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Mathematical analysis of a model of partial differential equations describing the adaptation of mosquitoes facing the usage of insecticides

Linlin Li

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THÈSE

PRÉSENTÉE À

L'UNIVERSITÉ DE BORDEAUX

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET D'INFORMATIQUE

par **Linlin LI**

POUR OBTENIR LE GRADE DE

DOCTEUR

SPÉCIALITÉ : Mathématiques

Analyse mathématique d'un modèle d'équations aux dérivées partielles décrivant l'adaptation des moustiques face à l'usage des insecticides

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Devant la commission d'examen composée de :

Mostafa ADIMY	INRIA Grenoble Rhône-Alpes	Rapporteur
Antoine PERASSO	Université de Franche-Comté	Rapporteur
Arnaud DUCROT	Université de Bordeaux	Examinateur
Marius TUCSNAK	Université de Bordeaux	Président du jury
Bedr'Eddine AINSEBA	Université de Bordeaux	Directeur de thèse

– 2018 –

Analyse mathématique d'un modèle d'équations aux dérivées partielles décrivant l'adaptation des moustiques face à l'usage des insecticides

(Mathematical analysis of a model of partial differential equations describing the mosquitoes plasticity)

Linlin Li

Institut de Mathématiques de Bordeaux UMR 5251
Université de Bordeaux
351, cours de la Libération - F 33 405 TALENCE

Résumé

Dans cette thèse on s'intéresse à un modèle mathématique décrivant l'adaptation du développement des populations de moustiques face à l'usage intensif des insecticides durant la nuit (moustiquaires imprégnées, répulsifs en spray, répulsifs avec diffuseur électrique, ...).

Le modèle proposé dans cette thèse est structuré en âge et dépend du temps/moment où le moustique pique pour prendre son repas. Ceci nous conduit à des modèles du type ultra parabolique. Le terme de renouvellement de la population de moustiques est non-local, comme pour tous les problèmes démographiques, mais comporte ici un noyau qui permet à la nouvelle génération d'adapter son temps de piqure (repas). Ceci est dû à la sélection de certains moustiques qui piquent plus tôt ou plus tard que les autres moustiques, suite à la pression imposée par l'usage intensif des pesticides à l'intérieur des habitats et en particulier durant la nuit. Les conditions aux bords par rapport au moment de piqure (repas) seront périodiques car selon les espèces, les moustiques prennent toujours leurs repas au même moment de la journée.

Le modèle non linéaire de diffusion par âge non local avec conditions aux limites périodiques est donné par:

$$\begin{cases} Dp - \delta\Delta p + \mu p = m(x)u(a,t,x)p, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases} \tag{0.1}$$

où $Q_{a_\dagger} = (0,a_\dagger) \times (0,T) \times (0,24)$ et

$$Dp(a,t,x) = \lim_{\varepsilon \to 0} \frac{p(a+\varepsilon, t+\varepsilon, x) - p(a,t,x)}{\varepsilon}$$

est la dérivée directionnelle de p par rapport à la direction $(1, 1, 0)$.

Les principaux résultats peuvent être classés dans 4 parties.

Dans une première partie, nous étudions le problème structuré par l'âge (0.1) modélisant la plasticité des moustiques dans un environnement naturel, c'est-à-dire $u \equiv 0$. Nous étudions d'abord le spectre d'un opérateur $\mathbb{A}$ qui est le générateur infinitésimal d'un C_0-semigroupe. On montre que l'opérateur n'a qu'une seule valeur propre réelle λ_0 algébriquement simple et est plus grande que n'importe quelle partie réelle des autres valeurs propres.
Par la théorie des semigroupes, on montre que le semigroupe $T(t)$ généré par $\mathbb{A}$ a l'expression asymptotique

$$\begin{aligned} T(t)\phi(a,x) =& e^{\lambda_0 t} e^{-\lambda_0 a} \mathcal{T}(0,a) C_{\lambda_0} \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) \int_0^a e^{-\lambda_0 (a-\delta)} \\ & \mathcal{T}(\delta,a)\phi(\delta,s) d\delta ds da + o(e^{(\lambda_0-\varepsilon)t}) \end{aligned}$$

où $\mathcal{T}(\tau,s) = e^{-\int_\tau^s \mu(\rho)d\rho} e^{\mathbb{B}(s-\tau)}$, $C_{\lambda_0} = \lim\limits_{\lambda \to \lambda_0} (\lambda - \lambda_0)(\mathbb{I} - \mathcal{B}_\lambda)^{-1}$ et ε est un nombre positif tel que $\sigma(\mathbb{A}) \cap \{\lambda | \lambda_0 - \varepsilon \le Re\lambda \le \lambda_0\} = \lambda_0$.

Grâce à ce développement asymptotique, nous montrons que si $\lambda_0 > 0$, il n'y a pas de solution d'équilibre stable non négative; quand $\lambda_0 = 0$, il existe une infinité de solutions d'équilibre stables non triviales; quand $\lambda_0 < 0$, seules des solutions d'équilibre régulières triviales 0 existent.

Enfin, nous présentons des simulations numériques pour la population de moustiques lors de l'usage de moustiquaires imprégnées d'insecticide. On peut voir que l'utilisation de moustiquaires imprégnées d'insecticide peut amener les moustiques à changer leurs habitudes et leur temps de piqure.

Dans la deuxième partie, nous étudions le problème de contrôle optimal de (0.1) avec $m(x) \equiv 1$. Notre objectif principal est de prouver qu'il existe un contrôle optimal $u(a,t,x)$ dans le cas d'usage limité d'insecticides, c'est-à-dire que $u(a,t,x)$ est limité par deux fonctions $\varsigma_1(a,t,x)$ et $\varsigma_2(a,t,x)$.
Le problème de contrôle optimal se formule comme suit:

$$(OH) \qquad Maximize \left\{ -\int_{Q_{a_\dagger}} u(a,t,x) p^u(a,t,x) dt dx da \right\},$$

soumis à $u(a,t,x) \in U$,

$$U = \{u(a,t,x) \in L^2(Q_{a_\dagger}) | \ \varsigma_1(a,t,x) \le u(a,t,x) \le \varsigma_2(a,t,x) \ a.e. \ in \ Q_{a_\dagger}\},$$

où $\varsigma_1(a,t,x), \ \varsigma_2(a,t,x) \in L^\infty(Q_{a_\dagger}), \ \varsigma_1(a,t,x) \le \varsigma_2(a,t,x) \le 0 \ a.e. \ in \ Q_{a_\dagger}$ et

$p^u(a,t,x)$ est la solution du modèle EDP.

Nous montrons d'abord l'existence de solutions ainsi que le principe de comparaison pour un système généralisé lié à notre problème d'âge structuré. Ensuite, nous prouvons l'existence du contrôle optimal pour la meilleure récolte (OH). Enfin, nous établissons les conditions nécessaires d'optimalité.

Dans la troisième partie, nous étudions la contrôlabilité exacte locale d'un problème structuré en âge modélisant la capacité des vecteurs du paludisme à changer leur temps de morsure et éviter les conditions environnementales stressantes engendrées par l'utilisation des pulvérisateurs à l'intérieur des habitations et des moustiquaires imprégnées d'insecticide. L'existence d'une solution stable non positive $p_s(a,x)$ a été obtenue par les résultats dans la première partie. Nous cherchons à trouver un contrôle tel que la solution $p(a,t,x)$ du problème structuré par âge puisse être égale à la solution stable $p_s(a,x)$ après un temps limité, disons $p(a,T,x) = p_s(a,x)$. Pour ce faire, nous transformons le problème en un problème de zéro exacte contrôlabilité, c'est-à-dire

$$\begin{cases} Dp - \delta\Delta p + \mu(a)p = m(x)u(a,t,x)(p+p_s), & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = \overline{p}_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases}$$

où $\overline{p}_0(a,x) = p_0(a,x) - p_s(a,x)$.

Nous établissons une nouvelle inégalité de Carleman pour (0.1). Ensuite, nous démontrons notre résultat en étudiant l'équation d'Euler-Lagrange. Finalement, nous faisons une simulation numérique en comparant la situation sans contrôle avec celle oú il y a un contrôle spécifique. Nous pouvons trouver que quand il n'y a pas de contrôle, la solution $p(a,t,x)$ ne peut pas être proche de la solution stable $p_s(a,x)$ et nous pouvons trouver un contrôle tel que la solution $p(a,t,x)$ est proche de $p_s(a,x)$ à $t = T$.

Dans la quatrième partie, nous nous intéressons au problème de la plasticité du moustique dans le cas de mortalité non linéaire, c'est-à-dire

$$\begin{cases} Dp - \delta\Delta p + \mu(a,w(t,x))p = u(a,w(t,x))p, & (a,t,x) \in Q, \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times \mathbb{R}. \end{cases}$$

où $Q = (0, a_\dagger) \times \mathbb{R}^+ \times \mathbb{R}$. Nous supposons que le taux de fécondité est important et que le taux de mortalité naturel est faible. Nous étudions ensuite le comportement en temps long de la population mature $w(t,x) = \int_{a_0}^{a_\dagger} p(a,t,x)da$ où a_0 est l'âge de maturité, sous différentes stratégies de contrôle. Premièrement, nous prouvons que s'il n'y a qu'un contrôle limité avec les insecticides, la population mature de moustiques ira à l'infini. D'un autre côté, si le contrôle insecticide est très important, cela signifie que le taux de mortalité total $\mu(a,w) - u(a,w)$ est important. Ensuite, la population va doucement tendre vers 0. En fait, une situation plus réaliste est que la population ne peut pas être infiniment grande ou très petite en raison de la limitation de la stratégie de contrôle par insecticides. Cela signifie que la population mature $w(t,x)$ peut atteindre certains états d'équilibré. Ainsi, dans le cas intermédiaire, nous dérivons un modèle temporel pour la population mature, c'est-à-dire

$$\begin{aligned} w_t - \delta\Delta w = &\int_{a_0}^{a_\dagger} (-\mu(a,w) + u(w))p(a,t,x)da \\ &+ M\beta \int_{-\infty}^{+\infty} \left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0, x-z-s)dz\right] f_{\delta a_0}(s)ds, \end{aligned}$$

pour $t > 0$, $x \in \mathbb{R}$ et

$$w(s,x) = \int_{a_0}^{a_\dagger} p_0(a+s,x)da, \text{ for } s \in [-a_0, 0] \text{ and } x \in \mathbb{R},$$

qui peut être gouverné par une sous-équation

$$\begin{aligned} w_t - \delta\Delta w \geq &- g(w) + u(w)w \\ &+ M\beta \int_{-\infty}^{+\infty} \left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0, x-z-s)dz\right] f_{\delta a_0}(s)ds, \end{aligned}$$

et une super-équation

$$\begin{aligned} w_t - \delta\Delta w \leq &u(w)w \\ &+ M\beta \int_{-\infty}^{+\infty} \left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0, x-z-s)dz\right] f_{\delta a_0}(s)ds. \end{aligned}$$

Nous prouvons l'existence de fronts de déplacement pour la sous-équation et l'utilisons pour prouver que la population mature atteindra finalement un équilibre entre les états positifs de la sous-équation et de la super-équation.

Pour conclure cette dernière section nous effectuons des simulations numériques

pour illustrer nos résultats théoriques.

Mots clés: modèle structuré en âge, comportement asymptotique, contrôlabilité, résistance comportementale, méthode adjointe, contrôle optimal

Abstract

This dissertation is concerned with an age structured problem modelling mosquito plasticity. The main results can be divided into four parts.

The first part presents an age structured problem modelling mosquito plasticity in a natural environment. We first investigate the analytical asymptotic solution through studying the spectrum of an operator $\mathbb{A}$ which is the infinitesimal generator of a C_0-semigroup. Additionally, we get the existence and nonexistence of nonnegative steady solutions under some conditions.

In the second part, we study the optimal control of an age structured problem. Firstly, we prove the existence of solutions and the comparison principle for a generalized system. Then, we prove the existence of the optimal control for the best harvesting. Finally, we establish necessary optimality conditions.

In the third part, we investigate the local exact controllability of an age structured problem modelling the ability of malaria vectors to shift their biting time to avoid the stressful environmental conditions generated by the use of indoor residual spraying (IRs) and insecticide-treated nets (ITNs). We establish a new Carleman's inequality for our age diffusive model with non local birth processus and periodic biting-time boundary conditions.

In the fourth part, we model a mosquito plasticity problem and investigate the large time behavior of matured population under different control strategies. Firstly, we prove that when the control is small, then the matured population will become large for large time and when the control is large, then the matured population will become small for large time. In the intermediate case, we derive a time-delayed model for the matured population which can be governed by a sub-equation and a super-equation. Finally, we prove the existence of traveling fronts for the sub-equation and use it to prove that the matured population will finally be between the positive states of the sub-equation and super-equation.

Keywords: age structured model, asymptotic behaviour, controllability, behavioural resistance, adjoint method, optimal control

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Contents

Chapter 1

Introduction

Throughout the human history, people have always been combating against many infectious diseases, such as malaria, dengue, yellow and Chikungunya fever, encephalitis and the diseases have caused uncounted mortality of mankind. One of the most studied diseases is malaria, which is mainly transmitted by Anopheles gambiae and Anopheles funestus and is caused by a species of parasite that belongs to the genus Plasmodium [18] . This pathology affects millions of people over the world, being predominant in equatorial region, e.g., Amazon rainforest, sub-saharan Africa and South East Asia. The Plasmodium is transmitted by female Anopheles mosquitoes when they bite and, thus, feed on human blood. As the statistical data show, malaria affects more than 100 tropical countries, placing 3.3 billion people at risk [98] and one African child's life is taken by malaria every minute [99].

During the past decades, many researchers studied the pathology of these infectious diseases and tried to control the transmission of them. To reduce human's suffering from malaria, people have been seeking efficient ways to control the malaria transmission for many years. The main strategies of controlling malaria are insecticide treated nets (ITNs) and indoor residual spraying (IRs) [18, 50, 68, 98]. Control mechanims acting on disease dynamics take into account the behaviourally characteristics of mosquito population, such as anthrophagy, endophily, endophagy, physiological susceptibility to pyrethroids, and night-biting preference. The effectiveness of these strategies depends on the susceptibility of the vector species to insecticides and their behaviour, ecology and population genetics [87]. Therefore, in the past decades, pyrethroid-treated bed nets are widely deployed in Africa, the estimated percentage of households with one impregnated net having increased from 3% in 2000 to 53% to 2012 [99]. By using these strategies, we see that the control of malaria has made slow but steady progress and the overall mortality rate has dropped by more than 25% since 2000 [68].

However, mosquitoes are adapting due to insecticide pressure related to (ITNs) and (IRs) usage. The recent reports on Malaria transmission shown that the long-term use of residual spraying (IRs) and insecticide-treated nets (ITNs) has been driving mosquito physiological and behavioural resistance [26, 28, 76, 87, 89, 96]. Many mosquito species exhibit high levels of phenotypic plasticity that can be expressed on host preference, biting activity, etc. Such heritable phenotypic plasticity allows individuals mosquitoes to flexibly adapt their behaviour according to the environmental conditions. The development of a crepuscular, outdoor feeding phenotype among anopheline population has been observed in areas of intensive use of IRs and ITNs. This adjust on biting time can jeopardize the success of Malaria control and promotes parasite evolution [38] and is more difficult to avoid than physiological ones that can be controlled by taking the form of rotation of a different class of insecticide [77].

Whereas, more realistic problems are always related to the bitting time. Thus, one of the most important thing we need to pay attention to is that the long-term use of residual spraying (IRs) and insecticide-treated nets (ITNs) leads to the emergence of insecticide-resistant Anopheles mosquitoes [28, 76, 96] which has great influence on the biting time of mosquitoes. Formerly, the maximum of anophelines aggressiveness was typically observed in the middle of night. In 1990, Fontenille [35] showed that the An. funestus biting peak occurred from 01:00 to 03:00 indoors and from 02:00 to 05:00 outdoors. But, after the implementation of long-lasting insecticidal nets (LLINs), Anopheles funestus showed a behavioural change in biting activity that An. funestus reached a peak of aggressiveness between 08:00 and 9:00, remaining anthropophilic and endophilic, while adopting diurnal feeding, essentially on humans [87]. Meanwhile, in the study [103], Anopheles arabiensis showed early biting activities in Ethiopia after use of (LLINs)both indoors and outdoors; and 80 percent of this vector were captured before 22:00 with a peak activity between 19:00 to 20:00. In southern Benin, the researchers obtained the similar results that 26.4% of An. funestus were caught after 06:00 by scaling up of universal coverage with (LLINs). And, in the recent papers [70, 74, 87, 103] showed substantial diurnal and early biting activity and more frequent outdoor biting. All these studies showed An. funestus has adapted its biting time to the new situation.

The emergence of the new biting time behavioural adaptation of mosquitoes in response to insecticidebased vector control interventions may make control tools ineffective, close awareness in the context of pre-elimination of malaria and also constitute a risk to people who are so accessible. Since mosquito behaviour is an essential component for assessing vectorial capacity to transmit malaria, the emergence of the new biting time behaviour can significantly increase the risk for malaria transmission and represents a new challenge for malaria control. One of the most challenging problems in science is to model biological phenomena. The great number of parameters involved in the dynamics of a biological population makes deduction of a general model quite difficult.

Hence, it is very important and necessary to consider the bitting time in the following researches of the control of mosquitoes and researches on the population dynamics of mosquitoes become essential. Here it is intended to develop a mathematical model based on partial differential equations to understand and study possible adaptations of Anopheles species.

Therefore, in this dissertation, we are going to model mosquito population adaption by additional vector control strategies about the bitting time. This model is an ultra-parabolic partial differential equation with nonlocal terms corresponding to birth and selection processus. The density of mosquitoes $p(a,t,x)$ will depend on time t, age a and also bitting time x. The introduction of the variable x in the system has the objective of illustrate mosquito biting behaviour, which will be of great importance in the following research on mosquitoes control. We suppose that the continuous usage of (ITNs) and (IRs) can reduce the fitness of mosquito populations by reducing its oviposition rate or increasing its mortality rate. The model also incorporates the idea that individuals have some amount of plastic adaptability that permits mosquito-adaptation to stressful situations and these stress-induced modifications are inherited. Selection also occurs in the renewal process allowing persistence of the adapted species and maximizing the mosquito fitness. The new generation of mosquitoes can adapt to ensure its survival and reproduction, changing the biting time in order to maximize its fitness. As for Chapters 2-4 of this dissertation, we first consider a linear model describing the dynamics of a single species population with age dependence, bitting time dependence and spatial structure as follows

$$\begin{cases} Dp - \delta\Delta p + \mu p = m(x)u(a,t,x)p, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases} \tag{1.1}$$

where $Q_{a_\dagger} = (0, a_\dagger) \times (0, T) \times (0, 24)$ and

$$Dp(a,t,x) = \lim_{\varepsilon \to 0} \frac{p\left(a+\varepsilon, t+\varepsilon, x\right) - p\left(a,t,x\right)}{\varepsilon}$$

is the directional derivative of p with respect to direction $(1,1,0)$. For $p(a,t,x)$ smooth enough, it is easy to know that

$$Dp(a,t,x) = \frac{\partial p(a,t,x)}{\partial t} + \frac{\partial p(a,t,x)}{\partial a}.$$

Here, $p(a,t,x)$ is the distribution of individuals of age $a \geq 0$ at time $t \geq 0$ and bitting at time $x \in [0, 24]$, $a_\dagger$ means the life expectancy of an individual and T is a positive constant. $m(x)$ is the characteristic function of ω, where $\omega \in (0, 24)$ is a nonempty open subset. As we announced, the mosquitoes can adapt their bitting time. Thus, we set their adapting model to be a Δ diffusion with a diffusive coefficient δ. Moreover, $\beta(a)$ denotes the natural fertility-rate of individual of age a and $\mu(a)$ denotes the natural death-rate (more assumptions will be made later). In fact, the new generation is also able to adapt the bitting time in order to maximize its fitness. Let η be the maximum bitting time difference which the new generation can reach and we model the adaption of the new generation by a kernel $K(x,s)$ as defined as below

$$K(x,s) = \begin{cases} (x-s)^2 e^{-(x-s)^2}, & s \in (0,24), \\ 0, & \text{else}. \end{cases}$$

The control function $u(a,t,x)$ represents the insecticidal effort, such as the use of (ITNs) and (RIs).

The study of population dynamics equations can be traced back to the works of Malthus [66] in 1798. Malthus introduced the simplest population dynamics model which he supposed that the rate of population growth is proportional to the size of the population, that is,

$$P'(t) = \alpha P(t), t > 0,$$

where $\alpha \in \mathbb{R}$ is the intrinsic growth constant. An more realistic improved model was proposed by Verhulst [92] in 1838, that is,

$$P'(t) = \alpha P(t) - \gamma P^2(t), t > 0,$$

where $\alpha \in \mathbb{R}$ is the intrinsic growth constant, while $\frac{\alpha}{\gamma}$ is called the environmental carrying capacity ($\gamma > 0$). But, one of the deficiencies of the above ordinary differential equation models is that they do not take into account an age structure which can influence population size and behaviour in realistic situations. In 1911, one of the most important improvements is that Sharpe and Lotka [81] first proposed the age-structured continuous model without diffusion as follows

$$Dp(a,t) + \mu(a)p(a,t) = 0, a \in (0, a_\dagger), t > 0,$$

where $a_\dagger$ is the maximal age for the population species and $\mu(a)$ is the mortality rate and depends only on age a. Whereas, more realistic problems are always related to some general continuous models with diffusion for age-structured populations. Therefore, in the following few decades, the study of age-dependent population dynamics with diffusion has been intensively developed by mathematical researchers. Some of the main results of this development need to be mentioned. In 1973, Gurtin and MacCamy [44] took into consideration diffusion for age-structured populations and extended in [45] after 4 years. Under appropriate boundary conditions, Gurtin and MacCamy investigated the following problem

$$Dp(a,t,x) + \mu(a, P(t,x))p(a,t,x) - k\Delta p(a,t,x) = 0,$$

$$P(t,x) = \int_0^{a_\dagger} p(a,t,x)da,$$

$$p(0,t,x) = \int_0^{a_\dagger} \beta(a, P(t,x))p(a,t,x)da,$$

where $(a,t,x) \in (0,a_\dagger) \times (0,+\infty) \times \Omega$, $\Omega \subset \mathbb{R}^n$. Another important study worth to mention is that a book of analysis and control of age-dependent population dynamics was written by Aniţa [13] in 2000. This book is the first book devoted to the control of continuous age structured population dynamics and it introduces the most important problems, approaches and results in the mathematical theory of age-dependent models. In this book, Aniţa [13] studied the following system from many interesting aspects

$$\begin{cases} Dp(a,t,x) - k\Delta p + \mu(a,t,x)p = f(a,t,x), & (a,t,x) \in Q_T, \\ \frac{\partial p}{\partial \nu}(a,t,x) = 0, & \text{on } \Sigma_T, \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p(a,t,x)da, & (t,x) \in (0,T) \times \Omega, \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times \Omega, \end{cases}$$

where $Q_T = (0,a_\dagger) \times (0,T) \times \Omega$, $\Sigma_T = (0,a_\dagger) \times (0,T) \times \partial\Omega$. In the recent years, there are many works devoted to an age structure model with and without diffusion [3, 4, 5, 6, 7, 13, 34, 105]. One important direction is the optimal control of an age structured problem which has been widely studied, one refers to [11, 12, 15, 34, 53, 58, 59, 105]. Meanwhile, the study of the controllability of partial differential equations of an age structure model also plays a very important role, one can refer to [16, 31, 36, 57, 90]. For an overview on age structured population dynamics models and their mathematical analysis, we refer to [37, 51, 55, 95].

In Chapters 2-4 of this dissertation, we mainly focus on the study of the following three aspects of system (1.1):

1 mathematical analysis of this model;
2 optimal control of this model;
3 local exact controllability of this model.

In Chapter 5 of this dissertation, we then adapt the model (1.1) to a non-linear one by setting that the natural death-rate μ and the control term u are depending on age a and the matured population $\int_{a_0}^{a_\dagger} p(a,t,x)da$, where a_0 is the matured age. In Chapter 5, we mainly focus on the large-time behavior of the matured population of mosquitoes by the usage of ITNs and IRs. We can see detailed presentation of the model and settings in Section 1.4.

In the following of this chapter, we state the main results of this dissertation.

1.1 Mathematical analysis of an age structured problem

It is well known that mathematical analysis of an age structured problem modeling phenotypic plasticity in mosquito behaviour in a natural environment plays an important role in the study of the control of mosquitoes. Therefore, in Chapter 2, we consider the system (1.1) corresponding to $u(a,t,x) \equiv 0$ to model the plasticity of mosquitoes in a natural environment, namely without any intervention of human activities, such as (IRs) and (ITNs). That is,

$$\begin{cases} Dp - \delta\Delta_x p + \mu(a)p = 0, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T)\times(0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger)\times(0,24). \end{cases} \tag{1.2}$$

Notice that in our model, the boundary condition is assumed to be periodic and the fertility term is nonlocal with the kernel $K(x,s)$. In fact, both Dirichlet boundary condition and local fertility term are very popular in mathematical modeling, such as dynamics population models of a single species with age dependence and spatial structure. We now review some known results about such models, that is, replacing the periodic boundary condition and the fertility term by the Dirichlet condition and $\int_0^{a_\dagger} \beta(a)p(a,t,x)da$ respectively. Chan and Guo [22] considered this model in the semigroup framework, by setting the fertility-rate β and the mortality-rate μ being independent of the space variable x, that is,

$$\begin{cases} Dp - K\Delta_x p + \mu(a)p = 0, & (a,t,x) \in (0,a_\dagger) \times (0,T) \times \Omega, \\ p(a,t,x)|_{\partial\Omega} = 0, & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a)p(a,t,x)da, & (t,x) \in (0,T) \times \Omega, \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times \Omega, \end{cases}$$

where Ω is a limited smooth domain in $\mathbb{R}^n$. They identified the infinitesimal generator and studied its spectral properties, which could be used to get the asymptotic behavior of the solutions. Then, Guo and Chan [42] removed the independence setting of β, μ, that is,

$$\begin{cases} Dp - K\Delta_x p + \mu(a,x)p = 0, & (a,t,x) \in (0,a_\dagger) \times (0,T) \times \Omega, \\ p(a,t,x)|_{\partial\Omega} = 0, & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a,x)p(a,t,x)da, & (t,x) \in (0,T) \times \Omega, \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times \Omega. \end{cases}$$

They got the asymptotic expression of the solution by analyzing the spectrum of the infinitesimal generator. We also refer to the works of Langlais [56], for the study of the long-time behaviour of the model where β and μ depend on the distribution p. The controllability problems on this model are also very attractive. Ainseba and Aniţa [5, 13] studied the local exact controllability of such model with the Dirichlet boundary condition and the local fertility term. The control problem with Neumann boundary condition can be referred to [6, 7].

We are interested in the ways on which Guo and Chan [22, 42] studied the asymptotic behaviour of the population model in [22, 42] throught the analysis of the spectrum of the infinitesimal generator and using some positive semigroup theories. In this dissertation, we mainly focus on the asymptotic behavior in Chapter 2. The key step for our paper is to find, for any initial $p_0(a,x) \in D(\mathbb{A})$, the asymptotic expression $p(a,t,x)$.

Before presenting our results, we need to introduce some useful notations. Let $X = L^2((0,a_\dagger)\times(0,24))$ with the usual norm and the operator $\mathbb{A} : X \longrightarrow X$ defined as

$$\mathbb{A}\phi(a,x) = -\frac{\partial\phi(a,x)}{\partial a} + \delta\Delta\phi(a,x) - \mu(a)\phi(a,x), \forall\phi(a,x) \in D(\mathbb{A}), \quad (1.3)$$

where

$$\begin{aligned} D(\mathbb{A}) = \{\phi(a,x)|\phi, \mathbb{A}\phi \in X, \phi(a,0) = \phi(a,24), \partial_x\phi(a,0) = \partial_x\phi(a,24), \\ \phi(0,x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)\phi(a,s)dsda\}. \end{aligned} \quad (1.4)$$

From the definition of the operator $\mathbb{A}$, the system (1.2) can be transformed into an evolutionary equation on the space X:

$$\begin{cases} \dfrac{dp(a,t,x)}{dt} = \mathbb{A}p(a,t,x), \\ p(a,0,x) = p_0(a,x). \end{cases}$$

For the following notations, we can refer to Marek [67, p.609] and Clement [25] for instance. If $\mathbb{A}$ is a linear operator from X into X, then $\rho(\mathbb{A})$ denotes the resolvent set of $\mathbb{A}$, that is, $\rho(\mathbb{A})$ is the set of all complex numbers λ for which $(\lambda\mathbb{I}-\mathbb{A})^{-1}$ is a bounded automorphism of $\mathbb{A}$ (let $R(\lambda, A) = (\lambda\mathbb{I}-\mathbb{A})^{-1}$ called the resolvent operator), where $\mathbb{I}$ denotes the identity operator. The complement of $\rho(\mathbb{A})$ in the complex plane is the spectrum of $\mathbb{A}$, and it is denoted by $\sigma(\mathbb{A})$. We denote by $\gamma(\mathbb{A})$ the spectral radius of $\mathbb{A}$, that is,

$$\gamma(\mathbb{A}) = \sup\{|\lambda| : \ \lambda \in \sigma(\mathbb{A})\}.$$

If $\mathbb{A}$ is an infinitesimal generator of a C_0-semigroup $T(t)$ on the space X, the spectral bound $s(\mathbb{A})$ can be denoted by

$$s(\mathbb{A}) = \sup\{|\lambda| : \ Re\lambda \in \sigma(\mathbb{A})\}.$$

And the growth bound of the semigroup $T(t)$ can be shown as

$$\omega(\mathbb{A}) = \inf_{t>0} \frac{1}{t} \log \|T(t)\|_{L^2((0,a_\dagger)\times(0,24))} = \lim_{t\to+\infty} \frac{1}{t} \log \|T(t)\|_{L^2((0,a_\dagger)\times(0,24))}.$$

We assume the following assumptions throughout Chapter 2:

(J1) $\mu(a) \in L^1_{loc}([0, a_\dagger))$ and $\displaystyle\int_0^{a_\dagger} \mu(\rho)d\rho = \infty$;
(J2) $\beta(a) \in L^\infty((0, a_\dagger))$, $\text{mes}\{a|a \in [0, a_\dagger], \beta(a) > 0\} > 0$;
(J3) $p_0(a, x) \in L^\infty((0, a_\dagger) \times (0, 24))$, $p_0(a, x) \geq 0$.

The following theorems are the main results of Chapter 2 and they will be proved in Chapter 2.

Theorem 1.1. *For any initial $p_0(a, x) \in D(\mathbb{A})$, the semigroup solution of (1.2) has the following asymptotic expression:*

$$p(a, t, x) = e^{\lambda_0 t} e^{-\lambda_0 a} \Upsilon(0, a) C_{\lambda_0} \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x, s) \int_0^a e^{-\lambda_0(a-\sigma)} \Upsilon(\sigma, a) \\ p_0(\sigma, s) ds da d\sigma + o(e^{(\lambda_0 - \varepsilon)t}),$$

where λ_0, C_{λ_0} and $\Upsilon(\tau, s)$ will be defined in Chapter 2.

The steady state of our model is very important, especially for our further researches about the control problem. The steady state of (1.2) is denoted by p_s, and should be a solution of

$$\begin{cases} \partial_a p_s(a, x) - \delta \Delta p_s(a, x) + \mu(a) p_s(a, x) = 0, & (a, x) \in (0, a_\dagger) \times (0, 24), \\ p_s(a, 0) = p_s(a, 24), & a \in (0, a_\dagger), \\ \partial_x p_s(a, 0) = \partial_x p_s(a, 24), & a \in (0, a_\dagger), \\ p_s(0, x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x, s) p_s(a, s) ds da, & x \in (0, 24). \end{cases} \tag{1.5}$$

Furthermore, $p_s(a, x)$ satisfies

$$p_s(a, x) \geq 0 \quad a.e. \ (a, x) \in (0, a_\dagger) \times (0, 24). \tag{1.6}$$

Theorem 1.2. *Consider (1.5) with λ_0 satisfying Theorem 1.1.*
(1) *If $\lambda_0 > 0$, then there is no nonnegative solution of (1.5) satisfying (1.6).*
(2) *If $\lambda_0 = 0$, then there exists infinitely many nontrivial solutions of (1.5) satisfying (1.6). Furthermore, for any nonzero steady state $p_s(a, x)$, there exists $\rho_0 > 0$ such that*

$$p_s(a, x) \geq \rho_0 > 0, \ \ a.e. \ (a, x) \in (0, a_1) \times (0, 24),$$

where $a_1 \in (0, a_\dagger)$.
(3) *If $\lambda_0 < 0$, then only trivial solutions p_s of (1.5) satisfying (1.6) exist, that is*

$$p_s(a, x) = 0 \quad a.e. \ (a, x) \in (0, a_\dagger) \times (0, 24).$$

The rest of Chapter 2 is organized as follows. In Section 2.1, we make some preparations which are necessary in what follows and we prove that $\mathbb{A}$ is an infinitesimal generator of a C_0-semigroup $T(t)$. In Section 2.2, we get the asymptotic behavior of (1.2) by analyzing the spectrum of the semigroup $T(t)$. Many abstract theories about semigroups used in this part can be referred to [25, 80, 104]. According to the asymptotic behaviour, we investigate the existence of steady states in Section 2.3. In Section 2.4, we present numerical simulations when there are insecticide-treated bed nets (ITNs).

1.2 Optimal control of an age structured problem

In Chapter 3, we focus on the study of (1.1) with $m(x) = 1$ for all $x \in (0, 24)$, that is,

$$\begin{cases} Dp - \delta\Delta p + \mu(a)p = u(a,t,x)p, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{1.7}$$

Our main goal is to prove that there exists an optimal control $u(a,t,x)$ in limited conditions, that is, $u(a,t,x)$ is bounded by two functions $\varsigma_1(a,t,x)$ and $\varsigma_2(a,t,x)$ such that the insecticidal efficiency reaches the best. Since the control function $u(a,t,x)$ is negative, it means that we can deal with the following optimal problem

$$(OH) \qquad Maximize \left\{ -\int_{Q_{a_\dagger}} u(a,t,x)p^u(a,t,x)dtdxda \right\},$$

subject to $u(a,t,x) \in U$,

$$U = \{u(a,t,x) \in L^2(Q_{a_\dagger}) |\ \varsigma_1(a,t,x) \le u(a,t,x) \le \varsigma_2(a,t,x) \ a.e.\ in\ Q_{a_\dagger}\},$$

where $\varsigma_1(a,t,x),\ \varsigma_2(a,t,x) \in L^\infty(Q_{a_\dagger}),\ \varsigma_1(a,t,x) \le \varsigma_2(a,t,x) \le 0\ a.e.\ in\ Q_{a_\dagger}$ and $p^u(a,t,x)$ is the solution of system (1.7). Here, we say that the control

$u^* \in U$ is optimal if

$$\int_{Q_{a_\dagger}} u^*(a,t,x)p^{u^*}(a,t,x)dtdxda \leq \int_{Q_{a_\dagger}} u(a,t,x)p^{u}(a,t,x)dtdxda,$$

for any $u(a,t,x) \in U$. The pair $(u^*(a,t,x), p^{u^*}(a,t,x))$ is an optimal pair and $\int_{Q_{a_\dagger}} u^*(a,t,x)p^{u^*}(a,t,x)dtdxda$ is the optimal value of the cost functional.

Let us recall some history about the optimal control researches. Since 1985 when Brokate [21] first proposed the optimal control of the population dynamical system with an age structure, it has been widely concerned and extensively studied by more and more researchers in the past few years. It is worth mentioning that the researches of Gurtin and Murphy [46, 47] about the optimal harvesting of age structured populations provide an important basis for subsequent researches of the optimal control problem. As is well known, the optimal harvesting problem governed by nonlinear age dependent population dynamics with diffusion was considered by Aniţa [13], where he mainly discussed the impact of the control in homogeneous Neuman boundary conditions. For more rich results about the optimal control of an age structure with non-periodic boundary conditions, one can refer to [11, 12, 34, 105] and references cited therein. Note that the above results are about non-periodic boundary conditions.

However, we have seen from the practical significance of biology that it is advantageous to consider age-structured models with periodic boundary conditions and nonlocal birth processes. We would like to refer to [10, 73] for some studies about the optimal control problem with periodic boundary conditions. We also refer to [15, 53, 59, 58] as reviewing references of the optimal control problem. Let us now mention some of our work about other aspects of system (1.7) with periodic boundary conditions and nonlocal birth processes. In [63], large time behaviour of the solution for such age structured population model was considered. Moreover, we considered the local exact controllability of such age structured problem in [64]. In Chapter 3, we study the optimal control of system (1.7).

From the biological point of view, we make the following hypotheses throughout Chapter 3:

(J1) $\mu(a) \in L^\infty_{loc}((0,a_\dagger))$, $\int_0^{a_\dagger} \mu(a)da = +\infty$ and $\mu(a) \geq 0$ $a.e.$ in $(0,a_\dagger)$;

(J2) $\beta(a) \in L^\infty((0,a_\dagger))$, $\beta(a) \geq 0$ $a.e.$ in $(0,a_\dagger)$;

(J3) $p_0(a,x) \in L^2((0,a_\dagger)\times(0,24))$, $p_0(a,x) \geq 0$ $a.e.$ in $(0,a_\dagger)\times(0,24)$.

Now we state our main results in Chapter 3.

Theorem 1.3. *For any $u(a,t,x) \in U$, there exists a unique solution $p^u(a,t,x) \in L^2(Q_{a_\dagger})$ of the system (1.7).*

Theorem 1.4. *Problem (OH) admits at least one optimal pair $(u^*(a,t,x), p^*(a,t,x))$.*

Theorem 1.5. *Let $(u^*(a,t,x), p^*(a,t,x))$ be an optimal pair for (OH) and $q(a,t,x)$ be the solution of the following system*

$$\begin{cases} Dq + \delta\Delta q - \mu(a)q + \beta(a)\displaystyle\int_{x-\eta}^{x+\eta} K(x,s)q(0,t,s)ds = -u^*q - u^*, \\ q(a,t,0) = q(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x q(a,t,0) = \partial_x q(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ q(a_\dagger,t,x) = 0, & (t,x) \in (0,T) \times (0,24), \\ q(a,T,x) = 0, & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{1.8}$$

Then, one has

$$u^*(a,t,x) = \begin{cases} \varsigma_1(a,t,x), & \text{if } q(a,t,x) > -1, \\ \varsigma_2(a,t,x), & \text{if } q(a,t,x) < -1. \end{cases}$$

Chapter 3 is organized as follows. In Section 3.1, we prove the existence of solutions and the comparison result for a linear model which is (1.7) in general settings. Section 3.2 is devoted to the proof of the existence of an optimal control of system (1.7) by Mazur's Theorem. Section 3.3 focuses on the necessary optimality conditions.

1.3 Local exact controllability of an age structured problem

In Chapter 4, we study the local exact controllability of an age structured problem modelling phenotypic plasticity in mosquito behaviour. That is, we prove the existence of a control $u(a,t,x)$ of system (1.1) such that the population of mosquitoes can reach a steady state.

We first recall some results of Theorem 1.2. Let $p_s(a,x)$ be a nontrivial steady-state of (1.1) and be a solution of (1.5). By a result of Theorem 1.2, we have steady states

$$p_s(a,x) \equiv 0, \ \ for \ \lambda_0 < 0,$$

$$p_s(a,x) > 0, \ \ for \ \lambda_0 = 0,$$

where λ_0 is the unique real eigenvalue of the infinitesimal generator governed by (1.1) with $u(a,t,x) \equiv 0$ (refer to Chapter 2 for the details).

In Chapter 4, we only consider the later case, that is, $p_s(a,x)$ such that

$$p_s(a,x) \geq \rho_0 > 0, \quad a.e. \ \ (a,x) \in (0,a_1) \times (0,24),$$

where ρ_0 is positive constant and $a_1 \in (0, a_\dagger)$. The objective of Chapter 4 is to prove the existence of a control $u(a,t,x)$ such that the solution of (1.1) satisfies

$$\begin{aligned} p(a,T,x) &= p_s(a,x), \quad && a.e. \ \ (a,x) \in (0,a_1) \times (0,24), \\ p(a,t,x) &\geq 0, && a.e. \ \ (a,t,x) \in Q_{a_\dagger}. \end{aligned}$$

Obviously, the last inequality is because biological reasons: $p(a,t,x)$ represents the density of a population. We can see that our main problem is equivalent to the exact null controllability problem of the following system

$$\begin{cases} Dp - \delta\Delta p + \mu(a)p = m(x)u(a,t,x)(p+p_s), & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = \overline{p}_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases} \tag{1.9}$$

where $\overline{p}_0(a,x) = p_0(a,x) - p_s(a,x)$.

In the last years many works were devoted to the controllability of partial differential equations of parabolic type [16, 31, 36, 57, 90]. For an overview on age structured population dynamics models and their mathematical analysis we refer to [37, 51, 55, 95]. The controllability of age structured problems modeling demographical processes was considered in [3, 4, 5, 6, 7, 13]. The nonlocal birth process in these models is nonlocal with respect to the age variable a, that is,

$$p(0,t) = \int_0^{a_\dagger} \beta(a)p(a,t)da.$$

The local exact controllability of the age structured problem with diffusion was established in [5]. The main proof is based on Carleman's inequality for the adjoint equation, one can refer to [4, 8, 13]. For the linear Lotka-McKendrick model without spatial structure, Viorel Barbu et al. established an observability inequality for the backward adjoint system [17] to get the exact controllability.

In Chapter 4, we discuss the local exact controllability of (1.1) with time-biting periodic boundary conditions and nonlocal birth process with respect to chronological age and biting-time. In Chapter 4, we also get a Carleman's inequality for our periodic boundary case.

Let $a_\dagger$ be a finite positive number. From the biological point of view, we make the following assumptions throughout Chapter 4:

(J1) $\mu(a) \in L^1_{loc}([0, a_\dagger))$ and $\int_0^{a_\dagger} \mu(\rho)d\rho = \infty$;

(J2) $\beta(a) \in L^\infty((0, a_\dagger))$, there exists $a_0, a_1 \in (0, a_\dagger)$ such that $\beta(a) = 0$ a.e. $a \in (0, a_0) \cup (a_1, a_\dagger)$, $\beta(a) > 0$ a.e. $a \in (a_0, a_1)$;

(J3) $p_0(a, x) \in L^\infty((0, a_\dagger) \times (0, 24))$, $p_0(a, x) \geq 0$ and $p_0(a, x) \not\equiv 0$ $a.e.(a, x) \in (0, a_\dagger) \times (0, 24)$.

The following theorem is the main result of Chapter 4.

Theorem 1.6. *If* $\|\overline{p}_0(a, x)\|_{L^\infty((0,a_\dagger)\times(0,24))}$ *is small enough, then there exists* $u(a, t, x) \in L^2(Q_{a_\dagger})$ *such that the solution of* (1.9) *satisfies*

$$\begin{aligned} p(a, T, x) &= 0, && a.e.(a, x) \in (0, a_\dagger) \times (0, 24), \\ p(a, t, x) &\geq -p_s(a, x), && a.e.(a, t, x) \in Q_{a_\dagger}. \end{aligned}$$

Chapter 4 is organized as follows. In Section 4.1, we give some preparations which are important and necessary in what follows. In particular, we get a Carleman inequality corresponding to our problem with periodic boundary condition. Section 4.2 focuses on the local exact controllability of (1.1), that is, we would prove Theorem 1.6 with the help of a system with an ordinary initial value. In Section 4.3, we present numerical simulations of (1.1).

1.4 Large-time behavior of an age structured model

In Chapter 5, we adapt our model (1.1) and investigate the dynamics of the mosquitoes population with the usage of (ITNs) and (IRs). Let $\mu(a, w) \geq 0$ be the natural death-rate of individuals of age a and the matured population

$$w(t, x) = \int_{a_0}^{a_\dagger} p(a, t, x)da.$$

We set that the (ITNs) and (IRs) are only useful to matured population, that is, $u(a, w) = 0$ for $a \in [0, a_0)$ and $u(a, w) = u(w)$ for $a \in [a_0, a_\dagger)$. Therefore, we model the mosquito plasticity problem as the following system

$$\begin{cases} Dp - \delta\Delta p + \mu(a, w(t,x))p = u(a, w(t,x))p, & (a,t,x) \in Q, \\ p(0,t,x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times \mathbb{R}. \end{cases} \tag{1.10}$$

where $Q = (0, a_\dagger) \times \mathbb{R}^+ \times \mathbb{R}$.

Let us recall some history about the single species with age structure researches. Before 1990, many researchers considered diffusion into a time delay model by simply adding a diffusion term to the corresponding delay ordinary differential equation model, see Memory [69] and Yoshida [102]. But, in the nature biology, individuals have not been at the same point in time at previous times. Thereupon, in 1990, Britton [20] first proposed to address the problem for a delayed Fisher equation on an infinite domain. More details in Chapter 5.2 will be given to see how we derive our problem into a time-delayed problem. Since it is so important for an age structure model to derive a reaction diffusion equation with time delay, more and more researchers have widely concerned and extensively studied about this problem in the past few years, see [40, 83, 85, 86].

Meanwhile, as for the reaction diffusion equations with time delay, there are rich results about local delay and nonlocal delay. For the reaction diffusion equations with local time delay, the KPP and bistable nonlinear diffusion equations with a discrete delay were considered by Schaaf [82]. In [97], more general reaction-diffusion systems with finite delay were studied by using the classical monotone iteration technique and the sub- and supersolutions method. For more rich results about the reaction-diffusion equations with discrete delay , one can refer to [23, 24, 29, 33, 84] and references cited therein. It is worth mentioning that the research of Ma and Zou [65] provided a more generalized method than Chen [23, 24] for a class of discrete reaction-diffusion monostable equation with delay.

By the practical significance of biology, it is advantageous to consider the reaction diffusion equations with nonlocal delays. We would like to mention the work of Britton [19, 20], since they first attempted to study the perodic traveling wave solutions in reaction-diffusion equations with nonlocal delays. Since then, there are many researchers devoted to proving the existence of traveling wave solutions of these type equations mainly by three methods: the perturbation theory [1, 40], the geometric singular perturbation theory [2, 14, 40, 78], the monotone iteration method [30, 72, 86, 93, 94]. In fact, the posterior results concerning about the existence of traveling wave solutions in Chapter 5 are due to the monotone iteration method [86, 94].

In Chapter 5, we consider that the bitting behavior of mosquitoes is periodic with 24 hours a day. It means that we consider the initial value $p_0(a,x)$ satisfying

$$p_0(a, x+24) = p_0(a,x), x \in \mathbb{R}$$

and the solution $p(a,t,x)$ satisfying

$$p(a,t,x+24) = p(a,t,x), a \in (0,a_\dagger), t \in \mathbb{R}^+, x \in \mathbb{R}.$$

We are interested in how the insecticidal control $u(a,w)$ (such as ITNs and IRs) effects the matured population of mosquitoes, that is, the large time behavior of $w(t,x)$. From the biological point of view, we make the following hypotheses throughout this paper:

(J1) The death rate $\mu(a,w) \geq 0$ satisfies that

$$\mu(a,w) = \begin{cases} \mu_1(a), & a \in [0,a_0), \\ \mu_2(a,w), & a \in [a_0,a_\dagger), \end{cases}$$

where $\mu_1(a) \in L^\infty(0,a_0)$, $\mu_2(a,w)$ is continuous with respect to a and w, $\mu_2(a,w) \in L^\infty_{loc}(a_0,a_\dagger)$ for every $w \geq 0$ and $\int_0^{a_\dagger} \mu(a,w)da = +\infty$ for every $w \geq 0$. As a matter of fact, the natural death population can not exceed the amount of matured population, that is, $0 \leq \int_{a_0}^{a_\dagger} \mu(a,w)p(a,t,x)da \leq w(t,x)$. Thus, we assume $0 \leq \int_{a_0}^{a_\dagger} \mu(a,w)p(a,t,x)da \leq g(w)$ for some smooth continuous function $g(w)$.

(J2) The birth rate $\beta(a)$ satisfies

$$\beta(a) = \begin{cases} 0, & a \in [0,a_0), \\ \beta, & a \in [a_0,a_\dagger), \end{cases}$$

where β is a positive constant.

(J3) The insecticidal control $u(a,w) \leq 0$ satisfies that

$$u(a,w) = \begin{cases} 0, & a \in [0,a_0), \\ u(w), & a \in [a_0,a_\dagger), \end{cases}$$

where $u(w)$ is a C^2 function in w.

(J4) $p_0(a,x) \in L^\infty([0,a_\dagger] \times \mathbb{R})$, $p_0(a,\cdot) \geq\not\equiv 0$ for every $a \in [0,a_\dagger]$ and $p_0(a,x+24) = p_0(a,x)$ for $x \in \mathbb{R}$.

(J5) We assume that $\sup_{w\geq 0}\mu(a,w) \leq \widetilde{\mu}(a)$ where $\widetilde{\mu}(a) \in L^{\infty}_{loc}([0,a_{\dagger}))$ and $\beta(a)$ is sufficiently large such that $\int_0^{a_{\dagger}} \beta(a)e^{-\int_0^a \widetilde{\mu}(\rho)d\rho}da$ is sufficiently large for every $w \geq 0$ which can ensure that there are mosquitos surviving forever.

We give some comments about these hypotheses. Notice that in (J1), the condition $\int_0^{a_{\dagger}} \mu(a,w)da = +\infty$ for every $w \geq 0$ ensures $a_{\dagger}$ being the life expectancy of an individual, that is,

$$p(a_{\dagger},t,x) = 0, t \in \mathbb{R}^{+}, x \in \mathbb{R}$$

(refer to [13]). From Theorem 1.7, the last assumption (J5) implies that if there is only less insecticidal control, the matured population of mosquitoes will go to infinity. In biological meaning, (J5) means that the fertility-rate is large and the natural death-rate is small and then the population will keep growing. On the other hand, if the insecticidal control is very large, it means that the total death-rate $\mu(a,w)+u(a,w)$ is large and $\int_0^{a_{\dagger}} \beta(a)e^{-\int_0^a(\mu(\rho,w)+u(\rho,w))d\rho}da$ is small in some sense. Then, the population will be decaying to 0 for large time. Such threshold can be more clear for some other population models, one can refer to [7].

Now we state the main results of Chapter 5.

Theorem 1.7. *(i) If* $\sup_{w\geq 0}|u(w)|$ *is small enough, then one has that*

$$w(t,x) \to +\infty, \ \text{as } t \to +\infty.$$

(ii) If $\inf_{w\geq 0}|u(w)|$ *is large enough, then one has that*

$$w(t,x) \to 0, \ \text{as } t \to +\infty.$$

In fact, a more realistic situation is that the population can not be infinite large or very small because of the limitation of insecticidal control strategy. It means that the matured population $w(t,x)$ may reach some balanced states. In Section 5.2, we will derive a time-delayed model for $w(t,x)$, that is, $w(t,x)$ satisfies

$$\begin{aligned} w_t - \delta\Delta w = &\int_{a_0}^{a_{\dagger}} (-\mu(a,w)+u(w))p(a,t,x)da \\ &+ M\beta \int_{-\infty}^{+\infty}\left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0,x-z-s)dz\right] f_{\delta a_0}(s)ds, \end{aligned} \tag{1.11}$$

for $t > 0$, $x \in \mathbb{R}$ and

$$w(s,x) = \int_{a_0}^{a_\dagger} p_0(a+s,x)da, \text{ for } s \in [-a_0, 0] \text{ and } x \in \mathbb{R}. \tag{1.12}$$

Since $0 \le \int_{a_0}^{a_\dagger} \mu(a,w)p(a,t,x)da \le g(w)$ by (J1), one has that $w(t,x)$ is governed by the following sub-equation and super-equation

$$\begin{aligned} w_t - \delta\Delta w \ge & - g(w) + u(w)w \\ & + M\beta \int_{-\infty}^{+\infty} \left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0, x-z-s)dz \right] f_{\delta a_0}(s)ds, \end{aligned} \tag{1.13}$$

and

$$w_t - \delta\Delta w \le u(w)w + M\beta \int_{-\infty}^{+\infty} \left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0, x-z-s)dz \right] f_{\delta a_0}(s)ds. \tag{1.14}$$

As we announced, the insecticidal control will be enhanced as the population of mosquitoes increasing and finally the population of mosquitoes will reach a balanced state, that is, the death-rate will offset the birth-rate in some sense. Therefore, we assume that both sub-equation and super-equation can reach a balanced state, that is,

(H1) there is $w_2 > 0$ such that $-g(w_2) + u(w_2)w_2 + M\beta M_1 w_2 = 0$, $-g(w) + u(w)w + M\beta M_1 w > 0$ for $0 < w < w_2$, $-g'(w_2) + u'(w_2)w_2 + u(w_2) + M\beta M_1 < 0$ and $g(0) = g'(0) = u(0) = 0$,

(H2) there is $w_3 > 0$ such that $u(w_3)w_3 + M\beta M_1 w_3 = 0$.

Here, $M_1 = \int_{-\infty}^{+\infty} \int_{-\eta}^{\eta} z^2 e^{-z^2} dz f_{\delta a_0}(s)ds$. Since $g(w) \ge 0$ for $w \ge 0$, it is obvious that $w_2 \le w_3$.

We then derive the existence of traveling fronts of the sub-equation. In fact, the traveling front can describe the invasion of one steady state to another. It implies that the area in which the matured population of mosquitoes is close to w_2 will invade the area with less mosquitoes.

Theorem 1.8. *Assume that (H1) hold. There exists a $c^* > 0$ such that for every $c > c^*$, the equation*

$$\begin{aligned} w_t - \delta\Delta w = & - g(w) + u(w)w \\ & + M\beta \int_{-\infty}^{+\infty} \left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w(t-a_0, x-z-s)dz \right] f_{\delta a_0}(s)ds \end{aligned} \tag{1.15}$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$*, admits a traveling front* $\phi(x+ct)$ *connecting* 0 *and* w_2*.*

Now, let us come back to the problem (1.11) with (1.12). Notice that $w(s,x+24) = w(s,x)$ for $s \in [-a_0, 0]$ and $x \in \mathbb{R}$ since $p_0(a,x+24) = p_0(a,x)$ for $a \in [0, a_\dagger]$ and $x \in \mathbb{R}$. We assume that the matured population is not large at the initial time, that is,

$$w(s,x) \leq w_3, \;\; s \in [-a_0, 0], \;\; x \in \mathbb{R}.$$

Assume further that $-g(w) + u(w)w$ satisfies

(H3) for every $\gamma \in (0,1)$, there exist $a = a(\gamma) > 0$ and $\alpha = \alpha(\gamma) > 0$ such that for any $\theta \in (0, \gamma]$ and $w \in [0, w_2]$,

$$(1-\theta)(-g(w)+u(w)w) - (-g((1-\theta)w) + u((1-\theta)w)(1-\theta)w) \leq -a\theta w^\alpha.$$

Theorem 1.9. *Assume that (H1), (H2) and (H3) hold. For problem* (1.11) *with the initial value* (1.12)*, it holds that for each* $\gamma \in (0,1)$*, there exist* $T > 0$*,* $\rho > 0$ *and* $\sigma > 0$ *such that for each* $\varepsilon \in [0, \gamma]$*, the following functions*

$$w(t,x) \geq (1 - \varepsilon e^{-\rho t})\phi(x + ct + \sigma\varepsilon e^{-\rho t}), \;\; \text{for } t \geq T,$$

$$w_2 \leq w(t,x) \leq w_3, \;\; \text{as } t \to +\infty.$$

Theorem 1.9 means that the population of matured mosquitoes will finally reach a balance between w_2 and w_3.

The rest of Chapter 5 is organized as follows. In Section 5.1, we analyze the large time behavior of the matured population when the control is small and when it is large, that is, we prove Theorem 1.7. Section 5.2 is devoted to the proof of the existence of traveling fronts for the sub-equation and the proof of Theorem 1.9. In Section 5.3, we present numerical simulations of (1.10).

Chapter 2

Mathematical analysis of an age structured problem

In this chapter, we first study the long time behaviour of the mosquito population without the intervention of human activities. It implies that in the general model (1.1), we set $u(a,t,x) = 0$. Our goal is to find an asymptotic expression for the mosquito population under no control, that is, Theorem 1.1. From the asymptotic expression, we can get the existence and nonexistence of the steady state. Finally, we do numerical simulations for the mosquito population when there are insecticide-treated bed nets (ITNs). One can see that the usage of ITNs can lead to a change in the mosquitoes bitting time.

2.1 Preliminaries

In this section, we give some auxiliary lemmas as a preparation for our main results that will be derived later. In fact, we have to prove that $\mathbb{A}$ is an infinitesimal generator of a C_0-semigroup $T(t)$.

At the beginning of this section, we study the following system

$$\begin{cases} Dp - \delta\Delta_x p + \mu(a)p = 0, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = C\displaystyle\int_0^{a_\dagger} \beta(a)p(a,t,x)da, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases} \tag{2.1}$$

where C can be any constant. Defining the operator $\mathbb{F} : X \to X$ as:

$$\mathbb{F}\phi(a,x) = -\frac{\partial \phi(a,x)}{\partial a} + \delta\Delta\phi(a,x) - \mu(a)\phi(a,x), \forall \phi(a,x) \in D(\mathbb{F}), \tag{2.2}$$

where

$$\begin{aligned} D(\mathbb{F}) = \{\phi(a,x) | \phi, \mathbb{A}\phi \in X, \phi(a,0) = \phi(a,24), \partial_x\phi(a,0) = \partial_x\phi(a,24), \\ \phi(0,x) = C\int_0^{a_\dagger} \beta(a)\phi(a,x)da\}, \end{aligned}$$

we can rewrite (2.1) as

$$\begin{cases} \dfrac{dp(a,t,x)}{dt} = \mathbb{F}p(a,t,x), \\ p(a,0,x) = p_0(a,x). \end{cases}$$

Define an operator

$$\mathcal{F}_\lambda = \int_0^{a_\dagger} C\beta(a)e^{-\lambda a}e^{-\int_0^a \mu(\rho)d\rho}e^{\mathbb{B}a}da, \tag{2.3}$$

where the operator $\mathbb{B} : L^2((0,24)) \longrightarrow L^2((0,24))$ is defined as

$$\mathbb{B}u(x) = \delta\Delta u(x),$$

for $u(x)$ satisfying

$$\begin{cases} u(0) = u(24), \\ u'(0) = u'(24). \end{cases}$$

Lemma 2.1. *The operator $\mathbb{F}$ defined by (2.2).*

(1) $\mathbb{F}$ has a real dominant eigenvalue $\widetilde{\lambda}_0$, that is, $\widetilde{\lambda}_0$ is greater than any real parts of the eigenvalues of $\mathbb{F}$.

(2) For the operator $\mathcal{F}_{\widetilde{\lambda}_0}$, 1 is an eigenvalue with an eigenfunction $\phi_0(x)$. Furthermore, $\gamma(\mathcal{F}_{\widetilde{\lambda}_0}) = 1$.

Proof. (1) We denote by $(\overline{\lambda}_i, \phi_i)_{i\geq 0}$ the eigenvalues and the eigenfunctions of the following problem

$$\begin{cases} -\delta\Delta\phi_i(x) = \overline{\lambda}_i\phi_i(x),\ x \in (0,24), \\ \phi_i(0) = \phi_i(24), \\ \partial_x\phi_i(0) = \partial_x\phi_i(24), \end{cases}$$

where $\int_0^{24} \phi_i^2(x)dx = 1,\ i \geq 0$, and $\phi_0(x) > 0$ with $x \in (0, 24)$. It is obvious that $\overline{\lambda}_0 = 0$ and $\phi_0(x)$ is a fixed positive constant. We also assume that $0 = \overline{\lambda}_0 < \overline{\lambda}_1 \leq \overline{\lambda}_2 \leq \cdots$.

Let F be the operator in $L^2(0, a_\dagger)$ defined as

$$F\phi(a) = -\frac{d\phi(a)}{da} - \mu(a)\phi(a),\ \forall \phi \in D(F),$$

where

$$D(F) = \{\phi(a)|\phi, F\phi \in L^2(0, 24), \phi(0) = C\int_0^{a_\dagger} \beta(a)\phi(a)da\}.$$

Let $\{\widehat{\lambda}_j\}_{j\geq 0}$ be the eigenvalues of F, that is, the solutions of the following equation

$$1 - C\int_0^{a_\dagger} \beta(a)e^{-\widehat{\lambda}_j a - \int_0^a \mu(\rho)d\rho}da = 0. \tag{2.4}$$

We assume that $\widehat{\lambda}_0 > Re\widehat{\lambda}_1 \geq Re\widehat{\lambda}_2 \geq \cdots$, even if it means re-arrange $\widehat{\lambda}_j$.

Now, we divide two steps to consider the following equation

$$(\lambda\mathbb{I} - \mathbb{F})\phi = \psi,\ \forall \psi \in X. \tag{2.5}$$

Step 1, for any $i,\ j \geq 0,\ \lambda + \overline{\lambda}_i \neq \widehat{\lambda}_j$, define

$$\phi(a, x) = \sum_{i=0}^{\infty} R(\lambda + \overline{\lambda}_i, F)\langle\psi(a, x), \phi_i(x)\rangle\phi_i(x),$$

where $\langle\psi(a, x), \phi_i(x)\rangle = \int_0^{24} \psi(a, x)\phi_i(x)dx,\ R(\lambda, F) = (\lambda\mathbb{I} - F)^{-1}$, the resolvent operator of F. Firstly, we prove that $\phi(a, x) \in X$ is well defined. Since F is the infinitesimal generator of a bounded strongly continuous semigroup from [52], there exist constants $M,\ \omega > 0$ such that

$$\|R(\lambda, F)\| \leq \frac{M}{Re\lambda - \omega},\ \text{for } Re\lambda > \omega.$$

Recalling that $\overline{\lambda}_i \to \infty$ as $i \to \infty$, there is a constant N such that $Re(\lambda+\overline{\lambda}_i) >$

ω when $i > N$. Then, one can compute that

$$
\begin{aligned}
&\sum_{i=0}^{\infty} \|R(\lambda+\overline{\lambda}_i, F)\langle\psi,\phi_i\rangle\|^2 \\
\leq &\sum_{i=0}^{N} \|R(\lambda+\overline{\lambda}_i, F)\langle\psi,\phi_i\rangle\|^2 + \left[\frac{M}{Re(\lambda+\overline{\lambda}_N)-\omega}\right]^2 \sum_{i=N+1}^{\infty} \|\langle\psi,\phi_i\rangle\| \\
\leq &\sum_{i=0}^{N} \|R(\lambda+\overline{\lambda}_i, F)\langle\psi,\phi_i\rangle\|^2 + \left[\frac{M}{Re(\lambda+\overline{\lambda}_N)-\omega}\right]^2 \|\psi\|^2 \\
< &\infty.
\end{aligned}
$$

It implies that $\phi(a,x) \in X$ is well defined. Secondly, we prove $\phi(a,x)$ is a solution of (2.5). For any $n > 0$,

$$
\begin{aligned}
&(\lambda\mathbb{I}-\mathbb{F})\sum_{i=0}^{n} R(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x) \\
=&\sum_{i=0}^{n}[\lambda R(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x) - \mathbb{F}R(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x)] \\
=&\sum_{i=0}^{n}[\lambda R(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x) - FR(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x) \\
&- R(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\delta\Delta\phi_i(x)] \\
=&\sum_{i=0}^{n}((\lambda+\overline{\lambda}_i)\mathbb{I}-F)R(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x) \\
=&\sum_{i=0}^{n}\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x) \\
\to&\psi(a,x),\ n\to\infty.
\end{aligned}
$$

Since F and Δ are both closed operators on X, one can infer that $\mathbb{F}$ is closed. Hence,

$$(\lambda\mathbb{I}-\mathbb{F})\phi = \psi.$$

That is, $\phi(a,x)$ is a solution of (2.5). Furthermore, it can be shown that ϕ is the unique solution of (2.5), and thus $\lambda \in \rho(\mathbb{F})$, the resolvent set of $\mathbb{F}$ and

$$R(\lambda,\mathbb{F})\psi = \sum_{i=0}^{\infty} R(\lambda+\overline{\lambda}_i, F)\langle\psi(a,x),\phi_i(x)\rangle\phi_i(x).$$

Step 2, for some i, j such that $\lambda+\overline{\lambda}_i = \widehat{\lambda}_j$, it is easy to check that $\phi(a,x) = e^{-(\lambda+\overline{\lambda}_i)a-\int_0^a \mu(\rho)d\rho}\phi_i(x)$ satisfies $(\lambda\mathbb{I}-\mathbb{F})\phi = 0$, that is, $\lambda = \widehat{\lambda}_j - \overline{\lambda}_i \in \sigma(\mathbb{F})$. In particular, $\widetilde{\lambda}_0 = \widehat{\lambda}_0 - \overline{\lambda}_0$ is the dominant eigenvalue of $\mathbb{F}$, with eigenfunction

$$\phi_{\widetilde{\lambda}_0}(a,x) = e^{-\widehat{\lambda}_0-\int_0^a \mu(\rho)d\rho}\phi_0(x).$$

It is easy to check that $C\phi_0(x)$ is the eigenfunction of the eigenvalue 1 of $\mathcal{F}_{\widetilde{\lambda}_0}$, where $\widetilde{\lambda}_0 = \widehat{\lambda}_0 - \overline{\lambda}_0$. Let any $\phi(x) \in L^2(0,24)$ be expanded as

$$\phi(x) = \sum_{i=0}^{\infty} \alpha_i \phi_i(x).$$

Then,

$$\begin{aligned}\mathcal{F}_{\widetilde{\lambda}_0}\phi(x) &= \sum_{i=0}^{\infty} \alpha_i \int_0^{a_\dagger} C\beta(a)e^{-\widetilde{\lambda}_0 a}e^{-\int_0^a \mu(\rho)d\rho}e^{\mathbb{B}a}\phi_i(x)da \\ &= \sum_{i=0}^{\infty} \alpha_i \int_0^{a_\dagger} C\beta(a)e^{-(\widetilde{\lambda}_0+\overline{\lambda}_i)a}e^{-\int_0^a \mu(\rho)d\rho}da\phi_i(x).\end{aligned}$$

Since $\overline{\lambda}_i \geq \overline{\lambda}_0$ and then $\widetilde{\lambda}_0 + \overline{\lambda}_i \geq \widehat{\lambda}_0$, it follows from (2.4) that

$$\int_0^{a_\dagger} C\beta(a)e^{-(\widetilde{\lambda}_0+\overline{\lambda}_i)a}e^{-\int_0^a \mu(\rho)d\rho}da \leq 1.$$

Thus, $\gamma(\mathcal{F}_{\widetilde{\lambda}_0}) = 1$. □

Following the proof of lemma 1 in [42] carefully, we can get the following lemma:

Lemma 2.2. *For any $0 \leq s_0 < a_+$, there exists a unique mild solution u(s,x), $0 \leq \tau \leq a_+ - s_0$ to the evolution equation on X for any initial function $\phi(x) \in L^2((0,24))$*

$$\begin{cases} \frac{\partial u(s,x)}{\partial s} = (-\mu(s_0+s)+\mathbb{B})u(s,x), \\ u(\tau,x) = \phi(x), \end{cases}$$

where the operator $\mathbb{B}_0$ is considered to be the Laplace operator with periodic boundary condition. Define solution operators of the initial value problem by

$$\mathcal{T}(s_0,\tau,s)\phi(x) = u(s,x), \qquad \forall\phi(x) \in L^2((0,24)),$$

then $\mathcal{T}(s_0,\tau,s)\phi(x)$ is a family of uniformly linear bounded compact positive *operators on X and is strongly continuous about τ,s. Furthermore,*

$$\mathcal{T}(s_0, \tau, s) = e^{-\int_\tau^s \mu(s_0+\rho)d\rho} e^{\mathbb{B}(s-\tau)},$$

where $e^{\mathbb{B}s}$ is the positive analytic semigroup generated by the operator $\mathbb{B}$.

Proof. Define an operator $\mathcal{H}_{s_0} : C([\tau, \overline{s}], L^2((0, 24))) \to C([\tau, \overline{s}], L^2((0, 24)))$ as:

$$\mathcal{H}_{s_0}(u(s,x)) = e^{\mathbb{B}(s-\tau)}\phi(x) - \int_\tau^s e^{\mathbb{B}(s-\sigma)}\mu(s_0+\sigma)u(\sigma, x)d\sigma,$$

where $0 < \overline{s} \le a_\dagger - s_0$. Then follow the proof of Lemma 1 in [42] step by step, we can get our result, so we omit the details here. □

Lemma 2.3. *The operator $\mathbb{A}$ defined by (1.3) and (1.4) is the infinitesimal generator of a C_0-semigroup $T(t)$ on the space X.*

Proof. First note that a C_0-semigroup $T(t)$ implies that there exists a constant ω and $M \ge 1$, so that

$$\|T(t)\| \le Me^{\omega t}, \qquad \forall t \ge 0.$$

Our strategy here is to apply the generalized Hille-Yoside Theorem (refer to Theorem 8.2.5 of [101] and Corrollary 3.8 of [75]), that is, to prove: (i) $\mathbb{A}$ is closed and $\overline{D(\mathbb{A})} = X$; (ii) for any $\lambda > \omega$, $\lambda \in \rho(\mathbb{A})$, and

$$\|R^n(\lambda, \mathbb{A})\| \le \frac{M}{(\lambda-\omega)^n}, \qquad n = 1, 2, 3 \cdots.$$

(i) One can compute that

$$\begin{aligned}
&\langle \mathbb{A}\phi(a,x), \phi(a,x)\rangle \\
=&\langle -\frac{\partial\phi(a,x)}{\partial a} + \delta\Delta\phi(a,x) - \mu(a)\phi(a,x), \phi(a,x)\rangle \\
=&-\int_0^{a_\dagger}\int_0^{24}\frac{\partial\phi}{\partial a}\phi\, da dx - \int_0^{a_\dagger}\int_0^{24}\mu(a)\phi^2 da dx + \delta\int_0^{a_\dagger}\int_0^{24}\Delta\phi\phi\, da dx \\
\le&-\int_0^{a_\dagger}\int_0^{24}\frac{\partial\phi}{\partial a}\phi\, da dx + \delta\int_0^{a_\dagger}\int_0^{24}\Delta\phi\phi\, da dx \\
\le&\frac{1}{2}\int_0^{24}\phi^2(0,x)dx
\end{aligned} \tag{2.6}$$

$$
\begin{aligned}
=&\frac{1}{2}\int_0^{24}\left(\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)\phi(a,s)dsdx\right)^2dx\\
\leq&\eta\int_0^{24}\left(\int_0^{a_\dagger}\beta^2(a)da\right)\left(\int_0^{a_\dagger}\int_{x-\eta}^{x+\eta}K^2(x,s)\phi^2(a,s)dads\right)dx\\
\leq&N\int_0^{a_\dagger}\beta^2(a)da\langle\phi(a,x),\phi(a,x)\rangle,
\end{aligned}
$$

for some constants $N>0$, which also implies that $\mathbb{A}$ is an m-dissipative operator when $\lambda\in\rho(\mathbb{A})$ for all sufficiently large $\lambda>0$. In fact, if this claim holds, $\mathbb{A}$ is a closed operator, and combining with the m-dissiptiveness of $\mathbb{A}$, we know that, for all sufficiently large λ, $(\mathbb{A}-\lambda\mathbb{I})$ is dissipative and $R(\mathbb{I}-(\mathbb{A}-\lambda\mathbb{I}))$ equals the whole space X. Thus from Theorem 4.6 in [75], it follows that $D(\mathbb{A}-\lambda\mathbb{I})$ is dense in X and so is $D(\mathbb{A})$, since X is a Hilbert space.

(ii) Now, we prove that $\lambda\in\rho(\mathbb{A})$ for all sufficiently large $\lambda>0$. In order to do this, we deal with the following equation

$$
(\lambda\mathbb{I}-\mathbb{A})\phi(a,x)=\psi(a,x),\qquad\forall\psi\in X,
$$

that is,

$$
\begin{cases}
\frac{\partial\phi(a,x)}{\partial a}=-(\lambda+\mu(a))\phi(a,x)+\delta\Delta\phi(a,x)+\psi(a,x),\\
\phi(0,x)=\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)\phi(a,s)dsda.
\end{cases}
$$

Let $\mathcal{T}(0,\tau,s)=\mathcal{T}(\tau,s)=e^{-\int_\tau^s\mu(\rho)d\rho}e^{\mathbb{B}(s-\tau)}$ and by Lemma 2.2, one has

$$
\phi(a,x)=e^{-\lambda a}\mathcal{T}(0,a)\phi(0,x)+\int_0^a e^{-\lambda(a-\delta)}\mathcal{T}(\delta,a)\psi(\delta,x)d\delta,
$$

and

$$
\begin{aligned}
&\phi(0,x)-\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)e^{-\lambda a}\mathcal{T}(0,a)\phi(0,s)dsda\\
&=\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)\int_0^a e^{-\lambda(a-\delta)}\mathcal{T}(\delta,a)\psi(\delta,s)d\delta dsda.
\end{aligned}\tag{2.7}
$$

Then define the operator $\mathcal{B}_\lambda:L^2((0,24))\to L^2((0,24))$ by

$$
\mathcal{B}_\lambda(\phi(x))=\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)e^{-\lambda a}\mathcal{T}(0,a)\phi(s)dsda.\tag{2.8}
$$

Here, notice that $\mathcal{B}_\lambda(\phi(x))$ is nonlocal in x with $\phi(x)$, since the part of the operation $\mathcal{B}_\lambda$ on $\phi(x)$ is the integral $\int_{x-\eta}^{x+\eta} K(x,s)\mathcal{T}(0,a)\phi(s)ds$. This is different of [22] and [42], whose related operators are local. Therefore, $\lambda \in \rho(\mathbb{A})$ if and only if $1 \in \rho(\mathcal{B}_\lambda)$. Furthermore, it follows from (2.7) and (2.8) that

$$\phi(0,x) =(\mathbb{I}-\mathcal{B}_\lambda)^{-1}\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)\int_0^a e^{-\lambda(a-\delta)}\mathcal{T}(\delta,a)\psi(\delta,s)d\delta ds da,$$

and

$$\begin{aligned}R(\lambda,\mathbb{A})\psi(a,x) =& e^{-\lambda a}\mathcal{T}(0,a)(\mathbb{I}-\mathcal{B}_\lambda)^{-1}\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)\int_0^a e^{-\lambda(a-\delta)}\\ &\mathcal{T}(\delta,a)\psi(\delta,s)d\delta ds da+\int_0^a e^{-\lambda(a-\delta)}\mathcal{T}(\delta,a)\psi(\delta,x)d\delta.\end{aligned} \tag{2.9}$$

By the definitions of $K(x,s)$ and $\mathcal{T}(0,a)$, we can show that

$$\|\mathcal{B}_\lambda\| \leq \|\int_0^{a_\dagger}\beta(a)e^{-\lambda a}e^{-\int_0^a\mu(\rho)d\rho}e^{\mathbb{B}a}da\|,$$

which implies that

$$\lim_{\lambda\to+\infty}\|\mathcal{B}_\lambda\| = 0.$$

Hence, for all sufficiently large $\lambda > 0$, $(\mathbb{I}-\mathcal{B}_\lambda)^{-1}$ exists and is bounded. Thus $1 \in \rho(\mathcal{B}_\lambda)$ which is equivalent to $\lambda \in \rho(\mathbb{A})$.

From (2.6) , one can obtain after some computations that

$$\|R^n(\lambda,\mathbb{A})\| \leq \frac{M}{(\lambda-\omega)^n}, \qquad n = 1,2,3\cdots.$$

This completes the proof. □

2.2 Asymptotic behavior

In this section, we study the asymptotic behavior of solutions of (1.2) by analyzing the spectrum of the semigroup. It means that we will prove Theorem 1.1.

Now, we state the asymptotic expression which indicates the asymptotic behavior.

Theorem 2.4. **(1)** *For the eigenvalues of the operator $\mathbb{A}$, there is only one real eigenvalue λ_0 which is algebraically simple and is larger than any real part of the other eigenvalues.*

(2) *The semigroup $T(t)$ has the asymptotic expression*

$$T(t)\phi(a,x) = e^{\lambda_0 t}e^{-\lambda_0 a}\mathcal{T}(0,a)C_{\lambda_0}\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)\int_0^a e^{-\lambda_0(a-\delta)} \mathcal{T}(\delta,a)\phi(\delta,s)d\delta ds da + o(e^{(\lambda_0-\varepsilon)t})$$

where $C_{\lambda_0} = \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)(\mathbb{I}-\mathcal{B}_\lambda)^{-1}$ and ε is any positive number such that $\sigma(\mathbb{A})\cap\{\lambda|\lambda_0-\varepsilon\le Re\lambda\le\lambda_0\}=\lambda_0$ holds.

Proof. (1) It will be done in two steps: (i) prove that $\mathbb{A}$ has only one real eigenvalue λ_0 and λ_0 is larger than any real part of the other eigenvalues; (ii) prove that λ_0 is algebraically simple by showing $T(t)$ is compact for $t\ge a_\dagger$.

(i) Define

$$E=\{\phi\in L^2([0,24])|\int_{x-\eta}^{x+\eta}K(x,s)\phi(s)ds\ge C\phi(x)\},$$

where $C>0$ is a sufficiently small constant.

Recall $\mathcal{F}_\lambda$ in (2.3) and denote the restrictions of $\mathcal{B}_\lambda$, $\mathcal{F}_\lambda$ on E by $\overline{\mathcal{B}_\lambda}, \overline{\mathcal{F}_\lambda}$ respectively. Then from (2.8) and (2.3),

$$\overline{\mathcal{B}_\lambda}\ge\overline{\mathcal{F}_\lambda}.$$

Given any nonnegative function $\phi(x)$, $\psi(x)\in L^2([0,24])$, both not identical to zero, then from [9] and [71], $\langle e^{\mathbb{B}a}\phi,\psi\rangle>0$ for all $a>0$. From the expression of $\mathcal{B}_\lambda$ and $K(x,s)$, it follows that

$$\langle\mathcal{B}_\lambda\phi,\psi\rangle>0,\ \text{for all real }\lambda>0. \tag{2.10}$$

Furthermore, if $\phi(x)\in E$, from assumption $(J1)$, $(J2)$ and the expression of $\mathcal{F}_\lambda$, we know that

$$\langle\overline{\mathcal{B}}_\lambda\phi,\psi\rangle\ge\langle\overline{\mathcal{F}}_\lambda\phi,\psi\rangle>0,\ \text{for all real }\lambda>0.$$

From Lemma 2.1, there is a $\widetilde{\lambda}_0$ such that $\gamma(\mathcal{F}_{\widetilde{\lambda}_0})=1$ and 1 is an eigenvalue of $\mathcal{F}_{\widetilde{\lambda}_0}$ with the eigenfunction $\phi_0(x)$. Remembering that $\phi_0(x)$ is a positive constant, it is easy to check that $\phi_0(x)\in E$, even if it means reducing C. Hence,

$$\overline{\mathcal{F}}_{\widetilde{\lambda}_0}\phi_0=\mathcal{F}_{\widetilde{\lambda}_0}\phi_0=\phi_0,$$

which implies

$$\gamma(\overline{\mathcal{F}}_{\widetilde{\lambda}_0})\ge 1.$$

Moreover, since $\gamma(\overline{\mathcal{F}}_{\widetilde{\lambda}_0}) \leq \gamma(\mathcal{F}_{\widetilde{\lambda}_0}) = 1$, one obtains that

$$\gamma(\overline{\mathcal{F}}_{\widetilde{\lambda}_0}) = 1.$$

Therefore we conclude that

$$\gamma(\mathcal{B}_{\widetilde{\lambda}_0}) \geq \gamma(\overline{\mathcal{B}}_{\widetilde{\lambda}_0}) \geq \gamma(\overline{\mathcal{F}}_{\widetilde{\lambda}_0}) = 1.$$

On the other hand, $\lim_{\lambda \to +\infty} \gamma(\mathcal{B}_\lambda) = 0$ and hence by continuity there exists a real λ_0 such that

$$\gamma(\mathcal{B}_{\lambda_0}) = 1.$$

Since $\mathcal{B}_{\lambda_0}$ is a compact positive operator, by Krein-Rutman Theorem there exists a nonnegative $\phi_{\lambda_0}(x) \in L^2(0, 24)$ such that

$$\mathcal{B}_{\lambda_0}\phi_{\lambda_0}(x) = \phi_{\lambda_0}(x), \tag{2.11}$$

it means that

$$\sigma(\mathcal{B}_{\lambda_0}) \neq \varnothing.$$

Since (2.10), the operator $\mathcal{B}_\lambda$ is semi-nonsupporting. From Theorem 4.3 of [67], we learn that $\gamma(\mathcal{B}_\lambda)$ is strictly monotone decreasing with respect to real λ. This is equivalent to the uniqueness of the real eigenvalue of operator $\mathbb{A}$. That is,

$$\sigma(\mathbb{A}) \neq \varnothing.$$

When $\lambda > \lambda_0$ and $\gamma(\mathcal{B}_\lambda) < \gamma(\mathcal{B}_{\lambda_0}) = 1$, $(\mathbb{I} - \mathcal{B}_\lambda)^{-1}$ exists and is positive, and hence $R(\lambda, \mathbb{A})$ is positive from (2.9). Thus, λ_0 is larger than any real part of the other eigenvalues.

(ii) Integrating along the characteristic, we obtain

$$p(a, t, x) = \begin{cases} \mathcal{T}(a - t, 0, t)p_0(a - t, x), \ \ a \geq t, \\ \mathcal{T}(0, 0, a) \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x, s)p(a, t - a, s)dsda, \ \ a < t. \end{cases}$$

When $t \geq a_\dagger$,

$$T(t)\phi(a, x) = \mathcal{T}(0, 0, a) \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x, s)[T(t - a)\phi](a, s)dsda.$$

Let ϕ_n weakly converge to ϕ in X. By the compactness of $\mathcal{T}(0,0,a)$, one has

$$\begin{aligned}&\|\mathcal{T}(0,0,a)\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)[T(t-a)(\phi_n-\phi)](a,s)dsda\|_{L^2([0,24])}\\ \to&0,\ n\to+\infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\|\mathcal{T}(0,0,a)\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)[T(t-a)(\phi_n-\phi)](a,s)dsda\|_{L^2([0,24])}\\ \leq&\|\mathcal{T}(0,0,a)\|\|\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)[T(t-a)(\phi_n-\phi)](a,s)dsda\|_{L^2([0,24])}\\ \leq&M\|\phi_n-\phi\|_{L^2([0,24])}\end{aligned}$$

is bounded. Using the dominant convergence theorem, we get

$$\lim_{n\to\infty}\|T(t)(\phi_n-\phi)\|=0.$$

That is, $T(t)\phi_n$ converge strongly to $T(t)\phi$. Thus, $T(t)$ is compact.

By the results of [25], the semigroup $T(t)$ generated by $\mathbb{A}$, is a positive semigroup and

$$\lambda_0=s(\mathbb{A})=\omega_0(\mathbb{A})$$

where $s(\mathbb{A})$, $\omega_0(\mathbb{A})$ denote the spectral bound of $\mathbb{A}$ and the growth bound of the semigroup $T(t)$ respectively. Since $T(t)$ is compact, it is known from [25] that

$$\omega_{ess}(\mathbb{A})=-\infty.$$

Furthermore, from Theorem 9.10 in [25], it is easy to get that

$$\lambda_0=\{\lambda|Re\lambda=s(\mathbb{A})\}.$$

It means that λ_0 is a pole of the resolvent of $R(\lambda,\mathbb{A})$. Thus,

$$\gamma(\mathcal{B}_{\lambda_0})=1$$

is a pole of $R(\lambda,\mathcal{B}_{\lambda_0})$. Moreover, by (2.10), one obtains that $\mathcal{B}_{\lambda_0}$ is a non-semisupporting operator. Since Theorem 1 in [80], one can obtain that

$$\gamma(\mathcal{B}_{\lambda_0})=1$$

is an algebraically simple eigenvalue of $\mathcal{B}_{\lambda_0}$. This is equivalent of λ_0 being an algebraically simple eigenvalue of $\mathbb{A}$.

(2) From (1), we have that

$$\sigma(\mathbb{A}) \cap \{\lambda|\lambda_0 - \varepsilon \leq Re\lambda \leq \lambda_0\} = \lambda_0,$$

and $T(t)$ is a compact operator. Then from Theorem 5 of [104], there are constants C and T_0, such that

$$\|T(t) - T(t)P_{\lambda_0}\| \leq Ce^{(\lambda_0-\varepsilon)t}, t \geq T_0,$$

where $T(t)$ is the semigroup generated by $\mathbb{A}$, P_{λ_0} is the mapping from X to B_{λ_0}, and B_{λ_0} is the eigenvalue space of λ_0 of $\mathbb{A}$. Furthermore,

$$T(t)\phi = T(t)P_{\lambda_0}\phi + o(e^{(\lambda_0-\varepsilon)t}). \tag{2.12}$$

Since λ_0 is an algebraically simple eigenvalue of $\mathbb{A}$, it is known from [49] that

$$P_{\lambda_0}\phi = \lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)R(\lambda, \mathbb{A})\phi. \tag{2.13}$$

Combining (2.12) and (2.13),

$$T(t)\phi = e^{\lambda_0 t}\lim_{\lambda\to\lambda_0}(\lambda - \lambda_0)R(\lambda, \mathbb{A})\phi + o(e^{(\lambda_0-\varepsilon)t}).$$

Then, using the expression (2.9) of $R(\lambda, \mathbb{A})\phi$,

$$\begin{aligned} T(t)\phi(a,x) =& e^{\lambda_0 t}e^{-\lambda_0 a}\mathfrak{T}(0,a)C_{\lambda_0}\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s) \\ & \int_0^a e^{-\lambda_0(a-\delta)}\mathfrak{T}(\delta,a)\phi(\delta,s)d\delta ds da \\ & + o(e^{(\lambda_0-\varepsilon)t}). \end{aligned}$$

□

Remark 2.5. *Here, we can see that Theorem 1.1 is a direct result of Theorem 2.4, so the proof of Theorem 1.1 is complete.*

2.3 Existence of steady states

As for the steady states (1.5) satisfying (1.6), our main result is Theorem 1.2. In this section, we prove Theorem 1.2 directly according to Theorem 2.4.

Proof. Firstly, let λ_0 be as defined in Theorem 2.4. Then, we look for the steady states (1.5) in the following three cases according to the sign of λ_0.

(1) When $\lambda_0 > 0$, we argue this case by a contradiction. Assume that $p_s(a,x)$ is a nonnegative solution of (1.5) satisfying (1.6). It is easy to see that $p_s(a,x) = p(a,t,x)$ is also a solution of the following system

$$\begin{cases} Dp(a,t,x) - \delta\Delta p(a,t,x) + \mu(a)p(a,t,x) = 0, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_s(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases}$$

Then by a result of Theorem 1.1, one can have the following asymptotic expression

$$\begin{aligned} p(a,t,x) =& e^{\lambda_0 t} e^{-\lambda_0 a} \mathcal{T}(0,a) C_{\lambda_0} \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) \\ & \int_0^a e^{-\lambda_0(a-\sigma)} \mathcal{T}(\sigma,a) p_0(\sigma,s) dsdad\sigma + o(e^{(\lambda_0-\varepsilon)t}). \end{aligned}$$

Thus,

$$\|p_s(a,x)\|_{L^2((0,a_\dagger)\times(0,24))} = \lim_{t\to+\infty} \|p(a,t,x)\|_{L^2((0,a_\dagger)\times(0,24))} = +\infty,$$

which is a contradiction. Thus, there is no nonnegative solution of (1.5) satisfying (1.6).

(2) When $\lambda_0 = 0$, it means that $0 \in \sigma(\mathbb{A})$. From the definition of $\mathbb{A}$, every eigenfunction related to 0 and its multiplications by any constant are solutions of (1.5).

Recalling (2.11) from the proof of Theorem 2.4, there is a nonnegative function $\phi_{\lambda_0}(x) \in L^2(0,24)$ such that

$$\mathcal{B}_{\lambda_0}(\phi_{\lambda_0}(x)) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) e^{-\lambda_0 a} \mathcal{T}(0,a) \phi_{\lambda_0}(s) dsda = \phi_{\lambda_0}(x).$$

By Lemma 2.2, one knows that $\mathcal{T}(0,a)$ is a bounded operator on X. Using

Cauchy-Schwarz inequality, for arbitrary $x_0 \in (0,24)$, one obtains

$$\begin{aligned}
&|\phi_{\lambda_0}(x) - \phi_{\lambda_0}(x_0)| \\
=&\left|\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) e^{-\lambda_0 a} \mathcal{T}(0,a)\phi_{\lambda_0}(s) ds da \right.\\
&\left. - \int_0^{a_\dagger} \beta(a) \int_{x_0-\eta}^{x_0+\eta} K(x_0,s) e^{-\lambda_0 a} \mathcal{T}(0,a)\phi_{\lambda_0}(s) ds da \right| \\
\le&\|\beta(a)\|_{L^\infty(0,a_\dagger)} \left|\int_0^{a_\dagger} \int_{x-\eta}^{x+\eta} K(x,s)\mathcal{T}(0,a)\phi_{\lambda_0}(s) ds da \right.\\
&\left. - \int_0^{a_\dagger} \int_{x_0-\eta}^{x_0+\eta} K(x_0,s)\mathcal{T}(0,a)\phi_{\lambda_0}(s) ds da \right| \\
\le&\|\beta(a)\|_{L^\infty(0,a_\dagger)} \left|\int_0^{a_\dagger} \int_{x-\eta}^{x+\eta} (K(x,s) - K(x_0,s))\mathcal{T}(0,a)\phi_{\lambda_0}(s) ds da \right| \\
&+ \|\beta(a)\|_{L^\infty(0,a_\dagger)} \left|\int_0^{a_\dagger} \int_{x_0-\eta}^{x+\eta} K(x_0,s)\mathcal{T}(0,a)\phi_{\lambda_0}(s) ds da \right| \\
&+ \|\beta(a)\|_{L^\infty(0,a_\dagger)} \left|\int_0^{a_\dagger} \int_{x_0+\eta}^{x+\eta} K(x_0,s)\mathcal{T}(0,a)\phi_{\lambda_0}(s) ds da \right| \\
\le&\|\beta(a)\|_{L^\infty(0,a_\dagger)} \|K(x,s) - K(x_0,s)\|_{L^2(x-\eta,x+\eta)} \|\mathcal{T}(0,a)\phi_{\lambda_0}(s)\|_{L^2((0,a_\dagger)\times(0,24))} \\
&+ \|\beta(a)\|_{L^\infty(0,a_\dagger)} \left(\int_{x_0-\eta}^{x+\eta} |K(x_0,s)|^2 ds\right)^{\frac{1}{2}} \|\mathcal{T}(0,a)\phi_{\lambda_0}(s)\|_{L^2((0,a_\dagger)\times(0,24))} \\
&+ \|\beta(a)\|_{L^\infty(0,a_\dagger)} \left(\int_{x_0+\eta}^{x+\eta} |K(x_0,s)|^2 ds\right)^{\frac{1}{2}} \|\mathcal{T}(0,a)\phi_{\lambda_0}(s)\|_{L^2((0,a_\dagger)\times(0,24))} \\
\le&C\|\beta(a)\|_{L^\infty(0,a_\dagger)} \|K(x,s) - K(x_0,s)\|_{L^2(x-\eta,x+\eta)} \|\phi_{\lambda_0}(s)\|_{L^2(0,24)} \\
&+ C\|\beta(a)\|_{L^\infty(0,a_\dagger)} \left(\int_{x_0-\eta}^{x+\eta} |K(x_0,s)|^2 ds\right)^{\frac{1}{2}} \|\phi_{\lambda_0}(s)\|_{L^2(0,24)} \\
&+ C\|\beta(a)\|_{L^\infty(0,a_\dagger)} \left(\int_{x_0+\eta}^{x+\eta} |K(x_0,s)|^2 ds\right)^{\frac{1}{2}} \|\phi_{\lambda_0}(s)\|_{L^2(0,24)} \\
\to&0, \text{ as } x \to x_0.
\end{aligned}$$

Thus, $\phi_{\lambda_0}(x)$ is continuous about x. Then, from the proof of Lemma 2.3, it is easy to check that

$$\phi(a,x) = \mathcal{T}(0,a)\phi_{\lambda_0}(x)$$

is an eigenfunction of the eigenvalue $\lambda_0 = 0$ of $\mathbb{A}$. Therefore, the steady states are

$$p_s(a,x) = c\mathcal{T}(0,a)\phi_{\lambda_0}(x) \ge 0, \text{ for any constant } c > 0.$$

By a result of Lemma 2.2, we know that $\mathcal{T}(0,a)$ is strongly continuous with respect to a. Hence, $p_s(a,x)$ is continuous about a, x in $(0,a_\dagger)\times(0,24)$.

Consider smooth function $v(a,x)$ such that

$$v(a,x)=e^{\int_0^a\mu(\rho)d\rho}p_s(a,x)\geq 0,\;\; a.e\;(a,x)\in(0,a_\dagger)\times(0,24).$$

Then, from (1.5), $v(a,x)$ satisfies

$$\begin{cases}\partial_a v-\delta\Delta v=0, & (a,x)\in(0,a_\dagger)\times(0,24),\\ v(a,0)=v(a,24), & a\in(0,a_\dagger),\\ \partial_x v(a,0)=\partial_x v(a,24), & a\in(0,a_\dagger),\\ v(0,x)=\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)e^{-\int_0^a\mu(\rho)d\rho}v(a,s)dsda, & x\in(0,24).\end{cases} \tag{2.14}$$

From the strong maximum principle,

$$v(a,x)>0,\;\;\text{for}(0,a_\dagger)\times(0,24).$$

Then,

$$v(0,x)=\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)e^{-\int_0^a\mu(\rho)d\rho}v(a,s)dsda>0,\;\;\text{for}x\in(0,24).$$

Assume by contradiction that v attains its minimum 0 at $(a_0,0)$, that is,

$$v(a_0,0)=0,\;\;for\;some\;a_0\in(0,a_\dagger).$$

Then,

$$\partial_a v(a_0,0)=0,\;\;and\;\partial_x v(a_0,0)\geq 0.$$

Since $v(a,o)=v(a,24)$ for $a\in(0,a_\dagger)$, one has that

$$v(a_0,24)=0,\;\;\partial_a v(a_0,24)=0and\;\partial_x v(a_0,24)\leq 0.$$

Since $\partial_x v(a,0)=\partial_x v(a,24)$ for $a\in(0,a_\dagger)$, we obtain that

$$\partial_x v(a_0,0)=\partial_x v(a_0,24)=0.$$

Since $v(a,x)>0$ for $(0,a_\dagger)\times(0,24)$,

$$\Delta v(a_0,0)=\partial_{xx}v(a_0,0)>0.$$

Thus,

$$(\partial_a v-\delta\Delta v)(a_0,0)<0$$

which is a contradiction of the first equation of (2.14). So that,

$$v(a,0), v(a,24) > 0, \;\; for \;\; a \in (0, a_\dagger).$$

By $v(0,x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) e^{-\int_0^a \mu(\rho)d\rho} v(a,s) dsda$, one also has that

$$v(0,0), v(0,24) > 0.$$

Therefore, we can conclude that for any $a_1 < a_\dagger$,

$$p_s(a,x) = e^{-\int_0^a \mu(\rho)d\rho} v(a,x) > 0, \;\; \text{a.e. in } [0,a_1] \times [0,24]$$

since $\int_0^a \mu(\rho)d\rho < \infty$ for $a < a_\dagger$. Finally, there exists $\rho_0 > 0$ such that

$$p_s(a,x) \geq \rho_0 > 0, \;\; \text{a.e. } (a,x) \in (0,a_1) \times (0,24).$$

(3) When $\lambda_0 < 0$, it follows from the arguments of (1) that

$$\|p_s(a,x)\|_{L^2((0,a_\dagger)\times(0,24))} = \lim_{t\to+\infty} \|p(a,t,x)\|_{L^2((0,a_\dagger)\times(0,24))} = 0.$$

Thus,

$$p_s(a,x) = 0 \;\; a.e. \; (a,x) \in (0,a_\dagger) \times (0,24).$$

□

2.4 Numerical simulations

In the following, we consider the mosquito plasticity with the usage of insecticide-treated bed nets (ITNs), that is, we consider

$$\begin{cases} Dp - \delta \Delta_x p + \mu(a) p = u(x) p, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p(a,t,s) dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{2.15}$$

We provide some numerical simulations to illustrate the interaction between the solution of (2.15) and the usage of ITNs. We assume that $a_\dagger = 1$, that is, $a \in [0,1)$. We consider system (2.15) with the parameters taking the values as follows

$$\delta = 0.001,\ \eta = 0.1\ \ \beta(a) = 50e^{-0.1(a-0.4)^2} \text{ and } \mu(a) = 0.5e^{2.4a}.$$

We set that the mosquitoes are alive between 21:00 pm and 3:00 am at the initial time and the mosquitoes reach the bitting peak at 24:00. That is, we set that

$$p_0(a,x) = 0.5e^{-(0.625(x-12))^2}e^{-10(a-0.4)^2}.$$

We consider that people use ITNs when they sleep from 22:00 pm to 2:00 am, that is, we set the usage of ITNs by

$$u(x) = \begin{cases} 25, & x \in [10,14], \\ 0, & \text{else.} \end{cases}$$

Then, in figures 1-3, we plot the solution of (2.15). We can see that the bitting peak of mosquitoes shifts to the 7:00 pm and 7:00 am.

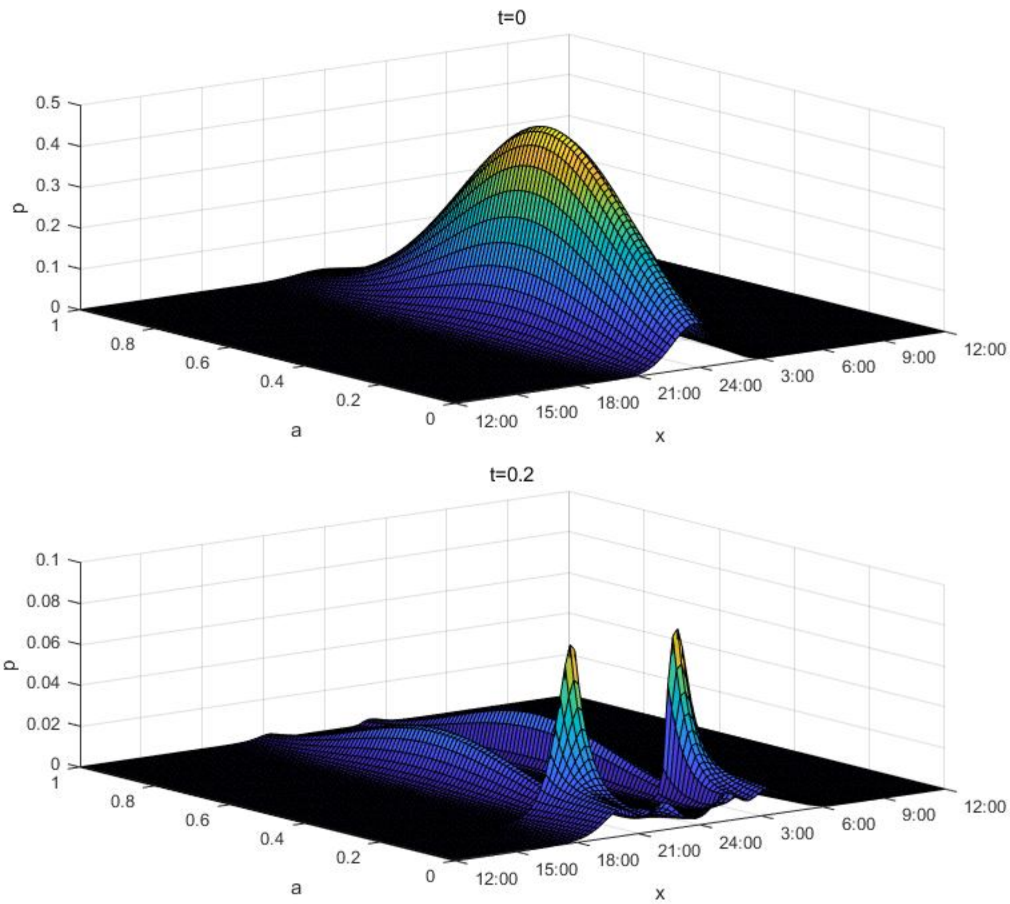

Figure 1: the solution $p(a,t,x)$ of (2.15) for $t=0$ and $t=0.2$ with ITNs.

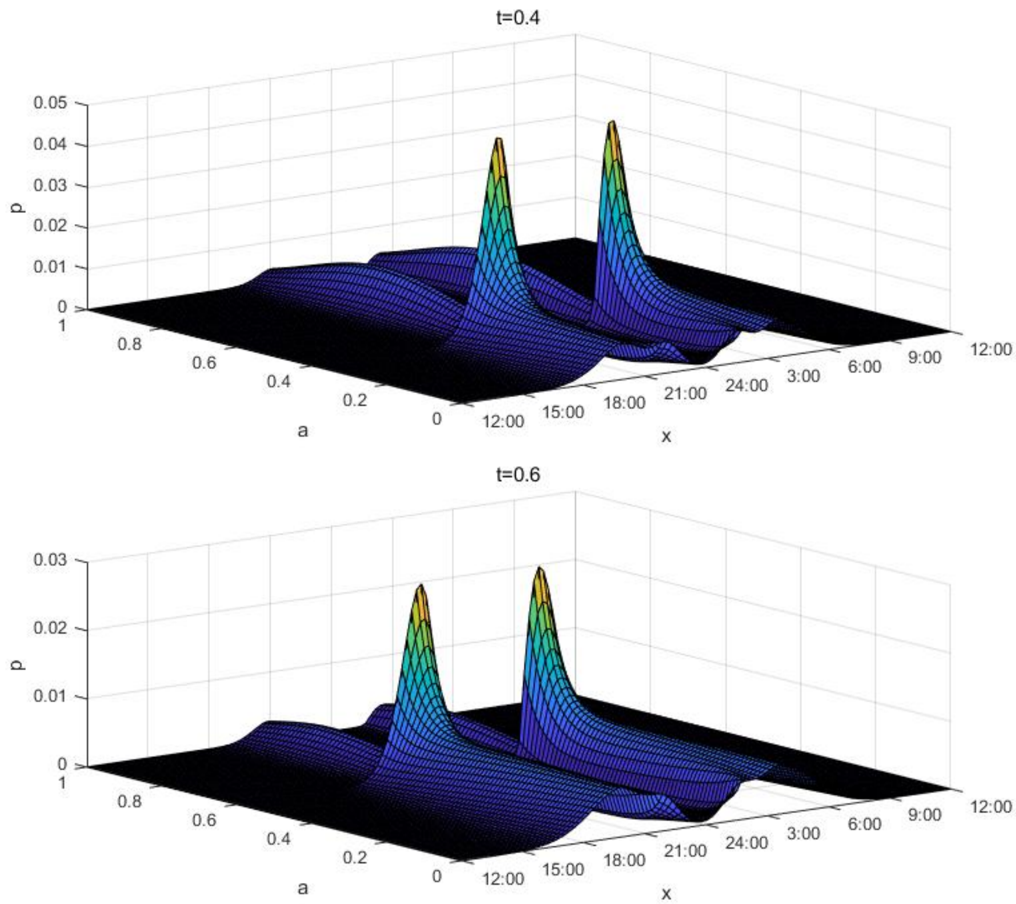

Figure 2: the solution $p(a,t,x)$ of (2.15) for $t = 0.4$ and $t = 0.6$ with ITNs.

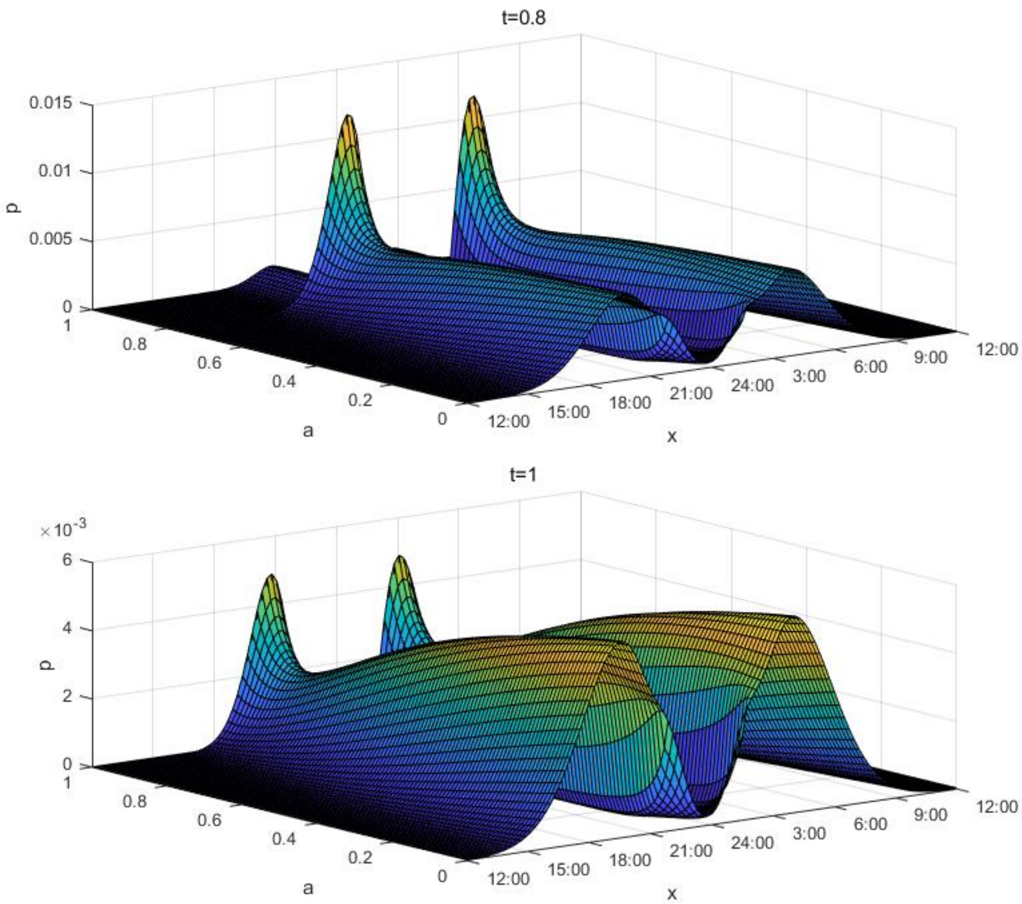

Figure 3: the solution $p(a, t, x)$ of (2.15) for $t = 0.8$ and $t = 1$ with ITNs.

Chapter 3

Optimal control of an age structured problem

In this chapter, we set the control term $u(a,t,x)$, such as ITNs and IRs, is bounded according to reality that the insecticide strategy can not be used unlimited. Our goal is to prove that there exist an optimal control $u(a,t,x)$ such that the insecticidal efficiency is the best. We also give the necessary optimality conditions.

3.1 Preliminaries

In this section, we study some properties of the following system, which is (1.7) in general settings,

$$\begin{cases} Dp - \delta\Delta p + \mu(a,t,x)p = f(a,t,x), & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases} \tag{3.1}$$

where β, p_0 are under the assumptions $(J2)$, $(J3)$, μ and f satisfy

$$\mu(a,t,x) \in L^\infty_{loc}([0,a_\dagger) \times [0,T] \times [0,24]),\ \mu(a,t,x) \geq 0 \text{ a.e. in } Q_{a_\dagger}, \tag{3.2}$$

$$f(a,t,x) \in L^2(Q_{a_\dagger}),\quad f(a,t,x) \geq 0 \text{ a.e. in } Q_{a_\dagger}.$$

Especially, we prove that there exists a unique solution of system (3.1) and the comparison principle for system (3.1).

Before going further, we need an auxiliary lemma, which can be proved by following the proof of [13, Lemma A2.7].

Lemma 3.1. *For any $y_0(x) \in L^2(0,24)$, $g(t,x) \in L^2((0,T)\times(0,24))$, there exists a unique solution*

$$y(t,x) \in L^2((0,T);H^1(0,24)) \cup L^2_{loc}((0,T);H^2(0,24))$$

of the following system

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) - \delta\Delta y(t,x) = g(t,x), & (t,x)\in(0,T)\times(0,24), \\ y(t,0) = y(t,24), & t\in(0,T), \\ y'(t,0) = y'(t,24), & t\in(0,T), \\ y(0,x) = y_0(x), & x\in(0,24). \end{cases}$$

Remark 3.2. *It is known that there exists an orthogonal basis $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2(0,24)$ and $\{\lambda_j\}\subset\mathbb{R}^+$, $\lambda_0 = 0$, $\lambda_j \to +\infty$ as $j\to+\infty$ such that*

$$\begin{cases} -\Delta\varphi_j(x) = \lambda_j\varphi_j(x), \text{ in } (0,24), \\ \varphi_j(0) = \varphi(24), \\ \varphi_j'(0) = \varphi_j'(24). \end{cases}$$

We can replace the basis in the proof of [13, Lemma A2.7] by our $\{\varphi_j\}_{j\in\mathbb{N}}$ and follow the same proof to get Lemma 3.1.

Let us first deal with the case when μ satisfies

(A) $\mu\in L^\infty(Q_{a_\dagger})$, $\mu(a,t,x)\geq 0$ a.e. in $Q_{a_\dagger}$.

Lemma 3.3. *For any fixed $f(a,t,x)\in L^2(Q_{a_\dagger})$, $b(t,x)\in L^2((0,T)\times(0,24))$, there exists a unique solution $p_b(a,t,x)\in L^2(Q_{a_\dagger})$ of the following system*

$$\begin{cases} Dp - \delta\Delta p + \mu(a,t,x)p = f(a,t,x), & (a,t,x)\in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t)\in(0,a_\dagger)\times(0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t)\in(0,a_\dagger)\times(0,T), \\ p(0,t,x) = b(t,x), & (t,x)\in(0,T)\times(0,24), \\ p(a,0,x) = p_0(a,x), & (a,x)\in(0,a_\dagger)\times(0,24), \end{cases} \tag{3.3}$$

where μ is under (A).

Proof. Fix any $q(a,t,x)\in L^2(Q_{a_\dagger})$, we first prove that the following system has a unique solution $p_{b,q}(a,t,x)$,

$$\begin{cases} Dp - \delta\Delta p + \mu q = f, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = b(t,x), & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{3.4}$$

Let S be an arbitrary characteristic line of equation

$$S = \{(a_0 + s, t_0 + s); s \in (0,\alpha)\},$$

where $(a_0,t_0) \in \{0\} \times (0,T) \cup (0,a_\dagger) \times \{0\}$ and $(a_0+\alpha, t_0+\alpha) \in \{a_\dagger\} \times (0,T) \cup (0,a_\dagger) \times \{T\}$ and define

$$\begin{cases} \widetilde{p}(s,x) = p(a_0+s, t_0+s, x), & (s,x) \in (0,\alpha) \times (0,24), \\ \widetilde{q}(s,x) = q(a_0+s, t_0+s, x), & (s,x) \in (0,\alpha) \times (0,24), \\ \widetilde{f}(s,x) = f(a_0+s, t_0+s, x), & (s,x) \in (0,\alpha) \times (0,24), \\ \widetilde{\mu}(s,x) = \mu(a_0+s, t_0+s, x), & (s,x) \in (0,\alpha) \times (0,24). \end{cases} \tag{3.5}$$

According to Lemma 3.1, the following system admits a unique solution

$$\widetilde{p} \in L^2((0,\alpha); H^1(0,24)) \cap L^2{}_{loc}((0,\alpha); H^2(0,24)),$$

$$\begin{cases} \frac{\partial \widetilde{p}}{\partial s} - \delta\Delta\widetilde{p} = \widetilde{f} - \widetilde{\mu}\widetilde{q}, & (s,x) \in (0,\alpha) \times (0,24), \\ \partial_x \widetilde{p}(s,0) = \partial_x \widetilde{p}(s,24), & s \in (0,\alpha), \\ \widetilde{p}(s,0) = \widetilde{p}(s,24), & s \in (0,\alpha), \\ \widetilde{p}(0,x) = \begin{cases} b(t_0,x), & a_0 = 0, \quad x \in (0,24), \\ p_0(a_0,x), & t_0 = 0, \quad x \in (0,24). \end{cases} \end{cases} \tag{3.6}$$

In fact, multiplying the first equation of system (3.6) by $\widetilde{p}$ and integrating on $(0,s) \times (0,24)$, one has

$$\begin{aligned} \|\widetilde{p}(s)\|^2_{L^2(0,24)} \leq & \|\widetilde{p}(0)\|^2_{L^2(0,24)} + \|\widetilde{f} - \widetilde{\mu}\widetilde{q}\|^2_{L^2((0,\alpha)\times(0,24))} \\ & + \int_0^s \|\widetilde{p}(\tau)\|^2_{L^2(0,24)} d\tau, \ \forall s \in [0,\alpha]. \end{aligned}$$

Then by a lemma from Bellman (see in Appendix) we get

$$\|\widetilde{p}(s)\|^2_{L^2(0,24)} \leq C(\|\widetilde{p}(0)\|^2_{L^2(0,24)} + \|\widetilde{f} - \widetilde{\mu}\widetilde{q}\|^2_{L^2((0,\alpha)(0,24))})e^{\alpha}, \ \forall s \in [0,\alpha]. \tag{3.7}$$

Now let us denote

$$p_{b,q}(a_0+s, t_0+s, x) = \widetilde{p}(s,x),\ (s,x) \in (0,\alpha)\times(0,24)$$

for any characteristic line S. It follows from Lemma 3.1 and (3.7) that

$$p_{b,q} \in L^2(S; H^1(0,24)) \cap L^2{}_{loc}(S; H^2(0,24))$$

for almost any characteristic line S, and $p_{b,q}$ satisfies

$$\begin{cases} Dp_{b,q} - \delta\Delta p_{b,q} + \mu(a,t,x)q = f(a,t,x), & (a,t,x) \in Q_{a_\dagger}, \\ p_{b,q}(a,t,0) = p_{b,q}(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ \partial_x p_{b,q}(a,t,0) = \partial_x p_{b,q}(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ p_{b,q}(0,t,x) = b(t,x), & (t,x) \in (0,T)\times(0,24), \\ p_{b,q}(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger)\times(0,24). \end{cases} \tag{3.8}$$

Now we prove that

$$p_{b,q}(a,t,x) \in L^2(Q_{a_\dagger}).$$

It is known that there exists an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}} \subset L^2(0,24)$ and $\{\lambda_j\} \subset \mathbb{R}^+$, $\lambda_0 = 0$, $\lambda_j \to +\infty$ as $j \to +\infty$ such that

$$\begin{cases} -\Delta\varphi_j(x) = \lambda_j\varphi_j(x),\ \text{in } (0,24), \\ \varphi_j(0) = \varphi(24), \\ \varphi_j'(0) = \varphi_j'(24). \end{cases}$$

Then, one has that

$$\begin{aligned} &f(a,t,x) - \mu(a,t,x)q(a,t,x) \\ =&\sum_{j=1}^{\infty} v^j(a,t)\varphi_j(x),\quad \text{in } L^2(0,24),\ \text{a.e. } (a,t) \in (0,a_\dagger)\times(0,T), \end{aligned}$$

$$b(t,x) = \sum_{j=1}^{\infty} b^j(t)\varphi_j(x),\ \text{in } L^2(0,24),\ \text{a.e. } t \in (0,T),$$

$$p_0(a,x) = \sum_{j=1}^{\infty} p_0^j(a)\varphi_j(x),\ \text{in } L^2(0,24),\ \text{a.e. } a \in (0,a_\dagger).$$

Furthermore, $p_{b,q}(a,t,x)$ has the following expression

$$p_{b,q}(a,t,x) := \sum_{j=1}^{\infty} p_{b,q}^j(a,t)\varphi_j(x),\ \text{in } L^2(0,24)\ \text{a.e. } (a,t) \in (0,a_\dagger)\times(0,T).$$

By substituting $p_{b,q}(a,t,x)$ into (3.8), one gets that $p^j_{b,q}(a,t)$ satisfies

$$\begin{cases} Dp^j_{b,q} + \lambda_j \delta p^j_{b,q} = v^j(a,t), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p^j_{b,q}(0,t) = b^j(t), & t \in (0,T) \\ p^j_{b,q}(a,0) = p^j_0(a), & a \in (0,a_\dagger). \end{cases}$$

One can follow the computation of Lemma 4.1 in Aniţa [13, p.113 − 114] and get that $p_{b,q}(a,t,x) \in L^2(Q_{a_\dagger})$ satisfies

$$\|p_{b,q}\|^2_{L^2(Q_{a_\dagger})} \leq e^T(\|p_0\|^2_{L^2((0,a_\dagger)\times(0,24))} + \|b\|^2_{L^2((0,T)\times(0,24))} + \|f-\mu q\|^2_{L^2(Q_{a_\dagger})}). \tag{3.9}$$

For an arbitrary $q(a,t,x) \in L^2(Q_{a_\dagger})$, we have obtained that system (3.4) has a solution $p_{b,q} \in L^2(Q_{a_\dagger})$. Define $\bar{p}_{b,q}(a,t,x) = 1/\lambda p_{b,q}(a,t,x)$ for a sufficiently large constant λ. Let us set a mapping $\Pi : L^2(Q_{a_\dagger}) \to L^2(Q_{a_\dagger})$ by

$$\Pi(q_i(a,t,x)) = \frac{1}{\lambda} p_{b,qi}(a,t,x) = \bar{p}_{b,qi}(a,t,x).$$

Take any two functions $q_1,\ q_2 \in L^2(Q_{a_\dagger})$ and then $\bar{p}_{b,q1} - \bar{p}_{b,q2}$ satisfies

$$\begin{cases} D(\bar{p}_{b,q1} - \bar{p}_{b,q2}) - \delta\Delta(\bar{p}_{b,q1} - \bar{p}_{b,q2}) + \frac{\mu}{\lambda}(q_1 - q_2) = 0,\ (a,t,x) \in Q_{a_\dagger}, \\ (\bar{p}_{b,q1} - \bar{p}_{b,q2})(a,t,0) = (\bar{p}_{b,q1} - \bar{p}_{b,q2})(a,t,24),\ (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x(\bar{p}_{b,q1} - \bar{p}_{b,q2})(a,t,0) = \partial_x(\bar{p}_{b,q1} - \bar{p}_{b,q2})(a,t,24),\ (a,t) \in (0,a_\dagger) \times (0,T), \\ (\bar{p}_{b,q1} - \bar{p}_{b,q2})(0,t,x) = 0,\ (t,x) \in (0,T) \times (0,24), \\ (\bar{p}_{b,q1} - \bar{p}_{b,q2})(a,0,x) = 0,\ (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{3.10}$$

By the result of (3.9), one has

$$\|\bar{p}_{b,q1} - \bar{p}_{b,q2}\|^2_{L^2(Q_{a_\dagger})} \leq \frac{e^T}{\lambda}(\|\mu(q_1-q_2)\|^2_{L^2(Q_{a_\dagger})}),\ in\ L^2(Q_{a_\dagger}).$$

Obviously, when T is small enough and since λ is sufficiently large, $\bar{p}_{b,q}(a,t,x)$ is a contraction mapping with respect to $q(a,t,x)$. Consequently, there exists a unique solution $p_b(a,t,x) = \lambda\bar{p}_b(a,t,x)$ of system (3.3) for sufficient small T. However, one can extend T by following previous steps for $t \in (T,2T)$. Thus, system (3.3) has a unique solution $p_b(a,t,x) \in L^2(Q_{a_\dagger})$. □

One can follow the same idea of the proof of [13, Lemma 4.1.2] to get the following Lemma.

Lemma 3.4. *For any* $b_1(t,x),\ b_2(t,x) \in L^2((0,T) \times (0,24)),\ 0 \leq b_1(t,x) \leq$

$b_2(t,x)$ *a.e. in* $(0,T)\times(0,24)$, *one has*

$$0 \le p_{b_1}(a,t,x) \le p_{b_2}(a,t,x), \ \text{a.e. in } Q_{a_\dagger},$$

where $p_{b_1}(a,t,x)$ *and* $p_{b_2}(a,t,x)$ *are the solutions of system* (3.3) *under* (A) *with* $b_1(a,t,x)$ *and* $b_2(a,t,x)$ *respectively.*

Lemma 3.5. *There exists a unique solution* $p(a,t,x)\in L^2(Q_{a_\dagger})$ *of system* (3.1) *under* (A).

Proof. Let us define an operator $\mathcal{F}: L^2((0,T)\times(0,24)) \to L^2((0,T)\times(0,24))$ by

$$(\mathcal{F}b)(t,x) = \int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)p_b(a,t,s)dsda, \quad \text{a.e. in } (0,T)\times(0,24).$$

For any fixed $b_i\in L^2((0,T)\times(0,24))$ $(i=1,2)$, let $p_{b_1}, p_{b_2}\in L^2(Q_{a_\dagger})$ be the solutions of system (3.1) with $b_1(a,t,x)$, $b_2(a,t,x)$ respectively. Let $v(a,t,x) = p_{b_1}(a,t,x) - p_{b_2}(a,t,x)$, then $v(a,t,x)$ satisfies

$$\begin{cases} Dv - \delta\Delta v + \mu(a,t,x)v = 0, & (a,t,x)\in Q_{a_\dagger}, \\ v(a,t,0) = v(a,t,24), & (a,t)\in(0,a_\dagger)\times(0,T), \\ \partial_x v(a,t,0) = \partial_x v(a,t,24), & (a,t)\in(0,a_\dagger)\times(0,T), \\ v(0,t,x) = b_1(t,x) - b_2(t,x), & (t,x)\in(0,T)\times(0,24), \\ v(a,0,x) = 0, & (a,x)\in(0,a_\dagger)\times(0,24). \end{cases} \tag{3.11}$$

Then it follows by the computation of Lemma 4.1 in Aniţa [13, $p.$116] that

$$\int_0^T e^{-\lambda t}\|v(t)\|^2_{L^2((0,a_\dagger)\times(0,24))}dt \le \frac{1}{\lambda}\int_0^T e^{-\lambda t}\|b_1(t)-b_2(t)\|^2_{L^2(0,24)}dt$$

for any $\lambda>0$. Consider $L^2((0,T)\times(0,24))$ with the norm $\|b\| = \left(\int_0^T e^{-\lambda t}\|b(t)\|^2_{L^2(0,24)}dt\right)^2$, for any $b\in L^2((0,T)\times(0,24))$. Then one has

$$\begin{aligned}
&\|\mathcal{F}b_1 - \mathcal{F}b_2\|^2 \\
=&\int_0^T e^{-\lambda t}\|\int_0^{a_\dagger}\beta(a)\int_{x-\eta}^{x+\eta}K(x,s)(p_{b_1}(a,t,s)-p_{b_2}(a,t,s))dsda\|^2_{L^2(0,24)}dt \\
\le& C\int_0^{a_\dagger}\beta^2(a)da\int_0^T e^{-\lambda t}\|v(t)\|^2_{L^2((0,a_\dagger)\times(0,24))}dt \\
\le& \frac{C}{\lambda}\int_0^{a_\dagger}\beta^2(a)da\|b_1-b_2\|^2,
\end{aligned}$$

where C is an appropriate positive constant related to $K(x,s)$. One can choose λ large such that

$$\lambda > C\int_0^{a_\dagger} \beta^2(a)da$$

and then $\mathcal{F}$ is a contraction mapping on $L^2((0,T)\times(0,24))$ with the norm $\|\cdot\|$. This completes the proof. □

From Lemma 3.5, one gets that the operator $\mathcal{F}$ is a contraction mapping. Moreover, combined with Lemma 3.4, one can follow the rest of the proof of [13, Lemma 4.1.1] to get the following comparison principle for (3.1).

Lemma 3.6. *If $p_i (i \in 1,2)$ are the solutions of the following systems*

$$\begin{cases} Dp_i - \delta\Delta p_i + \mu_i(a,t,x)p_i = f_i, & (a,t,x) \in Q_{a_\dagger}, \\ p_i(a,t,0) = p_i(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ \partial_x p_i(a,t,0) = \partial_x p_i(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ p_i(0,t,x) = \int_0^{a_\dagger} \beta_i(a)\int_{x-\eta}^{x+\eta} K(x,s)p_i(a,t,s)dsda, & (t,x) \in (0,T)\times(0,24), \\ p_i(a,0,x) = p_{0i}(a,x), & (a,x) \in (0,a_\dagger)\times(0,24), \end{cases}$$

where $\mu_1 \geq \mu_2$, $f_1 \leq f_2$, $\beta_1 \leq \beta_2$, $p_{01} \leq p_{02}$ and μ_1, μ_2 satisfy (A), then

$$0 \leq p_1(a,t,x) \leq p_2(a,t,x) \quad a.e. \text{ in } Q_{a_\dagger}.$$

By referring to the proof of [13, Theorem 4.1.3, Theorem 4.1.4] for the case when $\mu(a,t,x)$ satisfies (3.2), one can define

$$\mu^N(a,t,x) = \min\{\mu(a,t,x), N\}, \text{ for any } N \in \mathbb{N}^+,$$

and denote $p_N(a,t,x)$ to be the solution of system (3.1) with μ_N. Passing to the limit as $N \to +\infty$ for $p_N(a,t,x)$, one can get the solution of system (3.1). Then by the results of Lemma 3.5 and Lemma 3.6, we can get the following lemma.

Lemma 3.7. *There is a unique solution $p(a,t,x) \in L^2(Q_{a_\dagger})$ of system (3.1) with μ satisfying (3.2). If $p_i (i \in 1,2)$ are the solutions of system (3.1) with μ_1, f_1, β_1, p_{01} and μ_2, f_2, β_2, p_{02} respectively (μ_1, μ_2 satisfy (3.2)) and $\mu_1 \geq \mu_2$, $f_1 \leq f_2$, $\beta_1 \leq \beta_2$, $p_{01} \leq p_{02}$, then*

$$0 \leq p_1(a,t,x) \leq p_2(a,t,x) \quad a.e. \text{ in } Q_{a_\dagger}.$$

Remark 3.8. *According to Lemma 3.7 , we obtain the result of Theorem 1.3 directly.*

3.2 Existence of an optimal control

In this section, our main job is to obtain the existence of an optimal control of (1.7) by Mazur's Theorem, that is, we prove Theorem 1.4.

Proof of Theorem 1.4. Let $\Psi : U \to \mathbf{R}^+$ be defined by

$$\Psi(u) = \int_{Q_{a_\dagger}} u(a,t,x)p^u(a,t,x)dtdxda.$$

Then by the definition of $u(a,t,x)$, we have

$$\int_{Q_{a_\dagger}} \varsigma_1(a,t,x)\overline{p}(a,t,x)dtdxda \le \Psi(u) \le 0,$$

where $\overline{p}(a,t,x)$ is a solution of system (1.7) with $u \equiv 0$, $\mu \equiv 0$, $\beta = \|\beta(a)\|_{L^\infty(0,a_\dagger)}$, $p_0 = \|p_0\|_{L^\infty((0,a_\dagger)\times(0,24))}$, that is,

$$\begin{cases} Dp - \delta\Delta p = 0, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T)\times(0,24), \\ p(a,0,x) = p_0, & (a,x) \in (0,a_\dagger)\times(0,24). \end{cases} \tag{3.12}$$

Thus, we can assume that $d = \inf_{u\in U} \Psi(u)$, and there exists a sequence $\{u_N\} \in U, N \in N^*$ such that

$$\Psi(u_N) \to d, \quad N \to +\infty. \tag{3.13}$$

Since the result of Lemma 3.7, one obtains $0 \le p^{u_N}(a,t,x) \le \overline{p}(a,t,x)$ *a.e. in* $Q_{a_\dagger}$. Thus there exists a subsequence which still be denoted by $\{u_N\}$ such that

$$p^{u_N} \to p^* \quad \textit{weakly in } L^2(Q_{a_\dagger}).$$

Here, let p^{u_i} satisfy the following system

$$\begin{cases} Dp^{u_i} - \delta\Delta p^{u_i} = u_i p^{u_i}, & (a,t,x) \in Q_{a_\dagger}, \\ p^{u_i}(a,t,0) = p^{u_i}(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ \partial_x p^{u_i}(a,t,0) = \partial_x p^{u_i}(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,T), \\ p^{u_i}(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p^{u_i}(a,t,s)dsda, & (t,x) \in (0,T)\times(0,24), \\ p^{u_i}(a,0,x) = p_0, & (a,x) \in (0,a_\dagger)\times(0,24). \end{cases}$$

Then, by Mazur's Theorem, one has that $\forall\varepsilon > 0$, there exists $\lambda_i^N \geq 0$, $\sum_{i=N+1}^{k_N} \lambda_i^N = 1$ such that

$$\|p^* - \sum_{i=N+1}^{k_N} \lambda_i^N p^{u_i}\|_{L^2(Q_{a_\dagger})} \leq \varepsilon,\ k_N \geq N+1.$$

Denote $\widetilde{p}_N(a,t,x) = \sum_{i=N+1}^{k_N} \lambda_i^N p^{u_i}(a,t,x)$, we obtain

$$\widetilde{p}_N \to p^* \quad in\ L^2(Q_{a_\dagger}).$$

Now we consider the sequence $\{\widetilde{u}_N\}_{N\in N^*}$ of controls $\{u_i\}$. Here $\widetilde{u}_N(a,t,x)$ is defined by

$$\widetilde{u}_N(a,t,x) = \begin{cases} \dfrac{\sum_{i=N+1}^{k_N} \lambda_i^N u_i(a,t,x) p^{u_i}(a,t,x)}{\sum_{i=N+1}^{k_N} \lambda_i^N p^{u_i}}(a,t,x), & if \displaystyle\sum_{i=N+1}^{k_N} \lambda_i^N p^{u_i} \neq 0, \\ \varsigma_1(a,t,x), & if \displaystyle\sum_{i=N+1}^{k_N} \lambda_i^N p^{u_i} = 0. \end{cases}$$

It is easy to check that $\widetilde{u}_N \in U$. Thus, one learns that there exists a subsequence $\{\widetilde{u}_N\}_{N\in N^*}$ such that

$$\widetilde{u}_N \to u^* \quad weakly \ \ in \ \ L^2(Q_{a_\dagger}).$$

Obviously, $\widetilde{p}_N(a,t,x)$ is a solution of

$$\begin{cases} Dp - \delta\Delta p + \mu(a)p = \widetilde{u}_N(a,t,x)p, & (a,t,x) \in Q_{a_\dagger}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{3.14}$$

Passing to the limit in (3.14), we get

$$\begin{cases} Dp^* - \delta\Delta p^* + \mu(a)p^* = u^* p^*, & (a,t,x) \in Q_{a_\dagger}, \\ p^*(a,t,0) = p^*(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x p^*(a,t,0) = \partial_x p^*(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ p^*(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)p^*(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ p^*(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases}$$

It means that $p^*(a,t,x)$ is the solution of system (1.7) corresponding to $u^*(a,t,x)$. By the result of $\Psi(u_N)\to d,\ as\ N\to+\infty$, one has

$$\sum_{i=N+1}^{k_N}\lambda_i^N\Psi(u_i)=\sum_{i=N+1}^{k_N}\lambda_i^N d=d.$$

Therefore, we have

$$\begin{aligned}&\sum_{i=N+1}^{k_N}\lambda_i^N\Psi(u_i)\\ =&\sum_{i=N+1}^{k_N}\lambda_i^N\int_{Q_{a_\dagger}}u_i(a,t,x)p^{u_i}(a,t,x)dxdtda\\ =&\int_{Q_{a_\dagger}}\widetilde{u}_N(a,t,x)\widetilde{p}_N(a,t,x)dxdtda\\ \to&\int_{Q_{a_\dagger}}u^*(a,t,x)p^*(a,t,x)dxdtda\\ =&\Psi(u^*).\end{aligned}$$

Using (3.13) and the last equation, we can conclude that $d=\Psi(u^*)$. □

3.3 Necessary optimality conditions

In this section, our goal is to obtain the necessary optimality conditions of (OH) which is Theorem 1.5.

Proof of Theorem 1.5. First of all, we can get that system (1.8) has a unique solution $q(a,t,x)\in L^2(Q_{a_\dagger})$ by the same method as in the proof of the existence and uniqueness of solutions of system (1.7) in Section 2, so we omit the details here.

Since (u^*,p^*) is an optimal pair for (OH), we have

$$\int_{Q_{a_\dagger}}u^*p^{u^*}dtdxda\le\int_{Q_{a_\dagger}}(u^*+\varepsilon v)p^{u^*+\varepsilon v}dtdxda$$

for any $\varepsilon>0$ small enough, arbitrary $v(a,t,x)\in L^\infty(Q_{a_\dagger})$ such that

$$\begin{cases}v(a,t,x)\le 0, & \text{if}\ \ u^*(a,t,x)=\varsigma_2(a,t,x),\\ v(a,t,x)\ge 0, & \text{if}\ \ u^*(a,t,x)=\varsigma_1(a,t,x),\end{cases}$$

which implies

$$\int_{Q_{a_\dagger}} u^* \frac{p^{u^*+\varepsilon v} - p^{u^*}}{\varepsilon} dtdxda + \int_{Q_{a_\dagger}} v p^{u^*+\varepsilon v} dtdxda \geq 0. \tag{3.15}$$

Let

$$z^\varepsilon(a,t,x) = \frac{p^{u^*+\varepsilon v}(a,t,x) - p^{u^*}(a,t,x)}{\varepsilon},$$
$$y^\varepsilon(a,t,x) = \varepsilon z^\varepsilon(a,t,x),$$

then $y^\varepsilon(a,t,x)$ satisfies

$$\begin{cases} Dy^\varepsilon - \delta\Delta y^\varepsilon + \mu(a) y^\varepsilon = u^* y^\varepsilon + \varepsilon v p^{u^*+\varepsilon v}, & (a,t,x) \in Q_{a_\dagger}, \\ y^\varepsilon(a,t,0) = y^\varepsilon(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x y^\varepsilon(a,t,0) = \partial_x y^\varepsilon(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ y^\varepsilon(0,t,x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) y^\varepsilon(a,t,s) ds da, & (t,x) \in (0,T) \times (0,24), \\ y^\varepsilon(a,0,x) = 0, & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases}$$

Multiplying the first equation by y^ε and integrating on $Q_t = (0,a_\dagger) \times (0,t) \times (0,24)$, one obtains

$$\|y^\varepsilon(t)\|^2_{L^2((0,a_\dagger)\times(0,24))} \leq C \int_0^t \|y^\varepsilon(s)\|^2_{L^2((0,a_\dagger)\times(0,24))} ds + \varepsilon \int_{Q_t} |v| p^{u^*+\varepsilon v} |y^\varepsilon| dsdxda.$$

Then by the result of Lemma 3.7 and Bellman's Lemma (see in Appendix), we get

$$\begin{aligned} &\|y^\varepsilon(t)\|^2_{L^2((0,a_\dagger)\times(0,24))} \\ \leq &\varepsilon^2 \int_{Q_{a_\dagger}} |v|^2 \overline{p}^2 dtdxda + (1+C) \int_0^t \|y^\varepsilon(s)\|^2_{L^2((0,a_\dagger)\times(0,24))} ds \\ \leq &\varepsilon^2 e^{(1+C)t} \int_{Q_{a_\dagger}} |v|^2 \overline{p}^2 dtdxda \end{aligned}$$

where $\overline{p}(a,t,x)$ is a solution of system (3.12), $t \in [0,T]$ and C is a positive constant. This implies that

$$y^\varepsilon \to 0 \quad in \quad L^\infty(0,T; L^2((0,a_\dagger) \times (0,24))) \quad as \quad \varepsilon \to 0^+. \tag{3.16}$$

So the following convergence holds

$$p^{u^*+\varepsilon v} \to p^{u^*} \quad in \quad L^\infty(0,T; L^2((0,a_\dagger) \times (0,24))) \quad as \quad \varepsilon \to 0^+.$$

Recalling the definition of $z^\varepsilon(a,t,x)$, one has that $z^\varepsilon(a,t,x)$ satisfies

$$\begin{cases} Dz^\varepsilon - \delta\Delta z^\varepsilon + \mu(a)z^\varepsilon = u^* z^\varepsilon + vp^{u^*+\varepsilon v}, & (a,t,x) \in Q_{a_\dagger}, \\ z^\varepsilon(a,t,0) = z^\varepsilon(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x z^\varepsilon(a,t,0) = \partial_x z^\varepsilon(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ z^\varepsilon(0,t,x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) z^\varepsilon(a,t,s) ds da, & (t,x) \in (0,T) \times (0,24), \\ z^\varepsilon(a,0,x) = 0, & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases}$$

Let $h^\varepsilon(a,t,x) = z^\varepsilon(a,t,x) - z(a,t,x)$, where $z(a,t,x)$ is a solution of the following system

$$\begin{cases} Dz - \delta\Delta z + \mu(a)z = u^* z + vp^{u^*}, & (a,t,x) \in Q_{a_\dagger}, \\ z(a,t,0) = z(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x z(a,t,0) = \partial_x z(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ z(0,t,x) = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) z(a,t,s) ds da, & (t,x) \in (0,T) \times (0,24), \\ z(a,0,x) = 0, & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases}$$

Following the above proof step by step, we can get that

$$\|h^\varepsilon(t)\|^2_{L^2((0,a_\dagger)\times(0,24))} \leq e^{(1+C)t} \int_{Q_{a_\dagger}} |v|^2 |y^\varepsilon|^2 dt dx da.$$

Using (3.16), one obtains

$$z^\varepsilon \to z \ \ in \ \ L^\infty(0,T; L^2((0,a_\dagger)\times(0,24))) \ \ as \ \ \varepsilon \to 0^+.$$

Passing to the limit in (3.15), it follows

$$\int_{Q_{a_\dagger}} u^* z dt dx da + \int_{Q_{a_\dagger}} vp^{u^*} dt dx da \geq 0, \tag{3.17}$$

for arbitrary $v(a,t,x) \in L^\infty(Q_{a_\dagger})$ such that

$$\begin{cases} v(a,t,x) \leq 0, & \text{if } \ u^*(a,t,x) = \varsigma_2(a,t,x), \\ v(a,t,x) \geq 0, & \text{if } \ u^*(a,t,x) = \varsigma_1(a,t,x). \end{cases}$$

Multiplying the first equation of system (1.8) by $z(a,t,x)$ and integrating on

$Q_{a_\dagger}$, we get

$$\int_{Q_{a_\dagger}} \left(Dq + \delta\Delta q - \mu(a)q + \beta(a)\int_{x-\eta}^{x+\eta} K(x,s)q(0,t,s)ds \right) z(a,t,x)dtdxda$$
$$= \int_{Q_{a_\dagger}} (-u^* q - u^*)\, z(a,t,x)dtdxda.$$

Then by some calculations, one obtains

$$\int_{Q_{a_\dagger}} vp^{u^*} qdtdxda = \int_{Q_{a_\dagger}} u^* zdtdxda. \tag{3.18}$$

Combining (3.17) with (3.18), we learn that

$$\int_{Q_{a_\dagger}} vp^{u^*}(q+1)dtdxda \geq 0, \tag{3.19}$$

for arbitrary $v(a,t,x) \in L^\infty(Q_{a_\dagger})$ such that

$$\begin{cases} v(a,t,x) \leq 0, & \text{if } u^*(a,t,x) = \varsigma_2(a,t,x), \\ v(a,t,x) \geq 0, & \text{if } u^*(a,t,x) = \varsigma_1(a,t,x). \end{cases}$$

Now, for any $(a,t,x) \in Q_{a_\dagger}$, if $p^{u^*}(a,t,x) \neq 0$ holds, we can conclude that

$$u^*(a,t,x) = \begin{cases} \varsigma_1(a,t,x), & \text{if } q(a,t,x) > -1, \\ \varsigma_2(a,t,x), & \text{if } q(a,t,x) < -1. \end{cases}$$

In the rest of this section, we just to consider the set $B = \{(a,t,x) \in Q_{a_\dagger} | p^{u^*}(a,t,x) = 0\}$. Take any function $w(a,t,x) \in L^\infty(Q_{a_\dagger})$ such that $w(a,t,x) \neq 0$ for $(a,t,x) \in B$ and $w(a,t,x) \equiv 0$ for $(a,t,x) \in Q_{a_\dagger} - B$ and $u^* + w \in U$. Let $z(a,t,x) = p^{u^*+w} - p^{u^*}$ and then it satisfies

$$\begin{cases} Dz - \delta\Delta z + \mu(a)z = u^*(a,t,x)z + w(a,t,x)z, & (a,t,x) \in Q_{a_\dagger}, \\ z(a,t,0) = z(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ \partial_x z(a,t,0) = \partial_x z(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,T), \\ z(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s)z(a,t,s)dsda, & (t,x) \in (0,T) \times (0,24), \\ z(a,0,x) = 0, & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases}$$

By the uniqueness result, one can infer that $z(a,t,x) \equiv 0$ a.e. in $Q_{a_\dagger}$. This implies that we can change u^* in B with arbitrary values in $[\varsigma_1(a,t,x), \varsigma_2(a,t,x)]$

and the value of the related cost functional of (OH) remains the same. Then the conclusion is obvious and the proof is complete. □

Chapter 4

Local exact controllability of an age structured problem

In this chapter, we investigate the local exact controllability of the mosquito model. Our goal is to prove and numerically simulate that there exists a control $u(a,t,x)$ such that the mosquito population whose initial value is close to a positive steady state can reach the steady state at a finite time.

4.1 Preliminaries

Let us start with some nations. Here $T^* \in (0,+\infty)$, then we define

$$\begin{aligned} \alpha(t,x) &= \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C([0,24])}}}{t(T^*-t)}, \\ \phi(t,x) &= \frac{e^{\lambda\psi(x)}}{t(T^*-t)}, \end{aligned}$$

where λ is an appropriate positive constant and $\psi(x)$ will be defined in the following lemma.

In order to derive the Carleman inequality of our problem, we need the following lemma.

Lemma 4.1. *Let $\omega_0 \subset \omega$ be a nonempty bounded set of $(0,24)$. Then there exists a function $\psi \in C^2([0,24])$ such that*

$$\psi(x) \not\equiv 0,$$

$$\psi(0) = \psi(24) = 0,$$

$$\psi'(0) = \psi'(24),$$

$$|\psi'(x)| \geq \beta > 0, \quad \forall x \in (0,24) \setminus \omega_0, \tag{4.1}$$

where β is a positive constant.

Remark 4.2. *To prove Lemma 4.1, we construct a function $\psi(x)$ such that for any fixed ω_0 and $(a,b) \subset \omega_0$ for $0 < a < b < 24$:*

$$\psi(x) = \begin{cases} 1 - exp(\frac{C_1}{x - \frac{3a+b}{4}} + \sqrt{C_1}), & x \in [0, \frac{3a+b}{4}), \\ 1, & x = \frac{3a+b}{4}, \\ 1 - exp(2 - \frac{b-a}{2(x-\frac{3a+b}{4})}), & x \in (\frac{3a+b}{4}, \frac{a+b}{2}], \\ -1 + exp(2 - \frac{b-a}{2(\frac{3b+a}{4}-x)}), & x \in [\frac{a+b}{2}, \frac{3b+a}{4}), \\ -1, & x = \frac{3b+a}{4}, \\ -1 + exp(\frac{C_2}{\frac{3b+a}{4}-x} + \sqrt{C_2}), & x \in (\frac{3b+a}{4}, 24], \end{cases}$$

where $C_1 = (\frac{3a+b}{4})^2$ and $C_2 = (24 - \frac{3b+a}{4})^2$.

Now we check that $\psi(x) \in C^2([0,24])$ and it satisfies Lemma 4.1. By some calculations, one can obtain

$$\psi(0) = \psi(24) = 0,$$

$$\psi'(0) = \psi'(24) = 1,$$

$$\psi'(x) = \begin{cases} exp(\frac{C_1}{x-\frac{3a+b}{4}} + \sqrt{C_1})\frac{C_1}{(x-\frac{3a+b}{4})^2}, & x \in [0, \frac{3a+b}{4}), \\ -\frac{b-a}{2} exp(2 - \frac{b-a}{2(x-\frac{3a+b}{4})})\frac{1}{(x-\frac{3a+b}{4})^2}, & x \in (\frac{3a+b}{4}, \frac{a+b}{2}], \\ -\frac{b-a}{2} exp(2 - \frac{b-a}{2(\frac{3b+a}{4}-x)})\frac{1}{(x-\frac{3a+b}{4})^2}, & x \in [\frac{a+b}{2}, \frac{3b+a}{4}), \\ exp(\frac{C_2}{\frac{3b+a}{4}-x} + \sqrt{C_2})\frac{C_2}{(x-\frac{3a+b}{4})^2}, & x \in (\frac{3b+a}{4}, 24], \end{cases}$$

$$\psi''(x) = \begin{cases} exp(\frac{C_1}{x-\frac{3a+b}{4}} + \sqrt{C_1})(-\frac{{C_1}^2}{(x-\frac{3a+b}{4})^4} - \frac{2C_1}{(x-\frac{3a+b}{4})^3}), & x \in [0, \frac{3a+b}{4}), \\ -\frac{b-a}{2} exp(2 - \frac{b-a}{2(x-\frac{3a+b}{4})})(\frac{b-a}{2(x-\frac{3a+b}{4})^4} - \frac{2}{(x-\frac{3a+b}{4})^3}), & x \in (\frac{3a+b}{4}), \frac{a+b}{2}], \\ \frac{b-a}{2} exp(2 - \frac{b-a}{2(\frac{3b+a}{4}-x)})(\frac{b-a}{2(x-\frac{3a+b}{4})^4} - \frac{2}{(x-\frac{3a+b}{4})^3}), & x \in [\frac{a+b}{2}, \frac{3b+a}{4}), \\ exp(\frac{C_2}{\frac{3b+a}{4}-x} + \sqrt{C_2})(\frac{{C_2}^2}{(x-\frac{3a+b}{4})^4} + \frac{C_2}{(x-\frac{3a+b}{4})^3}), & x \in (\frac{3b+a}{4}, 24], \end{cases}$$

$$\lim_{x \to \frac{3a+b}{4}^-} \psi''(x) = \lim_{x \to \frac{3a+b}{4}^+} \psi''(x) = 0,$$

$$\lim_{x \to \frac{a+b}{2}^-} \psi''(x) = \lim_{x \to \frac{a+b}{2}^+} \psi''(x) = 0,$$

$$\lim_{x \to \frac{a+3b}{4}^-} \psi''(x) = \lim_{x \to \frac{a+3b}{4}^+} \psi''(x) = 0.$$

Then it is easy to see that the lemma holds.

Now, let $z(t,x), g(t,x) \in L^2((0,T^*)\times(0,24))$, we consider the following periodic boundary value problem.

Lemma 4.3. *For the solutions of the following problem,*

$$\begin{cases} z_t(t,x) + \delta\Delta z(t,x) = g(t,x), & (t,x)\in(0,T^*)\times(0,24),\\ z(t,0) = z(t,24), & t\in(0,T^*),\\ \partial_x z(t,0) = \partial_x z(t,24), & t\in(0,T^*), \end{cases}$$

there exist positive C, s_0 such that

$$\begin{aligned} &\int_0^{T^*}\int_0^{24}\Big(\frac{1}{s\phi}z_t^2 + \frac{1}{s\phi}z_{xx}^2 + s^3\phi^3 z^2 + s\phi z_x^2\Big)e^{2s\alpha}dxdt\\ \le& C\left(\int_0^{T^*}\int_\omega s^3\phi^3 z^2 e^{2s\alpha}dxdt + \|e^{s\alpha}g\|^2_{L^2((0,T^*)\times(0,24))}\right),\ \forall s > s_0. \end{aligned}$$

Proof. For convenience, we denote

$$w(t,x) = e^{s\alpha(t,x)}z(t,x).$$

Then $w(t,x)$ satisfies

$$\begin{aligned} &w_t - s\alpha_t w + \delta w_{xx} - 2\delta s\lambda\psi'\phi w_x + \delta s^2\lambda^2(\psi')^2\phi^2 w\\ &-(\delta s\lambda\psi''\phi + \delta s\lambda^2(\psi')^2\phi)w = e^{s\alpha(t,x)}g(t,x), \end{aligned} \tag{4.2}$$

and

$$w(T^*,x) = w(0,x) = 0, \quad x\in(0,24).$$

Define

$$L_1 w = \delta w_{xx} - s\alpha_t w + \delta s^2\lambda^2(\psi')^2\phi^2 w, \tag{4.3}$$

and

$$L_2 w = w_t - 2\delta s\lambda\psi'\phi w_x - 2\delta s\lambda^2(\psi')^2\phi w. \tag{4.4}$$

It follows that (4.2) is equivalent to

$$L_1 w + L_2 w = f_s(t,x),\ (t,x)\in(0,T^*)\times(0,24), \tag{4.5}$$

where

$$f_s(t,x) = g(t,x)e^{s\alpha(t,x)} - \delta s\lambda^2(\psi')^2\phi w + s\lambda\delta\psi''\phi w. \tag{4.6}$$

Taking L_2-norm of (4.5), we obtain

$$\begin{aligned}\|f_s\|^2_{L^2((0,T^*)\times(0,24))} =& \|L_1 w\|^2_{L^2((0,T^*)\times(0,24))} + \|L_2 w\|^2_{L^2((0,T^*)\times(0,24))} \\ &+ 2(L_1 w, L_2 w)_{L^2((0,T^*)\times(0,24))}. \end{aligned}\tag{4.7}$$

After some calculations, one gets

$$\begin{aligned}&(L_1 w, L_2 w)_{L^2((0,T^*)\times(0,24))}\\ &= \int_0^{T^*}\int_0^{24} \left(\delta^2 s^3\lambda^4(\psi')^4\phi^3 w^2 + 3\delta^2 s\lambda^2(\psi)'^2\phi w_x^2\right) dxdt + X_1 + J_1,\end{aligned}$$

where

$$\begin{aligned}X_1 = \int_0^{T^*}\int_0^{24} &(\frac{1}{2}s\alpha_{tt}w^2 - \delta s^2\lambda^2(\psi')^2\phi\phi_t w^2 + 3\delta^2 s^3\lambda^3(\psi')^2\psi''\phi^3 w^2 \\ &- \delta s^2\lambda\frac{\partial}{\partial x}(\alpha_t\psi'\phi)w^2 - 2\delta^2 s\lambda^2\frac{\partial}{\partial x}(\psi'\psi''\phi)w^2 - 3\delta^2 s\lambda^3(\psi')^2\psi''\phi w^2 \\ &-\delta^2 s\lambda^4(\psi')^4\phi w^2 + 2\delta s^2\lambda^2\alpha_t(\psi')^2\phi w^2 + \delta^2 s\lambda\psi''\phi w_x^2\big)\, dxdt,\end{aligned}$$

and

$$\begin{aligned}J_1 = \int_0^{T^*} &\big(\delta w_t w_x - \delta^2 s\lambda\psi'\phi w_x^2 - \delta^2 s^3\lambda^3(\psi')^3\phi^3 w^2 + \delta s^2\lambda\alpha_t\psi'\phi w^2 \\ &- 2\delta^2 s\lambda^2(\psi')^2\phi w w_x + \delta^2 s\lambda^2\frac{\partial}{\partial x}((\psi')^2\phi)w^2\big)\,|_0^{24}\, dt.\end{aligned}$$

From the definition of $\psi(x)$ and $z(t,0) = z(t,24)$, $z_x(t,0) = z_x(t,24)$, one gets that

$$w(t,0) = w(t,24), w_x(t,0) = w_x(t,24).$$

Then, after some calculations, we have

$$J_1 = 0,$$

and one can easily prove that

$$|X_1| \le C\int_0^{T^*}\int_0^{24}((s^3\lambda^3\phi^3 + s\lambda^4\phi)w^2 + s\lambda\phi w_x^2)dxdt,\ \ s\ge 1,\ \ \lambda\ge 1,$$

where the constant C is independent of s and λ. Combining (4.1) with (4.7), one obtains

$$\begin{aligned}
&\|f_s\|^2_{L^2((0,T^*)\times(0,24))}\\
=&\|L_1w\|^2_{L^2((0,T^*)\times(0,24))}+\|L_2w\|^2_{L^2((0,T^*)\times(0,24))}\\
&+2\int_0^{T^*}\int_0^{24}\left(\delta^2s^3\lambda^4(\psi')^4\phi^3w^2+3\delta^2s\lambda^2(\psi')^2\phi w_x^2\right)dxdt\\
&+2X_1\\
=&\|L_1w\|^2_{L^2((0,T^*)\times(0,24))}+\|L_2w\|^2_{L^2((0,T^*)\times(0,24))}\\
&+2\int_0^{T^*}\int_{\omega_0}\left(\delta^2s^3\lambda^4(\psi')^4\phi^3w^2+3\delta^2s\lambda^2(\psi')^2\phi w_x^2\right)dxdt\\
&+2\int_0^{T^*}\int_{[0,24]\setminus\omega_0}\left(\delta^2s^3\lambda^4(\psi')^4\phi^3w^2+3\delta^2s\lambda^2(\psi')^2\phi w_x^2\right)dxdt\\
&+2X_1\\
\geq&\|L_1w\|^2_{L^2((0,T^*)\times(0,24))}+\|L_2w\|^2_{L^2((0,T^*)\times(0,24))}\\
&+2\int_0^{T^*}\int_{[0,24]\setminus\omega_0}(\delta^2\beta^4s^3\lambda^4\phi^3w^2+3\delta^2\beta^2s\lambda^2\phi w_x^2)dxdt\\
&+2X_1.
\end{aligned}$$

Taking $\lambda>0$ sufficiently large, by virtue of (4.5) and (4.6), it follows that

$$\begin{aligned}
&\|L_1w\|^2_{L^2((0,T^*)\times(0,24))}+\|L_2w\|^2_{L^2((0,T^*)\times(0,24))}+\int_0^{T^*}\int_0^{24}(s^3\lambda^4\phi^3w^2\\
&+s\lambda^2\phi w_x^2)dxdt\\
\leq&C\left(\|e^{s\alpha}g\|^2_{L^2((0,T^*)\times(0,24))}+\int_0^{T^*}\int_{\omega_0}(s^3\lambda^4\phi^3w^2+s\lambda^2\phi w_x^2)dxdt\right),\ \forall s>s_0,
\end{aligned}\tag{4.8}$$

where C, s_0 are appropriate constants. Then it follows from (4.3), (4.4) and (4.8) that

$$\begin{aligned}
&\int_0^{T^*}\int_0^{24}\left(\frac{1}{s\phi}w_t^2+\frac{1}{s\phi}w_{xx}^2+s^3\lambda^4\phi^3w^2+s\lambda^2\phi w_x^2\right)dxdt\\
\leq&C\left(\|e^{s\alpha}g\|^2_{L^2((0,T^*)\times(0,24))}+\int_0^{T^*}\int_{\omega_0}(s^3\lambda^4\phi^3w^2+s\lambda^2\phi w_x^2\right)dxdt.
\end{aligned}$$

Recalling $w(t,x) = e^{s\alpha(t,x)}z(t,x)$, then

$$\begin{aligned}
&\int_0^{T^*}\int_0^{24}(\frac{1}{s\phi}z_t^2 + \frac{1}{s\phi}z_{xx}^2 + s^3\lambda^4\phi^3 z^2 + s\lambda^2\phi z_x^2)e^{2s\alpha}dxdt \\
\leq & C\left(\|e^{s\alpha}g\|^2_{L^2((0,T^*)\times(0,24))} + \int_0^{T^*}\int_{\omega_0}(s^3\lambda^4\phi^3 z^2 e^{2s\alpha} + s\lambda^2\phi z_x^2 e^{2s\alpha})dxdt\right), \\
& \forall s > s_0.
\end{aligned} \tag{4.9}$$

Multiplying the equation (4.2) by $s\lambda^2\phi z e^{2s\alpha}\rho(x)$ in $((0,T^*)\times(0,24))$, where $\rho(x) \in C_0^\infty(\omega), \rho(x) \equiv 1$ in ω_0 and integrating by parts with respect to t and x, one gets

$$\begin{aligned}
&\int_0^{T^*}\int_{\omega_0} s\lambda^2\phi z_x^2 e^{2s\alpha}dxdt \\
\leq & C\left(\int_0^{T^*}\int_{\omega} s^3\lambda^4\phi^3 z^2 e^{2s\alpha}dxdt + \|e^{s\alpha}g\|^2_{L^2((0,T^*)\times(0,24))}\right).
\end{aligned}$$

By using (4.9), we have

$$\begin{aligned}
&\int_0^{T^*}\int_0^{24}(\frac{1}{s\phi}z_t^2 + \frac{1}{s\phi}z_{xx}^2 + s^3\lambda^4\phi^3 z^2 + s\lambda^2\phi z_x^2)e^{2s\alpha}dxdt \\
\leq & C\left(\int_0^{T^*}\int_{\omega} s^3\lambda^4\phi^3 z^2 e^{2s\alpha}dxdt + \|e^{s\alpha}g\|^2_{L^2((0,T^*)\times(0,24))}\right), \forall s > s_0.
\end{aligned}$$

This completes the proof. □

According to Lemma 4.3 , we obtain the following estimates directly.

Corollary 4.4. *Let $z(t,x)$ be as defined in Lemma 4.3 and recall $\psi(x)$ which is obtained in Lemma 4.1, then there exist positive C, s_0 such that*

$$\begin{aligned}
&\int_0^{T^*}\int_0^{24}\left(\frac{t(T^*-t)e^{2s\alpha}}{s}\left(z_t^2 + z_{xx}^2\right) + \frac{s^3 e^{2s\alpha}}{t^3(T^*-t)^3}z^2 + \frac{s e^{2s\alpha}}{t(T^*-t)}z_x^2\right)dxdt \\
\leq & C\left(\int_0^{T^*}\int_{\omega}\frac{s^3 e^{2s\alpha}}{t^3(T^*-t)^3}z^2 dxdt + \|e^{s\alpha}g\|^2_{L^2((0,T^*)\times(0,24))}\right), \ \forall s > s_0.
\end{aligned}$$

4.2 The optimality system

In this section, we prove our main Theorem 1.6. Before the proof of Theorem 1.6, we first give a lemma which shows some estimates playing an important role in the proof of Theorem 1.6.

Let $T_0 \in (0, \min\{a_0, \frac{T}{2}, a_\dagger - a_1\})$, we define the following auxiliary areas and function:

$$\begin{aligned}
&D_{T_0} = (0, T_0) \times (0, 24), \\
&H = L^\infty((0, 2T_0) \times (0, 24)), \\
&Q_{2T_0} = (0, a_\dagger) \times (0, 2T_0) \times (0, 24), \\
&G = (0, a_\dagger) \times (0, T_0) \cup (0, T_0) \times (T_0, 2T_0), \\
&\Gamma_0 = \{T_0\} \times (T_0, 2T_0) \cup (T_0, a_\dagger - T_0) \times \{T_0\},
\end{aligned}$$

and

$$\varphi(a,t,x) = \begin{cases} e^{-2s\alpha(t,x)} t^3 (T_0 - t)^3, & if\ t \le a,\ (a,t) \in G, \\ e^{-2s\alpha(a,x)} a^3 (T_0 - a)^3, & if\ a < t,\ (a,t) \in G. \end{cases}$$

Before going further, we need the following auxiliary lemma.

Lemma 4.5. *Let $p^u(a,t,x)$ be the solution of the system*

$$\begin{cases} Dp - \delta\Delta p + \mu(a)p = m\widetilde{m}(a,t)u(a,t,x)p, & (a,t,x) \in Q_{2T_0}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0, 2T_0), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger) \times (0, 2T_0), \\ p(0,t,x) = b(t,x), & (t,x) \in (0, 2T_0) \times (0,24), \\ p(a,0,x) = p_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases} \tag{4.10}$$

where $b(t,x) \ge 0$, $p_0(a,x) \ge 0$, $\widetilde{m}(a,t)$ is the characteristic of G. Then, the following optimal control problem

$$Minimize \left\{ \int_G \int_0^{24} \varphi |u(a,t,x)p^u(a,t,x)|^2 dxdtda + \frac{1}{\varepsilon} \int_{\Gamma_0} \int_0^{24} |p^u - p_s|^2 dxdl \right\}$$

with $u \in U$, where

$$U = \{u \in L^2(Q_{2T_0}); \zeta_1(a,t,x) \le u(a,t,x) \le \zeta_2(a,t,x)\ a.e.\ in\ Q_{2T_0}\}$$

and ζ_1, $\zeta_2 \in L^\infty$, has an optimal pair (u^*, p^*).

Proof. Let

$$\Psi(u) := \int_G \int_0^{24} \varphi |u(a,t,x)p^u(a,t,x)|^2 dxdtda + \frac{1}{\varepsilon} \int_{\Gamma_0} \int_0^{24} |p^u - p_s|^2 dxdl.$$

Since $u \in U$ and by the comparison principle for (4.10) (which can be referred to [62, Lemma 2.7]), one has that $0 \le p^u(a,t,x) \le p^{\zeta_2}(a,t,x)$ in Q_{2T_0}. Then, there is $d \geq 0$ such that

$$d = \inf_{u \in U} \Psi(u).$$

Thus, there is a sequence $\{u_i\}_{i \in \mathcal{N}} \subset U$ such that

$$d \leq \Psi(u) < d + \frac{1}{i} \text{ and } \Psi(u_i) \to d.$$

By similar arguments as the proof of [62, Theorem 1.2], there exist $\widetilde{u}_N$ and $\widetilde{p}_N$ such that

$$\sum_{N+1}^{k_N} \lambda_i^N u_i(a,t,x) p^{u_i}(a,t,x) = \widetilde{u}_N(a,t,x)\widetilde{p}_N(a,t,x),$$

where $\sum_{N+1}^{k_N} \lambda_i^N = 1$ and there exist u^*, p^* such that

$$\widetilde{u}_N \to u^* \text{ weakly in } L^2(Q_{2T_0}),$$

$$\widetilde{p}_N \to p^* \text{ in } L^2(Q_{2T_0}).$$

Thus, one has that

$$\begin{aligned}
\sum_{N+1}^{k_N} \lambda_i^N \Psi(u_i) &= \sum_{N+1}^{k_N} \lambda_i^N \int_G \int_0^{24} \varphi |u_i(a,t,x)p_i(a,t,x)|^2 dxdtda \\
&+ \sum_{N+1}^{k_N} \lambda_i^N \frac{1}{\varepsilon} \int_{\Gamma_0} \int_0^{24} |p_i - p_s|^2 dxdl \\
&\geq \int_G \int_0^{24} \varphi |\sum_{N+1}^{k_N} \lambda_i^N u_i(a,t,x)p_i(a,t,x)|^2 dadtdx \\
&+ \frac{1}{\varepsilon} \int_{\Gamma_0} \int_0^{24} |\sum_{N+1}^{k_N} \lambda_i^N p_i - p_s|^2 dxdl \\
&= \int_G \int_0^{24} \varphi |\widetilde{u}_N(a,t,x)\widetilde{p}_N(a,t,x)|^2 dxdtda \\
&+ \frac{1}{\varepsilon} \int_{\Gamma_0} \int_0^{24} |\widetilde{p}_N - p_s|^2 dxdl \\
&\to \int_G \int_0^{24} \varphi |u^*(a,t,x)p^*(a,t,x)|^2 dxdtda \\
&+ \frac{1}{\varepsilon} \int_{\Gamma_0} \int_0^{24} |p^* - p_s|^2 dxdl.
\end{aligned}$$

Since $\Psi(u_i) \to d = \inf_{u\in U} \Psi(u)$, we have that

$$\int_G \int_0^{24} \varphi |u^*(a,t,x)p^*(a,t,x)|^2 dxdtda + \frac{1}{\varepsilon}\int_{\Gamma_0}\int_0^{24} |p^* - p_s|^2 dxdl = d.$$

This completes the proof. □

Let $b(t,x) \in H$ and $\|b\|_{L^\infty((0,2T_0)\times(0,24))}$ be small enough.

Lemma 4.6. *For the following system*

$$\begin{cases} Dp - \delta\Delta p + \mu(a)p = m\widetilde{m}(a,t)u(a,t,x)(p+p_s), & (a,t,x) \in Q_{2T_0}, \\ p(a,t,0) = p(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,2T_0), \\ \partial_x p(a,t,0) = \partial_x p(a,t,24), & (a,t) \in (0,a_\dagger)\times(0,2T_0), \\ p(0,t,x) = b(t,x), & (t,x) \in (0,2T_0)\times(0,24), \\ p(a,0,x) = \overline{p}_0(a,x), & (a,x) \in (0,a_\dagger)\times(0,24), \end{cases} \tag{4.11}$$

where $\widetilde{m}(a,t)$ is the characteristic of G, there exists a solution $p^u(a,t,x)$ which satisfies

$$p^u(a,t,x) = 0, \quad a.e.\ (a,t,x) \in \Gamma_0 \times (0,24),$$
$$\|p^u\|_{L^\infty(Q_{2T_0})} \le C(\|\bar{p}_0\|_{L^\infty((0,a_\dagger)\times(0,24))} + \|b\|_{L^\infty((0,a_\dagger)\times(0,24))}),$$

where C is an appropriate constant.

Proof. First of all, for any ε, we consider the optimal control problem related to (4.11),

$$\text{Minimize}\left\{\int_G\int_0^{24} \varphi|u(a,t,x)(p+p_s)|^2 dxdtda + \frac{1}{\varepsilon}\int_{\Gamma_0}\int_0^{24} |p|^2 dxdl\right\}. \tag{4.12}$$

$\Psi_\varepsilon(u)$ denote the value of the cost function in u. Since the cost function $\Psi_\varepsilon(u) : L^2(Q_{2T_0}) \longrightarrow R^+$ is continuous and

$$\lim_{\|u\|_{L^2(Q_{2T_0})}\to+\infty} \Psi_\varepsilon(u) = +\infty,$$

then it follows that $u(a,t,x)$ is bounded. Then by the result of lemma 4.5, we obtain that there exists at least one minimum point for Ψ_ε. Furthermore, there is an optimal pair $(u_\varepsilon(a,t,x), p_\varepsilon(a,t,x))$ for (4.12), which satisfies (4.11), that is,

$$\begin{cases} Dp_\varepsilon - \delta\Delta p_\varepsilon + \mu(a)p_\varepsilon = m(x)\widetilde{m}(a,t)u_\varepsilon(p_\varepsilon + p_s), & (a,t,x) \in Q_{2T_0}, \\ p_\varepsilon(a,t,0) = p_\varepsilon(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,2T_0), \\ \partial_x p_\varepsilon(a,t,0) = \partial_x p_\varepsilon(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,2T_0), \\ p_\varepsilon(0,t,x) = b(t,x), & (t,x) \in (0,2T_0) \times (0,24), \\ p_\varepsilon(a,0,x) = \overline{p}_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{4.13}$$

Now we consider the Euler-Lagrange Equation related to (4.11) and define

$$I(\tau) = \int_G \int_0^{24} \varphi|(u_\varepsilon + \tau\widehat{u}(a,t,x))(p_\tau + p_s)|^2 dxdtda + \frac{1}{\varepsilon}\int_{\Gamma_0}\int_0^{24} |p_\tau|^2 dxdl, \tag{4.14}$$

where τ is an arbitrary constant, $\widehat{u}(a,t,x) \in L^2([0,a_\dagger] \times [0,2T_0] \times [0,24])$ is an arbitrary function and $p_\tau(a,t,x)$ satisfies

$$\begin{cases} Dp_\tau - \delta\Delta p_\tau + \mu(a)p_\tau = m(x)\widetilde{m}(u_\varepsilon + \tau\widehat{u})(p_\tau + p_s), (a,t,x) \in Q_{2T_0}, \\ p_\tau(a,t,0) = p_\tau(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,2T_0), \\ \partial_x p_\tau(a,t,0) = \partial_x p_\tau(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,2T_0), \\ p_\tau(0,t,x) = b(t,x), & (t,x) \in (0,2T_0) \times (0,24), \\ p_\tau(a,0,x) = \overline{p}_0(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{4.15}$$

Note that

$$p_\varepsilon(a,t,x) = \lim_{\tau\to 0} p_\tau(a,t,x).$$

According to (4.13) and (4.15), we obtain

$$\begin{cases} D\frac{p_\tau - p_\varepsilon}{\tau} - \delta\Delta\frac{p_\tau - p_\varepsilon}{\tau} + \mu(a)\frac{p_\tau - p_\varepsilon}{\tau} = m(x)\widetilde{m}u_\varepsilon\frac{p_\tau - p_\varepsilon}{\tau} + m(x)\widetilde{m}\widehat{u}(p_\tau + p_s), \\ \frac{p_\tau - p_\varepsilon}{\tau}(a,t,0) = \frac{p_\tau - p_\varepsilon}{\tau}(a,t,24), \\ \partial_x\frac{p_\tau - p_\varepsilon}{\tau}(a,t,0) = \partial_x\frac{p_\tau - p_\varepsilon}{\tau}(a,t,24), \\ \frac{p_\tau - p_\varepsilon}{\tau}\tau(0,t,x) = 0, \\ \frac{p_\tau - p_\varepsilon}{\tau}(a,0,x) = 0, \end{cases}$$

where $a \in (0,a_\dagger), t \in (0,2T_0), x \in (0,24)$. Then defining

$$z(a,t,x) = \lim_{\tau\to 0}\frac{p_\tau - p_\varepsilon}{\tau}, \tag{4.16}$$

it is easy to verify that $z(a,t,x)$ satisfies

$$\begin{cases} Dz - \delta\Delta z + \mu(a)z = m\widetilde{m}u_\varepsilon z + m\widetilde{m}\widehat{u}(p_\varepsilon + p_s), & (a,t,x) \in Q_{2T_0}, \\ z(a,t,0) = z(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,2T_0), \\ \partial_x z(a,t,0) = \partial_x z(a,t,24), & (a,t) \in (0,a_\dagger) \times (0,2T_0), \\ z(0,t,x) = 0, & (t,x) \in (0,2T_0) \times (0,24), \\ z(a,0,x) = 0, & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases} \tag{4.17}$$

Here, recalling (4.14), (4.16) and deriving $I(\tau)$ with respect to τ at $\tau = 0$, one has

$$\begin{aligned} I'(0) =& 2\int_G \int_0^{24} \varphi(u_\varepsilon(a,t,x))^2 (p_\varepsilon + p_s) \lim_{\tau\to 0} \frac{p_\tau - p_\varepsilon}{\tau} dx dt da \\ &+ 2\int_G \int_0^{24} \varphi u_\varepsilon(a,t,x)\widehat{u}(a,t,x)(p_\varepsilon + p_s)^2 dx dt da \\ &+ \frac{2}{\varepsilon}\int_{\Gamma_0} \int_0^{24} p_\varepsilon \lim_{\tau\to 0} \frac{p_\tau - p_\varepsilon}{\tau} dx dl \\ =& 2\int_G \int_0^{24} \varphi(u_\varepsilon(a,t,x))^2 (p_\varepsilon + p_s) z(a,t,x) dx dt da \\ &+ 2\int_G \int_0^{24} \varphi u_\varepsilon(a,t,x)\widehat{u}(a,t,x)(p_\varepsilon + p_s)^2 dx dt da \\ &+ \frac{2}{\varepsilon}\int_{\Gamma_0} \int_0^{24} p_\varepsilon z(a,t,x) dx dl. \end{aligned}$$

Let $q_\varepsilon(a,t,x) \in Q_{2T_0}$ satisfying:

$$q_\varepsilon(a,t,x) = \varphi(a,t,x)u_\varepsilon(a,t,x)(p_\varepsilon(a,t,x) + p_s(a,x)).$$

Then, one gets:

$$\begin{aligned} I'(0) =& 2\int_G \int_0^{24} q_\varepsilon(a,t,x)m(x)\widetilde{m}(a,t)u_\varepsilon(a,t,x)z(a,t,x) dx dt da \\ &+ 2\int_G \int_0^{24} q_\varepsilon(a,t,x)m(x)\widetilde{m}(a,t)\widehat{u}(a,t,x)(p_\varepsilon + p_s) dx dt da \\ &+ \frac{2}{\varepsilon}\int_{\Gamma_0} \int_0^{24} p_\varepsilon z(a,t,x) dx dl \\ &+ 2\int_G \int_0^{24} (1 - m(x)\widetilde{m}(a,t))\varphi u_\varepsilon (p_\varepsilon + p_s)(u_\varepsilon z + \widehat{u}(p_\varepsilon + p_s)) dx dt da. \end{aligned}$$

By the first equation of (4.17), it follows after some calculations:

$$\begin{aligned}
I'(0) =&2\int_G\int_0^{24} z(a,t,x)(-Dq_\varepsilon - \delta\Delta q_\varepsilon + \mu(a)q_\varepsilon)dxdtda\\
&-2\int_G \delta(q_\varepsilon(a,t,24)\partial_x z(a,t,24) - q_\varepsilon(a,t,0)\partial_x z(a,t,0))dtda\\
&+2\int_G \delta(z(a,t,24)\partial_x q_\varepsilon(a,t,24) - z(a,t,0)\partial_x q_\varepsilon(a,t,0))dtda\\
&+2\int_{\Gamma_0}\int_0^{24} z(a,t,x)(q_\varepsilon(a,t,x) + \frac{1}{\varepsilon}p_\varepsilon(a,t,x))dxdl\\
&+2\int_{\Gamma\setminus\Gamma_0}\int_0^{24} z(a,t,x)q_\varepsilon(a,t,x)dxda\\
&+2\int_G\int_0^{24}(1-m(x)\widetilde{m}(a,t))\varphi u_\varepsilon(p_\varepsilon+p_s)(u_\varepsilon z + \widehat{u}(p_\varepsilon+p_s))dxdtda.
\end{aligned}$$

Since $z(a,t,x)$ and $\widehat{u}(a,t,x)$ are arbitrary, we can get that $q_\varepsilon(a,t,x)$ is the solution of

$$\begin{cases}
Dq(a,t,x) + \delta\Delta q(a,t,x) - \mu(a)q(a,t,x) = 0, & (a,t,x)\in G\times(0,24),\\
q(a,t,0) = q(a,t,24), & (a,t)\in G,\\
\partial_x q(a,t,0) = \partial_x q(a,t,24), & (a,t)\in G,\\
q(a,t,x) = 0, & (a,t,x)\in(\Gamma\setminus\Gamma_0)\times(0,24),\\
q(a,t,x) = -\frac{1}{\varepsilon}p_\varepsilon(a,t,x), & (a,t,x)\in\Gamma_0\times(0,24),
\end{cases} \tag{4.18}$$

where $\Gamma = (0,T_0)\times\{2T_0\}\cup\{a_\dagger\}\times(0,T_0)\cup\Gamma_0\cup(a_\dagger - T_0, a_\dagger)\times\{T_0\}$ and

$$(1-m(x)\widetilde{m}(a,t))\varphi u_\varepsilon(p_\varepsilon+p_s) = 0.$$

Thus one can get $u_\varepsilon(a,t,x)$ and $p_\varepsilon(a,t,x)$ satisfy

$$u_\varepsilon(p_\varepsilon + p_s(a,x)) = m(x)\widetilde{m}(a,t)q_\varepsilon\varphi^{-1}(a,t,x),\ a.e.(a,t,x)\in Q_{2T_0}. \tag{4.19}$$

Multiplying the first equation in (4.18) by p_ε and integrating on Q_{2T_0}, we obtain

$$\begin{aligned}
&\int_G\int_\omega \varphi(a,t,x)|u_\varepsilon(t)(p_\varepsilon(a,t,x)+p_s(a,x))|^2dxdtda + \frac{1}{\varepsilon}\int_{\Gamma_0}\int_0^{24}|p_\varepsilon(a,t,x)|^2dxdl\\
&= -\int_0^{T_0}\int_0^{24} b(t,x)q_\varepsilon(0,t,x)dxdt - \int_0^{a_\dagger - T_0}\int_0^{24}\overline{p}_0(a,x)q_\varepsilon(a,0,x)dxda.
\end{aligned}$$

Now, define

$$\begin{cases} \widetilde{u}_\varepsilon(t,x) = u_\varepsilon(\gamma+t, \theta+t, x), & (t,x) \in (0,T_0) \times (0,24), \\ \widetilde{p}_\varepsilon(t,x) = p_\varepsilon(\gamma+t, \theta+t, x), & (t,x) \in (0,T_0) \times (0,24), \\ \widetilde{p}_s(t,x) = p_s(\gamma+t, x), & (t,x) \in (0,T_0) \times (0,24), \\ \widetilde{q}_\varepsilon(t,x) = q_\varepsilon(\gamma+t, \theta+t, x), & (t,x) \in (0,T_0) \times (0,24), \\ \widetilde{\mu}(t) = \mu(\gamma+t), & t \in (0,T_0). \end{cases} \tag{4.20}$$

And, let S be an arbitrary characteristic line of equation

$$S = \{(\gamma+t, \theta+t); t \in (0,T_0)\}, (\gamma, \theta) \in (0, a_\dagger - T_0) \times \{0\} \cup \{0\} \times (0, T_0).$$

Then combining (4.11) with (4.20), we learn that $(\widetilde{u}_\varepsilon(t,x), \widetilde{p}_\varepsilon(t,x))$ satisfies

$$\begin{cases} \widetilde{p}_{\varepsilon t} - \delta\Delta\widetilde{p}_\varepsilon + \widetilde{\mu}(t)\widetilde{p}_\varepsilon = m(x)\widetilde{u}_\varepsilon(t,x)(\widetilde{p}_\varepsilon + \widetilde{p}_s), & (t,x) \in (0,T_0) \times (0,24), \\ \widetilde{p}_\varepsilon(t,0) = \widetilde{p}_\varepsilon(t,24), & t \in (0,T_0), \\ \partial_x\widetilde{p}_\varepsilon(t,0) = \partial_x\widetilde{p}_\varepsilon(t,24), & t \in (0,T_0), \\ \widetilde{p}_\varepsilon(0,x) = \begin{cases} b(\theta,x), & \gamma = 0, \quad x \in (0,24), \\ \overline{p}_0(\gamma,x), & \theta = 0, \quad x \in (0,24). \end{cases} \end{cases} \tag{4.21}$$

Furthermore, according to (4.19) and the definition of $\varphi(a,t,x)$, one has

$$\widetilde{u}_\varepsilon(t,x)(\widetilde{p}_\varepsilon + \widetilde{p}_s) = m(x)\widetilde{q}_\varepsilon \frac{e^{2s\alpha(t,x)}}{t^3(T_0-t)^3}, \quad a.e. \ (t,x) \in (0,T_0) \times (0,24), \tag{4.22}$$

where $\widetilde{q}_\varepsilon$ is the solution of

$$\begin{cases} \widetilde{q}_{\varepsilon t}(t,x) + \delta\Delta\widetilde{q}_\varepsilon(t,x) = \widetilde{\mu}(t)\widetilde{q}_\varepsilon(t,x), & (t,x) \in (0,T_0) \times (0,24), \\ \widetilde{q}_\varepsilon(t,0) = \widetilde{q}_\varepsilon(t,24), & t \in (0,T_0), \\ \partial_x\widetilde{q}_\varepsilon(t,0) = \partial_x\widetilde{q}_\varepsilon(t,24), & t \in (0,T_0), \\ \widetilde{q}_\varepsilon(T_0,x) = -\frac{1}{\varepsilon}\widetilde{p}_\varepsilon(T_0,x), & x \in (0,24). \end{cases} \tag{4.23}$$

Multiplying the first equation in (4.23) by $\widetilde{p}_\varepsilon(t,x)$ and integrating on D_{T_0},

$$\begin{aligned} &\int_0^{T_0}\int_\omega e^{-2s\alpha(t,x)} t^3(T_0-t)^3 |\widetilde{u}_\varepsilon(t,x)(\widetilde{p}_\varepsilon + \widetilde{p}_s)|^2 dxdt + \frac{1}{\varepsilon}\int_0^{24} |\widetilde{p}_\varepsilon(T_0,x)|^2 dx \\ &= -\int_0^{24} \widetilde{p}_\varepsilon(0,x)\widetilde{q}_\varepsilon(0,x)dx. \end{aligned} \tag{4.24}$$

Then, applying Corollary 4.4, for $s \geq \max(s_0, C\|\widetilde{\mu}\|^{2/3}_{C([0,a_\dagger - T_0])})$, we obtain

$$\begin{aligned}
&\int_{D_{T_0}} \left(\frac{t(T_0-t)e^{2s\alpha}}{s} \left(\widetilde{q}^2_{\varepsilon t} + |\Delta \widetilde{q}_\varepsilon|^2 \right) + \frac{s^3 e^{2s\alpha}}{t^3(T_0-t)^3} \widetilde{q}^2_\varepsilon + \frac{s e^{2s\alpha}}{t(T_0-t)} \widetilde{q}^2_{\varepsilon x} \right) dxdt \\
&\leq C \left(\int_0^{T_0} \int_\omega \frac{s^3 e^{2s\alpha}}{t^3(T_0-t)^3} \widetilde{q}^2_\varepsilon dxdt + \int_{D_{T_0}} e^{2s\alpha} \|\widetilde{\mu}\|^2_{C([0,T_0])} \widetilde{q}^2_\varepsilon dxdt \right) \\
&\leq C \int_0^{T_0} \int_\omega \frac{s^3 e^{2s\alpha}}{t^3(T_0-t)^3} \widetilde{q}^2_\varepsilon dxdt, \qquad (4.25)
\end{aligned}$$

where $\widetilde{q}_{\varepsilon t}$ and $\widetilde{q}_{\varepsilon x}$ are defined by the following equations respectively,

$$\widetilde{q}_{\varepsilon t} = \frac{\partial \widetilde{q}_\varepsilon}{\partial t},$$

$$\widetilde{q}_{\varepsilon x} = \frac{\partial \widetilde{q}_\varepsilon}{\partial x}.$$

On the other hand, multiplying the first equation in (4.23) by $\widetilde{q}_\varepsilon(t,x)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^{24} \widetilde{q}^2_\varepsilon(t,x)dx - \delta \int_0^{24} \widetilde{q}^2_{\varepsilon x}(t,x)dx - \int_0^{24} \widetilde{\mu}(t)\widetilde{q}^2_\varepsilon(t,x)dx = 0,$$

$$\frac{d}{dt}\int_0^{24} \widetilde{q}^2_\varepsilon(t,x)dx \geq 0, \text{ a.e. } t \in (0, T_0).$$

Then after some calculations, one gets

$$\int_0^{24} \widetilde{q}^2_\varepsilon(0,x)dx \leq C \int_0^{T_0} \int_0^{24} \frac{e^{2s\alpha}}{t^3(T_0-t)^3} \widetilde{q}^2_\varepsilon(t,x)dxdt.$$

By the result of (4.25), we have

$$\int_0^{24} \widetilde{q}^2_\varepsilon(0,x)dx \leq C \int_0^{T_0} \int_\omega \frac{e^{2s\alpha}}{t^3(T_0-t)^3} \widetilde{q}^2_\varepsilon(t,x)dxdt. \qquad (4.26)$$

Using Young's inequality, (4.22), (4.24) and (4.26), we obtain

$$\begin{aligned}
&\int_0^{T_0} \int_\omega e^{-2s\alpha(t,x)} t^3 (T_0-t)^3 |\widetilde{u}_\varepsilon(t,x)(\widetilde{p}_\varepsilon + \widetilde{p}_s)|^2 dxdt + \frac{1}{\varepsilon}\int_0^{24} |\widetilde{p}_\varepsilon(T_0,x)|^2 dx \\
&\leq C\|\widetilde{p}_\varepsilon(0,x)\|^2_{L^2((0,24))}. \qquad (4.27)
\end{aligned}$$

Then combining (4.25) with (4.27), we get

$$\int_{D_{T_0}} \left(\frac{t(T_0-t)e^{2s\alpha}}{s} \left(\widetilde{q}_{\varepsilon t}^2 + |\Delta \widetilde{q}_\varepsilon|^2 \right) + \frac{s^3 e^{2s\alpha}}{t^3(T_0-t)^3} \widetilde{q}_\varepsilon^2 + \frac{s e^{2s\alpha}}{t(T_0-t)} \widetilde{q}_{\varepsilon x}^2 \right) dxdt$$
$$\leq C \|\widetilde{p}_\varepsilon(0,x)\|^2_{L^2(0,24)}.$$

As a consequence,

$$\|\widetilde{v}_\varepsilon\|^2_{W_2^{1,2}((0,T_0)\times(0,24))} \leq C \|\widetilde{p}_\varepsilon(0,x)\|^2_{L^2(0,24)},$$

$$\widetilde{v}_\varepsilon(t,x) = \frac{e^{2s\alpha}}{t^3(T_0-t)^3} \widetilde{q}_\varepsilon(t,x), (t,x) \in (0,T_0)\times(0,24).$$

Then according to the standard theory, we know that

$$W_2^{1,2}((0,T_0)\times(0,24)) \subset L^\infty((0,T_0)\times(0,24)),$$

and for any $\varepsilon > 0$, we may infer that

$$\|\widetilde{v}_\varepsilon\|^2_{L^\infty((0,T_0)\times(0,24))} \leq C \|\widetilde{p}_\varepsilon(0,x)\|^2_{L^2(0,24)}. \tag{4.28}$$

The last estimate and the existence theory of parabolic boundary value problems in L^r (see [54]) imply that on a subsequence we have that

$$\begin{aligned} &\widetilde{v}_\varepsilon(t,x) \to \widetilde{v}(t,x) \quad &&\text{weakly in } L^\infty((0,T_0)\times(0,24)) \\ &\widetilde{p}_\varepsilon(t,x) \to \widetilde{p}^{\widetilde{v}}(t,x) \quad &&\text{weakly in } L^\infty((0,T_0)\times(0,24)), \end{aligned}$$

where $(\widetilde{v}(t,x), \widetilde{p}^{\widetilde{v}}(t,x))$ satisfies

$$\begin{cases} \widetilde{p}_t^{\widetilde{v}} - \delta\Delta\widetilde{p}^{\widetilde{v}} + \widetilde{\mu}\widetilde{p}^{\widetilde{v}} = m(x)\widetilde{v}, & (t,x) \in (0,T_0)\times(0,24), \\ \widetilde{p}^{\widetilde{v}}(t,0) = \widetilde{p}^{\widetilde{v}}(t,24), & t \in (0,T_0), \\ \partial_x \widetilde{p}^{\widetilde{v}}(t,0) = \partial_x \widetilde{p}^{\widetilde{v}}(t,24), & t \in (0,T_0), \\ \widetilde{p}^{\widetilde{v}}(0,x) = \begin{cases} b(\theta,x), & \gamma = 0, \quad x \in (0,24), \\ \overline{p}_0(\gamma,x), & \theta = 0, \quad x \in (0,24), \end{cases} \end{cases}$$

$$\widetilde{p}^{\widetilde{v}}(T_0,x) = 0 \quad \text{a.e. } x \in (0,24). \tag{4.29}$$

We claim that there exists a positive constant c_5 such that

$$\widetilde{p}_\varepsilon(t,x) + \widetilde{p}_s(t,x) \geq c_5 > 0, \ (t,x) \in (0,T_0)\times(0,24),$$

which will be proved in the last part of the proof. Since

$$\widetilde{u}_\varepsilon(t,x)(\widetilde{p}_\varepsilon(t,x)+\widetilde{p}_s(t,x)) = m(x)\widetilde{v}_\varepsilon(t,x),$$

we have that

$$\widetilde{u}_\varepsilon(t,x) \to \widetilde{u}(t,x) \qquad \text{weakly in } L^\infty_{loc}((0,T_0)\times(0,24)),$$

where $(\widetilde{u}(t,x), \widetilde{p}^{\widetilde{u}}(t,x))$ satisfies (4.21) and

$$\widetilde{u}(t,x) = \frac{m(x)\widetilde{v}(t,x)}{\widetilde{p}^{\widetilde{v}}(t,x)+\widetilde{p}_s(t,x)}, \quad (t,x)\in(0,T_0)\times(0,24).$$

From the uniqueness of the solution of (4.21), we known that

$$\widetilde{p}^{\widetilde{v}}(t,x) = \widetilde{p}^{\widetilde{u}}(t,x), \quad (t,x)\in(0,T_0)\times(0,24).$$

Furthermore, according to (4.21) and (4.29), we get

$$\widetilde{p}^{\widetilde{u}}(T_0,x) = 0 \quad \text{a.e. } x\in(0,24),$$

$$\|\widetilde{p}^{\widetilde{u}}\|^2_{L^\infty((0,T_0)\times(0,24))} \le C(\|\widetilde{p}_\varepsilon(0,x)\|^2_{L^\infty(0,24)} + \|m(x)\widetilde{v}\|^2_{L^\infty((0,T_0)\times(0,24))}).$$

Then recalling (4.28), one has

$$\|\widetilde{p}^{\widetilde{u}}\|^2_{L^\infty((0,T_0)\times(0,24))} \le C\|\widetilde{p}_\varepsilon(0,x)\|^2_{L^\infty(0,24)}.$$

For $(u(a,t,x), p^u(a,t,x))$ given by $(\widetilde{u}, \widetilde{p}^{\widetilde{u}})$ on each characteristic line, we get

$$u(a,t,x)\in L^2((0,a_\dagger)\times(0,2T_0)\times(0,24)),$$

and p^u is the solution of (4.11) satisfying

$$\begin{aligned} &p^u(a,t,x) = 0, \quad a.e.\ (a,t,x)\in\Gamma_0\times(0,24),\\ &\|p^u\|_{L^\infty(Q_{2T_0})} \le C(\|\bar{p}_0\|_{L^\infty((0,a_\dagger)\times(0,24))} + \|b\|_{L^\infty((0,2T_0)\times(0,24))}). \end{aligned} \tag{4.30}$$

Now, we prove that there exists a positive constant c_5 such that

$$\widetilde{p}_\varepsilon(t,x)+\widetilde{p}_s(t,x) \ge c_5 > 0 \ for\ (t,x)\in(0,T_0)\times(0,24).$$

Let $p^*(t,x) = \widetilde{p}_\varepsilon(t,x)+\widetilde{p}_s(t,x)$ and recall that $\widetilde{p}_\varepsilon$ satisfies (4.21), one has that

$p^*(t,x)$ satisfies

$$\begin{cases} \partial_t p^* - \delta\Delta p^* + \widetilde{\mu}(t)p^* = m(x)\widetilde{u}_\varepsilon p^*, & (t,x)\in(0,T_0)\times(0,24),\\ p^*(t,0)=p^*(t,24), & t\in(0,T_0),\\ \partial_x p^*(t,0)=\partial_x p^*(t,24), & t\in(0,T_0),\\ p^*(0,x)=\begin{cases} b(\theta,x)+\widetilde{p}_s(0,x), & \gamma=0,\quad x\in(0,24),\\ \overline{p}_0(\gamma,x)+\widetilde{p}_s(\gamma,x), & \theta=0,\quad x\in(0,24).\end{cases} \end{cases} \tag{4.31}$$

Therefore, our main goal is equivalent to prove that

$$p^*(t,x)>0,\ (t,x)\in[0,T_0]\times[0,24]$$

by two steps.

Step one, we prove

$$p^*(t,x)>0\ for\ (t,x)\in(0,T_0)\times(0,24).$$

Firstly, when $\gamma=0$, $p^*(0,x)=b(\theta,x)+\widetilde{p}_s(0,x)$. Since $p_s\geq\rho_0>0$ and $\|b\|_{L^\infty((0,2T_0)\times(0,24))}$ is small enough, one knows that there exists a constant c_1 such that

$$p^*(0,x)\geq c_1>0.$$

Secondly, when $\theta=0$, $p^*(0,x)=\overline{p}_0(\gamma,x)+\widetilde{p}_s(\gamma,x)$. Since $p_s\geq\rho_0>0$ and $\|\overline{p}_0(a,x)\|_{L^\infty((0,a_\dagger)\times(0,24))}$ is small enough, we obtain that there exists a constant c_2 such that

$$p^*(0,x)\geq c_2>0.$$

According to these, we learn that there exists a constant c_3 such that

$$p^*(0,x)\geq c_3>0.$$

By the strong maximum principle, it implies that

$$p^*(t,x)>0,\ (t,x)\in(0,T_0)\times(0,24).$$

Step two, we prove $p^*(t,x)>0$ in the following four cases where the variables (t,x) are on the boundary of $(0,T_0)\times(0,24)$. By the strong maximum principle, we have that $p^*(t,x)$ can reach the minimum at the boundary of $(0,T_0)\times(0,24)$. Case 1, if $p^*(t,x)$ attains its minimum at $(0,x_1)$, then $p^*(t,x)\geq p^*(0,x_1)$. Since step one, one knows $p^*(0,x)\geq c_3>0$ for $x\in(0,24)$.

Thus,

$$p^*(t,x) > 0.$$

Case 2, if $p^*(t,x)$ attains its minimum at (T_0, x_2), then $p^*(t,x) \geq p^*(T_0, x_2)$. Since $\widetilde{p}_\varepsilon(t,x) \to \widetilde{p}^{\widetilde{v}}(t,x)$ *weakly in* $L^\infty((0,T_0)\times(0,24))$ and $\widetilde{p}^{\widetilde{v}}(T_0,x) = 0$ *a.e.* $x \in (0,24)$, one has that $\widetilde{p}_\varepsilon(T_0,x) \to 0$ and there exists a constant c_4 such that $p^*(T_0,x) \geq c_4 > 0$. Thus,

$$p^*(t,x) > 0.$$

Case 3, if $p^*(t,x)$ attains its minimum at $(t_1, 0)$, then $p^*(t,x) \geq p^*(t_1,0)$. We assume by contradiction that $p^*(t_1,0) = 0$. Then, it follows from the strong maximum principle that $\partial_x p^*(t_1,24) < 0$. By the third equation of (4.31), we obtain that $\partial_x p^*(t_1,0) < 0$ which contradicts the minimum of $p^*(t,x)$ at $(t_1,0)$. Thus,

$$p^*(t,x) \geq p^*(t_1,0) > 0.$$

Case 4, if $p^*(t,x)$ attains its minimum at $(t_2,24)$, then $p^*(t,x) \geq p^*(t_2,24)$. By our periodic boundary condition, we have $p^*(t_2,24) = p^*(t_2,0)$. It follows from case 3 that

$$p^*(t,x) \geq p^*(t_2,0) > 0.$$

This completes the proof. □

In what follows, we are ready to prove our null exact controllability result. That is to say, we prove Theorem 1.6.

Proof of Theorem 1.6. For any $b(t,x) \in H$, we denote

$$\Phi(b) = \left\{ \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p^u(a,t,s)\,ds\,da \right\} \subset L^2((0,2T_0)\times(0,24)),$$

where $u(a,t,x) \in L^2((0,a_\dagger)\times(0,2T_0)\times(0,24))$, p^u is the solution of (4.11) satisfying (4.30) and $p^u(a,t,x) = 0,$ a.e. $(a,t,x) \in \Gamma_0 \times (0,24)$. There exists an element in $\Phi(b)$ which does not depend on b, and we divide it into two cases as follows:

- If $t > T_0$, then

$$\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p^u(a,t,s)\,ds\,da = 0.$$

This is because $\beta(a) = 0,$ a.e. $a \in (0,T_0)$ and $p^u(a,T_0,x) = 0,$ for $a \geq T_0$.

- If $t < T_0$, then

$$\int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p^u dsda = \int_{T_0}^{a_\dagger - T_0} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p^u dsda.$$

Thus, this depends only on $\bar{p}_0$ and not on b. We also have that

$$p^u(a, 2T_0, x) = 0, \text{ a.e. } \in (0, a_\dagger) \times (0, 24),$$

$$\int_{T_0}^{a_\dagger - T_0} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p^u(a,t,s) dsda \leq C\|\beta\|_{L^\infty(0,a_\dagger)} \cdot \|\bar{p}_0\|_{L^\infty((0,a_\dagger)\times(0,24))}. \tag{4.32}$$

So, for any $u(a,t,x)$ as above we can take

$$b(t,x) = \begin{cases} 0, & \text{a.e. } (t,x) \in (T_0, 2T_0) \times (0,24) \\ \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p^u(a,t,s) dsda, & \text{a.e. } (t,x) \in (0,T_0) \times (0,24) \end{cases}$$

a fixed point of the multi-valued function Φ. By (4.30) and (4.32) we have

$$\|p^u\|_{L^\infty(Q_{2T_0})} \leq C\|\bar{p}_0\|_{L^\infty(0,a_\dagger)}.$$

Thus, if $\|\bar{p}_0\|_{L^\infty(0,a_\dagger)}$ is small enough, there exists $u(a,t,x) \in L^2((0,a_\dagger) \times (0,2T_0) \times (0,24))$ and $p(a,t,x)$, such that the solution of (1.9) satisfies

$$p(a, 2T_0, x) = 0 \quad \text{a.e. } (a,x) \in (0,a_\dagger) \times (0,24),$$
$$\|p\|_{L^\infty(Q_{2T_0})} \leq C\|\bar{p}_0\|_{L^\infty(0,a_\dagger)} \leq \rho_0$$

and in conclusion

$$p(a,t,x) \geq -\rho_0, \quad a.e. \ (a,t,x) \in Q_{2T_0}.$$

Hence,

$$p(a,t,x) \geq -p_s(a,x), \quad a.e. \ (a,t,x) \in Q_{2T_0}.$$

Therefore, we conclude the controllability of (1.1). □

4.3 Numerical simulations

In the following, we provide some numerical simulations to illustrate the local exact controllability of (1.10). We rescale the bitting time variable $x \in [0, 24]$

intro $x \in [0,1]$ and we assume that $a_\dagger = 1$, that is, $a \in [0,1)$. We consider system (1.10) with the parameters taking the values as follows

$$\delta = 0.001,\ \eta = 0.1,\ \beta(a) = 3ce^{-0.1(a-0.4)^2} \text{ and } \mu(a) = 0.5e^{2.4a},$$

where c is a constant to be given.

We suppose that $p_s(a,x) = e^{-\int_0^a \mu(s)ds}$ is a steady solution. It implies that

$$1 = \int_0^{a_\dagger} \beta(a) \int_{x-\eta}^{x+\eta} K(x,s) p_s(a,s) ds da.$$

By numerical computation, we can get that

$$c \approx 15.314.$$

Then, $p_s(a,x)$ is a steady solution of (1.10), as shown in figure 4.

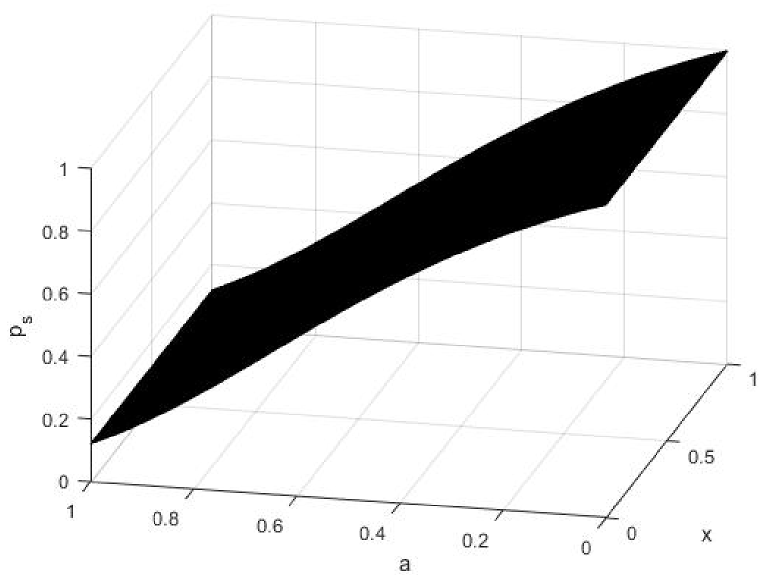

Figure 4: the steady solution $p_s(a,x) = e^{-\int_0^a \mu(s)ds}$.

Now, we set that

$$p_0(a,x) = p_s(a,x) + 0.5e^{-4a}\sin(2\pi x) \text{ and } u(a,t,x) = 0.$$

In following figures 5 and 6, we plot the solution of (1.10) without control, that is, $u(a,t,x) = 0$. We can see that the solution can not converge to the steady solution.

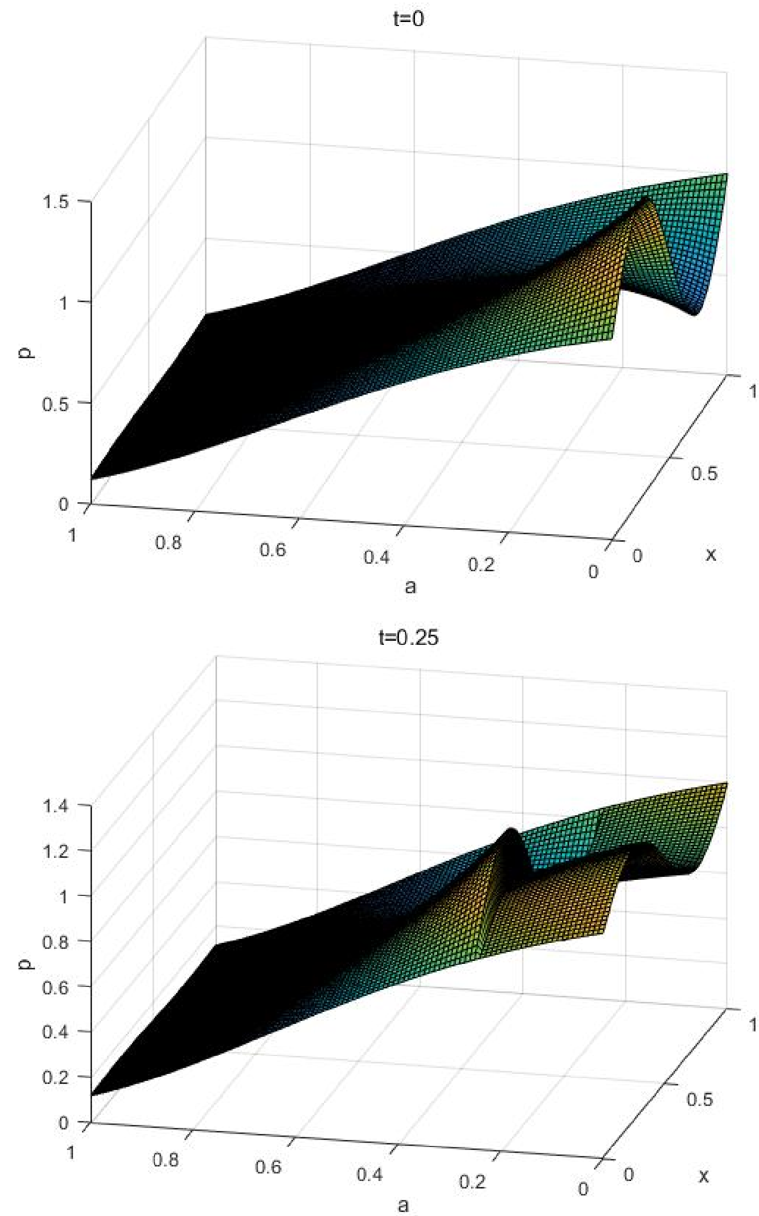

Figure 5: the solution $p(a,t,x)$ for $t=0$ and $t=0.25$ under no control.

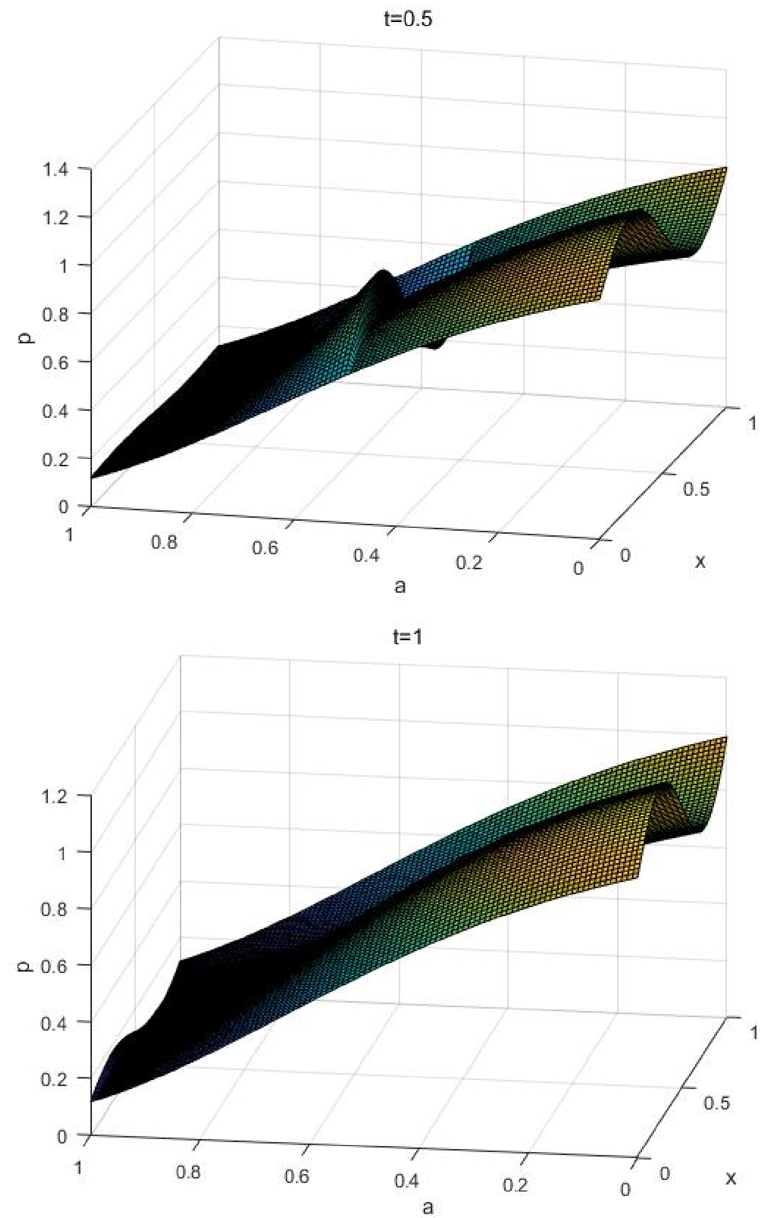

Figure 6: the solution $p(a,t,x)$ for $t=0.5$ and $t=1$ under no control.

By comparison, we set that

$$m(x) = 1 \text{ for } x \in [0, 1] \text{ and } u(a, t, x) = 1.2e^{-5t} \sin(2\pi x).$$

In the following figures 7 and 8, we plot the solution of (1.10) under control $u(a, t, x)$. We can see that the solution converges to the steady solution.

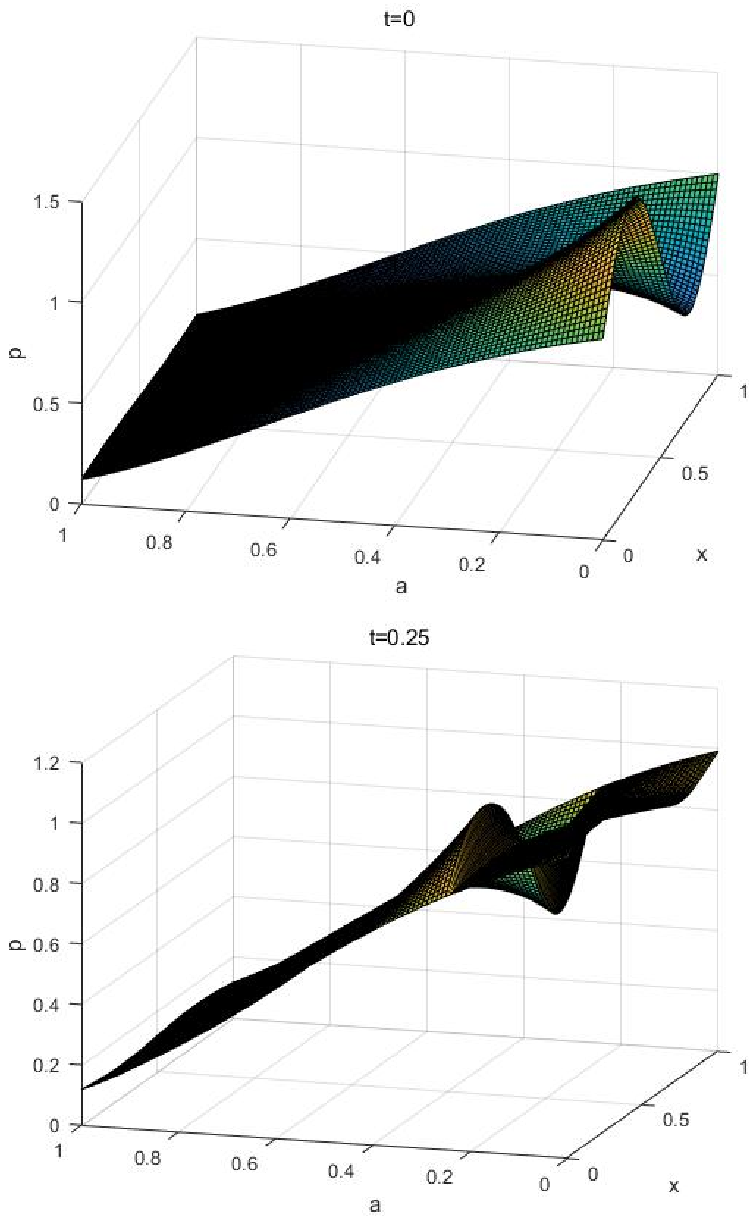

Figure 7: the solution $p(a,t,x)$ for $t=0$ and $t=0.25$ under control.

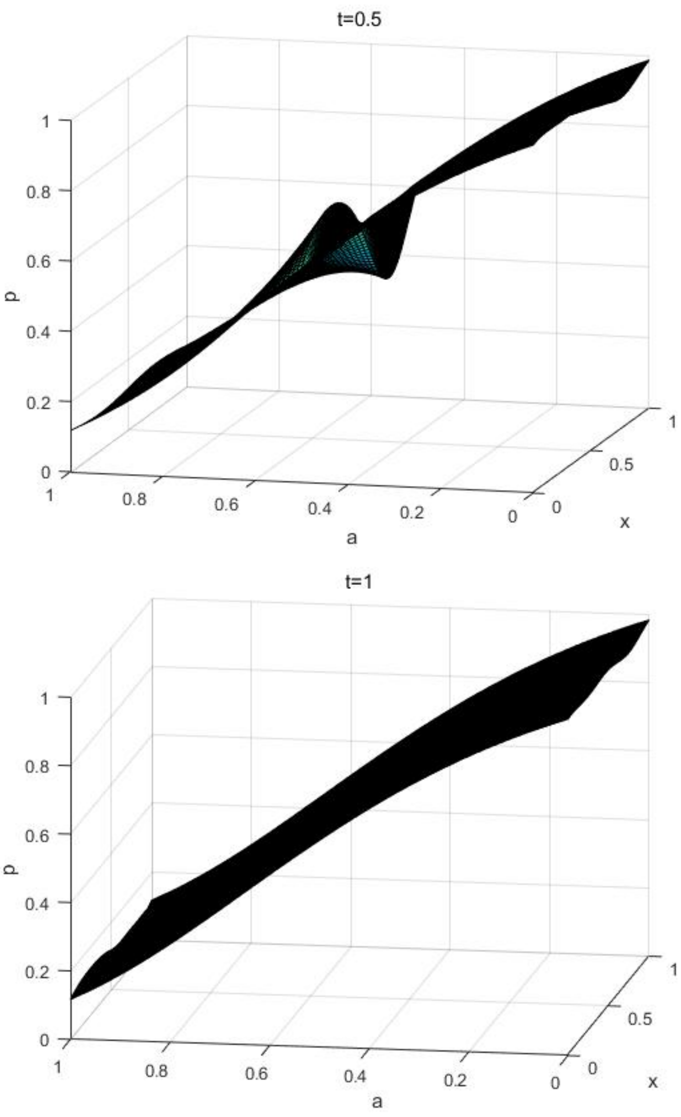

Figure 8: the solution $p(a, t, x)$ for $t = 0.5$ and $t = 1$ under control.

Chapter 5

Large-time behavior of an age structured model

In this chapter, we set that the insecticide strategy such as ITNs and IRs is only useful for matured mosquitoes according to reality. Our goal is to investigate the long time behaviour of the matured population of mosquitoes under different kinds of control $u(a,t,x)$. We deduce a time-delayed model for the matured population which also has its own interest. At last, we do numerical simulations for this problem.

5.1 Proof of Theorem 1.7

In this section, our main job is to study the behavior of the matured mosquitos population when the control $|u(w)|$ is small enough and large enough.

Let us start with the following assumptions:

(A1) $\mu^*(a) \in L^1_{loc}([0,a_\dagger))$, $\displaystyle\int_0^a \mu^*(\rho)d\rho < \infty$, where $a < a_\dagger$ and $\int_0^{a_\dagger} \mu^*(\rho)d\rho = +\infty$;

(A2) $\beta^*(a) \in L^\infty((0,a_\dagger))$, $\text{mes}\{a | a \in [0,a_\dagger], \beta^*(a) > 0\} > 0$;

(A3) $p_0^*(a,x) \in L^\infty((0,a_\dagger) \times (0,24))$, $p_0^*(a,x) \geq 0$.

Now, we first consider the following system

$$\begin{cases} Dq - \delta\Delta q + \mu^*(a)q = 0, & (a,t,x) \in Q_{a_\dagger}, \\ q(a,t,0) = q(a,t,24), & (a,t) \in (0,a_\dagger) \times \mathbb{R}^+, \\ \partial_x q(a,t,0) = \partial_x q(a,t,24), & (a,t) \in (0,a_\dagger) \times \mathbb{R}^+, \\ q(0,t,x) = C\displaystyle\int_0^{a_\dagger} \beta^*(a)q(a,t,x)da, & (t,x) \in \mathbb{R}^+ \times (0,24), \\ q(a,0,x) = p_0^*(a,x), & (a,x) \in (0,a_\dagger) \times (0,24), \end{cases} \tag{5.1}$$

where $Q_{a_\dagger} = (0, a_\dagger) \times \mathbb{R}^+ \times (0, 24)$ and C is a positive constant. Define the operator $\mathbb{F} : X \to X$ as:

$$\mathbb{F}\phi(a, x) = -\frac{\partial \phi(a, x)}{\partial a} + \delta\Delta\phi(a, x) - \mu^*(a)\phi(a, x), \forall \phi(a, x) \in D(\mathbb{F}), \quad (5.2)$$

where

$$D(\mathbb{F}) = \{\phi(a, x) | \phi, \mathbb{A}\phi \in X, \phi(a, 0) = \phi(a, 24), \partial_x\phi(a, 0) = \partial_x\phi(a, 24),$$
$$\phi(0, x) = C \int_0^{a_\dagger} \beta^*(a)\phi(a, x)da\}.$$

Then, we can write (5.1) as

$$\begin{cases} \dfrac{dq(a, t, x)}{dt} = \mathbb{F}q(a, t, x), \\ q(a, 0, x) = p_0^*(a, x). \end{cases}$$

Define an operator

$$\mathcal{F}_\lambda = \int_0^{a_\dagger} C\beta^*(a)e^{-\lambda a}e^{-\int_0^a \mu^*(\rho)d\rho}e^{\mathbb{B}a}da,$$

where the operator $\mathbb{B} : L^2((0, 24)) \longrightarrow L^2((0, 24))$ is defined as

$$\mathbb{B}u(x) = \delta\Delta u(x),$$

for $u(x)$ satisfying

$$\begin{cases} u(0) = u(24), \\ u'(0) = u'(24). \end{cases}$$

From [61, Lemma 2.1], one has the following lemma directly.

Lemma 5.1. *The operator $\mathbb{F}$ defined by (5.2).*

(1) *The operator $\mathbb{F}$ has a real dominant eigenvalue $\widetilde{\lambda}_0$, that is, $\widetilde{\lambda}_0$ is greater than any real part of the eigenvalues of $\mathbb{F}$.*

(2) *For the operator $\mathcal{F}_{\widetilde{\lambda}_0}$, 1 is an eigenvalue with the eigenfunction $\phi_0(x)$. Furthermore, $\gamma(\mathcal{F}_{\widetilde{\lambda}_0}) = 1$, where $\gamma(\mathcal{F}_{\widetilde{\lambda}_0})$ is the spectral radius of $\mathcal{F}_{\widetilde{\lambda}_0}$, that is, $\gamma(\mathcal{F}_{\widetilde{\lambda}_0}) = \sup\{|r| : r$ is an eigenvalue of $\mathcal{F}_{\widetilde{\lambda}_0}\}$.*

Furthermore, one can get the following lemma.

Lemma 5.2. *The eigenvalue $\widetilde{\lambda}_0$ obtained in Lemma 5.1 is such that*

(1) *if $C \int_0^{a_\dagger} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho}da > 1$, then $\widetilde{\lambda}_0 > 0$.*

(2) *if* $C\int_0^{a_\dagger}\beta^*(a)e^{-\int_0^a\mu^*(\rho)d\rho}da<1$, *then* $\widetilde{\lambda}_0<0$.

Proof. We denote by $(\overline{\lambda}_i,\phi_i)_{i\geq 0}$ the eigenvalues and eigenfunctions of the following problem

$$\begin{cases} -\delta\Delta\phi_i(x)=\overline{\lambda}_i\phi_i(x),\ x\in(0,24),\\ \phi_i(0)=\phi_i(24),\\ \partial_x\phi_i(0)=\partial_x\phi_i(24),\end{cases}$$

where $\int_0^{24}\phi_i^2(x)dx=1,\ i\geq 0$, and $\phi_0(x)>0$ in $(0,24)$. It is obvious that $\overline{\lambda}_0=0$ and $\phi_0(x)$ is a fixed positive constant. We also assume that $0=\overline{\lambda}_0<\overline{\lambda}_1\leq\overline{\lambda}_2\leq\cdots$.

Let $\mathbb{H}$ be the operator in $L^2(0,a_\dagger)$ defined as

$$\mathbb{H}\phi(a)=-\frac{d\phi(a)}{da}-\mu^*(a)\phi(a),\ \forall\phi\in D(\mathbb{H}),$$

where

$$D(\mathbb{H})=\{\phi(a)|\phi,\mathbb{H}\phi\in L^2(0,24),\phi(0)=C\int_0^{a_\dagger}\beta^*(a)\phi(a)da\}.$$

Let $\{\widehat{\lambda}_j\}_{j\geq 0}$ be the eigenvalues of $\mathbb{H}$, that is, the solutions of the following equation

$$1-C\int_0^{a_\dagger}\beta^*(a)e^{-\widehat{\lambda}_ja-\int_0^a\mu^*(\rho)d\rho}da=0.$$

We assume that $\widehat{\lambda}_0>Re\widehat{\lambda}_1\geq Re\widehat{\lambda}_2\geq\cdots$, even if it means re-arrange $\widehat{\lambda}_j$. From [61, Lemma 2.1], one knows that

$$\widetilde{\lambda}_0=\widehat{\lambda}_0-\overline{\lambda}_0=\widehat{\lambda}_0.$$

It is obvious that if $C\int_0^{a_\dagger}\beta^*(a)e^{-\int_0^a\mu^*(\rho)d\rho}da>1$, then

$$\widetilde{\lambda}_0>0.$$

If $C\int_0^{a_\dagger}\beta^*(a)e^{-\int_0^a\mu^*(\rho)d\rho}da<1$, then

$$\widetilde{\lambda}_0<0.$$

□

Lemma 5.3. *Let $q(a,t,x)$ be the solution of*

$$\begin{cases} Dq - \delta\Delta q + \mu^*(a)q = 0, & (a,t,x) \in Q_{a_\dagger}, \\ q(a,t,0) = q(a,t,24), & (a,t) \in (0,a_\dagger) \times \mathbb{R}^+, \\ \partial_x q(a,t,0) = \partial_x q(a,t,24), & (a,t) \in (0,a_\dagger) \times \mathbb{R}^+, \\ q(0,t,x) = \int_0^{a_\dagger} \beta^*(a) \int_{x-\eta}^{x+\eta} K_1(x,s)q(a,t,s)dsda, & (t,x) \in \mathbb{R}^+ \times (0,24), \\ q(a,0,x) = p_0^*(a,x), & (a,x) \in (0,a_\dagger) \times (0,24). \end{cases}$$

(1) *If $\int_0^{a_\dagger} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho}da$ is sufficiently large and*

$$K_1(x,s) = \begin{cases} (x-s)^2 e^{-(x-s)^2}, & (x,s) \in (0,24) \times (0,24), \\ 0, & else, \end{cases}$$

then

$$q(a,t,x) \to +\infty, \text{ as } t \to +\infty \text{ for every } a \in [0,a_1],\ x \in [0,24],$$

where $a_1 \in (a_0, a_\dagger)$.

(2) *If $\int_0^{a_\dagger} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho}da$ is sufficiently small and $K_1(x,s) \equiv K_1$ where K_1 is a positive constant, then*

$$q(a,t,x) \to 0, \text{ as } t \to +\infty \text{ for every } a \in [0,a_\dagger],\ x \in [0,24].$$

Proof. By referring to [61, Theorem 1.1], one knows that $q(a,t,x)$ has an asymptotic expression

$$\begin{aligned} q(a,t,x) =& e^{\lambda_0 t} e^{-\lambda_0 a} \mathfrak{T}(0,a) C_{\lambda_0} \int_0^{a_\dagger} \beta^*(a) \int_{x-\eta}^{x+\eta} K_1(x,s) \int_0^a e^{-\lambda_0(a-\sigma)} \mathfrak{T}(\sigma,a) \\ & p_0(\sigma,s)dsdad\sigma + o(e^{(\lambda_0-\varepsilon)t}). \end{aligned}$$

Here, λ_0 is the algebraically simple real eigenvalue of the operator $\mathbb{A} : L^2((0,a_\dagger) \times(0,24)) \longrightarrow L^2((0,a_\dagger) \times (0,24))$ defined as

$$\mathbb{A}\phi(a,x) = -\frac{\partial \phi(a,x)}{\partial a} + \delta\Delta\phi(a,x) - \mu^*(a)\phi(a,x), \forall \phi(a,x) \in D(\mathbb{A}),$$

$$\begin{aligned} D(\mathbb{A}) = \{\phi(a,x) | \phi, \mathbb{A}\phi \in X, \phi(a,0) = \phi(a,24), \partial_x\phi(a,0) = \partial_x\phi(a,24), \\ \phi(0,x) = \int_0^{a_\dagger} \beta^*(a) \int_{x-\eta}^{x+\eta} K_1(x,s)\phi(a,s)dsda\}. \end{aligned}$$

And λ_0 is larger than real part of the any other eigenvalues of the operator $\mathbb{A}$. $\mathcal{T}(\tau, s) = e^{-\int_\tau^s \mu^*(\rho)d\rho} e^{\mathbb{B}(s-\tau)}$. $C_{\lambda_0} = \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)(\mathbb{I} - \mathcal{B}_\lambda)^{-1}$, where the operator $\mathcal{B}_\lambda : L^2((0,24)) \to L^2((0,24))$ defined as

$$\mathcal{B}_\lambda(\phi(x)) = \int_0^{a_\dagger} \beta^*(a) \int_{x-\eta}^{x+\eta} K_1(x,s) e^{-\lambda a} \mathcal{T}(0,a)\phi(s) ds da.$$

(1) From the proof of [61, Theorem 3.1], one can take C in (5.1) sufficiently small such that

$$\lambda_0 \geq \widetilde{\lambda}_0.$$

Note that the choice of C depends on $K_1(x,s)$. Then, if $\int_0^{a_\dagger} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da$ is sufficiently large such that $C \int_0^{a_\dagger} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da > 1$, by the result of Lemma 5.2 (1), one has that

$$\lambda_0 \geq \widetilde{\lambda}_0 > 0.$$

From the asymptotic expression, one gets that

$$q(a,t,x) \to +\infty, \text{ as } t \to +\infty \text{ for every } a \in [0, a_0],\ x \in [0,24].$$

(2) Similar as the arguments of the proof of [61, Theorem 3.1], one can take C in (5.1) sufficiently large and prove that

$$\lambda_0 \leq \widetilde{\lambda}_0.$$

Then, if $\int_0^{a_\dagger} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da$ is sufficiently small such that

$$C \int_0^{a_\dagger} \beta^*(a) e^{-\int_0^a \mu^*(\rho)d\rho} da < 1,$$

by the result of Lemma 5.2 (2), one has that

$$\lambda_0 \leq \widetilde{\lambda}_0 < 0.$$

From the asymptotic expression, one has that

$$q(a,t,x) \to 0, \text{ as } t \to +\infty \text{ for every } a \in [0, a_0],\ x \in [0,24].$$

This completes the proof. □

Following the proof of [13, Lemma 4.2.2] carefully, we can get the following lemma:

Lemma 5.4. *If $p_i (i \in 1, 2)$ are the solutions of the following systems*

$$\begin{cases} Dp_i - \delta\Delta p_i + \mu_i(a, w)p_i = 0, & (a, t, x) \in Q_{a_\dagger}, \\ p_i(0, t, x) = \displaystyle\int_0^{a_\dagger} \beta_i(a) \int_{x-\eta}^{x+\eta} K_i(x, s)p_i(a, t, s)dsda, & (t, x) \in \mathbb{R}^+ \times (0, 24), \\ p_i(a, 0, x) = p_{0i}(a, x), & (a, x) \in (0, a_\dagger) \times (0, 24), \end{cases}$$

where $w(t, x) = \int_{a_0}^{a_\dagger} p(a, t, x)da$, $\mu_1(a, w)$, $\mu_2(a, w) \in L^\infty_{loc}([0, a_\dagger))$ for every $w \geq 0$ and are locally Lipschitz functions with respect to w, β_1, β_2 satisfy (A_2), K_1, $K_2 \in L^2([0, 24]^2)$, p_{01}, p_{02} satisfy (A_3) and $\mu_1 \geq \mu_2$, $\beta_1 \leq \beta_2$, $K_1 \leq K_2$, $p_{01} \leq p_{02}$, then

$$0 \leq p_1(a, t, x) \leq p_2(a, t, x) \quad a.e. \ in \ Q_{a_\dagger}.$$

Proof. Following [13, Theorem 4.2.2], one can easily get the previous comparison principle. Thus, we omit the details. □

We now extend $q(a, t, x)$ to $x \in \mathbb{R}$ periodically such that

$$\widetilde{q}(a, t, x) = q(a, t, x), \ x \in [0, 24],$$

$$\widetilde{q}(a, t, x + 24) = \widetilde{q}(a, t, x), \ x \in \mathbb{R}.$$

Then, $\widetilde{q}(a, t, x)$ satisfies

$$\begin{cases} D\widetilde{q} - \delta\Delta\widetilde{q} + \mu^*(a)\widetilde{q} = 0, & (a, t, x) \in Q, \\ \widetilde{q}(0, t, x) = \int_0^{a_\dagger} \beta^*(a) \int_{x-\eta}^{x+\eta} \widetilde{K}_1(x, s)\widetilde{q}(a, t, s)dsda, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \widetilde{q}(a, 0, x) = \widetilde{q}_0(a, x), & (a, x) \in (0, a_\dagger) \times \mathbb{R}, \end{cases} \tag{5.3}$$

where

$$\widetilde{q}_0(a, x) = p_0^*(a, x), x \in [0, 24],$$

$$\widetilde{q}_0(a, x + 24) = \widetilde{q}_0(a, x), x \in \mathbb{R},$$

$$\widetilde{K}_1(x, s) = K_1(x, s), x \in [0, 24], s \in [0, 24],$$

$$\widetilde{K}_1(x, s) = 0, x \in [0, 24], s < 0 \ and \ s > 24,$$

$$\widetilde{K}_1(x + 24, s + 24) = \widetilde{K}_1(x, s), x \in \mathbb{R}, s \in \mathbb{R}.$$

By Lemma 5.3, one has the following lemma.

Lemma 5.5. *Let $\widetilde{q}(a, t, x)$ be the solution of system (5.3).*

(1) *If* $\int_0^{a_\dagger} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho}da$ *is sufficiently large and*

$$K_1(x,s) = \begin{cases} (x-s)^2 e^{-(x-s)^2}, & (x,s) \in (0,24)\times(0,24), \\ 0, & \textit{else}, \end{cases} \tag{5.4}$$

then

$$\widetilde{q}(a,t,x) \to +\infty, \ \textit{as } t \to +\infty \textit{ for every } a \in [0,a_1] \textit{ and } x \in \mathbb{R}.$$

(2) *If* $\int_0^{a_\dagger} \beta^*(a)e^{-\int_0^a \mu^*(\rho)d\rho}da$ *is sufficiently small and* $K_1(x,s) \equiv K_1$ *where* K_1 *is a positive constant, then*

$$\widetilde{q}(a,t,x) \to 0, \ \textit{as } t \to +\infty \textit{ for every } a \in [0,a_\dagger] \textit{ and } x \in \mathbb{R}.$$

Proof of Theorem 1.7. (i) Let $p(a,t,x)$ be the solution of (1.10). From the assumption (J5), one can take $|u(w)|$ small enough such that

$$\mu(a,w) - u(w) \le \sup_{w\ge 0}(\mu(a,w) - u(a,w)) := \mu^*(a),$$

and $\int_0^{a_\dagger} \beta(a)e^{-\int_0^a \mu^*(\rho)d\rho}$ is sufficiently large. Let $\widetilde{q}(a,t,x)$ be the solution of system (5.3) with $\mu^*(a)$, $\beta(a)$, $\widetilde{K}_1(x,s)$ and $\widetilde{q}_0(a,x)$ where $\widetilde{K}_1(x,s)$ is the periodic extension of $K_1(x,s)$ defined by (5.4) and $\widetilde{q}_0(a,x) = p_0(a,x)$. Obviously, one has

$$\widetilde{K}_1(x,s) \le K(x,s).$$

By Lemma 5.4, one has that

$$p(a,t,x) \ge \widetilde{q}(a,t,x).$$

Then, from Lemma 5.5 (i), one has that

$$w(t,x) \ge \int_{a_0}^{a_\dagger} \widetilde{q}(a,t,x)da \ge \int_{a_0}^{a_1} \widetilde{q}(a,t,x)da \to +\infty, \ \text{as } t \to +\infty.$$

(ii) By the assumptions (J1), (J3), we know that

$$\mu(a,w) - u(a,w) \ge \inf_{w\ge 0} \mu(a,w) + \inf_{w\ge 0}(-u(a,w)).$$

Then let $\mu^*(a) = \inf_{w\ge 0}\mu(a,w) + \inf_{w\ge 0}(-u(a,w))$ and $|u(a,w)|$ be large enough such that $\int_0^{a_\dagger} \beta(a)e^{-\int_0^a \mu^*(\rho)d\rho}$ is small enough. Let $\widetilde{q}(a,t,x)$ be the solution of system (5.3) with $\mu^*(a)$, $\beta(a)$, $\widetilde{K}_1(x,s)$ and $\widetilde{q}_0(a,x)$ where $\widetilde{q}_0(a,x) =$

$p_0(a,x)$ and $\widetilde{K}_1(x,s) \equiv K_1 := \sup_{(x,s)\in\mathbb{R}^2} K(x,s)$. By Lemma 5.4, one has that

$$p(a,t,x) \le \widetilde{q}(a,t,x).$$

Then, from Lemma 5.5 (ii), one has that

$$w(t,x) \le \int_{a_0}^{a_\dagger} \widetilde{q}(a,t,x)da \to 0, \text{ as } t \to +\infty.$$

This completes the proof. □

5.2 Traveling fronts and Theorem 1.9

In the this section, we first derive a reaction-diffusion equation with time delay for the system (1.10) by using a similar method developed by So [86]. Then, we prove the existence of traveling fronts for the sub-equation (1.13)-(1.14) and Theorem 1.9.

5.2.1 Derivation of the model

In this subsection, we derive a model for the matured mosquitoes population of system (1.10) with two age classes and a fixed maturation period in a temporal unbounded domain. By integrating (1.10), one obtains

$$w_t - \delta\Delta w = p(a_0,t,x) + \int_{a_0}^{a_\dagger} (-\mu(a,w)+u(w))p(a,t,x)da, \ t>0, \ x\in\mathbb{R}. \tag{5.5}$$

In the following, we specify $p(a_0,t,x)$. For any fixed $s>0$, let $V^s(t,x) = p(t-s,t,x), \ s \le t \le a_0+s, \ x\in\mathbb{R}$, we have

$$\frac{\partial V^s(t,x)}{\partial t} = \delta\Delta V^s(t,x) - \mu_1(t-s)V^s(t,x), \ s\le t\le a_0+s, \ x\in\mathbb{R}. \tag{5.6}$$

By Fourier transform, let $V^s(t,x) = \int_{-\infty}^{+\infty} f(t,\varepsilon)e^{-i\varepsilon x}d\varepsilon$, one gets

$$\frac{\partial V^s(t,x)}{\partial t} = \int_{-\infty}^{+\infty} \frac{\partial f(t,\varepsilon)}{\partial t} e^{-i\varepsilon x}d\varepsilon, \ s\le t\le a_0+s, \ x\in\mathbb{R}. \tag{5.7}$$

$$\Delta V^s(t,x) = -\int_{-\infty}^{+\infty} \varepsilon^2 f(t,\varepsilon)e^{-i\varepsilon x}d\varepsilon, \ s\le t\le a_0+s, \ x\in\mathbb{R}. \tag{5.8}$$

By virtue of (5.6), (5.7), (5.8) and after some calculations, we obtain

$$f(t,\varepsilon) = f(s,\varepsilon)\exp\left[\int_0^{t-s}(-\delta\varepsilon^2 - \mu_1(\tau))d\tau\right],$$

$$V^s(t,x) = \int_{-\infty}^{+\infty} f(\tau,\varepsilon)\exp\left[\int_0^{t-s}(-\delta\varepsilon^2 - \mu_1(\tau))d\tau\right]e^{-i\varepsilon x}d\varepsilon. \tag{5.9}$$

Recalling the form of $p(t-s,t,x)$, we see

$$p(0,t,x) = V^s(s,x) = \int_{-\infty}^{+\infty} f(s,\varepsilon)e^{-i\varepsilon x}d\varepsilon.$$

Thus, $p(0,t,x)$ is the Fourier transform of $f(s,\varepsilon)$, it means that

$$f(s,\varepsilon) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} p(0,s,y)e^{i\varepsilon y}dy. \tag{5.10}$$

Using the results of (5.9) and (5.10), we have

$$\begin{aligned}
&p(a_0,t,x) = V^{t-a_0}(t,x)\\
=&\int_{-\infty}^{+\infty} f(t-a_0,\varepsilon)\exp\left[\int_0^{a_0}(-\delta\varepsilon^2 - \mu_1(\tau))d\tau\right]e^{-i\varepsilon x}d\varepsilon\\
=&\int_{-\infty}^{+\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty} p(0,t-a_0,y)e^{i\varepsilon y}dy\exp\left[\int_0^{a_0}(-\delta\varepsilon^2 - \mu_1(\tau))d\tau\right]e^{-i\varepsilon x}d\varepsilon\\
=&\frac{1}{\sqrt{4\pi\delta a_0}}e^{-\int_0^{a_0}\mu_1(\tau)d\tau}\\
&\int_{-\infty}^{+\infty}\left[\int_0^{a_\dagger}\beta(a)\int_{y-\eta}^{y+\eta}K(y,s)p(a,t-a_0,s)dsda\right]\exp\left[-\frac{(x-y)^2}{4\delta a_0}\right]dy.
\end{aligned}$$

By the definition of $K(x,s)$ and $(J2)$, one can get that

$$p(a_0,t,x) = M\beta\int_{-\infty}^{+\infty}\left[\int_{-\eta}^{\eta} z^2e^{-z^2}w(t-a_0,x-z-s)dz\right]f_{\delta a_0}(s)ds, \tag{5.11}$$

where $M = e^{-\int_0^{a_0}\mu_1(\tau)d\tau}$ and $f_{\delta a_0}(x) = \frac{1}{\sqrt{4\pi\delta a_0}}\exp(-\frac{x^2}{4\delta a_0})$. Then combining (5.11) with (5.5), for $t>0$, $x\in\mathbb{R}$, one gets

$$\begin{aligned}
w_t - \delta\Delta w = &\int_{a_0}^{a_\dagger}(-\mu(a,w)+u(w))p(a,t,x)da\\
&+ M\beta\int_{-\infty}^{+\infty}\left[\int_{-\eta}^{\eta} z^2e^{-z^2}w(t-a_0,x-z-s)dz\right]f_{\delta a_0}(s)ds.
\end{aligned}$$

On the other hand, we set the initial value

$$w(s,x)=\int_{a_0}^{a_\dagger} p(a,s,x)da=\int_{a_0}^{a_\dagger} p_0(a+s,x)da,\ \text{for } -a_0\le s\le 0.$$

Since $p_0(a,\cdot)\ge\not\equiv 0$ for every $a\in[0,a_\dagger]$, one has that

$$w(s,\cdot)\ge\not\equiv 0,\ \text{for every } s\in[-a_0,0].$$

Notice that $w(t,x+24)=w(t,x)$ since $p(a,t,x+24)=p(a,t,x)$.

5.2.2 Existence of traveling fronts

In this subsection, we study the sub-equation (1.15) and prove Theorem 1.8. The purpose of this section is to establish the existence of traveling fronts of (1.15).

A traveling front of (1.15) is a solution $w(t,x)=\phi(x+ct)$, where $c>0$ is the wave speed and $\phi\in C(\mathbb{R};\mathbb{R})$ is a non-decreasing function satisfying the following equation

$$\begin{aligned}c\phi'(\xi)=&\delta\phi''(\xi)-g(\phi)+u(\phi)\phi(\xi)\\&+M\beta\int_{-\infty}^{+\infty}\left[\int_{-\eta}^{\eta} z^2e^{-z^2}\phi(\xi-s-z-ca_0)dz\right]f_{\delta a_0}(s)ds,\end{aligned}\tag{5.12}$$

with the boundary conditions

$$\lim_{\xi\to-\infty}\phi(\xi)=0,\quad \lim_{\xi\to\infty}\phi(\xi)=w_2,\tag{5.13}$$

where $\xi=x+ct$. Then, for $\xi\in\mathbb{R}$, we define the following profile set

$$\begin{aligned}\Gamma=\{\phi\in C(\mathbb{R};\mathbb{R})|&(1)\ \phi(\xi)\ \text{ is non-decreasing;}\\&(2)\ \lim_{\xi\to-\infty}\phi(\xi)=0;\ \lim_{\xi\to\infty}\phi(\xi)=w_2\}.\end{aligned}$$

Notice that since $g(w)$, $u(w)$ and their derivatives are continuous, there exists a constant $b\ge 0$ such that

$$(-g(\phi)+u(\phi)\phi)-(-g(\psi)+u(\psi)\psi)+b(\phi-\psi)\ge 0,$$

where $\phi,\psi\in\Gamma$ and $0\le\psi(\xi)\le\phi(\xi)\le w_2$. Furthermore, define H :

$C(\mathbb{R};\mathbb{R}) \to C(\mathbb{R};\mathbb{R})$ by

$$H(\phi)(\xi) = b\phi(\xi) - g(\phi) + u(\phi)\phi(\xi) + M\int_{-\infty}^{+\infty}\beta\left[\int_{-\eta}^{\eta} z^2 e^{-z^2}\phi(\xi - s - z - ca_0)dz\right] f_{\delta a_0}(s)ds.$$

It follows that the equation of (5.12) involves the following nonhomogeneous system of ordinary differential equation

$$c\phi'(\xi) = \delta\phi''(\xi) - b\phi(\xi) + H(\phi)(\xi),\ \xi \in \mathbb{R}. \tag{5.14}$$

By exploring $H(\phi)$, we get the following lemma directly.

Lemma 5.6. *For any $\phi, \psi \in \Gamma$, we have*

(1) *$H(\phi)(\xi) \geq 0$, for all $\xi \in \mathbb{R}$.*

(2) *$H(\phi)(\xi)$ is non-decreasing in $\xi \in \mathbb{R}$.*

(3) *$H(\psi)(\xi) \leq H(\phi)(\xi)$, for all $\xi \in \mathbb{R}$, provided $\psi \in C(\mathbb{R};\mathbb{R})$ is such that $0 \leq \psi(\xi) \leq \phi(\xi) \leq w_2$, for all $\xi \in \mathbb{R}$.*

Proof. By some simple calculations, it is easy to get these results. Thus, we omit the proof. □

Now, we define subsolutions and supersolutions for (5.14) as follows.

Definition 5.7. *A function $\phi \in C(\mathbb{R};\mathbb{R})$ is called a supersolution of (5.14) if ϕ' and ϕ'' exist almost everywhere and are essentially bounded on $\mathbb{R}$, and ϕ satisfies*

$$c\phi'(\xi) \geq \delta\phi''(\xi) - b\phi(\xi) + H(\phi)(\xi),\ a.e\ in\ \mathbb{R}.$$

A subsolution of (5.14) is defined in a similar way by reversing the inequality in (5.7).

In the following, we first assume that there exists a pair $(\overline{\phi}, \underline{\phi})$, where $\overline{\phi} \in \Gamma$ is a supersolution and $\underline{\phi}$ is a subsolution of (5.14) (which is not necessarily in Γ), such that

(G1) $0 \leq \underline{\phi}(\xi) \leq \overline{\phi}(\xi) \leq w_2$, for all $\xi \in \mathbb{R}$;

(G2) $\underline{\phi}(\xi) \not\equiv 0$.

Our goal is to prove that the equation of (5.12) has a solution $\phi(\xi)$ satisfying the boundary conditions (5.13) by the iterative method. It is equivalent to verify that the equation of (5.14) has a solution $\phi(\xi)$ satisfying (5.13). Naturally, we start our iteration with a subsolution of (5.12) as the following iteration scheme:

$$c\phi_n'(\xi) = \delta\phi_n''(\xi) - b\phi_n(\xi) + H(\phi_{n-1})(\xi),\ \xi \in \mathbb{R},\quad n = 1, 2, \ldots, \tag{5.15}$$

with the boundary conditions

$$\lim_{\xi\to-\infty}\phi_n(\xi)=0,$$

$$\lim_{\xi\to\infty}\phi_n(\xi)=w_2,$$

where $\phi_0(\xi)=\overline{\phi}(\xi)$. Among all solutions of (5.15), we choose a special one and explore its properties as below

$$\begin{cases}\phi_n(\xi)=\frac{1}{\delta(\beta_1-\beta_2)}\left[\int_{-\infty}^{\xi}e^{\beta_1(\xi-s)}H(\phi_{n-1})(s)ds+\int_{\xi}^{\infty}e^{\beta_2(\xi-s)}H(\phi_{n-1})(s)ds\right],\\ \phi_0(\xi)=\overline{\phi}(\xi),\end{cases}$$

where $\xi\in\mathbb{R}, n=1,2,\ldots,$ and

$$\beta_1=\frac{c-\sqrt{c^2+4\delta b}}{2\delta},\quad \beta_2=\frac{c+\sqrt{c^2+4\delta b}}{2\delta}.$$

Following the proof of lemma 3.3, lemma 3.4 and proposition 3.5 in [97] as step by step, we can get the following Lemma.

Lemma 5.8. *The sequence of functions* $\{\phi_n(\xi)\}_{n=0}^{\infty}$ *satisfies*
(1) $\phi_n\in\Gamma$, *for all* $n=1,2,\ldots$;
(2) $0\le\underline{\phi}(\xi)\le\phi_n(\xi)\le\phi_{n-1}(\xi)\le\overline{\phi}(\xi)\le w_2$, *for all* $\xi\in\mathbb{R}$, $n=1,2,\ldots$;
(3) *Each* $\phi_n(\xi)$ *is a supersolution of* (5.12);
(4) $\phi(\xi)=\lim_{n\to\infty}\phi_n(\xi)$ *is a solution of* (5.12) *satisfying* (5.13).

Now, we summarize the above lemmas and obtain the following Theorem.

Theorem 5.9. *Suppose that* (5.12) *has a supersolution* $\overline{\phi}\in\Gamma$ *and a subsolution* $\underline{\phi}$ *(which is not necessarily in* Γ*) satisfying (G1), (G2). Then* (5.12) *has a solution satisfying the boundary conditions* (5.13)*. That is,* (5.13) *has a traveling wavefront solution* ϕ*, which connects* 0 *and the positive equilibrium* w_2.

We see that it is significant for us to prove the existence of a pair of supersolution and subsolution of (5.12) satisfying (G1), (G2). In the rest of this subsection, we will construct such a pair of supersolution and subsolution. Following the theory of Wang [94], we define the function

$$\Delta_c(\lambda)=\delta\lambda^2-c\lambda+\widetilde{M}_c(\lambda),\tag{5.16}$$

where $\widetilde{M}_c(\lambda) = M\beta e^{\delta a_0\lambda^2-\lambda c a_0}\int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z}dz$. Then, we can get the following lemma.

Lemma 5.10. *There exist c^* and λ^* such that*

(1) $\Delta_{c^*}(\lambda^*) = 0$ *and*
$$\frac{\partial}{\partial\lambda}\Delta_{c^*}(\lambda)|_{\lambda=\lambda^*} = 0;$$
(2) *If $0 < c < c^*$, then $\Delta_c(\lambda) > 0$ for any $\lambda > 0$;*
(3) *If $c > c^*$, then the equation $\Delta_c(\lambda) = 0$ has two positive real roots λ_1, λ_2 such that $0 < \lambda_1 < \lambda_2$ and*
$$\Delta_c(\lambda) = \begin{cases} > 0, & \lambda < \lambda_1, \\ < 0, & \lambda_1 < \lambda < \lambda_2, \\ < 0, & \lambda > \lambda_2. \end{cases}$$

Proof. By some calculations, we obtain
$$\begin{aligned}\frac{\partial}{\partial\lambda}\Delta_c(\lambda) =& 2\delta\lambda - c + M\beta e^{\delta a_0\lambda^2-\lambda c a_0}\int_{-\eta}^{\eta} z^2 e^{-z^2}(-z)e^{-\lambda z}dz \\ &+ M\beta e^{\delta a_0\lambda^2-\lambda c a_0}(\delta a_0\lambda - c a_0)\int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z}dz,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial\lambda^2}\Delta_c(\lambda) =& 2\delta + M\beta e^{\delta a_0\lambda^2-\lambda c a_0}\int_{-\eta}^{\eta} z^2 e^{-z^2}(z-(2\delta a_0\lambda - c a_0))^2 e^{-\lambda z}dz \\ &+ M\beta e^{\delta a_0\lambda^2-\lambda c a_0} 2\delta a_0\lambda\int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z}dz > 0,\end{aligned}$$

$$\frac{\partial}{\partial c}\Delta_c(\lambda) = -\lambda + M\beta(-\lambda a_0)e^{\delta a_0\lambda^2-\lambda c a_0}\int_{-\eta}^{\eta} z^2 e^{-z^2}(-z)e^{-\lambda z}dz < 0,$$

$$\Delta_c(0) = \widetilde{M}_c(0) = M\beta\int_{-\eta}^{\eta} z^2 e^{-z^2}dz > 0,$$

$$\frac{\partial}{\partial\lambda}\Delta_c(0) = -c + M\beta\int_{-\eta}^{\eta} z^2 e^{-z^2}(-z)dz + M\beta(-c a_0)\int_{-\eta}^{\eta} z^2 e^{-z^2}dz < 0,$$

$$\Delta_0(\lambda) = \delta\lambda^2 + \widetilde{M}_0(\lambda) = \delta\lambda^2 + M\beta e^{\delta a_0 \lambda^2} \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-\lambda z} dz > 0,$$

$$\lim_{\lambda \to +\infty} \Delta_c(\lambda) = +\infty.$$

Then it is easy to see that the lemma holds. □

Lemma 5.11. *Let c^*, λ_1 and λ_2 be defined as in Lemma 5.10, and choose $\rho > 0$ sufficiently small so that $\rho < \lambda_1 < \lambda_1 + \rho < \lambda_2$. Then for fix $c > c^*$, there exists a constant $L > 1$ such that the functions $\overline{\phi}$ and $\underline{\phi}$ defined by*

$$\overline{\phi}(\xi) = min\left\{w_2, w_2 e^{\lambda_1 \xi}\right\}, \xi \in \mathbb{R} \tag{5.17}$$

$$\underline{\phi}(\xi) = max\left\{0, w_2(1 - Le^{\rho\xi})e^{\lambda_1 \xi}\right\}, \xi \in \mathbb{R} \tag{5.18}$$

are a supersolution and a subsolution of (5.12), respectively.

Proof. First of all, we see that it is easy to verify that $\overline{\phi}, \underline{\phi}$ satisfy (G1), (G2). Now, We begin by proving that $\overline{\phi}$ and $\underline{\phi}$ are a pair of supersolution and subsolution of (5.12). Our strategy here is to prove this part into two steps: (i) $\overline{\phi}$ is a supersolution of (5.12) satisfying $\overline{\phi} \in \Gamma$; (ii) there exists a sufficiently large L such that $\underline{\phi}$ is a subsolution of (5.12).

Step(i): Note that $\overline{\phi} \in \Gamma$ is obvious. If $\xi \in (0, +\infty)$. Then $\overline{\phi}(\xi) = w_2$, $\overline{\phi}'(\xi) = \overline{\phi}''(\xi) = 0$. Since the definition of $\overline{\phi}(\xi)$, $0 \leq \overline{\phi}(\xi - s - z - ca_0) \leq w_2$. Recalling (A1), we have

$$c\overline{\phi}'(\xi) - \delta\overline{\phi}''(\xi) + b\overline{\phi}(\xi) - H(\overline{\phi})(\xi) \geq 0.$$

This is because that

$$\begin{aligned}
&c\overline{\phi}'(\xi) - \delta\overline{\phi}''(\xi) + b\overline{\phi}(\xi) - H(\overline{\phi})(\xi) \\
=&g(w_2) - u(w_2)w_2 - M\beta \int_{-\infty}^{+\infty} \left[\int_{-\eta}^{\eta} z^2 e^{-z^2} \overline{\phi}(\xi - s - z - ca_0) dz\right] f_{\delta a_0}(s) ds \\
\geq&g(w_2) - u(w_2)w_2 - M\beta M_1 w_2 \\
=&0.
\end{aligned}$$

If $\xi \in (-\infty, 0)$. Then $\overline{\phi}(\xi) = w_2 e^{\lambda_1 \xi}$, $\overline{\phi}'(\xi) = w_2 \lambda_1 e^{\lambda_1 \xi}$, $\overline{\phi}''(\xi) = w_2 \lambda_1^2 e^{\lambda_1 \xi}$.

Recalling (5.12), (5.16) and the assumptions (J1), (J3), one obtains

$$\begin{aligned}
&c\overline{\phi}'(\xi) - \delta\overline{\phi}''(\xi) + b\overline{\phi}(\xi) - H(\overline{\phi})(\xi)\\
=&cw_2\lambda_1 e^{\lambda_1\xi} - \delta w_2\lambda_1^2 e^{\lambda_1\xi} + g(\overline{\phi}) - u(\overline{\phi})w_2 e^{\lambda_1\xi}\\
&- M\beta\int_{-\infty}^{+\infty}\left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w_2 e^{\lambda_1(\xi-s-z-ca_0)}dz\right] f_{\delta a_0}(s)ds\\
\geq&(-\Delta_c(\lambda_1) + \widetilde{M}_c(\lambda_1))w_2 e^{\lambda_1\xi}\\
&- M\beta\int_{-\infty}^{+\infty}\left[\int_{-\eta}^{\eta} z^2 e^{-z^2} w_2 e^{\lambda_1(\xi-s-z-ca_0)}dz\right] f_{\delta a_0}(s)ds\\
=&\widetilde{M}_c(\lambda_1)w_2 e^{\lambda_1\xi} - \widetilde{M}_c(\lambda_1)w_2 e^{\lambda_1\xi}\\
=&0.
\end{aligned}$$

Therefore, $\overline{\phi} \in \Gamma$ is a supersolution of (5.12).

Step(ii): If $\xi \in (\frac{1}{\rho}\ln\frac{1}{L}, +\infty)$. Then $\underline{\phi}(\xi) = 0$, $\underline{\phi}'(\xi) = \underline{\phi}''(\xi) = 0$. Since the definition of $\underline{\phi}(\xi)$, $\underline{\phi}(\xi - s - z - ca_0) \geq 0$. Thus,

$$\begin{aligned}
&c\underline{\phi}'(\xi) - \delta\underline{\phi}''(\xi) + b\underline{\phi}(\xi) - H(\underline{\phi})(\xi)\\
=&- M\beta\int_{-\infty}^{+\infty}\left[\int_{-\eta}^{\eta} z^2 e^{-z^2}\underline{\phi}(\xi - s - z - ca_0)dz\right] f_{\delta a_0}(s)ds\\
\leq&0.
\end{aligned}$$

If $\xi \in (-\infty, \frac{1}{\rho}\ln\frac{1}{L})$. Then $\underline{\phi}(\xi) = w_2(1 - Le^{\rho\xi})e^{\lambda_1\xi}$, $\underline{\phi}'(\xi) = w_2(\lambda_1 - L(\rho + \lambda_1)e^{\rho\xi})e^{\lambda_1\xi}$ and $\underline{\phi}''(\xi) = w_2(\lambda_1^2 - L(\rho+\lambda_1)^2 e^{\rho\xi})e^{\lambda_1\xi}$. By Lemma 5.10, one obtains

$$\Delta_c(\rho + \lambda_1) = \delta(\rho + \lambda_1)^2 - c(\rho + \lambda_1) + \widetilde{M}_c(\rho + \lambda_1) < 0.$$

Notice from the Taylor expansion that $g(\underline{\phi}) - u(\underline{\phi})\underline{\phi} = g(0) + g'(0)\underline{\phi} + \frac{1}{2}g''(\theta_1)\underline{\phi}^2 - (u(0) + u'(\theta_2)\underline{\phi})\underline{\phi}$, where $0 \leq \theta_1, \theta_2 \leq \underline{\phi}$. Then by the assumption (H1), we have $g(\underline{\phi}) - u(\underline{\phi})\underline{\phi} \leq \widetilde{L}\underline{\phi}^2 \leq \widetilde{L}w_2^2 e^{2\lambda_1\xi}$ where $\widetilde{L} = \max_{w\in[0,w_2]}(|g''(w)| + |u'(w)|)$. Since that ρ is small such that $\rho < \lambda_1$, $g(\underline{\phi}) - u(\underline{\phi})\underline{\phi} \leq \widetilde{L}w_2^2 e^{(\rho+\lambda_1)\xi}$. Thus,

$$\begin{aligned}
&c\underline{\phi}'(\xi) - \delta\underline{\phi}''(\xi) + b\underline{\phi}(\xi) - H(\underline{\phi})(\xi)\\
=&cw_2(\lambda_1 - L(\rho + \lambda_1)e^{\rho\xi})e^{\lambda_1\xi} - \delta w_2(\lambda_1^2 - L(\rho + \lambda_1)^2 e^{\rho\xi})e^{\lambda_1\xi} + g(\underline{\phi})\\
&- u(\underline{\phi})\underline{\phi} - M\beta\int_{-\infty}^{+\infty}\int_{-\eta}^{\eta} z^2 e^{-z^2}\underline{\phi}(\xi - s - z - ca_0)dz f_{\delta a_0}(s)ds
\end{aligned}$$

$$
\begin{aligned}
\leq & c w_2(\lambda_1 - L(\rho+\lambda_1)e^{\rho\xi})e^{\lambda_1\xi} - \delta w_2(\lambda_1^2 - L(\rho+\lambda_1)^2 e^{\rho\xi})e^{\lambda_1\xi} + \widetilde{L} w_2^2 e^{(\rho+\lambda_1)\xi} \\
& - M\beta \int_{-\infty}^{+\infty}\int_{-\eta}^{\eta} z^2 e^{-z^2} w_2(1 - Le^{\rho(\xi-s-z-ca_0)})e^{\lambda_1(\xi-s-z-ca_0)} dz f_{\delta a_0}(s) ds \\
= & c\lambda_1 w_2 e^{\lambda_1\xi} - c w_2 L(\rho+\lambda_1)e^{(\rho+\lambda_1)\xi} - \delta w_2 \lambda_1^2 e^{\lambda_1\xi} + \delta w_2 L(\rho+\lambda_1)^2 e^{(\rho+\lambda_1)\xi} \\
& + \widetilde{L} w_2^2 e^{(\rho+\lambda_1)\xi} - M\beta w_2 e^{\lambda_1\xi} e^{\lambda_1^2\delta a_0 - \lambda_1 c a_0} \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-z\lambda_1} dz \\
& + M\beta w_2 L e^{(\rho+\lambda_1)\xi} e^{(\rho+\lambda_1)^2\delta a_0 - (\rho+\lambda_1)ca_0} \int_{-\eta}^{\eta} z^2 e^{-z^2} e^{-z(\rho+\lambda_1)} dz \\
= & - w_2 e^{\lambda_1\xi}\Delta_c(\lambda_1) + w_2 L e^{(\rho+\lambda_1)\xi}\Delta_c(\rho+\lambda_1) + \widetilde{L} w_2^2 e^{(\rho+\lambda_1)\xi} \\
= & w_2 e^{(\rho+\lambda_1)\xi}\Delta_c(\rho+\lambda_1)\left(L + \frac{w_2\widetilde{L}}{\Delta_c(\rho+\lambda_1)}\right) \\
< & 0.
\end{aligned}
$$

Here, L is a sufficiently large positive constant. Therefore, $\underline{\phi}$ is a subsolution of (5.12). The proof is complete. □

5.2.3 Proof of Theorem 1.9

In this subsection, we also assume that (H1), (H2), (H3) hold. In order to study the population of matured mosquitoes, we need the following two results which are established by Wang et al. [94].

Lemma 5.12. *[94, Theorem 3.3] Equation (1.15) has a unique mild solution $w(t,x)$ on $[0,+\infty]$ and $w(t,x)$ is a classical solution to (1.15) for $(t,x) \in (a_0,+\infty)\times\mathbb{R}$. Furthermore, for any pair of supersolution $\overline{w}(t,x)$ and subsolution $\underline{w}(t,x)$ of (1.15) on $[0,+\infty)$ with $0 \leq \underline{w}(t,x)$, $\overline{w}(t,x) \leq w_2$ for $t \in [-a_0,+\infty)$, $x \in \mathbb{R}$, and $\overline{w}(s,x) \geq \underline{w}(s,x)$ for $x\in\mathbb{R}$, $s\in[-a_0,0]$, there holds*

$$\overline{w}(t,x) \geq \underline{w}(t,x),\ \ x\in\mathbb{R},\ \ t\geq 0,$$

$$\overline{w}(t,x) - \underline{w}(t,x) \geq \Theta(J,t-t_0)\int_z^{z+1}\Big(\overline{w}(t_0,y) - \underline{w}(t_0,y)\Big)dy$$

for any $J\geq 0$, x and $z\in\mathbb{R}$ with $|x-z|\leq J$, and $t>t_0\geq 0$, where

$$\Theta(J,t) = \frac{1}{\sqrt{4\pi dt}}\exp\left(-L_1 t - \frac{(J+1)^2}{4dt}\right),\ \ J\geq 0,\ \ t>0$$

and $L_1 = \max_{0\leq w\leq w_2}|g'(w) + u'(w)w + u(w)|$. In particular, if there exists $x_0\in\mathbb{R}$ such that $\overline{w}(0,x_0) > \underline{w}(0,x_0)$, then $\overline{w}(t,x) > \underline{w}(t,x)$ for any $x\in\mathbb{R}$ and $t>0$.

Lemma 5.13. *[94, Lemma 3.7] For each $\gamma \in (0,1)$, there exist $\rho > 0$ and $\sigma > 0$ such that for each $\varepsilon \in [0,\gamma]$, the following function*

$$\underline{w}(t,x) = (1-\varepsilon e^{-\rho t})\phi(x+ct+\sigma\varepsilon e^{-\rho t}) \tag{5.19}$$

is a subsolution of (1.15), where $t \in \mathbb{R}$, $x \in \mathbb{R}$ and ϕ is a traveling front of (1.15).

Proof of Theorem 1.9. Recalling the equation (1.12) which is the definition of $w(s,x)$, we have $w(s,x) = \int_{a_0}^{a_\dagger} p_0(a+s,x)da \not\equiv 0$ for every $s \in [-a_0, 0]$ and $w(s,x) \le w_3$. Then, by Lemma 5.12, one has that

$$w(t,x) \le w_3, \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

$$w(t,x) > 0, \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

It follows that $\inf_{x\in\mathbb{R}} w(T+s,x) > 0$ for some fixed $T > 0$ and $s \in [-a_0, 0]$. Then, by Lemma 5.13, one can pick $\varepsilon > 0$ close 1 enough such that

$$\begin{aligned}\underline{w}(s,x) &= (1-\varepsilon e^{-\rho s})\phi(x+cs+\sigma\varepsilon e^{-\rho s}) \\ &\le (1-\varepsilon)\phi(x+cs+\sigma\varepsilon e^{-\rho s}) \\ &\le \inf_{x\in\mathbb{R}} w(T+s,x) \\ &\le w(T+s,x), \text{ for } s \in [-a_0,0] \text{ and } x \in \mathbb{R}.\end{aligned}$$

Therefore, by Lemma 5.12, it follows that

$$w(T+t,x) \ge \underline{w}(t,x), \text{ for } t \ge 0 \text{ and } x \in \mathbb{R}.$$

Then, using (5.19), one can obtain

$$w(T+t,x) \ge w_2, \text{ as } t \to +\infty.$$

This completes the proof. □

5.3 Numerical simulations

In the following, we provide some numerical simulations to illustrate the interaction between the matured population and the control. We rescale the bitting time variable $x \in [0,24]$ into $x \in [0,1]$ and we assume that $a_\dagger = 1$, that is, $a \in [0,1)$. We take the matured age $a_0 = 0.1$. We consider system (1.10) with the parameters taking the values as follows

$$\delta = 0.001, \;\; \eta = 0.1, \;\;\; \beta(a) = \begin{cases} 0, & a \in [0, a_0), \\ 200, & a \in [a_0, 1), \end{cases}$$

$$p_0(a, x) = 0.5e^{-10(x-0.5)^2}e^{-10(a-0.4)^2} \;\;\text{for } x \in [0, 1].$$

Firstly, we take that

$$\mu(a, w) = \begin{cases} 0.1a, & a \in [0, a_0), \\ 0.1a_0 + 0.5e^{2.4a}, & a \in [a_0, 1), \end{cases} \tag{5.20}$$

and consider (1.10) under no control, that is, $u(a, w) = 0$. Then, in figures 9 and 10, we plot the matured population $w(t, x)$. We can see that $w(t, x)$ becomes very large as time goes. It implies that if there is no control, the matured population will be very large.

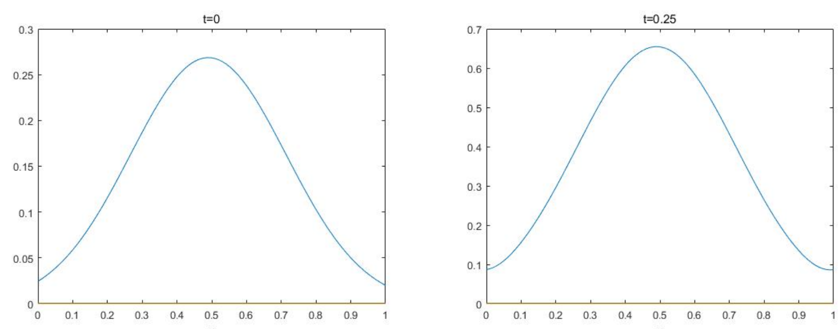

Figure 9: the matured population $w(t, x)$ for $t = 0$ and $t = 0.25$ with no control.

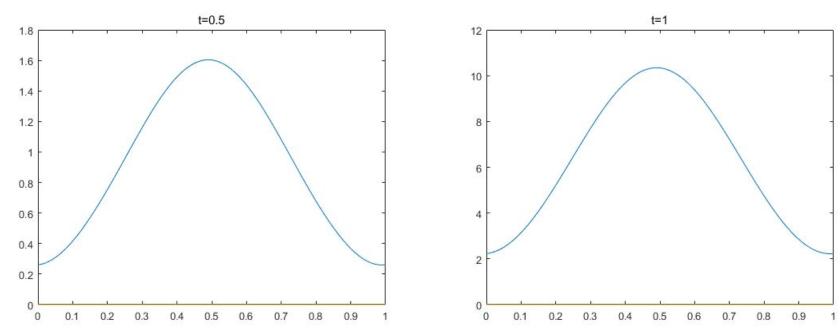

Figure 10: the matured population $w(t, x)$ for $t = 0.5$ and $t = 1$ with no control.

Now, we still take $\mu(a,w)$ be (5.20) and take the control large as

$$u(a,w)=\begin{cases}0, & a\in[0,a_0),\\ -w^2-95, & a\in[a_0,1),\end{cases}$$

Then, in following figures 11 and 12, we plot the matured population $w(t,x)$. We can see that $w(t,x)$ becomes very small as time goes. It means that under large control, the matured population will extinct.

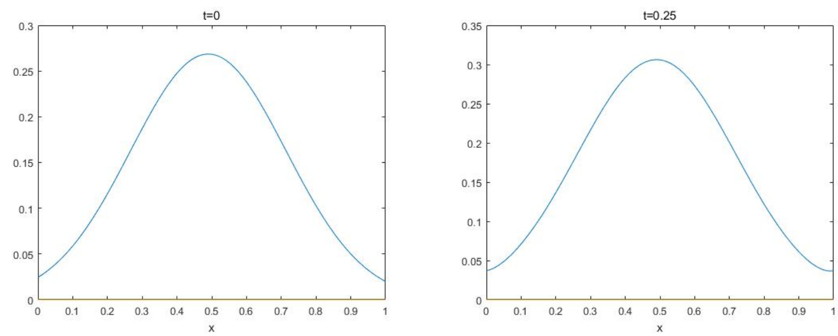

Figure 11: the matured population $w(t,x)$ for $t=0$ and $t=0.25$ with control $u(a,w)$.

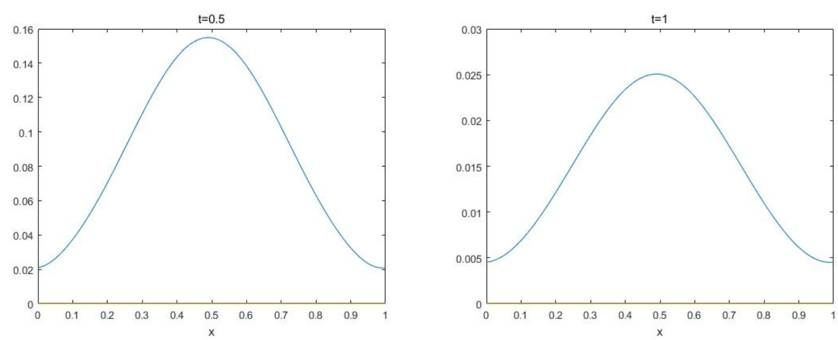

Figure 12: the matured population $w(t,x)$ for $t=0.5$ and $t=1$ with control $u(a,w)$.

Finally, we take that

$$\mu(a,w)=\begin{cases}0.1a, & a\in[0,a_0),\\ 0.1a_0+0.5e^{2.4a}w, & a\in[a_0,1),\end{cases} \text{ and } u(a,w)=\begin{cases}0, & a\in[0,a_0),\\ -w^2, & a\in[a_0,1).\end{cases}$$

In following figures 13, 14 and 15, we plot the matured population $w(t,x)$. We can see that $w(t,x)$ is in $[5,6]$ as time goes. It implies that under suitable control, the matured population will be controlled to be bounded and will not extinct.

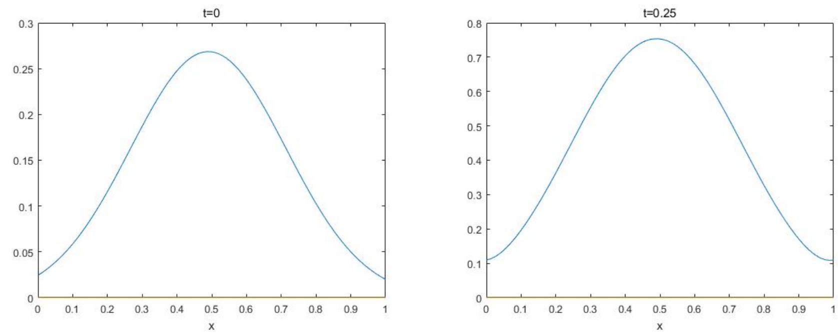

Figure 13: the matured population $w(t,x)$ for $t = 0$ and $t = 0.25$.

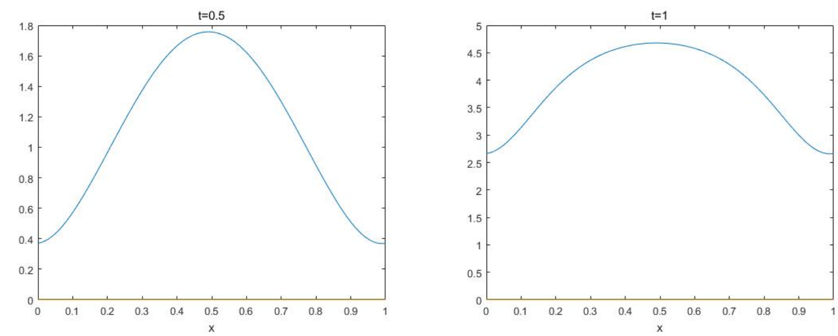

Figure 14: the matured population $w(t,x)$ for $t = 0.5$ and $t = 1$.

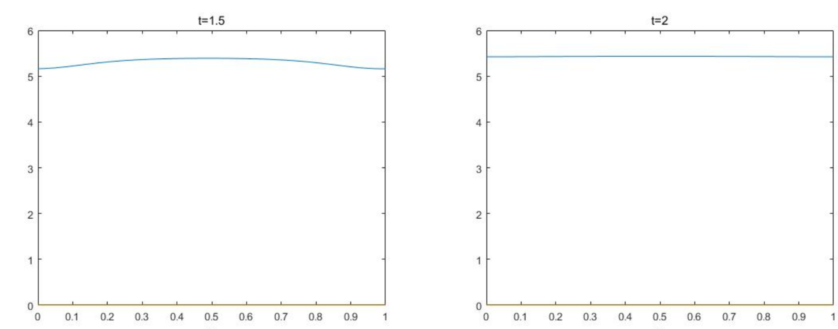

Figure 15: the matured population $w(t,x)$ for $t = 1.5$ and $t = 2$.

Appendix

Bellman's Lemma [13]: If $x \in C([a,b])$, $\psi \in L^1(a,b)$, $\psi(t) \geq 0$ $a.e.$ $t \in (a,b)$, $M \in R$ and for each $t \in [a,b]$,

$$x(t) \leq M + \int_a^t \psi(s)x(s)ds,$$

then

$$x(t) \leq M \exp\left(\int_a^t \psi(s)ds\right).$$

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