarity: KKT conditions, singular stationarity and stationarity of the feasibility measure. This algorithm maintains the good behaviors of SPDOPT-AL and SPDOPT in some neighborhood of a KKT point which are the quadratic convergence in the equality constrained case and the superlinear convergence in the general case. Moreover, we demonstrated that SPDOPT-ID improved the capability of SPDOPT-AL and SPDOPT in quickly detecting infeasibility. But when the feasibility measure becomes smaller than the feasibility tolerance ($\|c_k\| \leq \epsilon$, for some $k \in \mathbb{N}$), the new algorithm exhibits the same behavior as the original one. More precisely, in that case, the quadratic penalty parameter goes to zero and the multipliers associated to the equality constraints become unbounded. An open question would be to find an algorithm with a superlinear rate of convergence in any case, even if the sequence stays infeasible, but becomes nearly feasible (think about the realization of the constraint $e^x \leq 0$). In practice, we can always choose a feasibility tolerance "small enough", but from a conceptual point of view this not entirely satisfactory. Nevertheless, these chapters complete the local convergence analyses of an augmented Lagrangian method and of an interior point method for nonlinear optimization in the difficult case of infeasible problems. Note especially that no assumption on the linear independence of the gradient of active constraints is used in our analyses, contrary to the ones of Byrd *et al.* [40], Burke *et al.* [30] and Dai *et al.* [50]. This comes from the fact that in the infeasible case, the quadratic penalty parameter remains constant and thus provides a natural regularization of the matrix of the linear system to solve at each iteration.

In Chapter 5, we have proposed an algorithm which is based on a combination of the logarithmic barrier method and the proximal point method. We have shown that this method achieves superlinear convergence without the second order sufficient conditions in the case of a bound constrained optimization problem. A milder assumption related to a local error bound condition has been introduced. This local error bound condition can be seen as a natural extension of the one in unconstrained optimization. We note that this work can be extended to the more general case of a *nonlinear complementarity problem*: given a smooth map $G : \mathbb{R}^n \to \mathbb{R}^n$, find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad G(x) \geq 0 \quad \text{and} \quad 0 \leq x \perp G(x) \geq 0.$$

In addition, this local convergence analysis could also be extended to the case of the general nonlinear optimization problem (1.1). However, it will be necessary to state an error bound condition related to the original problem as Assumption 5.3. Some relations between the inverse of the regularized Jacobian matrix and its regularization parameter has been shown. These properties are useful to study the local behaviors of regularized methods.

Chapter 6 is devoted to answer the question raised by Armand and Omheni [12] about the quadratic convergence of SPDOPT-AL in degenerate problems. In addition, we proposed a rule to update the parameters of SPDOPT [13] to get the superlinear convergence in the problem (1.1) without any constraint qualification. Some local error bound conditions are deduced from remaining assumptions (Lipschitz continuity, second order sufficient conditions and strict complementarity). Similar to Chapter 5, the uniform boundedness of the inverse of the regularized Jacobian matrix by its regularization parameter was used to demonstrate the fast convergence of the algorithms in this chapter.

We now introduce two more general results than Lemmas 6.1 and 6.10. These results could be useful for the local convergence analysis of algorithms without the second order sufficient conditions and without the constraint qualification.

Lemma 7.1. *Let $w^* = (x^*, y^*)$ be a vector in $\mathbb{R}^N$, where $N = n + m$ and n, m are natural numbers. Let $H : \mathbb{R}^N \to \mathbb{R}^{n \times n}$ be a bounded function such that for all $w \in \mathbb{R}^N$, $H(w) = H(w)^\top$ and let $A : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be a Lipschitz continuous function. Let $\theta : \mathbb{R}^N \to \mathbb{R}_{++}$ and $\rho : \mathbb{R}^N \to \mathbb{R}_{++}$ be two bounded functions such that if ρ is not bounded away from zero, then $\rho(w)^l = \mathrm{O}(\theta(w))$ and $\rho(w) = \Omega(\|x - x^*\|^t)$, for all $w = (x, y) \in \mathbb{R}^N$, for some $l \in (0, 1)$ and $t \in (0, 1]$. For all $w \in \mathbb{R}^N$, let us* *define the matrix*

$$J(w) := \begin{pmatrix} H(w) + \theta(w)I & A(x) \\ A(x)^\top & -\rho(w)I \end{pmatrix}.$$

Assume that there exists $r > 0$ *such that for all* $w \in B(w^*, r)$*,*

$$H(w) + \frac{1}{\rho(w)} A(x)A(x)^\top \succeq 0. \tag{7.1}$$

Then, there exists $C > 0$ *such that for all* $w \in B(w^*, r)$*, the matrix* $J(w)$ *is nonsingular and* $\|J(w)^{-1}\| \leq \dfrac{C}{\min\{\theta(w), \rho(w)\}}$*.*

Proof. Let $w \in B(w^*, r)$ and let us define $\theta := \theta(w)$, $\rho := \rho(w)$. Let us show that the matrix $J(w)$ is nonsingular. By Proposition 2.7, we have $\det J(w) = \det\left(H(w) + \theta I + \frac{1}{\rho}A(x)A(x)^\top\right)\det(-\rho I)$. The assumption (7.1), $\theta > 0$ and $\rho > 0$ imply that $\det\left(H(w) + \theta I + \frac{1}{\rho}A(x)A(x)^\top\right) > 0$ and $\det(-\rho I) \neq 0$. It follows that the matrix $J(w)$ is nonsingular.

To prove the second assertion, let us consider the two following cases. The first case is when the function ρ is bounded away from zero. The conclusion will follow if the inverse of $J(w)$ is uniformly bounded by $\frac{1}{\theta(w)}$. For all $w \in B(w^*, r)$, let us define $M(w) = H(w) + \dfrac{1}{\rho(w)}A(x)A(x)^\top + \theta(w)I$. We deduce from (7.1) that for all $w \in B(w^*, r)$,

$$\|M(w)^{-1}\| \leq \frac{1}{\theta(w)}.$$

By noting that

$$J(w)^{-1} = \begin{pmatrix} M(w)^{-1} & \dfrac{1}{\rho(w)}M(w)^{-1}A(x) \\ \dfrac{1}{\rho(w)}A(x)^\top M(w)^{-1} & \dfrac{1}{\rho(w)^2}A(x)^\top M(w)^{-1}A(x) - \dfrac{1}{\rho(w)}I \end{pmatrix},$$

the boundedness of $\left(\|A(x)\|, \dfrac{1}{\rho(w)}, \theta(w)\right)$ and the above inequality imply that $\|J(w)^{-1}\| = \mathrm{O}\left(\dfrac{1}{\theta(w)}\right)$.

We now consider the second case in which there exists a sequence $\{w_k\}$ in $B(w^*, r)$ such that $\lim \rho(w_k) = 0$. To simplify the notation, let us denote $J_k := J(w_k)$, $H_k := H(w_k)$, $A_k := A(x_k)$, $\theta_k := \theta(w_k)$ and $\rho_k := \rho(w_k)$. The proof is based on a contradiction reasoning. We assume that $\lim \min\{\theta_k, \rho_k\}\|J_k^{-1}\| = \infty$. This implies that $\lim \rho_k\|J_k^{-1}\| = \infty$. Let us define $r = \operatorname{rank}(A^*) \leq \min\{m, n\}$, where $A^* := A(x^*)$. By Proposition 2.4, the matrix A^* can be expressed under the form

$$A^* = U\Sigma V^\top = \begin{pmatrix} U_\mathcal{I} & U_\mathcal{J} \end{pmatrix} \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_\mathcal{I}^\top \\ V_\mathcal{J}^\top \end{pmatrix}, \tag{7.2}$$

where $\Sigma_r = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$, $\sigma_1 \geq \ldots \geq \sigma_r > 0$ are the singular values of A^*, $U = \begin{pmatrix} U_\mathcal{I} & U_\mathcal{J} \end{pmatrix}$ and $V = \begin{pmatrix} V_\mathcal{I} & V_\mathcal{J} \end{pmatrix}$ are orthogonal matrices, $U_\mathcal{I} \in \mathbb{R}^{n\times r}, U_\mathcal{J} \in \mathbb{R}^{n\times(n-r)}, V_\mathcal{I} \in \mathbb{R}^{m\times r}, V_\mathcal{J} \in \mathbb{R}^{m\times(m-r)}$. For $k \in \mathbb{N}$, let us define

$$G_k := U^\top (A_k - A^*)V.$$

We then have for all k,

$$\begin{aligned} A_k &= A^* + UG_kV^\top = U(\Sigma + G_k)V^\top \\ &= \begin{pmatrix} U_\mathcal{I} & U_\mathcal{J} \end{pmatrix} \begin{pmatrix} \Sigma_r + G_k^{11} & G_k^{12} \\ G_k^{21} & G_k^{22} \end{pmatrix} \begin{pmatrix} V_\mathcal{I}^\top \\ V_\mathcal{J}^\top \end{pmatrix}, \end{aligned} \tag{7.3}$$

where $G_k^{11} \in \mathbb{R}^{r\times r}$, $G_k^{12} \in \mathbb{R}^{r\times(m-r)}$, $G_k^{21} \in \mathbb{R}^{(n-r)\times r}$, $G_k^{22} \in \mathbb{R}^{(n-r)\times(m-r)}$. From the definition of θ and ρ, the convergence to zero of $\{\rho_k\}$ and noting that $l < 1$, we have

$$\begin{aligned} \rho_k = \rho_k^l \rho_k^{1-l} &= \mathrm{O}(\theta_k)\rho_k^{1-l} \\ &= \mathrm{o}(\theta_k). \end{aligned} \tag{7.4}$$

From the Lipschitz continuity of A and $\|x_k - x^*\| = \mathrm{O}(\rho_k^{1/t}) = \mathrm{O}(\rho_k)$, for all $i, j = 1, 2$, we get

$$\|G_k^{ij}\| = \mathrm{O}(\rho_k). \tag{7.5}$$

This fact and (7.4) imply that

$$\|G_k^{ij}\| = \mathrm{o}(\theta_k) \quad \text{for all } i, j = 1, 2. \tag{7.6}$$

Since the l_2 norm is invariant under multiplication with orthogonal matrices, for all k, one has

$$\|J_k^{-1}\| = \|Q^\top J_k^{-1} Q\|,$$

where $Q = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ and $Q^\top Q = Q^\top Q = I$. From the definition of a matrix norm, there exists a sequence of unit vectors $\{v_k\} \subset \mathbb{R}^{2n}$ such that $\|J_k^{-1}\| = \|Q^\top J_k^{-1} Q\| = \|Q^\top J_k^{-1} Q v_k\|$. Define for all $k \in \mathbb{N}$, $u_k := Q^\top J_k^{-1} Q v_k / \|J_k^{-1}\|$. It follows that $\{u_k\}$ is a sequence of unit vectors with $\lim \frac{1}{\rho_k}\|Q^\top J_k Q u_k\| = 0$. Let $k \in \mathbb{N}$. By introducing the notation

$$u_k = \begin{pmatrix} a_k \\ b_k \\ c_k \\ d_k \end{pmatrix} \in \mathbb{R}^{2n} \quad \text{and} \quad \frac{1}{\rho_k} Q^\top J_k Q u_k = \begin{pmatrix} \alpha_k \\ \beta_k \\ \gamma_k \\ \delta_k \end{pmatrix}$$

and using (7.3), we have

$$\begin{aligned} U_{\mathcal{I}}^\top (H_k + \theta_k I) U_{\mathcal{I}} a_k + U_{\mathcal{I}}^\top (H_k + \theta_k I) U_{\mathcal{J}} b_k + (\Sigma_r + G_k^{11}) c_k + G_k^{12} d_k &= \rho_k \alpha_k, \\ U_{\mathcal{J}}^\top (H_k + \theta_k I) U_{\mathcal{I}} a_k + U_{\mathcal{J}}^\top (H_k + \theta_k I) U_{\mathcal{J}} b_k + G_k^{21} c_k + G_k^{22} d_k &= \rho_k \beta_k, \\ (\Sigma_r + (G_k^{11})^\top) a_k + (G_k^{21})^\top b_k - \rho_k c_k &= \rho_k \gamma_k, \\ (G_k^{12})^\top a_k + (G_k^{22})^\top b_k - \rho_k d_k &= \rho_k \delta_k, \end{aligned} \tag{7.7}$$

where the sequence $\{(\alpha_k, \beta_k, \gamma_k, \delta_k)\}$ converges to zero. From the assumption (7.1) and the orthogonality of the matrix U, we deduce that

$$\begin{aligned} b_k^\top U_{\mathcal{J}}^\top \left(H_k + \theta_k I + \frac{1}{\rho_k} A_k A_k^\top \right) U_{\mathcal{J}} b_k &\geq \theta_k \|U_{\mathcal{J}} b_k\|^2 \\ &\geq \theta_k \|b_k\|^2. \end{aligned} \tag{7.8}$$

From (7.3) and $U_{\mathcal{J}}^\top U = \begin{pmatrix} 0 & I \end{pmatrix}$, we get

$$\begin{aligned} U_{\mathcal{J}}^\top A_k &= \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \Sigma_r + G_k^{11} & G_k^{12} \\ G_k^{21} & G_k^{22} \end{pmatrix} V^\top \\ &= \begin{pmatrix} G_k^{21} & G_k^{22} \end{pmatrix} V^\top, \end{aligned}$$

which implies that

$$U_{\mathcal{J}}^\top A_k A_k^\top U_{\mathcal{J}} = G_k^{21} (G_k^{21})^\top + G_k^{22} (G_k^{22})^\top.$$

By substituting this equality to (7.8), we obtain

$$\theta_k \|b_k\|^2 \leq b_k^\top \left(U_{\mathcal{J}}^\top (H_k + \theta_k I) U_{\mathcal{J}} + \frac{1}{\rho_k} \left(G_k^{21} (G_k^{21})^\top + G_k^{22} (G_k^{22})^\top \right) \right) b_k. \tag{7.9}$$

We deduce from the third equation of (7.7) that

$$\frac{1}{\theta_k} \|a_k\| \leq \left\| \Sigma_r^{-1} \right\| \left(\frac{1}{\theta_k} \left(\|(G_k^{11})^\top\| \|a_k\| + \|(G_k^{21})^\top\| \|b_k\| \right) + \frac{\rho_k}{\theta_k} \|c_k\| + \frac{\rho_k}{\theta_k} \|\gamma_k\| \right).$$

From $\|\Sigma_r^{-1}\| = \frac{1}{\sigma_r}$, the boundedness of $\{(a_k, b_k, c_k, \gamma_k)\}$, (7.6) and (7.4) , we then get

$$\lim \frac{\|a_k\|}{\delta_k} = 0. \tag{7.10}$$

Let us show that

$$\mathcal{J} \neq \emptyset \quad \text{and} \quad \liminf b_k > 0. \tag{7.11}$$

Indeed, if this is not the case, we consider two following cases. The first case is when $\mathcal{J} = \emptyset$. From the first equation of (7.7), the boundedness of $\{(\|H_k\|, \theta_k, \|c_k\|, \|\alpha_k\|)\}$ and the fact that $\|U_{\mathcal{I}}^\top\| = \|U_{\mathcal{I}}\| \leq 1$, we get

$$\begin{aligned}
\|c_k\| &\leq \left\|\Sigma_r^{-1}\right\| (\|U_{\mathcal{I}}^\top\| \|H_k + \theta_k I\| \|U_{\mathcal{I}}\| \|a_k\| + \|G_k^{11}\| \|c_k\| + \rho_k \|\alpha_k\|) \\
&= \frac{1}{\theta_k} \Big(\mathrm{O}(\|a_k\|) + \mathrm{O}(\|G_k^{11}\|) + \mathrm{O}(\rho_k)\Big) .
\end{aligned}$$

From (7.10), (7.6) and (7.4), taking the limit on the both sides, we get

$$\lim c_k = 0,$$

which is in contradiction with the fact that $\{(a_k, c_k)\}$ is a sequence of unit vectors. Let us consider the second case where $\mathcal{J} \neq \emptyset$ and there exists an infinite subset $\mathcal{K} \subset \mathbb{N}$ such that $\lim_{k \in \mathcal{K}} b_k = 0$. By using the boundedness of $\{(\|H_k\|, \theta_k, c_k, d_k, \alpha_k)\}$, (7.5) and the fact that $\|U_{\mathcal{I}}^\top\| = \|U_{\mathcal{I}}\| \leq 1$, $\|U_{\mathcal{J}}\| \leq 1$ and $\|\Sigma_r^{-1}\| = \frac{1}{\sigma_r}$, the first and the fourth equations of (7.7) give us

$$\begin{aligned}
\|c_k\| &\leq \left\|\Sigma_r^{-1}\right\| \Big(\|U_{\mathcal{I}}^\top\| \|H_k + \theta_k I\| (\|U_{\mathcal{I}}\| \|a_k\| + \|U_{\mathcal{J}}\| \|b_k\|) \\
&\qquad + \|G_k^{11}\| \|c_k\| + \|G_k^{12}\| \|d_k\| + \rho_k \|\alpha_k\|\Big) \\
&= \mathrm{O}\left(\frac{1}{\theta_k} \|a_k\|\right) + \mathrm{O}(\|b_k\|) + \mathrm{O}(\rho_k), \\
\|d_k\| &\leq \frac{1}{\rho_k} \Big(\|(G_k^{12})^\top\| \|a_k\| + \|(G_k^{22})^\top\| \|b_k\|\Big) + \|\delta_k\| \\
&= \mathrm{O}\left(\frac{1}{\theta_k} \|a_k\|\right) + \mathrm{O}(\|b_k\|) + \mathrm{O}(\|\delta_k\|).
\end{aligned}$$

By using(7.10), the convergence to zero of $\{b_k\}_{\mathcal{K}}$ and $\{(\rho_k, \delta_k)\}$, we deduce that

$$\lim_{k \in \mathcal{K}} c_k = 0 \quad \text{and} \quad \lim_{k \in \mathcal{K}} d_k = 0,$$

which is in contradiction with the fact that $\{(a_k, b_k, c_k, d_k)\}$ is a sequence of unit vectors.

Premultiplying the second equation of (7.7) by $b_k^\top$, using (7.9) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\theta_k\|b_k\|^2 &\leq b_k^\top\left(U_{\mathcal{J}}^\top(H_k+\theta_k I)U_{\mathcal{J}}+\frac{1}{\rho_k}(G_k^{21}(G_k^{21})^\top+G_k^{22}(G_k^{22})^\top)\right)b_k\\ &= \|b_k\|\Big(\|U_{\mathcal{J}}^\top\|\|H_k+\theta_k I\|\|U_{\mathcal{I}}\|\|a_k\|+\|G_k^{21}\|\|c_k\|+\|G_k^{22}\|\|d_k\|+\rho_k\|\beta_k\|\Big)\\ &\quad+\frac{1}{\rho_k}\left(\|G_k^{21}\|^2+\|G_k^{22}\|^2\right)\|b_k\|^2,\end{aligned}$$

By reminding that $\|U_{\mathcal{J}}^\top\|\leq 1$, $\|U_{\mathcal{I}}\|\leq 1$ and the sequences $\{(b_k,c_k,d_k,\beta_k)\}$ and $\{H_k+\theta_k I\}$ are bounded, the above inequality and (7.5) imply that

$$\|b_k\| = \mathrm{O}\left(\frac{1}{\theta_k}\|a_k\|\right)+\mathrm{O}\left(\frac{\rho_k}{\theta_k}\right).$$

Taking the limit for $k\in\mathbb{N}$ in the above inequality, using (7.10) and (7.4), we obtain

$$\lim b_k = 0,$$

which is in contradiction with (7.11).

Hence, there exists $C>0$ such that for all $w\in B(w^*,r)$,

$$\|J(w)^{-1}\|\leq\frac{C}{\min\{\theta(w),\rho(w)\}}.$$

□

Lemma 7.2. *Let $w^*=(x^*,y^*,z^*)$ be a vector in $\mathbb{R}^N$ such that*

$$0\leq x^*\perp z^*\geq 0 \quad \textit{and} \quad a:=\min\{x_i^*+z_i^*|i=1,\dots n\}>0,$$

where $N=n+m+n$ and n,m are natural numbers. Let $A:\mathbb{R}^N\to\mathbb{R}^{n\times m}$ be a Lipschitz continuous function and $H:\mathbb{R}^N\to\mathbb{R}^{n\times n}$ be a bounded function such that for all $w\in\mathbb{R}^N$, $H(w)=H(w)^\top$. Let $\theta:\mathbb{R}^N\to\mathbb{R}_{++}$ and $\delta:\mathbb{R}^N\to\mathbb{R}_{++}$ be two bounded functions satisfying the following properties.

(i) If δ is bounded away from zero, then $\theta(w)=\Omega(\|x\circ z\|^t)$, for some $t\in(0,1)$.

(ii) Otherwise, if δ is not bounded away from zero, then for all $w=(x,y,z)\in\mathbb{R}^N$, $\delta(w)^l=\mathrm{O}(\theta(w))$ and $\delta(w)=\Omega\left(\|x-x^*\|^t\right)$ for some $l,t\in(0,1)$.

For all $w=(x,y,z)\in\mathbb{R}^N$, let us define the matrices $X=\mathrm{diag}(x)$, $Z=\mathrm{diag}(z)$

and

$$J(w) := \begin{pmatrix} H(w)+\theta(w)I & A(x) & -I \\ A(x)^\top & -\delta(w)I & 0 \\ Z & 0 & X \end{pmatrix}.$$

Assume that there exists $r \in (0, a)$ such that for all $w = (x, y, z) \in B(w^*, r)$ satisfying $v = (x, z) > 0$,

$$H(w) + X^{-1}Z + \frac{1}{\delta(w)} A(x)A(x)^\top \succeq 0. \tag{7.12}$$

Then, there exists $C > 0$ such that for all $w \in B(w^*, r)$ satisfying $v > 0$, the matrix $J(w)$ is nonsingular and $\|J(w)^{-1}\| \le \dfrac{C}{\min\{\theta(w), \delta(w)\}}$.

Proof. Let $w \in B(w^*, r)$ such that $v > 0$ and let us denote $\theta := \theta(w)$ and $\delta := \delta(w)$. Let us show that the matrix $J(w)$ is nonsingular. By Proposition 2.7, we have

$$\begin{aligned} \det J(w) &= \det X \det \begin{pmatrix} H(w)+X^{-1}Z & A(x) \\ A(x)^\top & -\delta I \end{pmatrix} \\ &= \det X \det(-\delta I) \det\left(H(w) + \theta I + X^{-1}Z + \frac{1}{\delta}A(x)A(x)^\top\right). \end{aligned}$$

From $x > 0$, $\delta > 0$, $\theta > 0$ and the assumption (7.12), we have $\det X > 0$, $\det(-\delta I) \neq 0$ and $\det\left(H(w) + \theta I + X^{-1}Z + \frac{1}{\delta}A(x)A(x)^\top\right) > 0$. It follows that the matrix $J(w)$ is nonsingular.

We now prove the uniform boundedness of the matrix $\min\{\theta(w), \delta(w)\}J(w)^{-1}$, for all $w \in B(w^*, r)$ such that $v > 0$, by a contradiction reasoning. Suppose that there exists a sequence $\{w_k\} \subset B(w^*, r)$ such that $v_k = (x_k, z_k) > 0$,

$$H_k + X_k^{-1}Z_k + \frac{1}{\delta_k}A_kA_k^\top \succeq 0, \tag{7.13}$$

but the sequence $\{\min\{\theta_k, \delta_k\}\|J_k^{-1}\|\}$ tends to infinity, where we use the notation $H_k := H(w_k)$, $A_k := A(x_k)$, $\theta_k := \theta(w_k)$, $\delta_k := \delta(w_k)$ and $J_k := J(w_k)$. The boundedness of the functions H, A, θ and δ imply that the sequences $\{H_k\}$, $\{A_k\}$, $\{\theta_k\}$ and $\{\delta_k\}$ are bounded. Let us define the set $\mathcal{J} = \{i \in \{1, \ldots, n\} : x_i^* = 0\}$ and $\mathcal{I} = \{1, \ldots, n\} \backslash \mathcal{J}$. The definition of a implies that $x_i^* \ge a$ for all $i \in \mathcal{I}$ and $z_i^* \ge a$, for all $i \in \mathcal{J}$. Let $\nu = a - r > 0$. For all $k \in \mathbb{N}$, we then have

$$[x_k]_i \ge x_i^* - \|w_k - w^*\| > \nu, \quad \text{for all } i \in \mathcal{I}$$

and

$$[z_k]_i \geq z_i^* - \|w_k - w^*\| > \nu, \quad \text{for all } i \in \mathcal{J},$$

which imply that $0 < \nu < \max\{x_k, z_k\}$ for all $k \in \mathbb{N}$. We consider two following cases. The first case is when the sequence $\{\delta_k\}$ is bounded away from zero. The assumption (i) implies that $\theta_k = \Omega(\|x_k \circ z_k\|^t)$, for some $t \in (0,1)$. For each $k \in \mathbb{N}$, let us define the matrix

$$\widetilde{J}_k := \begin{pmatrix} H_k + \theta_k I & -I & A_k \\ Z_k & X_k & 0 \\ A_k^\top & 0 & -\delta_k I \end{pmatrix}.$$

By applying Corollary 5.6 for the sequences $\{\widetilde{J}_k\}$, there exists $C > 0$ such that for all $k \in \mathbb{N}$, $\|\widetilde{J}_k^{-1}\| \leq \frac{C}{\theta_k}$. We note that for all $k \in \mathbb{N}$, $\|J_k^{-1}\| = \|\widetilde{J}_k^{-1}\| \leq \frac{C}{\theta_k}$ and $\frac{1}{\theta_k} \min\{\theta_k, \delta_k\} \leq 1$. Therefore, the sequence $\{\min\{\theta_k, \delta_k\}\|J_k^{-1}\|\}$ is bounded which is in contradiction with the assumption that this sequence tends to infinity. Hence, this case cannot happen.

Let us consider the second case in which there exists an infinite subset $\mathcal{K} \subset \mathbb{N}$ such that $\lim\limits_{k \in \mathcal{K}} \delta_k = 0$. The assumption (ii) implies that for all $k \in \mathbb{N}$, $\delta_k^l = \mathrm{O}(\theta_k)$ and $\delta_k = \Omega(\|x_k - x^*\|^t)$, for some $l, t \in (0,1)$. In addition, one has

$$\delta_k = \mathrm{O}(\min\{\theta_k, \delta_k\}). \tag{7.14}$$

By reordering the indices, we rewrite $x_k = (x_k^{\mathcal{I}}, x_k^{\mathcal{J}}) \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$, $z_k = (z_k^{\mathcal{I}}, z_k^{\mathcal{J}}) \in \mathbb{R}^{|\mathcal{I}| \times |\mathcal{J}|}$, the matrices H_k and A_k under the form

$$H_k = \begin{pmatrix} H_k^{\mathcal{I}\mathcal{I}} & H_k^{\mathcal{I}\mathcal{J}} \\ H_k^{\mathcal{J}\mathcal{I}} & H_k^{\mathcal{J}\mathcal{J}} \end{pmatrix} \quad \text{and} \quad A_k = \begin{pmatrix} A_k^{\mathcal{I}} \\ A_k^{\mathcal{J}} \end{pmatrix}.$$

From the definition of the sets $\mathcal{I}$ and $\mathcal{J}$, for all $k \in \mathbb{N}$, we have

$$x_k^{\mathcal{I}} \geq \nu, \tag{7.15}$$

and

$$z_k^{\mathcal{J}} \geq \nu. \tag{7.16}$$

The definition of a matrix norm implies that there exists a sequence of unit vectors $\{v_k\} \subset \mathbb{R}^n$ such that $\|J_k^{-1}\| = \|J_k^{-1} v_k\|$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, let us define $u_k := J_k^{-1} v_k / \|J_k^{-1}\|$. It follows that $\{u_k\}$ is a sequence of unit vectors and

$\lim \frac{1}{\delta_k}\|J_k u_k\| \leq \lim \frac{1}{\min\{\theta_k,\delta_k\}}\|J_k u_k\| = 0$. Let $k \in \mathcal{K}$. Introduce the notation

$$u_k = \begin{pmatrix} a_k^{\mathcal{I}} \\ a_k^{\mathcal{J}} \\ b_k \\ c_k^{\mathcal{I}} \\ c_k^{\mathcal{J}} \end{pmatrix} \quad \text{and} \quad \frac{1}{\delta_k} J_k u_k = \begin{pmatrix} \alpha_k^{\mathcal{I}} \\ \alpha_k^{\mathcal{J}} \\ \beta_k \\ \gamma_k^{\mathcal{I}} \\ \gamma_k^{\mathcal{J}} \end{pmatrix},$$

we then have

$$\begin{aligned}
&(H_k^{\mathcal{I}\mathcal{I}} + \theta_k I)a_k^{\mathcal{I}} + H_k^{\mathcal{I}\mathcal{J}} a_k^{\mathcal{J}} + A_k^{\mathcal{I}} b_k - c_k^{\mathcal{I}} = \delta_k \alpha_k^{\mathcal{I}}, \\
&H_k^{\mathcal{J}\mathcal{I}} a_k^{\mathcal{I}} + (H_k^{\mathcal{J}\mathcal{J}} + \theta_k I)a_k^{\mathcal{J}} + A_k^{\mathcal{J}} b_k - c_k^{\mathcal{J}} = \delta_k \alpha_k^{\mathcal{J}}, \\
&\left(A_k^{\mathcal{I}}\right)^\top a_k^{\mathcal{I}} + \left(A_k^{\mathcal{J}}\right)^\top a_k^{\mathcal{J}} - \delta_k b_k = \delta_k \beta_k, \\
&Z_k^{\mathcal{I}} a_k^{\mathcal{I}} + X_k^{\mathcal{I}} c_k^{\mathcal{I}} = \delta_k \gamma_k^{\mathcal{I}}, \\
&Z_k^{\mathcal{J}} a_k^{\mathcal{J}} + X_k^{\mathcal{J}} c_k^{\mathcal{J}} = \delta_k \gamma_k^{\mathcal{J}},
\end{aligned} \tag{7.17}$$

where the sequence $\left\{\left(\alpha_k^{\mathcal{I}}, \alpha_k^{\mathcal{J}}, \beta_k, \gamma_k^{\mathcal{I}}, \gamma_k^{\mathcal{J}}\right)\right\}$ converges to zero. From the fifth equation of (7.17), (7.16) and the fact that $\|c_k^{\mathcal{J}}\| \leq 1$, we then get

$$\begin{aligned}
\|a_k^{\mathcal{J}}\| &\leq \left\|\left(Z_k^{\mathcal{J}}\right)^{-1}\right\| \left(\|X_k^{\mathcal{J}}\|\|c_k^{\mathcal{J}}\| + \delta_k\|\gamma_k^{\mathcal{J}}\|\right) \\
&\leq \frac{1}{\nu}\delta_k \left(\delta_k^{-1}\|x_k^{\mathcal{J}}\| + \|\gamma_k^{\mathcal{J}}\|\right).
\end{aligned} \tag{7.18}$$

Since $\delta_k = \Omega(\|x_k - x^*\|^t)$ and $(x^*)^{\mathcal{J}} = 0$, one has

$$\delta_k^{-1}\|x_k^{\mathcal{J}}\| = \mathrm{O}(\delta_k^{-1}\|x_k - x^*\|) = \mathrm{O}\left(\delta_k^{1/t-1}\right).$$

Substituting the previous equality to (7.18) and reminding that the sequences $\{\delta_k\}_{\mathcal{K}}$ and $\{\gamma_k^{\mathcal{J}}\}$ converge to zero, $t \in (0,1)$, we deduce that

$$\lim_{k\in\mathcal{K}} \frac{1}{\delta_k}\|a_k^{\mathcal{J}}\| = 0. \tag{7.19}$$

Let us show that

$$\mathcal{I} \neq \emptyset \quad \text{and} \quad \liminf_{k\in\mathcal{K}} \left\|\left(a_k^{\mathcal{I}}, b_k\right)\right\| > 0. \tag{7.20}$$

Indeed, if this is not the case, there are two possibilities. The first one is $\mathcal{I} = \emptyset$. From the boundedness of $\{(\|H_k^{\mathcal{J}\mathcal{J}} + \theta_k I\|, \|A_k^{\mathcal{J}}\|, \delta_k)\}$, the second and the third equations of (7.17) imply that

$$\begin{aligned}
\left\|c_k^{\mathcal{J}}\right\| &\leq \|H_k^{\mathcal{J}\mathcal{J}}+\theta_k I\|\|a_k^{\mathcal{J}}\|+\|A_k^{\mathcal{J}}\|\|b_k\|+\delta_k\left\|\alpha_k^{\mathcal{J}}\right\| \\
&= \mathrm{O}\left(\frac{1}{\delta_k}\|a_k^{\mathcal{J}}\|\right)+\mathrm{O}(\|b_k\|)+\|\mathrm{O}(\|\alpha_k^{\mathcal{J}}\|), \\
\|b_k\| &\leq \frac{1}{\delta_k}\left\|\left(A_k^{\mathcal{J}}\right)^\top\right\|\|a_k^{\mathcal{J}}\|+\|\beta_k\| \\
&= \mathrm{O}\left(\frac{1}{\delta_k}\|a_k^{\mathcal{J}}\|\right)+\|\beta_k\|.
\end{aligned}$$

By virtue of (7.19) and the convergence to zero of $\{(\alpha_k^{\mathcal{J}},\beta_k)\}$, taking the limit for $k\in\mathcal{K}$, we then get

$$\lim_{k\in\mathcal{K}} b_k=0 \quad \text{and} \quad \lim_{k\in\mathcal{K}} c_k=0,$$

which is in contradiction with the fact that $\{(a_k^{\mathcal{J}},b_k,c_k)\}$ is a sequence of unit vectors. We consider the second possibility in which $\mathcal{I}\neq\emptyset$ but there exists an infinite subset $\mathcal{K}'\subset\mathcal{K}$ such that $\lim_{k\in\mathcal{K}'}\left\|\left(a_k^{\mathcal{I}},b_k\right)\right\|=0$. The boundedness of $\{(\|H_k+\theta_k I\|,\|A_k\|,\delta_k)\}$ and the first two equations of (7.17) give us

$$\begin{aligned}
\left\|\left(c_k^{\mathcal{I}},c_k^{\mathcal{J}}\right)\right\| &\leq \|H_k+\theta_k I\|\left\|\left(a_k^{\mathcal{I}},a_k^{\mathcal{J}}\right)\right\|+\|A_k\|\|b_k\|+\delta_k\left\|\left(\alpha_k^{\mathcal{I}},\alpha_k^{\mathcal{J}}\right)\right\| \\
&= \mathrm{O}\left(\frac{1}{\delta_k}\left\|a_k^{\mathcal{J}}\right\|\right)+\mathrm{O}\left(\left\|\left(a_k^{\mathcal{I}},b_k\right)\right\|\right)+\mathrm{O}\left(\left\|\left(\alpha_k^{\mathcal{I}},\alpha_k^{\mathcal{J}}\right)\right\|\right).
\end{aligned}$$

From (7.19) and the convergence to zero of the sequences $\left\{\left(a_k^{\mathcal{I}},b_k\right)\right\}_{\mathcal{K}'}$ and $\left\{\left(\alpha_k^{\mathcal{I}},\alpha_k^{\mathcal{J}}\right)\right\}$, the above inequality implies that

$$\lim_{k\in\mathcal{K}'} c_k^{\mathcal{I}}=\lim_{k\in\mathcal{K}'} c_k^{\mathcal{J}}=0,$$

which is again in contradiction with the fact that $\{u_k\}$ is a sequence of unit vectors.

By eliminating $c_k^{\mathcal{I}}$ in the first equation of (7.17), the first and the third equations of this system can be rewritten under the form

$$\begin{pmatrix} H_k^{\mathcal{I}\mathcal{I}}+\left(X_k^{\mathcal{I}}\right)^{-1}Z_k^{\mathcal{I}}+\theta_k I & A_k^{\mathcal{I}} \\ \left(A_k^{\mathcal{I}}\right)^\top & -\delta_k \end{pmatrix}\begin{pmatrix} a_k^{\mathcal{I}} \\ b_k \end{pmatrix}=\begin{pmatrix} -H_k^{\mathcal{I}\mathcal{J}}a_k^{\mathcal{J}}+\delta_k\left(\alpha_k^{\mathcal{I}}+\left(X_k^{\mathcal{I}}\right)^{-1}\gamma_k^{\mathcal{I}}\right) \\ -\left(A_k^{\mathcal{J}}\right)^\top a_k^{\mathcal{J}}+\delta_k\beta_k \end{pmatrix}. \tag{7.21}$$

Let us define for each $k\in\mathcal{K}$,

$$\hat{J}_k:=\begin{pmatrix} H_k^{\mathcal{I}\mathcal{I}}+\left(X_k^{\mathcal{I}}\right)^{-1}Z_k^{\mathcal{I}}+\theta_k I & A_k^{\mathcal{I}} \\ \left(A_k^{\mathcal{I}}\right)^\top & -\delta_k \end{pmatrix},\ \hat{v}_k:=\begin{pmatrix} -H_k^{\mathcal{I}\mathcal{J}}a_k^{\mathcal{J}}+\delta_k\left(\alpha_k^{\mathcal{I}}+\left(X_k^{\mathcal{I}}\right)^{-1}\gamma_k^{\mathcal{I}}\right) \\ -\left(A_k^{\mathcal{J}}\right)^\top a_k^{\mathcal{J}}+\delta_k\beta_k \end{pmatrix}.$$

On the one hand, by virtue of (7.15), the sequence $\left\{\left(X_k^{\mathcal{I}}\right)^{-1}\right\}$ is bounded. Combining this fact with the boundedness of the sequence $\left\{\left(\|H_k^{\mathcal{I}\mathcal{J}}\|, \|A_k^{\mathcal{J}}\|\right)\right\}$, the convergence to zero of the sequence $\left\{(\alpha_k^{\mathcal{I}}, \beta_k, \gamma_k^{\mathcal{I}})\right\}$ and (7.19), we deduce that

$$\lim_{k\in\mathcal{K}} \frac{\|\hat{v}_k\|}{\delta_k} = 0. \tag{7.22}$$

On the other hand, from (7.15) and the boundedness of $\{z_k\}$, the sequence $\left\{\left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}}\right\}$ is bounded . In addition, (7.13) implies that

$$H_k^{\mathcal{I}\mathcal{I}} + \left(X_k^{\mathcal{I}}\right)^{-1} Z_k^{\mathcal{I}} + \frac{1}{\delta_k} A_k^{\mathcal{I}} \left(A_k^{\mathcal{I}}\right)^{\top} \succeq 0.$$

By virtue of Lemma 7.1 and (7.14), there exists $C > 0$ such that for all $k \in \mathcal{K}$, $\|\widehat{J}_k^{-1}\| \leq \frac{C}{\delta_k}$. We then deduce from this fact and the equation (7.21) that

$$\left\|\left(a_k^{\mathcal{I}}, b_k\right)\right\| = \|\widehat{J}_k^{-1}\hat{v}_k\| \leq \frac{C}{\delta_k}\|\hat{v}_k\|.$$

Taking the limits for $k \in \mathcal{K}$ and using (7.22), we then get

$$\lim_{k\in\mathcal{K}} \left\|\left(a_k^{\mathcal{I}}, b_k\right)\right\| = 0,$$

which is in contradiction with (7.20).

In sum, for all $r \in (0, a)$ such that (7.12) holds, there exists $C > 0$ such that for all $w \in B(w^*, r)$ satisfying $v > 0$, $\min\{\theta(w), \delta(w)\}\|J(w)^{-1}\| \leq C$, which concludes the proof. □

In Lemma 7.2, if the matrix H is positive semidefinite, Friedlander and Orban [72, Corollary 5.2] shows a nearly similar upper bound on the inverse of the matrix J which is

$$\|J(w)^{-1}\| \leq \frac{1}{\min\{\theta(w), \delta(w)\}}.$$

In this thesis, the linear system at each iteration of algorithms is solved exactly. A future work is to consider a factorization-free approach, see, e.g., Arreckx and Orban [17].

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Titre thèse français: Détection de la non-réalisabilité et stratégies de régularisation en optimisation non linéaire

Résumé : Dans cette thèse, nous nous étudions des algorithmes d'optimisation non linéaire. D'une part nous proposons des techniques de détection rapide de la non-réalisabilité d'un problème à résoudre. D'autre part, nous analysons le comportement local des algorithmes pour la résolution de problèmes singuliers. Dans la première partie, nous présentons une modification d'un algorithme de lagrangien augmenté pour l'optimisation avec contraintes d'égalité. La convergence quadratique du nouvel algorithme dans le cas non-réalisable est démontrée théoriquement et numériquement. La seconde partie est dédiée à l'extension du résultat précédent aux problèmes d'optimisation non linéaire généraux avec contraintes d'égalité et d'inégalité. Nous proposons une modification d'un algorithme de pénalisation mixte basé sur un lagrangien augmenté et une barrière logarithmique. Les résultats théoriques de l'analyse de convergence et quelques tests numériques montrent l'avantage du nouvel algorithme dans la détection de la non-réalisabilité. La troisième partie est consacrée à étudier le comportement local d'un algorithme primal-dual de points intérieurs pour l'optimisation sous contraintes de borne. L'analyse locale est effectuée sans l'hypothèse classique des conditions suffisantes d'optimalité de second ordre. Celle-ci est remplacée par une hypothèse plus faible basée sur la notion de borne d'erreur locale. Nous proposons une technique de régularisation de la jacobienne du système d'optimalité à résoudre. Nous démontrons ensuite des propriétés de bornitude de l'inverse de ces matrices régularisées, ce qui nous permet de montrer la convergence superlinéaire de l'algorithme. La dernière partie est consacrée à l'analyse de convergence locale de l'algorithme primal-dual qui est utilisé dans les deux premières parties de la thèse. En pratique, il a été observé que cet algorithme converge rapidement même dans le cas où les contraintes ne vérifient l'hypothèse de qualification de Mangasarian-Fromovitz. Nous démontrons la convergence superlinéaire et quadratique de cet algorithme, sans hypothèse de qualification des contraintes.

Mots clés : optimisation nonlinéaire, detection de la non-réalisabilité, regularisation, dégénéré, méthode lagrangienne augmentée, méthode de point intérieur, méthodes primales-duales, borne d'erreur locale, convergence superlinéaire/quadratique.

Titre thèse anglais: Infeasibility Detection and Regularization Strategies in Nonlinear Optimization

Abstract: This thesis is devoted to the study of numerical algorithms for nonlinear optimization. On the one hand, we propose new strategies for the rapid infeasibility detection. On the other hand, we analyze the local behavior of primal-dual algorithms for the solution of singular problems. In the first part, we present a modification of an augmented Lagrangian algorithm for equality constrained optimization. The quadratic convergence of the new algorithm in the infeasible case is theoretically and numerically demonstrated. The second part is dedicated to extending the previous result to the solution of general nonlinear optimization problems with equality and inequality constraints. We propose a modification of a mixed logarithmic barrier-augmented Lagrangian algorithm. The theoretical convergence results and the numerical experiments show the advantage of the new algorithm for the infeasibility detection. In the third part, we study the local behavior of a primal-dual interior point algorithm for bound constrained optimization. The local analysis is done without the standard assumption of the second-order sufficient optimality conditions. These conditions are replaced by a weaker assumption based on a local error bound condition. We propose a regularization technique of the Jacobian matrix of the optimality system. We then demonstrate some boundedness properties of the inverse of these regularized matrices, which allow us to prove the superlinear convergence of our algorithm. The last part is devoted to the local convergence analysis of the primal-dual algorithm used in the first two parts of this thesis. In practice, it has been observed that this algorithm converges rapidly even in the case where the constraints do not satisfy the Mangasarian-Fromovitz constraint qualification. We demonstrate the superlinear and quadratic convergence of this algorithm without any assumption of constraint qualification.

Keywords: nonlinear optimization, infeasibility detection, regularization, degenerate, augmented Lagrangian method, interior point method, primal-dual methods, local error bound condition, superlinear/quadratic convergence.