Optimal control of evolution equations and its applications

Hawraa Nabolsi

Numéro d'ordre: 18/28

Université Libanaise

Thèse de doctorat

Pour obtenir le grade de Docteur de l'Université de

VALENCIENNES ET DU HAINAUT-CAMBRESIS

et Université LIBANAISE FACULTÉ DES SCIENCES

Mathématiques appliquées

présentée et soutenue par

NABOLSI Hawraa

le 17/07/2018, à Valenciennes

École doctorale: Science Pour l'Ingénieur (SPI)
Équipe de recherche, Laboratoire: Laboratoire de Mathématiques et ses Applications de Valenciennes (LAMAV)

Contrôle Optimal des Equations d'Evolution et ses Applications

Membres du jury:

Directeurs:	Pr. Luc Paquet	Université de Valenciennes et du Hainaut-Cambrésis
	Pr. Ali Wehbe	Université libanaise
Rapporteurs:	Pr. Eduardo CASAS	Université de Cantabria, Santander
	Pr. Jean-Pierre Raymond	Université Paul-Sabatier de Toulouse
Président du jury:	Pr. Serge Nicaise	Université de Valenciennes et du Hainaut-Cambrésis
Examinateurs:	Dr. Julie Valein	Université de Lorraine
	Dr. Amina Mortada	Université libanaise
	Dr. Hussein Fakih	Lebanese International University

Résumé

Dans cette thèse, tout d'abord , nous faisons l'Analyse Mathématique du modèle exact du chauffage radiatif d'un corps semi-transparent Ω par une source radiative noire qui l'entoure. Il s'agit donc d'étudier le couplage d'un système d'Equations de Transfert Radiatif avec condition au bord de réflectivité indépendantes avec une équation de conduction de la chaleur non linéaire avec condition limite non linéaire de type Robin. Nous prouvons l'existence et l'unicité de la solution et nous démontrons des bornes uniformes sur la solution et les intensités radiatives dans chaque bande de longueurs d'ondes pour laquelle le corps est semi-transparent, en fonction de bornes sur les données,

Deuxièmement, nous considérons le problème du contrôle optimal de la température absolue à l'intérieur du corps semi-transparent Ω en agissant sur la température absolue de la source radiative noire qui l'entoure. À cet égard, nous introduisons la fonctionnelle coût appropriée et l'ensemble des contrôles admissibles T_S, pour lesquels nous prouvons l'existence de contrôles optimaux. En introduisant l'espace des états et l'équation d'état, une condition nécessaire de premier ordre pour qu'un contrôle $T_S : t \to T_S(t)$ soit optimal, est alors dérivée sous la forme d'une inéquation variationnelle en utilisant le théorème des fonctions implicites et le problème adjoint.

Ensuite, nous considérons le problème de l'existence et de l'unicité d'une solution faible des équations de la thermoviscoélasticité dans une formulation mixte de type Hellinger-Reissner, la nouveauté par rapport au travail de M.E. Rognes et R. Winther (M3AS, 2010) étant ici l'apparition de la viscosité dans certains coefficients de l'équation constitutive, viscosité qui dépend dans ce contexte de la température absolue $T(x,t)$ et donc en particulier du temps t.

Enfin, nous considérons dans ce cadre le problème du contrôle optimal de la déformation du corps semi-transparent Ω, en agissant sur la température absolue de la source radiative noire qui l'entoure. Nous prouvons l'existence d'un contrôle optimal et nous calculons la dérivée Fréchet de la fonctionnelle coût réduite.

Mots-Clés

Fonction de Planck, équation de transfert radiatif avec condition aux limites de type réflexif, équation parabolique non-linéaire avec terme intégral de degré 0 et condition aux limites non-linéaire de type Robin, fonctionnelle coût, contrôle de la température à l'intérieur du corps par la température de la source radiative, fonctionnelle coût réduite, existence de contrôles optimaux, espace des états, continuité des états, équation d'état, différentiabilité Fréchet, théorème des fonctions implicites, problème adjoint, équation parabolique rétrograde, condition nécessaire d'optimalité du premier ordre, inéquation variationnelle, thermoviscoélasticité, modèle de Maxwell en thermoviscoélaticité, formulation mixte de type Hellinger-Reissner, existence et unicité de la solution, contrôle du champ de déplacements.

Abstract

This thesis begins with a rigorous mathematical analysis of the radiative heating of a semi-transparent body made of glass, by a black radiative source surrounding it. This requires the study of the coupling between quasi-steady radiative transfer boundary value problems with nonhomogeneous reflectivity boundary conditions (one for each wavelength band in the semi-transparent electromagnetic spectrum of the glass) and a nonlinear heat conduction evolution equation with a nonlinear Robin boundary condition which takes into account those wavelengths for which the glass behaves like an opaque body. We prove existence and uniqueness of the solution, and give also uniform bounds on the solution i.e. on the absolute temperature distribution inside the body and on the radiative intensities.

Now, we consider the temperature T_S of the black radiative source S surrounding the semi-transparent body Ω as the control variable. We adjust the absolute temperature distribution $(x,t) \mapsto T(x,t)$ inside the semi-transparent body near a desired temperature distribution $T_d(\cdot,\cdot)$ during the time interval of radiative heating $]0,t_f[$ by acting on T_S. In this respect, we introduce the appropriate cost functional and the set of admissible controls T_S, for which we prove the existence of optimal controls. Introducing the State Space and the State Equation, a first order necessary condition for a control $T_S : t \mapsto T_S(t)$ to be optimal is then derived in the form of a Variational Inequality by using the Implicit Function Theorem and the adjoint problem.

We come now to the goal problem which is the deformation of the semi-transparent body Ω by heating it with a black radiative source surrounding it. We introduce a weak mixed formulation of this thermoviscoelasticity problem and study the existence and uniqueness of its solution, the novelty here with respect to the work of M.E. Rognes et R. Winther (M3AS, 2010) being the apparition of the viscosity in some of the coefficients of the constitutive equation, viscosity which depends on the absolute temperature $T(x,t)$ and thus in particular on the time t.

Finally, we state in this setting the related optimal control problem of the deformation of the semi-transparent body Ω, by acting on the absolute temperature of the black radiative source surrounding it. We prove the existence of an optimal control and we compute the Fréchet derivative of the associated reduced cost functional.

Key words

Planck function, radiative transfer equation with the reflectivity boundary condition, nonlinear parabolic equations with an integral 0-order term, nonlinear boundary condition of the Robin type, cost functional, controlling the temperature inside the body by the temperature of the radiative source, reduced cost functional, existence of optimal controls, state space, continuity of the states, state equation, Fréchet differentiability, Implicit Function Theorem, adjoint Problem, backward parabolic equation, first order necessary optimality condition, variational inequality, Thermoviscoelasticity, Maxwell model in thermoviscoelasticity, mixed formulation of the Hellinger-Reissner type, existence and uniqueness of the solution, control of the field of displacements.

Acknowledgments

First and foremost, praises and thanks to the God, the Almighty, for his showers of blessings throughout my research work to complete the research successfully.
I would like to express my sincere gratitude to my advisors Prof. Luc Paquet and Pr. Ali Wehbe for the continuous support of my Ph.D study and related research, for their patience, motivation, and immense knowledge. Their guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisors and mentor for my Ph.D study.
Besides my advisors, I would also like to show gratitude to the rest of my thesis committee, including M. Eduardo Casas, M. Jean-Pierre Raymond, M. Serge Nicaise, Mme. Julie Valein and Mme. Amina Mortada, M. Hussein Fakih for their insightful comments and encouragement.
I am extremely grateful to the Lebanese University and Campus France for giving me a scholarship to continue my studies in France during my Ph.D.
I wish to express my thanks to Pr. Felix Ali Mehmeti, director of LAMAV and Mme Nabila Daifi the secretary of the Laboratory LAMAV for providing me with all the necessary facilities for the research.
I would like to say thanks to my friends and colleagues, in the laboratory KALMA, Fatima Abbas, Vivian Rizek, Sara Nasser, Rayan Nasser, Alaa Hayek, Mohamad Sammoury, Mohamad Akil, Mohamad Ghader, Sara Makki, Houssein Nasser El Dine and the secretary Wissam Berro. Also in the laboratory LAMAV, Tonia maria Alam, Elsa Ghandour, Thibault Defourneau, Jerome Tomezyk, Florent Dewez and specially my dear friend Fatiha Bekkouche who opened both her home and heart to me when I first arrived to Valenciennes.
I am extending my thanks to my friends, Nassab Yassine, Samar Zayat, Inas Amacha, Lara Saliba, Lynda chehami and Nejib Yacoubi for helping and encouraging me during my thesis.
My special thank goes to the manager of Sarafand High School, M. Haidar Khalife for

the keen interest shown to complete this thesis successfully.
Most importantly, none of this could have happened without my family, to my parents especially my father for their love, prayers, caring and sacrifices for educating and preparing me for my future. Also I express my thanks to my sisters, brothers, and my fiancé for their support and valuable prayers.
Finally, my thanks go to all the people who have supported me to complete the research work directly or indirectly.

Contents

1 Introduction en Français

1.1 Le sujet de ma thèse

Dans l'article "Chauffage radiatif d'une plaque de verre" publié dans la revue électronique "Mathematics in Action" (MSIA) en 2012 [50], le Pr. Luc Paquet, M. Raouf El Cheikh, le Pr. Dominique Lochegnies et le Dr. Norbert Siedow, ont considéré une plaque de verre horizontale infinie d'épaisseur l surmontée d'une tôle noire plane infinie S, à la température (absolue) $T_S(t)$ émettant des radiations thermiques dans sa direction et ont démontré l'existence et l'unicité de la solution du problème couplé dans la plaque de verre entre une équation de conduction de la chaleur non linéaire soumise à une condition aux limites de type Robin non-linéaire, et un système de problèmes aux limites indépendants pour l'équation de transfert radiatif résolus explicitement dans ce cas.

A sa suite, l'article "Contrôle du Chauffage Radiatif d'une Plaque de Verre" publié dans la revue Afrika Matematika (AFMA) en Juin 2016 [49], le Pr. Luc Paquet y a étudié le contrôle de la température absolue $T(x, t)$ suivant l'élévation x parcourant l'intervalle $[0, l]$ et l'instant t parcourant l'intervalle de temps fixé $]0, t_f[$ du chauffage radiatif de la plaque de verre, en agissant sur la température absolue $T_S(t)$ (notée $u(t)$ dans cet article) de la source radiative noire S. L'espace des états et l'application régissant l'équation d'état y ont été soigneusement définis pour que cette application $(T, T_S) \mapsto e(T, T_S)$ soit Fréchet différentiable et que sa dérivée par rapport à T au point (T, T_S) soit inversible. Ceci lui avait permis d'appliquer le théorème des fonctions implicites pour calculer la dérivée de la fonctionnelle coût réduite $\hat{J}(T_S)$ en un contrôle $T_S \in H^1(]0, t_f[)$, et de trouver la condition nécessaire du premier ordre pour qu'un contrôle soit optimal sous la forme d'une inéquation variationnelle.

Nous avons étendu le résultat d'existence et d'unicité de la solution de l'article MSIA (2012) [50], et la condition nécessaire du premier ordre pour qu'un contrôle soit optimal de l'article AFMA (juin 2016) [49], à un nouveau cadre, considérant cette fois pour corps en verre, un domaine borné Ω de $\mathbb{R}^3$ avec une frontière de classe C^1 (resp. $C^{1,1}$), entouré d'une source radiative noire S à la température absolue $T_S(t)$ à l'instant t.

Le but du chauffage radiatif du corps en verre Ω par la source radiative S est de déformer ce dernier le plus près possible d'une forme requise. Le problème du contrôle optimal de la déformation en agissant sur la température absolue $T_S(t)$ de la source de chaleur radiative S sera étudié dans le cadre des équations linéaires de la thermoviscoélasticité en ce qui concerne la déformation, la dépendance en la température absolue $T(x,t)$ du corps Ω dans l'équation constitutive de Maxwell via la viscosité, elle étant non-linéaire.

1.2 Présentation et discussion des résultats obtenus dans ma thèse

1.2.1 Chapitre 3: Couplage de l'équation de conduction thermique avec l'équation de transfert radiatif

Dans ce chapitre, nous analysons le problème mathématique exact, en modélisant le chauffage radiatif du corps semi-transparent Ω fait de verre, du temps initial 0 jusqu'au temps final de chauffage t_f, par une source radiative S: une surface noire S qui l'entoure, à la température absolue $T_S(t)$ $(0 \le t \le t_f)$. L'application principale de ce problème est, en chauffant le corps semi-transparent Ω, de lui permettre de se déformer en raison de son propre poids et de contrôler sa déformation (ou son champ de contraintes internes): voir par ex. les chapitres 4 et 5 de cette thèse, et nombre de papiers [3], [53], ..., pour n'en citer que quelques-uns. Nous devons donc analyser le couplage entre les problèmes aux limites de transfert radiatif:

$$\left\{\begin{array}{c} v\cdot\nabla_x I^k(x,t,v)+\kappa_k I^k(x,t,v)=\kappa_k B_g^k(T(x,t)),\ \textit{in}\ \Omega\times V,\\ I^k(x,t,v)=\rho_g(|\nu_x\cdot v|)I^k(x,t,v_i(x,v))+(1-\rho_g(|\nu_x\cdot v|))B_g^k(T_S(t)),\\ \textit{on}\ \Gamma_-,\end{array}\right. \tag{1.1}$$

où $\Gamma_- := \{(x,v)\in\partial\Omega\times V;\ \nu_x\cdot v<0\}$, et $v_i(x,v) := v-2(\nu_x\cdot v)\nu_x$, un problème pour chaque bande de longueur d'onde $[\lambda_k,\lambda_{k+1}[$ $(k=1,\dots,M)$ dans laquelle le verre se comporte comme un milieu semi-transparent, et le problème de conduction thermique nonstationnaire et non linéaire [50], [63]:

$$\begin{cases} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \Delta T(x,t) - \sum\limits_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum\limits_{k=1}^{M} \kappa_k \int\limits_V I^k(x,t,v)d\mu(v), \\ \quad in\ Q := \Omega\times]0,t_f[, \\ -k_h \frac{\partial T}{\partial \nu}(x,t) = h_c(T(x,t) - T_a) + \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda [B(T(x,t),\lambda) - B(T_S(t),\lambda)]d\lambda, \\ \quad on\ \Sigma := \partial\Omega\times]0,t_f[, \\ T(x,0) = T_0(x),\ \ on\ \Omega. \end{cases} \tag{1.2}$$

$T(x,t)$ désigne la température absolue au point x et à l'instant t. Dans les $(1.2)_{(i)}$ de conduction de la chaleur:

$$I^k(x,t,v) := \int\limits_{\lambda_k}^{\lambda_{k+1}} I(x,t,v,\lambda)d\lambda \tag{1.3}$$

radiative spectrale au point $x \in \Omega$, au temps $t \in [0,t_f]$, dans la direction (orientée) $v \in V := S_2$ (la sphère unité dans $\mathbb{R}^3$), et de longueur d'onde λ [46]. De même $B_g^k(T(x,t)) := n_g^2 \int\limits_{\lambda_k}^{\lambda_{k+1}} B(T(x,t),\lambda)d\lambda$, où $B(T,\lambda) := \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)}$,

est la fameuse fonction de Planck (voir [46], [68]), donnant l'intensité spectrale radiative à la longueur d'onde λ de tout corps noir à la température absolue T, intensité radiative qui dans ce cas est indépendante de la direction v. C_1 et C_2 sont des constantes positives dont les valeurs sont par exemple rappelées dans [50] et n_g l'indice de réfraction du matériau semi-transparent (pour le verre silicate sodo-calcique: $n_g \approx 1.46$ [60]). Pour des raisons relevant des mathématiques, dans [50], les auteurs ont étendu le domaine de définition de la fonction de Planck à tout nombre réel T en posant $B(T,\lambda) := 0$, $\forall T \leq 0$. Dans la condition aux limites $(1.2)_{(ii)}$, ν désigne le champ normal unitaire sortant le long de la frontière de Ω, et ν_x sa valeur au point $x \in \partial\Omega$. ρ_g dans la condition aux limites $(1.1)_{(ii)}$ désigne le coefficient de réflectivité: il s'agit d'une fonction continue, décroissante (au sens large), strictement positive, de l'intervalle $[0,1]$ dans lui-même égale à 1 près de 0. Pour plus de détails sur cette fonction, voir le chapitre 2 de cette thèse. Notons dans la condition aux limites $(1.1)_{(ii)}$, que l'expression $\rho_g(|\nu_x \cdot v|)$ dépend non seulement de x mais aussi de v, ce qui n'est pas le cas dans ([19], voir (33) p.258). En outre, notre problème ne rentre pas dans les exemples considérés dans [62, p.4].

Mentionnons, que pour que nos conditions aux limites non homogènes dans (1.1) et (1.2) soient physiquement réalistes, nous devrions supposer que notre domaine Ω est convexe. Cependant, nous étudierons le problème couplé (1.1) - (1.2) dans un domaine arbitraire Ω de $\mathbb{R}^3$ de classe C^1, nos conditions aux limites dans (1.1) et (1.2) ayant un sens avec cette hypothèse de régularité sur Ω.

Des détails supplémentaires concernant le problème (1.1) (resp. (1.2)), seront donné dans la sous-section 3.2.1 (resp. sous-section 3.3.1) de cette thèse, en particulier concernant la nomenclature.

De nombreux problèmes similaires ont été étudiés dans la littérature, principalement pour réduire la complexité de calcul du problème original (1.1) - (1.2), en considérant à la place des problèmes aux limites (1.1) pour les intensités radiatives $I^k(x,t,v)$, des problèmes aux limites elliptiques approchés pour les quantités d'intérêt dans (1.2), à savoir les rayonnements incidents [46] $\rho^k(.,.) := \int_V I^k(.,.,v)d\mu(v)$ ($d\mu$ indiquant dans cette formule la mesure de surface sur $V \equiv S_2$). En effet, ces quantités multipliées par κ_k, sont les sources volumiques de chaleur par unité de temps, apparaissant dans l'équation de conduction thermique non linéaire $(1.2)_{(i)}$. En particulier, les P_N approximations par harmoniques sphériques de $v \longmapsto I^k(x,t,v)$ ont été considérées par de nombreux auteurs [46]. Les approximations P_N simplifiées avec conditions aux limites appropriées dérivées des principes variationnels, les méthodes P_N de la théorie des réacteurs nucléaires [70], [4], ont été introduites dans la théorie du transfert radiatif de chaleur dans [40] et [38], mais ces méthodes approchées ne sont valables que pour des corps optiquement épais Ω. En particulier, la méthode SP_1 dont l'idée originale est l'approximation de $I^k(x,t,\mathbf{v})$ par une fonction affine en la variable $\mathbf{v}$ ([46], p.473) $I^k(x,t,\mathbf{v}) \approx a(x,t) + \mathbf{b}(x,t)\cdot\mathbf{v}$, avec $a(x,t) \in \mathbb{R}$ et $\mathbf{b}(x,t) \in \mathbb{R}^3$, a été utilisé dans de nombreux articles [39], [14], ..., pour n'en citer que quelques-uns. Dans [64], dans l'étude du refroidissement thermique d'un disque de verre mince circulaire, les intensités radiatives ont été calculées par la méthode du tracé en arrière de rayons thermiques jusqu'à un point suffisamment lointain, que pour avoir une influence négligeable sur l'intensité radiative calculée. Ces auteurs montrent numériquement, que même pour le verre mince, le rayonnement interne émis ou absorbé à l'intérieur du verre, représenté par le terme $-\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{M}\kappa_k \int_V I^k(x,t,v)d\mu(v)$ dans l'équation de conduction thermique $(1.2)_{(i)}$, ne peut pas être ignoré (une observation similaire a été faite pour les plaques de verre dans [48]). Dans [56], l'auteur considère les problèmes aux limites de transfert radiatif exact (1.1) couplés avec le problème aux limites pour l'équation de conduction thermique instationnaire (1.2) écrit en coordonnées cylindriques pour simuler le processus de refroidissement d'un tube de verre de quartz axisymétrique. La résolution numérique par la méthode des ordonnées discrètes, combinée avec le schéma par éléments finis SUPG, des équations de transfert radiatif tridimensionnel avec réflexion spéculaire à la frontière, contenant (1.1) comme un cas particulier, a été étudié dans [35], mais le couplage avec l'équation de conduction de la chaleur n'est considéré dans cet article. En 1-D, c'est-à-dire pour les plaques infinies, l'analyse mathématique d'un problème connexe lié à (1.1) - (1.2) a été faite dans [50]. L'estimation de l'erreur a priori pour le problème semi-discrétisé a été faite dans [48] et l'analyse mathématique du problème de contrôle optimal de la température à l'intérieur dans la plaque par la température $T_S(.)$ de la source radiative a été faite dans [49].

Cependant, à notre connaissance, l'analyse mathématique du problème exact (1.1) - (1.2) en dimension 3 n'a pas encore été faite. D'où le but du chapitre 2 de cette thèse de combler ce vide et de prouver l'existence et l'unicité de la solution pour le problème exact (1.1) - (1.2). Nous donnons aussi des bornes uniformes sur la solution en termes de bornes sur les données que sont la température ambiante de l'air sec environnant T_a apparaissant dans la condition aux limites $(1.2)_{(ii)}$, la condition initiale $T_0(\cdot)$ pour la température absolue dans $\Omega(1.2)$ $_{(iii)}$, et la température absolue $T_S(\cdot)$ en fonction du temps de la source radiative noire entourant Ω. En conséquence, nous obtenons également des bornes sur les intensités radiatives (voir Corollaire 3.4), ce qui pourrait être utile pour définir un critère d'arrêt, lors de l'utilisation de la méthode du tracé de rayons en arrière [64] pour résoudre numériquement nos problèmes aux limites (1.1) pour l'équation de transfert radiatif.

Dans la section 2, nous prouvons pour chaque $k = 1, \ldots, M$, l'existence et l'unicité de la solution dans l'espace $W^p(\Omega \times V)$ [19] (notée H^1_p in [1], Chapitre 2), du problème aux limites de transfert radiatif (1.1) avec la condition limite non homogène (1.1) $_{(ii)}$. En particulier, nous utilisons les théorèmes de trace de M. Cessenat pour les éléments de l'espace $W^p(\Omega \times V)$ [12], [13], [19, pp.252-253] et la théorie des opérateurs dissipatifs. Parmi les principales étapes, figure dans la sous-section 2.3, la preuve de la dissipativité et de la fermeture de l'opérateur $A = -v \cdot \nabla_x$ défini par (3.8). La preuve de la dissipativité de A est plutôt subtile et repose sur le fait que le coefficient de réflectivité $\rho_g(\cdot)$ est égal à 1 près de 0, ce qui implique que $\rho_g(|\nu_x \cdot v|) = 1$ si $|\nu_x \cdot v|$ est suffisamment petit, la ainsi-appelée "propriété de réflectivité totale pour les incidences rasantes"en Optique. Dans la sous-section 2.4, nous montrons que l'image de $\lambda I - A$ est égale à $L^p(\Omega \times V)$, $\forall \lambda > 0$, sans utiliser le résultat abstrait Théorème 3 in [19, p.254] vu que sa condition (22) semble difficile à vérifier. L'idée est de considérer un élément $g \in L^{p'}(\Omega \times V)$ "orthogonal à l'image de $(\lambda I - A)$ ", d'où l'on déduit que $(\lambda I - \hat{A})g = 0$ où $\hat{A}$ est un opérateur de définition similaire à celle de A dont on démontre en se ramenant à A qu'il est aussi disspatif. De l'inégalité de dissipativité suit que $g = 0$. Ceci démontre que l'image de $(\lambda I - A)$ est dense dans $L^p(\Omega \times V)$ et donc égale en fait à $L^p(\Omega \times V)$, A étant un opérateur fermé. Mentionnons, que le papier [62], ne donnerait pas dans le contexte L^2, la forme explicite du domaine (3.8) du générateur A, mais seulement le domaine exact d'un prégénérateur [62, p.9].

Dans la section 3, nous démontrons tout d'abord l'unicité de la solution $T(.,.)$ du problème couplé (2.2) - (2.1) dans l'espace de Sobolev $W(0, t_f)$ [18]. Ensuite, dans le but de prouver l'existence de la solution $T(.,.)$, supposant que les données sont bornées, on définit le convexe fermé S, sous-ensemble de $L^2(0, t_f; L^2(\Omega))$ défini par:

$$S := \{T \in L^2(0, t_f; L^p(\Omega)); \forall' t \in]0, t_f[: \forall' x \in \Omega : T_a \leq T(x,t) \leq T_S(t)\},$$

et aussi un problème de point fixe en remplaçant dans (1.2) T by $\tilde{T}$, sauf dans le terme volumique source de chaleur $h_T(x,t) := \sum_{k=1}^{M} \kappa_k \int_V I_T^k(x,t,v)d\mu(v)$. Cela nous permet de découpler notre problème (1.2) - (1.1), tout en nous réduisant à un problème initial aux limites de type parabolique semi-linéaire. Nous démontrons que pour tout $T \in S$, ce problème de point fixe admet une solution unique $\tilde{T} \in S$, et que sous l'hypothèse $T_S \in H^1(0,t_f)$ et $\frac{dT_S}{dt} \geq 0$ p.p sur $]0,t_f[$, que l'application $\Phi : T \mapsto \tilde{T}$ est continue de S dans S avec son image relativement compacte dans $L^2(Q)$. La preuve que l'application Φ laisse stable S nécessite d'établir des sortes de "principes du maximum "pour notre problème de point fixe (voir la sous-section (3.3.4)). La preuve de la continuité de Φ repose sur les bornes uniformes des intensités radiatives établies dans les Propositions 3.11 et 3.10, en conjonction avec l'estimation a priori (7.33) de [71, p.377] pour une équation parabolique semi-linéaire. En utilisant le théorème de Schauder, nous concluons à l'existence d'une solution faible du problème couplé (1.2) - (1.1), obtenant en outre des bornes uniformes sur la solution $T(x,t)$ et sur les intensités radiatives.

Ensuite, nous avons généralisé le résultat en supprimant la condition $\frac{dT_S}{dt} \geq 0$ sur T_S. La preuve procède en considérant un nouveau sous-ensemble convexe fermé $\mathcal{C}$ dans $L^2(Q) = L^2(0,t_f;L^2(\Omega))$ où $Q := \Omega \times]0,t_f[$ défini par

$$\mathcal{C} := \{T \in L^2(Q); \forall' t \in]0,t_f[: \forall' x \in \Omega : \underline{T} \leq T(x,t) \leq \bar{T}\}.$$

où les nombres fixes $\underline{T}$, $\bar{T} \in \mathbb{R}_+^*$ satisfont à

$$\begin{cases} \underline{T} < \bar{T},\ \underline{T} \leq T_a \leq \bar{T}, \\ \underline{T} \leq T_0(x) \leq \bar{T},\ \textit{for a.e. } x \in \Omega, \\ \underline{T} \leq T_S(t) \leq \bar{T},\ \textit{for a.e. } t \in]0,t_f[. \end{cases} \tag{1.4}$$

et nous prouvons l'existence d'une solution faible au problème (1.2) appartenant à $\mathcal{C}$.

Addendum: le 25 Juin 2018, suite à un e-mail de ResearchGate nous avons eu vent de travaux du Mathématicien Russe Andrey A. Amosov travaillant à la National Research University "Moscow Power Engineering Institute ". En particulier, nous avons pris connaissance de son travail intitulé: "Unique Solvability of a Nonstationary Problem of Radiative-Conductive Heat Exchange in a System of Semitransparent Bodies "publié dans le Russian Journal of Mathematical Physics en 2016 Vol. 23, $N°3$, pp.309-334. A noter que notre travail [51] relatif au chapitre 2 de cette thèse avait dèjà été soumis en 2016 au Journal "Mathematical Methods in the Applied Sciences ", plus précisément le 26 Août 2016. Cet article de A.A. Amosov est intimement lié au chapitre 2 de notre thèse mais fort heureusement les méthodes que nous avons utilisées diffèrent de celles utilisées par A.A. Amosov. De plus, concernant l'équation de conduction de la chaleur, A.A. Amosov considère une condition au bord de Neumann homogène (cfr. formule (1.3) p.311 de cet article), en particulier sans le terme intégral de notre condition au bord $(1.2)_{(ii)}$, terme prenant en compte les longueurs d'onde $\lambda \geq \lambda_0$ pour lesquelles le verre se comporte comme un corps opaque. La démonstration de l'existence de la solution du problème couplé dans cet article repose sur la méthode de Galerkin (cfr. Theorem 7.1 p.327). Quant à la démonstration de l'existence de la solution pour nos problèmes aux limites (1.1) relatifs à l'équation de transfert radiatif (ETR) l'auteur renvoit à un de ses articles précédents de 2013 publié dans le Journal of Mathematical Sciences United States, Vol. 191, $N^0 2$, May 2013, pp.101-149 intitulé "Boundary Value Problem for the Radiation Transfer Equation with Reflection and Refraction Conditions ". Sa démonstration procéde par une méthode de point fixe en se ramenant à la résolution d'une suite de problèmes de Dirichlet inhomogènes sur Γ_- pour l'ETR dont sa limite est la solution du problème aux limites avec la condition au bord de réflectivité. Cette méthode semble la mise en oeuvre dans le cadre de l'Analyse Mathématique de la méthode du tracé de rayons en arrière. Au contraire notre démonstration repose sur l'établissement de la maximale-dissipativité de l'opérateur A dont le domaine est défini par la condition de réflectivité homogène.

1.2.2 Chapitre 4: Contrôle du chauffage radiatif d'un corps semi-transparent

Au chapitre 3, j'ai travaillé sur l'extension à $3-d$ du résultat $1-d$ de l'article [49] publié dans AFMA en juin 2016 par mon superviseur le Pr. Luc Paquet, qui consiste à trouver la température la plus adéquate $T_S(t), t \in [0, t_f]$ de la source rayonnante noire S, pendant le processus de chauffage du corps semi-transparent Ω, un ouvert borné de classe C^1 de $\mathbb{R}^3$, afin d'obtenir la température $(x, t) \mapsto T(x, t)$ du corps semi-transparent Ω pendant le temps fixé de chauffage radiatif $0 \leq t \leq t_f$, aussi proche que possible d'une distribution de température désirée donnée $(x, t) \mapsto T_d(x, t)$. Afin de prendre également en compte le coût du contrôle T_S, nous considérons donc la fonctionnelle coût J définie par:

$$\begin{aligned} J : L^2(Q) \times H^1(]0, t_f[) \longrightarrow \mathbb{R} : (T, T_S) \mapsto & \frac{1}{2} \int_Q (T(x,t) - T_d(x,t))^2 dx \otimes dt \\ & + \frac{\delta_r}{2} \|T_S - T_{S,d}\|^2_{H^1(]0,t_f[)}, \end{aligned} \tag{1.5}$$

fonctionnelle que nous voulons minimiser, T étant assujettie à $T_S(.)$ par l'équation de conduction de la chaleur non-linéaire (1.2), et $T_S(.)$ assujetti à appartenir à l'ensemble des contrôles admissibles U_{ad} que nous définissons maintenant. Tout d'abord nous définissons notre espace des contrôles: $U := H^1(]0, t_f[)$, muni de la norme H^1. Nous considérons comme ensemble des contrôles admissibles U_{ad}:

$$U_{ad} := \{T_S \in H^1(]0, t_f[); \underline{T} \leq T_S(t) \leq \bar{T}, \ \forall t \in]0, t_f[\}. \tag{1.6}$$

where $\underline{T}$ and $\bar{T}$ satisfy (1.4).

U_{ad} est un sous-ensemble convexe et fermé de l'espace de Hilbert $U := H^1(]0, t_f[)$. Nous avons prouvé dans le chapitre précédent (voir aussi l'article [51]), que pour $T_S \in U_{ad}$, le problème initial aux valeurs frontières (1.2) possède une solution faible bornée unique

$$T(T_S) \in \{T \in L^2(0, t_f; H^1(\Omega)); \frac{dT}{dt} \in L^2(0, t_f; (H^1(\Omega))^*)\}. \tag{1.7}$$

Cela nous permet de définir le coût fonctionnel réduit:

$$\hat{J} : U_{ad} \to \mathbb{R} : T_S \longmapsto J(T(T_S), T_S). \tag{1.8}$$

Dans la section 3, nous démontrons l'existence d'un contrôle optimal çàd de $T_S \in U_{ad}$ tel que $\hat{J}(T_S) := \inf\limits_{v \in U_{ad}} \hat{J}(v)$.
Ensuite, dans la section 4, nous choisissons l'espace des états E et l'application contraignante $(T, T_S) \mapsto e(T, T_S)$ de manière judicieuse pour qu'elle soit Fréchet différentiable et que sa dérivée par rapport à la température T au point (T, T_S) soit inversible. Nous définissons l'espace des états E comme l'ensemble de tous les

$$T \in \left\{T \in L^2(0, t_f; H^1(\Omega)); \frac{dT}{dt} \in L^2(0, t_f; (H^1(\Omega))^*)\right\} \cap C(\bar{Q})$$

tels que $(c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T) \in L^r(Q)$ et $k_h \frac{\partial T}{\partial \nu} \in L^{s^*}(\Sigma)$ au sens des distributions, où $r \in]2.5, 2.72[$ et $s^* > 4$. Maintenant, nous donnons la définition précise de l'application contraignante $e(\cdot, \cdot)$:

$$\begin{aligned} e : E \times U &\longrightarrow L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega}) \\ (T, T_S) &\longmapsto e(T, T_S) = (e_1, e_2, e_3)(T, T_S) \end{aligned}$$

où

$$\begin{aligned} e_1(T, T_S) &= c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T - \psi(T) - h(T, T_S), \\ e_2(T, T_S) &= k_h \frac{\partial T}{\partial \nu} + h_c T + \Theta(T) - \Theta(T_S), \\ e_3(T, T_S) &= T(\cdot, 0). \end{aligned} \tag{1.9}$$

Les quantités $\psi(T), h(T, T_S), \Theta(T)$ et $\Theta(T_S)$ sont définies par les formules:

$$\begin{cases} \psi(T) := -\sum\limits_{k=1}^{M} 4\pi \kappa_k B_g^k(T), \ \forall T \in \mathbb{R}, \\ h_{T,T_S}(x, t) := \sum\limits_{k=1}^{M} \kappa_k \int\limits_V I_{T,T_S}^k(x, t, v) d\mu(v), \\ \Theta(T) := \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda B(T, \lambda) d\lambda, \ \forall T \in \mathbb{R}. \end{cases} \tag{1.10}$$

L'équation d'état s'écrit alors $e(T, T_S) = (0, h_c T_a, T_0)$. A la section 5, nous appliquons le théorème des fonctions implicites afin de calculer la dérivée du coût fonctionnel réduit $\hat{J}(T_S)$ en un contrôle $T_S \in H^1(]0, t_f[)$ et de donner la condition nécessaire d'optimalité du premier ordre pour qu'un contrôle T_S soit optimal par rapport au coût fonctionnel J, sous forme d'une inéquation variationnelle que voici:

$$\sum_{k=1}^{M} \kappa_k \int_0^{t_f} \Big\{ \int_\Omega \Big[\Big(4\pi - \int_V \kappa_k (\kappa_k - A)^{-1} 1_{\Omega \times V} d\mu(v) \Big) \eta_{Vol}(T_{T_S}; x, t) \Big] \, dx \Big\}$$
$$\frac{dB_g^k}{dT}(T_S(t))(v(t) - T_S(t))dt + \int_0^{t_f} (\int_{\partial\Omega} \eta_B(T_{T_S}; x, t) dS(x)) \frac{d\Theta}{dT}(T_S(t))(v(t) - T_S(t))dt$$
$$+\delta_r \int_0^{t_f} (T_S(t) - T_{S,d}(t))(v(t) - T_S(t))dt + \delta_r \int_0^{t_f} (\dot{T}_S(t) - \dot{T}_{Sd}(t))(\dot{v}(t) - \dot{T}_S(t))dt \geq 0, \forall v \in U_{ad}, \tag{1.11}$$

où η_{Vol} est la solution au sens faible du problème (4.39) et η_B est la trace de η_{Vol} sur $\Sigma = \partial\Omega \times]0, t_f[$.

Dans le cas $1 - d$ en espace [49], çàd celui d'une plaque de verre, où un problème de contrôle de ce type a été étudié, l'auteur de cet article utilise entre autres dans ce cas, une formule intégrale explicite pour h_{T,T_S}: la formule (6) [49], ce qui n'est pas possible en $3 - d$.

1.2.3 Chapitre 5: Les équations de la thermoviscoélasticité

Jusqu'à présent, nous avons étudié le problème du chauffage radiatif d'un corps en verre considéré comme un ensemble ouvert borné Ω de $\mathbb{R}^3$ avec une frontière de classe C^1 au moins. Nous avons également étudié le problème de contrôle optimal associé.

Après cela, nous arrivons à l'étude du problème but, qui consiste à déformer le volume de verre Ω en le chauffant avec une source radiative noire tout autour, et à contrôler sa déformation. A titre d'exemple industriel, la technologie de l'affaissement du verre sous son poids propre, qui consiste à chauffer le verre jusqu'à une température suffisamment élevée pour lui permettre de s'affaisser, est utilisé pour fabriquer des produits industriels en verre tels que pare-brises, miroirs ou lentilles.

Nous étudions l'existence et l'unicité de la déformation de Ω, résultant du chauffage radiatif du corps semi-transparent Ω par la source noire S. Dans le cadre de la théorie de la viscoélasticité, M.E.Rognes et R.Winther ont considéré dans leur publication [59], pour le modèle de Maxwell de la viscoélasticité, une formulation mixte de type Hellinger-Reissner avec relaxation de la symétrie en introduisant un multiplicateur de Lagrange, formulation mixte qui présente l'avantage d'éviter les équations intégrodifférentielles compliquées liant le champ des tenseurs des contraintes internes $(x, t) \longmapsto \sigma(x, t)$ au champ de tenseurs du taux de déformations. Dans cette formulation mixte, les inconnues sont le champ de tenseur des contraintes internes $(x, t) \longmapsto \sigma(x, t)$, le champ des vitesses

$(x,t) \longmapsto \dot{\mathbf{u}}(x,t)$ du champ de déplacements $(x,t) \longmapsto \mathbf{u}(x,t)$ (nous noterons aussi $\dot{\mathbf{u}}$ par $\mathbf{v}$) et la composante d'assymétrie $(x,t) \longmapsto \rho(x,t)$ du gradient de $\mathbf{u}$. La nouveauté, est qu'ici, nous considérons le problème de la thermoviscoélasticité au lieu du problème de la viscoélasticité [59], de sorte que nous avons maintenant toute une famille d'opérateurs $A_0(.,t)$ indexée par le temps $t \in [0,t_f]$:

$$\begin{aligned} A_0(.,t) : \left[L^2(\Omega)\right]^{3\times 3} &\to \left[L^2(\Omega)\right]^{3\times 3} \\ \sigma \longmapsto \tfrac{1+\nu}{\eta(T(.,t))}\sigma &- \tfrac{\nu}{\eta(T(.,t))} tr(\sigma) I_3, \end{aligned} \tag{1.12}$$

où $\eta(T(x,t))$ désigne la viscosité au point $x \in \Omega$ et au temps $t \in [0,t_f]$, qui dépend de la température absolue $T(x,t)$, et où I_3 dénote la matrice identité d'ordre 3. Au contraire, dans l'article de M.E.Rognes et R.Winther ([59], page 963), on suppose que A_0 est indépendant du temps t. Nous supposons que la viscosité $\eta(.)$ est une fonction positive strictement décroissante, définie et C^1 sur $\mathbb{R}_+^*$.

Nous dénotons par E le module de Young et par ν le coefficient de Poisson ($0 < \nu < \frac{1}{2}$). Désignant par $\varepsilon(u_E)$, le champ des tenseurs de déformations linéarisé correspondant au champ de déplacements élastiques u_E, nous avons:

$$\varepsilon(u_E) = \frac{1+\nu}{E}\sigma - \frac{\nu}{E} tr(\sigma) I_3,$$

où σ désigne le champ des tenseurs de contraintes. Le taux de déformations correspondant au taux des déplacements visqueux u_V, est donné par:

$$\varepsilon(\dot{u}_V) = \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)} tr(\sigma) I_3.$$

Maintenant, le champ de déplacement total est: $u = u_E + u_V$. Cela implique que:

$$\begin{aligned} \varepsilon(\dot{u}) &= \varepsilon(\dot{u}_E) + \varepsilon(\dot{u}_V) \\ &= \tfrac{1+\nu}{E}\dot{\sigma} - \tfrac{\nu}{E} tr(\dot{\sigma}) I_3 + \tfrac{1+\nu}{\eta(T)}\sigma - \tfrac{\nu}{\eta(T)} tr(\sigma) I_3. \end{aligned}$$

Ainsi, nous avons l'équation reliant la dérivée temporelle du champ de déplacements u, au champ des contraintes σ:

$$\varepsilon(\dot{u}) = \frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E} tr(\dot{\sigma}) I_3 + \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)} tr(\sigma) I_3, \tag{1.13}$$

appelée “loi de comportement de Maxwell ”. A cette équation, on joint l'équation d'équilibre:

$$-\operatorname{div}(\sigma) = g - \beta\, \nabla T, \tag{1.14}$$

où nous avons négligé les forces d'inertie, c'est-à-dire que nous considérons le problème “quasi-statique ”. $\beta := \frac{E}{1-2\nu}\alpha$, où α dénote le coefficient de dilatation linéaire. g indique un champ de force volumique agissant sur Ω, par exemple son poids volumique. Nous supposons au moins que $g \in L^2(Q)^3$, où Q désigne le "cylindre"$\Omega\times]0,t_f[$ dans l'espace-temps $\mathbb{R}^3 \times \mathbb{R}$. T étant la solution faible de l'équation de la conduction thermique, l'équation (1.2), $T \in L^2(]0,t_f[;H^1(\Omega))$, et donc $\nabla T \in L^2(Q)^3$. A ces deux équations, la loi constitutive (1.13) de Maxwell et l'équation d'équilibre (1.14), nous joignons une condition initiale pour le champ des contraintes σ:

$$\sigma(.,0) = \zeta. \tag{1.15}$$

Enfin, la frontière $\partial\Omega$ du domaine Ω est partitionnée en deux sous-ensembles ouverts disjoints Γ_D et Γ_N de $\partial\Omega$ tels que $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$. Sur Γ_D, nous imposons que le champ de vitesse des déplacements $\dot{u}$ soit égal à 0, et sur Γ_N, nous imposons: pas de tractions, c'est-à-dire que $(\sigma.\nu)_{|\Gamma_N} = 0$ où ν désigne le champ normal unitaire sortant le long de $\partial\Omega$. Nous écrivons maintenant une formulation faible pour ces deux équations et ces conditions aux limites. Considérant un champ de contraintes test $\tau \in H(\mathrm{div};\Omega)^3$ tel que $(\tau.\nu)_{|\Gamma_N} = 0$, mis à part cela arbitraire. Nous avons par la formule de Green:

$$\int_\Omega \varepsilon(v) : \tau \; dx = \langle \tau.\nu, v\rangle_\Gamma - \int_\Omega \mathrm{div}(\tau).v \; dx - \int_\Omega \rho(v) : \tau \; dx, \tag{1.16}$$

pour tout $v \in H^1(\Omega)^3$, où $\rho(v) := \frac{1}{2}(\nabla v - (\nabla v)^T)$. Appliquant la formule de Green (1.16) à $\varepsilon(\dot{u})$, en utilisant la condition aux limites $\dot{u}_{|\Gamma_D} = 0$, on obtient de la loi de comportement de Maxwell (1.13):

$$\begin{aligned}\int_\Omega [\tfrac{1+\nu}{\eta(T)}\sigma - \tfrac{\nu}{\eta(T)} tr(\sigma) I_3] : \tau \; dx + \int_\Omega [\tfrac{1+\nu}{E}\dot{\sigma} - \tfrac{\nu}{E} tr(\dot{\sigma}) I_3] : \tau \; dx \\ + \int_\Omega \mathrm{div}(\tau).\dot{u} \; dx + \int_\Omega \rho(\dot{u}) : \tau \; dx = 0,\end{aligned} \tag{1.17}$$

$\forall \tau \in H(\mathrm{div};\Omega)^3$ tel que $(\tau.\nu)_{|\Gamma_N} = 0$. Pour presque tous les $t \in]0,t_f[$, cette équation exprime sous une forme faible l'équation constitutive de Maxwell (1.13) et la condition aux limites $\dot{u}_{|\Gamma_D} = 0$ pour la vitesse de déplacements $\dot{u}$. Notons que, formellement au moins, si $u(.,.)$ satisfait à l'instant 0, $u(.,0)_{|\Gamma_D} = 0$, alors nous aurons $u(.,t)_{|\Gamma_D} = 0$, $\forall' t \in]0,t_f[$. La condition frontière $(\sigma.\nu)_{|\Gamma_N} = 0$ est une condition au bord essentielle de la la formulation mixte et est donc imposée dans la définition de l'espace fonctionnel $H_{\Gamma_N}(\mathrm{div};\Omega)^3$: nous essayons de trouver

$$\sigma(.,t) \in H_{\Gamma_N}(\mathrm{div};\Omega)^3,$$

où $H_{\Gamma_N}(\mathrm{div};\Omega)$ dénote le sous-ensemble fermé des champs de vecteurs de $H(\mathrm{div};\Omega)$ dont les traces normales sur Γ_N sont nulles. Maintenant à la forme faible de la loi de comportement de Maxwell et de la condition aux limites $\dot{u}|\Gamma_D = 0$, naturelle pour la formulation mixte, nous devons ajouter une forme faible de l'équation d'équilibre (1.14):

$$\int_\Omega \operatorname{div}(\sigma).w\ dx + \int_\Omega (g - \beta\ \nabla T).w\ dx = 0,\ \forall w \in \left[L^2(\Omega)\right]^3. \tag{1.18}$$

Nous devons par ailleurs exprimer sous une forme faible $\forall' t \in]0, t_f[$, la symétrie du tenseur des contraintes $\sigma(.,t)$, de sorte que nous obtenons l'équation:

$$\begin{gathered}\int_\Omega \operatorname{div}(\sigma).w\ dx + \int_\Omega (g - \beta\ \nabla T).w\ dx + \int_\Omega \sigma : \xi\ dx = 0,\ \forall w \in L^2(\Omega)^3, \\ \forall \xi \in [L^2(\Omega)]^{3\times 3}_{skew}, \end{gathered} \tag{1.19}$$

où

$$\left[L^2(\Omega)\right]^{3\times 3}_{skew} := \{\xi \in \left[L^2(\Omega)\right]^{3\times 3}; \xi + \xi^T = 0\}.$$

A ces formes faibles (1.17) (resp. (1.19)), des équations (1.13) (resp. (1.14) et de la symétrie de Σ), nous devons joindre la condition initiale (1.15). Supposant que $g \in H^1(0, t_f; L^2(\Omega)^3)$, que la température initiale dans Ω, T_0, est égale à la température ambiante T_a, que $T_S \in H^2(]0, t_f[)$ et satisfait à $T_S(0) = T_a$, que la condition initiale ζ pour σ vérifie $\zeta \in H_{\Gamma_N}(\operatorname{div};\Omega)^3$, $\zeta = \zeta^T$ et $\operatorname{div}\zeta = -(g(.,0) - \beta\nabla T(.,0))$, nous prouvons qu'il existe un unique triple

$$(\sigma, v, \rho) \in H^1(0, t_f; H_{\Gamma_N}(\operatorname{div};\Omega)^3) \times L^2(0, t_f; L^2(\Omega)^3) \times L^2(0, t_f; \left[L^2(\Omega)\right]^{3\times 3}_{skew})$$

tel que $\sigma(.,0) = \zeta$ et tel que $\forall' t \in]0, t_f[$ les équations (1.17) et (1.19) soient vérifiées, c'est-à-dire:

$$\left\{\begin{array}{l} \displaystyle\int_\Omega [\tfrac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \tfrac{\nu}{\eta(T(.,t))} tr(\sigma(.,t)) I_3] : \tau\ dx + \int_\Omega [\tfrac{1+\nu}{E}\dot\sigma(.,t) - \tfrac{\nu}{E} tr(\dot\sigma(.,t)) I_3] : \tau\ dx \\ \displaystyle\qquad + \int_\Omega \operatorname{div}(\tau).v(.,t)\ dx + \int_\Omega \rho(.,t) : \tau\ dx = 0,\ \forall \tau \in H_{\Gamma_N}(\operatorname{div};\Omega)^3, \\ \displaystyle\int_\Omega \operatorname{div}(\sigma(.,t)).w\ dx + \int_\Omega (g(.,t) - \beta\ \nabla T(.,t)).w\ dx + \int_\Omega \sigma(.,t) : \xi\ dx = 0,\ \forall w \in L^2(\Omega)^3, \\ \qquad\qquad \forall \xi \in [L^2(\Omega)]^{3\times 3}_{skew}. \end{array}\right.$$

1.2.4 Chapitre 6: Contrôle des déplacements dans le cadre de la thermoviscoélasticité

Dans ce chapitre, nous étudions le contrôle de la déformation de Ω, résultant du chauffage radiatif du corps semi-transparent Ω par la source radiative noire S. Tout d'abord, nous introduisons le problème de contrôle. Nous définissons $\forall t \in]0, t_f[$: $\vec{u}(t) := \int_0^t v(s)\ ds$. Par le théorème d'Ascoli ([20], p.143), l'application qui envoie $v \mapsto \vec{u}$ de $L^2(0,t_f;H^1(\Omega)^3)$ dans l'espace $C([0,t_f];L^2(\Omega)^3)$ est compacte. Ainsi, si nous avons une suite $(v_n)_{n\geq 1}$ qui converge faiblement dans $L^2(0,t_f;H^1(\Omega)^3)$, la suite correspondante $(\vec{u}_n)_{n\geq 1}$ convergera fortement dans $C([0,t_f];L^2(\Omega)^3)$ et a fortiori dans $L^2(0,t_f;L^2(\Omega)^3)$. Maintenant, nous définissons le nouvel ensemble de contrôles admissibles:

$$U_{ad} := \{T_S \in H^2(]0,t_f[); \underline{T} \leq T_S(t) \leq \bar{T},\ \forall t \in]0,t_f[, T_S(0) = T_a\}$$

et le coût fonctionnel

$$\begin{aligned} J : [L^2(Q)]^3 \times H^2(]0,t_f[) &\to \mathbb{R} \\ (\vec{u}, T_S) &\mapsto \frac{1}{2}\int_Q |\vec{u}(x,t) - \vec{u}_d(x,t)|^2 \ dx \otimes dt + \frac{\tilde{\delta}_r}{2}\left\|T_S - T_S^d\right\|^2_{H^2(]0,t_f[)}, \end{aligned}$$

où $\vec{u}_d(.,.) \in [L^2(Q)]^3$ désigne le champ désiré donné des déplacements et $T_S^d \in H^2(]0,t_f[)$ une évolution donnée de la la température absolue de la source radiative noire S de laquelle T_S ne devrait pas être “trop loin ”, la signification de ce “trop loin ”étant modulée par le coefficient strictement positif $\tilde{\delta}_r$ dans la définition du coût fonctionnel J, coefficient qui peut être choisi très petit. Comme nous l'avons dit plus haut, $\underline{T}$ et $\bar{T}$ dénotent deux nombres réels strictement positifs satisfaisant les conditions $(1.4)_{(i)}$ et $(1.4)_{(ii)}$. Observons que U_{ad} est un sous-ensemble convexe fermé de $H^2(]0,t_f[)$. Nous prouvons que l'application de l'ensemble ouvert $\tilde{U}_{ad} := \{T_S \in H^2(]0,t_f[); \frac{\underline{T}}{2} < T_S(t) < 2\bar{T}, \forall t \in [0,t_f], T_S(0) = T_a\}$ into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ qui envoie T_S sur $\vec{u}$ est continûment Fréchet différentiable. En conséquence, le coût réduit fonctionnel

$$\begin{aligned} \hat{J} : \tilde{U}_{ad} := \{T_S \in H^2(]0,t_f[); \tfrac{\underline{T}}{2} < T_S(t) < 2\bar{T},\ \forall t \in [0,t_f], T_S(0) = T_a\} \to \mathbb{R} \\ T_S \mapsto \tfrac{1}{2}\int_Q |\vec{u}(T_S)(x,t) - \vec{u}_d(x,t)|^2 \ dx \otimes dt + \tfrac{\tilde{\delta}_r}{2}\left\|T_S - T_S^d\right\|^2_{H^2(]0,t_f[)} \end{aligned}$$

est également continûment Fréchet différentiable. En particulier, si $\overline{T_S} \in U_{ad}$ est un contrôle optimal, il satisfait à l'inégalité variationnelle:

$$\hat{J}'(\overline{T_S})(T_S - \overline{T_S}) \geq 0,\ \forall T_S \in U_{ad}.$$

.

2 Introduction in English

2.1 The subject of my thesis

In the paper "Control of the Radiative Heating of a Glass Plate " published recently by Afrika Matematika (AFMA, June 2016)[49], Pr. Luc Paquet studied in it the control of the temperature $T(x,t)$ at time t along the flat glass thickness x during the fixed interval time of heating $]0,t_f[$, by acting on the temperature $u(t)$ of the black radiative source S, placed above the glass plate. A first order necessary condition in the form of a variational inequality has been derived for a control $u : t \to u(t)$ to be an optimal control. The state space and the constraining mapping have been carefully defined in order for the constraining mapping $(T,u) \mapsto e(T,u)$ to be Fréchet differentiable and its derivative with respect to the temperature T at a point (T,u) to be invertible. This had allowed him to apply the implicit function theorem in order to compute the derivative of the reduced cost functional $\hat{J}(u)$ at a control $u \in H^1(]0,t_f[)$, and to find the first order necessary condition for a control to be optimal.

In the paper published in the electronic journal "Mathematics in Action" (MSIA) in 2012 [50], entitled "Radiative Heating of a Glass Plate ", Pr. Luc Paquet, Mr. Raouf El Cheikh, Pr. Dominique Lochegnies and Dr. Norbert Siedow, have considered an infinite horizontal glass plate of width l with an infinite plane black sheet metal S, placed above it which emits thermal radiations and proved the existence and uniqueness of the solution of the coupled problem between a nonlinear heat conduction equation subject to a nonlinear Robin boundary condition, with the radiative transfer equation solved explicitly in this case.

We have extended the result of existence and uniqueness of the solution from the paper MSIA (2012)[50], and the first order necessary condition for an optimal control of the paper AFMA (June 2016)[49], to a new setting considering the domain Ω to be a bounded domain with a boundary of class C^1 (resp. $C^{1,1}$) of $\mathbb{R}^3$ surrounded by a black radiative source at absolute temperature $T_S(t) \equiv u(t)$ at time t.

The purpose of the radiative heating of a glass body is in fact to deform the body as near as possible to a required shape. The problem of the optimal control of the deformation by acting on the temperature of the radiative heating source will be studied in the framework of the linear viscoelasticity equations concerning the deformation.

2.2 Presentation and discussion of the results obtained in my thesis

2.2.1 Chapter 3: Coupling the Heat Conduction Equation with The Radiative Transfer Equation

In this Chapter, we analyze the exact mathematical problem, modeling the radiative heating of a semi-transparent body Ω made of a glass, from time 0 to the final time of heating t_f, by a black surface S surrounding it, at absolute temperature $T_S(t)$ $(0 \leq t \leq t_f)$. The main application of this problem is, by heating the semi-transparent body Ω, to allow it to deform due to its own weight and to control its deformation (or its stress-field): see e.g. Chapter 2 and 3, [3], [53], ..., to cite only a few number of papers. We must thus analyze the coupling between the quasi-steady radiative transfer boundary value problems:

$$\begin{cases} v \cdot \nabla_x I^k(x,t,v) + \kappa_k I^k(x,t,v) = \kappa_k B_g^k(T(x,t)), \ \ in \ \Omega \times V, \\ I^k(x,t,v) = \rho_g(|\nu_x \cdot v|) I^k(x,t,v_i(x,v)) + (1-\rho_g(|\nu_x \cdot v|)) B_g^k(T_S(t)), \\ on \ \Gamma_-, \end{cases} \tag{2.1}$$

where $\Gamma_- := \{(x,v) \in \partial\Omega \times V;\ \nu_x \cdot v < 0\}$, and $v_i(x,v) := v - 2(\nu_x \cdot v)\nu_x$, one for each wavelength band $[\lambda_k, \lambda_{k+1}[$ $(k = 1, \dots, M)$ in which the glass behaves like a semi-transparent medium and the linear absorption coefficient κ_k may be considered as constant, with the nonlinear heat conduction initial boundary value problem [50], [63]:

$$\begin{cases} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \Delta T(x,t) - \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{M} \kappa_k \int_V I^k(x,t,v) d\mu(v), \\ in \ Q := \Omega \times]0,t_f[, \\ -k_h \frac{\partial T}{\partial \nu}(x,t) = h_c(T(x,t) - T_a) + \pi \int_{\lambda_0}^{+\infty} \varepsilon_\lambda [B(T(x,t),\lambda) - B(T_S(t),\lambda)] d\lambda, \\ on \ \Sigma := \partial\Omega \times]0,t_f[, \\ T(x,0) = T_0(x), \ \ on \ \Omega. \end{cases} \tag{2.2}$$

$T(x,t)$ denotes the absolute temperature at point x and time t. In the radiative transfer boundary value problems (2.1) and in equation $(2.2)_{(i)}$:

$$I^k(x,t,v) := \int\limits_{\lambda_k}^{\lambda_{k+1}} I(x,t,v,\lambda)d\lambda \tag{2.3}$$

denotes the radiative intensity in the wavelength band $[\lambda_k, \lambda_{k+1}[$. In the integral (2.3), $I(x,t,v,\lambda)$ denotes the spectral radiative intensity at point $x \in \Omega$, time $t \in [0,t_f]$, (oriented) direction $v \in V := S_2$ the unit sphere in $\mathbb{R}^3$, and wavelength λ [46]. Similarly $B_g^k(T(x,t)) := n_g^2 \int\limits_{\lambda_k}^{\lambda_{k+1}} B(T(x,t),\lambda)d\lambda$, where $B(T,\lambda) := \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)}$ is the famous Planck function (see e.g. [46], [68]), giving the spectral radiative intensity at wavelength λ of any black body at absolute temperature T, which in this case is independent of the direction v. C_1 and C_2 are positive constants whose values are e.g. recalled in [50] and n_g denotes the refractive index of the semi-transparent material (for soda-lime-silicate glass: $n_g \approx 1.46$ [60]). For mathematical commodity, in [50], the authors have extended the domain of definition of the Planck function to all real number T by setting $B(T,\lambda) := 0$, $\forall T \leq 0$. In the boundary condition $(2.2)_{(ii)}$, ν denotes the unit exterior normal field along the boundary of Ω, and ν_x its value at point $x \in \partial\Omega$.

ρ_g in the boundary condition $(2.1)_{(ii)}$ denotes the reflectivity coefficient: it is a positive non-increasing continuous function from the interval $[0,1]$ into itself equal to 1 near 0. For more details about this function see Chapter 3. Let us remark in the boundary condition $(2.1)_{(ii)}$, that the expression $\rho_g(|\nu_x \cdot v|)$ depends not only on x but also on v, which is not the case in ([19], see (33) p.258). Also, our problem does not fit in the examples considered in [62, p.4]. Let us mention, that for our nonhomogeneous boundary conditions in (2.1) and (2.2) to be physically realistic, we should suppose our domain Ω to be convex. However, in this chapter, we will study the coupled problem (2.1)-(2.2) in an arbitrary bounded domain Ω of $\mathbb{R}^3$ of class C^1, our boundary conditions in (2.1) and (2.2) having sense with this regularity hypothesis on Ω.

Additional details concerning problem (2.1) (resp. (2.2)), will be given in subsection 3.2.1 (resp. subsection 3.3.1), in particular concerning the nomenclature.

Many related approximated problems have been studied in the literature, mainly to reduce the computational complexity of the original problem (2.1)-(2.2), considering instead of the radiative boundary value problems (2.1) for the radiative intensities $I^k(x,t,v)$, approximated elliptic boundary value problems for the quantities of interest in (2.2), the incident radiations [46] $\rho^k(.,.) := \int\limits_V I^k(.,.,v)d\mu(v)$ ($d\mu$ denoting the area measure on $V \equiv S_2$). Indeed, these quantities multiplied by κ_k, are the volumic sources of heat per unit of time, appearing in the nonlinear heat conduction equation $(2.2)_{(i)}$. In particular, P_N approximations by spherical harmonics of $v \longmapsto I^k(x,t,v)$ have been considered by many authors [46]. Simplified P_N approximations with appropriated boundary conditions derived from variational principles, the SP_N methods of nuclear reactor theory [70], [4], have been introduced in the field of radiative heat transfer in [40], and [38], but these approximated methods are only valid for optically thick body Ω. In particular, the SP_1-method whose original idea is the approximation of $I^k(x,t,\mathbf{v})$ by an affine function in the variable $\mathbf{v}$ ([46], p.473): $I^k(x,t,\mathbf{v}) \approx a(x,t) + \mathbf{b}(x,t)\cdot\mathbf{v}$, with $a(x,t) \in \mathbb{R}$ and $\mathbf{b}(x,t) \in \mathbb{R}^3$, has been used in many papers [39], [14],..., to cite only a few. In [64], in the study of the thermal cooling of a circular thin glass disk, the radiative intensities have been computed by tracing thermal rays back to a far away point, which has a negligible influence on the calculated radiative intensity. These authors show numerically, that even for thin glass, internal radiation emitted or absorbed inside the glass, represented by the term $-\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{M}\kappa_k \int_V I^k(x,t,v)d\mu(v)$ in the heat conduction equation $(2.2)_{(i)}$, can not be ignored (a similar observation has been made for slabs of glass in [48]). In [56], the author considers the exact radiative transfer boundary value problems (2.1) coupled with the heat conduction initial boundary value problem (2.2) written in cylindrical coordinates to simulate the cooling process of an axisymmetric quartz glass tube. The numerical resolution by the discrete ordinates method, combined with the SUPG finite element scheme, of 3-D radiative transfer equations with specular reflection at the boundary, containing (2.1) as a particular case, has been studied in [35], but no coupling with the heat conduction equation is considered in that paper. In 1-D, i.e. for slabs, the mathematical analysis of a related coupled problem to (2.1)-(2.2) has been made in [50], a priori error estimates for the semi-discrete problem have been derived in [48] and the mathematical analysis of the optimal control problem for the temperature inside the slab by the temperature $T_S(.)$ of the radiative source has been made in [49].

However, to our best knowledge, the mathematical analysis of the 3-D coupled exact problem (2.1)-(2.2) has not been done yet. Hence the purpose of this chapter is to fill this gap and to prove existence and uniqueness results for the exact problem (2.1)-(2.2). We give also uniform bounds on the solution in terms of bounds on the data which are the constant ambient temperature of the surrounding dry air T_a appearing in the boundary condition $(2.2)_{(ii)}$, the initial condition $T_0(\cdot)$ for the absolute temperature in Ω $(2.2)_{(iii)}$, and the absolute temperature $T_S(\cdot)$ versus time of the black radiative source surrounding Ω. As a consequence, we obtain also bounds on the radiative intensities (see Corollary 3.4), which could be useful in defining a stopping criterion, when using the backtracking method explained in [64] to solve numerically our boundary value problems (2.1).

In section 2, we prove for each $k = 1,\ldots,M$, the existence and uniqueness of the solution in the space $W^p(\Omega \times V)$ [19] (denoted H_p^1 in [1], Chapter 2), of the radiative transfer boundary value problem (2.1) with the nonhomogeneous boundary condition $(2.1)_{(ii)}$. In particular, we use the trace theorems of M. Cessenat for the elements of the space $W^p(\Omega \times V)$ [12], [13], [19, pp.252-253], and the theory of dissipative operators.

Among the main steps, is in subsection 2.3, the proof of the dissipativeness and closedness of the operator $A = -v \cdot \nabla_x$ defined by (3.8). The proof of the dissipativeness of A is rather subtle and relies on the fact that the reflectivity coefficient $\rho_g(\cdot)$ is equal to 1 near 0, which implies that $\rho_g(|\nu_x \cdot v|) = 1$ if $|\nu_x \cdot v|$ is sufficiently small, the so-called "property of total reflectivity for grazing incidences" in Optics. In subsection 2.4, we prove that the range of $\lambda I - A$ is equal to $L^p(\Omega \times V)$, $\forall \lambda > 0$, without using the abstract result Theorem 3 in [19, p.254] as its condition (22) seems difficult to verify. Let us mention, that the paper [62], would not give in the L^2-context, the explicit form of the domain (3.8) of the generator A, but only the exact domain of a pregenerator [62, p.9].

In section 3, we firstly prove the uniqueness of the solution $T(.,.)$ of the coupled problem (2.2)-(2.1) in the Sobolev space $W(0, t_f)$ [18]. Then, with the goal to prove the existence of the solution $T(.,.)$, supposing the data bounded, we define an appropriated closed convex subset S of $L^2(0, t_f; L^2(\Omega))$ defined by:

$$S := \{T \in L^2(0, t_f; L^p(\Omega)); \forall' t \in]0, t_f[: \forall' x \in \Omega : T_a \leq T(x, t) \leq T_S(t)\}.$$

, and also a fixed point problem by replacing in (2.2) T by $\tilde{T}$, except in the volumic source ot heat term $h_T(x,t) := \sum_{k=1}^{M} \kappa_k \int_V I_T^k(x, t, v) d\mu(v)$. This allows us to decouple our problem (2.2)-(2.1), reducing us at the same time to a semilinear parabolic initial boundary value problem. We prove that for all $T \in S$, this fixed point problem has a unique solution $\tilde{T} \in S$, and that under the hypothesis $T_S \in H^1(0, t_f)$ and $\frac{dT_S}{dt} \geq 0$ a.e on $]0, t_f[$, the mapping $\Phi : T \mapsto \tilde{T}$ is continuous from S into S with its range relatively compact in $L^2(Q)$. The proof that the mapping Φ operates in S requires to establish some kind of "maximum principles" for our fixed point problem (see subsection (3.3.4)). The proof of the continuity of Φ relies on the uniform bounds on the radiative intensities established in Propositions 3.11 and 3.10, in conjunction with the a priori estimate (7.33) of [71, p.377] for semilinear parabolic initial-boundary value problems. Using Schauder's theorem, we conclude to the existence of a weak solution to the coupled problem (2.2)-(2.1), obtaining moreover uniform bounds on the solution, and on the radiative intensities.

We have improved the result by removing the condition $\frac{dT_S}{dt} \geq 0$ on T_S. The proof proceeds by considering a new closed convex subset $\mathcal{C}$ in $L^2(Q) = L^2(0, t_f; L^2(\Omega))$ where $Q := \Omega \times]0, t_f[$ by

$$\mathcal{C} := \{T \in L^2(Q); \forall' t \in]0, t_f[: \forall' x \in \Omega : \underline{T} \leq T(x, t) \leq \bar{T}\}.$$

where the fixed numbers $\underline{T}$, $\bar{T} \in \mathbb{R}_+^*$ satisfy

$$\begin{cases} \underline{T} < \bar{T}, \ \underline{T} \leq T_a \leq \bar{T}, \\ \underline{T} \leq T_0(x) \leq \bar{T}, \ \textit{for a.e. } x \in \Omega, \\ \underline{T} \leq T_S(t) \leq \bar{T}, \ \textit{for a.e. } t \in]0, t_f[. \end{cases} \tag{2.4}$$

and we prove the existence of a weak solution to problem (2.2) belonging to $\mathcal{C}$

Addendum: On June 25, 2018, further to an e-mail of ResearchGate, we have learned about related papers of the Russian Mathematician Andrey A. Amosov working at the National Research University "Moscow Power Engineering Institute ". In particular, we have heard of his work entitled: "Unique Solvability of a Nonstationary Problem of Radiative-Conductive Heat Exchange in a System of Semitransparent Bodies "published in the Russian Journal of Mathematical Physics in 2016 Vol. 23, No. 3, pp.309-334. Let us mention that our work [51] related to chapter 3 of this thesis was already submitted in 2016 to the Journal "Mathematical Methods in the Applied Sciences ", more precisely on August 26, 2016. This article is closely related to chapter 3 of our thesis, but the methods we have used, are fortunately different from those used by A.A. Amosov. Moreover, concerning the equation of conduction of heat, A.A. Amosov considers a homogeneous Neumann boundary condition (see formula (1.3) p.311 of his article), thus in particular without the integral term of our boundary condition ($2.2_(ii)$), a term that takes into account the wavelengths $\lambda \geq \lambda_0$ for which the glass behaves like an opaque body. The proof of the existence of the solution in this article is based on Galerkin's method (see Theorem 7.1 p.327). To demonstrate the existence of the solution to our boundary value problems (2.1) for the radiative transfer equation (RTE) the author refers to one of his previous articles of 2013 published in the Journal of Mathematical Sciences United States, Vol. 191, $N^0$2, May 2013, pp.101-149 entitled "Boundary Value Problem for the Radiation Transfer Equation with Reflection and Refraction Conditions "; his proof proceeds by a fixed point method, reducing the resolution to a sequence of inhomogeneous Dirichlet problems on Γ_- for the RTE, whose limit is the solution of the boundary value problem for the RTE with the reflectivity boundary condition. This method seems to be related to the backward ray tracing method. On the contrary, our proof is based on the establishment of the maximal-dissipativity of the operator A for the RTE whose domain is defined by the homogeneous reflectivity boundary condition.

2.2.2 Chapter 4: Control of the Radiative Heating of a Semi-Transparent Body

In Chapter 4, I have worked on extending to $3-d$ the $1-d$ result of AFMA (June 2016) [49] which is to find the most adequate temperature $T_S(t), t \in [0, t_f]$ of the black radiating source S, during the heating process of a semi-transparent body Ω of class C^1 of $\mathbb{R}^3$, in order to obtain the temperature $T(x,t)$ of the semi-transparent body Ω at point x during the fixed time heating $0 < t < t_f$, as near as possible, to a given desired temperature distribution $(x,t) \mapsto T_d(x,t)$. We take also into account the cost of the control T_S. So, we consider the cost functional J defined by:

$$J : L^2(Q) \times H^1(]0,t_f[) \longrightarrow \mathbb{R} : (T,T_S) \mapsto \frac{1}{2}\int\limits_Q (T(x,t) - T_d(x,t))^2 dx \otimes dt \\ + \frac{\delta_r}{2}\|T_S - T_{S,d}\|^2_{H^1(]0,t_f[)}, \tag{2.5}$$

subject to the non linear heat conduction equation (2.2), and $T_S(.)$ to belong to the set of admissible controls that we now define. Firstly, we define our control space to be $U := H^1(]0,t_f[)$ endowed with H^1-norm and we define the closed subset of admissible controls:

$$U_{ad} := \{T_S \in H^1(]0,t_f[); \underline{T} \le T_S(t) \le \bar{T},\ \forall t \in]0,t_f[\}. \tag{2.6}$$

where $\underline{T}$ and $\bar{T}$ satisfy (2.4).

U_{ad} is a closed convex subset of the Hilbert space U. We prove in the Chapter 3 [51], that for $T_S \in U_{ad}$ the initial boundary value problem (2.2) possesses a unique bounded weak solution

$$T(T_S) \in \{T \in L^2(0,t_f;H^1(\Omega)), \frac{dT}{dt} \in L^2(0,t_f;(H^1(\Omega))^*)\}. \tag{2.7}$$

This allows us to define the reduced cost functional:

$$\hat{J} : U_{ad} \to \mathbb{R} : T_S \longmapsto J(T(T_S),T_S). \tag{2.8}$$

In section 3, we prove the existence of an optimal control that is of a $T_S \in U_{ad}$ such that $\hat{J}(T_S) := \inf\limits_{v \in U_{ad}} \hat{J}(v)$.
Then, in section 4 we choose the state space E and the constraining mapping $(T,T_S) \mapsto e(T,T_S)$ in order for the constraining mapping to be Fréchet differentiable and its derivative with respect to the temperature T at a point (T,T_S) to be invertible. We define the state space E as the set of all

$$T \in \left\{T \in L^2(0,t_f;H^1(\Omega)); \frac{dT}{dt} \in L^2(0,t_f;(H^1(\Omega))^*)\right\} \cap C(\bar{Q})$$

such that $(c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T) \in L^r(Q)$ and $k_h \frac{\partial T}{\partial \nu} \in L^{s^*}(\Sigma)$ in the sense of distributions, where $r \in]2.5, 2.72[$ and $s^* > 4$. Now, we define the constraining mapping by:

$$e : E \times U \longrightarrow L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$$
$$(T,T_S) \longmapsto e(T,T_S) = (e_1,e_2,e_3)(T,T_S)$$

where

$$\begin{aligned} e_1(T,T_S) &= c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T - \psi(T) - h(T,T_S), \\ e_2(T,T_S) &= k_h \frac{\partial T}{\partial \nu} + h_c T + \Theta(T) - \Theta(T_S), \\ e_3(T,T_S) &= T(\cdot,0) = T_0(\cdot), \end{aligned} \tag{2.9}$$

and the quantities $\psi(T), h(T,T_S), \Theta(T)$ and $\Theta(T_S)$ are defined by the formulas:

$$\begin{cases} \psi(T) := -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T), \ \forall T \in \mathbb{R}, \\ h_{T,T_S}(x,t) := \sum_{k=1}^{M} \kappa_k \int_V I_{T,T_S}^k(x,t,v)d\mu(v), \\ \Theta(T) := \pi \int_{\lambda_0}^{+\infty} \varepsilon_\lambda B(T,\lambda)d\lambda, \ \forall T \in \mathbb{R}. \end{cases} \tag{2.10}$$

Using the constraining mapping, the state equation writes down: $e(T,T_S) = (0, h_c T_a, T_0)$. In section 5, we apply the implicit function theorem in order to compute the derivative of the reduced cost functional $\hat{J}(T_S)$ at a control $T_S \in H^1(]0,t_f[)$ and so to give the first-order necessary optimality condition for a control T_S to be optimal with respect to the cost functional J which is

$$\begin{aligned} &\sum_{k=1}^{M} \kappa_k \int_0^{t_f} \Big\{ \int_\Omega \Big[\Big(4\pi - \int_V \kappa_k(\kappa_k - A)^{-1} 1_{\Omega\times V} d\mu(v)\Big) \eta_{Vol}(T_{T_S}; x,t) \Big] \ dx \Big\} \\ &\frac{dB_g^k}{dT}(T_S(t))(v(t) - T_S(t))dt + \int_0^{t_f} (\int_{\partial\Omega} \eta_B(T_{T_S}; x,t)dS(x)) \frac{d\Theta}{dT}(T_S(t))(v(t) - T_S(t))dt \\ &+\delta_r \int_0^{t_f} (T_S(t) - T_{S,d}(t))(v(t) - T_S(t))dt + \delta_r \int_0^{t_f} (\dot{T}_S(t) - \dot{T}_{Sd}(t))(\dot{v}(t) - \dot{T}_S(t))dt \geq 0, \forall v \in U_{ad}, \end{aligned} \tag{2.11}$$

where η_{Vol} is the solution in the weak sense of the backward parabolic boundary value problem (4.39) and η_B is the restriction of η_{Vol} to $\Sigma = \partial\Omega \times [0,t_f]$.

In a $1-d$ case in space, in [49] a control problem of this type is studied, using among others in that case, an explicit integral formula for h_{T,T_S} ([49], formula (6)).

2.2.3 Chapter 5: The Thermoviscoelasticity Equation

Until now, we have studied the problem of the radiative heating of the glass body considered as an open bounded set Ω in $\mathbb{R}^3$ with a boundary of class C^1 at least. We have also studied the related optimal control problem.

After that, we come to the goal problem which is to deform the volume of glass Ω by heating it with a black radiative source all around it. For example, glass sagging, which consists in heating glass until a temperature at which its viscosity is low enough to allow it to sag under its own weight due to gravity, is used to process glass industrial products such as windscreens, mirrors or lenses.

We study the existence and uniqueness of the deformation of Ω, resulting from the radiative heating of the semi-transparent body Ω by the black-source S. In the setting of the viscoelasticity theory, M. E. Rognes and R. Winther have considered in [59], a weak Hellinger-Reissner formulation for the Maxwell model of viscoelasticity, which presents the advantage to avoid the complicated integrodifferential equations linking the tensorfield of stress $(x,t) \longmapsto \sigma(x,t)$ to the tensorfield of the rate of deformations. In this formulation, the unknowns are the stress field $(x,t) \longmapsto \sigma(x,t)$, the velocity field $(x,t) \longmapsto \dot{\mathbf{u}}(x,t)$ of the displacement field $(x,t) \longmapsto \mathbf{u}(x,t)$ (we will also denote $\dot{\mathbf{u}}$ by $\mathbf{v}$) and the skew component $(x,t) \longmapsto \rho(x,t)$ of the gradient of $\mathbf{u}$. The novelty, is that here, we consider the thermoviscoelasticity problem instead of the viscoelasticity problem [59], so that we have now a whole family of operators $A_0(.,t)$ indexed by the time $t \in [0,t_f]$:

$$\begin{array}{c} A_0(.,t) : [L^2(\Omega)]^{3\times 3} \to [L^2(\Omega)]^{3\times 3} \\ \sigma \longmapsto \frac{1+\nu}{\eta(T(.,t))}\sigma - \frac{\nu}{\eta(T(.,t))} tr(\sigma) I_3, \end{array} \tag{2.12}$$

where $\eta(T(x,t))$ denotes the viscosity at point $x \in \Omega$ and at time $t \in [0,t_f]$, which depends on the absolute temperature $T(x,t)$, and I_3 denotes the identity matrix of order 3. On the contrary, in the paper of M.E. Rognes and R. Winther ([59], p. 963), it is supposed that A_0 is independent of the time t. We suppose that the viscosity $\eta(.)$ is a strictly positive decreasing function, defined and C^1 on $\mathbb{R}_+^*$.

We denote by E the Young modulus and by ν the Poisson coefficient $(0 < \nu < \frac{1}{2})$. Denoting by $\varepsilon(u_E)$ the linearized strainfield corresponding to the elastic displacement field u_E, we have:

$$\varepsilon(u_E) = \frac{1+\nu}{E}\sigma - \frac{\nu}{E} tr(\sigma) I_3,$$

where σ denotes the stressfield. The rate of deformations corresponding to the rate of the visco displacements u_V, is given by:

$$\varepsilon(\dot{u}_V) = \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)} tr(\sigma) I_3.$$

Now, the total displacement field $u = u_E + u_V$. This implies that:

$$\begin{array}{c} \varepsilon(\dot{u}) = \varepsilon(\dot{u}_E) + \varepsilon(\dot{u}_V) \\ = \frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E} tr(\dot{\sigma}) I_3 + \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)} tr(\sigma) I_3. \end{array}$$

Thus, we have the equation linking the time derivative of the displacement field u, to the stressfield σ:

$$\varepsilon(\dot{u}) = \frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E}tr(\dot{\sigma})I_3 + \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)}tr(\sigma)I_3, \tag{2.13}$$

often called the Maxwell constitutive law. To that equation, we join the equilibrium equation:

$$-\operatorname{div}(\sigma) = g - \beta\ \nabla T, \tag{2.14}$$

where we have neglected the inertial forces i.e. we consider the "quasi-static "problem. $\beta := \frac{E}{1-2\nu}\alpha$, where α denotes the linear coefficient of dilatation. g denotes a volumic forcefield acting on Ω, for example its volumic weight. We suppose at least that $g \in L^2(Q)^3$, where Q denotes the "cylinder" $\Omega\times]0,t_f[$ in the space-time $\mathbb{R}^3 \times \mathbb{R}$. T being the weak solution of the heat conduction equation (2.2), $T \in L^2(]0,t_f[;H^1(\Omega))$ and thus $\nabla T \in L^2(Q)^3$. To these two equations, the Maxwell constitutive law (2.13) and the equilibrium equation (2.14), we join an initial condition for the stressfield σ:

$$\sigma(.,0) = \zeta. \tag{2.15}$$

Lastly, the boundary $\partial\Omega$ of the domain Ω is partitioned into two disjoint open subsets Γ_D and Γ_N of $\partial\Omega$ such that $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$. Along Γ_D we impose the velocity field of displacements $\dot{u}$ to be 0, and on Γ_N we impose no tractions i.e. $(\sigma.\nu)_{|\Gamma_N} = 0$ where ν denotes the unit normal field along $\partial\Omega$ pointing outward of Ω . We now write a weak formulation for these two equations and boundary conditions. Considering a test stressfield $\tau \in H(\operatorname{div};\Omega)^3$ such that $(\tau.\nu)_{|\Gamma_N} = 0$, and we have by Green's formula:

$$\int_\Omega \varepsilon(v) : \tau\ dx = \langle \tau.\nu, v\rangle_\Gamma - \int_\Omega \operatorname{div}(\tau).v\ dx - \int_\Omega \rho(v) : \tau\ dx, \tag{2.16}$$

for every $v \in H^1(\Omega)^3$, where $\rho(v) := \frac{1}{2}(\nabla v - (\nabla v)^T)$. Applying Green's formula (2.16) to $\varepsilon(\dot{u})$, and using the boundary condition $\dot{u}_{|\Gamma_D} = 0$, we obtain from the Maxwell constitutive law (2.13):

$$\begin{aligned} \int_\Omega [\tfrac{1+\nu}{\eta(T)}\sigma - \tfrac{\nu}{\eta(T)}tr(\sigma)I_3] : \tau\ dx + \int_\Omega [\tfrac{1+\nu}{E}\dot{\sigma} - \tfrac{\nu}{E}tr(\dot{\sigma})I_3] : \tau\ dx \\ + \int_\Omega \operatorname{div}(\tau).\dot{u}\ dx + \int_\Omega \rho(\dot{u}) : \tau\ dx = 0, \end{aligned} \tag{2.17}$$

$\forall \tau \in H(\operatorname{div};\Omega)^3$ such that $(\tau.\nu)_{|\Gamma_N} = 0$. For almost every $t \in]0,t_f[$, this equation expresses in a weak form the Maxwell constitutive equation (2.13) and the boundary condition $\dot{u}_{|\Gamma_D} = 0$ for the velocity of displacements $\dot{u}$. Let us note, that formally at least, if $u(.,.)$ satisfies at time 0, $u(.,0)_{|\Gamma_D} = 0$, then we will have $u(.,t)_{|\Gamma_D} = 0$, $\forall' t \in]0,t_f[$. The boundary condition $(\sigma.\nu)_{|\Gamma_N} = 0$ is an essential one in the mixed formulation and is imposed in the definition of the functional space $H_{\Gamma_N}(\mathrm{div};\Omega)^3$: we try to find

$$\sigma(.,t) \in H_{\Gamma_N}(\mathrm{div};\Omega)^3,$$

where $H_{\Gamma_N}(\mathrm{div};\Omega)$ denotes the closed subset of vectorfields in $H(\mathrm{div};\Omega)$ whose normal traces are 0 on Γ_N. Now to the weak form of the Maxwell constitutive law, we must add a weak form of the equilibrium equation (2.14):

$$\int_\Omega \mathrm{div}(\sigma).w\ dx + \int_\Omega (g - \beta\ \nabla T).w\ dx = 0,\ \ \forall w \in \left[L^2(\Omega)\right]^3. \tag{2.18}$$

We must moreover express in a weak form $\forall' t \in]0, t_f[$, the symmetry of the stress tensor $\sigma(.,t)$, so that we obtain the equation:

$$\begin{gathered}\int_\Omega \mathrm{div}(\sigma).w\ dx + \int_\Omega (g - \beta\ \nabla T).w\ dx + \int_\Omega \sigma : \xi\ dx = 0,\ \ \forall w \in L^2(\Omega)^3,\\ \forall \xi \in [L^2(\Omega)]^{3\times 3}_{skew},\end{gathered} \tag{2.19}$$

where

$$\left[L^2(\Omega)\right]^{3\times 3}_{skew} := \{\xi \in \left[L^2(\Omega)\right]^{3\times 3}; \xi + \xi^T = 0\}.$$

To these weak forms (2.17) (resp. (2.19)), of the equations (2.13) (resp. (2.14)), we must join the initial condition (2.15). Supposing that $g \in H^1(0, t_f; L^2(\Omega)^3)$, that the initial temperature T_0 in Ω is equal to the ambient temperature T_a, that T_S belongs to $H^2(]0, t_f[)$ and satisfies $T_S(0) = T_a$, and that the initial condition ζ for σ verifies $\zeta \in H_{\Gamma_N}(\mathrm{div};\Omega)^3$, $\zeta = \zeta^T$ and $\mathrm{div}\,\zeta = -(g(.,0) - \beta \nabla T(.,0))$, we prove that there exists a unique triple

$$(\sigma, v, \rho) \in H^1(0, t_f; H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0, t_f; L^2(\Omega)^3) \times L^2(0, t_f; \left[L^2(\Omega)\right]^{3\times 3}_{skew})$$

such that $\sigma(.,0) = \zeta$ and such that $\forall' t \in]0, t_f[$ equations (2.17) and (2.19) are verified i.e.:

$$\left\{\begin{gathered}\int_\Omega [\tfrac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \tfrac{\nu}{\eta(T(.,t))} tr(\sigma(.,t)) I_3] : \tau\ dx + \int_\Omega [\tfrac{1+\nu}{E}\dot\sigma(.,t) - \tfrac{\nu}{E} tr(\dot\sigma(.,t)) I_3] : \tau\ dx\\ + \int_\Omega \mathrm{div}(\tau).v(.,t)\ dx + \int_\Omega \rho(.,t) : \tau\ dx = 0,\ \ \forall \tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3,\\ \int_\Omega \mathrm{div}(\sigma(.,t)).w\ dx + \int_\Omega (g(.,t) - \beta\ \nabla T(.,t)).w\ dx + \int_\Omega \sigma(.,t) : \xi\ dx = 0,\ \ \forall w \in L^2(\Omega)^3,\\ \forall \xi \in [L^2(\Omega)]^{3\times 3}_{skew}.\end{gathered}\right.$$

2.2.4 Chapter 6: Control of the Displacements in the Setting of the Thermoviscoelasticity

In this chapter, we study the control of the deformation of Ω, resulting from the radiative heating of the semi-transparent body Ω by the black radiative source S. Firstly, we introduce the control problem.

We define $\forall t \in]0,t_f[$: $\vec{u}(t) := \int_0^t v(s)\, ds$. By Ascoli's theorem ([20], p.143), the mapping which sends $v \mapsto \vec{u}$ from $L^2(0,t_f;H^1(\Omega)^3)$ into the space $C([0,t_f];L^2(\Omega)^3)$ is compact. Thus if we have a sequence $(v_n)_{n\geq 1}$ which converges weakly in $L^2(0,t_f;H^1(\Omega)^3)$, the corresponding sequence $(\vec{u}_n)_{n\geq 1}$ will converge strongly in $C([0,t_f];L^2(\Omega)^3)$ and a fortiori in $L^2(0,t_f;L^2(\Omega)^3)$. Now, we define the new set of admissible controls:

$$U_{ad} := \{T_S \in H^2(]0,t_f[); \underline{T} \leq T_S(t) \leq \bar{T},\ \forall t \in]0,t_f[, T_S(0) = T_a\}$$

and the cost functional

$$\begin{aligned} &J : [L^2(Q)]^3 \times H^2(]0,t_f[) \to \mathbb{R} \\ &\qquad (\vec{u}, T_S) \mapsto \frac{1}{2}\int_Q |\vec{u}(x,t) - \vec{u}_d(x,t)|^2\ dx \otimes dt + \frac{\tilde{\delta}_r}{2} \left\| T_S - T_S^d \right\|^2_{H^2(]0,t_f[)}, \end{aligned}$$

where $\vec{u}_d(.,.) \in [L^2(Q)]^3$ denotes the given desired field of displacements and $T_S^d \in H^2(]0,t_f[)$ a given evolution of the absolute temperature of the black source S to which T_S should not be "too far", the meaning of this "too far" being modulated by the strictly positive coefficient $\tilde{\delta}_r$ in the definition of the cost functional J, coefficient which is allowed to be chosen very small. As we say above, $\underline{T}$ and $\bar{T}$ denote two strictly positive real numbers satisfying conditions $(2.4)_{(i)}$ and $(2.4)_{ii}$. Let us observe that U_{ad} is a closed convex subset of $H^2(]0,t_f[)$. We prove that the mapping from the open set $\tilde{U}_{ad} := \{T_S \in H^2(]0,t_f[); \frac{\underline{T}}{3} < T_S(t) < 2\bar{T}, \forall t \in]0,t_f[, T_S(0) = T_a\}$ into $L^2(0,t_f;H^1_D)$ which sends T_S on $\vec{u}$ is continuously Fréchet differentiable. As a consequence the reduced cost functional

$$\begin{aligned} \hat{J} : \tilde{U}_{ad} &:= \{T_S \in H^2(]0,t_f[); \tfrac{\underline{T}}{2} < T_S(t) < 2\bar{T},\ \forall t \in]0,t_f[, T_S(0) = T_a\} \to \mathbb{R} \\ T_S &\mapsto \tfrac{1}{2}\int_Q |\vec{u}(T_S)(x,t) - \vec{u}_d(x,t)|^2\ dx \otimes dt + \tfrac{\tilde{\delta}_r}{2} \left\| T_S - T_S^d \right\|^2_{H^2(]0,t_f[)} \end{aligned}$$

is also continuously Fréchet differentiable. In particular, if $\overline{T_S} \in U_{ad}$ is an optimal control, it satisfies the variational inequality:

$$\hat{J}'(\overline{T_S})(T_S - \overline{T_S}) \geq 0,\ \forall T_S \in U_{ad}.$$

.

3

Coupling the Heat Conduction Equation with The Radiative Transfer Equation

Contents

3.1 Introduction

We consider the quasi-static problem for the radiative transfer equation (RTE) in a bounded semi-transparent body Ω of class C^1 of $\mathbb{R}^3$ exposed to the radiations of a black source S at absolute temperature $T_S(t)$ at time t and subjected to a nonhomogeneous reflectivity boundary condition, coupled with a nonlinear heat conduction evolution equation whose heat source is the absorbed thermal radiation and cooling also by emitting thermal radiations, equation subjected to a nonlinear boundary condition of the Robin type. For the RTE equation with the nonhomogeneous reflectivity boundary condition, the problem is posed in the Banach space $L^p(\Omega \times V)$ $(1 < p < +\infty)$ where V denotes the "unit sphere of directions". Using the theory of dissipative operators and the important trace theorems of M. Cessenat (1984, 1985) for the functions of the Sobolev spaces $W^p(\Omega \times V)$, we prove that the problem is well posed and that the solution is in fact essentially bounded using the arbitrariness of p $(1 < p < +\infty)$. Next, we consider the coupled problem. Firstly, we prove that the weak solution is at most unique. Then, we introduce two different closed convex subsets of $L^2(\Omega \times]0, t_f[)$, where t_f denotes the final time of radiative heating of Ω by the black source S and in each case, prove the existence of a weak solution belonging to it, by defining a fixed point problem and using Schauder's fixed point theorem.

3.2 Solvability of the RTE with the nonhomogeneous specular reflectivity boundary condition

3.2.1 Notations and scope of this section

In the remainder of this chapter, Ω will denote a bounded domain of class C^1 of $\mathbb{R}^3$, situated locally on one side of its boundary $\partial\Omega$. Let us recall that $V := S_2$ denotes the "unit sphere of directions in $\mathbb{R}^3$". We denote by dx the 3-d Lebesgue measure on Ω and by $d\mu$ the area measure on $V \equiv S_2 \subset \mathbb{R}^3$. Let us fix some $k \in \{1, \ldots, M\}$. Our purpose in this section is to prove that the quasi-steady boundary value problem for the radiative transfer equation (RTE):

$$v \cdot \nabla_x I^k(x,t,v) + \kappa_k I^k(x,t,v) = \kappa_k B_g^k(T(x,t)), \ \textit{a.e. in } \Omega \times V \tag{3.1}$$

with the nonhomogeneous specular reflectivity boundary condition

$$I^k(x,t,v) = \rho_g(|\nu_x \cdot v|) I^k(x,t,v_i) + (1 - \rho_g(|\nu_x \cdot v|)) B_g^k(T_S(t)), \ \textit{a.e. on } \Gamma_- \ , \tag{3.2}$$

is well posed in appropriate Banach spaces that will be precised later on. As in [19], Γ denotes $\partial\Omega \times V$,

$$\Gamma_+ := \{(x,v) \in \partial\Omega \times V; \ \nu_x \cdot v > 0\} \tag{3.3}$$

and

$$\Gamma_- := \{(x,v) \in \partial\Omega \times V; \ \nu_x \cdot v < 0\}. \tag{3.4}$$

In these last definitions, ν_x denotes the unit exterior normal vector at $x \in \partial\Omega$. Γ_+ and Γ_- are open subsets of Γ. In the boundary condition (4.12), $v_i = v_i(x, v)$ denotes the "incident vector" at point x on the "interior face" of the boundary $\partial\Omega$ admitting v as reflected vector i.e. $(x, v_i) \in \Gamma_+$ and

$$v_i = v_i(x, v) = v - 2(\nu_x \cdot v)\nu_x. \tag{3.5}$$

Let us observe that $\nu_x \cdot v_i = -\nu_x \cdot v$ and thus $|\nu_x \cdot v_i| = |\nu_x \cdot v|$. $\rho_g(\cdot)$ denotes the "reflectivity coefficient" at the interface of Ω with the ambient surrounding. This coefficient is given by Fresnel's relation (formula (2.95), [46, p.47]) in conjunction with Snell-Descartes' refraction law (formula (2.71), [46, p.44]). We will only retain that ρ_g is a positive, continuous, nonincreasing function, defined on the interval $[0, 1]$, taking values ≤ 1 and the constant value 1 on the subinterval $[0, \sqrt{\frac{n_g^2-1}{n_g^2}}]$.

Problem (3.1)-(3.2) will be set in the Banach space [19] (denoted H_p^1 in [1], Chapter 2):

$$W^p(\Omega \times V) = \{u \in L^p(\Omega \times V);\ v \cdot \nabla_x u = \mathrm{div}_x(u\ v) \in L^p(\Omega \times V)\}, \tag{3.6}$$

p being an arbitrary real number in $]1, +\infty[$. The full range of p is needed in the proof of Corollary 3.2 which yields that under certain hypotheses on the data, the solution of (3.1)-(3.2) is essentially bounded on $\Omega \times V$. In the definition of the space $W^p(\Omega \times V)$, $v \cdot \nabla_x u$ must be understood in the generalized sense as the unique element of the space $L^p(\Omega \times V)$ verifying:

$$< v \cdot \nabla_x u, \varphi >= -\iint\limits_{\Omega\times V} u(x, v)\ v \cdot \nabla_x \varphi(x, v)\ dx d\mu(v),\ \forall \varphi \in \mathcal{D}(\Omega \times V), \tag{3.7}$$

if it exists. An equivalent definition of $v \cdot \nabla_x u$ is given in [1, p.38] using less regular test functions. In this chapter the symbol ∇_x (resp. div_x), will always denote the gradient (resp. the divergence) with respect to the x variable only. The time variable t, plays the role of a parameter in equation (3.1) and its related boundary condition (3.2), and may be considered as fixed to some positive arbitrary value between 0 and t_f.

Firstly, we will consider the unbounded operator A defined in $L^p(\Omega \times V)$ by:

$$\begin{aligned} D(A) = \{u \in L^p(\Omega \times V);\ v \cdot \nabla_x u \in L^p(\Omega \times V),& \\ u(x, v) = \rho_g(|\nu_x \cdot v|)u(x, v_i),\ \textit{for a.e. } (x, v) \in \Gamma_-\},& \\ A : D(A) \to L^p(\Omega \times V) : u \longmapsto -v \cdot \nabla_x u.& \end{aligned} \tag{3.8}$$

The condition

$$u(x, v) = \rho_g(|\nu_x \cdot v|)u(x, v_i),\ \textit{for a.e. } (x, v) \in \Gamma_-, \tag{3.9}$$

with $v_i = v_i(x, v)$ (3.5) in the definition of $D(A)$, has to be understood in the sense of traces, knowing by [19, Corollary 1 p.220], that functions in $W^p(\Omega \times V)$ have traces in $L^p_{loc}(\Gamma_\mp)$. In subsections 2.3 and 2.4, we will show that the operator A is dissipative and closed, with dense domain, and that for all $\lambda > 0$, there exists $(\lambda I - A)^{-1} \in \mathcal{L}(L^p(\Omega \times V))$.

Taking $\lambda = \kappa_k$, we can now solve the quasi-stationary homogeneous boundary value problem for the RTE:

$$\begin{cases} v \cdot \nabla_x I^k(x,t,v) + \kappa_k I^k(x,t,v) = \kappa_k B_g^k(T(x,t)), \ \textit{for a.e.} \ (x,v) \in \Omega \times V, \\ I^k(x,t,v) = \rho_g(|\nu_x \cdot v|) I^k(x,t,v_i), \ \textit{for a.e.} \ (x,v) \in \Gamma_- \ . \end{cases} \tag{3.10}$$

Using, some change of unknown, we will deduce that the nonhomogeneous linear boundary value problem (3.1), (3.2) has a unique solution in $W^p(\Omega \times V)$ in the sense that, there exists one and only one solution $I^k(\cdot, t, \cdot)$ of (3.1) in the sense of distribution theory (3.7) and of (3.2) in the sense of traces (see Corollary 1 [19, p.220] or the sharper theorems of traces of M. Cessenat for the elements of the space $W^p(\Omega \times V)$ [12], [13], [19, pp.252-253]). We will also give uniform bounds on the radiative intensities $I^k(x,t,v)$. This will be the object of subsections 2.5 and 2.6.

Before going to the heart of this section, we shall need some preliminary lemmas and propositions.

3.2.2 Preliminary results

3.1 Lemma. *1°)* $\forall (x,v) \in \partial\Omega \times V$:

$$(v - 2(\nu_x \cdot v)\nu_x) \cdot \nu_x = -v \cdot \nu_x. \tag{3.11}$$

2°) The mapping

$$T : \partial\Omega \times V \to \partial\Omega \times V : (x,v) \mapsto (x, v - 2(\nu_x \cdot v)\nu_x) \tag{3.12}$$

is an involution.
3°) The measure $dS \otimes d\mu$ where dS (resp. $d\mu$) denotes the area measure on $\partial\Omega$ (resp. on V) is invariant by T.

Proof : The first and the second assertions are trivial. To prove the third assertion, we use Fubini's theorem. Let us consider a Borel set E contained in $\partial\Omega \times V$. Then one has:

$$(dS \otimes d\mu)(E) = \int_{\partial\Omega} \mu(\{v \in V; (x,v) \in E\}) \, dS(x)$$
$$= \int_{\partial\Omega} \mu(\{v \in V; (x,v) \in T^{-1}(E)\}) \, dS(x) = \int_{\partial\Omega} \left[\int_V \chi_{T^{-1}(E)}(x,v) \, d\mu(v) \right] \, dS(x)$$
$$= \iint_{\partial\Omega \times V} \chi_{T^{-1}(E)}(x,v) \, dS(x) d\mu(v) = (dS \otimes d\mu)(T^{-1}(E)),$$

what was to be proved, the equality from the first integral to the second one following from the fact that the mapping $v \mapsto v - 2(\nu_x \cdot v)\nu_x$ is the symmetry with respect to the vectorial plane of $\mathbb{R}^3$ orthogonal to ν_x, and thus leaves $d\mu$ invariant. $\square$

3.1 Remark. *In the following, it is important to well understand the meaning of $u \in W^p(\Omega \times V)$. Let us consider $u \in L^p(\Omega \times V)$. This implies that for a.e. $v \in V$, $u(\cdot, v) \in L^p(\Omega)$. Thus we may consider for a.e. $v \in V$, the distributional first order derivatives $\frac{\partial u(\cdot,v)}{\partial x_i} \in \mathcal{D}'(\Omega)$ of $u(\cdot, v)$ $(i = 1, 2, 3)$. $u \in W^p(\Omega \times V)$ means that in addition to $u \in L^p(\Omega \times V)$, the directional derivative in the v-direction in the sense of distributions of $u(\cdot, v)$, $\sum_{i=1}^{3} v_i \frac{\partial u(\cdot,v)}{\partial x_i} \in L^p(\Omega)$, for a.e. $v \in V$, and that the mapping $V \to L^p(\Omega) : v \mapsto \sum_{i=1}^{3} v_i \frac{\partial u(\cdot,v)}{\partial x_i}$ belongs to $L^p(V; L^p(\Omega))$. The space $W^p(\Omega \times V)$ is larger than $L^p(V; W^{1,p}(\Omega))$. For example if we take $p = 2$, and for Ω the unit ball B with center at the origin in $\mathbb{R}^3$, the function:*

$$f : \Omega \times V \to \mathbb{R} : (x, v) \mapsto \frac{1}{\sqrt{1 - \|x - (x|v)_{\mathbb{R}^3} v\|^2_{\mathbb{R}^3}}} \tag{3.13}$$

belongs to $W^2(\Omega \times V)$, but not to $L^2(V; H^1(\Omega))$.

We will need the following "Green's formula" (3.14) not only for smooth functions u, $g \in C^1(\bar{\Omega} \times V)$ (in which case this formula is well known and immediate to prove), but in the more general case in which the functions u (resp. g) are only in the space $W^p(\Omega \times V)$ (resp. $W^{p'}(\Omega \times V)$) and the product of their traces $u|_\Gamma.g|_\Gamma$ on $\Gamma := \partial\Omega \times V$ [12], [13], is only integrable with respect to the measure $|v.\nu(x)|\, dS(x)d\mu(v)$ on Γ. It is not possible to deduce Green's formula (3.14) in that general case from the "smooth case" by the density of the space $C^\infty(\bar{\Omega} \times V)$ in the spaces $W^p(\Omega \times V)$ (resp. $W^{p'}(\Omega \times V)$), using the trace theorems of [12], [13]. Indeed this argument is used in [1, Theorem 2.24], while the norm of the trace spaces denoted $\tilde{L}^p(\Gamma_\pm)$ in [1, p.40] do not allow that. In [62, p.5], for $p = 2$, "Green's formula" (3.14) is stated for $u, g \in W^2(\Omega \times V)$ such that their traces $u_{|\Gamma_+}$, $g_{|\Gamma_+} \in L^2(\Gamma_+)$, $u_{|\Gamma_-}$, $g_{|\Gamma_-} \in L^2(\Gamma_-)$ but without giving any proof. We make hereafter a weaker hypothesis than in Theorem 2.24 of [1]: we do not suppose that $u|_\Gamma \in L^p(\Gamma; |v.\nu(x)|\, dS(x)d\mu(v))$ (resp. $g|_\Gamma \in L^{p'}(\Gamma; |v.\nu(x)|\, dS(x)d\mu(v))$); we suppose only that $u|_\Gamma.g|_\Gamma \in L^1(\Gamma; |v.\nu(x)|\, dS(x)d\mu(v))$:

3.1 Proposition. *For all $u \in W^p(\Omega \times V)$ and $g \in W^{p'}(\Omega \times V)$ such that the product $u|_\Gamma.g|_\Gamma$ of their traces on $\Gamma := \partial\Omega \times V$ [12], [13], is integrable with respect to the measure $|v.\nu(x)|\, dS(x)d\mu(v)$ on Γ i.e. belongs to $L^1(\Gamma; |v.\nu(x)|\, dS(x)d\mu(v))$, we have the following Green's formula:*

$$\begin{aligned}\iint_{\Omega\times V} v \cdot \nabla_x u(x, v)\ g(x, v)dxd\mu(v) &= \iint_{\Gamma:=\partial\Omega\times V} u(x, v)g(x, v)v.\nu(x)dS(x)d\mu(v) \\ &\quad - \iint_{\Omega\times V} u(x, v)v \cdot \nabla_x g(x, v)dxd\mu(v).\end{aligned} \tag{3.14}$$

Proof : Firstly, let us remark that our hypotheses imply that for almost every $v \in V$ for the measure $d\mu(v)$, that the functions $u(\cdot,v) \in L^p(\Omega;dx)$, $v.\nabla_x u(\cdot,v) \in L^p(\Omega;dx)$, $g(\cdot,v) \in L^{p'}(\Omega;dx)$, $v.\nabla_x g(\cdot,v) \in L^{p'}(\Omega;dx)$ and also that the product of the traces of $u(\cdot,v)$ and $g(\cdot,v)$ on $\partial\Omega$ is integrable for the measure $|v.\nu(x)|\,dS(x)$ on every measurable subset of $\partial\Omega$. Let us fix such a $v \in V$. Let us denote by E_v the orthogonal vectorial 2- dimensional subspace of $\mathbb{R}^3$ to v for the ordinary scalar product of $\mathbb{R}^3$, and by P_v the orthogonal projection in $\mathbb{R}^3$ onto E_v, thus parallel to v. For $x \in \mathbb{R}^3$, we denote $P_v x \in E_v$ by x'_v and by $x_v = (x|v)_{\mathbb{R}^3}$. We will consider (x_v, x'_v) as the new coordinates of x, coordinates “adapted” to the v-direction. We proceed now like in the classical proof of the Ostrogradsky Theorem ([66], pp.216-219) (see also page 833 of [12]):

$$\begin{aligned}\int_\Omega v.\nabla_x u(x,v)g(x,v)dx &= \int_{P_v(\Omega)} dx'_v \int_{\{x_v\in\mathbb{R};(x_v,x'_v)\in\Omega\}} \frac{\partial u}{\partial x_v}(x_v,x'_v,v)g(x_v,x'_v,v)dx_v \\ &= \int_{P_v(\Omega)} dx'_v \sum_i \int_{J^i_{(x'_v,v)}} \frac{\partial u}{\partial x_v}(x_v,x'_v,v)g(x_v,x'_v,v)dx_v,\end{aligned}$$

where the $J^i_{(x'_v,v)}$ are the connected components of the bounded open subset $\{x_v \in \mathbb{R};(x_v,x'_v) \in \Omega\}$ of $\mathbb{R}$. It is easily seen that the number of such connected components is finite and uniformly upper bounded due to the fact that Ω is bounded and due to the existence of a tubular neighborhood of $\partial\Omega$ ([24], p.113) which implies that the distance between two such intervals is uniformly lower bounded by a strictly positive constant. Thus the $J^i_{(x'_v,v)}$ are open bounded subintervals of $\mathbb{R}$. For almost $x'_v \in P_v(\Omega)$, $u|_{J^i_{(x'_v,v)}} \in W^{1,p}(J^i_{(x'_v,v)}) \hookrightarrow C(\overline{J^i_{(x'_v,v)}})$ and $g|_{J^i_{(x'_v,v)}} \in W^{1,p'}(J^i_{(x'_v,v)}) \hookrightarrow C(\overline{J^i_{(x'_v,v)}})$ for every i. Let us denote by $x^{i-}(x'_v,v)$ (resp. $x^{i+}(x'_v,v)$) the left extremity (resp. the right extremity) of the interval $J^i_{(x'_v,v)}$. Thus:

$$\begin{aligned}\int_\Omega v.\nabla_x u(x,v)g(x,v)dx = &\int_{P_v(\Omega)} \sum_i [u(x^{i+}(x'_v,v),x'_v,v)g(x^{i+}(x'_v,v),x'_v,v) \\ &- u(x^{i-}(x'_v,v),x'_v,v)g(x^{i-}(x'_v,v),x'_v,v)]\ dx'_v \\ &- \int_{P_v(\Omega)} dx'_v \sum_i \int_{J^i_{(x'_v,v)}} u(x_v,x'_v,v)\frac{\partial g}{\partial x_v}(x_v,x'_v,v)dx_v \\ =&\sum_i \int_{P_v(\Omega)} u(x^{i+}(x'_v,v),x'_v,v)g(x^{i+}(x'_v,v),x'_v,v)\nu_x.v\ dS \\ &+ \sum_i \int_{P_v(\Omega)} u(x^{i-}(x'_v,v),x'_v,v)g(x^{i-}(x'_v,v),x'_v,v)\nu_x.v\ dS \\ &- \int_{P_v(\Omega)} dx'_v \int_{\{x_v\in\mathbb{R};(x_v,x'_v)\in\Omega\}} u(x_v,x'_v,v)\frac{\partial g}{\partial x_v}(x_v,x'_v,v)dx_v\end{aligned}$$

$$= \int_{\partial\Omega} u(x,v)g(x,v)\nu_x.v \; dS(x) - \int_{\Omega} u(x,v)v.\nabla_x g(x,v)dx,$$

owing to $dx'_v = |\nu_x.v| \;\; dS(x)$. Integrating both sides with respect to v on V for the measure $d\mu(v)$, we obtain:

$$\iint_{\Omega\times V} v.\nabla_x u(x,v)g(x,v)dxd\mu(v) = \iint_{\Gamma:=\partial\Omega\times V} u(x,v)g(x,v)v.\nu(x)dS(x)d\mu(v)$$
$$- \iint_{\Omega\times V} u(x,v)v.\nabla_x g(x,v)dxd\mu(v),$$

by Fubini's theorem applied to each of these three terms. What was to be proved. □

3.2 Remark. *In the preceding proof, we have said that for almost* $x'_v \in P_v(\Omega)$, $u(\cdot, x'_v, v)|_{J^i_{(x'_v,v)}} \in W^{1,p}(J^i_{(x'_v,v)})$ *and* $g(\cdot, x'_v, v)|_{J^i_{(x'_v,v)}} \in W^{1,p'}(J^i_{(x'_v,v)})$ *for every* i *which implies by the injection of* $W^{1,p}(J^i_{(x'_v,v)}) \hookrightarrow C(\overline{J^i_{(x'_v,v)}})$ *(resp.* $W^{1,p'}(J^i_{(x'_v,v)}) \hookrightarrow C(\overline{J^i_{(x'_v,v)}})$*) the existence of limit values of* $u(\cdot, x'_v, v)$ *(resp.* $g(\cdot, x'_v, v)$*) at the left extremity* $x^{i-}(x'_v, v)$ *(resp. the right extremity* $x^{i+}(x'_v, v)$*) of the interval* $J^i_{(x'_v,v)}$*. This essential fact for the existence of the limit values (thus of the traces) is said p.833 after formula (5) in the paper [12] of M. Cessenat, but not really proved and rather delicate. We give the proof in the following lemma:*

3.2 Lemma. *Let* $u \in W^p(\Omega \times V)$*. Then for almost every* $v \in V$*, for almost every* $x'_v \in P_v(\Omega)$, $u(\cdot, x'_v, v)|_{J^i_{(x'_v,v)}}$ *belongs to* $W^{1,p}(J^i_{(x'_v,v)})$ *where the* $J^i_{(x'_v,v)}$ *denote the connected components of the intersection with* Ω *of the straight line parallel to* v *passing through the point* $(0, x'_v)$*.*

Proof : Let us consider the function $u(\cdot, x'_v, v)|_{J^i_{(x'_v,v)}} : J^i_{(x'_v,v)} \to \mathbb{R} : x_v \mapsto u(x_v, x'_v, v)$. By Fubini's theorem, from $u \in W^p(\Omega \times V)$ follows that for almost every $v \in V$, the function $u(\cdot,\cdot,v) \in L^p(\Omega)$ and $v.\nabla_x u(\cdot,\cdot,v) \in L^p(\Omega)$. Also, $\forall\theta \in \mathcal{E}(V)$, C^∞ function on the compact manifold $V = S_2$, and $\forall\psi \in \mathcal{D}(\Omega)$, by the definition (3.7):

$$< v.\nabla_x u, \psi \otimes \theta >= -\int_V \theta(v) \int_\Omega u(x,v) \; v.\nabla_x \psi(x) \; dxd\mu(v), \;\; \forall\psi \in \mathcal{D}(\Omega), \forall\theta \in \mathcal{E}(V).$$

Thus $\forall\psi \in \mathcal{D}(\Omega)$:

$$\int_\Omega v.\nabla_x u(x,v)\psi(x) \; dx = -\int_\Omega u(x,v) \; v.\nabla_x \psi(x) \; dx, \;\; \forall' v \in V. \tag{3.15}$$

But $\mathcal{E}(\Omega)$ the Fréchet space of C^∞- functions in Ω is separable ([20], (17.1.2) p.227), from which follows that the subset of functions of $\mathcal{E}(\Omega)$ with compact supports contained in a fixed open relatively compact subset of Ω is also separable ([6], Prop. III.22 p.47 or [7], Prop. 3.25 page 73). Considering an exhaustion of Ω by a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of open relatively compact subsets of Ω such that $\bar{\Omega}_n \subset \Omega_{n+1}$, $\forall n \in \mathbb{N}$, it follows that the space

$\mathcal{D}(\Omega)$ of C^∞- functions with compact supports in Ω is also separable. Let thus E be a dense denumerable subset of $\mathcal{D}(\Omega)$. For every $\psi_n \in E$, (3.15) is true except possibly for a set V_n of $v \in V$ of μ-measure zero. The union $\bigcup\limits_n V_n$ being of μ-measure zero, for $v \in V$ except a set of measure zero, we have:

$$\int\limits_\Omega v.\nabla_x u(x,v)\psi(x)\ dx = -\int\limits_\Omega u(x,v)\ v.\nabla_x \psi(x)\ dx,\ \forall \psi \in \mathcal{D}(\Omega).$$

Let us fix such a $v \in V$ and the system of coordinates (x_v, x'_v) in $\mathbb{R}^3$, thus in particular in Ω. Let us consider "little" open cubes contained in Ω, whose sides are parallel to the axes in our coordinates system (x_v, x'_v). Ω can be written as the union of a denumerable family $\mathcal{F}$ of such cubes (e.g. by considering all cubes with sides parallel to the axes in our coordinates system (x_v, x'_v) with their center in Ω having rational coordinates and length of their half-sides equal to the distance of their respective centers to the boundary of Ω divided by $\sqrt{3}$) . Let $Q =]\alpha, \beta[\times C$ such an open cube belonging to our denumerable family of cubes $\mathcal{F}$, C denoting an open square whose sides have length $\beta - \alpha$. By Fubini's Theorem, follows that for almost every $x'_v \in C$, that the function $u(\cdot, x'_v, v) \in L^p(]\alpha, \beta[)$ and $v.\nabla_x u(\cdot, x'_v, v) \in L^p(]\alpha, \beta[)$. To verify that $\frac{du(\cdot, x'_v, v)}{dx_v}$ in the sense of distributions is equal to $v.\nabla_x u(\cdot, x'_v, v)$, it suffices to verify it locally. Thus, by the definition of the derivative in the sense of distributions, we have to verify that for almost every $x'_v \in C$ (as the union of a denumerable family of sets of measure zero being still a set of measure zero) :

$$\begin{aligned} < v.\nabla_x u(\cdot, x'_v, v), \varphi > &= \int\limits_\alpha^\beta v.\nabla_x u(x_v, x'_v, v)\varphi(x_v) dx_v \\ &= -\int\limits_\alpha^\beta u(x_v, x'_v, v)\frac{d\varphi}{dx_v}(x_v) dx_v, \quad \forall \varphi \in \mathcal{D}(]\alpha, \beta[). \end{aligned} \tag{3.16}$$

Taking in (3.15) for ψ any function of the form $(x_v, x'_v) \longmapsto \varphi(x_v)\tilde{\varphi}(x'_v)$ with $\varphi \in \mathcal{D}(]\alpha, \beta[)$ and $\tilde{\varphi} \in \mathcal{D}(C)$, equality (3.15) becomes:

$$\begin{aligned} &\int\limits_C dx'_v \tilde{\varphi}(x'_v) \int\limits_\alpha^\beta v.\nabla_x u(x_v, x'_v, v)\varphi(x_v) dx_v \\ &= -\int\limits_C dx'_v \tilde{\varphi}(x'_v) \int\limits_\alpha^\beta u(x_v, x'_v, v)\ \tfrac{d\varphi}{dx_v}(x_v) dx_v, \end{aligned} \tag{3.17}$$

which implies that for almost every $x'_v \in C$:

$$\int\limits_\alpha^\beta v.\nabla_x u(x_v, x'_v, v)\varphi(x_v) dx_v = -\int\limits_\alpha^\beta u(x_v, x'_v, v)\ \frac{d\varphi}{dx_v}(x_v) dx_v, \tag{3.18}$$

By the same reasonings than just after formula (3.15), follows that the space $\mathcal{D}(]\alpha,\beta[)$ of C^∞- functions with compact supports in $]\alpha,\beta[$ is also separable. Let thus D a dense denumerable subset of $\mathcal{D}(]\alpha,\beta[)$. For every $\varphi_n \in D$, (3.18) is true except possibly for a set E_n of $x'_v \in C$ of measure zero. The union $\bigcup\limits_n E_n$ being of measure zero, for $x'_v \in C$ except a set of measure zero, for every test function $\varphi \in \mathcal{D}(]\alpha,\beta[)$, (3.16) is true. $\square$

3.3 Remark. *The preceding lemma shows us that the space $W^p(\Omega\times V)$ can be defined as the space of those functions u belonging to $L^p(\Omega\times V)$ such that $\forall' v \in V$, $u(\cdot,x'_v,v)$ is $\forall' x'_v \in P_v(\Omega)$ absolutely continuous on the intersection of the straight line parallel to v passing through the point $(0,x'_v)\in\mathbb{R}^3$ with Ω and such that $\int\limits_V[\int\limits_\Omega|\frac{\partial u}{\partial x_v}(x_v,x'_v,v)|^p$ $dx_v dx'_v]d\mu(v) < +\infty$.*

3.2.3 Dissipativeness of the operator A

The aim of this subsection is to prove that the operator A is closed and dissipative. In the proof of the dissipativeness of A, we will use the property of the “reflectivity coefficient” $\rho_g(\cdot)$, to be equal to 1 near 0.

3.3 Lemma. *$\forall p \in]1,+\infty[$, $\forall h \in W^p(\Omega\times V)$, $v\cdot\nabla_x |h|^p = p\,|h|^{p-2}\,h\; v\cdot\nabla_x h$.*

Proof : 1°) For $h \in L^p(V;C^1(\Omega))\cap W^p(\Omega\times V)$, the formula follows from:

$$\begin{aligned} v.\nabla_x |h|^p &= v.\nabla_x (h^2)^{p/2} = \tfrac{p}{2}(h^2)^{p/2-1} v.\nabla_x h^2 \\ &= p(|h|^2)^{p/2-1} h\ v.\nabla_x h = p\,|h|^{p-2}\ \ h\ v.\nabla_x h. \end{aligned}$$

In particular, this formula is true for every $h \in \mathcal{D}(\overline{\Omega}\times V)$.
2°) Using the density of $\mathcal{D}(\overline{\Omega}\times V)$ in $W^p(\Omega\times V)$ (it is said in the proof of Theorem 1 in [19], pp.220-221), we now prove the formula for every function $h \in W^p(\Omega\times V)$. Let $h \in W^p(\Omega\times V)$ and $(h_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\overline{\Omega}\times V)$ converging to h for the norm of $W^p(\Omega\times V)$. For every h_n, $n\in\mathbb{N}$, we have using the preceding point and formula (3.7):

$$\begin{aligned} &< p\,|h_n|^{p-2}\,h_n\ v.\nabla_x h_n, \varphi > = < v.\nabla_x |h_n|^p, \varphi > \\ &= -\iint\limits_{\Omega\times V} |h_n|^p(x,v)\ v.\nabla_x\varphi(x,v)\ dxd\mu(v),\ \forall\varphi\in\mathcal{D}(\Omega\times V). \end{aligned} \tag{3.19}$$

Using Theorem IV.9 p. 58 of [6] or equivalently Theorem 4.9 p. 94 of [7], from the strong convergence in $L^p(\Omega\times V)$ of the sequence $(h_n)_{n\in\mathbb{N}}$ to h, follows that $(|h_n|^p)_{n\in\mathbb{N}}$ strongly converges to $|h|^p$ in $L^1(\Omega\times V)$ when $n\to+\infty$. Thus $\forall\varphi\in\mathcal{D}(\Omega\times V)$:

$$-\iint\limits_{\Omega\times V} |h_n|^p(x,v)\ v.\nabla_x\varphi(x,v)\ dxd\mu(v) \to -\iint\limits_{\Omega\times V} |h|^p(x,v)\ v.\nabla_x\varphi(x,v)\ dxd\mu(v),$$

when $n\to+\infty$. Using still Theorem IV.9 p. 58 of [6] or equivalently Theorem 4.9 p. 94 of [7], it follows that $p\,|h_n|^{p-2}\,h_n$ strongly converges to $p\,|h|^{p-2}\,h$ in $L^{p'}(\Omega\times V)$ $(\frac{1}{p'} :=$

$\frac{1}{p}-1)$ when $n \to +\infty$, which with $v.\nabla_x h_n$ strongly converges to $v.\nabla_x h$ in $L^p(\Omega \times V)$, implies that $p\,|h_n|^{p-2}\,h_n\;v.\nabla_x h_n$ strongly converges to $p\,|h|^{p-2}\,h\;v.\nabla_x h$ in $L^1(\Omega\times V)$ when $n \to +\infty$. Thus

$$\begin{aligned} &< p\,|h_n|^{p-2}\,h_n\;v.\nabla_x h_n, \varphi > \\ &= \iint\limits_{\Omega\times V} p\,|h_n|^{p-2}(x,v)\;h_n(x,v)\;v.\nabla_x h_n(x,v)\varphi(x,v)\;dxd\mu(v) \\ &\to \iint\limits_{\Omega\times V} p\,|h|^{p-2}(x,v)\;h(x,v)\;v.\nabla_x h(x,v)\varphi(x,v)\;dxd\mu(v), \end{aligned}$$

when $n \to +\infty$. Passing to the limit when $n \to +\infty$ in (3.19), we thus obtain:

$$\begin{aligned} < p\,|h|^{p-2}\,h\;v.\nabla_x h, \varphi > &= -\iint\limits_{\Omega\times V} |h|^p(x,v)\;v.\nabla_x\varphi(x,v)\;dxd\mu(v) \\ &=< v.\nabla_x\,|h|^p, \varphi >,\;\forall\varphi\in\mathcal{D}(\Omega\times V), \end{aligned}$$

which proves that $v.\nabla_x\,|h|^p = p\,|h|^{p-2}\,h\;v.\nabla_x h$ in the sense of distributions. □

3.4 Lemma. *Let us consider an odd C^1-function $\theta_1 : \mathbb{R}\to\mathbb{R}$, increasing (stricly) in the subinterval $[0,2]$, such that*

$$\theta_1 : \mathbb{R}\to\mathbb{R} : y \mapsto \begin{cases} y, & \text{if } y\in[0,1/2], \\ \leq y, & \text{if } y \in [1/2,2], \\ 1, & \text{if } y \geq 2. \end{cases}$$

We may also suppose that $0 \leq \theta_1' \leq 1$. Let us set $\forall n \in \mathbb{N}^* : \theta_n : \mathbb{R}\to\mathbb{R} : y \mapsto n\cdot\theta_1\left(\frac{y}{n}\right)$. Then $\forall u \in W^p(\Omega\times V)$: $\theta_n(u)\in W^p(\Omega\times V)$ and $v\cdot\nabla_x(\theta_n(u)) = \theta_n'(u)v\cdot\nabla_x u$. Consequently $\theta_n(u)\to u$ in $W^p(\Omega\times V)$ as $n\to+\infty$.

Proof : 1°) Firstly, let us prove that the sequence $\theta_n(u) \to u$ in $L^p(\Omega\times V)$ when $n \to +\infty$. From the definition of θ_n follows that $\theta_n(u(x,v)) = u(x,v)$, if $|u(x,v)| \leq \frac{n}{2}$. As $u \in L^p(\Omega\times V)$, $|u(x,v)| < +\infty$ a.e. and thus for almost $(x,v)\in\Omega\times V$, we will have $\theta_n(u(x,v)) = u(x,v)$, if n is sufficiently large. Thus $\theta_n(u)\to u$ a.e. in $\Omega\times V$ when $n\to+\infty$. From the definition of θ_n, we have also that $|\theta_n(u)| \leq |u|$ which belongs to $L^p(\Omega\times V)$. Thus by the Lebesgue's dominated convergence theorem ([54], p. 18), $\theta_n(u) \to u$ in $L^p(\Omega\times V)$ when $n\to+\infty$.
2°) Let us prove the formula $v.\nabla_x(\theta_n(u)) = \theta_n'(u)v.\nabla_x u$. Let us fix some $n\in\mathbb{N}^*$. We know that $\mathcal{D}(\overline{\Omega}\times V)$ is dense in $W^p(\Omega\times V)$ (see the proof of Theorem 1 in [19], pp.220-221). Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions in $\mathcal{D}(\overline{\Omega}\times V)$ which tends to u in $W^p(\Omega\times V)$. By the classical rule of derivation of composed functions: $v.\nabla_x(\theta_n(u_k)) = \theta_n'(u_k)v.\nabla_x u_k$. As $u_k \to u$ in $L^p(\Omega\times V)$ and $v.\nabla_x u_k \to v.\nabla_x u$ in $L^p(\Omega\times V)$, by using Theorem IV.9 p. 58 of [6] or equivalently Theorem 4.9 p. 94 of [7], there exists a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ such that $u_{k_l}\to u$ a.e. on $\Omega\times V$, $v.\nabla_x u_{k_l}\to v.\nabla_x u$ a.e. on $\Omega\times V$ and a function $h\in L^p(\Omega\times V)$

such that $|v.\nabla_x u_{k_l}| \leq h$ a.e. on $\Omega \times V$. $\theta_n'(\cdot)$ being continuous, we have also that $\theta_n'(u_{k_l}) \to \theta_n'(u)$ a.e. in $\Omega \times V$ when $l \to +\infty$. Thus $v.\nabla_x(\theta_n(u_{k_l})) = \theta_n'(u_{k_l})v.\nabla_x u_{k_l}$ tends to $\theta_n'(u)v.\nabla_x u$ a.e. in $\Omega \times V$ when $l \to +\infty$. Thus by the Lebesgue dominated convergence theorem in L^p, $v.\nabla_x(\theta_n(u_{k_l})) = \theta_n'(u_{k_l})v.\nabla_x u_{k_l}$ tends to $\theta_n'(u)v.\nabla_x u$ in $L^p(\Omega \times V)$ when $l \to +\infty$. As $\theta_n(u_{k_l}) \in C^1(\overline{\Omega} \times V)$, by the classical divergence theorem applied to $v.\nabla_x(\theta_n(u_{k_l})) = \mathrm{div}_x(\theta_n(u_{k_l}).v)$,

$$< v.\nabla_x(\theta_n(u_{k_l})), \varphi >= -\iint\limits_{\Omega\times V} \theta_n(u_{k_l}(x,v))\ v.\nabla_x\varphi(x,v)\ dxd\mu(v),\ \forall \varphi \in \mathcal{D}(\Omega \times V). \quad (3.20)$$

As $u_{k_l} \to u$ a.e., $\theta_n(\cdot)$ being continuous, we have also that $\theta_n(u_{k_l}) \to \theta_n(u)$ a.e. on $\Omega \times V$. $|\theta_n(u_{k_l})| \leq n$ which is fixed, thus by Lebesgue dominated convergence Theorem, the right hand-side in (3.20):

$$-\iint\limits_{\Omega\times V} \theta_n(u_{k_l}(x,v))\ v.\nabla_x\varphi(x,v)\ dxd\mu(v) \to -\iint\limits_{\Omega\times V} \theta_n(u(x,v))\ v.\nabla_x\varphi(x,v)\ dxd\mu(v),$$

when $l \to +\infty$. As $v.\nabla_x(\theta_n(u_{k_l})) \to \theta_n'(u)v.\nabla_x u$ in $L^p(\Omega \times V)$ when $l \to +\infty$, the left hand-side in (3.20): $< v.\nabla_x(\theta_n(u_{k_l})), \varphi >\to< \theta_n'(u)v.\nabla_x u, \varphi >$. Thus passing to the limit as $l \to +\infty$ in (3.20), we obtain:

$$< \theta_n'(u)v.\nabla_x u, \varphi >= -\iint\limits_{\Omega\times V} \theta_n(u(x,v))\ v.\nabla_x\varphi(x,v)\ dxd\mu(v),\ \forall \varphi \in \mathcal{D}(\Omega \times V).$$

Thus, in the sense of distributions: $v.\nabla_x(\theta_n(u)) = \theta_n'(u)v.\nabla_x u$.
3°) Thus it remains to prove that $\theta_n'(u)v.\nabla_x u \to v.\nabla_x u$ in $L^p(\Omega \times V)$ when $n \to +\infty$. As $0 \leq \theta_n' \leq 1$, $|\theta_n'(u)v.\nabla_x u| \leq |v.\nabla_x u| \in L^p(\Omega \times V)$, and $\theta_n'(u) \to 1$ a.e. in $\Omega \times V$ when $n \to +\infty$, by the Lebesgue dominated convergence Theorem ([54], p. 18), $\theta_n'(u)v.\nabla_x u \to v.\nabla_x u$ in $L^p(\Omega \times V)$ when $n \to +\infty$. Consequently, by the second point, $v.\nabla_x(\theta_n(u)) \to v.\nabla_x u$ in $L^p(\Omega \times V)$.
4°) From the first point and the third one, follows that $\theta_n(u) \to u$ in $W^p(\Omega \times V)$. □

3.5 Lemma. *Let $(X, \mathcal{A}, m)$ denotes a measure space. On $L^p(X, \mathcal{A}, m)$ $(1 < p < +\infty)$, we consider the semi-inner product (unique in fact [6, p.3], or [7, pp.3-4]) :*

$$[\cdot,\cdot]_p : L^p(X) \times L^p(X) \to \mathbb{R} : (f,g) \mapsto< f, Wg >_{L^p(X),L^{p'}(X)}, \quad (3.21)$$

where $Wg \in L^{p'}(X)$ $(\frac{1}{p'} := 1 - \frac{1}{p})$ is defined by $Wg := \frac{|g|^{p-2}g}{\|g\|_p^{p-2}}$ if $g \neq 0$, and $Wg := 0$ if $g = 0$. Then $[\cdot,\cdot]_p$ is a continuous mapping from $L^p(X) \times L^p(X) \to \mathbb{R}$.

Proof : It suffices to prove that the mapping $W : L^p(X) \to L^{p'}(X) : g \mapsto Wg$ is continuous. Firstly, let us consider a sequence $(g_n)_{n\geq 0}$ in $L^p(X)$ converging to 0 in $L^p(X)$.

$$\begin{aligned} \|Wg_n\|_{L^{p'}(X)} &= \left(\int\limits_X \left|\frac{|g_n|^{p-2}g_n}{\|g_n\|_p^{p-2}}\right|^{p'} dm\right)^{\frac{1}{p'}} \\ &= \frac{1}{\|g_n\|_p^{p-2}} \left(\int\limits_X |g_n|^{(p-1)p'} dm\right)^{\frac{1}{p'}} \\ &= \|g_n\|_{L^p(X)} . \end{aligned} \tag{3.22}$$

Thus $Wg_n \to 0$ in $L^{p'}(X)$. This proves the continuity of the mapping $W : L^p(X) \to L^{p'}(X)$ at point $0 \in L^p(X)$. Now, let us consider $g \in L^p(X)\backslash\{0\}$ and a sequence $(g_n)_{n\geq 0}$ in $L^p(X)$ converging to g in $L^p(X)$. By the partial converse to the Lebesgue dominated convergence theorem ([45], p.48), there exists a subsequence $(g_{n_k})_{k\geq 0}$ converging to g a.e. and a function $h \in L^p(X)$ such that $|g_{n_k}| \leq h$. This implies that $\left||g_{n_k}|^{p-2} g_{n_k}\right| \leq h^{p-1} \in L^{p'}(X)$. Thus by the Lebesgue dominated convergence theorem in $L^{p'}(X)$, $|g_{n_k}|^{p-2} g_{n_k} \to |g|^{p-2} g$ in $L^{p'}(X)$. Thus $Wg_{n_k} \to Wg$ in $L^{p'}(X)$. By a well known result of elementary General Topology, the sequence $(Wg_n)_{n\geq 0}$ itself converges to Wg itself in $L^{p'}(X)$. This proves that the mapping $W : L^p(X) \to L^{p'}(X) : g \mapsto Wg$ is continuous. □

3.2 Proposition. *The operator A defined by (3.8) is dissipative in $L^p(\Omega \times V)$.*

Proof : On $L^p(\Omega \times V)$, we consider the semi-inner product $[\cdot,\cdot]_p$ defined by (3.21) (with $X = \Omega \times V$ and $m = dx \otimes d\mu$). We have to prove that $[Au, u]_p \leq 0$, $\forall u \in D(A)\backslash\{0\}$. Let us consider some $u \in D(A)\backslash\{0\}$ and let us set $u_n := \theta_n(u)$. u_n belongs to $W^p(\Omega \times V)$ but in general $u_n \notin D(A)$ due to the homogeneous boundary condition in the definition of $D(A)$. u_n being bounded, its traces on $\Gamma_\pm$ belong to $L^\infty(\Gamma_\pm; \pm v.\nu(x)dS(x)d\mu(v))$ and thus a fortiori to $L^p(\Gamma_\pm; \pm v.\nu(x)dS(x)d\mu(v))$ respectively. Consequently, using Lemma 3.3 and applying Green's formula, Proposition 3.1, to $|u_n|^p$ and $1_{\Omega\times V}$ (formula $v\cdot\nabla_x|u_n|^p = p|u_n|^{p-2}u_n v \cdot \nabla_x u_n$, see Lemma 3.3, and u_n bounded show us that we may consider $|u_n|^p$ as an element of $W^p(\Omega \times V)$ in order to apply our Green's formula), we have:

$$\begin{aligned} &\|u_n\|_p^{p-2} \left[-v \cdot \nabla_x u_n, u_n\right]_p = -\iint\limits_{\Omega\times V} v \cdot \nabla_x u_n(x,v)|u_n(x,v)|^{p-2}u_n(x,v)dxd\mu(v) \\ &= -\tfrac{1}{p}\iint\limits_{\Omega\times V} v \cdot \nabla_x |u_n(x,v)|^p dxd\mu(v) = -\tfrac{1}{p}\iint\limits_{\partial\Omega\times V} |u_n(x,v)|^p v.\nu(x)dS(x)d\mu(v) \\ &= -\tfrac{1}{p}\iint\limits_{\Gamma_+} |u_n(x,v)|^p v.\nu(x)dS(x)d\mu(v) - \tfrac{1}{p}\iint\limits_{\Gamma_-} |u_n(x,v)|^p v.\nu(x)dS(x)d\mu(v), \end{aligned} \tag{3.23}$$

where $dS(x)$ denotes the area measure on $\partial\Omega$. But:

$$\iint_{\Gamma_-} |u_n(x,v)|^p v.\nu(x) dS(x) d\mu(v) = -\iint_{\Gamma_+} |u_n(x, v_r(x, v_i))|^p v_i.\nu(x) dS(x) d\mu(v_i), \tag{3.24}$$

by Lemma 3.1, where $v = v_r(x, v_i)$ denotes the reflected vector for which v_i is the incident vector at x. Renaming v the vector v_i, we obtain:

$$\|u_n\|_p^{p-2} [-v \cdot \nabla_x u_n, u_n]_p = -\frac{1}{p} \iint_{\Gamma_+} (|u_n(x,v)|^p - |u_n(x, v_r(x,v))|^p) v.\nu(x) dS(x) d\mu(v).$$

But $1 - \rho_g(\nu_x \cdot v)^p = 0$ *for* $0 \le \nu_x \cdot v \le \sqrt{\frac{n_g^2-1}{n_g^2}}$, as $\rho_g(\nu_x \cdot v) = 1$ *for* $0 \le \nu_x \cdot v \le \sqrt{\frac{n_g^2-1}{n_g^2}}$ ([50], p.8). Thus for $(x, v) \in \Gamma_+$:

$$u(x, v_r(x,v)) = u(x,v) \text{ for } \nu_x \cdot v \le \sqrt{\frac{n_g^2 - 1}{n_g^2}}$$

which implies

$$u_n(x, v_r(x,v)) = u_n(x,v) \text{ for } \nu_x \cdot v \le \sqrt{\frac{n_g^2 - 1}{n_g^2}}.$$

Consequently

$$\|u_n\|_p^{p-2} [-v \cdot \nabla_x u_n, u_n]_p = -\frac{1}{p} \iint_{K} (|u_n(x,v)|^p - |u_n(x, v_r(x,v))|^p) v.\nu(x) dS(x) d\mu(v),$$

where K denotes the set

$$K := \{(x,v) \in \partial\Omega \times V; \nu_x \cdot v \ge \sqrt{\frac{n_g^2 - 1}{n_g^2}}\}.$$

K being a compact subset of Γ_+ by the trace theorem, Theorem 1 of [19, p.220]: $u_n|_K \to u|_K$ in $L^p(K; dS(x)d\mu(v))$, which implies:

$$\iint_K |u_n(x,v)|^p v.\nu(x) dS(x) d\mu(v) \to \iint_K |u(x,v)|^p v.\nu(x) dS(x) d\mu(v).$$

The image of K by the continuous mapping (see Lemma 3.1):

$$T|_{\Gamma_+} : \Gamma_+ \to \Gamma_- : (x,v) \mapsto (x, v_r(x,v))$$

being a compact set in Γ_-, by the trace theorem, Theorem 1 of [19, p.220]: $u_n|_{T(K)} \to u|_{T(K)}$ in $L^p(K; dS(x)d\mu(v))$, which implies:

$$\iint_K |u_n(x, v_r(x,v))|^p v.\nu(x)dS(x)d\mu(v) \to \iint_K |u(x, v_r(x,v))|^p v.\nu(x)dS(x)d\mu(v).$$

Thus

$$\begin{aligned}\|u_n\|_p^{p-2}\,[-v\cdot\nabla_x u_n, u_n]_p &\to -\frac{1}{p}\iint_K (|u(x,v)|^p - |u(x,v_r(x,v))|^p)v.\nu(x)dS(x)d\mu(v),\\ &= -\frac{1}{p}\iint_{\Gamma_+} |u(x,v)|^p(1-\rho_g(\nu_x\cdot v)^p)v.\nu(x)dS(x)d\mu(v) \le 0,\end{aligned} \tag{3.25}$$

by (3.9). On the other hand, by Lemma 3.4, $u_n := \theta_n(u) \to u$ in $W^p(\Omega \times V)$ implies that $v\cdot\nabla_x u_n \to v\cdot\nabla_x u$ in $L^p(\Omega\times V)$. Thus, by the continuity of the semi-inner product on $L^p(\Omega\times V)$ (see Lemma 3.5):

$$\|u_n\|_p^{p-2}\,[-v\cdot\nabla_x u_n, u_n]_p \to \|u\|_p^{p-2}\,[Au, u]_p. \tag{3.26}$$

From (3.25),(3.26) follows that

$$\|u\|_p^{p-2}\,[Au,u]_p = -\frac{1}{p}\iint_{\Gamma_+} |u(x,v)|^p(1-\rho_g(\nu_x\cdot v)^p)v.\nu(x)dS(x)d\mu(v) \le 0.$$

Thus $[Au,u]_p \le 0$, $\forall u \in D(A)$ which proves that A is a dissipative operator. □

It is obvious that the operator A has a dense domain in $L^p(\Omega\times V)$ because $D(A) \supset \mathcal{D}(\Omega\times V)$ which is dense in $L^p(\Omega\times V)$. Let us show now that the operator A is a closed operator in $L^p(\Omega\times V)$.

3.3 Proposition. *A is a closed operator in $L^p(\Omega\times V)$.*

Proof : Let us consider a sequence $(u_n)_{n\ge1}$ in $D(A)$ converging to an element $u \in L^p(\Omega\times V)$ such that the sequence $(Au_n)_{n\ge1}$ converges to an element $f \in L^p(\Omega\times V)$. We must show that this implies that $u \in D(A)$ and that $Au = f$. Let us set $f_n = Au_n$, $\forall n \in \mathbb{N}$. By the definition of the operator A, we have that $f_n = -v\cdot\nabla_x u_n$ in the sense of distributions i.e.

$$\iint_{\Omega\times V} u_n(x,v)\; v\cdot\nabla_x\varphi(x,v)\; dxd\mu(v) = \iint_{\Omega\times V} f_n(x,v)\;\varphi(x,v)\; dxd\mu(v),\;\; \forall\varphi\in\mathcal{D}(\Omega\times V).$$

As $v\cdot\nabla_x\varphi \in L^\infty(\Omega\times V)$, $\Omega\times V$ is of finite measure, $u_n \to u$, and $f_n \to f$ in $L^p(\Omega\times V)$, we can pass to the limit in the previous identity and obtain:

$$\iint\limits_{\Omega\times V} u(x,v)\ v\cdot\nabla_x\varphi(x,v)\ dxd\mu(v) = \iint\limits_{\Omega\times V} f(x,v)\ \varphi(x,v)\ dxd\mu(v),\ \forall\varphi\in\mathcal{D}(\Omega\times V).$$

As $u\in L^p(\Omega\times V)$, $f\in L^p(\Omega\times V)$ and $-v\cdot\nabla_x u = f$ in the generalized sense, it follows that $u\in W^p(\Omega\times V)$ and $u_n\to u$ in the space $W^p(\Omega\times V)$. By the continuity of the trace operators (Corollary 1 of [19, p.220]) from $W^p(\Omega\times V)$ into $L^p_{loc}(\Gamma_\mp)$, $u_n|_{\Gamma_\mp}\to u|_{\Gamma_\mp}$ in $L^p_{loc}(\Gamma_\mp)$. This implies that some subsequence $\left(u_{n_k}|_{\Gamma_\mp}\right)_{k\geq 0}$ converges a.e. on $\Gamma_\mp$ to $u|_{\Gamma_\mp}$. In conclusion u satisfies also (3.9), and consequently, $u\in D(A)$ and $Au=f$. □

3.2.4 Surjectivity of $\lambda - A$, $\forall\lambda>0$

We want to prove that the range of $\lambda-A$, $R(\lambda-A)$ is equal to $L^p(\Omega\times V)$ for all $\lambda>0$. We are reduced to prove that $R(\lambda-A)$ is dense in $L^p(\Omega\times V)$. Indeed, using the closedness and the dissipativeness of A, the result will follow.

3.4 Proposition. *Let $\lambda>0$ be arbitrary. Let $g\in L^{p'}(\Omega\times V)$ be "orthogonal" to the range of $\lambda-A$ i.e.*

$$\iint\limits_{\Omega\times V} g(x,v)\ (\lambda-A)u(x,v)\ dxd\mu(v) = 0,\ \forall u\in D(A). \tag{3.27}$$

Then $v\cdot\nabla_x g=\lambda g$ in the generalized sense. Consequently $g\in W^{p'}(\Omega\times V)$.

Proof : The hypothesis implies in particular that:

$$< g, \lambda\varphi + v\cdot\nabla_x\varphi >_{L^{p'}(\Omega\times V),L^p(\Omega\times V)} = 0,\ \forall\varphi\in\mathcal{D}(\Omega\times V),$$

i.e.

$$\lambda < g,\varphi > = -\iint\limits_{\Omega\times V} g(x,v)v\cdot\nabla_x\varphi(x,v)dxd\mu(v),\ \forall\varphi\in\mathcal{D}(\Omega\times V).$$

Thus in view of (3.7), this amounts to say that in the generalized sense $v\cdot\nabla_x g=\lambda g\in L^{p'}(\Omega\times V)$. □

Now, let us define the unbounded operator $\hat{A}$ in $L^{p'}(\Omega\times V)$ by:

$$\begin{gathered} D(\hat{A}) = \{g\in W^{p'}(\Omega\times V); g(x,v)=\rho_g(|\nu_x\cdot v|)g(x,v_r),\ \textit{for a.e. } (x,v)\in\Gamma_+\}, \\ \textit{where } v_r = v_r(x,v) := v - 2(\nu_x\cdot v)\nu_x, \\ \hat{A} : D(\hat{A}) \to L^{p'}(\Omega\times V) : g\longmapsto v\cdot\nabla_x g. \end{gathered} \tag{3.28}$$

The notation $v_r=v_r(x,v)$ in (3.28) must be understood as the reflected vector corresponding to the incident vector v at point x on the interior face of $\partial\Omega$. We are going to prove that the operator $\hat{A}$ is also dissipative but in $L^{p'}(\Omega\times V)$, of course. To be more precise, the operator A (resp. $\hat{A}$) operating in the Banach space $L^p(\Omega \times V)$ (resp. $L^{p'}(\Omega \times V)$) will also be denoted A_p (resp. $\hat{A}_{p'}$) when necessary.

3.1 Definition. *In the following, we will denote by J the mapping*

$$J : \Omega \times V \to \Omega \times V : (x, v) \mapsto (x, -v).$$

3.5 Proposition. *$h \in D(\hat{A}_{p'})$ iff $h \circ J \in D(A_{p'})$ and*

$$\hat{A}_{p'} h = (A_{p'}(h \circ J)) \circ J. \tag{3.29}$$

Proof : Firstly $h \in D(\hat{A}_{p'})$ implies $h \in W^{p'}(\Omega \times V)$ and also $h \circ J \in W^{p'}(\Omega \times V)$. Now, for almost all $x \in \partial\Omega$ and $v \in V$ such that $(x, v) \in \Gamma_-$, we have:

$$\begin{aligned} (h \circ J)(x, v) &= h(x, -v) = \rho\left(|\nu_x \cdot v|\right) h(x, (-v)_r) \\ &= \rho\left(|\nu_x \cdot v|\right) (h \circ J)(x, -(-v)_r) \\ &= \rho\left(|\nu_x \cdot v|\right) (h \circ J)(x, -(-v + 2(v \cdot \nu_x)\, \nu_x)) \\ &= \rho\left(|\nu_x \cdot v|\right) (h \circ J)(x, v - 2(v \cdot \nu_x)\nu_x) \\ &= \rho\left(|\nu_x \cdot v|\right) (h \circ J)(x, v_i), \end{aligned} \tag{3.30}$$

where $v_i = v - 2(v \cdot \nu_x)\nu_x$ denotes the incident vector at point x on the interior face of $\partial\Omega$ admitting v as reflected vector. Thus $h \circ J \in D(A_{p'})$. Reciprocally, if $h \circ J \in D(A_{p'})$, h belongs to $W^{p'}(\Omega \times V)$ and $h \circ J(x, v) = \rho\left(|\nu_x \cdot v|\right)(h \circ J)(x, v_i)$, $\forall'(x, v) \in \Gamma_-$ which implies that $h(x, v) = \rho\left(|\nu_x \cdot v|\right) h(x, v_r)$, $\forall'(x, v) \in \Gamma_+$, and thus $h \in D(\hat{A}_{p'})$. Now, let us prove formula (3.29). $h \circ J : \Omega \times V \to \mathbb{R} : (x, v) \mapsto h(x, -v)$ and thus

$$\begin{aligned} \left(\hat{A}_{p'}(h \circ J)\right)(x, -v) &= -\sum_{k=1}^{3} (-v_k) \frac{\partial h}{\partial x_k}(x, v) \\ &= (v.\nabla_x h)(x, v) \\ &= (\hat{A}_{p'} h)(x, v). \end{aligned}$$

The proof of the proposition is complete. □

3.6 Proposition. *The operator $\hat{A} \equiv \hat{A}_{p'}$ defined by (3.28) is dissipative in $L^{p'}(\Omega \times V)$.*

Proof : On $L^{p'}(\Omega \times V)$, we consider the semi-inner product (unique in fact [6], p.3) :

$$[\cdot, \cdot] : L^{p'}(\Omega \times V) \times L^{p'}(\Omega \times V) \to \mathbb{R} : (f, h) \mapsto < f, Wh > \tag{3.31}$$

where $Wh \in L^p(\Omega \times V)$ ($\frac{1}{p} := 1 - \frac{1}{p'}$) is defined by: $Wh = \frac{|h|^{p'-2} h}{\|h\|_{p'}^{p'-2}}$ if $h \neq 0$ and 0 if $h = 0$. We have to prove that $[\hat{A}h, h] \leq 0$, $\forall h \in D(\hat{A})$. Let us consider some $h \in D(\hat{A}) \backslash \{0\}$. By

Proposition 3.5:

$$
\begin{aligned}
[\hat{A}h, h] &= [(J \circ A_{p'} \circ J)h, h] \\
&= \|h\|_{p'}^{2-p'} \iint\limits_{\Omega\times V} |h(x,v)|^{p'-2}\, h(x,v)(A_{p'}Jh)(x,-v)dxd\mu(v) \\
&= \|h\|_{p'}^{2-p'} \iint\limits_{\Omega\times V} |Jh(x,v)|^{p'-2}\, Jh(x,v)(A_{p'}Jh)(x,v)dxd\mu(v) \\
&= [A_{p'}Jh, Jh] \leq 0,
\end{aligned}
$$

by the change of variable $(x,v) \to (x,-v)$ in the double integral. Thus $[\hat{A}h, h] \leq 0$, $\forall h \in D(\hat{A})$ which proves that $\hat{A} \equiv \hat{A}_{p'}$ is also a dissipative operator like $A_{p'}$ in the Banach space $L^{p'}(\Omega \times V)$. □

3.4 Remark. *It can be proved that $\hat{A}_{p'} = (A_p)^*$. The proof uses in particular (3.29).*

3.7 Proposition. *Let $\lambda > 0$. Let $g \in L^{p'}(\Omega \times V)$ be "orthogonal" to the range of $\lambda - A$ i.e.*

$$\iint\limits_{\Omega\times V} g(x,v)\ (\lambda - A)u(x,v)\ dxd\mu(v) = 0,\ \ \forall u \in D(A).$$

Then $g \in W^{p'}(\Omega \times V)$ and

$$\int\limits_{V}\int\limits_{\partial\Omega} g(x,v)\ u(x,v)\ v.\nu(x)\ dS(x)d\mu(v) = 0, \tag{3.32}$$

$\forall u \in D(A)$, such that their traces on $\Gamma_\pm$, $u_\pm \in \mathcal{K}(\Gamma_\pm; \mathbb{R})$.

Proof : We know already by Proposition 3.4, that $g \in W^{p'}(\Omega \times V)$ and $v \cdot \nabla_x g = \lambda g$. As $u_\pm \in \mathcal{K}(\Gamma_\pm; \mathbb{R})$ by hypothesis, it follows by the trace theorem, Theorem 1 of [19, p.220], that the trace of $g.u$ on Γ is integrable with respect to the measure $|v.\nu(x)|\, dS(x)d\mu(v)$. Consequently, applying Green's formula (3.14), we obtain:

$$
\begin{aligned}
0 &= \iint\limits_{\Omega\times V} g(x,v)\ (\lambda - A)u(x,v)\ dxd\mu(v) = \iint\limits_{\Omega\times V} g(x,v)\ (\lambda u + v \cdot \nabla_x u)(x,v)\ dxd\mu(v) \\
&= \iint\limits_{\Omega\times V} u(x,v)(\lambda g - v \cdot \nabla_x g)(x,v)dxd\mu(v) + \iint\limits_{\partial\Omega\times V} g(x,v)u(x,v)v.\nu(x)dS(x)d\mu(v) \\
&= \iint\limits_{\partial\Omega\times V} g(x,v)u(x,v)v.\nu(x)dS(x)d\mu(v).
\end{aligned}
$$

□

3.8 Proposition. *Let $\lambda > 0$, and $g \in L^{p'}(\Omega \times V)$ satisfy (3.27). Denoting by $g_\pm$ (resp. $u_\pm$) the traces of $g \in W^{p'}(\Omega \times V)$ (resp. $u \in W^p(\Omega \times V)$) on $\Gamma_\pm$ (see Theorem 1 of [19,* *p.220]), we have* $\forall u \in D(A)$, *such that its traces on* $\Gamma_\pm$, $u_\pm \in \mathcal{K}(\Gamma_\pm;\mathbb{R})$*:*

$$\iint\limits_{\Gamma_+} u_+(x,v)(g_+(x,v) - \rho_g(|\nu_x \cdot v|)g_-(x,v_r(x,v)))\ v.\nu(x)dS(x)d\mu(v) = 0. \tag{3.33}$$

Proof : From (3.32) and $\partial\Omega \times V = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$ where

$$\Gamma_0 := \{(x,v) \in \partial\Omega \times V; v.\nu(x) = 0\},$$

we have $\forall u \in D(A)$ such that their traces on $\Gamma_\pm$, $u_\pm \in \mathcal{K}(\Gamma_\pm;\mathbb{R})$:

$$\begin{aligned} &\iint\limits_{\Gamma_+} g_+(x,v)\ u_+(x,v)\ v.\nu(x)\ dS(x)d\mu(v) - \\ &\iint\limits_{\Gamma_-} g_-(x,v)\ u_-(x,v)\ |v.\nu(x)|\ dS(x)d\mu(v) = 0. \end{aligned} \tag{3.34}$$

But due to $u_-(x,v) = \rho_g(|\nu_x \cdot v|)u_+(x,v_i(x,v))$ for a.e. $(x,v) \in \Gamma_-$, we have:

$$\begin{aligned} &\iint\limits_{\Gamma_-} g_-(x,v)\ u_-(x,v)\ |v.\nu(x)|\ dS(x)d\mu(v) \\ &= \iint\limits_{\Gamma_-} g_-(x,v)\ \rho_g(|\nu_x \cdot v|)u_+(x,v_i(x,v))\ |v.\nu(x)|\ dS(x)d\mu(v) \\ &= \iint\limits_{\Gamma_+} g_-(x,v_r(x,v_i))\ \rho_g(|\nu_x \cdot v_i|)u_+(x,v_i)\ v_i.\nu(x)\ dS(x)d\mu(v_i), \\ &= \iint\limits_{\Gamma_+} g_-(x,v_r(x,v))\ \rho_g(|\nu_x \cdot v|)u_+(x,v)\ v.\nu(x)\ dS(x)d\mu(v), \end{aligned} \tag{3.35}$$

by making the change of variables $(x,v) = (x,v_r(x,v_i))$ and using Lemma 3.1. From (3.34) and (3.35), we obtain (3.33). □

Now, we want to prove that (3.33) implies $g_+(x,v) = \rho_g(|\nu_x \cdot v|)g_-(x,v_r)$, for a.e. $(x,v) \in \Gamma_+$. This will result from the following lemma:

3.6 Lemma. *Given any* $u_+ \in \mathcal{K}(\Gamma_+;\mathbb{R})$*, there exists a function* $u \in D(A)$ *such that* $u|_{\Gamma_+} = u_+$*.*

Proof : Let us set:

$$u_-(x,v) = \rho_g(|\nu_x \cdot v|)u_+(x, v - 2(v_x \cdot v)v_x),\ \textit{for a.e.}\ (x,v) \in \Gamma_-\ .$$

Now, we have to prove that there exists a function $u \in W^p(\Omega \times V)$ such that $u|_{\Gamma_+} = u_+$ and $u|_{\Gamma_-} = u_-$ in the sense of traces theory. u_+ has a compact support in Γ_+. Consequently, the infimum of the positive continuous function

$$\tau|_{\Gamma_+} : \Gamma_+ \to \mathbb{R}_+^* : (x,v) \longmapsto \inf\{t > 0; x - tv \in \partial\Omega\}$$

on the support of u_+ is a strictly positive real number, which implies that $u_+ \in L^p(\Gamma_+; d\widetilde{\xi}^p_+)$ where $d\widetilde{\xi}^p_+ := (\tau|_{\Gamma_+})^{1-p}\nu_x \cdot v \; dS(x)d\mu(v)$ (for the given definition of $d\widetilde{\xi}^p_+$, see [12], [13] or [19] p.250). The mapping

$$\Gamma_+ \to \Gamma_- : (x, v) \longmapsto (x, v - 2(\nu_x \cdot v)\nu_x)$$

is a topological isomorphism from Γ_+ onto Γ_-. Consequently, the support of u_- is a compact set contained in Γ_-. This implies that the infimum of the positive continuous function

$$\tau|_{\Gamma_-} : \Gamma_- \to \mathbb{R}^*_+ : (x, v) \longmapsto \inf\{t > 0; x + tv \in \partial\Omega\}$$

on the compact support of u_- is strictly positive. Thus $u_- \in L^p(\Gamma_-; d\widetilde{\xi}^p_-)$ where $d\widetilde{\xi}^p_- := (\tau|_{\Gamma_-})^{1-p} \, |\nu_x \cdot v| \, dS(x)d\mu(v)$ (for the given definition of $d\widetilde{\xi}^p_-$, see [12], [13] or [19] p.251). Then by ([12] Theorem of trace 2, p.833, or [19] Theorem 1 p.252), there exists a function $u \in W^p(\Omega \times V)$ such that $u|_{\Gamma_+} = u_+$ and $u|_{\Gamma_-} = u_-$ in the sense of trace theory. □

3.1 Corollary. *Let $\lambda > 0$. Assume $g \in L^{p'}(\Omega \times V)$ satisfy (3.27). Then $g = 0$. Consequently $R(\lambda - A) = L^p(\Omega \times V)$, $\forall\lambda > 0$.*

Proof : Proposition 3.4 tells us, that $v \cdot \nabla_x g = \lambda g$ and consequently that $g \in W^{p'}(\Omega \times V)$. By (3.33) and Lemma 3.6,

$$(g_+(x, v) - \rho_g(|\nu_x \cdot v|)g_-(x, v_r)) \; v.\nu(x) = 0, \text{ for a.e. } (x, v) \in \Gamma_+ \, ,$$

for the measure $dS(x) \otimes d\mu(v)$. As $v.\nu(x) > 0$, $\forall(x, v) \in \Gamma_+$, we have:

$$g_+(x, v) = \rho_g(|\nu_x \cdot v|)g_-(x, v_r), \text{ for a.e. } (x, v) \in \Gamma_+ \, .$$

Thus in view of (3.28), $g \in D(\hat{A})$ and $(\lambda - \hat{A})g = 0$. But, by Proposition 3.6, we know that $\hat{A}$ is a dissipative operator in $L^{p'}(\Omega \times V)$. Thus:

$$\left\|(\lambda - \hat{A})g\right\|_{L^{p'}(\Omega\times V)} \geq \lambda \left\|g\right\|_{L^{p'}(\Omega\times V)}, \; \forall\lambda > 0.$$

This inequality and $(\lambda - \hat{A})g = 0$, implies $g = 0$. This means that the range of $\lambda - A$ is dense in $L^p(\Omega \times V)$, $\forall\lambda > 0$. As the dissipativeness and closedness of A implies that the range of $\lambda - A$ is closed, we deduce that $R(\lambda - A) = L^p(\Omega \times V)$, $\forall\lambda > 0$. □

Summarizing, we have established the following theorem:

3.1 Theorem. *A is a dissipative and closed operator such that every $\lambda > 0$ belongs to the resolvent set of A. Consequently:*

$$\left\|\lambda(\lambda - A)^{-1}\right\|_{\mathcal{L}(L^p(\Omega\times V))} \leq 1, \; \forall\lambda > 0.$$

3.5 Remark. *It follows from the previous theorem and the density of $D(A)$, using Lumer-Phillips' theorem [18, p.343], that A is the generator of a contraction C_0-semigroup (we will not need that property of A in this chapter).*

3.2.5 Solving the radiative transfer boundary value problem

In this subsection, we want to solve the radiative transfer boundary value problem (3.1) with the nonhomogeneous boundary condition (3.2), not only in $L^p(\Omega \times V)$, but also in $L^\infty(\Omega \times V)$, and to give bounds on the solution.

3.2 Theorem. *Assuming $T(\cdot, t) \in L^p(\Omega)$, the quasi-stationary radiative transfer boundary value problem (3.1)-(3.2) has a unique solution $I^k(\cdot, t, \cdot) \in W^p(\Omega \times V)$. Moreover:*

$$\left\| I^k(\cdot, t, \cdot) - B_g^k(T_S(t)) \right\|_{L^p(\Omega \times V)} \leq \sqrt[p]{4\pi} \left\| B_g^k(T(\cdot, t)) - B_g^k(T_S(t)) \right\|_{L^p(\Omega)}. \tag{3.36}$$

Proof : Equivalently, the new unknown $u_k(x, v) := I^k(x, t, v) - B_g^k(T_S(t))$ must satisfy in the weak sense:

$$\begin{cases} v \cdot \nabla_x u_k(x, v) + \kappa_k u_k(x, v) = \kappa_k(B_g^k(T(x, t)) - B_g^k(T_S(t))), & \textit{for a.e. } (x, v) \in \Omega \times V, \\ u_k(x, v) = \rho_g(|\upsilon_x \cdot v|) u_k(x, v_i), & \textit{for a.e. } (x, v) \in \Gamma_- . \end{cases}$$

Taking $\lambda = \kappa_k$ in Theorem 3.1: $u_k = \kappa_k(\kappa_k - A)^{-1}(B_g^k(T(\cdot, t)) - B_g^k(T_S(t)))$ is the solution and

$$\begin{aligned} \|u_k\|_{L^p(\Omega \times V)} &\leq \left\| B_g^k(T(\cdot, t)) - B_g^k(T_S(t)) \right\|_{L^p(\Omega \times V)} \\ &= \sqrt[p]{4\pi} \left\| B_g^k(T(\cdot, t)) - B_g^k(T_S(t)) \right\|_{L^p(\Omega)}. \end{aligned}$$

Thus $I^k(\cdot, t, \cdot)$ satisfies inequality (3.36). $\square$

3.6 Remark. *It follows immediately from the equation*

$$v \cdot \nabla_x u_k(x, v) + \kappa_k u_k(x, v) = \kappa_k(B_g^k(T(x, t)) - B_g^k(T_S(t))), \textit{ for a.e. } (x, v) \in \Omega \times V,$$

that one has also:

$$\begin{aligned} &\left\| v \cdot \nabla_x (I^k(\cdot, t, \cdot) - B_g^k(T_S(t))) \right\|_{L^p(\Omega \times V)} \\ &\qquad \leq 2\sqrt[p]{4\pi}\kappa_k \left\| B_g^k(T(\cdot, t)) - B_g^k(T_S(t)) \right\|_{L^p(\Omega)}, \end{aligned}$$

and thus

$$\begin{aligned} &\left\| I^k(\cdot, t, \cdot) - B_g^k(T_S(t)) \right\|_{W^p(\Omega \times V)} \\ &\qquad \leq \sqrt[p]{4\pi}(1 + 2\kappa_k) \left\| B_g^k(T(\cdot, t)) - B_g^k(T_S(t)) \right\|_{L^p(\Omega)}. \end{aligned} \tag{3.37}$$

3.2 Corollary. *Supposing that $T(\cdot,t) \in L^\infty(\Omega)$, the solution $I^k(\cdot,t,\cdot)$ of the quasi-stationary radiative transfer boundary value problem (4.4)-(4.12), belongs to $L^\infty(\Omega\times V)$ and satisfies the estimate:*

$$\left\|I^k(\cdot,t,\cdot) - B_g^k(T_S(t))\right\|_{L^\infty(\Omega\times V)} \leq \left\|B_g^k(T(\cdot,t)) - B_g^k(T_S(t))\right\|_{L^\infty(\Omega)}. \tag{3.38}$$

Consequently, the trace on $\Gamma := \partial\Omega\times V$ of $I^k(\cdot,t,\cdot)$ belongs to $L^\infty(\Gamma; |v.\nu(x)|\, dS(x)d\mu(v))$ and $v\cdot\nabla_x I^k(\cdot,t,\cdot) \in L^\infty(\Omega\times V)$.

Proof : Let us remark that in this case the solution $I^k(\cdot,t,\cdot)$ is the same for every $p \in]1,+\infty[$. By Theorem 1 of [73, p.34], for every $n\in\mathbb{N}^*$, it follows from estimate (3.36):

$$\begin{aligned}
&\left\|\min(\left|I^k(\cdot,t,\cdot) - B_g^k(T_S(t))\right|, n)\right\|_{L^\infty(\Omega\times V)}\\
&= \lim_{p\to+\infty}\left\|\min(\left|I^k(\cdot,t,\cdot) - B_g^k(T_S(t))\right|, n)\right\|_{L^p(\Omega\times V)}\\
&\leq \limsup_{p\to+\infty}\left\|I^k(\cdot,t,\cdot) - B_g^k(T_S(t))\right\|_{L^p(\Omega\times V)}\\
&\leq \limsup_{p\to+\infty}\left\|B_g^k(T(\cdot,t)) - B_g^k(T_S(t))\right\|_{L^p(\Omega)} = \left\|B_g^k(T(\cdot,t)) - B_g^k(T_S(t))\right\|_{L^\infty(\Omega)}.
\end{aligned}$$

Letting n goes to infinity, we deduce that (3.38) holds. □

3.2.6 Uniform bounds on the radiative intensities

In this subsection, we want to give uniform bounds on the radiative intensities $I^k(\cdot,t,\cdot)$ in function of uniform bounds on $T(\cdot,t)$ and $T_S(t)$, the quantities appearing as data in the radiative boundary value problem (3.1)-(3.2).

Firstly, supposing $T(\cdot,t) \in L^\infty(\Omega)$, and given $\bar{T}\in\mathbb{R}_+^*$ such that $T_S(t)\leq\bar{T}$ and $T(x,t)\leq\bar{T}$, for a.e. $x\in\Omega$, we want to prove that $I^k(x,t,v)\leq B_g^k(\bar{T})$. For that purpose, we will need the following lemmas:

3.7 Lemma. *Let us suppose that $T(\cdot,t)\in L^\infty(\Omega)$. Let $I^k(\cdot,t,\cdot)\in W^p(\Omega\times V)$, be the solution of the quasi-stationary radiative transfer equation (3.1), with the reflectivity boundary condition (3.2). Then*

$$\iint_{\Omega\times V} v.\nabla_x(I^k(x,t,v) - B_g^k(T_S(t)))(I^k(x,t,v) - B_g^k(T_S(t)))_+\, dx d\mu(v) \geq 0.$$

Proof : Firstly, let us remark that this integral has sense by Hölder's inequality because $v.\nabla_x(I^k(\cdot,t,\cdot) - B_g^k(T_S(t))) \in L^p(\Omega\times V)$, $(I^k(\cdot,t,\cdot) - B_g^k(T_S(t)))_+ \in L^\infty(\Omega\times V)$. We may suppose $p\geq 2$, in view of the argument of uniqueness invoked at the beginning of the proof of Corollary 3.2. Also $(I^k(\cdot,t,\cdot) - B_g^k(T_S(t)))_+ \in W^{p'}(\Omega\times V)$ as $p'\leq p$. Thus, by Corollary 3.2 and Green's formula, Proposition 3.1, we have:

$$
\begin{aligned}
&\iint_{\Omega\times V} v.\nabla_x(I^k(x,t,v)-B_g^k(T_S(t)))(I^k(x,t,v)-B_g^k(T_S(t)))_+\, dxd\mu(v) \\
&= \iint_{\partial\Omega\times V}(I^k(x,t,v)-B_g^k(T_S(t)))(I^k(x,t,v)-B_g^k(T_S(t)))_+\,\nu_x.v\; dS(x)d\mu(v) \\
&\quad -\iint_{\Omega\times V}(I^k(x,t,v)-B_g^k(T_S(t)))v.\nabla_x(I^k(x,t,v)-B_g^k(T_S(t)))_+\, dxd\mu(v) \\
&= \iint_{\partial\Omega\times V}(I^k(x,t,v)-B_g^k(T_S(t)))(I^k(x,t,v)-B_g^k(T_S(t)))_+\,\nu_x.v\; dS(x)d\mu(v) \\
&\quad -\iint_{\Omega\times V}(I^k(x,t,v)-B_g^k(T_S(t)))v.\nabla_x(I^k(x,t,v)-B_g^k(T_S(t))) \\
&\qquad 1_{\{(x,v);I^k(x,t,v)>B_g^k(T_S(t))\}}\; dxd\mu(v).
\end{aligned}
$$

Thus by using Lemma 3.1 and (4.12):

$$
\begin{aligned}
&\iint_{\Omega\times V} v.\nabla_x(I^k(x,t,v)-B_g^k(T_S(t)))(I^k(x,t,v)-B_g^k(T_S(t)))_+\, dxd\mu(v) \\
&= \tfrac{1}{2}\iint_{\partial\Omega\times V}(I^k(x,t,v)-B_g^k(T_S(t)))_+^2\;\nu_x.v\; dS(x)d\mu(v) \\
&= \tfrac{1}{2}\iint_{\Gamma_+}(I^k(x,t,v)-B_g^k(T_S(t)))_+^2\;\nu_x.v\; dS(x)d\mu(v) \\
&\quad +\tfrac{1}{2}\iint_{\Gamma_-}(I^k(x,t,v_r)-B_g^k(T_S(t)))_+^2\;\nu_x.v_r\; dS(x)d\mu(v_r) \\
&= \tfrac{1}{2}\iint_{\Gamma_+}(I^k(x,t,v)-B_g^k(T_S(t)))_+^2\;\nu_x.v\; dS(x)d\mu(v) \\
&\quad -\tfrac{1}{2}\iint_{\Gamma_+}\rho_g(|\nu_x\cdot v|)^2(I^k(x,t,v)-B_g^k(T_S(t)))_+^2\;\nu_x.v\; dS(x)d\mu(v)\geq 0.
\end{aligned}
$$

What was to be proved. □

As a variant of the preceding lemma, we have:

3.8 Lemma. *Supposing that $T(\cdot,t)\in L^\infty(\Omega)$ and given $\bar{T}\in\mathbb{R}_+^*$ such that $T_S(t)\leq\bar{T}$, we have:*

$$\iint_{\Omega\times V} v\cdot\nabla_x(I^k(x,t,v)-B_g^k(\bar{T}))(I^k(x,t,v)-B_g^k(\bar{T}))_+\, dxd\mu(v)\geq 0.$$

Proof : Firstly, let us remark that this integral is meaningful by Hölder's inequality, because $I^k(\cdot,t,\cdot)\in W^p(\Omega\times V)$, $\forall p>1$. By using Proposition 3.1, and the fact that

$I^k(\cdot,t,\cdot)_{|\Gamma} \in L^\infty(\Gamma)$ owing to Corollary 3.2, we obtain:

$$\begin{aligned}&\iint\limits_{\Omega\times V} v\cdot\nabla_x(I^k(x,t,v)-B_g^k(\bar{T}))(I^k(x,t,v)-B_g^k(\bar{T}))_+\,dxd\mu(v)\\ &\qquad=\tfrac{1}{2}\iint\limits_{\Gamma_+}(I^k(x,t,v)-B_g^k(\bar{T}))_+^2\;\nu_x.v\;dS(x)d\mu(v)\\ &\qquad+\tfrac{1}{2}\iint\limits_{\Gamma_-}(I^k(x,t,v)-B_g^k(\bar{T}))_+^2\;\nu_x.v\;dS(x)d\mu(v).\end{aligned}\tag{3.39}$$

Now for $(x,v)\in\Gamma_-$:

$$\begin{aligned}&I^k(x,t,v)-B_g^k(\bar{T})\\ &=\rho_g(|\nu_x\cdot v|)I^k(x,t,v_i)+(1-\rho_g(|\nu_x\cdot v|))B_g^k(T_S(t))-B_g^k(\bar{T})\\ &=\rho_g(|\nu_x\cdot v|)(I^k(x,t,v_i)-B_g^k(\bar{T}))+(1-\rho_g(|\nu_x\cdot v|))(B_g^k(T_S(t))-B_g^k(\bar{T})).\end{aligned}$$

Due to our hypothesis $(1-\rho_g(|\nu_x\cdot v|))(B_g^k(T_S(t))-B_g^k(\bar{T}))$ is non-positive and thus for $(x,v)\in\Gamma_-$:

$$I^k(x,t,v)-B_g^k(\bar{T})\le\rho_g(|\nu_x\cdot v|)(I^k(x,t,v_i)-B_g^k(\bar{T}))$$

which implies

$$(I^k(x,t,v)-B_g^k(\bar{T}))_+\le\rho_g(|\nu_x\cdot v|)(I^k(x,t,v_i)-B_g^k(\bar{T}))_+,$$

and thus also

$$(I^k(x,t,v)-B_g^k(\bar{T}))_+^2\le\rho_g(|\nu_x\cdot v|)^2(I^k(x,t,v_i)-B_g^k(\bar{T}))_+^2.$$

Thus from (3.39), it follows using Lemma 3.1, that:

$$\begin{aligned}&\iint\limits_{\Omega\times V} v\cdot\nabla_x(I^k(x,t,v)-B_g^k(\bar{T}))(I^k(x,t,v)-B_g^k(\bar{T}))_+\,dxd\mu(v)\\ &\ge\tfrac{1}{2}\iint\limits_{\Gamma_+}(1-\rho_g(|\nu_x\cdot v|)^2)(I^k(x,t,v)-B_g^k(\bar{T}))_+^2\;\nu_x.v\;dS(x)d\mu(v)\ge 0.\end{aligned}$$

□

3.7 Remark. *Lemma 3.7 is in fact a particular case of Lemma 3.8.*

3.9 Proposition. *Supposing $T(\cdot,t)\in L^\infty(\Omega)$, and $T(x,t)\le T_S(t)$, $\forall' x\in\Omega$, we have that $I^k(x,t,v)\le B_g^k(T_S(t))$, $\forall'(x,v)\in\Omega\times V$.*

Proof : $I^k(\cdot,t,\cdot)$ is solution of:

$$\begin{cases} v.\nabla_x I^k(x,t,v)+\kappa_k I^k(x,t,v)=\kappa_k B_g^k(T(x,t)),\ \forall'(x,v)\in\Omega\times V\\ I^k(x,t,v)=\rho_g(|\nu_x\cdot v|)I^k(x,t,v_i)+(1-\rho_g(|\nu_x\cdot v|))B_g^k(T_S(t)),\ \forall'(x,v)\in\Gamma_-.\end{cases}$$

Thus

$$\begin{aligned}
&v.\nabla_x(I^k(x,t,v) - B_g^k(T_S(t))) + \kappa_k(I^k(x,t,v) - B_g^k(T_S(t))) \\
&= \kappa_k \left(B_g^k(T(x,t)) - B_g^k(T_S(t)) \right), \ \forall'(x,v) \in \Omega \times V.
\end{aligned}$$

Multiplying both sides of this last equality by $(I^k(x,t,v) - B_g^k(T_S(t)))_+$, and integrating with respect to the measure $dx \otimes d\mu(v)$, we obtain:

$$\begin{aligned}
&\iint\limits_{\Omega\times V} v.\nabla_x(I^k(x,t,v) - B_g^k(T_S(t)))(I^k(x,t,v) - B_g^k(T_S(t)))_+ \ dxd\mu(v) \\
&+\kappa_k \iint\limits_{\Omega\times V} (I^k(x,t,v) - B_g^k(T_S(t)))(I^k(x,t,v) - B_g^k(T_S(t)))_+ \ dxd\mu(v) \\
&= \kappa_k \iint\limits_{\Omega\times V} (B_g^k(T(x,t)) - B_g^k(T_S(t)))(I^k(x,t,v) - B_g^k(T_S(t)))_+ \ dxd\mu(v).
\end{aligned}$$

By Lemma 3.7, the first term in the left-hand side of the previous equality is positive and due to our hypothesis, the right-hand side is negative. Thus

$$\begin{aligned}
&\kappa_k \iint\limits_{\Omega\times V} (I^k(x,t,v) - B_g^k(T_S(t)))(I^k(x,t,v) - B_g^k(T_S(t)))_+ \ dxd\mu(v) \\
&\qquad = \kappa_k \iint\limits_{\Omega\times V} (I^k(x,t,v) - B_g^k(T_S(t)))_+^2 \ dxd\mu(v)
\end{aligned}$$

is negative. Consequently $(I^k(x,t,v) - B_g^k(T_S(t)))_+ = 0$, $\forall'(x,v) \in \Omega \times V$, $dxd\mu(v)$, from which the result follows. □

Similarly, to the previous proposition, we have:

3.10 Proposition. *Supposing $T(\cdot,t) \in L^\infty(\Omega)$, and given $\bar{T} \in \mathbb{R}_+^*$ such that $T_S(t) \leq \bar{T}$ and $T(x,t) \leq \bar{T}$, for a.e. $x \in \Omega$, we have that $I^k(x,t,v) \leq B_g^k(\bar{T})$, for a.e. $(x,v) \in \Omega \times V$.*

Proof : From equation (3.1) follows that:

$$\begin{aligned}
&v \cdot \nabla_x(I^k(x,t,v) - B_g^k(\bar{T}) + \kappa_k(I^k(x,t,v) - B_g^k(\bar{T})) \\
&= \kappa_k \left(B_g^k(T(x,t)) - B_g^k(\bar{T}) \right), \ \textit{for a.e.} \ (x,v) \in \Omega \times V.
\end{aligned}$$

Multiplying both sides of this last equality by $(I^k(x,t,v) - B_g^k(\bar{T}))_+$, and integrating with respect to the measure $dx \otimes d\mu(v)$, we obtain:

$$\begin{aligned}
&\iint\limits_{\Omega\times V} v \cdot \nabla_x(I^k(x,t,v) - B_g^k(\bar{T}))(I^k(x,t,v) - B_g^k(\bar{T}))_+ \ dxd\mu(v) \\
&+\kappa_k \iint\limits_{\Omega\times V} (I^k(x,t,v) - B_g^k(\bar{T}))(I^k(x,t,v) - B_g^k(\bar{T}))_+ \ dxd\mu(v) \\
&= \kappa_k \iint\limits_{\Omega\times V} (B_g^k(T(x,t)) - B_g^k(\bar{T}))(I^k(x,t,v) - B_g^k(\bar{T}))_+ \ dxd\mu(v).
\end{aligned}$$

By Lemma 3.8, the first term in the left-hand side of the previous equality is non-negative and due to our hypothesis, the right-hand side is non-positive. Thus

$$\begin{aligned}\kappa_k \iint\limits_{\Omega\times V} &(I^k(x,t,v) - B_g^k(\bar{T}))(I^k(x,t,v) - B_g^k(\bar{T}))_+ \, dxd\mu(v) \\ &= \kappa_k \iint\limits_{\Omega\times V} (I^k(x,t,v) - B_g^k(\bar{T}))_+^2 \, dxd\mu(v)\end{aligned}$$

is non-positive. Consequently $(I^k(x,t,v) - B_g^k(\bar{T}))_+ = 0$, *for a.e.* $(x,v) \in \Omega \times V$, $dxd\mu(v)$, from which the result follows. ☐

3.8 Remark. *Proposition 3.9, is in fact a particular case of Proposition 3.10, by taking* $\bar{T} = T_S(t)$.

3.11 Proposition. *Supposing* $T(\cdot,t) \in L^\infty(\Omega)$, *and given* $\underline{T} \in \mathbb{R}_+^*$ *such that* $T(x,t) \geq \underline{T}$, *for a.e.* $x \in \Omega$, *and* $T_S(t) \geq \underline{T}$, *we have also that* $I^k(x,t,v) \geq B_g^k(\underline{T})$, *for a.e.* $(x,v) \in \Omega \times V$.

Proof : From

$$v.\nabla_x I^k(x,t,v) + \kappa_k I^k(x,t,v) = \kappa_k B_g^k(T(x,t)), \ \forall'(x,v) \in \Omega \times V,$$

follows that:

$$\begin{aligned} v.\nabla_x \left(I^k(x,t,v) - B_g^k(\underline{T})\right) + \kappa_k \left(I^k(x,t,v) - B_g^k(\underline{T})\right) = \\ \kappa_k \left(B_g^k(T(x,t)) - B_g^k(\underline{T})\right)\end{aligned}$$

$\forall'(x,v) \in \Omega \times V$. Multiplying both sides of this last equality by $(I^k(x,t,v) - B_g^k(\underline{T}))_-$, and integrating with respect to the measure $dx \otimes d\mu(v)$, we obtain:

$$\begin{aligned} &\kappa_k \iint\limits_{\Omega\times V} \left(I^k(x,t,v) - B_g^k(\underline{T})\right) \left(I^k(x,t,v) - B_g^k(\underline{T})\right)_- dxd\mu(v) \\ &= \kappa_k \iint\limits_{\Omega\times V} \left(B_g^k(T(x,t)) - B_g^k(\underline{T})\right) \left(I^k(x,t,v) - B_g^k(\underline{T})\right)_- dxd\mu(v) \\ &- \iint\limits_{\Omega\times V} v.\nabla_x \left(I^k(x,t,v) - B_g^k(\underline{T})\right) \left(I^k(x,t,v) - B_g^k(\underline{T})\right)_- dxd\mu(v). \end{aligned} \tag{3.40}$$

Due to our hypothesis, the first term in the right-hand side of the previous equality is positive. By Green's formula, Proposition 3.1, we have concerning the second term in the right-hand side of the previous equality:

$$\begin{aligned}
&-\iint\limits_{\Omega\times V} v.\nabla_x \left(I^k(x,t,v)-B_g^k(\underline{T})\right)\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_- dxd\mu(v)\\
&=-\iint\limits_{\partial\Omega\times V}\left(I^k(x,t,v)-B_g^k(\underline{T})\right)\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_- \nu_x.v\; dS(x)d\mu(v)\\
&\quad+\iint\limits_{\Omega\times V}\left(I^k(x,t,v)-B_g^k(\underline{T})\right) v.\nabla_x\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_- dxd\mu(v).\\
&=-\iint\limits_{\partial\Omega\times V}\left(I^k(x,t,v)-B_g^k(\underline{T})\right)\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_- \nu_x.v\; dS(x)d\mu(v)\\
&\quad+\iint\limits_{\Omega\times V}\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_- v.\nabla_x\left(I^k(x,t,v)-B_g^k(\underline{T})\right) dxd\mu(v),
\end{aligned}$$

as $\nabla_x\left(I^k(\cdot,t,\cdot)-B_g^k(\underline{T})\right)_- = -1_{\{I^k(\cdot,t,\cdot)<B_g^k(\underline{T})\}}\nabla_x\left(I^k(\cdot,t,\cdot)-B_g^k(\underline{T})\right)$. Thus:

$$\begin{aligned}
&-\iint\limits_{\Omega\times V} v.\nabla_x \left(I^k(x,t,v)-B_g^k(\underline{T})\right)\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_- dxd\mu(v)\\
&=-\frac{1}{2}\iint\limits_{\partial\Omega\times V}\left(I^k(x,t,v)-B_g^k(\underline{T})\right)\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_- \nu_x.v\; dS(x)d\mu(v)\\
&=\frac{1}{2}\iint\limits_{\Gamma_+}\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_-^2\; \nu_x.v\; dS(x)d\mu(v)\\
&\quad+\frac{1}{2}\iint\limits_{\Gamma_-}\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_-^2 \nu_x.v\; dS(x)d\mu(v)\\
&=\frac{1}{2}\iint\limits_{\Gamma_+}\left(I^k(x,t,v)-B_g^k(\underline{T})\right)_-^2\; \nu_x.v\; dS(x)d\mu(v)\\
&\quad-\frac{1}{2}\iint\limits_{\Gamma_+}\left(\rho_g(|\nu_x\cdot v|)I^k(x,t,v)+(1-\rho_g(|\nu_x\cdot v|))B_g^k(T_S(t))-B_g^k(\underline{T})\right)_-^2 \nu_x.v\; dS(x)d\mu(v),
\end{aligned} \tag{3.41}$$

by using Lemma 3.1. But

$$\begin{aligned}
&\rho_g(|\nu_x\cdot v|)I^k(x,t,v)+(1-\rho_g(|\nu_x\cdot v|))B_g^k(T_S(t))-B_g^k(\underline{T})\\
&=\rho_g(|\nu_x\cdot v|)(I^k(x,t,v)-B_g^k(\underline{T}))+(1-\rho_g(|\nu_x\cdot v|))(B_g^k(T_S(t))-B_g^k(\underline{T})).
\end{aligned}$$

By our hypotheses: $B_g^k(T_S(t))\geq B_g^k(\underline{T})$, which implies that

$$(1-\rho_g(|\nu_x\cdot v|))(B_g^k(T_S(t))-B_g^k(\underline{T}))$$

is also positive. Thus if

$$\rho_g(|\nu_x\cdot v|)I^k(x,t,v)+(1-\rho_g(|\nu_x\cdot v|))B_g^k(T_S(t))-B_g^k(\underline{T})$$

is negative, then $\rho_g(|\nu_x \cdot v|)(I^k(x,t,v) - B_g^k(\underline{T}))$ is negative and consequently $\geq (I^k(x,t,v) - B_g^k(\underline{T}))$. Thus, in this case

$$\begin{aligned}&\left(I^k(x,t,v) - B_g^k(\underline{T})\right)_- \\ &\geq \left(\rho_g(|\nu_x \cdot v|)I^k(x,t,v) + (1-\rho_g(|\nu_x \cdot v|))B_g^k(T_S(t)) - B_g^k(\underline{T})\right)_- .\end{aligned}$$

This implies by (3.41) that:

$$-\iint\limits_{\Omega\times V} v.\nabla_x \left(I^k(x,t,v) - B_g^k(\underline{T})\right)\left(I^k(x,t,v) - B_g^k(\underline{T})\right)_- dx d\mu(v) \geq 0.$$

Going back to (3.40), it follows that

$$\kappa_k \iint\limits_{\Omega\times V} \left(I^k(x,t,v) - B_g^k(\underline{T})\right)\left(I^k(x,t,v) - B_g^k(\underline{T})\right)_- dx d\mu(v) \geq 0,$$

implying that $\left(I^k(x,t,v) - B_g^k(\underline{T})\right)_- = 0$, $\forall'(x,v) \in \Omega \times V$, $dxd\mu(v)$. What was to be proved. $\square$

3.3 Coupling with the Heat Conduction Equation

3.3.1 Notations and scope of this section

We consider now the initial boundary value problem (2.2) for the Heat Conduction Equation in our bounded domain Ω of class C^1 of $\mathbb{R}^3$ with the nonlinear nonhomogeneous Robin boundary condition:

$$\begin{cases} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \Delta T(x,t) - \sum\limits_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum\limits_{k=1}^{M}\kappa_k \int\limits_V I_T^k(x,t,v)d\mu(v), \\ \qquad in\ Q := \Omega\times]0,t_f[, \\ -k_h \frac{\partial T}{\partial \nu}(x,t) = h_c(T(x,t) - T_a) + \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda [B(T(x,t),\lambda) - B(T_S(t),\lambda)]d\lambda, \\ \qquad on\ \Sigma := \partial\Omega\times]0,t_f[, \\ T(x,0) = T_0(x),\ \ on\ \Omega. \end{cases} \tag{3.42}$$

This nonlinear nonhomogeneous Robin boundary condition $(3.42)_{(ii)}$ takes into account the radiative heating at the boundary of Ω, for those wavelengths $\lambda \geq \lambda_0 \approx 5\ 10^{-6}$ of the black radiation emitted by the radiative source S surrounding Ω. For these wavelengths the glass behaves like an opaque body. In these equations (3.42), $c_p, m_g, k_h, h_c, T_a, \lambda_0, \varepsilon_\lambda$ are all positive constants. The constants c_p, m_g and k_h are named respectively the heat capacity, the mass density and the thermal conductivity of the glass. h_c denotes the convective heat transfer coefficient and T_a the constant absolute temperature of the dry air between our semi-transparent solid Ω and the black radiative surface S surrounding it. $\varepsilon_\lambda \approx 0.9$ is called the spectral hemispherical emittance [46, pp.21-22]. We suppose that it is a measurable function of λ defined on the interval $[\lambda_0, +\infty[$, with values in the interval $[0,1]$, independent of the temperature. For more explanations about ε_λ, see [50]. We write in this section $I_T^k(\cdot,t,\cdot)$ instead of simply $I^k(\cdot,t,\cdot)$ to underline the dependence of $I^k(\cdot,t,\cdot)$ solution of the radiative transfer boundary value problem (3.1)-(3.2), with respect to the distribution of the absolute temperature $T(\cdot,t)$ in Ω at time t (our problem (3.42) is in fact a coupled problem: the coupled problem (2.2)-(2.1) stated in the introduction). In (3.42), $T_0 \in L^2(\Omega)$ denotes the initial absolute temperature of Ω and $T_S(\cdot)$ the given absolute temperature of the black radiative source function of the time t only. We suppose that $T_S \in L^\infty_+(0,t_f)$. The incident radiations [46] $\rho^k(x,t) := \int\limits_V I_T^k(x,t,v)d\mu(v)$ at point $x \in \Omega$ and time $t \in]0,t_f[$ $(k=1,\dots,M)$, define functions in $L^2(Q)$ because it follows from inequality (3.36) that the solution $I_T^k(\cdot,\cdot,\cdot)$ to (3.1)-(3.2) belongs to $L^2(0,t_f;L^2(\Omega\times V))$.

Here, we want to prove existence and uniqueness of the weak solution to (3.42). Firstly, we must define the meaning of a weak solution $T \in W(0,t_f)$ to the initial boundary value problem (3.42). To alleviate the notations, similarly to [50, p.9], let us set:

$$\begin{cases} \psi(T) := -\sum\limits_{k=1}^{M} 4\pi\kappa_k B_g^k(T), \ \forall T \in \mathbb{R}, \\ h_T(x,t) := \sum\limits_{k=1}^{M} \kappa_k \int\limits_V I_T^k(x,t,v)d\mu(v), \\ \Theta(T) := \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda B(T,\lambda)d\lambda, \ \forall T \in \mathbb{R}. \end{cases} \tag{3.43}$$

3.2 Definition. *We shall say that $T \in W(0,t_f) := \{T \in L^2(0,t_f;H^1(\Omega)); \frac{dT}{dt} \in L^2(0,t_f;(H^1(\Omega))^*)\}$ [71, p.146] is a weak solution to (3.42) iff $\forall \varphi \in H^1(\Omega)$:*

$$\begin{aligned} c_p m_g \left\langle \tfrac{dT}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} &= \\ -k_h \int\limits_\Omega \nabla_x T(x,t) \nabla_x \varphi(x)dx + \int\limits_\Omega \psi(T(x,t))\varphi(x)dx &+ \int\limits_\Omega h_T(x,t)\varphi(x)dx \\ + \langle h_c(T_a - T(\cdot,t)) + \Theta(T_S(t)) - \Theta(T(\cdot,t)), \varphi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} &, \end{aligned} \tag{3.44}$$

where $H^{-1/2}(\Gamma) := \left(H^{1/2}(\Gamma)\right)^*$, and $T(\cdot,0) = T_0$. This last condition is meaningful due to $W(0,t_f) \hookrightarrow C([0,t_f];L^2(\Omega))$ ([71, p.148], [36, p.40]).

Firstly, we are going to prove that (3.42) has at most one weak solution. The proof of the existence of a weak solution, requiring previously to establish "maximum principles", will be given later.

3.3.2 Uniqueness result

3.12 Proposition. *The initial boundary value problem (3.42) has at most one weak solution.*

Proof : Let $T_1, T_2 \in W(0,t_f)$ be two weak solutions of (3.42) such that $T_1(\cdot,0) = T_2(\cdot,0) = T_0(.)$. Thus $T = T_1 - T_2$ satisfies:

$$c_p m_g \left\langle \tfrac{dT}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} = -k_h \int_\Omega \nabla_x T(x,t) \nabla_x \varphi(x) dx$$
$$+ \int_\Omega \left[\psi(T_1(x,t) - \psi(T_2(x,t)\right] \varphi(x) dx + \int_\Omega \left[h_{T_1}(x,t) - h_{T_2}(x,t)\right] \varphi(x) dx$$
$$+ h_c \left\langle T_2(\cdot,t) - T_1(\cdot,t), \varphi \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} + \left\langle \Theta(T_2(.,t)) - \Theta(T_1(\cdot,t)), \varphi \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)},$$
$$\forall \varphi \in H^1(\Omega).$$

As $T(.,t) \in H^1(\Omega)$ for a.e. $t \in]0,t_f[$, we may choose $\varphi = T(.,t)$ for such a $t \in]0,t_f[$, and obtain:

$$\tfrac{c_p m_g}{2} \tfrac{d}{dt} \int_\Omega T(x,t)^2 dx + k_h \int_\Omega \left|\nabla_x T(x,t)\right|^2 dx =$$
$$\int_\Omega \left[\psi(T_1(x,t) - \psi(T_2(x,t)\right] T(x,t) dx + \int_\Omega \left[h_{T_1}(x,t) - h_{T_2}(x,t)\right] T(x,t) dx$$
$$- h_c \left\langle T(\cdot,t), T(\cdot,t) \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} - \left\langle \Theta(T_1(.,t)) - \Theta(T_2(\cdot,t)), T(\cdot,t) \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}.$$

As $-\psi$ and Θ are increasing functions on $\mathbb{R}$, the first term and the last one, in the right-hand side are negative terms. Obviously, the last but one term in the right-hand side is also negative. Thus our last equation, implies the inequality:

$$\frac{c_p m_g}{2} \frac{d}{dt} \int_\Omega T(x,t)^2 dx + k_h \int_\Omega \left|\nabla_x T(x,t)\right|^2 dx \leq \int_\Omega \left[h_{T_1}(x,t) - h_{T_2}(x,t)\right] T(x,t) dx. \quad (3.45)$$

Let us bound the right-hand side of the previous inequality. By equations (3.1) and (3.2), $I^k(\cdot,t,\cdot) := I^k_{T_1}(\cdot,t,\cdot) - I^k_{T_2}(\cdot,t,\cdot)$ is solution of:

$$\begin{cases} v \cdot \nabla_x I^k(x,t,v) + \kappa_k I^k(x,t,v) = \kappa_k (B^k_g(T_1(x,t)) - B^k_g(T_2(x,t))), \\ \qquad \textit{for a.e. } (x,v) \in \Omega \times V, \\ I^k(x,t,v) = \rho_g(|v_x \cdot v|) I^k(x,t,v_i), \ \textit{for a.e. } (x,v) \in \Gamma_- . \end{cases}$$

By Proposition 3.2, for a.e. $t \in]0,t_f[$:

$$\begin{aligned} \left\|I^k(\cdot,t,\cdot)\right\|_{L^2(\Omega\times V)} &\leq \left\|B^k_g(T_1(\cdot,t)) - B^k_g(T_2(\cdot,t))\right\|_{L^2(\Omega\times V)} \\ &\leq \sqrt{4\pi} \left\|B^k_g(T_1(\cdot,t)) - B^k_g(T_2(\cdot,t))\right\|_{L^2(\Omega)}. \end{aligned}$$

But by [50, p.14], $\forall k = 1, \dots, M$, we have also that:

$$\left\| B_g^k(T_1(\cdot,t)) - B_g^k(T_2(\cdot,t)) \right\|_{L^2(\Omega)} \lesssim \| T_1(\cdot,t) - T_2(\cdot,t) \|_{L^2(\Omega)},$$

so that, by definition $(3.43)_{(ii)}$, follows that:

$$\| h_{T_1}(\cdot,t) - h_{T_2}(\cdot,t) \|_{L^2(\Omega)} \lesssim \| T_1(\cdot,t) - T_2(\cdot,t) \|_{L^2(\Omega)} = \| T(\cdot,t) \|_{L^2(\Omega)}. \tag{3.46}$$

Using (3.46) in the estimate (3.45), we get:

$$\frac{c_p m_g}{2} \frac{d}{dt} \int_\Omega T(x,t)^2 dx + k_h \int_\Omega |\nabla_x T(x,t)|^2 \, dx \lesssim \| T(\cdot,t) \|^2_{L^2(\Omega)}.$$

In particular, this implies that there exists a constant $C > 0$, such that:

$$\frac{d}{dt} \int_\Omega T(x,t)^2 dx \leq C \int_\Omega T(x,t)^2 dx.$$

Thus by Gronwall's inequality [25, p.624, (ii)], $\int_\Omega T(x,t)^2 dx = 0,\ \forall t \in [0, t_f]$.Thus:

$$T_1(\cdot,t) = T_2(\cdot,t),\ \forall t \in [0, t_f].$$

□

3.3.3 Definition of the fixed point problem

Now, we want to prove the existence of a weak solution to our initial boundary value problem (3.42). In this purpose, we now define a "fixed point problem", allowing to decouple our problem (3.42), and in turn reducing to a semilinear parabolic initial boundary value problem. Firstly, we define a closed convex set S of $L^2(Q)$:

$$S := \{ T \in L^2(0, t_f; L^2(\Omega)); \forall' t \in]0, t_f[: \forall' x \in \Omega : T_a \leq T(x,t) \leq T_S(t) \}. \tag{3.47}$$

We suppose $T_S(t) \geq T_a$, $\forall' t \in]0, t_f[$, so that $S \neq \emptyset$. Now, our fixed point problem is the following (for the notations, see (3.43)): given $T \in S$, find $\tilde{T} \in S$, weak solution of:

$$\left\{ \begin{array}{c} c_p m_g \frac{\partial \tilde{T}}{\partial t}(x,t) = k_h \Delta \tilde{T}(x,t) + \psi(\tilde{T}(x,t)) + h_T(x,t), \\ -k_h \frac{\partial \tilde{T}}{\partial \nu}(x,t) = h_c(\tilde{T}(x,t) - T_a) + \Theta(\tilde{T}(x,t)) - \Theta(T_S(t)), \\ \tilde{T}(x,0) = T_0(x),\ \forall' x \in \Omega. \end{array} \right. \tag{3.48}$$

By $\tilde{T}$ is a weak solution of (3.48), we mean that $\tilde{T} \in W(0,t_f)$ and satisfies

$$\begin{aligned} & c_p m_g \left\langle \frac{d\tilde{T}}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} \\ &= -k_h \int_\Omega \nabla_x \tilde{T}(x,t) \nabla_x \varphi(x) dx + \int_\Omega \psi(\tilde{T}(x,t)) \varphi(x) dx \\ &+ \int_\Omega h_T(x,t) \varphi(x) dx + h_c \left\langle T_a - \tilde{T}(\cdot,t), \varphi \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\ &+ \left\langle \Theta(T_S(t)) - \Theta(\tilde{T}(\cdot,t)), \varphi \right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}, \end{aligned} \tag{3.49}$$

$\forall \varphi \in H^1(\Omega)$ as well as $\tilde{T}(\cdot,0) = T_0(.)$. We must show that $\tilde{T}$ exists, is unique and satisfies

$$\forall' t \in]0,t_f[: \forall' x \in \Omega : T_a \leq \tilde{T}(x,t) \leq T_S(t)$$

to insure that $\tilde{T}$ still belongs to our closed convex set S. Let us note that, as $T \in S$ by hypothesis, we know by Proposition 3.9 and Proposition 3.11 that $\forall'(x,v) \in \Omega \times V$: $B_g^k(T_a) \leq I_T^k(x,t,v) \leq B_g^k(T_S(t))$. Let us note that every function $T \in S$ is positive and essentially bounded. Firstly, more generally, we have the following existence an uniqueness result for our problem (3.48):

3.13 Proposition. *For every $T \in L_+^\infty(Q)$, problem (3.48) possesses one and only one weak solution in the space $W(0,t_f)$.*

Proof : We want to apply Lemma 5.3 p.373 of [71] concerning the existence and uniqueness of the weak solution of a semilinear parabolic problem with a semilinear Robin boundary condition. Thus, we have to verify that the Assumptions 5.1 and 5.2 from p.266 of [71] hold. The function $d(y) := -\frac{\psi(y)}{c_p m_g}$ and the function $b(y) := h_c y + \Theta(y)$ are increasing with respect to y. By Corollary 3.7 p.14 (by the inequality in the proof of this corollary in fact) and (3.13) p.17 of [50], they are also globally Lipschitz continuous with respect to y. As $T_S \in L^\infty(]0,t_f[) \subset L^2(]0,t_f[)$, $\Theta(T_S) \in L^2(\Sigma)$ by Corollary 3.4 p.12 of [50]. Also by inequality (3.36), $h_T(\cdot,\cdot) \in L^2(Q)$. Thus by Lemma 5.3 p.373 of [71], for any initial condition $T_0 \in L^2(\Omega)$, problem (3.48) possesses one and only one weak solution in the space $W(0,t_f)$. □

3.3.4 Maximum principles

To prove that the mapping $T \mapsto \tilde{T}$ operates in the closed convex set, we need to establish some maximum principles:

3.14 Proposition. *Let $T \in L^\infty(Q)$ be given such that $T(x,t) \geq \underline{T}$, for a.e. $(x,t) \in Q$. Then $\tilde{T}(x,t) \geq \underline{T}$, for a.e. $(x,t) \in Q$, where $\underline{T} \in \mathbb{R}_+^*$ satisfies* $\begin{cases} \underline{T} \leq T_a, \\ \underline{T} \leq T_0(x), \text{ for a.e. } x \in \Omega, \\ \underline{T} \leq T_S(t), \text{ for a.e. } t \in]0,t_f[. \end{cases}$

Proof : Let

$$H : \mathbb{R} \to \mathbb{R} : y \mapsto \begin{cases} y^2/2, & \text{if } y < 0, \\ 0, & \text{if } y \geq 0. \end{cases}$$

Thus

$$H(\tilde{T}(x,t) - \underline{T}) = \begin{cases} \frac{1}{2}(\tilde{T}(x,t) - \underline{T})^2, & \text{if } \tilde{T}(x,t) < \underline{T}, \\ 0, & \text{if } \tilde{T}(x,t) \geq \underline{T}. \end{cases}$$

For every $t \in]0, t_f[$, let us set:

$$\tilde{\varphi}(t) := c_p m_g \int_\Omega H(\tilde{T}(\cdot, t) - \underline{T})\ dx.$$

As $\tilde{T} \in W(0,t_f) \hookrightarrow C([0,t_f]; L^2(\Omega))$, $\tilde{\varphi} \in C([0,t_f])$. Using the density of the space $\mathcal{D}([0,t_f]; C^1(\bar{\Omega}))$ in $W(0,t_f)$ endowed with its natural norm (consequently to [18], p.480 and Theorem 1.4.3.1 p.25 of [31] followed by a regularization [54] pp.46-47), and a sequence of functions of $\mathcal{D}([0,t_f]; C^1(\bar{\Omega}))$ approaching $\tilde{T}$ in $W(0,t_f)$, one can prove that $\forall' t \in]0, t_f[$:

$$\tilde{\varphi}'(t) = -c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot, t), (\tilde{T}(\cdot,t) - \underline{T})_- \right\rangle_{H^1(\Omega)^*, H^1(\Omega)} \tag{3.50}$$

in the sense of distributions. From that least formula and $\tilde{T} \in W(0,t_f)$, follows easily that $\frac{d\tilde{\varphi}}{dt} \in L^1(]0,t_f[)$. By formula (3.49):

$$\begin{aligned} \tilde{\varphi}'(t) &= -c_p m_g \left\langle \tfrac{d\tilde{T}}{dt}(\cdot, t), (\tilde{T}(\cdot,t) - \underline{T})_- \right\rangle_{H^1(\Omega)^*, H^1(\Omega)} \\ &= -k_h \int_\Omega \left| \nabla_x (\tilde{T}(x,t) - \underline{T})_- \right|^2 dx - \int_\Omega \psi(\tilde{T}(x,t))(\tilde{T}(x,t) - \underline{T})_- dx \\ &- \int_\Omega h_T(x,t)(\tilde{T}(x,t) - \underline{T})_- dx - h_c \int_\Gamma (T_a - \tilde{T}(x,t))\ (\tilde{T}(x,t) - \underline{T})_-\ dS(x) \\ &+ \int_\Gamma \left[\Theta(\tilde{T}(x,t)) - \Theta(T_S(t)) \right] (\tilde{T}(x,t) - \underline{T})_-\ dS(x). \end{aligned} \tag{3.51}$$

The first and fourth terms in the right-hand side of equation (3.51) are obviously negative. The last term is also negative because $\Theta(\cdot)$ is an increasing function and if $\tilde{T}(x,t) \leq \underline{T}$, then due to our hypothesis $T_S(t) \geq \underline{T}$, $\forall t \in]0,t_f[$, we have $T_S(t) \geq \tilde{T}(x,t)$, $\forall' x \in \Omega$. Thus:

$$\tilde{\varphi}'(t) \leq -\int_\Omega \left[h_T(x,t) + \psi(\tilde{T}(x,t)) \right] (\tilde{T}(x,t) - \underline{T})_- dx.$$

But by Proposition 3.11, $I^k(x,t,v) \geq B_g^k(\underline{T})$, $\forall'(x,v) \in \Omega \times V$. Thus:

$$\begin{aligned} h_T(x,t) &:= \sum_{k=1}^{M} \kappa_k \int_V I_T^k(x,t,v) d\mu(v) \\ &\geq \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(\underline{T}). \end{aligned}$$

On the other hand if $\tilde{T}(x,t) \geq \underline{T}$, then $(\tilde{T}(x,t) - \underline{T})_- = 0$, and if $\tilde{T}(x,t) < \underline{T}$, then

$$-\psi(\tilde{T}(x,t)) = \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(\tilde{T}(x,t)) \leq \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(\underline{T}).$$

Thus:

$$\tilde{\varphi}'(t) \leq \int_\Omega \left[\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(\underline{T}) - h_T(x,t) \right] (\tilde{T}(x,t) - \underline{T})_- dx \leq 0.$$

In conclusion, $\tilde{\varphi}$ is an absolutely continuous nonnegative function on the interval $[0, t_f]$, null for $t = 0$ due to our hypothesis that $T_0(x) \geq \underline{T}$, $\forall' x \in \Omega$, and decreasing. Thus $\tilde{\varphi} = 0$. In view of its definition, this implies $\tilde{T}(x,t) \geq \underline{T}$, $\forall'(x,t) \in Q$. What was to be proved. □

3.15 Proposition. *Let us suppose that $T \in L_+^\infty(Q)$. Let us suppose that the initial condition $T_0(\cdot)$ satisfies $\forall t \in]0, t_f[$: $T_a \leq T_0(x) \leq T_S(t)$, $\forall' x \in \Omega$, that $T_S \in H^1(]0, t_f[)$, that $\frac{dT_S}{dt} \geq 0$ a.e., and that $T(x,t) \leq T_S(t)$, $\forall'(x,t) \in Q$. Under these hypotheses, we have also that: $\tilde{T}(x,t) \leq T_S(t)$, $\forall'(x,t) \in Q$.*

Proof : Let

$$\check{H} : \mathbb{R} \to \mathbb{R} : y \mapsto \begin{cases} y^2/2, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0. \end{cases}$$

Thus

$$\check{H}(\tilde{T}(x,t) - T_S(t)) = \begin{cases} \frac{1}{2}(\tilde{T}(x,t) - T_S(t))^2, & \text{if } \tilde{T}(x,t) > T_S(t), \\ 0, & \text{if } \tilde{T}(x,t) \leq T_S(t). \end{cases}$$

For every $t \in]0, t_f[$, let us set:

$$\tilde{\varphi}(t) := c_p m_g \int_\Omega \check{H}(\tilde{T}(\cdot,t) - T_S(t))\, dx.$$

As $\tilde{T} \in W(0,t_f) \hookrightarrow C([0,t_f];L^2(\Omega))$, $\tilde{\varphi} \in C([0,t_f])$. Using the same arguments that in the previous proposition to prove formula (3.50), we have $\forall' t \in]0,t_f[$:

$$\begin{aligned}\tilde{\varphi}'(t) &= c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot,t) - \frac{dT_S}{dt}(t), (\tilde{T}(\cdot,t) - T_S(t))_+ \right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \\ &= c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot,t), (\tilde{T}(\cdot,t) - T_S(t))_+ \right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \\ &- c_p m_g \frac{dT_S}{dt}(t) \int_\Omega (\tilde{T}(x,t) - T_S(t))_+ dx,\end{aligned} \tag{3.52}$$

in the sense of distributions. From that least formula and $\tilde{T} \in W(0,t_f)$, follows easily that $\frac{d\tilde{\varphi}}{dt} \in L^1(]0,t_f[)$. Also, it is clear from our hypothesis $\frac{dT_S}{dt} \geq 0$ a.e., that the second term in the right-hand side of formula (3.52) is negative a.e.. By formula (3.49):

$$\begin{aligned} &c_p m_g \left\langle \tfrac{d\tilde{T}}{dt}(\cdot,t), (\tilde{T}(\cdot,t) - T_S(t))_+ \right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \\ &= -k_h \int_\Omega \nabla_x \tilde{T}(x,t).\nabla_x (\tilde{T}(x,t) - T_S(t))_+ dx \\ &+ h_c \int_\Gamma (T_a - \tilde{T}(x,t))(\tilde{T}(x,t) - T_S(t))_+ \, dS(x) \\ &+ \int_\Gamma \left[\Theta(T_S(t)) - \Theta(\tilde{T}(x,t))\right] (\tilde{T}(x,t) - T_S(t))_+ \, dS(x) \\ &+ \int_\Omega \psi(\tilde{T}(x,t))(\tilde{T}(x,t) - T_S(t))_+ dx + \int_\Omega h_T(x,t)(\tilde{T}(x,t) - T_S(t))_+ dx \end{aligned} \tag{3.53}$$

As

$$-k_h \int_\Omega \nabla_x \tilde{T}(x,t).\nabla_x (\tilde{T}(x,t) - T_S(t))_+ dx = -k_h \int_\Omega \left| \nabla_x (\tilde{T}(x,t) - T_S(t))_+ \right|^2 dx,$$

the first term in the right-hand side of equation (3.53) is negative. Due to our hypotheses, the second term in the right-hand side of equation (3.53) is obviously negative. The third term also is negative, as $\Theta(\cdot)$ is an increasing function. Let us examine now the last two terms in the right-hand side of equation (3.53). By our hypotheses and Proposition 3.9:

$$\begin{aligned} h_T(x,t) &:= \sum_{k=1}^{M} \kappa_k \int_V I_T^k(x,t,v) d\mu(v) \\ &\leq \sum_{k=1}^{M} 4\pi \kappa_k B_g^k(T_S(t)). \end{aligned}$$

Thus

$$\begin{aligned}
&\int_\Omega (h_T(x,t)+\psi(\tilde{T}(x,t)))(\tilde{T}(x,t)-T_S(t))_+dx\\
&\quad\le \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T_S(t))\int_\Omega(\tilde{T}(x,t)-T_S(t))_+dx\\
&\quad-\sum_{k=1}^{M} 4\pi\kappa_k\int_\Omega B_g^k(\tilde{T}(x,t))(\tilde{T}(x,t)-T_S(t))_+dx\\
=&\sum_{k=1}^{M}4\pi\kappa_k\int_\Omega\left[B_g^k(T_S(t))-B_g^k(\tilde{T}(x,t))\right](\tilde{T}(x,t)-T_S(t))_+\,dx\le 0.
\end{aligned}$$

Thus $\tilde{\varphi}'(t)\le 0$. In conclusion, $\tilde{\varphi}$ is an absolutely continuous nonnegative function on the interval $[0,t_f]$, null for $t=0$ due to our hypothesis on $T_0(\cdot)$ and decreasing. Thus $\tilde{\varphi}=0$. In view of its definition, this implies $\tilde{T}(x,t)\le T_S(t)$, $\forall'(x,t)\in Q$. What was to be proved. □

Alternatively, to the previous proposition, we have also:

3.16 Proposition. *Let us suppose that $T\in L^\infty_+(Q)$ and that $T(x,t)\le\bar{T}$, for a.e. $(x,t)\in Q:=\Omega\times]0,t_f[$. Then, we have also that: $\tilde{T}(x,t)\le\bar{T}$, for a.e. $(x,t)\in Q:=\Omega\times]0,t_f[$, where $\bar{T}\in\mathbb{R}^*_+$ satisfy* $\begin{cases} T_a\le\bar{T}, \\ T_0(x)\le\bar{T},\ \textit{for a.e. } x\in\Omega,\\ T_S(t)\le\bar{T},\ \textit{for a.e. } t\in]0,t_f[.\end{cases}$

Proof : Let

$$\check{H}:\mathbb{R}\to\mathbb{R}:y\mapsto\begin{cases} y^2/2, & \text{if } y>0,\\ 0, & \text{if } y\le 0.\end{cases}$$

Thus

$$\check{H}(\tilde{T}(x,t)-\bar{T})=\begin{cases}\frac{1}{2}(\tilde{T}(x,t)-\bar{T})^2, & \text{if } \tilde{T}(x,t)>\bar{T},\\ 0, & \text{if } \tilde{T}(x,t)\le\bar{T}.\end{cases}$$

For every $t\in]0,t_f[$, let us set:

$$\tilde{\varphi}(t):=c_pm_g\int_\Omega\check{H}(\tilde{T}(\cdot,t)-\bar{T})\,dx.$$

As $\tilde{T}\in W(0,t_f)\hookrightarrow C([0,t_f];L^2(\Omega))$, $\tilde{\varphi}\in C([0,t_f])$. Using the same arguments as in the proof of formula (3.50), we have $\forall' t\in]0,t_f[$:

$$\tilde{\varphi}'(t)=c_pm_g\left\langle\frac{d\tilde{T}}{dt}(\cdot,t),(\tilde{T}(\cdot,t)-\bar{T})_+\right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \tag{3.54}$$

in the sense of distributions. From that least formula and $\tilde{T}\in W(0,t_f)$, follows easily that $\frac{d\tilde{\varphi}}{dt} \in L^1(]0, t_f[)$. By formula (3.49):

$$\begin{aligned} & c_p m_g \left\langle \frac{d\tilde{T}}{dt}(\cdot, t), (\tilde{T}(\cdot, t) - \bar{T})_+ \right\rangle_{H^1(\Omega)^*, H^1(\Omega)} \\ & = -k_h \int_\Omega \nabla_x \tilde{T}(x,t).\nabla_x (\tilde{T}(x,t) - \bar{T})_+ dx \\ & + h_c \int_\Gamma (T_a - \tilde{T}(x,t))(\tilde{T}(x,t) - \bar{T})_+ \, dS(x) \\ & + \int_\Gamma \left[\Theta(T_S(t)) - \Theta(\tilde{T}(x,t))\right] (\tilde{T}(x,t) - \bar{T})_+ \, dS(x) \\ & + \int_\Omega \psi(\tilde{T}(x,t))(\tilde{T}(x,t) - \bar{T})_+ dx + \int_\Omega h_T(x,t)(\tilde{T}(x,t) - \bar{T})_+ dx \end{aligned} \tag{3.55}$$

As

$$-k_h \int_\Omega \nabla_x \tilde{T}(x,t).\nabla_x (\tilde{T}(x,t) - \bar{T})_+ dx = -k_h \int_\Omega \left|\nabla_x (\tilde{T}(x,t) - \bar{T})_+\right|^2 dx,$$

the first term in the right-hand side of equation (3.55) is negative. Due to our hypotheses, the second and the third terms in the right-hand side of equation (3.55) are obviously negative. For the third term, we must also use that $\Theta(\cdot)$ is an increasing function. Let us examine now the last two terms in the right-hand side of equation (3.55). By Proposition 3.10, $\forall' t \in]0, t_f[$:

$$\begin{aligned} h_T(x,t) &:= \sum_{k=1}^{M} \kappa_k \int_V I_T^k(x,t,v) d\mu(v) \\ &\leq \sum_{k=1}^{M} 4\pi \kappa_k B_g^k(\bar{T}), \ \ \forall' x \in \Omega. \end{aligned}$$

Thus

$$\begin{aligned} & \int_\Omega (h_T(x,t) + \psi(\tilde{T}(x,t)))(\tilde{T}(x,t) - \bar{T})_+ dx \\ & \leq \sum_{k=1}^{M} 4\pi \kappa_k B_g^k(\bar{T}) \int_\Omega (\tilde{T}(x,t) - \bar{T})_+ dx \\ & - \sum_{k=1}^{M} 4\pi \kappa_k \int_\Omega B_g^k(\tilde{T}(x,t))(\tilde{T}(x,t) - \bar{T})_+ dx \\ & = \sum_{k=1}^{M} 4\pi \kappa_k \int_\Omega \left[B_g^k(\bar{T}) - B_g^k(\tilde{T}(x,t))\right] (\tilde{T}(x,t) - \bar{T})_+ \, dx \leq 0. \end{aligned}$$

Thus $\tilde{\varphi}'(t) \leq 0$. In conclusion, $\tilde{\varphi}$ is an absolutely continuous nonnegative function on the interval $[0, t_f]$, null for $t = 0$ due to our hypothesis on $T_0(\cdot)$ and decreasing. Thus $\tilde{\varphi} = 0$. In view of its definition, this implies $\tilde{T}(x,t) \leq \bar{T}$, $\forall' (x,t) \in Q$. What was to be proved. □

3.3.5 Existence of a fixed point

Under the hypotheses

$$\begin{cases} T_S \in H^1(]0,t_f[) \\ \forall t \in]0,t_f[: T_a \leq T_0(x) \leq T_S(t), \forall' x \in \Omega \\ \frac{dT_S}{dt} \geq 0 \quad a.e. \quad on \quad]0,t_f[\end{cases} \tag{3.56}$$

We now know by the Propositions 3.15 and 3.14 with $\underline{T} = T_a$, that the mapping Φ which sends T onto $\tilde{T}$ the weak solution of problem (3.48), operates into S. We must prove that $\Phi : S \to S$ verify the hypotheses of Schauder's theorem (see e.g. [50, p.21]), to derive the existence of a fixed point to Φ, thus of a weak solution to our coupled problem (3.42).

3.17 Proposition. *Under the hypotheses (3.56), Φ is a continuous mapping from S into S.*

Proof : Let us consider thus a sequence $(T_n)_{n\in\mathbb{N}}$ in S converging to $T \in S$. Let us set $\tilde{T}_n = \Phi(T_n)$, $\forall n \in \mathbb{N}$. We must show that the sequence $(\tilde{T}_n)_{n\in\mathbb{N}}$ converges to $\tilde{T} = \Phi(T)$ for the strong topology of $L^2(Q) = L^2(0,t_f;L^2(\Omega))$. By estimate (7.33) p.377 of [71] for semilinear parabolic initial-boundary value problems, we have:

$$\left\|\tilde{T}_n\right\|_{W(0,t_f)} \lesssim \|h_{T_n}\|_{L^2(Q)} + \|\Theta(T_S) + h_c T_a\|_{L^2(\Sigma)} + \|T_0\|_{L^2(\Omega)}. \tag{3.57}$$

Let us show that $\|h_{T_n}\|_{L^2(Q)}$ is bounded independently of n.

$$h_{T_n}(x,t) = \sum_{k=1}^{M} \kappa_k \int_V I^k_{T_n}(x,t,v)d\mu(v)$$

and we know by Proposition 3.9 and Proposition 3.11 with $\underline{T} = T_a$, that $\forall'(x,t,v) \in \Omega\times]0,t_f[\times V$:

$$B^k_g(T_a) \leq I^k_{T_n}(x,t,v) \leq B^k_g(T_S(t)) \leq B^k_g(T_S(t_f)). \tag{3.58}$$

Thus $\forall'(x,t) \in \Omega\times]0,t_f[$:

$$4\pi \sum_{k=1}^{M} \kappa_k B^k_g(T_a) \leq h_{T_n}(x,t) \leq 4\pi \sum_{k=1}^{M} \kappa_k B^k_g(T_S(t_f)), \tag{3.59}$$

which implies that $\|h_{T_n}\|_{L^2(Q)}$ is bounded independently of n. It now follows from estimate (3.57), that $\left\|\tilde{T}_n\right\|_{W(0,t_f)}$ is bounded independently of n. Thus, there exists a subsequence $(\tilde{T}_{n_k})_{k\in\mathbb{N}}$ such that $\tilde{T}_{n_k} \rightharpoonup \tilde{T}$ weakly in $L^2(0,t_f;H^1(\Omega))$ and $\frac{d\tilde{T}_{n_k}}{dt} \rightharpoonup \frac{d\tilde{T}}{dt}$ weakly in $L^2(0,t_f;(H^1(\Omega))^*)$. We must now prove that $\tilde{T} = \Phi(T)$. Multiplying both sides of equation (3.49) by an arbitrary function $\xi \in L^2(]0,t_f[)$ and integrating from 0 to t_f, we obtain $\forall \xi \in L^2(]0,t_f[)$, $\forall \varphi \in H^1(\Omega)$:

$$\begin{aligned} c_p m_g \int_0^{t_f} \left\langle \frac{d\tilde{T}_{n_k}}{dt}(\cdot,t),\varphi \right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \xi(t)dt = -k_h \int_0^{t_f}\int_\Omega \nabla_x \tilde{T}_{n_k}(x,t).\nabla_x \varphi(x)\xi(t)dxdt \\ +\int_0^{t_f}\int_\Omega \psi(\tilde{T}_{n_k}(x,t))\varphi(x)\xi(t)dxdt + \int_0^{t_f}\int_\Omega h_{T_{n_k}}(x,t)\varphi(x)\xi(t)dxdt \\ -h_c\int_0^{t_f} \left\langle \tilde{T}_{n_k}(\cdot,t),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt + h_c \int_0^{t_f} \langle T_a,\varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\xi(t)dt \\ +\int_0^{t_f} \langle \Theta(T_S(t)),\varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\xi(t)dt - \int_0^{t_f} \left\langle \Theta(\tilde{T}_{n_k}(\cdot,t)),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt. \end{aligned} \tag{3.60}$$

Now, we want to pass to the limit in (3.60) as $k \to +\infty$.
. By similar reasonings which have lead us to inequality (4.32), we obtain:

$$\|h_{T_n}(\cdot,t) - h_T(\cdot,t)\|_{L^2(\Omega)} \lesssim \|T_n(\cdot,t) - T(\cdot,t)\|_{L^2(\Omega)},$$

which implies:

$$\int_0^{t_f} \|h_{T_n}(\cdot,t) - h_T(\cdot,t)\|^2_{L^2(\Omega)} dt \lesssim \int_0^{t_f} \|T_n(\cdot,t) - T(\cdot,t)\|^2_{L^2(\Omega)} dt,$$

so that $h_{T_n} \to h_T$ in $L^2(Q) = L^2(0,t_f;L^2(\Omega))$. Thus $\forall \xi \in L^2(]0,t_f[)$, $\forall \varphi \in H^1(\Omega)$:

$$\int_0^{t_f}\int_\Omega h_{T_{n_k}}(x,t)\varphi(x)\xi(t)dxdt \to \int_0^{t_f}\int_\Omega h_T(x,t)\varphi(x)\xi(t)dxdt. \tag{3.61}$$

As $\tilde{T}_{n_k} \rightharpoonup \tilde{T}$ weakly in $L^2(0,t_f;H^1(\Omega))$, the traces on Γ: $\tilde{T}_{n_k}|_\Gamma \rightharpoonup \tilde{T}|_\Gamma$ weakly in $L^2(0,t_f;H^{1/2}(\Gamma))$ and thus a fortiori in $L^2(0,t_f;H^{-1/2}(\Gamma))$. Thus:

$$\int_0^{t_f} \left\langle \tilde{T}_{n_k}(\cdot,t),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt \to \int_0^{t_f} \left\langle \tilde{T}(\cdot,t),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt.$$

. Now, by Theorem 16.1 pp.99-101 of [42], the continuous injection from $H^1(\Omega) \hookrightarrow H^{1/2+\varepsilon}(\Omega)$ is compact, $\forall \varepsilon \in]0,\frac{1}{2}[$. Using the compacity Theorem 5.1 p.58 of [41], it follows that the continuous embedding from the space

$$W(0,t_f) \hookrightarrow L^2(0,t_f;H^{1/2+\varepsilon}(\Omega)) \tag{3.62}$$

is also compact. Thus $\tilde{T}_{n_k}|_\Gamma \longrightarrow \tilde{T}|_\Gamma$ strongly in $L^2(0,t_f;H^\varepsilon(\Gamma))$ and a fortiori in $L^2(0,t_f;L^2(\Gamma)) \simeq L^2(\Gamma\times]0,t_f[)$. The nonlinear mapping $\Theta : \mathbb{R} \to \mathbb{R}$ being Lipschitz continuous ([50], (3.13) p.17), we have also that $\Theta(\tilde{T}_{n_k}|_\Gamma) \longrightarrow \Theta(\tilde{T}|_\Gamma)$ strongly in $L^2(\Gamma\times]0,t_f[)$ and thus

$$\int_0^{t_f} \left\langle \Theta(\tilde{T}_{n_k}(\cdot,t)),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt$$
$$\to \int_0^{t_f} \left\langle \Theta(\tilde{T}(\cdot,t)),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt.$$

Also from ([50], Proposition 3.16) which implies that the nonlinear mapping $\psi:\mathbb{R}\to\mathbb{R}$ is also Lipschitz continuous and the compacity of the embedding from $W(0,t_f)$ into $L^2(Q)$, which results from (3.62), we have also that $\psi(\tilde{T}_{n_k}) \longrightarrow \psi(\tilde{T})$ strongly in $L^2(Q)$ and thus

$$\int_0^{t_f}\int_\Omega \psi(\tilde{T}_{n_k}(x,t))\varphi(x)\xi(t)dxdt \to \int_0^{t_f}\int_\Omega \psi(\tilde{T}(x,t))\varphi(x)\xi(t)dxdt.$$

Passing to the limit in (3.60), we obtain:

$$\begin{aligned}
c_p m_g \int_0^{t_f} \left\langle \tfrac{d\tilde{T}}{dt}(\cdot,t),\varphi \right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \xi(t)dt &= -k_h \int_0^{t_f}\int_\Omega \nabla_x \tilde{T}(x,t).\nabla_x\varphi(x)\xi(t)dxdt \\
&+ \int_0^{t_f}\int_\Omega \psi(\tilde{T}(x,t))\varphi(x)\xi(t)dxdt + \int_0^{t_f}\int_\Omega h_T(x,t)\varphi(x)\xi(t)dxdt \\
-h_c\int_0^{t_f} \left\langle \tilde{T}(\cdot,t),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt &+ h_c\int_0^{t_f} \left\langle T_a,\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt \\
+\int_0^{t_f} \left\langle \Theta(T_S(t)),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt &- \int_0^{t_f} \left\langle \Theta(\tilde{T}(\cdot,t)),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt.
\end{aligned} \tag{3.63}$$

Moreover, as $\tilde{T}_{n_k} \rightharpoonup \tilde{T}$ weakly in $W(0,t_f)$, we have also by

$$W(0,t_f) \hookrightarrow C([0,t_f];L^2(\Omega))$$

([36], Theorem 1.32, p.40) that $\tilde{T}_{n_k} \rightharpoonup \tilde{T}$ weakly in $C([0,t_f];L^2(\Omega))$ and thus also $\tilde{T}_{n_k}(\cdot,0) \rightharpoonup \tilde{T}(.,0)$ weakly in $L^2(\Omega)$. As $\tilde{T}_{n_k}(\cdot,0) = T_0(\cdot)$, $\forall k\in\mathbb{N}$, we have also $\tilde{T}(\cdot,0) = T_0(\cdot)$. Thus $\tilde{T} = \Phi(T)$. Thus $\tilde{T}_{n_k} \to \tilde{T}$ strongly in $L^2(Q) = L^2(0,t_f;L^2(\Omega))$. In conclusion, Φ is a continuous mapping from S into S. What was to be proved. □

3.18 Proposition. *Under the hypotheses (3.56), the range of Φ is a relatively compact set in $L^2(Q)$.*

Proof : We have seen in the preceding proof that $\|h_T\|_{L^2(Q)}$ is bounded for $T\in S$. By estimate (7.33) p.377 of [71] for semilinear parabolic initial-boundary value problems, we have:

$$\|\Phi(T)\|_{W(0,t_f)} \lesssim \|h_T\|_{L^2(Q)} + \|\Theta(T_S) + h_c T_a\|_{L^2(\Sigma)} + \|T_0\|_{L^2(\Omega)},$$

$\forall T \in S$. Thus the set $\{\Phi(T); T \in S\}$ is bounded in $W(0,t_f)$. Using the compacity Theorem 5.1 p.58 of [41], it follows that the continuous embedding from the space $W(0,t_f)$ into the space $L^2(Q) = L^2(0,t_f; L^2(\Omega))$ is compact. Thus, our claim follows. □

3.3 Theorem. *Under the hypotheses (3.56), the initial boundary value problem for the heat conduction equation (3.42) has a unique weak solution.*

Proof : We have already proved previously that the solution is at most unique. From Schauder's fixed point theorem as stated in A.Friedman's book ([27], p.171) and the two preceding propositions, it follows that our problem (3.48) possesses a fixed point in S, from which follows the existence of a weak solution to our initial boundary value problem for the heat conduction equation (3.42). □

As an immediate corollary of the fact that the weak solution of problem (3.42) belongs to S, we have:

3.3 Corollary. *Under the hypotheses (3.56), the weak solution of our initial boundary value problem for the heat conduction equation (3.42) satisfies to the bounds:* $T_a \leq T(x,t) \leq T_S(t)$, $\forall' x \in \Omega$, $\forall' t \in]0,t_f[$.

3.3.6 Improving the Results

The hypotheses of the preceding corollary or of the preceding theorem, are rather restrictive on T_S. In particular we have supposed that $T_S \in H^1(]0,t_f[)$ and that $\frac{dT_S}{dt} \geq 0$, a.e. on $]0,t_f[$. But of course, we have the interesting property that $T(x,t) \leq T_S(t)$, $\forall' t \in]0,t_f[$ and $\forall' x \in \Omega$ for the weak solution of the heat conduction equation (3.42). Instead, as an alternative to the closed convex set S of $L^2(Q)$, considered in (3.47), we now choose as closed convex subset of $L^2(Q)$, the closed convex set $\mathcal{C}$:

$$\mathcal{C} := \{T \in L^2(Q); \forall' t \in]0,t_f[: \forall' x \in \Omega : \underline{T} \leq T(x,t) \leq \bar{T}\}, \tag{3.64}$$

where the fixed numbers $\underline{T}$, $\bar{T} \in \mathbb{R}_+^*$ satisfy

$$\begin{cases} \underline{T} < \bar{T},\ \underline{T} \leq T_a \leq \bar{T}, \\ \underline{T} \leq T_0(x) \leq \bar{T},\ \textit{for a.e. } x \in \Omega, \\ \underline{T} \leq T_S(t) \leq \bar{T},\ \textit{for a.e. } t \in]0,t_f[. \end{cases} \tag{3.65}$$

Also, we suppose only now that $T_S \in L^\infty(]0,t_f[)$. By Propositions 3.14 and 3.16, the mapping which sends T from $\mathcal{C}$ onto $\tilde{T}$, the weak solution of problem (3.48) operates into $\mathcal{C}$. Analogously to Proposition 3.17, we have:

3.19 Proposition. *The mapping* $\Phi : T \mapsto \tilde{T}$ *is a continuous mapping from* $\mathcal{C}$ *into* $\mathcal{C}$.

Proof : The proof is similar to the proof of Proposition 3.17. The only difference in the proof is that instead of inequalities (3.58) and (3.59), we have now by Propositions 3.11 and 3.10, that $\forall'(x,t,v) \in \Omega\times]0,t_f[\times V$:

$$B_g^k(\underline{T}) \leq I_{T_n}^k(x,t,v) \leq B_g^k(\bar{T}). \tag{3.66}$$

and $\forall'(x,t) \in \Omega\times]0,t_f[$:

$$4\pi\sum_{k=1}^{M}\kappa_k B_g^k(\underline{T}) \leq h_{T_n}(x,t) \leq 4\pi\sum_{k=1}^{M}\kappa_k B_g^k(\bar{T}). \tag{3.67}$$

Using inequality (3.57), we deduce that $(\tilde{T}_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W(0,t_f)$ and the proof pursues as in the proof of Proposition 3.17. □

Now, similarly to Proposition 3.18, we have also:

3.20 Proposition. *The range of Φ is a relatively compact set in $L^2(Q)$.*

The proof is similar to the proof of Proposition 3.18. As a consequence, using Schauder's fixed point Theorem, we have also the following Theorem:

3.4 Theorem. *The initial boundary value problem (3.42) has a unique weak solution, which moreover belongs to the closed convex subset $\mathcal{C}$ of $L^2(Q)$.*

The proof is similar to the proof of Theorem 3.3

3.4 Corollary. *The weak solution $T(.,.)$ to the coupled problem (2.2)-(2.1) satisfies the bounds $\underline{T} \leq T(.,.) \leq \bar{T}$ and the radiative intensities I_T^k the bounds $B_g^k(\underline{T}) \leq I_T^k(x,t,v) \leq B_g^k(\bar{T})$, for a.e. $(x,t,v) \in \Omega\times]0,t_f[\times V$ ($\forall k = 1,\ldots,M$).*

These latest inequalities result immediately from Propositions 3.10 and 3.11.

4 Control of the Radiative Heating of a Semi-Transparent Body

Contents

4.1 Introduction

In Chapter 3 we have studied the existence and uniqueness of the solution of the coupling between radiative transfer equation (RTE) with the Heat Conduction Equation (see below (4.2), (4.4). Our aim in this chapter is the mathematical analysis of the control problem of the temperature $T(\cdot,\cdot)$ inside our semi-transparent body made of glass Ω by acting on the temperature of the black radiative source S surrounding it. Many authors have considered diffusive approximations of the RTE and have made the analysis for the corresponding approximate problem [55], [14], [36]. Nevertheless, the mathematical analysis of the exact optimal control problem remains open. We consider the semi-transparent body Ω as an open bounded set of class C^1 of $\mathbb{R}^3$. We want to find the most adequate absolute temperature functions (optimal controls) $T_S :]0, t_f[\to \mathbb{R}_+^* : t \mapsto T_S(t)$ of the black radiating source S surrounding the semi-transparent body Ω , during its radiative heating up to the fixed final time t_f, in order to obtain its distribution of temperature $(x,t) \mapsto T(x,t)$ near, in the mean square sense, to a given desired temperature distribution $(x,t) \mapsto T_d(x,t)$. We assume that $T_d(\cdot,\cdot)$ belongs to $L^2(Q)$ where $Q := \Omega \times]0, t_f[$, and we take also into account the cost of the control T_S into our cost functional J. More precisely, we want to prove existence of optimal controls and to give first-order necessary optimality conditions for the following cost functional J defined by:

$$\begin{aligned} J : L^2(Q) \times H^1(]0,t_f[) \longrightarrow \mathbb{R} : (T, T_S) \mapsto & \frac{1}{2} \int_Q (T(x,t) - T_d(x,t))^2 dx \otimes dt \\ & + \frac{\delta_r}{2} \|T_S - T_{S,d}\|^2_{H^1(]0,t_f[)}, \end{aligned} \tag{4.1}$$

subject to:

the nonlinear heat conduction equation with a non linear Robin boundary condition on $\partial\Omega$ of class C^1 of $\mathbb{R}^3$

$$\begin{cases} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \Delta T(x,t) - \sum_{k=1}^{M} 4\pi \kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{M} \kappa_k \int_V I^k_{T,T_S}(x,t,v) d\mu(v) \\ -k_h \frac{\partial T}{\partial \nu}(x,t) = h_c(T(x,t) - T_a) + \pi \int_{\lambda_0}^{+\infty} \varepsilon_\lambda [B(T(x,t),\lambda) - B(T_S(t),\lambda)] d\lambda \\ T(x,0) = T_0(x), \ \forall' x \in \Omega, \end{cases} \tag{4.2}$$

and to

$$T_S \in U_{ad} := \{T_S \in H^1(]0,t_f[); \underline{T} \le T_S(t) \le \bar{T}, \ \forall t \in]0, t_f[\}. \tag{4.3}$$

U_{ad} is the closed convex subset of admissible controls in our space of controls $U := H^1(]0,t_f[)$. Here, $\underline{T}$ and $\bar{T}$ denote two strictly positive real numbers such that $\underline{T} < \bar{T}$, $\underline{T} \le T_a \le \bar{T}$ and $\underline{T} \le T_0(x) \le \bar{T}$, $\forall' x \in \Omega$. In (4.1), δ_r is a strictly positive constant and $T_{S,d} \in U_{ad}$ is a given "pattern" control. In these equations, $T_a, c_p, m_g, k_h, h_c, \lambda_0, \varepsilon_\lambda$ are all strictly positive constants. T_a denotes the absolute temperature of the surrounding medium to Ω (e.g. air). The constants c_p, m_g and k_h are named respectively heat capacity, mass density and thermal conductivity of glass. h_c denotes the convective heat transfer coefficient. In (4.2), V denotes the "sphere of directions" i.e. the unit sphere S_2 in $\mathbb{R}^3$, and $v \in V$ an arbitrary direction.

$$B_g^k(T(x,t)) := n_g^2 \int_{\lambda_k}^{\lambda_{k+1}} B(T(x,t),\lambda)d\lambda,$$

where $B(T,\lambda) := \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}}-1)}$ is the famous Planck function, the spectral radiative intensity of the emitted thermal radiation of any black body, depending only of its absolute temperature T and of the wavelength λ of the radiation but the same in any direction $v \in V$. C_1 and C_2 are constants and for their precise values see [50]. n_g denotes the refractive index of the semi-transparent material, glass in occurrence, which is approximately 1.46 [50]. κ_k is the (linear) absorption constant coefficient of glass in the wavelength band $[\lambda_k, \lambda_{k+1}]$, $k = 1, \ldots, M$, where the M intervals $[\lambda_k, \lambda_{k+1}]$ form a partition of the region of the electromagnetic wave spectrum in which glass is semi-transparent. $I^k_{T,T_S}(x,t,v)$ in equation $(4.2)_{(i)}$ (or $I^k(T,T_S)(x,t,v)$) denotes the "spectral radiative intensity in the wavelength interval $[\lambda_k, \lambda_{k+1}]$" at point $x \in \Omega$ and in direction $v \in V$, at time $t \in]0, t_f[$. For fixed $t \in]0, t_f[$, the function $(x, v) \mapsto I^k_{T,T_S}(x,t,v)$ is solution of the radiative transfer equation (RTE) with the nonhomogeneous reflectivity boundary condition :

$$\begin{cases} v.\nabla_x I^k(x,t,v) + \kappa_k I^k(x,t,v) = \kappa_k B_g^k(T(x,t)), \ \forall'(x,v) \in \Omega \times V, \\ I^k(x,t,v) = \rho_g(|\nu_x \cdot v|)I^k(x,t,v_i) + (1-\rho_g(|\nu_x \cdot v|))B_g^k(T_S(t)), \ \forall'(x,v) \in \Gamma_- \ . \end{cases} \tag{4.4}$$

In (4.4), $T(\cdot,\cdot)$ appears in the right-hand side of the equation and $T_S(\cdot)$ our control, appears in the r.h.s. of the boundary condition. In our parabolic initial boundary value problem (4.2), $T_S(\cdot)$ our control, appears also in the r.h.s. of the boundary condition, $I^k_{T,T_S}(x,t,v)$ appears in the r.h.s. of the equation and $T_0(\cdot)$ denotes the initial condition. To alleviate the notations, we introduce the condensed notations:

$$\begin{cases} \psi(T) := -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T), \ \forall T \in \mathbb{R}, \\ h_{T,T_S}(x,t) := \sum_{k=1}^{M} \kappa_k \int_V I^k_{T,T_S}(x,t,v)d\mu(v), \\ \Theta(T) := \pi \int_{\lambda_0}^{+\infty} \varepsilon_\lambda B(T,\lambda)d\lambda, \ \forall T \in \mathbb{R}, \end{cases} \tag{4.5}$$

where $B_g^k(T) := n_g^2 \int_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)d\lambda$ if $T > 0$ and 0 if $T \leq 0$. In the preceding formula, ϵ_λ is a positive constant called the spectral hemispherical emittance ([46], p.63); like in ([65], p.70), we have supposed that the spectral hemispherical absorptance is equal to the spectral hemispherical emittance for wavelength λ belonging to the glass opaque region in the electromagnetic wave spectrum $[\lambda_0, +\infty[$, and ϵ_λ independent of the absolute temperature. We also suppose that as a function of the wavelength $\varepsilon. \in L^\infty([\lambda_0, +\infty[)$. We have proved in (Chapter , Theorem 3.4), that given $T_S \in U_{ad}$ and $T_0(\cdot) \in L^2(\Omega)$ satisfying the conditions $\underline{T} \leq T_0(x) \leq \bar{T}$, $\forall' x \in \Omega$, that the initial boundary value problem (4.2) possesses a unique weak solution

$$T_{T_S} := T(T_S) \in W(0, t_f) := \{T \in L^2(0, t_f; H^1(\Omega)), \frac{dT}{dt} \in L^2(0, t_f; H^1(\Omega)^*)\}. \tag{4.6}$$

Moreover, by that theorem, T_{T_S} satisfies the inequalities $\underline{T} \leq T_{T_S}(x,t) \leq \bar{T}$, $\forall'(x,t) \in \Omega\times]0, t_f[$. This allows us to define the reduced cost functional:

$$\hat{J} : U_{ad} \to \mathbb{R} : T_S \longmapsto J(T_{T_S}, T_S). \tag{4.7}$$

Also by Corollary 3.4 of Chapter 3, we have $\forall T_S \in U_{ad}$:

$$B_g^k(\underline{T}) \leq I^k_{T_{T_S}, T_S} \leq B_g^k(\bar{T}). \tag{4.8}$$

The spectral radiative intensities $I^k_{T_{T_S}, T_S}$ are thus uniformly bounded for T_S running over U_{ad}.

Let us now describe briefly the contents of this chapter.

In section 2, we prove that under the hypothesis that the initial condition $T_0 \in C(\bar{\Omega})$, that the solution $T(\cdot, \cdot)$ of our coupled thermal problem (4.2), $T \in C(\bar{Q})$.

In section 3, we prove that the reduced cost functional $\hat{J}(\cdot)$ is weakly lower semi-continuous on U_{ad} (defined by formula (4.3)), endowed with the weak topology of $U := H^1(]0, t_f[)$, result from which we infer the existence of an optimal control $T_S(\cdot)$ that is of an admissible control $T_S(\cdot) \in U_{ad}$ such that $\hat{J}(T_S) = \inf_{v \in U_{ad}} \hat{J}(v)$.

In section 4, we introduce the state space E and the state equation $e(T, T_S) = (0, h_c T_a, T_0)$, where $e(\cdot, \cdot)$ denotes the constraining mapping. We prove that the constraining mapping

$$e : E \times U \to L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$$

(see Definition 4.4) is continuously Fréchet differentiable on $E \times U$ and we prove that the linear continuous mapping $D_T e(T, T_S)$ is an isomorphism between E and $L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$. Using the Implicit Function Theorem, we deduce that the mapping $T : C \to E : T_S \mapsto T(T_S)$, where C (see Definition 4.6) denotes some open neighbourhood of U_{ad}, is continuously Fréchet differentiable and we give the expression of its Fréchet derivative.

In section 5, we derive from the previous result the expression of the Fréchet derivative of the reduced cost functional $\hat{J}$. To avoid the computation of $(D_T e(T(T_S), T_S))^{-1}$, we introduce the adjoint system (4.37). Its solution can be obtained by solving the linear backward nonlocal parabolic problem (4.39). Finally, we derive the first order necessary optimality condition (4.46) in the form of a variational inequality for a control $T_S(\cdot)$ to be an optimal control.

4.2 Continuity of the Solution

Like in Chapter 3, we suppose in this chapter that Ω is a bounded domain of class C^1 in $\mathbb{R}^3$. On the initial condition T_0, we henceforth suppose moreover that $T_0 \in C(\bar{\Omega})$. In this short section, we prove that under this additional hypothesis, the solution T_{T_S} of the parabolic initial boundary value problem (4.2), belongs to $C(\bar{Q})$, $\forall T_S \in U_{ad}$. Let us note that the hypotheses $T_0 \in C(\bar{\Omega})$, is trivially a necessary condition for that property of the solution to be true.

4.1 Lemma. *$T_{T_S} \in L^{\bar{r}}(Q)$ for any $\bar{r} > 2.5$ $(2.5 = \frac{n}{2} + 1$, where $n = 3$ is the dimension).*

Proof : As by Corollary 3.4 of Chapter 3, $\underline{T} \leq T_{T_S}(x,t) \leq \bar{T}$, $\forall'(x,t) \in Q := \Omega\times]0, t_f[$, $T_{T_S} \in L^\infty(Q)$ and thus a fortiori to any $L^{\bar{r}}(Q)$. □

4.1 Proposition. *Under the additional hypothesis that the initial condition $T_0 \in C(\bar{\Omega})$, $\forall T_S \in U_{ad}$, the weak solution of our initial boundary value problem (4.2), $T_{T_S} \in C(\bar{Q})$, where $Q := \Omega\times]0, t_f[$.*

Proof : By (4.8) and $\underline{T} \leq T_{T_S}(x,t) \leq \bar{T}$, $\forall'(x,t) \in Q := \Omega\times]0, t_f[$, the function

$$f : (x,t) \mapsto \frac{1}{c_p m_g}\psi(T_{T_S})(x,t) + \sum_{k=1}^{M} \frac{\kappa_k}{c_p m_g} \int_V I^k_{T_{T_S},T_S}(x,t,v)d\mu(v)$$

belongs to $L^\infty(Q)$ and thus to any $L^{\bar{r}}(Q)$ in particular for any $\bar{r} > 2.5$. As T_{T_S} is bounded, its trace on the lateral boundary $\Sigma := \partial\Omega\times]0, t_f[$ is essentially bounded. T_S being also bounded, it follows from the inequality

$$B(T,\lambda) := \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}} - 1)} \leq \frac{2C_1}{C_2}\frac{T}{\lambda^4} \tag{4.9}$$

for the Planck's function $B(\cdot,\cdot)$ ([50], p.10), that the function

$$g : (x,t) \mapsto \frac{h_c}{k_h}(T_{T_S}(x,t) - T_a) + \frac{\pi}{k_h}\int_{\lambda_0}^{+\infty} \varepsilon_\lambda[B(T_{T_S}(x,t),\lambda) - B(T_S(t),\lambda)]d\lambda,$$

defined on the lateral boundary $\Sigma := \partial\Omega\times]0, t_f[$, belongs to $L^\infty(\Sigma)$ and thus to $L^s(\Sigma)$ for any $s > 4$ $(4 = n + 1$, where $n = 3$ is the dimension of $\Omega)$. Then applying Lemma 7.12, p. 378 of [71], we obtain that $T_{T_S} \in C(\bar{Q})$. □

4.1 Remark. *From Lemma 7.12, p.378 of [71], we have also the bound:*

$$\begin{aligned}&\|T_{T_S}\|_{L^2(0,t_f;H^1(\Omega))} + \left\|\tfrac{dT_{T_S}}{dt}\right\|_{L^2(0,t_f;(H^1(\Omega))^*)} + \|T_{T_S}\|_{C(\bar{Q})}\\ &\lesssim \|\psi(T_{T_S}) + \textstyle\sum_{k=1}^{M} \kappa_k \int\limits_V I^k_{T_{T_S},T_S}(\cdot,\cdot,v)d\mu(v)\|_{L^r(Q)}\\ &+\|h_c(T_{T_S} - T_a) + \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda[B(T_{T_S},\lambda) - B(T_S(\cdot),\lambda)]d\lambda\|_{L^s(\Sigma)} + \|T_0\|_{C(\bar{\Omega})}\\ &\leq C,\end{aligned}$$

where the constant C depends only on r, on s, on $\|T_0\|_{C(\bar{\Omega})}$, and on $\bar{T}$, by using moreover (4.3), Corollary 3.4 of Chapter 3, and by inequality (4.9) for Planck's function.

4.3 Existence of an Optimal Control

In this section, we want to prove the existence of optimal controls *i.e.* $T_S \in U_{ad}$ such that

$$\hat{J}(T_S) := \inf_{v\in U_{ad}} \hat{J}(v).$$

In that purpose, we firstly prove some continuity dependence of the weak solutions of problem (4.2) with respect to the control T_S. More precisely, we prove the following result:

4.2 Proposition. *The mapping $T_S \longmapsto T(T_S)$ is continuous from U_{ad} endowed with the weak topology inherited from $H^1(]0,t_f[)$ into $L^2(Q)$ endowed with the strong topology.*

For the proof of this proposition, we need to recall some definitions and results. Let us recall that the Planck's function is defined by:

$$B(T,\lambda) := \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T}} - 1)}$$

for $T > 0$ and $\lambda > 0$; if $T \leq 0$, we set $B(T,\lambda) = 0$ (see [50]). Then, we have the following result (see Lemma 3.6 in [50]):

4.2 Lemma. *Let us fix some $\lambda > 0$. Then the function $\mathbb{R} \longrightarrow \mathbb{R} : \hat{T} \longmapsto B(\hat{T},\lambda)$ is Lipschitzian with constant $\frac{2C_1}{C_2\lambda^4}$.*

It follows that (see Corollary 3.7 in [50]):

4.1 Corollary. *Let $T_1, T_2 \in L^2(\Omega)$. Then:*

$$\|B^k_g(T_1(\cdot)) - B^k_g(T_2(\cdot))\|_{L^2(\Omega)} \leq 2n_g^2\frac{C_1}{C_2}\frac{\lambda^3_{k+1} - \lambda^3_k}{3\lambda^3_k\lambda^3_{k+1}}\|T_1(\cdot) - T_2(\cdot)\|_{L^2(\Omega)}.$$

Now by changing Ω by Q in the previous corollary, we deduce the following result:

4.3 Lemma. *The mapping from $L^2(Q)$ into $L^2(Q)$ which associates T to $B_g^k(T)$ is Lipschitzian and thus a fortiori continuous.*

Finally, we have the following trace result:

4.4 Lemma. *The mapping from $L^2(\Sigma)$ into $L^2(\Sigma)$ which associates T to $\Theta(T) := \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda B(T,\lambda)d\lambda$, is Lipschitzian and thus a fortiori continuous.*

Proof : By using Lemma 4.2 we have:

$$\begin{aligned} |\Theta(T_1(x,t)) - \Theta(T_2(x,t))| &= \pi \left| \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda (B(T_1(x,t),\lambda) - B(T_2(x,t),\lambda))d\lambda \right| \\ &\leq \pi \int\limits_{\lambda_0}^{+\infty} \frac{2C_1}{C_2\lambda^4} |T_1(x,t) - T_2(x,t)|d\lambda \lesssim |T_1(x,t) - T_2(x,t)|, \end{aligned}$$

and so $\int\limits_\Sigma |\Theta(T_1) - \Theta(T_2)|^2 dS(x)\ dt \lesssim \int\limits_\Sigma |T_1 - T_2|^2 dS(x)\ dt$.
Hence $\|\Theta(T_1) - \Theta(T_2)\|_{L^2(\Sigma)} \lesssim \|T_1 - T_2\|_{L^2(\Sigma)}$ and we get the result. □
To finish, we also recall some definitions and results of Chapter 3. Firstly, let us recall that we have defined in Chapter 3, the unbounded operator A_p in $L^p(\Omega\times V)$ by:

$$\begin{aligned} D(A_p) = \{&u \in L^p(\Omega\times V);\ v.\nabla_x u \in L^p(\Omega\times V),\\ &u(x,v) = \rho_g(|\nu_x\cdot v|)u(x, v - 2(\nu_x\cdot v)\nu_x),\ \forall'(x,v)\in\Gamma_-\},\\ &A_p : D(A_p) \to L^p(\Omega\times V) : u \longmapsto -v.\nabla_x u, \end{aligned} \tag{4.10}$$

where $v.\nabla_x u$ must be understood in the weak sense Chapter 3. We also have defined in Chapter 3 the unbounded operator $\hat{A}_{p'}$ in $L^{p'}(\Omega\times V)$ by:

$$\begin{aligned} D(\hat{A}_{p'}) = \{&g \in L^{p'}(\Omega\times V); v.\nabla_x g \in L^{p'}(\Omega\times V),\\ &g(x,v) = \rho_g(|\nu_x\cdot v|)g(x, v - 2(\nu_x\cdot v)\nu_x),\ \forall'(x,v)\in\Gamma_+\},\\ &\hat{A}_{p'} : D(\hat{A}_{p'}) \to L^{p'}(\Omega\times V) : g \longmapsto v.\nabla_x g, \end{aligned} \tag{4.11}$$

where $v.\nabla_x g$ must be understood in the weak sense. It is proved in Chapter 3, that A_p (resp. $\hat{A}_{p'}$) is dissipative in $L^p(\Omega\times V)$ (resp. $L^{p'}(\Omega\times V)$). When p' is the conjugate of p, $\hat{A}_{p'} = (A_p)^*$. Also if $1 < p_1 < p_2 < +\infty$, $A_{p_2} \subset A_{p_1}$ and similarly if $1 < p_1' < p_2' < +\infty$, $\hat{A}_{p_2'} \subset \hat{A}_{p_1'}$. Supported by this last fact, we will very often write in the following simply A, respectively $\hat{A}$, the context indicating the value of p, respectively of p'.

We are able to prove our main proposition.

Proof of Proposition 4.2. The proof is divided into three steps.

Step 1. Let $(T_{S,n})_{\in\mathbb{N}}$ be a weakly convergent sequence in U_{ad} and let us call T_S its weak limit. T_S belongs to U_{ad} since a closed convex set is also weakly closed. The injection from $H^1(]0,t_f[)$ into $C([0,t_f])$ being compact [6], the sequence $(T_{S,n})_{n\in\mathbb{N}}$ also strongly converges to T_S in the space $C([0,t_f])_{,\|\cdot\|_\infty}$. That convergence with $T_{S,n}\in U_{ad}$, $\forall n\in\mathbb{N}$, implies that $\underline{T} \leq T_S(t) \leq \bar{T}$. From Remark 4.1 follows that the sequence $\left(T_{T_{S,n}}\right)_{n\in\mathbb{N}}$ is a bounded sequence in the space $W(0,t_f)$ (4.8). Consequently, the sequence $(T_{T_{S,n}})_{n\in\mathbb{N}}$ possesses a weakly convergent subsequence $(T_{T_{S,n_j}})_{j\in\mathbb{N}}$ in the space $W(0,t_f)$. Let T be the weak limit of $(T_{T_{S,n_j}})_{j\in\mathbb{N}}$ in the space $W(0,t_f)$. By the compacity result, Theorem 5.1 pp.57-58 in [41], we get that the injection from $W(0,t_f)$ into $L^2(0,t_f;L^2(\Omega))$ is a compact mapping and thus the sequence $(T_{T_{S,n_j}})_{j\in\mathbb{N}}$ is also strongly convergent to T in $L^2(0,t_f;L^2(\Omega))$. $L^2(0,t_f;L^2(\Omega))$ being isomorphic to $L^2(Q)$, $(T_{T_{S,n_j}})_{j\in\mathbb{N}}$ converges also strongly to T in $L^2(Q)$, and then by Lemma 4.3, $(B_g^k(T_{T_{S,n_j}}))_{j\in\mathbb{N}}$ converges strongly to $B_g^k(T)$ in $L^2(Q)$. This implies by formula $(4.5)_{(i)}$ that $\psi(T_{T_{S,n_j}})$ strongly converges to $\psi(T)$ in $L^2(Q)$. Also, by Lemma 4.2, $(B_g^k(T_{S,n_j}))_{j\in\mathbb{N}}$ converges strongly to $B_g^k(T_S)$ in $L^2(]0,t_f[)$.

Step 2. (Convergence of $h_{T_{T_{S,n_j}},T_{S,n_j}}$)
We have that $I^k_{T_{T_{S,n_j}},T_{S,n_j}}(\cdot,t,\cdot)$ is the solution of the boundary value problem (4.4) for the radiative transfer equation and so by looking to the proof of Theorem 3.2 in Chapter 3:

$$I^k_{T_{T_{S,n_j}},T_{S,n_j}}(\cdot,t,\cdot) = \kappa_k(\kappa_k - A)^{-1}(B_g^k(T_{T_{S,n_j}}(.,t)) - B_g^k(T_{S,n_j}(t))) + B_g^k(T_{S,n_j}(t)).$$

Then, by the dissipativity of the operator A in $L^2(\Omega\times V)$ (see Chapter 3 which implies that $\|\kappa_k(\kappa_k - A)^{-1}\|_{\mathcal{L}(L^2(\Omega\times V))} \leq 1$, we have:

$$\begin{aligned}
&\left(\int_V \left\|I^k_{T_{T_{S,n_j}},T_{S,n_j}} - I^k_{T,T_S}\right\|_{L^2(Q)} d\mu(v)\right)^2 \\
&\leq 4\pi \int_V \left\|I^k_{T_{T_{S,n_j}},T_{S,n_j}} - I^k_{T,T_S}\right\|^2_{L^2(Q)} d\mu(v) \\
&= 4\pi \int_V d\mu(v) \int_0^{t_f}\int_\Omega (I^k_{T_{T_{S,n_j}},T_{S,n_j}}(x,t,v) - I^k_{T,T_S}(x,t,v))^2 dxdt \\
&= 4\pi \int_0^{t_f}\int_{\Omega\times V} (I^k_{T_{T_{S,n_j},T_{S,n_j}}}(x,t,v) - I^k_{T,T_S}(x,t,v))^2 dxd\mu(v)dt \\
&= 4\pi \int_0^{t_f} \|I^k_{T_{T_{S,n_j},T_{S,n_j}}}(\cdot,t,\cdot) - I^k_{T,T_S}(\cdot,t,\cdot)\|^2_{L^2(\Omega\times V)}dt \\
&\leq 32\pi^2 \int_0^{t_f} \|B_g^k(T_{T_{S,n_j}}(\cdot,t)) - B_g^k(T(\cdot,t))\|^2_{L^2(\Omega)}dt + 128\pi^2 \int_0^{t_f} |B_g^k(T_{S,n_j}(t)) - B_g^k(T_S(t))|^2 dt \\
&= 32\pi^2\|B_g^k(T_{T_{S,n_j}}(\cdot,\cdot)) - B_g^k(T(\cdot,\cdot))\|^2_{L^2(Q)} + 128\pi^2\|B_g^k(T_{S,n_j}) - B_g^k(T_S)\|^2_{L^2(0,t_f)}.
\end{aligned}$$

Thus:

$$\begin{aligned}
\|h_{T_{T_{S,n_j}},T_{S,n_j}} - h_{T,T_S}\|_{L^2(Q)} &= \|\sum_{k=1}^{M} \kappa_k \int_V (I^k_{T_{T_{S,n_j}},T_{S,n_j}} - I^k_{T,T_S}) d\mu(v)\|_{L^2_(Q)} \\
&\leq \sum_{k=1}^{M} \kappa_k \int_V \|I^k_{T_{T_{S,n_j}},T_{S,n_j}} - I^k_{T,T_S}\|_{L^2(Q)} d\mu(v) \\
&\leq 8\pi \sum_{k=1}^{M} \kappa_k \|B^k_g(T_{T_{S,n_j}}) - B^k_g(T)\|_{L^2(Q)} \\
&+16\pi \sum_{k=1}^{M} \kappa_k \|B^k_g(T_{S,n_j}) - B^k_g(T_S)|_{L^2(0,t_f)}.
\end{aligned}$$

So, by Corollary 4.1, we deduce that $h_{T_{T_{S,n_j}},T_{S,n_j}}$ converges strongly to h_{T,T_S} in $L^2(Q)$, where h_{T,T_S} is defined by formula $(4.5)_{(ii)}$.

Step 3. (Convergence of $T_{T_{S,n_j}}$)
By the continuous embedding of the space

$$\{T \in L^2(0,t_f;H^1(\Omega)); \frac{dT}{dt} \in L^2(0,t_f;(H^1(\Omega))^*)\} \hookrightarrow C([0,t_f];L^2(\Omega)),$$

the subsequence $(T_{T_{S,n_j}})_{j\in\mathbb{N}}$ converges weakly to T also in the space $C([0,t_f];L^2(\Omega))$. This implies that $T_{T_{S,n_j}}(\cdot,0) \xrightarrow{w} T(\cdot,0)$ in $L^2(\Omega)$ so that $T(\cdot,0) = T_0$.
On the other hand, $T_{T_{S,n_j}} \xrightarrow{w} T$ in $W(0,t_f)$, and we have $\forall\epsilon \in]0,1]$, $H^1(\Omega) \overset{c}{\hookrightarrow} H^{1-\varepsilon}(\Omega) \hookrightarrow (H^1(\Omega))^*$. Thus applying the Lemma of compacity from ([41], pp.57-58), $W(0,t_f) \overset{c}{\hookrightarrow} L^2(0,t_f;H^{1-\varepsilon}(\Omega))$ which implies $T_{T_{S,n_j}} \xrightarrow{s} T$ in $L^2(0,t_f;H^{1-\varepsilon}(\Omega))$. Now, $\forall\epsilon \in]0,\frac{1}{2}[$, the trace operator is continuous from $H^{1-\varepsilon}(\Omega) \to H^{\frac{1}{2}-\varepsilon}(\partial\Omega)$, so that the traces on $\Sigma = \partial\Omega\times]0,t_f[$ of the $T_{T_{S,n_j}}$ converge strongly to the trace of T in $L^2(0,t_f;L^2(\partial\Omega)) = L^2(\Sigma)$. And so by Lemma 4.4, $\Theta(T_{T_{S,n_j}})$ strongly converges to $\Theta(T)$ in $L^2(\Sigma)$, where $\Theta(T)$ is defined by formula $(4.5)_{(iii)}$.
We need to prove that $T = T(T_S)$. Now by an equivalence similar to that of Theorem 1.33, p.42 of [36], $T_{T_{S,n_j}}$ satisfies the equivalent weak formulation,

$$\begin{aligned}
&c_p m_g \int_0^{t_f} \left\langle \frac{dT_{T_{S,n_j}}}{dt}(.,t),\varphi \right\rangle_{(H^1(\Omega))^*,H^1(\Omega)} \xi(t)dt = \\
&\int_0^{t_f}\int_\Omega h_{T_{T_{S,n_j}},T_{S,n_j}}(x,t)\varphi(x)\xi(t)dx\otimes dt - k_h \int_0^{t_f}\int_\Omega \nabla_x T_{T_{S,n_j}}(x,t)\nabla_x\varphi(x)\xi(t)dx\otimes dt \\
&+\int_0^{t_f}\int_\Omega \psi(T_{T_{S,n_j}}(x,t))\varphi(x)\xi(t)dx\otimes dt + h_c\int_0^{t_f} \left\langle T_a - T_{T_{S,n_j}}(\cdot,t),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt \\
&+\int_0^{t_f} \left\langle \Theta(T_{S,n_j}(t)) - \Theta(T_{T_{S,n_j}}(\cdot,t)),\varphi \right\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \xi(t)dt,
\end{aligned}$$

$\forall\varphi \in H^1(\Omega), \forall\xi \in L^2(]0,t_f[)$. Using all the previous convergence properties to pass to the limit in the above equation as $j \to +\infty$, we obtain that $T = T(T_S)$. Thus $(T_{T_{S,n_j}})_{j\in\mathbb{N}}$ strongly converges to $T_{T_S} = T(T_S)$ in $L^2(Q)$.
A standard argument of general topology shows us now that the sequence $(T_{T_{S,n}})_{n\in\mathbb{N}}$ itself

strongly converges to $T_{T_S} = T(T_S)$ in $L^2(Q)$, what was to be proved.

Now, we prove the aim of this section

4.1 Theorem. *There exists $T_S \in U_{ad}$ such that $\hat{J}(T_S) = \inf_{v\in U_{ad}} \hat{J}(v)$.*

Proof : Let us set $L := \inf_{v\in U_{ad}} \hat{J}(v)$. There exists a sequence $(T_{S,n})_{n\in\mathbb{N}} \subset U_{ad}$ such that $\hat{J}(T_{S,n}) \to L$. The sequence $(T_{S,n})_{n\in\mathbb{N}}$ is bounded in $U_{ad} \subset H^1(]0,t_f[)$, because if it was not, there would exist a subsequence $\left(T_{S,n_j}\right)_{j\in\mathbb{N}}$ such that $\left\|T_{S,n_j}\right\|_{H^1(]0,t_f[)} \to +\infty$ as $j \to +\infty$. This would imply that $\hat{J}(T_{S,n_j}) \to +\infty$ in contradiction with $\hat{J}(T_{S,n}) \to L$. The sequence $(T_{S,n})_{n\in\mathbb{N}}$ being bounded in $U_{ad} \subset H^1(]0,t_f[)$, possess a subsequence $(T_{S,n_l})_{l\in\mathbb{N}}$ weakly convergent to some element $T_S \in U_{ad}$. By Proposition 4.2 and formula (4.1), the norm of $H^1(]0,t_f[)$ being weakly lower semi-continuous ([71], p.47), we have: $\hat{J}(T_S) \leq \liminf_{l\to+\infty} \hat{J}(T_{S,n_l})$ and thus $\hat{J}(T_S) = L = \inf_{v\in U_{ad}} \hat{J}(v)$, what was to be proved. □

4.4 Continuous Fréchet differentiability dependence of the state with respect to the control

The aim of this section is to prove that the mapping $T_S \mapsto T(T_S)$ from an open neighborhood of U_{ad} into an appropriate state space E is continuously Fréchet differentiable. We divide this section into four subsections.

4.4.1 State Space and State Equation

Our purpose now is to define the state space and the state equation. Firstly, we need to define the meaning of $(c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T) \in L^r(Q)$ and $k_h \frac{\partial T}{\partial \nu} \in L^{s^*}(\Sigma)$ for

$$T \in W(0,t_f) := \left\{T \in L^2(0,t_f;H^1(\Omega));\, \frac{dT}{dt} \in L^2(0,t_f;(H^1(\Omega))^*)\right\}.$$

4.1 Definition. *Let $T \in W(0,t_f)$, we say that $(c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T) \in L^r(Q)$ in the sense of distributions iff there exists some function $g \in L^r(Q)$ such that $\forall \varphi \in L^2(0,t_f;\mathring{H}^1(\Omega))$*

$$\begin{aligned} c_p m_g \int_0^{t_f} \left\langle \frac{dT}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + k_h \int_0^{t_f}\int_\Omega \nabla_x T(x,t) \cdot \nabla_x \varphi(x,t) dx \otimes dt \\ = \int_0^{t_f}\int_\Omega g(x,t)\varphi(x,t) dx \otimes dt,\ \forall \varphi \in L^2(0,t_f;H^1(\Omega)). \end{aligned} \tag{4.12}$$

Then we set: $c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T := g$. *Equation (4.12) is equivalent to* $\forall' t \in]0, t_f[$:

$$c_p m_g \left\langle \tfrac{dT}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} + k_h \int_\Omega \nabla_x T(x,t) \cdot \nabla_x \varphi(x) dx = \int_\Omega g(x,t)\varphi(x) dx, \quad \forall \varphi \in \mathring{H}^1(\Omega). \tag{4.13}$$

4.2 Definition. *Let* $T \in W(0,t_f)$*, such that* $c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T$ *in the sense of distributions belongs to* $L^r(Q)$*. Let us set* $g := c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T$*. We say that* $k_h \frac{\partial T}{\partial \nu} \in L^{s^*}(\Sigma)$ *in the sense of distributions iff there exists some function* $h \in L^{s^*}(\Sigma)$ *such that:*

$$\begin{aligned} c_p m_g \int_0^{t_f} \left\langle \frac{dT}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} dt + k_h \int_0^{t_f}\int_\Omega \nabla_x T(x,t) \cdot \nabla_x \varphi(x,t) dx \otimes dt \\ - \int_0^{t_f}\int_\Omega g(x,t)\varphi(x,t) dx \otimes dt = \int_0^{t_f}\int_{\partial\Omega} h(x,t)\varphi(x,t) dS(x) dt, \\ \forall \varphi \in L^2(0,t_f; H^1(\Omega)). \end{aligned} \tag{4.14}$$

We set $k_h \frac{\partial T}{\partial \nu} = h$. *Equation (4.14) is equivalent to* $\forall' t \in]0, t_f[$:

$$\begin{aligned} c_p m_g \left\langle \tfrac{dT}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} + k_h \int_\Omega \nabla_x T(x,t) \cdot \nabla_x \varphi(x) dx \\ - \int_\Omega g(x,t)\varphi(x) dx = \int_{\partial\Omega} h(x,t)\varphi(x) dS(x), \quad \forall \varphi \in H^1(\Omega). \end{aligned} \tag{4.15}$$

4.3 Definition. *(State Space) Let us fix some real numbers* $r \in]2.5, 2.72[$ *and* $s^* > 4$*. By the state space* E*, we mean the set of all* $T \in W(0,t_f) \cap C(\bar{Q})$ *such that* $(c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T) \in L^r(Q)$ *and* $k_h \frac{\partial T}{\partial \nu} \in L^{s^*}(\Sigma)$ *in the sense of distributions.*
On that real vectorial space, we define the norm:

$$\begin{aligned} \|T\|_E := \|T\|_{L^2(0,t_f;H^1(\Omega))} + \|\frac{dT}{dt}\|_{L^2(0,t_f;H^1(\Omega)^*)} + \|T\|_{\infty,\bar{Q}} \\ + \|c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T\|_{L^r(Q)} + \|k_h \frac{\partial T}{\partial \nu}\|_{L^{s^*}(\Sigma)}. \end{aligned}$$

Endowed with that norm, the state space E *becomes a Banach space.*

Under previous definitions, we have the following result:

4.5 Lemma. *Let* $(T, T_S) \in E \times U$*, then* $\psi(T)$ *and* $h_{T,T_S} \in L^r(Q)$*,* $\forall r > 2.5$*. Also,* $\Theta(T_{|\Sigma})$ *and* $\Theta(T_S) \in L^{s^*}(\Sigma)$*,* $\forall s^* > 4$*.*

Proof : We have that $T \in C(\bar{Q})$ and thus a fortiori $T \in L^r(Q)$, $\forall r > 2.5$. We have also $B(T,\lambda) \le \frac{2C_1}{C_2}\frac{T}{\lambda^4}$ by ([50], p.10). Thus,

$$B^k(T) = \int\limits_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda)d\lambda \le \frac{2C_1}{C_2}T \int\limits_{\lambda_k}^{\lambda_{k+1}} \frac{d\lambda}{\lambda^4} \lesssim T.$$

Consequently, $B_g^k(T) \lesssim n_g^2 T$, $\forall k = 1, ..., M$. As $T \in L^r(Q)$, $\forall r > 2.5$, it follows from these inequalities that $\psi(T) \in L^r(Q)$, $\forall r > 2.5$.
Now as T and T_S are bounded, we get by Corollary 3.4 of Chapter 3 that the I^k_{T,T_S} are bounded and thus belong to $L^r(Q \times V)$, $\forall r > 2.5$ and so $h_{T,T_S} \in L^r(Q)$, $\forall r > 2.5$.
As $T \in C(\bar{Q})$ and $T_S \in C([0,t_f])$, it follows from Lemma 4.2 that for each $\lambda > 0$, the function $B(T,\lambda) \in C(\bar{Q})$ and $B(T_S,\lambda) \in C([0,t_f])$ respectively. Using the bound $B(T,\lambda) \le \frac{2C_1}{C_2}\frac{T}{\lambda^4}$ and applying the Lebesgue Continuity Theorem to $\Theta(T) := \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda B(T,\lambda)d\lambda$, it follows that $\Theta(T) \in C(\bar{Q})$ and $\Theta(T_S) \in C([0,t_f])$ respectively. A fortiori $\Theta(T_{|\overline{\Sigma}}) \in C(\overline{\Sigma})$ and thus belongs to $L^{s^*}(\Sigma)$ for all $s^* > 4$. Also if we look to $\Theta(T_S)$, as a function defined on $\overline{\Sigma}$ which is constant in x, $x \in \Gamma$, we may consider $\Theta(T_S)$ as an element of $C(\overline{\Sigma})$. A fortiori, view in that way, $\Theta(T_S)$ belongs to $L^{s^*}(\Sigma)$ for all $s^* > 4$. □
This lemma leads to the following definition:

4.4 Definition. *By the constraining mapping $e(\cdot,\cdot)$, we mean the mapping:*

$$\begin{aligned} e : E \times U &\longrightarrow L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega}) \\ (T,T_S) &\longmapsto e(T,T_S) = (e_1,e_2,e_3)(T,T_S) \end{aligned}$$

where

$$\begin{aligned} e_1(T,T_S) &= c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T - \psi(T) - h_{T,T_S}, \\ e_2(T,T_S) &= k_h(\frac{\partial T}{\partial \nu})_{|\Sigma} + h_c T_{|\Sigma} + \Theta(T_{|\Sigma}) - \Theta(T_S), \\ e_3(T,T_S) &= T(\cdot,0), \end{aligned}$$

and the quantities $\psi(T), h_{T,T_S}, \Theta(T)$ and $\Theta(T_S)$ are defined by formulas (4.5).

Then, we have the following result:

4.3 Proposition. *Let $f \in L^r(Q), g \in L^{s^*}(\Sigma)$ and $T_0 \in C(\bar{\Omega})$. Let $T_S \in U$, then there exists T belonging to the state space E such that $e(T,T_S) = (f,g,T_0(.))$ iff T is the weak solution if it exists of:*

$$\begin{cases} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \Delta T(x,t) + \psi(T(x,t)) + h_{T,T_S}(x,t) + f(x,t), \\ -k_h \frac{\partial T}{\partial \nu}(x,t) = h_c(T(x,t)) + \Theta(T(x,t)) - \Theta(T_S(t)) - g(x,t), \\ T(x,0) = T_0(x), \ \forall' x \in \Omega, \end{cases}$$

i.e. satisfies:

$$
\begin{gathered}
c_p m_g \left\langle \frac{dT}{dt}(.,t), \varphi \right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} = -k_h \int_\Omega \nabla_x T(x,t) \nabla_x \varphi(x) dx \\
+ \int_\Omega \psi(T(x,t))\varphi(x) dx + \langle \Theta(T_S(t)) - \Theta(T(\cdot,t)), \varphi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\
+ \int_\Omega h_{T,T_S}(x,t)\varphi(x) dx - h_c \langle T(\cdot,t), \varphi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} \\
+ \int_\Omega f(x,t)\varphi(x) dx + \int_{\partial\Omega} g(x,t)\varphi(x) dS(x), \\
\forall \varphi \in H^1(\Omega), \forall' t \in]0, t_f[,
\end{gathered}
$$

and $T(x,0) = T_0(x)$, $\forall x \in \Omega$.

The proof follows from the definition of the constraining mapping $e(.,.)$, definitions 4.1 and 4.2, by proceeding like in the proof of Proposition 10 of [49], considering test functions $\varphi \in L^2(0,t_f; H^1(\Omega))$ which are tensor products of functions of $H^1(\Omega))$ with functions of $L^2(0,t_f)$, functions which form a dense subspace of $L^2(0,t_f; H^1(\Omega))$.

4.2 Corollary. *For $T_S \in U_{ad}$, $e(T,T_S) = (0, h_c T_a, T_0)$ is equivalent to T is the weak solution of the nonlinear heat conduction equation with the nonlinear Robin boundary condition, which we know it exists and it is unique by Chapter 3.*

Finally, we have the following definition:

4.5 Definition. *(State Equation) By the state equation, we will mean in the following, the equation*

$$e(T,T_S) = (0, h_c T_a, T_0).$$

This name is justified by the previous corollary.

4.4.2 Fréchet differentiability of the constraining mapping

This subsection is devoted to prove that the nonlinear constraining mapping e defined in Definition 4.4 is continuously Fréchet differentiable. More precisely, we will prove the following theorem:

4.2 Theorem. *The nonlinear constraining mapping (see Definition 4.4)*

$$e : E \times U \to L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega}),$$

is continuously Fréchet differentiable on $E \times U$.

For this aim, we need to prove several lemmas.

4.6 Lemma. *The mapping which to each $T \in C(\bar{Q})$ associates $B_g^k(T)$ is a continuously Fréchet differentiable mapping from $C(\bar{Q})$ into $L^p(Q)$, $\forall p \in [1, +\infty[$.*

Proof : We know that $B_g^k(T) = n_g^2 B^k(T)$; so it suffices to prove the corresponding property for $B^k(T)$, where B^k is defined by:

$$B^k : C(\bar{Q}) \to L^2(Q) : T \mapsto B^k(T) := \int_{\lambda_k}^{\lambda_{k+1}} B(T,\lambda) d\lambda,$$

where $B : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R} : (T,\lambda) \mapsto B(T,\lambda)$ with

$$B(T,\lambda) := \begin{cases} \frac{2C_1}{\lambda^5 (e^{\frac{C_2}{\lambda T}} - 1)}, & \text{if } T > 0 \\ 0, & \text{if } T \leq 0 \end{cases}$$

denotes the Planck function ([50], p.2 and p.5). An elementary computation shows that ([50], p.14):

$$\frac{\partial B}{\partial T}(T,\lambda) = \begin{cases} \frac{2C_1}{C_2 \lambda^4} \frac{\left(\frac{C_2}{\lambda T}\right)^2 e^{\frac{C_2}{\lambda T}}}{(e^{\frac{C_2}{\lambda T}} - 1)^2}, & \text{if } T > 0 \\ 0, & \text{if } T \leq 0 \end{cases} \tag{4.16}$$

(0 if $T \leq 0$) so,

$$\frac{dB^k}{dT}(T) = \int_{\lambda_k}^{\lambda_{k+1}} \frac{\partial B}{\partial T}(T,\lambda) d\lambda = \frac{2C_1}{C_2} \int_{\lambda_k}^{\lambda_{k+1}} \frac{(\frac{C_2}{\lambda T})^2 e^{\frac{C_2}{\lambda T}}}{(e^{\frac{C_2}{\lambda T}} - 1)^2} \frac{d\lambda}{\lambda^4}, \forall T \in \mathbb{R} \tag{4.17}$$

(0 if $T \leq 0$) and the inequality

$$\sup_{\mu > 0} \frac{\mu^2 e^\mu}{(e^\mu - 1)^2} < +\infty \tag{4.18}$$

shows that $\frac{dB^k}{dT}$ is bounded. It follows from (4.17) and (4.18) by the Lebesgue continuity theorem that the mapping $\mathbb{R} \to \mathbb{R} : T \mapsto \frac{dB^k}{dT}(T)$ is continuous. Now, let us investigate if the directional derivative of B^k at point $T \in C(\bar{Q})$ in the direction $\delta T \in C(\bar{Q})$ exists ([2] p.25-26). Let $s \in \mathbb{R}_+^*$,

$$\frac{B^k(T + s\delta T) - B^k(T)}{s} = \int_{\lambda_k}^{\lambda_{k+1}} \frac{B(T + s\delta T, \lambda) - B(T,\lambda)}{s} d\lambda.$$

By Lagrange theorem on finite increments:

$$\begin{aligned}\frac{B^k(T(x,t)+s\delta T(x,t))-B^k(T(x,t))}{s} &= \frac{1}{s}\int_{T(x,t)}^{T(x,t)+s\delta T(x,t)} \frac{dB^k}{dT}(T)dT \\ &= \Big(\int_0^1 \frac{dB^k}{dT}(T(x,t)+rs\delta T(x,t))dr\Big)\delta T(x,t).\end{aligned} \tag{4.19}$$

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers tending to 0. By (4.19), the continuity and the boundedness of $\frac{dB^k}{d\xi}$, we have that:

$$\frac{B^k(T(x,t)+s_n\delta T(x,t))-B^k(T(x,t))}{s_n} \xrightarrow[n\to+\infty]{} \frac{dB^k}{dT}(T(x,t))\delta T(x,t), \forall(x,t)\in Q.$$

Also, by (4.19):

$$\Big|\frac{(B^k(T(x,t))+s_n\delta T(x,t))-B^k(T(x,t))}{s_n}\Big|^p$$

is bounded by a constant independent of $n, x,$ and t. Thus by the Lebesgue dominated convergence theorem, the sequence $\left(\frac{B^k(T(\cdot,\cdot)+s_n\delta T(\cdot,\cdot))-B^k(T(\cdot,\cdot))}{s_n}\right)_{n\in\mathbb{N}}$ converges to $\frac{dB^k}{dT}(T(\cdot,\cdot))\delta T(\cdot,\cdot)$ in $L^p(Q)$. This proves that $B^k(\cdot)$ is directionally derivable at T. As $\frac{dB^k}{dT}(T(\cdot,\cdot))$ is also a bounded function, the mapping

$$C(\bar{Q})\to L^p(Q):\delta T(\cdot,\cdot)\mapsto\frac{dB^k}{dT}(T(\cdot,\cdot))\delta T(\cdot,\cdot)$$

is a linear continuous mapping. Thus $B^k: C(\bar{Q})\to L^p(Q)$ is a Gâteaux differentiable at every $T\in C(\bar{Q})$ ([2], p.26) and its Gâteaux derivative at point $T\in C(\bar{Q})$ is the linear continuous mapping

$$DB^k(T): C(\bar{Q})\to L^p(Q):\delta T(\cdot,\cdot)\mapsto\frac{dB^k}{dT}(T(\cdot,\cdot))\delta T(\cdot,\cdot)$$

Now let us prove that $B^k: C(\bar{Q})\to L^p(Q)$ is continuously Gâteaux differentiable on $C(\bar{Q})$. Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $C(\bar{Q})$ converging to T in $C(\bar{Q})$. We have that

$$\|DB^k(T_n)-DB^k(T)\|_{\mathcal{L}(C(\bar{Q});L^p(Q))}\le\|\frac{dB^k}{dT}(T_n(\cdot,\cdot))-\frac{dB^k}{dT}(T(\cdot,\cdot))\|_{L^p(Q)}. \tag{4.20}$$

Thus it suffices to prove that $\frac{dB^k}{dT}(T_n(\cdot,\cdot))\to\frac{dB^k}{dT}(T(\cdot,\cdot))$ in $L^p(Q)$. As $\frac{dB^k}{dT}$ is a continuous function and $T_n\to T$ in $C(\bar{Q})$ implies that $T_n(x,t)\to T(x,t),\forall(x,t)\in Q$ as $n\to+\infty$, it follows that $\frac{dB^k}{dT}(T_n(x,t))\to\frac{dB^k}{dT}(T(x,t)),\forall(x,t)\in Q$, as $n\to+\infty$. $\frac{dB^k}{dT}$ being a bounded function, by the Lebesgue dominated convergence theorem, the sequence $\frac{dB^k}{dT}(T_n(\cdot,\cdot))\to$

$\frac{dB^k}{dT}(T(\cdot,\cdot))$ in $L^p(Q)$, and thus by (4.20), $DB_k(T_n) \to DB^k(T)$ in $\mathcal{L}(C(\bar{Q}), L^p(Q))$. Thus $B^k : C(\bar{Q}) \to L^p(Q)$ is continuously Gâteaux differentiable on $C(\bar{Q})$. Consequently, by Lemma 2.8, p.28 [2] (or 5b of exercise 3.4, p.160 of [33]), $B^k : C(\bar{Q}) \to L^p(Q)$ is continuously Fréchet differentiable on $C(\bar{Q})$. The proof is thus complete. $\square$

4.7 Lemma. *The mapping $E \times U \to L^p(Q) : (T, T_S) \mapsto h_{T,T_S}$ defined by $(4.5)_{(ii)}$, is a continuously Fréchet differentiable mapping from $E \times U$ into $L^p(Q)$, $\forall p \in [1, +\infty[$.*

Proof : We have $h_{T,T_S}(x,t) := \sum_{k=1}^{M} \kappa_k \int_V I^k_{T,T_S}(x,t,v)d\mu(v)$, so it suffices to prove that $\int_V I^k_{T,T_S}(\cdot,\cdot,v)d\mu(v)$ is Fréchet differentiable from $E \times U$ into $L^p(Q)$.
From the proof of Theorem 3.2 of Chapter 3, we have that:

$$I^k_{T,T_S}(\cdot,t,\cdot) = \kappa_k(\kappa_k - A)^{-1}(B^k_g(T(\cdot,t)) - B^k_g(T_S(t))) + B^k_g(T_S(t)),$$

where the unbounded operator A has been defined Chapter 3 and its definition recalled in this chapter (see formula (4.10)). We know that $E \hookrightarrow C(\bar{Q})$.
We have proved in Lemma 4.6 that the mapping: $C(\bar{Q}) \to L^p(Q) : T \mapsto B^k_g(T)$ is continuously Fréchet differentiable.
Let us set $B_k = \kappa_k(\kappa_k - A)^{-1} \in \mathcal{L}(L^p(\Omega \times V))$ and let us denote by $\tilde{B}_k$ its extension as a linear continuous operator in $L^p(Q \times V) = L^p(]0,t_f[; L^p(\Omega \times V))$ by setting $\forall f \in L^p(Q \times V)$:

$$(\tilde{B}_k f)(t) = B_k(f(t)), \forall' t \in]0,t_f[. \tag{4.21}$$

So :

$$\int_0^{t_f} \|(\tilde{B}_k f)(t)\|^p dt = \int_0^{t_f} \|B_k(f(t))\|^p dt \leq \int_0^{t_f} \|f(t)\|^p dt,$$

where $\|.\|$ denotes the norm in $L^p(\Omega \times V)$. Thus $\tilde{B}_k$ is a linear continuous operator from $L^p(Q \times V)$ into $L^p(Q \times V)$ extending $B_k : L^p(\Omega \times V) \to L^p(\Omega \times V)$.
So $I^k_{T,T_S} = \tilde{B}_k B^k_g(T) + B^k_g(T_S) - \tilde{B}_k B^k_g(T_S)$, implies that:

$$(D_T I^k_{T,T_S})(T,T_S) = \tilde{B}_k \circ i \circ DB^k_g(T) \in \mathcal{L}(C(\bar{Q}); L^p(Q \times V)), \tag{4.22}$$

where $i : L^p(Q) \hookrightarrow L^p(Q \times V)$ is the continuous injection which to each function in $L^p(Q)$ associates the function in $L^p(Q \times V)$ constant with respect to $v \in V$, and that:

$$(D_{T_S} I^k_{T,T_S})(T,T_S) = j \circ DB^k_g(T_S) - \tilde{B}_k \circ j \circ DB^k_g(T_S) \in \mathcal{L}(C([0,t_f]); L^p(Q \times V)), \tag{4.23}$$

where $DB^k_g(T_S) : C([0,t_f]) \to L^p(]0,t_f[)$ and $j : L^p(]0,t_f[) \hookrightarrow L^p(Q \times V)$ which to each function in $L^p(]0,t_f[)$ associates the function in $L^p(Q \times V)$ constant with respect to $x \in \Omega$ and $v \in V$.

Now by Lemma 4.6 and formulas (4.22) and (4.23), we have that the mapping $(T, T_S) \to (D_T I^k_{T,T_S})(T, T_S)$ is continuous from $C(\bar{Q}) \times C([0, t_f]) \to \mathcal{L}(C(\bar{Q}); L^p(Q \times V))$, and that the mapping $(T, T_S) \to (D_{T_S} I^k_{T,T_S})(T, T_S)$ is continuous from $C(\bar{Q}) \times C([0, t_f]) \to \mathcal{L}(C([0, t_f]); L^p(Q \times V))$. Thus by Proposition (8.9.1), p.173 [23], the mapping

$$\begin{aligned} C(\bar{Q}) \times C([0, t_f]) &\to L^p(Q \times V) \\ (T, T_S) &\mapsto I^k(T, T_S) \end{aligned} \tag{4.24}$$

is continuously Fréchet differentiable. To obtain $\int\limits_V I^k_{T,T_S}(\cdot, \cdot, v) d\mu(v)$ it suffices to integrate I^k_{T,T_S} with respect to v. This integration defines a linear continuous mapping from $L^p(Q \times V)$ into $L^p(Q)$. Thus, the mapping $(T, T_S) \mapsto \int\limits_V I^k_{T,T_S}(\cdot, \cdot, v) d\mu(v)$ being the composition of this integration operator with respect to v with the mapping (4.24), it follows that the mapping

$$\begin{aligned} C(\bar{Q}) \times C([0, t_f]) &\to L^p(Q) \\ (T, T_S) &\mapsto \int\limits_V I^k_{T,T_S}(\cdot, \cdot, v) d\mu(v) \end{aligned}$$

is continuously differentiable. Hence, as $h(T, T_S)$ is a linear combination of these mappings, it is also a continuously Fréchet differentiable mapping. □

4.8 Lemma. *The nonlinear mapping*

$$\Theta : C(\bar{\Sigma}) \to L^{s^*}(\Sigma) : v \mapsto \Theta(v),$$

where

$$\Theta(v) : \Sigma \to \mathbb{R} : (x, t) \mapsto \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda B(v(x, t), \lambda) d\lambda,$$

is continuously Fréchet differentiable and its derivative at $v \in C(\bar{\Sigma})$ is the linear continuous mapping

$$D\Theta(v) : C(\bar{\Sigma}) \to L^{s^*}(\Sigma) : \delta v \to \frac{d\Theta}{dv}(v)\delta v,$$

where

$$\frac{d\Theta}{dv}(v) = \pi \int\limits_{\lambda_0}^{+\infty} \varepsilon_\lambda \frac{\partial B}{\partial v}(v, \lambda) d\lambda = \pi \frac{2C_1}{C_2} \int\limits_{\lambda_0}^{+\infty} \frac{\varepsilon_\lambda \left(\frac{C_2}{\lambda v}\right)^2 e^{\frac{C_2}{\lambda v}}}{(e^{\frac{C_2}{\lambda v}} - 1)^2} \frac{d\lambda}{\lambda^4}, \forall v \in \mathbb{R}. \tag{4.25}$$

Proof : The proof is similar to the proof of Lemma 4.6. □

Proof of Theorem 4.2. We start with the first component of e which is e_1. According to the definition of the state space E, the map $(T, T_S) \mapsto c_p m_g \frac{\partial T}{\partial t} - k_h \Delta T$ is a linear continuous mapping from $E \times U$ into $L^r(Q)$ $(r > 2.5)$. Thus this map is obviously

continuously Fréchet differentiable.
Now the map $(T, T_S) \mapsto \psi(T) = -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T)$ is continuously Fréchet differentiable as we know, by Lemma 4.6 that the map $T \mapsto B_g^k(T)$ is continuously Fréchet differentiable from $C(\bar{Q})$ into $L^r(Q)$. For the mapping $(T, T_S) \mapsto h_{T,T_S}$, we know by Lemma 4.7, that it is continuously Fréchet differentiable from $E \times U$ into $L^r(Q)$. So, by Definition 4.4, summing the pieces, we obtain that e_1 is continuously Fréchet differentiable on $E \times U$.
We want now to prove that e_2 is continuously Fréchet differentiable on $E \times U$. We have that $k_h \frac{\partial T}{\partial \nu} + h_c T$ is a linear and continuous mapping according to the definition of the state space E, so that it is a continuously Fréchet differentiable on $E \times U$. By the previous proposition we have that the mapping $\Theta : C(\bar{\Sigma}) \to L^{s^*}(\Sigma)$ is continuously Fréchet differentiable on $C(\bar{\Sigma})$, and by composition to the right with the projection of $E \times U$ onto E, followed by the trace mapping from E into $C(\bar{\Sigma})$, we obtain using $\Theta(T)_{|\bar{\Sigma}} = \Theta(T_{|\bar{\Sigma}})$, that the mapping $(T, T_S) \mapsto \Theta(T)_{|\bar{\Sigma}}$ is continuously Fréchet differentiable from $E \times U$ into $L^{s^*}(\Sigma)$. That the mapping $E \times U \to L^{s^*}(\Sigma) : (T, T_S) \mapsto \Theta(T_S)$ is continuously Fréchet differentiable, follows from $U := H^1(]0, t_f[) \hookrightarrow C([0, t_f])$, and the preceding proposition.
Summing the pieces, we obtain that e_2 is continuously Fréchet differentiable on $E \times U$.
e_3 being linear and continuous is obviously continuously Fréchet differentiable on $E \times U$.
Hence, by proposition 8.1.5 of [23] , the mapping e is continuously Fréchet differentiable on $E \times U$.

4.4.3 Property of the partial derivative of the constraining mapping

In this subsection, we prove that the partial derivative at point (T, T_S) with respect to T of the constraining mapping e is an isomorphism from E into $L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$.
Firstly, from Theorem 4.2, we have the following result:

4.4 Proposition. *The partial derivative at point (T, T_S) with respect to T of the continuously Fréchet differentiable mapping:*

$$e : E \times U \to L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$$

is the linear and continuous mapping $D_T e(T, T_S) : E \to L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$ with

$$\begin{aligned} D_T e_1(T, T_S)\delta T &= c_p m_g \frac{\partial \delta T}{\partial t} - k_h \Delta \delta T + \sum_{k=1}^{M} 4\pi\kappa_k \frac{dB_g^k}{dT}(T)\delta T \\ &\quad - \sum_{k=1}^{M} \kappa_k \int_V [(D_T I^k)(T, T_S)\delta T](\cdot, \cdot, v) d\mu(v), \\ D_T e_2(T, T_S)\delta T &= k_h (\frac{\partial \delta T}{\partial \nu})_{|\Sigma} + h_c (\delta T)_{|\Sigma} + \frac{d\Theta}{dT}(T_{|\Sigma})(\delta T)_{|\Sigma}, \\ D_T e_3(T, T_S)\delta T &= \delta T(\cdot, 0), \end{aligned}$$

where $\frac{dB_g^k}{dT}(T) = n_g^2 \frac{dB^k}{dT}(T)$ *and* $\frac{dB^k}{dT}(T)$ *is given by formula (4.17),* $\frac{d\Theta}{dT}(T)$ *by formula (4.25), and* $(D_T I^k)(T, T_S)$ *by formula (4.22).*

Next, we will prove the following main result of this subsection:

4.5 Proposition. *The linear continuous mapping* $D_T e(T, T_S)$ *is an isomorphism between* E *and* $L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$.

For the proof we need some lemmas.

4.9 Lemma. *Let* $T \in W(0, t_f)$. *Then* $T \in L^r(Q)$ *for some* $r > 2.72$.

Proof : As $W(0, t_f) \hookrightarrow C([0, t_f]; L^2(\Omega))$, we know that $T \in C([0, t_f]; L^2(\Omega))$. A fortiori $T \in L^p(0, t_f; L^2(\Omega))$, $\forall 1 \leq p \leq +\infty$. We know also that $T \in L^2(0, t_f; L^q(\Omega))$ for $q \in [2, 6]$ as $H^1(\Omega) \hookrightarrow L^q(\Omega))$ for $q \in [2, 6]$. By using these two informations on T, we are going to deduce by interpolation that $T \in L^r(Q)$ for some $r > 2.5$. We have by Theorem p.8 in P. Grivard's thesis [30], $[L^{p_0}(\Omega_{Gr}, A_0); L^{p_1}(\Omega_{Gr}, A_1)]_\theta = L^s(\Omega_{Gr}, [A_0, A_1]_\theta)$, where Ω_{Gr} denotes an arbitrary open set in some $\mathbb{R}^n$ space (the statement in [30] is even more general), $0 < \theta < 1$, $\frac{1}{s} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and (A_0, A_1) is a couple of interpolation i.e two Banach spaces with continuous injections in a vectorial Hausdorff space (p.15, [69]). Let us apply that result of interpolation by taking $\Omega_{Gr} =]0, t_f[$, $p_0 = p$, $A_0 = L^2(\Omega)$, $p_1 = 2$, $A_1 = L^q(\Omega)$. Thus we want to interpolate between $L^p(0, t_f; L^2(\Omega))$ and $L^2(0, t_f; L^q(\Omega))$. We may choose p arbitrary in $]1, +\infty[$ and q arbitrary in $[2, 6]$. Let us choose $\theta = 0.4$ so that $\frac{1}{s} = 0.6 \times \frac{1}{p} + 0.4 \times \frac{1}{2}$. Thus $\frac{1}{s} = \frac{0.6+0.2+p}{p}$ which implies $s = \frac{1}{0.2+\frac{0.6}{p}}$. Taking p very large, we can approach $s \approx 5$, by lower values. Thus s will be strictly bigger than 2.5, if we take p large enough. $[A_0, A_1]_\theta = L^{\bar{r}}(\Omega)$ where $\frac{1}{\bar{r}} = 0.6 \times \frac{1}{2} + 0.4 \times \frac{1}{q}$ which implies $\bar{r} = \frac{1}{0.3+\frac{0.4}{q}}$. If we take $q = 6$, then we will have $\bar{r} > 2.72$.
Thus,

$$\begin{aligned} T &\in L^s(]0, t_f[; [L^2(\Omega), L^q(\Omega)]_{0.4}) \\ &= L^s(]0, t_f[; L^{\bar{r}}(\Omega)) \subset L^r(]0, t_f[; L^r(\Omega)) \\ &= L^r(Q) \text{ for some } r > 2.5. \end{aligned}$$

□

4.10 Lemma. *The linear continuous operator* $D_T I^k_{T,T_S} \equiv D_T I^k(T, T_S)$ *from* $C(\bar{Q})$ *into* $L^r(Q \times V)$ *extends by density and continuity to a linear continuous operator* $[D_T I^k(T, T_S)]^{ext}$ *from* $L^r(Q)$ *into* $L^r(Q \times V)$, $\forall 1 < r < +\infty$.

Proof : Let us take an element $\delta T \in C(\bar{Q})$. By formula (4.22), we have $\forall (x, t, v) \in Q \times V$:

$$\left(D_T I^k(T, T_S).\delta T \right)(x, t, v) = \left[\kappa_k (\kappa_k - A)^{-1} \frac{dB_g^k}{dT}(T(., t)) \delta T(., t) \right](x, v).$$

By [50], $\frac{dB_g^k}{dT} \in L^\infty(\mathbb{R})$ and by Chapter 3, $\kappa_k(\kappa_k - A)^{-1} \in \mathcal{L}(L^r(\Omega \times V))$. The result follows. □

4.3 Corollary. *The linear continuous operator $D_T h(T, T_S)$ from $C(\bar{Q})$ into $L^r(Q)$ extends by density and continuity to a linear continuous operator $[D_T h(T, T_S)]^{ext}$ from $L^r(Q)$ into $L^r(Q)$, $\forall 1 < r < +\infty$.*

Proof : This follows from the previous lemma and formula $(4.5)_{(ii)}$. □

Now following the definition of D_T, we define, for all $t \in [0, t_f]$, the following bilinear form on $H^1(\Omega) \times H^1(\Omega)$:

$$\begin{aligned} a(t; \delta T, \delta\tau) = \frac{1}{c_p m_g} \Big\{ & k_h \int_\Omega \nabla_x(\delta T)(x) \nabla_x(\delta\tau)(x) dx \\ & - \int_\Omega \frac{d\psi}{dT}(T(x,t)) \delta T(x) \delta\tau(x) dx \\ & - \sum_{k=1}^{M} \kappa_k \int_\Omega \int_V ([D_T I^k(T, T_S)]^{ext} \delta T)(x, t, v) \delta\tau(x) d\mu(v) dx \\ & + \int_{\partial\Omega} h_T^c(x,t) \delta T(x) \delta\tau(x) dS(x) \Big\}, \end{aligned} \tag{4.26}$$

where $h_T^c(x,t) := h_c + \frac{d\Theta}{dT}(T(x,t)), \forall (x,t) \in \Sigma$.

4.11 Lemma. *The function $t \mapsto a(t, \delta T, \delta\tau)$ is measurable and there exists a constant M depending only of t_f such that*

$$|a(t, \delta T, \delta\tau)| \leq M \|\delta T\| \|\delta\tau\|, \quad \forall, \delta T, \delta\tau \in H^1(\Omega).$$

Proof : As $T \in E$ implies $T \in C(\bar{Q})$, the mapping

$$(x,t) \mapsto \frac{d\psi}{dT}(T(x,t))$$

is also continuous on $\bar{Q}$ and thus bounded, which implies:

$$\begin{aligned} \left| \int_\Omega \frac{d\psi}{dT}(T(x,t)) \delta T(x) \delta\tau(x) dx \right| & \leq c \|\delta T\|_{L^2(\Omega)} \|\delta\tau\|_{L^2(\Omega)} \\ & \leq c \|\delta T\|_{H^1(\Omega)} \|\delta\tau\|_{H^1(\Omega)}, \end{aligned}$$

with a constant c independent of t. We have $\frac{dB_g^k}{dT}(T(\cdot,\cdot)) \in C(\bar{Q})$, and $\kappa_k(\kappa_k - A)^{-1}$ is a linear and continuous operator with $\|\kappa_k(\kappa_k - A)^{-1}\|_{\mathcal{L}(L^2(\Omega \times V))} \leq 1$ (see Chapter 3), so that by formula (4.22):

$$\begin{aligned}&\left|\int_\Omega\int_V([D_T I^k(T,T_S)]^{ext}\delta T)(x,t,v)\delta\tau(x)d\mu(v)dx\right|\\&\leq c_1\left\|\frac{dB_g^k}{dT}(T(\cdot,\cdot))\right\|_{\infty,\bar{Q}}\|\delta T\|_{H^1(\Omega)}\|\delta\tau\|_{H^1(\Omega)},\end{aligned} \tag{4.27}$$

with a constant c_1 independent of t.

Now by formula (4.25), inequality (4.18) and the Lebesgue continuity Theorem, $\frac{d\Theta}{dT}$ is continuous on $\mathbb{R}$ which implies that the mapping:

$(x,t)\mapsto h_T^c(x,t) := h_c + \frac{d\Theta}{dT}(T(x,t))$ is continuous on $\bar{\Sigma}$ and thus bounded. Consequently:

$$\left|\int_{\partial\Omega} h_T^c(x,t)\delta T(x)\delta\tau(x)dS(x)\right| \lesssim \|h_T^c\|_\infty.\|\delta T\|_{H^1(\Omega)}.\|\delta\tau\|_{H^1(\Omega)},$$

with a constant independent of t. Thus for each $t\in[0,t_f], a(t;\cdot,\cdot)$ is a bilinear continuous form on $H^1(\Omega)\times H^1(\Omega)$ with a norm bounded independently of t. □

4.12 Lemma. *Given $u_0\in L^2(\Omega)$ and $f\in L^2(0,t_f;[H^1(\Omega)]^*)$, the initial boundary value problem*

$$\begin{cases}\frac{d}{dt}(u(.),v)+a(.,u(.),v)=(f(.),v), & \forall v\in H^1(\Omega),\\ u(0)=u_0,\end{cases}$$

possesses a unique solution $u\in W(0,t_f)$, where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$ and $\frac{d}{dt}$ is in the sense of $\mathcal{D}'(0,t_f)$

Proof : In view of Lemma 4.11 it is sufficient to verify hypothesis (3.15), p.511 of [18], *i.e.* to prove that the bilinear form $a(t;\cdot,\cdot)$ is uniformly coercive on $H^1(\Omega)$ with respect to $L^2(\Omega)$ i.e. that $\exists\alpha>0$, $\exists\lambda\geq 0$ such that $\forall' t\in]0,t_f[$

$$a(t;u,u)+\lambda\|u\|^2_{L^2(\Omega)}\geq\alpha\|u\|^2_{H^1(\Omega)},\ \forall u\in H^1(\Omega).$$

By inequality (4.27), we have the following bound:

$$\left|\iint_{\Omega\times V}([D_T I^k(T,T_S)]^{ext}\delta\tau)(x,t,v)\delta\tau(x)d\mu(v)dx\right|\leq c_1\left\|\frac{dB_g^k}{dT}(T(\cdot,\cdot))\right\|_{\infty,\bar{Q}}\|\delta\tau\|^2_{L^2(\Omega)}.$$

As $T\mapsto B(T,\lambda)$ is an increasing function, $-\frac{d\psi}{dT}$ and $\frac{d\Theta}{dT}$ are positive functions, and as h_c is a positive constant, then $h_T^c(\cdot,\cdot)$ is a positive function. Thus by (4.26) and the preceding inequality, $\forall\delta\tau\in H^1(\Omega)$:

$$a(t;\delta\tau,\delta\tau)\geq\frac{k_h}{c_p mg}\int_\Omega|\nabla_x(\delta\tau)|^2(x)dx-c_1\Big(\sum_{k=1}^M\kappa_k\left\|\frac{dB_g^k}{dT}(T(\cdot,\cdot))\right\|_{\infty,\bar{Q}}\Big)\|\delta\tau\|^2_{L^2(\Omega)}.$$

Hence, there exists $\lambda > 0, \alpha > 0$ such that $\forall \delta\tau \in H^1(\Omega)$, we have:

$$a(t, \delta\tau, \delta\tau) + \lambda \|\delta\tau\|^2_{L^2(\Omega)} \geq \alpha \|\delta\tau\|^2_{H^1(\Omega)}.$$

$\square$

Now we are able to prove the main result of this subsection:

Proof of Proposition 4.5.

.

Firstly, we must verify that $\forall t \in [0, t_f]$, that the right-hand side of (4.26) defines a bilinear continuous form.

We want to apply Theorem 1, p.512-513 and Theorem 2, p.513 of [18] for uniqueness and existence of the solution to the problem: g

Let $f \in L^r(Q)$ and $g \in L^{s^*}(\Sigma)$ be given. We know, by the definition of the state space, Definition 4.3, that $r \in]2.5, 2.72[$ and that $s^* > 4$. Let us define f_{DL} by:

$$\langle f_{DL}(t), \delta\tau \rangle_{H^1(\Omega)^*, H^1(\Omega)} := \int_\Omega f(x,t)\delta\tau(x)dx + \int_{\partial\Omega} g(x,t)\delta\tau(x)dS(x).$$

Now, we must verify hypothesis (3.20), p.512 of [18], i.e that

$$f_{DL} \in L^2(0, t_f; H^1(\Omega)^*).$$

We have:

$$\begin{aligned}
|\langle f_{DL}(t), \delta\tau \rangle_{H^1(\Omega)^*, H^1(\Omega)}| &= \left| \int_\Omega f(x,t)\delta\tau(x)dx + \int_{\partial\Omega} g(x,t)\delta\tau(x)dS(x) \right| \\
&\leq \int_\Omega |f(x,t)||\delta\tau(x)|dx + \int_{\partial\Omega} |g(x,t)||\delta\tau(x)|dS(x) \\
&\leq \|f(\cdot,t)\|_{L^2(\Omega)}.\|\delta\tau\|_{L^2(\Omega)} + \|g(\cdot,t)\|_{L^2(\partial\Omega)}.\|\delta\tau\|_{L^2(\partial\Omega)} \\
&\lesssim \Big(\|f(\cdot,t)\|_{L^2(\Omega)} + \|g(\cdot,t)\|_{L^2(\partial\Omega)} \Big) \|\delta\tau\|_{H^1(\Omega)},
\end{aligned}$$

which implies: $\|f_{DL}\|_{H^1(\Omega)^*} \lesssim \|f(\cdot,t)\|_{L^2(\Omega)} + \|g(\cdot,t)\|_{L^2(\partial\Omega)}$. Integrating from 0 to t_f the square of both sides of the previous inequality, we obtain:

$$\begin{aligned}
\int_0^{t_f} \|f_{DL}(t)\|^2_{H^1(\Omega)^*} dt &\leq 2 \int_0^{t_f} \|f(\cdot,t)\|^2_{L^2(\Omega)} dt + 2 \int_0^{t_f} \|g(\cdot,t)\|^2_{L^2(\partial\Omega)} dt \\
&\lesssim \|f\|^2_{L^2(0,t_f;L^2(\Omega))} + \|g\|^2_{L^2(0,t_f,L^2(\partial\Omega))} \\
&\lesssim \|f\|^2_{L^r(Q)} + \|g\|^2_{L^{s^*}(\Sigma)},
\end{aligned}$$

which shows us clearly that $f_{DL} \in L^2(0, t_f; H^1(\Omega)^*)$.

Then by Theorem 1, p.512 and Theorem 2, p.513 of [18] for uniqueness and existence, $\forall\ \delta T_0 \in L^2(\Omega)$, there exists one and only one

$$\delta T \in W(0,t_f) := \{\delta\tilde{T} \in L^2(]0,t_f[;H^1(\Omega)); \frac{d(\delta\tilde{T})}{dt} \in L^2(]0,t_f[;H^1(\Omega)^*)\} \tag{4.28}$$

such that $\forall\ \delta\tau \in H^1(\Omega)$, we have:

$$\frac{d}{dt}(\delta T(\cdot)|\delta\tau)_{L^2(\Omega)} + a(\cdot,\delta T(\cdot),\delta\tau) = \frac{1}{c_p m_g}\langle f_{DL}(\cdot),\delta\tau\rangle$$

in the sense of $\mathcal{D}'(0,t_f)$, and $\delta T(\cdot,0) = \delta T_0$. This is a fortiori true, if the initial condition $\delta T_0 \in C(\bar{\Omega})$. As $\delta\tau \in W(0,t_f)$, this is equivalent to $\forall' t \in]0,t_f[$:

$$\begin{aligned}
&c_p m_g \left\langle \frac{d\delta T}{dt}(t),\delta\tau\right\rangle + k_h \int_\Omega \nabla_x(\delta T)(x,t)\nabla_x(\delta\tau)(x)dx - \int_\Omega \frac{d\psi}{dT}(T(x,t))\\
&\times \delta T(x,t)\delta\tau(x)dx - \int_\Omega [D_T h(T,T_S)]^{ext}(x,t)\delta T(x,t)\delta\tau(x)dx\\
&+ h_c \int_{\partial\Omega} \delta T(x,t)\delta\tau(x)dS(x) + \int_{\partial\Omega} \frac{d\Theta}{dT}(T(x,t))\delta T(x,t)\delta\tau(x)dS(x)\\
&= \int_\Omega f(x,t)\delta\tau(x)dx + \int_{\partial\Omega} g(x,t)\delta\tau(x)dS(x),\ \ \forall\delta\tau \in H^1(\Omega).
\end{aligned}$$

By equations (4.13) and (4.15) this is equivalent to:

$$c_p m_g \frac{\partial\delta T}{\delta t} - k_h\Delta\delta T - \frac{d\psi}{dT}(T(\cdot,\cdot))\delta T - [D_T h(T,T_S)]^{ext}\delta T = f \tag{4.29}$$

and

$$k_h\frac{\partial\delta T}{\partial\nu} + h_c\delta T + \frac{d\Theta}{dT}(T(\cdot,\cdot))\delta T = g. \tag{4.30}$$

From equation (4.29), Lemma 4.9 applied to δT and Corollary 4.3, we get:

$$c_p m_g \frac{\partial\delta T}{\delta t} - k_h\Delta\delta T \in L^r(Q). \tag{4.31}$$

On the other hand, $h_T^c(x,\cdot) = h_c + \frac{d\Theta}{dT}(T(x,\cdot))$ is bounded as $\frac{d\Theta}{dT}$ is bounded. Set

$$\hat{f} := f + \frac{d\psi}{dT}(T(\cdot,\cdot))\delta T + [D_T h(T,T_S)]^{ext}\delta T,$$

we can view δT as a weak solution of the following initial boundary value problem:

$$\begin{cases}
c_p m_g \frac{\partial\delta T}{\delta t}(x,t) - k_h\Delta\delta T(x,t) = \hat{f}(x,t),\ \ \forall(x,t)\in Q\\
k_h\frac{\partial\delta T}{\partial\nu}(x,t) + h_c^T(x,t)\delta T(x,t) = g(x,t),\ \ \forall(x,t)\in\Sigma\\
\delta T(x,0) = \delta T_0(x),\ \ \forall x\in\Omega.
\end{cases} \tag{4.32}$$

We know that $g \in L^{s^*}(\Sigma)$ and that $\hat{f} \in L^r(Q)$ as $r < 2.72$ by the definition of the state space E and $\delta T \in W(0,t_f) \hookrightarrow L^r(Q)$ by Lemma 4.9. We know also that $h_c^T(\cdot,\cdot)$ is continuous on $\bar{\Sigma}$ and thus measurable and bounded on $\bar{\Sigma}$. Using Lemma 7.12 p.378 of [69] , Theorem 5.1 p.1306 of [11] and the superposition principle, it follows that $\delta T \in C(\bar{Q})$. Going back to the boundary condition (4.30), we get also $k_h \frac{\partial \delta T}{\partial \nu} \in L^{s^*}(\Sigma)$. Thus, we achieve that $\delta T \in E$. This allows us now to say that:

$$[D_T h(T,T_S)]^{ext} \delta T = D_T h(T,T_S) \delta T,$$

so that by Proposition 4.4, equations (4.29) and (4.30) amounts to say that

$$D_T e_1(T,T_S).\delta T = f,$$

and $D_T e_2(T,T_S).\delta T = g$. Also $\delta T(\cdot,0) = \delta T_0 \in C(\bar{\Omega})$, which is $D_T e_3(T,T_S).\delta T = \delta T_0$. This proves that $D_T e(T,T_S)$ is an algebraic isomorphism between E and $L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$

4.4.4 Proof of the Fréchet differentiability of T

This subsection is devoted to the proof of the main result of section four *i.e.* we will prove that the mapping $T_S \to T(T_S)$ is continuously Fréchet differentiable from an open neighborhood C of U_{ab} into the state space E. Firstly, we define C by:

4.6 Definition. *By C we denote the open set of the Hilbert space U defined by:*

$$C = \{T_S \in U; \frac{\underline{T}}{2} < T_S(t) < 2\,\overline{T},\ \forall t \in [0,t_f]\}$$

where $\underline{T}$ and $\overline{T}$ satisfy the conditions (2.4) in Chapter 3.

Let us remark that C is an open neighborhood of the closed convex subset of admissible controls U_{ad} in the Hilbert space $U = H^1(]0,t_f[)$ due to the continuous injection of $U = H^1(]0,t_f[) \hookrightarrow C([0,t_f])$. All the previous results remain true if we replace U_{ad} by $\overline{C}$ which amounts to replace $\underline{T}$ by $\frac{\underline{T}}{2}$ and $\overline{T}$ by $2\,\overline{T}$ because $\frac{\underline{T}}{2}$ and $2\,\overline{T}$ satisfy the hypotheses $\frac{\underline{T}}{2} < 2\,\overline{T}$, $\frac{\underline{T}}{2} \leq T_a \leq 2\,\overline{T}$ and $\frac{\underline{T}}{2} \leq T_0(x) \leq 2\,\overline{T}$, $\forall x \in \Omega$. In particular $\forall T_S \in \overline{C}$, and $\forall T_0 \in C(\bar{\Omega})$ satisfying $\frac{\underline{T}}{2} \leq T_0(x) \leq 2\,\overline{T}$, $\forall x \in \Omega$, there exists one and only one weak solution $T(T_S) \in W(0,t_f)$ of the initial boundary value problem (4.2). Moreover $T(T_S)$ satisfy the inequalities

$$\frac{\underline{T}}{2} \leq T(T_S)(x,t) \leq 2\,\overline{T},\ \forall (x,t) \in \overline{Q},$$

$T(T_S) \in E$ and $e(T(T_S),T_S) = (0, h_c T_a, T_0)$. This allows us in particular, to extend the domain of definition of the reduced cost functional, from U_{ad} to the open neighborhood

C of U_{ad}:

$$\hat{J} : C \to \mathbb{R} : T_S \longmapsto J(T(T_S), T_S). \tag{4.33}$$

Now we are able to prove our main result:

4.3 Theorem. *The mapping $T : C \to E : T_S \mapsto T(T_S)$ is continuously Fréchet differentiable with Fréchet derivative:*

$$DT(T_S) = -(D_T e(T(T_S), T_S))^{-1} \circ D_{T_S} e(T(T_S), T_S), \tag{4.34}$$

$\forall T_S \in C$.

Proof : $\forall T_S \in C$, $T(T_S) \in E$ and satisfies:

$$\begin{cases} e_1(T(T_S), T_S) = c_p m_g \frac{\partial T(T_S)}{\partial t} - k_h \Delta T(T_S) - \psi(T(T_S)) - h(T(T_S), T_S) = 0, \\ e_2(T(T_S), T_S) = k_h (\frac{\partial T(T_S)}{\partial \nu})_{|\Sigma} + h_c (T(T_S))_{|\Sigma} + \Theta((T(T_S))_{|\Sigma}) - \Theta(T_S) = h_c T_a, \\ e_3(T(T_S), T_S) = T_0(\cdot), \end{cases}$$

i.e. $e(T(T_S), T_S) - (0, h_c T_a, T_0) = (0, 0, 0)$, where T_0 is the given initial data function in $C(\bar{\Omega})$.
By Proposition 4.5, the linear continuous mapping $D_T e(T, T_S)$ is an isomorphism between E and $L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$. In particular $(D_T e(T, T_S))^{-1}$ exists. Thus by the implicit function theorem ([23], p.270), applied to the function

$$E \times C \to E : (T, T_S) \mapsto e(T, T_S) - (0, h_c T_a, T_0),$$

$\forall T_S \in C$, there exists open neighborhoods $V_U \subset C$ of T_S in U, and $V_E(T(T_S))$ of $T(T_S)$ in E, and a unique continuous mapping

$$\breve{T} : V_U(T_S) \to E : \breve{T}_S \mapsto \breve{T}(\breve{T}_S)$$

such that:

(i) $\breve{T}(T_S) = T(T_S)$,

(ii) $\forall \breve{T}_S \in V_U(T_S)$, there exists exactly one $\breve{T} \in V_E(T(T_S))$ such that $e(\breve{T}, \breve{T}_S) = (0, h_c T_a, T_0)$, namely $\breve{T} = \breve{T}(\breve{T}_S)$.

Moreover, the mapping $\breve{T} : V_U(T_S) \to E$ is continuously Fréchet differentiable with derivative:

$$D(\breve{T})(\breve{T}_S) = -(D_{\breve{T}} e(\breve{T}(\breve{T}_S), \breve{T}_S))^{-1} \circ D_{\breve{T}_S} e(\breve{T}(\breve{T}_S), \breve{T}_S),$$

$\forall \breve{T}_S \in V_U(T_S)$. As $\forall \breve{T}_S \in V_U(T_S)$, $T(\breve{T}_S) = \breve{T}(\breve{T}_S)$ by uniqueness, the result follows. □

4.5 First Order Necessary Optimality Condition

The purpose of this section is to derive a necessary condition in the form of a variational inequality for an admissible control to be an optimal control. We suppose $r \in]2.5, 2.72[$ and $s^* > 4$.
We divide this section into three subsections.

4.5.1 Derivative of the reduced cost functional and the adjoint system

Let us recall that the reduced cost functional $\hat{J}$, is the mapping:

$$\hat{J} : C \to \mathbb{R} : T_S \mapsto J(T(T_S), T_S) = \frac{1}{2} \int_Q (T(T_S) - T_d)^2(x,t) dx \otimes dt + \frac{\delta_r}{2} \|T_S - T_{S,d}\|^2_{H^1(]0,t_f[)}.$$

4.6 Proposition. *Let us set $Z := L^r(Q) \times L^{s^*}(\Sigma) \times C(\bar{\Omega})$. Let us denote by $\mathcal{I}_{r'}$ the isomorphism from $L^{r'}(Q)$ onto $(L^r(Q))^*$, where $\frac{1}{r'} + \frac{1}{r} = 1$ and by $i_2 : E \hookrightarrow L^2(Q)$, respectively $i_r : E \hookrightarrow L^r(Q)$. Let $T_S \in C$ and $\delta T_S \in U$. Then, the mapping $\hat{J} : C \to \mathbb{R}$ is continuously Fréchet differentiable and:*

$$D\hat{J}(T_S).\delta T_S = -\langle \xi^*, D_{T_S} e(T(T_S), T_S).\delta T_S \rangle_{Z^*,Z} + \delta_r (T_S - T_{S,d} | \delta T_S)_U, \tag{4.35}$$

where $\xi^* \in Z^*$ denotes the solution of the so called "adjoint system":

$$[D_T e(T(T_S), T_S)]^* \xi^* = (i_r^* \circ \mathcal{I}_{r'})(T(T_S) - T_d).$$

In the right-hand side of the adjoint equation, we consider $T(T_S)$ and T_d as elements of the space $L^{r'}(Q)$.

Proof : From Theorem 4.3, we deduce that the mapping $C \to L^2(Q) : T_S \mapsto T(T_S) - T_d$ is also continuously Fréchet differentiable, and thus also, by composition, the map which to each $T_S \in C$ associates $(T(T_S) - T_d | T(T_S) - T_d)_{L^2(Q)}$ ([23], (8.1.4) p.150). Also by the same reference, the mapping which to each $T_S \in C$ associates $(T_S - T_{S,d} | T_S - T_{S,d})_U$ is also continuously Fréchet differentiable. Thus the mapping $\hat{J} : C \to \mathbb{R}$ is also continuously Fréchet differentiable. Using the previous results, especially formula (4.34), we obtain:

$$D\hat{J}(T_S).\delta T_S = -\int_Q (T(T_S) - T_d)(i_2 \circ (D_T e(T(T_S), T_S))^{-1} \circ D_{T_S} e(T(T_S), T_S)) \cdot \delta T_S \, dx \otimes dt + \delta_r (T_S - T_{S,d} | \delta T_S)_U. \tag{4.36}$$

But,

$$\begin{aligned}
&\int_Q (T(T_S) - T_d)(i_2 \circ (D_T e(T(T_S), T_S))^{-1} \circ D_{T_S} e(T(T_S), T_S)).\delta T_S \; dx \otimes dt \\
&= (T(T_S) - T_d | i_2 \circ (D_T e(T(T_S), T_S))^{-1} \circ D_{T_S} e(T(T_S), T_S).\delta T_S)_{L^2(Q)} \\
&= \left\langle \mathcal{I}_{r'}(T(T_S) - T_d), i_r \circ (D_T e(T(T_S), T_S))^{-1} \circ D_{T_S} e(T(T_S), T_S).\delta T_S \right\rangle_{L^r(Q)^* \times L^r(Q)} \\
&= \left\langle ([(D_T e(T_{T_S}, T_S))^{-1}]^* \circ i_r^* \circ \mathcal{I}_{r'})(T(T_S) - T_d), D_{T_S} e(T(T_S), T_S).\delta T_S \right\rangle_{Z^*, Z}.
\end{aligned}$$

Thus, if we set:

$$\xi^* := ([(D_T e(T(T_S), T_S))^{-1}]^* \circ i_r^* \circ \mathcal{I}_{r'})(T(T_S) - T_d) \in Z^*,$$

equation (4.36) becomes:

$$D\hat{J}(T_S).\delta T_S = -\langle \xi^*, D_{T_S} e(T(T_S), T_S).\delta T_S \rangle_{Z^*, Z} + \delta_r (T_S - T_{S,d} | \delta T_S)_U,$$

where $\xi^* \in Z^*$ is the solution of the "adjoint system":

$$(D_T e(T(T_S), T_S))^* \xi^* = (i_r^* \circ \mathcal{I}_{r'})(T(T_S) - T_d) \tag{4.37}$$

as $[(D_T e(T(T_S), T_S))^{-1}]^* = [(D_T e(T(T_S), T_S))^*]^{-1}$. □

Let us explicit the Fréchet derivative $D_{T_S} e(T, T_S)$ which appears in the right-hand side of our previous formula (4.35) for the Fréchet derivative of the reduced cost functional.

4.7 Proposition. *Let $T \in E$ and $T_S, \delta T_S \in U = H^1(]0, t_f[)$. Then,*

$$D_{T_S} e(T, T_S).\delta T_S = \begin{pmatrix} -\sum_{k=1}^{M} \kappa_k \left[4\pi - \int_V \kappa_k (\kappa_k - A)^{-1} 1_{\Omega \times V} \; d\mu(v) \right] \frac{dB_g^k}{dT}(T_S) \delta T_S \\ -\frac{d\Theta}{dT}(T_S) \; \delta T_S \\ 0 \end{pmatrix}, \tag{4.38}$$

where the unbounded operator A in $L^r(\Omega \times V)$ has been defined in Chapter 3 and its precise definition recalled in this chapter (see in section 3, formula (4.10) with $p = r$)

Proof : It follows from Definition 4.4, that:

$$D_{T_S} e(T, T_S).\delta T_S = \begin{pmatrix} D_{T_S} e_1(T, T_S).\delta T_S = -(D_{T_S} h)(T, T_S).\delta T_S \\ D_{T_S} e_2(T, T_S).\delta T_S = -(D_{T_S} \Theta)(T_S).\delta T_S \\ D_{T_S} e_3(T, T_S).\delta T_S = 0 \end{pmatrix}.$$

Applying now formula (4.23) of Lemma 4.7, and Lemma 4.8, we obtain formula (4.38). □

Now, our objective is to give a suitable solution of the adjoint system (4.37).

4.5.2 Solving the adjoint system

4.13 Lemma. *Supposing that $T_d \in L^r(Q)$, the following linear final boundary value problem with homogeneous boundary condition and final zero condition at the final time of radiative heating t_f: find $\eta_{Vol} \in W(0, t_f)$ such that:*

$$\begin{cases} c_p m_g \frac{\partial \eta_{Vol}}{\partial t}(x,t) + k_h \Delta \eta_{Vol}(x,t) - \sum\limits_{k=1}^{M} 4\pi\kappa_k \dfrac{dB_g^k}{dT}(T_{T_S}(x,t))\eta_{Vol}(x,t) \\ \quad + \sum\limits_{k=1}^{M} \kappa_k^2 \dfrac{dB_g^k}{dT}(T_{T_S}(x,t) \int\limits_V \left[(\kappa_k - \hat{A})^{-1}\eta_{Vol}(\cdot,t)\right](x,v)d\mu(v) \\ \quad = T_d(x,t) - T_{T_S}(x,t), \quad \forall (x,t) \in Q, \\ \\ k_h \frac{\partial \eta_{Vol}}{\partial \nu}(x,t) + h^c_{T_{T_S}}(x,t)\eta_{Vol}(x,t) = 0, \, \forall (x,t) \in \Sigma, \\ \\ \eta_{Vol}(x,t_f) = 0, \forall x \in \Omega, \end{cases} \tag{4.39}$$

where the unbounded operator $\hat{A}$ in $L^2(\Omega \times V)$ has been defined in Chapter 3 and its precise definition recalled in this chapter (see formula (4.11) with $p' = 2$), and where $h^c_{T_{T_S}}(x,t) := h_c + \frac{d\Theta}{dT}(T_{T_S}(x,t))$, possesses one and only one solution in the weak sense. Moreover η_{Vol} belongs to $C(\overline{Q})$.

Proof : By η_{Vol} is a weak solution of (4.39, we mean that $\eta_{Vol}(\cdot,\cdot) \in W(0,t_f) := \{\rho \in L^2(0,t_f;H^1(\Omega));\, \frac{d\rho}{dt} \in L^2(0,t_f,H^1(\Omega)^*)\}$, that $\eta_{Vol}(\cdot,t_f) = 0$ and that $\forall \rho \in L^2(0,t_f;H^1(\Omega))$:

$$\begin{aligned} & c_p m_g \int\limits_0^{t_f} \left\langle \frac{d\eta_{Vol}}{dt}(\cdot,t), \rho(\cdot,t) \right\rangle_{H^1(\Omega)^*, H^1(\Omega)} dt - k_h \int\limits_0^{t_f} \int\limits_\Omega \nabla \eta_{Vol}(x,t) \nabla \rho(x,t) dx \otimes dt \\ & - \sum_{k=1}^{M} 4\pi \int\limits_Q \frac{dB_g^k}{dT}(T_{T_S}(x,t))\eta_{Vol}(x,t)\rho(x,t)dx \otimes dt \\ & + \sum_{k=1}^{M} \kappa_k^2 \int\limits_Q \frac{dB_g^k}{dT}(T_{T_S}(x,t) \int\limits_V \left[(\kappa_k - \hat{A})^{-1}\eta_{Vol}(\cdot,t)\right](x,v)d\mu(v)\rho(x,t)dx \otimes dt \\ & - \int\limits_0^{t_f} \int\limits_{\partial\Omega} h^c_{T_{T_S}}(x,t)\eta_{Vol}(x,t)\rho(x,t)dS(x) \otimes dt \\ & = \int\limits_Q (T_d(x,t) - T_{T_S}(x,t))\rho(x,t)dx \otimes dt. \end{aligned} \tag{4.40}$$

By the same proof as in Proposition 4.5, it follows that $\eta_{Vol}(\cdot,\cdot)$ exists, is unique and belongs to $C(\overline{Q})$. A fortiori $\eta_{Vol} \in (L^r(Q))^*$. □

4.14 Lemma. *Assume that $T_d \in L^r(Q)$, and $\xi^* = (\eta_{Vol}, \eta_B, \gamma)$ the solution of the adjoint system (4.37), where η_{Vol} is the solution in the weak sense of the problem (4.39), η_B and γ are deduced from η_V by the equations:*

$$\begin{cases} \eta_B = \eta_{Vol}|_\Sigma, \\ \gamma(dx) = c_p m_g \eta_{Vol}(\cdot, 0)dx, \end{cases} \tag{4.41}$$

then $\xi^* \in Z^*$.

Proof : By Lemma 4.13, $\eta_{Vol} \in C(\overline{Q})$. Thus, a fortiori $\eta_{Vol} \in (L^r(Q))^*$. Also, trivially $\eta_B := \eta_{Vol}|_\Sigma \in C(\bar{\Sigma}) \subset (L^{s^*}(\Sigma))^*$. γ is a signed measure of totally finite variation on $\bar{\Omega}$ because

$$|\gamma|(\bar{\Omega}) = c_p m_g \int\limits_{\bar{\Omega}} |\eta_{Vol}(x, 0)| dx < +\infty.$$

Consequently, γ defines a continuous linear form on $C(\bar{\Omega})$ i.e. belongs to $(C(\bar{\Omega}))^*$. Defining $\xi^* := (\eta_{Vol}, \eta_B, \gamma)$, it is now obvious that $\xi^* \in Z^*$. □

4.8 Proposition. *Suppose that $T_d \in L^r(Q)$, then $\xi^* = (\eta_{Vol}, \eta_B, \gamma)$ is solution of the adjoint system (4.37), where η_{Vol}, η_B, and γ are defined in Lemma (4.14).*

Proof : By Lemma 4.14, we know that $\xi^* = (\eta_{Vol}, \eta_B, \gamma) \in Z^*$. To say that $\xi^* = (\eta_{Vol}, \eta_B, \gamma) \in Z^*$ is solution of the adjoint system (4.37) i.e. of $(D_T e(T(u), u))^* \xi^* = (i_r^* \circ \mathcal{I}_{r'})(T(u) - T_d)$ amounts to say that $\forall \delta T \in E$:

$$\langle \xi^*, D_T e(T(u), u).\delta T \rangle_{Z^*, Z} = \langle T(u) - T_d, \delta T \rangle_{L^{r'}(Q), L^r(Q)},$$

i.e., that $\xi^* = (\eta_{Vol}, \eta_B, \gamma)$ satisfies $\forall \delta T \in E$:

$$\begin{aligned} &\langle \eta_{Vol}, D_T e_1(T(u), u).\delta T \rangle_{L^r(Q)^*, L^r(Q)} + \langle \eta_B, D_T e_2(T(u), u).\delta T \rangle_{(L^{s^*}(\Sigma))^*, L^{s^*}(\Sigma)} \\ &\quad + \langle \gamma, D_T e_3(T(u), u).\delta T \rangle_{(C(\bar{\Omega}))^*, C(\bar{\Omega})} = \langle T(u) - T_d, \delta T \rangle_{L^{r'}(Q), L^r(Q)}. \end{aligned}$$

Equivalently, for $\xi^* = (\eta_{Vol}, \eta_B, \gamma) \in Z^*$ to be solution of the adjoint system (4.37), we must have $\forall \delta T \in E$:

$$\begin{aligned} &\int\limits_Q (c_p m_g \frac{\partial \delta T}{\partial t} - k_h \Delta \delta T(x,t)) \eta_{Vol}(x,t) dx \otimes dt + \sum_{k=1}^M 4\pi\kappa_k \int\limits_Q \frac{dB_g^k}{dT}(T_{T_S}(x,t)) \delta T(x,t) \\ &\times \eta_{Vol}(x,t) dx \otimes dt - \int\limits_Q \sum_{k=1}^M \kappa_k \int\limits_V (D_{T_{T_S}} I^k_{T_{T_S}, T_S} \cdot \delta T)(x,t,v) \eta_{Vol}(x,t) d\mu(v) dx \otimes dt \\ &+ \int\limits_0^{t_f} \int\limits_{\partial\Omega} (k_h \frac{\partial \delta T}{\partial \nu}(x,t) + h^c_{T_{T_S}}(x,t) \delta T(x,t)) \eta_B(x,t) dS(x) dt + \int\limits_\Omega \delta T(x,0) d\gamma(x) \\ &= \int\limits_Q (T_{T_S}(x,t) - T_d(x,t)) \delta T(x,t) dx \otimes dt. \end{aligned} \tag{4.42}$$

Using Definitions 4.1 and 4.2, let us transform the left hand side of (4.42):

$$\begin{aligned}
&\int\limits_Q (c_p m_g \frac{\partial \delta T}{\partial t} - k_h \Delta \delta T(x,t))\eta_{Vol}(x,t)dx \otimes dt + \sum_{k=1}^{M} 4\pi\kappa_k \int\limits_Q \frac{dB_g^k}{dT}(T_{T_S}(x,t))\delta T(x,t) \\
&\times \eta_{Vol}(x,t)dx \otimes dt - \int\limits_Q \sum_{k=1}^{M} \kappa_k \int\limits_V (D_{T_{T_S}} I^k_{T_{T_S},T_S} \cdot \delta T)(x,t,v)d\mu(v)\eta_{Vol}(x,t)dx \otimes dt \\
&+ \int\limits_0^{t_f}\int\limits_{\partial\Omega} (k_h \frac{\partial \delta T}{\partial \nu}(x,t) + h^c_{T_{T_S}}(x,t)\delta T(x,t))\eta_B(x,t)dS(x)dt + \int\limits_\Omega \delta T(x,0)d\gamma(x) \\
&= c_p m_g \int\limits_0^{t_f} \left\langle \frac{d\delta T}{dt}(\cdot,t), \eta_{Vol}(\cdot,t)\right\rangle_{H^1(\Omega)^*,H^1(\Omega)} dt + k_h \int\limits_0^{t_f}\int\limits_\Omega \nabla\delta T(x,t)\nabla\eta_{Vol}(x,t) \\
&\times dx \otimes dt - \int\limits_0^{t_f}\int\limits_{\partial\Omega} k_h \frac{\partial \delta T}{\partial \nu}(x,t)\eta_{Vol}(x,t)dx \otimes dt + \sum_{k=1}^{M} 4\pi\kappa_k \int\limits_Q \frac{dB_g^k}{dT}(T_{T_S}(x,t)).\delta T(x,t) \\
&\eta_{Vol}(x,t)dx \otimes dt - \sum_{k=1}^{M}\int\limits_Q \kappa_k \int\limits_V (D_{T_{T_S}} I^k_{T_{T_S},T_S} .\delta T)(x,t,v).\eta_{Vol}(x,t)d\mu(v)dx \otimes dt \\
&+ \int\limits_0^{t_f}\int\limits_{\partial\Omega} (k_h \frac{\partial \delta T}{\partial \nu}(x,t) + h^c_{T_{T_S}}(x,t)\delta T(x,t))\eta_B(x,t)dS(x)dt + \int\limits_\Omega \delta T(x,0)d\gamma(x).
\end{aligned} \tag{4.43}$$

As $\eta_B(x,t) = \eta_{Vol}(x,t)$, $\forall (x,t) \in \Sigma$, the third term in the right-hand side simplifies with the first term in the last but one integral. Now by the formula of integration by parts in $W(0,t_f)$, Theorem 3.11, p. 148 of [71], we have:

$$\begin{aligned}
&c_p m_g \int\limits_0^{t_f} \left\langle \frac{d\delta T}{dt}(\cdot,t), \eta_{Vol}(\cdot,t)\right\rangle_{H^1(\Omega)^*,H^1(\Omega)} dt = \\
&- c_p m_g \int\limits_0^{t_f} \left\langle \frac{d\eta_{Vol}}{dt}(\cdot,t), \delta T(\cdot,t)\right\rangle_{H^1(\Omega)^*,H^1(\Omega)} dt - \int\limits_\Omega \delta T(x,0)d\gamma(x).
\end{aligned} \tag{4.44}$$

due to $\eta_{Vol}(\cdot,t_f) = 0$ and the definition of the Radon measure γ (see formula 4.41). Also by Chapter 3:

$$\begin{aligned}
&\sum_{k=1}^{M} \kappa_k \int\limits_{\Omega\times V} (D_T I^k_{T_S,T_S} .\delta T)(x,t,v)\ \eta_{Vol}(x,t)dx \\
&= \sum_{k=1}^{M} \kappa_k^2 \int\limits_{\Omega\times V} [(\kappa_k - A)^{-1} \cdot (\tfrac{dB_g^k}{dT}(T_{T_S}(\cdot,t))\delta T(\cdot,t)\ )](x,v)\eta_{Vol}(x,t)dxd\mu(v) \\
&= \sum_{k=1}^{M} \kappa_k^2 \int\limits_{\Omega\times V} \tfrac{dB_g^k}{dT}(T_{T_S}(x,t))\delta T(x,t)\ [(\kappa_k - A^*)^{-1} \cdot \eta_{Vol}(\cdot,t)](x,v)dxd\mu(v) \\
&= \int\limits_{\Omega\times V} \sum_{k=1}^{M} \kappa_k^2 \tfrac{dB_g^k}{dT}(T_{T_S}(x,t))[(\kappa_k - \hat{A})^{-1} \cdot \eta_{Vol}(\cdot,t)](x,v)d\mu(v)\delta T(x,t)dx.
\end{aligned} \tag{4.45}$$

From equations (4.43), (4.44), (4.45) and (4.40), the left hand side of (4.42) is equal to:

$$
\begin{aligned}
&- c_p m_g \int_0^{t_f} \left\langle \frac{d\eta_{Vol}}{dt}(\cdot,t), \delta T(\cdot,t) \right\rangle_{H^1(\Omega)^*,H^1(\Omega)} dt + k_h \int_0^{t_f}\int_\Omega \nabla \delta T(x,t) \nabla \eta_{Vol}(x,t) \\
&\times dx \otimes dt + \sum_{k=1}^{M} 4\pi\kappa_k \int_Q \frac{dB_g^k}{dT}(T_{T_S}(x,t)).\delta T(x,t)\eta_{Vol}(x,t) dx \otimes dt \\
&- \sum_{k=1}^{M} \int_Q \kappa_k^2 \frac{dB_g^k}{dT}(T_{T_S}(x,t))\delta T(x,t) \int_V [(\kappa_k - \hat{A})^{-1} \cdot \eta_{Vol}(\cdot,t)](x,v) d\mu(v) dx \otimes dt \\
&+ \int_0^{t_f}\int_{\partial\Omega} h^c_{T_{T_S}}(x,t)\delta T(x,t)\eta_B(x,t) dS(x) dt \\
&= \int_Q (T_{T_S}(x,t) - T_d(x,t))\delta T(x,t) dx \otimes dt
\end{aligned}
$$

Thus equality (4.42) is true with η_{Vol} defined by (4.40). The proof is complete. $\square$

4.2 Remark. *In the following to underline the dependence on $\eta_{Vol}(\cdot,\cdot)$ with respect to T_{T_S}, we will write $\eta_{Vol}(T_{T_S};\cdot,\cdot)$ instead of $\eta_{Vol}(\cdot,\cdot)$. For the same reason, we will write also $\eta_B(T_{T_S};\cdot,\cdot)$ and $\mu(T_{T_S};\cdot)$.*

4.5.3 The variational inequality

We are now able to state our first order necessary optimality condition:

4.4 Theorem. *Let $T_S \in U_{ad}$, be an optimal control i.e, satisfying $\hat{J}(T_S) := \inf_{\nu\in U_{ad}} \hat{J}(\nu)$, which we know to exist.*
Then T_S satisfies the following variational inequality: $\forall \nu \in U_{ad}$:

$$
\begin{aligned}
&\sum_{k=1}^{M} \kappa_k \int_0^{t_f} \left\{ \int_\Omega \left[\left(4\pi - \int_V \kappa_k(\kappa_k - A)^{-1} 1|_{\Omega\times V} d\mu(v) \right) \eta_{Vol}(T_{T_S};x,t) \right] dx \right\} \\
&\tfrac{dB_g^k}{dT}(T_S(t))(\nu(t) - T_S(t))dt + \int_0^{t_f} (\int_{\partial\Omega} \eta_B(T_{T_S};x,t) dS(x)) \frac{d\Theta}{dT}(T_S(t))(\nu(t) - T_S(t))dt \\
&+\delta_r \int_0^{t_f} (T_S(t) - T_{S,d}(t))(\nu(t) - T_S(t))dt + \delta_r \int_0^{t_f} (\dot{T}_S(t) - \dot{T}_{Sd}(t))(\dot{\nu}(t) - \dot{T}_S(t))dt \geq 0.
\end{aligned}
\tag{4.46}
$$

Proof : By Theorem 1.46, p.66 [36], if T_S is an optimal control, then T_S satisfies $\hat{J}'(T_S)(\nu - T_S) \geq 0$, $\forall \nu \in U_{ad}$. So, we need to compute $\hat{J}'(T_S)(\nu - T_S)$, by formula (4.35) and Proposition 4.8, $\forall \nu \in H^1(]0, t_f[)$:

$$
\hat{J}'(T_S)(\nu - T_S) = - \langle \xi^*, D_{T_S} e(T(T_S), T_S)(\nu - T_S) \rangle_{Z^*,Z} + \delta_r (T_S - T_{S,d}|\nu - T_S)_U,
$$

where $\xi^* = (\eta_{Vol}(T_{T_S};\cdot,\cdot), \eta_B(T_{T_S};\cdot,\cdot), \gamma(T_{T_S};dx))$; $\eta_{Vol}(T_{T_S};\cdot,\cdot)$ is the weak solution of the "adjoint problem" (4.39), $\eta_B(T_{T_S};\cdot,\cdot) = \eta_{Vol}(T_{T_S},\cdot,\cdot)|_\Sigma$ and $\gamma(T_{T_S};dx) = c_p m_g \eta_{Vol}(T_{T_S};\cdot,0)dx$. Then, by using formula (4.38) for $D_{T_S}e(T_{T_S},T_S).(\nu - T_S)$, we have:

$$\hat{J}'(T_S)(\nu - T_S) =$$

$$= \sum_{k=1}^{M} \kappa_k \int_0^{t_f} \Big\{ \int_\Omega \Big[\Big(4\pi - \int_V \kappa_k(\kappa_k - A)^{-1} 1|_{\Omega\times V} d\mu(v)\Big)\eta_{Vol}(T_{T_S};x,t)\Big]\ dx\Big\}$$

$$\frac{dB_g^k}{dT}(T_S(t))(\nu(t) - T_S(t))dt + \int_0^{t_f} (\int_{\partial\Omega} \eta_B(T_{T_S};x,t)dS(x))\frac{d\Theta}{dT}(T_S(t))(\nu(t) - T_S(t))dt$$

$$+\delta_r \int_0^{t_f} (T_S(t) - T_{S,d}(t))(\nu(t) - T_S(t))dt + \delta_r \int_0^{t_f} (\dot{T}_S(t) - \dot{T}_{S,d}(t))(\dot{\nu}(t) - \dot{T}_S(t))dt.$$

and so we get the result. $\square$

5

The Thermoviscoelasticity Equation

Contents

5.1 Introduction

Firstly, we prove the existence and uniqueness of the solution of the equations of thermoviscoelasticity. Then, in the second chapter we will state and study a related control problem. Let us now introduce the thermoviscoelasticity problem. Let Ω be a bounded domain of class $C^{1,1}$ in $\mathbb{R}^3$, representing a semi-transparent body, and let us denote by $T(x,t)$ the absolute temperature at point $x \in \Omega$ and at time t consequently to the radiative heating of the semi-transparent body Ω by a black-source S at absolute temperature $T_S(t)$ at time t surrounding Ω. This problem of the existence and uniqueness of the solution of the radiative heating problem of the semi-transparent body Ω, has been studied in Chapter 3, and the related control problem of the distribution of temperature in Ω during the whole time interval of radiative heating $[0, t_f]$, has been studied in Chapter 3. Here, we study the existence and uniqueness, and the control, of the deformation of Ω, resulting from the radiative heating of the semi-transparent body Ω by the black-source S. In the setting of the viscoelasticity theory, M. E. Rognes and R. Winther have considered in [59], a weak Hellinger-Reissner formulation for the Maxwell model of viscoelasticity, which presents the advantage to avoid the complicated integrodifferential equations linking the tensor field of stress $(x,t) \longmapsto \sigma(x,t)$ to the tensor field of the rate of deformations. In this formulation, the unknowns are the stress field $(x,t) \longmapsto \sigma(x,t)$, the velocity field

$(x,t) \longmapsto \dot{\mathbf{u}}(x,t)$ of the displacement field $(x,t) \longmapsto \mathbf{u}(x,t)$ (we will also denote $\dot{\mathbf{u}}$ by $\mathbf{v}$) and the skew component $(x,t) \longmapsto \rho(x,t)$ of the gradient of $\mathbf{u}$. The novelty, is that here, we consider the thermoviscoelasticity problem instead of the viscoelasticity problem, so that we have now a whole family of operators $A_0(.,t)$ indexed by the time $t \in [0,t_f]$:

$$\begin{array}{c} A_0(.,t) : [L^2(\Omega)]^{3\times 3} \to [L^2(\Omega)]^{3\times 3} \\ \sigma \longmapsto \frac{1+\nu}{\eta(T(.,t))}\sigma - \frac{\nu}{\eta(T(.,t))}tr(\sigma)I_3, \end{array} \tag{5.1}$$

where $\eta(T(x,t))$ denotes the viscosity at point $x \in \Omega$ and at time $t \in [0,t_f]$, which depends on the absolute temperature $T(x,t)$, and I_3 denotes the identity matrix of order 3. On the contrary, in the paper ([59], p. 963), it is supposed that A_0 is independent of the time t. We suppose that the viscosity $\eta(.)$ is a strictly positive decreasing function, defined and C^1 on $\mathbb{R}_+^*$.

5.1.1 Solving the Thermoviscoelasticity Equations for the Maxwell Model

Firstly, let us explain what is the Maxwell model. Let us denote by E the Young modulus and by ν the Poisson coefficient ($0 < \nu < \frac{1}{2}$). Denoting by $\varepsilon(u_E)$ the linearized strain-field corresponding to the elastic displacement field u_E, we have:

$$\varepsilon(u_E) = \frac{1+\nu}{E}\sigma - \frac{\nu}{E}tr(\sigma)I_3,$$

where σ denotes the stressfield. The rate of deformations corresponding to the rate of the visco displacements u_V, is given by:

$$\varepsilon(\dot{u}_V) = \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)}tr(\sigma)I_3.$$

Now, the total displacement field $u = u_E + u_V$. This implies that:

$$\begin{array}{c} \varepsilon(\dot{u}) = \varepsilon(\dot{u}_E) + \varepsilon(\dot{u}_V) \\ = \frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E}tr(\dot{\sigma})I_3 + \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)}tr(\sigma)I_3. \end{array}$$

Thus, we have the equation linking the time derivative of the displacement field u, to the stressfield σ:

$$\varepsilon(\dot{u}) = \frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E}tr(\dot{\sigma})I_3 + \frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)}tr(\sigma)I_3, \tag{5.2}$$

often called the Maxwell constitutive law. Let us remark, that if we introduce the deviatoric $\sigma^D := \sigma - \frac{1}{3}tr(\sigma)I_3$ of the stressfield σ, and $\varepsilon^D := \varepsilon - \frac{1}{3}tr(\varepsilon)I_3$ the deviatoric of the strainfield ε, this last equation is equivalent to the two equations

$$\dot{\sigma}^D + \frac{E}{\eta(T)}\sigma^D = \frac{E}{1+\nu}\varepsilon^D(\dot{u})$$

and

$$tr(\dot{\sigma}) + \frac{E}{\eta(T)} tr(\sigma) = \frac{E}{1-2\nu} tr\varepsilon(\dot{u}),$$

which can be solved easily giving us:

$$tr\sigma(x,t) = tr\sigma(x,0)e^{-E\int_0^t \frac{ds}{\eta(T(x,s))}} + \frac{E}{1-2\nu}\int_0^t e^{-E\int_{t'}^t \frac{ds}{\eta(T(x,s))}} tr\varepsilon(\dot{u}(x,t'))dt',$$

and

$$\sigma^D(x,t) = \sigma^D(x,0)e^{-E\int_0^t \frac{ds}{\eta(T(x,s))}} + \frac{E}{1+\nu}\int_0^t e^{-E\int_{t'}^t \frac{ds}{\eta(T(x,s))}} \varepsilon^D(\dot{u}(x,t'))dt'.$$

To equation (5.2), we join the equilibrium equation:

$$-\operatorname{div}(\sigma) = g - \beta\ \nabla T, \tag{5.3}$$

where we have neglected the inertial forces: we consider the quasi-static problem. $\beta := \frac{E}{1-2\nu}\alpha$, where α denotes the linear coefficient of dilatation. g denotes a volumic forcefield acting on Ω, for example its volumic weight. We suppose at least that $g \in L^2(Q)^3$, where Q denotes the "cylinder "$\Omega\times]0,t_f[$ in space-time $\mathbb{R}^3 \times \mathbb{R}$. We will suppose more on g in section 5.3. T being the weak solution of the heat conduction equation 3.42 of Chapter 3, $T \in L^2(]0,t_f[;H^1(\Omega))$ and thus also $\nabla T \in L^2(Q)^3$. To these two equations, the Maxwell constitutive law (5.2) and the equilibrium equation (5.3), we join an initial condition for the stressfield σ:

$$\sigma(.,0) = \zeta. \tag{5.4}$$

Lastly, the boundary $\partial\Omega$ of the domain Ω is partitioned into two disjoint open subsets Γ_D and Γ_N such that $\bar{\Gamma}_D \cup \bar{\Gamma}_N = \partial\Omega$. Along Γ_D we impose the velocity field of displacements $\dot{u}$ to be 0, and on Γ_N we impose no tractions i.e. $(\sigma.\nu)_{|\Gamma_N} = 0$ where ν denotes the unit normal field along $\partial\Omega$ pointing outward of Ω . We now write a weak formulation for these two equations and boundary conditions. Considering a test stressfield $\tau \in H(\operatorname{div};\Omega)^3$ such that $(\tau.\nu)_{|\Gamma_N} = 0$, we have by Green's formula:

$$\int_\Omega \varepsilon(v) : \tau\ dx = -\int_\Omega \operatorname{div}(\tau).v\ dx - \int_\Omega \rho(v) : \tau\ dx, \tag{5.5}$$

for every $v \in H^1_{\Gamma_D}(\Omega)^3$, where $\rho(v) := \frac{1}{2}(\nabla v - \nabla v^T)$. Applying Green's formula (5.5) to $\varepsilon(\dot{u})$, and using the boundary condition $\dot{u}_{|\Gamma_D} = 0$, we obtain from the Maxwell constitutive law (5.2):

$$\begin{aligned} \int_\Omega [\tfrac{1+\nu}{\eta(T)}\sigma - \tfrac{\nu}{\eta(T)} tr(\sigma) I_3] : \tau \; dx + \int_\Omega [\tfrac{1+\nu}{E}\dot{\sigma} - \tfrac{\nu}{E} tr(\dot{\sigma}) I_3] : \tau \; dx \\ + \int_\Omega \operatorname{div}(\tau).\dot{u} \; dx + \int_\Omega \rho(\dot{u}) : \tau \; dx = 0, \end{aligned} \tag{5.6}$$

$\forall \tau \in H(\operatorname{div};\Omega)^3$ such that $(\tau.\nu)_{|\Gamma_N} = 0$. For almost every $t \in]0, t_f[$, this equation expresses in a weak form the Maxwell constitutive equation (5.2) and the boundary condition $\dot{u}_{|\Gamma_D} = 0$ for the velocity of displacements $\dot{u}$. Let us note, that formally at least, if $u(.,.)$ satisfies at time 0, $u(.,0)_{|\Gamma_D} = 0$, then we will have $u(.,t)_{|\Gamma_D} = 0$, $\forall' t \in]0, t_f[$. The boundary condition $(\sigma.\nu)_{|\Gamma_N} = 0$ is an essential one in the mixed formulation and is imposed in the definition of the functional space $H_{\Gamma_N}(\operatorname{div};\Omega)^3$: we try to find

$$\sigma(.,t) \in H_{\Gamma_N}(\operatorname{div};\Omega)^3,$$

where $H_{\Gamma_N}(\operatorname{div};\Omega)$ denotes the closed subset of vector-fields in $H(\operatorname{div};\Omega)$ whose normal traces are 0 on Γ_N. Now to the weak form of the Maxwell constitutive law, we must add a weak form of the equilibrium equation (5.3):

$$\int_\Omega \operatorname{div}(\sigma).w \; dx + \int_\Omega (g - \beta \; \nabla T).w \; dx = 0, \;\; \forall w \in \left[L^2(\Omega)\right]^3. \tag{5.7}$$

We must moreover express in a weak form $\forall' t \in]0, t_f[$, the symmetry of the stress tensor $\sigma(.,t)$, so that we obtain the equation:

$$\begin{aligned} \int_\Omega \operatorname{div}(\sigma).w \; dx + \int_\Omega (g - \beta \; \nabla T).w \; dx + \int_\Omega \sigma : \xi \; dx = 0, \;\; \forall w \in L^2(\Omega)^3, \\ \forall \xi \in [L^2(\Omega)]^{3\times 3}_{skew}, \end{aligned} \tag{5.8}$$

where

$$\left[L^2(\Omega)\right]^{3\times 3}_{skew} := \{\xi \in \left[L^2(\Omega)\right]^{3\times 3}; \xi + \xi^T = 0\}.$$

To these weak forms (5.6) (resp. (5.8)), of the equations (5.2) (resp. (5.3)), we must join the initial condition (5.4). Supposing that $g \in H^1(0, t_f; L^2(\Omega)^3)$, that the hypotheses on the initial temperature T_0 in Ω and on the temperature T_S of the black-source S done in Proposition 5.3 are verified, and that the initial condition ζ for σ verifies $\zeta \in H_{\Gamma_N}(\operatorname{div};\Omega)^3$, $\zeta = \zeta^T$ and $\operatorname{div}\zeta = -(g(.,0) - \beta \nabla T(.,0))$, we prove that there exists a unique triple

$$(\sigma, v, \rho) \in H^1(0, t_f; H_{\Gamma_N}(\operatorname{div};\Omega)^3) \times L^2(0, t_f; L^2(\Omega)^3) \times L^2(0, t_f; \left[L^2(\Omega)\right]^{3\times 3}_{skew})$$

such that $\sigma(.,0)=\zeta$ and such that $\forall' t\in]0,t_f[$ equations (5.6) and (5.8) are verified i.e.:

$$\begin{cases} \int_\Omega [\frac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \frac{\nu}{\eta(T(.,t))}tr(\sigma(.,t))I_3] : \tau \; dx + \int_\Omega [\frac{1+\nu}{E}\dot{\sigma}(.,t) - \frac{\nu}{E}tr(\dot{\sigma}(.,t))I_3] : \tau \; dx \\ \qquad + \int_\Omega \mathrm{div}(\tau).v(.,t) \; dx + \int_\Omega \rho(.,t) : \tau \; dx = 0, \;\; \forall \tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3, \\ \int_\Omega \mathrm{div}(\sigma(.,t)).w \; dx + \int_\Omega (g(.,t) - \beta \; \nabla T(.,t)).w \; dx + \int_\Omega \sigma(.,t) : \xi \; dx = 0, \;\; \forall w \in L^2(\Omega)^3, \\ \qquad \forall \xi \in [L^2(\Omega)]^{3\times 3}_{skew} . \end{cases}$$

It follows in fact from these equations that $v\in L^2(0,t_f;H^1(\Omega)^3)$.

5.2 Regularity results for the absolute temperature

5.1 Proposition. *Let T be the weak solution of the heat conduction equation 3.42 of Chapter 3 i.e. of:*

$$\begin{cases} c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \Delta T(x,t) - \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{M} \kappa_k \int_V I_T^k(x,t,v)d\mu(v), \\ \qquad \forall'(x,t) \in Q := \Omega \times]0,t_f[, \\ -k_h \frac{\partial T}{\partial \nu}(x,t) = h_c(T(x,t) - T_a) + \pi \int_{\lambda_0}^{+\infty} \varepsilon_\lambda [B(T(x,t),\lambda) - B(T_S(t),\lambda)] d\lambda \\ \qquad \forall'(x,t) \in \Sigma := \partial\Omega \times]0,t_f[, \\ \qquad T(x,0) = T_0(x), \;\; \forall' x \in \Omega. \end{cases}$$

Supposing that the initial condition $T_0\in C(\bar{\Omega})\cap H^1(\Omega)$ and that $T_S\in U_{ad}$, we have that $T\in H^{2,1}(Q)$.

Proof : From equation $3.42_{(i)}$ of Chapter 3, we have:

$$c_p m_g \frac{\partial T}{\partial t}(x,t) = k_h \Delta T(x,t) - \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{M} \kappa_k \int_V I_T^k(x,t,v)d\mu(v). \tag{5.9}$$

We have proved in (Chapter 3, Proposition 4.1), that under the hypotheses that the initial condition $T_0\in C(\bar{\Omega})$ and $T_S\in U_{ad}$, that $T\equiv T_{T_S}\in C(\bar{Q})$. $T_S\in H^1(]0,t_f[)\hookrightarrow C([0,t_f])$ implying $B_g^k(T_S)\in L^\infty(]0,t_f[)$ and $T\equiv T_{T_S}$ belonging a fortiori to $L^\infty(Q)$ by Theorem 3.4 of Chapter 2, we have $I^k_{T_{T_S},T_S}(\cdot,\cdot,\cdot)\in L^\infty(Q\times V)$ ($I_T^k(\cdot,\cdot,\cdot)\equiv I^k_{T_{T_S},T_S}(\cdot,\cdot,\cdot)$) (Chapter 3). Thus $\sum_{k=1}^{M}\kappa_k\int_V I_T^k(\cdot,\cdot,v)d\mu(v)\in L^\infty(Q)$.

$\cdot$ $B_g^k(T)\in L^\infty(Q)$ because $\left|B_g^k(T)\right|\lesssim |T|$ ([50], p.10, Lemma 3.1).

Thus the “right-hand side” f in equation (5.9):

$$f := -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(\cdot,\cdot)) + \sum_{k=1}^{M}\kappa_k \int_V I_T^k(\cdot,\cdot,v)d\mu(v) \tag{5.10}$$

belongs to $L^\infty(Q)$.
Let us now look to the boundary condition 3.42 of Chapter 3:

$$-k_h\frac{\partial T}{\partial\nu}(x,t) = h_c(T(x,t)-T_a) + \pi\int_{\lambda_0}^{+\infty}\varepsilon_\lambda[B(T(x,t),\lambda)-B(T_S(t),\lambda)]d\lambda. \tag{5.11}$$

It is also clear that the right-hand side

$$\begin{aligned}&g : \Sigma \to \mathbb{R},\\ &(x,t)\mapsto h_c(T(x,t)-T_a) + \pi\int_{\lambda_0}^{+\infty}\varepsilon_\lambda[B(T(x,t),\lambda)-B(T_S(t),\lambda)]d\lambda,\end{aligned}$$

in this boundary condition belongs to $L^\infty(\Sigma)$ where $\Sigma := \partial\Omega\times]0,t_f[$. Let us prove that the “right-hand sides” f, g and the initial condition $T_0(\cdot)$ satisfy the hypotheses of the regularity result in ([43], p.78, with $s=-\frac{1}{4}$):
. $f\in L^\infty(Q)$ and thus a fortiori to $L^2(Q)$. Thus $f\in\mathcal{H}^{-1/2,-1/4}(Q) = \left(H^{1/2,1/4}(Q)\right)^*$ (this is only a notation used p.78 in [43]).
. $g\in L^\infty(\Sigma)$ and thus a fortiori to $L^2(\Sigma)$. Thus $g\in\mathcal{H}^{0,0}(\Sigma) := L^2(\Sigma)$ (this is only a notation used p.68 in [43]).
. $T_0\in\mathcal{H}^{2(-1/4+1/2)}(\Omega) = \mathcal{H}^{1/2}(\Omega) := H^{1/2}(\Omega)$ (this is only a notation used p.68 in [43]).
Thus by the regularity result at the bottom of p.78 in [43] with $s=-\frac{1}{4}$, the solution $T(\cdot,\cdot)$ of the Heat equation 3.42 of Chapter 3 belongs to the space $H^{3/2,3/4}(Q)$. Using now the “First Trace Theorem” p.9 of [43], it follows that the trace of T on Σ, $T_{|\Sigma}\in H^{1,1/2}(\Sigma)$. Now:

$$H^{1,1/2}(\Sigma) := L^2(0,t_f;H^1(\partial\Omega))\cap H^{1/2}(0,t_f;L^2(\partial\Omega))$$

by definition. But:

$$H^{1,1/2}(\Sigma)\subset L^2(0,t_f;H^{1/2}(\partial\Omega))\cap H^{1/4}(0,t_f;L^2(\partial\Omega)) := H^{1/2,1/4}(\Sigma).$$

Thus a fortiori $T_{|\Sigma}\in H^{1/2,1/4}(\Sigma)$. We want to prove that $g\in H^{1/2,1/4}(\Sigma)$. Let us prove that this implies that the mapping

$$\Theta\circ T:\Sigma\to\mathbb{R}:(x,t)\mapsto\Theta(T(x,t)) = \pi\int_{\lambda_0}^{+\infty}\varepsilon_\lambda B(T(x,t),\lambda)\,d\lambda,$$

where

$$\Theta : \mathbb{R} \to \mathbb{R} : z \mapsto \Theta(z) = \pi \int_{\lambda_0}^{+\infty} \varepsilon_\lambda B(z,\lambda)\, d\lambda, \tag{5.12}$$

is also in the Sobolev space $H^{1/2,1/4}(\Sigma)$. $\Theta(\cdot)$ being a lipschitzian function [50], we have:

$$\begin{aligned}
&\int_0^{t_f}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|\Theta(T(x,t)) - \Theta(T(y,t))|^2}{|x-y|^{2+(1/2)\times 2}} dS(x)\, dS(y)\, dt \\
&\lesssim \int_0^{t_f}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|T(x,t) - T(y,t)|^2}{|x-y|^{2+(1/2)\times 2}} dS(x)\, dS(y)\, dt \\
&< +\infty,
\end{aligned}$$

which proves that $\Theta \circ T$ also belongs to $L^2(0,t_f; H^{1/2}(\partial\Omega))$. On the other hand:

$$\int_0^{t_f}\int_0^{t_f} \frac{\|\Theta(T(\cdot,t)) - \Theta(T(\cdot,s))\|^2_{L^2(\partial\Omega)}}{|t-s|^{3/2}}\; ds\; dt \lesssim \int_0^{t_f}\int_0^{t_f} \frac{\|T(\cdot,t) - T(\cdot,s)\|^2_{L^2(\partial\Omega)}}{|t-s|^{3/2}}\; ds\; dt < +\infty,$$

which proves that $\Theta \circ T$ also belongs to $H^{1/4}(0,t_f; L^2(\partial\Omega))$. Thus $\Theta \circ T \in H^{1/2,1/4}(\Sigma)$. To conclude that g belongs to $H^{1/2,1/4}(\Sigma)$, it remains to prove that $\Theta \circ T_S \in H^{1/4}(]0,t_f[)$ for this function does not depend on the variable x. We have:

$$\int_0^{t_f}\int_0^{t_f} \frac{|\Theta(T_S(t)) - \Theta(T_S(s))|^2}{|t-s|^{3/2}}\; ds\; dt \lesssim \int_0^{t_f}\int_0^{t_f} \frac{|T_S(t) - T_S(s)|^2}{|t-s|^{3/2}}\; ds\; dt < +\infty.$$

In conclusion $g \in H^{1/2,1/4}(\Sigma)$. Using our hypothesis that $T_0 \in H^1(\Omega)$ and applying now Theorem 6.2 p.37 of [43] with $r = 0$, we obtain $T \in H^{2,1}(Q)$. □

5.1 Lemma. *Let T be the weak solution of the Heat conduction equation 3.42 and let us suppose that the initial condition $T_0 \in C(\bar{\Omega}) \cap H^1(\Omega)$. Then $\frac{\partial f}{\partial t} \in L^2(Q)$ where f has been defined by formula (5.10) above. Moreover:*

$$\left\|\frac{\partial f}{\partial t}\right\|_{L^2(Q)} \lesssim \left\|\frac{\partial T}{\partial t}\right\|_{L^2(Q)} + \|T_S\|_{H^1(]0,t_f[)}\, .$$

Proof : 1°) Setting $\psi := -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(\cdot,\cdot))$, we have:

$$\frac{\partial(\psi \circ T)}{\partial t}(x,t) = -\sum_{k=1}^{M} 4\pi\kappa_k \frac{dB_g^k}{dT}(T(x,t))\frac{\partial T}{\partial t}(x,t).$$

By ([50], p.14), $\frac{dB_g^k}{dT}(T(\cdot,\cdot)) \in L^\infty(Q)$. By the preceding proposition, we know that $T \in H^{2,1}(Q)$. In particular $\frac{\partial T}{\partial t} \in L^2(Q)$. Thus $\frac{\partial(\psi\circ T)}{\partial t} \in L^2(Q)$.

2°) $\frac{\partial}{\partial t}(\sum_{k=1}^{M}\kappa_k\int_V I_T^k(x,t,v)d\mu(v)) = \sum_{k=1}^{M}\kappa_k\int_V \frac{\partial I_T^k}{\partial t}(x,t,v)d\mu(v)$. Derivating our boundary value problem for the radiative intensity:

$$\begin{cases} v.\nabla_x I^k(x,t,v) + \kappa_k I^k(x,t,v) = \kappa_k B_g^k(T(x,t)),\ \forall'(x,v) \in \Omega\times V, \\ I^k(x,t,v) = \rho_g(|\nu_x\cdot v|)I^k(x,t,v_i) + (1-\rho_g(|\nu_x\cdot v|))B_g^k(T_S(t)),\ \forall'(x,v) \in \Gamma_-\ . \end{cases}$$

with respect to time, we have:

$$\begin{cases} v.\nabla_x \dot I^k(x,t,v) + \kappa_k \dot I^k(x,t,v) = \kappa_k \frac{dB_g^k}{dT}(T(x,t))\dot T(x,t),\ \forall'(x,v) \in \Omega\times V, \\ \dot I^k(x,t,v) = \rho_g(|\nu_x\cdot v|)\dot I^k(x,t,v_i) + (1-\rho_g(|\nu_x\cdot v|))\frac{dB_g^k}{dT}(T_S(t))\dot T_S(t),\ \forall'(x,v) \in \Gamma_-\ . \end{cases} \tag{5.13}$$

As $T_S \in H^1(]0,t_f[)$, $\dot T_S \in L^2(]0,t_f[)$ and thus the mapping "$t \longmapsto \frac{dB_g^k}{dT}(T_S(t))\dot T_S(t)$"$\in L^2(]0,t_f[)$. Let us set $J^k(x,t,v) := \dot I^k(x,t,v) - \frac{dB_g^k}{dT}(T_S(t))\dot T_S(t)$. We have:

$$\begin{cases} v.\nabla_x J^k(x,t,v) + \kappa_k J^k(x,t,v) = \kappa_k(\frac{dB_g^k}{dT}(T(x,t))\dot T(x,t) - \frac{dB_g^k}{dT}(T_S(t))\dot T_S(t)),\ \forall'(x,v) \in \Omega\times V, \\ J^k(x,t,v) = \rho_g(|\nu_x\cdot v|)J^k(x,t,v_i),\ \forall'(x,v) \in \Gamma_-\ . \end{cases}$$

By the dissipativity of the operator $-v.\nabla_x$ with the reflectivity boundary condition (see Proposition 3.2 of Chapter 3) in $L^2(\Omega\times V)$:

$$\left\|J^k(\cdot,t,\cdot)\right\|_{L^2(\Omega\times V)} \leq \sqrt{4\pi}\left\|\frac{dB_g^k}{dT}(T(\cdot,t))\dot T(\cdot,t) - \frac{dB_g^k}{dT}(T_S(t))\dot T_S(t)\right\|_{L^2(\Omega)}.$$

Thus

$$\left\|\dot I^k(\cdot,t,\cdot)\right\|_{L^2(\Omega\times V)} \lesssim \left\|\frac{dB_g^k}{dT}(T(\cdot,t))\dot T(\cdot,t)\right\|_{L^2(\Omega)} + \left|\frac{dB_g^k}{dT}(T_S(t))\right|\left|\dot T_S(t)\right|.$$

Integrating $\dot I^k(\cdot,t,\cdot)$ on the sphere of directions V with respect to the area measure $d\mu(v)$ on V, we get by the previous inequality:

$$\begin{aligned} \left\|\int_V \dot I^k(\cdot,t,v)\ d\mu(v)\right\|_{L^2(\Omega)} &\leq \int_V \left\|\dot I^k(\cdot,t,v)\right\|_{L^2(\Omega)}\ d\mu(v) \\ \lesssim \left[\int_V \left\|\dot I^k(\cdot,t,v)\right\|_{L^2(\Omega)}^2\ d\mu(v)\right]^{1/2} &= \left[\int_{\Omega\times V}\left|\dot I^k(x,t,v)\right|^2\ dx\ d\mu(v)\right]^{1/2} \\ = \left\|\dot I^k(\cdot,t,\cdot)\right\|_{L^2(\Omega\times V)} &\lesssim \left\|\tfrac{dB_g^k}{dT}(T(\cdot,t))\dot T(\cdot,t)\right\|_{L^2(\Omega)} + \left|\tfrac{dB_g^k}{dT}(T_S(t))\right|\left|\dot T_S(t)\right|. \end{aligned}$$

Setting $h := \sum_{k=1}^{M} \kappa_k \int_V I_T^k(\cdot,\cdot,v)d\mu(v)$, we obtain using the previous inequality:

$$\left\|\frac{\partial h}{\partial t}\right\|_{L^2(Q)}^2 = \int_0^{t_f} \left\|\frac{\partial h}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)}^2 dt \lesssim \sum_{k=1}^{M} \int_0^{t_f} \left\| \int_V \dot{I}^k(\cdot,t,v)\, d\mu(v) \right\|_{L^2(\Omega)}^2 dt$$
$$\lesssim \int_0^{t_f} \left\|\dot{T}(\cdot,t)\right\|_{L^2(\Omega)}^2 dt + \int_0^{t_f} \left|\dot{T}_S(t)\right|^2 dt$$
$$\lesssim \left\|\frac{\partial T}{\partial t}\right\|_{L^2(Q)}^2 + \|T_S\|_{H^1(]0,t_f[)}^2,$$

as $\frac{dB_g^k}{dT}$ is bounded ([50], p.14).
3°) As $f := \psi + h$, $\frac{\partial f}{\partial t} = \frac{\partial \psi}{\partial t} + \frac{\partial h}{\partial t}$, and thus by the two preceding points: $\frac{\partial f}{\partial t} \in L^2(Q)$ and $\left\|\frac{\partial f}{\partial t}\right\|_{L^2(Q)} \lesssim \left\|\frac{\partial T}{\partial t}\right\|_{L^2(Q)} + \|T_S\|_{H^1(]0,t_f[)}$. □

5.2 Lemma. *Let us suppose that $T_0 \in H^2(\Omega)$. Then, the following initial boundary value problem*

$$\begin{cases} c_p m_g \frac{\partial v}{\partial t}(x,t) = k_h \Delta v(x,t) + \frac{\partial f}{\partial t}(x,t), \qquad x \in \Omega,\ 0 < t < t_f\ , \\ -k_h \frac{\partial v}{\partial \nu}(x,t) = h_c v(x,t) + \Theta'(\int_0^t v(x,\tau)d\tau + T_0(x))v(x,t) - \Theta'(T_S(t))\dot{T}_S(t), \\ \qquad\qquad x \in \partial\Omega,\ 0 < t < t_f\ , \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0) + k_h \Delta T_0(x)), \quad \forall' x \in \Omega,\ (I.C.\ at\ t=0), \end{cases} \tag{5.14}$$

where

$$f(x,0) := -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T_0(x)) + \sum_{k=1}^{M} \kappa_k \int_V I_{T_0}^k(x,v)d\mu(v),\ \forall x \in \Omega, \tag{5.15}$$

possesses at least one weak solution. In (5.14), f denotes the function defined by formula (5.10).

Proof : Let us firstly precise what we mean by a weak solution of the initial boundary value problem (5.14). By a weak solution, we mean an element $v \in W(0,t_f) := \{v \in L^2(0,t_f;H^1(\Omega)); \dot{v} \in L^2(0,t_f;H^1(\Omega)^*)\}$ such that $\forall' t \in]0,t_f[$, $\forall \varphi \in H^1(\Omega)$:

$$\begin{cases} c_p m_g \frac{d}{dt} \langle v(\cdot,t), \varphi \rangle_{H^1(\Omega)^*,H^1(\Omega)} = -k_h \int_\Omega \nabla v(x,t) \cdot \nabla \varphi(x) dx + \int_\Omega \frac{\partial f}{\partial t}(x,t)\varphi(x)dx \\ \quad -h_c \int_{\partial\Omega} v(x,t)\varphi(x)dS(x) - \int_{\partial\Omega} \Theta'(\int_0^t v(x,\tau)d\tau + T_0(x))v(x,t)\varphi(x)dS(x) \\ \quad +\Theta'(T_S(t))\dot{T}_S(t) \int_{\partial\Omega} \varphi(x)dS(x), \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0) + k_h \Delta T_0(x)),\ \forall' x \in \Omega. \end{cases} \tag{5.16}$$

As f and also $\frac{df}{dt} \in L^2(0,t_f;L^2(\Omega))$ by the previous lemma, $f \in C([0,t_f];L^2(\Omega))$, and thus $f(\cdot,0) \in L^2(\Omega)$. Thus the right-hand side in the initial condition of the initial boundary value problem (5.16) belongs to $L^2(\Omega)$. To prove the existence of a solution to (5.16), we apply Faedo's Galerkin method. Let us consider $(w_j)_{j\geq 1}$ an orthonormal basis in $L^2(\Omega)$, which belong to $H^1(\Omega)$ and satisfy $\int\limits_\Omega \nabla w_j(x)\cdot\nabla w_k(x)dx = 0$ for $j \neq k$ (such a basis may be obtained by considering for example the eigenfunctions of the operator $-\Delta$ with the homogeneous Neumann boundary condition). Let us consider the vectorial subspace of dimension m, generated by $w_1,\ldots,w_m$, and let us try to find $v_m(\cdot,t) = \sum\limits_{j=1}^m (v_m(\cdot,t)|w_j)_{L^2(\Omega)}w_j$ solution of the following Cauchy problem for the nonlinear system of (integro-) differential equations:

$$\begin{cases} c_p m_g(\dot{v}_m(\cdot,t)|w_j)_{L^2(\Omega)} + k_h\int\limits_\Omega \nabla v_m(x,t)\cdot\nabla w_j(x)dx \\ \quad = \int\limits_\Omega \frac{\partial f}{\partial t}(x,t)w_j(x)dx - h_c\int\limits_{\partial\Omega} v_m(x,t)w_j(x)dS(x) \\ \quad -\int\limits_{\partial\Omega}\Theta'(\int\limits_0^t v_m(x,\tau)d\tau + T_0(x))v_m(x,t)w_j(x)dS(x) \\ \quad +\Theta'(T_S(t))\dot{T}_S(t)\int\limits_{\partial\Omega} w_j(x)dS(x), \qquad \forall j=1,\ldots,m, \\ (v_m(\cdot,0)|w_j)_{L^2(\Omega)} = \frac{1}{c_p m_g}(f(\cdot,0)+k_h\Delta T_0(\cdot)|w_j)_{L^2(\Omega)}, \ \forall j=1,\ldots,m. \end{cases} \tag{5.17}$$

As is usual to prove existence of a solution to a system of differential equations, we transform this system of differential equations in a system of nonlinear integral equations:

$$\begin{cases} c_p m_g(v_m(\cdot,t)|w_j)_{L^2(\Omega)} + k_h\int\limits_0^t\int\limits_\Omega \nabla v_m(x,s)\cdot\nabla w_j(x)dx\ ds \\ \quad = (f(\cdot,t)|w_j)_{L^2(\Omega)} - h_c\int\limits_0^t\int\limits_{\partial\Omega} v_m(x,s)w_j(x)dS(x)ds \\ \quad -\int\limits_{\partial\Omega}\Theta(\int\limits_0^t v_m(x,s)ds + T_0(x))w_j(x)dS(x) + \int\limits_{\partial\Omega}\Theta(T_0(x))w_j(x)dS(x) \\ \quad +(\Theta(T_S(t))-\Theta(T_S(0)))\int\limits_{\partial\Omega} w_j(x)dS(x) + k_h(\Delta T_0|w_j)_{L^2(\Omega)}, \ \forall j=1,\ldots,m. \end{cases} \tag{5.18}$$

As $v_m(\cdot,t) = \sum_{k=1}^{m}(v_m(\cdot,t)|w_k)_{L^2(\Omega)}w_k$,

$$\begin{aligned} & k_h\int_\Omega \nabla v_m(x,s)\cdot\nabla w_j(x)dx \\ &= k_h\sum_{k=1}^{m}(v_m(\cdot,s)|w_k)_{L^2(\Omega)}\int_\Omega \nabla w_k(x)\cdot\nabla w_j(x)dx \\ &= k_h(v_m(\cdot,s)|w_j)_{L^2(\Omega)}\left|w_j\right|^2_{H^1(\Omega)}. \end{aligned}$$

In particular: $k_h\int_0^t\int_\Omega \nabla v_m(x,s)\cdot\nabla w_j(x)dx\ ds = k_h\left|w_j\right|^2_{H^1(\Omega)}\int_0^t(v_m(\cdot,s)|w_j)_{L^2(\Omega)}ds$. Let us set: $v_m^{(j)}(t) := (v_m(\cdot,t)|w_j)_{L^2(\Omega)}$, $\forall j = 1,\ldots,m$. The above nonlinear system of m integral equations may be rewritten in the matrix form:

$$\begin{aligned} c_p m_g \begin{pmatrix} v_m^{(1)}(t) \\ \vdots \\ v_m^{(m)}(t)\end{pmatrix} &= -k_h\begin{pmatrix} \left|w_1\right|^2_{H^1(\Omega)}\int_0^t v_m^{(1)}(s)ds \\ \vdots \\ \left|w_m\right|^2_{H^1(\Omega)}\int_0^t v_m^{(m)}(s)ds \end{pmatrix} \\ &+ \begin{pmatrix} (f(.,t)|w_1)_{L^2(\Omega)} \\ \vdots \\ (f(.,t)|w_m)_{L^2(\Omega)} \end{pmatrix} \\ &-h_c\sum_{k=1}^{m}\int_0^t v_m^{(k)}(s)ds \begin{pmatrix} \int_{\partial\Omega} w_k(x)w_1(x)dS(x) \\ \vdots \\ \int_{\partial\Omega} w_k(x)w_m(x)dS(x) \end{pmatrix} \\ +&\begin{pmatrix} \int_{\partial\Omega}[\Theta(T_0(x)) - \Theta(\sum_{k=1}^{m}\int_0^t v_m^{(k)}(s)ds\ w_k(x) + T_0(x))]w_1(x)dS(x) \\ \vdots \\ \int_{\partial\Omega}[\Theta(T_0(x)) - \Theta(\sum_{k=1}^{m}\int_0^t v_m^{(k)}(s)ds\ w_k(x) + T_0(x))]w_m(x)dS(x) \end{pmatrix} \\ +&(\Theta\left(T_S(t)\right) - \Theta\left(T_S(0)\right))\begin{pmatrix} \int_{\partial\Omega} w_1(x)dS(x) \\ \vdots \\ \int_{\partial\Omega} w_m(x)dS(x) \end{pmatrix} + k_h\begin{pmatrix} (\Delta T_0|w_1)_{L^2(\Omega)} \\ \vdots \\ (\Delta T_0|w_m)_{L^2(\Omega)} \end{pmatrix}. \end{aligned} \tag{5.19}$$

Now, to prove that this system of m nonlinear scalar integral equations in the m unknowns $t \mapsto v_m^{(j)}(t)$, $j = 1,\ldots,m$, possesses one and only one solution, we proceed as in the proof of Cauchy's theorem. Firstly, let us introduce the Banach space $E = C([0,t_f];\mathbb{R}^m)$ endowed with the sup norm. Let us define the mapping $T : E \to E : (v_m^{(1)},\ldots,v_m^{(m)}) \mapsto T(v_m^{(1)},\ldots,v_m^{(m)})$, where $T(v_m^{(1)},\ldots,v_m^{(m)})$ denotes the transposed of the right-hand side of equation (5.19). Now, we are going to prove that some power of the mapping T for the composition law is a contraction. Inequality (3.6) page 13 of [50] about Planck function, implies that $\Theta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function. Thus there exists a positive constant C such that:

$$\left|T(v_m^{(1)},\ldots,v_m^{(m)})\,(t) - T(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\,(t)\right|_{\mathbb{R}^m} \leq C\sum_{j=1}^{m}\int_0^t \left|v_m^{(j)}(s) - \tilde{v}_m^{(j)}(s)\right| ds. \tag{5.20}$$

Applying Cauchy-Schwarz inequality in $\mathbb{R}^m$, we obtain:

$$\begin{aligned}[c]c &\left|T(v_m^{(1)},\ldots,v_m^{(m)})\,(t) - T(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\,(t)\right|_{\mathbb{R}^m} \\ &\leq C\sqrt{m}\int_0^t \left|\underline{v}_m(s) - \underline{\tilde{v}}_m(s)\right|_{\mathbb{R}^m} ds \\ &\leq C\sqrt{m}\; t \;\left\|\underline{v}_m - \underline{\tilde{v}}_m\right\|_E \ ,\end{aligned} \tag{5.21}$$

where $\underline{v}_m(s) := (v_m^{(1)},\ldots,v_m^{(m)})\,(s)$ and $\underline{\tilde{v}}_m(s) := (\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\,(s)$. This inequality implies that:

$$\left\|T(v_m^{(1)},\ldots,v_m^{(m)}) - T(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\right\|_E \leq C\sqrt{m}\; t_f \left\|\underline{v}_m - \underline{\tilde{v}}_m\right\|_E . \tag{5.22}$$

Let us see now what these inequalities imply for $T^{\circ 2} := T \circ T$. By inequality (5.20), Cauchy-Schwarz inequality in $\mathbb{R}^m$ and inequality (5.21), we obtain:

$$\begin{aligned} &\left|T^{\circ 2}(v_m^{(1)},\ldots,v_m^{(m)})\,(t) - T^{\circ 2}(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\,(t)\right|_{\mathbb{R}^m} \\ &= \left|T(T(v_m^{(1)},\ldots,v_m^{(m)})\,(t)) - T(T(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\,(t))\right|_{\mathbb{R}^m} \\ &\leq C\sum_{j=1}^{m}\int_0^t \left|T(v_m^{(1)},\ldots,v_m^{(m)})^{(j)}(s) - T(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})^{(j)}(s)\right| ds \\ &\leq C\sqrt{m}\int_0^t \left|T(v_m^{(1)},\ldots,v_m^{(m)})\,(s) - T(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\,(s)\right|_{\mathbb{R}^m} ds \\ &\leq \frac{\left(C\sqrt{m}\; t\right)^2}{2}\left\|v_m - \tilde{v}_m\right\|_E , \end{aligned}$$

inequality which implies:

$$\left\|T^{\circ 2}(v_m^{(1)},\ldots,v_m^{(m)}) - T^{\circ 2}(\tilde{v}_m^{(1)},\ldots,\tilde{v}_m^{(m)})\right\|_E \leq \frac{\left(C\sqrt{m}t_f\right)^2}{2}\left\|v_m - \tilde{v}_m\right\|_E .$$

Iterating, we obtain $\forall l \in \mathbb{N}^*$ that

$$\left\| T^{\circ l}(v_m^{(1)}, \dots, v_m^{(m)}) - T^{\circ l}(\tilde{v}_m^{(1)}, \dots, \tilde{v}_m^{(m)}) \right\|_E \leq \frac{\left(C\sqrt{m}t_f\right)^l}{l!} \left\| v_m - \tilde{v}_m \right\|_E .$$

Now $e^{C\sqrt{m}t_f} = \sum_{l=1}^{+\infty} \frac{\left(C\sqrt{m}t_f\right)^l}{l!}$ and as the general term of a convergent series tends to zero, $\frac{\left(C\sqrt{m}t_f\right)^l}{l!}$ tends to zero as $l \to +\infty$. Thus for l sufficiently large $T^{\circ l} : E \to E$ is a contraction and possesses thus one and only one fixed point $(v_m^{(1)}, \dots, v_m^{(m)}) \in E = C([0, t_f]; \mathbb{R}^m)$. As $T(v_m^{(1)}, \dots, v_m^{(m)})$ is also a fixed point of $T^{\circ l}$, it follows by unicity of the fixed point of $T^{\circ l}$ that $T(v_m^{(1)}, \dots, v_m^{(m)}) = (v_m^{(1)}, \dots, v_m^{(m)})$. Thus, $(v_m^{(1)}, \dots, v_m^{(m)})$ is also a fixed point of T. If T would possess another fixed point, it would be also a fixed point of $T^{\circ l}$ which possesses one and only one fixed point. Thus, T possesses one and only one fixed point. Thus, we know now that the Cauchy problem for the system of m nonlinear differential equations (5.17) possesses one and only one solution on the time interval $[0, t_f]$. Now, we have to pass to the limit as $m \to +\infty$. In that purpose, we are going to prove that the sequence $(v_m)_{m \geq 1}$ is bounded in $W(0, t_f)$ and for that some energy estimates. As $v_m(t)$ is for a fixed t a linear combination of the functions w_j, $j = 1, \dots, m$, it follows from (5.17) that:

$$\begin{gathered} c_p m_g (\dot{v}_m(\cdot, t) | v_m(\cdot, t))_{L^2(\Omega)} + k_h \left| v_m(\cdot, t) \right|^2_{H^1(\Omega)} \\ = \int_\Omega \frac{\partial f}{\partial t}(x, t) v_m(x, t) dx - h_c \int_{\partial\Omega} v_m(x, t)^2 dS(x) - \int_{\partial\Omega} \Theta'(\int_0^t v_m(x, \tau) d\tau + T_0(x)) v_m(x, t)^2 dS(x) \\ + \Theta'(T_S(t)) \dot{T}_S(t) \int_{\partial\Omega} v_m(x, t) dS(x). \end{gathered}$$

As $\Theta'(\cdot) \geq 0$, it follows from the previous equality, the inequality: $\forall \varepsilon > 0$:

$$\begin{gathered} \tfrac{c_p m_g}{2} \tfrac{d}{dt} \left\| v_m(\cdot, t) \right\|^2_{L^2(\Omega)} + k_h \left| v_m(\cdot, t) \right|^2_{H^1(\Omega)} \leq (\tfrac{\partial f}{\partial t}(\cdot, t) | v_m(\cdot, t))_{L^2(\Omega)} \\ + \tfrac{1}{\varepsilon^2} \left| (\Theta \circ u)'(t) \right|^2 + (\varepsilon^2 \left| \partial\Omega \right| - h_c) \ \left\| v_m(\cdot, t) \right\|^2_{L^2(\partial\Omega)} . \end{gathered}$$

Putting the last term of the right-hand side of the previous inequality in the left-hand side, and applying Lemma 2.5 p.35 of [71] (more usually called Poincaré's inequality), we obtain: $\exists C > 0$ such that:

$$\begin{gathered} \tfrac{c_p m_g}{2} \tfrac{d}{dt} \left\| v_m(\cdot, t) \right\|^2_{L^2(\Omega)} + C \left\| v_m(\cdot, t) \right\|^2_{H^1(\Omega)} \leq (\tfrac{\partial f}{\partial t}(\cdot, t) | v_m(\cdot, t))_{L^2(\Omega)} \\ + \tfrac{1}{\varepsilon^2} \left| (\Theta \circ T_S)'(t) \right|^2 . \end{gathered}$$

By

$$(\frac{\partial f}{\partial t}(\cdot, t) | v_m(\cdot, t))_{L^2(\Omega)} \leq \delta^2 \left\| v_m(\cdot, t) \right\|^2_{L^2(\Omega)} + \frac{1}{\delta^2} \left\| \frac{\partial f}{\partial t}(\cdot, t) \right\|^2_{L^2(\Omega)} , \ \forall \delta > 0,$$

$\exists C_1, C_2 > 0$ such that:

$$\frac{c_p m_g}{2}\frac{d}{dt}\|v_m(\cdot,t)\|^2_{L^2(\Omega)} + C_1\|v_m(\cdot,t)\|^2_{H^1(\Omega)} \leq C_2\left\|\frac{\partial f}{\partial t}(\cdot,t)\right\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon^2}\left|(\Theta\circ T_S)'(t)\right|^2.$$

Integrating both sides from 0 to t, we obtain $\forall t \in [0,t_f]$:

$$\begin{aligned}
&\frac{c_p m_g}{2}\|v_m(\cdot,t)\|^2_{L^2(\Omega)} + C_1\int_0^t \|v_m(\cdot,s)\|^2_{H^1(\Omega)}\,ds \\
&\leq C_2\left\|\frac{\partial f}{\partial t}\right\|^2_{L^2(Q)} + \frac{1}{\varepsilon^2}\int_0^t \left|(\Theta\circ T_S)'(s)\right|^2\,ds + \frac{c_p m_g}{2}\|v_m(\cdot,0)\|^2_{L^2(\Omega)} \\
&\leq C_2\left\|\frac{\partial f}{\partial t}\right\|^2_{L^2(Q)} + \frac{1}{\varepsilon^2}\int_0^{t_f} \left|(\Theta\circ T_S)'(s)\right|^2\,ds + \frac{1}{2c_p m_g}\|f(\cdot,0)+k_h\Delta T_0\|^2_{L^2(\Omega)}.
\end{aligned}$$

In particular, this shows that the sequence $(v_m)_{m\geq1}$ is bounded in $C([0,t_f];L^2(\Omega))$ and also in $L^2(0,t_f;H^1(\Omega))$. To be able to deduce that the sequence $(v_m)_{m\geq1}$ is bounded in $W(0,t_f)$, we still have to prove that the sequence $(\dot{v}_m)_{m\geq1}$ is bounded in $L^2(0,t_f;[H^1(\Omega)]^*)$. Let $w \in H^1(\Omega)$ be such that $\|w\|_{H^1(\Omega)} \leq 1$. Let us decompose w orthogonally into a part $\tilde{w}_1 \in span(w_1,\ldots,w_m)$ and another part $\tilde{w}_2$ belonging to the orthogonal of the finite dimensional vector subspace $span(w_1,\ldots,w_m)$ in $H^1(\Omega)$. This implies that $\|\tilde{w}_1\|_{H^1(\Omega)} \leq 1$. Now:

$$\begin{aligned}
\langle \dot{v}_m(t), w\rangle_{H^1(\Omega)^*,H^1(\Omega)} &= (\dot{v}_m(t)|w)_{L^2(\Omega)} \\
&= (\dot{v}_m(t)|\tilde{w}_1)_{L^2(\Omega)} + (\dot{v}_m(t)|\tilde{w}_2)_{L^2(\Omega)}.
\end{aligned}$$

But $\tilde{w}_2 = \sum_{k=m+1}^{+\infty}(w|\frac{w_k}{\|w_k\|_{H^1(\Omega)}})_{H^1(\Omega)}\frac{w_k}{\|w_k\|_{H^1(\Omega)}}$, this series being convergent in $H^1(\Omega)$ and thus a fortiori in $L^2(\Omega)$ as $H^1(\Omega) \hookrightarrow L^2(\Omega)$. This implies that for $j \in \{1,\ldots,m\}$:

$$(w_j|\tilde{w}_2)_{L^2(\Omega)} = \sum_{k=m+1}^{+\infty}(w|\frac{w_k}{\|w_k\|_{H^1(\Omega)}})_{H^1_L(]0,l[)}\frac{(w_j|w_k)_{L^2(\Omega)}}{\|w_k\|_{H^1(\Omega)}} = 0.$$

As $\dot{v}_m(t)$ is a linear combination of $w_1,\ldots,w_m$, it follows that $(\dot{v}_m(t)|\tilde{w}_2)_{L^2(\Omega)} = 0$. Thus $\langle\dot{v}_m(t),w\rangle_{H^1(\Omega)^*,H^1(\Omega)} = (\dot{v}_m(t)|\tilde{w}_1)_{L^2(\Omega)}$. By the first equation of (5.17), the continuity of the trace operator $H^1(\Omega) \to L^2(\partial\Omega)$ and the boundedness of $\Theta'(\cdot)$ ([50], p.14):

$$\begin{aligned}
&c_p m_g\langle\dot{v}_m(t),w\rangle_{H^1(\Omega)^*,H^1(\Omega)} = c_p m_g(\dot{v}_m(t)|\tilde{w}_1)_{L^2(\Omega)} = \\
&-k_h\int_\Omega \nabla v_m(x,t)\cdot\nabla\tilde{w}_1(x)dx + \int_\Omega \frac{\partial f}{\partial t}(x,t)\tilde{w}_1(x)dx - h_c\int_{\partial\Omega} v_m(x,t)\tilde{w}_1(x)dS(x) \\
&-\int_{\partial\Omega}\Theta'(\int_0^t v_m(x,\tau)d\tau + T_0(x))v_m(x,t)\tilde{w}_1(x)dS(x) + \Theta'(T_S(t))\dot{T}_S(t)\int_{\partial\Omega}\tilde{w}_1(x)dS(x) \\
&\lesssim \left\|\frac{\partial f}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)} + \|v_m(\cdot,t)\|_{H^1(\Omega)} + \left|\dot{T}_S(t)\right|,
\end{aligned}$$

with some constant independent of m. Replacing w by $-w$, we obtain:

$$\left|\langle \dot{v}_m(t), w\rangle_{H^1(\Omega)^*,H^1(\Omega)}\right| \lesssim \left\|\frac{\partial f}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)} + \|v_m(\cdot,t)\|_{H^1(\Omega)} + |\dot{u}(t)|\,,$$

for every $w \in H^1(\Omega)$ such that $\|w\|_{H^1(\Omega)} \leq 1$. Thus:

$$\|\dot{v}_m(t)\|_{H^1(\Omega)^*} \lesssim \left\|\frac{\partial f}{\partial t}(\cdot,t)\right\|_{L^2(\Omega)} + \|v_m(\cdot,t)\|_{H^1(\Omega)} + |\dot{u}(t)|\,.$$

Integrating both sides from 0 to t_f, we obtain:

$$\int_0^{t_f} \|\dot{v}_m(t)\|^2_{H^1(\Omega)^*}\ dt \lesssim \int_0^{t_f} \left\|\frac{\partial f}{\partial t}(\cdot,t)\right\|^2_{L^2(\Omega)} dt + \int_0^{t_f} \|v_m(\cdot,t)\|^2_{H^1(\Omega)}\,dt + \int_0^{t_f} |\dot{u}(t)|^2\,dt.$$

Thus the sequence $(\dot{v}_m)_{m\geq 1}$ is bounded in $L^2(0,t_f;[H^1(\Omega)]^*)$. In conclusion, the sequence $(v_m)_{m\geq 1}$ is bounded in $W(0,t_f)$ and in $C([0,t_f];L^2(\Omega))$. Thus, there exists a subsequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ of the sequence $(v_m)_{m\geq 1}$ which converges weakly to some element v in $L^2(0,t_f;H^1(\Omega))$, weakly star in $L^\infty(0,t_f;L^2(\Omega))$ and such that the sequence of their derivatives $\left(\dot{v}_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to $\dot{v}$ in $L^2(0,t_f;H^1(\Omega)^*)$. Let ξ be an arbitrary function belonging to the space $L^2(]0,t_f[)$.

$$\int_0^{t_f} c_p m_g (\dot{v}_{m_{\tilde{l}}}(t)|w_j)_{L^2(\Omega)}\xi(t)dt \to \int_0^{t_f} c_p m_g \left\langle \dot{v}(t), w_j\right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \xi(t)dt$$

as $\tilde{l} \to +\infty$ as the mapping

$$L^2(0,t_f;H^1(\Omega)^*) \to \mathbb{R} : \psi \mapsto \int_0^{t_f} c_p m_g \left\langle \psi(t), w_j\right\rangle_{H^1(\Omega)^*,H^1(\Omega)} \xi(t)dt$$

is a continuous linear form on $L^2(0,t_f;H^1(\Omega)^*)$.

$$\int_0^{t_f} k_h(\nabla v_{m_{\tilde{l}}}(t)|\nabla w_j)_{L^2(\Omega)^3}\xi(t)dt \to \int_0^{t_f} k_h(\nabla v(t)|\nabla w_j)_{L^2(\Omega)^3}\xi(t)dt$$

as the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $L^2(0,t_f;H^1(\Omega))$ and the mapping

$$L^2(0,t_f;H^1(\Omega)) \to \mathbb{R} : \psi \mapsto \int_0^{t_f} k_h(\nabla\psi(t)|\nabla w_j)_{L^2(\Omega)^3}\xi(t)dt$$

is a continuous linear form on $L^2(0,t_f;H^1(\Omega))$. As the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $L^2(0,t_f;H^1(\Omega))$, and the mapping

$$L^2(0,t_f;H^1(\Omega)) \to L^2(]0,t_f[;L^2(\partial\Omega)) : w \mapsto w(.,t)_{|\partial\Omega}$$

is linear and continuous, it follows that the sequence of the traces of the functions $v_{m_{\tilde{l}}}$ on $\Sigma := \partial\Omega\times]0,t_f[$ converges weakly to $v_{|\Sigma}$ in $L^2(\Sigma)$ as $\tilde{l}\to+\infty$. This implies that

$$-h_c\int_0^{t_f}\int_{\partial\Omega} v_{m_{\tilde{l}}}(x,t)w_j(x)dS(x)\xi(t)dt \to -h_c\int_0^{t_f}\int_{\partial\Omega} v(x,t)(l)w_j(x)dS(x)\xi(t)dt$$

as $\tilde{l}\to+\infty$. Let us now prove that:

$$\begin{gathered}-\int_0^{t_f}\int_{\partial\Omega}\Theta'(\int_0^t v_{m_{\tilde{l}}}(x,\tau)d\tau+T_0(x))v_{m_{\tilde{l}}}(x,t)w_j(x)dS(x)\xi(t)dt \to \\ -\int_0^{t_f}\int_{\partial\Omega}\Theta'(\int_0^t v(x,\tau)d\tau+T_0(x))v(x,t)w_j(x)dS(x)\xi(t)dt\end{gathered} \tag{5.23}$$

as $\tilde{l}\to+\infty$. We know that the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $W(0,t_f)$. By the compacity lemma p. 57 of [41], $\forall\varepsilon\in]0,\frac{1}{2}[$, the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges strongly to v in $L^2(0,t_f;H^{1-\varepsilon}(\Omega))$ and thus the sequence of the traces on the lateral boundary $\Sigma=\partial\Omega\times]0,t_f[$ of the $v_{m_{\tilde{l}}}$ converges strongly to $v_{|\Sigma}$ in $L^2(0,t_f;H^{\frac{1}{2}-\varepsilon}(\partial\Omega))$ and thus a fortiori in $L^2(0,t_f;L^2(\partial\Omega))=L^2(\Sigma)$. To prove (5.23), we decompose the modulus of the difference of both sides of (5.23) into two terms. By Proposition 4.7.5 p.226 and Proposition 4.6.5 p.217 of [44], $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ possesses a subsequence that we still denote by $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$, such that for almost every $x\in\partial\Omega$, $v_{m_{\tilde{l}}}(x,.)$ converges to $v(x,.)$ in $L^2(]0,t_f[)$. In particular for almost every $x\in\partial\Omega$, $\int_0^t v_{m_{\tilde{l}}}(x,\tau)d\tau$ converges to $\int_0^t v(x,,\tau)d\tau$, $\forall t\in]0,t_f[$. A fortiori $\int_0^t v_{m_{\tilde{l}}}(x,\tau)d\tau$ converges to $\int_0^t v(x,,\tau)d\tau$, for almost every $(x,t)\in\Sigma$. $\Theta'(\cdot)$ being bounded, $\left|\Theta'(\int_0^t v_{m_{\tilde{l}}}(x,\tau)d\tau+T_0(x))\right|\leq\|\Theta'(\cdot)\|_{\infty,\mathbb{R}}$. Thus, firstly:

$$\begin{gathered}\left|\int_\Sigma[\Theta'(\int_0^t v_{m_{\tilde{l}}}(x,\tau)d\tau+T_0(x))-\Theta'(\int_0^t v(x,\tau)d\tau+T_0(x))]v(x,t)w_j(x)\xi(t)dS(x)dt\right| \\ \leq\int_\Sigma\left|[\Theta'(\int_0^t v_{m_{\tilde{l}}}(x,\tau)d\tau+T_0(x))-\Theta'(\int_0^t v(x,\tau)d\tau+T_0(x))]\right||v(x,t)|\,|w_j(x)|\,|\xi(t)|\,dS(x)dt \\ \to 0,\end{gathered}$$

by the continuity of $\Theta'(\cdot)$ and Lebesgue Dominated Convergence Theorem. Secondly:

$$\left|\int\limits_{\Sigma}\Theta'(\int\limits_{0}^{t}v_{m_{\tilde{l}}}(x,\tau)d\tau+T_0(x))(v_{m_{\tilde{l}}}(x,t)-v(x,t))w_j(x)\xi(t)dS(x)dt\right|$$
$$\lesssim\int\limits_{\Sigma}\left|v_{m_{\tilde{l}}}(x,t)-v(x,t)\right|\left|w_j(x)\right|\left|\xi(t)\right|dS(x)dt$$
$$\to 0,$$

for the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges strongly to v in $L^2(0,t_f;L^2(]0,l[))$. Thus (5.23) is proved. We are now in a position to pass to the limit as $\tilde{l}\to+\infty$ in the first equation of (5.17) multiplied by $t\mapsto\xi(t)$ and integrated from 0 to t_f, obtaining $\forall j=1,\ldots,m$:

$$\begin{gathered}c_pm_g\int\limits_{0}^{t_f}(\dot{v}_m(\cdot,t)|w_j)_{L^2(\Omega)}\xi(t)\;dt+k_h\int\limits_{0}^{t_f}\int\limits_{\Omega}\nabla v_m(x,t)\cdot\nabla w_j(x)\xi(t)\;dx\;dt\\
=\int\limits_{0}^{t_f}\int\limits_{\Omega}\tfrac{\partial f}{\partial t}(x,t)w_j(x)\xi(t)\;dx\;dt-h_c\int\limits_{\Sigma}v_m(x,t)w_j(x)\xi(t)\;dS(x)\;dt\\
-\int\limits_{\Sigma}\Theta'(\int\limits_{0}^{t}v_m(x,\tau)d\tau+T_0(x))v_m(x,t)w_j(x)\xi(t)dS(x)\;dt+\int\limits_{0}^{t_f}\Theta'(T_S(t))\dot{T}_S(t)\int\limits_{\partial\Omega}w_j(x)dS(x)\xi(t)\;dt,\end{gathered}\tag{5.24}$$

ξ being arbitrary in $L^2(]0,t_f[)$, we obtain the first equation of (5.16). As the sequence $\left(v_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ converges weakly to v in $L^2(0,t_f;H^1(\Omega))$, and their time derivatives $\left(\dot{v}_{m_{\tilde{l}}}\right)_{\tilde{l}\geq 1}$ weakly to $\dot{v}$ in $L^2(0,t_f;[H^1(\Omega)]^*)$, the sequence $\left(v_{m_{\tilde{l}}}(\cdot,0)\right)_{\tilde{l}\geq 1}$ converges weakly to $v(\cdot,0)$ in $L^2(\Omega)$. Passing to the limit in the second equation of (5.17), we obtain $v(\cdot,0)=\frac{1}{c_pm_g}(f(\cdot,0)+k_h\Delta T_0)$ i.e. the second equation of (5.16). We have thus proved the existence of at least one weak solution to the initial boundary value problem (5.14). $\square$

5.3 Lemma. *Supposing that the initial condition $T_0(\cdot)$ for the initial boundary value problem 3.42 i.e. of*

$$\begin{cases}c_pm_g\frac{\partial T}{\partial t}(x,t)=k_h\Delta T(x,t)-\sum\limits_{k=1}^{M}4\pi\kappa_kB_g^k(T(x,t))+\sum\limits_{k=1}^{M}\kappa_k\int\limits_{V}I_T^k(x,t,v)d\mu(v),\;(x,t)\in\Omega\times]0,t_f[,\\
-k_h\frac{\partial T}{\partial\nu}(x,t)=h_c(T(x,t)-T_a)+\Theta(T(x,t))-\Theta(T_S(t)),\;(x,t)\in\partial\Omega\times]0,t_f[,\\
T(x,0)=T_0(x),\;x\in\Omega,\end{cases}\tag{5.25}$$

in which Θ is defined by formula (5.12), belongs to $H^2(\Omega)$, and satisfies the compatibility condition on $\partial\Omega$ with T_S:

$$k_h\frac{\partial T_0}{\partial\nu}(x)+h_c(T_0(x)-T_a)+\Theta(T_0(x))=\Theta(T_S(0)),\;\forall' x\in\partial\Omega,\tag{5.26}$$

we have $\frac{dT}{dt}=v$, where v denotes a weak solution of the initial boundary value problem

(5.14) which consequently is unique.

Proof : Let us consider a weak solution v of the initial boudary value problem (5.14) which exists by the preceding lemma. Thus $v \in W(0,t_f) := \{v \in L^2(0,t_f;H^1(\Omega)); \dot{v} \in L^2(0,t_f;H^1(\Omega)^*)\}$ and satisfies $\forall' t \in]0,t_f[$, $\forall \varphi \in H^1(\Omega)$:

$$\begin{cases} c_p m_g \frac{d}{dt}\left\langle v(\cdot,t),\varphi\right\rangle_{H^1(\Omega)^*,H^1(\Omega)} = \\ -k_h \int\limits_{\Omega} \nabla v(x,t)\cdot\nabla\varphi(x)dx + \int\limits_{\Omega} \frac{\partial f}{\partial t}(x,t)\varphi(x)dx - h_c \int\limits_{\partial\Omega} v(x,t)\varphi(x)dS(x) \\ -\int\limits_{\partial\Omega} \Theta'(\int\limits_0^t v(x,\tau)d\tau + T_0(x))v(x,t)\varphi(x)dS(x) + \Theta'(T_S(t))\dot{T}_S(t)\int\limits_{\partial\Omega}\varphi(x)dS(x), \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0)+k_h\Delta T_0(x)),\ \forall' x\in\Omega. \end{cases} \tag{5.27}$$

where f is defined by formula (5.10) i.e.

$$f(x,t) := -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T(x,t)) + \sum_{k=1}^{M}\kappa_k \int\limits_V I_T^k(x,v)d\mu(v),\ \forall x\in\Omega. \tag{5.28}$$

Let us set $\tilde{T}(\cdot,t) := \int\limits_0^t v(\cdot,\tau)d\tau + T_0(\cdot)$. It follows easily from the theory of the Bochner integral, that T so defined belongs to $W(0,t_f)$ and that $\frac{d\tilde{T}}{dt}(\cdot,t) = v(\cdot,t)$, $\forall' t\in]0,t_f[$. In particular $\frac{d\tilde{T}}{dt} \in L^2(0,t_f;H^1(\Omega))$. Integrating both sides of (5.27) from 0 to t, we obtain:

$$\begin{aligned} c_p m_g \int\limits_\Omega \frac{d\tilde{T}}{dt}(x,t)\varphi(x)dx - c_p m_g \int\limits_\Omega v(x,0)\varphi(x)dx = -k_h \int\limits_\Omega (\nabla\tilde{T}(x,t) - \nabla T_0(x))\cdot\nabla\varphi(x)dx \\ + \int\limits_\Omega (f(x,t)-f(x,0))\varphi(x)dx - h_c\int\limits_{\partial\Omega}(\tilde{T}(x,t)-T_0(x))\varphi(x)dS(x) \\ -\int\limits_{\partial\Omega}(\Theta(\tilde{T}(x,t))-\Theta(T_0(x)))\varphi(x)dS(x) + (\Theta(T_S(t))-\Theta(T_S(0)))\int\limits_{\partial\Omega}\varphi(x)dS(x). \end{aligned}$$

Using the initial condition $v(x,0) = \frac{1}{c_p m_g}(f(x,0)+k_h\Delta T_0(x))$, $\forall' x\in\Omega$, for the initial boundary value problem (5.14), we obtain:

$$\begin{aligned} c_p m_g \int\limits_\Omega \frac{d\tilde{T}}{dt}(x,t)\varphi(x)dx - k_h\int\limits_\Omega \Delta T_0(x)\varphi(x)dx = -k_h\int\limits_\Omega (\nabla\tilde{T}(x,t)\cdot\nabla\varphi(x)dx \\ +k_h\int\limits_\Omega \nabla T_0(x)).\nabla\varphi(x)dx + \int\limits_\Omega f(x,t)\varphi(x)dx - h_c\int\limits_{\partial\Omega}(\tilde{T}(x,t)-T_0(x))\varphi(x)dS(x) \\ -\int\limits_{\partial\Omega}(\Theta(\tilde{T}(x,t))-\Theta(T_0(x)))\varphi(x)dS(x) + (\Theta(T_S(t))-\Theta(T_S(0)))\int\limits_{\partial\Omega}\varphi(x)dS(x). \end{aligned}$$

T_0 belonging to $H^2(\Omega)$ and φ belonging to $H^1(\Omega)$, using Green's formula (Lemma 1.5.3.7, p. 59 of [31]), we obtain after simplification:

$$c_p m_g \int_\Omega \frac{d\tilde{T}}{dt}(x,t)\varphi(x)dx - k_h \int_{\partial\Omega} \frac{\partial T_0}{\partial \nu}(x)\varphi(x)dS(x) = -k_h \int_\Omega \nabla \tilde{T}(x,t)\cdot\nabla\varphi(x)dx$$
$$+\int_\Omega f(x,t)\varphi(x)dx - h_c \int_{\partial\Omega} (\tilde{T}(x,t) - T_0(x))\varphi(x)dS(x)$$
$$-\int_{\partial\Omega} (\Theta(\tilde{T}(x,t)) - \Theta(T_0(x)))\varphi(x)dS(x) + (\Theta(u(t)) - \Theta(u(0)))\int_{\partial\Omega}\varphi(x)dS(x).$$

Now, using the compatibility condition on $\partial\Omega$ (5.26) to further simplify, we obtain:

$$c_p m_g \int_\Omega \frac{d\tilde{T}}{dt}(x,t)\varphi(x)dx = -k_h \int_\Omega \nabla \tilde{T}(x,t)\cdot\nabla\varphi(x)dx$$
$$+\int_\Omega f(x,t)\varphi(x)dx - h_c \int_{\partial\Omega} (\tilde{T}(x,t) - T_a)\varphi(x)dS(x)$$
$$-\int_{\partial\Omega} \Theta(\tilde{T}(x,t))\varphi(x)dS(x) + \Theta(T_S(t))\int_{\partial\Omega}\varphi(x)dS(x).$$

Considering the difference $T - \tilde{T}$

$$c_p m_g \left\langle \frac{d(T-\tilde{T})}{dt}(.,t), \varphi\right\rangle_{(H^1(\Omega))^*, H^1(\Omega)} = -k_h \int_\Omega \nabla_x (T-\tilde{T})(x,t)\cdot\nabla_x\varphi(x)dx$$
$$-h_c \left\langle T(\cdot,t) - \tilde{T}(.,t), \varphi\right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} - \left\langle \Theta(T(\cdot,t)) - \Theta(\tilde{T}(\cdot,t)), \varphi\right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)},$$
$$\forall \varphi \in H^1(\Omega).$$

Considering $\varphi = (T-\tilde{T})(\cdot,t)$ which belongs to $H^1(\Omega)$, $\forall' t \in]0,t_f[$, we obtain:

$$\frac{d}{dt}\int_\Omega (T-\tilde{T})^2(x,t)\ dx \le 0,\ \forall' t \in]0,t_f[,$$

as $\Theta(\cdot)$ is an increasing function. The absolutely continuous function

$$[0,t_f] \to \mathbb{R} : t \mapsto \int_\Omega (T-\tilde{T})^2(x,t)\ dx$$

is thus decreasing, positive and being null for $t = 0$, is null. Thus $T(\cdot,t) = \tilde{T}(\cdot,t)$, $\forall' t \in]0,t_f[$. Consequently $\frac{dT}{dt} = v$. □

5.1 Remark. *Let us remark that if the initial condition is equal to the ambient temperature i.e.* $T_0(x) = T_a$, $\forall x \in \Omega$, *then the compatibility condition (5.26) is verified if* $T_S(0) = T_a$.

5.2 Proposition. *Supposing that the initial condition* $T_0(\cdot)$ *of the initial boundary value problem 3.42 i.e. of the initial boundary value problem (5.25) belongs to* $H^2(\Omega)$ *and sat-*

isfies the compatibility condition (5.26), then the weak solution T of that initial boundary value problem (5.25) satisfies the regularity property on its time derivative that $\frac{dT}{dt} \in L^2(0, t_f; H^1(\Omega))$.

Proof : As by the preceding lemma $\frac{dT}{dt} = v$, where v denotes a weak solution of the initial boundary value problem (5.14), it follows immediately from $v \in W(0, t_f) := \{v \in L^2(0, t_f; H^1(\Omega)); \dot{v} \in L^2(0, t_f; H^1(\Omega)^*)\}$, that

$$\frac{dT}{dt} \in L^2(0, t_f; H^1(\Omega)). \tag{5.29}$$

$\square$

5.3 Proposition. *Under the same hypotheses on the initial condition $T_0(\cdot)$ for the initial boundary value problem (5.25) than in our previous Proposition 5.2, supposing also that $T_S \in H^2(]0, t_f[)$ and that*

$$\begin{gathered} x \mapsto f(x,0) + k_h(\Delta T_0)(x) \\ = -\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T_0)(x) + \sum_{k=1}^{M} \kappa_k \int_V I^k_{T_0, T_S(0)}(x, v) d\mu(v) + k_h(\Delta T_0)(x) \end{gathered} \tag{5.30}$$

belongs to $C(\bar{\Omega})$, the solution T of the initial boundary value problem (5.25) satisfies the regularity property that $\frac{dT}{dt} \in C(\bar{Q})$, where Q denotes the cylinder $Q := \Omega \times]0, t_f[$.

Proof : We know that $\frac{dT}{dt} = v$ is a weak solution of the initial boundary value problem

$$\begin{cases} c_p m_g \frac{\partial v}{\partial t}(x,t) = k_h \Delta v(x,t) + \frac{\partial f}{\partial t}(x,t), \qquad x \in \Omega,\ 0 < t < t_f\ , \\ -k_h \frac{\partial v}{\partial \nu}(x,t) = h_c v(x,t) + \Theta'(\int_0^t v(x,\tau)d\tau + T_0(x))v(x,t) - \Theta'(T_S(t))T_S'(t), \\ \qquad\qquad\qquad\qquad\qquad x \in \partial\Omega,\ 0 < t < t_f\ , \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0) + k_h \Delta T_0(x)), \qquad \forall' x \in \Omega,\ (\text{I.C. at } t = 0). \end{cases} \tag{5.31}$$

Let us set $a(x,t) := \Theta'(\int_0^t v(x,\tau)d\tau + T_0(x))$ and $b(t) := -\Theta'(T_S(t))$. $a \in L^\infty(\Sigma)$ and $b \in L^\infty(]0, t_f[)$. With these notations, the preceding initial boundary value problem can be rewritten:

$$\begin{cases} c_p m_g \frac{\partial v}{\partial t}(x,t) = k_h \Delta v(x,t) + \frac{\partial f}{\partial t}(x,t), & x \in \Omega,\ 0 < t < t_f\ , \\ -k_h \frac{\partial v}{\partial \nu}(x,t) = h_c v(x,t) + a(x,t)v(x,t) + b(t)T_S'(t), & x \in \partial\Omega,\ 0 < t < t_f\ , \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0) + k_h \Delta T_0(x)), & \forall' x \in \Omega,\ (\text{I.C. at } t = 0). \end{cases} \tag{5.32}$$

To be able to apply Theorem 1.40 p.49 of [36] to the preceding initial boundary value problem, we must firstly prove that $\frac{\partial f}{\partial t} \in L^r(Q)$, for some $r > \frac{3}{2} + 1 = 2.5$. In view of formula (5.28) for $f(\cdot,\cdot)$, we have to prove that $\frac{\partial T}{\partial t} = v \in L^r(Q)$, for some $r > 2.5$. By the regularity result at the bottom of p.78 in [43] with $s = -\frac{1}{2}$, the solution $v(\cdot,\cdot)$ of the initial boundary value problem (5.32) belongs to the space $H^{1,1/2}(Q)$. Now $H^{1,1/2}(Q) = [H^{2,1}(Q), L^2(Q)]_{1/2} \hookrightarrow [H^1(Q), L^2(Q)]_{1/2} = H^{1/2}(Q)$. As $H^{1/2}(Q) \hookrightarrow L^r(Q)$ for $r < \frac{8}{3} > 2.5$, $v \in L^r(Q)$ for some $r > 2.5$. From

$$\frac{\partial f}{\partial t} = -\sum_{k=1}^{M} 4\pi\kappa_k \frac{dB_g^k}{dT}(T(\cdot,\cdot))v + \sum_{k=1}^{M}\kappa_k \int_V \frac{\partial I_T^k}{\partial t}(\cdot,\cdot,v)d\mu(v),$$

using $v \in L^r(Q)$, and some results of Chapter 3, we deduce that $\frac{\partial f}{\partial t} \in L^r(Q)$. Applying now Theorem 1.40 p.49 of [36] to the initial boundary value problem (5.32), we deduce that $v = \frac{\partial T}{\partial t} \in C(\bar{Q})$. □

5.2 Remark. *1°) As the initial condition T_0 is supposed to belong to $H^2(\Omega)$, by the Sobolev embedding theorem ([31], p.27): $H^2(\Omega) \hookrightarrow C^{0,1/2}(\bar{\Omega})$ and thus a fortiori $T_0 \in C(\bar{\Omega})$. $B_g^k(T_0)(x) = n_g^2 \int_{\lambda_k}^{\lambda_{k+1}} B(T_0(x),\lambda)\, d\lambda$ where $B(T_0(x),\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T_0(x)}}-1)}$, C_1 and C_2 are constants and n_g denotes the refractive index of the semi-transparent material, glass in occurrence [50]. It is proved in [50] p.14 that:*

$$B(T_0(x),\lambda) = \frac{2C_1}{\lambda^5(e^{\frac{C_2}{\lambda T_0(x)}}-1)} \le \frac{2C_1}{C_2}\frac{T_0(x)}{\lambda^4},$$

if $T_0(x)$ is positive ($B(T_0(x),\lambda) = 0$, if $T_0(x)$ is negative). Thus by the Lebesgue continuity theorem, $B_g^k(T_0)$ is continuous on $\bar{\Omega}$. Consequently, condition (5.30), is equivalent to the continuity on $\bar{\Omega}$ of the function

$$x \longmapsto \sum_{k=1}^{M}\kappa_k \int_V I^k_{T_0,T_S(0)}(x,v)d\mu(v) + k_h(\Delta T_0)(x).$$

2°) If $T_0(x) = T_a$, $\forall x \in \Omega$, and $T_S(0) = T_a$, we have already remarked that the compatibility condition (5.26) is verified. In that case $I^k_{T_0,T_S(0)}(x,v) = I^k_{T_a,T_a}(x,v) = B_g^k(T_a)$ and thus $\kappa_k \int_V I^k_{T_0,T_S(0)}(x,v)d\mu(v) = 4\pi\kappa_k B_g^k(T_a)$ is a constant function thus trivially continuous on $\bar{\Omega}$. Consequently, in that case condition (5.30) is also verified.

5.3 Existence and Uniqueness Result

In the following, we will suppose that $g \in H^1(]0,t_f[;L^2(\Omega)^3)$ and that the hypotheses on T_0 and T_S done in Proposition 5.3 are verified. We introduce for every time $t \in [0,t_f]$, the auxiliary problem: find $\sigma_e(.,t) \in H_{\Gamma_N}(\text{div};\Omega)^3$, $u_e(.,t) \in L^2(\Omega)^3$, $\rho_e(.,t) \in [L^2(\Omega)]^{3\times 3}_{skew}$

such that:

$$\begin{cases} \int\limits_{\Omega} [\frac{1+\nu}{\eta(T(.,t))}\sigma_e(.,t) - \frac{\nu}{\eta(T(.,t))} tr(\sigma_e(.,t))I] : \tau \; dx + \int\limits_{\Omega} \operatorname{div}(\tau).u_e(.,t) \; dx + \int\limits_{\Omega} \rho_e(.,t) : \tau \; dx = 0, \\ \qquad\qquad \forall \tau \in H_{\Gamma_N}(\operatorname{div};\Omega)^3, \\ \int\limits_{\Omega} \operatorname{div}(\sigma_e(.,t)).w \; dx + \int\limits_{\Omega} \sigma_e(.,t) : \xi \; dx + \int\limits_{\Omega} (g(.,t) - \beta \; \nabla T(.,t)).w \; dx = 0, \\ \qquad\qquad \forall w \in L^2(\Omega)^3, \forall \xi \in [L^2(\Omega)]^{3\times 3}_{skew}. \end{cases} \tag{5.33}$$

Let us set $\mu(.,t) := \frac{\eta(T(.,t))}{2(1+\nu)}$ and $\lambda(.,t) := \frac{\eta(T(.,t))\nu}{(1+\nu)(1-2\nu)}$. Problem (5.33) that has been just defined is equivalent to: find

$$u_e(.,t) \in H^1_{\Gamma_D}(\Omega)^3 := \{w \in H^1(\Omega)^3; w = 0 \text{ on } \Gamma_D\}, \tag{5.34}$$

such that $\forall v \in H^1_{\Gamma_D}(\Omega)^3$:

$$\begin{cases} \int\limits_{\Omega} (2\mu(.,t)\varepsilon(u_e(.,t)) : \varepsilon(v) + \lambda(.,t) \operatorname{div}(u_e(.,t)) \operatorname{div}(v)) \; dx \\ \qquad = \int\limits_{\Omega} (g(.,t) - \beta \; \nabla T(.,t)).v \; dx, \end{cases} \tag{5.35}$$

$\sigma_e(.,t)$ and $\rho_e(.,t)$ being then defined in terms of $u_e(.,t)$ by the formulas:

$$\begin{cases} \sigma_e(.,t) := 2\mu(.,t)\varepsilon(u_e(.,t)) + \lambda(.,t) \operatorname{div}(u_e(.,t))I_3, \\ \rho_e(.,t) := \frac{1}{2}(\nabla u_e(.,t) - (\nabla u_e)^T(.,t)). \end{cases} \tag{5.36}$$

Let us note that $\operatorname{div}(u_e(.,t))$ (resp. $\operatorname{div}(v)$) is the same as $tr\varepsilon(u_e(.,t))$ (resp. $tr\varepsilon(v)$). The proof of this equivalence is somewhat similar to the proof of Theorem 3.2 p. 327 of [26]. It uses Green's formula (5.5). By Corollary 3.4 of Chapter 3, $\underline{T} \leq T(.,.) \leq \bar{T}$, and as the viscosity $\eta(.)$ is a positive decreasing function of the temperature: $\eta(\bar{T}) \leq \eta(T(.,.)) \leq \eta(\underline{T})$. This implies that the positive coefficients $\lambda(.,.)$ and $\mu(.,.)$ are bounded. In particular, there exists a constant $C > 0$ such that $\mu(.,.) \geq C > 0$.

5.4 Proposition. *There exists one and only one $u_e \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ verifying (5.35) for almost every $t \in]0,t_f[$.*

Proof : We want to apply Lax-Milgram's lemma in the space $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$. Let us define the bilinear form:

$$\begin{aligned} a(.,.) : L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3) \times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3) \to \mathbb{R} \\ (u,v) \longmapsto \int\limits_0^{t_f}\int\limits_{\Omega} (2\mu(.,.)\varepsilon(u(.,.)) : \varepsilon(v(.,.)) + \lambda(.,.) \operatorname{div}(u(.,.)) \operatorname{div}(v(.,.))) \; dx \; dt. \end{aligned} \tag{5.37}$$

Using the fact that the coefficients $\lambda(.,.)$ and $\mu(.,.)$ are bounded, it is easy to verify that it is a continuous bilinear form. By Korn's inequality (Corollary (11.2.22) p.285 of [5]), there exists a constant $C > 0$ such that:

$$\|\varepsilon(v(.,t))\|_{[L^2(\Omega)]^{3\times 3}} \geq C\,\|v(.,t)\|_{H^1_{\Gamma_D}(\Omega)^3}\,.$$

Thus

$$a(v,v) \gtrsim \int_0^{t_f} \|v(.,t)\|^2_{H^1_{\Gamma_D}(\Omega)^3}\,dt = \|v\|^2_{L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)}\,.$$

Thus the bilinear form $a(.,.)$ is also coercive on the space $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$. Thus by "Lax-Milgram's lemma" (Theorem (2.7.7) p.62 of [5]), there exists one and only one $u_e \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ such that $\forall v \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$:

$$a(u_e,v) = \int_0^{t_f}\int_\Omega (g(x,t) - \beta\,\nabla T(x,t)).v(x,t)\;dx\;dt.$$

Considering functions $v \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ of the particular form $v = w\otimes\zeta$, where $w \in H^1_{\Gamma_D}(\Omega)^3$ and $\zeta \in L^2(0,t_f)$, we obtain:

$$\begin{aligned}\int_0^{t_f}\left[\int_\Omega (2\mu(x,t)\varepsilon(u_e(x,t)) : \varepsilon(w(x)) + \lambda(x,t)\,\mathrm{div}(u_e(x,t))\,\mathrm{div}(w(x)))\;dx\right]\zeta(t)\;dt \\ = \int_0^{t_f}\left[\int_\Omega (g(x,t) - \beta\,\nabla T(x,t))w(x)\;dx\right]\;\zeta(t)dt,\end{aligned}$$

$\forall\zeta \in L^2(0,t_f)$. Thus $\forall w \in H^1_{\Gamma_D}(\Omega)^3$, we have for almost every $t \in]0,t_f[$:

$$\begin{aligned}\int_\Omega (2\mu(x,t)\varepsilon(u_e(x,t)) : \varepsilon(w(x)) + \lambda(x,t)\,\mathrm{div}(u_e(x,t))\,\mathrm{div}(w(x)))\;dx \\ = \int_\Omega (g(x,t) - \beta\,\nabla T(x,t)).w(x)\;dx.\end{aligned}$$

As $H^1_{\Gamma_D}(\Omega)^3$ is separable and the countable union of sets of measure 0 is still a set of measure 0, our claim follows. $\square$

We want now to prove more, that in fact $u_e \in H^1(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$. Let us denote by

$$\begin{aligned}L_t : H^1_{\Gamma_D}(\Omega)^3 \to \left[H^1_{\Gamma_D}(\Omega)^3\right]^* : z \mapsto \\ v \mapsto \int_\Omega (2\mu(x,t)\varepsilon(z)(x) : \varepsilon(v)(x) + \lambda(x,t)(\mathrm{div}\,z)(x)(\mathrm{div}\,v)(x)\;dx.\end{aligned} \tag{5.38}$$

By Korn's inequality and Lax-Milgram's lemma, $\forall t \in [0,t_f]$, the continuous linear operator $L_t : H^1_{\Gamma_D}(\Omega)^3 \to \left[H^1_{\Gamma_D}(\Omega)^3\right]^*$ is an isomorphism for $\mu(.,t) := \frac{\eta(T(.,t))}{2(1+\nu)}$ is lower bounded by $\frac{\eta(\bar{T})}{2(1+\nu)} > 0$.

5.4 Lemma. *The mapping* $L : [0,t_f] \to \mathcal{L}(H^1_{\Gamma_D}(\Omega)^3; \left[H^1_{\Gamma_D}(\Omega)^3\right]^*)$ *is derivable at* $\forall t \in]0,t_f[$ *and*

$$\begin{gathered}\dot{L}_t : H^1_{\Gamma_D}(\Omega)^3 \to \left[H^1_{\Gamma_D}(\Omega)^3\right]^* : u \mapsto \\ v \mapsto \int\limits_{\Omega} (2\dot{\mu}(x,t)\varepsilon(u)(x) : \varepsilon(v)(x) + \dot{\lambda}(x,t)(\operatorname{div} u)(x)(\operatorname{div} v)(x)\ dx.\end{gathered} \tag{5.39}$$

Moreover L *is continuously derivable on the interval* $[0,t_f]$.

Proof : Being a function of one variable, the derivability amounts to prove that $\lim\limits_{h\to 0} \frac{L_{t+h}-L_t}{h}$ exists for every $t \in [0,t_f]$. We are going to prove that in fact $\lim\limits_{h\to 0} \frac{L_{t+h}-L_t}{h} = \dot{L}_t$, $\forall t \in [0,t_f]$, where $\dot{L}_t$ is defined by formula (5.39). For $h \in \mathbb{R}\backslash\{0\}$, such that $t+h \in [0,t_f]$, one has that:

$$\begin{gathered}\frac{L_{t+h}-L_t}{h} : H^1_{\Gamma_D}(\Omega)^3 \to \left[H^1_{\Gamma_D}(\Omega)^3\right]^* : u \mapsto \\ v \mapsto \int\limits_{\Omega} (2\tfrac{\mu(x,t+h)-\mu(x,t)}{h}\varepsilon(u)(x) : \varepsilon(v)(x) + \tfrac{\lambda(x,t+h)-\lambda(x,t)}{h}(\operatorname{div} u)(x)(\operatorname{div} v)(x))\ dx.\end{gathered}$$

But:

$$\begin{aligned}[c]c\frac{\mu(x,t+h)-\mu(x,t)}{h} - \dot{\mu}(x,t) &= \frac{1}{h}\int\limits_t^{t+h} \dot{\mu}(x,s)\ ds - \dot{\mu}(x,t) \\ &= \frac{1}{h}\int\limits_t^{t+h} (\dot{\mu}(x,s) - \dot{\mu}(x,t))\ ds.\end{aligned}$$

implying that:

$$\left|\frac{\mu(x,t+h)-\mu(x,t)}{h} - \dot{\mu}(x,t)\right| \leq \frac{1}{h}\int\limits_{[t,t+h]} |\dot{\mu}(x,s) - \dot{\mu}(x,t)|\ ds.$$

Thus $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall h \in]-\delta,\delta[\backslash\{0\}$

$$\left|\frac{\mu(x,t+h)-\mu(x,t)}{h} - \dot{\mu}(x,t)\right| \leq \varepsilon$$

for $\dot{\mu}(.,.) : \bar{Q} \to \mathbb{R} : (x,t) \mapsto \frac{\eta'(T(x,t))\dot{T}(x,t)}{2(1+\nu)}$ is uniformly continuous on $\bar{Q}$, $T(.,.)$ and $\dot{T}(.,.)$ being continuous on $\bar{Q}$ by Proposition 5.3. The same is also true for $\lambda(.,.)$ by the same arguments. Consequently: $\left\|\frac{L_{t+h}-L_t}{h} - \dot{L}_t\right\|_{\mathcal{L}(H^1_{\Gamma_D}(\Omega)^3;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} \lesssim \varepsilon$ for $h \in]-\delta,\delta[\backslash\{0\}$.This proves 5.39.

By the uniform continuity of the functions $\dot{\lambda}$ and $\dot{\mu}$ on $\bar{Q}$, it follows that the mapping $[0,t_f] \to \mathcal{L}(H^1_{\Gamma_D}(\Omega)^3; \left[H^1_{\Gamma_D}(\Omega)^3\right]^*) : t \mapsto \dot{L}_t$ is also continuous. $\square$

By Korn's inequality and Lax-Milgram's lemma in the space $H^1_{\Gamma_D}(\Omega)^3$, the operators $L_t \in \mathcal{L}(H^1_{\Gamma_D}(\Omega)^3; \left[H^1_{\Gamma_D}(\Omega)^3\right]^*)$ are invertible. For the mapping $[0,t_f] \to \mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*; H^1_{\Gamma_D}(\Omega)^3) : t \mapsto L_t^{-1}$, we have:

5.5 Proposition. *The mapping $[0,t_f] \to \mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*; H^1_{\Gamma_D}(\Omega)^3) : t \mapsto L_t^{-1}$ is continuously derivable on the interval $[0,t_f]$ and*

$$\frac{d}{dt} L_t^{-1} = -L_t^{-1} \circ \dot{L}_t \circ L_t^{-1}. \tag{5.40}$$

Proof : This follows from ([20] (8.3.1) pp. 155-156). □

5.6 Proposition. *$\forall' t \in]0,t_f[$:*

$$\begin{aligned} \tfrac{du_e}{dt}(.,t) = &-L_t^{-1} \circ \dot{L}_t \circ L_t^{-1}(g(.,t) - \beta\ \nabla T(.,t)) \\ &+L_t^{-1}(\dot{g}(.,t) - \beta\ \nabla \dot{T}(.,t)). \end{aligned} \tag{5.41}$$

Proof : This follows from formula $u_e(.,t) = L_t^{-1}(g(.,t) - \beta\ \nabla T(.,t))$ using the preceding proposition and from ([9], Appendix). □

5.7 Proposition. *The mapping $[0,t_f] \to H^1_{\Gamma_D}(\Omega)^3 : t \mapsto u_e(.,t)$ is absolutely continuous.*

Proof : The mapping $[0,t_f] \to \mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*; H^1_{\Gamma_D}(\Omega)^3) : t \mapsto L_t^{-1}$ being continuously derivable on the interval $[0,t_f]$ is a fortiori absolutely continuous. As $\hat{g} := g - \beta\ \nabla T \in H^1(0,t_f; \left[H^1_{\Gamma_D}(\Omega)^3\right]^*)$, the mapping $[0,t_f] \to \left[H^1_{\Gamma_D}(\Omega)^3\right]^* : t \longmapsto \hat{g}(.,t)$ is also absolutely continuous ([9], Appendix). Thus $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for any finite family of non-overlapping intervals $(]a_i,b_i[)_{1\leq i\leq n}$ contained in the interval $[0,t_f]$ such that $\sum_{i=1}^{n}(b_i - a_i) < \delta$, we have:

$$\begin{aligned}
&\sum_{i=1}^{n} \left\| L_{b_i}^{-1}(\hat{g}(.,b_i)) - L_{a_i}^{-1}(\hat{g}(.,a_i)) \right\|_{H^1_{\Gamma_D}(\Omega)^3} \\
&\leq \sum_{i=1}^{n} \left\| L_{b_i}^{-1} \right\|_{\mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*;H^1_{\Gamma_D}(\Omega)^3)} \left\| \hat{g}(.,b_i) - \hat{g}(.,a_i) \right\|_{\left[H^1_{\Gamma_D}(\Omega)^3\right]^*} \\
&+\sum_{i=1}^{n} \left\| L_{b_i}^{-1} - L_{a_i}^{-1} \right\|_{\mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*;H^1_{\Gamma_D}(\Omega)^3)} \left\| \hat{g}(.,a_i) \right\|_{\left[H^1_{\Gamma_D}(\Omega)^3\right]^*} \\
&\leq (\sup_{t\in[0,t_f]} \left\| L_t^{-1} \right\|_{\mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*;H^1_{\Gamma_D}(\Omega)^3)}) \sum_{i=1}^{n} \left\| \hat{g}(.,b_i) - \hat{g}(.,a_i) \right\|_{\left[H^1_{\Gamma_D}(\Omega)^3\right]^*} \\
&+(\sup_{t\in[0,t_f]} \left\| \hat{g}(.,t) \right\|_{\left[H^1_{\Gamma_D}(\Omega)^3\right]^*} \sum_{i=1}^{n} \left\| L_{b_i}^{-1} - L_{a_i}^{-1} \right\|_{\mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*;H^1_{\Gamma_D}(\Omega)^3)} \\
&\lesssim \varepsilon.
\end{aligned}$$

Thus by the formula $u_e(.,t) = L_t^{-1}(g(.,t) - \beta\ \nabla T(.,t))$, $\forall t \in [0,t_f]$, the mapping $[0,t_f] \to H^1_{\Gamma_D}(\Omega)^3 : t \mapsto u_e(.,t)$ is also absolutely continuous. □

5.8 Proposition. *Supposing $g \in H^1(0,t_f;L^2(\Omega)^3)$ (it suffices in fact to suppose that $g \in H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)$), that the initial condition $T_0 \in H^2(\Omega)$ and satisfies the compatibility condition (5.26) on $\partial\Omega$ with T_S, then $u_e \in H^1(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$.*

Proof : By formula (5.39), the mapping $[0,t_f] \to \mathcal{L}(H^1_{\Gamma_D}(\Omega)^3;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*) : t \mapsto \dot{L}_t$ is continuous on $[0,t_f]$ and thus bounded. The mapping $[0,t_f] \to \mathcal{L}(\left[H^1_{\Gamma_D}(\Omega)^3\right]^*;H^1_{\Gamma_D}(\Omega)^3) : t \mapsto L_t^{-1}$ is continuous on $[0,t_f]$ and thus also bounded. Thus by formula (5.41):

$$\begin{aligned}\left\|\tfrac{du_e}{dt}(.,t)\right\|_{H^1_{\Gamma_D}(\Omega)^3} \lesssim \|g(.,t)-\beta\ \nabla T(.,t)\|_{\left[H^1_{\Gamma_D}(\Omega)^3\right]^*} \\ +\left\|\dot{g}(.,t)-\beta\ \nabla \dot{T}(.,t)\right\|_{\left[H^1_{\Gamma_D}(\Omega)^3\right]^*},\end{aligned} \tag{5.42}$$

which implies by Proposition 5.2 that $\int_0^{t_f} \left\|\frac{du_e}{dt}(.,t)\right\|^2_{H^1_{\Gamma_D}(\Omega)^3}\ dt < +\infty$. Using moreover Proposition 5.7, this implies that $u_e \in H^1(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ by ([9], Appendix) and the reflexivity of the Banach space $H^1_{\Gamma_D}(\Omega)^3$. □

Now, that the auxilary problem has been solved, let us come back to our problem: we have to find $\sigma \in H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3)$ time dependent tensorfield, $v \in L^2(0,t_f;L^2(\Omega)^3)$ and $\rho \in L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})$ verifying equations (5.6), (5.8) and the initial condition (5.4) i.e. such that

$$\begin{cases}\int_\Omega [\frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)}tr(\sigma)I_3] : \tau\ dx + \int_\Omega [\frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E}tr(\dot{\sigma})I_3] : \tau\ dx \\ +\int_\Omega \mathrm{div}(\tau).v\ dx + \int_\Omega \rho : \tau\ dx = 0,\ \forall \tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3, \\ \int_\Omega \mathrm{div}(\sigma).w\ dx + \int_\Omega (g-\beta\ \nabla T).w\ dx + \int_\Omega \sigma : \xi\ dx = 0,\ \forall w \in L^2(\Omega)^3, \forall \xi \in [L^2(\Omega)]^{3\times3}_{skew}, \\ \sigma(.,0) = \zeta.\end{cases} \tag{5.43}$$

In particular $\sigma(.,t)$, must verify the equilibrium equation: $-\mathrm{div}(\sigma(.,t)) = g(.,t) - \beta\ \nabla T(.,t), \forall t \in [0,t_f]$. $g \in H^1(0,t_f;L^2(\Omega)^3)$ and $\nabla T \in H^1(0,t_f;L^2(\Omega)^3)$ also by Proposition 5.2. Thus, in particular $g-\beta\ \nabla T \in C([0,t_f];L^2(\Omega)^3)$. We suppose that the initial datum ζ for the stressfield σ belongs to $H_{\Gamma_N}(\mathrm{div};\Omega)^3$ and verifies $-\mathrm{div}\,\zeta = g(.,0)-\beta\nabla T(.,0)$. Let us introduce the new unknown $\sigma_0(.,t) := \sigma(.,t) - \sigma_e(.,t)$, $\forall t \in [0,t_f]$. $\sigma_0(\cdot,t)$ must verify:

$$\mathrm{div}\,\sigma_0(.,t) = \mathrm{div}\,\sigma(.,t) - \mathrm{div}\,\sigma_e(.,t) = 0,\ \forall t \in [0,t_f], \tag{5.44}$$

and:

$$\sigma_0(.,t).\nu_{|\Gamma_N} = \sigma(.,t).\nu_{|\Gamma_N} - \sigma_e(.,t).\nu_{|\Gamma_N} = 0,\ \forall t \in [0,t_f]. \tag{5.45}$$

Similarly to [59], let us introduce the closed subspace of $[L^2(\Omega)]^{3\times3}$:

$$H_0 = \{\tau \in \left[L^2(\Omega)\right]^{3\times3}; \tau = \tau^T,\ \mathrm{div}\,\tau = 0 \text{ and } \tau.\nu_{|\Gamma_N} = 0\}. \tag{5.46}$$

In this definition the operator divergence div must be understood in the weak sense (i.e. in the sense of distributions). From equations (5.44) and (5.45), it is clear that $\sigma_0(.,t)$ must belong to H_0. Let us consider the linear continuous operator:

$$\begin{array}{c} A_1 : [L^2(\Omega)]^{3\times 3} \to [L^2(\Omega)]^{3\times 3} \\ \sigma \longmapsto \frac{1+\nu}{E}\sigma - \frac{\nu}{E} tr(\sigma) I_3, \end{array} \tag{5.47}$$

and let us denote by P_{H_0} the orthogonal projection from $[L^2(\Omega)]^{3\times 3}$ onto H_0. By equation $(5.33)_{(i)}$, we have:

$$\int_\Omega [\frac{1+\nu}{\eta(T(.,t))}\sigma_e(.,t) - \frac{\nu}{\eta(T(.,t))} tr(\sigma_e(.,t))I] : \tau \; dx \; = 0, \; \forall \tau \in H_0.$$

Using the notation (5.1), this latest equation can be rewritten:

$$\langle A_0(.,t)\sigma_e(.,t), \tau\rangle = 0, \; \forall \tau \in H_0.$$

By equation (5.6), we must have:

$$\langle A_0(.,t)\sigma(.,t), \tau\rangle + \langle A_1\dot{\sigma}(.,t), \tau\rangle = 0, \; \forall \tau \in H_0. \tag{5.48}$$

From these latest two equations and $\sigma(.,t) = \sigma_e(.,t) + \sigma_0(.,t)$, this amounts to impose $\sigma_0(.,t)$ to also verify:

$$\langle A_0(.,t)\sigma_0(.,t), \tau\rangle + \langle A_1\dot{\sigma}_0(.,t), \tau\rangle = -\langle A_1\dot{\sigma}_e(.,t), \tau\rangle, \; \forall \tau \in H_0. \tag{5.49}$$

From $u_e \in H^1(0, t_f; H^1_{\Gamma_D}(\Omega)^3)$ follows that $\sigma_e \in H^1(0, t_f; [L^2(\Omega)]^{3\times 3})$ and thus $\dot{\sigma}_e \in L^2(0, t_f; [L^2(\Omega)]^{3\times 3})$. This implies also that $\sigma_e \in C([0, t_f]; [L^2(\Omega)]^{3\times 3})$ and thus $\sigma_e(0)$ has sense and belongs to $[L^2(\Omega)]^{3\times 3}$. We have $\zeta := \sigma(.,0) \in H_{\Gamma_N}(\mathrm{div};\Omega)^3$ and $\mathrm{div}\,\zeta = -(g(.,0) - \beta\nabla T(.,0))$ by hypothesis, so that $\sigma_0(.,0) = \zeta - \sigma_e(.,0) \in H_{\Gamma_N}(\mathrm{div};\Omega)^3$ and $\mathrm{div}\,\sigma_0(.,0) = 0$ by the equilibrium equation $(5.33)_{(ii)}$ for σ_e. $\sigma_0(.,0)$ belonging to $H_{\Gamma_N}(\mathrm{div};\Omega)^3$, we have also $\sigma_0(.,0).\nu = 0$, on Γ_N. Thus $\sigma_0(.,0) \in H_0$. Using the orthogonal projection operator P_{H_0} on H_0, equation (5.49) and its initial condition may be rewritten:

$$\left\{ \begin{array}{c} (P_{H_0}A_1)\dot{\sigma}_0(.,t) + (P_{H_0}A_0(.,t))\,\sigma_0(.,t) = -(P_{H_0}A_1)\,\dot{\sigma}_e(.,t), \; \forall' t \in]0, t_f[\;, \\ \sigma_0(.,0) = \zeta - \sigma_e(.,0). \end{array} \right. \tag{5.50}$$

We have to prove the existence of a unique $\sigma_0 \in H^1(0, t_f; H_0)$ satisfying (5.50). Firstly, we prove:

5.9 Proposition. *$(P_{H_0}A_1)|_{H_0}$ is an isomorphism from H_0 onto H_0.*

Proof : We are going to prove that $(P_{H_0}A_1)|_{H_0} : H_0 \to H_0$ verifies the hypotheses of

Lax-Milgram's lemma. Firstly $\forall \sigma, \tau \in [L^2(\Omega)]^{3\times 3}$:

$$\begin{aligned}\left|(A_1\sigma|\tau)_{[L^2(\Omega)]^{3\times3}}\right| &= \left|(\frac{1+\nu}{E}\sigma - \frac{\nu}{E}tr(\sigma)I_3|\tau)_{[L^2(\Omega)]^{3\times3}}\right| \\ &\leq \left|(\frac{1+\nu}{E}\sigma|\tau)_{[L^2(\Omega)]^{3\times3}}\right| + \frac{\nu}{E}\left\|tr(\sigma)\right\|_{L^2(\Omega)}\left\|tr(\tau)\right\|_{L^2(\Omega)} \\ &\leq \frac{1+\nu}{E}\left\|\sigma\right\|_{[L^2(\Omega)]^{3\times3}}\left\|\tau\right\|_{[L^2(\Omega)]^{3\times3}} + 3\frac{\nu}{E}\left\|\sigma\right\|_{[L^2(\Omega)]^{3\times3}}\left\|\tau\right\|_{[L^2(\Omega)]^{3\times3}} \\ &= \frac{1+4\nu}{E}\left\|\sigma\right\|_{[L^2(\Omega)]^{3\times3}}\left\|\tau\right\|_{[L^2(\Omega)]^{3\times3}},\end{aligned} \tag{5.51}$$

which show us that the mapping $(\sigma,\tau) \mapsto (A_1\sigma|\tau)_{[L^2(\Omega)]^{3\times3}}$ is a bilinear continuous form on $[L^2(\Omega)]^{3\times3}$. Let us now prove that this continuous bilinear form is also coercive on $[L^2(\Omega)]^{3\times3}$:

$$\begin{aligned}(A_1\sigma|\sigma)_{[L^2(\Omega)]^{3\times3}} &= (\frac{1+\nu}{E}\sigma - \frac{\nu}{E}tr(\sigma)I_3|\sigma)_{[L^2(\Omega)]^{3\times3}} \\ &\geq \frac{1+\nu}{E}\left\|\sigma\right\|^2_{[L^2(\Omega)]^{3\times3}} - \frac{\nu}{E}\left\|tr(\sigma)\right\|^2_{L^2(\Omega)} \\ &\geq \frac{1+\nu}{E}\left\|\sigma\right\|^2_{[L^2(\Omega)]^{3\times3}} - 3\frac{\nu}{E}\left\|\sigma\right\|^2_{[L^2(\Omega)]^{3\times3}} \\ &= \frac{1-2\nu}{E}\left\|\sigma\right\|^2_{[L^2(\Omega)]^{3\times3}} \\ &\geq C\left\|\sigma\right\|^2_{[L^2(\Omega)]^{3\times3}},\end{aligned} \tag{5.52}$$

where C is a strictly positive constant for the Poisson coefficient ν satisfies $0 < \nu < \frac{1}{2}$. For $(P_{H_0}A_1)|_{H_0}$, (5.51) implies that $\forall \sigma, \tau \in H_0$:

$$\begin{aligned}\left|(P_{H_0}A_1\sigma|\tau)_{[L^2(\Omega)]^{3\times3}}\right| &= \left|(A_1\sigma|P_{H_0}\tau)_{[L^2(\Omega)]^{3\times3}}\right| \\ &= \left|(A_1\sigma|\tau)_{[L^2(\Omega)]^{3\times3}}\right| \\ &\leq \frac{1+4\nu}{E}\left\|\sigma\right\|_{[L^2(\Omega)]^{3\times3}}\left\|\tau\right\|_{[L^2(\Omega)]^{3\times3}},\end{aligned}$$

which show us that the mapping $(\sigma,\tau) \mapsto ((P_{H_0}A_1)|_{H_0}\sigma|\tau)_{[L^2(\Omega)]^{3\times3}}$ is a bilinear continuous form on H_0. On the other hand, inequality (5.52) implies that $\forall \sigma \in H_0$:

$$(P_{H_0}A_1\sigma|\sigma)_{[L^2(\Omega)]^{3\times3}} = (A_1\sigma|\sigma)_{[L^2(\Omega)]^{3\times3}} \geq C\left\|\sigma\right\|^2_{[L^2(\Omega)]^{3\times3}},$$

which show us that the bilinear continuous form $(\sigma,\tau) \mapsto ((P_{H_0}A_1)|_{H_0}\sigma|\tau)_{[L^2(\Omega)]^{3\times3}}$ is also coercive on H_0. Thus by Lax-Milgram's lemma, $(P_{H_0}A_1)|_{H_0}$ is an isomorphism from H_0 onto H_0. $\square$

The preceding proposition allows us to rewrite our Cauchy problem (5.50) on H_0 in the normalized form:

$$\begin{cases} \dot{\sigma}_0(.,t) + ((P_{H_0}A_1)|_{H_0})^{-1}(P_{H_0}A_0(.,t))\,\sigma_0(.,t) \\ \qquad\qquad = -((P_{H_0}A_1)|_{H_0})^{-1}(P_{H_0}A_1)\,\dot{\sigma}_e(.,t),\ \forall' t \in]0,t_f[\ , \\ \sigma_0(.,0) = \zeta - \sigma_e(.,0). \end{cases} \tag{5.53}$$

Let us set $A(t) := -((P_{H_0}A_1)|_{H_0})^{-1}(P_{H_0}A_0(.,t))$ and $b(t) := -((P_{H_0}A_1)|_{H_0})^{-1}(P_{H_0}A_1)\,\dot{\sigma}_e(.,t)$. This allows us to rewrite our Cauchy problem (5.52) in the form:

$$\begin{cases} \frac{d\sigma_0}{dt}(t) = A(t)\sigma_0(t) + b(t),\ \forall' t \in]0,t_f[\ , \\ \sigma_0(0) = \zeta - \sigma_e(0). \end{cases} \tag{5.54}$$

As $T \in C(\bar{Q})$ and is thus uniformly continuous, it follows from fromula (5.1), that the mapping $[0,t_f] \to \mathcal{L}([L^2(\Omega)]^{3\times 3}) : t \mapsto A_0(.,t)$ is continuous, and thus that the mapping $[0,t_f] \to \mathcal{L}(H_0) : t \mapsto A(t)$ is also continuous. From $u_e \in H^1(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ follows that $\dot{\sigma}_e \in L^2(0,t_f;[L^2(\Omega)]^{3\times 3})$ implying that $b \in L^2(0,t_f;H_0)$. Following ([10], p.128), one introduces the homogeneous differential equation

$$\frac{dR}{dt}(t) = A(t) \circ R(t), \tag{5.55}$$

where the unknown function $t \mapsto R(t)$ takes its values in $\mathcal{L}(H_0)$. Denoting by $t \longmapsto R(t,s)$ the C^1-solution on the interval $[0,t_f]$ of that differential equation which takes the value I_{H_0} for $t = s \in [0,t_f]$, we have that $R(t,s) \in Isom(H_0;H_0)$, $R(t,s) = R(t,s') \circ R(s',s)$, $\forall t,s,s' \in [0,t_f]$. In terms of that family of operators $(R(t,s))_{t,s\in[0,t_f]}$, the solution of the Cauchy problem for the inhomogeneous differential equation (5.54) can be written ([10], p.132):

$$\sigma_0(t) := R(t,0).(\zeta - \sigma_e(0)) + \int_0^t R(t,\tau).b(\tau)\ d\tau,\ \forall t \in [0,t_f]. \tag{5.56}$$

It is in fact easily verified that σ_0 defined by formula (5.56) belongs to $C([0,t_f];H_0)$, that its time-derivative verifies equation $(5.54)_{(i)}$, $\forall' t \in]0,t_f[$, and belongs to $L^2(0,t_f;H_0)$ due to that equation and $b \in L^2(0,t_f;H_0)$. Thus $\sigma_0 \in H^1(0,t_f;H_0)$. If we set $\sigma := \sigma_0 + \sigma_e$, then by Proposition 5.8, $\sigma \in H^1(0,t_f;[L^2(\Omega)]^{3\times 3})$. Moreover, σ is a symmetric tensorfield $\sigma_{|\Gamma_N} = 0$, and satisfies $-\operatorname{div}\sigma = 0 - \operatorname{div}\sigma_e = g - \beta\ \nabla T$. Thus σ verifies equation $(5.8)\equiv(5.43)_{(ii)}$. By equation (5.48), σ verifies also:

$$\langle A_0(.,t)\sigma(.,t),\tau\rangle + \langle A_1\dot{\sigma}(.,t),\tau\rangle = 0,\ \forall \tau \in H_0,$$

i.e.

$$\int_\Omega [\frac{1+\nu}{\eta(T)}\sigma - \frac{\nu}{\eta(T)}tr(\sigma)I_3] : \tau\ dx + \int_\Omega [\frac{1+\nu}{E}\dot{\sigma} - \frac{\nu}{E}tr(\dot{\sigma})I_3] : \tau\ dx = 0,\ \forall \tau \in H_0. \tag{5.57}$$

Now to determine the velocity $v \equiv \dot{u}$, we must use equation (5.6)≡(5.43)$_{(i)}$. For almost every $t \in [0, t_f]$, the mapping

$$\begin{aligned} & H_{\Gamma_N}(\mathrm{div};\Omega)^3 \to \mathbb{R} \\ & \qquad \tau \longmapsto \int_\Omega [\frac{1+\nu}{\eta(T(.,t))}\sigma(.,t) - \frac{\nu}{\eta(T(.,t))} tr(\sigma(.,t)) I_3] : \tau \; dx \\ & \qquad\qquad + \int_\Omega [\frac{1+\nu}{E}\dot{\sigma}(.,t) - \frac{\nu}{E} tr(\dot{\sigma}(.,t)) I_3] : \tau \; dx, \end{aligned} \tag{5.58}$$

is a continuous linear form on $H_{\Gamma_N}(\mathrm{div};\Omega)^3$, whose restriction to the closed subspace H_0 is null. Let us set $X := H_{\Gamma_N}(\mathrm{div};\Omega)^3$ and let us denote by

$$V^0 := \{g \in X^*; \langle g, \tau\rangle = 0, \; \forall \tau \in H_0\}. \tag{5.59}$$

The continuous linear form (5.58) on X belongs to V^0. Let us denote by $M := [L^2(\Omega)]^3 \times [L^2(\Omega)]^{3\times 3}_{skew}$. Let us denote by $B : X := H_{\Gamma_N}(\mathrm{div};\Omega)^3 \to M^*$, the linear continuous operator which to each $\tau \in X$ associates the continuous linear form on M defined by:

$$\langle B\tau, (v, \eta)\rangle_{M^*,M} := \int_\Omega \mathrm{div}\, \tau . v \; dx + \int_\Omega \tau : \eta \; dx. \tag{5.60}$$

The adjoint operator $B^* : M \to X^*$ is the continuous linear operator which to each $(v, \eta) \in M := [L^2(\Omega)]^3 \times [L^2(\Omega)]^{3\times 3}_{skew}$ associates the continuous linear form on X :

$$X \to \mathbb{R} : \tau \longmapsto \int_\Omega \mathrm{div}\, \tau . v \; dx + \int_\Omega \tau : \eta \; dx. \tag{5.61}$$

5.10 Proposition. *Let us consider the bilinear continuous form* $b : X \times M = H_{\Gamma_N}(\mathrm{div};\Omega)^3 \times ([L^2(\Omega)]^3 \times [L^2(\Omega)]^{3\times 3}_{skew}) \to \mathbb{R}$ *defined by:*

$$b(\tau, (v, \eta)) = \int_\Omega \mathrm{div}\, \tau . v \; dx + \int_\Omega \tau : \eta \; dx. \tag{5.62}$$

Then $b(\cdot,\cdot)$ *satisfies the uniform inf-sup condition:* $\exists \beta > 0$ *such that* $\forall v \in [L^2(\Omega)]^3$ *and* $\forall \eta \in [L^2(\Omega)]^{3\times 3}_{skew}$:

$$\sup_{\tau \in X} \frac{b(\tau, (v, \eta))}{\|\tau\|_{X = H_{\Gamma_N}(\mathrm{div};\Omega)^3}} \geq \frac{1}{\beta}(\|v\|_{[L^2(\Omega)]^3} + \|\eta\|_{[L^2(\Omega)]^{3\times 3}_{skew}}). \tag{5.63}$$

Proof : 1°) Let us fix μ^* and λ^* two strictly positive real numbers. Let $\xi \in H^1_{\Gamma_D}(\Omega)^3$

be the unique solution of the problem:

$$\begin{cases} \operatorname{div}(2\mu^*\varepsilon(\xi) + \lambda^* \operatorname{div}(\xi)\delta) = v \text{ in } \Omega, \\ \xi = 0 \text{ on } \Gamma_D, \\ (2\mu^*\varepsilon(\xi) + \lambda^* \operatorname{div}(\xi)\delta) \cdot \nu = 0 \text{ on } \Gamma_N. \end{cases}$$

By Korn's inequality, ξ satisfies: $\|\xi\|_{\left[H^1_{\Gamma_D}(\Omega)\right]^3} \lesssim \|v\|_{[L^2(\Omega)]^3}$. Setting $\tau_1 := 2\mu^*\varepsilon(\xi) + \lambda^* \operatorname{div}(\xi)\delta$, we have $b(\tau_1, (v, \eta)) = \|v\|^2_{[L^2(\Omega)]^3}$ because η is skew-symmetric and τ_1 is symmetric. Moreover $\|v\|_{[L^2(\Omega)]^3} \lesssim \|\tau_1\|_{X=H_{\Gamma_N}(\operatorname{div};\Omega)^3} \lesssim \|v\|_{[L^2(\Omega)]^3}$.
2°) Let $(\eta_k)_{k\geq 1} \in [\mathcal{D}(\Omega)]^{3\times 3}_{skew}$ be a sequence in $[\mathcal{D}(\Omega)]^{3\times 3}_{skew}$ converging to η in $[L^2(\Omega)]^{3\times 3}_{skew}$. Let $\zeta_k \in H^1_{\Gamma_D}(\Omega)^3$ be the unique solution of the problem:

$$\begin{cases} \operatorname{div}(2\mu^*\varepsilon(\zeta_k) + \lambda^* \operatorname{div}(\zeta_k)\delta) = -\operatorname{div}(\eta_k) \text{ in } \Omega, \\ \zeta_k = 0 \text{ on } \Gamma_D, \\ (2\mu^*\varepsilon(\zeta_k) + \lambda^* \operatorname{div}(\zeta_k)\delta) \cdot \nu = 0 \text{ on } \Gamma_N. \end{cases}$$

The variational formulation of this problem is:

$$\int_\Omega (2\mu^*\varepsilon(\zeta_k) : \varepsilon(v) + \lambda^* \operatorname{div}(\zeta_k) \operatorname{div}(v))\, dx = \int_\Omega \eta_k : \nabla v\, dx,\ \forall v \in H^1_{\Gamma_D}(\Omega)^3.$$

From this variational equation follows that $\|\zeta_k\|_{\left[H^1_{\Gamma_D}(\Omega)\right]^3} \lesssim \|\eta_k\|_{[L^2(\Omega)]^{3\times 3}}$ and also

$$\|\zeta_k - \zeta_l\|_{\left[H^1_{\Gamma_D}(\Omega)\right]^3} \lesssim \|\eta_k - \eta_l\|_{[L^2(\Omega)]^3}\,. \tag{5.64}$$

Setting $\tau_{2,k} := 2\mu^*\varepsilon(\zeta_k) + \lambda^* \operatorname{div}(\zeta_k)\delta + \eta_k$, we have $\operatorname{div}(\tau_{2,k}) = 0$, $\tau_{2,k} = 0$ on Γ_D and $(2\mu^*\varepsilon(\tau_{2,k}) + \lambda^* \operatorname{div}(\tau_{2,k})\delta) \cdot \nu = 0$ on Γ_N. Inequality (5.64) implies that the sequence $(\tau_{2,k})_{k\geq 1}$ is a Cauchy sequence in $\left[L^2(\Omega)\right]^{3\times 3}$ and as $\operatorname{div}(\tau_{2,k}) = 0$, $\forall k \geq 1$, it is also a Cauchy sequence in $[H(\operatorname{div};\Omega)]^3$. Thus this sequence converges in $[H(\operatorname{div};\Omega)]^3$. Let us denote by τ_2 its limit in $[H(\operatorname{div};\Omega)]^3$. τ_2 satisfies:

$$\begin{cases} \operatorname{div}(\tau_2) = 0, \\ \tau_2 - \tau_2^T = \eta - \eta^T, \\ \tau_2 \cdot \nu = 0 \text{ on } \Gamma_N. \end{cases}$$

Moreover $\|\tau_2\|_{[L^2(\Omega)]^{3\times 3}} \lesssim \|\eta\|_{[L^2(\Omega)]^{3\times 3}}$ if $\eta \neq 0$; if $\eta = 0$, then it suffices to take $\tau_2 = 0$.
3°) Let us now set: $\tau = \tau_1 + \tau_2$.

$$\begin{aligned} b(\tau, (v, \eta)) &= b(\tau_1, (v, \eta)) + b(\tau_2, (v, \eta)) \\ &= \|v\|^2_{[L^2(\Omega)]^3} + \|\eta\|^2_{[L^2(\Omega)]^{3\times 3}}\,, \end{aligned}$$

and

$$\begin{aligned}\|\tau\|_{X=H_{\Gamma_N}(\mathrm{div};\Omega)^3} &\leq \|\tau_1\|_{X=H_{\Gamma_N}(\mathrm{div};\Omega)^3} + \|\tau_2\|_{X=H_{\Gamma_N}(\mathrm{div};\Omega)^3} \\ &\lesssim \|v\|_{[L^2(\Omega)]^3} + \|\eta\|_{[L^2(\Omega)]^{3\times 3}} .\end{aligned}$$

These two latest inequalities imply the result. □

5.3 Remark. *The proof given in the previous proposition is an adaptation to the 3-d case of the proof given for the 2-d case in ([8], pp. 9-11).*

From Lemma 4.1 p. 54 of [29], follows from the previous proposition that:

5.11 Proposition. *The operator $B^* : M \to X^*$ is an isomorphism from M onto V^0 and $\forall (v, \eta) \in M$:*

$$\|(v,\eta)\|_{[L^2(\Omega)]^3 \times [L^2(\Omega)]^{3\times 3}_{skew}} \leq \frac{1}{\beta} \|B^*(v,\eta)\|_{X^*=(H_{\Gamma_N}(\mathrm{div};\Omega)^3)^*} . \tag{5.65}$$

Applying the previous proposition to the linear continuous form (5.58), we obtain that $\forall' t \in]0, t_f[$, $\exists! (v(t), \rho(t)) \in M := L^2(\Omega)^3 \times [L^2(\Omega)]^{3\times 3}_{skew}$ such that

$$\begin{aligned}< B^*(v(t),\rho(t)), \tau >= -\{\int_\Omega [\tfrac{1+\nu}{\eta(T(.,t))}\sigma(.,t) - \tfrac{\nu}{\eta(T(.,t))} tr(\sigma(.,t)) I_3] : \tau \, dx \\ + \int_\Omega [\tfrac{1+\nu}{E}\dot\sigma(.,t) - \tfrac{\nu}{E} tr(\dot\sigma(.,t)) I_3] : \tau \, dx\}, \ \forall \tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3,\end{aligned}$$

i.e.

$$\begin{aligned}\int_\Omega \mathrm{div}\, \tau . v(t) \, dx + \int_\Omega \tau : \rho(t) \, dx = \\ -\{\int_\Omega [\tfrac{1+\nu}{\eta(T(.,t))}\sigma(.,t) - \tfrac{\nu}{\eta(T(.,t))} tr(\sigma(.,t)) I_3] : \tau \, dx \\ + \int_\Omega [\tfrac{1+\nu}{E}\dot\sigma(.,t) - \tfrac{\nu}{E} tr(\dot\sigma(.,t)) I_3] : \tau \, dx\}, \ \forall \tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3,\end{aligned} \tag{5.66}$$

thus verifying equation $(5.43)_{(i)} \equiv (5.6)$. Raising to the square boths sides of inequality (5.65) for $(v, \eta) = (v(t), \rho(t))$, t being a fixed real number between 0 and t_f, and then integrating both sides with respect to t from 0 to t_f, we obtain that $v(.) \in L^2(0, t_f; L^2(\Omega)^3)$, $\rho(.) \in L^2(0, t_f; [L^2(\Omega)]^{3\times 3}_{skew})$ and that

$$\|v\|_{L^2(0,t_f;L^2(\Omega)^3)} + \|\rho\|_{L^2(0,t_f;[L^2(\Omega)]^{3\times 3}_{skew})} \lesssim \|\sigma\|_{H^1(0,t_f;[L^2(\Omega)]^{3\times 3})} , \tag{5.67}$$

by (5.66). We have thus obtained the following theorem:

5.1 Theorem. *Let us suppose that $g \in H^1(0, t_f; L^2(\Omega)^3)$, that the hypotheses on T_0 and T_S done in Proposition 5.3 are verified and that the initial condition ζ for σ verifies $\zeta \in H_{\Gamma_N}(\mathrm{div};\Omega)^3$, $\zeta = \zeta^T$ and $\mathrm{div}\, \zeta = -(g(.,0) - \beta \nabla T(.,0))$. Then, there exists a unique* *triple*

$$(\sigma, v, \rho) \in H^1(0,t_f; H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0,t_f;L^2(\Omega)^3) \times L^2(0,t_f;\left[L^2(\Omega)\right]^{3\times3}_{skew})$$

such that $\sigma(.,0) = \zeta$ *and such that* $\forall' t \in]0,t_f[$*:*

$$\left\{\begin{array}{c} \int_\Omega [\frac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \frac{\nu}{\eta(T(.,t))}tr(\sigma(.,t))I_3] : \tau \; dx + \int_\Omega [\frac{1+\nu}{E}\dot{\sigma}(.,t) - \frac{\nu}{E}tr(\dot{\sigma}(.,t))I_3] : \tau \; dx \\ + \int_\Omega \mathrm{div}(\tau).v(.,t) \; dx + \int_\Omega \rho(.,t) : \tau \; dx = 0, \;\; \forall \tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3, \\ \int_\Omega \mathrm{div}(\sigma(.,t)).w \; dx + \int_\Omega (g(.,t) - \beta \; \nabla T(.,t)).w \; dx + \int_\Omega \sigma(.,t) : \xi \; dx = 0, \;\; \forall w \in L^2(\Omega)^3, \\ \forall \xi \in [L^2(\Omega)]^{3\times3}_{skew}. \end{array}\right. \tag{5.68}$$

Moreover from inequalities (5.42) and (5.67), and equations (5.56) and (5.53), we have:

$$\begin{array}{c} \|v\|_{L^2(0,t_f;L^2(\Omega)^3)} + \|\rho\|_{L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})} + \|\sigma\|_{H^1(0,t_f;[L^2(\Omega)]^{3\times3})} \\ \lesssim \|g - \beta\nabla T\|_{H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} + \|\zeta\|_{[L^2(\Omega)]^{3\times3}}. \end{array} \tag{5.69}$$

5.1 Corollary. *Taking the first derivatives with respect to* x_1*,* x_2*,* x_3*, in the sense of distributions, we have also* $\forall' t \in]0,t_f[$*:*

$$\left\{\begin{array}{c} \varepsilon(v(.,t)) := \frac{1}{2}(\nabla v(.,t) + \nabla v(.,t)^T) \\ = [\frac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \frac{\nu}{\eta(T(.,t))}tr(\sigma(.,t))I_3] + [\frac{1+\nu}{E}\dot{\sigma}(.,t) - \frac{\nu}{E}tr(\dot{\sigma}(.,t))I_3], \\ \rho(.,t) = \frac{1}{2}(\nabla v(.,t) - \nabla v(.,t)^T). \end{array}\right. \tag{5.70}$$

Consequently, we have also:

$$\|v\|_{L^2(0,t_f;H^1(\Omega)^3)} \lesssim \|g - \beta\nabla T\|_{H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} + \|\zeta\|_{[L^2(\Omega)]^{3\times3}}. \tag{5.71}$$

Also $\forall' t \in]0,t_f[$*:* $v(.,t)|_{\Gamma_D} = 0$*, where* $\Gamma_D := \Gamma \backslash \Gamma_N$*.*

Proof : Considering in equation $(5.68)_{(i)}$, arbitrary test tensor fields $\tau \in \mathcal{D}(\mathrm{div};\Omega)^3$, we have $\forall' t \in]0,t_f[$:

$$\begin{array}{c} < \nabla v(.,t), \tau > = \int_\Omega [\frac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \frac{\nu}{\eta(T(.,t))}tr(\sigma(.,t))I_3] : \tau \; dx \\ + \int_\Omega [\frac{1+\nu}{E}\dot{\sigma}(.,t) - \frac{\nu}{E}tr(\dot{\sigma}(.,t))I_3] : \tau \; dx + \int_\Omega \rho(.,t) : \tau \; dx. \end{array}$$

Thus, we have for $\nabla v(.,t)$ considered in the sense of distributions:

$$\begin{aligned}\nabla v(.,t) = &[\tfrac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \tfrac{\nu}{\eta(T(.,t))}tr(\sigma(.,t))I_3]\\ &+[\tfrac{1+\nu}{E}\dot\sigma(.,t) - \tfrac{\nu}{E}tr(\dot\sigma(.,t))I_3] + \rho(.,t).\end{aligned} \tag{5.72}$$

$[\frac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \frac{\nu}{\eta(T(.,t))}tr(\sigma(.,t))I_3] + [\frac{1+\nu}{E}\dot\sigma(.,t) - \frac{\nu}{E}tr(\dot\sigma(.,t))I_3]$ being a symmetric tensor belonging to $[L^2(\Omega)]^{3\times3}$and $\rho(.,t)$ a skew symmetric one, we obtain that $\nabla v(.,t) \in [L^2(\Omega)]^{3\times3}$,

$$\begin{aligned}\tfrac{1}{2}(\nabla v(.,t) + \nabla v(.,t)^T) = &[\tfrac{1+\nu}{\eta(T(.,t))}\sigma(.,t)) - \tfrac{\nu}{\eta(T(.,t))}tr(\sigma(.,t))I_3]\\ &+[\tfrac{1+\nu}{E}\dot\sigma(.,t) - \tfrac{\nu}{E}tr(\dot\sigma(.,t))I_3],\end{aligned}$$

and

$$\rho(.,t) = \frac{1}{2}(\nabla v(.,t) - \nabla v(.,t)^T). \tag{5.73}$$

This proves equations (5.70). From equality (5.72), and inequality (5.69), which implies:

$$\|\sigma\|_{H^1(0,t_f;[L^2(\Omega)]^{3\times3})} \lesssim \|g-\beta\nabla T\|_{H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} + \|\zeta\|_{[L^2(\Omega)]^{3\times3}}$$

and

$$\|\rho\|_{L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})} \lesssim \|g-\beta\nabla T\|_{H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} + \|\zeta\|_{[L^2(\Omega)]^{3\times3}},$$

follows inequality (5.71). To derive the last property that $\forall' t \in]0,t_f[$: $v(.,t)|_{\Gamma_D} = 0$, let us consider in equation $(5.68)_{(i)}$, arbitrary symmetric test tensor fields $\tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3$. We know from equation (5.73), that $\rho(.,t)$ is skewsymmetric. Using Green's formula in $H(\mathrm{div};\Omega)^3$ ([28], p.24) and equation (5.72), we obtain: $\int_{\partial\Omega} v(x,t)(\tau.\nu)(x)\ dS(x) = 0$, $\forall\tau \in H_{\Gamma_N}(\mathrm{div};\Omega)^3$. This implies that $\int_{\partial\Omega} v(x,t)\zeta(x)\ dS(x) = 0$, $\forall\zeta \in L^2(\partial\Omega)^3$ which is 0 on Γ_N, and thus that $v(.,t)|_{\Gamma_D} = 0$. $\square$

6

Control of the Displacements in the Setting of the Thermoviscoelasticity

Contents

6.1 Introduction

In this chapter, we study the control of the deformation of Ω, resulting from the radiative heating of the semi-transparent body Ω by the black-source S.

We define $\forall t \in]0, t_f[$: $\vec{u}(t) := \int\limits_0^t v(s)\ ds$. By Corollary 5.1 in the previous Chapter, $\vec{u} \in C([0, t_f]; H^1(\Omega)^3)$, and by more specifically inequality (5.71), we have:

$$\|\vec{u}\|_{C([0,t_f];H^1(\Omega)^3)} \lesssim \|g - \beta\nabla T\|_{H^1\left(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*\right)} + \|\zeta\|_{[L^2(\Omega)]^{3\times 3}}. \tag{6.1}$$

By Ascoli's theorem ([20], p.143), the mapping which sends $v \mapsto \vec{u}$ from $L^2(0, t_f; H^1(\Omega)^3)$ into the space $C([0, t_f]; L^2(\Omega)^3)$ is compact. Thus if we have a sequence $(v_n)_{n\geq 1}$ which converges weakly in $L^2(0, t_f; H^1(\Omega)^3)$, the corresponding sequence $(\vec{u}_n)_{n\geq 1}$ will converge strongly in $C([0, t_f]; L^2(\Omega)^3)$ and a fortiori in $L^2(0, t_f; L^2(\Omega)^3)$. Now, we define the new set of admissible controls:

$$U_{ad} := \{T_S \in H^2(]0, t_f[); \underline{T} \leq T_S(t) \leq \bar{T},\ \forall t \in]0, t_f[, T_S(0) = T_a\} \tag{6.2}$$

and the cost functional

$$J : [L^2(Q)]^3 \times H^2(]0,t_f[) \to \mathbb{R} : (\vec{u}, T_S) \mapsto \frac{1}{2}\int\limits_Q |\vec{u}(x,t) - \vec{u}_d(x,t)|^2 \; dx \otimes dt \\ + \frac{\tilde{\delta}_r}{2} \left\| T_S - T_S^d \right\|^2_{H^2(]0,t_f[)}, \tag{6.3}$$

where $\vec{u}_d(.,.) \in [L^2(Q)]^3$ denotes the given desired field of displacements and $T_S^d \in H^2(]0,t_f[)$ a given evolution of the absolute temperature of the black source S to which T_S should not be "too far", the meaning of this "too far" being modulated by the strictly positive coefficient $\tilde{\delta}_r$ in the definition of the cost functional J, coefficient which is allowed to be chosen very small. As in Chapter 3, $\underline{T}$ and $\bar{T}$ denote two strictly positive real numbers such that $\underline{T} < \bar{T}$, $\underline{T} \leq T_a \leq \bar{T}$ and $\underline{T} \leq T_0(x) \leq \bar{T}$, $\forall' x \in \Omega$. Let us observe that U_{ad} is a closed convex subset of $H^2(]0,t_f[)$ and thus weakly closed ([71], p.47).

6.2 Existence of an Optimal Control

In this section, we want to prove the existence of an optimal control, i.e . There exists $\hat{T}_S \in U_{ad}$ such that $\hat{J}(\hat{T}_S) = \inf\limits_{T_S \in U_{ad}} \hat{J}(T_S)$.

6.1 Proposition. *Let T be the weak solution of the heat conduction equation 3.42 i.e. of the parabolic equation (5.9) with the nonlinear Robin boundary condition (5.11) and the initial condition $T(.,0) = T_0$. Supposing that the initial condition $T_0 \in C(\bar{\Omega}) \cap H^1(\Omega)$ and that $T_S \in U_{ad}$, we have that $T \in H^{2,1}(Q)$ and moreover that*

$$\|T\|_{H^{2,1}(Q)} \leq C(\underline{T}, \bar{T}, \|T_0\|_{H^1(\Omega)}), \tag{6.4}$$

where $C(\underline{T}, \bar{T}, \|T_0\|_{H^1(\Omega)})$ denotes a constant depending only on $\underline{T}$, $\bar{T}$ and $\|T_0\|_{H^1(\Omega)}$. In particular:

$$\left\| \frac{\partial T}{\partial t} \right\|_{L^2(Q)} \leq C(\underline{T}, \bar{T}, \|T_0\|_{H^1(\Omega)}). \tag{6.5}$$

Proof : This follows by a meticulous inspection of the proof of Proposition 5.1. □

6.1 Condition. *To simplify our statements,* ***we henceforth suppose that the initial condition*** $T_0 = T_a$.

6.2 Proposition. *Let T be the weak solution of the heat conduction equation 3.42 i.e. of the parabolic equation (5.9) with the nonlinear Robin boundary condition (5.11) and the initial condition $T(.,0) = T_0 = T_a$. We have that $T \in H^{2,1}(Q)$ and that*

$$\left\| \frac{\partial \nabla T}{\partial t} \right\|_{L^2(Q)^3} \lesssim C(\underline{T}, \bar{T}, T_a) + \|T_S\|_{H^1(]0,t_f[)}. \tag{6.6}$$

Also:

$$\|\nabla T\|_{L^2(Q)^3} + \left\|\frac{\partial \nabla T}{\partial t}\right\|_{L^2(Q)^3} \leq C(\underline{T}, \bar{T}, T_a) + \|T_S\|_{H^1(]0,t_f[)}. \tag{6.7}$$

A fortiori

$$\|\nabla T\|_{H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} \leq C(\underline{T}, \bar{T}, T_a) + \|T_S\|_{H^1(]0,t_f[)}. \tag{6.8}$$

Proof : It follows immediately by the previous proposition that $T \in H^{2,1}(Q)$. By Lemma 5.3, we know that $\frac{dT}{dt} = v$ where v is the weak solution of the initial boundary value problem

$$\begin{cases} c_p m_g \frac{\partial v}{\partial t}(x,t) = k_h \Delta v(x,t) + \frac{\partial f}{\partial t}(x,t), \quad & x \in \Omega,\ 0 < t < t_f\ , \\ -k_h \frac{\partial v}{\partial \nu}(x,t) = h_c v(x,t) + \Theta'(\int\limits_0^t v(x,\tau)d\tau + T_0(x))v(x,t) - \Theta'(T_S(t))\dot{T}_S(t), & \\ & x \in \partial\Omega,\ 0 < t < t_f\ , \\ v(x,0) = \frac{1}{c_p m_g}(f(x,0) + k_h \Delta T_0(x)), & \forall' x \in \Omega,\ (\text{I.C. at } t = 0). \end{cases} \tag{6.9}$$

Let us remark that with our initial condition $T_0(.) = T_a$ and $T_S(0) = T_a$ that

$$\begin{aligned} v(.,0) &= \tfrac{1}{c_p m_g}(f(.,0) + k_h \Delta T_0) = v(.,0) = \tfrac{1}{c_p m_g} f(.,0) \\ &= \tfrac{1}{c_p m_g}[-\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T_a) + \sum_{k=1}^{M} \kappa_k \int\limits_V I^k_{T_0,T_S(0)}(x,v)d\mu(v)] \\ &= \tfrac{1}{c_p m_g}[-\sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T_a) + \sum_{k=1}^{M} 4\pi\kappa_k B_g^k(T_a)] = 0,\ \forall x \in \Omega. \end{aligned}$$

Multiplying equation $(6.9)_{(i)}$ by v, then integrating both sides on $\Omega\times]0,t_f[$ and using the boundary condition $(6.9)_{(ii)}$, we obtain:

$$\begin{aligned} &\tfrac{c_p m_g}{2}\int\limits_0^{t_f}\int\limits_\Omega \tfrac{\partial v}{\partial t}(x,t)v(x,t)dxdt + k_h \int\limits_0^{t_f}\int\limits_\Omega |\nabla v(x,t)|^2 dxdt \\ &= \int\limits_0^{t_f}\int\limits_\Omega \tfrac{\partial f}{\partial t}(x,t)v(x,t)dxdt - \int\limits_0^{t_f}\int\limits_{\partial\Omega} \Theta'(\int\limits_0^t v(x,\tau)d\tau + T_0(x))v(x,t)^2 dS(x)dt \\ &\quad - h_c \int\limits_0^{t_f}\int\limits_{\partial\Omega} v(x,t)^2 dS(x)dt + \int\limits_0^{t_f}\int\limits_{\partial\Omega} v(x,t)^2 \Theta'(T_S(t))\dot{T}_S(t)dS(x)dt. \end{aligned}$$

As $\Theta'(.) \geq 0$, the following inequality follows: $\forall\varepsilon > 0$, $\forall\delta > 0$:

$$\begin{aligned} &\tfrac{c_p m_g}{2}\|v(.,t_f)\|^2_{L^2(\Omega)} + k_h \|\nabla v\|^2_{L^2(0,t_f;L^2(\Omega)^3)} + h_c \|v\|^2_{L^2(0,t_f;L^2(\partial\Omega))} \leq \tfrac{\delta}{2}\|v\|^2_{L^2(Q)} \\ &+ \tfrac{1}{2\delta}\left\|\tfrac{\partial f}{\partial t}\right\|^2_{L^2(Q)} + \tfrac{\varepsilon}{2}\|v\|^2_{L^2(0,t_f;L^2(\partial\Omega))} + \tfrac{1}{2\varepsilon}|\partial\Omega|\,\|\Theta'\|^2_{\infty,\mathbb{R}}\left\|\dot{T}_S\right\|^2_{L^2(]0,t_f[)}. \end{aligned}$$

Using Poincaré's inequality, $\frac{dT}{dt} = v$, Lemma 5.1 and inequality (6.5) it follows from the previous inequality that: $\left\|\frac{dT}{dt}\right\|_{L^2(0,t_f;H^1(\Omega))} \lesssim \|T_S\|_{H^1(]0,t_f[)}$, from which inequality (6.6) follows. Inequality (6.7) follows from inequality (6.6) and inequality (6.4). $\square$

6.3 Proposition. *The mapping from* $U_{ad} := \{T_S \in H^2(]0,t_f[); \underline{T} \leq T_S(t) \leq \bar{T},\ \forall t \in]0,t_f[, T_S(0) = T_a\}$ *into* $F := H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0,t_f;H^1(\Omega)^3) \times L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})$ *which sends* T_S *onto* $(\sigma,v,\rho) \in F$*, the unique triple such that* $\sigma(.,0) = \zeta$ *and such that* $\forall' t \in]0,t_f[$ *the variational system (5.68) is satisfied, is continuous from* U_{ad} *endowed with the weak topology into* $H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0,t_f;L^2(\Omega)^3) \times L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})$ *endowed with the weak topology.*

Proof : Let us consider a sequence $(T_{S,n})_{n\in\mathbb{N}}$ in U_{ad} converging weakly to T_S in $H^2(]0,t_f[)$. This implies that $(T_{S,n})_{n\in\mathbb{N}}$ converges strongly to T_S in $C([0,t_f])$ and thus T_S still belongs to U_{ad}. Also the weak convergence of the sequence $(T_{S,n})_{n\in\mathbb{N}}$ implies that it is a bounded sequence in $H^2(]0,t_f[)$. Let T_n (resp. T) be the weak solution of the Heat conduction equation 3.42 i.e. of the parabolic equation (5.9) with datum $T_{S,n}$ (resp. T_S) in the nonlinear Robin boundary condition (5.11) and the initial condition $T_n(.,0) = T_0 = T_a$, $\forall n \in \mathbb{N}$ (resp. $T(.,0) = T_0 = T_a$). By the proof of Proposition 4.2 in Chapter 4, the sequence $(T_n)_{n\in\mathbb{N}}$ converges strongly to T in $L^2(Q)$. By Corollary 3.4 in Chapter 3, we have also that $\underline{T} \leq T_n(.,.) \leq \bar{T}$. Let us denote by $(\sigma_n,v_n,\rho_n) \in H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0,t_f;L^2(\Omega)^3) \times L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})$ the unique triple such that $\sigma_n(.,0) = \zeta$ and such that $\forall' t \in]0,t_f[$ the variational system (5.68) is satisfied. By Corollary 5.1, we know that v_n belongs in fact to $L^2(0,t_f;H^1(\Omega)^3)$. By the estimates (5.69), (5.71):

$$\begin{aligned}
&\|v_n\|_{L^2(0,t_f;H^1(\Omega)^3)} + \|\rho_n\|_{L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})} + \|\sigma_n\|_{H^1(0,t_f;[L^2(\Omega)]^{3\times3})} \\
&\lesssim \|g - \beta\nabla T_n\|_{H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} + \|\zeta\|_{[L^2(\Omega)]^{3\times3}} \\
&\lesssim C(\underline{T},\bar{T},T_a) + \|T_{S,n}\|_{H^1(]0,t_f[)} + \|g\|_{H^1(0,t_f;\left[H^1_{\Gamma_D}(\Omega)^3\right]^*)} + \|\zeta\|_{[L^2(\Omega)]^{3\times3}} \\
&\leq Cst.
\end{aligned}$$

This proves that the sequence $((\sigma_n,v_n,\rho_n))_{n\in\mathbb{N}}$ is bounded in $H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0,t_f;H^1(\Omega)^3) \times L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})$ by remarking also that $-\mathrm{div}(\sigma_n) = g - \beta\nabla T_n$ and using inequality (6.7). By inequality (6.4) and (6.7), the sequence $(T_n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,t_f;H^1(\Omega)^3)$. Thus $(T_n)_{n\in\mathbb{N}}$ possesses a subsequence $(T_{n_k})_{k\in\mathbb{N}}$ which converges also weakly to T in $H^1(0,t_f;H^1(\Omega)^3)$. The sequence $((\sigma_n,v_n,\rho_n))_{n\in\mathbb{N}}$ being bounded possesses a subsequence that after some renaming we may call also $((\sigma_{n_k},v_{n_k},\rho_{n_k}))_{k\in\mathbb{N}}$, which converges weakly to some element $(\sigma,v,\rho) \in H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0,t_f;H^1(\Omega)^3) \times L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})$. Let us prove that (σ,v,ρ) is the unique triple in $H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times L^2(0,t_f;H^1(\Omega)^3) \times L^2(0,t_f;[L^2(\Omega)]^{3\times3}_{skew})$ such that $\sigma(.,0) = \zeta$, and such that $\forall' t \in]0,t_f[$ the variational system (5.68) is satisfied. The triple $(\sigma_{n_k},v_{n_k},\rho_{n_k}) \in H^1(0,t_f;H_{\Gamma_N}(\mathrm{div};\Omega)^3) \times$

$L^2(0,t_f;H^1(\Omega)^3)\times L^2(0,t_f;[L^2(\Omega)]^{3\times 3}_{skew})$ satisfies $\forall' t\in]0,t_f[$:

$$\begin{cases} \int\limits_\Omega [\frac{1+\nu}{\eta(T_{n_k}(x,t))}\sigma_{n_k}(x,t)) - \frac{\nu}{\eta(T_{n_k}(x,t))} tr(\sigma_{n_k}(x,t))I_3] : \tau(x)\ dx \\ \quad + \int\limits_\Omega [\frac{1+\nu}{E}\dot{\sigma}_{n_k}(x,t) - \frac{\nu}{E} tr(\dot{\sigma}_{n_k}(x,t))I_3] : \tau(x)\ dx \\ + \int\limits_\Omega (\operatorname{div}\tau)(x).v_{n_k}(x,t)\ dx + \int\limits_\Omega \rho_{n_k}(x,t) : \tau(x)\ dx = 0,\ \forall \tau\in H_{\Gamma_N}(\operatorname{div};\Omega)^3, \\ \int\limits_\Omega \operatorname{div}(\sigma_{n_k}(x,t)).w(x)\ dx + \int\limits_\Omega (g(.,t) - \beta\ \nabla T_{n_k}(.,t)).w(x)\ dx \\ \quad + \int\limits_\Omega \sigma_{n_k}(x,t) : \xi(x)\ dx = 0,\ \forall w\in L^2(\Omega)^3, \\ \quad \forall \xi\in [L^2(\Omega)]^{3\times 3}_{skew}. \end{cases} \tag{6.10}$$

This is equivalent to:

$$\begin{cases} \int\limits_0^{t_f}\int\limits_\Omega [\frac{1+\nu}{\eta(T_{n_k}(x,t))}\sigma_{n_k}(x,t)) - \frac{\nu}{\eta(T_{n_k}(x,t))} tr(\sigma_{n_k}(x,t))I_3] : \tau(x)\ dx\ \varphi(t)\ dt \\ \quad + \int\limits_0^{t_f}\int\limits_\Omega [\frac{1+\nu}{E}\dot{\sigma}_{n_k}(x,t) - \frac{\nu}{E} tr(\dot{\sigma}_{n_k}(x,t))I_3] : \tau(x)\ dx\ \varphi(t)\ dt \\ + \int\limits_0^{t_f}\int\limits_\Omega (\operatorname{div}\tau)(x).v_{n_k}(x,t)\ dx\ \varphi(t)\ dt + \int\limits_0^{t_f}\int\limits_\Omega \rho_{n_k}(x,t) : \tau(x)\ dx\ \varphi(t)\ dt = 0, \\ \quad \forall \tau\in H_{\Gamma_N}(\operatorname{div};\Omega)^3, \forall\varphi\in L^2(]0,t_f[), \\ \int\limits_0^{t_f}\int\limits_\Omega \operatorname{div}(\sigma_{n_k}(x,t)).w(x)\ dx\ \varphi(t)\ dt + \int\limits_0^{t_f}\int\limits_\Omega (g(x,t) - \beta\ \nabla T_{n_k}(x,t)).w(x)\ dx\ \varphi(t)\ dt \\ \quad + \int\limits_0^{t_f}\int\limits_\Omega \sigma_{n_k}(x,t) : \xi(x)\ dx\ \varphi(t)\ dt = 0,\ \forall w\in L^2(\Omega)^3, \\ \quad \forall \xi\in [L^2(\Omega)]^{3\times 3}_{skew}, \forall\varphi\in L^2(]0,t_f[). \end{cases} \tag{6.11}$$

Let us show that we can pass to the limit in these two equations. Firstly:

$$\begin{aligned}
&\int_0^{t_f}\int_\Omega \left|\tfrac{1+\nu}{\eta(T_{n_k}(x,t))}\sigma_{n_k}(x,t) : \tau(x)\ \varphi(t) - \tfrac{1+\nu}{\eta(T(x,t))}\sigma(x,t) : \tau(x)\ \varphi(t)\right| dx\ dt \\
&\quad\lesssim \int_0^{t_f}\int_\Omega \left|\sigma_{n_k}(x,t) : \tau(x)\ \varphi(t) - \sigma(x,t) : \tau(x)\ \varphi(t)\right| \tfrac{1}{\eta(T_{n_k}(x,t))} dx\ dt \\
&\qquad + \int_0^{t_f}\int_\Omega \left|\sigma(x,t) : \tau(x)\ \varphi(t)\right| \left|\tfrac{1}{\eta(T_{n_k}(x,t))} - \tfrac{1}{\eta(T(x,t))}\right| dx\ dt \\
&\quad\lesssim \tfrac{1}{\eta(\bar{T})}\int_0^{t_f}\int_\Omega \left|\sigma_{n_k}(x,t) : \tau(x)\ \varphi(t) - \sigma(x,t) : \tau(x)\ \varphi(t)\right| dx\ dt \\
&\qquad + \int_0^{t_f}\int_\Omega \left|\sigma(x,t) : \tau(x)\ \varphi(t)\right| \left|\tfrac{1}{\eta(T_{n_k}(x,t))} - \tfrac{1}{\eta(T(x,t))}\right| dx\ dt.
\end{aligned}$$

The first term in the right-hand side of the last inequality tends to 0 as a fortiori $(\sigma_{n_k})_{k\in\mathbb{N}}$ converges weakly to σ in $L^2(Q)$. For the second term, we may use Lebesgue dominated convergence theorem as the function $(x,t) \mapsto \sigma(x,t) : \tau(x)\ \varphi(t)$ is in $L^1(Q)$ and $\left|\tfrac{1}{\eta(T_{n_k}(x,t))} - \tfrac{1}{\eta(T(x,t))}\right| \le \tfrac{2}{\eta(\bar{T})}$ so that:

$$\left|\sigma(x,t) : \tau(x)\ \varphi(t)\right| \left|\frac{1}{\eta(T_{n_k}(x,t))} - \frac{1}{\eta(T(x,t))}\right| \le \frac{2}{\eta(\bar{T})} \left|\sigma(x,t) : \tau(x)\ \varphi(t)\right|,$$

where the right-hand side in this inequality is an integrable function on Q. Thus $\int_0^{t_f}\int_\Omega \left|\sigma(x,t) : \tau(x)\ \varphi(t)\right| \left|\tfrac{1}{\eta(T_{n_k}(x,t))} - \tfrac{1}{\eta(T(x,t))}\right| dx\ dt$ tends to 0 as $k \to +\infty$. In conclusion,

$$\int_0^{t_f}\int_\Omega \left|\frac{1+\nu}{\eta(T_{n_k}(x,t))}\sigma_{n_k}(x,t) : \tau(x)\ \varphi(t) - \frac{1+\nu}{\eta(T(x,t))}\sigma(x,t) : \tau(x)\ \varphi(t)\right| dx\ dt$$

tends to 0 as $k \to +\infty$. Secondly:

$$\begin{aligned}
&\int_0^{t_f}\int_\Omega [\tfrac{1+\nu}{E}\dot{\sigma}_{n_k}(x,t) - \tfrac{\nu}{E}tr(\dot{\sigma}_{n_k}(x,t))I_3] : \tau(x)\ dx\ \varphi(t)\ dt \\
&\qquad \to \int_0^{t_f}\int_\Omega [\tfrac{1+\nu}{E}\dot{\sigma}(x,t) - \tfrac{\nu}{E}tr(\dot{\sigma}(x,t))I_3] : \tau(x)\ dx\ \varphi(t)\ dt,
\end{aligned}$$

as $k \to +\infty$, because $(\dot{\sigma}_{n_k})_{k\in\mathbb{N}}$ converges to $\dot{\sigma}$ weakly in $L^2(0,t_f; H_{\Gamma_N}(\mathrm{div};\Omega)^3)$ and thus a fortiori in $L^2(0,t_f;L^2(\Omega)^3)$, and $\tau\otimes\varphi \in [L^2(Q)]^{3\times 3}$. Thirdly $\int_0^{t_f}\int_\Omega (\mathrm{div}\,\tau)(x).v_{n_k}(x,t)\ dx\ \varphi(t)\ dt$

tends to $\int_0^{t_f}\int_\Omega (\operatorname{div}\tau)(x).v(x,t)\,dx\,\varphi(t)\,dt$ because v_{n_k} converges weakly to v in $L^2(0,t_f;H^1(\Omega)^3)$, and thus a fortiori in $L^2(0,t_f;L^2(\Omega)^3)$, and $\operatorname{div}\tau\otimes\varphi\in L^2(Q)$. Lastly, $\int_0^{t_f}\int_\Omega \rho_{n_k}(x,t):\tau(x)\,dx\,\varphi(t)\,dt$ tends to $\int_0^{t_f}\int_\Omega \rho(x,t):\tau(x)\,dx\,\varphi(t)\,dt$ as ρ_{n_k} converges weakly to ρ in $L^2(0,t_f;[L^2(\Omega)]^{3\times 3}_{skew})$, and $\tau\otimes\varphi\in[L^2(Q)]^{3\times 3}$. Thus, we can pass to the limit in the first equation of the variational system (6.11). The second equation of the variational system (6.11) tells us that σ_{n_k} is symmetric and that $-\operatorname{div}(\sigma_{n_k})=g-\beta\nabla T_{n_k}$. σ_{n_k} converges weakly to σ in $H^1(0,t_f;H_{\Gamma_N}(\operatorname{div};\Omega)^3)$. A fortiori $-\operatorname{div}(\sigma_{n_k})$ converges weakly to $-\operatorname{div}(\sigma)$ in $L^2(0,t_f;L^2(\Omega)^3)$. We know also that ∇T_{n_k} converges weakly to ∇T in $L^2(0,t_f;L^2(\Omega)^3)$. Thus, we may pass to the limit in the second equation of the variational system (6.11), and we obtain:

$$\int_0^{t_f}\int_\Omega \operatorname{div}(\sigma(x,t)).w(x)\,dx\,\varphi(t)\,dt+\int_0^{t_f}\int_\Omega (g(x,t)-\beta\,\nabla T(x,t)).w(x)\,dx\,\varphi(t)\,dt$$
$$+\int_0^{t_f}\int_\Omega \sigma(x,t):\xi(x)\,dx\,\varphi(t)\,dt=0,\ \forall w\in L^2(\Omega)^3,$$
$$\forall\xi\in[L^2(\Omega)]^{3\times 3}_{skew},\forall\varphi\in L^2(]0,t_f[).$$

We have also $\sigma(.,0)=\zeta$, as the evaluation at time $t=0$, is a linear continuous form on $H^1(0,t_f;H_{\Gamma_N}(\operatorname{div};\Omega)^3)$ and σ_{n_k} converges weakly to σ in $H^1(0,t_f;H_{\Gamma_N}(\operatorname{div};\Omega)^3)$. By a standard argument of General Topology for metric spaces (we can reduce us to a metric space as in "a separable Hilbert space, every closed ball is a metrizable space for the weak topology", [22] (12.15.10) p.75), it follows that the sequence $((\sigma_n,v_n,\rho_n))_{n\in\mathbb{N}}$ itself converges weakly to (σ,v,ρ) in $H^1(0,t_f;H_{\Gamma_N}(\operatorname{div};\Omega)^3)\times L^2(0,t_f;H^1(\Omega)^3)\times L^2(0,t_f;[L^2(\Omega)]^{3\times 3}_{skew})$. □

6.1 Corollary. *The mapping from* $U_{ad}:=\{T_S\in H^2(]0,t_f[);\underline{T}\le T_S(t)\le\bar{T},\ \forall t\in]0,t_f[,T_S(0)=T_a\}$ *into* $L^2(0,t_f;L^2(\Omega)^3)$ *which sends* $T_S(.)$ *onto* $\vec{u}:[0,t_f]\to\mathbb{R}^3:t\longmapsto\int_0^t v(s)\,ds$ *is continuous from* U_{ad} *endowed with the weak topology into* $L^2(0,t_f;L^2(\Omega)^3)$ *endowed with the strong topology.*

Proof : From the previous proposition, we know that the mapping which sends $T_S(.)$ onto v is continuous from U_{ad} endowed with the weak topology into $L^2(0,t_f;H^1(\Omega)^3)$ endowed with the weak topology. The injection from $C([0,t_f];L^2(\Omega)^3)$ into $L^2(0,t_f;L^2(\Omega)^3)$ is of course continuous. It suffices thus to prove that the mapping which sends v onto $\vec{u}$ is a compact mapping from $L^2(0,t_f;H^1(\Omega)^3)$ into $C([0,t_f];L^2(\Omega)^3)$. We must thus prove that the set

$$E := \{\vec{u} : [0, t_f] \to \mathbb{R}^3 : t \longmapsto \int_0^t v(s)\, ds; \|v\|_{L^2(0,t_f;H^1(\Omega)^3)} \leq 1\}$$

is a relatively compact subset of $C([0, t_f]; L^2(\Omega)^3)$. We must thus verify the hypotheses of Ascoli's theorem ([20], p.143). Firstly, let us verify that E is equicontinuous at every point t_0 of the compact metric space $[0, t_f]$. Let $\vec{u} \in E$ and t a point in the interval $[0, t_f]$.

$$\begin{aligned}
\|\vec{u}(t) - \vec{u}(t_0)\|_{H^1(\Omega)^3} = \left\| \int_{t_0}^{t} v(s)\, ds \right\|_{H^1(\Omega)^3} &\leq \int_{\min(t_0,t)}^{\max(t_0,t)} \|v(s)\|_{H^1(\Omega)^3}\, ds \\
&\leq |t - t_0|^{1/2} \left(\int_{\min(t_0,t)}^{\max(t_0,t)} \|v(s)\|^2_{H^1(\Omega)^3}\, ds \right)^{1/2} \\
&\leq |t - t_0|^{1/2} \left(\int_0^{t_f} \|v(s)\|^2_{H^1(\Omega)^3}\, ds \right)^{1/2} \\
&\leq |t - t_0|^{1/2} \|v\|_{L^2(0,t_f;H^1(\Omega)^3)} \leq |t - t_0|^{1/2}.
\end{aligned}$$

Thus $\forall \varepsilon > 0$, $\forall t, t_0 \in [0, t_f]$: $|t - t_0| < \varepsilon^2$ implies $\|\vec{u}(t) - \vec{u}(t_0)\|_{H^1(\Omega)^3} < \varepsilon$. This proves that E is equicontinuous at every point t_0 of the compact metric space $[0, t_f]$. Secondly, let us verify that $\forall t_0 \in [0, t_f]$: $E(t_0) := \{\vec{u}(t_0); \vec{u} \in E\}$ is a relatively compact subset in $L^2(\Omega)^3$. Bounding like in the previous inequality, we have:

$$\|\vec{u}(t_0)\|_{H^1(\Omega)^3} = \left\| \int_0^{t_0} v(s)\, ds \right\|_{H^1(\Omega)^3} \leq \sqrt{t_0},$$

showing us that $E(t_0)$ is a bounded subset of $H^1(\Omega)^3$. As the injection from $H^1(\Omega)^3$ into $L^2(\Omega)^3$ is compact, $E(t_0)$ is a relatively compact subset in $L^2(\Omega)^3$. Thus the hypotheses of Ascoli's theorem ([20], p.143) are verified implying that E is a relatively compact subset of $C([0, t_f]; L^2(\Omega)^3)$. The mapping which sends v onto $\overrightarrow{u}$ from $L^2(0, t_f; H^1(\Omega)^3)$ into $C([0, t_f]; L^2(\Omega)^3)$ is thus a linear compact mapping, from which the result follows. □

Let us introduce the reduced cost functional:

$$\begin{aligned}
\hat{J} : U_{ad} := \{T_S \in H^2(]0, t_f[); \underline{T} \leq T_S(t) \leq \bar{T},\ \forall t \in]0, t_f[, T_S(0) = T_a\} \to \mathbb{R} \\
T_S \mapsto \tfrac{1}{2} \int_Q |\vec{u}(T_S)(x,t) - \vec{u}_d(x,t)|^2\ dx \otimes dt + \tfrac{\tilde{\delta}_r}{2} \left\| T_S - T_S^d \right\|^2_{H^2(]0,t_f[)}
\end{aligned} \tag{6.12}$$

We are now in a position to prove the existence of an optimal control for the field of displacements:

6.2 Theorem. *There exists $\hat{T}_S \in U_{ad}$ such that $\hat{J}(\hat{T}_S) = \inf_{T_S \in U_{ad}} \hat{J}(T_S)$.*

Proof : Let us set $L := \inf\limits_{T_S \in U_{ad}} \hat{J}(T_S)$. There exists a sequence $(T_{S,n})_{n\in\mathbb{N}} \subset U_{ad}$ such that $\hat{J}(T_{S,n}) \to L$. The sequence $(T_{S,n})_{n\in\mathbb{N}}$ is bounded in $U_{ad} \subset H^2(]0, t_f[)$, because if it was not, there would exist a subsequence $(T_{S,n_k})_{k\in\mathbb{N}}$ such that $\|T_{S,n_k}\|_{H^2(]0,t_f[)} \to +\infty$ as $k \to +\infty$, implying $\hat{J}(T_{S,n_k}) \to +\infty$ in contradiction with $\hat{J}(T_{S,n_k}) \to L$. The sequence $(T_{S,n})_{n\in\mathbb{N}}$ being bounded in $U_{ad} \subset H^2(]0, t_f[)$, possesses a subsequence $(T_{S,n_k})_{k\in\mathbb{N}}$ weakly convergent to some element $\hat{T}_S \in U_{ad}$. Using the fact that the norm mapping being continuous and convex is weakly lower semicontinuous ([71], p.47), and the preceding corollary implying that $\vec{u}(T_{S,n_k})$ tends to $\vec{u}(\hat{T}_S)$ in the $L^2(Q)^3$, we obtain that $\hat{J}(\hat{T}_S) \leq \liminf\limits_{k\to+\infty} \hat{J}(T_{S,n_k})$, and thus $\hat{J}(\hat{T}_S) = L = \inf\limits_{T_S\in U_{ad}} \hat{J}(T_S)$, what was to be proved. $\square$

6.3 Continuous Fréchet Differentiability of the Reduced Cost Functional

The aim of this section is to prove the continuous Fréchet differentiability of the reduced cost functional, which allows us to write the variational inequality. Let us define U_{T_a} the closed affine subspace of $U := H^2(]0, t_f[)$:

$$U_{T_a} := \{T_S \in U := H^2(]0, t_f[); T_S(0) = T_a\}. \tag{6.13}$$

Also, U_0 denotes the closed vectorial subspace of $U := H^2(]0, t_f[)$:

$$U_0 := \{T_S \in U := H^2(]0, t_f[); T_S(0) = 0\}. \tag{6.14}$$

6.3.1 Continuous Fréchet Differentiability

It follows immediately from Theorem 4.3 of Chapter 4, that the mapping from $U_{T_a} \to C\left(\bar{Q}\right) : T_S \to T\left(T_S\right)$ is continuously Fréchet differentiable (U_{T_a} being in particular an affine normed space, in the definition of the Fréchet derivative at a point $T_S \in U_{T_a}$, one considers increments δT_S belonging to the associated guiding vector space of U_{T_a} i.e. increments $\delta T_S \in U$ such that $\delta T_S(0) = 0$; see [61] (III,3;13) p.197).

6.1 Lemma. *The mapping from the open set $A := \{T \in C\left(\bar{Q}\right); \frac{\bar{T}}{2} < T(x,t) < 2\bar{T},\ \forall (x,t) \in \bar{Q}\}$ of $C\left(\bar{Q}\right)$ into $C\left(\bar{Q}\right)$, which sends T onto $\eta(T)$ is continuously Fréchet differentiable.*

Proof : Let $T \in A$ and $\delta T \in C\left(\bar{Q}\right)$, with $\|\delta T\|_\infty$ sufficiently small so that $T + \delta T$ still belongs to A. Let $(x,t) \in \bar{Q} = \bar{\Omega} \times [0, t_f]$ ($Q := \Omega \times]0, t_f[$).

$$\begin{aligned}
&\eta(T(x,t) + \delta T(x,t)) - \eta(T(x,t)) - \eta'(T(x,t))\delta T(x,t) \\
&\quad = \int_{T(x,t)}^{T(x,t)+\delta T(x,t)} (T(x,t) + \delta T(x,t) - s)\eta''(s)\ ds \\
&\quad = \int_0^{\delta T(x,t)} (\delta T(x,t) - r)\eta''(T(x,t) + r)\ dr.
\end{aligned}$$

We have:

$$\begin{aligned}
&|\eta(T(x,t) + \delta T(x,t)) - \eta(T(x,t)) - \eta'(T(x,t))\delta T(x,t)| \\
&\quad = \left| \int_0^{\delta T(x,t)} (\delta T(x,t) - r)\eta''(T(x,t) + r)\ dr \right| \\
&\quad \leq \sup_{s \in [\frac{T}{2}, 2\bar{T}]} \left|\eta''(s)\right| \int_{[0, \delta T(x,t)]} |\delta T(x,t) - r|\ dr \\
&\quad \leq \sup_{s \in [\frac{T}{2}, 2\bar{T}]} \left|\eta''(s)\right| |\delta T(x,t)|^2,\ \forall (x,t) \in \bar{Q}.
\end{aligned}$$

Thus

$$\|\eta(T + \delta T) - \eta(T) - \eta'(T)\delta T\|_{\infty,\bar{Q}} \leq \sup_{s \in [\frac{T}{2}, 2\bar{T}]} \left|\eta''(s)\right| \|\delta T\|^2_{\infty,\bar{Q}},$$

implying that:

$$\|\eta(T + \delta T) - \eta(T) - \eta'(T)\delta T\|_{\infty,\bar{Q}} = \circ(\|\delta T\|_{\infty,\bar{Q}}),$$

proving that the mapping $A \to C\left(\bar{Q}\right) : T \longmapsto \eta(T) \equiv \eta \circ T$ is Fréchet differentiable at every $T \in A$ and that:

$$D(T \in A \longmapsto \eta(T))(T) : C\left(\bar{Q}\right) \to C\left(\bar{Q}\right) : \delta T \mapsto \eta'(T)\delta T, \tag{6.15}$$

$\forall T \in A$. η' being continuous on the compact interval $[\frac{T}{2}, 2\bar{T}]$ is uniformly continuous, which implies that the mapping

$$A \subset C\left(\bar{Q}\right) \to \mathcal{L}(C\left(\bar{Q}\right); C\left(\bar{Q}\right)) : T \mapsto \eta'(T)$$

is continuous. $\square$

6.2 Corollary. *Let $C := \{T_S \in U_{T_a}; \frac{T}{2} < T_S(t) < 2\bar{T},\ \forall t \in [0, t_f]\}$. The mapping from $H^1(0, t_f; [H_{\Gamma_N}(\mathrm{div}; \Omega)]^3_{sym}) \times C$ into $L^2(0, t_f; [L^2(\Omega)]^{3\times 3})$ which sends (σ, T_S) onto $\frac{1+\nu}{\eta(T(T_S))}\sigma$ is continuously Fréchet differentiable.*

Proof : Let $T_S \in C$ be fixed. We have also that $\frac{T}{2} < T(T_S) < 2\bar{T}$. As a result, the function $\frac{1+\nu}{\eta(T(T_S))}$ is a continuous and bounded function on $Q =]0, t_f[\times \Omega$. Thus, the mapping which sends $\sigma \in H^1(0, t_f; [H_{\Gamma_N}(\mathrm{div}; \Omega)]^3_{sym})$ onto $\frac{1+\nu}{\eta(T(T_S))}\sigma \in L^2(0, t_f; [L^2(\Omega)]^{3\times 3})$ is linear and continuous. Consequently, its partial dervative with respect to the variable σ at any point in $H^1(0, t_f; [H_{\Gamma_N}(\mathrm{div}; \Omega)]^3_{sym}) \times C$, is itself. It is easily seen that this partial derivative is continuous $H^1(0,t_f;[H_{\Gamma_N}(\mathrm{div};\Omega)]^3_{sym}) \times C$. On the other hand, by Theorem 4.3 of Chapter 4 and the previous lemma, the mapping from C into $C\left(\bar{Q}\right)$, which sends T_S onto $\eta(T(T_S))$ is continuously Fréchet differentiable. We have also that $\eta(2\bar{T}) < \eta(T(T_S(t))) < \eta(\frac{T}{2})$, $\forall t \in [0,t_f]$. Let us denote by

$$B := \{f \in C\left(\bar{Q}\right); \eta(2\bar{T}) < f(x,t) < \eta(\frac{T}{2}), \forall (x,t) \in \bar{Q}\}.$$

If $f \in B$, we have $\eta(2\bar{T}) < \inf\limits_{(x,t)\in\bar{Q}} f(x,t) \leq \sup\limits_{(x,t)\in\bar{Q}} < \eta(\frac{T}{2})$, and thus $f+h \in B$ for $h \in C\left(\bar{Q}\right)$ with $\|h\|_{\infty,\bar{Q}}$ sufficiently small. Thus B is an open set in $C\left(\bar{Q}\right)$. For such a h: $\frac{1}{f+h} - \frac{1}{f} + \frac{h}{f^2} = \frac{h^2}{(f+h)f^2}$, which implies that $\left\|\frac{1}{f+h} - \frac{1}{f} + \frac{h}{f^2}\right\|_{\infty,\bar{Q}} = \left\|\frac{h^2}{(f+h)f^2}\right\|_{\infty,\bar{Q}} \leq \frac{\|h\|^2_{\infty,\bar{Q}}}{\eta(2\bar{T})^3}$. Thus the mapping from B into $C\left(\bar{Q}\right)$ which sends f onto $\frac{1}{f}$ is differentiable at every point of B and $D(\frac{1}{\cdot})(f).h = -\frac{h}{f^2}$. One has also that, if $f_1,\ f_2 \in B$:

$$D(\frac{1}{\cdot})(f_2).h - D(\frac{1}{\cdot})(f_1).h = h(\frac{1}{f_1^2} - \frac{1}{f_2^2}) = h\frac{(f_2-f_1)(f_2+f_1)}{f_1^2 f_2^2},$$

and thus:

$$\left\|D(\frac{1}{\cdot})(f_2) - D(\frac{1}{\cdot})(f_1)\right\|_{\mathcal{L}(C(\bar{Q});C(\bar{Q}))} \leq 2\frac{\eta(\frac{T}{2})}{\eta(2\bar{T})^4}\left\|f_2 - f_1\right\|_{\infty,\bar{Q}},$$

which show us that moreover the mapping from B into $C\left(\bar{Q}\right)$ which sends f onto $\frac{1}{f}$ is continuously differentiable on B. In conclusion, the mapping from C into $C\left(\bar{Q}\right)$, which sends T_S onto $\frac{1}{\eta(T(T_S))}$ is continuously Fréchet differentiable. Consequently, the partial dervative of the mapping $(\sigma,T_S) \mapsto \frac{1+\nu}{\eta(T(T_S))}\sigma$ from $H^1(0,t_f;[H_{\Gamma_N}(\mathrm{div};\Omega)]^3_{sym}) \times C$ into $L^2(0,t_f;[L^2(\Omega)]^{3\times 3})$ with respect to the variable T_S exists at any point in $H^1(0,t_f;[H_{\Gamma_N}(\mathrm{div};\Omega)]^3_{sym})\times C$, and it is easily seen that this partial derivative is also continuous $H^1(0,t_f;[H_{\Gamma_N}(\mathrm{div};\Omega)]^3_{sym})\times C$. Using (8.9.1) p.175 of [20], the result follows. □

6.1 Definition. *For every $\xi \in C\left(\bar{Q}\right)$, we introduce the bilinear continuous form:*

$$a_\xi(.,.) : L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3) \times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3) \to \mathbb{R}$$
$$(u,v) \mapsto \int_0^{t_f}\int_\Omega (\tfrac{\xi(x,t)}{1+\nu}\varepsilon(u(x,t)) : \varepsilon(v(x,t)) + \tfrac{\xi(x,t)\nu}{(1+\nu)(1-2\nu)}\,\mathrm{div}(u(x,t))\,\mathrm{div}(v(x,t)))\ dx\otimes dt. \tag{6.16}$$

By A_ξ, we moreover denote the associated linear continuous operator from $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*$ such that:

$$< A_\xi u, v >_{L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*,L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)} = a_\xi(u,v), \tag{6.17}$$

for every $u, v \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$.

6.4 Proposition. *For* $\xi \in C\left(\bar{Q}\right)$, $\xi > 0$, *the linear continuous operator* A_ξ *is a linear isomorphism from* $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ *onto* $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*$.

Proof : This result from Lax-Milgram's lemma applied to the bilinear continuous form $a_\xi(.,.)$, using $\inf_Q \xi > 0$ and Korn's inequality to deduce its coercivity, like in Proposition 5.4. □

6.5 Proposition. *The mapping from* $C\left(\bar{Q}\right)$ *into* $\mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3);L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*)$ *which sends* $\xi \in C\left(\bar{Q}\right)$ *onto* A_ξ *is a linear continuous mapping. Consequently, its Fréchet derivative at any point* $\xi \in C\left(\bar{Q}\right)$ *is itself i.e. the linear continuous mapping from* $C(\bar{Q})$ *into* $\mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3);L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*)$ *which sends* $\delta\xi \in C\left(\bar{Q}\right)$ *onto* $A_{\delta\xi}$.

Proof : The trilinear form:

$$a(.,.,.) : C\left(\bar{Q}\right) \times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3) \times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3) \to \mathbb{R} : (\xi,u,v) \mapsto$$

$$a_\xi(u,v) = \int_0^{t_f}\int_\Omega \left(\tfrac{\xi(x,t)}{1+\nu}\varepsilon(u(x,t)) : \varepsilon(v(x,t)) + \tfrac{\xi(x,t)\nu}{(1+\nu)(1-2\nu)}\operatorname{div}(u(x,t))\operatorname{div}(v(x,t))\right) dx \otimes dt, \tag{6.18}$$

is continuous. Consequently, the mapping which sends $\xi \in C\left(\bar{Q}\right)$ onto A_ξ is a linear continuous mapping. The second part of the statement follows immediately. □

6.3 Corollary. *The mapping from the open set* $A := \{T \in C\left(\bar{Q}\right); \frac{\bar{T}}{2} < T(x,t) < 2\bar{T},\ \forall (x,t) \in \bar{Q}\}$ *of* $C\left(\bar{Q}\right)$ *into* $\mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3);L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*)$ *which sends* $T \mapsto A_{\eta(T)}$ *is continuously differentiable in the Fréchet sense and for every* $T \in A$, *its Fréchet derivative at point* T *is the linear continuous mapping:*

$$C\left(\bar{Q}\right) \to \mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3);L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*) : \delta T \mapsto A_{\eta'(T)\delta T}\ . \tag{6.19}$$

Proof : By Lemma 6.1, the mapping from A into $C\left(\bar{Q}\right)$, which sends T onto $\eta(T)$ is continuously Fréchet differentiable and its Fréchet derivative at point $T \in A$ is the linear continuous mapping

$$C\left(\bar{Q}\right) \to C\left(\bar{Q}\right) : \delta T \mapsto \eta'(T)\delta T.$$

It suffices now to apply the previous proposition ant the rule of derivation of composite mappings ((8.2.1) p.153 of [20]). □

6.4 Corollary. *The mapping from the open set* $A := \{T \in C\left(\bar{Q}\right); \frac{\bar{T}}{2} < T(x,t) < 2\bar{T},\ \forall (x,t) \in \bar{Q}\}$ *of* $C\left(\bar{Q}\right)$ *into* $\mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*;L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3))$ *which sends* $T \mapsto A^{-1}_{\eta(T)}$ *is continuously differentiable in the Fréchet sense and for every* $T \in A$, *its Fréchet derivative at point* T *is the linear continuous mapping:*

$$\begin{gathered} C\left(\bar{Q}\right) \to \mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*;L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)) \\ \delta T \mapsto -A^{-1}_{\eta(T)} \circ A_{\eta'(T)\delta T} \circ A^{-1}_{\eta(T)}. \end{gathered} \tag{6.20}$$

Proof : The mapping $\eta(T) \in C\left(\bar{Q}\right)$ and $\eta(T)$ is positive on $\bar{Q}$. Consequently by Proposition 6.4, $A_{\eta(T)}$ is a linear isomorphism from $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ onto $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*$. The result now follows from ([20] (8.3.2) pp. 155-156 or better Theorem 5.4.3 p.73 in [10]). $\square$

6.6 Proposition. *Let $E \subset C\left(\bar{Q}\right) \cap L^2(0,t_f;H^1(\Omega))$ denotes the state space defined in Definition 4.3 of Chapter 4. Then the mapping from $E \cap A$ (for the definition of A, see the preceding corollary) into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$, which sends T onto $u_e = A^{-1}_{\eta(T)}(g-\beta\nabla T)$ is continuously differentiable on $E\cap A$ and its Fréchet derivative at the point $T \in E\cap A$ is the continuous linear mapping from E into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ which sends $\delta T \in E$ onto*

$$\delta u_e := -(A^{-1}_{\eta(T)} \circ A_{\eta'(T)\delta T} \circ A^{-1}_{\eta(T)})(g-\beta\nabla T) - \beta A^{-1}_{\eta(T)}\nabla\delta T. \tag{6.21}$$

Proof : The mapping from $E\cap A$ into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$, which sends T onto $u_e = A^{-1}_{\eta(T)}(g-\beta\nabla T)$, may be seen as the composition of the mapping from $E\cap A$ into the cartesian product $\mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*;L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)) \times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*$ which sends T onto $(A_{\eta(T)}, g-\beta\nabla T)$ followed by the bilinear continuous mapping from cartesian product $\mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*;L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3))\times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*$ into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ which sends $(B,f)\mapsto B(f)$. By ((8.1.4) p.152 of [20]), the Fréchet derivative of this bilinear continuous mapping at the point

$$(B,f) = (A^{-1}_{\eta(T)}, g-\beta\nabla T) \in \mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*;L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3))\times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*$$

is the linear continuous mapping which sends $(\delta B,\delta f) \in \mathcal{L}(L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*;L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3))\times L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)^*$ onto $A^{-1}_{\eta(T)}\delta f + \delta B(g-\beta\nabla T)$. To compute the derivative of the composite function which sends T onto $u_e = A^{-1}_{\eta(T)}(g-\beta\nabla T)$ at the point T, we must consider in the preceding formula $\delta f = -\beta\nabla\delta T$ and $\delta B = -(A^{-1}_{\eta(T)} \circ A_{\eta'(T)\delta T} \circ A^{-1}_{\eta(T)})$, which gives us the required result. $\square$

To compute the derivative of the composite function which sends $T_S \in U_{T_a} \cap \{T_S\,;\frac{\bar{T}}{2} < T_S(t) < 2\bar{T},\ \forall t\in[0,t_f]\}$ onto $u_e(T(T_S))$, it suffices to use the preceding proposition and formula (4.34) of Chapter 4 for $DT(T_S)$:

$$DT(T_S) = -\{[D_T e(T(T_S),T_S)]^{-1} \circ D_{T_S} e(T(T_S),T_S)\}. \tag{6.22}$$

6.5 Corollary. *The derivative of the function from $\tilde{U}_{ad} := \{T_S \in H^2(]0,t_f[);\frac{\bar{T}}{2} < T_S(t) < 2\bar{T},\ \forall t \in]0,t_f[, T_S(0) = T_a\}$ into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$, which sends T_S onto $u_e = A^{-1}_{\eta(T(T_S))}(g-\beta\nabla T(T_S))$ is continuously differentiable. Its derivative at the point $T_S \in \tilde{U}_{ad}$ is the linear continuous mapping from $H^2(]0,t_f[)$ into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$, which sends $\delta T_S \in U_0$ where $U_0 := \{T_S \in U; T_S(0) = 0\}$ onto*

$$\delta u_e := -(A^{-1}_{\eta(T(T_S))} \circ A_{\eta'(T(T_S))\delta T} \circ A^{-1}_{\eta(T(T_S))})(g-\beta\nabla T(T_S)) - \beta A^{-1}_{\eta(T(T_S))}\nabla\delta T \tag{6.23}$$

taking for δT in that formula:

$$\delta T := DT(T_S).\delta T_S = -\{[D_T e(T(T_S), T_S)]^{-1} \circ D_{T_S} e(T(T_S), T_S)\}(\delta T_S). \tag{6.24}$$

Now, let us consider the mapping which sends $T_S \in \tilde{U}_{ad}$ onto $\sigma_e(.,.) := 2\mu(.,.)\varepsilon(u_e(.,.)) + \lambda(.,.) \operatorname{div}(u_e(.,.))I_3 \in L^2(0, t_f; L^2(\Omega)^{3\times3})$ where $\mu(.,.) := \frac{\eta(T(T_S)(.,.))}{2(1+\nu)}$ and $\lambda(.,.) := \frac{\eta(T(T_S)(.,.))\nu}{(1+\nu)(1-2\nu)}$. We are going to prove that this mapping is also Fréchet differentiable.

6.7 Proposition. *The mapping from $\tilde{U}_{ad}$ into $L^2(0, t_f; L^2(\Omega)^{3\times3})$ which sends T_S onto σ_e is also continuously Fréchet differentiable. Its Fréchet derivative at point T_S is given by formula (6.25).*

Proof : Let us firstly consider the term:

$$2\mu\ \varepsilon(u_e) : (x, t) \longmapsto \frac{\eta(T(x, t))}{1+\nu}\frac{1}{2}(\nabla_x u_e(x, t) + \nabla_x u_e(x, t)^T).$$

The gradient with respect to x, ∇_x, and consequently $\varepsilon(.)$ defines a continuous linear mapping from $L^2(0, t_f; H^1_{\Gamma_D}(\Omega)^3)$ into $L^2(0, t_f; L^2(\Omega)^{3\times3})$. By the preceding corollary, and the theorem on the derivative of composite functions, the mapping $T_S \mapsto \varepsilon(u_e)(T(T_S))$ is thus continuously differentiable from $\tilde{U}_{ad}$ into $L^2(0, t_f; L^2(\Omega)^{3\times3})$. On the other hand, by Theorem 4.3 of Chapter 4 and Lemma 6.1, the mapping from $\tilde{U}_{ad}$ into $C\left(\bar{Q}\right)$, which sends T_S onto $\eta(T(T_S))$ is continuously Fréchet differentiable. The mapping:

$$C\left(\bar{Q}\right) \times L^2(0, t_f; L^2(\Omega)^{3\times3}) \to L^2(0, t_f; L^2(\Omega)^{3\times3}) : (\eta, \varepsilon) \mapsto \eta.\varepsilon$$

is bilinear and continuous. By ((8.1.4) p.152 of [20]), the Fréchet derivative of this bilinear continuous mapping at the point $(\eta, \varepsilon) \in C\left(\bar{Q}\right) \times L^2(0, t_f; L^2(\Omega)^{3\times3})$ is the linear continuous mapping which sends $(\delta\eta, \delta\varepsilon)$ belonging to $C\left(\bar{Q}\right) \times L^2(0, t_f; L^2(\Omega)^{3\times3})$ onto $\eta.\delta\varepsilon + \delta\eta.\varepsilon$ belonging to $L^2(0, t_f; L^2(\Omega)^{3\times3})$. Similar reasonings apply to the term $\lambda\ \operatorname{div}(u_e(.,.))I_3$. Writing an explicit formula for the Fréchet derivative at the point T_S is somewhat heavy: it is the linear continuous mapping from U_{T_a} into $L^2(0, t_f; L^2(\Omega)^{3\times3})$ which sends arbitrary $\delta T_S \in U_0$ onto

$$\begin{aligned} \delta\sigma_e := & \tfrac{\eta(T(T_S))}{(1+\nu)}\ \varepsilon(\delta u_e) + \tfrac{\nu\eta(T(T_S))}{(1+\nu)(1-2\nu)}\ \operatorname{div}(\delta u_e(.,.))I_3 \\ & + \tfrac{\eta'(T(T_S))\delta T}{(1+\nu)}\ \varepsilon(u_e) + \tfrac{\nu\eta'(T(T_S))\delta T}{(1+\nu)(1-2\nu)}\ \operatorname{div}(u_e(.,.))I_3, \end{aligned} \tag{6.25}$$

where δT and δu_e are defined by the formulas (6.24) and (6.23). $\square$

Now, we are going to prove that the mapping from $\tilde{U}_{ad}$ which sends T_S onto σ_0 defined by the Cauchy problem (5.53) for the differential equation $(5.53)_{(i)}$, is also a continuously Fréchet differentiable function with values in $L^2(0, t_f; L^2(\Omega)^{3\times3})$. In that purpose, let us introduce the new unknown

$$\xi(., t) := \sigma_0(., t) + ((P_{H_0} A_1)|_{H_0})^{-1} (P_{H_0} A_1)\, \sigma_e(., t). \tag{6.26}$$

$\xi \in L^2(0, t_f; H_0)$. It follows from (5.53), that this new dependent variable is the solution of the Cauchy problem for the differential equation in H_0:

$$\left\{ \begin{array}{c} \frac{d\xi}{dt}(.,t) = -\left((P_{H_0}A_1)|_{H_0}\right)^{-1}(P_{H_0}A_0(.,t))\,\xi(.,t) \\ +\left((P_{H_0}A_1)|_{H_0}\right)^{-1}(P_{H_0}A_0(.,t))\left((P_{H_0}A_1)|_{H_0}\right)^{-1}(P_{H_0}A_1)\,\sigma_e(.,t), \\ \xi(.,0) = \zeta - \sigma_e(.,0) + \left((P_{H_0}A_1)|_{H_0}\right)^{-1}(P_{H_0}A_1)\,\sigma_e(.,0). \end{array} \right. \tag{6.27}$$

Let us set:

$$\zeta_e(.,t) := \left((P_{H_0}A_1)|_{H_0}\right)^{-1}(P_{H_0}A_0(.,t))\left((P_{H_0}A_1)|_{H_0}\right)^{-1}(P_{H_0}A_1)\,\sigma_e(.,t), \tag{6.28}$$

and as previously $A(t) := -\left((P_{H_0}A_1)|_{H_0}\right)^{-1}(P_{H_0}A_0(.,t))$ (see equation (5.54)). $A(.) \in C([0,t_f];\mathcal{L}(H_0))$ and $\zeta_e \in L^2(0,t_f;H_0)$. Let us observe that $A(.)$ and ζ_e depend on T via $t \mapsto A_0(.,t)$. Using these notations, the differential equation $(6.27)_{(i)}$ may be rewritten:

$$\frac{d\xi}{dt}(.,t) = A(t)\xi(.,t) + \zeta_e(.,t). \tag{6.29}$$

Integrating both sides from 0 to t, we obtain:

$$\xi(.,t) = \xi(.,0) + \int_0^t A(s)\xi(.,s)\;ds + \int_0^t \zeta_e(.,s)\;ds. \tag{6.30}$$

Now, let us introduce the operator $\tilde{A} \in \mathcal{L}(L^2(0,t_f;H_0))$:

$$\begin{aligned} \tilde{A} : L^2(0,t_f;H_0) &\to L^2(0,t_f;H_0) \\ \xi &\mapsto \text{“}]0,t_f[\to H_0 : t \mapsto \int_0^t A(s)\xi(.,s)\;ds\text{”}. \end{aligned} \tag{6.31}$$

6.8 Proposition. *Denoting by $\mathcal{I}$ the identity operator on $L^2(0,t_f;H_0)$, the operator $\mathcal{I}-\tilde{A}$ is invertible in $\mathcal{L}(L^2(0,t_f;H_0))$.*

Proof : From the definition (6.31) of $\tilde{A}$, we have $\forall \xi \in L^2(0,t_f;H_0)$ and $\forall t \in [0,t_f]$:

$$\begin{aligned} \left\|(\tilde{A}\xi)(t)\right\|_{H_0} &\leq \int_0^t \|A(s)\|_{\mathcal{L}(H_0)}\|\xi(s)\|_{H_0}\;ds \leq \|A\|_{C([0,t_f];\mathcal{L}(H_0))}\int_0^t \|\xi(s)\|_{H_0}\;ds \\ &\leq \|A\|_{C([0,t_f];\mathcal{L}(H_0))}\int_0^{t_f}\|\xi(s)\|_{H_0}\;ds = \|A\|_{C([0,t_f];\mathcal{L}(H_0))}\|\xi\|_{L^1(0,t_f;H_0)}\,. \end{aligned}$$

We have of course that $\|\xi\|_{L^1(0,t_f;H_0)} \leq \sqrt{t_f}\,\|\xi\|_{L^2(0,t_f;H_0)}$. Now, let us consider $\tilde{A}^{\circ 2} := \tilde{A}\circ\tilde{A}$.

We have $\forall \xi \in L^2(0,t_f;H_0)$ and $\forall t \in [0,t_f]$:

$$\begin{aligned}
\left\|(\tilde{A}^{\circ 2}\xi)(t)\right\|_{H_0} &\leq \int_0^t \|A(s)\|_{\mathcal{L}(H_0)} \left\|(\tilde{A}\xi)(s)\right\|_{H_0} \, ds \\
&\leq \|A\|_{C([0,t_f];\mathcal{L}(H_0))} \int_0^t \left\|(\tilde{A}\xi)(s)\right\|_{H_0} \, ds \\
&\leq \|A\|^2_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^1(0,t_f;H_0)} \; t.
\end{aligned} \tag{6.32}$$

For $\tilde{A}^{\circ 3}$, we have $\forall \xi \in L^2(0,t_f;H_0)$ and $\forall t \in [0,t_f]$:

$$\begin{aligned}
\left\|(\tilde{A}^{\circ 3}\xi)(t)\right\|_{H_0} &\leq \int_0^t \|A(s)\|_{\mathcal{L}(H_0)} \left\|(\tilde{A}^{\circ 2}\xi)(s)\right\|_{H_0} \, ds \leq \|A\|_{C([0,t_f];\mathcal{L}(H_0))} \int_0^t \left\|(\tilde{A}^{\circ 2}\xi)(s)\right\|_{H_0} \, ds \\
&\leq \|A\|^3_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^1(0,t_f;H_0)} \int_0^t s \, ds = \|A\|^3_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^1(0,t_f;H_0)} \frac{t^2}{2}.
\end{aligned}$$

Continuing our recurrence, we obtain for $\tilde{A}^{\circ n}$:

$$\begin{aligned}
\left\|(\tilde{A}^{\circ n}\xi)(t)\right\|_{H_0} &\leq \|A\|^n_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^1(0,t_f;H_0)} \frac{t^{n-1}}{(n-1)!} \\
&\leq \|A\|^n_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^1(0,t_f;H_0)} \frac{t^{n-1}}{(n-1)!} \\
&\leq \sqrt{t_f}\, \|A\|^n_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^2(0,t_f;H_0)} \frac{t^{n-1}}{(n-1)!},
\end{aligned}$$

$\forall \xi \in L^2(0,t_f;H_0)$, $\forall t \in [0,t_f]$ and $\forall n \in \mathbb{N}^*$. This latest inequality implies that:

$$\begin{aligned}
\left\|\tilde{A}^{\circ n}\xi\right\|_{L^2(0,t_f;H_0)} &\leq \sqrt{t_f}\, \|A\|^n_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^2(0,t_f;H_0)} \frac{\left(\int_0^{t_f} t^{2n-2} \, dt\right)^{\frac{1}{2}}}{(n-1)!} \\
&\leq \sqrt{t_f}\, \|A\|^n_{C([0,t_f];\mathcal{L}(H_0))} \|\xi\|_{L^2(0,t_f;H_0)} \frac{t_f^{n-\frac{1}{2}}}{\sqrt{2n-1}\;(n-1)!} \\
&= \frac{\left(\|A\|_{C([0,t_f];\mathcal{L}(H_0))} \; t_f\right)^n}{\sqrt{2n-1}\;(n-1)!} \|\xi\|_{L^2(0,t_f;H_0)}\,, \\
&\quad \forall \xi \in L^2(0,t_f;H_0).
\end{aligned}$$

Thus $\left\|\tilde{A}^{\circ n}\right\|_{\mathcal{L}(L^2(0,t_f;H_0))} \leq \frac{t_f \; \|A\|_{C([0,t_f];\mathcal{L}(H_0))}}{\sqrt{2n-1}} \frac{\left(t_f \; \|A\|_{C([0,t_f];\mathcal{L}(H_0))}\right)^{n-1}}{(n-1)!}$, $\forall n \in \mathbb{N}^*$. Thus the series:

$$\mathcal{I} + \tilde{A} + \tilde{A}^2 + \cdots + \tilde{A}^n + \cdots$$

is absolutely convergent in $\mathcal{L}(L^2(0,t_f;H_0))$. One verifies immediately that:

$$(\mathcal{I}+\tilde{A}+\tilde{A}^2+\cdots+\tilde{A}^n+\cdots)(\mathcal{I}-\tilde{A})=\mathcal{I}$$

and that

$$(\mathcal{I}-\tilde{A})(\mathcal{I}+\tilde{A}+\tilde{A}^2+\cdots+\tilde{A}^n+\cdots)=\mathcal{I}.$$

Thus the operator $\mathcal{I}-\tilde{A}$ is invertible in $\mathcal{L}(L^2(0,t_f;H_0))$. $\square$

Consequently, equation (6.30) can be rewritten:

$$\xi=(\mathcal{I}-\tilde{A})^{-1}(\xi(.,0)+\int_0^{.}\zeta_e(.,s)\,ds). \tag{6.33}$$

6.9 Proposition. *The mapping from the open set $A:=\{T\in C\left(\bar{Q}\right);\frac{\bar{T}}{2}<T(x,t)<2\bar{T},\ \forall(x,t)\in\bar{Q}\}$ of $C\left(\bar{Q}\right)$, which sends T onto $\tilde{A}\equiv\tilde{A}(T)\in\mathcal{L}(L^2(0,t_f;H_0))$ is continuously Fréchet differentiable from the open set $A\subset C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0))$. Consequently, $(\mathcal{I}-\tilde{A})^{-1}$ is also continuously Fréchet differentiable from the open set $A\subset C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0))$.*

Proof : By (8.3.2) pp. 155-156 in [20] or better Theorem 5.4.3 p.73 in [10] and the preceding proposition, it suffices to prove that the mapping which sends T onto $\tilde{A}(T)$ is continuously Fréchet differentiable from the open set $A\subset C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0))$. Let us denote by B the linear continuous operator

$$B:L^2(0,t_f;H_0)\to L^2(0,t_f;L^2(\Omega)^{3\times3}):\sigma\mapsto(1+\nu)\sigma-\nu.tr(\sigma)I_3 \tag{6.34}$$

The mapping from $C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0);L^2(0,t_f;L^2(\Omega)^{3\times3}))$ which sends $f\in C\left(\bar{Q}\right)$ onto $f.B\in\mathcal{L}(L^2(0,t_f;H_0);L^2(0,t_f;L^2(\Omega)^{3\times3}))$ is linear and continuous and thus also the mapping from $C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0))$ which sends $f\in C\left(\bar{Q}\right)$ onto $-\left((P_{H_0}A_1)|_{H_0}\right)^{-1}P_{H_0}\circ f.B\in\mathcal{L}(L^2(0,t_f;H_0))$. Consequently this mapping is also continuously Fréchet differentiable from $C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0))$. Now, by Lemma 6.1, the mapping from the open set $A\subset C\left(\bar{Q}\right)$ into $C\left(\bar{Q}\right)$ which sends $T\in A$ onto $f:=\frac{1}{\eta(T(.,.))}$ is continuously Fréchet differentiable from the open set $A\subset C\left(\bar{Q}\right)$ into $C\left(\bar{Q}\right)$. It results by composition that the mapping which sends $T\in A\subset C\left(\bar{Q}\right)$ onto $-\left((P_{H_0}A_1)|_{H_0}\right)^{-1}P_{H_0}\circ\frac{1}{\eta(T(.,.))}.B\in\mathcal{L}(L^2(0,t_f;H_0))$ is also continuously Fréchet differentiable from $A\subset C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0))$. Composing now with the linear continuous mapping

$$L^2(0,t_f;H_0)\to L^2(0,t_f;H_0)$$
$$g\mapsto \text{“}]0,t_f[\to H_0:t\mapsto\int_0^t g(.,s)\,ds\text{”},$$

it follows that the mapping which sends T onto $\tilde{A}(T)$ from the open set $A \subset C\left(\bar{Q}\right)$ into $\mathcal{L}(L^2(0,t_f;H_0))$ is continuously Fréchet differentiable. $\square$

6.10 Proposition. *The mapping from the open set $A \cap E$, which sends $T \in A \cap E$ onto $\zeta_e = \zeta_e(T)$ is continuously Fréchet differentiable from the open set $A \cap E \subset E$ into $L^2(0,t_f;H_0)$.*

Proof : By Proposition 6.7, the mapping $T \mapsto \sigma_e = \sigma_e(T)$ is a continuously Fréchet differentiable mapping from the open set $A \cap E \subset E$ into $L^2(0,t_f;L^2(\Omega)^{3\times 3})$, and thus also the mapping $T \mapsto B\left((P_{H_0}A_1)|_{H_0}\right)^{-1} P_{H_0}A_1\sigma_e(T)$, where the linear continuous operator $B \in \mathcal{L}(L^2(0,t_f;H_0)); L^2(0,t_f;L^2(\Omega)^{3\times 3}))$ has been defined by formula (6.34). By the proof of Corollary 6.2, the mapping which sends T onto $\frac{1}{\eta(T(.,.))}$ is a continuously Fréchet differentiable mapping from the open set $A \cap E \subset E$ into $C(\bar{Q})$. Composing with the bilinear continuous mapping

$$C(\bar{Q}) \times L^2(0,t_f;L^2(\Omega)^{3\times 3}) \to L^2(0,t_f;L^2(\Omega)^{3\times 3}) : (f,\alpha) \mapsto f.\alpha,$$

which is thus a continuously Fréchet differentiable mapping by ((8.1.4) p.152 of [20]), it follows that the mapping from the open set $A \cap E \subset E$ into $L^2(0,t_f;L^2(\Omega)^{3\times 3})$, which sends T onto

$$A_0(.,.)B\left((P_{H_0}A_1)|_{H_0}\right)^{-1} P_{H_0}A_1\sigma_e(T) = \frac{1}{\eta(T(.,.))}[$$
$$(1+\nu)\left((P_{H_0}A_1)|_{H_0}\right)^{-1} P_{H_0}A_1\sigma_e(T) - \nu.tr(\left((P_{H_0}A_1)|_{H_0}\right)^{-1} P_{H_0}A_1\sigma_e(T)).I_3],$$

is a continuously Fréchet differentiable mapping from the open set $A \cap E \subset E$ into $L^2(0,t_f;L^2(\Omega)^{3\times 3})$. Composing still with $\left((P_{H_0}A_1)|_{H_0}\right)^{-1} P_{H_0}$, the result follows by formula (6.28). $\square$

6.6 Corollary. *The mapping from the open set $A \cap E$, which sends $T \in A \cap E$ onto the mapping*

$$\text{“}]0,t_f[\to H_0 : t \mapsto \int_0^t \zeta_e(T)(s)\ ds\text{”}$$

is continuously Fréchet differentiable from the open set $A \cap E \subset E$ into $C([0,t_f];H_0)$ and thus a fortiori as a mapping with values in $L^2(0,t_f;H_0)$.

Proof : It suffices to compose the mapping which sends $T \in A \cap E$ onto $\zeta_e = \zeta_e(T) \in L^2(0,t_f;H_0)$, with the linear continuous mapping

$$L^2(0,t_f;H_0) \to C([0,t_f];H_0) : f \mapsto \text{“}]0,t_f[\to H_0 : t \mapsto \int_0^t f(s)\ ds\text{”}.$$

$\square$

6.7 Corollary. *The mapping from the open set $A \cap E$, which sends $T \in A \cap E$ onto $\xi = \xi(T) \in L^2(0,t_f;H_0)$ is continuously Fréchet differentiable from the open set $A \cap E \subset E$ into $L^2(0,t_f;H_0)$.*

Proof : It suffices to compose the mapping which sends $T \in A \cap E$ onto the couple $((\mathcal{I}-\tilde{A})^{-1}, \xi(.,0) + \int_0^{\cdot} \zeta_e(.,s)\ ds)$ belonging to

$$\mathcal{L}(L^2(0,t_f;H_0)) \times L^2(0,t_f;H_0)$$

with the bilinear continuous mapping

$$\begin{gathered}\mathcal{L}(L^2(0,t_f;H_0)) \times L^2(0,t_f;H_0) \to L^2(0,t_f;H_0)\\ (B,g) \mapsto Bg.\end{gathered}$$

□

6.8 Corollary. *The mapping from the open set $A \cap E$, which sends $T \in A \cap E$ onto $\sigma_0 \in L^2(0,t_f;H_0)$ is continuously Fréchet differentiable from the open set $A \cap E \subset E$ into $L^2(0,t_f;H_0)$. As a consequence, the mapping from $\tilde{U}_{ad}$ into $L^2(0,t_f;L^2(\Omega)^{3\times3})$ which sends T_S onto $\sigma = \sigma_0 + \sigma_e$ is also continuously Fréchet differentiable.*

Proof : The first assertion follows from the previous corollary, formula

$$\sigma_0(.,t) = \xi(.,t) - ((P_{H_0}A_1)|_{H_0})^{-1}(P_{H_0}A_1)\,\sigma_e(.,t),$$

which follows from formula (6.26) and Proposition 6.7. The second assertion follows from Proposition 6.7. □

Now, let us turn to $\overrightarrow{u}$. Recalling that $\vec{u}(t) := \int_0^t v(s)\ ds$, $\forall t \in [0,t_f]$ and integrating both sides of the first equation of (5.70) from 0 to t, we obtain:

$$\begin{aligned}\varepsilon(\overrightarrow{u}(.,t)) &= \tfrac{1+\nu}{E}\sigma(.,t) - \tfrac{1+\nu}{E}\zeta(.) - \tfrac{\nu}{E}tr(\sigma(.,t))I_3 + \tfrac{\nu}{E}tr(\zeta(.))I_3\\ &+(1+\nu)\int_0^t \tfrac{\sigma(.,s)}{\eta(T(.,s))}\ ds - \nu(\int_0^t \tfrac{tr(\sigma(.,s))}{\eta(T(.,s))}\ ds)\ I_3.\end{aligned} \tag{6.35}$$

Consequently, we have the following:

6.11 Proposition. *The mapping from the open set $A \cap E$ in E, which sends $T \in A \cap E$ onto $\varepsilon(\overrightarrow{u}) \in L^2(0,t_f;L^2(\Omega)^{3\times3})$ is continuously Fréchet differentiable from the open set $A \cap E \subset E$ into $L^2(0,t_f;L^2(\Omega)^{3\times3})$. As a consequence, the mapping from $\tilde{U}_{ad}$ into $L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ which sends T_S onto $\overrightarrow{u} \in L^2(0,t_f;H^1_{\Gamma_D}(\Omega)^3)$ is also continuously Fréchet differentiable.*

Proof : The first assertion is a consequence of the previous corollary and equality (6.35). The second assertion is a consequence of Korn's inequality ([5], (11.2.23) p.285 and Remark (11.2.27) p.286), which implies that the mapping which sends $\overrightarrow{w} \in H^1_{\Gamma_D}(\Omega)^3$ onto $\varepsilon(\overrightarrow{w})$ defines an isomorphism from $H^1_{\Gamma_D}(\Omega)^3$ onto a closed subspace of $L^2(\Omega)^{3\times 3}$. $\square$

6.9 Corollary. *The reduced cost functional*

$$\begin{aligned} \hat{J} : \tilde{U}_{ad} := \{T_S \in H^2(]0,t_f[); \tfrac{\bar{T}}{2} < T_S(t) < 2\bar{T},\ \forall t \in]0,t_f[, T_S(0) = T_a\} \to \mathbb{R} \\ T_S \mapsto \tfrac{1}{2}\int\limits_Q |\vec{u}(T_S)(x,t) - \vec{u}_d(x,t)|^2\ dx \otimes dt + \tfrac{\tilde{\delta}_r}{2} \left\| T_S - T_S^d \right\|^2_{H^2(]0,t_f[)} \end{aligned}$$

is also continuously Fréchet differentiable. In particular, if $\overline{T_S} \in U_{ad}$ is an optimal control, it satisfies the variational inequality:

$$\hat{J}'(\overline{T_S})(T_S - \overline{T_S}) \geq 0,\ \forall T_S \in U_{ad}.$$

Bibliography

[1] V. Agoshkov, “Boundary Value Problems for Transport Equations”, Springer Science + Business Media, LLC (1998).

[2] F. Bonnans, “Optimisation continue cours et problèmes corrigés ”, Dunod, Paris (2006).

[3] M.B. Boubaker, B. Le Corre, Y. Meshaka, G. Jeandel, “Finite Element simulation of the slumping process of a glass plate using 3-D generalized viscoelastic Maxwell Model”, Journal of Non-Crystalline Solids, Vol.405, pp.45-54 (2014).

[4] P.S. Brantley, E.W. Larsen, “The Simplified P_3 Approximation”, Nuclear Science and Engineering: **134**, 1-21 (2000).

[5] S.C. Brenner, L.R. Scott, *The Mathematical Theory of Finite Element Methods,* Second Edition, Texts in Applied Mathematics 15, Springer-Verlag (2002).

[6] H. Brézis, “Analyse fonctionnelle Théorie et applications”, Masson (1993).

[7] H. Brézis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer (2011).

[8] L. Boulaajine, *Méthode des éléments finis mixte duale pour les problèmes de l’élasticité et de l’élastodynamique: analyse d’erreur a priori et a posteriori*, Thèse de doctorat de l’Université de Valenciennes et du Hainaut-Cambrésis, France, soutenue le 10 juillet 2006.

[9] H. Brézis, *Opérateurs Maximaux Monotones et semi-groupes de contractions dans les espaces de Hilbert,* North-Holland Mathematical Studies 5, Notas de Matemática (50) Editor: Leopoldo Nachbin (1973).

[10] H. Cartan, *Calcul Différentiel,* Collection Méthodes, Hermann, Paris (1971).

[11] E. Casas, “Pontryagin’s Principle for State-Constrained Boundary Control Problems of Semilinear Parabolic Equations ”, SIAM J. Control. Optim. Vol. 35, No. 4, pp. 1297-1327, July 1997.

[12] M. Cessenat, “Théorèmes de trace L^p pour des espaces de fonctions de la neutronique”, C.R. Acad. Sc. Paris, t.299, Série I, n°16, 1984, pp. 831-834.

[13] M. Cessenat, “Théorèmes de trace L^p pour des espaces de fonctions de la neutronique”, C.R. Acad. Sc. Paris, t.300, Série I, n°3, 1985, pp. 89-92.

[14] D. Clever, J. Lang, “Optimal Control of Radiative Heat Transfer in Glass Cooling with Restrictions on the Temperature Gradient”, Optim. Control Appl. Methods 33, 157–175 (2012).

[15] D. Clever, J. Lang, “Optimal Control of Radiative Heat Transfer in Glass Cooling with Restrictions on the Temperature Gradient ”, Optim. Control Appl. Meth. (2011), Published online in Wiley Online Library, DOI:10|1002|oca.974.

[16] R. Dautray, J.-L. Lions, “Mathematical Analysis and Numerical Methods for Science and Technology Volume 1 Physical Origins and Classical Methods”, Springer-Verlag (2000).

[17] R. Dautray, J.-L. Lions, “Mathematical Analysis and Numerical Methods for Sciences and Technology, Vol. 2, Functional and Variational Methods ”, Springer-Verlag (2000).

[18] R. Dautray, J.-L. Lions, “Mathematical Analysis and Numerical Methods for Sciences and Technology, Vol. 5, Evolution Problems I ”, Springer-Verlag (2000).

[19] R. Dautray, J.-L. Lions, “Mathematical Analysis and Numerical Methods for Science and Technology Volume 6 - Evolution Problems II", Springer-Verlag (2000).

[20] J. Dieudonné, “Eléments d’Analyse, Tome III”, Chapitres XVI et XVII, Gauthier-Villars Editeur (1970).

[21] J. Dieudonné, *éléments d’analyse 1 / fondements de l’analyse moderne,* Gauthier-Villars, Editeur, Cahiers Scientifiques publiés sous la direction de M. Gaston Julia, Fascicule XXVIII, Paris (1968).

[22] J. Dieudonné, *éléments d’analyse 2,* Gauthier-Villars, Editeur, Cahiers Scientifiques publiés sous la direction de M. Gaston Julia, Fascicule XXXI, Paris (1969).

[23] J. Dieudonné, “Foundations of Modern Analysis Enlarged and Corrected Printing ”, Academic Press, New York (1969).

[24] M. P. do Carmo, “Differential Geometry of Curves and Surfaces”, Prentice-Hall (1976).

[25] L.C. Evans, “Partial Differential Equations”, GSM Volume 19, American Mathematical Society Providence, Rhode Island (1998).

[26] M. Farhloul, L. Paquet, *Refined Mixed Finite Element Method for the Elasticity Problem in a Polygonal Domain,* Numer. Methods Partial Differential Eq., 18, pp. 323-339 (2002).

[27] A. Friedman, “Partial Differential Equations”, Robert E. Krieger Publishing Company (1976).

[28] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method Theory and Applications,* SpringerBriefs in Mathematics, Springer (2014).

[29] V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations Theory and Algorithms,* SCM 5, Springer-Verlag (1986).

[30] P. Grisvard, “Commutativité de deux foncteurs d’interpolation et applications ”, J. Math. Pures Appliquées , 45, 143-290, (1966).

[31] P. Grisvard, *Elliptic Problems in Nonsmooth Domains,* Pitman (1985).

[32] J.-P. Henry, F. Parsy, “Cours d’élasticité”, Dunod Université (1982).

[33] B. Héron, F. Issard-Roch, C. Picard, “Analyse numérique Exercices et problèmes corrigés ”, Dunod, Paris (1999).

[34] B. Le Corre, “Thermoformage du verre Développement numérique d’un modèle thermomécanique”, Laboratoire d’Energétique et de Mécanique Théorique et Appliquée (UMR 7563), ED 409, Université de Lorraine (2014).

[35] D. Le Hardy, Y. Favennec, B. Rousseau, F. Hecht, “Specular reflection treatment for the 3-D radiative transfer equation solved with the discrete ordinates method”, Journal of Computational Physics 334 (2017), pp.541-572.

[36] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich, “*Optimization with PDE Constraints*”, Springer (2009).

[37] D. Kinderlehrer, G. Stampacchia, “An Introduction to Variational Inequalities and their Applications”, Academic Press (1980).

[38] A. Klar, J. Lang, M. Seaïd, “Adaptative solution of SP_N-approximations to radiative heat transfer in glass”, International Journal of Thermal Science, 44:1013-1023, (2005).

[39] J. Lang, “Adaptive computation for boundary control of radiative heat tranfer in glass”, Journal of Computational and Applied Mathematics 183 (2005), 312-326.

[40] E. Larsen, G. Thömmes, A. Klar, M. Seaïd, T. Götz, “Simplified P_N-approximations to the Equations of Radiative Heat Transfer and Applications”, Journal of Computational Physics, 183, pp. 652-675 (2002).

[41] J.L. Lions, “*Quelques méthodes de résolution des problèmes aux limites non linéaires*”, Dunod (1969).

[42] J.L. Lions, E. Magenes, “*Non-Homogeneous Boundary Value Problems and Applications*”, Volume I, Springer-Verlag (1972).

[43] J.L. Lions, E. Magenes, “*Non-Homogeneous Boundary Value Problems and Applications*”, Volume II, Springer-Verlag (1972).

[44] C.-M. Marle, Mesures et probabilités, Collection Enseignement des sciences 19, Hermann (1974).

[45] D. Mitrović, D. Žubrinić, “Fundamentals of applied functional analysis Distributions - Sobolev spaces - nonlinear elliptic equations”, Pitman Monographs and Surveys in Pure and Applied Mathematics 91, Addison Wesley (1998).

[46] M.F. Modest, “Radiative Heat Transfer”, second edition, Academic Press (2003).

[47] J. Nečas, “Les Méthodes Directes en Théorie des Equations Elliptiques”, Masson et C^{ie} (1967).

[48] L. Paquet, “Radiative Heating of a Glass Plate: the semi-discrete problem”, accepted for publication in Afrika Matematika (13 January 2018), published online first (24 January 2018) https://doi.org/10.1007/s13370-018-0542-z.

[49] L. Paquet, “Control of the Radiative Heating of a Glass Plate”, Afrika Matematika 27, 673-699 (2016).

[50] L. Paquet, R. El Cheikh, D. Lochegnies, N. Siedow, “*Radiative Heating of a Glass Plate*”, Mathematics In Action, Vol. 5, 1-30 (2012).

[51] L. Paquet and H. Nabolsi, *The Radiative Transfer Equation with a Nonhomogeneous Reflectivity Boundary Condition Coupled with the Heat Conduction Equation*, submitted to Mathematical Methods in the Applied Sciences the 26th of August 2016.

[52] L. Paquet, H. Nabolsi and A. Wehbe, *Control of the Radiative Heating of a Semi-Transparent Body*, preprint of the LAMAV, UVHC, July 11, 2016.

[53] M.H. Parsa, M. Rad, M.R. Shahhosseini, M.H. Shahhosseini, “Simulation of windscreens bending using viscoplastic formulation”, Journal of Materials Processing Technology 170, pp. 298-303, (2005).

[54] D. Pascali, S. Sburlan, “Nonlinear mappings of monotone type”, Editura Academiei, Bucuresti, Romania, Sijthoff & Noordhoff International Publishers (1978).

[55] R. Pinnau, “Analysis of optimal boundary control for radiative heat ransfer modeled by the SP1-system ”, Communications in Mathematical Sciences, 5,4:951-969, (2007).

[56] W. Potze, “Radiation heat transfer in axisymmetric quartz glass tubes”, Journal of Quantitative Spectroscopy & Radiative Transfer 84 (2004), pp. 575-586.

[57] M. Renardy, R.C. Rogers, “An Introduction to Partial Differential Equations”, Texts in Applied Mathematics 13, Springer-Verlag (1993).

[58] P. Reuss, “La Neutronique”, Presses Universitaires de France, “Que sais-je ?” 3307, (1998).

[59] M. E. Rognes and R. Winther, *Mixed Finite Element Methods for Linear Viscoelasticity using Weak Symmetry,* Mathematical Models and Methods in the Applied Sciences, 20(6), pp. 955-985 (2010).

[60] G.W. Scherer, “Relaxation in Glass and Composites”, Krieger Publishing Company, Malabar, Florida (1992).

[61] L. Schwartz, *Cours d’analyse I*, Hermann, Paris (1967).

[62] R. Sentis, “Equations de transport avec conditions aux limites de type réflexion”, Rapport de Recherche N°162, Septembre 1982, HAL Id: inria-00076397.

[63] N. Siedow, T. Grosan, D. Lochegnies, E. Romero, “Application of a New Method for Radiative Heat Tranfer to Flat Glass Tempering”, *J. Am. Ceram. Soc.,* **88** [8] 2181-2187 (2005).

[64] N. Siedow, D. Lochegnies, F. Béchet, P. Moreau, H. Wakatsuki, N. Inoue “Axisymmetric modeling of the thermal cooling, including radiation, of a circular glass disk”, *International Journal of Heat and Mass Transfer,* **89** 414-424 (2015).

[65] L. Soudre, “Etude numérique et expérimentale du thermoformage d’une plaque de verre”, thèse soutenue le 9 décembre 2008, Nancy-Université Henri Poincaré, laboratoire LEMTA, tel-00382905, version 1-11 May 2009.

[66] V. Smirnov, “Cours de Mathématiques Supérieures, Tome II”, Editions Mir, Moscou (1970).

[67] S.S. Sritharan (editor), "Optimal Control of Viscous Flow", Society for Industrial and Applied Mathematics, siam (1998).

[68] J. Taine, F. Enguehard, E. Iacona, Transferts thermiques Introduction aux transferts d'énergie, 5^e édition, Dunod (2014).

[69] H. Triebel, "Interpolation theory, Function spaces, Differential Operators ", North-Holland Mathematical Library, Volume 18, North-Holland Publishing Company (1978).

[70] D.I. Tomašević, E.W. Larsen, "The Simplified P_2 Approximation", Nuclear Science and Engineering: **122**, 309-325 (1996).

[71] F. Tröltzsch, "Optimal Control of Partial Differential Equations Theory, Methods and Applications", Translated from the German text "Optimale Steuerung Partieller Differentialgleichungen" (2005) by Jürgen Sprekels, GSM Volume 112, Applied Mathematics, AMS Providence, Rhode Island (2010).

[72] V.S. Vladimirov, "Mathematical Problems in the One-Velocity Theory of Particle Tranport", Trudy Matematischeskogo Institute imeni V.A. Steklova, LXI (1961), translated from Transactions of the V.A. Steklov Mathematical Institute, 61 (1961), A Translation by Atomic Energy of Canada Limited.

[73] K. Yosida, "Functional Analysis", Third Edition, Springer-Verlag (1971).