Analysis of some nonlocal models in population dynamics

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Abstract

This thesis is mainly devoted to the mathematical analysis of some nonlocal models arising in population dynamics. In general, the study of these models meets with numerous difficulties owing to the lack of compactness and of regularizing effects. In this respect, their analysis requires new tools, both theoretical and qualitative. We present several results in this direction.

In the first part, we develop a functional analytic toolbox which allows one to handle some quantities arising in the study of these models. In the first place, we extend the characterization of Sobolev spaces due to Bourgain, Brezis and Mironescu to low regularity function spaces of Besov type. This results in a new theoretical framework that is more adapted to the study of some nonlocal equations of Fisher-KPP type. In the second place, we study the regularity of the restrictions of these functions to hyperplanes. We prove that, for a large class of Besov spaces, a surprising loss of regularity occurs. Moreover, we obtain an optimal characterization of the regularity of these restrictions in terms of spaces of so-called "generalized smoothness".

In the second part, we study qualitative properties of solutions to some nonlocal reaction-diffusion equations set in (possibly) heterogeneous domains. In collaboration with J. Coville, F. Hamel and E. Valdinoci, we consider the case of a perforated domain which consists of the Euclidean space to which a compact set, called an "obstacle", is removed. When the latter is convex (or close to being convex), we prove that the solutions are necessarily constant. In a joint work with J. Coville, we study in greater detail the influence of the geometry of the obstacle on the classification of the solutions. Using tools of the type of those developed in the first part of this thesis, we construct a family of counterexamples when the obstacle is no longer convex. Lastly, in a work in collaboration with S. Dipierro, we study qualitative properties of solutions to nonlinear elliptic systems in variational form. We establish various monotonicity results in a fairly general setting that covers both local and fractional operators.

Keywords: nonlocal reaction-diffusion equations, rigidity results, Besov spaces, calculus of variations, perforated domains, function space theory.

Résumé

Cette thèse est consacrée principalement à l'analyse mathématique de modèles non-locaux issus de la dynamique des populations. En général, l'étude de ces modèles se heurte à de nombreuses difficultés dues à l'absence de compacité et d'effets régularisants. A ce titre, leur analyse requiert de nouveaux outils tant théoriques que qualitatifs. Nous présentons des résultats recouvrant ces deux aspects.

Dans une première partie, nous développons une "boîte à outils" destinée à traiter certaines quantités récurrentes dans l'étude de ces modèles. En premier lieu, nous étendons la caractérisation des espaces de Sobolev due à Bourgain, Brezis et Mironescu à des espaces de fonctions moins réguliers de type Besov, offrant ainsi un cadre théorique plus adapté à l'étude de certaines équations du type Fisher-KPP. En second lieu, nous étudions la régularité de ces fonctions par restriction sur des hyperplans. Nous montrons que, pour une large classe d'espaces de Besov, une surprenante perte de régularité a lieu. En outre, nous obtenons une caractérisation optimale de la régularité de ces restrictions via des espaces dits à "régularité généralisée".

Dans une seconde partie, nous nous intéressons aux propriétés qualitatives des solutions d'équations de réaction-diffusion non-locales posées dans des domaines possiblement hétérogènes. En collaboration avec J. Coville, F. Hamel et E. Valdinoci, nous considérons le cas d'un domaine perforé consistant en l'espace euclidien privé d'un ensemble compact appelé "obstacle". Lorsque ce dernier est convexe (ou presque convexe), nous montrons que les solutions sont nécessairement constantes. Dans un travail conjoint avec J. Coville, nous étudions plus en détail l'influence de la géométrie de l'obstacle sur la classification des solutions. En utilisant des outils du type de ceux développés dans la première partie de cette thèse, nous construisons une famille de contre-exemples lorsque l'obstacle n'est plus convexe. Enfin, dans un travail en collaboration avec S. Dipierro, nous étudions les propriétés qualitatives des solutions de systèmes d'équations elliptiques non-linéaires sous forme variationnelle. Nous y démontrons plusieurs résultats de monotonicité dans un cadre très général qui couvre à la fois le cas des opérateurs locaux et fractionnaires.

Mots-clés : équations de réaction-diffusion non-locales, résultats de rigidité, espaces de Besov, calcul des variations, domaines perforés, théorie des fonctions.

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Hâtez-vous lentement, et sans perdre courage,
Vingt fois sur le métier remettez votre ouvrage,
Polissez-le sans cesse, et le repolissez,
Ajoutez quelquefois, et souvent effacez.

N. Boileau, *L'art poétique*, Chant I.

Contents

General introduction

Introduction

1 Foreword

Over past decades, it has become clear that some phenomena in population dynamics cannot be described by local equations. Inter alia, the experimental data have shown that the dispersal of biological populations often presents “nonlocal features”. This has led mathematicians and biologists to consider *nonlocal models*. But if they share some resemblances with the classical models, their treatment is often a delicate matter. In general, these models enjoy neither good compactness properties nor good regularizing effects. To make matters worse, there may not even be such thing as (fractional) Sobolev spaces to cope with situations where they would be the natural tool to use. The usual techniques of proof are therefore more complex to implement. One thus expects new approaches to be helpful to handle these issues.

This thesis is intended as a contribution in this direction and, more generally, to the field of nonlocal equations. Of course, this is a very wide topic and we will broach only a small part of it. We will be primarily interested in equations involving a “nonlocal diffusion term” and a “nonlinear reaction term”.

This manuscript consists of two parts which have different styles. In Part I, we address *functional analytic issues* and we provide new results regarding smoothness and function spaces in connection with nonlocal equations; whereas, in Part II, we study *qualitative properties* of solutions to some nonlocal reaction-diffusion problems, with special emphasis on rigidity results.

The motivation of this thesis and the outline of our main contributions are explained in greater details in the next two sections.

2 State of art

2.1 A brief history of population dynamics

Before going to the heart of this thesis, it is worth taking a small walk through history. As a matter of fact, population dynamics is a much older topic than usually thought. The very first to show preoccupations of this kind was Plato (ca. 427-347 BC) in his

Laws (*Νόμοι* in Greek). In the fifth book of this monumental essay, he wondered what should be the optimal organization of a city-state in order for it to remain stable over the course of time. Although this might seem quite remote from population dynamics, he raised there important questions on the relationship of the population to the environment. Based on both ecological and governance grounds, he argued that the total number of citizens should be kept constant equal to 5,040 because it can be divided into many lesser parts which should, in his view, be very convenient to optimize both the organization of the *polis* and the demographic needs.[1] To maintain this number constant he also pointed out the need for a demographic legislation, foreshadowing Malthusian ideas on demography.[2]

It is striking to see that Plato, 2,400 years ago, already foresaw the importance of population dynamics and the possibility of a mathematical approach (although rather fancy[3]) to address the problem!

More than 1,500 years later, Fibonacci (1175-1250) in his 1202 book *Liber Abaci* introduced the famous sequence named after him. What is however less known, is that it was introduced to solve a problem which one would nowadays call a typical problem in population dynamics. Here is an English translation of the original statement (in Latin) of the problem:

> *A certain man had one pair of rabbits together in a certain enclosed place. One wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair and in the second month those born to bear also.*

Interestingly enough, the solution to this problem, the Fibonacci sequence $(F_n)_{n\geqslant 0}$, grows like $\varphi^n/\sqrt{5}$ as $n \to \infty$, where $\varphi = (1+\sqrt{5})/2$ is the golden ratio (see e.g. [149, Theorem 12, p.27]). In other words, it involves an *exponential growth* of the population. In a certain sense — that will be made clear in the next page — this

[1] "We must fix at the right total the number of citizens; next, we must agree about the distribution of them, into how many sections, and each of what size they are to be divided; and among these sections we must distribute, as equally as we can, both the land and the houses. [...] Of land we need as much as is capable of supporting so many inhabitants of temperate habits. [...] The number 5,040 is here chosen because, for a number of moderate size, it has the greatest possible number of divisors (59), including all the digits from 1 to 10. [...] And in order that these things may remain in this state forever, these further rules must be observed: the number of hearths, as now appointed by us, must remain unchanged, and must never become either more or less." Plato, *Laws*, Book V.

[2] "The magistrates [...] shall consider how to deal with the excess or deficiency in families, and contrive means as best they can to secure that the 5,040 households shall remain unaltered. There are many contrivances possible: where the fertility is great, there are methods of inhibition, and contrariwise there are methods of encouraging and stimulating the birth-rate, by means of honours and dishonours." Plato, *op. cit.*

[3] It is also amusing to notice that $7! = 1\times2\times3\times4\times5\times6\times7 = 5{,}040$ which suggests an influence of Pythagoreanism.

feature somehow instates the Fibonacci sequence as a forerunner of some discoveries which would play an important role in the establishment of population dynamics.

But modern population dynamics really start to expand from the seminal work of Malthus (1766-1834). In his celebrated *Essay on the Principle of Population* (1798), he developed what is now regarded as the "first principle" of population dynamics. In substance, it says that

1. all life forms have a propensity to exponential population growth when the resources are abundant;[4]
2. the population growth is limited by the availability of the resources.[5]

This amounts to say that, for a population with unlimited resources, the size $u(t)$ of the population at time t is given by $u(t) = u_0 e^{\lambda t}$, where u_0 is the initial population size and $\lambda > 0$ is the growth rate of the population. In other words, u obeys

$$\begin{cases} u'(t) = \lambda u(t) & \text{for } t > 0, \\ u(0) = u_0. \end{cases} \tag{2.1}$$

This is known as the *Malthusian growth model*.

However, this model is generally unrealistic: it does not take into account the competition between individuals for the resource, which thereby yields exponential growth of the population. A couple of decades later, Verhulst [146] (1804-1849), inspired by the work of Malthus, proposed a new model that adjusted (2.1) to take into account the resource limitation phenomenon. This model, known as the *logistic growth model*, takes the form

$$u'(t) = \lambda u(t) \left(1 - \frac{u(t)}{\kappa}\right), \tag{2.2}$$

where $\lambda > 0$ is the growth rate of the population and $\kappa > 0$ is a constant corresponding to the maximum population size of the species that the environment can sustain indefinitely given the availability of the resource, called the carrying capacity. Of course, if $\kappa \to \infty$, then this boils down to the Malthusian growth model.

Equation (2.2) gives rise to an unstable equilibrium at 0 and a stable equilibrium at κ. Indeed, this is because $u'(t) = 0$ when $u(t) \in \{0, \kappa\}$ while $u'(t) > 0$ in $\{0 < u < \kappa\}$ and $u'(t) < 0$ in $\{u > \kappa\}$. Therefore, κ attracts any function starting from $u(0) \in (0, \kappa)$. That is, $u(t) \to \kappa$ as $t \to \infty$. For this reason, the nonlinear function of u on the right-hand side of (2.2) is sometimes referred to as *monostable*.

[4] "Population, when unchecked, increases in a geometrical ratio. [...] Taking the population of the world at any number, a thousand millions, for instance, the human species would increase in the ratio of 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, etc." Malthus, *Essay on the Principle of Population*.

[5] "The increase of population is necessarily limited by the means of subsistence", Malthus, *op. cit.*

But the mathematical theory of population dynamics really took flight in the 1930's. Until then, investigations were based on the Malthusian principle (exponential growth) and on the Verhulst model (logistic growth). Two fundamental discoveries would change this old paradigm.

The first one is due to Allee (1885-1955). According to the Verhulst model, the per capita growth rate of the population is all the greater when the population is small. That is, the less individuals there are, the less mortality there is, as the available resource per individual is higher. In his 1931 book *Animal Aggregations: A Study in General Sociology*, Allee discovered that this might not always be the case. Another phenomenon must be taken into account: cooperation between individuals. He pointed out that

> there is a positive correlation between population size and the individual fitness (or per capita growth rate) of the population.

This is called the "Allee effect", but some authors refer to it as the "second principle" of population dynamics. It manifests through different mechanisms such as mate limitation or cooperative feeding. Loosely speaking, the higher the population size is, the more likely the individuals are to cooperate, which then results in a demographic increase. Contrariwise, the lower the population size is, the less likely the individuals are to cooperate, which then results in a demographic decrease. From the mathematical point of view, this can be modelled by introducing a *critical threshold* above which the population tends to increase and below which it tends to decrease. It is usually translated into models of the type

$$u'(t) = \lambda u(t) \left(1 - \frac{u(t)}{\kappa}\right) (u(t) - \theta), \tag{2.3}$$

where λ and κ have the same meaning as in (2.2) and $\theta > 0$ is the critical size of the population.

By contrast with the Verhulst model, there are now two stable equilibria at 0 and κ, respectively, and an unstable equilibrium at θ. For this reason, the nonlinear function of u on the right-hand side of (2.3) is called *bistable*. This is indeed consistent with Allee's observations: if it holds that $u(0) \in (0, \theta)$, then, after some large time, the population will decline to extinction, namely $u(t) \to 0$ as $t \to \infty$. Conversely, if $u(0) \in (\theta, \kappa)$, then the population will tend to reach the maximal amount of individuals that the environment is able to sustain, namely $u(t) \to \kappa$ as $t \to \infty$.

The second milestone in the history of population dynamics arose with the discovery, in 1937, by Fisher [73] and, independently, by Kolmogorov, Petrovskii and Piskunov [93] of the so-called Fisher-KPP equation:

$$\frac{\partial u}{\partial t} = \rho \Delta u + \lambda u \left(1 - \frac{u}{\kappa}\right), \tag{2.4}$$

where $\rho > 0$ is a diffusion coefficient analogous to that used in physics. The principal novelty of this model is that it takes into account the *spatial interactions* between individuals. All the models considered hitherto were not spatialized, meaning that, in each of the populations considered, each individual was assumed to interact with all the other individuals. However, in most situations, the heterogeneity of the spatial distribution cannot be neglected. The population size $u(t)$ is thus not sufficiently informative and one must look for its density $u(t, x)$ instead. This difficulty is overcome by considering that the movement of individuals is approximated by a random motion which then results in a *reaction-diffusion equation* (see the next section for more explanations).

Since these seminal works, much attention has been paid to models taking both into account the diffusion of the species and general demographic variations, such as

$$\frac{\partial u}{\partial t} = \Delta u + f(u), \tag{2.5}$$

where f is some nonlinear reaction term, see e.g. [6, 70, 102, 153, 154].

It is worth mentioning that, over the past century, equations of the type of (2.5) have emerged from a number of seemingly remote areas, most notably in the study of combustion phenomena and phase transitions in liquid crystals, superconductivity and material sciences in general. In the latter case, one typically has

$$f(X) = X(1 - X^2),$$

and (2.5) is then called the *Allen-Cahn* or *Ginzburg-Landau* equation (see e.g. [21, 22]). As a matter of fact, mechanistic models in ecology often intertwine with other branches of natural sciences. Although the mathematical theories of the above-mentioned fields are rather disjoint, the tools and methods developed for either one of them often turn out to find applications in the other. These equations are customary gathered together under the umbrella term "reaction-diffusion problems".

We refer the interested reader to [10, 123, 148] (and references therein) for further historical remarks and a comprehensive study of the above models.

2.2 From local to nonlocal models: towards a new paradigm

Since the seminal work of Fisher [73], Kolmogorov, Petrovskii and Piskunov [93], the study of spatial distribution of individuals has become a fundamental cornerstone of population dynamics. In fact, the patterns formed by the individuals at the microscopic level can be shown to determine the diffusion at the macroscopic level. Therefore a better knowledge of the former is expected to improve our understanding of the latter. From our perspective, this means that one can recover the Fisher-KPP equation starting from microscopic considerations only. Indeed, interpreting the trajectories of the individuals as realizations of a Lévy process $(X_t)_{t\geqslant 0}$, we can then consider its associated Feller semi-group $(P_t)_{t\geqslant 0}$. In turn, this semi-group is generated by an operator L given by

$$Lu(x) = \lim_{t\to 0^+} \frac{P_t u(x) - u(x)}{t}.$$

This operator, called the *infinitesimal generator* of $(P_t)_{t\geqslant 0}$, describes the diffusion of the species at the macroscopic level. Since diffusion is accompanied by demographic variations, we then obtain — at least formally — an equation of the type

$$\frac{\partial u}{\partial t} = Lu + f(u), \tag{2.6}$$

where f is some nonlinear reaction term. If $(X_t)_{t\geqslant 0}$ is a Brownian motion, then the operator L boils down to the Laplacian, namely

$$Lu = \frac{1}{2}\Delta u,$$

(see e.g. [5, Example 3.3.4, p.141]) and we recover essentially (2.4).

The recent development of GPS technologies has allowed the detailed tracking of individuals within a given species. This has been the source of a new revolution in the development of population dynamics and has shed new lights on how to describe diffusion phenomena. If the collected sets of data highly support the assumption that the trajectories follow Lévy processes (see e.g. [11, 12, 104, 121, 128]), it has been observed that the Brownian motion does *not* always account for the patterns formed by individuals. In particular, they may exhibit *long-range jumps* which are not accounted by Brownian motions. In this case, dispersion can occur over large distances and may exhibit "nonlocal features".

From the PDE standpoint, this means that, depending on the specific behavior one wishes to describe, different operators from the Laplacian may arise.

In the case of marine predators, the Brownian motion is relevant where prey is abundant, but *a Lévy type behavior occurs when prey is sparsely distributed* [80, 82, 122, 130]. In this situation, random patterns of individuals are best described by processes with long-range jumps. Typical examples of such processes are α-stable Lévy processes with $0 < \alpha < 2$. Although they are somehow related to the Brownian motion (which may be seen as a 2-stable Lévy process) their infinitesimal generator happen to be of a completely different nature. It is given by the singular integral

$$Lu(x) = -(-\Delta)^{\alpha/2} u(x) := C(N,\alpha)\ \text{p.v.} \int_{\mathbb{R}^N} \frac{u(y) - u(x)}{|x-y|^{N+\alpha}} \mathrm{d}y, \tag{2.7}$$

where $C(N,\alpha) > 0$ is a normalization constant and "p.v." stands for the Cauchy principal value (see e.g. [74, 113, 145]). This operator is called the *fractional Laplacian*.

Another example of processes which arise in this situation are compound Poisson processes. Their infinitesimal generator is, again, of a completely different nature. Precisely, one has

$$Lu(x) = \int_{\mathbb{R}^N} J(x-y)(u(y)-u(x))\mathrm{d}y, \tag{2.8}$$

where $J \in L^1(\mathbb{R}^N)$ is a nonnegative kernel with $J(x-y)$ encoding the probability to jump from a location x to a location y (see e.g. [5, Example 3.3.7, p.141]).

What is here striking is that both (2.7) and (2.8) are *nonlocal integral operators*.[6] This has strong consequences and induces strange and surprising phenomena.

To name only a few, Dipierro, Savin and Valdinoci [58] have shown that *every reasonably smooth function can be locally approximated by $\frac{\alpha}{2}$-harmonic functions*, i.e. such that $(-\Delta)^{\alpha/2}u = 0$. This highly contrasts with the rigidity of classical harmonic functions and is a purely nonlocal feature. In the same vein, Cafferelli, Dipierro and Valdinoci [44] proved that a similar "density result" holds with nonlocal Fisher-KPP type equations of the form

$$\frac{\partial u}{\partial t} = -(-\Delta)^{\alpha/2}u + u(\sigma - u),$$

where σ is a function to be thought of as a resource producing a birth rate proportional to it (see also [59]). This means that *equations of the type of* (2.6) *may enjoy very different properties depending on the type of diffusion considered*.

The fractional Laplacian shares common features with the classical Laplacian. In particular, compactness properties are preserved and the function spaces naturally associated to it, although of a nonlocal nature, have similar properties. A result due to Caffarelli and Silvestre [43] shows that it may even be localized by adding a new variable.

By contrast with the fractional Laplacian, convolution operators of the type of (2.8) *lack of strong compactness properties* which makes their analytical treatment much more involved. In particular, there are *no a priori regularity results* in general and *no powerful functional analytic framework* similar to that provided by the (fractional) Sobolev spaces.

But the nonlocal setting presents a further difficulty. As a matter of fact, boundary value problems, for both (2.7) and (2.8), cannot be handled in the same way as in the local case because of the contribution of the diffusion coming from outside the domain. Moreover, if various results are known for general equations of the type

[6] It is worth pointing out that the equation that was *actually* derived by Kolmogorov, Petrovskii and Piskunov in [93] was a nonlocal equation of the type of (2.6) with L of the form (2.8) (see [93, Formula (7.32), p.174]). In fact, the local equation (2.4) was obtained as a sort of approximation using a formal Taylor expansion, which suggests that nonlocal equations are natural in this context.

of (2.6) in the whole space $\mathbb{R}^N$ (see e.g. [13, 45, 50, 69, 155]), numerous questions remain open when it comes to heterogeneous domains. Such problems are a major issue in population dynamics. In this regard, let us quote Turchin [144] who, in his 1998 monograph *Quantitative Analysis of Movement*, wrote that

> "the spatial dimension and, in particular, the interplay between environmental heterogeneity and individual movement, is an extremely important aspect of ecological dynamics."

The purpose of this thesis is to study some of these nonlocal features, especially those arising when considering operators of convolution type as in (2.8).

3 Contributions

The topics addressed in this thesis are quite different from a chapter to another, with the notable exception of Chapter 4 which can be seen as a follow-up to Chapter 3. A significant part of this thesis is dedicated to the study of models of the type

$$\frac{\partial u}{\partial t}(t,x) = \int_\Omega J(x-y)(u(t,y)-u(t,x))\mathrm{d}y + f(u(t,x)),$$

set in some domain $\Omega \subset \mathbb{R}^N$, albeit we sometimes emphasize more on the functional analytic aspects behind it than on the equation itself. Some of our contributions are thus of interest *per se* even outside the realm of population dynamics.

Our results sometimes overlap with other areas such as phase transitions in physics and material sciences (in particular Chapter 5 and, to some extent, Chapter 2). But, as pointed out at the end of Section 2.1 above, these topics are closely related to population dynamics.

This thesis is divided into five (mutually independent) chapters, split into two parts. The first part deals with *regularity issues* arising in nonlocal reaction-diffusion problems. The second part is mainly concerned with *qualitative properties* of solutions to some nonlocal problems with special emphasis on rigidity results. Possible extensions of our results are briefly discussed at the end of the thesis.

We present below the content of each chapter and our main contributions.

3.1 A functional analytic toolbox

Part I contains two chapters which are adapted from the following papers:

$\ast$ J. Brasseur: A Bourgain-Brezis-Mironescu characterization of higher order Besov-Nikol'skii spaces, *Ann. Inst. Fourier* (2017), (forthcoming).

$\ast$ J. Brasseur: On restrictions of Besov functions, *Nonlinear Anal.* **170** (2018), p. 197-225.

3.1.1 A Bourgain-Brezis-Mironescu characterization of Besov-Nikol'skii spaces

In Chapter 1, we study a problem in functional analysis that emerges in the study of some nonlocal Fisher-KPP type equations. Precisely, we consider the equation

$$\frac{1}{\varepsilon^m}(J_\varepsilon * u(x) - u(x)) + u(x)(a(x) - u(x)) = 0 \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

where $\varepsilon > 0$, $m \in [0, 2]$, u is the density of a given population, $J_\varepsilon(z) := \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right)$, with $J \in C \cap L^1(\mathbb{R}^N)$, is a symmetric positive dispersal kernel with unit mass having finite m-th order moment, and $a \in C^2(\mathbb{R}^N)$ is a function satisfying

$$\limsup_{|x|\to\infty} a(x) < 0.$$

The parameter ε is a measure of the spread of dispersal of the species and $1/\varepsilon^m$ is a rate of dispersal which arises when considering a "cost function" (see [16]).

As an example, one may think of a population of trees producing and dispersing seeds. Several dispersal strategies are then possible: either it disperses few seeds but over large distances ($\varepsilon \gg 1$) or it disperses many of them but over smaller distances ($\varepsilon \ll 1$). The parameter m measures the influence of the cost function on the different possible strategies.

Studying persistence of the population amounts to seeking for a positive solution to (3.1). Of particular interest is the asymptotic behavior of solutions, which allows to determine whether one of the extreme strategies ($\varepsilon \ll 1$ or $\varepsilon \gg 1$) yield persistence or extinction. But if we have a clear picture when $\varepsilon \to \infty$ (see [16, Theorems 1.3 and 1.4]) it is only poorly understood when $\varepsilon \to 0^+$ and $0 < m < 2$.

The best result in this direction is due to Berestycki, Coville and Vo [16]. They have proved that if J behaves sufficiently well, for example if it is compactly supported with $J(0) > 0$, and if a is such that $\max\{a, 0\} \not\equiv 0$, then, when $\varepsilon \to 0^+$, u_ε converges pointwise almost everywhere to some non-negative bounded function v satisfying

$$v(x)(a(x) - v(x)) = 0 \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

Unfortunately, this equation admits infinitely many solutions, so one cannot directly infer a persistence strategy for that case.

However, it is known that solutions to (3.1) satisfy

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^m}\mathrm{d}x\mathrm{d}y \leqslant C \quad \text{for all } \varepsilon > 0, \tag{3.3}$$

where $\rho_\varepsilon(z) = \varepsilon^{-m}|z|^m J_\varepsilon(z)$, see [16, Lemma 5.1(ii)]. This inequality is in fact the key tool which allows to handle the case $m = 2$. Indeed, (3.3) together with the recent characterization of Sobolev spaces derived by Bourgain, Brezis, Mironescu [27] and Ponce [118] implies that $(u_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and that it converges along a subsequence to some function v satisfying

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(x-y)\frac{|v(x)-v(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y\leqslant C,$$

which, by the result of Bourgain et al. [27, Theorem 2], is equivalent to saying that v belongs to the Sobolev space $H^1(\mathbb{R}^N)$. Then, relying on standard elliptic theory, it can be shown that v is the unique nontrivial solution of

$$\beta\,\Delta v(x)+v(x)(a(x)-v(x))=0,$$

where $\beta>0$ is some constant depending on N and J.

Unfortunately, there is no such characterization when $0<m<2$, which prevents from using this strategy. Whence, a detailed study of functionals of the type of (3.3) is needed. This is the main purpose of Chapter 1.

More precisely, given $s\in(0,1]$ and $p\in[1,\infty)$, we study the properties of functions $f\in L^p(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(x-y)\frac{|f(x)-f(y)|^p}{|x-y|^{sp}}\mathrm{d}x\mathrm{d}y\leqslant C\quad\text{as }\varepsilon\to 0^+,\tag{3.4}$$

where $(\rho_\varepsilon)_{\varepsilon>0}\subset L^1(\mathbb{R}^N)$ is a standard sequence of mollifiers, i.e. such that

$$\begin{cases}\rho_\varepsilon\geqslant 0\ \text{ a.e. in }\mathbb{R}^N\text{ for any }\varepsilon>0,\\ \displaystyle\int_{\mathbb{R}^N}\rho_\varepsilon(z)\mathrm{d}z=1\ \text{ for any }\varepsilon>0,\\ \displaystyle\lim_{\varepsilon\to 0^+}\int_{|z|\geqslant\delta}\rho_\varepsilon(z)\mathrm{d}z=0\ \text{ for all }\delta>0.\end{cases}\tag{3.5}$$

In their paper, Berestycki et al. [16] were expecting (3.4) with $0<s<1$ to provide a description of fractional Sobolev spaces. However, the underlying space happens to be of a different nature. Precisely, we show that it coincides with the so-called Besov-Nikol'skii space $B^s_{p,\infty}(\mathbb{R}^N)$ (see Definition 3.3) provided that $(\rho_\varepsilon)_{\varepsilon>0}$ satisfies

$$\rho_\varepsilon(z)=\frac{1}{\varepsilon^N}\,\rho\left(\frac{z}{\varepsilon}\right)\ \text{ for some }\rho\in L^1(\mathbb{R}^N).\tag{3.6}$$

That is, we prove the following

Theorem — *Let $s\in(0,1)$ and $p\in[1,\infty)$. Let $(\rho_\varepsilon)_{\varepsilon>0}\subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying* (3.5) *and* (3.6). *Then, $f\in L^p(\mathbb{R}^N)$ satisfies* (3.4) *if, and only if, $f\in B^s_{p,\infty}(\mathbb{R}^N)$. Moreover,*

$$[f]^p_{B^s_{p,\infty}(\mathbb{R}^N)}\sim\sup_{\varepsilon>0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(x-y)\frac{|f(x)-f(y)|^p}{|x-y|^{sp}}\mathrm{d}x\mathrm{d}y.$$

The main difficulty in establishing this result is that the techniques of proof used in [27, 33, 118] to handle the case of Sobolev spaces do *not* adapt. There are at least two reasons for this: on the one hand, many properties of the growth rate which arises in (3.4) when $s = 1$ are lost in the fractional case $s \in (0, 1)$ and, on the other hand, smooth functions are *not* dense in $B^s_{p,\infty}(\mathbb{R}^N)$. To cope with this, we develop a new strategy relying essentially on elementary arguments.

Further, we prove that an important property does no longer hold: compactness. When $s = 1$, it is known that any bounded sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\frac{|f_\varepsilon(x) - f_\varepsilon(y)|^p}{|x-y|^p}\mathrm{d}x\mathrm{d}y \leqslant C \quad \text{as } \varepsilon \to 0^+,$$

must be relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ provided that $(\rho_\varepsilon)_{\varepsilon>0}$ satisfies some mild symmetry properties (see e.g. [27, 118]).

We show that, surprisingly, this does *not* extend to the fractional setting.

Theorem — *Let $s \in (0, 1)$ and $p \in [1, \infty)$. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a sequence of mollifiers of the form* (1.8) *with $\rho \in L^1(\mathbb{R}^N)$ satisfying*

$$\int_{\mathbb{R}^N} \rho(z)|z|^{p(1-s)}\mathrm{d}z < \infty.$$

Then, there exists a bounded sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\frac{|f_\varepsilon(x) - f_\varepsilon(y)|^p}{|x-y|^{sp}}\mathrm{d}x\mathrm{d}y \leqslant C \quad \text{as } \varepsilon \to 0^+,$$

but which is not relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

Chapter 1 is in fact concerned with more general functionals than (3.4). Beyond theoretical interest, this extra degree of generality allows one to handle both the (delicate) case $p = \infty$ and the higher order case $s \geqslant 1$ (see Theorem 2.3). A remarkable consequence of this is that it enables to derive new characterizations for both Lipschitz and Zygmund-Hölder spaces (see Theorem 2.5). So that, in the end, we obtain a unified approach for representing as diverse scales as Sobolev, BV, Lipschitz, Besov-Nikol'skii and Zygmund-Hölder spaces.

On another note, we exhibit several consequences of our results which allow to clarify the relationship between the usual Besov spaces $B^s_{p,q}(\mathbb{R}^N)$ with $q < \infty$ and the Nikol'skii spaces $B^s_{p,\infty}(\mathbb{R}^N)$, see in particular Examples 2.10, 2.11 and 2.19.

Chapter 1 contains several other results with different flavors. In particular, we extend the by-now celebrated limiting embedding:

$$q^{-1/q}\|\nabla f\|_{L^p(\mathbb{R}^N)} \sim \lim_{r\to 1^-}(1-r)^{1/q}\|f\|_{B^r_{p,q}(\mathbb{R}^N)} \quad \text{for } 1 < p, q < \infty,$$

to the Lipschitz case $p = \infty$ (see Theorem 2.12). In the same vein, we investigate the analogue of this for Besov-Nikol'skii spaces. Namely, we ask wether it holds that

$$\sup_{0<r<s} (s-r)^{1/q} \|f\|_{B^r_{p,q}(\mathbb{R}^N)},$$

yields a equivalent semi-norm on $B^s_{p,\infty}(\mathbb{R}^N)$. It turns out that this is not true (see Theorem 2.14) which suggests that, differently from the integer order case, the restriction to (3.6) in Theorem 3.1 cannot be removed.

3.1.2 On restrictions of Besov functions

Chapter 2 is certainly our most theoretical contribution. There, we study the "restriction property" in the context of Besov spaces $B^s_{p,q}(\mathbb{R}^N)$. Function spaces $X(\mathbb{R}^N)$ having this property are those which admit that

$$f(\cdot, y) \in X(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d},$$

whenever $f \in X(\mathbb{R}^N)$. This property, which holds true on the vast majority of functions spaces, plays a fundamental role in lifting theory and, by extension, in some reaction-diffusion problems of Ginzburg-Landau type, see [27, 108].

With moderate work, it can be shown to hold in $B^s_{p,q}(\mathbb{R}^N)$ whenever $q \leqslant p$ in the whole range of relevant parameters, that is: $0 < q \leqslant p \leqslant \infty$ and $s > \sigma_p$ where

$$\sigma_p := N \left(\frac{1}{p} - 1 \right)_+ .$$

We prove that this is no longer the case when $p < q$.

Namely, we obtain the following

Theorem — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$ and let $s > \sigma_p$. Then, there exists a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot, y) \notin B^s_{p,q}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}.$$

This result is doubly surprising since *it has no comparable antecedents* and because Besov spaces are mere microscopic modifications of fractional Sobolev spaces which are known to satisfy this property.

Our construction is nontrivial and requires advanced decomposition techniques, most notably subatomic decompositions.

One of the major property of these spaces (and what makes them so useful) is that they embed in nice spaces. This is particularly relevant for applications as it allows to grasp some additional regularity. For example, the standard embedding theorem (which holds *independently* of the value of q) says that

$$B^s_{p,q}(\mathbb{R}^N) \hookrightarrow A^{s,p}(\mathbb{R}^N), \tag{3.7}$$

where $A^{s,p}(\mathbb{R}^N)$ stands for

$$C^{s-\frac{N}{p}}(\mathbb{R}^N), \;\; \mathrm{BMO}(\mathbb{R}^N) \;\text{ and }\; L^{\frac{Np}{N-sp},\infty}(\mathbb{R}^N),$$

when respectively $sp > N$, $sp = N$ and $sp < N$ (the precise definitions will be given in Chapter 2, see Definition 2.19). In particular, since $B^s_{p,q}(\mathbb{R}^N)$ enjoys the restriction property when $q \leqslant p$, it is easily seen that

$$f(\cdot, y) \in A^{s,p}(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d}. \tag{3.8}$$

whenever $f \in B^s_{p,q}(\mathbb{R}^N)$. It is quite natural to ask wether (3.8) still holds when $p < q$. But we prove that even this weaker property *fails*.

Theorem — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$ and let $s > \sigma_p$. Then, there exists a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot, y) \notin A^{s,p}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}.$$

Furthermore, we establish a positive result about the regularity of these restrictions. Precisely, we exhibit a function space $X(\mathbb{R}^N)$ that is intermediate between $B^s_{p,q}(\mathbb{R}^N)$ and $B^{s'}_{p,q}(\mathbb{R}^N)$ for all $0 < s' < s$ and such that

$$\forall f \in B^s_{p,q}(\mathbb{R}^N), \quad f(\cdot, y) \in X(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d}.$$

This space X is a so-called *Besov space of generalized smoothness*, usually denoted by $B^{(s,\Psi)}_{p,q}$ (we refer to Definition 2.11 for the definition of these spaces). In this setting, s remains the dominant smoothness parameter and Ψ is a positive function of log-type, called *admissible*, which allows encoding more general types of smoothness.

Of course, this depends on the interplay between Ψ and the parameters p and q. We prove that restrictions of Besov functions to almost every hyperplanes belong to the space $B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$, whenever

$$\sum_{j\geqslant 0} \Psi(2^{-j})^{\chi} < \infty, \tag{3.9}$$

where $\chi = \frac{qp}{q-p}$ (resp. $\chi = p$ if $q = \infty$).

Theorem — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$, $s > \sigma_p$ and let Ψ be an admissible function satisfying* (3.9). *Suppose that $f \in B^s_{p,q}(\mathbb{R}^N)$. Then,*

$$f(\cdot, y) \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}.$$

We also prove that the condition (3.9) is optimal, at least when $q = \infty$. When $q < \infty$, it still can be shown to be sharp but under the additional requirement that

$$\chi < \frac{1}{c_\infty} \quad \text{where } \; c_\infty := \sup_{0<t\leqslant 1} \log_2 \frac{\Psi(t)}{\Psi(t^2)}, \tag{3.10}$$

and χ is as in (3.9). In other words, we arrive at a sharp characterization of the aforementioned loss of regularity.

THEOREM — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$, $s > \sigma_p$ and let Ψ be an admissible function that does not satisfy* (3.9). *If $q < \infty$ and Ψ is increasing suppose, in addition, that* (3.10) *holds true. Then, there is a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot, y) \notin B^{(s,\Psi)}_{p,q}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}.$$

In the course of Chapter 2 we also establish the analogue of the above results in the setting of Besov spaces of generalized smoothness. This is of independent interest given their relevance in stochastic calculus and in the theory of pseudo-differential operators (where they appear in a natural way), see e.g. [1, 85, 99, 101, 126].

These results may find applications in lifting theory and in the study of some turbulence phenomena in fluid mechanics, where the need for this type of property recently appeared, see [86] and references therein. Also, they suggest an implicit link with other results related to fractal geometry and may lead to further developments in this direction, see [9, 60, 61].

3.2 Rigidity results

Part II contains three chapters which are adapted from the following papers:

∗ J. Brasseur, J. Coville, F. Hamel & E. Valdinoci: Liouville type results for a nonlocal obstacle problem, *hal-01672149* (2017).

∗ J. Brasseur & J. Coville: A counterexample to the Liouville property of some nonlocal problems, *hal-01769598* (2018).

∗ J. Brasseur & S. Dipierro: Some monotonicity results for general systems of nonlinear elliptic PDEs, *J. Diff. Equations* **261** (2016), no. 5, p. 2854-2880.

3.2.1 Liouville type results for a nonlocal obstacle problem

Chapter 3 is a work in collaboration with J. Coville, F. Hamel and E. Valdinoci. To a certain extent, it may be seen as a contribution towards the following problem:

how does the geometry of the environment affects the evolution of a population with nonlocal dispersal? (3.11)

The focus here is on perforated domains, that is when the environment possesses some inaccessible regions. To be more specific, Chapter 3 is concerned with qualitative properties of solutions to nonlocal reaction-diffusion equations of the form

$$Lu + f(u) = 0 \quad \text{in } \mathbb{R}^N \setminus K, \tag{3.12}$$

where $K \subset \mathbb{R}^N$ is a compact "obstacle", L is the nonlocal operator given by

$$Lu(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)(u(y)-u(x))\mathrm{d}y, \tag{3.13}$$

and $f \in C^1([0,1])$ is a bistable nonlinearity. To clarify the ideas, suppose that

$$f(u) = \lambda u(1-u)(u-\theta) \text{ for some } \theta \in (0,1) \text{ and } \lambda > 0.$$

Before we go any further, some comments are in order. This problem may be thought of as a nonlocal version of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ \nabla u \cdot \nu = 0 & \text{on } \partial K, \end{cases} \tag{3.14}$$

where ν is the outward unit vector normal to K (assuming that K is smooth enough).

The local problem (3.14) was first studied by Berestycki, Hamel and Matano in [17]. There, it is shown that there exist an entire solution $u(t,x)$ to the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \mathbb{R} \times \mathbb{R}^N \setminus K, \\ \nabla u \cdot \nu = 0 & \text{on } \mathbb{R} \times \partial K, \end{cases} \tag{3.15}$$

satisfying $0 < u(t,x) < 1$ for all $(t,x) \in \mathbb{R} \times \overline{\mathbb{R}^N \setminus K}$, and a classical solution $u_\infty(x)$ to the elliptic problem

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \nabla u_\infty \cdot \nu = 0 & \text{on } \partial K, \\ 0 \leqslant u_\infty \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u_\infty(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{3.16}$$

This latter solution is obtained as the large-time limit of $u(t,x)$ in the sense that

$$u(t,x) \to u_\infty(x) \text{ as } t \to \infty \text{ locally uniformly in } x \in \overline{\mathbb{R}^N \setminus K}.$$

Moreover, this result is *independent* of the geometry of K. Of course, this does not mean that the geometry does not play a role: its influence is *encoded* in the stationary solution u_∞. Berestycki et al. proved that if the obstacle K satisfies some "good" geometrical properties, for example if K is starshaped, then the solution u_∞ to (3.16) must be identically equal to 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$ (see [17, Theorem 6.1]). Per contra, if K is no longer starshaped but merely simply connected, they show that this Liouville type property may fail.

If we interpret u as the density of a population with bistable growth this means that, after some large time, *whether the population tends to occupy the whole space depends on the geometry of* K.

In Chapter 3, we deal with qualitative properties of solutions to equation (3.12), together with some asymptotic limiting conditions at infinity similar to those appearing in (3.16). Namely, we will be concerned with solutions to

$$\begin{cases} Lu + f(u) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ 0 \leqslant u \leqslant 1 & \text{in } \mathbb{R}^N \setminus K, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{3.17}$$

By analogy, it is natural to expect the influence of the environment in the nonlocal setting to be encoded by (3.17). Therefore, a first step towards answering (3.11) lies in the study of solutions to (3.17).

The goal of Chapter 3 is to find geometrical conditions on K ensuring that the solutions to (3.17) are identically equal to 1 in the whole set $\mathbb{R}^N \setminus K$.

We show that this holds true whenever K is convex.

Theorem — *Let $K \subset \mathbb{R}^N$ be a compact convex set and let $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$ be a function satisfying*

$$\begin{cases} Lu + f(u) \leqslant 0 & \textit{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \textit{as } |x| \to +\infty. \end{cases} \tag{3.18}$$

Then, $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$.

Whereas the solutions to (3.16) are automatically classical C^2 solutions (by standard elliptic theory), there is, in general, no smoothing effect for (3.18) and *the solutions may even not be continuous at all.* This is the reason why we require some a priori continuity.

Nonetheless, if we ask for a solution instead of a super-solution, we prove that it is possible to get rid of this a priori assumption provided that the nonlinearity does not vary "too much". Precisely,

Theorem — *Let $K \subset \mathbb{R}^N$ be a compact convex set. Suppose that*

$$\max_{[0,1]} f' < \frac{1}{2}. \tag{3.19}$$

Let $u : \mathbb{R}^N \setminus K \to [0,1]$ be a measurable function satisfying

$$\begin{cases} Lu + f(u) = 0 & \text{a.e. } \textit{in } \mathbb{R}^N \setminus K, \\ u(x) \to 1 & \textit{as } |x| \to +\infty. \end{cases}$$

Then, $u = 1$ a.e. in $\mathbb{R}^N \setminus K$.

We prove several improvement of these results. In particular, we prove that when J is compactly supported and square integrable, then the requirement on the asymptotic behavior of u can be weakened to

$$\sup_{\mathbb{R}^N\setminus K} u = 1,$$

which is even stronger than what is known in the local case (see Theorem 2.4).

Chapter 3 also contains a "robustness" result. Namely, we prove that if a compact set K is sufficiently close to a compact convex set (in the $C^{0,\alpha}$ topology), then the Liouville property still holds (see Theorem 2.6). In fact, this result is a sort of dichotomy: if the Liouville property can be shown to hold for a given set K, then it still holds for all compact sets sufficiently close it (in the $C^{0,\alpha}$ topology).

Our results are mainly based on sub- and super-solutions techniques and, to this end, we prove various comparison principles which are of independent interest (see Lemmata 4.1, 4.2 and 4.3).

3.2.2 A counterexample to the Liouville property of some nonlocal problems

Chapter 4 is a work in collaboration with J. Coville. It is meant as a follow-up to Chapter 3, although it may be read independently. The purpose here is to better pinpoint the geometrical conditions under which the Liouville property established at the previous chapter remains valid. An angle of this question consists in finding an obstacle K for which it does not hold. Chapter 4 provides an answer to this question. Precisely, we establish the following

Theorem — *There are (non-starshaped) simply connected compact obstacles K and data f and J for which problem*

$$\begin{cases} Lu + f(u) = 0 & \textit{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \textit{as } |x| \to +\infty, \end{cases}$$

has a solution $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$ which is not identically equal to 1 in $\overline{\mathbb{R}^N \setminus K}$.

The main difficulty in our construction lies in the lack of compactness. We show how to circumvent this difficulty relying on the Bourgain-Brezis-Mironescu characterization of Sobolev spaces. This enables us to obtain a priori estimates which allow the use of variational methods.

Remarkably, our construction is flexible enough to handle broader classes of nonlocal operators where the dispersal process need not be isotropic but instead depends on the geodesic distance between points in $\overline{\mathbb{R}^N \setminus K}$. For example, we prove that the same result holds when L is replaced by

$$L_{\mathrm{g}}u(x) := \int_{\mathbb{R}^N\setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))(u(y)-u(x))\mathrm{d}y, \tag{3.20}$$

where $d_{\mathrm{g}}(\cdot,\cdot)$ is the geodesic distance on $\overline{\mathbb{R}^N \setminus K}$ and $\widetilde{J} \in L^1_{\mathrm{loc}}(0,\infty)$ is a locally integrable kernel such that

$$\sup_{x\in\mathbb{R}^N\setminus K} \int_{\mathbb{R}^N\setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))\mathrm{d}y < \infty.$$

Operators of the type of (3.20) are actually of interest in their own right. They give an alternative way to describe the evolution of individuals in a heterogeneous medium which, in some situations, may be regarded as more realistic. Their most interesting feature is that the individuals can no longer travel through the obstacle: they are compelled to bypass it as if it were a material obstacle.

3.2.3 Some monotonicity results for general systems of nonlinear elliptic PDEs

Chapter 5 is a work in collaboration with S. Dipierro. It is concerned with symmetry and monotonicity properties of general systems of nonlinear elliptic partial differential equations. Although the topic is rather old, recent years have seen a renewed interest in these questions. A typical example which has raised a lot of attention is:

$$\begin{cases} \Delta u = uv^2, \\ \Delta v = vu^2, \\ u, v > 0. \end{cases} \tag{3.21}$$

This system emerges in the study of phase separation phenomena for Bose-Einstein condensates with multiple states. A natural question to ask is:

$$\text{under which conditions do } u \text{ and } v \text{ enjoy monotonicity properties?} \tag{3.22}$$

Such rigidity property plays an important role in the study of (3.21), inter alia, it allows the classification of solutions. In this perspective, it is also of interest to ask for De Giorgi type results. For example:

$$\text{under which conditions are } u \text{ and } v \text{ one-dimensional?} \tag{3.23}$$

Some answers to (3.22) and (3.23) are known, see for example [18, 20, 62, 152].

However, if (3.21) has a relatively “simple” form, things can get significantly tougher when considering more general nonlinearities and (possibly anomalous) diffusion processes. For example, one may want to address similar questions for systems of the form:

$$\begin{cases} (-\Delta)^{s_1} u = F_1(u,v), \\ (-\Delta)^{s_2} v = F_2(u,v), \end{cases} \tag{3.24}$$

where $s_1, s_2 \in (0, 1]$ and F_1, F_2 are the derivatives with respect to the first and the second variable, respectively, of some function $F \in C^{1,1}(\mathbb{R}^2)$. Systems of the type of (3.24) have recently been investigated (see e.g. [133, 134, 147, 152]) and some symmetry results have been obtained under various assumptions (see e.g. [56, 67]).

The purpose of Chapter 5 is to find a method providing an answer to (3.22) and (3.23) in a general setting (including (3.21) and (3.24) as particular cases).

Precisely, the goal of Chapter 5 is to show that minima and stable solutions of general energy functionals of the form

$$\mathscr{E}(u, v) = \int_\Omega F(\nabla u, \nabla v, u, v, x) \mathrm{d}x,$$

enjoy some monotonicity properties, under an assumption on the growth at infinity of the energy, which we call the "stability inequality" (see (3.26) below).

This setting allows not only to handle systems with general nonlinearities, but also with quite general diffusion operators, possibly of degenerate type. The most distinguished examples are the p-Laplacian and the mean curvature operator. By the Caffarelli-Silvestre extension theorem [43], this covers also fractional (and, hence, nonlocal) operators such as the fractional Laplacian.

The usual approach to obtain such rigidity results is to apply some stability inequality to a cut-off function. But this approach is generally difficult to implement because it requires to work with the precise form of the energy (which can be quite complicated).

Our strategy, inspired by [124, 125], relies on a completely different argument. Loosely speaking, it consists in comparing the energies of (u, v) and a perturbed translation of itself. Then, using the stability inequality and a contradiction argument, we can prove that the solutions indeed enjoy some rigidity properties. In doing so, we *do not need* to work with the precise form of the potential F and we can therefore deal with general energy functionals.

Our working hypotheses are the following. First, we assume that both the domain Ω and the potential F are invariant under translation in the e_N-direction, namely

$$\Omega = \mathscr{V} \times \mathbb{R} \quad \text{for some} \quad \mathscr{V} \subseteq \mathbb{R}^{N-1},$$

and F does not depend on the x_N-coordinate. We also suppose that

$$F = F(p_1, p_2, z_1, z_2, x') \in C(\mathbb{R}^{2N} \times \mathbb{R}^2 \times \mathscr{V}), \tag{3.25}$$

and that F is C^2 and convex with respect to the first two variables. Also, we assume that there exists a constant $C > 0$ such that, for any $p_1, p_2, q_1, q_2 \in \mathbb{R}^N$ with $|q_1| \leqslant |p_1 \cdot e_N|/4$ and $|q_2| \leqslant |p_2 \cdot e_N|/4$, it holds that

$$|F_{p_1 p_1}(p_1 + q_1, p_2, z_1, z_2, x')| \leqslant C |F_{p_1 p_1}(p_1, p_2, z_1, z_2, x')|,$$

$$|F_{p_2p_2}(p_1, p_2+q_2, z_1, z_2, x')| \leqslant C|F_{p_2p_2}(p_1, p_2, z_1, z_2, x')|,$$
$$|F_{p_1p_2}(p_1+q_1, p_2+q_2, z_1, z_2, x')| \leqslant C|F_{p_1p_2}(p_1, p_2, z_1, z_2, x')|.$$

Notice that these assumptions are fairly general and apply, in particular, to (3.21) and (3.24).

The first result of Chapter 5 deals with minimizers. Namely, we obtain the following

THEOREM — *Let $u, v \in C^1(\Omega)$ be such that (u, v) is a minimizer of $\mathscr{E}$ and that the growth condition*

$$\int_{\Omega\cap B_R} |F_{p_1p_1}||\nabla u|^2 + |F_{p_2p_2}||\nabla v|^2 + |F_{p_1p_2}||\nabla u||\nabla v| \,\mathrm{d}x \lesssim R^2, \tag{3.26}$$

is satisfied for large enough $R > 0$, where the derivatives of F are evaluated at $(\nabla u, \nabla v, u, v, x')$. Then, u and v are monotone on each line in the e_N-direction, i.e., for any $x \in \Omega$, either $u_N(x+te_N) \geqslant 0$ or $u_N(x+te_N) \leqslant 0$, and either $v_N(x+te_N) \geqslant 0$ or $v_N(x+te_N) \leqslant 0$, for any $t \in \mathbb{R}$. In particular, u and v are one-dimensional.

Moreover, we show that a similar result holds for *stable solutions* up to a slight additional regularity requirement on F and (u, v).

THEOREM — *Suppose that $F \in C^{3,\alpha}(\mathbb{R}^{2N} \times \mathbb{R}^2 \times \mathscr{V})$. Let (u, v) be such that either $u, v \in C^{0,1}(\Omega)$ are convex or $u, v \in C^{1,1}(\Omega)$. Moreover, suppose that (u, v) is a stable solution of $\mathscr{E}$, and that the growth condition* (3.26) *holds true. Then, u and v are monotone in the e_N-direction, i.e. either $u_N \geqslant 0$ or $u_N \leqslant 0$ and either $v_N \geqslant 0$ or $v_N \leqslant 0$ in Ω.*

The results contained in Chapter 5 are, in fact, slightly more general. They still hold when replacing the notions of stable and minimal solutions by weaker ones. Indeed, we prove that it suffices to consider minimizers and stable solutions among the class of functions which are obtained by piecewise Lipschitz deformations in the e_N-direction (see Theorems 1.5 and 1.10).

Several applications of our results are presented at the end of Chapter 5.

Part I

A functional analytic toolbox

Chapter 1

A Bourgain-Brezis-Mironescu characterization of higher order Besov-Nikol'skii spaces

This chapter is inspired by the paper [28], accepted for publication in *Annales de l'Institut Fourier*.

1 Introduction

1.1 A brief state of art

Let $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ be a sequence of mollifiers, i.e. a sequence satisfying

$$\begin{cases} \rho_\varepsilon \geqslant 0 \ \text{ a.e. in } \mathbb{R}^N \text{ for any } \varepsilon > 0, \\ \displaystyle\int_{\mathbb{R}^N} \rho_\varepsilon(z)\mathrm{d}z = 1 \ \text{ for any } \varepsilon > 0, \\ \displaystyle\lim_{\varepsilon\to 0^+} \int_{|z|\geqslant\delta} \rho_\varepsilon(z)\mathrm{d}z = 0 \ \text{ for all } \delta > 0. \end{cases} \tag{1.1}$$

Let $M \in \mathbb{N}^*$, $1 \leqslant p < \infty$ and $s \in (0, M]$. We are interested in the properties of functions $f \in L^p(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega \left(\int_{\mathbb{R}^N} \frac{|\Delta_h^M f(x)|^p}{|h|^{sp}} \mathrm{d}x \right) \mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+, \tag{1.2}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, concave function and $\Delta_h^M f(x)$ stands for the usual M-th order forward difference of f given by

$$\Delta_h^M f(x) := \sum_{j=0}^{M} (-1)^{M-j} \binom{M}{j} f(x + hj), \quad x, h \in \mathbb{R}^N. \tag{1.3}$$

The assumptions on ω will be made precise later on.

Functionals of the type of (1.2) were initially introduced by Bourgain, Brezis and Mironescu [27, 33] to obtain a new characterization of the Sobolev space $W^{1,p}(\mathbb{R}^N)$. Namely, for $M = s = 1$ and $\omega(t) = t$, (1.2) reads

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(h) \frac{|f(x+h) - f(x)|^p}{|h|^p} \mathrm{d}x\mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+, \tag{1.4}$$

and the result of Bourgain, Brezis and Mironescu states that, any $f \in L^p(\mathbb{R}^N)$ satisfying (1.4) belongs to the Sobolev space $W^{1,p}(\mathbb{R}^N)$ if $1 < p < \infty$, or to $BV(\mathbb{R}^N)$ if $p = 1$, provided $(\rho_\varepsilon)_{\varepsilon>0}$ is radial. More precisely, they have shown that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(h) \frac{|f(x+h) - f(x)|^p}{|h|^p} \mathrm{d}x\mathrm{d}h = K_{p,N} \|\nabla f\|^p_{L^p(\mathbb{R}^N)},$$

where

$$K_{p,N} := \int_{\mathbb{S}^{N-1}} |\sigma \cdot e|^p \mathrm{d}\mathscr{H}^{N-1}(\sigma), \quad e \in \mathbb{S}^{N-1}.$$

As a result, they were able to establish the following limiting embedding

$$\lim_{r \to 1^-} (1-r)p\|f\|^p_{W^{r,p}(\mathbb{R}^N)} = K_{p,N}\|\nabla f\|^p_{L^p(\mathbb{R}^N)}. \tag{1.5}$$

Since this work, numerous new characterizations of the Sobolev spaces $W^{k,p}(\mathbb{R}^N)$ or $BV(\mathbb{R}^N)$ have been obtained [24, 25, 54, 68, 118, 119, 131] and various asymptotic formulae characterizing the Sobolev norms in terms of fractional norms have been derived [89, 94, 105, 142]. For instance, Maz'ya and Shaposhnikova [105] obtained the counterpart of (1.5) in the critical case $r \to 0^+$, that is

$$\lim_{r \to 0^+} rp\|f\|^p_{W^{r,p}(\mathbb{R}^N)} = 2\sigma_N \|f\|^p_{L^p(\mathbb{R}^N)},$$

whenever $f \in \bigcup_{0<r<1} W^{r,p}(\mathbb{R}^N)$ and where σ_N stands for the superficial measure of the unit sphere $\mathbb{S}^{N-1}$.

Also, let us mention the work of Ponce [119] who was the first to obtain a characterization of the space $BV(\mathbb{R}^N)$ in terms of a class of functions in $L^1(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \Omega\left(\frac{|f(x+h) - f(x)|}{|h|}\right) \mathrm{d}x\mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+, \tag{1.6}$$

under suitable growth assumptions on $\Omega \in C(\mathbb{R}_+, \mathbb{R}_+)$.

More recently, such type of characterizations were extended by Borghol [25], Bojarski, Ihnatsyeva, Kinnunen [24] and Ferreira, Kreisbeck and Ribeiro [68], who considered the cases $1 < p < \infty$ in higher order Sobolev spaces. Typically, in [68] it is shown that the spaces $W^{k,p}(\mathbb{R}^N)$, with $p \in (1,\infty)$ and $k \in \mathbb{N}^*$, can be characterized by quantities of the type

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(h)\ \Omega\left(\frac{|\Delta_h^k f(x)|}{|h|^k}\right) \mathrm{d}x\mathrm{d}h, \tag{1.7}$$

where $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, convex function such that

$$m_1 t^p \leqslant \Omega(t) \leqslant m_2 t^p,$$

for all $t \geqslant 0$ and some positive constants $0 < m_1 < m_2$.

To our knowledge, very few is known in the case $0 < s < M$. Nonetheless, recent works of Lamy and Mironescu [98] suggest a connection between expressions of the type of (1.2) and Besov spaces. In [98], the authors prove the following

Theorem 1.1 (Lamy, Mironescu, [98]). — *Let $s > 0$, $p, q \in [1,\infty]$ and let $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ satisfying* (1.1) *and such that*

$$\rho_\varepsilon(h) = \frac{1}{\varepsilon^N}\,\rho\left(\frac{h}{\varepsilon}\right) \text{ for some } \rho \in L^1(\mathbb{R}^N). \tag{1.8}$$

Then,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)} + \left\|\frac{1}{\varepsilon^s}\|f * \rho_\varepsilon - f\|_{L^p(\mathbb{R}^N)}\right\|_{L^q((0,1),\frac{\mathrm{d}\varepsilon}{\varepsilon})}. \tag{1.9}$$

The converse of this holds under some additional moment condition on ρ (see [98] for further details). In fact, the case $q = \infty$ is not properly stated nor explicitly proven in [98]. To fill this gap, we shall give some additional details at the end of the chapter. A consequence of this, which has not been noticed in [98], is the following

Proposition 1.2. — *Let $s \in (0,1)$, $p \in [1,\infty)$ and $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ satisfying* (1.1) *and* (1.8). *Then, the following statements are equivalent:*

(i) $f \in B^s_{p,\infty}(\mathbb{R}^N)$,

(ii) $f \in L^p(\mathbb{R}^N)$ *satisfies*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(h)\frac{|f(x+h) - f(x)|^p}{|h|^{sp}}\mathrm{d}x\mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+. \tag{1.10}$$

Moreover,

$$\|f\|^p_{B^s_{p,\infty}(\mathbb{R}^N)} \sim \|f\|^p_{L^p(\mathbb{R}^N)} + \sup_{\varepsilon\in(0,1)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(h)\frac{|f(x+h) - f(x)|^p}{|h|^{sp}}\mathrm{d}x\mathrm{d}h. \tag{1.11}$$

It is worth noticing that, by contrast with the representation of $B^s_{p,\infty}(\mathbb{R}^N)$ obtained in [98], no moment condition on ρ_ε is needed. Moreover, since ρ_ε does not need to be radial, some directions may be privileged, yet with no impact on the resulting norm. This is in clear contrast with the case $s = 1$ (see also [46, Remark 10] or [119, Corollary 3, p.232]).

This sheds new lights on how to describe smoothness and could be of potential interest in some problems of the calculus of variations and in the study of some integro-differential equations (see e.g. [2, 8, 16, 76, 77, 135]).

Also, in view of Proposition 1.2, it is natural to ask for corresponding assertions of (1.6) and (1.7) in the framework of the fractional Besov-Nikol'skii spaces $B^s_{p,\infty}(\mathbb{R}^N)$. For example: what can be said about the limiting behavior of (1.10) when $\varepsilon \to 0^+$? Can one describe higher order Besov-Nikol'skii spaces via expressions of the type (1.2)? It is the main concern of this chapter to deal with these issues.

1.2 Main Motivation

This work originates in a problem raised in [16]. Consider the heterogeneous Fisher-KPP equation:

$$\frac{1}{\varepsilon^m}\big(J_\varepsilon * u(x) - u(x)\big) + f(x,u) = 0, \quad u = u_\varepsilon, \quad x \in \mathbb{R}^N, \quad \varepsilon > 0, \tag{1.12}$$

where $m \in [0,2]$, u is the density of a given population, $J_\varepsilon(z) := \frac{1}{\varepsilon^N} J\big(\frac{z}{\varepsilon}\big)$, with $J \in C \cap L^1(\mathbb{R}^N)$ a symmetric positive dispersal kernel with unit mass and having finite m-th order moment, and $f \in C^{1,\alpha}(\mathbb{R}^{N+1})$ is a heterogeneous KPP type non-linearity, that is:

$$\begin{cases} f(\cdot,0) = 0, \\ \text{for all } x \in \mathbb{R}^N,\ f(x,s)/s \text{ is decreasing with respect to } s \in (0,\infty), \\ \text{there exists } S(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ such that } f(\cdot, S(\cdot)) \leqslant 0. \end{cases}$$

For the sake of simplicity, we restrict our attention to non-linearities of the form

$$f(x,s) = s(a(x) - s), \quad \text{with} \quad \limsup_{|x|\to\infty} a(x) < 0.$$

Roughly speaking, f models the growth rate of the population and J the probability to jump from one location to another. The parameter ε is a measure of the spread of dispersal of the species. The scaling term $\frac{1}{\varepsilon^m}$ can be interpreted as the rate of dispersal of the species. It arises when considering a cost function (see [16, Section 2] for a more detailed explanation on the matter). Consider for instance a tree producing and dispersing seeds. Then, $\varepsilon \ll 1$ represents a strategy where the dispersal rate is large but the seeds are spread over smaller distances, and $\varepsilon \gg 1$ represents the opposite strategy (i.e. smaller dispersal rate but the seeds are spread over larger distances). As for the parameter m, it measures the influence of the cost function on the different strategies.

Existence of positive solutions to (1.12) is naturally expected to provide a persistence criteria for the population under consideration. Nonetheless, if the asymptotic behavior of solutions to (1.12) is quite well understood when $\varepsilon \to \infty$ (see [16]), it is not the case when $\varepsilon \to 0^+$ and $0 < m < 2$. Berestycki et al. [16] were able to prove the following result.

THEOREM 1.3 (Berestycki, Coville, Vo, [16]). — *Assume that J is compactly supported with $J(0) > 0$, that $m \in (0, 2)$, that $\max\{a, 0\} \not\equiv 0$ and that $a \in C^2(\mathbb{R}^N)$. Then, when $\varepsilon \to 0^+$, the solution u_ε of (1.12) converges almost everywhere to some non-negative bounded function v satisfying*

$$v(x)(a(x) - v(x)) = 0 \quad \textit{in } \mathbb{R}^N. \tag{1.13}$$

Unfortunately, equation (1.13) admits infinitely many solutions, so it may happen that $v \equiv 0$ (extinction) or that $v = a_+\mathbb{1}_K$ for some compact $K \subset \operatorname{supp}(a_+)$ (persistence in a given area of the ecological niche). Whence, one cannot directly infer a persistence strategy for that case.

However, it is known that solutions to (1.12), when they exist, satisfy

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^m}\mathrm{d}x\mathrm{d}y \leqslant C \quad \text{for all } \varepsilon > 0, \tag{1.14}$$

with $\rho_\varepsilon(z) = \varepsilon^{-m}|z|^m J_\varepsilon(z)$ a smooth mollifier satisfying (1.1) (see [16, Lemma 5.1(ii)] for a proof).

To quote Berestycki et al.: "*If for the case $m = 2$ we could rely on elliptic regularity and the new description of Sobolev spaces developed in Bourgain et al. [27], Brezis [33], Ponce [118, 119] to get some compactness, this characterization does not allow us to treat the case $m < 2$. We believe that a new characterization of fractional Sobolev spaces in the spirit of the work of Bourgain, Brezis and Mironescu [27, 33] will be helpful to resolve this issue.*"

This motivates the study of general classes of functions of the type of (1.2), in particular the forthcoming Theorems 2.3 and 2.15.

1.3 Comments

If (1.10) is very similar to (1.4), the underlying spaces, $W^{1,p}(\mathbb{R}^N)$ and $B^s_{p,\infty}(\mathbb{R}^N)$, are very different in nature and one has to cope with some technicalities. Among others, it is not clear anymore whether the limit of (1.10) as $\varepsilon \to 0^+$ exists nor, even if it does, whether it provides an equivalent semi-norm. In the integer order case, things are not too controversial in the sense that

$$\|\nabla f\|_{L^p(\mathbb{R}^N)} \sim \limsup_{|h|\to 0} \frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|} = \sup_{h\neq 0} \frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|}. \tag{1.15}$$

(see e.g. [142] or Lemma 7.1), while the counterpart of (1.15) in the fractional case $s \in (0,1)$ is not true in general. Indeed, every nontrivial function $f \in C_c^\infty(\mathbb{R}^N)$ satisfies

$$\lim_{|h|\to 0} \frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|^s} = 0 < \sup_{h\neq 0} \frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|^s} = [f]_{B^s_{p,\infty}(\mathbb{R}^N)}, \tag{1.16}$$

whenever $s \in (0,1)$, $p \in [1,\infty]$. Finiteness of either or both the two first terms in the left-hand side of (1.16) equally describes $B^s_{p,\infty}(\mathbb{R}^N)$ in the sense that they define the same set of functions. But the respective (semi-)norms induced by these quantities are not equivalent (see Section 5). For these reasons, at some places, it will be more convenient to state our results in terms of suprema as in (1.11) instead of limits.

On the other hand, smooth functions are not dense in $B^s_{p,\infty}(\mathbb{R}^N)$, so that the arguments used in the integer case do not simply adapt. We show how to do this in a way that allows, not only to give a meaning, but also to handle the tricky case $p = \infty$ in both the integer and the fractional case, using only elementary arguments. Also, in the particular case where ρ is radially symmetric, we improve (1.11) to a semi-norm equivalence at all orders $s > 0$. More general quantities are also investigated as well as compactness in the case of a sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}^N)$.

At the end, this yields a common nonlocal description for the Besov-Nikol'skii spaces $B^s_{p,\infty}(\mathbb{R}^N)$, the Hölder-Zygmund spaces $\mathscr{C}^s(\mathbb{R}^N)$, the $BV(\mathbb{R}^N)$ space, the Sobolev spaces $W^{k,p}(\mathbb{R}^N)$ and the Lipschitz space $C^{0,1}(\mathbb{R}^N)$. As a by-product, we obtain new characterizations for these spaces and a new limiting embedding between Lipschitz and Besov spaces which extends the previous known results.

2 Main results

2.1 A new characterization of Besov-Nikol'skii spaces

To state our results, we shall introduce some notations and terminology.

Definition 2.1. — *A function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be roughly subadditive if there exists a constant $A > 0$ such that,*

$$\omega(t_1 + t_2) \leqslant A \{\omega(t_1) + \omega(t_2)\},$$

for every $t_1, t_2 \in \mathbb{R}_+$. If $A = 1$, then we say that ω is subadditive.

To shorten our statements, it will be more convenient to call C_{inc} the set of all continuous, increasing functions $\omega : [0,\infty) \to [0,\infty)$ satisfying $\omega(0) = 0$ and

$$\lim_{t\to\infty} \omega(t) = \infty.$$

Also, we set

$$C^+_{\text{inc}} := \Big\{ \omega \in C_{\text{inc}} \text{ such that } \omega \text{ is roughly subadditive} \Big\}.$$

Remark 2.2. — Observe that if $\omega_1, \omega_2 \in C^+_{\text{inc}}$, then $\omega_1 \circ \omega_2 \in C^+_{\text{inc}}$.

Typical examples of functions in C^+_{inc} are:

(i) $\omega_1(t) = t^\alpha$ with $\alpha > 0$,

(ii) $\omega_2(t) = \ln(1+t)$,

(iii) $\omega_3(t) = t \tanh(t)$,

(iv) $\omega_4(t) = \operatorname{arsinh}(t)$, ...

More generally, if $\omega \in C_{\text{inc}}$ is concave, then $\omega \in C^+_{\text{inc}}$ (see e.g. [36, Theorem 5]). As indicated by the example of t^α with $\alpha > 1$, C^+_{inc} contains also some convex functions as long as they do not increase too fast. Indeed, a direct computation shows that if $\omega : [0,\infty) \to [0,\infty)$ is a continuous, convex function with $\omega(0) = 0$ and if $\omega(2t) \leqslant \kappa\, \omega(t)$, for all $t \geqslant 0$ and some constant $\kappa > 0$ (independent of t), then $\omega \in C^+_{\text{inc}}$.

Our first result reads as follows

Theorem 2.3. — *Let $M \in \mathbb{N}^*$, $s \in (0, M)$ and $p \in [1,\infty]$. Let $\omega \in C^+_{\text{inc}}$ and $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying* (1.1) *and* (1.8). *Then, the following statements are equivalent:*

(i) $f \in B^s_{p,\infty}(\mathbb{R}^N)$,

(ii) $f \in L^p(\mathbb{R}^N)$ *is such that*

$$\int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+. \tag{2.1}$$

Moreover,

$$\omega\left([f]_{B^s_{p,\infty}(\mathbb{R}^N)}\right) \sim \sup_{\varepsilon>0} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \mathrm{d}h.$$

Remark 2.4. — It is noteworthy that the assumptions of Theorem 2.3 are somehow self-improving. For example, if $\omega \in C_{\text{inc}}$ is such that

$$\alpha_1\, \underline{\omega} \leqslant \omega \leqslant \alpha_2\, \overline{\omega} \quad \text{a.e. in } \mathbb{R}^N,$$

for some $\underline{\omega}$, $\overline{\omega} \in C^+_{\text{inc}}$ and $\alpha_1, \alpha_2 > 0$, then ω still characterizes $B^s_{p,\infty}(\mathbb{R}^N)$. Note also that the Jensen inequality allows to extend this result to convex $\omega \in C_{\text{inc}}$.

Moreover, the conclusion of Theorem 2.3 still holds under the slightly weaker assumption that $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ satisfies (1.1) and (1.8) with $\rho \in L^1(\mathbb{R}^N)$ such that there exists a number $\delta > 0$ and a nonnegative radial function φ with $\rho \geqslant \varphi$ a.e. in B_δ and $\int_{B_\delta} \varphi > 0$.

Also, when $1 \leqslant p < \infty$, the fact that $\omega \in C^+_{\text{inc}}$ allows one to replace (2.1) by

$$\int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\int_{\mathbb{R}^N} \Omega\left(\frac{|\Delta_h^M f(x)|}{|h|^s}\right) \mathrm{d}x\right) \mathrm{d}h, \tag{2.2}$$

for any continuously increasing $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Omega(0) = 0$ and

$$m_1 t^p \leqslant \Omega(t) \leqslant m_2 t^p, \tag{2.3}$$

for all $t \geqslant 0$ and some $0 < m_1 \leqslant m_2$.

By the same token, we obtain the following counterpart for the Lipschitz space.

Theorem 2.5. — *Let $\omega \in C^+_{\text{inc}}$ and $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying* (1.1) *and* (1.8)*. Then, the following statements are equivalent:*

(i) $f \in C^{0,1}(\mathbb{R}^N)$,

(ii) $f \in L^\infty(\mathbb{R}^N)$ *is such that*

$$\int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\frac{\|f(\cdot + h) - f\|_{L^\infty(\mathbb{R}^N)}}{|h|}\right) \mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+.$$

Moreover,

$$\omega\left([f]_{C^{0,1}(\mathbb{R}^N)}\right) \sim \limsup_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\frac{\|f(\cdot + h) - f\|_{L^\infty(\mathbb{R}^N)}}{|h|}\right) \mathrm{d}h.$$

In fact, our proof also allows to cover first order Sobolev spaces. For example, in view of (2.2), we have the

Theorem 2.6. — *Let $1 \leqslant p < \infty$, $(\omega, \Omega) \in C^+_{\text{inc}} \times C_{\text{inc}}$ with Ω satisfying* (2.3) *and $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying* (1.1) *and* (1.8)*. Then, the following statements are equivalent:*

(i) $f \in W^{1,p}(\mathbb{R}^N)$ *(resp. $f \in BV(\mathbb{R}^N)$ if $p = 1$),*

(ii) $f \in L^p(\mathbb{R}^N)$ *is such that*

$$\int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\int_{\mathbb{R}^N} \Omega\left(\frac{|f(x + h) - f(x)|}{|h|}\right) \mathrm{d}x\right) \mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+.$$

Moreover,

$$\omega\left(\|\nabla f\|^p_{L^p(\mathbb{R}^N)}\right) \sim \limsup_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\int_{\mathbb{R}^N} \Omega\left(\frac{|f(x + h) - f(x)|}{|h|}\right) \mathrm{d}x\right) \mathrm{d}h. \tag{2.4}$$

Note that the limit superior in the right-hand side of (2.4) may not necessarily coincide with the limit inferior, depending on the choices of ω and Ω.

Remark 2.7. — If $\omega(t) = t$ and Ω is convex, then the corresponding assertion still holds in higher order Sobolev spaces, see [68] for a proof.

Here are some straightforward consequences of Theorem 2.3.

Example 2.8. — Let $M \in \mathbb{N}^*$, $s \in (0, M)$ and $J \in L^1(\mathbb{R}^N)$ be a radial function such that

$$\mathscr{J} := \int_{\mathbb{R}^N} J(z)|z|^{sq}\mathrm{d}h < \infty \quad \text{for some } 1 \leqslant q < \infty.$$

Then, choosing

$$\rho_\varepsilon(h) = \frac{1}{\mathscr{J}}\frac{|h|^{sq}}{\varepsilon^{sq}}J_\varepsilon(h),$$

and $\omega(t) = t^q$ we obtain

$$[f]_{B^s_{p,\infty}(\mathbb{R}^N)} \sim \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon^{sq}}\int_{\mathbb{R}^N} J_\varepsilon(h)\|\Delta_h^M f\|^q_{L^p(\mathbb{R}^N)}\mathrm{d}h\right)^{1/q}. \tag{2.5}$$

Remark 2.9. — Notice that the quantity (1.14) appearing in the study of the nonlocal Fisher-KPP equation (1.12) can be seen as a particular case of (2.5).

Other choices of ρ_ε highlight interesting links with the more classical Besov spaces $B^s_{p,q}(\mathbb{R}^N)$ with $1 \leqslant q < \infty$ (see Definition 3.3 on Section 3 for the definition of these spaces).

Example 2.10. — Given $1 \leqslant q < \infty$, the choice $\omega(t) = t^q$ and

$$\rho_\varepsilon(h) = \frac{1}{C|h|^N}\mathbb{1}_{(\varepsilon,2\varepsilon)}(|h|), \tag{2.6}$$

where $C = \sigma_N \ln(2)$, yields

$$[f]_{B^s_{p,\infty}(\mathbb{R}^N)} \sim \sup_{\varepsilon>0}\left(\int_{\varepsilon<|h|<2\varepsilon}\frac{\|\Delta_h^M f\|^q_{L^p(\mathbb{R}^N)}}{|h|^{N+sq}}\mathrm{d}h\right)^{1/q}.$$

Example 2.11. — Given $1 \leqslant q < \infty$, the choice $\omega(t) = t^q$ and

$$\rho_\varepsilon(h) = \frac{1}{\sigma_N\varepsilon^{(s-r)q}}\frac{(s-r)q}{|h|^{N-(s-r)q}}\mathbb{1}_{(0,\varepsilon)}(|h|),$$

for some $r \in (0, s)$, gives

$$q^{-1/q}[f]_{B^s_{p,\infty}(\mathbb{R}^N)} \sim \sup_{\varepsilon>0} \frac{(s-r)^{1/q}}{\varepsilon^{s-r}}\left(\int_{|h|<\varepsilon}\frac{\|\Delta_h^M f\|^q_{L^p(\mathbb{R}^N)}}{|h|^{N+rq}}\mathrm{d}h\right)^{1/q}.$$

2.2 Limits of Besov norms

Following the original result of Bourgain, Brezis and Mironescu in [27]; Karadzhov, Milman, Xiao [89] and Triebel [142] proved the following limiting embedding

$$q^{-1/q}\|\nabla f\|_{L^p(\mathbb{R}^N)} \sim \lim_{r\to 1^-} (1-r)^{1/q}[f]_{B^r_{p,q}(\mathbb{R}^N)} \quad \text{for } 1<p,q<\infty. \tag{2.7}$$

See e.g. [142] where higher order derivatives are also studied.

The counterpart of Example 2.11 for the Lipschitz space leads one to ask wether (2.7) still holds in the critical case $p=\infty$ (recall $W^{1,\infty}(\mathbb{R}^N)$ is the same as $C^{0,1}(\mathbb{R}^N)$). However, because of the restriction to (1.8) in Theorem 2.5 one cannot directly infer that this is the case. In addition, spaces of the type $W^{1,\infty}(\mathbb{R}^N)$ or $B^s_{\infty,q}(\mathbb{R}^N)$ do not admit spaces such as $C^\infty_c(\mathbb{R}^N)$ as dense subset (they are not even separable) and they inherit from the "bad" properties of $L^\infty(\mathbb{R}^N)$. This makes the validity of (2.7) in the case $p=\infty$ rather unclear.

We prove that a weaker version of (2.7) still holds when $p=\infty$.

Theorem 2.12. — *Let $q\in[1,\infty)$ and assume $f\in L^\infty(\mathbb{R}^N)$ is such that*

$$\limsup_{r\to 1^-} (1-r)^{1/q}\|f\|_{B^r_{\infty,q}(\mathbb{R}^N)} < \infty. \tag{2.8}$$

Then, $f\in C^{0,1}(\mathbb{R}^N)$. Moreover,

$$q^{-1/q}[f]_{C^{0,1}(\mathbb{R}^N)} \sim \limsup_{r\to 1^-} (1-r)^{1/q}\|f\|_{B^r_{\infty,q}(\mathbb{R}^N)}. \tag{2.9}$$

Remark 2.13. — Due to the lack of continuity of the translations in $L^\infty(\mathbb{R}^N)$ it is not clear wether the lim sup in (2.8) (resp. in (2.9)) can be replaced by a lim inf.

The proof can be carried out using subadditivity and monotonicity arguments via an improvement of the Chebychev inequality due to Bourgain, Brezis and Mironescu [27] together with Theorem 2.3.

However, in the fractional case, one lose the aforementioned monotonicity and the arguments fail. In view of Example 2.11 and $\mathscr{C}^s(\mathbb{R}^N) = B^s_{\infty,\infty}(\mathbb{R}^N)$ it is natural to ask wether or not the counterpart holds for $B^s_{p,\infty}(\mathbb{R}^N)$.

Using subatomic decompositions we were able to show that this is not the case.

Theorem 2.14. — *Let $s>0$, $p\in[1,\infty)$ and $q\in[1,\infty)$. Then, there exists a function $f\in L^p(\mathbb{R}^N)\setminus B^s_{p,\infty}(\mathbb{R}^N)$ such that*

$$\sup_{0<r<s} (s-r)^{1/q}\|f\|_{\mathbf{B}^r_{p,q}(\mathbb{R}^N)} < \infty.$$

Here, "$\|f\|_{\mathbf{B}^r_{p,q}(\mathbb{R}^N)}$" stands for the $B^r_{p,q}(\mathbb{R}^N)$-norm of f in the sense of subatomic decomposition theory (see Definition 4.2 below).

In particular, this suggests that the restriction to (1.8) in Theorems 1.2 and 2.3 (and actually also in (1.9) when $q=\infty$) is not far from being optimal.

2.3 A non-compactness result

In the integer case $s = 1$, it is known that any bounded sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(h)\frac{|f_\varepsilon(x+h) - f_\varepsilon(x)|^p}{|h|^p}\mathrm{d}x\mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+,$$

must be relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ provided $(\rho_\varepsilon)_{\varepsilon>0}$ is a suitable sequence of mollifiers (e.g. nonincreasing if $N = 1$ [27] or radially symmetric if $N \geqslant 2$ [118]).

Per contra, we show that this phenomenon does not extend to $s \in \mathbb{R}_+ \setminus \mathbb{N}$, at least if ρ_ε exhibits a reasonable decay at infinity.

Theorem 2.15. — *Let $M \in \mathbb{N}^*$, $s \in (0, M)$ and $p \in [1, \infty)$. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a sequence of mollifiers of the form* (1.8) *with $\rho \in L^1(\mathbb{R}^N)$ satisfying the moment condition*

$$\int_{\mathbb{R}^N} \rho(z)|z|^{p(M-s)}\mathrm{d}z < \infty.$$

Then, there exists a bounded sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(h)\frac{|\Delta_h^M f_\varepsilon(x)|^p}{|h|^{sp}}\mathrm{d}x\mathrm{d}h \leqslant C \quad \text{as } \varepsilon \to 0^+,$$

but which is not relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

Remark 2.16. — In some particular cases it is possible to get rid of assumption (1.8). For instance, if the ρ_ε's are non-increasing and supported in some ball of the form $B_{r\varepsilon}$ for all $\varepsilon > 0$ and some $r > 0$, then the result still holds. Notice also that the conclusion of Theorem 2.15 still holds for slightly more general functionals in the spirit of (2.1) with, say, $\omega = |\cdot|^{q/p}$, $\Omega = |\cdot|^p$, for any $q \geqslant 1$.

In the same vein, we obtain the following

Theorem 2.17. — *Let $s > 0$, $p \in [1, \infty)$ and $q \in [1, \infty)$. Then, there exists a bounded sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}^N)$ satisfying*

$$\limsup_{\varepsilon\to 0^+} \varepsilon\|f_\varepsilon\|^q_{B^{s-\varepsilon}_{p,q}(\mathbb{R}^N)^*} < \infty, \tag{2.10}$$

but which is not relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

The subscript "$*$" in (2.10) means that the $B^{s-\varepsilon}_{p,q}$-norm of f_ε is calculated using $(\lfloor s\rfloor + 1)$-th order finite differences (according to Definition 3.3). This is no longer true if, instead, we use smaller order differences. For example, if $(f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\mathbb{R}^N)$, then

$$\limsup_{\varepsilon\to 0^+} \varepsilon \|f_\varepsilon\|^p_{W^{1-\varepsilon,p}(\mathbb{R}^N)} < \infty,$$

implies that $(f_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, while

$$\limsup_{\varepsilon\to 0^+} \varepsilon \|f_\varepsilon\|^p_{B^{1-\varepsilon}_{p,p}(\mathbb{R}^N)^*} < \infty,$$

does not. Evidently, this restriction is immaterial if $0 < s \notin \mathbb{N}$.

2.4 An approximation criteria

It is well-known that neither $C_c^\infty(\mathbb{R}^N)$ nor $\mathscr{S}(\mathbb{R}^N)$ are dense in $B^s_{p,\infty}(\mathbb{R}^N)$. If the question of how to approximate a given $f \in B^s_{p,q}(\mathbb{R}^N)$ in a "suitable manner" has already been well-studied (see e.g. [95, 114, 132]), to the author's knowledge it seems, however, that no criterion to recognize a function $f \in B^s_{p,\infty}(\mathbb{R}^N)$ which can be approximated by smooth functions in its natural (strong) topology is available in the literature.

An interesting consequence of (the proof of) Theorem 2.3 is that it gives such a criterion.

Corollary 2.18. — *Let $M \in \mathbb{N}^*$, $s \in (0, M)$, $p \in [1,\infty)$. Let $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ be a sequence of radial functions satisfying* (1.1) *and* (1.8), *and let $\omega \in C^+_{\mathrm{inc}}$. Then, the following statements are equivalent:*

(i) $f \in L^p(\mathbb{R}^N)$ is such that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \mathrm{d}h = 0,$$

(ii) $f \in B^s_{p,\infty}(\mathbb{R}^N)$ and there exists $(f_n)_{n\geqslant 0} \subset C_c^\infty(\mathbb{R}^N)$ such that

$$\|f - f_n\|_{B^s_{p,\infty}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

A noteworthy consequence of Corollary 2.18 is the following

Example 2.19. — Let $s \in (0,1)$ and $p \in [1,\infty)$. Then, with the choice (2.6) and $\omega(t) = t^p$ we find that condition (ii) above is equivalent to

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \int_{\varepsilon<|x-y|<2\varepsilon} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y = 0,$$

or, more generally, to

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|h|<2\varepsilon}\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}^q}{|h|^{N+sq}}\mathrm{d}h=0,$$

in the higher order case.

In Sections 3 and 4 we detail all our notations and useful definitions. In Section 5, we show some preliminary estimates which aims to simultaneously open the way to Corollary 2.18 and to explain why it is more convenient to represent $B^s_{p,\infty}(\mathbb{R}^N)$ in terms of the supremum of (2.1) rather than in terms of its limits. Section 6 is devoted to the proof of Theorem 2.3 and Section 7 to that of Theorems 2.5, 2.6, and 2.12. In Section 8 we establish Theorem 2.14. In Section 9, we prove Theorems 2.15 and 2.17. Finally, in the Appendix, we discuss Proposition 1.2.

3 Notations and definitions

Throughout this chapter we will make use of the following notations.

$\mathbb{S}^{N-1}$:	is the unit sphere of $\mathbb{R}^N$;
$\mathscr{H}^{N-1}$:	is the $(N-1)$-dimensional Hausdorff measure;
$\lvert K\rvert$:	is the Lebesgue measure of the set K;
$\mathbb{1}_K$:	is the characteristic function of the set K;
B_R :	is the ball of radius $R>0$ centered at the origin;
$B_R(x)$:	is the ball of radius $R>0$ centered at $x\in\mathbb{R}^N$;
τ_h :	is the translation operator $\tau_h f(x)=f(x+h)$, $x,h\in\mathbb{R}^N$;
$f*g$:	is the convolution of f and g;
$\lesssim$:	is the "approximatively-less-than" symbol: $a\lesssim b\Leftrightarrow a\leqslant Cb$;
$\sim$:	is the equivalence symbol: $a\sim b\Leftrightarrow a\lesssim b$ and $b\lesssim a$;
$⨍_A$:	is the integral mean symbol: $⨍_A f=\frac{1}{\lvert A\rvert}\int_A f$.

We denote by $L^p(\mathbb{R}^N)$ the *Lebesgue space* of (equivalence classes of) functions for which the p-th power of the absolute value is Lebesgue integrable (resp. essentially bounded functions when $p=\infty$); by $C_c^\infty(\mathbb{R}^N)$ the space of smooth compactly supported functions; by $\mathscr{S}(\mathbb{R}^N)$ the *Schwartz space* of rapidly decaying functions; and, by $\mathscr{S}'(\mathbb{R}^N)$, its dual, the space of tempered distributions. The *Lipschitz space* $C^{0,1}(\mathbb{R}^N)$ is the space of functions $f\in L^\infty(\mathbb{R}^N)$ for which the semi-norm

$$[f]_{C^{0,1}(\mathbb{R}^N)}:=\sup_{h\neq 0}\frac{\|\tau_h f-f\|_{L^\infty(\mathbb{R}^N)}}{|h|}, \tag{3.1}$$

is finite. The space $C^{0,1}(\mathbb{R}^N)$ is a Banach space for the norm

$$\|f\|_{C^{0,1}(\mathbb{R}^N)}:=\|f\|_{L^\infty(\mathbb{R}^N)}+[f]_{C^{0,1}(\mathbb{R}^N)}.$$

The number (3.1) is called the Lipschitz constant of f. For the sake of clarity, we recall some further definitions.

Definition 3.1. — *Let $p \in [1,\infty)$ and $k \in \mathbb{N}^*$. The k-th order Sobolev space $W^{k,p}(\mathbb{R}^N)$ is defined as the closure of $C_c^\infty(\mathbb{R}^N)$ under the norm*

$$\|f\|_{W^{k,p}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)} + \bigg(\sum_{1\leqslant|\alpha|\leqslant k} \|D^\alpha f\|^p_{L^p(\mathbb{R}^N)} \bigg)^{1/p}.$$

Definition 3.2. — *The space of functions of bounded variation, denoted by $BV(\mathbb{R}^N)$, is the space of all $f \in L^1(\mathbb{R}^N)$ such that*

$$[f]_{BV(\mathbb{R}^N)} := \sup \left\{ \int_{\mathbb{R}^N} f(x) \operatorname{div} \phi(x)\, \mathrm{d}x : \phi \in C^1_c(\mathbb{R}^N), \quad \|\phi\|_{L^\infty(\mathbb{R}^N)} \leqslant 1 \right\} < \infty,$$

naturally endowed with the norm

$$\|f\|_{BV(\mathbb{R}^N)} := \|f\|_{L^1(\mathbb{R}^N)} + [f]_{BV(\mathbb{R}^N)}.$$

Definition 3.3. — *Let $M \in \mathbb{N}^*$, $s \in (0, M)$ and $p, q \in [1,\infty]$. The Besov space $B^s_{p,q}(\mathbb{R}^N)$ consists of all functions $f \in L^p(\mathbb{R}^N)$ such that*

$$[f]_{B^s_{p,q}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \|\Delta^M_h f\|^q_{L^p(\mathbb{R}^N)} \frac{\mathrm{d}h}{|h|^{N+sq}} \right)^{\frac{1}{q}} < \infty, \tag{3.2}$$

which, in the case $q = \infty$, is to be understood as

$$[f]_{B^s_{p,\infty}(\mathbb{R}^N)} := \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|\Delta^M_h f\|_{L^p(\mathbb{R}^N)}}{|h|^s} < \infty,$$

where $\Delta^M_h f$ is given by (1.3). The space $B^s_{p,q}(\mathbb{R}^N)$ is naturally endowed with the norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)} + [f]_{B^s_{p,q}(\mathbb{R}^N)}.$$

Remark 3.4. — Of course, if one denotes the semi-norm (3.2) by $[f]^{(M)}_{B^s_{p,q}(\mathbb{R}^N)}$, then for $M_1, M_2 \in \mathbb{N}^*$ with $M_1 < M_2$ and $s \in (0, M_1)$ it holds

$$[f]^{(M_1)}_{B^s_{p,q}(\mathbb{R}^N)} \sim [f]^{(M_2)}_{B^s_{p,q}(\mathbb{R}^N)},$$

(similarly when $q = \infty$), so that the definition above is indeed consistent. We refer to [136] (e.g. estimate (45) on p.99) or Lemma 6.3 for further details.

Remark 3.5. — The integral in (3.2) can be indifferently replaced by an integral over $\{|h| \leqslant \delta\}$ for any $\delta > 0$, or on the whole $\mathbb{R}^N$ since the singular part in h in the integral arises when h is close to zero, while the integral on $\{|h| > \delta\}$ can always be dominated by the L^p-norm of f.

Of special interest are the cases $q = p$, $p = \infty$ and/or $q = \infty$. The *fractional Sobolev spaces* $W^{s,p}(\mathbb{R}^N)$ (sometimes also called *Slobodeckij*, *Gagliardo*, or *Aronszajn spaces*) is defined by $W^{s,p}(\mathbb{R}^N) = B^s_{p,p}(\mathbb{R}^N)$ for $s \notin \mathbb{N}$. In this context, the semi-norm (3.2) when $s \in (0,1)$ is often referred to as the Gagliardo semi-norm.

When $q = \infty$, the space $B^s_{p,\infty}(\mathbb{R}^N)$ is called the *Nikol'skii space*. This scale gives another interesting way to measure the convergence rate of the translate of a given function to itself. It is well-known that, for any $p, q \in [1, \infty)$ and $s > 0$,

$$B^s_{p,q}(\mathbb{R}^N) \hookrightarrow B^s_{p,\infty}(\mathbb{R}^N),$$

where "$\hookrightarrow$" stands for the continuous imbedding symbol. We refer to [129, 138] for a proof of this fact. When $p = q = \infty$, then the space $B^s_{\infty,\infty}(\mathbb{R}^N)$ coincides with the Hölder-Zygmund space $\mathscr{C}^s(\mathbb{R}^N)$.

Moreover, by contrast with $W^{s,p}(\mathbb{R}^N)$ (see e.g. [72] for a simple proof of this fact) or, more generally, with the spaces $B^s_{p,q}(\mathbb{R}^N)$ with $p, q \in (1, \infty)$, neither $C^\infty_c(\mathbb{R}^N)$ nor $\mathscr{S}(\mathbb{R}^N)$ are dense in $B^s_{p,\infty}(\mathbb{R}^N)$, see e.g. [138, Theorem 2.3.2 (a), p.172]. The Nikol'skii spaces are Banach spaces but, unlike, say, $W^{s,p}(\mathbb{R}^N)$ with $1 < p < \infty$, neither reflexive [138, Remark 2, p.199] nor separable [138, Theorem 2.11.2 (d), p.237].

4 Subatomic decompositions

There exists many ways to decompose a function $f \in B^s_{p,q}(\mathbb{R}^N)$ into "building blocks". The theory of subatomic (or quarkonial) decompositions developed by Triebel in [137, 139] is one of them of particular interest because, unlike related decompositions of atomic or, say, Littlewood-Paley type, it yields a decomposition of any function $f \in B^s_{p,q}(\mathbb{R}^N)$ on a suitable system of functions which is independent of f and the resulting coefficients are linearly dependent on f. In such a framework, the search for a function amounts, roughly speaking, to seeking for a discrete sequence of numbers.

For the convenience of the reader we recall some basic definitions.

Definition 4.1. — *Let $\nu \geqslant 0$, $m \in \mathbb{Z}^N$ and $\psi \in C^\infty_c(\mathbb{R}^N)$ be a non-negative function with* $\operatorname{supp}(\psi) \subset B_{2^r}$ *for some $r \geqslant 0$ and*

$$\sum_{k \in \mathbb{Z}^N} \psi(x - k) = 1, \quad \textit{if } x \in \mathbb{R}^N.$$

Let $Q_{\nu,m}$ be the cube of sides parallel to the coordinate axis with side-length $2^{-\nu}$ and centered at $2^{-\nu}m$. Let $s \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$, $\beta \in \mathbb{N}^N$ and

$$\psi^\beta(x) = x^\beta \psi(x) := x_1^{\beta_1} \dots x_N^{\beta_N} \psi(x).$$

Then,

$$(\beta \mathrm{qu})_{\nu,m}(x) := 2^{-\nu(s-\frac{N}{p})} \psi^\beta(2^\nu x - m), \quad x \in \mathbb{R}^N,$$

is called an (s,p)-β-quark relative to $Q_{\nu,m}$.

Definition 4.2. — *Let $s > 0$, $1 \leqslant p, q \leqslant \infty$ and $(\beta\mathrm{qu})_{\nu,m}$ be (s,p)-β-quarks according to Definition 2.4. Let $\varrho > r$ where r has the same meaning as in Definition 2.4. For all $\lambda = \{\lambda^\beta_{\nu,m} \in \mathbb{C} : (\nu, m, \beta) \in \mathbb{N} \times \mathbb{Z}^N \times \mathbb{N}^N\}$ we set*

$$\|\lambda\|_{\varrho,p,q} := \sup_{\beta \in \mathbb{N}^N} 2^{\varrho|\beta|} \left(\sum_{\nu \geqslant 0} \left(\sum_{m \in \mathbb{Z}^N} |\lambda^\beta_{\nu,m}|^p \right)^{q/p} \right)^{1/q},$$

with obvious modification if $p = \infty$ and/or $q = \infty$.

We call $\mathbf{B}^s_{p,q}(\mathbb{R}^N)$ the collection of all $f \in \mathscr{S}'(\mathbb{R}^N)$ which can be represented as

$$f(x) = \sum_{\beta \in \mathbb{N}^N} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^N} \lambda^\beta_{\nu,m} (\beta\mathrm{qu})_{\nu,m}(x), \tag{4.1}$$

endowed with the norm

$$\|f\|_{\mathbf{B}^s_{p,q}(\mathbb{R}^N)} := \inf \|\lambda\|_{\varrho,p,q}, \tag{4.2}$$

where the infinimum is taken over all admissible representations (4.1).

The standard fact of subatomic decompositions states as follows

Theorem 4.3. — *Let $s > 0$ and $1 \leqslant p, q \leqslant \infty$. Then,* (4.2) *does not depend upon the choice of ϱ nor on ψ, and $\mathbf{B}^s_{p,q}(\mathbb{R}^N)$ is a Banach space which coincides with the space $B^s_{p,q}(\mathbb{R}^N)$ introduced in Definition 3.3. Moreover,*

$$\|f\|_{\mathbf{B}^s_{p,q}(\mathbb{R}^N)} \sim \|f\|_{B^s_{p,q}(\mathbb{R}^N)}.$$

We refer to [139] and references therein for a proof of this. In fact, there are *optimal subatomic coefficients*, i.e. coefficients $\lambda^\beta_{\nu,m}(f)$ realizing the infinimum in (4.2) and which can be obtained as a dual pairing of the form $\langle f, \Psi^{\beta,\varrho}_{\nu,m} \rangle_{\mathscr{S}',\mathscr{S}}$ where $(\Psi^{\beta,\varrho}_{\nu,m}) \subset \mathscr{S}(\mathbb{R}^N)$ is an appropriate sequence of functions. We refer to [139] for further details.

5 The space $N^{s,p}(\mathbb{R}^N)$

The aim of this section is twofold. On the one hand, we point out that, even though the spaces $B^s_{p,\infty}(\mathbb{R}^N)$ can be characterized as limits superior (see Proposition 5.2 below), it does not yield an equivalent norm (as it does for the Sobolev spaces $W^{1,p}(\mathbb{R}^N)$ with $p > 1$, see e.g. Lemma 7.1). As will become clear in the next section, this is the reason why $B^s_{p,\infty}(\mathbb{R}^N)$ is more conveniently described as the supremum of (2.1) rather than as its limit superior. On the other hand, we provide some preliminary results towards Corollary 2.18. For simplicity, we consider only first order differences $\Delta^1_h f = \tau_h f - f$ but all the results of this section also hold for higher order differences.

For the sake of convenience, we define a "new" function space which, in fact, is merely another way to look at the Nikol'skii space $B^s_{p,\infty}(\mathbb{R}^N)$ as shown hereafter.

Definition 5.1. — *Let $s \in (0,1)$ and $p \in [1,\infty]$. Then, the space $N^{s,p}(\mathbb{R}^N)$ consists of all functions $f \in L^p(\mathbb{R}^N)$ such that*

$$[f]_{N^{s,p}(\mathbb{R}^N)} := \limsup_{|h|\to 0} \frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} < \infty.$$

It is endowed with the following norm:

$$\|f\|_{N^{s,p}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)} + [f]_{N^{s,p}(\mathbb{R}^N)}.$$

In addition, we also define

$$N^{s,p}_0(\mathbb{R}^N) := \Big\{ f \in N^{s,p}(\mathbb{R}^N) : [f]_{N^{s,p}(\mathbb{R}^N)} = 0 \Big\}.$$

As expected, we have the

Proposition 5.2. — *Let $s \in (0,1)$ and $p \in [1,\infty]$. Then,*

$$B^s_{p,\infty}(\mathbb{R}^N) = N^{s,p}(\mathbb{R}^N).$$

Remark 5.3. — The equality here holds in the sense of sets: the topology of both are not precisely equivalent as shown below. In fact, "$[\cdot]_{N^{s,p}(\mathbb{R}^N)}$" is a quite crude way to characterize the Nikol'skii space. For these reasons (and in order not to mix with both topologies) we shall write $B^s_{p,\infty}(\mathbb{R}^N) = (B^s_{p,\infty}(\mathbb{R}^N), \|\cdot\|_{B^s_{p,\infty}(\mathbb{R}^N)})$ and $N^{s,p}(\mathbb{R}^N) = (B^s_{p,\infty}(\mathbb{R}^N), \|\cdot\|_{N^{s,p}(\mathbb{R}^N)})$.

Proof. — Let $f \in B^s_{p,\infty}(\mathbb{R}^N)$. Then, for all $\delta > 0$, we have

$$[f]_{B^s_{p,\infty}(\mathbb{R}^N)} := \sup_{h\in\mathbb{R}^N\setminus\{0\}} \frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \geqslant \sup_{0<|h|<\delta} \frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s}.$$

Letting $\delta \to 0^+$, we get

$$[f]_{B^s_{p,\infty}(\mathbb{R}^N)} \geqslant \limsup_{|h|\to 0} \frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} =: [f]_{N^{s,p}(\mathbb{R}^N)}, \tag{5.1}$$

and so $f \in N^{s,p}(\mathbb{R}^N)$. Conversely, let $f \in N^{s,p}(\mathbb{R}^N)$. Then, for all $\eta > 0$ there is a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have

$$\left| \sup_{0<|h|<\delta} \frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} - [f]_{N^{s,p}(\mathbb{R}^N)} \right| < \eta.$$

Now fix such η and δ. By the triangle inequality we obtain

$$\sup_{0<|h|<\delta} \frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} < \eta + [f]_{N^{s,p}(\mathbb{R}^N)} < \infty.$$

On the other hand,

$$\sup_{\delta\leqslant|h|} \frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant \frac{2}{\delta^s}\|f\|_{L^p(\mathbb{R}^N)} < \infty.$$

Therefore, $f \in B^s_{p,\infty}(\mathbb{R}^N)$. □

Proposition 5.4. — *Let $s \in (0,1)$ and $p \in (1, \infty]$. Then,*

$$W^{1,p}(\mathbb{R}^N) \subset N^{s,p}_0(\mathbb{R}^N) \quad \text{and} \quad BV(\mathbb{R}^N) \subset N^{s,1}_0(\mathbb{R}^N).$$

Proof. — First, let $f \in W^{1,p}(\mathbb{R}^N)$ (resp. $f \in BV(\mathbb{R}^N)$ if $p = 1$). Then,

$$\frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant |h|^{1-s}\|\nabla f\|_{L^p(\mathbb{R}^N)}, \quad \forall h \in \mathbb{R}^N.$$

Taking the limit superior as $|h| \to 0$ gives $f \in N^{s,p}_0(\mathbb{R}^N)$. □

Proposition 5.5. — *Let $s \in (0,1)$, $p \in [1,\infty)$ and $\mathring{N}^{s,p}(\mathbb{R}^N)$ denote the closure of $C^\infty_c(\mathbb{R}^N)$ in $N^{s,p}(\mathbb{R}^N)$. Then,*

$$\mathring{N}^{s,p}(\mathbb{R}^N) = N^{s,p}_0(\mathbb{R}^N).$$

In particular, $N^{s,p}_0(\mathbb{R}^N)$ is a closed subspace of $N^{s,p}(\mathbb{R}^N)$.

Proof. — "$\subset$": By definition, $C_c^\infty(\mathbb{R}^N)$ is dense in $\mathring{N}^{s,p}(\mathbb{R}^N)$, whence the inclusion $\mathring{N}^{s,p}(\mathbb{R}^N) \subset N_0^{s,p}(\mathbb{R}^N)$ is straightforward.

"$\supset$": Let $f \in N_0^{s,p}(\mathbb{R}^N)$ and let $(f_n)_{n\geqslant 0} \subset C_c^\infty(\mathbb{R}^N)$ be such that

$$\|f - f_n\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

Then, clearly,

$$\begin{aligned} \|f - f_n\|_{N^{s,p}(\mathbb{R}^N)} &:= \|f - f_n\|_{L^p(\mathbb{R}^N)} + [f - f_n]_{N^{s,p}(\mathbb{R}^N)} \\ &\leqslant \|f - f_n\|_{L^p(\mathbb{R}^N)} + [f]_{N^{s,p}(\mathbb{R}^N)} + [f_n]_{N^{s,p}(\mathbb{R}^N)} \\ &= \|f - f_n\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Whence, $f \in \mathring{N}^{s,p}(\mathbb{R}^N)$. Moreover, the map

$$\Theta : f \in N^{s,p}(\mathbb{R}^N) \mapsto [f]_{N^{s,p}(\mathbb{R}^N)}$$

is continuous. Therefore $N_0^{s,p}(\mathbb{R}^N) = \Theta^{-1}(\{0\})$ is closed in $N^{s,p}(\mathbb{R}^N)$. □

Proposition 5.6. — *Let $s \in (0,1)$, $p \in [1,\infty)$ and $\mathring{B}^s_{p,\infty}(\mathbb{R}^N)$ (resp. $\mathring{N}^{s,p}(\mathbb{R}^N)$) denote the closure of $C_c^\infty(\mathbb{R}^N)$ in $B^s_{p,\infty}(\mathbb{R}^N)$ (resp. $N^{s,p}(\mathbb{R}^N)$). Then,*

$$f \in \mathring{N}^{s,p}(\mathbb{R}^N) \ \text{ if, and only if, } \ f \in \mathring{B}^s_{p,\infty}(\mathbb{R}^N).$$

Proof. — Let $f \in \mathring{N}^{s,p}(\mathbb{R}^N)$ and $(f_n)_{n\geqslant 0} \subset C_c^\infty(\mathbb{R}^N)$ be such that

$$f_n \to f \quad \text{in } N^{s,p}(\mathbb{R}^N) \text{ as } n \to \infty.$$

Thus, for all $\eta > 0$ there exists $n_0 = n_0(\eta) \geqslant 0$ and $\delta_0 = \delta_0(\eta) > 0$ such that

$$n \geqslant n_0, \quad \delta \in (0, \delta_0) \quad \Longrightarrow \quad \sup_{|h|<\delta} \frac{\|\Delta_h^1(f - f_n)\|_{L^p(\mathbb{R}^N)}}{|h|^s} < \eta.$$

Now, fix such η, δ and n_0. On the other hand, for all $\eta > 0$ and all $\delta > 0$ there is a $n_1 = n_1(\eta, \delta) \geqslant 0$ such that

$$n \geqslant n_1 \quad \Longrightarrow \quad \sup_{|h|\geqslant\delta} \frac{\|\Delta_h^1(f - f_n)\|_{L^p(\mathbb{R}^N)}}{|h|^s} < \eta.$$

Indeed, this is because

$$\sup_{|h|\geqslant\delta} \frac{\|\Delta_h^1(f - f_n)\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant \frac{2}{\delta^s} \|f - f_n\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

Therefore, for all $n \geqslant \max\{n_0, n_1\}$,

$$\sup_{h \neq 0} \frac{\|\Delta_h^1(f - f_n)\|_{L^p(\mathbb{R}^N)}}{|h|^s} < \eta.$$

Summing up, we find that, for all $\eta > 0$, there exists $M \geqslant 0$ such that

$$n \geqslant M \quad \Longrightarrow \quad [f - f_n]_{B^s_{p,\infty}(\mathbb{R}^N)} < \eta.$$

Thus, $f \in \mathring{B}^s_{p,\infty}(\mathbb{R}^N)$.

Conversely, let $f \in \mathring{B}^s_{p,\infty}(\mathbb{R}^N)$ and $(f_n)_{n \geqslant 0} \subset C_c^\infty(\mathbb{R}^N)$ be such that $f_n \to f$ in $B^s_{p,\infty}(\mathbb{R}^N)$. Using (5.1) we find

$$\begin{aligned} [f]_{N^{s,p}(\mathbb{R}^N)} &\leqslant [f - f_n]_{N^{s,p}(\mathbb{R}^N)} + [f_n]_{N^{s,p}(\mathbb{R}^N)} \\ &= [f - f_n]_{N^{s,p}(\mathbb{R}^N)} \\ &\leqslant [f - f_n]_{B^s_{p,\infty}(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $f \in \mathring{N}^{s,p}(\mathbb{R}^N)$. □

6 Characterization of Besov-Nikol'skii spaces

6.1 Preliminary

For the sake of clarity we shall introduce the following short notation

$$\mathscr{D}_\omega(\rho_\varepsilon, f) := \int_{\mathbb{R}^N} \rho_\varepsilon(h)\ \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \mathrm{d}h.$$

First, an easy observation.

Proposition 6.1. — *Let $M \in \mathbb{N}^*$, $s > 0$, $p \in [1, \infty]$ and $(\rho_\varepsilon)_{\varepsilon > 0}$ be a sequence of mollifiers. Assume $\omega \in C^+_{\text{inc}}$. Then,*

$$\limsup_{\varepsilon \to 0^+} \mathscr{D}_\omega(\rho_\varepsilon, f) \leqslant\ \omega\left(\limsup_{|h| \to 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right),$$

and

$$\sup_{\varepsilon > 0} \mathscr{D}_\omega(\rho_\varepsilon, f) \leqslant\ \omega\left(\sup_{h \neq 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right).$$

Proof. — Let $\eta > 0$ be any fixed number. Then, we have

$$\mathscr{D}_\omega(\rho_\varepsilon, f) = \left(\int_{0 \leqslant |h| \leqslant \eta} + \int_{|h| > \eta} \right) \rho_\varepsilon(h)\ \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right) \mathrm{d}h.$$

On the one hand,

$$\begin{aligned} \int_{0 \leqslant |h| \leqslant \eta} \rho_\varepsilon(h)\ \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right) \mathrm{d}h & \\ &\leqslant \sup_{0 \leqslant |h| \leqslant \eta} \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right) \int_{0 \leqslant |h| \leqslant \eta} \rho_\varepsilon(h) \mathrm{d}h \\ &\leqslant \sup_{0 \leqslant |h| \leqslant \eta} \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right). \end{aligned}$$

On the other hand, since ω is non-decreasing

$$\begin{aligned} \int_{|h| > \eta} \rho_\varepsilon(h)\ \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right) \mathrm{d}h &\leqslant \omega\left(\frac{2^M \|f\|_{L^p(\mathbb{R}^N)}}{\eta^s} \right) \int_{|h| > \eta} \rho_\varepsilon(h) \mathrm{d}h \\ &\longrightarrow 0 \quad \text{as } \varepsilon \to 0^+. \end{aligned}$$

Therefore,

$$\limsup_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\ \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right) \mathrm{d}h \leqslant \sup_{0 \leqslant |h| \leqslant \eta} \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right).$$

Taking now the limit as $\eta \to 0^+$ and using $\omega \in C^+_{\mathrm{inc}}$ we obtain

$$\limsup_{\varepsilon \to 0^+} \mathscr{D}_\omega(\rho_\varepsilon, f) \leqslant \omega\left(\limsup_{|h| \to 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \right).$$

The remaining inequality follows by a direct application of Hölder's inequality. $\square$

Here is another estimate we shall need.

Lemma 6.2. — *Let $p \in [1, \infty]$, $M \in \mathbb{N}^*$, $h_1, h_2 \in \mathbb{R}^N$ and $h = h_1 + h_2$. Then,*

$$\|\Delta_h^{2M} f\|_{L^p(\mathbb{R}^N)} \lesssim \|\Delta_{h_1}^M f\|_{L^p(\mathbb{R}^N)} + \|\Delta_{h_2}^M f\|_{L^p(\mathbb{R}^N)},$$

for all $f \in L^p(\mathbb{R}^N)$.

This is essentially covered by [136, Estimate (16), p.112] but, for the sake of completeness, we choose to provide the details.

Proof. — Let $f \in \mathscr{S}(\mathbb{R}^N)$. Since translations $\tau_h f$ have Fourier transform $e^{ih\cdot\xi}\widehat{f}$, the Fourier transform of $\Delta_h^M f$ writes

$$\mathscr{F}[\Delta_h^M f](\xi) = \widehat{f}(\xi) \sum_{j=0}^{M} \binom{M}{j} (-1)^{M-j} (e^{ih\cdot\xi})^j.$$

And so, by applying the binomial formula and taking the inverse Fourier transform of the result one gets

$$\Delta_h^M f = \mathscr{F}^{-1}[(e^{ih\cdot\xi} - 1)^M \widehat{f}].$$

Now let $h_1, h_2 \in \mathbb{R}^N$ and $h = h_1 + h_2$. Notice that we have

$$e^{ih\cdot\xi} - 1 = e^{ih_1\cdot\xi}(e^{ih_2\cdot\xi} - 1) + e^{ih_1\cdot\xi} - 1.$$

Let $P \in \mathbb{C}[X, Y]$ be the polynomial defined by

$$P(X, Y) = (X(Y-1) + (X-1))^{2M}.$$

By the binomial formula one may find $Q_1, Q_2 \in \mathbb{C}[X, Y]$ such that

$$P(X, Y) = (X-1)^M Q_1(X, Y) + X^M (Y-1)^M Q_2(X, Y).$$

This holds for any $X, Y \in \mathbb{C}$. In particular

$$(e^{ih\cdot\xi} - 1)^{2M} = (e^{ih_1\cdot\xi} - 1)^M Q_1(e^{ih_1\cdot\xi}, e^{ih_2\cdot\xi}) + e^{iMh_1\cdot\xi}(e^{ih_2\cdot\xi} - 1)^M Q_2(e^{ih_1\cdot\xi}, e^{ih_2\cdot\xi}).$$

Multiplying this equality by $\widehat{f}(\xi)$ and taking the inverse Fourier transform of the result, we obtain:

$$\begin{aligned}\Delta_h^{2M} f = \mathscr{F}^{-1}&\left[\sum_{k,\ell=0}^{M} \alpha_{k,\ell}(e^{ih_1\cdot\xi} - 1)^M \mathscr{F}\left[f(\cdot + kh_1 + \ell h_2)\right]\right] \\ &+ \mathscr{F}^{-1}\left[\sum_{k,\ell=0}^{M} \beta_{k,\ell} e^{iMh_1\cdot\xi}(e^{ih_2\cdot\xi} - 1)^M \mathscr{F}\left[f(\cdot + kh_1 + \ell h_2)\right]\right],\end{aligned}$$

where $\alpha_{k,\ell}$ and $\beta_{k,\ell}$ are the respective coefficients of Q_1 and Q_2. Otherwise said,

$$\Delta_h^{2M} f = \sum_{k,\ell=0}^{M} \alpha_{k,\ell} \Delta_{h_1}^M f\big(\cdot + kh_1 + \ell h_2\big) + \sum_{k,\ell=0}^{M} \beta_{k,\ell} \Delta_{h_2}^M f\big(\cdot + (k+M)h_1 + \ell h_2\big).$$

We therefore obtain that, for each $f \in \mathscr{S}(\mathbb{R}^N)$

$$\|\Delta_h^{2M} f\|_{L^p(\mathbb{R}^N)} \leqslant C\left(\|\Delta_{h_1}^M f\|_{L^p(\mathbb{R}^N)} + \|\Delta_{h_2}^M f\|_{L^p(\mathbb{R}^N)}\right),$$

for some constant $C > 0$ depending only on M, Q_1 and Q_2. Since $\mathscr{S}(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for $p < \infty$ the result follows for every $f \in L^p(\mathbb{R}^N)$. When $p = \infty$, the above still holds in the $\mathscr{S}'$ sense and, thus, extends to $L^\infty(\mathbb{R}^N)$ as well. $\square$

Also, we recall the following well-known fact.

Lemma 6.3. — *Let $M \in \mathbb{N}^*$, $s \in (0, M)$ and $f \in L^p(\mathbb{R}^N)$. Then,*

$$\sup_{h \neq 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant C(s, M) \sup_{h \neq 0} \frac{\|\Delta_h^{2M} f\|_{L^p(\mathbb{R}^N)}}{|h|^s}, \tag{6.1}$$

for some constant $C(s, M) > 0$ depending only on s and M. Similarly,

$$\limsup_{|h| \to 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant C(s, M) \limsup_{|h| \to 0} \frac{\|\Delta_h^{2M} f\|_{L^p(\mathbb{R}^N)}}{|h|^s}. \tag{6.2}$$

This is a consequence of [136, Estimate (45), p.99], but the proof being very short we chose to provide all the details.

Proof. — Let $f \in L^p(\mathbb{R}^N)$ and $P \in \mathbb{C}[X]$ be the unique polynomial such that

$$P(X)(X - 1) = 1 - \left(\frac{X + 1}{2} \right)^M. \tag{6.3}$$

Note that P exists because $X - 1$ is a divisor of the right-hand side of (6.3). In particular, we have that

$$(X - 1)^M = \frac{1}{2^M} (X^2 - 1)^M + (X - 1)^{M+1} P(X).$$

Hence, for every $h, \xi \in \mathbb{R}^N$ we have

$$(e^{ih \cdot \xi} - 1)^M = \frac{1}{2^M} (e^{i2h \cdot \xi} - 1)^M + (e^{ih \cdot \xi} - 1)^{M+1} P(e^{ih \cdot \xi}).$$

Whence, reasoning as in Lemma 6.2, we obtain

$$\Delta_h^M f(x) = \frac{1}{2^M} \Delta_{2h}^M f(x) + \Delta_h^{M+1} \left(\sum_{\ell \in L} a_\ell f(x + h\ell) \right),$$

for some finite set of indices $L \subset \mathbb{N}$ and coefficients a_ℓ depending on P. Thus, for every $s \in (0, M)$, $h \neq 0$ and $f \in L^p(\mathbb{R}^N)$ it holds,

$$\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant \frac{1}{2^{M-s}} \frac{\|\Delta_{2h}^M f\|_{L^p(\mathbb{R}^N)}}{|2h|^s} + C \frac{\|\Delta_h^{M+1} f\|_{L^p(\mathbb{R}^N)}}{|h|^s}.$$

We obtain that

$$\left(1 - \frac{1}{2^{M-s}} \right) \sup_{h \neq 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant C \sup_{h \neq 0} \frac{\|\Delta_h^{M+1} f\|_{L^p(\mathbb{R}^N)}}{|h|^s}.$$

Therefore (6.1) follows by induction. The proof of (6.2) is similar. □

6.2 Proof of Theorem 2.3

Let $M \in \mathbb{N}^*$, $s > 0$, $p \in [1, \infty]$, $\omega \in C^+_{\mathrm{inc}}$ and $(\rho_\varepsilon)_{\varepsilon>0}$ be as in Theorem 2.3. Here again, we will make use of the short notation

$$\mathscr{D}_\omega(\rho_\varepsilon, f) := \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \mathrm{d}h.$$

In addition, we call $\mathfrak{M}(\mathbb{R}^N)$ the set of mollifiers $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ satisfying (1.8) for some $\rho \in L^1(\mathbb{R}^N)$ such that there exists a number $\delta > 0$ and a nonnegative, nondecreasing, radial function $\Psi \in C(\mathbb{R}^N)$ with

$$\rho(h) \geqslant \Psi(h) \quad \text{for a.e. } h \in B_\delta \text{ and } \int_{B_{\delta/4}} \Psi > 0. \tag{6.4}$$

Note that, by Proposition 6.1, we only need to establish a one-sided inequality.

We begin with a few preliminary facts (Claim A and Claim B) showing that the proof of Theorem 2.3 reduces to the case where $(\rho_\varepsilon)_{\varepsilon>0} \in \mathfrak{M}(\mathbb{R}^N)$.

Claim A. — *It is enough to establish Theorem 2.3 for radial ρ's such that*

$$\operatorname*{ess\,inf}_{\mathscr{A}} \rho > 0, \tag{6.5}$$

for some annulus $\mathscr{A} \subset \mathbb{R}^N$ centered at zero.

Proof of Claim A. — Let $\rho \in L^1(\mathbb{R}^N)$ be a nonnegative radial function with unit mass. Then, there is a nonnegative function $\widetilde{\rho} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ with $\rho(z) = \widetilde{\rho}(|z|)$. In particular, we may find some $0 < c_1 < c_2$ such that

$$\int_{c_1}^{c_2} \widetilde{\rho}(\theta)\mathrm{d}\theta > 0.$$

Let $0 < \theta_0 < 1$ be such that $c_1 > c_2\theta_0$ and let ρ^* be the radial function given by

$$\rho^*(z) := C \fint_{\theta_0}^1 \rho(\theta z)\mathrm{d}\theta = C \fint_{\theta_0}^1 \widetilde{\rho}(\theta|z|)\mathrm{d}\theta \quad \text{for } z \in \mathbb{R}^N,$$

where $C > 0$ is given by

$$C := (1-\theta_0)\left(\int_{\theta_0}^1 \frac{\mathrm{d}\theta}{\theta^N}\right)^{-1}.$$

Notice that, by the Fubini theorem, $\rho^* \in L^1(\mathbb{R}^N)$ and ρ^* has unit mass. Indeed, this is because

$$\|\rho^*\|_{L^1(\mathbb{R}^N)} = \frac{C}{1-\theta_0}\int_{\theta_0}^1 \int_{\mathbb{R}^N} \rho(\theta z)\mathrm{d}z\mathrm{d}\theta = \frac{C}{1-\theta_0}\int_{\theta_0}^1 \frac{\mathrm{d}\theta}{\theta^N} = 1.$$

Furthermore, one easily checks that ρ^* satisfies (6.5). Indeed, we have

$$\operatorname*{ess\,inf}_{|z|\in\left[c_2,\frac{c_1}{\theta_0}\right]} \rho^*(z) = \frac{C}{1-\theta_0} \operatorname*{ess\,inf}_{|z|\in\left[c_2,\frac{c_1}{\theta_0}\right]} \int_{\theta_0|z|}^{|z|} \widetilde{\rho}(\theta)\frac{\mathrm{d}\theta}{|z|} \geqslant \frac{C\theta_0}{c_1(1-\theta_0)} \int_{c_1}^{c_2} \widetilde{\rho}(\theta)\mathrm{d}\theta > 0.$$

On the other hand, we have

$$\rho_\varepsilon(\theta\,\cdot) = \theta^{-N}\rho_{\varepsilon/\theta} \leqslant \theta_0^{-N}\rho_{\varepsilon/\theta} \quad \text{for any } \theta\in[\theta_0,1].$$

Hence,

$$\frac{1}{C}\,\mathscr{D}_\omega(\rho_\varepsilon^*, f) = \fint_{\theta_0}^1 \mathscr{D}_\omega(\rho_\varepsilon(\theta\,\cdot), f)\mathrm{d}\theta \leqslant \theta_0^{-N} \sup_{\theta_0\leqslant\theta\leqslant 1} \mathscr{D}_\omega(\rho_{\varepsilon/\theta}, f). \tag{6.6}$$

Assuming that Theorem 2.3 holds for mollifiers with ρ satisfying (6.5), we finally obtain

$$\omega\left([f]_{B^s_{p,\infty}(\mathbb{R}^N)}\right) \lesssim \sup_{\varepsilon>0} \mathscr{D}_\omega(\rho_\varepsilon, f).$$

Thus, the claim follows. $\square$

Claim B. — *It is enough to establish Theorem 2.3 for* $(\rho_\varepsilon)_{\varepsilon>0}\in\mathfrak{M}(\mathbb{R}^N)$.

Proof of Claim B. — Let $\rho\in L^1(\mathbb{R}^N)$ be a nonnegative radial function with unit mass. Then, there is a nonnegative function $\widetilde{\rho}\in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ with $\rho(z)=\widetilde{\rho}(|z|)$. On account of Claim A, we may assume that there are some $0<r_1<r_2$ and some $\alpha>0$ with

$$\widetilde{\rho}(t) \geqslant \alpha\mathbb{1}_{(r_1,r_2)}(t) =: \Psi(t) \quad \text{for a.e. } t\geqslant 0.$$

If $r_1<\frac{r_2}{4}$, then $(\rho_\varepsilon)_{\varepsilon>0}\in\mathfrak{M}(\mathbb{R}^N)$ and the claim is trivial. Hence, we may assume that $r_1\geqslant\frac{r_2}{4}$. To show that the latter case reduces to the former, we simply clip together rescaled copies of Ψ as follows. For any $j\geqslant 0$, define

$$\theta_j := \left(\frac{r_1}{r_2}\right)^j \quad\text{and}\quad \Psi_{\theta_j}(t) := \theta_j^{-N}\Psi\left(\frac{t}{\theta_j}\right).$$

Observe that, by construction, $\theta_j\to 0$ as $j\to\infty$ and

$$0<\cdots<\theta_{j+1}r_1<\theta_{j+1}r_2=\theta_j r_1<\theta_j r_2<\cdots<\theta_1 r_2 = r_1 < r_2.$$

Thus, the supports of the Ψ_{θ_j}'s are mutually disjoint and they form a countable partition of $[0,r_2]$. Now take an integer $k\in\mathbb{N}$ such that

$$k > \frac{\ln\left(\frac{1}{5}\right)}{\ln\left(\frac{r_1}{r_2}\right)}.$$

By construction, this guarantees that $\theta_k < \frac{1}{5}$ and, in turn, that

$$\operatorname{supp}(\Psi_{\theta_k}) \subset \left[0, \frac{r_2}{5}\right].$$

In particular, we have

$$\left[\frac{r_2}{5}, r_2\right) \subset \bigcup_{j=0}^{k} \operatorname{supp}(\Psi_{\theta_j}).$$

Fix such a $k \in \mathbb{N}$ and set $J = [\![0, k]\!]$. Then, the function

$$\eta^*(t) := \sum_{j \in J} \Psi_{\theta_j}(t), \quad \text{for } t \geqslant 0,$$

is bounded and

$$\left[\frac{r_2}{5}, r_2\right) \subset \operatorname{supp}(\eta^*) \subset [0, r_2].$$

Moreover, η^* satisfies the following monotonicity property:

$$\eta^*(t_1) \geqslant \eta^*(t_2) \geqslant \alpha \quad \text{whenever} \quad \frac{r_2}{5} < t_1 < t_2 < r_2.$$

Thus, there is a nondecreasing function $\Phi^* \in C(\mathbb{R}_+)$ with

$$\eta^* \geqslant \Phi^* \quad \text{a.e. in } [0, r_2] \text{ and } \int_0^{r_2/4} \Phi^*(t)\, t^{N-1} \mathrm{d}t > 0. \tag{6.7}$$

Indeed, it suffices to take e.g.

$$\Phi^*(t) := \frac{5\alpha}{4r_2}\left(t - \frac{r_2}{5}\right) \mathbb{1}_{(\frac{r_2}{5}, \infty)}(t).$$

See Figure 1.1 for a visual evidence. Now, set

$$\Phi(x) := \frac{1}{c}\, \Phi^*(|x|) \quad \text{and} \quad \eta(x) := \frac{1}{c}\, \eta^*(|x|) \quad \text{where} \quad c := \int_{B_{r_2}} \eta^*(|x|) \mathrm{d}x.$$

By construction, $\eta \in L^1(\mathbb{R}^N)$ and η has unit mass. Moreover, by (6.7), we have

$$\eta \geqslant \Phi \quad \text{a.e. in } B_{r_2} \text{ and } \int_{B_{r_2/4}} \Phi > 0.$$

Whence, $(\eta_\varepsilon)_{\varepsilon>0} \in \mathfrak{M}(\mathbb{R}^N)$. On the other hand,

$$c\, \mathscr{D}_\omega(\eta_\varepsilon, f) = \sum_{j \in J} \mathscr{D}_\omega\big(\Psi_{\theta_j \varepsilon}(|\cdot|), f\big) \leqslant \sum_{j \in J} \mathscr{D}_\omega(\rho_{\theta_j \varepsilon}, f). \tag{6.8}$$

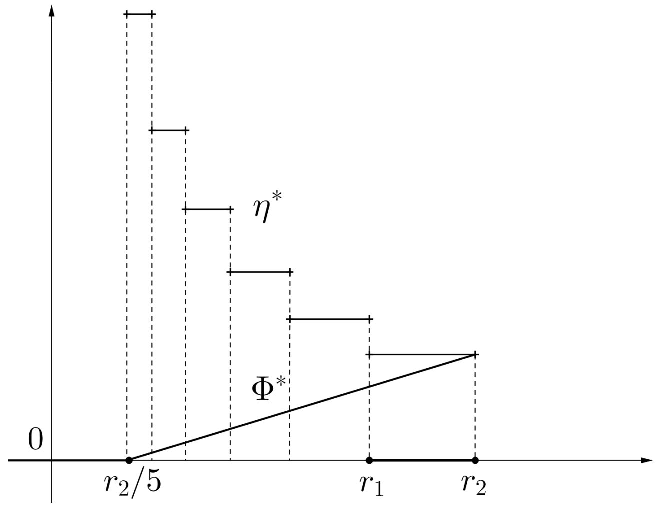

Figure 1.1. Construction of η^* and Φ^*.

Hence, one obtains

$$\sup_{\varepsilon>0} \mathscr{D}_\omega(\eta_\varepsilon, f) \leqslant \frac{\#J}{c} \sup_{\varepsilon>0} \mathscr{D}_\omega(\rho_\varepsilon, f).$$

Assuming that Theorem 2.3 holds for mollifiers belonging to $\mathfrak{M}(\mathbb{R}^N)$, we finally obtain

$$\omega\left([f]_{B^s_{p,\infty}(\mathbb{R}^N)}\right) \lesssim \sup_{\varepsilon>0} \mathscr{D}_\omega(\rho_\varepsilon, f).$$

Thus, the claim follows. $\square$

Remark 6.4. — By (6.6) and (6.8) we also have that

$$\omega\left(\limsup_{|h|\to 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \lesssim \limsup_{\varepsilon\to 0^+} \mathscr{D}_\omega(\rho_\varepsilon, f),$$

holds for any radial $(\rho_\varepsilon)_{\varepsilon>0}$ satisfying (1.1) and (1.8) whenever it holds for any $(\rho_\varepsilon)_{\varepsilon>0}$ belonging to $\mathfrak{M}(\mathbb{R}^N)$.

We may now complete the proof of Theorem 2.3.

Step 1: case $M = 1$ and $s \in (0,1)$.

Let $p \in [1,\infty]$, $(\rho_\varepsilon)_{\varepsilon>0} \in \mathfrak{M}(\mathbb{R}^N)$, $\omega \in C^+_{\mathrm{inc}}$ and $f \in L^p(\mathbb{R}^N)$ satisfying (2.1). Let $h \in \mathbb{R}^N$ (to be fixed later) and let $z \in \mathbb{R}^N$. Then, we have

$$\tau_z f - f = \tau_h f - f + \tau_h(\tau_{z-h} f - f). \tag{6.9}$$

This implies

$$\|\tau_h f - f\|_{L^p(\mathbb{R}^N)} \leqslant \|\tau_z f - f\|_{L^p(\mathbb{R}^N)} + \|\tau_{z-h} f - f\|_{L^p(\mathbb{R}^N)}. \tag{6.10}$$

Now, choose $z \in B_{|h|}(h)$. Then, since z and $z-h$ belong to $B_{2|h|}$, it comes

$$\frac{1}{2^s}\frac{\|\tau_h f - f\|_{L^p(\mathbb{R}^N)}}{|h|^s} \leqslant \frac{\|\tau_z f - f\|_{L^p(\mathbb{R}^N)}}{|z|^s} + \frac{\|\tau_{z-h} f - f\|_{L^p(\mathbb{R}^N)}}{|z-h|^s}. \tag{6.11}$$

Since ω is roughly subadditive, there exists a constant $A_\omega > 0$ depending only on ω and such that, for every $x, y \in \mathbb{R}_+$,

$$\omega(x+y) \leqslant A_\omega \left\{\omega(x) + \omega(y)\right\}. \tag{6.12}$$

Remark 6.5. — Note that (6.12) implies $\omega(2x) \leqslant 2A_\omega \omega(x)$ and, since $\omega \in C^+_{\text{inc}}$, it is increasing, thus $\omega(2^s x) \leqslant 2A_\omega\omega(x)$ for $s \leqslant 1$. Similarly, when $s \leqslant M \in \mathbb{N}^*$, one has $\omega(2^s x) \leqslant (2A_\omega)^M \omega(x)$.

From (1.8), Remark 6.5 and since $s \leqslant 1$, using the short notation $\Delta^1_h f = \tau_h f - f$ we have

$$\begin{aligned}
\omega\left(\frac{\|\Delta^1_h f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) &\leqslant \omega\left(2^s \frac{\|\Delta^1_z f\|_{L^p(\mathbb{R}^N)}}{|z|^s} + 2^s \frac{\|\Delta^1_{z-h} f\|_{L^p(\mathbb{R}^N)}}{|z-h|^s}\right) \\
&\leqslant A_\omega \left\{\omega\left(2^s \frac{\|\Delta^1_z f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) + \omega\left(2^s \frac{\|\Delta^1_{z-h} f\|_{L^p(\mathbb{R}^N)}}{|z-h|^s}\right)\right\} \\
&\leqslant 2A^2_\omega \left\{\omega\left(\frac{\|\Delta^1_z f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) + \omega\left(\frac{\|\Delta^1_{z-h} f\|_{L^p(\mathbb{R}^N)}}{|z-h|^s}\right)\right\}.
\end{aligned} \tag{6.13}$$

Using $(\rho_\varepsilon)_{\varepsilon>0} \in \mathfrak{M}(\mathbb{R}^N)$ we know there exist a radially nondecreasing $\Psi \in C(\mathbb{R}^N)$ and a number $\delta > 0$ such that

$$\rho_\varepsilon(z) \geqslant \Psi_\varepsilon(|z|) \quad \text{for a.e. } z \in B_{\varepsilon\delta} \text{ and all } \varepsilon > 0. \tag{6.14}$$

As seen in Figure 1.2, we clearly have

$$\Psi_\varepsilon(|z-h|) \leqslant \Psi_\varepsilon(|z|) \quad \text{for all } h \in B_{\varepsilon\delta/2},\ z \in B_{|h|/2}(h) \text{ and } \varepsilon > 0. \tag{6.15}$$

Let $h \in B_{\varepsilon\delta/2}$. Multiplying (6.13) by $\Psi_\varepsilon(|z-h|)$ and using (6.14)-(6.15) we obtain

$$\begin{aligned}
&\Psi_\varepsilon(|z-h|)\ \omega\left(\frac{\|\Delta^1_h f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \\
&\qquad \leqslant 2A^2_\omega \left\{\rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta^1_z f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) + \rho_\varepsilon(z-h)\ \omega\left(\frac{\|\Delta^1_{z-h} f\|_{L^p(\mathbb{R}^N)}}{|z-h|^s}\right)\right\},
\end{aligned}$$

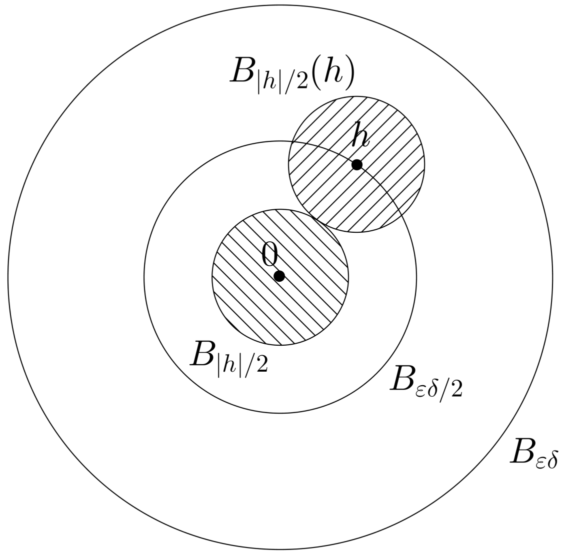

Figure 1.2. Spatial conditions on h and z.

and this holds for all $h \in B_{\varepsilon\delta/2}$ and a.e. $z \in B_{|h|/2}(h)$. So, taking $|h| = \delta\varepsilon/2$ and integrating over $z \in B_{|h|/2}(h)$, yields:

$$\begin{aligned} C(\Psi,\delta)\ \omega\left(\frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) &\leqslant 2A_\omega^2 \int_{B_{|h|/2}(h)} \rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta_z^1 f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) \mathrm{d}z \\ &\quad + 2A_\omega^2 \int_{B_{|h|/2}(h)} \rho_\varepsilon(z-h)\ \omega\left(\frac{\|\Delta_{z-h}^1 f\|_{L^p(\mathbb{R}^N)}}{|z-h|^s}\right) \mathrm{d}z \\ &\leqslant 4A_\omega^2 \int_{\mathbb{R}^N} \rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta_z^1 f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) \mathrm{d}z, \end{aligned}$$

where

$$C(\Psi,\delta) := \int_{B_{|h|/2}(h)} \Psi_\varepsilon(|z-h|)\mathrm{d}z = \int_{B_{\delta/4}} \Psi(|z|)\mathrm{d}z > 0.$$

Whence,

$$\omega\left(\frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \leqslant \frac{4A_\omega^2}{C(\Psi,\delta)} \int_{\mathbb{R}^N} \rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta_z^1 f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) \mathrm{d}z. \tag{6.16}$$

Passing to the limit superior as $|h| \to 0$ in (6.16) it follows

$$\omega\big([f]_{N^{s,p}(\mathbb{R}^N)}\big) \leqslant \frac{4A_\omega^2}{C(\Psi,\delta)} \limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta_z^1 f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) \mathrm{d}z,$$

where we used the continuity of ω. This, together with Proposition 6.1 yields

$$\omega\big([f]_{N^{s,p}(\mathbb{R}^N)}\big) \sim \limsup_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta_z^1 f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) \mathrm{d}z. \tag{6.17}$$

Similarly, taking the supremum over $h \neq 0$ in (6.16), we obtain

$$\omega\left([f]_{B^s_{p,\infty}(\mathbb{R}^N)}\right) \sim \sup_{\varepsilon>0} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\ \omega\left(\frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \mathrm{d}h.$$

Remark 6.6. — Estimate (6.17) together with Proposition 5.6 and Remark 6.4 prove Corollary 2.18 for $1 \leqslant p < \infty$ and $s \in (0,1)$ (recall we have assumed $\omega(0) = 0$).

Step 2: case $M \geqslant 2$ and $s \in (0, M)$.

The assumption $s \in (0,1)$ being artificial (by Remark 6.5) the above still holds for general $s > 0$. In particular, replacing (6.10) by the estimate of Lemma 6.2, for $f \in L^p(\mathbb{R}^N)$, one obtains

$$\omega\left(\frac{\|\Delta_h^{2M} f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \leqslant C(M,\rho,\omega) \int_{\mathbb{R}^N} \rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta_z^M f\|_{L^p(\mathbb{R}^N)}}{|z|^{sp}}\right) \mathrm{d}z, \tag{6.18}$$

for any $s \in (0, M)$. Taking the supremum over $\varepsilon > 0$ (i.e. over $|h| > 0$) and recalling that ω is a continuous, non-decreasing function, we find that

$$\omega\left([f]_{B^s_{p,\infty}(\mathbb{R}^N)}\right) \lesssim \sup_{\varepsilon>0} \int_{\mathbb{R}^N} \rho_\varepsilon(z)\ \omega\left(\frac{\|\Delta_z^M f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) \mathrm{d}z.$$

This is because the space $B^s_{p,\infty}(\mathbb{R}^N)$ with $s \in (0, M)$ is characterized by finite differences of order M, i.e.

$$[f]_{B^s_{p,\infty}(\mathbb{R}^N)} \sim \sup_{|h|\neq 0} \frac{\|\Delta_h^{2M} f\|_{L^p(\mathbb{R}^N)}}{|h|^s}, \quad \forall s \in (0, M),$$

Indeed, recall Lemma 6.3 and $\|\Delta_h^{2M} f\|_{L^p(\mathbb{R}^N)} \leqslant C(M)\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}$.

Remark 6.7. — As above, we still have

$$\omega\left(\limsup_{|h|\to 0} \frac{\|\Delta_h^M f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \sim \limsup_{\varepsilon\to 0^+} \mathscr{D}_\omega(\rho_\varepsilon, f).$$

So that Corollary 2.18 follows in that case too.

Remark 6.8. — Note that, when $(\rho_\varepsilon)_{\varepsilon>0} \in \mathfrak{M}(\mathbb{R}^N)$ (with corresponding Ψ and δ), we have actually proved a stronger estimate than needed. Indeed, we have shown that for any $h \in \mathbb{R}^N \setminus \{0\}$, $s \in (0,1]$, $p \in [1,\infty]$ and $(\rho_\varepsilon)_{\varepsilon>0} \in \mathfrak{M}(\mathbb{R}^N)$ it holds

$$\omega\left(\frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|^s}\right) \leqslant C(\Psi,\delta,A_\omega) \int_{\mathbb{R}^N} \rho_{\varepsilon(|h|)}(z)\; \omega\left(\frac{\|\Delta_z^1 f\|_{L^p(\mathbb{R}^N)}}{|z|^s}\right) \mathrm{d}z,$$

where $\varepsilon(t) = \frac{2t}{\delta}$ and A_ω is as in Definition 2.1.

Step 3: proof of Remark 2.4.

Let $1 \leqslant p < \infty$, $\omega \in C^+_{\mathrm{inc}}$ and $\Omega \in C_{\mathrm{inc}}$ satisfying (2.3). Then, we have

$$\begin{aligned}\omega\left(\int_{\mathbb{R}^N} \Omega\left(\frac{|\Delta_h^M f(x)|}{|h|^s}\right) \mathrm{d}x\right) &\geqslant \omega\left(m_1 \frac{\|\Delta_h^M f\|^p_{L^p(\mathbb{R}^N)}}{|h|^{sp}}\right) \\ &\geqslant K_1(m_1, A_\omega)\, \omega\left(\frac{\|\Delta_h^M f\|^p_{L^p(\mathbb{R}^N)}}{|h|^{sp}}\right),\end{aligned}$$

where $K_1(m_1, A_\omega) > 0$ and $A_\omega > 0$ is a number such that ω satisfies the condition of Definition 2.1 with $A = A_\omega$. Similarly, for some $K_2(m_2, A_\omega) > 0$,

$$\omega\left(\int_{\mathbb{R}^N} \Omega\left(\frac{|\Delta_h^M f(x)|}{|h|^s}\right) \mathrm{d}x\right) \leqslant K_2(m_2, A_\omega)\, \omega\left(\frac{\|\Delta_h^M f\|^p_{L^p(\mathbb{R}^N)}}{|h|^{sp}}\right).$$

Now, since $\widetilde{\omega} = \omega \circ |\cdot|^p$ lies in C^+_{inc} (by Remark 2.2) we obtain the desired claim, i.e. that

$$\omega\left([f]^p_{B^s_{p,\infty}(\mathbb{R}^N)}\right) \sim \sup_{\varepsilon>0} \int_{\mathbb{R}^N} \rho_\varepsilon(h)\, \omega\left(\int_{\mathbb{R}^N} \Omega\left(\frac{|\Delta_h^M f(x)|}{|h|^s}\right) \mathrm{d}x\right) \mathrm{d}h.$$

The remaining claims of Remark 2.4 follow by a similar argument of comparison.

7 Characterization of Sobolev and BV spaces

We begin with a preliminary result.

Lemma 7.1. — *Let $p \in [1,\infty]$, $f \in L^p(\mathbb{R}^N)$ and let*

$$A := \sup_{h\neq 0} \frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|}.$$

If A is finite, then,

$$A = \limsup_{|h|\to 0} \frac{\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}}{|h|}.$$

Although our argument is much simpler, a proof of a similar result (involving moduli of continuity) may be found in [142]. However, the argument in [142] heavily relies on the continuity of $\|\Delta_h^1 f\|_{L^p(\mathbb{R}^N)}$ and, thus, does not cover the case $p = \infty$. We show that, in fact, it is enough to ask only for some kind of subadditivity.

Proof. — Let $f \in L^p(\mathbb{R}^N)$, $1 \leqslant p \leqslant \infty$. For any $t \in \mathbb{R}$, define

$$F(t) := \sup_{\sigma \in \mathbb{S}^{N-1}} \|\Delta_{\sigma t}^1 f\|_{L^p(\mathbb{R}^N)}.$$

Clearly, F is measurable. Now, let $t_1, t_2 \in \mathbb{R}$. Specializing (6.10) in $h = (t_1 + t_2)\sigma$ and $z = t_1\sigma$, for some $\sigma \in \mathbb{S}^{N-1}$, yields

$$\|\Delta_{\sigma(t_1+t_2)}^1 f\|_{L^p(\mathbb{R}^N)} \leqslant \|\Delta_{\sigma t_1}^1 f\|_{L^p(\mathbb{R}^N)} + \|\Delta_{-\sigma t_2}^1 f\|_{L^p(\mathbb{R}^N)} \leqslant F(t_1) + F(t_2).$$

Consequently, $F(t_1 + t_2) \leqslant F(t_1) + F(t_2)$ for all $t_1, t_2 \in \mathbb{R}$. Whence, $F : \mathbb{R} \to [0, \infty)$ is a measurable, subadditive function. Now suppose that

$$A := \sup_{t>0} \frac{F(t)}{t} < \infty.$$

Then, by the limit theorem of subadditive functions [96, Theorem 16.3.3, p.467],

$$A = \lim_{t \to 0^+} \frac{F(t)}{t} = \lim_{t \to 0^+} \sup_{\sigma \in \mathbb{S}^{N-1}} \frac{\|\Delta_{\sigma t}^1 f\|_{L^p(\mathbb{R}^N)}}{t}.$$

This proves the lemma. □

7.1 Proofs of Theorems 2.5 and 2.6

Proof. — The proof follows from a straightforward adaptation of the proof of Theorem 2.3 in the case $M = 1$ and $s \in (0, 1)$ with Lemma 7.1 and the fact that, using for example [27, Theorem 2, Theorem 3'],

$$\|\nabla f\|_{L^p(\mathbb{R}^N)} \sim \limsup_{|h| \to 0} \frac{\|f(\cdot + h) - f\|_{L^p(\mathbb{R}^N)}}{|h|}, \tag{7.1}$$

for all $1 \leqslant p < \infty$ (when $p = 1$ the left-hand side of (7.1) is to be understood in the BV-sense, i.e. as the total mass of the Radon measure ∇f).

The case $p = \infty$ follows from the arguments above and the definition of the Lipschitz semi-norm. □

7.2 A limiting embedding between Lipschitz and Besov spaces

This subsection is devoted to the proof of Theorem 2.12. To this end, we recall the following improvement of the Chebychev inequality due to Bourgain, Brezis and Mironescu [27].

Lemma 7.2 (Bourgain, Brezis, Mironescu, [27]). — *Let $g, h : (0, \delta) \to \mathbb{R}_+$. Assume that $g(t) \leqslant g(t/2)$ for all $t \in (0, \delta)$ and that h is non-increasing. Then, for some constant $C = C(N) > 0$,*

$$\delta^{-N} \int_0^\delta t^{N-1} g(t) \mathrm{d}t \int_0^\delta t^{N-1} h(t) \mathrm{d}t \leqslant C \int_0^\delta t^{N-1} g(t) h(t) \mathrm{d}t.$$

We are now ready to prove Theorem 2.12.

Proof of Theorem 2.12. — Let $q \in [1, \infty)$ and let $(\rho_\varepsilon)_{\varepsilon \in (0,1]}$ be defined by

$$\rho_\varepsilon(t) := \frac{1}{|B_1|} \frac{\varepsilon^{1-\varepsilon}}{t^{N-\varepsilon}} \mathbb{1}_{(0,\varepsilon)}(t) \quad \text{for all } \varepsilon \in (0,1] \text{ and all } t \geqslant 0.$$

Note that

$$\int_0^\infty \rho_\varepsilon(t)\, t^{N-1} \mathrm{d}t = \frac{1}{|B_1|} \quad \text{for all } \varepsilon \in (0,1]. \tag{7.2}$$

In addition, we also set

$$\eta_\varepsilon(h) := \varepsilon^{-N}\, C_2 \frac{|h|}{\varepsilon} \mathbb{1}_{B_\varepsilon}(h) \quad \text{for all } \varepsilon \in (0,1] \text{ and all } h \in \mathbb{R}^N,$$

where $C_2 > 0$ is a constant such that η_ε has unit mass for each ε. Notice that $(\eta_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ is a sequence of radial functions satisfying (1.1) and (1.8). In particular, by Theorem 2.5, we know that

$$[f]_{C^{0,1}(\mathbb{R}^N)} \lesssim \limsup_{\varepsilon \to 0^+} \int_{B_\varepsilon} \eta_\varepsilon(h) \frac{\|\Delta_h^1 f\|_{L^\infty(\mathbb{R}^N)}}{|h|} \mathrm{d}h. \tag{7.3}$$

Next, for every $t > 0$, define

$$F(t) := \int_{\mathbb{S}^{N-1}} \|\Delta_{\sigma t}^1 f\|_{L^\infty(\mathbb{R}^N)} \mathrm{d}\mathscr{H}^{N-1}(\sigma).$$

By the triangle inequality we have $F(2t) \leqslant 2F(t)$ so that if we let

$$g(t) := \frac{F(t)}{t},$$

we have $g(t) \leqslant g(t/2)$. In these notations, we have the identity:

$$\int_{B_\varepsilon} \rho_\varepsilon(|h|) \frac{\|\Delta_h^1 f\|_{L^\infty(\mathbb{R}^N)}}{|h|} \mathrm{d}h = \int_0^\varepsilon t^{N-1} \rho_\varepsilon(t) g(t) \mathrm{d}t.$$

Invoking Lemma 7.2 and (7.2) we deduce that, for every $\varepsilon \in (0,1]$,

$$\begin{aligned}
\int_{B_\varepsilon} \rho_\varepsilon(|h|) \frac{\|\Delta_h^1 f\|_{L^\infty(\mathbb{R}^N)}}{|h|} \mathrm{d}h &\gtrsim \varepsilon^{-N} \int_0^\varepsilon t^{N-1} \rho_\varepsilon(t) \mathrm{d}t \int_0^\varepsilon t^{N-1} g(t) \mathrm{d}t \\
&= \fint_{B_\varepsilon} \frac{\|\Delta_h^1 f\|_{L^\infty(\mathbb{R}^N)}}{|h|} \mathrm{d}h \\
&\geqslant \varepsilon^{-1} \fint_{B_\varepsilon} \|\Delta_h^1 f\|_{L^\infty(\mathbb{R}^N)} \mathrm{d}h \\
&\gtrsim \int_{B_\varepsilon} \eta_\varepsilon(h) \frac{\|\Delta_h^1 f\|_{L^\infty(\mathbb{R}^N)}}{|h|} \mathrm{d}h.
\end{aligned}$$

Whence, using (7.3), we come up with

$$[f]_{C^{0,1}(\mathbb{R}^N)} \lesssim \limsup_{\varepsilon \to 0^+} \int_{B_\varepsilon} \rho_\varepsilon(|h|) \frac{\|\Delta_h^1 f\|_{L^\infty(\mathbb{R}^N)}}{|h|} \mathrm{d}h.$$

Now, use the Jensen inequality to deduce that

$$\begin{aligned}
[f]^q_{C^{0,1}(\mathbb{R}^N)} &\lesssim \limsup_{\varepsilon \to 0^+} \int_{B_\varepsilon} \rho_\varepsilon(|h|) \frac{\|\Delta_h^1 f\|^q_{L^\infty(\mathbb{R}^N)}}{|h|^q} \mathrm{d}h \\
&\lesssim \limsup_{\varepsilon \to 0^+} \varepsilon^{-\varepsilon} \left(\varepsilon \int_{B_\varepsilon} \frac{\|\Delta_h^1 f\|^q_{L^\infty(\mathbb{R}^N)}}{|h|^{N+q-\varepsilon}} \mathrm{d}h \right) \\
&\lesssim \limsup_{\varepsilon \to 0^+} \varepsilon \int_{\mathbb{R}^N} \frac{\|\Delta_h^1 f\|^q_{L^\infty(\mathbb{R}^N)}}{|h|^{N+q-\varepsilon}} \mathrm{d}h.
\end{aligned}$$

Define $\sigma \in (1 - \frac{1}{q}, 1)$ by the relation $\varepsilon = q(1-\sigma)$. Then,

$$[f]^q_{C^{0,1}(\mathbb{R}^N)} \lesssim \limsup_{\sigma \to 1^-} q(1-\sigma) [f]^q_{B^\sigma_{\infty,q}(\mathbb{R}^N)}.$$

The converse of this is covered by Proposition 6.1. □

8 A non-limiting embedding theorem

This section is devoted to the proof of Theorem 2.14. The idea of the proof is very similar to that of Theorem 4.4 (ii) on p.36 in [136] (see in particular pp.39-40 there).

Nevertheless, we choose to give more details in order to make the dependence of the constants involved on s, p and q as explicit as possible.

We will need some preliminary estimates.

Lemma 8.1. — *Let $(u_j)_{j\geqslant 0}$ be the sequence defined by*

$$u_j := \begin{cases} k & \text{if } j = 2^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

Then, $(u_j)_{j\geqslant 0} \notin \ell^\infty(\mathbb{N})$ and

$$\sup_{\varepsilon>0} \varepsilon \sum_{j\geqslant 0} 2^{-j\varepsilon} u_j \leqslant \frac{2}{e\ln(2)}.$$

Proof. — Let $\varepsilon > 0$ and set

$$A_\varepsilon := \sum_{j\geqslant 0} \varepsilon 2^{-j\varepsilon} u_j = \sum_{k\geqslant 0} \varepsilon 2^{-2^k\varepsilon} k.$$

Using the (trivial) estimate $e^{-x} \leqslant 1/(ex)$, we have $2^{-x} \leqslant 1/(ex\ln(2))$. Thus,

$$A_\varepsilon \leqslant \frac{1}{e\ln(2)} \sum_{k\geqslant 0} k 2^{-k}.$$

Recalling the well-known identity $\sum k x^k = x/(1-x)^2$ (for $0 \leqslant x < 1$), we finally obtain

$$A_\varepsilon \leqslant \frac{2}{e\ln(2)}.$$

Since this holds for every $\varepsilon > 0$, we obtain the desired claim. □

Lemma 8.2. — *Let $M \in \mathbb{N}^*$ and $(u_k)_{k\geqslant 1}$ be a sequence of non-negative numbers. Let $\psi \in C_c^\infty(\mathbb{R})$ be such that ψ is not a polynomial of degree less than or equal to $M-1$, and such that*

$$\operatorname{supp}(\psi) \subset [-\eta, \eta] \quad \text{for some } \eta \geqslant 1,$$

and set

$$f(x_1, \dots, x_N) = \sum_{k\geqslant 1} u_k \,\psi\left(\frac{x_1 - 2(M+\eta)k}{2^{-k}}\right) \dots \psi\left(\frac{x_N - 2(M+\eta)k}{2^{-k}}\right).$$

Then, for any fixed $j \geqslant 1$, we have

$$\sup_{\frac{1}{2^{j+1}} \leqslant |h| \leqslant \frac{1}{2^j}} \|\Delta_h^M f\|_{L^p(\mathbb{R}^N)} \geqslant c\, u_j 2^{-j\frac{N}{p}},$$

for some constant $c > 0$ depending only on N, p, M and ψ.

Proof. — We begin with the case $N = 1$. Fix any $j \geqslant 1$ and let $|h| \leqslant 2^{-j}$. Let us set

$$x_j := 2(M+\eta)j \quad \text{and} \quad R_j := 2^{-j}(M+\eta).$$

Since $\operatorname{supp}(\psi) \subset [-\eta, \eta]$ and $|h| \leqslant 2^{-j}$, for any $\ell \in [\![0, M]\!]$, we have

$$\begin{aligned} x \in \operatorname{supp} \psi\left(\frac{\cdot + \ell h - x_j}{2^{-j}}\right) &\Leftrightarrow x + h\ell \in \left[x_j - \eta 2^{-j}, x_j + \eta 2^{-j}\right] \\ &\Leftrightarrow x \in \left[x_j - h\ell - \eta 2^{-j}, x_j - h\ell + \eta 2^{-j}\right] =: B_{\ell,j}. \end{aligned}$$

And, clearly

$$\operatorname{supp}\left(\Delta_h^M \psi\left(\frac{\cdot - x_j}{2^{-j}}\right)\right) \subset \bigcup_{\ell \in [\![0,M]\!]} B_{\ell,j}.$$

Thus,

$$\operatorname{supp}\left(\Delta_h^M \psi\left(\frac{\cdot - x_j}{2^{-j}}\right)\right) \subset [x_j - R_j, x_j + R_j] =: \mathscr{B}_j.$$

Furthermore,

$$R_{j+1} + R_j = 2^{-j}(M+\eta)\left(1 + \frac{1}{2}\right) < 2(M+\eta) = x_{j+1} - x_j,$$

and so, the $\mathscr{B}_j$'s are mutually disjoint. Therefore, given any fixed $j \geqslant 1$ and $\varepsilon > 0$ a small parameter less than R_j, we have

$$\begin{aligned} \|\Delta_h^M f\|_{L^p(\mathbb{R})}^p &= \left\| \sum_{k \geqslant 1} u_k \Delta_h^M \psi\left(\frac{\cdot - x_k}{2^{-k}}\right)\right\|_{L^p(\mathbb{R})}^p \\ &\geqslant \int_{x_j - \varepsilon}^{x_j + \varepsilon} \left| \sum_{k \geqslant 1} u_k \Delta_h^M \psi\left(\frac{x - x_k}{2^{-k}}\right)\right|^p \mathrm{d}x \\ &= u_j^p \int_{x_j - \varepsilon}^{x_j + \varepsilon} \left| \Delta_h^M \psi\left(\frac{x - x_j}{2^{-j}}\right)\right|^p \mathrm{d}x \\ &= u_j^p 2^{-j} \int_{-\varepsilon/2^{-j}}^{\varepsilon/2^{-j}} |\Delta_{h/2^{-j}}^M \psi(x)|^p \mathrm{d}x. \end{aligned}$$

Whence, writing $K_j := \overline{B_{2^{-j}} \setminus B_{2^{-(j+1)}}}$ for $j \geqslant 0$ we have

$$\begin{aligned} \sup_{h \in K_j} \|\Delta_h^M f\|_{L^p(\mathbb{R})}^p &\geqslant u_j^p 2^{-j} \sup_{h \in K_j} \int_{-\varepsilon}^{\varepsilon} |\Delta_{h/2^{-j}}^M \psi(x)|^p \mathrm{d}x \\ &= c_\varepsilon^p u_j^p 2^{-j}. \end{aligned}$$

where

$$c_\varepsilon = c_\varepsilon(M,p,\psi) := \sup_{\frac{1}{2}\leqslant |h|\leqslant 1} \left(\int_{-\varepsilon}^{\varepsilon} |\Delta_h^M \psi(x)|^p \mathrm{d}x\right)^{1/p}.$$

Since $\varepsilon > 0$ is an arbitrary small parameter and ψ is not a polynomial of degree less than or equal to $M-1$, we may find a number $\varepsilon_0 > 0$ such that $c_{\varepsilon_0} > 0$.

The proof when $N \geqslant 2$ follows by a straightforward adaptation of the case $N = 1$ using the product structure $\psi(x_1)\dots\psi(x_N)$ and Fubini's theorem which gives the result with $c = c_{\varepsilon_0}^N$. □

We are now ready to prove Theorem 2.14.

Proof of Theorem 2.14. — Let $M \in \mathbb{N}^*$ be such that $s \in (0, M)$ and let u_j be the sequence of Lemma 8.1. Also, we choose $\psi \in C_c^\infty(\mathbb{R}^N)$ such that

$$\operatorname{supp}(\psi) \subset B_2 \quad\text{and}\quad \sum_{m\in\mathbb{Z}^N} \psi(x-m) = 1 \quad \text{for any } x \in \mathbb{R}^N.$$

In addition, we suppose that ψ has the product structure

$$\psi(x) = \Psi(x_1)\dots\Psi(x_N),$$

for some $\Psi \in C_c^\infty(\mathbb{R})$ different from a polynomial of degree less than or equal to $M-1$. Then, we set

$$m_j := 2(M+2)j\,\mathbf{1} \in \mathbb{Z}^N \quad \text{with } \mathbf{1} := (1,\dots,1) \in \mathbb{Z}^N,$$

and we define

$$\begin{aligned} f(x) &:= \sum_{j\geqslant 1} u_j^{1/q} 2^{-j(s-\frac{N}{p})} \psi(2^j(x-m_j)) \\ &= \sum_{j\geqslant 1} \big(u_j^{1/q} 2^{-j\varepsilon}\big) 2^{-j(s-\varepsilon-\frac{N}{p})} \psi(2^j(x-m_j)). \end{aligned}$$

where $x \in \mathbb{R}^N$. It follows from Definition 2.4 that

$$2^{-j(s-\varepsilon-\frac{N}{p})}\psi(2^j(x-m_j))$$

can be interpreted as $(s-\varepsilon,p)$-0-quarks. Accordingly, by Definition 4.2, we have that

$$\varepsilon \|f\|^q_{\mathbf{B}^{s-\varepsilon}_{p,q}(\mathbb{R}^N)} \leqslant \varepsilon \sum_{j\geqslant 1} \big(2^{-j\varepsilon} u_j^{1/q}\big)^q.$$

Using Lemma 8.1 we obtain that

$$\varepsilon \|f\|^q_{\mathbf{B}^{s-\varepsilon}_{p,q}(\mathbb{R}^N)} \leqslant \frac{2q^{-1}}{e\ln(2)} < \infty, \quad \forall \varepsilon \in (0,s).$$

In particular, recalling Theorem 4.3, $f \in L^p(\mathbb{R}^N)$. Also, for all $j \geqslant 1$, we write

$$K_j := \{2^{-(j+1)} \leqslant |h| \leqslant 2^{-j}\}.$$

Recall that

$$\|f\|_{B^s_{p,\infty}(\mathbb{R}^N)} \sim \|f\|_{L^p(\mathbb{R}^N)} + \sup_{j\geqslant 1} \; 2^{js} \sup_{h\in K_j} \|\Delta^M_h f\|_{L^p(\mathbb{R}^N)},$$

is an equivalent norm on $B^s_{p,\infty}(\mathbb{R}^N)$ (this is a discretized version of Theorem 2.5.12 on p.110 in [136]). Using this together with Lemma 8.2 we get

$$\|f\|_{B^s_{p,\infty}(\mathbb{R}^N)} \geqslant c \sup_{j\geqslant 1} u_j^{1/q} = \infty.$$

Here $c = c(N,p,M,\psi) > 0$. Thus $f \notin B^s_{p,\infty}(\mathbb{R}^N)$. This completes the proof. □

9 Non-compactness results

This section is devoted to the proofs of Theorem 2.15 and Theorem 2.17. We begin with the former one.

Proof of Theorem 2.15. — For simplicity, we replace $\varepsilon > 0$ by $1/n$ with $n \geqslant 1$ and write ρ_n instead of $\rho_{1/n}$. We write

$$x = (x_1, \dots, x_N) \in \mathbb{R}^N,$$
$$y = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1},$$

and, for all $n \geqslant 1$, we let

$$f_n(x) := n^{\frac{M-s}{Mp}}\, \Phi(n^{\frac{M-s}{M}} x_N)\, \varphi(y),$$

for some arbitrary $\Phi \in C^\infty_c(\mathbb{R})$ and $\varphi \in C^\infty_c(\mathbb{R}^{N-1})$ (if $N=1$, replace φ by 1) with

$$\max\left\{\|\Phi\|_{W^{M,p}(\mathbb{R})}, \|\varphi\|_{W^{M,p}(\mathbb{R}^{N-1})}\right\} \leqslant C_0. \tag{9.1}$$

Note that $f_n \to 0$ a.e. in $\mathbb{R}^N$. Further, from Fubini's theorem we infer that

$$\int_{\mathbb{R}^N} |f_n(x)|^p \mathrm{d}x = \left(\int_{\mathbb{R}^{N-1}} |\varphi(y)|^p \mathrm{d}y\right)\left(n^{\frac{M-s}{M}} \int_{\mathbb{R}} |\Phi(n^{\frac{M-s}{M}} x_N)|^p \mathrm{d}x_N\right)$$

$$= \|\varphi\|^p_{L^p(\mathbb{R}^{N-1})}\|\Phi\|^p_{L^p(\mathbb{R})} = C_1.$$

On the one hand, we observe that (9.1) gives

$$\|D^\alpha f_n\|^p_{L^p(\mathbb{R}^N)} \lesssim 1 \quad \text{for each } \alpha = (\alpha_1, \dots, \alpha_{N-1}, 0) \in \mathbb{N}^N \text{ with } |\alpha| \leqslant M.$$

While, on the other hand, for all $j \in [\![1, M]\!]$,

$$\begin{aligned}
\left\| \frac{\partial^j f_n}{\partial x_N^j} \right\|^p_{L^p(\mathbb{R}^N)} &= n^{\frac{(M-s)}{M} jp} \|\varphi\|^p_{L^p(\mathbb{R}^{N-1})} \, n^{\frac{M-s}{M}} \int_{\mathbb{R}} |\Phi^{(j)}(n^{\frac{M-s}{M}} x_N)|^p \mathrm{d}x_N \\
&= n^{\frac{(M-s)}{M} jp} \|\varphi\|^p_{L^p(\mathbb{R}^{N-1})} \|\Phi^{(j)}\|^p_{L^p(\mathbb{R})} \\
&\leqslant n^{(M-s)p} \|\varphi\|^p_{L^p(\mathbb{R}^{N-1})} \max_{j \in [\![1,M]\!]} \|\Phi^{(j)}\|^p_{L^p(\mathbb{R})}.
\end{aligned}$$

Whence, using the product structure of f_n we get

$$\sup_{|\alpha| \leqslant M} \|D^\alpha f_n\|^p_{L^p(\mathbb{R}^N)} \leqslant C_2 n^{(M-s)p} \quad \text{for all } n \geqslant 1.$$

Moreover, for all $h \neq 0$, it holds

$$\|\Delta_h^M f_n\|^p_{L^p(\mathbb{R}^N)} \leqslant |h|^{Mp} \sup_{|\alpha| \leqslant M} \|D^\alpha f_n\|^p_{L^p(\mathbb{R}^N)} \leqslant C_2 |h|^{Mp} n^{(M-s)p}.$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_n(h) \frac{|\Delta_h^M f_n(x)|^p}{|h|^{sp}} \mathrm{d}x \mathrm{d}h &\lesssim \int_{\mathbb{R}^N} \rho_n(h) |h|^{p(M-s)} n^{p(M-s)} \mathrm{d}h \\
&= \int_{\mathbb{R}^N} \rho(h) |h|^{p(M-s)} \mathrm{d}h.
\end{aligned}$$

We thus conclude that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \rho_n(h) \frac{|\Delta_h^M f_n(x)|^p}{|h|^{sp}} \mathrm{d}x \mathrm{d}h \leqslant C_3 \quad \text{for any } n \geqslant 1.$$

Yet, $(f_n)_{n \geqslant 1}$ is not relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. □

The proof of Theorem 2.17 is as follows.

Proof of Theorem 2.17. — The proof in this case is very similar to that of Theorem 2.15. We let $M \in \mathbb{N}^*$, $s \in (0, M)$ and pick a slightly different sequence of functions, for example

$$f_n(x) := n^{\frac{\gamma}{Mp}} \, \Phi(n^{\frac{\gamma}{M}} x_N) \, \varphi(y),$$

where $0 \leqslant \gamma \leqslant \frac{1}{q}$, $\Phi \in C_c^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty(\mathbb{R}^{N-1})$. Also, we set

$$\rho_n(h) := \frac{1}{n\sigma_N |h|^{N-1/n}} \mathbb{1}_{(0,1)}(|h|).$$

As above, one has

$$\|f_n\|_{L^p(\mathbb{R}^N)} = \|\Phi\|_{L^p(\mathbb{R})} \|\varphi\|_{L^p(\mathbb{R}^{N-1})} \quad \text{and} \quad \sup_{|\alpha| \leqslant M} \|D^\alpha f_n\|_{L^p(\mathbb{R}^N)} \lesssim n^\gamma.$$

Whence, we arrive at

$$\begin{aligned}
\int_{B_1} \rho_n(h) \frac{\|\Delta_h^M f_n\|_{L^p(\mathbb{R}^N)}^q}{|h|^{sq}} \mathrm{d}h &\lesssim \int_{B_1} \rho_n(h) |h|^{(M-s)q} n^{\gamma q} \mathrm{d}h \\
&= n^{\gamma q - 1} \int_0^1 \frac{\mathrm{d}r}{r^{1-(q(M-s)+1/n)}} \\
&= \frac{n^{\gamma q}}{1 + (M-s)qn} \lesssim \frac{1}{n^{1-\gamma q}}.
\end{aligned}$$

Since $0 \leqslant \gamma \leqslant \frac{1}{q}$, we obtain

$$\int_{B_1} \rho_n(h) \frac{\|\Delta_h^M f_n\|_{L^p(\mathbb{R}^N)}^q}{|h|^{sq}} \mathrm{d}h \leqslant C \quad \text{for all } n \geqslant 1.$$

However, $(f_n)_{n \geqslant 1}$ is not relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. □

Appendix

In [98], Lamy and Mironescu proved the

Theorem 9.1 (Lamy, Mironescu, [98]). — *Let $s > 0$, $p \in [1, \infty)$ and $(\rho_\varepsilon)_{\varepsilon > 0}$ satisfying* (1.1) *and* (1.8)*. Then,*

$$\|f\|_{B^s_{p\infty}(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^N)} + \sup_{\varepsilon \in (0,1)} \frac{\|\rho_\varepsilon * f - f\|_{L^p(\mathbb{R}^N)}}{\varepsilon^s}. \tag{9.2}$$

Since Theorem 9.1 is not properly stated in [98] nor its proof, we shall give a brief sketch of the proof in order to justify that their result indeed applies to the scale $B^s_{p,\infty}(\mathbb{R}^N)$.

Sketch of the proof. — It is well-known that each tempered distribution $f \in \mathscr{S}'(\mathbb{R}^N)$ can be decomposed as

$$f = \sum_{j \geqslant 0} f_j, \tag{9.3}$$

where $f_0 = f * \zeta$, $f_j = f * \varphi_{2^{1-j}}$, $j \geqslant 1$, and $\zeta, \varphi \in \mathscr{S}(\mathbb{R}^N)$ are functions satisfying

(i) $\operatorname{supp}(\widehat{\zeta}) \subset B_2$ and $\widehat{\zeta} \equiv 1$ in a neighborhood of $\bar{B}_1$,

(ii) $\varphi := \zeta_{1/2} - \zeta$ with $\widehat{\varphi} = \widehat{\zeta}(\cdot/2) - \widehat{\zeta}$ and $\operatorname{supp}(\widehat{\varphi}) \subset B_4 \setminus \bar{B}_1$.

where the subscript φ_k means $k^{-N}\varphi(\cdot/k)$ and $\widehat{\varphi}$ stands for the Fourier transform of φ (similarly for ζ). Formula (9.3) is called the Littlewood-Paley decomposition of f. Furthermore, it is known that each function in $B^s_{p,\infty}(\mathbb{R}^N)$ is a tempered distribution, so that this decomposition makes sense here and may even serve to formulate an equivalent norm on this space via the formula

$$\|f\|_{B^s_{p,\infty}(\mathbb{R}^N)} \sim \sup_{j\geqslant 0} 2^{js}\|f_j\|_{L^p(\mathbb{R}^N)}.$$

To see that Theorem 9.1 holds it suffices to discretize the last term on the right-hand side of (9.2) as

$$\sup_{\varepsilon\in(1/2,1)} \sup_{j\geqslant 0} 2^{js}\|f - f * \rho_{2^{-j}\varepsilon}\|_{L^p(\mathbb{R}^N)}.$$

At this stage, all the estimates obtained in [98] directly apply because it is the terms $\|f_j\|_{L^p(\mathbb{R}^N)}$ which are estimated there (and not their sum nor their integral) in terms of the quantity $\|f - f * \rho_{2^{-j}\varepsilon}\|_{L^p(\mathbb{R}^N)}$. □

Using this result, Proposition 1.2 can be proved by arguing as follows.

Proof of Proposition 1.2. — Suppose without loss of generality that the ρ_ε's are compactly supported and that $\operatorname{supp}(\rho) \subset B_1$. Also, up to replace ρ_ε by $\frac{\rho_\varepsilon(h)+\rho_\varepsilon(-h)}{2}$, we can always assume that each ρ_ε is even. Then, by the Jensen inequality,

$$\begin{aligned}
\frac{\|\rho_\varepsilon * f - f\|^p_{L^p(\mathbb{R}^N)}}{\varepsilon^{sp}} &= \frac{1}{\varepsilon^{sp}}\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\rho_\varepsilon(h)[f(x-h)-f(x)]\mathrm{d}h\right|^p \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^N}\int_{B_\varepsilon}\rho_\varepsilon(-h)\frac{|f(x+h)-f(x)|^p}{\varepsilon^{sp}}\mathrm{d}h\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^N}\int_{B_\varepsilon}\rho_\varepsilon(h)\frac{|f(x+h)-f(x)|^p}{|h|^{sp}}\mathrm{d}h\mathrm{d}x.
\end{aligned}$$

Whence,

$$\sup_{\varepsilon\in(0,1)}\frac{\|\rho_\varepsilon * f - f\|^p_{L^p(\mathbb{R}^N)}}{\varepsilon^{sp}} \lesssim \sup_{\varepsilon\in(0,1)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(h)\frac{|f(x+h)-f(x)|^p}{|h|^{sp}}\mathrm{d}h\mathrm{d}x.$$

And so, by Theorem 9.1, $f \in B^s_{p,\infty}(\mathbb{R}^N)$. The proof when ρ is not compactly supported follows by a simple comparison argument: cutting off ρ as $\widetilde{\rho} := \rho\mathbb{1}_{B_R}$ for some $R > 0$ with $|B_R \cap \operatorname{supp}(\rho)| > 0$, we clearly have $\rho \geqslant \widetilde{\rho}$ and (1.10) implies that the same property holds for $\widetilde{\rho}$ instead of ρ (up to some multiplicative factor $\|\rho\|_{L^1(B_R)}$ to make $\widetilde{\rho}_\varepsilon$ a sequence of mollifiers), i.e. that $f \in B^s_{p,\infty}(\mathbb{R}^N)$. □

Chapter 2

On restrictions of Besov functions

This chapter is inspired by the paper [29], published in *Nonlinear Analysis*.

1 Introduction

In this chapter, we address the following question: given a function $f \in B^s_{p,q}(\mathbb{R}^N)$,

what can be said about the smoothness of $f(\cdot, y)$ *for a.e.* $y \in \mathbb{R}^{N-d}$ *?*

In order to formulate this as a meaningful question, one is naturally led to restrict oneself to $1 \leqslant d < N$, $0 < p, q \leqslant \infty$ and $s > \sigma_p$, where

$$\sigma_p = N\left(\frac{1}{p} - 1\right)_+, \tag{1.1}$$

since otherwise $f \in B^s_{p,q}(\mathbb{R}^N)$ need not be a regular distribution.

Let us begin with a simple observation. If $f \in L^p(\mathbb{R}^N)$ for some $0 < p \leqslant \infty$, then

$$f(\cdot, y) \in L^p(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d}.$$

This is a straightforward consequence of Fubini's theorem. Using similar Fubini-type arguments, one can show that, if $f \in W^{s,p}(\mathbb{R}^N)$ for some $0 < p \leqslant \infty$ and $\sigma_p < s \notin \mathbb{N}$, then we have $f(\cdot, y) \in W^{s,p}(\mathbb{R}^d)$ for a.e. $y \in \mathbb{R}^{N-d}$. We say that these spaces have the *restriction property*.

Unlike their cousins, the Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^N)$, Besov spaces do not enjoy the Fubini property unless $p = q$, that is

$$\sum_{j=1}^{N} \left\| \| f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_N) \|_{B^s_{p,q}(\mathbb{R})} \right\|_{L^p(\mathbb{R}^{N-1})},$$

is an equivalent quasi-norm on $B^s_{p,q}(\mathbb{R}^N)$ if, and only if, $p = q$; while the counterpart for $F^s_{p,q}(\mathbb{R}^N)$ holds for any given values of p and q where it makes sense (see [139, Theorem 4.4, p.36] for a proof). In particular, $B^s_{p,p}(\mathbb{R}^N)$ and $F^s_{p,q}(\mathbb{R}^N)$ have the restriction property. It is natural to ask wether or not this feature holds in $B^s_{p,q}(\mathbb{R}^N)$ for an arbitrary $q \neq p$.

Let us recall some known facts.

Fact 1.1. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < q \leqslant p \leqslant \infty$, $s > \sigma_p$ and $f \in B^s_{p,q}(\mathbb{R}^N)$. Then, it holds that*

$$f(\cdot, y) \in B^s_{p,q}(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d}.$$

(A proof of a slightly more general result will be given in the sequel, see Proposition 5.1.)

In fact, there is a weaker version of Fact 1.1, which shows that this stays "almost" true when $p < q$. This can be stated as follows

Fact 1.2. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$, $s > \sigma_p$ and $f \in B^s_{p,q}(\mathbb{R}^N)$. Then, it holds that*

$$f(\cdot, y) \in \bigcap_{s' < s} B^{s'}_{p,q}(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d}.$$

See e.g. [86, Theorem 1] or [9, Theorem 1.1].

Mironescu [107] suggested that it might be possible to construct a counterexample to Fact 1.1 when $p < q$. We prove that this is indeed the case. This is quite remarkable since, to our knowledge, the list of properties of the spaces $B^s_{p,q}$ where q plays a crucial role is rather short.

Our first result is the following

Theorem 1.3. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$ and let $s > \sigma_p$. Then, there exists a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot, y) \notin B^s_{p,\infty}(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d}.$$

Remark 1.4. — Note that this is actually stronger than what we initially asked for, since $B^s_{p,q} \hookrightarrow B^s_{p,\infty}$.

Remark 1.5. — We were informed that, concomitant to our work, a version of Theorem 1.3 for $N = 2$ and $p \geqslant 1$ was proved by Mironescu, Russ and Sire in [108]. We present another proof independent of it with different techniques. In fact, we will even prove a generalized version of Theorem 1.3 that incorporates other related function spaces (see Theorem 6.1) which is of independent interest.

Despite the negative conclusion of Theorem 1.3, one may ask if something weaker than Fact 1.1 still holds when $p < q$. For example, by standard embeddings, we know that

$$B^s_{p,q}(\mathbb{R}^N) \hookrightarrow A^{s,p}(\mathbb{R}^N) \quad \text{for any } 0 < q < \infty,$$

where $A^{s,p}(\mathbb{R}^N)$ stands for

$$C^{s-\frac{N}{p}}(\mathbb{R}^N), \;\; \mathrm{BMO}(\mathbb{R}^N) \;\text{ and }\; L^{\frac{Np}{N-sp},\infty}(\mathbb{R}^N), \tag{1.2}$$

when respectively $sp > N$, $sp = N$ and $sp < N$ (see Subsection 2.3). In particular, we may infer from Fact 1.1 that if $q \leqslant p$, then for every $f \in B^s_{p,q}(\mathbb{R}^N)$ it holds

$$f(\cdot, y) \in A^{s,p}(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d}.$$

It is tempting to ask wether the same is true when $p < q$. But, as it turns out, even this fails to hold. This is the content of our next result.

Theorem 1.6. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$ and let $s > \sigma_p$. Then, there exists a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot, y) \notin A^{s,p}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}.$$

It is nonetheless possible to refine the conclusions of Fact 1.2 and Theorem 1.3. We find that a natural way to characterize such restrictions is to look at a more general scale of functions known as *Besov spaces of generalized smoothness*, denoted by $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ (see Definition 2.11). This type of spaces was first introduced by the Russian school in the mid-seventies (see e.g. [88, 97, 103]) and was shown to be useful in various problems ranging from Black-Scholes equations [126] to the study of pseudo-differential operators [1, 85, 99, 101]. A comprehensive state of art covering both old and recent material can be found in [65]. Several versions of these spaces were studied in the literature, from different points of view and different degrees of generality. We choose to follow the point of view initiated by Edmunds and Triebel in [60] (see also [48, 61, 100, 111, 139]), which seems better suited to our purposes. Here, s remains the dominant smoothness parameter and Ψ is a positive function of log-type called *admissible* (see Definition 2.9). That admissible function is a finer tuning that allows encoding more general types of smoothness. The simplest example is the function $\Psi \equiv 1$ for which one has $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N) = B^s_{p,q}(\mathbb{R}^N)$.

More generally, the spaces $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ are intercalated scales between $B^{s-\varepsilon}_{p,q}(\mathbb{R}^N)$ and $B^{s+\varepsilon}_{p,q}(\mathbb{R}^N)$. For example: if Ψ is increasing, then we have

$$B^s_{p,q}(\mathbb{R}^N) \hookrightarrow B^{(s,\Psi)}_{p,q}(\mathbb{R}^N) \hookrightarrow B^{s'}_{p,q}(\mathbb{R}^N) \quad \text{for every } s' < s,$$

see [111, Proposition 1.9(vi)].

We prove that restrictions of Besov functions to almost every hyperplanes belong to $B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$, whenever Ψ satisfies the following growth condition

$$\sum_{j\geqslant 0} \Psi(2^{-j})^{\chi} < \infty, \tag{1.3}$$

with $\chi = \frac{qp}{q-p}$ (resp. $\chi = p$ if $q = \infty$).

More precisely, we prove the following

Theorem 1.7. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$, $s > \sigma_p$ and let Ψ be an admissible function satisfying* (1.3). *Suppose that $f \in B^s_{p,q}(\mathbb{R}^N)$. Then,*

$$f(\cdot, y) \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}.$$

It turns out that the condition (1.3) on Ψ in Theorem 1.7 is optimal, at least when $q = \infty$. In other words, we obtain a sharp characterization of the aforementioned loss of regularity.

Theorem 1.8. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$, $s > \sigma_p$ and let Ψ be an admissible function that does not satisfy* (1.3). *If $q < \infty$ and Ψ is increasing suppose, in addition, that*

$$\frac{qp}{q-p} < \frac{1}{c_\infty} \quad \textit{where} \quad c_\infty := \sup_{0<t\leqslant 1} \log_2 \frac{\Psi(t)}{\Psi(t^2)}. \tag{1.4}$$

Then, there is a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that

$$f(\cdot, y) \notin B^{(s,\Psi)}_{p,q}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}.$$

Remark 1.9. — Notice that condition (1.4) is sufficient and also not far from being necessary to ensure that (1.3) does not hold, as it happens that for some particular choices of Ψ, (1.3) is equivalent to $\frac{qp}{q-p} > \frac{1}{c_\infty}$.

A fine consequence of Theorem 1.7 is that it provides a substitute for $A^{s,p}(\mathbb{R}^d)$ when $p < q$ (in Theorem 1.6), which could be of interest in some applications (see e.g. [27, 108]). For example, if $sp > d$, $p < q$ and (1.3) is satisfied, then by Theorem 1.7 and [40, Proposition 3.4] we have

$$\forall f \in B^s_{p,q}(\mathbb{R}^N), \quad f(\cdot, y) \in C^{(s-\frac{d}{p},\Psi)}(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d},$$

where $C^{(\alpha,\Psi)}(\mathbb{R}^d)$ is the generalized Hölder space $B^{(\alpha,\Psi)}_{\infty,\infty}(\mathbb{R}^d)$ (see Remark 2.21 below).

Remark 1.10. — It is actually possible to formulate Theorems 1.7 and 1.8 in terms of the space $B^{w(\cdot)}_{p,q}(\mathbb{R}^d)$ introduced by Ansorena and Blasco in [3, 4], even though their results do not allow to handle higher orders $s \geqslant 1$ and neither the case $0 < p < 1$ nor $0 < q < 1$. Nevertheless, this is merely another side of the same coin and we wish to avoid unnecessary complications. Beyond technical matters, our approach is motivated by the relevance of the scale $B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$ in physical problems and in fractal geometry (see e.g. [60, 61, 111, 137, 139]).

In the course of the chapter we will also address the corresponding problem with $f \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ instead of $f \in B^{s}_{p,q}(\mathbb{R}^N)$ which is of independent interest. In fact, as we will show, our techniques allow to extend Theorems 1.3, 1.7 and 1.8 to this generalized setting with almost no modifications, see Theorems 6.1, 7.1, 7.2 and Remark 7.3.

Chapter 2 is organized as follows. In the forthcoming Section 2 we recall some useful definitions and results related to Besov spaces. In Section 3, we give some preliminary results on sequences which will be needed for our purposes. In Section 4, we establish some general estimates within the framework of subatomic decompositions and, in Section 5, we use these estimates to prove a generalization of Fact 1.1 which will be used to prove Theorem 1.7. In Section 6, we prove at a stroke Theorems 1.3 and 1.6 using the results collected at Section 3. Finally, in Section 7, we prove Theorems 1.7 and 1.8.

2 Notations and definitions

For the convenience of the reader, we specify below some notations used all along this chapter.

As usual, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{C}$ the set of all complex numbers and $\mathbb{Z}$ the collection of all integers. The set of all *nonnegative* integers $\{0, 1, 2, ...\}$ will be denoted by $\mathbb{N}$, and the set of all *positive* integers $\{1, 2, ...\}$ will be denoted by $\mathbb{N}^*$.

The N-dimensional real Euclidean space will be denoted by $\mathbb{R}^N$. Similarly, $\mathbb{N}^N$ (resp. $\mathbb{Z}^N$) stands for the lattice of all points $m = (m_1, \dots, m_N) \in \mathbb{R}^N$ with $m_j \in \mathbb{N}$ (resp. $m_j \in \mathbb{Z}$).

Given a real number $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ its integral part and by x_+ its positive part $\max\{0, x\}$. By analogy, we write $\mathbb{R}_+ := \{x_+ : x \in \mathbb{R}\}$.

The cardinal of a discrete set $E \subset \mathbb{Z}$ will be denoted by $\#E$. Given two integers $a, b \in \mathbb{Z}$ with $a < b$ we denote by $[\![a, b]\!]$ the set of all integers belonging to the segment line $[a, b]$, namely

$$[\![a, b]\!] := [a, b] \cap \mathbb{Z}.$$

We will sometimes make use of the approximatively-less-than symbol "$\lesssim$", that is we write $a \lesssim b$ for $a \leqslant C\, b$ where $C > 0$ is a constant independent of a and b.

Similarly, $a \gtrsim b$ means that $b \lesssim a$. Also, we write $a \sim b$ whenever $a \lesssim b$ and $b \lesssim a$.

We will denote by $\mathscr{L}_N$ the N-dimensional Lebesgue measure and by B_R the N-dimensional ball of radius $R > 0$ centered at zero.

The characteristic function of a set $E \subset \mathbb{R}^N$ will be denoted by $\mathbb{1}_E$.

We recall that a *quasi-norm* is similar to a norm in that it satisfies the norm axioms, except that the triangle inequality is replaced by

$$\|x + y\| \leqslant K(\|x\| + \|y\|) \quad \text{for some } K > 0.$$

Given two quasi-normed spaces $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$, we say that $A \hookrightarrow B$ when $A \subset B$ with continuous embedding, i.e. when

$$\|f\|_B \lesssim \|f\|_A \quad \text{for all } f \in A.$$

Further, we denote by $\ell^p(\mathbb{N})$, $0 < p < \infty$, the space of sequences $u = (u_j)_{j \geqslant 0}$ such that

$$\|u\|_{\ell^p(\mathbb{N})} := \left(\sum_{j \geqslant 0} |u_j|^p \right)^{1/p} < \infty,$$

and by $\ell^\infty(\mathbb{N})$ the space of bounded sequences.

As usual, $\mathscr{S}(\mathbb{R}^N)$ denotes the (Schwartz) space of rapidly decaying functions and $\mathscr{S}'(\mathbb{R}^N)$ its dual, the space of tempered distributions.

Given $0 < p \leqslant \infty$, we denote by $L^p(\mathbb{R}^N)$ the space of measurable functions f in $\mathbb{R}^N$ for which the p-th power of the absolute value is Lebesgue integrable (resp. f is essentially bounded when $p = \infty$), endowed with the quasi-norm

$$\|f\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |f(x)|^p \mathrm{d}x \right)^{1/p},$$

(resp. the essential sup-norm when $p = \infty$).

We collect below the different representations of Besov spaces which will be in use in this chapter.

2.1 Classical Besov spaces

Perhaps the simplest (and the most intuitive) way to define Besov spaces is through finite differences. This can be done as follows.

Let f be a function in $\mathbb{R}^N$. Given $M \in \mathbb{N}^*$ and $h \in \mathbb{R}^N$, let

$$\Delta_h^M f(x) = \sum_{j=0}^{M} (-1)^{M-j} \binom{M}{j} f(x + hj),$$

be the iterated difference operator.

Within these notations, Besov spaces can be defined as follows.

Definition 2.1. — *Let $M \in \mathbb{N}^*$, $0 < p, q \leqslant \infty$ and $s \in (0, M)$ with $s > \sigma_p$ where σ_p is given by* (1.1)*. The Besov space $B^s_{p,q}(\mathbb{R}^N)$ consists of all functions $f \in L^p(\mathbb{R}^N)$ such that*

$$[f]_{B^s_{p,q}(\mathbb{R}^N)} := \left(\int_{|h| \leqslant 1} \|\Delta^M_h f\|^q_{L^p(\mathbb{R}^N)} \frac{\mathrm{d}h}{|h|^{N+sq}} \right)^{1/q} < \infty,$$

which, in the case $q = \infty$, is to be understood as

$$[f]_{B^s_{p,\infty}(\mathbb{R}^N)} := \sup_{|h| \leqslant 1} \frac{\|\Delta^M_h f\|_{L^p(\mathbb{R}^N)}}{|h|^s} < \infty.$$

The space $B^s_{p,q}(\mathbb{R}^N)$ is naturally endowed with the quasi-norm

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)} + [f]_{B^s_{p,q}(\mathbb{R}^N)}. \tag{2.1}$$

Remark 2.2. — Different choices of M in (2.1) yield equivalent quasi-norms.

Remark 2.3. — If $p, q \geqslant 1$, then $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^N)}$ is a norm. However, if either $0 < p < 1$ or $0 < q < 1$, then the triangle inequality is no longer satisfied and it is only a quasi-norm. Nevertheless, we have the following useful inequality

$$\|f + g\|_{B^s_{p,q}(\mathbb{R}^N)} \leqslant \left(\|f\|^\eta_{B^s_{p,q}(\mathbb{R}^N)} + \|g\|^\eta_{B^s_{p,q}(\mathbb{R}^N)} \right)^{1/\eta},$$

where $\eta := \min\{1, p, q\}$, which compensates the absence of a triangle inequality.

For our purposes, we shall require a more abstract apparatus which will be provided by the so-called *subatomic* (or *quarkonial*) *decompositions*. This provides a way to decompose any $f \in B^s_{p,q}(\mathbb{R}^N)$ along elementary building blocks (essentially made up of a single function independent of f) and to, somehow, reduce it to a sequence of numbers (depending linearly on f). This type of decomposition first appeared in the monograph [137] of Triebel and was further developed in [139] (see also [84, 92, 112, 140]). We outline below the basics of the theory.

Given $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^N$, we denote by $Q_{\nu,m} \subset \mathbb{R}^N$ the cube with sides parallel to the coordinate axis, centered at $2^{-\nu}m$ and with side-length $2^{-\nu}$.

Definition 2.4. — *Let $\psi \in C^\infty(\mathbb{R}^N)$ be a nonnegative function with*

$$\operatorname{supp}(\psi) \subset \left\{ y \in \mathbb{R}^N : |y| < 2^r \right\},$$

for some $r \geqslant 0$ and

$$\sum_{k \in \mathbb{Z}^N} \psi(x - k) = 1 \quad \textit{for any } x \in \mathbb{R}^N.$$

Let $s > 0$, $0 < p \leqslant \infty$, $\beta \in \mathbb{N}^N$ and $\psi^\beta(x) = x^\beta \psi(x)$. Then, for $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^N$, the function

$$(\beta\mathrm{qu})_{\nu,m}(x) := 2^{-\nu(s-\frac{N}{p})}\psi^\beta(2^\nu x - m) \quad \text{for } x \in \mathbb{R}^N, \tag{2.2}$$

is called an (s,p)-β-quark relative to the cube $Q_{\nu,m}$.

Remark 2.5. — When $p = \infty$, (2.2) means $(\beta\mathrm{qu})_{\nu,m}(x) := 2^{-\nu s}\psi^\beta(2^\nu x - m)$.

Definition 2.6. — *Given $0 < p, q \leqslant \infty$, we define $b_{p,q}$ as the space of sequences $\lambda = (\lambda_{\nu,m})_{\nu \geqslant 0, m \in \mathbb{Z}^N}$ such that*

$$\|\lambda\|_{b_{p,q}} := \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^N} |\lambda_{\nu,m}|^p \right)^{q/p} \right)^{1/q} < \infty.$$

For the sake of convenience we will make use of the following notations

$$\lambda = \left\{ \lambda^\beta : \beta \in \mathbb{N}^N \right\} \quad \text{with} \quad \lambda^\beta = \left\{ \lambda^\beta_{\nu,m} \in \mathbb{C} : (\nu, m) \in \mathbb{N} \times \mathbb{Z}^N \right\}.$$

Then, we have the

Theorem 2.7. — *Let $0 < p, q \leqslant \infty$, $s > \sigma_p$ and $(\beta\mathrm{qu})_{\nu,m}$ be (s,p)-β-quarks according to Definition 2.4. Let $\varrho > r$ (where r has the same meaning as in Definition 2.4). Then, $B^s_{p,q}(\mathbb{R}^N)$ coincides with the collection of all $f \in \mathscr{S}'(\mathbb{R}^N)$ which can be represented as*

$$f(x) = \sum_{\beta \in \mathbb{N}^N} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^N} \lambda^\beta_{\nu,m} (\beta\mathrm{qu})_{\nu,m}(x), \tag{2.3}$$

where $\lambda^\beta \in b_{p,q}$ is a sequence such that

$$\|\lambda\|_{b_{p,q,\varrho}} := \sup_{\beta \in \mathbb{N}^N} 2^{\varrho|\beta|} \|\lambda^\beta\|_{b_{p,q}} < \infty.$$

Moreover,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \sim \inf_{(2.3)} \|\lambda\|_{b_{p,q,\varrho}}, \tag{2.4}$$

where the infimum is taken over all admissible representations (2.3)*. In addition, the right hand side of* (2.4) *is independent of the choice of ψ and $\varrho > r$.*

An elegant proof of this result may be found in [137, Section 14.15, pp.101-104] (see also [139, Theorem 2.9, p.15]).

Remark 2.8. — It is known that, given $f \in B^s_{p,q}(\mathbb{R}^N)$ and a fixed $\varrho > r$, there is a decomposition $\lambda^\beta_{\nu,m}$ (depending on the choice of $(\beta\mathrm{qu})_{\nu,m}$ and ϱ) realizing the infimum in (2.4) and which is said to be an *optimal subatomic decomposition* of f. We refer to [139] (especially Corollary 2.12 on p.23) for further details.

2.2 Besov spaces of generalized smoothness

Before we define what we mean by "Besov space of generalized smoothness", we first introduce some necessary definitions.

Definition 2.9. — *A real function* Ψ *on the interval* $(0,1]$ *is called admissible if it is positive and monotone on* $(0,1]$, *and if*

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}) \quad \text{for any } j \in \mathbb{N}.$$

Example 2.10. — Let $0 < c < 1$ and $b \in \mathbb{R}$. Then,

$$\Psi(x) := |\log_2(cx)|^b \quad \text{for } x \in (0,1],$$

is an example of admissible function. Another example is

$$\Psi(x) := (\log_2|\log_2(cx)|)^b \quad \text{for } x \in (0,1].$$

Roughly speaking, admissible functions are functions having at most logarithmic growth or decay near zero. They may be seen as particular cases of a class of functions introduced by Karamata in the mid-thirties [91, 90] known as *slowly varying functions* (see e.g. [23, Definition 1.2.1, p.6] and also [34, Section 3, p.226] where the reader may find a discussion on how these two notions relate to each other as well as various examples and references).

We refer the interested reader to [111, 139] for a detailed review of the properties of admissible functions.

Definition 2.11. — *Let* $M \in \mathbb{N}^*$, $0 < p, q \leqslant \infty$, $s \in (0, M)$ *with* $s > \sigma_p$ *and let* Ψ *be an admissible function. The Besov space of generalized smoothness* $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ *consists of all functions* $f \in L^p(\mathbb{R}^N)$ *such that*

$$[f]_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} := \left(\int_0^1 \sup_{|h| \leqslant t} \|\Delta_h^M f\|^q_{L^p(\mathbb{R}^N)} \frac{\Psi(t)^q}{t^{1+sq}} \, \mathrm{d}t \right)^{1/q} < \infty,$$

which, in the case $q = \infty$, *is to be understood as*

$$[f]_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^N)} := \sup_{0 < t \leqslant 1} t^{-s} \Psi(t) \sup_{|h| \leqslant t} \|\Delta_h^M f\|_{L^p(\mathbb{R}^N)} < \infty.$$

The space $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ *is naturally endowed with the quasi-norm*

$$\|f\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} := \|f\|_{L^p(\mathbb{R}^N)} + [f]_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)}. \tag{2.5}$$

Remark 2.12. — Different choices of M in (2.5) yield equivalent quasi-norms.

Remark 2.13. — Observe that, by taking $\Psi \equiv 1$, we recover the usual Besov spaces, that is we have

$$\|f\|_{B^{(s,1)}_{p,q}(\mathbb{R}^N)} \sim \|f\|_{B^s_{p,q}(\mathbb{R}^N)},$$

see [136, Theorem 2.5.12, p.110] for a proof of this.

Remark 2.14. — As already mentioned in the introduction, these spaces were introduced by Triebel and Edmunds in [60, 61] to study some fractal pseudo-differential operators, but the first comprehensive studies go back to Moura [111] (see also [35, 40, 39, 81, 92, 140, 141]). In the literature these spaces are usually defined from the Fourier-analytical point of view (e.g. in [111, 139]) but, as shown in [81, Theorem 2.5, p.161], the two approaches are equivalent.

Remark 2.15. — Notice that, here as well, the triangle inequality fails to hold when either $0 < p < 1$ or $0 < q < 1$, but, in virtue of the Aoki-Rolewicz lemma, we have the same kind of compensation as in the classical case, see [87, Lemma 1.1, p.3]. That is, there exists $\eta \in (0,1]$ and an equivalent quasi-norm $\|\cdot\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N),*}$ with

$$\|f+g\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N),*} \leqslant \Big(\|f\|^\eta_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N),*} + \|g\|^\eta_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N),*}\Big)^{1/\eta}.$$

A fine property of these spaces is that they admit subatomic decompositions. In fact, it suffices to modify the definition of (s,p)-β-quarks to this generalized setting in the following way.

Definition 2.16. — *Let r, ψ and ψ^β with $\beta \in \mathbb{N}^N$ be as in Definition 2.4. Let $s > 0$ and $0 < p \leqslant \infty$. Let Ψ be an admissible function. Then, in generalization of* (2.2),

$$(\beta\mathrm{qu})_{\nu,m}(x) := 2^{-\nu(s-\frac{N}{p})}\Psi(2^{-\nu})^{-1}\psi^\beta(2^\nu x - m) \quad \text{for } x \in \mathbb{R}^N,$$

is called an (s,p,Ψ)-β-quark.

Then, we have the following

Theorem 2.17. — *Let $0 < p,q \leqslant \infty$, $s > \sigma_p$ and Ψ be an admissible function. Let $(\beta\mathrm{qu})_{\nu,m}$ be (s,p,Ψ)-β-quarks according to Definition 2.16. Let $\varrho > r$ (where r has the same meaning as in Definition 2.16). Then, $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ coincides with the collection of all $f \in \mathscr{S}'(\mathbb{R}^N)$ which can be represented as*

$$f(x) = \sum_{\beta\in\mathbb{N}^N}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^N} \lambda^\beta_{\nu,m}(\beta\mathrm{qu})_{\nu,m}(x), \tag{2.6}$$

where $\lambda^\beta \in b_{p,q}$ is a sequence such that

$$\|\lambda\|_{b_{p,q,\varrho}} := \sup_{\beta\in\mathbb{N}^N} 2^{\varrho|\beta|}\|\lambda^\beta\|_{b_{p,q}} < \infty.$$

Moreover,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \sim \inf_{(2.6)} \|\lambda\|_{b_{p,q,\varrho}}\,, \tag{2.7}$$

where the infimum is taken over all admissible representations (2.6). *In addition, the right hand side of* (2.7) *is independent of the choice of* ψ *and* $\varrho > r$.

This result can be found in [111, Theorem 1.23, pp.35-36] (see also [92, Theorem 10, p.284]).

Remark 2.18. — The counterpart of Remark 2.8 for $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ remains valid, see [111, Remark 1.26, p.48].

2.3 Related spaces and embeddings

Let us now say a brief word about embeddings. Given a locally integrable function f in $\mathbb{R}^N$ and a set $B \subset \mathbb{R}^N$ having finite nonzero Lebesgue measure, we let

$$f_B := \fint_B f(y)\,\mathrm{d}y = \frac{1}{\mathscr{L}_N(B)} \int_B f(y)\,\mathrm{d}y,$$

be the average of f on B.

Moreover, we denote by $f^* : \mathbb{R}_+ \to \mathbb{R}_+$ the *decreasing rearrangement* of f, given by

$$f^*(t) := \inf\left\{\lambda \geqslant 0 : \mu_f(\lambda) \leqslant t\right\},$$

for all $t \geqslant 0$, where

$$\mu_f(\lambda) := \mathscr{L}_N\left(\left\{x \in \mathbb{R}^N : |f(x)| > \lambda\right\}\right),$$

is the so-called *distribution function* of f.

Definition 2.19. — *Let* $s > 0$ *and* $0 < p < \infty$.

(i) *The Zygmund-Hölder space* $C^s(\mathbb{R}^N)$ *is the Besov space* $B^s_{\infty,\infty}(\mathbb{R}^N)$.

(ii) *The space of functions of bounded mean oscillation, denoted by* $\mathrm{BMO}(\mathbb{R}^N)$, *consists of all locally integrable functions* f *such that*

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^N)} := \sup_B \fint_B |f(x) - f_B|\,\mathrm{d}x < \infty, \tag{2.8}$$

where the supremum in (2.8) *is taken over all balls* $B \subset \mathbb{R}^N$.

(iii) *The weak* L^p*-space, denoted by* $L^{p,\infty}(\mathbb{R}^N)$, *consists of all measurable functions* f *such that*

$$\|f\|_{L^{p,\infty}(\mathbb{R}^N)} := \sup_{t>0} t^{1/p} f^*(t) < \infty,$$

where f^* *is the decreasing rearrangement of* f.

Let us now state the following

THEOREM 2.20 (Sobolev embedding theorem for $B^s_{p,q}$). — *Let $0 < p, q < \infty$ and let $s > \sigma_p$. Then, the following hold true:*

(i) *if $sp > N$, then $B^s_{p,q}(\mathbb{R}^N) \hookrightarrow C^{s-\frac{N}{p}}(\mathbb{R}^N)$;*

(ii) *if $sp = N$, then $B^s_{p,q}(\mathbb{R}^N) \hookrightarrow \mathrm{BMO}(\mathbb{R}^N)$;*

(iii) *if $sp < N$, then $B^s_{p,q}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np}{N-sp},\infty}(\mathbb{R}^N)$.*

In particular, $B^s_{p,q}(\mathbb{R}^N) \hookrightarrow A^{s,p}(\mathbb{R}^N)$ where $A^{s,p}(\mathbb{R}^N)$ is as in (1.2).

Proof. — The cases (i), (ii) and (iii) are respectively covered by [136, Formula (12), p.131], [108, Lemma 6.5] and [117, Théorème 8.1, p.301]. □

Remark 2.21. — Let us briefly mention that a corresponding result holds for the spaces $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$. As already mentioned in the introduction, the space $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ is embedded in a generalized version of the Hölder space when $sp > N$. When $sp < N$, it is shown in [40] that $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ embeds in a weighted version of $L^{\frac{Np}{N-sp},\infty}(\mathbb{R}^N)$. Yet, when $sp = N$, the corresponding substitute for BMO does not seem to have been clearly identified nor considered in the literature, see however [39, 78, 79, 110] where some partial results are given.

3 Preliminaries

In this section, we study the properties of some discrete sequences which will play an important role in the sequel. More precisely, we will be interested in the convergence of series of the type

$$\sum_{j\geqslant 0} 2^j |\lambda_{j,\lfloor 2^j x\rfloor}| \quad \text{for } x > 0,$$

where $\lambda = (\lambda_{j,k})_{j,k\geqslant 0}$ is an element of some Besov sequence space, say, $b_{1,q}$ with $q > 1$.

3.1 Some technical lemmata

Let us start with a famous result due to Cauchy.

THEOREM 3.1 (Cauchy's condensation test). — *Let $\lambda \in \ell^1(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then,*

$$\sum_{j\geqslant 1} \lambda_j \leqslant \sum_{j\geqslant 0} 2^j \lambda_{2^j} \leqslant 2 \sum_{j\geqslant 1} \lambda_j.$$

Remark 3.2. — The monotonicity assumption on λ is central here. Indeed, there exist nonnegative sequences $\lambda \in \ell^1(\mathbb{N})$ which are not nonincreasing and such that $\sum_{j\geqslant 0} 2^j \lambda_{2^j} = \infty$. Take for example:

$$\lambda_j = \begin{cases} 1/k^2 & \text{if } j = 2^k \text{ and } k \neq 0, \\ 2^{-j} & \text{else.} \end{cases}$$

Then, clearly, $\lambda \in \ell^1(\mathbb{N})$. However, $2^j \lambda_{2^j} = \frac{2^j}{j^2}$ when $j \geqslant 1$, so that $(2^j \lambda_{2^j})_{j\geqslant 0} \notin \ell^1(\mathbb{N})$. A simple consequence of Cauchy's condensation test is the following

Lemma 3.3. — *Let $\lambda \in \ell^1(\mathbb{N})$ be a nonnegative, nonincreasing sequence. Then,*

$$\sum_{j\geqslant 0} 2^j \lambda_{\lfloor 2^j x \rfloor} \leqslant \phi(x) \sum_{j\geqslant 1} \lambda_j \quad \textit{for any } x > 0,$$

where $\phi(x) := \frac{4}{|x|}\left(\mathbb{1}_{[1,\infty)}(x) + (1-\log_2|x|)\mathbb{1}_{(0,1)}(x)\right)$.

Proof. — Let $k \in \mathbb{N}$ and $2^k \leqslant x \leqslant 2^{k+1}$. Then, by Cauchy's condensation test

$$\sum_{j\geqslant 0} 2^j \lambda_{\lfloor 2^j x \rfloor} \leqslant \sum_{j\geqslant 0} 2^j \lambda_{2^{k+j}} = 2^{-k} \sum_{j\geqslant k} 2^j \lambda_{2^j} \leqslant 2^{-k} \sum_{j\geqslant 0} 2^j \lambda_{2^j} \leqslant \frac{4}{x} \sum_{j\geqslant 1} \lambda_j.$$

In like manner, for $2^{-(k+1)} \leqslant x \leqslant 2^{-k}$, we have

$$\sum_{j\geqslant 0} 2^j \lambda_{\lfloor 2^j x \rfloor} \leqslant \sum_{j\geqslant 0} 2^j \lambda_{\lfloor 2^{j-k-1} \rfloor} = 2^{k+1} \sum_{j\geqslant -k-1} 2^j \lambda_{\lfloor 2^j \rfloor} \leqslant 2^{k+1}(k+1) \sum_{j\geqslant 0} 2^j \lambda_{2^j}.$$

Finally, invoking again Cauchy's condensation test, we have

$$\sum_{j\geqslant 0} 2^j \lambda_{\lfloor 2^j x \rfloor} \leqslant 2^{k+2}(k+1) \sum_{j\geqslant 1} \lambda_j \leqslant \frac{4}{x}\big(1 - \log_2(x)\big) \sum_{j\geqslant 1} \lambda_j.$$

This completes the proof. □

In some sense, this "functional version" of Cauchy's condensation test may be generalized to sequences which are not necessarily nonincreasing.

Indeed, one can show that

$$\mathscr{L}_1\left(\left\{x \in \mathbb{R}_+ : \sum_{j\geqslant 0} 2^j |\lambda_{\lfloor 2^j x \rfloor}| = +\infty\right\}\right) = 0,$$

whenever $\lambda \in \ell^1(\mathbb{N})$. This is due to the fact that ℓ^p-spaces can be seen as "amalgams" of $L^p(1,2)$ and a weighted version of ℓ^p.

More precisely, we have

LEMMA 3.4. — *Let $0 < p < \infty$ and let $\lambda \in \ell^p(\mathbb{N})$. Then,*

$$\Big(\sum_{j\geqslant 1} |\lambda_j|^p\Big)^{1/p} = \Big(\int_{[1,2]} \sum_{j\geqslant 0} 2^j |\lambda_{\lfloor 2^j x\rfloor}|^p \,\mathrm{d}x\Big)^{1/p}.$$

Proof. — It suffices to assume $p = 1$ and that λ is nonnegative. Then,

$$\frac{1}{2^k} \sum_{2^k \leqslant j < 2^{k+1}} \lambda_j = \fint_{[2^k,2^{k+1}]} \lambda_{\lfloor x\rfloor} \,\mathrm{d}x = \frac{1}{2^k} \int_{[1,2]} \lambda_{\lfloor 2^k y\rfloor} 2^k \,\mathrm{d}y = \int_{[1,2]} \lambda_{\lfloor 2^k y\rfloor} \,\mathrm{d}y,$$

which yields

$$\sum_{j\in\mathbb{N}^*} \lambda_j = \sum_{k\in\mathbb{N}} \sum_{2^k\leqslant j<2^{k+1}} \lambda_j = \sum_{k\in\mathbb{N}} 2^k \int_{[1,2]} \lambda_{\lfloor 2^k x\rfloor} \,\mathrm{d}x = \int_{[1,2]} \Big(\sum_{k\in\mathbb{N}} \lambda_{\lfloor 2^k x\rfloor} 2^k\Big) \mathrm{d}x.$$

The proof is complete. □

We now establish a technical inequality which will be needed in the sequel.

LEMMA 3.5. — *Let $N \geqslant 1$ and $0 < p < \infty$. Let $\lambda = (\lambda^\beta_{j,k})_{(j,\beta,k)\in\mathbb{N}\times\mathbb{N}^N\times\mathbb{N}^N}$ be a sequence such that the partial sequences $(\lambda^\beta_{j,k})_{k\in\mathbb{N}^N}$ belong to $\ell^p(\mathbb{N}^N)$ for all $(j,\beta) \in \mathbb{N}\times\mathbb{N}^N$. Then, for any positive $(\alpha_j)_{j\geqslant 0} \in \ell^1(\mathbb{N})$ there exists $C = C(\lambda,\alpha,N,d) > 0$ such that for any $(j,\beta) \in \mathbb{N}\times\mathbb{N}^N$,*

$$2^{jN} |\lambda^\beta_{j,\lfloor 2^j x\rfloor}|^p \leqslant C \,\frac{\max\{1, |\beta|^{N+1}\}}{\alpha_j} \sum_{k\in\mathbb{N}^N} |\lambda^\beta_{j,k}|^p,$$

holds for a.e. $x = (x_1,\dots,x_N) \in [1,2]^N$ where

$$\lfloor 2^j x\rfloor = (\lfloor 2^j x_1\rfloor, \dots, \lfloor 2^j x_N\rfloor) \in \mathbb{N}^N.$$

Proof. — For the sake of convenience, we use the following notations

$$U_{j,\beta}(x) := 2^{jN} |\lambda^\beta_{j,\lfloor 2^j x\rfloor}|^p \quad \text{and} \quad U^{j,\beta} := \sum_{k\in\mathbb{N}^N} |\lambda^\beta_{j,k}|^p.$$

We have to prove that

$$U_{j,\beta}(x) \leqslant C \,\frac{\max\{1, |\beta|^{N+1}\}}{\alpha_j}\, U^{j,\beta},$$

for a.e. $x \in [1,2]^N$ and any $(j,\beta) \in \mathbb{N}\times\mathbb{N}^N$. By iterated applications of Lemma 3.4, we have

$$\int_{[1,2]^N} U_{j,\beta}(x) \,\mathrm{d}x \leqslant U^{j,\beta}. \tag{3.1}$$

Now, define

$$\Gamma_{j,\beta} := \left\{ x \in [1,2]^N : U_{j,\beta}(x) \geqslant \frac{\max\{1, |\beta|^{N+1}\}}{\alpha_j} U^{j,\beta} \right\}.$$

Then, applying Markov's inequality and using (3.1), we have

$$\mathscr{L}_N(\Gamma_{j,\beta}) \leqslant \frac{\alpha_j}{\max\{1, |\beta|^{N+1}\}} \quad \text{for any } (j,\beta) \in \mathbb{N} \times \mathbb{N}^N.$$

In turn, this gives

$$\sum_{\beta \in \mathbb{N}^N} \sum_{j \geqslant 0} \mathscr{L}_N(\Gamma_{j,\beta}) < \infty.$$

Therefore, we can apply the Borel-Cantelli lemma and deduce that there exists $j_0, \beta_0 \geqslant 0$ such that

$$U_{j,\beta}(x) \leqslant \frac{\max\{1, |\beta|^{N+1}\}}{\alpha_j} U^{j,\beta},$$

for any $j > j_0$ and/or $|\beta| > \beta_0$ and a.e. $x \in [1,2]^N$. On the other hand, for any $j \leqslant j_0$ and $|\beta| \leqslant \beta_0$ we have

$$U_{j,\beta}(x) \leqslant 2^{j_0 N} \max\{1, |\beta|^{N+1}\} \frac{\max_{0 \leqslant j \leqslant j_0} \alpha_j}{\alpha_j} U^{j,\beta}.$$

This completes the proof. □

3.2 Some useful sequences

We now construct some key sequences which will be at the heart of the proofs of Theorems 1.3, 1.6 and 1.8.

Lemma 3.6. — *There exists a sequence $(\zeta_k)_{k \geqslant 0} \subset \mathbb{R}_+$ satisfying*

$$\sup_{j \geqslant 0} \frac{1}{2^j} \sum_{2^j \leqslant k < 2^{j+1}} \zeta_k \leqslant 1, \tag{3.2}$$

and such that

$$\sup_{j \geqslant 0} \zeta_{\lfloor 2^j x \rfloor} = \infty \quad \text{for all } x \in [1,2). \tag{3.3}$$

Proof. — Let us first construct an auxiliary sequence satisfying (3.2).

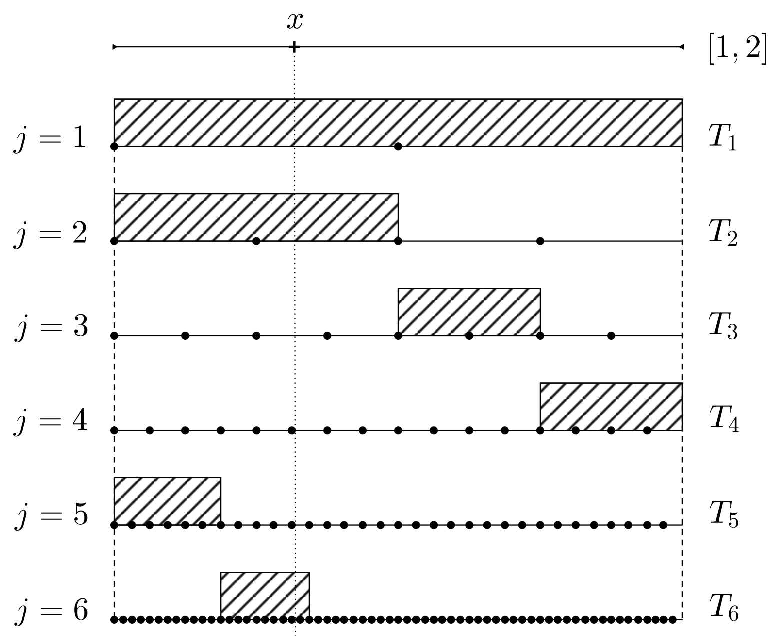

Figure 2.1. Construction of the first terms of $(\zeta_k)_{k\geqslant 0}$.
The hatched zone corresponds to the values of x for which $\zeta_{\lfloor 2^j x\rfloor}$ takes nonzero values.

Let $(\lambda_k)_{k\geqslant 0}$ be a sequence such that $\lambda_0 = \lambda_1 = 0$ and such that, for any $j \geqslant 1$, the $\lfloor \frac{2^j}{j} \rfloor$ first terms of the sequence $(\lambda_k)_{k\geqslant 0}$ on the discrete interval $[\![2^j, 2^{j+1}-1]\!]$ have value j and the remaining terms are all equal to zero. Then, for any $j \geqslant 1$, we have

$$\frac{1}{2^j}\sum_{2^j\leqslant k<2^{j+1}} \lambda_k = \frac{j+\cdots+j+0+\cdots+0}{2^j} = \frac{j\lfloor \frac{2^j}{j}\rfloor}{2^j} \leqslant 1.$$

For the sake of convenience, we set

$$T_j := [\![2^j, 2^{j+1}-1]\!] \quad \text{for any } j \geqslant 0.$$

We will construct a sequence $(\zeta_k)_{k\geqslant 0}$ satisfying both (3.2) and (3.3) by rearranging the terms of $(\lambda_k)_{k\geqslant 0}$. To this end, we follow the following procedure. For $k \in [\![0, 2^3-1]\!]$ we impose $\zeta_k = \lambda_k$. For $j = 3$, we shift the values of $(\lambda_k)_{k\geqslant 0}$ on T_3 in such a way that the smallest $x \in [1,2)$ such that $\zeta_{\lfloor 2^3 x\rfloor}$ is nonzero coincides with the limit superior of the set of all $z \in [1,2)$ such that $\zeta_{\lfloor 2^2 z\rfloor}$ is nonzero. For $j = 4$, we shift the values of $(\lambda_k)_{k\geqslant 0}$ on T_4 in such a way that the smallest $x \in [1,2]$ such that $\zeta_{\lfloor 2^4 x\rfloor}$ is nonzero coincides with the limit superior of the set of all $z \in [1,2)$ such that $\zeta_{\lfloor 2^3 z\rfloor}$ is nonzero, and so on. When the range of nonzero terms has reached the last term on T_j for some $j \geqslant 1$, we start again from T_{j+1} and set $\zeta_k = \lambda_k$ on T_{j+1}, and we repeat the above procedure. See Figure 2.1 for a visual illustration.

If, for some $j \geqslant 0$, it happens that the above shifting of the λ_k's on T_j exceeds T_j, then we shift the λ_k's on T_j in such a way that the limit superior of the set of all $x \in [1, 2]$ for which $\zeta_{\lfloor 2^j x \rfloor}$ is nonzero coincides with $x = 2$.

Note that this procedure is well-defined because the proportion of nonzero terms on each T_j is $2^{-j} \lfloor \frac{2^j}{j} \rfloor$ which has a divergent series thus allowing us to fill as much "space" as needed.

Then, by construction, for any $x \in [1, 2)$ there are infinitely many values of $j \geqslant 0$ such that $\zeta_{\lfloor 2^j x \rfloor} = j$. Consequently, (3.3) holds. Moreover, (3.2) is trivially satisfied.

This completes the proof. $\square$

As an immediate corollary, we have

Corollary 3.7. — *Let $0 < p < q \leqslant \infty$. Then, there exists a sequence $(\lambda_{j,k})_{j,k \geqslant 0} \subset \mathbb{R}_+$ satisfying*

$$\left(\sum_{j \geqslant 0} \left(\sum_{k \geqslant 0} \lambda_{j,k}^p \right)^{q/p} \right)^{1/q} < \infty,$$

(with the usual modification if $q = \infty$) and such that

$$\sup_{j \geqslant 0} 2^{j/p} \lambda_{j, \lfloor 2^j x \rfloor} = \infty \quad \textit{for all } x \in [1, 2).$$

Remark 3.8. — By "usual modification if $q = \infty$" we mean that, when $q = \infty$, the quasi-norm $\|\cdot\|_{\ell^q(\mathbb{N})} = (\sum_{j \geqslant 0} |\cdot|^q)^{1/q}$ is replaced by $\|\cdot\|_{\ell^\infty(\mathbb{N})} = \sup_{j \geqslant 0} |\cdot|$.

Proof. — When $q = \infty$, it suffices to set

$$\lambda_{j,k} = \begin{cases} 2^{-j/p} \zeta_k^{1/p} & \text{if } \ 2^j \leqslant k < 2^{j+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.4}$$

where $(\zeta_k)_{k \geqslant 0}$ is the sequence constructed at Lemma 3.6.

When $q < \infty$, we simply replace $(\zeta_k)_{k \geqslant 0}$ in (3.4) by $(\xi_k)_{k \geqslant 0}$ where

$$\xi_k = j^{-\sqrt{\frac{p}{q}}} \zeta_k \quad \text{for any } k \in [\![2^j, 2^{j+1} - 1]\!] \text{ with } j \geqslant 1,$$

and $\xi_0 = \xi_1 = 0$. Then, we obtain

$$\sum_{j \geqslant 1} \left(\frac{1}{2^j} \sum_{2^j \leqslant k < 2^{j+1}} \xi_k \right)^{q/p} = \sum_{j \geqslant 1} \left(j^{-\sqrt{\frac{p}{q}}} \cdot \frac{1}{2^j} \sum_{2^j \leqslant k < 2^{j+1}} \zeta_k \right)^{q/p} \leqslant \sum_{j \geqslant 1} j^{-\sqrt{\frac{q}{p}}} < \infty.$$

Moreover, by construction of $(\zeta_k)_{k\geqslant 0}$, for any $x \in [1,2)$, there is a countably infinite set $J_x \subset \mathbb{N}$ such that $\zeta_{\lfloor 2^j x\rfloor} = j$ for any $j \in J_x$. In particular,

$$\xi_{\lfloor 2^j x\rfloor} = j^\alpha \quad \text{for any } j \in J_x \text{ and } x \in [1,2),$$

where $\alpha = 1 - \sqrt{p/q} > 0$. Thus,

$$\sup_{j\geqslant 0} 2^j \lambda^p_{j,\lfloor 2^j x\rfloor} = \sup_{j\geqslant 0} \xi_{\lfloor 2^j x\rfloor} \geqslant \sup_{j\in J_x} j^\alpha = \infty \quad \text{for any } x \in [1,2),$$

which is what we had to show. □

We conclude this section by a weighted version of Corollary 3.7.

Lemma 3.9. — *Let $0 < p < q \leqslant \infty$. Let Ψ be an admissible function that does not satisfy* (1.3). *If $q < \infty$ and Ψ is increasing assume, in addition, that*

$$\chi = \frac{qp}{q-p} < \frac{1}{c_\infty},$$

where c_∞ is as in Theorem 1.8. Then, there exists a sequence $(\lambda_{j,k})_{j,k\geqslant 0} \subset \mathbb{R}_+$ such that

$$\left(\sum_{j\geqslant 0}\left(\sum_{k\geqslant 0}\lambda^p_{j,k}\right)^{q/p}\right)^{1/q} < \infty, \tag{3.5}$$

(with the usual modification if $q = \infty$) and

$$\left(\sum_{j\geqslant 0} 2^{j\frac{q}{p}} \lambda^q_{j,\lfloor 2^j x\rfloor} \Psi(2^{-j})^q\right)^{1/q} = \infty \quad \textit{for all } x \in [1,2), \tag{3.6}$$

(with the usual modification if $q = \infty$).

Proof. — The proof is essentially the same as in the unweighted case with minor changes that we shall now detail.

Let us begin with the case $q = \infty$. Let $\beta_j := \Psi(2^{-j})^p$. Since $\beta_j > 0$ and $(\beta_j)_{j\geqslant 0} \notin \ell^1(\mathbb{N})$ we may find another positive sequence $(\gamma_j)_{j\geqslant 0}$ which has a divergent series and such that

$$\frac{\beta_j}{\gamma_j} \to \infty \quad \text{as } j \to \infty,$$

i.e. $(\gamma_j)_{j\geqslant 0}$ diverges slower than $(\beta_j)_{j\geqslant 0}$. Take, for example

$$\gamma_j = \frac{\beta_j}{\sum_{k=0}^j \beta_k},$$

see e.g. [7]. Note that $0 < \gamma_j \leqslant 1$ for all $j \geqslant 0$. Let $(\varrho_k)_{k\geqslant 0}$ be a sequence such that $\varrho_0 = \varrho_1 = 0$ and such that, for any $j \geqslant 1$, the $\lfloor 2^j\gamma_j \rfloor$ first terms of the sequence $(\varrho_k)_{k\geqslant 0}$ on the discrete interval $T_j := [\![2^j, 2^{j+1}-1]\!]$ have value $\frac{1}{\gamma_j}$ and the remaining terms are all equal to zero. Then, for any $j \geqslant 1$, we have

$$\frac{1}{2^j}\sum_{2^j\leqslant k<2^{j+1}} \varrho_k = \frac{\frac{1}{\gamma_j}+\cdots+\frac{1}{\gamma_j}+0+\cdots+0}{2^j} = \frac{\frac{1}{\gamma_j}\lfloor 2^j\gamma_j\rfloor}{2^j} \leqslant 1.$$

Now, since the proportion of nonzero terms on each T_j is $2^{-j}\lfloor 2^j\gamma_j\rfloor$ which has a divergent series, we may apply to $(\varrho_j)_{j\geqslant 0}$ the same rearrangement as in the proof of Lemma 3.6. That is, we can construct a sequence $(\varrho_j^*)_{j\geqslant 0}$ such that

$$\frac{1}{2^j}\sum_{2^j\leqslant k<2^{j+1}} \varrho_k^* \leqslant 1 \quad \text{for all } j \geqslant 0,$$

and for any $x \in [1,2)$ there is a countably infinite set $J_x \subset \mathbb{N}$ such that

$$\beta_j\varrho^*_{\lfloor 2^j x\rfloor} = \frac{\beta_j}{\gamma_j} = \sum_{k=0}^{j}\beta_k \quad \text{for all } j \in J_x,$$

i.e. we have

$$\sup_{j\geqslant 0}\,\beta_j\varrho^*_{\lfloor 2^j x\rfloor} \geqslant \lim_{\substack{j\to\infty\\ j\in J_x}}\sum_{k=0}^{j}\beta_k = \infty.$$

Therefore, letting

$$\lambda_{j,k} = \begin{cases} 2^{-j/p}(\varrho_k^*)^{1/p} & \text{if } 2^j \leqslant k < 2^{j+1}, \\ 0 & \text{otherwise}, \end{cases}$$

we obtain a sequence satisfying both (3.5) and (3.6).

Let us now prove the lemma when $q < \infty$. Notice that if Ψ is either constant or decreasing there is nothing to prove since the result is a consequence of Corollary 3.7. Hence, we may assume that Ψ is increasing. By our assumptions, we have

$$\frac{1}{p}\log_2\frac{\beta_j}{\beta_{2j}} \leqslant c_\infty < \frac{1}{\chi},$$

which implies that

$$\beta_j \leqslant 2^{c_\infty p}\beta_{2j} \leqslant \cdots \leqslant 2^{kc_\infty p}\beta_{2^k j} \quad \text{for any } k \in \mathbb{N}. \tag{3.7}$$

By Cauchy's condensation test, we have

$$2\sum_{j\geqslant 0}\beta_j^{\frac{q}{q-p}} \geqslant \sum_{j\geqslant 0} 2^j \beta_{2^j}^{\frac{q}{q-p}} \geqslant \beta_1^{\frac{q}{q-p}} \sum_{j\geqslant 0} 2^{j(1-c_\infty\chi)} = \infty.$$

Thus, we may infer as above that the following positive sequence has divergent series:

$$\widetilde{\gamma}_j = \frac{\beta_j^{\frac{q}{q-p}}}{\sum_{k=0}^{j}\beta_k^{\frac{q}{q-p}}}.$$

Notice that $\widetilde{\gamma}_j \leqslant 1/(j+1)$. Now define $(\tau_j)_{j\geqslant 0}$ by $\tau_j := \widetilde{\gamma}_j/\beta_j$. Since $2^{-c_\infty p}\beta_1 \leqslant j^{c_\infty p}\beta_j$ for any $j \geqslant 1$ (by (3.7) and the monotonicity of β_j), our assumptions on χ and c_∞ then imply

$$\sum_{j\geqslant 0}\tau_j^{q/p} \leqslant \tau_0^{q/p} + \sum_{j\geqslant 1}\frac{\beta_j^{-q/p}}{(j+1)^{q/p}} \leqslant \tau_0^{q/p} + \sum_{j\geqslant 1}\frac{2^{c_\infty q}\beta_1^{-q/p}}{j^{q(1/p-c_\infty)}} < \infty.$$

The conclusion now follows by letting $\widetilde{\lambda}_{j,k} := \tau_j^{1/p}\lambda_{j,k}$ where $\lambda_{j,k}$ is the sequence constructed above with $\widetilde{\gamma}_j$ instead of γ_j. Indeed, we have

$$\sum_{j\geqslant 0}\left(\sum_{k\geqslant 0}\widetilde{\lambda}_{j,k}^p\right)^{q/p} \leqslant \sum_{j\geqslant 0}\tau_j^{q/p} < \infty,$$

and, for each $x \in [1,2)$, there is a countably infinite set $\widetilde{J}_x \subset \mathbb{N}$ such that

$$2^{j/p}\widetilde{\lambda}_{j,\lfloor 2^j x\rfloor}\beta_j^{1/p} = 2^{j/p}\,\tau_j^{1/p}\left(\frac{\beta_j}{\widetilde{\gamma}_j}\right)^{1/p} 2^{-j/p} = 1 \quad \text{for any } j \in \widetilde{J}_x.$$

Therefore, $(2^{j/p}\widetilde{\lambda}_{j,\lfloor 2^j x\rfloor}\beta_j^{1/p})_{j\geqslant 0} \notin \ell^q(\mathbb{N})$. This completes the proof. $\square$

4 General estimates

Throughout this section we will write $x \in \mathbb{R}^N$ as $x = (x_1, \dots, x_N) = (x', x'')$ with $x' \in \mathbb{R}^d$, $x'' \in \mathbb{R}^{N-d}$ and, similarly, $m = (m', m'') \in \mathbb{Z}^N$ and $\beta = (\beta', \beta'') \in \mathbb{N}^N$. Also, we set

$$\mathscr{D} := \{0,1\}^{N-d}.$$

Let $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\operatorname{supp}(\psi) \subset B_1$ and that

$$2^{-\nu(s-\frac{N}{p})}\psi^\beta(2^\nu x - m),$$

are (s,p)-β-quarks. Also, we assume that ψ has the product structure

$$\psi(x_1,\dots,x_N) = \psi(x_1)\dots\psi(x_N). \tag{4.1}$$

Let $\varrho > 0$ and $f \in B^s_{p,q}(\mathbb{R}^N)$. Then, by Theorem 2.7, there are coefficients $\lambda^\beta_{\nu,m}$ such that

$$f(x) = \sum_{\beta\in\mathbb{N}^N}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^N}\lambda^\beta_{\nu,m}2^{-\nu(s-\frac{N}{p})}\psi^\beta(2^\nu x - m). \tag{4.2}$$

We can further assume that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \sim \sup_{\beta\in\mathbb{N}^N} 2^{\varrho|\beta|}\left(\sum_{\nu\geqslant 0}\left(\sum_{m\in\mathbb{Z}^N}|\lambda^\beta_{\nu,m}|^p\right)^{q/p}\right)^{1/q}, \tag{4.3}$$

i.e. that $\lambda^\beta_{\nu,m} = \lambda^\beta_{\nu,m}(f)$ is an *optimal subatomic decomposition* of f. Note, however, that the optimality of the decomposition $\lambda^\beta_{\nu,m}(f)$ depends on the choice of $\varrho > 0$ (this can be seen from [139, Corollary 2.12, p.23]). Of course, by Theorem 2.7, we still have

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \lesssim \sup_{\beta\in\mathbb{N}^N} 2^{\varrho'|\beta|}\left(\sum_{\nu\geqslant 0}\left(\sum_{m\in\mathbb{Z}^N}|\lambda^\beta_{\nu,m}|^p\right)^{q/p}\right)^{1/q},$$

for any positive $\varrho' \neq \varrho$. Using (4.1) and (4.3), we can decompose $f(\cdot, x'')$ as

$$f(x',x'') = \sum_{\nu\geqslant 0}\sum_{\beta'\in\mathbb{N}^d}\sum_{m'\in\mathbb{Z}^d} b^{\beta'}_{\nu,m'}(\lambda,x'')2^{-\nu(s-\frac{d}{p})}\psi^{\beta'}(2^\nu x' - m'),$$

where we have set

$$b^{\beta'}_{\nu,m'}(\lambda,x'') := 2^{\nu\frac{N-d}{p}}\sum_{\beta''\in\mathbb{N}^{N-d}}\sum_{m''\in\mathbb{Z}^{N-d}}\lambda^\beta_{\nu,m}\psi^{\beta''}(2^\nu x'' - m''). \tag{4.4}$$

Then, defining

$$J^\varrho_{p,q}(\lambda,x'') := \sup_{\beta'\in\mathbb{N}^d} 2^{\varrho|\beta'|}\left(\sum_{\nu\geqslant 0}\left(\sum_{m'\in\mathbb{Z}^d}|b^{\beta'}_{\nu,m'}(\lambda,x'')|^p\right)^{q/p}\right)^{1/q}, \tag{4.5}$$

we obtain

$$\|f(\cdot,x'')\|_{B^s_{p,q}(\mathbb{R}^d)} \lesssim J^\varrho_{p,q}(\lambda,x'').$$

In fact, we also have

$$\|f(\cdot,x'')\|_{B^s_{p,q}(\mathbb{R}^d)} \lesssim J^{\varrho'}_{p,q}(\lambda,x''), \tag{4.6}$$

for any $\varrho' > 0$. For the sake of convenience, we introduce some further notations. Given any $\delta \in \mathscr{D}$, we set

$$b^{\beta',\delta}_{\nu,m'}(\lambda, x'') := 2^{\nu\frac{N-d}{p}} \sum_{\beta'' \in \mathbb{N}^{N-d}} |\lambda^{\beta}_{\nu,m',\lfloor 2^\nu x_{d+1}\rfloor+\delta_{d+1},\dots,\lfloor 2^\nu x_N\rfloor+\delta_N}|, \tag{4.7}$$

$$J^{\varrho,\delta}_{p,q}(\lambda, x') := \sup_{\beta' \in \mathbb{N}^d} 2^{\varrho|\beta'|} \Bigg(\sum_{\nu \geqslant 0} \Bigg(\sum_{m' \in \mathbb{Z}^d} |b^{\beta',\delta}_{\nu,m'}(\lambda, x'')|^p \Bigg)^{q/p} \Bigg)^{1/q}. \tag{4.8}$$

Notice that since $\operatorname{supp}(\psi^\beta) \subset B_1$, we have

$$\psi^{\beta''}(2^\nu x'' - m'') \neq 0 \implies m_i \in \{\lfloor 2^\nu x_i \rfloor, \lfloor 2^\nu x_i \rfloor + 1\} \quad \text{for all } i \in [\![d+1, N]\!].$$

And so, using (4.4) and (4.5), we can derive the following bounds

$$b^{\beta'}_{\nu,m'}(\lambda, x'') \leqslant \sum_{\delta \in \mathscr{D}} b^{\beta',\delta}_{\nu,m'}(\lambda, x''),$$

and

$$J^{\varrho}_{p,q}(\lambda, x'') \leqslant c \sum_{\delta \in \mathscr{D}} J^{\varrho,\delta}_{p,q}(\lambda, x''), \tag{4.9}$$

for some $c > 0$ depending only on $\#\mathscr{D}$, p and q.

As a consequence of (4.6) and (4.9), to estimate $\|f(\cdot, x'')\|_{B^s_{p,q}(\mathbb{R}^d)}$ from above one only needs to estimate the terms (4.8) from above, for each $\delta \in \mathscr{D}$.

Within these notations, we have the following

Lemma 4.1. — *Let $N \geqslant 2$, $0 < p, q \leqslant \infty$, $\delta \in \mathscr{D}$ and $0 < \varrho' < \varrho_0$. Then, with the notations above*

$$J^{\varrho',\delta}_{p,q}(\lambda, x'') \lesssim \sup_{\beta \in \mathbb{N}^N} 2^{\varrho_0|\beta|} \Bigg(\sum_{\nu \geqslant 0} \Bigg(\sum_{m' \in \mathbb{Z}^d} |\lambda^{\beta}_{\nu,m',\lfloor 2^\nu x''\rfloor+\delta}|^p 2^{\nu(N-d)} \Bigg)^{q/p} \Bigg)^{1/q},$$

for a.e. $x'' \in \mathbb{R}^{N-d}$ (with the usual modification if $p = \infty$ and/or $q = \infty$), where

$$\lfloor 2^\nu x'' \rfloor + \delta = (\lfloor 2^\nu x_{d+1} \rfloor + \delta_{d+1}, \dots, \lfloor 2^\nu x_N \rfloor + \delta_N) \in \mathbb{Z}^{N-d}.$$

Proof. — Suppose first that $p, q < \infty$. For simplicity, we will write

$$m''_{\nu,\delta} := \lfloor 2^\nu x'' \rfloor + \delta.$$

Using (4.7) and (4.8) we get

$$J^{\varrho',\delta}_{p,q}(\lambda, x'') \lesssim \sup_{\beta' \in \mathbb{N}^d} 2^{\varrho'|\beta'|} \Bigg(\sum_{\nu \geqslant 0} \Bigg(\sum_{m' \in \mathbb{Z}^d} \Bigg(\sum_{\beta'' \in \mathbb{N}^{N-d}} |\lambda^{\beta}_{\nu,m',m''_{\nu,\delta}}| \Bigg)^p 2^{\nu(N-d)} \Bigg)^{q/p} \Bigg)^{1/q}.$$

Write $\lambda^\beta_{\nu,m} = 2^{-\varrho_0|\beta|}\Lambda^\beta_{\nu,m}$ and let $a := \varrho_0 - \varrho'$. Then,

$$2^{\varrho'|\beta'|}|\lambda^\beta_{\nu,m}| = 2^{\varrho'|\beta'|-\varrho_0|\beta|}|\Lambda^\beta_{\nu,m}| \leqslant 2^{-a|\beta|}|\Lambda^\beta_{\nu,m}| \leqslant 2^{-a|\beta''|}|\Lambda^\beta_{\nu,m}|.$$

Hence, by Hölder's inequality we have

$$\begin{aligned} J^{\varrho',\delta}_{p,q}(\lambda, x'') &\lesssim \sup_{\beta'\in\mathbb{N}^d}\Bigg(\sum_{\nu\geqslant 0}\Bigg(\sum_{m'\in\mathbb{Z}^d}\Bigg(\sum_{\beta''\in\mathbb{N}^{N-d}} 2^{-a|\beta''|}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|\Bigg)^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q} \\ &\leqslant K_{a/2}\sup_{\beta'\in\mathbb{N}^d}\Bigg(\sum_{\nu\geqslant 0}\Bigg(\sum_{m'\in\mathbb{Z}^d}\sup_{\beta''\in\mathbb{N}^{N-d}} 2^{-p\frac{a}{2}|\beta''|}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q}, \end{aligned}$$

where we have used the notation

$$K_\alpha = \sum_{\beta''\in\mathbb{N}^{N-d}} 2^{-\alpha|\beta''|} \quad \text{for } \alpha > 0.$$

Since the ℓ^p spaces are increasing with p, by successive applications of the Hölder inequality, we have

$$\begin{aligned} J^{\varrho',\delta}_{p,q}(\lambda, x'') &\lesssim K_{a/2}\sup_{\beta'\in\mathbb{N}^d}\Bigg(\sum_{\nu\geqslant 0}\Bigg(\sum_{\beta''\in\mathbb{N}^{N-d}} 2^{-p\frac{a}{2}|\beta''|}\sum_{m'\in\mathbb{Z}^d}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q} \\ &\leqslant K_{a/2}K^{1/p}_{p\frac{a}{4}}\sup_{\beta'\in\mathbb{N}^d}\Bigg(\sum_{\nu\geqslant 0}\Bigg(\sup_{\beta''\in\mathbb{N}^{N-d}} 2^{-p\frac{a}{4}|\beta''|}\sum_{m'\in\mathbb{Z}^d}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q} \\ &= K_{a/2}K^{1/p}_{p\frac{a}{4}}\sup_{\beta'\in\mathbb{N}^d}\Bigg(\sum_{\nu\geqslant 0}\sup_{\beta''\in\mathbb{N}^{N-d}} 2^{-q\frac{a}{4}|\beta''|}\Bigg(\sum_{m'\in\mathbb{Z}^d}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q} \\ &\leqslant K_{a/2}K^{1/p}_{p\frac{a}{4}}\sup_{\beta'\in\mathbb{N}^d}\Bigg(\sum_{\nu\geqslant 0}\sum_{\beta''\in\mathbb{N}^{N-d}} 2^{-q\frac{a}{4}|\beta''|}\Bigg(\sum_{m'\in\mathbb{Z}^d}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q} \\ &= K_{a/2}K^{1/p}_{p\frac{a}{4}}\sup_{\beta'\in\mathbb{N}^d}\Bigg(\sum_{\beta''\in\mathbb{N}^{N-d}} 2^{-q\frac{a}{4}|\beta''|}\sum_{\nu\geqslant 0}\Bigg(\sum_{m'\in\mathbb{Z}^d}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q} \\ &\leqslant K_{a/2}K^{1/p}_{p\frac{a}{4}}K^{1/q}_{q\frac{a}{4}}\sup_{\beta\in\mathbb{N}^N}\Bigg(\sum_{\nu\geqslant 0}\Bigg(\sum_{m'\in\mathbb{Z}^d}|\Lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q}. \end{aligned}$$

Letting $K_{a,p,q} := K_{a/2}K^{1/p}_{p\frac{a}{4}}K^{1/q}_{q\frac{a}{4}}$ and recalling $\lambda^\beta_{\nu,m} = 2^{-\varrho_0|\beta|}\Lambda^\beta_{\nu,m}$ we get

$$J^{\varrho',\delta}_{p,q}(\lambda, x'') \leqslant K_{a,p,q}\sup_{\beta\in\mathbb{N}^N} 2^{\varrho_0|\beta|}\Bigg(\sum_{\nu\geqslant 0}\Bigg(\sum_{m'\in\mathbb{Z}^d}|\lambda^\beta_{\nu,m',m''_{\nu,\delta}}|^p 2^{\nu(N-d)}\Bigg)^{q/p}\Bigg)^{1/q},$$

which is the desired estimate. The proof when $p = \infty$ and/or $q = \infty$ is similar but technically simpler. $\square$

Remark 4.2. — Of course, when $p = \infty$, the term "$2^{\nu(N-d)}$" disappears (recall Remark 2.5) so that, in this case, Fact 1.1 follows directly from the above lemma.

Remark 4.3. — The same kind of estimate holds in the setting of Besov spaces of generalized smoothness. That is, given a function $f \in B^s_{p,q}(\mathbb{R}^N)$ decomposed as above by (4.2) with (4.1) and (4.3), we can estimate the $B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$-quasi-norm of its restrictions to almost every hyperplanes $f(\cdot, x'')$ exactly in the same fashion. It suffices to replace the (s,p)-β-quarks $(\beta\mathrm{qu})_{\nu,m}$ in the decomposition of $f(\cdot, x'')$ by $\Psi(2^{-\nu})^{-1}(\beta\mathrm{qu})_{\nu,m}$ in order to get (s,p,Ψ)-β-quarks. From here, we can reproduce the same reasoning as in Lemma 4.1 with $\Psi(2^{-\nu})\lambda^\beta_{\nu,m}$ instead of $\lambda^\beta_{\nu,m}$ and we obtain

$$\|f(\cdot, x'')\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)} \lesssim \sum_{\delta \in \mathscr{D}} \widetilde{J}^{\varrho_0,\delta}_{p,q}(\lambda, x''),$$

with

$$\widetilde{J}^{\varrho_0,\delta}_{p,q}(\lambda, x'') := \sup_{\beta \in \mathbb{N}^N} 2^{\varrho_0|\beta|} \left(\sum_{\nu \geqslant 0} \left(\Psi(2^{-\nu})^p \sum_{m' \in \mathbb{Z}^d} |\lambda^\beta_{\nu,m',\lfloor 2^\nu x'' \rfloor + \delta}|^p 2^{\nu(N-d)} \right)^{q/p} \right)^{1/q}.$$

Similarly, given a function $f \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$, we can estimate the $B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$-quasi-norm of its restrictions $f(\cdot, x'')$ in the same spirit. This is done up to a slight modification in the discussion above. It suffices to multiply the (s,p)-β-quarks considered above by a factor of $\Psi(2^{-\nu})^{-1}$ and to take $\eta^\beta_{\nu,m}$, the optimal subatomic decomposition of $f \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ with respect to these new quarks. Then, the $B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$-quasi-norm of f and the $B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$-quasi-norm of its restrictions $f(\cdot, x'')$ satisfy the same relations as when $\Psi \equiv 1$ with $\eta^\beta_{\nu,m}$ instead of $\lambda^\beta_{\nu,m}$. That is, we still have

$$\|f\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} \sim \sup_{\beta \in \mathbb{N}^N} 2^{\varrho|\beta|} \left(\sum_{\nu \geqslant 0} \left(\sum_{m \in \mathbb{Z}^N} |\eta^\beta_{\nu,m}|^p \right)^{q/p} \right)^{1/q},$$

and

$$\|f(\cdot, x'')\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)} \lesssim \sum_{\delta \in \mathscr{D}} J^{\varrho_0,\delta}_{p,q}(\eta, x''),$$

where $\varrho, \varrho_0 > 0$ and $J^{\varrho_0,\delta}_{p,q}(\eta, x'')$ is as in (4.8).

5 The case $q \leqslant p$

This section is concerned with Fact 1.1 (Fact 1.2 being only a consequence of Theorem 1.7). We will use subatomic decompositions together with the estimate given at Lemma 4.1 to get the following generalization of Fact 1.1.

PROPOSITION 5.1. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < q \leqslant p \leqslant \infty$ and $s > \sigma_p$. Let Ψ be an admissible function. Let $K \subset \mathbb{R}^{N-d}$ be a compact set and let $f \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$. Then, it holds that*

$$\left(\int_K \|f(\cdot,x'')\|^q_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)} \mathrm{d}x''\right)^{1/q} \leqslant C\|f\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)},$$

for some constant $C = C(K,N,d,p,q) > 0$ (with the usual modification if $q = \infty$).

Proof. — Without loss of generality, we may consider the case $K = [1,2]^{N-d}$ only (the general case follows from standard scaling arguments). Also, we can suppose that $p < \infty$ since otherwise, when $p = \infty$, the desired result is a simple consequence of Lemma 4.1 (recall Remark 4.2). Let us first prove Lemma 5.1 for $\Psi \equiv 1$ (it will be clear at the end why this is enough to deduce the general case).

Let $f \in B^s_{p,q}(\mathbb{R}^N)$. Given the (s,p)-β-quarks $(\beta\mathrm{qu})_{\nu,m}$ and $\varrho > r$ defined at Section 4 we let $\lambda^\beta_{\nu,m} = \lambda^\beta_{\nu,m}(f)$ be the corresponding optimal subatomic decomposition. In particular

$$f(x) = \sum_{\beta\in\mathbb{N}^N}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^N} \lambda^\beta_{\nu,m}(\beta\mathrm{qu})_{\nu,m}(x),$$

with

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \sim \sup_{\beta\in\mathbb{N}^N} 2^{\varrho|\beta|}\left(\sum_{\nu\geqslant 0}\left(\sum_{m\in\mathbb{Z}^N}|\lambda^\beta_{\nu,m}|^p\right)^{q/p}\right)^{1/q}.$$

By the discussion in Section 4, we have that

$$\|f(\cdot,x'')\|_{B^s_{p,q}(\mathbb{R}^d)} \lesssim \sum_{\delta\in\mathscr{D}} J^{\varrho',\delta}_{p,q}(\lambda,x''), \tag{5.1}$$

for all $\varrho' \in (0,\varrho)$, where $J^{\varrho',\delta}_{p,q}(\lambda,x'')$ is given by (4.8). Define

$$\Lambda^\beta_{\nu,m''} := \left(\sum_{m'\in\mathbb{Z}^d}|\lambda^\beta_{\nu,m',m''}|^p\right)^{1/p}.$$

In particular,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \sim \sup_{\beta\in\mathbb{N}^N} 2^{\varrho|\beta|}\left(\sum_{\nu\geqslant 0}\left(\sum_{m''\in\mathbb{Z}^{N-d}}|\Lambda^\beta_{\nu,m''}|^p\right)^{q/p}\right)^{1/q}.$$

Then, the conclusion of Lemma 4.1 rewrites

$$J^{\varrho',\delta}_{p,q}(\lambda,x'')^q \lesssim \sup_{\beta\in\mathbb{N}^N} 2^{\varrho_0 q|\beta|}\sum_{\nu\geqslant 0} 2^{\nu q\frac{N-d}{p}}|\Lambda^\beta_{\nu,\lfloor 2^\nu x''\rfloor+\delta}|^q \quad \text{for all } \delta\in\mathscr{D},$$

and some $\varrho_0 \in (\varrho', \varrho)$. Integration over $[1,2]^{N-d}$ yields

$$\begin{aligned} I_\delta := \int_{[1,2]^{N-d}} J_{p,q}^{\varrho',\delta}(\lambda, x'')^q \mathrm{d}x'' &\lesssim \int_{[1,2]^{N-d}} \sup_{\beta\in\mathbb{N}^N} 2^{\varrho_0 q|\beta|} \sum_{\nu\geqslant 0} 2^{\nu q \frac{N-d}{p}} |\Lambda^\beta_{\nu,\lfloor 2^\nu x''\rfloor+\delta}|^q \mathrm{d}x'' \\ &\leqslant \sum_{\beta\in\mathbb{N}^N} 2^{\varrho_0 q|\beta|} \sum_{\nu\geqslant 0} \int_{[1,2]^{N-d}} 2^{\nu q \frac{N-d}{p}} |\Lambda^\beta_{\nu,\lfloor 2^\nu x''\rfloor+\delta}|^q \mathrm{d}x''. \end{aligned}$$

Now, we observe that

$$2^{\nu q \frac{N-d}{p}} |\Lambda^\beta_{\nu,\lfloor 2^\nu x''\rfloor+\delta}|^q \leqslant \Bigg(\sum_{k\in\mathbb{N}^{N-d}} |\Lambda^\beta_{\nu,\lfloor 2^{k_{d+1}} x_{d+1}\rfloor+\delta_{d+1},\cdots,\lfloor 2^{k_N} x_N\rfloor+\delta_N}|^p 2^{k_{d+1}+\cdots+k_N} \Bigg)^{q/p}.$$

Hence, using the fact that $q \leqslant p$ and applying $N-d$ times Lemma 3.4, we get

$$\begin{aligned} I_\delta &\lesssim \sum_{\beta\in\mathbb{N}^N} 2^{\varrho_0 q|\beta|} \sum_{\nu\geqslant 0} \Bigg(\sum_{k\in\mathbb{N}^{N-d}} |\Lambda^\beta_{\nu,k+\delta}|^p \Bigg)^{q/p} \\ &\leqslant \sum_{\beta\in\mathbb{N}^N} 2^{(\varrho_0-\varrho)q|\beta|} \sup_{\beta\in\mathbb{N}^N} 2^{\varrho q|\beta|} \sum_{\nu\geqslant 0} \Bigg(\sum_{k\in\mathbb{N}^{N-d}} |\Lambda^\beta_{\nu,k+\delta}|^p \Bigg)^{q/p} \\ &= K_{\varrho,N,q} \sup_{\beta\in\mathbb{N}^N} 2^{\varrho q|\beta|} \sum_{\nu\geqslant 0} \Bigg(\sum_{k\in\mathbb{N}^{N-d}} |\Lambda^\beta_{\nu,k+\delta}|^p \Bigg)^{q/p} \\ &\leqslant K_{\varrho,N,q} \sup_{\beta\in\mathbb{N}^N} 2^{\varrho q|\beta|} \|\lambda^\beta\|^q_{b_{p,q}}. \end{aligned}$$

Thus, recalling (5.1), we arrive at

$$\left(\int_{[1,2]^{N-d}} \|f(\cdot, x'')\|^q_{B^s_{p,q}(\mathbb{R}^d)} \mathrm{d}x'' \right)^{1/q} \lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^N)}.$$

Now, having in mind Remark 4.3, we can reproduce exactly the same proof when $\Psi \not\equiv 1$ with almost no modifications. This completes the proof. $\square$

6 The case $p < q$

In this section we prove, at a stroke, Theorem 1.3 and Theorem 1.6. As will become clear, the proof of Theorem 1.6 will easily follow from that of Theorem 1.3.

Let us begin with the following more general result:

Theorem 6.1. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$ and $s > \sigma_p$. Let Ψ be an admissible function. Then, there exists a function $f \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot, x'') \notin B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^d) \quad \text{for a.e. } x'' \in \mathbb{R}^{N-d}.$$

Proof. — We will essentially follow two steps.

Step 1: case $d = N-1$.

We will construct a function satisfying the requirements of Theorem 6.1 (and hence of Theorem 1.3) via its subatomic coefficients.

Let Ψ be an admissible function. Let $0 < p < q \leqslant \infty$, $s > \sigma_p$, $M = \lfloor s \rfloor + 1$ and let $(\lambda_{j,k})_{j,k\geqslant 0} \in b_{p,q}$ be the sequence constructed at Corollary 3.7.

Also, we let $\psi \in C_c^\infty(\mathbb{R}^N)$ be a function such that

$$\operatorname{supp}(\psi) \subset [-2,2]^N, \quad \inf_{z\in[0,1]^N} \psi(z) > 0 \text{ and } \sum_{m\in\mathbb{Z}^N} \psi(\cdot - m) \equiv 1. \tag{6.1}$$

In addition, we will suppose that ψ has the product structure

$$\psi(x) = \psi(x_1) \dots \psi(x_N). \tag{6.2}$$

Notice that such a ψ always exists.[1] Then, we define

$$\begin{aligned} f(x) = \sum_{j\geqslant 0}\sum_{k\geqslant 0} \lambda_{j,k} 2^{-j(s-\frac{N}{p})} \Psi(2^{-j})^{-1} \\ \times \psi(2^j(x_1 - C_M j)) \dots \psi(2^j(x_{N-1} - C_M j))\psi(2^j x_N - k), \end{aligned} \tag{6.3}$$

where $C_M = 2(M+2)$. It follows from Definition 2.16 that

$$\Psi(2^{-j})^{-1} 2^{-j(s-\frac{N}{p})} \psi(2^j x - m) \quad \text{for } x \in \mathbb{R}^N,$$

with

$$m = (C_M 2^j j, \dots, C_M 2^j j, k) \in \mathbb{Z}^N,$$

can be interpreted as (s,p,Ψ)-0-quarks relative to the cube $Q_{j,m}$. Consequently, by Theorem 2.17 and Corollary 3.7, we have

$$\|f\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} \leqslant c \left(\sum_{j\geqslant 0} \left(\sum_{k\geqslant 0} \lambda^p_{j,k} \right)^{q/p} \right)^{1/q} < \infty,$$

(modification if $q = \infty$). Therefore, $f \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$. In particular, the sum in the right-hand side of (6.3) converges in $L^p(\mathbb{R}^N)$ and is unconditionally convergent for

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[1] Here is an example. Let $u(t) := e^{-1/t^2} \mathbb{1}_{(0,\infty)}(t)$ (extended by 0 in $(-\infty, 0]$) and let $v(t) = u(1+t)u(1-t)$. Then,

$$\psi(x) := \prod_{j=1}^N \frac{1}{2} \psi_0\left(\frac{x_j}{2}\right) \quad \text{where} \quad \psi_0(t) = \frac{v(t)}{v(t-1) + v(t) + v(t+1)},$$

is a smooth function satisfying (6.1) and (6.2).

a.e. $x \in \mathbb{R}^N$ (notice the terms involved are all nonnegative) and, by Fubini, $f(\cdot, x_N)$ also converges in $L^p(\mathbb{R}^{N-1})$ for a.e. $x_N \in \mathbb{R}$. Thus, letting

$$\Lambda_j(x_N) := \sum_{k \geqslant 0} \lambda_{j,k} 2^{j/p} \psi(2^j x_N - k),$$

we may rewrite (6.3) as

$$f(x', x_N) = \sum_{j \geqslant 0} \Lambda_j(x_N)\, 2^{-j(s - \frac{N-1}{p})} \Psi(2^{-j})^{-1} \psi(2^j(x_1 - C_M j)) \ldots \psi(2^j(x_{N-1} - C_M j)).$$

Notice that assumption (6.1) implies that there is a $c_0 > 0$ such that

$$\psi(2^j x_N - \lfloor 2^j x_N \rfloor) \geqslant c_0 > 0 \quad \text{for all } x_N \in [1,2] \text{ and } j \geqslant 0.$$

In particular, we have

$$\Lambda_j(x_N) \geqslant c_0\, \lambda_{j, \lfloor 2^j x_N \rfloor} 2^{j/p}. \tag{6.4}$$

Now, for all $j \geqslant 0$, we write

$$K_j := \left\{ h \in \mathbb{R}^{N-1} : 2^{-(j+1)} \leqslant |h| \leqslant 2^{-j} \right\}. \tag{6.5}$$

By [28, Lemma 8.2] (in fact in [28] it is implicitly supposed that $1 \leqslant p < \infty$ but the proof still works when $0 < p < 1$) and (6.4), we have

$$\begin{aligned} \sup_{h \in K_j} \|\Delta_h^M f(\cdot, x_N)\|_{L^p(\mathbb{R}^{N-1})} &\geqslant c\, 2^{-js} \Psi(2^{-j})^{-1} \Lambda_j(x_N) \\ &\geqslant c'\, 2^{-js} \Psi(2^{-j})^{-1}\, 2^{j/p} \lambda_{j, \lfloor 2^j x_N \rfloor}, \end{aligned} \tag{6.6}$$

for any $j \geqslant 0$ and some $c' > 0$ independent of j. Recall that

$$\|g\|_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} \sim \|g\|_{L^p(\mathbb{R}^{N-1})} + \sup_{j \geqslant 1} 2^{js} \Psi(2^{-j}) \sup_{h \in K_j} \|\Delta_h^M g\|_{L^p(\mathbb{R}^{N-1})},$$

is an equivalent quasi-norm on $B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})$ (this is a discretized version of Definition 2.11). This together with (6.6) and Corollary 3.7 gives

$$\|f(\cdot, x_N)\|_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} \gtrsim \sup_{j \geqslant 0} 2^{j/p} \lambda_{j, \lfloor 2^j x_N \rfloor} = \infty \quad \text{for a.e. } x_N \in [1,2].$$

Therefore, $f(\cdot, x_N) \notin B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})$ for a.e. $x_N \in [1,2]$.

We will show that one can construct a function satisfying the requirements of Theorem 6.1 by considering a weighted sum of translates of the function f constructed above. To this end, we let

$$f_l(x', x_N) := f(x', x_N + l) \quad \text{for } l \in \mathbb{Z},$$

and we define

$$g := \sum_{l\in\mathbb{Z}} 2^{-|l|} f_l.$$

Then, by the triangle inequality for Besov quasi-norms, we have

$$\|g\|^\eta_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} \leqslant \sum_{l\in\mathbb{Z}} 2^{-\eta|l|} \|f_l\|^\eta_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} \leqslant c_\eta \|f\|^\eta_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} < \infty,$$

for some $0 < \eta \leqslant 1$. Hence, $g \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$. To complete the proof we need to show that

$$g(\cdot, x_N) \notin B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1}) \quad \text{for a.e. } x_N \in \mathbb{R}. \tag{6.7}$$

Let $m \in \mathbb{Z}$. Then, by the triangle inequality for Besov quasi-norms we have

$$\begin{aligned} 2^{-\eta|m|} \|f_m(\cdot, x_N)\|^\eta_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} &\leqslant \|g(\cdot, x_N)\|^\eta_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} + \sum_{l\neq m} 2^{-\eta|l|} \|f_l(\cdot, x_N)\|^\eta_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} \\ &\leqslant \|g(\cdot, x_N)\|^\eta_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} + c_\eta \sup_{l\neq m} \|f_l(\cdot, x_N)\|^\eta_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})}. \end{aligned} \tag{6.8}$$

Clearly, the left-hand side of (6.8) is infinite for a.e. $x_N \in [1-m, 2-m]$. Thus, to prove (6.7), one only needs to make sure that the last term on the right-hand side of (6.8) is finite for a.e. $x_N \in [1-m, 2-m]$. For it, we notice that, by construction, it is necessary to have

$$j \geqslant 1 \quad \text{and} \quad 2^j \leqslant k < 2^{1+j}, \tag{6.9}$$

for $\lambda_{j,k} \neq 0$ to hold. In particular, $\Lambda_0 \equiv 0$ and $\Lambda_j(x_N)$ consists only in finitely many terms for a.e. $x_N \in \mathbb{R}$. In addition, by our assumptions on the support of ψ, we have $\psi(2^j x_N - k) \neq 0$ provided

$$\left| x_N - \frac{k}{2^j} \right| \leqslant 2^{1-j}. \tag{6.10}$$

By (6.9) and (6.10), we deduce that if $x_N \in \mathbb{R} \setminus [1, 2]$, then there are only finitely many values of $j \geqslant 1$ such that $\Lambda_j(x_N) \not\equiv 0$. In particular,

$$f(\cdot, x_N + l) \in B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1}) \quad \text{for a.e. } x_N \in [1, 2] \text{ and all } l \in \mathbb{Z} \setminus \{0\}. \tag{6.11}$$

Moreover, a consequence of (6.9) and (6.10) is that

$$j \geqslant 1 \quad \text{and} \quad x_N \in \operatorname{supp}(\Lambda_j) \quad \Longrightarrow \quad 1 - 2^{1-j} \leqslant x_N < 2 + 2^{1-j}.$$

In turn, this implies that the support of Λ_j is included in $[0, 3]$. Therefore,

$$f(\cdot, x_N + l) \equiv 0 \quad \text{for a.e. } x_N \in [1, 2] \text{ and all } l \in \mathbb{Z} \text{ with } |l| \geqslant 2. \tag{6.12}$$

Hence, by (6.11) and (6.12), we infer that

$$\max_{l\neq 0} \|f_l(\cdot, x_N)\|_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} < \infty \quad \text{for a.e. } x_N \in [1,2].$$

In like manner, for every $m \in \mathbb{Z}$, we have

$$\max_{l\neq m} \|f_l(\cdot, x_N)\|_{B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^{N-1})} < \infty \quad \text{for a.e. } x_N \in [1-m, 2-m].$$

This proves the theorem for $d = N-1$.

Step 2: case $1 \leqslant d < N-1$.

By the above, we know that Theorem 1.3 holds for any $N \geqslant 2$ and $d = N-1$. In particular, there exists a function $f \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^{d+1})$ such that $f(\cdot, x_{d+1}) \notin B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^d)$ for a.e. $x_{d+1} \in \mathbb{R}$. Now, pick a function $w \in \mathscr{S}(\mathbb{R}^{N-d-1})$ with $w > 0$ on $\mathbb{R}^{N-d-1}$ and set

$$g(x) = g(x_1, \dots, x_N) = f(x_1, \dots, x_d, x_{d+1}) w(x_{d+2}, \dots, x_N).$$

It is standard that $g \in L^{\overline{p}}(\mathbb{R}^N)$ where $\overline{p} := \max\{1, p\}$. Then, letting $M = \lfloor s \rfloor + 1$ and using [136, Formula (16), p.112], we have that

$$\begin{aligned}\sup_{|h|\leqslant t} \|\Delta_h^{2M} g\|_{L^p(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})} &\sup_{|h''|\leqslant t} \|\Delta_{h''}^M w\|_{L^p(\mathbb{R}^{N-d-1})} \\ &+ \|w\|_{L^p(\mathbb{R}^{N-d+1})} \sup_{|h'|\leqslant t} \|\Delta_{h'}^M f\|_{L^p(\mathbb{R}^{d+1})},\end{aligned}$$

for any $h = (h', h'') \in \mathbb{R}^N \setminus \{0\}$ with $h' = (h_1, \dots, h_{d+1})$ and $h'' = (h_{d+2}, \dots, h_N)$. In particular, recalling Remark 2.12, we see that this implies

$$\|g\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})} \|w\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^{N-d-1})} + \|w\|_{L^p(\mathbb{R}^{N-d-1})} \|f\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^{d+1})}.$$

Hence, $g \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^N)$. Moreover, it is easily seen that

$$g(\cdot, x_{d+1}, \dots, x_N) = f(\cdot, x_{d+1}) w(x_{d+2}, \dots, x_N) \notin B^{(s,\Psi)}_{p,\infty}(\mathbb{R}^d),$$

for a.e. $(x_{d+1}, \dots, x_N) \in \mathbb{R}^{N-d}$. This completes the proof. $\square$

The function we have constructed above (in the proof of Theorem 6.1) turns out to verify the conclusion of Theorem 1.6.

Proof of Theorem 1.6. — For simplicity, we outline the proof for $N = 2$ and $d = 1$ only (the general case follows from the same arguments as above). Let f be the function constructed in the proof of Theorem 6.1 with $\Psi \equiv 1$, namely

$$f(x_1, x_2) := \sum_{j\geqslant 0} \Lambda_j(x_2)\, 2^{-j(s-\frac{1}{p})} \psi(2^j(x_1 - C_M j)),$$

with

$$\Lambda_j(x_2) := \sum_{k\geqslant 0} \lambda_{j,k} 2^{j/p} \psi(2^j x_2 - k),$$

where ψ, C_M and $(\lambda_{j,k})_{j,k\geqslant 0}$ are as in the proof of Theorem 6.1. Clearly,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^2)} \leqslant c \left(\sum_{j\geqslant 0} \left(\sum_{k\geqslant 0} \lambda^p_{j,k} \right)^{q/p} \right)^{1/q} < \infty.$$

Hence, $f \in B^s_{p,q}(\mathbb{R}^2)$. We now distinguish the cases $sp > 1$, $sp = 1$ and $sp < 1$.

Step 1: case $sp > 1$.

This case works as in Theorem 6.1. Indeed, by the supports of the functions involved, we have for a.e. $x_2 \in [1, 2]$,

$$\|f(\cdot, x_2)\|_{C^{s-\frac{1}{p}}(\mathbb{R})} \sim \sup_{j\geqslant 0} 2^{j(s-\frac{1}{p})} \sup_{h\in K_j} \|\Delta^M_h f(\cdot, x_2)\|_{L^\infty(\mathbb{R})} \gtrsim \sup_{j\geqslant 0} 2^{j/p} \lambda_{j,\lfloor 2^j x_2 \rfloor} = \infty,$$

where K_j is given by (6.5). We may now conclude as in the proof of Theorem 6.1.

Step 2: case $sp = 1$.

It suffices to notice that, for any $k \geqslant 0$, we have

$$\begin{aligned}
\|f(\cdot, x_2)\|_{\mathrm{BMO}(\mathbb{R})} &\geqslant \fint_{C_M k-2^{-k}}^{C_M k+2^{-k}} \left| \fint_{C_M k-2^{-k}}^{C_M k+2^{-k}} \big[f(x, x_2) - f(z, x_2)\big] \mathrm{d}z \right| \mathrm{d}x \\
&= \fint_{C_M k-2^{-k}}^{C_M k+2^{-k}} \left| \sum_{j\geqslant 0} \Lambda_j(x_2) \fint_{C_M k-2^{-k}}^{C_M k+2^{-k}} \big[\psi(2^j(x - C_M j)) - \psi(2^j(z - C_M j))\big] \mathrm{d}z \right| \mathrm{d}x.
\end{aligned}$$

Hence, by the support of the functions involved we deduce that

$$\begin{aligned}
&\|f(\cdot, x_2)\|_{\mathrm{BMO}(\mathbb{R})} \\
&\quad \geqslant \Lambda_k(x_2) \fint_{C_M k-2^{-k}}^{C_M k+2^{-k}} \left| \fint_{C_M k-2^{-k}}^{C_M k+2^{-k}} \big[\psi(2^k(x - C_M k)) - \psi(2^k(z - C_M k))\big] \mathrm{d}z \right| \mathrm{d}x \\
&\quad = \Lambda_k(x_2) \fint_{-1}^{1} \left| \fint_{-1}^{1} \big[\psi(x) - \psi(z)\big] \mathrm{d}z \right| \mathrm{d}x \geqslant c' \Lambda_k(x_2).
\end{aligned}$$

Therefore, we have

$$\|f(\cdot, x_2)\|_{\mathrm{BMO}(\mathbb{R})} \gtrsim \sup_{j\geqslant 0} \Lambda_j(x_2) = \sup_{j\geqslant 0} 2^{j/p} \lambda_{j,\lfloor 2^j x_2 \rfloor} = \infty \quad \text{for a.e. } x_2 \in [1, 2].$$

Thus, we may again conclude as in the proof of Theorem 6.1.

Step 3: case $sp < 1$.

Define $r := \frac{p}{1-sp}$ and rewrite f as

$$f(x_1, x_2) := \sum_{j \geqslant 0} c_j(x_2)\, 2^{j/r} f_j(x_1),$$

where we have set $f_j(x_1) := \psi(2^j(x_1 - C_M j))$ and

$$c_j(x_2) := 2^{-j(s-\frac{2}{p})}\, 2^{-j/r} \sum_{k \geqslant 0} \lambda_{j,k} \psi(2^j x_2 - k).$$

Since the f_j's have mutually disjoint support we find that

$$f(\cdot, x_2)^*(t) \geqslant c_j(x_2)\, 2^{j/r} f_j^*(t) \qquad \text{for any } t \geqslant 0 \text{ and } j \geqslant 0.$$

Moreover, it is easy to see that $f_j^*(t) = \psi^*(2^j t)$. In turn, this implies that

$$\|f(\cdot, x_2)\|_{L^{r,\infty}(\mathbb{R})} \geqslant c_j(x_2)\, 2^{j/r} \sup_{t>0} t^{1/r} \psi^*(2^j t) = c_j(x_2) \|\psi\|_{L^{r,\infty}(\mathbb{R})} \gtrsim 2^{j/p} \lambda_{j,\lfloor 2^j x_2 \rfloor}.$$

Hence, for a.e. $x_2 \in [1, 2]$,

$$\|f(\cdot, x_2)\|_{L^{r,\infty}(\mathbb{R})} \gtrsim \sup_{j \geqslant 0} 2^{j/p} \lambda_{j,\lfloor 2^j x_2 \rfloor} = \infty.$$

This completes the proof. $\square$

7 Characterization of restrictions of Besov functions

In this section, we prove that Besov spaces of generalized smoothness are the natural scale in which to look for restrictions of Besov functions. More precisely, we will prove Theorems 1.7 and 1.8. We present several results, with different assumptions and different controls on the norm of $f(\cdot, x'')$.

Let us begin with the following

THEOREM 7.1. — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q \leqslant \infty$ and let $s > \sigma_p$. Let $K \subset \mathbb{R}^{N-d}$ be a compact set. Let Φ and Ψ be two admissible functions such that*

$$\sum_{j \geqslant 0} \Phi(2^{-j})^{-p} \Psi(2^{-j})^p < \infty. \tag{7.1}$$

Let $f \in B^{(s,\Phi)}_{p,q}(\mathbb{R}^N)$. Then, there exists a constant $C > 0$ such that

$$\|f(\cdot, x'')\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)} \leqslant C\|f\|_{B^{(s,\Phi)}_{p,q}(\mathbb{R}^N)} \quad \textit{for a.e. } x'' \in K.$$

Moreover, the constant C is independent of x'' but may depend on f, K, N, d, p, q, Φ and Ψ.

Proof. — Let us first suppose that $q < \infty$ and that $\Phi \equiv 1$. Without loss of generality we may take $K = [1,2]^{N-d}$, the general case being only a matter of scaling. Here again, we use the following short notation

$$\lfloor 2^\nu x'' \rfloor = (\lfloor 2^\nu x_{d+1} \rfloor, \dots, \lfloor 2^\nu x_N \rfloor) \in \mathbb{N}^{N-d}.$$

Let $f \in B^s_{p,q}(\mathbb{R}^N)$ and write its subatomic decomposition as

$$f(x) = \sum_{\beta \in \mathbb{N}^N} \sum_{\nu \geqslant 0} \sum_{m \in \mathbb{Z}^N} \lambda^\beta_{\nu,m} (\beta\mathrm{qu})_{\nu,m}(x),$$

where the (s,p)-β-quarks $(\beta\mathrm{qu})_{\nu,m}$ are as in Section 4 and $\lambda^\beta_{\nu,m} = \lambda^\beta_{\nu,m}(f)$ is the optimal subatomic decomposition of f, i.e. such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^N)} \sim \sup_{\beta \in \mathbb{N}^N} 2^{\varrho|\beta|} \|\lambda^\beta\|_{b_{p,q}}. \tag{7.2}$$

Let $\varepsilon > 0$ be small. Rewriting f as in the discussion at Section 4 and using Lemma 4.1 together with Remark 4.3 we have

$$\|f(\cdot, x'')\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)} \lesssim \sum_{\delta \in \mathscr{D}} \widetilde{J}^{\varrho-\varepsilon,\delta}_{p,q}(\lambda, x''), \tag{7.3}$$

where

$$\widetilde{J}^{\varrho-\varepsilon,\delta}_{p,q}(\lambda, x'') := \sup_{\beta \in \mathbb{N}^N} 2^{(\varrho-\varepsilon)|\beta|} \left(\sum_{\nu \geqslant 0} \left(2^{\nu(N-d)} \Psi(2^{-\nu})^p \sum_{m' \in \mathbb{Z}^d} |\lambda^\beta_{\nu,m',\lfloor 2^\nu x''\rfloor + \delta}|^p \right)^{q/p} \right)^{1/q}.$$

By Lemma 3.5, we know that for any positive sequence $(\alpha_\nu)_{\nu \geqslant 0} \in \ell^1(\mathbb{N})$ there is a constant $C = C(\lambda, \alpha, N, d) > 0$ such that

$$2^{\nu(N-d)} \sum_{m' \in \mathbb{Z}^d} |\lambda^\beta_{\nu,m',\lfloor 2^\nu x''\rfloor + \delta}|^p \leqslant C \frac{\max\{1, |\beta|^{N-d+1}\}}{\alpha_\nu} \sum_{m \in \mathbb{Z}^N} |\lambda^\beta_{\nu,m}|^p,$$

for a.e. $x'' \in [1,2]^{N-d}$ and any $(\nu, \beta) \in \mathbb{N} \times \mathbb{N}^N$. In particular, we have

$$\widetilde{J}^{\varrho-\varepsilon,\delta}_{p,q}(\lambda, x'') \lesssim \sup_{\beta \in \mathbb{N}^N} 2^{(\varrho-\varepsilon)|\beta|} \max\left\{1, |\beta|^{\frac{N-d+1}{p}}\right\} \left(\sum_{\nu \geqslant 0} \left(\frac{\Psi(2^{-\nu})^p}{\alpha_\nu} \sum_{m \in \mathbb{Z}^N} |\lambda^\beta_{\nu,m}|^p \right)^{q/p} \right)^{1/q}.$$

Now, by assumption (7.1), we can choose $\alpha_\nu = \Psi(2^{-\nu})^p$. Therefore, recalling (7.3), we have

$$\|f(\cdot, x'')\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)} \lesssim \sup_{\beta\in\mathbb{N}^N} 2^{(\varrho-\varepsilon)|\beta|} \max\left\{1, |\beta|^{\frac{N-d+1}{p}}\right\} \left(\sum_{\nu\geqslant 0}\left(\sum_{m\in\mathbb{Z}^N} |\lambda^\beta_{\nu,m}|^p\right)^{q/p}\right)^{1/q}$$
$$\leqslant \sup_{\beta\in\mathbb{N}^N} 2^{\varrho|\beta|}\left(\sum_{\nu\geqslant 0}\left(\sum_{m\in\mathbb{Z}^N} |\lambda^\beta_{\nu,m}|^p\right)^{q/p}\right)^{1/q}.$$

Finally, recalling (7.2), we have

$$\|f(\cdot, x'')\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{B^s_{p,q}(\mathbb{R}^N)} \quad \text{for a.e. } x'' \in [1,2]^{N-d}.$$

The proof when $q = \infty$ and/or $\Phi \not\equiv 1$ is similar (recall Remark 4.3). It this latter case, one only have to adjust the (s,p)-β-quarks by a factor of $\Phi(2^{-\nu})^{-1}$ and to replace $\lambda^\beta_{\nu,m}$ by $\eta^\beta_{\nu,m}$, the optimal decomposition of $f \in B^{(s,\Phi)}_{p,q}(\mathbb{R}^N)$ along these (s,p,Φ)-β-quarks. Then, as in Remark 4.3, it suffices to replace $\Psi(2^{-\nu})\lambda^\beta_{\nu,m}$ in the estimate (7.3) of $\|f(\cdot,x'')\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)}$ by $\Psi(2^{-\nu})/\Phi(2^{-\nu})\,\eta^\beta_{\nu,m}$ and the same proof yields the desired conclusion. □

We carry on with the following generalization of Theorem 1.7.

Theorem 7.2. — *Let $N \geqslant 3$, $1 \leqslant d < N$, $0 < r \leqslant p < q \leqslant \infty$, $s > \sigma_p$ and let $\chi = \frac{qr}{q-r}$ (resp. $\chi = r$ if $q = \infty$). Let Φ and Ψ be two admissible functions such that*

$$\sum_{j\geqslant 0} \Phi(2^{-j})^{-\chi}\Psi(2^{-j})^{\chi} < \infty. \tag{7.4}$$

Let $K \subset \mathbb{R}^{N-d}$ be a compact set and suppose that $f \in B^{(s,\Phi)}_{p,q}(\mathbb{R}^N)$. Then,

$$\left(\int_K \|f(\cdot,x'')\|^r_{B^{(s,\Psi)}_{p,r}(\mathbb{R}^d)} \mathrm{d}x''\right)^{1/r} \leqslant C\|f\|_{B^{(s,\Phi)}_{p,q}(\mathbb{R}^N)},$$

for some constant $C = C(K,N,d,p,q,\Phi,\Psi) > 0$.

Proof. — By Hölder's inequality, the definition of the norms involved and our assumptions on s, p, q, r, χ, we see that (7.4) implies that $B^{(s,\Phi)}_{p,q}(\mathbb{R}^N) \subset B^{(s,\Psi)}_{p,r}(\mathbb{R}^N)$ continuously, i.e.

$$\|f\|_{B^{(s,\Psi)}_{p,r}(\mathbb{R}^N)} \leqslant \left(\sum_{j\geqslant 0}\Phi(2^{-j})^{-\chi}\Psi(2^{-j})^{\chi}\right)^{1/\chi} \|f\|_{B^{(s,\Phi)}_{p,q}(\mathbb{R}^N)}.$$

Using now Proposition 5.1, we obtain the desired conclusion. □

We now prove Theorem 1.8.

Proof of Theorem 1.8. — The proof works exactly as in Theorem 1.3 and, here again, it suffices to prove the result for $N \geqslant 2$, $d = N-1$. We prove the case $N = 2$ only but the general case $N \geqslant 2$ is similar. Let $(\lambda_{j,k})_{j,k\geqslant 0}$ be the sequence constructed at Lemma 3.9. Let $M \in \mathbb{N}^*$ with $s < M$. We consider the following function

$$f(x_1, x_2) = \sum_{j\geqslant 0} \sum_{k\geqslant 0} \lambda_{j,k} 2^{-j(s-\frac{2}{p})} \psi(2^j(x_1 - C_M j))\psi(2^j x_2 - k),$$

where ψ and C_M are as in the proof of Theorem 1.3. There, we have shown that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^2)} \leqslant c\|\lambda\|_{b_{p,q}},$$

(thus implying $f \in B^s_{p,q}(\mathbb{R}^2)$) and

$$\sup_{h\in K_j} \|\Delta_h^M f(\cdot, x_2)\|_{L^p(\mathbb{R})} \geqslant c\, 2^{-js} 2^{j/p} \lambda_{j,\lfloor 2^j x_2\rfloor},$$

for any $j \geqslant 0$ and a.e. $x_2 \in [1, 2]$, where K_j is as in (6.5). From this it follows that

$$2^{js}\Psi(2^{-j}) \sup_{h\in K_j} \|\Delta_h^M f(\cdot, x_2)\|_{L^p(\mathbb{R})} \geqslant c\, \Psi(2^{-j}) 2^{j/p} \lambda_{j,\lfloor 2^j x_2\rfloor}.$$

Now, by Lemma 3.9 and since

$$\|g\|_{B^{(s,\Psi)}_{p,q}(\mathbb{R})} \sim \|g\|_{L^p(\mathbb{R})} + \left(\sum_{j\geqslant 0} 2^{jsq}\Psi(2^{-j})^q \sup_{h\in K_j} \|\Delta_h^M g\|^q_{L^p(\mathbb{R})}\right)^{1/q},$$

is an equivalent quasi-norm on $B^{(s,\Psi)}_{p,q}(\mathbb{R})$, we have $f \in B^s_{p,q}(\mathbb{R}^2)$ and $f(\cdot, x_2) \notin B^{(s,\Psi)}_{p,q}(\mathbb{R})$ for a.e. $x_2 \in [1, 2]$. Then, arguing exactly as in Theorem 1.3, we obtain a function satisfying the requirements of Theorem 1.8. This completes the proof. □

Remark 7.3. — This can be generalized in the spirit of Theorem 7.1. Indeed, repeating the arguments of the proof of Theorem 6.1, we can prove that if (7.4) is violated, then there is a function $f \in B^{(s,\Phi)}_{p,q}(\mathbb{R}^N)$ such that $f(\cdot, x'') \notin B^{(s,\Psi)}_{p,q}(\mathbb{R}^d)$ for a.e. $x'' \in \mathbb{R}^{N-d}$.

Part II

Rigidity results

Chapter 3

Liouville type results for a nonlocal obstacle problem

This chapter is inspired by the paper [31] written in collaboration with J. Coville, F. Hamel and E. Valdinoci.

1 Introduction

A classical topic in applied analysis consists in the study of diffusive processes in media with an obstacle: roughly speaking, a dispersal follows a Brownian motion in an environment that possess an inaccessible region. At the level of partial differential equations, this translates into a reaction-diffusion equation that is defined outside a set K, which acts as an impenetrable obstacle and along which Neumann conditions are prescribed.

One of the cornerstones in the study of these processes lies in suitable rigidity results of Liouville-type, which allow the classification of stationary solutions, at least under some geometric assumption on the obstacle K.

In this chapter, we will study a nonlocal version of a diffusion equation and provide a series of Liouville-type results (whose precise statements will be given in Section 2). Not only the results obtained have a theoretical interest in the development of the theory of nonlocal equations, but they also possess several potential applications (especially in mathematical biology, where the dispersal of biological populations often presents nonlocal features, see e.g. formula (1) in [49], or in [19]).

Concretely, we will suppose that the diffusion operator arises by convolution with an integrable kernel and we will show that solutions of bistable stationary equations with fixed behavior at infinity are necessarily constant, at least when the obstacle is convex or "close to being convex" (we also observe that similar rigidity results do not hold in general for nonconvex obstacles).

Interestingly, in the nonlocal case, the boundary conditions along the obstacle do not need to be prescribed a priori (differently from the classical case).

In addition, the nonlocal operator that we consider here is not "regularizing", so some care is needed in our case to deal with a possible lack of regularity of the solutions.

We now provide the detailed mathematical description of the problem that we take into account.

1.1 A nonlocal obstacle problem

Throughout this chapter, K denotes a compact set of $\mathbb{R}^N$ with $N \geqslant 2$, and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$. We are interested in qualitative properties of bounded solutions to the following nonlocal semilinear equation

$$Lu + f(u) = 0 \quad \text{in } \mathbb{R}^N \setminus K, \tag{1.1}$$

where L is the nonlocal diffusion operator given by

$$Lu(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)(u(y) - u(x))\mathrm{d}y. \tag{1.2}$$

The kernel $J \in L^1(\mathbb{R}^N)$ is a radially symmetric non-negative function with unit mass and f is a C^1 "bistable" nonlinearity (precise assumptions on f and J will be given later on).

This problem may be thought of (see the next page for more explanations) as a nonlocal version of the following problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ \nabla u \cdot \nu = 0 & \text{on } \partial K, \end{cases} \tag{1.3}$$

where ν is the outward unit vector normal to K, assuming for (1.4) that K is smooth enough. For problem (1.4) with the local diffusion operator Δu, it was shown in [17] that there exist a time-global classical solution $u(t,x)$ to the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \mathbb{R} \times \overline{\mathbb{R}^N \setminus K}, \\ \nabla u \cdot \nu = 0 & \text{on } \mathbb{R} \times \partial K, \end{cases} \tag{1.4}$$

satisfying $0 < u(t,x) < 1$ for all $(t,x) \in \mathbb{R} \times \overline{\mathbb{R}^N \setminus K}$, and a classical solution $u_\infty(x)$ to the elliptic problem

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \nabla u_\infty \cdot \nu = 0 & \text{on } \partial K, \\ 0 \leqslant u_\infty \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u_\infty(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{1.5}$$

The function u_∞ is a stationary solution of (1.4) and it is actually obtained as the large time limit of $u(t,x)$, in the sense that $u(t,x) \to u_\infty(x)$ as $t \to +\infty$ locally uniformly in $x \in \overline{\mathbb{R}^N \setminus K}$. Under some geometric conditions on K (e.g. if K is starshaped or directionally convex, see [17] for precise assumptions) it is shown in [17, Theorems 6.1 and 6.4] that solutions to (1.5) are actually identically equal to 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$. This Liouville property shows that the solutions $u(t,x)$ of (1.4) constructed in [17] then satisfy

$$u(t,x) \underset{t\to+\infty}{\longrightarrow} 1 \quad \text{locally uniformly in } x \in \overline{\mathbb{R}^N \setminus K}. \tag{1.6}$$

To some extent, this result can be given an ecological interpretation. Consider a population with trajectories describing a Brownian motion in an environment consisting of the whole space $\mathbb{R}^N$ with a compact obstacle K, and suppose that f represents the demographic rate of the population. Then, the solution $u(t,x)$ to (1.4) can be understood as the density of the population at time t and location x. In this context, (1.6) means that, at large time, the population tends to occupy the whole space.

Assuming now that the trajectories follow, say, a compound Poisson process, then the diffusion phenomena are better described by a convolution-type operator such as (1.2). The reaction-diffusion equation $\frac{\partial u}{\partial t} = \Delta u + f(u)$ is then replaced by the equation

$$\frac{\partial u}{\partial t} = Lu + f(u),$$

with the nonlocal dispersion operator L, see [69, 83]. In this chapter, we deal with qualitative properties of the stationary solutions of equation (1.1), together with some asymptotic limiting conditions at infinity similar to those appearing in (1.5). Namely, we will be mainly concerned with solutions of

$$\begin{cases} Lu + f(u) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ 0 \leqslant u \leqslant 1 & \text{in } \mathbb{R}^N \setminus K, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{1.7}$$

It is expected that (1.5) and (1.7) share some common properties. One of the goals of the present chapter is, as for (1.5), to find some geometric conditions on K which guarantee that the solutions u to (1.7) are identically equal to 1. Moreover, as in [26] for (1.5), we will also show the robustness of the Liouville type results for (1.7).

We notice however that, whereas the solutions of (1.5) are automatically classical C^2 solutions in $\overline{\mathbb{R}^N \setminus K}$ if the boundary ∂K is smooth enough (by standard interior and boundary elliptic estimates), there is in general no smoothing effect for the nonlocal problems (1.1) or (1.7). The solutions u may even not be continuous in general. Yet some regularity results (uniform or Hölder continuity) will be shown here under additional assumptions on the data J and f. Actually, one of the difficulties and novelties of this chapter, as compared to [17], is to deal with this a priori *lack of regularity* in general.

We observe also that, in (1.1) or (1.7), we *do not ask for any additional boundary condition on* ∂K. To understand why this is so, let us give some heuristic motivation. First of all, the most intuitive nonlocal counterpart of (1.5) would be to replace Δu in (1.5) by $\widetilde{L}_\varepsilon u$ with $\varepsilon > 0$ small, where

$$\widetilde{L}_\varepsilon u(x) := \frac{1}{\beta\varepsilon^2}\int_{\mathbb{R}^N\setminus K}\widetilde{J}_\varepsilon(x-y)(u(y)-u(x))\mathrm{d}y,$$

and $\widetilde{J}_\varepsilon(z) = \varepsilon^{-N}\widetilde{J}(\varepsilon^{-1}z)$, $\widetilde{J}$ being a radially symmetric kernel with

$$\beta = (2N)^{-1}\int_{\mathbb{R}^N}\widetilde{J}(z)|z|^2\mathrm{d}z.$$

In other words, the nonlocal dispersion operator Lu in (1.1) or (1.7) would be replaced by $\widetilde{L}_\varepsilon u$ and the kernel J would be given by $(\beta\varepsilon^2)^{-1}\widetilde{J}_\varepsilon$. Furthermore, using for example [27], the associated energy of (1.7), with $\widetilde{L}_\varepsilon$ in place of L, can be thought of as an approximation of that of (1.5) (see [2] and also [16, 19, 83] where similar quantities are considered in a biological framework). Now, to see how the Neumann boundary condition in (1.5) can be recovered from (1.1) or (1.7) with $\widetilde{L}_\varepsilon$ as $\varepsilon \to 0^+$, let us consider for simplicity the case where ∂K is of class C^1 with unit normal ν and the bounded function u is of class $C^1(\overline{\mathbb{R}^N\setminus K})$ and is extended as a $C^1(\mathbb{R}^N)$ function still denoted by u. Formula (1.1) then also holds by continuity in $\overline{\mathbb{R}^N\setminus K}$ and, for every $x, y \in \overline{\mathbb{R}^N\setminus K}$, there exists a point $c_{x,y} \in [x,y]$ such that $u(y)-u(x) = \nabla u(c_{x,y})\cdot(y-x)$. It follows that, for every $x \in \overline{\mathbb{R}^N\setminus K}$,

$$-f(u(x)) = \frac{1}{\beta\varepsilon}\int_{\mathbb{R}^N\setminus K}\widehat{J}_\varepsilon(x-y)\,\nabla u(c_{x,y})\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y,$$

where $\widehat{J}_\varepsilon(z) = \varepsilon^{-N}\widehat{J}(\varepsilon^{-1}z)$ and $\widehat{J}(z) = \widetilde{J}(z)|z|$. Then, for all $x \in \partial K$, a formal computation leads to

$$\gamma\,\nabla u(x)\cdot\nu = \lim_{\varepsilon\to0^+}\int_{\mathbb{R}^N\setminus K}\widehat{J}_\varepsilon(x-y)\,\nabla u(c_{x,y})\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y = \lim_{\varepsilon\to0^+}\big(-\varepsilon\beta f(u(x))\big) = 0,$$

where $\gamma = (1/2)\int_{\mathbb{R}^N}\widetilde{J}(z)|z_1|\mathrm{d}z > 0$. Hence, $\nabla u\cdot\nu = 0$ on ∂K and (1.7) is then a reasonable nonlocal counterpart for (1.5). The above calculation justifies, at least formally, why no additional boundary condition on ∂K is required in (1.1) or (1.7).

1.2 General assumptions, notations and definitions

Let us now specify the detailed assumptions made throughout the chapter. As already mentioned above, we suppose that f is of "bistable" type and J is a radially symmetric kernel. More precisely, we will assume that

$$f \in C^1([0,1]), \quad f(0) \geqslant 0, \quad f'(1) < 0, \tag{1.8}$$

$$\begin{cases} J \in L^1(\mathbb{R}^N) \text{ is a non-negative, radially symmetric kernel with unit mass,} \\ \text{there are } 0 \leqslant r_1 < r_2 \text{ such that } J(x) > 0 \text{ for a.e. } x \text{ with } r_1 < |x| < r_2, \end{cases} \tag{1.9}$$

and there exists a function $\phi \in C(\mathbb{R})$ satisfying

$$\begin{cases} J_1 * \phi - \phi + f(\phi) \geqslant 0 \text{ in } \mathbb{R}, \\ \phi \text{ is increasing in } \mathbb{R}, \quad \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \end{cases} \tag{1.10}$$

where $J_1 \in L^1(\mathbb{R})$ is the non-negative even function with unit mass given for a.e. $x \in \mathbb{R}$ by

$$J_1(x) := \int_{\mathbb{R}^{N-1}} J(x, y_2, \cdots, y_N)\, \mathrm{d}y_2 \cdots \mathrm{d}y_N.$$

We notice that, in addition to the first property in (1.9), the second one is immediately fulfilled if J is assumed to be continuous. Moreover, we notice that condition (1.10) implies immediately that $0 < \phi < 1$ in $\mathbb{R}$. As is well-known (see e.g. [13, 50]), condition (1.10) is satisfied if, in addition to (1.8) and (1.9), the following assumptions are made on f and J:

$$\begin{cases} \exists\, \theta \in (0,1), \quad f(0) = f(\theta) = f(1) = 0, \quad f < 0 \text{ in } (0,\theta), \quad f > 0 \text{ in } (\theta, 1), \\ \displaystyle\int_0^1 f > 0, \quad f'(0) < 0, \quad f'(\theta) > 0, \quad f'(1) < 0, \quad f' < 1 \text{ in } [0,1], \\ \displaystyle\int_{\mathbb{R}} J_1(x)|x| \mathrm{d}x < +\infty \quad \text{and} \quad J \in W^{1,1}(\mathbb{R}^N). \end{cases} \tag{1.11}$$

Let us also list in this subsection a few notations and definitions that will be used all along the chapter:

$\lvert E\rvert$:	is the Lebesgue measure of the measurable set E;
$\mathbb{1}_E$:	is the characteristic function of the set E;
B_R :	is the open Euclidean ball of radius $R > 0$ centered at the origin;
$B_R(x)$:	is the open Euclidean ball of radius $R > 0$ centered at $x \in \mathbb{R}^N$;
$\mathscr{A}(R_1, R_2)$:	is the open annulus $B_{R_2} \setminus \overline{B_{R_1}}$ for $0 \leqslant R_1 < R_2$, by setting $\overline{B_0} = \{0\}$;
$\mathscr{A}(x, R_1, R_2)$:	is the open annulus $x + \mathscr{A}(R_1, R_2)$;
$g * h$:	is the convolution of g and h;
g^+ :	is the positive part of g, i.e. $g^+ := \max\{0, g\}$.

Given $\Omega \subset \mathbb{R}^N$ and $p \in [1, \infty]$, we denote by $L^p(\Omega)$ the Lebesgue space of (equivalence classes of) measurable functions g for which the p-th power of the absolute value is Lebesgue integrable when $p < \infty$ (resp. essentially bounded when $p = \infty$). When the context is clear, we will write $\|g\|_p$ instead of $\|g\|_{L^p(\Omega)}$. Given $\alpha \in (0, 1]$ and $p \in [1, \infty]$, $B^\alpha_{p,\infty}(\mathbb{R}^N)$ stands for the Nikol'skii space consisting in all measurable functions $g \in L^p(\mathbb{R}^N)$ such that

$$[g]_{B^\alpha_{p,\infty}(\mathbb{R}^N)} := \sup_{h \neq 0} \frac{\|g(\cdot + h) - g\|_{L^p(\mathbb{R}^N)}}{|h|^\alpha} < +\infty.$$

We note that, when $p = \infty$, the space $B^\alpha_{\infty,\infty}(\mathbb{R}^N)$ coincides with the classical Hölder space $C^{0,\alpha}(\mathbb{R}^N)$. For a set $E \subset \mathbb{R}^N$ and $g : E \to \mathbb{R}$, we set

$$[g]_{C^{0,\alpha}(E)} = \sup_{x \in E,\, y \in E,\, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\alpha}.$$

Let us finally recall some useful notions of regularity of a compact set K.

Definition 1.1. — *Let $\alpha \in (0, 1]$. We say that a compact set $K \subset \mathbb{R}^N$ has $C^{0,\alpha}$ boundary if there exist $r > 0$, $p \in \mathbb{N}$, p rotations $(R_i)_{1 \leqslant i \leqslant p}$ of $\mathbb{R}^N$, p points $(z_i)_{1 \leqslant i \leqslant p}$ of ∂K and p functions $(\psi_i)_{1 \leqslant i \leqslant p}$ defined in the $(N - 1)$-dimensional ball $B^{N-1}_r = \{x' \in \mathbb{R}^{N-1};\, |x'| < r\}$ of class $C^{0,\alpha}(B^{N-1}_r)$ and such that*

$$\partial K = \bigcup_{1 \leqslant i \leqslant p} R_i\Big(\big\{x_N = \psi_i(x');\, x' \in B^{N-1}_r\big\}\Big), \tag{1.12}$$

and

$$\mathring{K} \cap B_r(z_i) = R_i\Big(\big\{x_N > \psi_i(x');\, x' \in B^{N-1}_r\big\}\Big) \cap B_r(z_i), \tag{1.13}$$

for every $1 \leqslant i \leqslant p$.

Definition 1.2. — *Let $\alpha \in (0, 1]$, let $K \subset \mathbb{R}^N$ be a compact convex set with non-empty interior (∂K is then automatically of class $C^{0,\alpha}$) and let $(K_\varepsilon)_{0 < \varepsilon \leqslant 1} \subset \mathbb{R}^N$ be a family of compact, simply connected sets having $C^{0,\alpha}$ boundary. We say that $(K_\varepsilon)_{0 < \varepsilon \leqslant 1}$ is a family of $C^{0,\alpha}$ deformations of K if the following conditions are fulfilled:*

(i) $K \subset K_{\varepsilon_1} \subset K_{\varepsilon_2}$ for all $0 < \varepsilon_1 \leqslant \varepsilon_2 \leqslant 1$;

(ii) $K_\varepsilon \to K$ as $\varepsilon \downarrow 0$ in $C^{0,\alpha}$, in the sense that there exist $r > 0$, $p \in \mathbb{N}$, p rotations $(R_i)_{1 \leqslant i \leqslant p}$ of $\mathbb{R}^N$, p points $(z_i)_{1 \leqslant i \leqslant p}$ of ∂K, p functions $(\psi_i)_{1 \leqslant i \leqslant p}$ and p families of functions $(\psi_{i,\varepsilon})_{1 \leqslant i \leqslant p,\, 0 < \varepsilon \leqslant 1}$ of class $C^{0,\alpha}(B^{N-1}_r)$ describing ∂K and ∂K_ε as in (1.12) and (1.13) above, and such that

$$\|\psi_i - \psi_{i,\varepsilon}\|_{C^{0,\alpha}(B^{N-1}_r)} \to 0 \quad \text{as } \varepsilon \downarrow 0, \quad \text{for every } 1 \leqslant i \leqslant p.$$

2 Main results

The Liouville property for the local problem (1.5) says that $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$ under some geometric conditions on K, in particular when K is convex, see [17]. When the obstacle K is convex, we prove that this Liouville property still holds for (1.7) with the nonlocal operator L. We will actually prove several results, which correspond to various assumptions on the solutions u and the data f and J. We will also show the robustness of the Liouville type property with respect to small deformations of the obstacle K. The assumptions (1.8), (1.9) and (1.10) will be common assumptions of almost all results. In some statements, assumption (1.10) is replaced by the stronger assumption (1.11).

2.1 A first rough Liouville type result

Under rather mild additional assumptions on K, we first state a "rough" Liouville type property for the solutions of (1.7), if f is assumed to be non-negative on the range of u.

Proposition 2.1. — *Let $K \subset \mathbb{R}^N$ be a compact set such that $\mathbb{R}^N \setminus K$ is connected. Assume that $f \in C^1([0,1])$ and J satisfies* (1.9). *Let $\theta \in [0,1)$ and assume that $f \geqslant 0$ in $[\theta, 1]$. Let $u : \overline{\mathbb{R}^N \setminus K} \to [\theta, 1]$ be a continuous solution of*

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{2.1}$$

Then, $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$.

One of the main goals of the chapter is to understand under which conditions on K this Liouville type property still holds or does not hold when u ranges in the whole interval $[0, 1]$.

2.2 Liouville type properties for convex obstacles

Our first main theorem is the following result dealing with continuous super-solutions to $L(u) + f(u) = 0$ ranging in $[0, 1]$.

Theorem 2.2. — *Let $K \subset \mathbb{R}^N$ be a compact convex set. Assume* (1.8), (1.9), (1.10) *and let*

$$u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1]), \tag{2.2}$$

be a function satisfying

$$\begin{cases} Lu + f(u) \leqslant 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{2.3}$$

Then, $u = 1$ *in* $\overline{\mathbb{R}^N \setminus K}$.

If we ask for a solution of (1.7) instead of a super-solution, it turns out that the regularity or limiting conditions required on u to obtain a Liouville type result can be considerably weakened, by strengthening the assumptions made on f and/or J.

Firstly, the continuity assumption (2.2) can be relaxed provided the nonlinearity does not vary too much. More precisely, we will prove the following result.

THEOREM 2.3. — *Let* $K \subset \mathbb{R}^N$ *be a compact convex set. Assume* (1.8), (1.9), (1.10) *and suppose that*

$$\max_{[0,1]} f' < \frac{1}{2}. \tag{2.4}$$

Let $u : \mathbb{R}^N \setminus K \to [0,1]$ *be a measurable function satisfying*

$$\begin{cases} Lu + f(u) = 0 & \text{a.e. in } \mathbb{R}^N \setminus K, \\ u(x) \to 1 & \text{as } |x| \to +\infty. \end{cases}$$

Then, $u = 1$ *a.e. in* $\mathbb{R}^N \setminus K$.

Secondly, assuming that f and J satisfy (1.11) instead of (1.10), that $J \in L^2(\mathbb{R}^N)$ and is compactly supported, and that f does not vary too much or u is a priori uniformly continuous, then the assumptions on the asymptotic behaviour of u at infinity can be noticeably weakened. More precisely, the following result holds.

THEOREM 2.4. — *Let* $K \subset \mathbb{R}^N$ *be a compact convex set and assume that* f *and* J *satisfy* (1.8), (1.9) *and* (1.11). *Assume further that* J *is compactly supported and* $J \in L^2(\mathbb{R}^N)$. *If* $u : \overline{\mathbb{R}^N \setminus K} \to [0,1]$ *is uniformly continuous in* $\overline{\mathbb{R}^N \setminus K}$ *and obeys*

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \sup\limits_{\mathbb{R}^N \setminus K} u = 1, \end{cases} \tag{2.5}$$

then $u = 1$ *in* $\overline{\mathbb{R}^N \setminus K}$. *Similarly, if* (2.4) *holds and if* $u : \mathbb{R}^N \setminus K \to [0,1]$ *is a measurable function satisfying*

$$\begin{cases} Lu + f(u) = 0 & \text{a.e. in } \mathbb{R}^N \setminus K, \\ \operatorname*{ess\,sup}\limits_{\mathbb{R}^N \setminus K} u = 1, \end{cases} \tag{2.6}$$

then $u = 1$ *a.e. in* $\mathbb{R}^N \setminus K$.

Remark 2.5. — Condition (2.4) ensures that u actually has a uniformly continuous representative in $\overline{\mathbb{R}^N \setminus K}$ (see Lemma 2.2 and Remark 3.3). However, if u is already known to be uniformly continuous, then Theorem 2.4 provides the same conclusion without assumption (2.4) (see Lemma 7.2 for further details).

2.3 Robustness of the Liouville property for nearly convex obstacles K

Under some flatness assumptions on f, and following a line of ideas in [26], it turns out that the Liouville property is still available under small Hölder perturbations of a given convex obstacle K. Namely, the following result holds.

Theorem 2.6. — *Let $\alpha \in (0,1]$, let $K \subset \mathbb{R}^N$ be a compact convex set with non-empty interior and let $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$ be a family of $C^{0,\alpha}$ deformations of K. Assume* (1.8), (1.9), (1.11) *and suppose that $J \in B^{\alpha}_{1,\infty}(\mathbb{R}^N)$ and*

$$\max_{[0,1]} f' < \inf_{0<\varepsilon\leqslant 1}\ \inf_{x\in\mathbb{R}^N\setminus K_\varepsilon} \|J(x-\cdot)\|_{L^1(\mathbb{R}^N\setminus K_\varepsilon)}.$$

For $0<\varepsilon\leqslant 1$, let L_ε be the operator given by, for every $v \in L^\infty(\mathbb{R}^N\setminus K_\varepsilon)$,

$$L_\varepsilon v(x) := \int_{\mathbb{R}^N\setminus K_\varepsilon} J(x-y)(v(y)-v(x))\mathrm{d}y.$$

Then there exists $\varepsilon_0 \in (0,1]$ such that for all $\varepsilon \in (0,\varepsilon_0]$ the unique measurable solution u_ε of

$$\begin{cases} L_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0 & \text{a.e. in } \mathbb{R}^N\setminus K_\varepsilon,\\ 0\leqslant u_\varepsilon \leqslant 1 & \text{a.e. in } \mathbb{R}^N\setminus K_\varepsilon,\\ u_\varepsilon(x)\to 1 & \text{as } |x|\to+\infty,\end{cases} \tag{2.7}$$

is $u_\varepsilon = 1$ a.e. in $\mathbb{R}^N\setminus K_\varepsilon$.

Remark 2.7. — It should be noted that the monotonicity assumption (i) in Definition 1.2 has been made for simplicity and is not necessary for our purposes. Moreover, the conclusion of Theorem 2.6 remains true whenever $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$ is a family of $C^{0,\alpha}$ deformations of any compact set K for which the conclusion of Theorem 2.3 is valid. Since there exist some smooth, compact, non-convex and simply connected sets which are $C^{0,\alpha}$ close to a smooth, compact and convex set, Theorem 2.6 implies that the Liouville property holds for some smooth, compact and non-convex obstacles, and then also for their $C^{0,\alpha}$ perturbations. Finally, we conjecture that the Liouville property of Theorem 2.3 holds for any starshaped compact obstacle as well.

However, as in the local case (see [17, Theorem 6.5]), the above Liouville type properties cannot be expected for general obstacles. For example, one can easily find counterexamples if K is no longer simply connected. Take for instance $K = \overline{\mathscr{A}(1,2)} = \overline{B_2}\setminus B_1$ and suppose that J is supported in $B_{1/2}$. Then, the function u defined by

$$u(x) = \begin{cases} 1 & \text{if } x\in\mathbb{R}^N\setminus B_2,\\ 0 & \text{if } x\in\overline{B_1},\end{cases}$$

is a continuous solution of (1.1). Yet, u is not identically 1 in the whole set $\mathbb{R}^N\setminus K$.

Outline of the chapter. The following first sections are concerned with general results on the solutions to problems (1.1) or (1.7). Namely, in Section 3, we show that the solutions are uniformly continuous, more precisely they have a uniformly continuous representative, if rather mild assumptions are made on f. In Section 4, we give several comparison principles that fit our purposes. We then use these comparison principles in Section 5 to construct a radially symmetric lower bound for the solutions. In Section 6, we study an auxiliary problem which will enable us to pave the way towards the proof of Theorem 2.4. The remaining part of the chapter is devoted to the proofs of our main results. In Section 7, we prove, at a stroke, Theorems 2.2 and 2.3, and with more work we show how to relax the assumptions on u when the kernel J is compactly supported, that is we prove Theorem 2.4. In Section 8, as a preliminary result we prove the rough Liouville-type result Proposition 2.1 and then we establish our robustness result Theorem 2.6.

3 Some auxiliary regularity results

Throughout this section, K is any compact subset of $\mathbb{R}^N$, f is any $C^1(\mathbb{R})$ function, and J is any $L^1(\mathbb{R}^N)$ non-negative and radially symmetric kernel with unit mass. For $x \in \mathbb{R}^N$, we write

$$\mathscr{J}(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)\mathrm{d}y.$$

Notice that $\mathscr{J}$ is a uniformly continuous function in $\mathbb{R}^N$. In the sequel, for any $\delta > 0$, we will denote K_δ the closed thickening of K with width δ, defined by

$$K_\delta := K + \overline{B_\delta}.$$

We now prove that, when J is compactly supported and f' is not too large in $[0,1]$, then the measurable solutions u to (1.7) are continuous far away from the obstacle.

Lemma 3.1. — *Suppose that $K \subset \mathbb{R}^N$ is a compact set and that J is supported in the ball B_δ for some $\delta > 0$. Suppose that*

$$\max_{[0,1]} f' < 1. \tag{3.1}$$

Let $u \in L^\infty(\mathbb{R}^N \setminus K, [0,1])$ be a solution of $Lu + f(u) = 0$ a.e. in $\mathbb{R}^N \setminus K$. Then u is uniformly continuous in $\overline{\mathbb{R}^N \setminus K_\delta}$, in the sense that u has a representative in its class of equivalence that is uniformly continuous in $\overline{\mathbb{R}^N \setminus K_\delta}$. If, in addition, $J \in B^\alpha_{1,\infty}(\mathbb{R}^N)$ for some $\alpha \in (0,1]$, then $u \in C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\delta})$ and $\big(1 - \max_{[0,1]} f'\big)[u]_{C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\delta})} \leqslant [J]_{B^\alpha_{1,\infty}(\mathbb{R}^N)}$.

Proof. — For every x and y in $\mathbb{R}^N \setminus K_\delta$, we have

$$\begin{aligned} Lu(x) - Lu(y) &= \int_{\mathbb{R}^N\setminus K} u(z)\big(J(x-z) - J(y-z)\big)\mathrm{d}z \\ &\quad - u(x)\int_{\mathbb{R}^N\setminus K} J(x-z)\,\mathrm{d}z + u(y)\int_{\mathbb{R}^N\setminus K} J(y-z)\mathrm{d}z \\ &= -\mathscr{J}(x)u(x) + \mathscr{J}(y)u(y) + \int_{\mathbb{R}^N\setminus K} u(z)\big(J(x-z) - J(y-z)\big)\mathrm{d}z. \end{aligned}$$

Since J has unit mass and is supported in B_δ, we get that $\mathscr{J}(x) = \mathscr{J}(y) = 1$. Therefore,

$$Lu(x) - Lu(y) + u(x) - u(y) = \int_{\mathbb{R}^N\setminus K} u(z)\big(J(x-z) - J(y-z)\big)\mathrm{d}z.$$

Now, remember that u is a solution to $Lu + f(u) = 0$ a.e. in $\mathbb{R}^N \setminus K$. In particular, there exists a measurable negligible set E such that $Lu(z) + f(u(z)) = 0$ (and $u(z) \in [0,1]$) for all $z \in \mathbb{R}^N \setminus (K \cup E)$. Hence, letting

$$g(t) := t - f(t),$$

for $t \in [0,1]$, we obtain that

$$g(u(x)) - g(u(y)) = \int_{\mathbb{R}^N\setminus K} u(z)\big(J(x-z) - J(y-z)\big)\mathrm{d}z =: h(x,y), \tag{3.2}$$

for all $x, y \in \mathbb{R}^N \setminus (K_\delta \cup E)$. Notice that, since $J \in L^1(\mathbb{R}^N)$ and $u \in L^\infty(\mathbb{R}^N \setminus K)$, the function h defined by the right-hand side of the previous equation can actually be defined in $\mathbb{R}^N \times \mathbb{R}^N$ and it is uniformly continuous in $\mathbb{R}^N \times \mathbb{R}^N$. Furthermore, by (3.1), the function $g \in C^1([0,1])$ is such that $g' > 0$ in $[0,1]$. It is then a C^1 diffeomorphism from $[0,1]$ to $[g(0), g(1)] = [-f(0), 1 - f(1)]$. Let us denote $g^{-1} : [g(0), g(1)] \to [0,1]$ its reciprocal.

Fix $y_0 \in \mathbb{R}^N \setminus (K_\delta \cup E)$. For every $x \in \mathbb{R}^N \setminus (K_\delta \cup E)$, (3.2) yields

$$g(u(y_0)) + h(x, y_0) = g(u(x)) \in [g(0), g(1)].$$

Since the function $x \mapsto g(u(y_0)) + h(x, y_0)$ is continuous (in the whole $\mathbb{R}^N$) and since E is negligible, it follows that $g(u(y_0)) + h(x, y_0) \in [g(0), g(1)]$ for all $x \in \mathbb{R}^N \setminus K_\delta$ (since any point of the open set $\mathbb{R}^N \setminus K_\delta$ is the limit of a sequence of points in $\mathbb{R}^N \setminus (K_\delta \cup E)$). Define now

$$\widetilde{u}(x) = g^{-1}\big(g(u(y_0)) + h(x, y_0)\big) \quad \text{for } x \in \overline{\mathbb{R}^N \setminus K_\delta}.$$

By (3.2), one has $\widetilde{u} = u$ in $\mathbb{R}^N \setminus (K_\delta \cup E)$. Furthermore, $\widetilde{u}$ is uniformly continuous in $\overline{\mathbb{R}^N \setminus K_\delta}$ owing to its definition, since h is uniformly continuous in $\mathbb{R}^N \times \mathbb{R}^N$ and g^{-1} is C^1 hence Lipschitz continuous in $[g(0), g(1)]$.

Even if it means redefining u by $\widetilde{u}$ in $\overline{\mathbb{R}^N \setminus K_\delta}$, it follows that u is uniformly continuous in $\overline{\mathbb{R}^N \setminus K_\delta}$ and that (3.2) holds, by continuity, for all $x, y \in \overline{\mathbb{R}^N \setminus K_\delta}$. In particular, since $0 \leqslant u \leqslant 1$ in $\mathbb{R}^N \setminus K$, we get that

$$\forall\, x, y \in \overline{\mathbb{R}^N \setminus K_\delta}, \quad |g(u(x)) - g(u(y))| \leqslant \|J(\cdot + x - y) - J\|_{L^1(\mathbb{R}^N)}. \tag{3.3}$$

Finally, if $J \in B^\alpha_{1,\infty}(\mathbb{R}^N)$ with $\alpha \in (0, 1]$, then (3.3) yields $g(u) \in C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\delta})$ and

$$[g(u)]_{C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\delta})} \leqslant \sup_{h \neq 0} \frac{\|J(\cdot + h) - J\|_{L^1(\mathbb{R}^N)}}{|h|^\alpha} = [J]_{B^\alpha_{1,\infty}(\mathbb{R}^N)}.$$

Since $\max_{[g(0),g(1)]} |(g^{-1})'| \leqslant (1 - \max_{[0,1]} f')^{-1}$ and $0 \leqslant u \leqslant 1$, one concludes that $u \in C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\delta})$ and $\big(1 - \max_{[0,1]} f'\big)[u]_{C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\delta})} \leqslant [J]_{B^\alpha_{1,\infty}(\mathbb{R}^N)}$. □

We now establish a regularity result for u in the whole set $\overline{\mathbb{R}^N \setminus K}$ for flatter nonlinearities.

Lemma 3.2. — *Suppose that $K \subset \mathbb{R}^N$ is a compact set and that*

$$\max_{[0,1]} f' < \inf_{\mathbb{R}^N \setminus K} \mathscr{J}. \tag{3.4}$$

Let $u \in L^\infty(\mathbb{R}^N \setminus K, [0, 1])$ be a solution of $Lu + f(u) = 0$ a.e. in $\mathbb{R}^N \setminus K$. Then, u can be redefined up to a negligible set and extended as a uniformly continuous function in $\overline{\mathbb{R}^N \setminus K}$. If, in addition, $J \in B^\alpha_{1,\infty}(\mathbb{R}^N)$ for some $\alpha \in (0, 1]$, then $u \in C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K})$ and

$$\Big(\inf_{\mathbb{R}^N \setminus K} \mathscr{J} - \max_{[0,1]} f'\Big)\, [u]_{C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K})} \leqslant 2\, [J]_{B^\alpha_{1,\infty}(\mathbb{R}^N)}.$$

Remark 3.3. — In the case of a compact convex obstacle K, then the conclusion of Lemma 2.2 still holds if (2.5) is replaced by

$$\max_{[0,1]} f' < \frac{1}{2}.$$

Indeed, $\mathscr{J} \geqslant 1/2$ in $\mathbb{R}^N \setminus K$ when K is convex (remember also that J is always assumed to be a non-negative radially symmetric kernel with unit mass). The bound $1/2$ is somehow optimal, since K can be as large as desired, still in the class of compact convex sets. However, this bound deteriorates considerably if K is only starshaped, as the infimum of $\mathscr{J}$ in $\mathbb{R}^N \setminus K$ can become arbitrarily small. Roughly speaking, the less convex the obstacle K, the flatter the nonlinearity f needs to be to insure (2.5) and the interior continuity of the solution u.

Proof of Lemma 2.2. — Reasoning exactly as in the proof of Lemma 3.1, there exists a measurable negligible set E such that

$$\forall\, x, y \in \mathbb{R}^N \setminus (K \cup E), \quad G(x, u(x)) - G(y, u(y)) = h(x, y), \tag{3.5}$$

where $h(x, y)$ is defined in (3.2) (remember also that h is uniformly continuous in $\mathbb{R}^N \times \mathbb{R}^N$) and

$$G(x, s) = \mathscr{J}(x)\, s - f(s) \ \text{ for } (x, s) \in \mathbb{R}^N \times [0, 1].$$

By (2.5) and the continuity of $\mathscr{J}$, the function G is such that $\partial_s G(x, s) > 0$ for all $(x, s) \in \overline{\mathbb{R}^N \setminus K} \times [0, 1]$. For every $x \in \overline{\mathbb{R}^N \setminus K}$, the function $G(x, \cdot)$ is then a C^1 diffeomorphism from $[0, 1]$ to $[G(x, 0), G(x, 1)]$. Let us denote $H_x : [G(x, 0), G(x, 1)] \to [0, 1]$ its reciprocal, that is, $H_x(G(x, t)) = t$ for all $x \in \overline{\mathbb{R}^N \setminus K}$ and $t \in [0, 1]$.

Fix $y_0 \in \mathbb{R}^N \setminus (K \cup E)$. For every $x \in \mathbb{R}^N \setminus (K \cup E)$, (3.5) yields

$$G(y_0, u(y_0)) + h(x, y_0) = G(x, u(x)) \in [G(x, 0), G(x, 1)].$$

Since the function $x \mapsto G(y_0, u(y_0)) + h(x, y_0)$ is continuous (in the whole space $\mathbb{R}^N$), since G is itself continuous in $\mathbb{R}^N \times [0, 1]$ and since E is negligible, it follows that $G(y_0, u(y_0)) + h(x, y_0) \in [G(x, 0), G(x, 1)]$ for all x in the open set $\mathbb{R}^N \setminus K$. Define now

$$\widetilde{u}(x) = H_x\big(G(y_0, u(y_0)) + h(x, y_0)\big) \quad \text{for } x \in \overline{\mathbb{R}^N \setminus K}.$$

By (3.5), one has $\widetilde{u} = u$ in $\mathbb{R}^N \setminus (K \cup E)$. Furthermore, $\widetilde{u}$ is continuous in $\overline{\mathbb{R}^N \setminus K}$ owing to its definition, since h is continuous in $\mathbb{R}^N \times \mathbb{R}^N$ and $(x, s) \mapsto H_x(s)$ is continuous in the set $\big\{(x, s) \in \overline{\mathbb{R}^N \setminus K} \times \mathbb{R};\ s \in [G(x, 0), G(x, 1)]\big\}$. Even if it means redefining u by $\widetilde{u}$ in $\mathbb{R}^N \setminus K$ and extending it by $\widetilde{u}$ in $\overline{\mathbb{R}^N \setminus K}$, it follows that u is continuous in $\overline{\mathbb{R}^N \setminus K}$ and that (3.5) holds, by continuity, for all x, y in the open set $\mathbb{R}^N \setminus K$ and then in $\overline{\mathbb{R}^N \setminus K}$. In particular, since $0 \leqslant u \leqslant 1$ in $\overline{\mathbb{R}^N \setminus K}$, we get that

$$\forall\, x, y \in \overline{\mathbb{R}^N \setminus K}, \quad |G(x, u(x))) - G(y, u(y))| \leqslant \|J(\cdot + x - y) - J\|_{L^1(\mathbb{R}^N)}. \tag{3.6}$$

Finally, define

$$\beta := \inf_{\mathbb{R}^N \setminus K} \mathscr{J} - \max_{[0,1]} f' > 0,$$

the positivity of β resulting from (2.5). From (3.6) together with the definition of G and the inequalities $0 \leqslant u \leqslant 1$ in $\overline{\mathbb{R}^N \setminus K}$, one infers that, for all $x, y \in \overline{\mathbb{R}^N \setminus K}$,

$$\begin{aligned}
&\big|\mathscr{J}(x)\,(u(x) - u(y)) - (f(u(x)) - f(u(y)))\big| \\
&\qquad\qquad \leqslant \|J(\cdot + x - y) - J\|_{L^1(\mathbb{R}^N)} + \big|u(y)\,(\mathscr{J}(x) - \mathscr{J}(y))\big| \\
&\qquad\qquad \leqslant 2\,\|J(\cdot + x - y) - J\|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

It follows from the mean value theorem and the above definition of β that

$$\beta\,|u(x)-u(y)|\leqslant 2\,\|J(\cdot+x-y)-J\|_{L^1(\mathbb{R}^N)},$$

for all $x,y\in\overline{\mathbb{R}^N\setminus K}$. In particular, the function u is uniformly continuous in $\overline{\mathbb{R}^N\setminus K}$. Furthermore, if $J\in B^{\alpha}_{1,\infty}(\mathbb{R}^N)$ for some $\alpha\in(0,1]$, then $u\in C^{0,\alpha}(\overline{\mathbb{R}^N\setminus K})$ and $\beta\,[u]_{C^{0,\alpha}(\overline{\mathbb{R}^N\setminus K})}\leqslant 2\,[J]_{B^{\alpha}_{1,\infty}(\mathbb{R}^N)}$. The proof of Lemma 2.2 is thereby complete. $\square$

4 Comparison principles

In this section, we collect some comparison principles that fit for our purposes. Throughout this section, K is any compact subset of $\mathbb{R}^N$, f is any $C^1(\mathbb{R})$ function, and J is any $L^1(\mathbb{R}^N)$ non-negative and radially symmetric kernel with unit mass.

We start with a weak maximum principle.

Lemma 4.1 (Weak maximum principle). — *Assume that*

$$f'\leqslant -c_1 \text{ in } [1-c_0,+\infty),\ \text{ for some } c_0>0,\ c_1>0. \tag{4.1}$$

Let $H\subset\mathbb{R}^N$ be an open affine half-space such that $K\subset H^c=\mathbb{R}^N\setminus H$. Let $u,v\in L^\infty(\mathbb{R}^N\setminus K)$ be such that

$$u,\ v\in C\big(\overline{H}\big), \tag{4.2}$$

and

$$\begin{cases} Lu+f(u)\leqslant 0 & \text{in } \overline{H},\\ Lv+f(v)\geqslant 0 & \text{in } \overline{H}.\end{cases} \tag{4.3}$$

Assume also that

$$u\geqslant 1-c_0\quad \text{in } \overline{H}, \tag{4.4}$$

that

$$\limsup_{|x|\to+\infty}\big(v(x)-u(x)\big)\leqslant 0, \tag{4.5}$$

and that

$$v\leqslant u\quad \text{a.e. in } H^c\setminus K. \tag{4.6}$$

Then, $v\leqslant u$ a.e. in $\mathbb{R}^N\setminus K$.

Proof. — We let $w:=v-u$. We want to prove that $w\leqslant 0$ a.e. in $\mathbb{R}^N\setminus K$. From (4.6), we only have to show that $w\leqslant 0$ in $\overline{H}$ (remember that from (4.2) both functions u and v are assumed to be continuous in $\overline{H}$). We argue by contradiction and we suppose that $\sup_{\overline{H}} w > 0$. Then, thanks to (4.6), one has $\sup_{\mathbb{R}^N\setminus K} w = \sup_{\overline{H}} w > 0$ and there exists a sequence $(x_j)_{j\in\mathbb{N}}$ in $\overline{H}$ such that

$$\lim_{j\to+\infty} w(x_j) = \sup_{\mathbb{R}^N\setminus K} w > 0.$$

It follows then from (4.5) that the sequence $(x_j)_{j\in\mathbb{N}}$ is bounded. Thus, up to extraction of a subsequence, there exists a point $\bar{x} \in \overline{H}$ such that $x_j \to \bar{x}$ as $j \to +\infty$, hence $w(\bar{x}) = \lim_{j\to+\infty} w(x_j) > 0$ by (4.2). As a consequence, (4.3) yields

$$\begin{aligned} Lw(\bar{x}) = Lv(\bar{x}) - Lu(\bar{x}) &\geqslant -f(v(\bar{x})) + f(u(\bar{x})) \\ &= -w(\bar{x}) \int_0^1 f'(tv(\bar{x}) + (1-t)u(\bar{x}))\mathrm{d}t. \end{aligned} \tag{4.7}$$

Moreover, combining (4.4) and $w(\bar{x}) > 0$, we obtain that $v(\bar{x}) = w(\bar{x}) + u(\bar{x}) > u(\bar{x}) \geqslant 1 - c_0$, and so $tv(\bar{x}) + (1-t)u(\bar{x}) \geqslant 1 - c_0$ for all $t \in [0,1]$. From this and (4.1), we conclude that $f'\big(tv(\bar{x}) + (1-t)u(\bar{x})\big) \leqslant -c_1 < 0$ for all $t \in [0,1]$. This inequality, together with $w(\bar{x}) > 0$, yields

$$-w(\bar{x}) \int_0^1 f'(tv(\bar{x}) + (1-t)u(\bar{x}))\mathrm{d}t > 0.$$

By inserting this information into (4.7), we get $Lw(\bar{x}) > 0$. That is, recalling (1.2) and the nonnegativity of J,

$$\begin{aligned} 0 < Lw(\bar{x}) &= \int_{\mathbb{R}^N\setminus K} J(\bar{x}-y)(w(y) - w(\bar{x}))\mathrm{d}y \\ &= \int_{\mathbb{R}^N\setminus K} J(\bar{x}-y)\Big(w(y) - \sup_{\mathbb{R}^N\setminus K} w\Big)\mathrm{d}y \leqslant 0. \end{aligned}$$

This is a contradiction, and so the desired result is established. $\square$

The next lemma is concerned with a strong maximum principle.

Lemma 4.2 (Strong maximum principle). — *Assume that J satisfies* (1.9), *with $0 \leqslant r_1 < r_2$. Let $H \subset \mathbb{R}^N$ be an open affine half-space such that $K \subset H^c$. Let $u, v \in L^\infty(\mathbb{R}^N \setminus K)$ satisfy* (4.2) *and* (4.3). *Assume also that*

$$v \leqslant u \quad \text{a.e. in } \mathbb{R}^N \setminus K, \tag{4.8}$$

and that there exists $\bar{x} \in \overline{H}$ such that $v(\bar{x}) = u(\bar{x})$. Then,

$$v = u \quad \text{a.e. in } (H + B_{r_2}) \setminus K.$$

Proof. — We let $w := v - u$. Notice that $w(\bar{x}) = 0$. As a consequence, using (4.3), we can write

$$Lw(\bar{x}) = Lv(\bar{x}) - Lu(\bar{x}) \geqslant -f(v(\bar{x})) + f(u(\bar{x})) = 0.$$

On the other hand, $w(y) \leqslant 0 = w(\bar{x})$ for a.e. $y \in \mathbb{R}^N \setminus K$, thanks to (4.8), and therefore

$$Lw(\bar{x}) = \int_{\mathbb{R}^N \setminus K} J(\bar{x} - y)(w(y) - w(\bar{x}))\mathrm{d}y \leqslant 0.$$

Hence, $Lw(\bar{x}) = 0$ and

$$0 = \int_{\mathbb{R}^N \setminus K} J(\bar{x} - y)(w(y) - w(\bar{x}))\mathrm{d}y = \int_{\mathbb{R}^N \setminus K} J(\bar{x} - y)w(y)\mathrm{d}y.$$

From our assumptions, we have $w \leqslant 0$ a.e. in $\mathbb{R}^N \setminus K$. Accordingly, since J is such that $J > 0$ a.e. in the annulus $\mathscr{A}(r_1, r_2)$ from the general assumption (1.9), it follows that

$$w(x) = 0 \;\text{ i.e. } v(x) = u(x) \;\text{ for a.e. } x \in \mathscr{A}(\bar{x}, r_1, r_2) \cap \mathbb{R}^N \setminus K.$$

In particular, since u and v are continuous in $\overline{H}$ and $H \subset \mathbb{R}^N \setminus K$, we get that

$$v(x) = u(x) \;\text{ for all } x \in \overline{\mathscr{A}(\bar{x}, r_1, r_2)} \cap \overline{H} =: \Omega_1(\bar{x}).$$

Applying the same arguments as above to the new set of contact points $\Omega_1(\bar{x})$ we obtain that $v(x) = u(x)$ for all $x \in \overline{\mathscr{A}(x_1, r_1, r_2)} \cap \overline{H}$ and for all $x_1 \in \Omega_1(\bar{x})$. As a consequence, $v(x) = u(x)$ for all $x \in \overline{B_\mu(\bar{x})} \cap \overline{H}$ with $\mu := r_2 - r_1$. Iterating this procedure over again implies that $v(x) = u(x)$ for each x in $\overline{B_{2\mu}(\bar{x})} \cap \overline{H}$ and so on in $\overline{B_{k\mu}(\bar{x})} \cap \overline{H}$ for any $k \in \mathbb{N}$. Hence, $v = u$ in $\overline{H}$.

Therefore, as in the beginning of the proof, it follows that $Lw(x) = 0$ for all $x \in \overline{H}$ and

$$v = u \;\text{ a.e. in } \big(\overline{H} + \mathscr{A}(r_1, r_2)\big) \cap \big(\mathbb{R}^N \setminus K\big) = (H + B_{r_2}) \setminus K.$$

The proof of Lemma 4.2 is thereby complete. □

Finally, we derive a sweeping-type result in the spirit of Serrin's sweeping theorem [127] (see also [106], and page 29 in [120] for a very clear explanation of the method).

Lemma 4.3 (Sweeping principle). — *Assume that J satisfies (1.9), with $0 \leqslant r_1 < r_2$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, let a, b, s_1, s_2, s_3, s_4 be some real numbers such that $a \leqslant b$ and $r_2 \leqslant s_1 \leqslant s_2 < s_3 \leqslant s_4$. Let $u \in C(\overline{\mathscr{A}(s_1, s_4)})$ satisfy*

$$\int_{\mathscr{A}(s_1, s_4)} J(x - y)u(y)\mathrm{d}y - u(x) + g(u(x)) \leqslant 0 \;\text{ for all } x \in \overline{\mathscr{A}(s_1, s_4)}, \tag{4.9}$$

and

$$\int_{\mathscr{A}(s_1,s_4)} J(x-y)u(y)\mathrm{d}y - u(x) + g(u(x)) < 0 \quad \textit{for all } x \in \mathscr{A}(s_2,s_3). \tag{4.10}$$

Let $(w_\tau)_{\tau\in[a,b]}$ *be a continuous family in* $C(\overline{\mathscr{A}(s_1,s_4)})$ *such that*

$$\int_{\mathscr{A}(s_1,s_4)} J(x-y)w_\tau(y)\mathrm{d}y - w_\tau(x) + g(w_\tau(x)) \geqslant 0 \quad \textit{for all } x \in \overline{\mathscr{A}(s_1,s_4)}. \tag{4.11}$$

Assume further that there exists $\tau_0 \in [a,b]$ *such that* $w_{\tau_0} \leqslant u$ *in* $\overline{\mathscr{A}(s_1,s_4)}$. *Then* $w_\tau \leqslant u$ *in* $\overline{\mathscr{A}(s_1,s_4)}$ *for every* $\tau \in [a,b]$.

Proof. — Let us define $\Sigma \subset [a,b]$ to be the following set:

$$\Sigma := \Big\{\tau \in [a,b];\ w_\tau \leqslant u \text{ in } \overline{\mathscr{A}(s_1,s_4)}\Big\}.$$

To prove the theorem, we will show that Σ is a non-empty open and closed set relatively to $[a,b]$. It will then follow that $\Sigma = [a,b]$ and the theorem will be proved. First of all, by definition, Σ is a closed subset of $[a,b]$ and $\tau_0 \in \Sigma$. To finish our proof, it remains to show that Σ is an open set relatively to $[a,b]$. So let us pick $\tau \in \Sigma$. We have $w_\tau \leqslant u$ in $\overline{\mathscr{A}(s_1,s_4)}$. By continuity of u and w_τ in the compact set $\overline{\mathscr{A}(s_1,s_4)}$, either $\max_{\overline{\mathscr{A}(s_1,s_4)}}(w_\tau - u) < 0$ or there exists $z \in \overline{\mathscr{A}(s_1,s_4)}$ such that $w_\tau(z) = u(z)$. In the latter case, using $w_\tau \leqslant u$ in $\overline{\mathscr{A}(s_1,s_4)}$ together with (4.9) and (4.11) at the point z, we get that

$$0 \leqslant \int_{\mathscr{A}(s_1,s_4)} J(z-y)(u(y) - w_\tau(y))\mathrm{d}y \leqslant 0.$$

Using the continuity of both u and w_τ and the fact that $J > 0$ a.e. in $\mathscr{A}(r_1,r_2)$ for some $0 \leqslant r_1 < r_2$ by (1.9), it follows that $w_\tau = u$ in $\overline{\mathscr{A}(z,r_1,r_2)} \cap \overline{\mathscr{A}(s_1,s_4)}$ (which is nonempty since $r_2 \leqslant s_1$) and then $w_\tau = u$ in $\overline{\mathscr{A}(z',r_1,r_2)} \cap \overline{\mathscr{A}(s_1,s_4)}$ for all $z' \in \overline{\mathscr{A}(z,r_1,r_2)} \cap \overline{\mathscr{A}(s_1,s_4)}$. In particular, it is easy to see that there exists $r > 0$ such that $w_\tau = u$ in $\overline{B_r(z)} \cap \overline{\mathscr{A}(s_1,s_4)}$. As a consequence, the non-empty set $\big\{x \in \overline{\mathscr{A}(s_1,s_4)};\ w_\tau(x) = u(x)\big\}$ is both (obviously) closed and open relatively to the (connected) set $\overline{\mathscr{A}(s_1,s_4)}$ and it is thus equal to $\overline{\mathscr{A}(s_1,s_4)}$. In other words, $w_\tau = u$ in $\overline{\mathscr{A}(s_1,s_4)}$, hence

$$\int_{\mathscr{A}(s_1,s_4)} J(x-y)u(y)\mathrm{d}y - u(x) + g(u(x)) = 0 \quad \text{ for all } x \in \overline{\mathscr{A}(s_1,s_4)},$$

contradicting (4.10) in $\mathscr{A}(s_2,s_3)$. Therefore, we must have $\max_{\overline{\mathscr{A}(s_1,s_4)}}(w_\tau - u) < 0$. Since w_τ is continuous with respect to τ in the uniform norm, there exists $\delta > 0$ such that $w_{\tau'} \leqslant u$ in $\overline{\mathscr{A}(s_1,s_4)}$ for all $\tau' \in (\tau-\delta,\tau+\delta)\cap[a,b]$. Hence, $(\tau-\delta,\tau+\delta)\cap[a,b] \subset \Sigma$, which shows that Σ is open relatively to $[a,b]$. $\square$

Remark 4.4. — The previous arguments immediately show that, when $r_1 = 0$ in (1.9), the sweeping principle holds in any compact connected set F. Namely, if J satisfies (1.9) with $r_1 = 0$, if $u \in C(F)$ satisfies (4.9) with F instead of $\overline{\mathscr{A}(s_1, s_4)}$ and the strict inequality somewhere in F, if $(w_\tau)_{\tau\in[a,b]}$ is a continuous family in $C(F)$ satisfying (4.11) with F instead of $\overline{\mathscr{A}(s_1, s_4)}$ and if $w_{\tau_0} \leqslant u$ in F for some $\tau_0 \in [a, b]$, then $w_\tau \leqslant u$ in F for every $\tau \in [a, b]$.

5 Construction of radially symmetric lower bounds

In this section, we derive a first lower bound on continuous non-negative super-solutions u of (2.3) that we constantly use along this chapter. Throughout this section, K is any compact subset of $\mathbb{R}^N$, f is any $C^1(\mathbb{R})$ function, and J is any $L^1(\mathbb{R}^N)$ non-negative and radially symmetric kernel with unit mass. We recall that J_1 is the non-negative even $L^1(\mathbb{R})$ kernel with unit mass defined for a.e. $y_1 \in \mathbb{R}$ by

$$J_1(y_1) := \int_{\mathbb{R}^{N-1}} J(y_1, y_2, \cdots, y_N)\,\mathrm{d}y_2 \cdots \mathrm{d}y_N,$$

and that assumption (1.10) means the existence of a continuous increasing function $\phi : \mathbb{R} \to (0, 1)$ such that

$$\begin{cases} \displaystyle\int_{\mathbb{R}} J_1(\tau - \sigma)(\phi(\sigma) - \phi(\tau))\mathrm{d}\sigma + f(\phi(\tau)) \geqslant 0 \ \text{ for all } \ \tau \in \mathbb{R}, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1. \end{cases} \tag{5.1}$$

Then, for such ϕ, we establish the following lemma:

Lemma 5.1. — *Assume that f and J satisfy* (4.1) *and* (1.10), *let $\gamma \in (0, 1]$ and let $u \in C\big(\overline{\mathbb{R}^N \setminus K},\ [\gamma, 1]\big)$ be a function satisfying* (2.3). *Then, there exists $r_0 > 0$ such that*

$$\phi(|x| - r_0) \leqslant u(x) \ \text{ for all } x \in \overline{\mathbb{R}^N \setminus K}.$$

Proof. — Since $u(x) \to 1$ as $|x| \to +\infty$, there exists $R_0 > 0$ so large that $K \subset B_{R_0}$ and $u \geqslant 1 - c_0$ in $\mathbb{R}^N \setminus B_{R_0}$, where $c_0 > 0$ is given in (4.1). By (5.1), there exists $A > 0$ such that $\phi \leqslant \gamma$ in $(-\infty, -A]$. Define

$$r_0 = R_0 + A > 0,$$

and let us check that the conclusion of Lemma 5.1 holds with this real number r_0.

Let e be any unit vector of $\mathbb{R}^N$, that is, $e \in \partial B_1 = \mathbb{S}^{N-1}$. For $r \in \mathbb{R}$, let $\phi_{r,e}$ be the function defined by

$$\phi_{r,e}(x) := \phi(e \cdot x - r) \ \text{ for } x \in \mathbb{R}^N,$$

where $e \cdot x$ stands for the standard inner product in $\mathbb{R}^N$. Let $(e_1, \cdots, e_N)$ be the canonical basis of $\mathbb{R}^N$ and let $\mathscr{R}$ be a rotation such that $e = \mathscr{R}e_1$. Set now $\widetilde{e}_i := \mathscr{R}e_i$ for all $i \in \{2, \cdots, N\}$ and, for x, $y \in \mathbb{R}^N$ and $r \in \mathbb{R}$, let us define $x^* = x - re$, $y^* = y - re$ and

$$\begin{cases} X &= (X_1, \cdots, X_N) &= (x^* \cdot e, x^* \cdot \widetilde{e}_2, \cdots, x^* \cdot \widetilde{e}_N) &= \mathscr{R}^{-1}y^*, \\ Y &= (Y_1, \cdots, Y_N) &= (y^* \cdot e, y^* \cdot \widetilde{e}_2, \cdots, y^* \cdot \widetilde{e}_N) &= \mathscr{R}^{-1}y^*. \end{cases}$$

Since J is rotationally invariant, we deduce from (5.1) that, for all $x \in \mathbb{R}^N$ and $r \in \mathbb{R}$,

$$\begin{aligned} L_{\mathbb{R}^N}\phi_{r,e}(x) &:= \int_{\mathbb{R}^N} J(x-y)(\phi_{r,e}(y) - \phi_{r,e}(x))\mathrm{d}y \\ &= \int_{\mathbb{R}} J_1(X_1 - Y_1)(\phi(Y_1) - \phi(X_1))\mathrm{d}Y_1 \\ &\geqslant -f(\phi(X_1)) = -f(\phi(x \cdot e - r)) = -f(\phi_{r,e}(x)). \end{aligned} \tag{5.2}$$

Set $H_e := \{x \in \mathbb{R}^N;\ x \cdot e > R_0\}$ (notice that $\overline{H_e} \cap K = \emptyset$). We remark that, if $r \geqslant r_0$ and $x \in H_e^c \setminus K$, then

$$\phi_{r,e}(x) = \phi(x \cdot e - r) \leqslant \phi(R_0 - r_0) = \phi(-A) \leqslant \gamma \leqslant u(x).$$

Furthermore, if $r \geqslant r_0$, $y \in K$ and $x \in \overline{H_e}$, then $y \cdot e - r \leqslant |y| - r \leqslant R_0 - r$ and

$$\phi_{r,e}(x) = \phi(x \cdot e - r) \geqslant \phi(R_0 - r) \geqslant \phi(y \cdot e - r) = \phi_{r,e}(y).$$

Accordingly, by (5.2) and the definition of H_e, for any $r \geqslant r_0$ and $x \in \overline{H_e}$,

$$L\phi_{r,e}(x) = L_{\mathbb{R}^N}\phi_{r,e}(x) - \int_K J(x-y)(\phi_{r,e}(y) - \phi_{r,e}(x))\mathrm{d}y \geqslant -f(\phi_{r,e}(x)). \tag{5.3}$$

Consequently, we can exploit the weak comparison principle of Lemma 4.1 (used here with $H = H_e \subset \mathbb{R}^N \setminus K$ and $v = \phi_{r_0,e}$) and deduce that

$$\phi(x \cdot e - r_0) = \phi_{r_0,e}(x) \leqslant u(x),$$

for every $x \in \mathbb{R}^N \setminus K$ and also for every $x \in \overline{\mathbb{R}^N \setminus K}$ by continuity. This inequality holds for every $e \in \partial B_1$, while $r_0 > 0$ does not depend on e. In particular, taking into account the possible choice of $e = x/|x|$ if $x \neq 0$ (and any $e \in \partial B_1$ if $x = 0$), we conclude that

$$\phi(|x| - r_0) \leqslant u(x) \quad \text{ for all } x \in \overline{\mathbb{R}^N \setminus K}.$$

This proves Lemma 5.1. $\square$

Remark 5.2. — If $\mathbb{R}^N \setminus K$ is connected, if $f(0) \geqslant 0$ and if J satisfies (1.9) with $r_1 = 0$ (for instance, if J is continuous at the origin with $J(0) > 0$), then Lemma 5.1 holds with $\gamma = 0$. Indeed, these additional assumptions imply that $\inf_{\mathbb{R}^N \setminus K} u > 0$. If not, then by continuity of u and the limiting conditions in (2.3), there exists $x_0 \in \overline{\mathbb{R}^N \setminus K}$ such that $u(x_0) = 0$. Thus, by (2.3) and $f(0) \geqslant 0$,

$$0 \geqslant Lu(x_0) = \int_{\mathbb{R}^N \setminus K} J(x_0 - y)(u(y) - u(x_0))\mathrm{d}y,$$

and $u(y) = u(x_0) = 0$ for all $y \in \overline{B_{r_2}(x_0)} \cap \overline{\mathbb{R}^N \setminus K}$. Therefore, $u(y) = 0$ for all $y \in \overline{\mathbb{R}^N \setminus K}$ by repeating this argument and by connectedness of $\mathbb{R}^N \setminus K$. This contradicts the limit $u(y) \to 1$ as $|y| \to +\infty$. Finally, $\inf_{\mathbb{R}^N \setminus K} u > 0$ and the conclusion of Lemma 5.1 holds.

6 Construction of solutions in large balls

We recall that $B_R(x)$ denotes the open Euclidean ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and of radius $R > 0$, and that $B_R = B_R(0)$. Throughout this section we suppose that f and J satisfy (1.8), (1.9) and (1.11). Here, for any $R > 0$ large enough and any $x_0 \in \mathbb{R}^N$, we will construct and study the properties of positive continuous solutions of the following auxiliary problem

$$\mathscr{L}_{B_R(x_0)}[v](x) - v(x) + f(v(x)) = 0 \quad \text{for } x \in \overline{B_R(x_0)}, \tag{6.1}$$

where

$$\mathscr{L}_{B_R(x_0)}[v](x) := \int_{B_R(x_0)} J(x-y)v(y)\mathrm{d}y. \tag{6.2}$$

Besides the own interest of (6.1), the properties of some particular solutions v of (6.1) are essential in the proof of Theorem 2.4, as they will provide key estimates ensuring to derive the asymptotic behaviour of the solutions u of (2.5) or (2.6). So in Sections 6.1 and 6.2, our main concern will be to establish, for any $x_0 \in \mathbb{R}^N$ and $R > 0$ large enough, the existence of a positive maximal solution $v_{x_0,R}$ to (6.1), such that $v_{x_0,R} \to 1$ locally uniformly in $\mathbb{R}^N$ as $R \to +\infty$. Based on the construction of these solutions in closed balls $\overline{B_R(x_0)}$, we will next show in Section 6.3 the existence of continuous and compactly supported sub-solutions in $\mathbb{R}^N$.

6.1 Existence of a positive solution in $\overline{B_R(x_0)}$

This section is devoted to the proof of the existence of a positive continuous solution of (6.1) in $\overline{B_R(x_0)}$, for any $R > 0$ large enough and any $x_0 \in \mathbb{R}^N$.

Lemma 6.1. — *Assume that f and J satisfy* (1.8), (1.9) *and* (1.11). *Assume further that J is compactly supported and $J \in L^2(\mathbb{R}^N)$. Then there exists $d_0 = d_0(f,J) > 0$ such that for every $x_0 \in \mathbb{R}^N$ and $R \geqslant d_0$, problem* (6.1) *admits a positive continuous solution $v : \overline{B_R(x_0)} \to (0,1)$ such that $\max_{\overline{B_R(x_0)}} v > \theta$, where $\theta \in (0,1)$ is defined in* (1.11).

Proof. — Let $x_0 \in \mathbb{R}^N$ be fixed, let also $R_J > 0$ be fixed (independently of x_0) such that

$$\operatorname{supp}(J) \subset B_{R_J},$$

and pick any $R > R_J$. To construct a solution, we adapt the strategy used in [47] for the construction of a solution of a local reaction-diffusion equation. The proof is divided into three main steps.

Step 1: definition and elementary properties of an energy functional $\mathscr{E}$

In the proof of Lemma 6.1, let us extend f by $f'(1)(s-1)$ for $s \geqslant 1$ and by $-f(-s)$ for $s \leqslant 0$ and denote $\widetilde{f}$ this extension. Now, define

$$F(t) := \int_0^t \widetilde{f}(s)\mathrm{d}s \quad \text{for } t \in \mathbb{R}, \quad c(x) := 1 - \int_{B_R(x_0)} J(x-y)\mathrm{d}y \in [0,1] \quad \text{for } x \in \mathbb{R}^N,$$

and consider the following energy functional

$$\begin{aligned}\mathscr{E}(u) := \frac{1}{4}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)(u(y)-u(x))^2\mathrm{d}x\mathrm{d}y \\ + \frac{1}{2}\int_{B_R(x_0)} c(x)u^2(x)\mathrm{d}x - \int_{B_R(x_0)} F(u(x))\mathrm{d}x,\end{aligned} \tag{6.3}$$

defined for $u \in L^2(B_R(x_0))$. Since $J \in L^1(\mathbb{R}^N)$, $\mathscr{E}$ is well defined in $L^2(B_R(x_0))$. Moreover, using the definition of F and the oddness of $\widetilde{f}$, we have

$$\int_{B_R(x_0)} F(u(x))\mathrm{d}x = \int_{B_R(x_0)} F(|u(x)|)\mathrm{d}x, \tag{6.4}$$

for any $u \in L^2(B_R(x_0))$, while elementary computations yield

$$\begin{aligned}\mathscr{E}(u) = -\frac{1}{2}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)u(x)u(y)\mathrm{d}x\mathrm{d}y \\ + \frac{1}{2}\int_{B_R(x_0)} u^2(x)\mathrm{d}x - \int_{B_R(x_0)} F(u(x))\mathrm{d}x.\end{aligned} \tag{6.5}$$

From the last two formulas, one infers that, for any $u \in L^2(B_R(x_0))$,

$$\mathscr{E}(|u|) = -\frac{1}{2}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)|u(y)||u(x)|\mathrm{d}x\mathrm{d}y$$

$$+\frac{1}{2}\int_{B_R(x_0)}|u(x)|^2\mathrm{d}x-\int_{B_R(x_0)}F(|u(x)|)\mathrm{d}x\leqslant\mathscr{E}(u).$$

To complete Step 1, let us check that the functional $\mathscr{E}$ is bounded from below in $L^2(B_R(x_0))$. From (1.11) and (6.4), the definition of F and $\widetilde{f}$, and since $\widetilde{f}(s)\leqslant 0$ for $s\geqslant 1$, we see that, for any $u\in L^2(B_R(x_0))$,

$$\int_{B_R(x_0)}F(u(x))\mathrm{d}x\leqslant\int_{B_R(x_0)}\int_0^{\min\{1,|u(x)|\}}\widetilde{f}(s)\mathrm{d}s\leqslant R^N|B_1|\int_0^1 f(s)\mathrm{d}s,$$

where $|B_1|$ denotes the Lebesgue measure of the unit ball. Setting

$$C_0:=|B_1|\int_0^1 f(s)\mathrm{d}s>0,$$

we thus get that

$$\mathscr{E}(u)\geqslant -C_0R^N\quad\text{for any } u\in L^2(B_R(x_0)). \tag{6.6}$$

Hence, the quantity

$$\gamma:=\inf_{u\in L^2(B_R(x_0))}\mathscr{E}(u), \tag{6.7}$$

is finite.

Step 2: the infimum of $\mathscr{E}$ in $L^2(B_R(x_0))$ is achieved

We shall now see that γ is achieved for some $v\in L^2(B_R(x_0))$. So, let $(u_n)_{n\in\mathbb{N}}$ be a minimising sequence. From the inequality $\mathscr{E}(|u|)\leqslant\mathscr{E}(u)$, we may assume without loss of generality that the functions u_n are all non-negative.

Let us first check that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(B_R(x_0))$. To do so, we recall the definition (6.2) of $\mathscr{L}_{B_R(x_0)}$ and we notice that the principal eigenvalue λ_p of the operator $\mathscr{L}_{B_R(x_0)}-\mathrm{Id}$ is negative (see for example [2, 15, 51, 75] for a precise definition of λ_p and some of its properties) and satisfies

$$-\lambda_p=\inf_{\|\varphi\|_{L^2(B_R(x_0))}=1}\widetilde{\mathscr{E}}(\varphi),$$

where

$$\widetilde{\mathscr{E}}(\varphi):=\frac{1}{2}\int_{B_R(x_0)}\int_{B_R(x_0)}J(x-y)(\varphi(y)-\varphi(x))^2\mathrm{d}x\mathrm{d}y+\int_{B_R(x_0)}c(x)\,\varphi^2(x)\mathrm{d}x.$$

As a consequence, from (6.6), we get

$$\mathscr{E}(u_n)\geqslant -\frac{\lambda_p}{2}\int_{B_R(x_0)}u_n^2(x)\mathrm{d}x-C_0R^N,$$

for all $n \in \mathbb{N}$. Therefore the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(B_R(x_0))$ since it is a minimising sequence and since $\lambda_p < 0$. Up to extraction of a subsequence, the sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly in $L^2(B_R(x_0))$ to a non-negative function $v \in L^2(B_R(x_0))$.

We actually claim that

$$\mathscr{E}(v) = \gamma. \tag{6.8}$$

Due to the lack of compactness in this non-local minimisation problem, we cannot expect to get a strong convergence in $L^2(B_R(x_0))$ for the minimising subsequence and therefore passing to the limit in the energy (6.3) is not immediate. To overcome this difficulty, let us observe that by introducing the function

$$G(t) := \int_0^t (s - \widetilde{f}(s))\,\mathrm{d}s = \frac{t^2}{2} - F(t),$$

we get from (6.5) that, for any $n \in \mathbb{N}$,

$$\mathscr{E}(u_n) = -\frac{1}{2}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)u_n(x)u_n(y)\mathrm{d}x\mathrm{d}y + \int_{B_R(x_0)} G(u_n(x))\mathrm{d}x,$$

and therefore

$$\begin{aligned}\mathscr{E}(u_n) - \mathscr{E}(v) = &-\frac{1}{2}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)[u_n(x)u_n(y) - v(x)v(y)]\mathrm{d}x\mathrm{d}y \\ &+ \int_{B_R(x_0)} [G(u_n(x)) - G(v(x))]\mathrm{d}x.\end{aligned} \tag{6.9}$$

Observe that the double integral in the right-hand side of (6.9) can be rewritten as

$$\begin{aligned}&\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)u_n(x)u_n(y)\mathrm{d}x\mathrm{d}y \\ &\qquad = \int_{B_R(x_0)} u_n(x)\left(\int_{B_R(x_0)} J(x-y)[u_n(y)-v(y)]\mathrm{d}y\right)\mathrm{d}x \\ &\qquad + \int_{B_R(x_0)} v(y)\left(\int_{B_R(x_0)} J(x-y)u_n(x)\mathrm{d}x\right)\mathrm{d}y.\end{aligned}$$

Using Lebesgue's dominated convergence theorem, together with the assumption $J \in L^2(\mathbb{R}^N)$ and the $L^2(B_R(x_0))$ weak convergence of the sequence $(u_n)_{n\in\mathbb{N}}$, it is easy to see that

$$\lim_{n\to+\infty}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)u_n(x)u_n(y)\mathrm{d}x\mathrm{d}y = \int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)v(x)v(y)\mathrm{d}x\mathrm{d}y,$$

and therefore

$$\lim_{n\to+\infty} -\frac{1}{2}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)[u_n(x)u_n(y)-v(x)v(y)]\mathrm{d}x\mathrm{d}y = 0. \tag{6.10}$$

On the other hand, since $\widetilde{f}'(s) < 1$ for all $s \in \mathbb{R}$ (by assumption), the function G is convex and, for all $n \in \mathbb{N}$, we get

$$\int_{B_R(x_0)} [G(u_n(x)) - G(v)(x)]\mathrm{d}x \geqslant \int_{B_R(x_0)} G'(v(x))[u_n(x) - v(x)]\mathrm{d}x.$$

From the definition of G and $\widetilde{f}$, together with the fact that $v \in L^2(B_R(x_0))$, we infer that $G'(v) \in L^2(B_R(x_0))$. Using the $L^2(B_R(x_0))$ weak convergence of $(u_n)_{n\in\mathbb{N}}$ to v, it follows that

$$\liminf_{n\to+\infty} \int_{B_R(x_0)} [G(u_n(x)) - G(v(x))]\mathrm{d}x \geqslant 0. \tag{6.11}$$

Thus passing to the limit in (6.9), and using (6.10) and (6.11), we obtain $\gamma - \mathscr{E}(v) \geqslant 0$. Together with the definition (6.7) of γ, this shows that v is a minimiser of the energy $\mathscr{E}$, that is, (6.8) holds.

Step 3: v is a continuous positive solution u of (6.1)

We first show in this step that v is a solution to (6.1) with $\widetilde{f}$ instead of f. From (6.8), v is a critical point of $\mathscr{E}$ and in particular, it follows from the formulation (6.5) of $\mathscr{E}$ that v is a non-negative weak solution of $\mathscr{L}_{B_R(x_0)}[v] - v + \widetilde{f}(v) = 0$ in $B_R(x_0)$. Since all functions $\mathscr{L}_{B_R(x_0)}[v]$, v and $\widetilde{f}(v)$ belong to $L^2(B_R(x_0))$, the function v satisfies $\mathscr{L}_{B_R(x_0)}[v](x) - v(x) + \widetilde{f}(v(x)) = 0$ for a.e. $x \in B_R(x_0)$. Furthermore, since $J \in L^2(\mathbb{R}^N)$, the Cauchy-Schwarz inequality implies that $\mathscr{L}_{B_R(x_0)}[v] \in L^\infty(B_R(x_0))$. Therefore, since by (1.11) the function $s \mapsto s - \widetilde{f}(s)$ is a C^1 diffeomorphism from $\mathbb{R}^+$ onto $\mathbb{R}^+$ and since v is non-negative, it follows from the equation $\mathscr{L}_{B_R(x_0)}[v] - v + \widetilde{f}(v) = 0$ a.e. in $B_R(x_0)$ that $v \in L^\infty(B_R(x_0))$. Thus by reproducing the arguments of the proof of Lemma 3.1 we deduce that v has a representative, still denoted by v, which is continuous in $\overline{B_R(x_0)}$ and satisfies

$$\mathscr{L}_{B_R(x_0)}[v](x) - v(x) + \widetilde{f}(v(x)) = 0 \quad \text{ for all } x \in \overline{B_R(x_0)}. \tag{6.12}$$

Remember now that, from (1.9), $J > 0$ a.e. in $\mathscr{A}(r_1, r_2)$ with $0 \leqslant r_1 < r_2$, and that $R \geqslant R_J \geqslant r_2 > r_1$, with $\operatorname{supp}(J) \subset B_{R_J}$. As a consequence, if there exists a point $x \in \overline{B_R(x_0)}$ such that $v(x) = 0$, then, arguing as in the proof of the strong maximum principle (Lemma 4.2) or as in the proof of the sweeping principle (Lemma 4.3), it follows that $v = 0$ in $\mathscr{A}(x, r_1, r_2) \cap \overline{B_R(x_0)}$, hence $v = 0$ in $\mathscr{A}(y, r_1, r_2) \cap \overline{B_R(x_0)}$ for all $y \in \mathscr{A}(x, r_1, r_2) \cap \overline{B_R(x_0)}$ and finally $v = 0$ in $\overline{B_r(x)} \cap \overline{B_R(x_0)}$ for some $r > 0$.

Therefore, the non-empty set $\left\{x \in \overline{B_R(x_0)};\ v(x) = 0\right\}$ is both (obviously) closed and open relatively to $\overline{B_R(x_0)}$ and is thus equal to $\overline{B_R(x_0)}$. As a consequence, either $v \equiv 0$ in $\overline{B_R(x_0)}$ or $v > 0$ in $\overline{B_R(x_0)}$.

In this paragraph, we prove that the solution v constructed is a solution of (6.1), namely we just need to show that $v \leqslant 1$ in $\overline{B_R(x_0)}$. To do so, define $M = \max_{\overline{B_R(x_0)}} v \geqslant 0$ and let $\bar{x} \in \overline{B_R(x_0)}$ be such that $v(\bar{x}) = M$. Assume by contradiction that $M > 1$. By evaluating (6.12) at $\bar{x}$ and using the definition of $\widetilde{f}$, we get that

$$\int_{B_R(x_0)} J(\bar{x} - y)v(y)\mathrm{d}y = \mathscr{L}_{B_R(x_0)}[v](\bar{x}) = M - \widetilde{f}(M) > M.$$

Since $v \leqslant M$ in $\overline{B_R(x_0)}$, this leads to a contradiction. Hence $M \leqslant 1$ and thus v is a non-negative continuous solution of (6.1) in $\overline{B_R(x_0)}$. Furthermore, as for the positivity of v, one gets that either $v \equiv 1$ in $\overline{B_R(x_0)}$ or $v < 1$ in $\overline{B_R(x_0)}$. The former case is impossible since $\mathscr{L}_{B_R(x_0)}[v] \not\equiv 1$ in $\overline{B_R(x_0)}$ (indeed, $\int_{B_R(x_0)} J(x-y)\mathrm{d}y < 1$ for all $x \in \partial B_R(x_0)$). Thus, $0 \leqslant v < 1$ in $\overline{B_R(x_0)}$.

Finally, let us verify that the solution v constructed is not the trivial solution. To do so, it is enough to show that $\mathscr{E}(v) \neq \mathscr{E}(0) = 0$. We claim that, for $R > R_J$ large enough, $\mathscr{E}(v) < 0$. Indeed, let us consider the test function $\varphi := \mathbb{1}_{B_R(x_0)} \in L^2(B_R(x_0))$. We have

$$\begin{aligned}
\mathscr{E}(\varphi) &= \frac{1}{4}\int_{B_R(x_0)}\int_{B_R(x_0)} J(x-y)\big(\varphi(y) - \varphi(x)\big)^2 \mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{2}\int_{B_R(x_0)} c(x)\varphi^2(x)\mathrm{d}x - \int_{B_R(x_0)} F(\varphi(x))\mathrm{d}x \\
&= \frac{1}{2}\int_{B_R(x_0)} c(x)\mathrm{d}x - R^N|B_1|\int_0^1 f(s)\mathrm{d}s \\
&= \frac{1}{2}\int_{B_R(x_0)}\int_{\mathbb{R}^N\setminus B_R(x_0)} J(x-y)\mathrm{d}x\mathrm{d}y - R^N|B_1|\int_0^1 f(s)\mathrm{d}s.
\end{aligned}$$

Since $\mathrm{supp}(J) \subset B_{R_J}$, the above equality yields

$$\begin{aligned}
\mathscr{E}(\varphi) &= \frac{1}{2}\int_{B_R(x_0)\setminus B_{R-R_J}(x_0)}\int_{\mathbb{R}^N\setminus B_R(x_0)} J(x-y)\mathrm{d}x\mathrm{d}y - R^N|B_1|\int_0^1 f(s)\mathrm{d}s, \\
&\leqslant \frac{1}{2}|B_1|\left(R^N - (R-R_J)^N\right) - R^N|B_1|\int_0^1 f(s)\mathrm{d}s.
\end{aligned}$$

Thus, since $\int_0^1 f(s)\mathrm{d}s > 0$, there exists $d_0 = d_0(J,f) \in (R_J, +\infty)$, independent of x_0, such that, for every $R \geqslant d_0$, the right-hand side of the above inequality is negative and thus $\mathscr{E}(v) \leqslant \mathscr{E}(\varphi) < 0$, which proves our claim. Furthermore, since $0 \leqslant v < 1$ in $\overline{B_R(x_0)}$ and $F \leqslant 0$ in $[0,\theta]$, one infers that $\max_{\overline{B_R(x_0)}} v > \theta$, hence $v > 0$ in $\overline{B_R(x_0)}$ (remember that v was either positive or identically equal to 0 in $\overline{B_R(x_0)}$).

As a conclusion, for every R such that $R \geqslant d_0$, there exists a solution $v \in C(\overline{B_R(x_0)}, (0,1))$ to (6.1) with $\max_{\overline{B_R(x_0)}} v > \theta$. The point $x_0 \in \mathbb{R}^N$ being arbitrary and the constant d_0 being independent of x_0, the proof of Lemma 6.1 is thereby complete. □

6.2 Existence and properties of the maximal solution in $\overline{B_R(x_0)}$

Let us now look more closely at the properties of positive solutions of (6.1) and in particular at the maximal solution, if any. To this end, let us in this subsection extend continuously f by $f'(1)(s-1)$ for $s \geqslant 1$ and by 0 for $s \leqslant 0$. To simplify our presentation let us still denote f this extension.

Let us first recall the notion of maximal solution for problem (6.1).

Definition 6.2. — *Let $x_0 \in \mathbb{R}^N$ and $R > 0$. A function $v \in C(\overline{B_R(x_0)}, [0,1])$ is called a maximal solution to* (6.1) *in $\overline{B_R(x_0)}$ if any solution $w \in C(\overline{B_R(x_0)}, [0,1])$ satisfies $w \leqslant v$ in $\overline{B_R(x_0)}$.*

The following lemma provides the existence and uniqueness of a maximal solution to the problem (6.1) when $R > 0$ is large enough.

Lemma 6.3. — *Assume that f and J satisfy* (1.8), (1.9) *and* (1.11). *Assume further that J is compactly supported and $J \in L^2(\mathbb{R}^N)$. Then there exists $d_0 = d_0(f,J) > 0$, given as in Lemma* 6.1, *such that for every $x_0 \in \mathbb{R}^N$ and $R \geqslant d_0$, problem* (6.1) *admits a unique maximal solution $v_{x_0,R}$ in $\overline{B_R(x_0)}$ and $v_{x_0,R}$ satisfies $0 < v_{x_0,R} < 1$ in $\overline{B_R(x_0)}$.*

Proof. — Let $x_0 \in \mathbb{R}^N$ be fixed and let R be fixed such that $R \geqslant d_0$, where $d_0 = d_0(f,J) > 0$ is given in Lemma 6.1. We will check that the conclusion holds with this quantity d_0. First of all, the uniqueness of the maximal solution in $\overline{B_R(x_0)}$, if any, is a trivial consequence of its definition.

Let us then focus on the construction of a maximal solution. From Lemma 6.1, there exists a solution $v \in C(\overline{B_R(x_0)}, (0,1))$ to (6.1). Now, remember that 1 is a strict super-solution to (6.1). Therefore, since f is Lipschitz continuous, it follows that we can construct a maximal solution $v_{x_0,R} \in C(\overline{B_R(x_0)}, (0,1))$ to (6.1) such that

$$0 < v \leqslant v_{x_0,R} < 1 \ \text{ in } \overline{B_R(x_0)},$$

by using standard monotone iterative scheme as in [51, Theorem A.1]. For the sake of completeness, let us describe this scheme in the next paragraph.

First, let us observe that, from the assumptions on J, the linear operator $\mathscr{L}_{B_R(x_0)}$ is a continuous operator in $C(\overline{B_R(x_0)})$. Next let us choose a real number $k>0$ large enough such that the function $s \mapsto -ks - f(s)$ is decreasing in $\mathbb{R}$. We can increase further k if necessary to ensure that $k+1 \in \rho(\mathscr{L}_{B_R(x_0)})$, where $\rho(\mathscr{L}_{B_R(x_0)})$ denotes the resolvent of the operator $\mathscr{L}_{B_R(x_0)}$. We note that, by this choice of k, the operator $\mathscr{L}_{B_R(x_0)} - (k+1)$ satisfies a comparison principle, in the sense that if $w \in C(\overline{B_R(x_0)})$ satisfies $\mathscr{L}_{B_R(x_0)}[w] - (k+1)w \geqslant 0$ in $\overline{B_R(x_0)}$ then $w \leqslant 0$ in $\overline{B_R(x_0)}$ (see [51, 53]). Now, set $v_0 = 1$ and let $v_1 \in C(\overline{B_R(x_0)})$ be the solution of the following linear problem

$$\mathscr{L}_{B_R(x_0)}[v_1](x) - (k+1)v_1(x) = -kv_0(x) - f(v_0(x)) \quad \text{for } x \in \overline{B_R(x_0)}. \tag{6.13}$$

The function v_1 is well defined, since by construction the continuous operator $\mathscr{L}_{B_R(x_0)} - (k+1)$ is invertible. We claim that $v \leqslant v_1 \leqslant v_0$ in $\overline{B_R(x_0)}$. Indeed, since $v\,(\leqslant 1)$ and $v_0 = 1$ are respectively a solution and a super-solution of (6.1), we have, for $x \in \overline{B_R(x_0)}$,

$$\begin{aligned}
&\mathscr{L}_{B_R(x_0)}[v_1 - v_0](x) - (k+1)(v_1(x) - v_0(x)) = -\mathscr{L}_{B_R(x_0)}[1](x) + 1,\\
&\mathscr{L}_{B_R(x_0)}[v_1 - v](x) - (k+1)(v_1(x) - v(x)) = -kv_0(x) - f(v_0(x)) + kv(x) + f(v(x)).
\end{aligned}$$

Thus implying that

$$\begin{aligned}
\mathscr{L}_{B_R(x_0)}[v_1 - v_0](x) - (k+1)(v_1(x) - v_0(x)) &\geqslant 0,\\
\mathscr{L}_{B_R(x_0)}[v_1 - v](x) - (k+1)(v_1(x) - v(x)) &\leqslant 0.
\end{aligned}$$

So, the inequality $v \leqslant v_1 \leqslant v_0$ in $\overline{B_R(x_0)}$ follows from the comparison principle satisfied by the operator $\mathscr{L}_{B_R(x_0)} - (k+1)$. In particular, $0 < v_1 \leqslant 1$ in $\overline{B_R(x_0)}$. Now let $v_2 \in C(\overline{B_R(x_0)})$ be the solution of (6.13) with v_2 instead of v_1 in the left-hand side and v_1 instead of v_0 in the right-hand side. From the monotonicity of $s \mapsto -ks - f(s)$ and from the comparison principle, we have $v \leqslant v_2 \leqslant v_1 \leqslant v_0$ in $\overline{B_R(x_0)}$. By induction, we can construct a non-increasing sequence of functions $(v_n)_{n\in\mathbb{N}}$ in $C(\overline{B_R(x_0)})$ satisfying $v \leqslant v_{n+1} \leqslant v_n \leqslant v_0$ in $\overline{B_R(x_0)}$ and

$$\mathscr{L}_{B_R(x_0)}[v_{n+1}](x) - (k+1)v_{n+1}(x) = -kv_n(x) - f(v_n(x)) \quad \text{for } x \in \overline{B_R(x_0)}. \tag{6.14}$$

Since the sequence is non-increasing and bounded from below, the quantity

$$v_{x_0,R}(x) := \inf_{n\in\mathbb{N}} v_n(x) = \lim_{n\to+\infty} v_n(x) \;\in [v(x), 1] \;(\subset (0,1])$$

is well defined for every $x \in \overline{B_R(x_0)}$. Moreover, by passing to the limit in the equation (6.14) and using Lebesgue's dominated convergence theorem, it follows that $v_{x_0,R}$ is a solution of (6.1). As in the proof of Lemma 3.1, we infer that $v_{x_0,R}$ is continuous in $\overline{B_R(x_0)}$ and, as in the proof of Lemma 6.1, we get that $v_{x_0,R} < 1$ in $\overline{B_R(x_0)}$. To sum up, $v_{x_0,R}$ is a solution of (6.1) belonging to $C(\overline{B_R(x_0)}, (0,1))$.

We finally claim that $v_{x_0,R}$ is a maximal solution to (6.1). Indeed, let $w \in C(\overline{B_R(x_0)}, [0,1])$ be any solution to (6.1). By replacing v with w in the arguments of the previous paragraph and using the fact that the sequence $(v_n)_{n\in\mathbb{N}}$ is defined with the same initial value $v_0 = 1$, we get that $w \leqslant v_n$ in $\overline{B_R(x_0)}$ for every $n \in \mathbb{N}$, hence $w \leqslant v_{x_0,R}$ in $\overline{B_R(x_0)}$. The proof of Lemma 6.3 is thereby complete. □

The maximal solutions $v_{x_0,R}$ possess some important properties, in particular they are monotone non-decreasing with respect to the domains.

LEMMA 6.4. — *Let us assume that f and J satisfy* (1.8), (1.9) *and* (1.11). *Assume further that J is compactly supported and $J \in L^2(\mathbb{R}^N)$. Let $d_0 = d_0(f,J) > 0$ be given as in Lemmata* 6.1 *and* 6.3. *The following properties hold:*

(i) for every $x_1, x_2 \in \mathbb{R}^N$ and $d_0 \leqslant R_1 \leqslant R_2$ such that $B_{R_1}(x_1) \subset B_{R_2}(x_2)$, then

$$v_{x_1,R_1}(x) \leqslant v_{x_2,R_2}(x) \quad \text{for all } x \in \overline{B_{R_1}(x_1)};$$

(ii) for every $x_0 \in \mathbb{R}^N$ and $R \geqslant d_0$, the function $v_{0,R}(\cdot - x_0)$ defined in $\overline{B_R(x_0)}$ satisfies $v_{0,R}(\cdot - x_0) = v_{x_0,R}$ in $\overline{B_R(x_0)}$;

(iii) for every $x_0 \in \mathbb{R}^N$ and $R \geqslant d_0$,

$$\min_{\overline{B_R(x_0)}} v_{x_0,4R} \geqslant \max_{\overline{B_R(x_0)}} v_{x_0,2R}.$$

Proof. — The proof of (i) is straightforward. Indeed, let $x_1, x_2 \in \mathbb{R}^N$ and $d_0 \leqslant R_1 \leqslant R_2$ be such that $B_{R_1}(x_1) \subset B_{R_2}(x_2)$. Recall from the proof of Lemma 6.3 that $v_{x_2,R_2} \in C(\overline{B_{R_2}(x_2)}, (0,1))$ can be defined as $v_{x_2,R_2} = \lim_{n\to+\infty} v_n$, where $(v_n)_{n\in\mathbb{N}}$ is the sequence of positive functions in $C(\overline{B_{R_2}(x_2)}, (0,1])$ defined by induction by $v_0 = 1$ in $\overline{B_{R_2}(x_2)}$ and, for $n \in \mathbb{N}$,

$$\mathscr{L}_{B_{R_2}(x_2)}[v_{n+1}](x) - (k+1)v_{n+1}(x) = -kv_n(x) - f(v_n(x)) \quad \text{for } x \in \overline{B_{R_2}(x_2)}.$$

Here $k > 0$ is such that $k+1 \in \rho(\mathscr{L}_{B_{R_2}(x_2)})$ and the function $s \mapsto -ks - f(s)$ is decreasing. By increasing k if necessary we may assume that $k+1 \in \rho(\mathscr{L}_{B_{R_2}(x_2)}) \cap \rho(\mathscr{L}_{B_{R_1}(x_1)})$. Now observe that, for any $n \in \mathbb{N}$, v_n satisfies

$$\mathscr{L}_{B_{R_1}(x_1)}[v_{n+1}](x) - (k+1)v_{n+1}(x) \leqslant -kv_n(x) - f(v_n(x)) \quad \text{for } x \in \overline{B_{R_1}(x_1)}, \tag{6.15}$$

that is, the function v_{n+1} is a super-solution to problem (6.14) in $\overline{B_{R_1}(x_1)}$. We claim that, for every $n \in \mathbb{N}$,

$$v_{x_1,R_1}(x) \leqslant v_n(x) \ \text{ for all } x \in \overline{B_{R_1}(x_1)}.$$

To do so, we proceed by induction. By construction of v_{x_1,R_1} and the definition of v_0, we know that $v_{x_1,R_1}(x) \leqslant v_0(x)$ for all $x \in \overline{B_{R_1}(x_1)}$. For $n \in \mathbb{N}$, assume that $v_{x_1,R_1}(x) \leqslant v_n(x)$ for all $x \in \overline{B_{R_1}(x_1)}$, and let us prove that $v_{x_1,R_1}(x) \leqslant v_{n+1}(x)$ for all $x \in \overline{B_{R_1}(x_1)}$. Let $w := v_{x_1,R_1} - v_{n+1}$ in $\overline{B_{R_1}(x_1)}$. From (6.15), since the function $s \mapsto -ks - f(s)$ is decreasing and $v_{x_1,R_1}(x) \leqslant v_n(x)$ for all $x \in \overline{B_{R_1}(x_1)}$, we see that w satisfies

$$\mathscr{L}_{B_{R_1}(x_1)}[w](x) - (k+1)w(x) \geqslant -kv_{x_1,R_1}(x) - f(v_{x_1,R_1}(x)) + kv_n(x) + f(v_n(x)) \geqslant 0,$$

for $x \in \overline{B_{R_1}(x_1)}$. Since the operator $\mathscr{L}_{B_{R_1}(x_1)} - (k+1)$ satisfies the maximum principle we then deduce that $w \leqslant 0$ in $\overline{B_{R_1}(x_1)}$, that is, $v_{x_1,R_1}(x) \leqslant v_{n+1}(x)$ for all $x \in \overline{B_{R_1}(x_1)}$. Therefore, for every $x \in \overline{B_{R_1}(x_1)}$, we have $v_{x_1,R_1}(x) \leqslant \lim_{n\to+\infty} v_n(x) = v_{x_2,R_2}(x)$.

(ii) follows from the following observations. For any $x_0 \in \mathbb{R}^N$, the function $v_{0,R}(\cdot - x_0) \in C(\overline{B_R(x_0)}, (0,1))$ satisfies

$$\mathscr{L}_{B_R(x_0)}[v_{0,R}(\cdot - x_0)](x) - v_{0,R}(x - x_0) + f(v_{0,R}(x - x_0)) = 0 \quad \text{for all } x \in \overline{B_R(x_0)}.$$

Therefore, by the maximality of $v_{x_0,R}$, it follows that $v_{0,R}(\cdot - x_0) \leqslant v_{x_0,R}$ in $\overline{B_R(x_0)}$. Similarly, one can show that $v_{x_0,R}(\cdot + x_0) \leqslant v_{0,R}$ in $\overline{B_R}$. Finally, $v_{0,R}(\cdot - x_0) = v_{x_0,R}$ in $\overline{B_R(x_0)}$.

To prove (iii), we simply observe that, for any $x_1 \in \overline{B_{2R}(x_0)}$, one has $B_{2R}(x_1) \subset B_{4R}(x_0)$ and, by (i), $v_{x_0,4R} \geqslant v_{x_1,2R}$ in $\overline{B_{2R}(x_1)}$. Property (ii) yields $v_{x_0,2R}(\cdot - (x_1 - x_0)) = v_{x_1,2R}$ in $\overline{B_{2R}(x_1)}$, hence

$$v_{x_0,4R}(x) \geqslant v_{x_0,2R}(x - (x_1 - x_0)) \quad \text{for all } x_1 \in \overline{B_{2R}(x_0)} \text{ and } x \in \overline{B_{2R}(x_1)}.$$

Now, since for every $x, y \in \overline{B_R(x_0)}$ there exists (a unique) $x_1 \in \overline{B_{2R}(x_0)}$ such that $y = x - (x_1 - x_0)$ and $x \in \overline{B_R(x_1)} \subset \overline{B_{2R}(x_1)}$, the latter inequality implies that

$$v_{x_0,4R}(x) \geqslant v_{x_0,2R}(x - (x_1 - x_0)) = v_{x_0,2R}(y),$$

for all $x, y \in \overline{B_R(x_0)}$, which completes the proof. □

We can now state our last property about the maximal solution.

Lemma 6.5. — *Let us assume that f and J satisfy* (1.8), (1.9) *and* (1.11). *Assume further that J is compactly supported and $J \in L^2(\mathbb{R}^N)$. Then, for every $x_0 \in \mathbb{R}^N$, $v_{x_0,R} \to 1$ as $R \to +\infty$ locally uniformly in $\mathbb{R}^N$.*

Proof. — Let $x_0 \in \mathbb{R}^N$ be fixed. Consider any non-decreasing sequence $(R_n)_{n\in\mathbb{N}}$ in $[d_0, +\infty)$ and converging to $+\infty$, where $d_0 = d_0(f, J) > 0$ is given in Lemmata 6.1

and 6.3 (we recall that $d_0 > R_J$, where $\mathrm{supp}(J) \subset B_{R_J}$). Thanks to part (i) of Lemma 6.4, the sequence $(v_{x_0,R_n})_{n\in\mathbb{N}}$ is non-decreasing, in the sense that $v_{x_0,R_n} \leqslant v_{x_0,R_p}$ in $\overline{B_{R_n}(x_0)}$ for all $n \leqslant p$. Moreover, $0 < v_{x_0,R_n} < 1$ in $\overline{B_{R_n}(x_0)}$ for each $n \in \mathbb{N}$. As a consequence, the sequence $(v_{x_0,R_n})_{n\in\mathbb{N}}$ converges pointwise in $\mathbb{R}^N$ to a function $0 < \bar{v} \leqslant 1$ which, thanks to Lebesgue's dominated convergence theorem, satisfies

$$J * \bar{v}(x) - \bar{v}(x) + f(\bar{v}(x)) = 0 \quad \text{for all } x \in \mathbb{R}^N. \tag{6.16}$$

As in the proof of Lemma 3.1, the function $\bar{v}$ can be viewed as a uniformly continuous function and therefore the limit $v_{x_0,R_n} \to \bar{v}$ holds locally uniformly in $\mathbb{R}^N$.

Consider now any $x_1 \in \mathbb{R}^N$ and any $\delta \in [d_0, +\infty)$. We can then extract a subsequence of $(R_n)_{n\in\mathbb{N}}$ that we still denote $(R_n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $B_\delta(x_1) \subset B_{R_n}(x_0)$ and $R_{n+1} \geqslant 4R_n$. By Lemma 6.1 and parts (i) and (iii) of Lemma 6.4, we get that

$$\begin{aligned} 1 > \min_{\overline{B_\delta(x_1)}} v_{x_0,R_{n+1}} \geqslant \min_{\overline{B_{R_n}(x_0)}} v_{x_0,R_{n+1}} \geqslant \min_{\overline{B_{R_n}(x_0)}} v_{x_0,4R_n} \geqslant \max_{\overline{B_{R_n}(x_0)}} v_{x_0,2R_n} \geqslant \cdots \\ \cdots \geqslant \max_{\overline{B_{R_n}(x_0)}} v_{x_0,R_n} \geqslant \max_{\overline{B_\delta(x_1)}} v_{x_0,R_n} \geqslant \max_{\overline{B_\delta(x_1)}} v_{x_1,\delta} > \theta. \end{aligned} \tag{6.17}$$

Taking the limit as $n \to +\infty$ in the inequality

$$\min_{\overline{B_\delta(x_1)}} v_{x_0,R_{n+1}} \geqslant \max_{\overline{B_\delta(x_1)}} v_{x_0,R_n},$$

we obtain that

$$\min_{\overline{B_\delta(x_1)}} \bar{v} \geqslant \max_{\overline{B_\delta(x_1)}} \bar{v}.$$

Hence, $\bar{v}$ is equal to a constant $C_{x_1,\delta}$ in $\overline{B_\delta(x_1)}$ and, thanks to (6.17), there holds $\theta < C_{x_1,\delta} \leqslant 1$. Furthermore, since $x_1 \in \mathbb{R}^N$ is arbitrary, it follows that $\bar{v}$ is equal to a constant $C \in (\theta, 1]$ in $\mathbb{R}^N$.

Lastly, (6.16) yields $f(C) = 0$. Since f satisfies (1.11) and $\theta < C \leqslant 1$, we infer that $C = 1$. Therefore, $\bar{v} = 1$ in $\mathbb{R}^N$ and thus the sequence $(v_{x_0,R_n})_{n\in\mathbb{N}}$ converges to 1 locally uniformly in $\mathbb{R}^N$ as $n \to +\infty$. Since the non-decreasing sequence $(R_n)_{n\in\mathbb{N}}$ converging to $+\infty$ is arbitrary, and so is $\delta \in [d_0, +\infty)$, it follows that $v_{x_0,R}$ converges to 1 locally uniformly in $\mathbb{R}^N$ as $R \to +\infty$. The proof of Lemma 6.5 is thereby complete. □

6.3 Compactly supported continuous sub-solutions in $\mathbb{R}^N$

In this section, we construct compactly supported continuous sub-solutions from $\mathbb{R}^N$ to $[0, 1]$ of problems of the type (6.1). Such continuous sub-solutions will then serve as a building block of some lower bounds in the proof of Theorem 2.4.

Let us first introduce some useful notations. For $x_0 \in \mathbb{R}^N$, $R > 0$ and $x \in \mathbb{R}^N$, let $\mathscr{P}_{x_0,R}(x)$ be the projection of x on the closed convex set $\overline{B_R(x_0)}$, that is, $\mathscr{P}_{x_0,R}(x) \in \overline{B_R(x_0)}$ and

$$|x - \mathscr{P}_{x_0,R}(x)| = \mathrm{dist}(x, B_R(x_0)) = \min_{y \in \overline{B_R(x_0)}} |x - y|.$$

LEMMA 6.6. — *Assume that f and J satisfy* (1.8), (1.9) *and* (1.11). *Assume further that J is compactly supported and $J \in L^2(\mathbb{R}^N)$. Let $d_0 = d_0(f, J) > 0$ be given as in Lemmata* 6.1 *and* 6.3 *and, for any $x_0 \in \mathbb{R}^N$ and $R \geqslant d_0$, let $v_{x_0,R} \in C(\overline{B_R(x_0)}, (0,1))$ be the maximal solution of* (6.1). *Then there exists $\delta_0 > 0$ such that, for any $x_0 \in \mathbb{R}^N$, $R \geqslant d_0$ and $\delta \in (0, \delta_0]$, the continuous function $w_{x_0,R,\delta} : \mathbb{R}^N \to [0,1)$ defined by*

$$w_{x_0,R,\delta}(x) = \max\big\{v_{x_0,R}(\mathscr{P}_{x_0,R}(x)) - \delta^{-1}\,|x - \mathscr{P}_{x_0,R}(x)|, 0\big\}, \tag{6.18}$$

satisfies

$$\underbrace{\int_{B_{R'}(x_0)} J(x-y) w_{x_0,R,\delta}(y)\,\mathrm{d}y}_{=\mathscr{L}_{B_{R'}(x_0)}[w_{x_0,R,\delta}](x)} - w_{x_0,R,\delta}(x) + f(w_{x_0,R,\delta}(x)) \geqslant 0, \tag{6.19}$$

for all $x \in \mathbb{R}^N$ and all $R' \geqslant R + \delta$.

Proof. — In view of (6.18), we see that, for every $x_0 \in \mathbb{R}^N$, $R \geqslant d_0$ and $\delta > 0$, the function $w_{x_0,R,\delta}$ is continuous $\mathbb{R}^N$, that $0 \leqslant w_{x_0,R,\delta} < 1$ in $\mathbb{R}^N$, that $w_{x_0,R,\delta} = v_{x_0,R}$ in $\overline{B_R(x_0)}$ and that $w_{x_0,R,\delta} = 0$ in $\mathbb{R}^N \setminus B_{R+\delta}(x_0)$.

We set $g(s) := s - f(s)$ for $s \in [0,1]$. From (1.11), we see that

$$\gamma := \min_{[0,1]} g' > 0. \tag{6.20}$$

We recall that, by (1.11), J is assumed to belong to $W^{1,1}(\mathbb{R}^N)$, and set

$$\delta_0 := \gamma \cdot \left(\int_{\mathbb{R}^N} |\nabla J(z)|\,\mathrm{d}z\right)^{-1} > 0. \tag{6.21}$$

Let us now fix any $x_0 \in \mathbb{R}^N$, $R \geqslant d_0$, $\delta \in (0, \delta_0]$ and let us check that (6.19) holds for any $R' \geqslant R + \delta$. Since both $w_{x_0,R,\delta}$ and J are non-negative and since $w_{x_0,R,\delta} = 0$ in $\mathbb{R}^N \setminus B_{R+\delta}(x_0)$, recalling also that $f(0) = 0$ due to (1.11), we see that it is sufficient to show (6.19) for $x \in B_{R+\delta}(x_0)$. Furthermore, by monotonicity of the integral with respect to R', it is enough to show (6.19) for $R' = R + \delta$.

For any $x \in B_{R+\delta}(x_0)$, there holds

$$\int_{B_{R+\delta}(x_0)} J(x-y) w_{x_0,R,\delta}(y)\,\mathrm{d}y = \int_{B_{R+\delta}(x_0)\setminus B_R(x_0)} J(x-y) w_{x_0,R,\delta}(y)\,\mathrm{d}y$$

$$+\int_{B_R(x_0)} J(x-y)v_{x_0,R}(y)\mathrm{d}y.$$

Therefore, it follows from the above equality and the definitions of $v_{x_0,R}$ and $w_{x_0,R,\delta}$ that, for $x \in \overline{B_R(x_0)}$,

$$\int_{B_{R+\delta}(x_0)} J(x-y)w_{x_0,R,\delta}(y)\mathrm{d}y - w_{x_0,R,\delta}(x) + f(w_{x_0,R,\delta}(x)) = \int_{B_{R+\delta}(x_0)\setminus B_R(x_0)} J(x-y)w_{x_0,R,\delta}(y)\mathrm{d}y \geqslant 0.$$

To complete our proof, we have to show that the above inequality holds also for $x \in B_{R+\delta}(x_0) \setminus \overline{B_R(x_0)}$. To this end, let us consider $x \in B_{R+\delta}(x_0) \setminus \overline{B_R(x_0)}$ and set

$$s(x) := v_{x_0,R}(\mathscr{P}_{x_0,R}(x)) \quad \text{and} \quad \tau(x) := \mathrm{dist}(x, B_R(x_0)) = |x - \mathscr{P}_{x_0,R}(x)| > 0,$$

that is, $w_{x_0,R,\delta}(x) = \max\{s(x) - \delta^{-1}\tau(x), 0\}$. From the nonnegativity of J and $w_{x_0,R,\delta}$ and the fact that $w_{x_0,R,\delta} = v_{x_0,R}$ in $\overline{B_R(x_0)}$, we have

$$\begin{aligned}\int_{B_{R+\delta}(x_0)} & J(x-y)w_{x_0,R,\delta}(y)\mathrm{d}y - w_{\delta,R,x_0}(x) + f(w_{\delta,R,x_0}(x)) \\ &\geqslant \int_{B_R(x_0)} J(x-y)v_{x_0,R}(y)\mathrm{d}y - \max\{s(x)-\delta^{-1}\tau(x), 0\} + f(\max\{s(x)-\delta^{-1}\tau(x), 0\}).\end{aligned} \tag{6.22}$$

Now, two situations may occur: either $s(x) \leqslant \delta^{-1}\tau(x)$ (that is, $w_{x_0,R,\delta}(x) = 0$), or $s(x) > \delta^{-1}\tau(x)$ (that is, $w_{x_0,R,\delta}(x) > 0$). In the first situation, we easily conclude that

$$\int_{B_{R+\delta}(x_0)} J(x-y)w_{x_0,R,\delta}(y)\mathrm{d}y - w_{x_0,R,\delta}(x) + f(w_{x_0,R,\delta}(x)) \geqslant \int_{B_R(x_0)} J(x-y)v_{x_0,R}(y)\mathrm{d}y \geqslant 0.$$

So let us now assume that $s(x) > \delta^{-1}\tau(x)$, that is,

$$0 < w_{x_0,R,\delta}(x) = s(x) - \delta^{-1}\tau(x) \leqslant s(x) = v_{x_0,R}(\mathscr{P}_{x_0,R}(x)) < 1. \tag{6.23}$$

Let us rewrite the first integral in the right-hand side of (6.22) as

$$\begin{aligned}\int_{B_R(x_0)} J(x-y)v_{x_0,R}(y)\mathrm{d}y &= \int_{B_R(x_0)} [J(x-y) - J(\mathscr{P}_{x_0,R}(x)-y)]v_{x_0,R}(y)\mathrm{d}y \\ &\quad + \int_{B_R(x_0)} J(\mathscr{P}_{x_0,R}(x)-y)v_{x_0,R}(y)\mathrm{d}y.\end{aligned}$$

Since $v_{x_0,R}$ solves (6.1) in $\overline{B_R(x_0)}$, since

$$\mathscr{P}_{x_0,R}(x) \in \overline{B_R(x_0)} \quad \text{and} \quad s(x) = v_{x_0,R}(\mathscr{P}_{x_0,R}(x)),$$

and since $J \in W^{1,1}(\mathbb{R}^N)$, the above equality yields

$$\begin{aligned}\int_{B_R(x_0)} J(x-y)v_{x_0,R}(y)\mathrm{d}y &\geqslant s(x) - f(s(x)) - \int_{B_R(x_0)} |J(x-y) - J(\mathscr{P}_{x_0,R}(x) - y)|\mathrm{d}y. \\ &\geqslant s(x) - f(s(x)) - \int_{\mathbb{R}^N} |J(x-y) - J(\mathscr{P}_{x_0,R}(x) - y)|\mathrm{d}y. \\ &\geqslant s(x) - f(s(x)) - \tau(x) \times \int_{\mathbb{R}^N} |\nabla J(z)|\,\mathrm{d}z.\end{aligned}$$

Combining now the above inequality with (6.22) and $s(x) - \delta^{-1}\tau(x) > 0$, and using (4.38), (4.39) and (4.40), we get

$$\begin{aligned}&\int_{B_{R+\delta}(x_0)} J(x-y)w_{x_0,R,\delta}(y)\mathrm{d}y - w_{x_0,R,\delta}(x) + f(w_{x_0,R,\delta}(x)) \\ &\qquad \geqslant g(s(x)) - g(s(x) - \delta^{-1}\tau(x)) - \gamma\,\delta_0^{-1}\,\tau(x) \geqslant (\gamma\,\delta^{-1} - \gamma\,\delta_0^{-1})\tau(x) \geqslant 0.\end{aligned}$$

This is the desired inequality and the proof of Lemma 6.6 is thereby complete. □

7 The case of convex obstacles: proofs of the main Liouville type results

In this section, we prove our main results. We first consider in Section 7.1 the case where J is a general kernel satisfying (1.9), namely we prove Theorems 2.2 and 2.3. Once this is done, we consider in Section 7.2 kernels having compact support and we prove Theorem 2.4. Section 7.3 is devoted to the proof of a lemma used in the proof of Theorem 2.4. Throughout Section 7, we always assume that K is a compact convex set and that f and J satisfy the conditions (1.8), (1.9) and (1.10).

7.1 General kernels: proofs of Theorems 2.2 and 2.3

Let us start our proof of Theorem 2.2 with the following simple observation.

Lemma 7.1. — *Let $K \subset \mathbb{R}^N$ be a compact convex set and assume* (1.8) *and* (1.9). *Let $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$ satisfy* (2.3), *that is,*

$$Lu + f(u) \leqslant 0 \quad \text{in } \overline{\mathbb{R}^N \setminus K}, \tag{7.1}$$

$$u(x) \to 1 \quad \text{as } |x| \to +\infty. \tag{7.2}$$

Then there exists $\gamma \in (0,1]$ such that $\gamma \leqslant u \leqslant 1$ in $\overline{\mathbb{R}^N \setminus K}$.

Proof. — We proceed by contradiction. Suppose that the conclusion does not hold. Then, by continuity of u and (7.2), there exists a point $x_0 \in \overline{\mathbb{R}^N \setminus K}$, such that $u(x_0) = 0$. Arguing as in the proof of the strong maximum principle (Lemma 4.2) or in the proof of the sweeping principle (Lemma 4.3), we get that $u = 0$ in $\overline{\mathscr{A}(x_0, r_1, r_2)} \cap \overline{\mathbb{R}^N \setminus K}$, where $0 \leqslant r_1 < r_2$ are given in (1.9), and then $u = 0$ in $\overline{\mathscr{A}(x_1, r_1, r_2)} \cap \overline{\mathbb{R}^N \setminus K}$ for all $x_1 \in \overline{\mathscr{A}(x_0, r_1, r_2)} \cap \overline{\mathbb{R}^N \setminus K}$. Since K is convex, it follows in particular that $u = 0$ in $\overline{B_r(x_0)} \cap \overline{\mathbb{R}^N \setminus K}$ for some $r > 0$. Finally, the non-empty set $\{x \in \overline{\mathbb{R}^N \setminus K};\ u(x) = 0\}$ is both (obviously) closed and open relatively to the connected set $\overline{\mathbb{R}^N \setminus K}$. Hence $u = 0$ in $\overline{\mathbb{R}^N \setminus K}$, contradicting (7.2). □

We now turn to the proof of Theorem 2.2.

Proof of Theorem 2.2. — Let K, f, J and u be as in Theorem 2.2. Firstly, without loss of generality, one can assume by (1.8) that f is extended to a $C^1(\mathbb{R})$ function satisfying (4.1). Secondly, by (2.3) and the boundedness of K, there exists $R_0 > 0$ large enough so that $K \subset B_{R_0}$ and $u \geqslant 1 - c_0$ in $\mathbb{R}^N \setminus B_{R_0}$, where $c_0 > 0$ is given in (4.1).

We proceed the proof by contradiction, and suppose that

$$\inf_{\mathbb{R}^N \setminus K} u < 1. \tag{7.3}$$

From (2.3) and (7.3), together with the continuity of u, there exists then $x_0 \in \overline{\mathbb{R}^N \setminus K}$ such that

$$u(x_0) = \min_{\overline{\mathbb{R}^N \setminus K}} u \in [0, 1).$$

We observe that, by Lemma 7.1, one has $u(x_0) > 0$. Now, since K is convex, there exists $e \in \partial B_1$ such that $K \subset H_e^c$, where H_e is the open affine half-space defined by

$$H_e := x_0 + \{x \in \mathbb{R}^N;\ x \cdot e > 0\}.$$

In light of assumption (1.10), there exists an increasing function $\phi \in C(\mathbb{R})$ such that

$$\begin{cases} J_1 * \phi - \phi + f(\phi) \geqslant 0 \ \text{ in } \mathbb{R}, \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1. \end{cases}$$

Let us also define the function

$$\varphi_r(x) := \phi_{r,e}(x) = \phi(x \cdot e - r), \quad x \in \mathbb{R}^N,$$

and the following quantity

$$r_* := \inf\left\{r \in \mathbb{R}\,;\ \varphi_r \leqslant u \text{ in } \overline{\mathbb{R}^N \setminus K}\right\}.$$

From Lemmata 5.1 and 7.1, we know that $r_* \in [-\infty, r_0]$, where $r_0 > 0$ is given in Lemma 5.1.

We claim that in fact

$$r_* = -\infty. \tag{7.4}$$

The proof of (7.4) is by contradiction. We assume that $r_* \in \mathbb{R}$. Then, there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ of positive real numbers such that $\varphi_{r_*+\varepsilon_j}(x) = \phi(x\cdot e - r_* - \varepsilon_j) \leqslant u(x)$ for all $x \in \overline{\mathbb{R}^N \setminus K}$ and $\varepsilon_j \to 0$ as $j \to +\infty$. Thus passing to the limit as $j \to +\infty$, we obtain that

$$\varphi_{r_*}(x) \leqslant u(x) \quad \text{ for all } x \in \overline{\mathbb{R}^N \setminus K}.$$

Let us denote H the open affine half-space

$$H = \big\{x \in \mathbb{R}^N;\ x \cdot e > R_0\big\}.$$

Notice that $\overline{H} \cap K = \emptyset$ and that u is well defined and continuous in $\overline{H}$. We also observe that, by construction,

$$\sup_{H^c} \varphi_{r_*} < 1. \tag{7.5}$$

Two cases may occur.

Case 1: $\inf_{H^c\setminus K}(u - \varphi_{r_*}) > 0$. In this situation, thanks to the uniform continuity of ϕ, there exists $\varepsilon > 0$ such that

$$\inf_{H^c\setminus K}(u - \varphi_{r_*-\varepsilon}) > 0.$$

Now, we observe that u and $\varphi_{r_*-\varepsilon}$ satisfy

$$\begin{cases} Lu + f(u) \leqslant 0 & \text{in } \overline{H}, \\ L\varphi_{r_*-\varepsilon} + f(\varphi_{r_*-\varepsilon}) \geqslant 0 & \text{in } \overline{H} \quad \text{(by (5.3))}, \\ u \geqslant \varphi_{r_*-\varepsilon} & \text{in } H^c \setminus K, \end{cases}$$

together with $u \geqslant 1 - c_0$ in $\mathbb{R}^N \setminus B_{R_0} \supset \overline{H}$ and $\lim_{|x|\to+\infty} u(x) = 1$, while $\varphi_{r_*-\varepsilon} \leqslant 1$ in $\mathbb{R}^N$. Thus, by the weak maximum principle (Lemma 4.1) and the continuity of u and $\varphi_{r_*-\varepsilon}$ in $\overline{\mathbb{R}^N \setminus K}$, we get that $u \geqslant \varphi_{r_*-\varepsilon}$ in $\overline{\mathbb{R}^N \setminus K}$. This contradicts the minimality of r_* and therefore Case 1 is ruled out.

Case 2: $\inf_{H^c\setminus K}(u - \varphi_{r_*}) = 0$. In this situation, by (7.2) and (7.5), and by continuity of u and φ_{r_*}, there exists a point $\bar{x} \in \overline{H^c \setminus K}$ such that $u(\bar{x}) = \varphi_{r_*}(\bar{x})$. Note that $\bar{x} \in \overline{H_e}$, since otherwise $\bar{x} \in \mathbb{R}^N \setminus \overline{H_e}$, namely $\bar{x} \cdot e < x_0 \cdot e$, and the chain of inequalities

$$u(\bar{x}) = \varphi_{r_*}(\bar{x}) < \varphi_{r_*}(x_0) \leqslant u(x_0) = \min_{\overline{\mathbb{R}^N\setminus K}} u$$

leads to a contradiction. Therefore, we have $\varphi_{r_*} \leqslant u$ in $\overline{\mathbb{R}^N \setminus K}$ with equality at a point $\bar{x} \in \overline{\mathbb{R}^N \setminus K} \cap \overline{H_e}$. Since $K \subset H_e^c$ and φ_{r*} and u satisfy respectively

$$\begin{cases} Lu + f(u) \leqslant 0 & \text{in } \overline{H_e}, \\ L\varphi_{r_*} + f(\varphi_{r_*}) \geqslant 0 & \text{in } \overline{H_e} \ \ \text{(by (5.3))}, \end{cases}$$

it follows in particular from the strong maximum principle (Lemma 4.2) that $\varphi_{r_*} = u$ in $\overline{H_e}$. Thus, for any $e^\perp \in \partial B_1$ such that $e^\perp \cdot e = 0$, one infers from (7.2) and the definition of φ_{r_*} that

$$1 = \lim_{t \to +\infty} u(x_0 + t\, e^\perp) = \lim_{t \to +\infty} \varphi_{r_*}(x_0 + t\, e^\perp) = \varphi_{r_*}(x_0) < 1.$$

This contradiction rules out Case 2 too.

Hence (7.4) holds true and as a consequence we have that $\varphi_r \leqslant u$ in $\overline{\mathbb{R}^N \setminus K}$ for any $r \in \mathbb{R}$. In particular, recalling that $\phi(+\infty) = 1$, we get that

$$1 > u(x_0) \geqslant \lim_{r \to -\infty} \varphi_r(x_0) = \lim_{r \to -\infty} \phi(x_0 \cdot e - r) = 1,$$

a contradiction. Therefore, (7.3) can not hold. In other words, $\inf_{\overline{\mathbb{R}^N \setminus K}} u = 1$, i.e. $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$. The proof of Theorem 2.2 is thereby complete. □

We observe that, by the same token, we obtain Theorem 2.3.

Proof of Theorem 2.3. — By Lemma 2.2, Remark 3.3 and our assumptions on f, we know that u has a (uniformly) continuous representative $u^* \in C(\overline{\mathbb{R}^N \setminus K})$ in its class of equivalence and we can identify u with u^*. The desired result now follows as a consequence of Theorem 2.2. □

7.2 Compactly supported kernels: proof of Theorem 2.4

In this subsection we prove Theorem 2.4. That is, provided some additional assumptions on f and J are satisfied, we show that the Liouville result obtained in Theorem 2.2 holds true when the uniform limit of u as $|x| \to +\infty$, namely condition (7.2), is replaced by the following weaker condition

$$\operatorname*{ess\,sup}_{\mathbb{R}^N \setminus K} u = 1, \tag{7.6}$$

where $u : \mathbb{R}^N \setminus K \to [0, 1]$ is a measurable solution of $Lu + f(u) = 0$ a.e. in $\mathbb{R}^N \setminus K$. The condition (7.6) can be rewritten as

$$\sup_{\mathbb{R}^N \setminus K} u = 1, \tag{7.7}$$

if u is already assumed to be uniformly continuous in $\overline{\mathbb{R}^N \setminus K}$. Note that the extra assumptions (2.4) made on f (namely $f' < 1/2$ in $[0,1]$) actually guarantees that u has a uniformly continuous representative in its class of equivalence, as follows from Lemma 2.2 and Remark 3.3. As a consequence, in the proof of Theorem 2.4 we can assume without loss of generality that $u : \overline{\mathbb{R}^N \setminus K} \to [0,1]$ is uniformly continuous and satisfies (7.7). Notice immediately that the same arguments as in the proof of Lemma 7.1 imply that

$$u > 0 \ \text{ in } \overline{\mathbb{R}^N \setminus K}. \tag{7.8}$$

Otherwise u would be identically equal to 0 in $\overline{\mathbb{R}^N \setminus K}$, contradicting the assumption (7.7).

The key-point in the proof of Theorem 2.4 is the following lemma.

Lemma 7.2. — *Let $K \subset \mathbb{R}^N$ be a compact set and assume that f and J satisfy* (1.8), (1.9) *and* (1.11). *Assume further that J is compactly supported and $J \in L^2(\mathbb{R}^N)$. Let $u : \overline{\mathbb{R}^N \setminus K} \to [0,1]$ be a uniformly continuous solution of* (2.5). *Then, $u(x) \to 1$ as $|x| \to +\infty$.*

The proof of Lemma 7.2 is postponed in Section 7.3. In this section, we complete the proof of Theorem 2.4.

Proof of Theorem 2.4. — From the previous paragraphs, the function u can be assumed to be uniformly continuous in $\overline{\mathbb{R}^N \setminus K}$ without loss of generality. Then, since the condition (1.11), together with (1.8) and (1.9), implies the condition (1.10), the assumptions of Theorem 2.2 are all fulfilled, thanks to Lemma 7.2. Therefore $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$, completing the proof of Theorem 2.4. □

7.3 Proof of Lemma 7.2

This section is devoted to the proof of Lemma 7.2. It is divided into four main steps. To prove Lemma 7.2, it suffices to show that for any $\varepsilon > 0$ small enough there exists $R(\varepsilon) > 0$ such that $u \geqslant 1 - \varepsilon$ in $\mathbb{R}^N \setminus B_{R(\varepsilon)}$. To obtain such a lower bound, our strategy relies on the existence of continuous families of continuous sub-solutions w_τ which satisfy $w_\tau \geqslant 1 - \varepsilon$ in $\overline{B_1(x_\tau)}$ for some $x_\tau \in \mathbb{R}^N$ (these sub-solutions are drawn from Section 6.3). Then, we use the sweeping principle to propagate the lower bound satisfied by the w_τ's to a lower bound for u.

Step 1: the solution u is close to 1 *in some large balls*

In this step, we show that, for any $\varepsilon > 0$, $\ell > 0$, and $R > 0$, there exists a point $x^* \in \mathbb{R}^N \setminus K$ such that

$$|x^*| > \ell, \quad B_R(x^*) \subset \mathbb{R}^N \setminus K, \ \text{ and } \ u \geqslant 1 - \varepsilon \text{ in } \overline{B_R(x^*)}. \tag{7.9}$$

To do so, notice first that, from (2.5) and the continuity of u in $\overline{\mathbb{R}^N \setminus K}$, two situations may occur: namely, either there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N \setminus K$ such that

$$\lim_{n\to+\infty} |x_n| = +\infty \quad \text{and} \quad \lim_{n\to+\infty} u(x_n) = 1, \tag{7.10}$$

or there exists a point $\bar{x} \in \overline{\mathbb{R}^N \setminus K}$ such that $u(\bar{x}) = 1$. In the latter case, since $f(u(\bar{x})) = f(1) = 0$, we get, as in the proof of Lemma 7.1, that $u = 1$ in $\overline{\mathbb{R}^N \setminus K}$: the claim (7.9) is therefore trivial in this case.

Thus, it suffices to treat the former case (7.10) only. Consider the functions u_n defined in $\overline{\mathbb{R}^N \setminus K} - x_n$ by

$$u_n(x) = u(x + x_n).$$

Since u is uniformly continuous in $\overline{\mathbb{R}^N \setminus K}$ and since K is compact and $\lim_{n\to+\infty} |x_n| = +\infty$, it follows that, for every $r > 0$, the functions u_n's, ranging in $[0, 1]$, are defined in $\overline{B_r}$ for all n large enough and are uniformly equicontinuous in $\overline{B_r}$. From Arzela-Ascoli theorem and the diagonal extraction process, there exists a continuous function $u_\infty : \mathbb{R}^N \to [0, 1]$ such that, up to extraction of a subsequence, $u_n \to u_\infty$ locally uniformly in $\mathbb{R}^N$ as $n \to +\infty$. Furthermore, $u_\infty(0) = 1$ by (7.10). On the other hand, the functions u_n's satisfy

$$\int_{(\mathbb{R}^N\setminus K)-x_n} J(x-y)u_n(y)\mathrm{d}y - \left(\int_{(\mathbb{R}^N\setminus K)-x_n} J(x-y)\mathrm{d}y\right)u_n(x) + f(u_n(x)) = 0,$$

for all $x \in \overline{\mathbb{R}^N \setminus K} - x_n$. Lebesgue's dominated convergence theorem implies that

$$J * u_\infty - u_\infty + f(u_\infty) = 0 \ \text{ in } \mathbb{R}^N.$$

Since $f(u_\infty(0)) = f(1) = 0$ and $u_\infty \leqslant 1$ in $\mathbb{R}^N$, we get as in the proof of Lemma 7.1 that $u_\infty = 1$ in $\mathbb{R}^N$. In particular, for any fixed $\varepsilon > 0$, $\ell > 0$, and $R > 0$, it follows that, for every $n \in \mathbb{N}$ large enough, there holds $|x_n| > \ell$, $B_R(x_n) \subset \mathbb{R}^N \setminus K$ and $u_n \geqslant 1 - \varepsilon$ in $\overline{B_R}$, that is, $u \geqslant 1 - \varepsilon$ in $\overline{B_R}(x_n)$. In other words, the claim (7.9) holds with $x^* = x_n$ and n large enough.

Step 2: a sub-solution in a ball

Fix $\varepsilon > 0$ small enough so that $f' < 0$ in $[1 - \varepsilon, 1]$, and let us now establish a lower bound for u in a ball far away from K, by using a sub-solution drawn from Section 6.3. We recall here that $R_J > 0$ is such that $\mathrm{supp}(J) \subset B_{R_J}$.

We first claim that there exist $x^* \in \mathbb{R}^N$, $0 < R_J \leqslant R_K \leqslant R$ and a function $w \in C(\mathbb{R}^N, [0, 1))$ such that

$$\begin{cases} B_{R+1}(x^*) \ \subset \ \mathbb{R}^N\setminus B_{R_K} \ \subset \ \mathbb{R}^N\setminus K, \quad u \geqslant 1-\varepsilon \text{ in } \overline{B_{R+1}(x^*)}, \\ \mathscr{L}_{B_{R+1}(x^*)}[w] - w + f(w) \geqslant 0 \text{ in } \mathbb{R}^N, \quad w \geqslant 1-\varepsilon \text{ in } \overline{B_1(x^*)}, \quad w = 0 \text{ in } \mathbb{R}^N\setminus B_{R+1}(x^*). \end{cases} \tag{7.11}$$

To show this claim, let $R_K \geqslant \max\{1, R_J\}$ be such that $K \subset B_{R_K}$. Then choose $R \geqslant \max\{R_K, d_0\} \geqslant 1$ ($d_0 > 0$ is given as in Lemmata 6.1 and 6.3) such that the maximal solution $v_{0,R} \in C(\overline{B_R}, (0,1))$ to problem (6.1) in $\overline{B_R}$ satisfies

$$v_{0,R} \geqslant 1 - \varepsilon \quad \text{in } \overline{B_1}. \tag{7.12}$$

Note that such a real number R exists according to Lemmata 6.3 and 6.5. On the one hand, as far as the first line in (7.11) is concerned, formula (7.9), applied here with $\ell = R+1+R_K > 0$ and $R+1 > 0$ in place of R, yields the existence of $x^* \in \mathbb{R}^N$ such that

$$|x^*| > R + 1 + R_K, \tag{7.13}$$

(hence, $B_{R+1}(x^*) \subset \mathbb{R}^N \backslash B_{R_K} \subset \mathbb{R}^N \backslash K$) and

$$u \geqslant 1 - \varepsilon \quad \text{in } \overline{B_{R+1}(x^*)}. \tag{7.14}$$

Thanks to (7.12) and part (ii) of Lemma 6.4, the maximal solution

$$v_{x^*,R} \in C(\overline{B_R(x^*)}, (0,1)),$$

to problem (6.1) in $\overline{B_R(x^*)}$ satisfies $v_{x^*,R} \geqslant 1 - \varepsilon$ in $\overline{B_1(x^*)}$. On the other hand, as far as the second line in (7.11) is concerned, Lemma 6.6 provides the existence of a function $w \in C(\mathbb{R}^N, [0,1))$ such that

$$\mathscr{L}_{B_{R+1}(x^*)}[w] - w + f(w) \geqslant 0 \text{ in } \mathbb{R}^N, \quad w = v_{x^*,R} \text{ in } \overline{B_R(x^*)} \supset \overline{B_1(x^*)},$$

and $w = 0$ in $\mathbb{R}^N \setminus B_{R+1}(x^*)$. As a consequence, x^*, R_K, R and w fulfill (7.11).

We then claim that

$$w \leqslant u \quad \text{in } \overline{\mathbb{R}^N \setminus K}. \tag{7.15}$$

Since $w = 0$ in $\mathbb{R}^N \setminus B_{R+1}(x^*)$ and $u \geqslant 0$ in $\overline{\mathbb{R}^N \setminus K}$, we only need to show that $w \leqslant u$ in $\overline{B_{R+1}(x^*)}$ ($\subset \overline{\mathbb{R}^N \setminus K}$). Denote

$$z := w - u,$$

in $\overline{B_{R+1}(x^*)}$ and assume that

$$\max_{\overline{B_{R+1}(x^*)}} z = z(\bar{x}) > 0,$$

for some $\bar{x} \in \overline{B_{R+1}(x^*)}$. Since $\overline{B_{R+1}(x^*)} \subset \overline{\mathbb{R}^N \setminus K}$ and u and J are non-negative with J having a unit mass in $L^1(\mathbb{R}^N)$, it follows from the equation $Lu + f(u) = 0$ satisfied by u in $\overline{\mathbb{R}^N \setminus K}$ that

$$\mathscr{L}_{B_{R+1}(x^*)}[u](\bar{x}) - u(\bar{x}) + f(u(\bar{x})) \leqslant 0.$$

Together with the first inequality of the second line of (7.11) applied at $\bar{x}$, we get that

$$\mathscr{L}_{B_{R+1}(x^*)}[z](\bar{x}) - z(\bar{x}) + f(w(\bar{x})) - f(u(\bar{x})) \geqslant 0. \tag{7.16}$$

Since $z \leqslant z(\bar{x})$ in $\overline{B_{R+1}(x^*)}$, one has $\mathscr{L}_{B_{R+1}(x^*)}[z](\bar{x}) - z(\bar{x}) \leqslant 0$. Furthermore, remembering (7.14) and the choice of ε, there holds $1 - \varepsilon \leqslant u(\bar{x}) = w(\bar{x}) - z(\bar{x}) < w(\bar{x}) < 1$ and $f' < 0$ in $[1-\varepsilon, 1]$, hence $f(w(\bar{x})) - f(u(\bar{x})) < 0$. This contradicts (7.16). Therefore, $\max_{\overline{B_{R+1}(x^*)}} z \leqslant 0$, that is, $w \leqslant u$ in $\overline{B_{R+1}(x^*)}$ and then in $\overline{\mathbb{R}^N \setminus K}$.

Step 3: a lower bound in annuli with large inner radii

Let us now construct some families of sub-solutions and exploit the sweeping principle (Lemma 4.3) to get a lower bound of u in some annuli. To do so, let $x^* \in \mathbb{R}^N$, $0 < R_J \leqslant R_K \leqslant R$ and $w \in C(\mathbb{R}^N, [0,1))$ be as in (7.11). Consider any orthonormal basis $(e_1, \cdots, e_N)$ of $\mathbb{R}^N$ and, for $\tau \in [0, 2\pi]$, let $\mathscr{R}_\tau$ be the rotation of angle τ in the plane spanned by (e_1, e_2) (that is, $\mathscr{R}_\tau e_1 = (\cos\tau)e_1 + (\sin\tau)e_2$ and $\mathscr{R}_\tau e_2 = -(\sin\tau)e_1 + (\cos\tau)e_2$) and leaving invariant the vectors $e_3, \cdots, e_N$. We set

$$A := \mathscr{A}(|x^*| - R - 1, |x^*| + R + 1) = B_{|x^*|+R+1} \setminus \overline{B_{|x^*|-R-1}}.$$

From (7.11), note that $\overline{A} \subset \mathbb{R}^N \setminus B_{R_K} \subset \mathbb{R}^N \setminus K$ (hence, $\overline{A} \cap K = \emptyset$). Now for each $\tau \in [0, 2\pi]$ and $x \in \mathbb{R}^N$, we set

$$w_\tau(x) := w(\mathscr{R}_\tau x).$$

Thanks to the rotational invariance of J and A, and since $B_{R+1}(x^*) \subset A$ and both J and w are non-negative, it follows from (7.11) that each function w_τ satisfies

$$\mathscr{L}_A[w_\tau] - w_\tau + f(w_\tau) \geqslant 0 \quad \text{in } \mathbb{R}^N.$$

On the other hand, it follows from (2.5) that the function u obeys

$$\begin{aligned}\mathscr{L}_A[u](x) - &u(x) + f(u(x)) \\ &= -\int_{\mathbb{R}^N \setminus (K \cup A)} J(x-y)u(y)\mathrm{d}y - u(x)\left(1 - \int_{\mathbb{R}^N \setminus K} J(x-y)\mathrm{d}y\right) \leqslant 0,\end{aligned}$$

for all $x \in \overline{\mathbb{R}^N \setminus K}$ and therefore for all $x \in \overline{A}$. In addition, thanks to positivity of u in $\overline{\mathbb{R}^N \setminus K}$ (remember (7.8)) and the fact that $J > 0$ a.e. in $\mathscr{A}(r_1, r_2)$ with $0 \leqslant r_1 < r_2 \leqslant R_J \leqslant R_K \leqslant R$ (remember (1.9) and $\mathrm{supp}(J) \subset B_{R_J}$), one infers that

$$\int_{\mathbb{R}^N \setminus (K \cup A)} J(x-y)u(y)\mathrm{d}y > 0 \ \text{ for all } x \in A' := \mathscr{A}(|x^*|+R+1-r_2, |x^*|+R+1) \ (\subset A),$$

hence

$$\mathscr{L}_A[u](x) - u(x) + f(u(x)) < 0 \ \text{ for all } x \in A'.$$

Since $w \leqslant u$ in $\overline{A}\,(\subset \overline{\mathbb{R}^N \setminus K})$ by (7.15) and $r_2 \leqslant R_K \leqslant |x^*|-R-1$ by (7.13), it follows from the sweeping principle (Lemma 4.3) applied to u, to the family $(w_\tau)_{\tau\in[0,2\pi]}$ and to

$$(s_1, s_2, s_3, s_4) = (|x^*| - R - 1, |x^*| + R + 1 - r_2, |x^*| + R + 1, |x^*| + R + 1),$$

that

$$w_\tau \leqslant u \ \text{ in } \overline{A} \ \text{ for every } \tau \in [0, 2\pi]. \tag{7.17}$$

Notice also (even if the following inequalities will not explicitly be used in the next step) that, since $w \geqslant 1 - \varepsilon$ in $\overline{B_1(x^*)}$ by (7.11), the family of estimates in (7.17) implies in particular that $u \geqslant 1 - \varepsilon$ in $\bigcup_{\tau\in[0,2\pi]} \overline{B_1(\mathscr{R}_\tau^{-1}x^*)}$. Since the previous arguments are independent of the choice of the orthonormal basis $(e_1, \dots, e_N)$, we also get that $u \geqslant 1 - \varepsilon$ in $\overline{\mathscr{A}(|x^*| - 1, |x^*| + 1)}$.

Step 4: conclusion

Let us now finish our argument. To complete the proof of Lemma 7.2, we will again construct an adequate family of sub-solutions and use the sweeping principle to push further the estimates obtained in the previous step. To do so, pick some $\rho > 0$ and consider the domain

$$A_\rho := \mathscr{A}(|x^*| - R - 1, |x^*| + R + 1 + \rho),$$

where $R > 0$ is defined in Steps 2 and 3. From (7.11), we note that $\overline{A_\rho} \subset \mathbb{R}^N \setminus B_{R_K} \subset \mathbb{R}^N \setminus K$ (hence, $\overline{A_\rho} \cap K = \emptyset$). Next, consider any rotation $\mathscr{R}$ of $\mathbb{R}^N$, let $e := x^*/|x^*| \in \partial B_1$ and, for each $\sigma \in [0, \rho]$ and $x \in \mathbb{R}^N$, denote

$$W_\sigma(x) := w(\mathscr{R}\,x - \sigma\,e).$$

As in the previous step, from the rotational invariance of J and A_ρ, and since $B_{R+1}(x^* + \sigma e) \subset A_\rho$ for every $\sigma \in [0, \rho]$ and both J and w are non-negative, it follows from (7.11) that each function W_σ satisfies

$$\mathscr{L}_{A_\rho}[W_\sigma] - W_\sigma + f(W_\sigma) \geqslant 0 \ \text{ in } \mathbb{R}^N.$$

Similarly, it follows from (2.5) that the function u obeys

$$\mathscr{L}_{A_\rho}[u] - u + f(u) \leqslant 0 \ \text{ in } \overline{\mathbb{R}^N \setminus K},$$

(and therefore in $\overline{A_\rho}$), while

$$\mathscr{L}_{A_\rho}[u] - u + f(u) < 0 \ \text{ in } \mathscr{A}(|x^*| + R + 1 + \rho - r_2, |x^*| + R + 1 + \rho)\ (\subset A_\rho).$$

From the inequality (7.17) of the previous step (which holds for every $\tau \in [0, 2\pi]$ and for every orthonormal basis $(e_1, \cdots, e_N)$), we have $W_0 \leqslant u$ in $\overline{A}$ and then in

$\overline{\mathbb{R}^N \setminus K}$ (since $W_0 = 0$ in $\mathbb{R}^N \setminus A$ and $u > 0$ in $\overline{\mathbb{R}^N \setminus K}$. As a consequence, $W_0 \leqslant u$ in $\overline{A_\rho}$. Finally, it follows from the sweeping principle (Lemma 4.3) applied to u, to the family $(W_\sigma)_{\sigma\in[0,\rho]}$ and to

$$(s_1, s_2, s_3, s_4) = (|x^*| - R - 1, |x^*| + R + 1 + \rho - r_2, |x^*| + R + 1 + \rho, |x^*| + R + 1 + \rho),$$

that $W_\sigma \leqslant u$ in $\overline{A_\rho}$ for every $\sigma \in [0, \rho]$. Since $w \geqslant 1 - \varepsilon$ in $\overline{B_1(x^*)}$ by (7.11), we obtain in particular that

$$u \geqslant 1 - \varepsilon \quad \text{in} \bigcup_{\sigma\in[0,\rho]} \overline{B_1(\mathscr{R}^{-1}(x^* + \sigma e))}.$$

The previous arguments being independent of the choice of $\rho > 0$ and the rotation $\mathscr{R}$ of $\mathbb{R}^N$, we conclude that

$$u(x) \geqslant 1 - \varepsilon \quad \text{for all } |x| \geqslant |x^*| - 1.$$

Since $\varepsilon > 0$ can be arbitrarily small, the proof of Lemma 7.2 is thereby complete.

8 The case of small perturbations of convex obstacles

In this section, we explore further the validity of the Liouville Theorem 2.2 and we prove Theorem 2.6, a kind of stability result for the Liouville property. In the spirit of the results of Bouhours [26], we show that the Liouville property obtained in Theorem 2.2 still holds true for small perturbations of convex obstacles, provided some additional assumptions are made on f and J. To do so, we adapt to our problem the arguments developed in [26] and, in particular, we will rely on the following

Lemma 8.1. — *Assume all hypotheses of Theorem* 2.6. *Then, for every* $\delta \in (0, 1)$, *there exists a real number* $R_\delta > 0$ *such that, for any* $\varepsilon \in (0, 1]$ *and any measurable solution* $u_\varepsilon : \mathbb{R}^N \setminus K_\varepsilon \to [0, 1]$ *of* (2.7), *there holds* $u_\varepsilon(x) \geqslant 1 - \delta$ *for a.e.* $|x| \geqslant R_\delta$.

Before proving Lemma 8.1, let us first establish a preliminary "rough" Liouville-type result, namely Proposition 2.1.

Proof of Proposition 2.1. — We recall that $f \in C^1([0, 1])$, that J is assumed to satisfy (1.9), that K is a compact set such that $\mathbb{R}^N \setminus K$ is connected, and that $u : \overline{\mathbb{R}^N \setminus K} \to [\theta, 1]$ is a continuous solution of (1.1) such that $f \geqslant 0$ on $[\theta, 1]$. Let us set

$$m = \inf_{\overline{\mathbb{R}\setminus K}} u \ \in [\theta, 1].$$

Suppose, by contradiction, that $m < 1$. Let $(x_n)_{n\in\mathbb{N}} \subset \overline{\mathbb{R}^N \setminus K}$ be a sequence such that $u(x_n) \to m$ as $n \to +\infty$. Since $u(x) \to 1$ as $|x| \to +\infty$, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded and, up to extraction of a subsequence, we may assume that it converges to some $\bar{x} \in \overline{\mathbb{R}^N \setminus K}$. Evaluating the equation satisfied by u at x_n, we obtain

$$\int_{\mathbb{R}^N\setminus K} J(x_n - y)(u(y) - u(x_n))\mathrm{d}y + f(u(x_n)) = 0.$$

By assumption, $f(u(x)) \geqslant 0$ for all $x \in \overline{\mathbb{R}^N \setminus K}$ and therefore

$$\int_{\mathbb{R}^N\setminus K} J(x_n - y)(u(y) - u(x_n))\mathrm{d}y \leqslant 0.$$

Since $J \in L^1(\mathbb{R}^N)$ passing to the limit in the above inequality results in

$$0 \leqslant \int_{\mathbb{R}^N\setminus K} J(\bar{x} - y)(u(y) - m)\mathrm{d}y \leqslant 0.$$

Thus, arguing as in Section 4 and using (1.9) and the connectedness of $\mathbb{R}^N \setminus K$, we obtain that $u = m$ (< 1) in $\overline{\mathbb{R}^N \setminus K}$. Since $u(x) \to 1$ as $|x| \to +\infty$, we get a contradiction. The proof of Proposition 2.1 is thereby complete. □

Let us now turn our attention to the proof of Lemma 8.1.

Proof of Lemma 8.1. — First of all, in virtue of Lemma 2.2, we know that, for every $\varepsilon \in (0, 1]$, every measurable solution $u_\varepsilon : \overline{\mathbb{R}^N \setminus K_\varepsilon} \to [0, 1]$ of (2.7) possesses a Hölder continuous representative $u_\varepsilon^* \in C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\varepsilon})$. Consequently, we are allowed to identify u_ε with u_ε^*. For simplicity, we omit the superscript $*$ and write simply u_ε instead of u_ε^*.

Let us then continuously extend f by $f'(0)s$ for $s \leqslant 0$ and by $f'(1)(s-1)$ for $s \geqslant 1$ and still denote f this extension. We also observe that, since $(K_\varepsilon)_{0<\varepsilon\leqslant 1}$ is a family of (at least) $C^{0,\alpha}$ deformations of K in the sense of Definition 1.2, there exists a real number $R_0 > 0$ such that

$$K_\varepsilon \subset B_{R_0} \quad \text{for all } 0 < \varepsilon \leqslant 1. \tag{8.1}$$

Notice now that it is sufficient to show the conclusion of Lemma 8.1 for $\delta > 0$ small enough. For any $\delta > 0$ small enough, we are going to consider an auxiliary problem whose solutions will provide an appropriate lower bound for u_ε, allowing us to prove the desired uniform convergence as $|x| \to +\infty$. To this end, for $\delta \in (0, 1)$, denote

$$f_\delta(s) := f(s) - f(1 - \delta/2) \text{ for } s \in \mathbb{R}, \quad \text{and} \quad s_\delta := \frac{f(1-\delta/2)}{f'(0)}.$$

It is immediate to check that there exists $\delta_1 \in (0,1)$ such that, for every $\delta \in (0,\delta_1)$, one has $s_\delta < 0 < 1-\delta/2 < 1$ and

$$\begin{cases} f_\delta \leqslant f \text{ in } \mathbb{R}, \quad f'_\delta = f' < 1/2 \text{ in } \mathbb{R}, \\ f_\delta(s_\delta) = 0, \quad f'_\delta(s_\delta) < 0, \quad f_\delta(1-\delta/2) = 0, \quad f'_\delta(1-\delta/2) < 0, \quad \int_{s_\delta}^{1-\delta/2} f_\delta(r)\mathrm{d}r > 0, \\ f_\delta \text{ vanishes only once in } (s_\delta, 1-\delta/2). \end{cases}$$

Using the results obtained in [13, 45, 50, 155], we know that, for every $\delta \in (0,\delta_1)$, there exists a continuous function $\phi_\delta : \mathbb{R} \to (s_\delta, 1-\delta/2)$ satisfying

$$\begin{cases} L_{\mathbb{R}}\phi_\delta + f_\delta(\phi_\delta) = J_1 * \phi_\delta - \phi_\delta + f_\delta(\phi_\delta) \geqslant 0 \ \text{ in } \mathbb{R}, \\ \phi_\delta \text{ is increasing in } \mathbb{R}, \\ \phi_\delta(-\infty) = s_\delta, \quad \phi_\delta(0) = 0, \quad \phi_\delta(+\infty) = 1-\delta/2. \end{cases}$$

Fix in the sequel any $\delta \in (0,\delta_1)$, any $\varepsilon \in (0,1]$ and any (Hölder-continuous) function $u_\varepsilon : \overline{\mathbb{R}^N \setminus K_\varepsilon} \to [0,1]$ solving (2.7). For $A > 0$, we let $\Phi_{\delta,A}$ be the function defined in $\mathbb{R}^N$ by

$$\Phi_{\delta,A}(x) := \phi_\delta(|x| - A).$$

We observe that, by construction, we have

$$\Phi_{\delta,R_0}(x) \leqslant 0 \leqslant u_\varepsilon \quad \text{ for all } x \in \overline{B_{R_0} \setminus K_\varepsilon}. \tag{8.2}$$

Our aim is to extend the above relation to all $x \in \mathbb{R}^N \setminus \overline{B_{R_0}}$. Since $u_\varepsilon(x) \to 1$ as $|x| \to +\infty$, there exists $R_\varepsilon > R_0$ such that $u_\varepsilon(x) \geqslant \max(1-c_0, 1-\delta/2)$ for all $|x| \geqslant R_\varepsilon$ where $c_0 > 0$ is such that $f' < 0$ in $[1-c_0, +\infty)$. Then, reasoning as in Lemma 5.1 (or using directly that $\Phi_{\delta,A} \to s_\delta < 0$ as $A \to +\infty$ locally uniformly in $\mathbb{R}^N$ and $\Phi_{\delta,A} < 1-\delta/2 < 1$ in $\mathbb{R}^N$), we obtain that, for some $A_\varepsilon > 0$,

$$\Phi_{\delta,A_\varepsilon} \leqslant u_\varepsilon \ \text{ in } \overline{\mathbb{R}^N \setminus K_\varepsilon}.$$

Consequently, it makes sense to define

$$A^* := \inf\Big\{A \in \mathbb{R}\,;\ \Phi_{\delta,A} \leqslant u_\varepsilon \text{ in } \overline{\mathbb{R}^N \setminus K_\varepsilon}\Big\} \leqslant A_\varepsilon.$$

We claim that

$$A^* \leqslant R_0, \tag{8.3}$$

We argue by contradiction and assume that $A^* > R_0$. From the definition of A^* and the continuity of ϕ_δ, we have

$$\Phi_{\delta,A^*} \leqslant u_\varepsilon \ \text{ in } \overline{\mathbb{R}^N \setminus K_\varepsilon}. \tag{8.4}$$

If $\min_{\overline{B_{R_\varepsilon}\setminus K_\varepsilon}}(u_\varepsilon - \Phi_{\delta,A^*}) > 0$, then from the uniform continuity of ϕ_δ, there exists $\tau > 0$ small enough such that $\Phi_{\delta,A^*-\tau} \leqslant u_\varepsilon$ in $\overline{B_{R_\varepsilon} \setminus K_\varepsilon}$. On the other hand, $\Phi_{\delta,A^*-\tau} < 1-\delta/2 \leqslant u_\varepsilon$ in $\mathbb{R}^N \setminus B_{R_\varepsilon}$. Hence, $\Phi_{\delta,A^*-\tau} \leqslant u_\varepsilon$ in $\overline{\mathbb{R}^N \setminus K_\varepsilon}$, a contradiction with the definition of A^*. Therefore, $\min_{\overline{B_{R_\varepsilon}\setminus K_\varepsilon}}(u_\varepsilon - \Phi_{\delta,A^*}) = 0$. Since u_ε and Φ_{δ,A^*} are continuous, there exists $x_0 \in \overline{B_{R_\varepsilon} \setminus K_\varepsilon}$ such that

$$\Phi_{\delta,A^*}(x_0) = u_\varepsilon(x_0).$$

Since $A^* > R_0$ by assumption, it follows from (8.2) and the strict monotonicity of $\Phi_{\delta,A}$ with respect to A that $x_0 \in \overline{B_{R_\varepsilon} \setminus B_{R_0}}$. Let us set $e_0 = x_0/|x_0|$ and define the open affine half-space

$$H := \big\{x \in \mathbb{R}^N\,;\ x\cdot e_0 > R_0\big\}\ (\subset \mathbb{R}^N \setminus K_\varepsilon).$$

From (8.4) and the definition of Φ_{δ,A^*}, we have

$$u_\varepsilon(x) \geqslant \varphi(x) := \phi_\delta(x\cdot e_0 - A^*) \quad \text{for all } x \in \overline{\mathbb{R}^N \setminus K_\varepsilon}.$$

Reasoning as in Lemma 5.1 and recalling the assumptions on f_δ, we have that

$$\begin{cases} L_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0 & \text{in } \overline{H},\\ L_\varepsilon\varphi + f(\varphi) \geqslant 0 & \text{in } \overline{H} \ \ \text{(as in (5.3))},\\ u_\varepsilon \geqslant \varphi & \text{in } \overline{\mathbb{R}^N \setminus K_\varepsilon},\\ u_\varepsilon(x_0) = \varphi(x_0) & \text{with } x_0 \in \overline{H}. \end{cases}$$

Applying the strong maximum principle (Lemma 4.2) we obtain in particular that $u_\varepsilon = \varphi$ in $\overline{H}$. This is impossible since $u_\varepsilon(x) \to 1$ as $|x| \to +\infty$, while $\varphi < 1-\delta/2 < 1$ in $\mathbb{R}^N$. As a consequence, the claim (8.3) holds true.

From (8.3) and the monotonicity of $\Phi_{\delta,A}$ with respect to A, we then deduce that

$$\Phi_{\delta,R_0} \leqslant \Phi_{\delta,A^*} \leqslant u_\varepsilon \ \text{ in } \overline{\mathbb{R}^N \setminus K_\varepsilon}.$$

Since $\varepsilon \in (0,1]$ and $u_\varepsilon : \overline{\mathbb{R}^N \setminus K_\varepsilon} \to [0,1]$ solving (2.7) were arbitrary, since $R_0 > 0$ verifying (8.1) was independent of ε, and since $\phi_\delta(+\infty) = 1-\delta/2 > 1-\delta$, the desired conclusion follows. □

We are now ready to prove Theorem 2.6.

Proof of Theorem 2.6. — First of all, as in the proof of Lemma 8.1, it follows from Lemma 2.2 that, for every $\varepsilon \in (0,1]$, every measurable solution $u_\varepsilon : \mathbb{R}^N \setminus K_\varepsilon \to [0,1]$

of (2.7) can be identified with its Hölder continuous $C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K_\varepsilon})$ representative. Furthermore, Lemma 2.2 yields

$$[u_\varepsilon]_{C^{0,\alpha}(\overline{\mathbb{R}^N\setminus K_\varepsilon})} \leqslant A := \frac{2[J]_{B^\alpha_{1,\infty}(\mathbb{R}^N)}}{\inf\limits_{0<\eta\leqslant 1} \inf\limits_{x\in\mathbb{R}^N\setminus K_\eta} \|J(x-\cdot)\|_{L^1(\mathbb{R}^N\setminus K_\eta)} - \max\limits_{[0,1]} f'}.$$

Note that A is independent of ε. In particular, for every $\varepsilon_* \in (0,1]$ and every $R \geqslant R_0$, where $R_0 > 0$ is chosen as in (8.1), the family $(u_\varepsilon)_{0<\varepsilon\leqslant\varepsilon_*}$ is uniformly bounded in $C^{0,\alpha}(\overline{B_R \setminus K_{\varepsilon_*}})$. Recalling that $K_\varepsilon \to K$ as $\varepsilon \to 0^+$ in the $C^{0,\alpha}$ sense, there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \in (0,1]$ converging to 0^+ and a function $u_0 \in C^{0,\alpha}(\overline{\mathbb{R}^N \setminus K})$ such that, for all $R \geqslant R_0$ and $\beta \in (0,\alpha)$,

$$\|u_{\varepsilon_j} - u_0\|_{C^{0,\beta}(\overline{B_R\setminus K_{\varepsilon_j}})} \to 0 \quad \text{as } j \to +\infty. \tag{8.5}$$

Notice that $0 \leqslant u_0 \leqslant 1$ in $\overline{\mathbb{R}^N \setminus K}$. By Lemma 8.1 we know that $u_\varepsilon(x) \to 1$ uniformly in $\varepsilon > 0$ as $|x| \to +\infty$. Consequently,

$$u_0(x) \to 1 \quad \text{as } |x| \to +\infty. \tag{8.6}$$

Now, we claim that

$$Lu_0(x) + f(u_0(x)) = 0 \quad \text{in } \overline{\mathbb{R}^N \setminus K}, \tag{8.7}$$

where L is given by (1.2). This can be seen as follows. First, fix x in the open set $\mathbb{R}^N \setminus K$ and an integer j_0 large enough such that $x \in \mathbb{R}^N \setminus K_{\varepsilon_j}$ for all $j \geqslant j_0$. Notice that $f(u_{\varepsilon_j}(x)) \to f(u_0(x))$ as $j \to +\infty$ since f is continuous. Next, for all $j \geqslant j_0$ we have

$$\begin{aligned} L_{\varepsilon_j} u_{\varepsilon_j}(x) - Lu_0(x) &= \int_{\mathbb{R}^N\setminus K_{\varepsilon_j}} J(x-y)\big[(u_{\varepsilon_j}-u_0)(y) - (u_{\varepsilon_j}-u_0)(x)\big]\mathrm{d}y \\ &\quad - \int_{K_{\varepsilon_j}\setminus K} J(x-y)(u_0(y)-u_0(x))\mathrm{d}y. \end{aligned}$$

For every $R \geqslant R_0$ and $j \geqslant j_0$, there holds

$$\begin{aligned} |L_{\varepsilon_j} u_{\varepsilon_j}(x) - Lu_0(x)| &\leqslant 2\int_{K_{\varepsilon_j}\setminus K} J(x-y)\,\mathrm{d}y + 2\int_{\mathbb{R}^N\setminus B_R} J(x-y)\,\mathrm{d}y \\ &\quad + \|u_{\varepsilon_j} - u_0\|_{L^\infty(B_R\setminus K_{\varepsilon_j})} + |u_{\varepsilon_j}(x) - u_0(x)|. \end{aligned}$$

Since $K_{\varepsilon_j} \to K$ in the $C^{0,\alpha}$ sense and $J \in L^1(\mathbb{R}^N)$, we have in particular that the first term in the right-hand side converges to 0 as $j \to +\infty$. Recalling (8.5) and letting first $j \to +\infty$ and then $R \to +\infty$, we find that

$$L_{\varepsilon_j} u_{\varepsilon_j}(x) - Lu_0(x) \to 0 \;\text{ as } j \to +\infty.$$

Therefore, (8.7) holds for all $x \in \mathbb{R}^N \setminus K$ and finally for all $x \in \overline{\mathbb{R}^N \setminus K}$ by continuity and boundedness of u_0 in $\overline{\mathbb{R}^N \setminus K}$.

Remember now that $u_0 \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$. By (8.6), (8.7) and Theorem 2.2, we infer that $u_0 = 1$ in $\overline{\mathbb{R}^N \setminus K}$. This also shows that the limit of the functions u_{ε_j} is unique and, hence, $u_\varepsilon \to 1$ as $\varepsilon \to 0^+$ in the sense of (8.5), not only along a subsequence.

We conclude by contradiction. Suppose then that there exists countably infinitely many numbers in $(0,1]$, which we label in decreasing order as $(\varepsilon_j)_{j\in\mathbb{N}}$, such that $\varepsilon_j \to 0^+$ as $j \to +\infty$ and

$$\forall\, j \in \mathbb{N}, \quad \exists\, x_j \in \overline{\mathbb{R}^N \setminus K_{\varepsilon_j}}, \quad u_{\varepsilon_j}(x_j) = \min_{\overline{\mathbb{R}^N \setminus K_{\varepsilon_j}}} u_{\varepsilon_j} < 1. \tag{8.8}$$

Note that this makes sense since, without loss of generality, we have identified the functions u_{ε_j} with their continuous representatives in $\overline{\mathbb{R}^N \setminus K_{\varepsilon_j}}$. We observe that (1.11), (8.8) and Proposition 2.1 yield that

$$u_{\varepsilon_j}(x_j) < \theta \quad \text{for all } j \in \mathbb{N}.$$

Now, since the functions u_{ε_j} converge uniformly to 1 as $|x| \to +\infty$ (by Lemma 8.1), the sequence $(x_j)_{j\in\mathbb{N}}$ is bounded. Hence, up to extraction of a subsequence, we may assume that $x_j \to \bar{x}$ as $j \to +\infty$, for some $\bar{x} \in \overline{\mathbb{R}^N \setminus K}$. Furthermore, since the functions u_{ε_j} converge to $u_0 \equiv 1$ as $j \to +\infty$ in the sense of (8.5), we obtain that

$$1 > \theta > u_{\varepsilon_j}(x_j) \underset{j\to+\infty}{\longrightarrow} u_0(\bar{x}) = 1.$$

This is a contradiction. Therefore, there exists an $\varepsilon_0 \in (0,1]$ such that $u_\varepsilon = 1$ in $\overline{\mathbb{R}^N \setminus K_\varepsilon}$ for every $\varepsilon \in (0, \varepsilon_0)$ and for every measurable solution $u_\varepsilon : \mathbb{R}^N \setminus K_\varepsilon \to [0,1]$ of (2.7) (after identification with its continuous representative). The proof of Theorem 2.6 is thereby complete. □

Chapter 4

A counterexample to the Liouville property of some nonlocal problems

This chapter is inspired by the paper [30] written in collaboration with J. Coville.

1 Introduction

1.1 A nonlocal problem in heterogeneous media

Let K be a compact set of $\mathbb{R}^N$ with $N \geqslant 2$, and let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^N$. We are interested in the qualitative properties of positive solutions u to the following problem

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ 0 \leqslant u \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty, \end{cases} \tag{1.1}$$

where f is a bistable nonlinearity with $f(0) = f(1) = 0$ and L is the nonlocal operator

$$Lu(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)(u(y) - u(x))\mathrm{d}y, \tag{1.2}$$

with $J \in L^1(\mathbb{R}^N)$ a non-negative kernel with unit mass. The precise assumptions on f and J will be given later on.

This type of model naturally arises in the study of the behavior of particles evolving in a heterogeneous medium. The typical kind of problem we have in mind comes from population dynamics. In this setting, the movement of the individuals is modelled by a stochastic process that is defined in a domain that possesses several inaccessible regions (reflecting the heterogeneity of the environment). At the macroscopic level, the corresponding density of population $u(t,x)$ satisfies a reaction-diffusion equation that is defined outside a set K, which acts as an obstacle. When the individuals follow isotropic Poisson jump processes, this reaction-diffusion equation is given by

$$\frac{\partial u}{\partial t} = Lu + f(u) \quad \text{in } \mathbb{R} \times \mathbb{R}^N \setminus K, \tag{1.3}$$

and the solutions to (1.1) are particular stationary solutions to (1.3).

In recent years, much attention has been paid to the case of Brownian diffusion. In this situation, the reaction-diffusion equation takes the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \mathbb{R} \times \mathbb{R}^N \setminus K, \\ \nabla u \cdot \nu = 0 & \text{on } \mathbb{R} \times \partial K. \end{cases} \tag{1.4}$$

This problem was first studied by Berestycki, Hamel and Matano in [17]. There, it is shown that there exists a solution to (1.4) that satisfies $0 < u(t,x) < 1$ for all $(t,x) \in \mathbb{R} \times \overline{\mathbb{R}^N \setminus K}$, as well as a classical solution, u_∞, to

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \nabla u_\infty \cdot \nu = 0 & \text{on } \partial K, \\ 0 \leqslant u_\infty \leqslant 1 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u_\infty(x) \to 1 & \text{as } |x| \to +\infty. \end{cases} \tag{1.5}$$

This latter solution is actually obtained as the large time limit of $u(t,x)$; more precisely:

$$u(t,x) \to u_\infty(x) \text{ as } t \to \infty, \text{ locally uniformly in } x \in \overline{\mathbb{R}^N \setminus K}.$$

In addition, they were able to classify the solutions u_∞ to (1.5) under some geometric assumptions on K. When the obstacle K is either starshaped of directionally convex (see [17, Definition 1.2]), they prove that the solutions to (1.5) are actually identically equal to 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$. This was further extended to more complex obstacles by Bouhours who showed a sort of "stability" of this Liouville type property with respect to small regular perturbations of the obstacle, see [26]. From the biological standpoint, this means that, after some large time, *the population tends to occupy the whole space*.

Yet, when the domain is no longer starshaped nor directionally convex but merely simply connected, it is shown in [17] that *this Liouville type property may fail.*

In other words, the geometry of the domain may force the population to diffuse heterogeneously in space, even after some large time.

It is expected that (1.1) and (1.5) share some common properties. In particular, some of the results obtained for (1.5) should, to some extent, hold true as well for (1.1).

Recently, Brasseur et al. [31] have shown that (1.1) enjoys a similar Liouville type property when K is convex (or close to being convex) and when the data f and J satisfy some rather mild assumptions. That is, any solution u to (1.1) is identically equal to 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$. They also point out that this cannot be expected for general obstacles since one can easily find counterexamples when K is no longer simply connected. Indeed, take for instance $K = \overline{\mathscr{A}(1,2)} = \overline{B_2} \setminus B_1$ and suppose that J is supported in $B_{1/2}$. Then, the function u defined by

$$u(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^N \setminus B_2, \\ 0 & \text{if } x \in \overline{B_1}, \end{cases}$$

is a continuous solution to (1.1); yet, u is not identically 1 in the whole set $\overline{\mathbb{R}^N \setminus K}$. In view of this, it is natural to ask:

> *what are the optimal geometric assumptions on K ensuring that* (1.1) *enjoys such a Liouville property?*

So far, this question remains open.

In this chapter, our main concern is to find out whether it is possible to construct a *nontrivial simply connected* obstacle K, as well as data f and J, for which (1.1) has a continuous solution u which is not identically equal to 1.

Note that this is actually a quite reasonable question. Indeed, since the Liouville property does not hold true on annuli it is quite natural to expect counterexamples on simply connected obstacles which are "ε-close" to an annulus. We will see that this is indeed the case. Precisely, we will construct a family of simply connected compact sets K_ε and data f_ε and J_ε for which the solution to (1.1) need not be identically equal to 1.

1.2 Main results

Before we state our main results, let us first specify the assumptions made all along this chapter. We will assume that J is such that

$$\begin{cases} J \in L^1(\mathbb{R}^N) \text{ is a non-negative, radially symmetric kernel with unit mass,} \\ \text{there are } 0 \leqslant r_1 < r_2 \text{ such that } J(x) > 0 \text{ for a.e. } x \text{ with } r_1 < |x| < r_2, \\ M_1(J) := \displaystyle\int_{\mathbb{R}^N} J(x)|x| \mathrm{d}x < +\infty \text{ and } J \in W^{1,1}(\mathbb{R}^N), \end{cases} \tag{1.6}$$

and that $f \in C^1([0,1])$ is a "bistable" nonlinearity, namely

$$\begin{cases} \exists\, \theta \in (0,1), \quad f(0) = f(\theta) = f(1) = 0, \quad f < 0 \text{ in } (0,\theta), \quad f > 0 \text{ in } (\theta,1), \\ \displaystyle\int_0^1 f(s)\mathrm{d}s > 0, \quad f'(0) < 0, \quad f'(\theta) > 0, \quad f'(1) < 0, \quad f' < 1 \text{ in } [0,1]. \end{cases} \tag{1.7}$$

Our first result reads as follows

Theorem 1.1. — *Let $N \geqslant 2$. Then, there are smooth (non-starshaped) simply connected compact obstacles K and data f and J satisfying* (1.6) *and* (1.7) *for which problem* (1.1) *has a positive nonconstant solution $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$.*

The obstacles constructed in Theorem 1.1 are almost of the same nature as those given in [17] for the local case. Namely, we consider an annulus $\mathscr{A}$ into which a small channel is pierced, see Figure 4.1 below for a visual illustration.

By contrast with the classical reaction-diffusion, the operator L does *not* enjoy strong compactness properties and has no regularising effects. So our construction is *not* a simple adaptation of the techniques of proof used for the local problem (1.5). One of the novelties of this chapter is that we show how to circumvent these issues. As we shall explain in the sequel, our argument is in fact general enough to recover the local problem as a limit case (see our remarks below).

Let us briefly describe our approach. Our strategy relies essentially on two ingredients. First, we take advantage of the fact that the kernel J and the nonlinearity f may be chosen at our convenience. That is, instead of considering the problem (1.1), we can consider a rescaled version of (1.1) given an appropriate choice of J. In our setting, J will be such that

$$\mathrm{supp}(J) = B_r \text{ for some } r > 0, \text{ and } J \in L^2(\mathbb{R}^N) \text{ is radially } \textit{non-increasing}. \tag{1.8}$$

Then, given a small parameter ε, we look for a nonconstant positive solution u_ε to

$$\int_{\mathbb{R}^N \setminus K} J_\varepsilon(x-y)(u_\varepsilon(y) - u_\varepsilon(x))\mathrm{d}y + f_\varepsilon(u_\varepsilon(x)) = 0 \quad \text{in } \overline{\mathbb{R}^N \setminus K}, \tag{1.9}$$

that further satisfies $0 \leqslant u_\varepsilon \leqslant 1$ in $\overline{\mathbb{R}^N \setminus K}$ and $u_\varepsilon(x) \to 1$ as $|x| \to +\infty$, where

$$f_\varepsilon(s) := \varepsilon^2 f(s) \quad \text{and} \quad J_\varepsilon(z) = \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right).$$

In order to prove Theorem 1.1, we only need to show that, for some $\varepsilon > 0$, there is some obstacle K_ε such that (1.9) admits a positive nonconstant solution u_ε.

Second, we consider a well-chosen family of smooth simply connected obstacles $(K_\varepsilon)_{0<\varepsilon<1}$ that look like an annulus with a tiny channel of diameter of the order of $\varepsilon^{N/(N-1)}$ pierced in it (see the Figure 4.1). Given such a family, we prove that, for ε small enough, (1.9) indeed admits a positive nonconstant continuous solution. More precisely, we prove the following

Theorem 1.2. — *Let $N \geqslant 2$. Let J and f be such that* (1.6), (1.7) *and* (1.8) *hold. Then, there exist $\varepsilon_* > 0$ and a family of smooth simply connected obstacles $(K_\varepsilon)_{0<\varepsilon<1} \subset \mathbb{R}^N$ such that, for all $0 < \varepsilon < \varepsilon^*$, there is a positive nonconstant solution $u_\varepsilon \in C(\overline{\mathbb{R}^N \setminus K_\varepsilon}, [0,1])$ to* (1.9).

Due to the lack of a strong regularising property of (1.9), the construction of u_ε relies essentially on elementary arguments. In particular, we obtain a solution u_ε to (1.9) using an adequate monotone iterative scheme and elementary estimates. The main difficulty in our proof lies in the construction of an adequate pair of ordered continuous sub- and super-solution in a context where the equation (1.1) does not allow the use of traditional schemes based on compactness arguments. To cope with this major difficulty, we make a detailed construction of the obstacle K_ε and design it in such a way that we still can obtain standard L^2-estimates by elementary means. This requires a detailed analysis of all the parameters involved at each steps of our construction, especially when we construct our super-solution. To construct our super-solution we rely on the fact that a solution u_ε to (1.9) satisfies in particular

$$\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)(u_\varepsilon(y)-u_\varepsilon(x))\mathrm{d}x + f(u_\varepsilon(x)) = 0, \tag{1.10}$$

and, from there, relying essentially on the Bourgain-Brezis-Mironescu characterisation of Sobolev spaces (see e.g. [27, 118]), we can interpret the first term on the left-hand side as a *nonlocal approximation* of Δu in the sense that its energy approximates the L^2-variation of u. This, in turn, with a pertinent choice of K_ε and a well-chosen auxiliary problem, allows one to derive *a priori* bounds to construct a super-solution by means of variational methods.

A striking consequence of our construction is that it adapts almost straightforwardly to other situations. For example, it applies to the standard reaction-diffusion equation (1.5) providing so an alternative proof of the existence of a counterexample. But it also extends to broader classes of nonlocal operators where the dispersal process need not be isotropic but instead depends on the geodesic distance between points in $\overline{\mathbb{R}^N \setminus K}$. Indeed, our proof also adapts (with almost no changes) to operators of the form

$$L_{\mathrm{g}}u(x) := \int_{\mathbb{R}^N \setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))(u(y)-u(x))\mathrm{d}y, \tag{1.11}$$

where $d_{\mathrm{g}}(\cdot,\cdot)$ is the geodesic distance on $\overline{\mathbb{R}^N \setminus K}$ and $\widetilde{J} \in L^1_{\mathrm{loc}}(0,\infty)$ is such that

$$\sup_{x\in\mathbb{R}^N \setminus K} \int_{\mathbb{R}^N \setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))\mathrm{d}y < \infty, \tag{1.12}$$

and $z \mapsto \widetilde{J}(|z|)$ satisfies (1.6). More precisely, we have

Theorem 1.3. — *Let $N \geqslant 2$. Then, there are smooth (non-starshaped) simply connected compact obstacles K and data f and $\widetilde{J}$ satisfying* (1.7) *and* (1.12) *for which the problem*

$$\begin{cases} L_{\mathrm{g}} u + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ u(x) \to 1 & \text{as } |x| \to +\infty, \end{cases} \tag{1.13}$$

has a solution $u \in C(\overline{\mathbb{R}^N \setminus K}, [0,1])$ which is not identically equal to 1 *in $\overline{\mathbb{R}^N \setminus K}$.*

The obstacle K and the data f and $J\,(=\widetilde{J}(|\cdot|))$ constructed at Theorem 1.3 are exactly the same as in Theorem 1.1.

Problem (1.13) is of interest in its own right. It gives an alternative way to describe the evolution of particles within a perforated domain which, in some situations, may be regarded as more realistic. The point here is that particles *cannot* travel through K (as is it the case for problem (1.1)). Instead, they are compelled to "bypass" K as if it was a material obstacle. This particularity may be helpful to study the dynamics of some species (such as worms or spores) for which this behavior is well-suited.

When needed we will state in side remarks the necessary changes to make to the proofs in order to handle this type of dispersal processes.

Remark 1.4. — It turns out that the techniques of proof used in [31] to establish the Liouville property of (1.1) for convex domains also apply to this modified setting (at least when J is non-increasing), but we leave this to a subsequent paper.

The chapter is organized as follows. After describing our notations, we recall some results from the literature in Section 2. In Section 3, given a pair (J, f) we construct an adequate family of obstacles. Then, in Section 4, we construct some particular super-solutions to the problem (1.9). Finally, in Section 5, we use the super-solution constructed at Section 4 to prove Theorem 1.2.

Notations

Let us list a few notations that will be used throughout the chapter.

As usual, $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$ and $B_R(x)$ the open Euclidean ball of radius $R > 0$ centred at $x \in \mathbb{R}^N$ (when $x = 0$, we simply write B_R). We denote by $\mathscr{A}(R_1, R_2)$ the open annulus $B_{R_2} \setminus \overline{B_{R_1}}$.

For a compact set $\Omega \subset \mathbb{R}^N$, we denote by $\operatorname{diam}(\Omega)$ its diameter, given by

$$\operatorname{diam}(\Omega) := \sup_{x,y \in \Omega} |x - y|.$$

The N-dimensional Hausdorff measure will be denoted by $\mathscr{H}^N$. For a measurable set $E \subset \mathbb{R}^N$, we denote by $|E|$ its Lebesgue measure and by $\mathbb{1}_E$ its characteristic function. If $0 < |E| < \infty$ and if $g : \mathbb{R}^N \to \mathbb{R}$ is locally integrable, we denote by

$$\fint_E g(x)\mathrm{d}x = \frac{1}{|E|} \int_E g(x)\mathrm{d}x,$$

the average of g in the set E. Also, we denote by $L^p(E)$, $1 \leqslant p \leqslant \infty$, the Lebesgue space of (equivalence classes of) measurable functions g for which the p-th power of the absolute value is Lebesgue integrable when $p < \infty$ (resp. essentially bounded when $p = \infty$).

2 Preliminaries

In this section, we recall some known results that will be used throughout the chapter. In most cases, we will omit their proofs and point the interested reader to the relevant references.

We first state a general existence result.

Lemma 2.1. — *Assume that f and J satisfy* (1.6) *and* (1.7). *Let $K \subset \mathbb{R}^N$ be a compact set and let $\underline{u}, \overline{u} \in C(\mathbb{R}^N \setminus K)$ be such that*

$$\begin{cases} L\overline{u} + f(\overline{u}) \leqslant 0 & \text{in } \mathbb{R}^N \setminus K, \\ L\underline{u} + f(\underline{u}) \geqslant 0 & \text{in } \mathbb{R}^N \setminus K. \end{cases}$$

Assume, in addition, that

$$\limsup_{|x|\to\infty} \underline{u}(x) = \lim_{|x|\to\infty} \overline{u}(x) = 1, \tag{2.1}$$

and that

$$0 \leqslant \underline{u} \leqslant \overline{u} \leqslant 1 \quad \text{in } \mathbb{R}^N \setminus K. \tag{2.2}$$

Then, there exists $u \in L^\infty(\mathbb{R}^N \setminus K)$ such that

$$\begin{cases} Lu + f(u) = 0 & \text{in } \mathbb{R}^N \setminus K, \\ \underline{u} \leqslant u \leqslant \overline{u} & \text{in } \mathbb{R}^N \setminus K. \end{cases}$$

Although the proof of Lemma 2.1 relies on rather standard arguments it is not straightforward. For this reason, we will give a detailed proof (which is postponed to the Appendix at the end of the chapter).

Next, we recall a regularity result for nonlocal equations of the form

$$\int_{\Omega\setminus K} J(x-y)u(y)\mathrm{d}y - \mathscr{J}(x)u(x) + f(u(x)) = 0 \quad \text{in } \Omega \setminus K, \tag{2.3}$$

where

$$\mathscr{J}(x) := \int_{\mathbb{R}^N \setminus K} J(x-y)\mathrm{d}y. \tag{2.4}$$

Precisely,

LEMMA 2.2. — *Assume that $f \in C^1([0,1])$ and that J satisfies (1.6). Let $\Omega \subset \mathbb{R}^N$ be an open set having C^1 boundary. Suppose that $K \subset \Omega$ is a compact set and that*

$$\max_{[0,1]} f' < \inf_{\Omega \setminus K} \mathscr{J}. \tag{2.5}$$

Let $u \in L^\infty(\Omega \setminus K, [0,1])$ be a solution to (2.3) a.e. in $\Omega \setminus K$. Then, u can be redefined up to a negligible set and extended as a uniformly continuous function in $\overline{\Omega \setminus K}$.

For a detailed proof, we refer to [31, Lemma 3.2] (see also [13, 19]).

Remark 2.3. — Note that Ω need not be bounded. In particular, Lemma 2.2 holds when $\Omega = \mathbb{R}^N$.

Finally, we recall the following result

LEMMA 2.4. — *Let $K \subset \mathbb{R}^N$ is a compact set and suppose that f and J satisfy (1.6) and (1.7). Assume further that J is compactly supported and that $J \in L^2(\mathbb{R}^N)$. Let $u \in C(\mathbb{R}^N \setminus K, [0,1])$ be a solution to*

$$\begin{cases} Lu + f(u) = 0 & \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \sup\limits_{\mathbb{R}^N \setminus K} u = 1, \end{cases} \tag{2.6}$$

Then, $u(x) \to 1$ as $|x| \to \infty$.

The proof may be found in [31, Lemma 7.2].

Remark 2.5. — The above results still hold when $J(x-y)$ is replaced by $\widetilde{J}(d_{\mathrm{g}}(x,y))$. For the validity of Lemma 2.1 in this case, we refer to Remark 5.4 in the Appendix. On the other hand, a careful inspection of the proof of [31, Lemma 3.2] shows that the condition (2.6) with $\mathscr{J}$ replaced by

$$\widetilde{\mathscr{J}}(x) := \int_{\mathbb{R}^N \setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))\mathrm{d}y, \tag{2.7}$$

still implies the continuity of solutions to

$$\int_{\Omega \setminus K} \widetilde{J}(d_{\mathrm{g}}(x,y))u(y)\mathrm{d}y - \widetilde{\mathscr{J}}(x)u(x) + f(u(x)) = 0,$$

in $\overline{\Omega \setminus K}$. Similarly, Lemma 2.4 holds as well with L_{g} (as given by (1.11)) instead of L since its proof requires only estimates on convex regions on which it trivially holds that $d_{\mathrm{g}}(x,y) = |x-y|$.

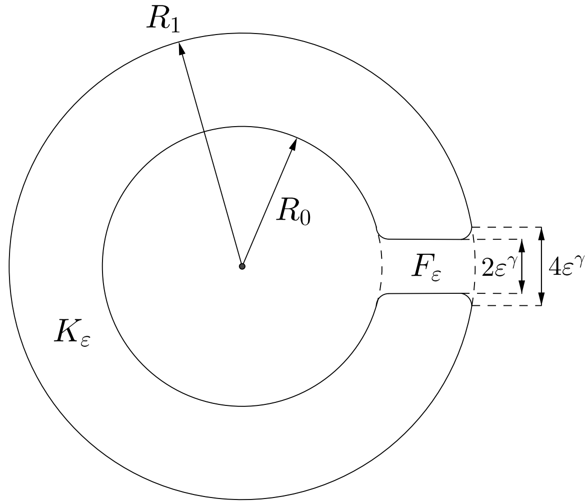

Figure 4.1. Illustration of K_ε in dimension 2.

3 Construction of a family of obstacles

This section is devoted to the construction of an appropriate family of obstacles $(K_\varepsilon)_{0<\varepsilon<1}$. Our construction will depend on the interplay with the datum (J, f). As mentioned in the introduction, we will assume that J satisfies (1.6) and (1.8) and that f satisfies (1.7). However, before constructing $(K_\varepsilon)_{0<\varepsilon<1}$, we need to define some important quantities depending on f and J. We will call $C_0 > 0$ and $M_2(J) > 0$ the constants respectively defined by

$$\begin{cases} C_0 := \max_{s\in[0,1]} f(s), & (3.1) \\ M_2(J) := \int_{\mathbb{R}^N} J(z)|z|^2 \mathrm{d}z. & (3.2) \end{cases}$$

Note that the assumptions (1.6) and (1.7) guarantee that these two numbers are well-defined. Furthermore, we introduce two quantities, $C_{N,J}$ and R_0^*, respectively defined by

$$\begin{cases} C_{N,J} := \dfrac{\pi^2 M_2(J)}{32N}, & (3.3) \\ R_0^*(J,f) := \sqrt{\dfrac{\theta C_{N,J}}{5C_0}}. & (3.4) \end{cases}$$

Let us now start the construction of the obstacle. Fix some $R_1 > 2$ and let $0 < R_0 < R_0^*(J,f)$ (where $R_0^*(J,f)$ is as in (3.4)). Let $0 < \varepsilon < 1$ be a small parameter and set $\gamma := \frac{N}{N-1}$. We call $\mathscr{A}$ the annulus $\mathscr{A} := \mathscr{A}(R_0, R_1)$ and we consider a smooth compact simply connected set $K_\varepsilon \subset \overline{\mathscr{A}}$ satisfying the following properties:

(i) $\overline{\mathscr{A}} \cap \{x \in \mathbb{R}^N\,;\ x_1 \leqslant 0\} \subset K_\varepsilon$,

(ii) $\overline{\mathscr{A}} \cap \{x \in \mathbb{R}^N\,;\ x_1 > 0, |x'| > 2\varepsilon^\gamma\} \subset K_\varepsilon$,

(iii) $K_\varepsilon \subset \big(\overline{\mathscr{A}} \cap \{x \in \mathbb{R}^N\,;\ x_1 \leqslant 0\}\big) \cup \big(\overline{\mathscr{A}} \cap \{x \in \mathbb{R}^N\,;\ x_1 > 0, |x'| \geqslant \varepsilon^\gamma\}\big)$,

(iv) $\mathscr{A}(R_0 + \varepsilon^\gamma/4, R_1 - \varepsilon^\gamma/4) \cap \{x \in \mathbb{R}^N; x_1 > 0, |x'| \geqslant \varepsilon^\gamma\} \subset K_\varepsilon$,

where $x = (x_1, x')$ and $x' = (x_2, \ldots, x_N)$ (see Figure 4.1).

Furthermore, we define the following open set:

$$F_\varepsilon := \mathscr{A} \setminus K_\varepsilon.$$

We will refer to $(K_\varepsilon)_{0<\varepsilon<1}$ as the *family of obstacles associated to the pair* (J, f).

Let us also list in this section a preparatory lemma.

Proposition 3.1. — *Let $N \geqslant 2$ and let (J, f) be a pair satisfying* (1.7) *and* (1.8). *Let $(K_\varepsilon)_{0<\varepsilon<1}$ be the family of obstacles associated to the pair (J, f). Let*

$$f_\varepsilon(s) := \varepsilon^2 f(s) \ \ \text{and} \ \ J_\varepsilon(z) := \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right). \tag{3.5}$$

Then, there exists some $\varepsilon_0 > 0$ depending only on N, R_0, J and f', such that

$$\max_{[0,1]} f'_\varepsilon < \inf_{x \in \mathbb{R}^N \setminus K_\varepsilon} \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}x, \ \text{for all } \varepsilon \in (0, \varepsilon_0). \tag{3.6}$$

Proposition 3.1 will play an important role in the sequel. Inter alia, it guarantees that the solutions of some nonlocal equations defined in the sequel are *continuous*.

Proof. — By assumption (1.8), up to rescale J, we may assume without loss of generality that $\mathrm{supp}(J) = B_{1/2}$. Now, let $0 < \varepsilon < \varepsilon_1 := \min\{1, R_0/2\}$, $x \in \mathbb{R}^N \setminus K_\varepsilon$ and define

$$\widetilde{F}_\varepsilon := \{z \in \mathbb{R}^N; R_0 < z_1 < R_1,\ |z'| < \varepsilon^\gamma\} \text{ and } \Lambda_\varepsilon(x) := B_{\varepsilon/2} \cap (\widetilde{F}_\varepsilon - x).$$

We will estimate from below the integral in the right-hand side of (3.6). For it, we will treat separately the case where $x \in \widetilde{F}_\varepsilon$ and the case where $x \in \mathbb{R}^N \setminus (K_\varepsilon \cup \widetilde{F}_\varepsilon)$.

Step 1: Lower bound in $\widetilde{F}_\varepsilon$

Let $x \in \widetilde{F}_\varepsilon$. Since J_ε is radially non-increasing, non-negative and supported in $B_{\varepsilon/2}$, there is some $\widetilde{J}_\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $J_\varepsilon(z) = \widetilde{J}_\varepsilon(|z|)$ and $\mathrm{supp}(\widetilde{J}_\varepsilon) = [0, \varepsilon/2)$.

Thus, passing to polar coordinates, the mass carried by $J_\varepsilon(x-\cdot)$ in $\widetilde{F}_\varepsilon$ can be written as

$$\int_{\widetilde{F}_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y = \int_{\Lambda_\varepsilon(x)} J_\varepsilon(y)\mathrm{d}y = \int_{\mathbb{S}^{N-1}} \left(\int_0^{\varepsilon/2} \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t)\widetilde{J}_\varepsilon(t)t^{N-1}\mathrm{d}t\right) \mathrm{d}\mathscr{H}^{N-1}(\sigma).$$

Notice that $\Lambda_\varepsilon(x)$ is a convex set and that $0 \in \Lambda_\varepsilon(x)$. In particular, both $t \mapsto \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t)$ and $t \mapsto \widetilde{J}_\varepsilon(t)$ are non-increasing functions. Hence, using Chebyshev's integral inequality (see e.g. [109, Theorem 2.5.10, p.40]), we have

$$\int_{\widetilde{F}_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \frac{N}{(\varepsilon/2)^N}\int_{\mathbb{S}^{N-1}} \left(\int_0^{\varepsilon/2} \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t)t^{N-1}\mathrm{d}t\int_0^{\varepsilon/2} \widetilde{J}_\varepsilon(t)t^{N-1}\mathrm{d}t\right) \mathrm{d}\mathscr{H}^{N-1}(\sigma).$$

Since J_ε has unit mass and $\mathrm{supp}(J_\varepsilon) = B_{\varepsilon/2}$, one has

$$\int_0^{\varepsilon/2} \widetilde{J}_\varepsilon(t)t^{N-1}\mathrm{d}t = \sigma_N^{-1} = \big(N|B_1|\big)^{-1},$$

where $\sigma_N = \mathscr{H}^{N-1}(\mathbb{S}^{N-1})$. Ergo,

$$\begin{aligned}\int_{\widetilde{F}_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y &\geqslant \frac{1}{|B_{\varepsilon/2}|}\int_{\mathbb{S}^{N-1}} \left(\int_0^{\varepsilon/2} \mathbb{1}_{\Lambda_\varepsilon(x)}(\sigma t)t^{N-1}\mathrm{d}t\right) \mathrm{d}\mathscr{H}^{N-1}(\sigma)\\ &= \frac{1}{|B_{\varepsilon/2}|}\int_{B_{\varepsilon/2}} \mathbb{1}_{\Lambda_\varepsilon(x)}(y)\mathrm{d}y.\end{aligned}$$

Since $\widetilde{F}_\varepsilon \subset \mathbb{R}^N \setminus K_\varepsilon$ and $\Lambda_\varepsilon(x) = B_{\varepsilon/2} \cap (\widetilde{F}_\varepsilon - x)$, we get

$$\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \frac{|B_{\varepsilon/2}(x)\cap \widetilde{F}_\varepsilon|}{|B_{\varepsilon/2}|}, \text{ for any } x \in \widetilde{F}_\varepsilon. \tag{3.7}$$

Let us now estimate the quantity $|B_{\varepsilon/2}(x) \cap \widetilde{F}_\varepsilon|$. Observe that for ε small enough, say when $0 < \varepsilon < \varepsilon_2 := 4^{-(N-1)}$, one has $\varepsilon/2 > 2\varepsilon^\gamma$. In particular, this implies that $B_{\varepsilon/2}(x) \cap \widetilde{F}_\varepsilon$ always contains an hyper-rectangle of the form $\mathscr{T}\big((0,\varepsilon/4)\times(0,2\varepsilon^\gamma)\times \cdots \times (0,2\varepsilon^\gamma)\big)$ for some translation $\mathscr{T}$ of $\mathbb{R}^N$, so that

$$|B_{\varepsilon/2}(x)\cap \widetilde{F}_\varepsilon| \geqslant (\varepsilon/4)\times(2\varepsilon^\gamma)^{N-1} = 2^{N-3}\varepsilon^{N+1}.$$

Therefore, recalling (3.7), we obtain that, for all $0 < \varepsilon < \varepsilon_2$ and all $x \in \widetilde{F}_\varepsilon$, it holds

$$\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant C_1\varepsilon, \tag{3.8}$$

for some $C_1 > 0$ depending on N only.

Step 2: Lower bound in $\mathbb{R}^N \setminus (K_\varepsilon \cup \widetilde{F}_\varepsilon)$

Let us now consider the case where $x \in \mathbb{R}^N \setminus (K_\varepsilon \cup \widetilde{F}_\varepsilon)$. For it, we first note that, since $0 < \varepsilon^\gamma < \varepsilon < R_0/2$ (remember $0 < \varepsilon < \varepsilon_1$), the point $x_0 := (R_0, \varepsilon^\gamma, 0, \cdots, 0) \in \partial \widetilde{F}_\varepsilon$ satisfies

$$|x_0|^2 = R_0^2 + \varepsilon^{2\gamma} < R_0^2 + R_0\varepsilon^\gamma/2 < (R_0 + \varepsilon^\gamma/4)^2,$$

which implies that $\widetilde{F}_\varepsilon \cap B_{R_0+\varepsilon^\gamma/4} \neq \emptyset$. On the other hand, it is clear from the definition of $\widetilde{F}_\varepsilon$ that $\widetilde{F}_\varepsilon \setminus B_{R_1-\varepsilon^\gamma/4} \neq \emptyset$. A consequence of this is that

$$\widetilde{F}_\varepsilon \cap \mathscr{A}(R_0+\varepsilon^\gamma/4, R_1-\varepsilon^\gamma/4) = \mathscr{A}(R_0+\varepsilon^\gamma/4, R_1-\varepsilon^\gamma/4) \cap \big\{z \in \mathbb{R}^N; z_1 > 0, |z'| < \varepsilon^\gamma\big\}.$$

Whence, recalling properties (i) and (iv) in the definition of K_ε, we deduce that

$$\mathscr{A}(R_0 + \varepsilon/4, R_1 - \varepsilon/4) \subset \mathscr{A}(R_0 + \varepsilon^\gamma/4, R_1 - \varepsilon^\gamma/4) \subset K_\varepsilon \cup \widetilde{F}_\varepsilon,$$

where, in the left-hand side, we have used the fact that $\varepsilon^\gamma < \varepsilon$. In turn, this implies that

$$x \in \mathbb{R}^N \setminus \mathscr{A}(R_0 + \varepsilon/4, R_1 - \varepsilon/4).$$

In particular, since $0 < \varepsilon < R_0/2 < R_0$, we may find a point $z \in \mathbb{R}^N$ such that

$$|x - z| = \frac{3\varepsilon}{8} \text{ and } B_{\varepsilon/8}(z) \subset B_{\varepsilon/2}(x) \setminus \overline{\mathscr{A}} \subset \mathbb{R}^N \setminus K_\varepsilon. \tag{3.9}$$

Indeed, when $x \in \mathbb{R}^N \setminus B_{R_1-\varepsilon/4}$, this follows from the convexity of B_{R_1}; and, when $x \in B_{R_0+\varepsilon/4}$, the constraint $0 < \varepsilon < R_0/2$ allows one to choose z on the diagonal of $B_{R_0+\varepsilon/4}$ containing x. On account of this, we may write

$$\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \int_{B_{\varepsilon/8}(z)} J_\varepsilon(x-y)\mathrm{d}y = \int_{B_{\varepsilon/8}(z-x)} J_\varepsilon(y)\mathrm{d}y = \int_{B_{1/8}\left(\frac{z-x}{\varepsilon}\right)} J(y)\mathrm{d}y.$$

Now, by (3.9), we have $(z - x)/\varepsilon \in \partial B_{3/8}$. Thus,

$$\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \int_{B_{1/8}(e_x)} J(y)\mathrm{d}y =: M_J(e_x) \text{ for some } e_x \in \partial B_{3/8}.$$

Notice that $B_{1/8}(e_x) \subset B_{1/2} = \mathrm{supp}(J)$ (because $e_x \in \partial B_{3/8}$) which implies $M_J(e_x) > 0$. Moreover, since J is radially symmetric, the quantity $M_J(e_x)$ does not depend on the choice of $e_x \in \partial B_{3/8}$, namely

$$M_J(e_x) = M_J(e) \equiv M_J > 0, \text{ for every } e \in \partial B_{3/8},$$

and some constant M_J depending on J only.

Therefore, for any $0 < \varepsilon < \varepsilon_1$ and $x \in \mathbb{R}^N \setminus (K_\varepsilon \cup \widetilde{F}_\varepsilon)$, it holds

$$\int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant M_J > 0. \tag{3.10}$$

Step 3: Conclusion

Since $\mathbb{R}^N \setminus K_\varepsilon = \widetilde{F}_\varepsilon \cup (\mathbb{R}^N \setminus (\widetilde{F}_\varepsilon \cup K_\varepsilon))$, by (3.8) and (3.10), we obtain

$$\inf_{x\in\mathbb{R}^N\setminus K_\varepsilon} \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \geqslant \min\{M_J, C_1\}\,\varepsilon,$$

for any $0 < \varepsilon < \varepsilon_3 := \min\{\varepsilon_1, \varepsilon_2\}$. Whence, letting

$$\varepsilon_0 := \min\left\{\varepsilon_3, \frac{\min\{M_J, C_1\}}{\max_{[0,1]} f'}\right\},$$

and recalling that $f_\varepsilon(s) = \varepsilon^2 f(s)$, we obtain

$$\max_{[0,1]} f_\varepsilon' < \inf_{x\in\mathbb{R}^N\setminus K_\varepsilon} \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}x \ \text{ for any } \varepsilon \in (0, \varepsilon_0),$$

which is the desired inequality. □

Remark 3.2. — Since J_ε is radially non-increasing and satisfies (1.6), there is some non-increasing $\widetilde{J}_\varepsilon \in L^1_{\mathrm{loc}}(0,\infty)$ satisfying $J_\varepsilon(z) = \widetilde{J}_\varepsilon(|z|)$. In particular, since $d_{\mathrm{g}}(x,y) \geqslant |x-y|$, it holds that

$$\sup_{x\in\mathbb{R}^N\setminus K_\varepsilon} \int_{\mathbb{R}^N \setminus K_\varepsilon} \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\mathrm{d}y \leqslant \sup_{x\in\mathbb{R}^N\setminus K_\varepsilon} \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\,\mathrm{d}y = 1,$$

thus implying that $\widetilde{J}_\varepsilon$ satisfies (1.12). Moreover, Proposition 3.1 still holds when $J_\varepsilon(x-y)$ is replaced by $\widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))$, i.e. we still have

$$\max_{[0,1]} f_\varepsilon' < \inf_{x\in\mathbb{R}^N\setminus K_\varepsilon} \int_{\mathbb{R}^N \setminus K_\varepsilon} \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\mathrm{d}y. \tag{3.11}$$

Indeed, this is because our proof reduces to estimate the mass carried by $J_\varepsilon(x-\cdot)$ on *convex* sub-domains of $\mathbb{R}^N \setminus K_\varepsilon$ and, in this case, the geodesic distance coincides with the Euclidean distance, namely it holds that $J_\varepsilon(x-y) = \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))$.

4 Construction of a global super-solution

In this section we construct a global super-solution to (1.9). Precisely, given a pair (J, f) satisfying (1.7) and (1.8) and given the family of obstacles $(K_\varepsilon)_{0<\varepsilon<1}$ associated to (J, f) (as defined in Section 3), we construct a global super-solution $\bar{u}_\varepsilon$ to

$$\int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)(\bar{u}_\varepsilon(y)-\bar{u}_\varepsilon(x))\mathrm{d}y + f_\varepsilon(\bar{u}_\varepsilon(x)) \leqslant 0 \quad \text{for } x \in \mathbb{R}^N \setminus K_\varepsilon, \tag{4.1}$$

that further satisfies

$$\bar{u}_\varepsilon \equiv 1 \quad \text{for } x \in \mathbb{R}^N \setminus B_R, \tag{4.2}$$

for some large $R > 0$, where f_ε and J_ε are as in (3.5). More precisely, we prove the following

Lemma 4.1. — *Let $N \geqslant 2$ and let (J, f) be a pair satisfying* (1.7) *and* (1.8). *Let $(K_\varepsilon)_{0<\varepsilon<1}$ be the family of obstacles associated to the pair (J, f) (as defined in Section 3). Let f_ε and J_ε be as in* (3.5). *Then, there exists $R^* > 0$ and $\varepsilon^* > 0$ such that, for all $0 < \varepsilon < \varepsilon^*$ and all $R \geqslant R^*$, there is a continuous positive nonconstant function $\bar{u}_\varepsilon$ satisfying* (4.1) *and* (4.2).

The proof of Lemma 4.1 follows essentially two steps. In the first step, we construct a positive solution to a suitable auxiliary problem defined in $B_R \setminus K_\varepsilon$ for some large R. Then, in a second step, we regularise this solution to obtain a super-solution that satisfies both (4.1) and (4.2). To simplify the presentation each step of the proof corresponds to a subsection.

4.1 An auxiliary problem in $B_R \setminus K_\varepsilon$

Let us first construct an adequate auxiliary problem. To do so, we define a new nonlinearity, $\widetilde{f}$, satisfying

$$\widetilde{f}(s) := \begin{cases} -\kappa s & \text{for } s \leqslant \frac{3\theta}{4}, \\ f_0(s) & \text{for } \frac{3\theta}{4} < s < \theta, \\ f(s) & \text{for } \theta \leqslant s \leqslant 1, \\ f'(1)(s-1) & \text{for } s > 1, \end{cases} \tag{4.3}$$

where $\theta \in (0, 1)$ is as in (1.7), $\kappa > 0$ is a small number and f_0 is a smooth function such that $\widetilde{f} \in C^1(\mathbb{R})$. From (1.7), we can choose $\kappa > 0$ and f_0 such that

$$f \leqslant \widetilde{f} \ \text{ in } [0,1], \ \max_{[0,1]} \widetilde{f}(s) = \max_{[0,1]} f \text{ and } \sup_{\mathbb{R}} \widetilde{f}' \leqslant \sup_{[0,1]} f'. \tag{4.4}$$

Now, for $R > R_1 + 2$, we let $L_{R,\varepsilon}$ be the operator given by

$$L_{R,\varepsilon}w(x) := \int_{B_R \setminus K_\varepsilon} J_\varepsilon(x-y)(w(y)-w(x))\mathrm{d}y, \tag{4.5}$$

and we consider the following problem

$$L_{R,\varepsilon}u_{\varepsilon,R}(x) + c_\varepsilon(x)(1-u_{\varepsilon,R}(x)) + \widetilde{f}_\varepsilon(u_{\varepsilon,R}(x)) = 0 \quad \text{for all } x \in \overline{B_R \setminus K_\varepsilon}, \tag{4.6}$$

where

$$\widetilde{f}_\varepsilon(s) = \varepsilon^2 \widetilde{f}(s) \text{ for } s \in \mathbb{R} \text{ and } c_\varepsilon(x) := \int_{\mathbb{R}^N \setminus B_R} J_\varepsilon(x-y)\mathrm{d}y \text{ for } x \in B_R \setminus K_\varepsilon. \tag{4.7}$$

Our goal in this step is to show that, for each $\varepsilon \in (0,1)$ small enough, there exists a continuous function $u_{\varepsilon,R} : \overline{B_R \setminus K_\varepsilon} \to (0,1)$ satisfying (4.6).

Remark 4.2. — Observe that, by construction (remember (4.4)), the function

$$\widehat{u}_{\varepsilon,R} := \begin{cases} u_{\varepsilon,R} & \text{in } \overline{B_R \setminus K}, \\ 1 & \text{in } \mathbb{R}^N \setminus \overline{B}_R, \end{cases}$$

provides a *discontinuous* super-solution to (4.1) satisfying (4.2). We are thus on the right track to construct the required super-solution.

For it, we observe that, by setting $v_{\varepsilon,R} := 1 - u_{\varepsilon,R}$, (4.6) rewrites

$$L_{R,\varepsilon}v_{\varepsilon,R}(x) - c_\varepsilon(x)v_{\varepsilon,R}(x) + g_\varepsilon(v_{\varepsilon,R}(x)) = 0 \quad \text{for } x \in \overline{B_R \setminus K_\varepsilon}, \tag{4.8}$$

with $g_\varepsilon(s) := -\varepsilon^2 \widetilde{f}(1-s)$. Therefore, to construct $u_{\varepsilon,R}$ it suffices to construct a positive solution $v_{\varepsilon,R} : \overline{B_R \setminus K} \to (0,1)$ to (4.8). As in [17], this will be done using a variational argument. To do so, we define

$$g(s) := -\widetilde{f}(1-s), \quad G(t) := \int_0^t g(s)\mathrm{d}s \quad \text{and} \quad G_\varepsilon(t) := \varepsilon^2 G(t),$$

for all $s, t \in \mathbb{R}$ and $\varepsilon \in (0,1)$. Now, for any $\varepsilon \in (0,1)$ and any domain $\Omega \subset B_R \setminus K_\varepsilon$, we consider the following energy functional

$$\mathscr{E}_{\varepsilon,\Omega}(w) := \frac{1}{4}\int_\Omega\int_\Omega J_\varepsilon(x-y)(w(x)-w(y))^2\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_\Omega c_\varepsilon(x)w^2(x)\mathrm{d}x - \int_\Omega G_\varepsilon(w(x))\mathrm{d}x, \tag{4.9}$$

for $w \in L^2(\Omega)$. Observe that for any $\varepsilon > 0$ and any domain $\Omega \subset B_R \setminus K_\varepsilon$, the null function $w \equiv 0$ is a *global* minimiser of $\mathscr{E}_{\varepsilon,\Omega}$. Therefore, we have to construct a *local* minimiser. However, unlike its local analogue, the energy functional $\mathscr{E}_{\varepsilon,\Omega}$ does *not* posses strong compactness properties, rendering this type of approach very delicate to implement.

With this in mind, we will show that, for the family K_ε constructed in Section 3 and ε small enough, the above energy has indeed a nontrivial local minimiser when $\Omega = B_R \setminus K_\varepsilon$.

Following the scheme of construction introduced in [17], we first show that the function $w_0 := \mathbb{1}_{B_{R_0}}$ is a strict minimiser of the functional $\mathscr{E}_{\varepsilon,B_{R_0}}$ when $\varepsilon \in (0,1)$ is small enough.

More precisely,

PROPOSITION 4.3. — *Let $N \geqslant 2$, $0 < R_0 < R_0^*(J,f)$ (where $R_0^*(J,f)$ is given by (3.4)) and let $w_0 := \mathbb{1}_{B_{R_0}}$. Then, there exists $\kappa_0 > 0$, $0 < \varepsilon_1(J,N,R_0) < 1$ and $0 < \delta_0(R_0) < |B_{R_0}|^{1/2}$ such that, for each $0 < \varepsilon < \varepsilon_1$, it holds that*

$$\mathscr{E}_{\varepsilon,B_{R_0}}(w) - \mathscr{E}_{\varepsilon,B_{R_0}}(w_0) \geqslant \kappa_0 \varepsilon^2 \|w - w_0\|^2_{L^2(B_{R_0})},$$

for all $w \in L^2(B_{R_0})$ such that $\|w - w_0\|_{L^2(B_{R_0})} \leqslant \delta_0$.

Proof. — Let us begin with some preliminary observations. First, we notice that since g is linear around 1 (because $\widetilde{f}$ is linear around 0), the function G_ε is smooth in a neighborhood of 1. In particular, there exists $\tau_0(\widetilde{f}) > 0$ such that

$$G_\varepsilon(t) = G_\varepsilon(1) + G_\varepsilon'(1)(t-1) + \frac{1}{2} G_\varepsilon''(1)(t-1)^2 \quad \text{for any } |t-1| < \tau_0.$$

But since $G_\varepsilon'(1) = \varepsilon^2 G'(1) = \varepsilon^2 g(1) = 0$ and $G_\varepsilon''(1) = \varepsilon^2 G''(1) = \varepsilon^2 g'(1) = -\varepsilon^2 \widetilde{f}'(0) = -\varepsilon^2 \kappa$, this expansion can be rewritten as

$$G_\varepsilon(t) = \varepsilon^2 G(1) - \frac{\kappa \varepsilon^2}{2}(t-1)^2 \quad \text{for any } |t-1| < \tau_0. \tag{4.10}$$

Using the number τ_0, we define

$$\delta_0 := \min\left\{\frac{\theta}{4}, \frac{C_0}{\kappa}, \frac{\tau_0}{2}\right\} |B_{R_0}|^{1/2}, \tag{4.11}$$

where θ, C_0 and κ are as in (1.7), (3.1) and (4.3); and we let $w \in L^2(B_{R_0})$ be any function such that

$$\|w - w_0\|_{L^2(B_{R_0})} \leqslant \delta_0. \tag{4.12}$$

Second, denoting by $\langle w_0 \rangle := \operatorname{span}_{L^2(B_{R_0})}(w_0)$ the vector space spanned by w_0 and letting $\langle w_0 \rangle^\perp$ be its orthogonal with respect to the standard scalar product of $L^2(B_{R_0})$, we can write the space $L^2(B_R)$ as the direct sum $L^2(B_{R_0}) = \langle w_0 \rangle \oplus \langle w_0 \rangle^{\perp}$. This means that we may always find a constant $\alpha \in \mathbb{R}$ and a function $h \in \langle w_0 \rangle^{\perp}$ such that w decomposes as $w = \alpha w_0 + h$. In particular, the orthogonality of h with respect to w_0 implies that

$$\int_{B_{R_0}} h(x)\mathrm{d}x = 0 \quad \text{and} \quad \|w - w_0\|^2_{L^2(B_{R_0})} = (1-\alpha)^2 \|w_0\|^2_{L^2(B_{R_0})} + \|h\|^2_{L^2(B_{R_0})}. \tag{4.13}$$

In view of this, assumption (4.12) gives

$$-\frac{\delta_0}{|B_{R_0}|^{1/2}} \leqslant (1-\alpha) \leqslant \frac{\delta_0}{|B_{R_0}|^{1/2}} \quad \text{and} \quad \|h\|_{L^2(B_{R_0})} \leqslant \delta_0. \tag{4.14}$$

This fact will be abundantly used in the sequel.

This being said, we are now in position to prove Proposition 4.3. For it, we observe that, since $w_0 \equiv 1$ in B_{R_0}, we have that

$$\mathscr{E}_{\varepsilon, B_{R_0}}(w_0) = -\int_{B_{R_0}} G_\varepsilon(w_0(x))\mathrm{d}x = -G_\varepsilon(1)|B_{R_0}| = -\varepsilon^2\, G(1)|B_{R_0}|.$$

Furthermore, thanks to $R > R_0 + 2$ and $\operatorname{supp}(J_\varepsilon) \subset B_{\frac{\varepsilon}{2}}$, we have that $c_\varepsilon(x) \equiv 0$ in B_{R_0}, for any $0 < \varepsilon < 1$. Consequently, $\mathscr{E}_{\varepsilon, B_{R_0}}(w)$ rewrites

$$\mathscr{E}_{\varepsilon, B_{R_0}}(w) \;=\; \underbrace{\frac{1}{4}\int_{B_{R_0}}\int_{B_{R_0}} J_\varepsilon(x-y)(w(x)-w(y))^2 \mathrm{d}x\mathrm{d}y}_{II} \; \underbrace{-\int_{B_{R_0}} G_\varepsilon(w(x))\mathrm{d}x}_{I}.$$

Let us first estimate II. In view of the Bourgain-Brezis-Mironescu representation of $H^1(B_{R_0})$ (see [27]), one can interpret II as a nonlocal approximation of $\|\nabla w\|^2_{L^2(B_{R_0})}$. The crux of our strategy is that, as shown by Ponce [118, Theorem 1.1], this nonlocal approximation enjoys a Poincaré-type inequality. Let us now proceed. Let $(\rho_\varepsilon)_{0<\varepsilon<1}$ be the family of radially symmetric mollifiers defined by

$$\rho_\varepsilon(z) := M_2(J)^{-1} J_\varepsilon(z)|z|^2 \varepsilon^{-2} \quad \text{for } \varepsilon \in (0,1),$$

where $M_2(J)$ is given by (3.2). Notice that, by construction, it satisfies

$$\rho_\varepsilon \geqslant 0 \;\text{ a.e. in }\; \mathbb{R}^N, \quad \int_{\mathbb{R}^N} \rho_\varepsilon(z)\mathrm{d}z = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \int_{|z| \geqslant \tau} \rho_\varepsilon(z)\mathrm{d}z = 0,$$

for each $0 < \varepsilon < 1$ and each $\tau > 0$. Moreover, II can be rewritten as

$$II = \varepsilon^2 \frac{M_2(J)}{4} \int_{B_{R_0}} \int_{B_{R_0}} \rho_\varepsilon(x-y) \frac{|w(x)-w(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y.$$

Now, by [118, Theorem 1.1], we know that there exists some $\varepsilon_1 = \varepsilon_1(J, N, R_0) > 0$ such that the following Poincaré-type inequality

$$\left\| w - \fint_{B_{R_0}} w \right\|^2_{L^2(B_{R_0})} \leqslant \frac{2A_0}{K_{2,N}} \int_{B_{R_0}} \int_{B_{R_0}} \rho_\varepsilon(x-y) \frac{|w(x)-w(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y$$
$$= \frac{8A_0\varepsilon^{-2}}{K_{2,N} M_2(J)} \times II,$$

holds for all $\varepsilon \in (0, \varepsilon_1)$ and all $w \in L^2(B_{R_0})$. Here,

$$K_{2,N} := \int_{\mathbb{S}^{N-1}} (\sigma \cdot e_1)^2 \, \mathrm{d}\mathscr{H}^{N-1}(\sigma) = \frac{1}{N},$$

and $A_0 > 0$ is the smallest constant such that the standard Poincaré-Wirtinger inequality holds. That is, A_0 is the smallest positive constant such that

$$\left\| w - \fint_{B_{R_0}} w \right\|^2_{L^2(B_{R_0})} \leqslant A_0 \|\nabla w\|^2_{L^2(B_{R_0})},$$

holds for any $w \in H^1(B_{R_0})$. In our case, A_0 satisfies the upper bound:

$$A_0 \leqslant \frac{\mathrm{diam}(B_{R_0})^2}{\pi^2} = \frac{4R_0^2}{\pi^2},$$

see [14, Theorem 3.2] (see also [116]). In particular, this gives

$$\varepsilon^2 \frac{\pi^2 M_2(J)}{32 N R_0^2} \left\| w - \fint_{B_{R_0}} w \right\|^2_{L^2(B_{R_0})} \leqslant II.$$

Now, since $w = \alpha w_0 + h$, since $w_0 \equiv 1$ on B_{R_0} and since h is integral free (by (4.13)) we have

$$II \geqslant \varepsilon^2 \frac{C_{N,J}}{R_0^2} \|h\|^2_{L^2(B_{R_0})}, \tag{4.15}$$

where $C_{N,J}$ is given by (3.3). We are now left to estimate I. For it, we rewrite I as follows

$$I = \mathscr{E}_{\varepsilon, B_{R_0}}(w_0) - \int_{B_{R_0}} \big[G_\varepsilon(w(x)) - G_\varepsilon(w_0(x))\big] \mathrm{d}x. \tag{4.16}$$

To estimate the last integral, we split it into two parts, I_1 and I_2, where

$$I_1 := -\int_{B_{R_0}} \big[G_\varepsilon(w_0 + (\alpha-1)w_0 + h) - G_\varepsilon(w_0 + (\alpha-1)w_0)\big],$$

$$I_2 := -\int_{B_{R_0}} \left[G_\varepsilon(w_0+(\alpha-1)w_0)-G_\varepsilon(w_0)\right].$$

Let us first estimate I_2. Using (4.11), (4.12) and (4.14) we have in particular that $|1-\alpha| < \tau_0$. This, together with (4.10), gives

$$I_2 = -\int_{B_{R_0}} \left[G_\varepsilon(w_0+(\alpha-1)w_0)-G_\varepsilon(w_0)\right] = \frac{\kappa}{2}\,\varepsilon^2|B_{R_0}|(\alpha-1)^2.$$

Therefore, recalling (4.16), we get

$$I = \mathscr{E}_{\varepsilon,B_{R_0}}(w_0) + \frac{\kappa}{2}\,\varepsilon^2|B_{R_0}|(\alpha-1)^2 + I_1. \tag{4.17}$$

Let us now estimate I_1. On account of (4.14), we may write

$$\alpha = 1-\eta \quad \text{for some } |\eta| \leqslant \frac{\delta_0}{|B_{R_0}|^{1/2}}. \tag{4.18}$$

Then, a standard change of variables yields

$$I_1 = \varepsilon^2 \int_{B_{R_0}} \int_{1-\eta}^{1-\eta+h(x)} \widetilde{f}(1-\tau)\,\mathrm{d}\tau\mathrm{d}x = -\varepsilon^2 \int_{B_{R_0}} \int_0^{-h(x)} \widetilde{f}(\tau+\eta)\,\mathrm{d}\tau\mathrm{d}x.$$

Now, we set

$$\Sigma := \left\{x \in B_{R_0}; -h(x) > \frac{\theta}{2}\right\},$$

and we decompose I_1 as

$$I_1 = -\varepsilon^2 \left(\int_\Sigma \int_0^{-h(x)} \widetilde{f}(\tau+\eta)\,\mathrm{d}\tau\mathrm{d}x + \int_{B_{R_0}\setminus\Sigma} \int_0^{-h(x)} \widetilde{f}(\tau+\eta)\,\mathrm{d}\tau\mathrm{d}x \right). \tag{4.19}$$

We will estimate these two integrals separately. In view of (4.11) and (4.18), we have that $|\eta| \leqslant \theta/4$. In turn, this implies that

$$-h(x) + |\eta| \leqslant \frac{3\theta}{4} \quad \text{for any } x \in B_{R_0} \setminus \Sigma.$$

Since, by construction, $\widetilde{f}$ is linear in $(-\infty, 3\theta/4]$, we get

$$\begin{aligned} \int_{B_{R_0}\setminus\Sigma} \int_0^{-h(x)} \widetilde{f}(\tau+\eta)\,\mathrm{d}\tau\mathrm{d}x &= -\kappa \int_{B_{R_0}\setminus\Sigma} \left(\int_0^{\eta-h(x)} \tau\,\mathrm{d}\tau - \int_0^{\eta} \tau\,\mathrm{d}\tau \right) \mathrm{d}x \\ &= -\frac{\kappa}{2} \int_{B_{R_0}\setminus\Sigma} h^2(x)\,\mathrm{d}x + \kappa\eta \int_{B_{R_0}\setminus\Sigma} h(x)\,\mathrm{d}x \end{aligned}$$

$$= -\frac{\kappa}{2}\int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x - \kappa\eta\int_\Sigma h(x)\mathrm{d}x,$$

where, in the last equality, we have used the fact that h is integral free, that is:

$$\int_{B_{R_0}\setminus\Sigma} h(x)\mathrm{d}x + \int_\Sigma h(x)\mathrm{d}x = 0.$$

Using now the Cauchy-Schwarz inequality, we get

$$\int_{B_{R_0}\setminus\Sigma}\int_0^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant -\frac{\kappa}{2}\int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x + \kappa|\eta|\sqrt{|\Sigma|}\|h\|_{L^2(\Sigma)}.$$

By the Bienaymé-Chebyshev inequality, we have

$$|\Sigma| \leqslant \left(\frac{2}{\theta}\right)^2 \|h\|^2_{L^2(\Sigma)}, \tag{4.20}$$

and thus

$$\int_{B_{R_0}\setminus\Sigma}\int_0^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant -\frac{\kappa}{2}\int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x + \frac{2\kappa|\eta|}{\theta}\|h\|^2_{L^2(\Sigma)}. \tag{4.21}$$

Thanks to (4.11) and (4.18), (4.21) reduces to

$$\int_{B_{R_0}\setminus\Sigma}\int_0^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant -\frac{\kappa}{2}\int_{B_{R_0}\setminus\Sigma} h^2(x)\mathrm{d}x + \frac{2C_0}{\theta}\|h\|^2_{L^2(\Sigma)}. \tag{4.22}$$

Let us now estimate the first integral on the right-hand side of (4.19). For it, we observe that

$$\tau+\eta \geqslant -|\eta| \geqslant \frac{-\delta_0}{|B_{R_0}|^{1/2}} \geqslant -\frac{C_0}{\kappa} \quad \text{for any } x\in\Sigma \text{ and any } \tau\in(0,-h(x)).$$

Recalling (4.4), we then obtain

$$\sup_{x\in\Sigma}\sup_{\tau\in(0,-h(x))} \widetilde{f}(\tau+\eta) \leqslant \sup_{s\geqslant -\frac{C_0}{\kappa}} \widetilde{f}(s) = \max_{s\in[0,1]} \widetilde{f}(s) = C_0.$$

This, together with the Cauchy-Schwarz inequality, gives

$$\int_\Sigma\int_0^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant C_0\int_\Sigma |h(x)|\mathrm{d}x \leqslant C_0\sqrt{|\Sigma|}\|h\|_{L^2(\Sigma)},$$

Using the Bienaymé-Chebyshev inequality (4.20), we finally get

$$\int_{\Sigma}\int_{0}^{-h(x)} \widetilde{f}(\tau+\eta)\mathrm{d}\tau\mathrm{d}x \leqslant \frac{2C_0}{\theta}\|h\|^2_{L^2(\Sigma)}. \tag{4.23}$$

Collecting (4.15), (4.17), (4.22) and (4.23), we obtain that

$$\begin{aligned}\mathscr{E}_{\varepsilon,B_{R_0}}(w) &- \mathscr{E}_{\varepsilon,B_{R_0}}(w_0)\\ &\geqslant \varepsilon^2\left(\frac{\kappa}{2}|B_{R_0}|(\alpha-1)^2 + \frac{\kappa}{2}\|h\|^2_{L^2(B_{R_0}\setminus\Sigma)} - \frac{4C_0}{\theta}\|h\|^2_{L^2(\Sigma)} + \frac{C_{N,J}}{R_0^2}\|h\|^2_{L^2(B_{R_0})}\right),\end{aligned}$$

for all $0<\varepsilon<\varepsilon_1$ and all $w\in L^2(B_{R_0})$ with $\|w-w_0\|_{L^2(B_{R_0})}\leqslant\delta_0$. Recalling that $0<R_0\leqslant R_0^*(J,f)$ and using (3.4), we have $C_{N,J}/R_0^2\geqslant 5C_0/\theta$. This, together with the above inequality, yields

$$\mathscr{E}_{\varepsilon,B_{R_0}}(w) - \mathscr{E}_{\varepsilon,B_{R_0}}(w_0) \geqslant \varepsilon^2\left(\frac{\kappa}{2}|B_{R_0}|(\alpha-1)^2 + \frac{C_0}{\theta}\|h\|^2_{L^2(B_{R_0})}\right).$$

Therefore, letting

$$\kappa_0 := \inf\left\{\frac{\kappa}{2},\frac{C_0}{\theta}\right\},$$

and recalling (4.13), we obtain

$$\mathscr{E}_{\varepsilon,B_{R_0}}(w) - \mathscr{E}_{\varepsilon,B_{R_0}}(w_0) \geqslant \varepsilon^2\kappa_0\|w-w_0\|^2_{L^2(B_{R_0})},$$

for all $0<\varepsilon<\varepsilon_1$ and all $w\in L^2(B_{R_0})$ with $\|w-w_0\|_{L^2(B_{R_0})}\leqslant\delta_0$. □

Remark 4.4. — Note that the proof of Proposition 4.3 relies only on L^2-estimates and on a Poincaré-type inequality. Remarkably, this allows one to adapt straightforwardly our arguments to the local analogue of $\mathscr{E}_{\varepsilon,B_{R_0}}$.

Using Proposition 4.3, we now prove the following

PROPOSITION 4.5. — *Let $N\geqslant 2$, and let $\mathscr{E}_{\varepsilon,R}$ be the energy functional defined by (4.9) with $\Omega = B_R\setminus K_\varepsilon$. Then, there exists $C^*>0$, $0<\delta_0<|B_{R_0}|^{1/2}$ and $0<\varepsilon_{\delta_0}<1$ such that, for any $0<\varepsilon<\varepsilon_{\delta_0}$ and any $w\in L^2(B_R\setminus K_\varepsilon)$ with $\|w-w_0\|_{L^2(B_R\setminus K_\varepsilon)}=\delta_0$, it holds that*

$$\mathscr{E}_{\varepsilon,R}(w) - \mathscr{E}_{\varepsilon,R}(w_0) > C^*\varepsilon^2.$$

Proof. — Let us first notice that our assumptions on $\widetilde{f}$ imply that there is some $\kappa_1>0$ such that

$$-G(t)\geqslant\kappa_1 t^2 \text{ for every } t\in\mathbb{R}. \tag{4.24}$$

Let us now compute the energy of w_0. Since $\mathrm{supp}(J_\varepsilon) = B_{\varepsilon/2}$ and $R_1 - R_0 > \varepsilon$, a straightforward calculation yields

$$\mathscr{E}_{\varepsilon,R}(w_0) = \mathscr{E}_{\varepsilon,B_{R_0}}(w_0) + \frac{1}{2}\int_{F_\varepsilon}\int_{B_{R_0}} J_\varepsilon(x-y)\mathrm{d}x\mathrm{d}y.$$

In addition, elementary computations yield

$$\begin{aligned}\frac{1}{2}\int_{F_\varepsilon}\int_{B_{R_0}} J_\varepsilon(x-y)\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_{F_\varepsilon\cap B_{R_0+\frac{\varepsilon}{2}}}\left(\int_{B_{R_0}} J_\varepsilon(x-y)\mathrm{d}x\right)\mathrm{d}y\\ &\leqslant \frac{|F_\varepsilon \cap B_{R_0+\frac{\varepsilon}{2}}|}{2} \leqslant C\varepsilon^{N+1},\end{aligned}$$

for come constant $C = C(N) > 0$. As a consequence, we obtain

$$\mathscr{E}_{\varepsilon,R}(w_0) \leqslant \mathscr{E}_{\varepsilon,B_{R_0}}(w_0) + C\varepsilon^{N+1}. \tag{4.25}$$

Next, developing $\mathscr{E}_{\varepsilon,R}(w)$, we get

$$\begin{aligned}\mathscr{E}_{\varepsilon,R}(w) = {}& \mathscr{E}_{\varepsilon,B_{R_0}}(w) + \mathscr{E}_{\varepsilon,F_\varepsilon}(w) + \mathscr{E}_{\varepsilon,B_R\setminus B_{R_1}}(w)\\ &+ \frac{1}{2}\int_{F_\varepsilon}\left(\int_{B_{R_0}} + \int_{B_R\setminus B_{R_1}}\right) J_\varepsilon(x-y)(w(x)-w(y))^2\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Using (4.24) we obtain that $\mathscr{E}_{\varepsilon,\Omega}(w) \geqslant \kappa_1\varepsilon^2\|w\|^2_{L^2(\Omega)}$ for any domain $\Omega \subset B_R \setminus K_\varepsilon$. In particular, since $w_0 = 0$ in $F_\varepsilon \cup B_R \setminus B_{R_1}$ we have

$$\mathscr{E}_{\varepsilon,B_R\setminus K_\varepsilon}(w) \geqslant \mathscr{E}_{\varepsilon,B_{R_0}}(w) + \kappa_1\varepsilon^2\|w - w_0\|^2_{L^2(F_\varepsilon\cup B_R\setminus B_{R_1})}. \tag{4.26}$$

Gluing together (4.25) and (4.26), we obtain

$$\begin{aligned}\mathscr{E}_{\varepsilon,R}(w) - \mathscr{E}_{\varepsilon,R}(w_0) \geqslant {}& \mathscr{E}_{\varepsilon,B_{R_0}}(w) - \mathscr{E}_{\varepsilon,B_{R_0}}(w_0)\\ &+ \kappa_1\varepsilon^2\|w - w_0\|^2_{L^2(F_\varepsilon\cup B_R\setminus B_{R_1})} - C\varepsilon^{N+1}.\end{aligned} \tag{4.27}$$

Now, by Proposition 4.3, there exists $\kappa_0 > 0$, $0 < \delta_0 < |B_{R_0}|^{1/2}$ and $\varepsilon_1 > 0$ such that, for any $0 < \varepsilon < \varepsilon_1$ and any $w \in L^2(B_{R_0})$ with $\|w - w_0\|_{L^2(B_{R_0})} \leqslant \delta_0$, we have

$$\mathscr{E}_{\varepsilon,B_{R_0}}(w) - \mathscr{E}_{\varepsilon,B_{R_0}}(w_0) \geqslant \kappa_0\varepsilon^2\|w - w_0\|^2_{L^2(B_{R_0})}. \tag{4.28}$$

Letting $\bar\kappa := \min\{\kappa_1, \kappa_0\}$ and combining (4.28) and (4.27), we obtain

$$\mathscr{E}_{\varepsilon,R}(w) - \mathscr{E}_{\varepsilon,R}(w_0) \geqslant \varepsilon^2\bar\kappa\|w - w_0\|^2_{L^2(B_R\setminus K_\varepsilon)} - C\varepsilon^{N+1} = \varepsilon^2\left(\bar\kappa\delta_0^2 - C\varepsilon^{N-1}\right), \tag{4.29}$$

for all $0 < \varepsilon < \varepsilon_1$ and all $w \in L^2(B_R \setminus K_\varepsilon)$ with $\|w - w_0\|_{L^2(B_R \setminus K_\varepsilon)} = \delta_0$. The conclusion now follows from (4.29) and the choice

$$C^* = \frac{\bar{\kappa}\delta_0^2}{2} \quad \text{and} \quad \varepsilon_{\delta_0} := \min\left\{\varepsilon_1, \left(\frac{\bar{\kappa}\delta_0^2}{2C}\right)^{\frac{1}{N-1}}\right\}.$$

The proof is thereby complete. □

We are now in position to construct a positive solution to (4.8).

Proposition 4.6. — *Let $N \geqslant 2$ and let (J, f) be a pair satisfying* (1.7) *and* (1.8). *Let $(K_\varepsilon)_{0<\varepsilon<1}$ be the family of obstacles associated to the pair (J, f) (as defined in Section 3). Let $\widetilde{f}$ be the extension of f given by* (4.3) *and let $\widetilde{f}_\varepsilon$ and J_ε be respectively given by* (4.7) *and* (3.5). *Then, there exists $\bar{\varepsilon} > 0$ such that, for all $0 < \varepsilon < \bar{\varepsilon}$, there is a function $v_{\varepsilon,R} \in C(\overline{B_R \setminus K_\varepsilon})$ satisfying* (4.8) *and $0 < v_{\varepsilon,R} < 1$ in $\overline{B_R \setminus K_\varepsilon}$.*

Proof. — Let $w_0 := \mathbb{1}_{B_{R_0}}$ and let $0 < \delta_0 < |B_{R_0}|^{1/2}$ and $0 < \varepsilon_{\delta_0} < 1$ be quantities constructed in the proof of Proposition 4.5, namely such that

$$\mathscr{E}_{\varepsilon,R}(w) - \mathscr{E}_{\varepsilon,R}(w_0) > C^*\varepsilon^2,$$

holds for some constant $C^* > 0$ and for any $0 < \varepsilon < \varepsilon_{\delta_0}$ and any $w \in L^2(B_R \setminus K_\varepsilon)$ with $\|w - w_0\|_{L^2(B_R \setminus K_\varepsilon)} = \delta_0$. Let us fix $0 < \varepsilon < \bar{\varepsilon} := \min\{\varepsilon_0, \varepsilon_{\delta_0}\}$ where ε_0 is as in Proposition 3.1. Further, we denote by $\mathbb{B}_{\delta_0}(w_0)$ the following set:

$$\mathbb{B}_{\delta_0}(w_0) := \Big\{w \in L^2(B_R \setminus K_\varepsilon)\,;\ \|w - w_0\|_{L^2(B_R \setminus K_\varepsilon)} \leqslant \delta_0\Big\},$$

and we define

$$m := \inf_{w \in \mathbb{B}_{\delta_0}(w_0)} \mathscr{E}_{\varepsilon,R}(w).$$

Note that m is well-defined since $\mathscr{E}_{\varepsilon,R}$ is a non-negative continuous functional in $L^2(B_R \setminus K_\varepsilon)$.

Using Lemma 4.5, we will show that there is a *local minimum* $v_{\varepsilon,R}$ of the energy $\mathscr{E}_{\varepsilon,R}$ in the ball $\mathbb{B}_{\delta_0}(w_0)$ which is also a solution to (4.8). However, it must be noted that $\mathscr{E}_{\varepsilon,R}$ lacks of strong compactness properties and passing to the limit along a subsequence is not straightforward. So let us first show that m is achieved in $\mathbb{B}_{\delta_0}(w_0)$.

Take a minimising sequence $(v_j)_{j\in\mathbb{N}} \subset \mathbb{B}_{\delta_0}(w_0)$. Notice that $|w| \in \mathbb{B}_{\delta_0}(w_0)$ for all $w \in \mathbb{B}_{\delta_0}(w_0)$. Moreover, a straightforward computation shows that $\mathscr{E}_{\varepsilon,R}(|v_j|) \leqslant \mathscr{E}_{\varepsilon,R}(v_j)$ for all $j \geqslant 0$. Thus, we may assume that the v_j's are a.e. non-negative for every $j \geqslant 0$. By (4.24), we have $-G_\varepsilon(t) \geqslant \kappa_1\varepsilon^2 t^2$ for all $t \in \mathbb{R}$. In particular, $\mathscr{E}_{\varepsilon,R}(v_j) \geqslant \kappa_1\varepsilon^2\|v_j\|^2_{L^2(B_R \setminus K_\varepsilon)}$ for all $j \geqslant 0$. Therefore $(v_j)_{j\in\mathbb{N}}$ is bounded in $L^2(B_R \setminus$

K_ε). Whence, up to extract a subsequence, we obtain that v_j converges weakly in $L^2(B_R \setminus K_\varepsilon)$ to some $v_{\varepsilon,R} \in \mathbb{B}_{\delta_0}(w_0)$ (notice that $\mathbb{B}_{\delta_0}(w_0)$ is *closed* in $L^2(B_R \setminus K_\varepsilon)$). Let us check that $v_{\varepsilon,R}$ is indeed a minimiser of $\mathscr{E}_{\varepsilon,R}$ in $\mathbb{B}_{\delta_0}(w_0)$. To this end, we shall introduce the following notations

$$\mathscr{J}_\varepsilon(x) := \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\mathrm{d}y \quad \text{and} \quad H_\varepsilon(x,s) := \int_0^s \big(\mathscr{J}_\varepsilon(x)\tau - g_\varepsilon(\tau)\big)\mathrm{d}\tau.$$

Since $0 < \varepsilon < \varepsilon_0$, by Proposition 3.1, we have

$$\max_{[0,1]} f_\varepsilon' < \inf_{\mathbb{R}^N \setminus K_\varepsilon} \mathscr{J}_\varepsilon.$$

Therefore, from the construction of g_ε (remember (4.4)), we have

$$g_\varepsilon'(s) = \varepsilon^2 \widetilde{f}'(1-s) \leqslant \max_{\mathbb{R}} \widetilde{f}_\varepsilon' \leqslant \max_{[0,1]} f_\varepsilon' < \inf_{\mathbb{R}^N \setminus K_\varepsilon} \mathscr{J}_\varepsilon \quad \text{for any } s \in \mathbb{R}. \tag{4.30}$$

Whence, $H_\varepsilon(x,\cdot)$ is convex for each fixed x. Developing the terms involved in the definition of $\mathscr{E}_{\varepsilon,R}$ we arrive at

$$\mathscr{E}_{\varepsilon,R}(w) = -\frac{1}{2}\int_{B_R \setminus K_\varepsilon}\int_{B_R \setminus K_\varepsilon} J_\varepsilon(x-y)w(x)w(y)\mathrm{d}x\mathrm{d}y + \int_{B_R \setminus K_\varepsilon} H_\varepsilon(x, w(x))\mathrm{d}x.$$

Using the weak convergence of $(v_j)_{j\in\mathbb{N}}$ towards $v_{\varepsilon,R}$ and the dominated convergence theorem, we can pass to the limit in the double integral and get that

$$\begin{aligned}\lim_{j\to+\infty}\int_{B_R \setminus K_\varepsilon}\int_{B_R \setminus K_\varepsilon} & J_\varepsilon(x-y)v_j(x)v_j(y)\mathrm{d}x\mathrm{d}y \\ &= \int_{B_R \setminus K_\varepsilon}\int_{B_R \setminus K_\varepsilon} J_\varepsilon(x-y)v_{\varepsilon,R}(x)v_{\varepsilon,R}(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Moreover, since $H_\varepsilon(x,\cdot)$ is convex, we have

$$\int_{B_R \setminus K_\varepsilon}\big[H_\varepsilon(x, v_j(x)) - H_\varepsilon(x, v_{\varepsilon,R}(x))\big]\mathrm{d}x \geqslant \int_{B_R \setminus K_\varepsilon} \partial_s H_\varepsilon(x, v_{\varepsilon,R}(x))(v_j(x) - v_{\varepsilon,R}(x))\mathrm{d}x.$$

From the definition of H_ε, g_ε and from (4.30) a quick computation shows that $|\partial_s H_\varepsilon(x,s)| = |\mathscr{J}_\varepsilon(x)s - g_\varepsilon(s)| \leqslant A|s|$ for all $s \in \mathbb{R}$ and some constant $A > 0$. Since $v_{\varepsilon,R} \in L^2(B_R \setminus K_\varepsilon)$, it follows that $\partial_s H_\varepsilon(\cdot, v_{\varepsilon,R}(\cdot)) \in L^2(B_R \setminus K_\varepsilon)$. Therefore, using the previous two displayed formulas and the weak convergence of v_j towards $v_{\varepsilon,R}$, we obtain $\lim_{j\to\infty}[\mathscr{E}_{\varepsilon,R}(v_j) - \mathscr{E}_{\varepsilon,R}(v_{\varepsilon,R})] \geqslant 0$. Since, on the other hand, $\lim_{j\to\infty} \mathscr{E}_{\varepsilon,R}(v_j) = m \leqslant \mathscr{E}_{\varepsilon,R}(v_{\varepsilon,R})$, we finally obtain

$$\mathscr{E}_{\varepsilon,R}(v_{\varepsilon,R}) = m = \inf_{w \in \mathbb{B}_{\delta_0}(w_0)} \mathscr{E}_{\varepsilon,R}(w) \leqslant \mathscr{E}_{\varepsilon,R}(w_0).$$

Now, thanks to Proposition 4.5, we deduce that $v_{\varepsilon,R} \in \mathbb{B}_{\delta_0}(w_0)$ is a local minimiser and, as such, $v_{\varepsilon,R}$ solves (4.8) almost everywhere in $\overline{B_R \setminus K_\varepsilon}$.

Let us now check that $v_{\varepsilon,R}$ is a continuous solution to (4.8) in the whole set $\overline{B_R \setminus K_\varepsilon}$. Since $J_\varepsilon \in L^2(\mathbb{R}^N)$ and $v_{\varepsilon,R} \in L^2(B_R \setminus K_\varepsilon)$, it follows from the equation (4.8) satisfied by $v_{\varepsilon,R}$ that $N_\varepsilon(\cdot, v_{\varepsilon,R}(\cdot)) \in L^\infty(B_R \setminus K_\varepsilon)$ where $N_\varepsilon(x,s) := \mathscr{J}_\varepsilon(x)\, s - g_\varepsilon(s)$. By (4.30), the map $N_\varepsilon(x,\cdot)$ is bijective and thus $v_{\varepsilon,R} \in L^\infty(B_R \setminus K_\varepsilon)$. Using now Lemma 2.2 and (4.30) we may further infer that $v_{\varepsilon,R}$ is continuous in $\overline{B_R \setminus K_\varepsilon}$.

To complete the proof it remains to show that $0 < v_{\varepsilon,R} < 1$. Let us first prove that $v_{\varepsilon,R} < 1$. Suppose, by contradiction, that $\|v_{\varepsilon,R}\|_\infty \geqslant 1$. Then, by continuity of $v_{\varepsilon,R}$, there must be a point $\bar{x} \in \overline{B_R \setminus K_\varepsilon}$ at which $v_{\varepsilon,R}$ attains its maximum, i.e. $v_{\varepsilon,R}(\bar{x}) = \|v_{\varepsilon,R}\|_\infty$. Using now the equation satisfied by $v_{\varepsilon,R}$, we have

$$0 \geqslant \int_{B_R \setminus K_\varepsilon} J_\varepsilon(\bar{x} - y)(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(\bar{x}))\mathrm{d}y = c_\varepsilon(\bar{x}) v_{\varepsilon,R}(\bar{x}) - g_\varepsilon(v_{\varepsilon,R}(\bar{x})) \geqslant 0.$$

Thus, since $\mathrm{supp}(J_\varepsilon) = B_{\varepsilon/2}$, we have $v_{\varepsilon,R}(y) = v_{\varepsilon,R}(\bar{x})$ for any $y \in B_{\varepsilon/2}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon}$. Note that $B_{\varepsilon/2}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon}$ is nonempty whence we may iterate this reasoning over again and obtain that $v_{\varepsilon,R} \equiv v_{\varepsilon,R}(\bar{x}) \geqslant 1$. Now choose $x_0 \in \Omega_\varepsilon$ such that $c_\varepsilon(x_0) > 0$. Then, evaluating (4.8) at x_0, one obtains

$$0 = \int_{B_R \setminus K_\varepsilon} J_\varepsilon(x_0 - y)(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(x_0))\mathrm{d}y = c_\varepsilon(x_0) v_{\varepsilon,R}(x_0) - g_\varepsilon(v_{\varepsilon,R}(x_0)) \geqslant c_\varepsilon(x_0) > 0,$$

which is a contradiction.

Therefore $v_{\varepsilon,R} < 1$. Since, by construction, we have that $v_{\varepsilon,R} \geqslant 0$, it remains to check that $v_{\varepsilon,R}$ cannot cancel. Assume, by contradiction, that this is the case, namely that there exists a point $x_0 \in \overline{B_R \setminus K_\varepsilon}$ such that $v_{\varepsilon,R}(x_0) = 0$. Then, by (4.8), we have that

$$\int_{B_R \setminus K_\varepsilon} J_\varepsilon(x_0 - y)(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(x_0))\mathrm{d}y = 0,$$

and, as above, this implies that $v_{\varepsilon,R} \equiv 0$. However, since $v_{\varepsilon,R} \in \mathbb{B}_{\delta_0}(w_0)$ and $\delta_0 < |B_{R_0}|^{1/2}$, we have $\delta_0 \geqslant \|v_{\varepsilon,R} - w_0\|_{L^2(B_R \setminus K_\varepsilon)} = \|w_0\|_{L^2(B_R \setminus K_\varepsilon)} = |B_{R_0}|^{1/2} > \delta_0$, which is a contradiction. The proof of Proposition 4.6 is thereby complete. $\square$

From now on (and until the end of Section 4), ε will be fixed and taken so small that $0 < \varepsilon < \bar{\varepsilon}$, where $\bar{\varepsilon}$ is as defined in Proposition 4.6.

4.2 An extension procedure

Let us now complete the proof of Lemma 4.1. We will modify the function $v_{\varepsilon,R}$ constructed above in order to get a continuous super-solution to (4.1) satisfying

(4.2). Let us briefly explain our strategy. Since, by construction, $v_{\varepsilon,R}$ satisfies (4.8), the function $u_{\varepsilon,R} = 1 - v_{\varepsilon,R}$ verifies (4.6) and, as already noted above, extending the function $u_{\varepsilon,R}$ by 1 outside B_R, we obtain a (discontinuous) super-solution to (4.1) that satisfies (4.2). The aim of this section is to find the right extension of $u_{\varepsilon,R}$ that provides the desired super-solution.

To do so, we first introduce some useful notations. Given $R > 0$ and $x \in \mathbb{R}^N$, we let $\mathscr{P}_R(x)$ be the projection of x to the ball $\overline{B_R}$, that is

$$\mathscr{P}_R(x) \in \overline{B_R} \quad \text{and} \quad |x - \mathscr{P}_R(x)| = \operatorname{dist}(x, B_R) = \min_{y \in \overline{B_R}} |x - y|.$$

For $\sigma > 0$, we let $\overline{u}_{\varepsilon,\sigma} \in C(\overline{\mathbb{R}^N \setminus K_\varepsilon})$ be the following function

$$\overline{u}_{\varepsilon,\sigma}(x) := \min\left\{u_{\varepsilon,R}(\mathscr{P}_R(x)) + \sigma^{-1}\,|x - \mathscr{P}_R(x)|, 1\right\}. \tag{4.31}$$

We shall see that, for well-chosen σ, the function $\overline{u}_{\varepsilon,\sigma}$ will satisfy

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant 0 \quad \text{for all } x \in \mathbb{R}^N \setminus K_\varepsilon, \tag{4.32}$$

where L_ε is the nonlocal operator given by

$$L_\varepsilon w(x) := \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)(w(y) - w(x))\mathrm{d}y. \tag{4.33}$$

Namely, we claim

Claim 4.7. — *There exists $\sigma_\varepsilon > 0$ such that $\overline{u}_{\varepsilon,\sigma}$ satisfies* (4.32) *for all $\sigma < \sigma_\varepsilon$.*

Observe that by proving Claim 4.7, we end the proof of Lemma 4.1. Indeed, by construction, we have $f \leqslant \widetilde{f}$ so that $\overline{u}_{\varepsilon,\sigma}$ trivially satisfies (4.1). As for condition (4.2) it is also satisfied (by construction of $\overline{u}_{\varepsilon,\sigma}$) provided that R is taken sufficiently large.

Proof. — Define $\mathscr{A}_R := \mathbb{R}^N \setminus \overline{B_R}$. As in the previous section, we set

$$\mathscr{J}_\varepsilon(x) = \int_{\mathbb{R}^N \setminus K_\varepsilon} J(x-y)\mathrm{d}y \quad \text{and} \quad c_\varepsilon(x) = \int_{\mathbb{R}^N \setminus B_R} J_\varepsilon(x-y)\mathrm{d}y.$$

Then, in view of (4.31), we have

$$\begin{aligned} L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant \int_{B_R \setminus K_\varepsilon} & J_\varepsilon(x-y)(u_{\varepsilon,R}(y) - \overline{u}_{\varepsilon,\sigma}(x))\mathrm{d}y \\ & + c_\varepsilon(x)(1 - \overline{u}_{\varepsilon,\sigma}(x)) + \widetilde{f}(\overline{u}_{\varepsilon,\sigma}(x)). \end{aligned} \tag{4.34}$$

Since $\overline{u}_{\varepsilon,\sigma}(x) = u_{\varepsilon,R}(x)$ for all $x \in \overline{B_R \setminus K_\varepsilon}$, using (4.6) we easily get that

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant 0 \quad \text{for } x \in \overline{B_R \setminus K_\varepsilon}. \tag{4.35}$$

To complete the proof, it remains to show that $\overline{u}_{\varepsilon,\sigma}$ satisfies (4.32) in the set $\mathscr{A}_R$. We shall consider two sub-domains, Π^+ and Π^-, defined as follows

$$\begin{aligned} \Pi^- &:= \mathscr{A}_R \cap \big\{\overline{u}_{\varepsilon,\sigma} < 1\big\}, \\ \Pi^+ &:= \mathscr{A}_R \cap \big\{\overline{u}_{\varepsilon,\sigma} = 1\big\}. \end{aligned}$$

Note that since $\overline{u}_{\varepsilon,\sigma}(x) = 1$ for all $x \in \Pi^+$, it follows directly from (4.34) that

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) = \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)(u_{\varepsilon,R}(y) - 1)\mathrm{d}y \leqslant 0, \tag{4.36}$$

for any $x \in \Pi^+$. Thus, to conclude the proof we need only to check that (4.36) still holds in Π^-. To this end, for any $x \in \Pi^-$ and any $s \in [0,1]$, we set

$$g_R(x,s) := \mathscr{J}_\varepsilon(\mathscr{P}_R(x))s - \widetilde{f}_\varepsilon(s). \tag{4.37}$$

Now, since $0 < \varepsilon < \varepsilon_0$, it follows from Proposition 3.1 that there exists a $\gamma > 0$ such that

$$\inf_{z \in \mathbb{R}^N \setminus K_\varepsilon} \min_{s \in [0,1]} \partial_s g_R(z,s) > \gamma. \tag{4.38}$$

Next, since $J \in W^{1,1}(\mathbb{R}^N)$ (by (1.6)) we may set

$$\sigma_\varepsilon := \varepsilon\gamma \cdot \left(\int_{\mathbb{R}^N} |\nabla J(z)|\mathrm{d}z\right)^{-1} > 0. \tag{4.39}$$

Let us also set

$$s(x) := u_{\varepsilon,R}(\mathscr{P}_R(x)) \quad \text{and} \quad \tau(x) := \mathrm{dist}(x, B_R) = |x - \mathscr{P}_R(x)| > 0.$$

Then, $\overline{u}_{\varepsilon,\sigma}$ rewrites $\overline{u}_{\varepsilon,\sigma}(x) = s(x) + \sigma^{-1}\tau(x)$ and

$$0 < s(x) + \sigma^{-1}\tau(x) < 1 \quad \text{for any } x \in \Pi^-. \tag{4.40}$$

Now, using (4.31) and the definition of L_ε, we can rewrite $L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x)$ as

$$\begin{aligned} L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) = L_\varepsilon \overline{u}_{\varepsilon,\sigma}(\mathscr{P}_R(x)) &+ \int_{\mathbb{R}^N \setminus K_\varepsilon} [J_\varepsilon(x-y) - J_\varepsilon(\mathscr{P}_R(x)-y)](\overline{u}_{\varepsilon,\sigma}(y) - \overline{u}_{\varepsilon,\sigma}(x))\mathrm{d}y \\ &- \frac{\tau(x)}{\sigma} \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(\mathscr{P}_R(x) - y)\mathrm{d}y. \end{aligned}$$

Since $\mathscr{P}_R(x) \in \overline{B_R \setminus K_\varepsilon}$, since $J \in W^{1,1}(\mathbb{R}^N)$ and since $J \geqslant 0$ a.e. in $\mathbb{R}^N$, by (4.35) we obtain

$$\begin{aligned}
L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) &\leqslant -\frac{\tau(x)}{\sigma}\mathscr{J}_\varepsilon(\mathscr{P}_R(x)) - \widetilde{f}_\varepsilon(s(x)) + \int_{\mathbb{R}^N \setminus K_\varepsilon} |J_\varepsilon(x-y) - J_\varepsilon(\mathscr{P}_R(x)-y)|\,\mathrm{d}y. \\
&\leqslant -\frac{\tau(x)}{\sigma}\mathscr{J}_\varepsilon(\mathscr{P}_R(x)) - \widetilde{f}_\varepsilon(s(x)) + \int_{\mathbb{R}^N} |J_\varepsilon(x-y) - J_\varepsilon(\mathscr{P}_R(x)-y)|\,\mathrm{d}y. \\
&\leqslant -\frac{\tau(x)}{\sigma}\mathscr{J}_\varepsilon(\mathscr{P}_R(x)) - \widetilde{f}_\varepsilon(s(x)) + \frac{\tau(x)}{\varepsilon}\int_{\mathbb{R}^N} |\nabla J(z)|\mathrm{d}z.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(s(x)+\sigma^{-1}\tau(x)) \leqslant \Big(&\widetilde{f}_\varepsilon(s(x)+\sigma^{-1}\tau(x)) - \widetilde{f}_\varepsilon(s(x))\Big) \\
&- \frac{\tau(x)}{\sigma}\mathscr{J}_\varepsilon(\mathscr{P}_R(x)) + \frac{\tau(x)}{\varepsilon}\int_{\mathbb{R}^N} |\nabla J(z)|\mathrm{d}z.
\end{aligned}$$

By adding and subtracting $s(x)\mathscr{J}_\varepsilon(\mathscr{P}_R(x))$ on the right hand side of the above inequality and recalling (4.37), we obtain

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant \big(g_R\left(x, s(x)\right) - g_R\left(x, s(x)+\sigma^{-1}\tau(x)\right)\big) + \gamma\sigma_\varepsilon^{-1}\tau(x).$$

where we have used (4.39). By (4.38), (4.40) and the mean value theorem, we deduce that there exists some

$$\xi \in \left[s(x), s(x)+\sigma^{-1}\tau(x)\right] \subset [0,1],$$

such that

$$g_R\left(x, s(x)\right) - g_R\left(x, s(x)+\sigma^{-1}\tau(x)\right) = -\partial_s g_R(x,\xi)\,\sigma^{-1}\tau(x) \leqslant -\gamma\sigma^{-1}\tau(x).$$

Therefore, for every $0 < \sigma < \sigma_\varepsilon$, we obtain that

$$L_\varepsilon \overline{u}_{\varepsilon,\sigma}(x) + \widetilde{f}_\varepsilon(\overline{u}_{\varepsilon,\sigma}(x)) \leqslant \gamma\tau(x)\left(\frac{1}{\sigma_\varepsilon} - \frac{1}{\sigma}\right) < 0 \quad \text{for any } x \in \Pi^-.$$

The proof of Claim 4.7 is thereby complete. □

Remark 4.8. — An analogue version of Lemma 4.1 holds when $J_\varepsilon(x-y)$ is replaced by $\widetilde{J}_\varepsilon(d_g(x,y))$ where $\widetilde{J}_\varepsilon$ is a locally integrable function such that $\widetilde{J}_\varepsilon(|z|) = J_\varepsilon(z)$ and $d_{\mathrm{g}}(x,y)$ is the geodesic distance on $\overline{\mathbb{R}^N \setminus K_\varepsilon}$. Indeed, the only places where the structure of the radial kernel J_ε came into place is when we used the Poincaré-type inequality [118, Theorem 1.1] in Proposition 4.5, when we asserted that the solutions to (4.8) satisfying $\max_{[0,1]} f_\varepsilon' < \inf_{B_R\setminus K_\varepsilon} \mathscr{J}_\varepsilon$ are continuous and when we made our extension procedure. But the Poincaré inequality was only needed in the ball B_{R_0} and, by convexity, it trivially holds that $J_\varepsilon(x-y) = \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))$ for any $(x,y) \in B_{R_0} \times B_{R_0}$. Similarly, the extension procedure required only to evaluate the new function on the annulus $B_{R+\sigma} \setminus B_R$ but, since $R - R_1 > 0$ is large and ε is small, it still holds that $J_\varepsilon(x-y) = \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))$ for any $x \in B_{R+\sigma} \setminus B_R$ and any $y \in \mathbb{R}^N \setminus K_\varepsilon$. Moreover, as already noted in Remark 2.5, condition (3.11) still implies the continuity of solutions to the corresponding auxiliary problem:

$$\int_{B_R \setminus K_\varepsilon} \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(x))\mathrm{d}y - \widetilde{c}_\varepsilon(x) v_{\varepsilon,R} + g_\varepsilon(v_{\varepsilon,R}(x)) = 0,$$

for $x \in \overline{B_R \setminus K_\varepsilon}$, where, by analogy, we have set

$$\widetilde{c}_\varepsilon(x) := \int_{\mathbb{R}^N \setminus B_R} \widetilde{J}_\varepsilon(d_{\mathrm{g}}(x,y))\mathrm{d}y.$$

In fact, the only place where some care should be taken is when justifying that if

$$\int_{B_R \setminus K_\varepsilon} \widetilde{J}_\varepsilon(d_{\mathrm{g}}(\bar{x},y))(v_{\varepsilon,R}(y) - v_{\varepsilon,R}(\bar{x}))\mathrm{d}y = 0, \tag{4.41}$$

where $\bar{x} \in \overline{B_R \setminus K_\varepsilon}$ is a point at which $v_{\varepsilon,R}$ reaches an extremum, then it holds that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in \overline{B_R \setminus K_\varepsilon}$ (which is needed to establish the analogue of Proposition 4.6). But, fortunately, the geometry of K_ε is simple enough to ensure that this is still the case. Indeed, (4.41) implies that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in \Pi_1(\bar{x}) := \{z \in \overline{B_R \setminus K_\varepsilon}; d_{\mathrm{g}}(\bar{x},z) < \varepsilon/2\}$. By iteration, one finds that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in \Pi_j(\bar{x})$ and any $j \geqslant 1$, where $\Pi_j(\bar{x})$ is given by

$$\Pi_{j+1}(\bar{x}) := \bigcup_{y \in \Pi_j(\bar{x})} \left\{ z \in \overline{B_R \setminus K_\varepsilon};\ d_{\mathrm{g}}(y,z) < \varepsilon/2 \right\}, \text{ for any } j \geqslant 1.$$

Then, one can show that, for some $j_0 \geqslant 1$ (independent of $\bar{x}$), it holds that $B_{\varepsilon/4}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon} \subset \Pi_{j_0}(\bar{x})$. Whence, iterating the same reasoning over again, one gets that $v_{\varepsilon,R}(y) \equiv v_{\varepsilon,R}(\bar{x})$ for any $y \in B_{k\varepsilon/4}(\bar{x}) \cap \overline{B_R \setminus K_\varepsilon}$ and any $k \in \mathbb{N}$; which then gives the desired result.

5 Construction of continuous global solutions

In this final section we construct a positive nonconstant solution to (1.9). Our goal will be to find an ordered pair of global continuous sub- and super-solution. That is, given $0 < \varepsilon < \varepsilon^*$ (where ε^* has the same meaning as in Lemma 4.1), we aim to construct two functions, $\underline{u}_\varepsilon$ and $\overline{u}_\varepsilon$, such that

$$\begin{cases} L_\varepsilon \overline{u}_\varepsilon + f_\varepsilon(\overline{u}_\varepsilon) \leqslant 0 & \text{in } \mathbb{R}^N \setminus K_\varepsilon, \\ L_\varepsilon \underline{u}_\varepsilon + f_\varepsilon(\underline{u}_\varepsilon) \geqslant 0 & \text{in } \mathbb{R}^N \setminus K_\varepsilon, \\ 0 \leqslant \underline{u}_\varepsilon \leqslant \overline{u}_\varepsilon \leqslant 1 & \text{in } \mathbb{R}^N \setminus K_\varepsilon, \end{cases}$$

(where L_ε is as in (4.33)) and which further satisfy

$$\lim_{x_1 \to +\infty} \underline{u}_\varepsilon(x) = 1 \quad \text{and} \quad \lim_{|x| \to +\infty} \overline{u}_\varepsilon(x) = 1. \tag{5.1}$$

Here, $x_1 = x \cdot e_1$ where $e_1 := (1, 0, \cdots, 0) \in \mathbb{S}^{N-1}$. Then, by Lemmata 2.1 and 2.2 we automatically obtain the existence of a continuous solution u_ε to

$$L_\varepsilon u_\varepsilon + f_\varepsilon(u_\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \setminus K_\varepsilon, \tag{5.2}$$

satisfying $0 \leqslant \underline{u}_\varepsilon \leqslant u_\varepsilon \leqslant \overline{u}_\varepsilon \leqslant 1$. This, together with (5.1), yields a continuous solution to (5.2) satisfying $0 < u_\varepsilon < 1$ and $u_\varepsilon(x) \to 1$ as $x_1 \to \infty$. In particular, we have $\sup_{x \in \mathbb{R}^N \setminus K_\varepsilon} u_\varepsilon(x) = 1$. Since (1.6), (1.7) are satisfied, u_ε is continuous, J_ε is compactly supported and $J_\varepsilon \in L^2(\mathbb{R}^N)$ (by (1.8)), we may apply Lemma 2.4 and we obtain that $\lim_{|x| \to +\infty} u_\varepsilon(x) = 1$, which proves that u_ε satisfies the requirements of Theorem 1.2 and thus Theorem 1.1 is proved.

Therefore, to complete the proof of Theorem 1.2, we need only to prove the following lemma.

Lemma 5.1. — *Let (J, f) be a pair satisfying (1.7) and (1.8). Let $(K_\varepsilon)_{0<\varepsilon<1}$ be the family of obstacles associated to the pair (J, f) (as defined in Section 3). Let $(J_\varepsilon, f_\varepsilon)$ be as in (3.5) and let $\varepsilon^* > 0$ be as in Lemma 4.1. Then, there exists $r_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon^*$, there is*

(i) a continuous global sub-solution $\underline{u}_\varepsilon$ to (5.2) satisfying $\underline{u}_\varepsilon \equiv 0$ in $\{x_1 \leqslant r_0\}$ and $\underline{u}_\varepsilon(x) \to 1$ as $x_1 \to \infty$,

(ii) a continuous global nonconstant super-solution $\overline{u}_\varepsilon$ to (5.2) satisfying $\overline{u}_\varepsilon \equiv 1$ in $\mathbb{R}^N \setminus B_{r_0}$ and $0 < \overline{u}_\varepsilon \leqslant 1$.

In particular, $0 \leqslant \underline{u}_\varepsilon < \overline{u}_\varepsilon \leqslant 1$.

Proof. — By Lemma 4.1, we know that there exists some $R^* > 0$ and some $0 < \varepsilon^* < 1$ such that, for all $0 < \varepsilon < \varepsilon^*$, there is a nonconstant super-solution $\overline{u}_\varepsilon \in C(\overline{\mathbb{R}^N \setminus K_\varepsilon})$ to (5.2) that satisfies $\overline{u}_\varepsilon \equiv 1$ in $\mathbb{R}^N \setminus B_{R^*}$. So, we are left to prove that there exists a sub-solution $\underline{u}_\varepsilon$ to (5.2) satisfying (i) and such that $\underline{u}_\varepsilon \leqslant \overline{u}_\varepsilon$.

To do so, let us extend f outside $[0, 1]$ by $f'(0)s$ when $s \geqslant 0$ and $f'(1)(s-1)$ for $s \geqslant 1$. For simplicity, we still denote by f this extension. Now, we take $\delta \in (0, 1)$

and we let f_δ be a C^1 function defined in $\mathbb{R}$ such that

$$\begin{cases} f_\delta \leqslant f \text{ in } \mathbb{R}, \text{ and } f_\delta(s) = f(s) \text{ for } s \geqslant \theta, \\ f_\delta \text{ has only one zero, } \theta_\delta = \theta, \text{ in } (-\delta, 1), \\ f_\delta\,(-\delta) = 0, \quad f_\delta\,(1) = 0, \\ f'_\delta(s) < 1 \text{ for any } s \in [-\delta, 1] \text{ and } f'_\delta\,(-\delta)\,,\, f'_\delta\,(1) < 0, \\ \int_{-\delta}^1 f_\delta(s)\mathrm{d}s > 0. \end{cases}$$

Since $f \in C^1(\mathbb{R})$ satisfies (1.7) such a function $f_\delta \in C^1\big(\mathbb{R}\big)$ always exists provided that δ is taken sufficiently small, say if $0 < \delta < \delta_1$ for some small $\delta_1 > 0$.

Let $f_{\varepsilon,\delta}(s) := \varepsilon^2 f_\delta(s)$ and let $L_{\mathbb{R}^N}$ be the operator given by

$$L_{\mathbb{R}^N} u(x) := \int_{\mathbb{R}^N} J_\varepsilon(x-y)(u(y) - u(x))\mathrm{d}y. \tag{5.3}$$

Since J_ε is radially symmetric (because J is), using the results obtained in [13, 45, 50, 155], we know that, for any $0 < \varepsilon < 1$, there exists an increasing function $\phi_{\varepsilon,\delta} \in C^1(\mathbb{R})$ and a number $c_{\varepsilon,\delta} > 0$ such that the function $\varphi_{\varepsilon,\delta}(x) := \phi_{\varepsilon,\delta}(x \cdot e_1)$ satisfies

$$\begin{cases} L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta}(x) + f_{\varepsilon,\delta}(\varphi_{\varepsilon,\delta}(x)) = c_{\varepsilon,\delta}\phi'_{\varepsilon,\delta}(x_1) \geqslant 0 \quad \text{for all } x \in \mathbb{R}^N, \\ \varphi_{\varepsilon,\delta}(-\infty) = -\delta, \quad \varphi_{\varepsilon,\delta}(\infty) = 1 \quad \text{and} \quad \varphi_{\varepsilon,\delta} = 0 \quad \text{in } H_{e_1}, \end{cases} \tag{5.4}$$

where H_{e_1} is the hyperplane $H_{e_1} := \{x_1 = 0\}$. Now, for any $r_0 > 0$, we let $\varphi_{\varepsilon,\delta,r_0}$ be the function defined by

$$\varphi_{\varepsilon,\delta,r_0}(x) := \varphi_{\varepsilon,\delta}(x - r_0).$$

By construction, for every $r_0 > 0$, we have

$$L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta,r_0} + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}) \geqslant L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta,r_0} + f_{\varepsilon,\delta}(\varphi_{\varepsilon,\delta,r_0}) \geqslant 0 \quad \text{in } \mathbb{R}^N. \tag{5.5}$$

Now, we set

$$\underline{u}_\varepsilon(x) := \max\big\{0, \varphi_{\varepsilon,\delta,r_0}(x)\big\} \quad \text{and} \quad H_* := \big\{x \in \mathbb{R}^N\,;\ x_1 \geqslant r_0\big\}.$$

Note that, for all $0 < \varepsilon < \varepsilon^*$, it holds that $K_\varepsilon \subset \mathbb{R}^N \setminus H_*$ provided that r_0 is chosen sufficiently large. Let us now prove that, for r_0 large enough, $\underline{u}_\varepsilon$ is a sub-solution to (5.2).

First, if $x \in \mathbb{R}^N \setminus (K \cup H_*)$, then $\underline{u}_\varepsilon(x) = 0$ and

$$L_\varepsilon \underline{u}_\varepsilon(x) + f_\varepsilon(\underline{u}_\varepsilon(x)) = \int_{\mathbb{R}^N \setminus K_\varepsilon} J_\varepsilon(x-y)\underline{u}_\varepsilon(y)\mathrm{d}y \geqslant 0. \tag{5.6}$$

Next, if $x \in H_*$, then, since J_ε is compactly supported, we have

$$\bigcup_{x\in H_*} \big(x + \mathrm{supp}(J_\varepsilon)\big) \subset \mathbb{R}^N \setminus K_\varepsilon,$$

provided that r_0 is chosen sufficiently large. From this and (5.5), we deduce that

$$\begin{aligned} L_\varepsilon \underline{u}_\varepsilon(x) + f_\varepsilon(\underline{u}_\varepsilon(x)) &= \int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)(\underline{u}_\varepsilon(y) - \varphi_{\varepsilon,\delta,r_0}(x))\mathrm{d}y + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}(x)) \\ &\geqslant \int_{\mathbb{R}^N\setminus K_\varepsilon} J_\varepsilon(x-y)(\varphi_{\varepsilon,\delta,r_0}(y) - \varphi_{\varepsilon,\delta,r_0}(x))\mathrm{d}y + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}(x)) \\ &= L_{\mathbb{R}^N}\varphi_{\varepsilon,\delta,r_0}(x) + f_\varepsilon(\varphi_{\varepsilon,\delta,r_0}(x)) \geqslant 0. \end{aligned}$$

Together with (5.6), we obtain that $\underline{u}_\varepsilon$ is a global sub-solution to (5.2) which, by (5.4), satisfies $\underline{u}_\varepsilon(x) \to 1$ as $x_1 \to \infty$ and $\underline{u}_\varepsilon(x) = 0$ if $x_1 \leqslant r_0$. By increasing r_0 to R^* (if necessary) we then achieve $\underline{u}_\varepsilon < \overline{u}_\varepsilon$ when $0 < \varepsilon < \varepsilon^*$. The proof of Lemma 5.1 is thereby complete. □

Remark 5.2. — Observe that, on account of Remarks 2.5, 3.2 and 4.8, the *same* proof as above yields an analogous result with L_{g} in place of L. To see this, it suffices to notice that our arguments are essentially focused on what is happening *far away* from K and, since the kernel we consider is compactly supported, the operator L_{g} will then coincide with L (possibly up to take R sufficiently large). In like manner, as already mentioned in Remark 2.5, the fact that "$\sup_{\mathbb{R}^N\setminus K_\varepsilon} u = 1$" implies that "$\lim_{|x|\to\infty} u(x) = 1$" still holds with L_{g} in place of L since, here as well, the proof relies only on estimates of the behaviour of u far away from K_ε.

Appendix

In this appendix, we prove Lemma 2.1. Our strategy closely follows [31, 52] and relies on the well-known monotone iterative method. Before doing so, we first state a preliminary lemma.

Lemma 5.3. — *Let $K \subset \mathbb{R}^N$ be a compact set and assume that J satisfies* (1.6). *Let $k > 0$ and let $w \in C(\mathbb{R}^N \setminus K)$ be such that*

$$Lw - kw \geqslant 0 \quad \text{in } \mathbb{R}^N \setminus K, \tag{5.7}$$

and that

$$\limsup_{|x|\to\infty} w(x) \leqslant 0. \tag{5.8}$$

Then,

$$w \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus K.$$

Proof. — Suppose, by contradiction, that $\sup_{\mathbb{R}^N\setminus K} w > 0$. Then, by assumption (5.8), there exists a number $r > 0$ with $K \subset B_r$ and a sequence $(x_j)_{j\geqslant 0} \subset B_r \setminus K$ such that

$$\lim_{j\geqslant 0} w(x_j) = \sup_{B_r\setminus K} w = \sup_{\mathbb{R}^N\setminus K} w > 0. \tag{5.9}$$

Since $(x_j)_{j\geqslant 0}$ is bounded, up to extraction of a subsequence, there exists a point $\bar{x} \in \overline{B_r \setminus K}$ such that $x_j \to \bar{x}$ as $j \to \infty$. Moreover, since w is continuous and (5.7) is satisfied everywhere in $\mathbb{R}^N \setminus K$, it makes sense to evaluate (5.7) at x_j for any $j \geqslant 0$. That is, we have

$$\int_{\mathbb{R}^N\setminus K} J(x_j - y)(w(y) - w(x_j))\mathrm{d}y \geqslant kw(x_j) \quad \text{for any } j \geqslant 0.$$

But, since $k > 0$, using (5.9) and the dominated convergence theorem, we obtain

$$0 \geqslant \int_{\mathbb{R}^N\setminus K} J(\bar{x} - y)\left(w(y) - \sup_{\mathbb{R}^N\setminus K} w\right)\mathrm{d}y \geqslant k \sup_{\mathbb{R}^N\setminus K} w > 0,$$

which is a contradiction. The proof is thereby complete. □

We are now in position to prove Lemma 2.1.

Proof of Lemma 2.1. — Let us first observe that, from the assumptions made on J, the operator L is linear and continuous on $(C_0(\mathbb{R}^N \setminus K), \|\cdot\|_\infty)$, where

$$C_0(\mathbb{R}^N \setminus K) := \left\{w \in C(\mathbb{R}^N \setminus K);\ \lim_{|x|\to\infty} w(x) = 0\right\}.$$

Indeed, this is because, given any $w \in C_0(\mathbb{R}^N \setminus K)$, we have

$$Lw(x) = \int_{\mathbb{R}^N} J(y)\left(\mathbb{1}_{x-\mathbb{R}^N\setminus K}(x) w(x-y)\right)\mathrm{d}y - \mathscr{J}(x)w(x),$$

where $\mathscr{J}$ is as in (2.4), and, by the dominated convergence theorem, we have that $Lw(x) \to 0$ as $|x| \to \infty$. The continuity of Lw is a mere consequence of the continuity of translations in $L^1(\mathbb{R}^N)$ and of the continuity of w, as is easily seen from the (trivial) inequality

$$|Lw(x_1) - Lw(x_2)| \leqslant 2\|w\|_\infty \int_{\mathbb{R}^N} |J(y+x_1-x_2) - J(y)|\mathrm{d}y + |w(x_1) - w(x_2)|, \tag{5.10}$$

which holds for any $x_1, x_2 \in \mathbb{R}^N \setminus K$. So that L indeed maps $C_0(\mathbb{R}^N \setminus K)$ into itself. Moreover, the continuity of the operator L follows from the fact that

$$\|Lw\|_\infty \leqslant 2\|w\|_\infty \quad \text{for any } w \in C_0(\mathbb{R}^N \setminus K).$$

Next, we let $k > 0$ be a number large enough so that the map $s \mapsto -ks - f(s)$ is decreasing in $[0, 1]$ and that $k \in \rho(L)$ where $\rho(L)$ denotes the resolvent of the operator L.

Let $\underline{u}$ and $\overline{u}$ be continuous global sub- and super-solutions to

$$Lu + f(u) = 0 \quad \text{in } \mathbb{R}^N \setminus K, \tag{5.11}$$

satisfying (2.1) and (2.2).

We will construct a solution u to (5.11) satisfying $\underline{u} \leqslant u \leqslant \overline{u}$ using a monotone iterative scheme. That is, we will construct u as the limit of an appropriate sequence of functions. The main tool behind our construction is the comparison principle Lemma 5.3. To this end, we have to make sure that the sequence we construct has the right asymptotic behavior as $|x| \to \infty$ (as required by Lemma 5.3). With this aim in mind, we first construct an appropriate sequence of auxiliary functions. Namely, we define $v_0 \equiv 0$ and, for $x \in \mathbb{R}^N \setminus K$ and $j \geqslant 0$, we let

$$Lv_{j+1}(x) - kv_{j+1}(x) = -kv_j(x) - f(\overline{u}(x) + v_j(x)) - L\overline{u}(x). \tag{5.12}$$

Let us check that the v_j's are well-defined elements of $C_0(\mathbb{R}^N \setminus K)$. Since $k \in \rho(L)$ and $0 \equiv v_0 \in C_0(\mathbb{R}^N \setminus K)$, v_1 is a well-defined element of $C_0(\mathbb{R}^N \setminus K)$ as soon as

$$f(\overline{u}(\cdot)) + L\overline{u}(\cdot) \in C_0(\mathbb{R}^N \setminus K),$$

which is the case since $f(1) = 0$, f is continuous, $\overline{u}(x) \to 1$ as $|x| \to \infty$ and $L\overline{u} \in C_0(\mathbb{R}^N \setminus K)$ (because $\overline{u} \in C(\mathbb{R}^N \setminus K)$) and

$$L\overline{u}(x) = \int_{\mathbb{R}^N} J(y)\mathbb{1}_{x-\mathbb{R}^N\setminus K}(y)(\overline{u}(x-y) - \overline{u}(x))\mathrm{d}y \underset{|x|\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} J(y)(1-1)\mathrm{d}y = 0.$$

Similarly, if, for some $j \geqslant 0$, it holds that $v_j \in C_0(\mathbb{R}^N \setminus K)$, then, given that $k \in \rho(L)$ and that $L\overline{u} \in C_0(\mathbb{R}^N \setminus K)$, v_{j+1} is a well-defined element of $C_0(\mathbb{R}^N \setminus K)$ as soon as

$$f(\overline{u}(\cdot) + v_j(\cdot)) \in C_0(\mathbb{R}^N \setminus K),$$

which trivially holds since f is continuous, $f(1) = 0$ and $\overline{u}(x) \to 1, v_j(x) \to 0$ as $|x| \to \infty$. Whence, by induction, we infer that the v_j's are, indeed, well-defined elements of $C_0(\mathbb{R}^N \setminus K)$.

Let us now define a sequence $(u_j)_{j\geqslant 0} \subset C(\mathbb{R}^N \setminus K)$ by setting $u_j := \overline{u} + v_j$. Then, by construction, for any $x \in \mathbb{R}^N \setminus K$ and $j \geqslant 0$, we have

$$Lu_{j+1}(x) - ku_{j+1}(x) = -ku_j(x) - f(u_j(x)), \tag{5.13}$$

and the u_j's satisfy the limit condition

$$\lim_{|x|\to\infty} u_j(x) = 1 \quad \text{for any } j \geqslant 0. \tag{5.14}$$

We will show that the desired solution to (5.11) can be obtained as the pointwise limit of $(u_j)_{j\geqslant 0}$. Let us proceed step by step. First, when $j = 0$, we have

$$Lu_1(x) - ku_1(x) = -ku_0(x) - f(u_0(x)) \quad \text{for } x \in \mathbb{R}^N \setminus K. \tag{5.15}$$

We claim that $\underline{u} \leqslant u_1 \leqslant u_0 = \overline{u}$ in $\mathbb{R}^N \setminus K$. Indeed, we have

$$\begin{cases} L(u_1 - u_0)(x) - k(u_1 - u_0) = -Lu_0(x) - f(u_0(x)), \\ L(u_1 - \underline{u})(x) - k(u_1 - \underline{u}) \leqslant f(\underline{u}(x)) + k\underline{u}(x) - f(u_0(x)) - ku_0(x). \end{cases}$$

Since $u_0 = \overline{u}$ is a super-solution to (5.11), $\underline{u} \leqslant \overline{u}$ and $s \mapsto -ks - f(s)$ is decreasing, we obtain that

$$\begin{cases} L(u_1 - u_0)(x) - k(u_1 - u_0) \geqslant 0, \\ L(u_1 - \underline{u})(x) - k(u_1 - \underline{u}) \leqslant 0. \end{cases} \tag{5.16}$$

By construction of u_1 (remember (2.1) and (5.14)), we have

$$\lim_{|x|\to\infty} (u_1 - u_0)(x) = 0 \quad \text{and} \quad \liminf_{|x|\to\infty} (u_1 - \underline{u})(x) \geqslant 0. \tag{5.17}$$

This, together with Lemma 5.3, then gives that $\underline{u} \leqslant u_1 \leqslant u_0 = \overline{u}$ in $\overline{\mathbb{R}^N \setminus K}$. Similarly, by (5.13), the function $u_2 \in C(\mathbb{R}^N \setminus K)$ solves (5.15) with u_2 in place of u_1 and u_1 in place of u_0. Thus, from (2.1), (5.14) and the monotonicity of $s \mapsto -ks - f(s)$, we deduce that (5.16) and (5.17) still hold with u_2 instead of u_1 and u_1 instead of u_0. We may then apply the comparison principle Lemma 5.3 and we deduce that $\underline{u} \leqslant u_2 \leqslant u_1 \leqslant u_0 = \overline{u}$ in $\mathbb{R}^N \setminus K$. By induction, we infer that the u_j's satisfy the monotonicity relation

$$\underline{u} \leqslant \cdots \leqslant u_{j+1} \leqslant u_j \leqslant \cdots \leqslant u_2 \leqslant u_1 \leqslant u_0 = \overline{u}.$$

Since $(u_j)_{j\geqslant 0}$ is non-increasing and bounded from below by $\underline{u}$, the function

$$u(x) := \lim_{j\to\infty} u_j(x) \in [\underline{u}(x), \overline{u}(x)], \tag{5.18}$$

is well-defined for any $x \in \mathbb{R}^N \setminus K$. In particular, since $0 \leqslant \underline{u} \leqslant \overline{u} \leqslant 1$, it follows from (5.18) that $u \in L^\infty(\mathbb{R}^N \setminus K)$. It remains only to check that the function u is a solution to (5.11). For it, it suffices to let $j \to \infty$ in (5.13) (using the dominated convergence theorem), which then gives

$$Lu(x) + f(u(x)) = 0 \quad \text{for any } x \in \mathbb{R}^N \setminus K.$$

The proof is thereby complete. $\square$

Remark 5.4. — The same arguments also apply when the operator L is replaced by L_{g} provided that $J = \widetilde{J}(|\cdot|)$ satisfies (1.8), since it still holds that if $w(x) \to \ell \in \mathbb{R}$ as $|x| \to \infty$, then $L_{\mathrm{g}}w(x) \to 0$ as $|x| \to \infty$. Moreover, the continuity of w still implies the continuity of $L_{\mathrm{g}}w$ but the proof is less obvious since one can no longer rely on the continuity of translations in $L^1(\mathbb{R}^N)$. For the sake of completeness, we state a last lemma below which justifies why this is true.

LEMMA 5.5. — *Let $K \subset \mathbb{R}^N$ be a compact set and assume that $\widetilde{J}$ satisfies* (1.12) *and that $\widetilde{J}$ is supported in the segment line* $[0, r]$ *for some* $r > 0$. *Let $w \in C(\mathbb{R}^N \setminus K)$. Then, $L_{\mathrm{g}}w \in C(\mathbb{R}^N \setminus K)$.*

Proof. — Let $x_1, x_2 \in \mathbb{R}^N \setminus K$ with x_1 fixed and x_2 arbitrarily close to x_1. For $w \in C(\mathbb{R}^N \setminus K)$, the analogue of (5.10) is here:

$$\begin{aligned}|L_{\mathrm{g}}w(x_1) - L_{\mathrm{g}}w(x_2)| \leqslant 2\|w\|_\infty \left|\int_{\mathbb{R}^N} [\widetilde{J}(d_{\mathrm{g}}(x_1, y)) - \widetilde{J}(d_{\mathrm{g}}(x_2, y))]\mathrm{d}y\right| \\ + \|\widetilde{\mathscr{J}}\|_\infty |w(x_1) - w(x_2)|,\end{aligned}$$

where $\widetilde{\mathscr{J}}$ is as in (2.7). Since $w \in C(\mathbb{R}^N \setminus K)$, the delicate part is to show that the first term on the right-hand side vanishes as $x_2 \to x_1$. This can be done as follows. Let $\delta > 0$ be small enough so that $x_2 \in B_{\delta/2}(x_1) \subset B_\delta(x_1) \subset \mathbb{R}^N \setminus K$. Then, we may write

$$\begin{aligned}&\left|\int_{\mathbb{R}^N \setminus K} [\widetilde{J}(d_{\mathrm{g}}(x_1, y)) - \widetilde{J}(d_{\mathrm{g}}(x_2, y))]\mathrm{d}y\right| \\ &\qquad \leqslant \int_{\mathbb{R}^N \setminus (B_\delta(x_1) \cup K)} |\widetilde{J}(d_{\mathrm{g}}(x_1, y)) - \widetilde{J}(d_{\mathrm{g}}(x_2, y))|\mathrm{d}y \\ &\qquad\quad + \int_{B_{\delta(x_1)}} |\widetilde{J}(d_{\mathrm{g}}(x_1, y)) - \widetilde{J}(d_{\mathrm{g}}(x_2, y))|\mathrm{d}y \\ &\qquad =: I_1(x_1, x_2) + I_2(x_1, x_2).\end{aligned}$$

Since $d_{\mathrm{g}}(x_i, y) = |x_i - y|$ for any $i \in \{1, 2\}$ and $y \in B_\delta(x_1)$, we have

$$I_2(x_1, x_2) \leqslant \|J(\cdot + x_1 - x_2) - J\|_{L^1(\mathbb{R}^N)} \underset{x_2 \to x_1}{\longrightarrow} 0.$$

On the other hand, since J is radially symmetric, $\operatorname{supp}(J) = B_r$ and $J \in W^{1,1}(B_r)$, by [71, Theorems 1.1 and 2.3], we have that $\widetilde{J} \in W^{1,1}((0, r), t^{N-1})$, $\widetilde{J}$ is almost everywhere equal to a continuous function, $\widetilde{J}'$ exists almost everywhere and

$$\int_0^r |\widetilde{J}'(t)| t^{N-1}\mathrm{d}t \leqslant C_1 \int_{B_r} |\nabla J(z)|\mathrm{d}z. \tag{5.19}$$

Therefore, using the fact that $d_{\mathrm{g}}(x_i, y) \geqslant |x_i - y| \geqslant \delta/2$ for any $y \in \mathbb{R}^N \setminus (B_\delta(x_1) \cup K)$, we have

$$I_1(x_1, x_2) \leqslant \int_{\mathbb{R}^N \setminus (B_\delta(x_1) \cup K)} \int_{d_{\mathrm{g}}(x_2,y)}^{d_{\mathrm{g}}(x_1,y)} |\widetilde{J}'(t)| \mathrm{d}t \mathrm{d}y$$
$$\leqslant \left(\frac{2}{\delta}\right)^{N-1} \int_{\mathbb{R}^N \setminus (B_\delta(x_1) \cup K)} \int_{d_{\mathrm{g}}(x_2,y)}^{d_{\mathrm{g}}(x_1,y)} |\widetilde{J}'(t)| t^{N-1} \mathrm{d}t \mathrm{d}y. \tag{5.20}$$

Now, since $x_1, x_2 \in B_{\delta/2}(x_1) \subset \mathbb{R}^N \setminus K$ and $d_{\mathrm{g}}(\cdot, \cdot)$ is a distance, we have

$$|d_{\mathrm{g}}(x_1, y) - d_{\mathrm{g}}(x_2, y)| \leqslant d_{\mathrm{g}}(x_1, x_2) = |x_1 - x_2| \underset{x_2 \to x_1}{\longrightarrow} 0.$$

Therefore, using (5.19), (5.20) and the dominated convergence theorem, we obtain that

$$I_1(x_1, x_2) \to 0 \quad \text{as } x_2 \to x_1.$$

This completes the proof. $\square$

Chapter 5

Some monotonicity results for general systems of nonlinear elliptic PDEs

This chapter is inspired by the paper [32] written in collaboration with S. Dipierro, published in the *Journal of Differential Equations*.

1 Introduction

1.1 Monotonicity and 1-dimensional symmetry for systems

In this chapter, we study monotonicity properties of minima and stable solutions of general energy functionals of the type

$$\int_{\Omega} F(\nabla u, \nabla v, u, v, x)\, \mathrm{d}x, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$.

Recent years have seen numerous ongoing research activities in investigating symmetry properties of systems of PDEs. A typical example is

$$\begin{cases} \Delta u = uv^2, \\ \Delta v = vu^2, \\ \quad u, v > 0. \end{cases} \tag{1.2}$$

which arise in phase separation for Bose-Einstein condensates with multiple states. The interested reader may refer to [18, 20] (and references therein) for a derivation of this phase separation model.

In particular, in [18], the authors establish the existence, symmetry and non-degeneracy of solutions to (1.2) in one dimension. They show that entire solutions are *reflectionally symmetric*, i.e. there exists $x_0 \in \mathbb{R}$ such that $u(x - x_0) = v(x_0 - x)$ for any $x \in \mathbb{R}$. They also establish a result which may be seen as the analogue of a celebrated conjecture of De Giorgi for problem (1.2) in dimension 2. More precisely, they show that monotone solutions to (1.2) in dimension 2 have one-dimensional symmetry under the growth condition

$$u(x) + v(x) \leqslant C(1 + |x|),$$

for some $C > 0$. On the other hand, the linear growth is the lowest possible for positive solutions to (1.2). Namely, if there exists $\alpha \in (0, 1)$ such that

$$u(x) + v(x) \leqslant C(1 + |x|)^{\alpha},$$

then both u and v are constants, and at least one of them is 0, see [115].

In dimension 2, Farina [62] improves the result in [18] showing that if (u, v) is a monotone solution to (1.2) and has at most algebraic growth at infinity, then it must be one-dimensional. It turns out that the monotonicity condition can be weakened in order to get the one-dimensional symmetry for solutions to (1.2). Indeed, it has been proved in [20] that *stable solutions* to (1.2) are one-dimensional in $\mathbb{R}^2$.

Let us also mention [66] where symmetry results for systems in dimension 2 involving more general nonlinearities have been obtained. Quasilinear (possibly degenerate) elliptic systems in $\mathbb{R}^2$ have also been considered in [55]. As for dimensions higher than 2, in [64], the authors prove that if (u, v) has algebraic growth and

$$\lim_{x_N \to \pm\infty} \big(u(x', x_N) - v(x', x_N)\big) = \pm\infty \quad \text{uniformly in } x' \in \mathbb{R}^{N-1},$$

then (u, v) depends on x_N only. Also, in [150, 151], it has been proved that if (u, v) has linear growth, then it is one-dimensional.

Recently, the nonlocal counterpart of (1.2) has also been investigated, see e.g. [133, 134, 147, 152], and some symmetry results have been obtained in [56] for a quite general system of nonlocal PDEs of the form

$$\begin{cases} (-\Delta)^{s_1} u = F_1(u, v), \\ (-\Delta)^{s_2} v = F_2(u, v), \end{cases}$$

where F_1 and F_2 denote the derivatives of a function $F \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$ with respect to the first and the second variable, respectively; s_1, $s_2 \in (0, 1)$ and, given $s \in (0, 1)$, $(-\Delta)^s$ stands for the fractional Laplace operator

$$(-\Delta)^s u(x) := \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, \mathrm{d}y,$$

where p.v. denotes the Cauchy principal value (see [113] for the definition and further details). See also [67] for symmetry results for nonlocal systems of equations.

Our aim in this chapter is to provide monotonicity results for minima and stable solutions of energy functionals of the form (1.1) (thus embracing all the systems considered above). We will consider the case in which both the domain Ω and the functional F are invariant under translations in the e_N-direction, and we will deal with symmetry properties of minima or stable solutions in the class of functions which are obtained by piecewise Lipschitz domain deformations in the e_N-direction (see Definitions 1.1 and 1.2). We will also require some rather mild assumptions on F and a growth condition (see (1.5)).

The key tool of our proofs relies on a technique introduced in [124] to study the regularity of fractional minimal surfaces in dimension 2 (see [41] where these object were introduced), and developed in [125] to work in a more general setting. See also [42, 57] where these techniques have been used in the context of free boundary problems.

Here we adapt the new strategy of [125] to the case of general systems of equations. The idea is to look at the stability inequality *without dealing with its precise form* (which, in some cases, can be very hard to handle), and to simply compare the energy of the couple (u, v) to that of its translations. To this end, one also needs to modify the "translated" couple at infinity to make it a compact perturbation of (u, v). Here the growth condition comes into play and ensures that the energy of the perturbed couple can be made arbitrarily close to the energy of (u, v). Then, if u and v are not monotone in the e_N-direction, one can modify locally the perturbed couple in order to get lower energy, but this is in contradiction with the minimality of (u, v). This strategy allows to deal with quite general form of energy functionals.

We now introduce the setting in which we will work and give precise definitions and statements of our results.

1.2 The mathematical setting

We consider a domain $\Omega \subset \mathbb{R}^N$ and study the symmetry properties of minima for functionals of the type

$$\mathscr{E}(u, v) := \int_\Omega F(\nabla u, \nabla v, u, v, x)\mathrm{d}x.$$

We assume that the domain Ω and the functional F are invariant under translations in the e_N-direction, namely

$$\Omega = \mathscr{V} \times \mathbb{R}, \quad \mathscr{V} \subseteq \mathbb{R}^{N-1},$$

and F does not depend on the x_N-coordinate. We will denote a point in Ω by $x = (x', x_N) \in \mathscr{V} \times \mathbb{R}$. We also suppose that F is convex with respect to the first two variables. Precisely, we assume that

$$F = F(p_1, p_2, z_1, z_2, x') \in C(\mathbb{R}^{2N} \times \mathbb{R}^2 \times \mathscr{V}), \tag{1.3}$$

and F is C^2 and uniformly convex in p_1 and p_2 at all p_1, p_2 with

$$p_1 \cdot e_N \neq 0 \;\text{ and }\; p_2 \cdot e_N \neq 0, \;\text{ for all } (z_1, z_2, x') \in \mathbb{R}^2 \times \mathscr{V}.$$

Finally, we assume that there exists a constant $C > 0$ such that

$$\begin{aligned} |F_{p_1p_1}(p_1 + q_1, p_2, z_1, z_2, x')| &\leqslant C|F_{p_1p_1}(p_1, p_2, z_1, z_2, x')|, \\ |F_{p_2p_2}(p_1, p_2 + q_2, z_1, z_2, x')| &\leqslant C|F_{p_2p_2}(p_1, p_2, z_1, z_2, x')|, \\ |F_{p_1p_2}(p_1 + q_1, p_2 + q_2, z_1, z_2, x')| &\leqslant C|F_{p_1p_2}(p_1, p_2, z_1, z_2, x')|, \end{aligned} \tag{1.4}$$

for any $p_1, p_2, q_1, q_2 \in \mathbb{R}^N$ with $|q_1| \leqslant |p_1 \cdot e_N|/4$ and $|q_2| \leqslant |p_2 \cdot e_N|/4$. We will make these assumptions throughout the chapter.

Given $R > 0$, we consider the following energy functional

$$\mathscr{E}_R(u, v) := \int_{\Omega \cap B_R} F(\nabla u(x), \nabla v(x), u(x), v(x), x') \mathrm{d}x,$$

where, as usual, B_R denotes the ball of radius R centered at the origin.

Following the line in [125], we want to study the symmetry properties of minimal or stable solutions of the above functional among suitable perturbations, obtained by domain deformations in the direction given by e_N. To do this, we recall the following definitions introduced in [125].

Definition 1.1. — *Given a function w, an e_N-Lipschitz deformation of w in B_R is a function $\overline{w}$ defined by*

$$\overline{w}(x) = w(x + \varphi(x) e_N) \quad \textit{for any } x \in \Omega,$$

where $\varphi \in C^{0,1}(\mathbb{R}^N)$ is a function supported in B_R with $\|\partial_N \varphi\|_{L^\infty(\mathbb{R}^N)} < 1$.

Definition 1.2. — *Given a function $w \in C^{0,1}(\Omega)$, a piecewise e_N-Lipschitz deformation of w in B_R is a function $\overline{w} \in C^{0,1}(\Omega)$ defined by*

$$\overline{w}(x) = \overline{w}^{(i)}(x) \quad \textit{for some } i \textit{ (depending on } x \in \Omega),$$

where $\overline{w}^{(1)}, \ldots, \overline{w}^{(m)}$ constitute a finite number of e_N-Lipschitz deformations of w in B_R. In this case, we write $\overline{w} \in D_R(w)$.

Also, if all $\overline{w}^{(i)}$ satisfy

$$\overline{w}^{(i)}(x) = w(x + \varphi^{(i)}(x) e_N) \quad \textit{with } \|\varphi^{(i)}\|_{C^{0,1}(\Omega)} \leqslant \delta$$

for some δ, we write $\overline{w} \in D^\delta_R(w)$.

We recall also some elementary properties of piecewise e_N-Lipschitz deformations, which follow easily from the definitions above.

Proposition 1.3. — *The following properties hold true:*

(i) *if* $\overline{w}^1, \overline{w}^2 \in D_R^\delta(w)$, *then* $\min\{\overline{w}^1, \overline{w}^2\}, \max\{\overline{w}^1, \overline{w}^2\} \in D_R^\delta(w)$;

(ii) *if* $\overline{w} \in D_R^\delta(w), \widetilde{w} \in D_R^\delta(\overline{w})$, *then* $\widetilde{w} \in D_R^{3\delta}(w)$;

(iii) *if* $\overline{w} \in D_R^\delta(w)$, *then* $\|\overline{w} - w\|_{L^\infty(\Omega)} \leqslant C\delta\|w\|_{C^{0,1}(\Omega)}$;

(iv) *if* $\overline{w} \in D_R^\delta(w), w \in C^{1,1}(\Omega)$, *then* $\|\overline{w} - w\|_{C^{0,1}(\Omega)} \leqslant C\delta\|w\|_{C^{1,1}(\Omega)}$.

In the sequel, we will also assume a growth condition on the functional $\mathscr{E}$. Namely, we suppose that there exists a constant $C > 0$ such that, for R sufficiently large,

$$\begin{aligned}\int_{\Omega\cap B_R} |F_{p_1p_1}(\nabla u, \nabla v, u, v, x')||\nabla u|^2 + |F_{p_2p_2}(\nabla u, \nabla v, u, v, x')||\nabla v|^2 \\ + |F_{p_1p_2}(\nabla u, \nabla v, u, v, x')||\nabla u||\nabla v| \mathrm{d}x \leqslant CR^2.\end{aligned} \tag{1.5}$$

1.3 A monotonicity result for minimizers of the energy functional

The first result here deals with e_N-minimizers of $\mathscr{E}$. To state it, we give the following:

Definition 1.4. — *We say that* (u, v), *with* $u, v \in C^{0,1}(\Omega)$, *is an* e_N*-minimizer for* $\mathscr{E}$ *if, for any* $R > 0$, *we have that* $\mathscr{E}_R(u, v)$ *is finite and*

$$\mathscr{E}_R(u, v) \leqslant \mathscr{E}_R(\overline{u}, \overline{v}),$$

for any $\overline{u} \in D_R(u)$ *and any* $\overline{v} \in D_R(v)$.

We are now in the position to state our first monotonicity result.

Theorem 1.5. — *Let* $u, v \in C^1(\Omega)$ *and let* F *satisfy* (1.3) *and* (1.4). *Suppose that* (u, v) *is an* e_N*-minimizer for the energy* $\mathscr{E}$ *and that the growth condition* (1.5) *is satisfied. Then,* u *and* v *are monotone on each line in the* e_N*-direction, i.e., for any* $x \in \Omega$, *either* $u_N(x + te_N) \geqslant 0$ *or* $u_N(x + te_N) \leqslant 0$, *and either* $v_N(x + te_N) \geqslant 0$ *or* $v_N(x + te_N) \leqslant 0$, *for any* $t \in \mathbb{R}$.

Remark 1.6. — We notice that if a continuous function w is monotone on each line in $\mathbb{R}^N$, then it is one-dimensional, that is there exist a function $w_0 : \mathbb{R} \to \mathbb{R}$ and a unit direction $\omega \in \mathbb{S}^{N-1}$ such that $w(x) = w_0(\omega \cdot x)$ (see [125, Section 9] for a proof of this fact).

1.4 A monotonicity result for stable solutions

The second result that we state concerns stable critical points of the energy instead of e_N-minimizers. We give a definition of the stability condition that involves the second variation of the functional $\mathscr{E}$ for deformations of the solution both in the e_N-direction and in the vertical e_{N+1}-direction. Precisely, we give the following:

Definition 1.7. — *Given a function w, a piecewise Lipschitz deformation of w in the $\{e_N, e_{N+1}\}$-directions is a function $\widetilde{w}$ defined as*

$$\widetilde{w} = \overline{w} + \phi,$$

where $\overline{w} \in D_R^\delta(w)$ and ϕ is a Lipschitz function with compact support in $\Omega \cap B_R$, and $\|\phi\|_{C^{0,1}(\Omega)} \leqslant \delta$. In this case, we write $w \in \mathscr{D}_R^\delta(w)$.

Definition 1.8. — *We say that (u, v) is an $\{e_N, e_{N+1}\}$-stable solution of $\mathscr{E}$ if for every $R > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ depending on R, ε, u and v such that, for any $t \in (0, \delta)$, we have that $\mathscr{E}_R(u, v)$ is finite and*

$$\mathscr{E}_R(\widetilde{u}, \widetilde{v}) - \mathscr{E}_R(u, v) \geqslant -\varepsilon t^2, \tag{1.6}$$

for any $\widetilde{u} \in \mathscr{D}_R^t(u)$ and any $\widetilde{v} \in \mathscr{D}_R^t(v)$.

Remark 1.9. — As we shall see later on (see Lemma 2.1), this notion of $\{e_N, e_{N+1}\}$-stable solution shares intimate links with the classical notion of stability. In fact, this follows from the same arguments as in the case of a single equation (see [125]). Actually, we can infer directly from the definition that

(i) if (u, v) is a classical minimizer of $\mathscr{E}$, then (u, v) is $\{e_N, e_{N+1}\}$-stable for $\mathscr{E}$;

(ii) if (u, v) is $\{e_N, e_{N+1}\}$-stable for $\mathscr{E}$, then (u, v) is a critical point for $\mathscr{E}$.

This is because we allow perturbation in the e_{N+1}-direction.

Next, we state our monotonicity result for $\{e_N, e_{N+1}\}$-stable solutions.

Theorem 1.10. — *Let $F \in C^{3,\alpha}(\mathbb{R}^{2N} \times \mathbb{R}^2 \times \mathscr{V})$ satisfy (1.4) and let (u, v) be such that either $u, v \in C^{0,1}(\Omega)$ are convex or $u, v \in C^{1,1}(\Omega)$. Moreover, suppose that (u, v) is an $\{e_N, e_{N+1}\}$-stable solution of $\mathscr{E}$, and that the growth condition (1.5) holds true. Then, u and v are monotone in the e_N-direction, i.e. either $u_N \geqslant 0$ or $u_N \leqslant 0$ and either $v_N \geqslant 0$ or $v_N \leqslant 0$ in Ω.*

By Theorem 1.10 and Remark 1.6 one obtains that an $\{e_N, e_{N+1}\}$-stable solution of $\mathscr{E}$ is one-dimensional, in the sense that there exist $\overline{u}$, $\overline{v} : \mathbb{R} \to \mathbb{R}$ and ω_u, $\omega_v \in \mathbb{S}^{N-1}$ such that $(u(x), v(x)) = \big(\overline{u}(\omega_u \cdot x), \overline{v}(\omega_v \cdot x)\big)$. We remark that, in general, it is not possible to conclude that u and v have the same direction of monotonicity. It is not true, for instance, for an uncoupled system of PDEs, such as

$$\begin{cases} \Delta u = 0, \\ \Delta v = 0. \end{cases}$$

See also [55, Remark 1.4] for a discussion on this fact.

Nonetheless, there are cases in which it is possible to obtain that u and v have the same direction of monotonicity, even for quite general systems, see e.g. [55] and [56].

1.5 Organization of the chapter

In Section 2 we show that, under suitable assumptions, the notion of $\{e_N, e_{N+1}\}$-stability is equivalent to the notion of classical stability. In Section 3 we perform a local analysis and, in Section 4, we estimate the energy of the perturbation at infinity, collecting the basic facts in order to prove Theorems 1.5 and 1.10 in Sections 5 and 6, respectively. Finally, in Section 7, we investigate some applications of our results in some concrete cases.

2 Stability property

The following lemma justifies Remark 1.9.

LEMMA 2.1. — *If $\Omega = \mathbb{R}^N$, $F \in C^2$ and u, $v \in C^2(\mathbb{R}^N)$, then (u, v) is a classical stable solution for the energy functional $\mathscr{E}$ if, and only if, (u, v) is an $\{e_N, e_{N+1}\}$-stable solution of $\mathscr{E}$ according to Definition 1.8.*

Proof. — Suppose first that (u, v), with u, $v \in C^2(\mathbb{R}^N)$, is a classical stable solution. That is, by definition, for any $\varphi^1, \varphi^2 \in C_0^{0,1}(B_R)$, we have

$$\liminf_{t \to 0} \frac{\mathscr{E}_R(u + t\varphi^1, v + t\varphi^2) - \mathscr{E}_R(u, v)}{t^2} \geqslant 0. \tag{2.1}$$

In addition, setting

$$\Delta^1_{\varphi^1,\varphi^2} F := F(\nabla u + t\nabla\varphi^1, \nabla v + t\nabla\varphi^2, u + t\varphi^1, v + t\varphi^2, x') - F(\nabla u, \nabla v, u, v, x'), \tag{2.2}$$

we have

$$\Delta^1_{\varphi^1,\varphi^2} F = tH(\varphi^1, \varphi^2) + \frac{t^2}{2} L(\varphi^1, \varphi^2) + o(t^2), \tag{2.3}$$

where

$$H(\varphi^1, \varphi^2) := F_{(p_1)_i}\varphi^1_i + F_{z_1}\varphi^1 + F_{(p_2)_i}\varphi^2_i + F_{z_2}\varphi^2,$$

$$
\begin{aligned}
L(\varphi^1,\varphi^2) := {} & F_{(p_1)_i(p_1)_j}\varphi^1_i\varphi^1_j + F_{z_1z_1}(\varphi^1)^2 + 2F_{(p_1)_iz_1}\varphi^1\varphi^1_i \\
& + F_{(p_2)_i(p_2)_j}\varphi^2_i\varphi^2_j + F_{z_2z_2}(\varphi^2)^2 + 2F_{(p_2)_iz_2}\varphi^2\varphi^2_i \\
& + F_{(p_1)_i(p_2)_j}\varphi^1_i\varphi^2_j + 2F_{z_2z_1}\varphi^1\varphi^2 + 2F_{(p_2)_iz_1}\varphi^2\varphi^2_i \\
& + F_{(p_2)_i(p_1)_j}\varphi^2_i\varphi^1_j + 2F_{(p_1)_iz_2}\varphi^2\varphi^1_i,
\end{aligned}
$$

and the derivatives of F are evaluated at $(\nabla u, \nabla v, u, v, x')$.

Now, we observe that, since (u, v) is a critical point of $\mathscr{E}$, it holds that

$$\int_{B_R} H(\varphi^1,\varphi^2) = 0.$$

So integrating (2.3) and recalling (2.2), we obtain that

$$\mathscr{E}_R(u + t\varphi^1, v + t\varphi^2) - \mathscr{E}_R(u, v) = \frac{t^2}{2}\int_{B_R} L(\varphi^1,\varphi^2)\mathrm{d}x + o(t^2). \tag{2.4}$$

Dividing by t^2 and recalling (2.1) we find that

$$\int_{B_R} L(\varphi^1,\varphi^2)\mathrm{d}x \geqslant 0.$$

Thus, (2.4) yields

$$\mathscr{E}_R(u + t\varphi^1, v + t\varphi^2) - \mathscr{E}_R(u, v) \geqslant o(t^2). \tag{2.5}$$

Given $\widetilde{u} \in \mathscr{D}^t_R(u)$ and $\widetilde{v} \in \mathscr{D}^t_R(v)$ we may choose $\varphi^1 := \frac{\widetilde{u}-u}{t}$ and $\varphi^2 := \frac{\widetilde{v}-v}{t}$. By construction, the Lipschitz norm of φ^1 and φ^2 is bounded by a quantity that does not depend on t. Therefore (2.5) implies (1.6), and so it follows that (u, v) is $\{e_N, e_{N+1}\}$-stable.

Reciprocally, let (u, v) be an $\{e_N, e_{N+1}\}$-stable solution of $\mathscr{E}$ (and so, a critical point). In this case, we choose $\widetilde{u} := u + t\varphi^1$ and $\widetilde{v} := v + t\varphi^2$ in (1.6). Then, taking ε arbitrarily small, we prove (2.1). Therefore (u, v) is a stable solution of $\mathscr{E}$. This completes the proof. □

3 Local analysis

In this section, we consider local perturbations of (u, v) in the e_N-direction. More precisely, we will show that we can perturb the couples

$$\big(\max\{u(x), u(x) + te_N\}, v(x)\big) \quad \text{and} \quad \big(u(x), \max\{v(x), v(x) + te_N\}\big)$$

in such a way that the energy of the "perturbed couples" decreases.

We first show that if one of the two elements of a couple (u, v) is a maximum of two functions that form an angle at an intersection point, then it cannot be an e_N-minimizer for $\mathscr{E}$.

Lemma 3.1. — *Suppose that* $0 \in \Omega$, *and that* u, u^1, v, v^1 *are* C^1 *functions which satisfy*

$$u(0) = u^1(0), \quad u^1_N(0) < 0 < u_N(0), \tag{3.1}$$

and

$$v(0) = v^1(0), \quad v_N(0) < 0 < v^1_N(0).$$

Then, the couples

$$g_1 := (\max\{u, u^1\}, v) \ \text{ and } \ g_2 := (u, \max\{v, v^1\}),$$

are not e_N*-minimizers for* $\mathscr{E}$ *in any ball* B_η *with* $\eta > 0$.

Proof. — We prove the lemma for the couple g_1, the proof for g_2 being similar with obvious modifications.

We argue by contradiction and we assume that g_1 is an e_N-minimizer in a ball B_η, for some $\eta > 0$. We define $F_0(p_1, p_2) := F(p_1, p_2, 0, 0, 0)$. Moreover, we observe that if we subtract a linear functional from F then this does not affect the minimality. Indeed, if we consider

$$\widetilde{F}(p_1, p_2, z_1, z_2, x) := F(p_1, p_2, z_1, z_2, x) - p_0 \cdot p_1,$$

and if $\widetilde{\mathscr{E}}$ is the associated energy functional, then we have

$$\mathscr{E}_R(u, v) - \widetilde{\mathscr{E}}_R(u, v) = \int_{B_R} p_0 \cdot \nabla u \, \mathrm{d}x = \int_{\partial B_R} p_0 \, u \cdot \nu \, \mathrm{d}\sigma.$$

This means that the difference between $\mathscr{E}_R(u, v)$ and $\widetilde{\mathscr{E}}_R(u, v)$ is a term which depends only on the boundary value of u. Therefore, if (u, v) is an e_N-minimizer for $\mathscr{E}$, then it is an e_N-minimizer for $\widetilde{\mathscr{E}}$ as well. In view of this, we may assume that

$$F_0(\nabla u(0), \nabla v(0)) = F_0(\nabla u^1(0), \nabla v(0)). \tag{3.2}$$

Moreover, we can suppose that $u(0) = u^1(0) = 0$ (up to translate F in the variable z_1). Now, for small $r > 0$, we define the rescaled functions

$$u_r(x) := r^{-1} u(rx), \quad u^1_r(x) := r^{-1} u^1(rx) \ \text{ and } \ v_r(x) := r^{-1} v(rx).$$

We also set $g_r := (\max\{u_r, u^1_r\}, v_r)$. If

$$F_r(p_1, p_2, z_1, z_2, x) := F(p_1, p_2, r z_1, r z_2, r x),$$

is the rescaled functional, we have that g_r is an e_N-minimizer for F_r in $B_{\eta/r}$. Sending $r \to 0^+$, we obtain the following limits, which are uniform on compact sets:

$$F_r \to F_0(p_1, p_2), \qquad \begin{array}{ll} u_r(x) \to u_0(x) := \nabla u(0) \cdot x, & \nabla u_r \to \nabla u_0, \\ u^1_r(x) \to u^1_0(x) := \nabla u^1(0) \cdot x, & \nabla u^1_r \to \nabla u^1_0, \\ v_r(x) \to v_0(x) := \nabla v(0) \cdot x, & \nabla v_r \to \nabla v_0. \end{array} \tag{3.3}$$

Now, we set $f_0 := \max\{u_0, u_0^1\}$ and we consider the couple $g_0 := (f_0, v_0)$. The fact that F is strictly convex in the variable p_1 implies that

$$g_0 \text{ is not a minimizer for } F_0. \tag{3.4}$$

To see this, given the function

$$\beta_R(x') := c\,\delta \max\{0, |x'| - R\},$$

with R large and c, δ small, we define

$$h_0 := 1 + \alpha\, u_0 + (1-\alpha)\, u_0^1 - \beta_R(x'), \tag{3.5}$$

where $\alpha \in (0,1)$ is small. Then, we have that

$$\text{the couple } \big(\max\{f_0, h_0\}, v_0\big) \text{ agrees with } g_0 \text{ outside the ball } B_{R+R_1}. \tag{3.6}$$

On the other hand, in B_R, considering h_0, we cut the graphs of two transversal linear functions by a single one. Therefore, if we take R sufficiently large,

$$\text{the couple } \big(\max\{f_0, h_0\}, v_0\big) \text{ has lower energy for } F_0 \text{ than } g_0. \tag{3.7}$$

To see this, we observe that, by definition, ∇f_0 coincides with either ∇u_0 or ∇u_0^1, and so from (3.2) we have that

$$F_0(\nabla u_0, \nabla v_0) = F_0(\nabla u_0^1, \nabla v_0) = F_0(\nabla f_0, \nabla v_0).$$

Therefore, by the strict convexity of F_0 with respect to the first variable, we have that, fixed α and δ,

$$\begin{aligned} F_0(\nabla h_0, \nabla v_0) &= F_0\big((\alpha \nabla u_0 + (1-\alpha)\nabla u_0^1, \nabla v_0\big) \\ &\leqslant \alpha F_0(\nabla u_0, \nabla v_0) + (1-\alpha) F_0(\nabla u_0^1, \nabla v_0) - \eta \\ &= F_0(\nabla f_0, \nabla v_0) - \eta, \end{aligned} \tag{3.8}$$

in $B_R \cap \{h_0 > f_0\}$, for some $\eta > 0$. On the other hand, the set $\{h_0 > f_0\}$ is contained in a strip in the e_N-direction. This and (3.6) imply that, when we integrate (3.8) in B_R, then the energy of $(\max\{f_0, h_0\}, v_0)$ is less than the energy of g_0 minus a term of order $\underline{C}\eta R^{N-1}$, whereas, when we integrate in B_{R+R_1}, the energy of $(\max\{f_0, h_0\}, v_0)$ is less than the energy of g_0 minus a term of order $\overline{C}\eta R^{N-2}$. All in all, if R is large enough we obtain (3.7). In turn, this implies (3.4).

Now, we set

$$f_r := \max\{u_r, u_r^1\}. \tag{3.9}$$

Since the convergence in (3.3) is uniform, we have that

$$h_r := \big(\max\{f_r, h_0\}, v_r\big),$$

has lower energy for F_r than g_r as well.

Hence, we can scale back and obtain that $h_*(x) := rh_r(x/r)$ has lower energy for F in $B_{r(R+R_1)} \subseteq B_\eta$ than g_1.

In order to get a contradiction, it remains to prove that h_* is an allowed perturbation of g_1 (according to Definition 1.2). Notice that this is equivalent to check that h_r is an allowed perturbation of g_r. That is, recalling (3.9), we have to prove that

$$\begin{gathered}\max\{f_r, h_0\} \text{ is a piecewise Lipschitz domain deformation of } f_r\\ \text{with Lipschitz norm bounded by } \delta.\end{gathered} \tag{3.10}$$

To do this, we recall the uniform convergence of u_r and u_r^1 to u_0 and u_0^1 respectively (as given by (3.3)) and the definition of h_0 given in (3.5) to obtain that

$$h_0(x) = 1 + \alpha\, u_r(x) + (1-\alpha)\, u_r^1(x) - \beta_R(x') + \omega_r(x), \tag{3.11}$$

where $\omega_r \to 0$ as $r \to 0^+$ locally uniformly, together with its derivatives.

Now we notice that our hypothesis in (3.1) gives that $\nabla u_0^1 \cdot e_N < 0 < \nabla u_0 \cdot e_N$. This, together with the uniform convergence in (3.3), implies that we can apply the Implicit Function Theorem, and we have that the part of the graph of $\max\{f_r, h_0\}$ where $h_0 > f_r$ can be obtained from u_r by a Lipschitz domain deformation with Lipschitz norm less than δ, provided that we take α sufficiently small. Indeed, fixed x' (and so looking at the 1-dimensional problem in the last variable only) and recalling (3.11), we obtain that

$$u_r(x', x_N + \varphi(x)) = h_0(x) = 1 + \alpha\, u_r(x) + (1-\alpha)\, u_r^1(x) - \beta_R(x') + \omega_r(x),$$

thanks to the Implicit Function Theorem in 1-dimension. Furthermore, if α and r are sufficiently small, the perturbation function φ has norm bounded by $\delta/2$. This shows (3.10) and finishes the proof of Lemma 3.1. $\square$

Now we deal with perturbations of the couples

$$\big(\max\{u(x), u(x) + te_N\}, v(x)\big) \quad \text{and} \quad \big(u(x), \max\{v(x), v(x) + te_N\}\big)$$

with lower energy.

Lemma 3.2. — *Let $u, v \in C^2(\Omega)$ be such that (u, v) is a critical point for the functional $\mathscr{E}$ in a neighborhood of the origin, and let F be C^2 in a neighborhood of*

$$(\nabla u(0), \nabla v(0), u(0), v(0), 0).$$

Suppose that

$$u_N(0) = 0, \quad \nabla u_N(0) \neq 0,$$

and set

$$w^1(x) := \max\{u(x), u(x+te_N)\}. \tag{3.12}$$

Then, for every $\eta > 0$, there exists a function ψ^1 which is Lipschitz and has compact support in B_η such that, for any small t,

$$\mathscr{E}_\eta(w^1 + t\psi^1, v) - \mathscr{E}_\eta(w^1, v) \leqslant -c\,t^2,$$

where $c > 0$ is a small constant that depends on F, η and u.

Remark 3.3. — We point out that we have an analogous result if we consider the function v instead of u. Precisely, if we assume that

$$v_N(0) = 0, \quad \nabla v_N(0) \neq 0,$$

and we consider

$$w^2(x) := \max\{v(x), v(x+te_N)\},$$

then, for every $\eta > 0$ there exists a function ψ^2 which is Lipschitz and has compact support in B_η such that, for any small t,

$$\mathscr{E}_\eta(u, w^2 + t\psi^2) - \mathscr{E}_\eta(u, w^2) \leqslant -ct^2,$$

where $c > 0$ is a small constant depending on F, η and v.

Proof of Lemma 3.2. — We define

$$u^1(x) := \frac{u(x+te_N) - u(x)}{t} \tag{3.13}$$

and we observe that

$$\|u^1 - u_N\|_{C^{0,1}(B_\eta)} = o(1) \quad \text{as } t \to 0. \tag{3.14}$$

We consider a Lipschitz function g_1 and, using the fact that $F \in C^2$ in the variables p_1 and z_1, we compute

$$\begin{aligned} F(\nabla u + t\nabla g_1, \nabla v, u + tg_1, v, x') &= F(\nabla u, \nabla v, u, v, x') + t\,(F_{p_1}\nabla g_1 + F_{z_1} g_1) \\ &\quad + t^2 \Big((\nabla g_1)^T F_{p_1 p_1} \nabla g_1 + F_{z_1 z_1} g_1^2 + 2 g_1 F_{p_1 z_1} \cdot \nabla g_1 \Big) + o(t^2), \end{aligned}$$

where the derivatives of F are evaluated at $(\nabla u, \nabla v, u, v, x')$ and the constant in the error term $o(t^2)$ depends on u, F and $\|g_1\|_{C^{0,1}(B_\eta)}$. Hence, we obtain

$$\mathscr{E}_\eta(u + tg_1, v) = \mathscr{E}_\eta(u, v) + t\,L(g_1) + t^2 Q(g_1) + o(t^2),$$

with

$$L(g_1) := \int_{B_\eta} (F_{p_1} \cdot \nabla g_1 + F_{z_1} g_1)\, \mathrm{d}x$$

and

$$\begin{aligned} Q(g_1) &:= \int_{B_\eta} G(\nabla g_1, g_1, x) \mathrm{d}x \\ &= \int_{B_\eta} \Big((\nabla g_1)^T F_{p_1 p_1} \nabla g_1 + F_{z_1 z_1} g_1^2 + 2 g_1 F_{p_1 z_1} \cdot \nabla g_1 \Big)\, \mathrm{d}x. \end{aligned}$$

Now, we take a Lipschitz function ψ^1 with compact support in B_η, and we use the fact that (u, v) is a critical point for $\mathscr{E}$ to obtain

$$\mathscr{E}_\eta(u + tu_+^1 + t\psi^1, v) - \mathscr{E}_\eta(u + tu_+^1, v) = t^2 \left(Q(u_+^1 + \psi^1) - Q(u_+^1) \right) + o(t^2). \quad (3.15)$$

Using (3.12) and (3.13), we can write

$$w^1 = u + tu_+^1. \quad (3.16)$$

Now, we claim that, for η sufficiently small,

$$Q(u_+^1) - Q((u_N)_+) = o(1) \quad \text{and} \quad Q(u_+^1 + \psi^1) - Q((u_N)_+ + \psi^1) = o(1), \quad (3.17)$$

as $t \to 0$. We focus on the first equality in (3.17), since the second is similar. To prove it, fixed $\mu > 0$, we define

$$\mathscr{B}_\mu^1 := B_\eta \cap \{|u_N| \leqslant \mu\} \quad \text{and} \quad \mathscr{B}_\mu^2 := B_\eta \cap \{|u_N| > \mu\}.$$

Notice that (3.14) implies that

$$\lim_{t \to 0} \|u_+^1 - (u_N)_+\|_{C^{0,1}(\mathscr{B}_\mu^2)} = \lim_{t \to 0} \|u^1 - u_N\|_{C^{0,1}(\mathscr{B}_\mu^2)} = 0. \quad (3.18)$$

As for the contribution coming from $\mathscr{B}_\mu^1$, we observe that, since $\nabla u_N(0) \neq 0$, the measure of $\mathscr{B}_\mu^1$ is at most of the order of μ. This, together with (3.18), gives that

$$\lim_{t \to 0} |Q(u_+^1) - Q((u_N)_+)| \leqslant C\mu,$$

for some $C > 0$. Since μ can be taken arbitrarily small, this implies (3.17).

Formula (3.17) means that, for η sufficiently small, we can write $(u_N)_+$ instead of u_+^1 in the right hand side of (3.15). Hence, recalling also (3.16), we have that

$$\mathscr{E}_\eta(w^1 + t\psi^1, v) - \mathscr{E}_\eta(w^1, v) = t^2 \left(Q((u_N)_+ + \psi^1) - Q((u_N)_+) \right) + o(t^2). \quad (3.19)$$

Now, we notice that u_N, 0 and G satisfy the hypotheses of Remark 4.3 in [125], and therefore $(u_N)_+$ is not a minimizer of Q. So we can choose the function ψ^1 in such a way that

$$Q((u_N)_+ + \psi^1) \leqslant Q((u_N)_+) - c$$

for some small c, which may depend on u, F and η. This together with (3.19) gives that, for small t,

$$\mathscr{E}_\eta(w^1 + t\psi^1, v) - \mathscr{E}_\eta(w^1, v) \leqslant -ct^2$$

and this concludes the proof. □

Now we show that, under an additional regularity hypothesis on F, the non-degeneracy condition $\nabla u_N \neq 0$ in Lemma 3.2 is satisfied. For this we use the Hopf Lemma, by adapting the proof of Lemma 4.6 in [125] to the slightly more delicate case of a system of equations.

LEMMA 3.4. — *Assume that*

$$\begin{aligned} &\textit{either } u, v \in C^{0,1}(\Omega) \textit{ are convex,} \\ &\textit{or } u, v \in C^{1,1}(\Omega). \end{aligned}$$

Furthermore, assume that (u, v) is a critical point for $\mathscr{E}$ and $F \in C^{3,\alpha}$ in a neighborhood of $(\nabla u(0), \nabla v(0), u(0), v(0), 0)$. Then, u and v are of class $C^{3,\alpha}$ in a neighborhood of 0. If, in addition, $u_N(0) = 0$ (resp. $v_N(0) = 0$) and u_N (resp. v_N) does not vanish identically in a neighborhood V_0 of 0, then there exists a point $x_0 \in V_0$ (resp. $x_1 \in V_0$) such that

$$u_N(x_0) = 0, \quad \nabla u_N(x_0) \neq 0 \quad \textit{and} \quad v_N(x_1) = 0, \quad \nabla v_N(x_1) \neq 0.$$

Proof. — Since (u, v) is a critical point for $\mathscr{E}$ it satisfies the elliptic system of equation

$$\begin{cases} G^1(\nabla^2 u, \nabla^2 v, \nabla u, \nabla v, u, v, x') := \mathrm{div}_x F_{p_1}(M) - F_{z_1}(M) = 0, \\ G^2(\nabla^2 u, \nabla^2 v, \nabla u, \nabla v, u, v, x') := \mathrm{div}_x F_{p_2}(M) - F_{z_2}(M) = 0, \end{cases}$$

where $M := (\nabla u, \nabla v, u, v, x')$. Consider the first equation of the above system, with v fixed. If $v \in C^{0,1}(\Omega)$ is convex, then $\nabla v \in L^\infty(\Omega)^N$ and $\nabla^2 v \in L^\infty(\Omega)^{N\times N}$ exist almost everywhere (this follows from Rademacher's theorem and Alexandrov's theorem). On the other hand, if $v \in C^{1,1}(\Omega)$, then $\nabla^2 v \in L^\infty(\Omega)^{N\times N}$ also exists almost everywhere (this follows from Rademacher's theorem). Thus, at fixed v, the first equation (in the variable $u = u_v$) is satisfied in the classical sense and the corresponding solution u_v belongs to $C^{3,\alpha}(\Omega)$ (this follows from Theorem 2.1 in [143], the Schauder estimates and the fact that $F \in C^{3,\alpha}$). In like manner, for any fixed u, the solution v_u of the second[1] equation belongs to $C^{3,\alpha}(\Omega)$. Therefore, $(u,v) \in C^{3,\alpha}(\Omega) \times C^{3,\alpha}(\Omega)$. This shows the first claim of Lemma 3.4.

Now, we focus on the second claim of Lemma 3.4. To prove it, we observe that

$$G^1 = G^1(q_1, q_2, p_1, p_2, z_1, z_2, x') \in C^{1,\alpha}. \tag{3.20}$$

Hence, differentiating in the e_N-direction we see that $w := u_N \in C^{2,\alpha}(\Omega)$ satisfies the linearized equation (in the viscosity sense)

$$G^1_{(q_1)_{ij}} w_{ij} + G^1_{(p_1)_i} w_i + G^1_{z_1} w = -G^1_{(q_2)_{ij}} (v_N)_{ij} - G^1_{(p_2)_i} (v_N)_i - G^1_{z_2} v_N. \tag{3.21}$$

Here, the derivatives of G^1 are evaluated at $(\nabla^2 u, \nabla^2 v, \nabla u, \nabla v, u, v, x')$. For the convenience of the reader, we rewrite (3.21) as

$$L^1 w = f_1,$$

where

$$\begin{aligned} L^1 w &:= G^1_{(q_1)_{ij}} w_{ij} + G^1_{(p_1)_i} w_i + G^1_{z_1} w, \\ f_1 &:= -G^1_{(q_2)_{ij}} (v_N)_{ij} - G^1_{(p_2)_i} (v_N)_i - G^1_{z_2} v_N. \end{aligned}$$

Moreover, we set

$$\begin{aligned} P &:= \{x \in V_0 : f_1(x) > 0\}, \\ E &:= \{x \in V_0 : f_1(x) = 0\}, \\ N &:= \{x \in V_0 : f_1(x) < 0\}. \end{aligned}$$

In addition, denote by $(P_j)_{j \in J_1}$ (resp. $(E_j)_{j \in J_2}$) the connected components of P (resp. E). This makes sense since $f_1 \in C^{0,\alpha}(V_0)$ (recall (3.20) and the fact that v is $C^{3,\alpha}$ in V_0).

Note that if $P = N = \varnothing$ then $f_1 \equiv 0$, and so the conclusion follows as in Lemma 4.6 in [125]. Thus, we may suppose now that either P or N are nonempty. Moreover, we can assume that P is nonempty, otherwise it is enough to replace w by $-w$.

Also, we observe that

$$\begin{aligned} &\text{if there exists a compact set } \Sigma \Subset V_0 \text{ such that } w_{|\Sigma} \equiv 0, \\ &\text{then, necessarily, } \Sigma \subset E. \end{aligned} \tag{3.22}$$

Now we take[2] $x_0 \in \{w = 0\} \cap P$. Then, there exists $j_1 \in J_1$ such that $x_0 \in P_{j_1}$. It follows from (3.22) and the continuity of f_1 that w cannot vanish identically in P_{j_1}.

[1] We may actually consider indifferently the first or the second equation, as only one equation is needed.

[2] We use here the short notation $\{w = 0\} = \{x \in V_0 : w(x) = 0\}$, and similarly for $\{w < 0\}$ and $\{w > 0\}$.

Thus, we can apply the Hopf Lemma to w at the point x_0, which admits a tangent ball from either $\{w < 0\} \cap P_{j_1}$ or $\{w > 0\} \cap P_{j_2}$, and the conclusion follows.

If $x_0 \in \{w = 0\} \cap N$, we replace w with $-w$ and we reason as above.

If $x_0 \in \{w = 0\} \cap E$, we let E_{j_2}, for some $j_2 \in J_2$, be the connected component of E such that $x_0 \in E_{j_2}$. By the continuity of f_1, there exists $j_1 \in J_1$ such that $K := E_{j_2} \cup P_{j_1}$ is connected (up to exchanging w with $-w$). Again, (3.22) implies that w cannot vanish identically in K, and so we can apply the Hopf Lemma to w at x_0, which admits a tangent ball from either $\{w < 0\} \cap K$ or $\{w > 0\} \cap K$. The same arguments apply to v_N. This completes the proof of Lemma 3.4. □

Remark 3.5. — The above result states that the non-degeneracy hypothesis $\nabla u_N \neq 0$ of Lemma 3.2 is always satisfied under a slightly stronger hypothesis on the potential F. On the other hand, notice that we obtain a $C^{3,\alpha}$ regularity for u and v near 0, even though we asked only for, say, a $C^{1,1}$ regularity. Thus, Lemma 3.2 is consistent.

4 Perturbations at infinity

In this section we modify the couples

$$\big(\max\{u(x), u(x) + te_N\}, v(x)\big) \quad \text{and} \quad \big(u(x), \max\{v(x), v(x) + te_N\}\big)$$

at infinity in such a way that they become compact perturbations of the couple (u, v). For this, for any $R > 1$ we define the function $\varphi_R : \mathbb{R} \to \mathbb{R}$, which is Lipschitz, even and with compact support, as

$$\varphi_R(s) := \begin{cases} 1 & \text{if } 0 \leqslant s \leqslant \sqrt{R}, \\ 2\,\frac{\log R - \log s}{\log R} & \text{if } \sqrt{R} < s < R, \\ 0 & \text{if } s \geqslant R. \end{cases} \tag{4.1}$$

We have that

$$\varphi_R'(s) := \begin{cases} 0 & \text{if } s \in (0, \sqrt{R}) \cup (R, +\infty), \\ \frac{-2}{s \log R} & \text{if } s \in (\sqrt{R}, R). \end{cases} \tag{4.2}$$

Also, for any $t \in (0, \sqrt{R}/4]$, we consider the following bi-Lipschitz change of coordinates:

$$x \mapsto y(x) := x + t\varphi_R(|x|)e_N. \tag{4.3}$$

In these new coordinates, we define the functions $u^+_{R,t}$ and $v^+_{R,t}$ as

$$u^+_{R,t}(y) := u(x) \quad \text{and} \quad v^+_{R,t}(y) := v(x).$$

We observe that $u^+_{R,t}(x)$ and $v^+_{R,t}$ coincide with $u(x-te_N)$ and $v(x-te_N)$ respectively in $B_{\sqrt{R}/2}$ and with $u(x)$ and $v(x)$ respectively outside B_R.

We also define $u^-_{R,t}$ and $v^-_{R,t}$ by replacing t with $-t$ in (4.3).

Now, we want to obtain an estimate of $\mathscr{E}_R(u^+_{R,t}, v^+_{R,t})$ and $\mathscr{E}_R(u^-_{R,t}, v^-_{R,t})$ in terms of $\mathscr{E}_R(u,v)$. For this, we notice that

$$D_x y = I + A,$$

where

$$A(x) := t\,\varphi_R'(|x|) \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1}{|x|} & \frac{x_2}{|x|} & \dots & \frac{x_N}{|x|} \end{pmatrix},$$

and that $\|A\| \leqslant t\,|\varphi_R'(|x|)| \ll 1$. Moreover,

$$D_y x = (I+A)^{-1} = I - \frac{A}{1+\mathrm{tr}(A)}.$$

Furthermore, the following relations hold:

$$\nabla_y u^+_{R,t} = \nabla_x u\, D_y x, \quad \nabla_y v^+_{R,t} = \nabla_x v\, D_y x \quad \text{and} \quad \mathrm{d}y = \big(1+\mathrm{tr}(A)\big)\mathrm{d}x.$$

Hence, we have

$$\begin{aligned}
&\int_{\Omega\cap B_R} F\left(\nabla_y u^+_{R,t}, \nabla_y v^+_{R,t}, u^+_{R,t}, v^+_{R,t}, y'\right)\mathrm{d}y \\
&= \int_{\Omega\cap B_R} F\left(\nabla_x u\left(I - \frac{A}{1+\mathrm{tr}(A)}\right), \nabla_x v\left(I - \frac{A}{1+\mathrm{tr}(A)}\right), u, v, x'\right)\big(1+\mathrm{tr}(A)\big)\mathrm{d}x.
\end{aligned}$$

Now we use the hypothesis (1.4) for F to bound the right hand side from above: more precisely, since $|(p_1 A)| \leqslant |p_1\cdot e_N|/4$ and $|(p_2 A)| \leqslant |p_2\cdot e_N|/4$, we have that

$$\begin{aligned}
F&\left(p_1\left(I - \frac{A}{1+\mathrm{tr}(A)}\right), p_2\left(I - \frac{A}{1+\mathrm{tr}(A)}\right), z_1, z_2, x'\right)\big(1+\mathrm{tr}(A)\big) \\
&\leqslant F(p_1,p_2,z_1,z_2,x')\big(1+\mathrm{tr}(A)\big) \\
&\quad - F_{p_1}(p_1,p_2,z_1,z_2,x')\cdot(p_1A) - F_{p_2}(p_1,p_2,z_1,z_2,x')\cdot(p_2A) \\
&\quad + C\big(\,|F_{p_1p_1}(p_1,p_2,z_1,z_2,x')|\,|p_1A|^2 + |F_{p_2p_2}(p_1,p_2,z_1,z_2,x')|\,|p_2A|^2 \\
&\quad + 2\,|F_{p_1p_2}(p_1,p_2,z_1,z_2,x')|\,|p_1A|\,|p_2A|\,\big).
\end{aligned}$$

If we consider $u^-_{R,t}$ and $v^-_{R,t}$, we can write the same inequality with A replaced by $-A$. Therefore, we obtain

$$\begin{aligned}
&\mathscr{E}_R(u^+_{R,t}, v^+_{R,t}) + \mathscr{E}_R(u^-_{R,t}, v^-_{R,t}) - 2\mathscr{E}_R(u,v) \\
&\qquad \leqslant C\int_{\Omega\cap B_R}\Big(|F_{p_1p_1}|\,|\nabla u|^2|A|^2 + |F_{p_2p_2}|\,|\nabla v|^2|A|^2 + 2\,|F_{p_1p_2}|\,|\nabla u||\nabla v||A|^2\Big)\mathrm{d}x \\
&\leqslant \frac{Ct^2}{(\log R)^2}\int_{\Omega\cap\left(B_R\setminus B_{\sqrt{R}}\right)}\Big(|F_{p_1p_1}||\nabla u|^2 + |F_{p_2p_2}||\nabla v|^2 + |F_{p_1p_2}||\nabla u||\nabla v|\Big)\frac{\mathrm{d}x}{|x|^2}, \quad (4.4)
\end{aligned}$$

where the derivatives of F are evaluated at $(\nabla u, \nabla v, u, v, x')$. Now we set

$$h(r) := \int_{\Omega\cap B_r}\big(|F_{p_1p_1}||\nabla u|^2 + |F_{p_2p_2}||\nabla v|^2 + |F_{p_1p_2}||\nabla u||\nabla v|\big)\,\mathrm{d}x.$$

Recalling (1.5), we have that $h(r) \leqslant Cr^2$. Also, we see that

$$\int_{\sqrt{R}}^{R}\frac{h'(r)}{r^2}\,\mathrm{d}r \leqslant \frac{h(R)}{R^2} + 2\int_{\sqrt{R}}^{R}\frac{h(r)}{r^3}\,\mathrm{d}r \leqslant C\log R. \tag{4.5}$$

Passing to polar coordinates in the last integral in (4.4) and using (4.5) we get

$$\limsup_{R\to+\infty}\sup_{t\in(0,\sqrt{R}/4)}\frac{\mathscr{E}_R(u^+_{R,t}, v^+_{R,t}) + \mathscr{E}_R(u^-_{R,t}, v^-_{R,t}) - 2\mathscr{E}_R(u,v)}{t^2} \leqslant 0.$$

Remark 4.1. — We point out that in a similar way, by perturbing only one of the element of the couple (u, v), one can obtain

$$\limsup_{R\to+\infty}\sup_{t\in(0,\sqrt{R}/4)}\frac{\mathscr{E}_R(u^+_{R,t}, v) + \mathscr{E}_R(u^-_{R,t}, v) - 2\mathscr{E}_R(u,v)}{t^2} \leqslant 0, \tag{4.6}$$

and

$$\limsup_{R\to+\infty}\sup_{t\in(0,\sqrt{R}/4)}\frac{\mathscr{E}_R(u, v^+_{R,t}) + \mathscr{E}_R(u, v^-_{R,t}) - 2\mathscr{E}_R(u,v)}{t^2} \leqslant 0. \tag{4.7}$$

We conclude this section recalling the following integral formulas:

$$\begin{aligned}
\mathscr{E}_R\left(\max\left\{u^-_{R,t}, u\right\}, v\right) + \mathscr{E}_R\left(\min\left\{u^-_{R,t}, u\right\}, v\right) &= \mathscr{E}_R(u^-_{R,t}, v) + \mathscr{E}_R(u,v), \\
\mathscr{E}_R\left(u, \max\left\{v^-_{R,t}, v\right\}\right) + \mathscr{E}_R\left(u, \min\left\{v^-_{R,t}, v\right\}\right) &= \mathscr{E}_R(u, v^-_{R,t}) + \mathscr{E}_R(u,v).
\end{aligned} \tag{4.8}$$

5 Proof of Theorem 1.5

We recall the notation introduced at the beginning of Section 4, and we observe that, since (u, v) is an e_N-minimizer,

$$\mathscr{E}_R(u, v) \leqslant \mathscr{E}_R(u^+_{R,t}, v).$$

From this and (4.6) we obtain that

$$\lim_{R\to+\infty} \mathscr{E}_R(u^-_{R,t}, v) - \mathscr{E}_R(u, v) = 0, \tag{5.1}$$

at t fixed. Using again the minimality of (u, v), we have that

$$\mathscr{E}_R(u, v) \leqslant \mathscr{E}_R\left(\min\left\{u^-_{R,t}, u\right\}, v\right),$$

which, together with the first relation in (4.8), implies that

$$\mathscr{E}_R\left(\max\left\{u^-_{R,t}, u\right\}, v\right) - \mathscr{E}_R(u^-_{R,t}, v) = \mathscr{E}_R(u, v) - \mathscr{E}_R\left(\min\left\{u^-_{R,t}, u\right\}, v\right) \leqslant 0. \tag{5.2}$$

Putting together (5.1) and (5.2) we obtain

$$\lim_{R\to+\infty} \mathscr{E}_R\left(\max\left\{u^-_{R,t}, u\right\}, v\right) - \mathscr{E}_R(u, v) = 0. \tag{5.3}$$

We set

$$f_{R,t} := \max\left\{u^-_{R,t}, u\right\}, \tag{5.4}$$

and we observe that

$$f_{R,t} = \max\left\{u(x), u(x + te_N)\right\},$$

and $f_{R,t} \in D^t_R(u)$.

Now, we argue by contradiction, assuming that $u \in C^1(\Omega)$ is not monotone on a line in the direction e_N. This implies that we can take $t > 0$ in such a way that $u(x)$ and $u(x + te_N)$ satisfy the hypotheses of Lemma 3.1, say at some point $x_0 \in \Omega$ (see Remark 4.2 in [125]). Therefore, we have that $g_{R,t} := (f_{R,t}, v)$ is not an e_N-minimizer for $\mathscr{E}$. Hence, in a neighborhood of x_0, we can perturb $g_{R,t}$ into $\widetilde{g}_{R,t}$ in such a way that

$$\mathscr{E}_R(\widetilde{g}_{R,t}) \leqslant \mathscr{E}_R(g_{R,t}) - c,$$

for some $c > 0$ which depends only on (u, v). From the last inequality and (5.3) we reach a contradiction with the minimality of (u, v) as $R \to +\infty$.

If we assume that $v \in C^1(\Omega)$ is not monotone on a line in the e_N-direction, we get again a contradiction with the fact that (u, v) is an e_N-minimizer. Indeed we can repeat the same argument as above, using (4.7) and the second integral formula in (4.8). This concludes the proof of Theorem 1.5.

6 Proof of Theorem 1.10

We recall the notation introduced at the beginning of Section 4. From (4.6) we deduce that, for $\varepsilon > 0$, we can take R large such that

$$\mathscr{E}_R(u^+_{R,t}, v) + \mathscr{E}_R(u^-_{R,t}, v) - 2\mathscr{E}_R(u, v) \leqslant \varepsilon t^2. \tag{6.1}$$

Moreover, we know that (u, v) is $\{e_N, e_{N+1}\}$-stable, and so

$$\mathscr{E}_R(w_1, v) - \mathscr{E}_R(u, v) \geqslant -\varepsilon t^2 \quad \text{for any } w_1 \in \mathscr{D}^t_R(u), \tag{6.2}$$

for every t sufficiently small (see Definition 1.8).

Now, recalling the definition of $f_{R,t}$ in (5.4), we use the first integral formula in (4.8) to obtain that

$$\begin{aligned}
\mathscr{E}_R(f_{R,t}, v) - \mathscr{E}_R(u, v) &= \mathscr{E}_R(u^-_{R,t}, v) - \mathscr{E}_R\left(\min\left\{u^-_{R,t}, u\right\}, v\right) \\
&= \mathscr{E}_R(u^-_{R,t}, v) + \mathscr{E}_R(u^+_{R,t}, v) - 2\mathscr{E}_R(u, v) + \mathscr{E}_R(u, v) \\
&\qquad - \mathscr{E}_R\left(\min\left\{u^-_{R,t}, u\right\}, v\right) + \mathscr{E}_R(u, v) - \mathscr{E}_R(u^+_{R,t}, v) \\
&\leqslant 3\varepsilon t^2,
\end{aligned}$$

where we have used (6.1) and (6.2).

Suppose, by contradiction, that u_N changes sign in Ω. Then, by Lemma 3.4, there exists a point $x_0 \in \Omega$ such that the hypotheses of Lemma 3.2 are satisfied in a neighborhood of x_0. Hence, we have that we can perturb $f_{R,t}$ into $\widetilde{f}_{R,t}$ near x_0 in such a way that

$$\mathscr{E}_R(\widetilde{f}_{R,t}, v) \leqslant \mathscr{E}_R(f_{R,t}, v) - ct^2,$$

where $\widetilde{f}_{R,t} \in \mathscr{D}^{Ct}_R(u)$, for some c, $C > 0$ which depend only on u. Then, we obtain

$$\mathscr{E}_R(\widetilde{f}_{R,t}, v) \leqslant \mathscr{E}_R(u, v) + (3\varepsilon - c)t^2,$$

which gives a contradiction with the stability inequality (1.6) if we take $\varepsilon \ll c$.

If we assume that v_N changes sign in Ω, we reason in a similar way, using (4.7), the second integral formula in (4.8) and Remark 3.3 to reach the same contradiction. Therefore, either $u_N \geqslant 0$ or $u_N \leqslant 0$ and either $v_N \geqslant 0$ or $v_N \leqslant 0$ in Ω. This completes the proof of Theorem 1.10.

7 Some applications

7.1 Two-stated mixture of Bose-Einstein condensate

Here, we consider the following system

$$\begin{cases} \Delta u = uv^2, \\ \Delta v = vu^2, \\ \quad u, v > 0. \end{cases} \tag{7.1}$$

As already mentioned in the Introduction, the above system arises in the analysis of phase separation phenomena in binary mixtures of Bose-Einstein condensates with multiple states.

The energy associated to (7.1) is the following:

$$\mathscr{E}(u,v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + u^2v^2 \,\mathrm{d}x.$$

Under a suitable growth condition, we show here that stable solutions of (7.1) are one-dimensional, i.e. there exist $\overline{u}$, $\overline{v} : \mathbb{R} \to \mathbb{R}$ and ω_u, $\omega_v \in \mathbb{S}^{N-1}$ such that

$$(u(x), v(x)) = \big(\overline{u}(\omega_u \cdot x), \overline{v}(\omega_v \cdot x)\big).$$

Precisely:

Proposition 7.1. — *Let $u, v \in C^2(\mathbb{R}^N)$. Suppose that (u,v) is a stable solution to* (7.1)*, and that the following growth condition holds true: there exists a constant $C > 0$ such that, for R sufficiently large,*

$$\int_{B_R} |\nabla u|^2 + |\nabla v|^2 \,\mathrm{d}x \leqslant CR^2. \tag{7.2}$$

Then, (u,v) possesses one-dimensional symmetry.

Proof. — We define

$$F(p_1, p_2, z_1, z_2, x') := \frac{1}{2}\Big(|p_1|^2 + |p_2|^2 + z_1^2 z_2^2\Big).$$

Notice that F is convex in p_1 and p_2 and satisfies (1.4). Moreover $F_{p_1p_1} = 1 = F_{p_2p_2}$ and $F_{p_1p_2} = 0$, therefore the growth condition (1.5) is ensured by (7.2). Then, we can apply Theorem 1.10 (recall also Lemma 2.1) and we obtain that (u,v) is one-dimensional. □

As a corollary, we obtain the one-dimensional symmetry in dimension 2 for stable solutions to (7.1) which have linear growth at infinity (see [20]).

Corollary 7.2. — *Let $N = 2$ and let $u, v \in C^2(\mathbb{R}^2)$. Suppose that (u,v) is a stable solution to* (7.1)*, and that the following growth condition holds true: there exists a constant $C > 0$ such that*

$$|u(x)| + |v(x)| \leqslant C(1 + |x|). \tag{7.3}$$

Then, (u,v) possesses one-dimensional symmetry.

Proof. — We observe that, by Theorem 5.1 in [150], if (u,v) satisfies (7.3) then $|\nabla u|$ and $|\nabla v|$ are bounded in $\mathbb{R}^2$. Therefore, condition (7.2) is trivially satisfied, and so we get the desired result thanks to Proposition 7.1. □

7.2 General systems with p-Laplacian type operators

Theorem 1.10 actually applies to a broader class of operators and nonlinearities. Indeed, with the notation introduced in [55] (see in particular pages 3474-3475 there), one can consider

$$\begin{cases} \operatorname{div}\big(a(|\nabla u|)\nabla u\big) = \widetilde{F}_1(u,v), \\ \operatorname{div}\big(b(|\nabla v|)\nabla v\big) = \widetilde{F}_2(u,v), \end{cases} \tag{7.4}$$

where $\widetilde{F}$ is a $C^{1,1}_{\mathrm{loc}}$ function on $\mathbb{R}^2$ (it corresponds to the function F introduced in [55], here denoted as $\widetilde{F}$ to avoid confusion), and $\widetilde{F}_1$, $\widetilde{F}_2$ denote the derivatives of $\widetilde{F}$ with respect to the first and the second variable respectively.

Then, we have the following:

Proposition 7.3. — *Let $u \in C^1(\mathbb{R}^N) \cap C^2(\{\nabla u \neq 0\})$ and $v \in C^1(\mathbb{R}^N) \cap C^2(\{\nabla v \neq 0\})$. Suppose that (u,v) is a stable solution to (7.4), and that conditions $(B1)$ and $(B2)$ in [63] are satisfied for a, b, A and B. Also, assume that the following growth condition holds true: there exists a constant $C > 0$ such that, for R sufficiently large,*

$$\int_{B_R} \Lambda_2(|\nabla u|) + \Gamma_2(|\nabla v|) \leqslant CR^2, \tag{7.5}$$

where Λ_2 and Γ_2 are as in [55]. Then, (u,v) possesses one-dimensional symmetry.

Proof. — We define

$$F(p_1,p_2,z_1,z_2,x') := \Lambda_2(|p_1|) + \Gamma_2(|p_2|) + \widetilde{F}(z_1,z_2),$$

and we verify that

$$\int_{\mathbb{R}^N} F(\nabla u, \nabla v, u, v, x')\,\mathrm{d}x$$

satisfies the hypotheses needed to apply Theorem 1.10. Observe, first, that being stable for the above energy is the same a being stable for (7.4) (see Definition 1.2 in [55]). Now, recalling the notations used in [55], we derive

$$\begin{aligned} F_{(p_1)_i}(p_1,p_2,z_1,z_2,x') &= \lambda_2(|p_1|)_{(p_1)_i} = a(|p_1|)(p_1)_i, \\ F_{(p_2)_i}(p_1,p_2,z_1,z_2,x') &= \gamma_2(|p_2|)_{(p_2)_i} = b(|p_2|)(p_2)_i, \\ F_{(p_1)_i(p_1)_j}(p_1,p_2,z_1,z_2,x') &= a(|p_1|)\,\delta_{ij} + a'(|p_1|)\,|p_1|^{-1}\,(p_1)_i(p_1)_j = A_{ij}(p_1), \\ F_{(p_2)_i(p_2)_j}(p_1,p_2,z_1,z_2,x') &= b(|p_2|)\,\delta_{ij} + b'(|p_2|)\,|p_2|^{-1}\,(p_2)_i(p_2)_j = B_{ij}(p_2), \\ F_{(p_1)_i(p_2)_j}(p_1,p_2,z_1,z_2,x') &= 0. \end{aligned}$$

Therefore, Lemma 2.1 in [55] implies the desired convexity properties on F. Moreover, it also gives that $|F_{p_1p_1}(p_1,p_2,z_1,z_2,x')|$ and $|F_{p_2p_2}(p_1,p_2,z_1,z_2,x')|$ are bounded from above and below by $(\lambda_1(|p_1|) + \lambda_2(|p_1|))$ and $(\gamma_1(|p_2|) + \gamma_2(|p_2|))$, respectively, up to multiplicative constants. Furthermore, since conditions $(B1)$ and $(B2)$ in [63] are satisfied for a, b, A and B, one can use Lemma 4.2 there to see that, if $|p_1|$, $|p_2| \leqslant M$, $|q_1| \leqslant |p_1|/2$ and $|q_2| \leqslant |p_2|/2$, then

$$\begin{aligned} \lambda_1(|p_1|) &\leqslant C_M \lambda_2(|p_1|), & \lambda_2(|p_1+q_1|) &\leqslant C_M \lambda_2(|p_1|), \\ \gamma_1(|p_2|) &\leqslant C_M \gamma_2(|p_2|), & \gamma_2(|p_2+q_2|) &\leqslant C_M \gamma_2(|p_2|), \end{aligned}$$

for some $C_M > 0$. As a consequence, we have

$$\begin{aligned} |F_{p_1p_1}(p_1,p_2,z_1,z_2,x')| &\leqslant C_M \lambda_2(|p_1|), \\ |F_{p_2p_2}(p_1,p_2,z_1,z_2,x')| &\leqslant C_M \gamma_2(|p_2|), \end{aligned} \tag{7.6}$$

up to rename C_M. Therefore, recalling also formulae (1.4)-(1.7) in [55], we get

$$\begin{aligned} |F_{p_1p_1}(p_1+q_1,p_2,z_1,z_2,x')| &\leqslant C_M \lambda_2(|p_1+q_1|) \\ &\leqslant C_M \lambda_2(|p_1|) \leqslant C_M |F_{p_1p_1}(p_1,p_2,z_1,z_2,x')|, \end{aligned}$$

and

$$\begin{aligned} |F_{p_2p_2}(p_1,p_2+q_2,z_1,z_2,x')| &\leqslant C_M \gamma_2(|p_2+q_2|) \\ &\leqslant C_M \gamma_2(|p_2|) \leqslant C_M |F_{p_2p_2}(p_1,p_2,z_1,z_2,x')|, \end{aligned}$$

for any $2|q_1| \leqslant |p_1| \leqslant M$ and $2|q_2| \leqslant |p_2| \leqslant M$. This means that (1.4) is satisfied.

It remains to check the growth condition (1.5). For this, notice that (7.6) and formula (4.13) in [63] give that

$$\begin{aligned} |F_{p_1p_1}(p_1,p_2,&z_1,z_2,x')|\,|p_1|^2 + |F_{p_2p_2}(p_1,p_2,z_1,z_2,x')|\,|p_2|^2 \\ &\leqslant C_M\Big(\lambda_2(|p_1|)|p_1|^2 + \gamma_2(|p_2|)|p_2|^2\Big) \\ &= C_M\Big(a(|p_1|)|p_1|^2 + b(|p_2|)|p_2|^2\Big) \\ &\leqslant C_M\Big(\Lambda_2(|p_1|) + \Gamma_2(|p_2|)\Big). \end{aligned}$$

This and (7.5) imply that

$$\begin{aligned} \int_{B_R} |F_{p_1p_1}(\nabla u,\nabla v,u,v,x')|\,|\nabla u|^2 + |F_{p_2p_2}(\nabla u,\nabla v,u,v,x')|\,|\nabla v|^2 \,\mathrm{d}x \\ \leqslant C_M \int_{B_R} (\Lambda_2(|\nabla u|) + \Gamma_2(|\nabla v|))\,\mathrm{d}x \leqslant C_M R^2, \end{aligned}$$

up to rename C_M. This shows (1.5) holds true. Hence, we can apply Theorem 1.10, thus obtaining the desired monotonicity property. □

As paradigmatic examples in Proposition 7.3 one can take the p-Laplacian, with $p \in (1, +\infty)$ if $\{\nabla u = 0\} = \varnothing$ and $p \in [2, +\infty)$ if $\{\nabla u = 0\} \neq \varnothing$ (in this case $a(t) = t^{p-2}$), or the mean curvature operator (in this case $a(t) = (1+t^2)^{-1/2}$). We stress on the fact that one can also take different operators a and b satisfying the hypotheses (for instance, one can take a to be the p-Laplacian and b an operator of mean curvature type).

Furthermore, we observe that condition (7.5) is satisfied, for instance, when $N = 2$ and ∇u and ∇v are bounded (thanks to the hypotheses on a and b, see page 3474 in [55]), This means that we recover Theorem 7.1 of [55] for stable solutions, without requiring conditions on the sign of $\widetilde{F}_{12}$.

7.3 Systems involving the fractional Laplacian

The general setting of our results allows us to treat also the case of nonlocal systems of equations, i.e.

$$\begin{cases} (-\Delta)^{s_1} u = \widetilde{F}_1(u, v), \\ (-\Delta)^{s_2} v = \widetilde{F}_2(u, v), \end{cases} \tag{7.7}$$

where s_1, $s_2 \in (0, 1)$, $\widetilde{F}$ is a $C^{1,1}_{\mathrm{loc}}$ function on $\mathbb{R}^2$ and $\widetilde{F}_1$ and $\widetilde{F}_2$ denote the derivatives of $\widetilde{F}$ with respect to the first and the second variable respectively.

As a matter of fact, as in [125, Remark 2.12], we observe that one can generalize the functional considered in (1.1) to the following functional

$$\int_\Omega F(\nabla u, \nabla v, u, v, x')\,\mathrm{d}x + \int_{\partial\Omega} G(u, v, x')\,\mathrm{d}\mathscr{H}^{N-1}, \tag{7.8}$$

where G satisfies the same regularity assumptions as F. Furthermore, the growth condition in (1.5) can be weakened in the following way. We define $\ell_0(R) := R$ and

$$\ell_k(R) := \log\big(\ell_{k-1}(R)\big) = \underbrace{\log \circ \ldots \circ \log}_{k \text{ times}} R \qquad \text{for any } k \in \mathbb{N}^*.$$

We also set

$$\pi_k(R) := \prod_{j=0}^{k} \ell_j(R).$$

Then, one can require that, for some $k \in \mathbb{N}$,

$$\begin{aligned} \int_{\Omega\cap B_R} &|F_{p_1p_1}(\nabla u, \nabla v, u, v, x')||\nabla u|^2 + |F_{p_2p_2}(\nabla u, \nabla v, u, v, x')||\nabla v|^2 \\ &+ |F_{p_1p_2}(\nabla u, \nabla v, u, v, x')||\nabla u||\nabla v|\,\mathrm{d}x \leqslant CR\pi_k(R), \end{aligned} \tag{7.9}$$

instead of (1.5). Notice that if $k = 0$ then (7.9) corresponds to (1.5). See Remark 2.13 and Section 9 in [125] for the proof and further discussion on this fact.

We also recall that, using the extension result in [43], one can localize problem (7.7) by adding one variable, see e.g. formula (1.7) in [56]. We will denote by U and V the extension functions of u and v, respectively (see e.g. formulae (2.3) and (2.4) in [56]).

So one has to deal with an energy functional like (7.8) with

$$\begin{cases} \Omega := (0, +\infty) \times \mathbb{R}^N, \qquad \mathscr{V} := (0, +\infty) \times \mathbb{R}^{N-1}, \\ F(p_1, p_2, z_1, z_2, x') := x_1^{1-2s_1} |p_1|^2 + x_1^{1-2s_2} |p_2|^2, \\ G(z_1, z_2, x') := \widetilde{F}(z_1, z_2), \end{cases} \tag{7.10}$$

(notice that in this application one has to replace N by $N+1$ to apply Theorem 1.10).

With this, we can prove the following:

PROPOSITION 7.4. — *Let $u, v \in C^2(\mathbb{R}^N)$. Suppose that (u, v) is a stable solution to* (7.7). *Also, assume that the following growth condition holds true in the extension: there exist a constant $C > 0$ and $k \in \mathbb{N}$ such that, for R sufficiently large,*

$$\int_{B_R} x_1^{1-2s_1} |\nabla U|^2 + x_1^{1-2s_2} |\nabla V|^2 \, \mathrm{d}x_1 \cdots \mathrm{d}x_{N+1} \leqslant C R \pi_k(R). \tag{7.11}$$

Then, (u, v) possesses one-dimensional symmetry.

Proof. — With the notation introduced in (7.10), we observe that the thesis simply follows from (7.11) (that ensures the growth condition (7.9)) and Theorem 1.10. □

We remark that (7.11) is a reasonable energy growth condition, since it is satisfied for instance in the case of a single equation when $N = 2$ for any $s \in (0, 1)$ and when $N = 3$ for $s \in (1/2, 1)$, see [37, 38]. Moreover, if $s_1 = s_2 = 1/2$, it can be checked as in [37] with suitable modifications under an additional assumption on the bound of ∇U and ∇V (see in particular formula (1.16) and Section 4 in [37]). We finally remark that, differently from [56], *we do not need* here any sign assumption on $\widetilde{F}$.

Some open problems

In this last chapter, we list a few possible extensions of our results.

On Chapter 1

A natural question is whether Theorem 2.3 extends to the case of Besov-Nikol'skii spaces on domains, at least under some regularity assumptions.

Open problem — *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain (possibly unbounded). Let $s \in (0,1)$, $p \in [1,\infty)$ and let $(\rho_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^N)$ be a family of mollifiers of the form $\rho_\varepsilon(z) = \varepsilon^{-N}\rho(\varepsilon^{-1}z)$. Then, is it true that*

$$[f]^p_{B^s_{p,\infty}(\Omega)} \sim \sup_{\varepsilon>0} \int_\Omega \int_\Omega \rho_\varepsilon(x-y) \frac{|f(x)-f(y)|^p}{|x-y|^{sp}} \mathrm{d}x\mathrm{d}y\,?$$

Our guess is that this holds true for every domain $\Omega \subset \mathbb{R}^N$ whose complement satisfies the measure density condition:

$$\exists c > 0, \quad |B_r(x) \cap {}^c\Omega| \geqslant c r^N \quad \text{for any } (r,x) \in (0,1] \times {}^c\Omega.$$

Although this condition is quite weak it seems however sufficiently strong to avoid "cusp" like singularities (such as a disk with a slit).

A related question would be to investigate whether functionals of the form

$$\int_\Omega \int_\Omega \rho_\varepsilon(d_{\mathrm{g}}(x,y)) \frac{|f(x)-f(y)|^p}{d_{\mathrm{g}}(x,y)^{sp}} \mathrm{d}x\mathrm{d}y,$$

where $d_{\mathrm{g}}(\cdot,\cdot)$ is the geodesic distance on Ω, can provide a description of $B^s_{p,\infty}(\Omega)$ and, if so, under which conditions on Ω. This could pave the way towards a new description of Besov-Nikol'skii spaces in general metric spaces and may be useful in problems of the type of those addressed at Chapters 3 and 4.

Another interesting perspective of research would be to figure out whether there are some reasonable conditions ensuring that a bounded sequence $(f_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}^N)$

satisfying

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\rho_\varepsilon(x-y)\frac{|f_\varepsilon(x)-f_\varepsilon(y)|^p}{|x-y|^{sp}}\mathrm{d}x\mathrm{d}y \leqslant C \quad \text{as } \varepsilon\downarrow 0,$$

is relatively compact in $L^p_{\text{loc}}(\mathbb{R}^N)$.

On Chapter 2

In Chapter 2, we have shown that, for every $f \in B^s_{p,q}(\mathbb{R}^N)$ with $p < q$, it holds that

$$f(\cdot,y) \in B^{(s,\Psi)}_{p,q}(\mathbb{R}^d) \quad \text{for a.e. } y \in \mathbb{R}^{N-d},$$

provided that Ψ satisfies the growth condition

$$\left(\sum_{j\geqslant 0}\Psi(2^{-j})^\chi\right)^{1/\chi} < \infty. \tag{7.12}$$

If we could show that this latter condition is optimal when $q = \infty$, our results are less satisfying when $q < \infty$ since we required the further assumption that

$$\frac{qp}{q-p} < \frac{1}{c_\infty}.$$

Is it possible to get rid of this extra condition?

Open problem — *Let $N \geqslant 2$, $1 \leqslant d < N$, $0 < p < q < \infty$, $s > \sigma_p$ and let Ψ be an admissible function that does not satisfy* (7.12). *Then, does there exist a function $f \in B^s_{p,q}(\mathbb{R}^N)$ such that*

$$f(\cdot,y) \notin B^{(s,\Psi)}_{p,q}(\mathbb{R}^d) \quad \textit{for a.e. } y \in \mathbb{R}^{N-d}\ ?$$

On Chapters 3 and 4

In Chapter 3, we have shown that the requirement on the asymptotic behavior of u can be replaced by a condition on the supremum of u provided that J is square integrable. However, this assumption is quite unnatural and it would be interesting to find out whether it is possible to get rid of it. The reason why we had to make this hypothesis is because we had to cope with the lack of compactness which forced us to make some additional assumptions on J. In some sense, it is a bit counterintuitive because it says that J cannot be too singular, while, in the case of the fractional Laplacian, it is precisely the singularity which yields compactness.

Open problem — *Let $K \subset \mathbb{R}^N$ be a compact convex set. Suppose that J is compactly supported. If $u : \overline{\mathbb{R}^N \setminus K} \to [0,1]$ is uniformly continuous in $\overline{\mathbb{R}^N \setminus K}$ and obeys*

$$\begin{cases} Lu + f(u) \; = \; 0 \;\; \text{in } \overline{\mathbb{R}^N \setminus K}, \\ \displaystyle\sup_{\mathbb{R}^N \setminus K} u \; = \; 1, \end{cases}$$

then, is it true that $u = 1$ in the whole set $\overline{\mathbb{R}^N \setminus K}$?

It is also tempting to ask for a *characterization* of the set of all obstacles K for which the Liouville property holds (however, this is not known even for the local case which is a priori simpler).

And, of course, it remains the question of the existence and large-time convergence of solutions to the evolution problem

$$\frac{\partial u}{\partial t} = Lu + f(u) \quad \text{in } \mathbb{R} \times \mathbb{R}^N \setminus K.$$

On Chapter 5

If the results of Chapter 5 are fairly general, they still require a variational structure. One therefore cannot handle "prey-predator" type systems such as

$$\begin{cases} (-\Delta)^s u = u(\alpha - \beta v), \\ (-\Delta)^s v = -v(\gamma - \delta u). \end{cases}$$

It would be interesting to find a general method that could handle *at least* some systems in non-variational form.

Bibliography

[1] A. Almeida & A. Caetano, "Real interpolation of generalized Besov-Hardy spaces and applications", *J. Fourier Anal. Appl.* **17** (2011), p. 691-719.

[2] F. Andreu-Vaillo, J. M. Mazon, J. D. Rossi & J. J. Toledo-Melero, *Nonlocal diffusion problems*, Mathematical Surveys and Monographs, no. 165, AMS, 2010.

[3] J. L. Ansorena & O. Blasco, "Atomic decomposition of weighted Besov spaces", *J. Lond. Math. Soc.* **53** (1996), no. 1, p. 127-140.

[4] ———, "Characterization of weighted Besov spaces", *Math. Nachr.* **171** (2006), no. 1, p. 5-17.

[5] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge Univ. Press, 2004.

[6] D. G. Aronson & H. F. Weinberger, "Multidimensional nonlinear diffusions arising in population genetics", *Adv. Math.* **30** (1978), p. 33-76.

[7] J. M. Ash, "Neither a worst convergent nor a best divergent series exists", *College Math. J.* **28** (1997), p. 296-297.

[8] G. Aubert & P. Kornprobst, "Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful to solve variational problems?", *SIAM J. Numer. Anal.* **47** (2009), no. 2, p. 844-860.

[9] J.-M. Aubry, D. Maman & S. Seuret, "Local behavior of traces of Besov functions: prevalent results", *J. Funct. Anal.* **264** (2013), no. 3, p. 631-660.

[10] N. Bacaër, *A short history of mathematical population dynamics*, Springer-Verlag London, 2011.

[11] F. Bartumeus, "Lévy processes in animal movement: an evolutionary hypothesis", *Fractals* **15** (2007), no. 2, p. 151-162.

[12] ———, "Behavioral intermittence, Lévy patterns, and randomness in animal movement", *Oikos* **118** (2009), no. 4, p. 488-494.

[13] P. W. Bates, P. C. Fife, X. Ren & X. Wang, "Travelling waves in a convolution model for phase transitions", *Arch. Ration. Mech. Anal.* **138** (1997), p. 105-136.

[14] M. Bebendorf, "A note on the Poincaré inequality for convex domains", *Z. Anal. Anwend.* **22** (2003), no. 3, p. 751-756.

[15] H. Berestycki, J. Coville & H.-H. Vo, "On the definition and the properties of the principal eigenvalue of some nonlocal operators", *J. Funct. Anal.* **271** (2016), p. 2701-2751.

[16] ———, "Persistence criteria for populations with non-local dispersion", *J. Math. Biol.* **72** (2016), no. 7, p. 1693-1745.

[17] H. Berestycki, F. Hamel & H. Matano, "Bistable travelling waves around an obstacle", *Comm. Pure Appl. Math.* **62** (2009), p. 729-788.

[18] H. Berestycki, T.-C. Lin, J. Wei & C. Zhao, "On phase-separation models: asymptotics and qualitative properties", *Arch. Ration. Mech. Anal.* **208** (2013), no. 1, p. 163-200.

[19] H. Berestycki & N. Rodiguez, "A non-local bistable reaction-diffusion equation with a gap", *Disc. Cont. Dyn. Syst. A* **37** (2017), p. 685-723.

[20] H. Berestycki, S. Terracini, K. Wang & J. Wei, "On entire solutions of an elliptic system modeling phase separations", *Adv. Math.* **243** (2013), p. 102-126.

[21] F. Bethuel, H. Brezis & F. Hélein, *Ginzburg-landau vortices*, Birkhäuser, 1994.

[22] ———, *Mathematical problems related to liquid crystals, superconductors and superfluids*, Physics on Manifolds, Conference dedicated to Y. Choquet-Bruhat, no. 15 (M. Flato, R. Kerner and A. Lichnerowicz eds.), Kluwer, p. 11-21, 1994.

[23] N. H. Bingham, C. M. Goldie & J. L. Teugels, *Regular variation*, Cambridge Univ. Press, 1987.

[24] B. Bojarski, L. Ihnatsyeva & J. Kinnunen, "How to recognize polynomials in higher order Sobolev spaces", *Math. Scand.* **112** (2013), no. 2, p. 161-181.

[25] R. Borghol, “Some properties of Sobolev spaces”, *Asymptot. Anal.* **51** (2007), no. 3, p. 303-318.

[26] J. Bouhours, “Robustness for a Liouville type theorem in exterior domains”, *J. Dyn. Diff. Equations* **27** (2015), p. 297-306.

[27] J. Bourgain, H. Brezis & P. Mironescu, “Another look at Sobolev spaces”, *Optimal Control and Partial Differential Equations (J. L. Menaldi, E. Rofman and A. Sulem, eds.) A volume in honour of A. Bensoussan’s 60th birthday, IOS Press* (2001), p. 439-455.

[28] J. Brasseur, “A Bourgain-Brezis-Mironescu characterization of higher order Besov-Nikol’skii spaces”, *Ann. Inst. Fourier (to appear)* (2017).

[29] ———, “On restrictions of Besov functions”, *Nonlinear Anal.* **170** (2018), p. 197-225.

[30] J. Brasseur & J. Coville, “A counterexample to the Liouville property of some nonlocal problems”, *Preprint. hal-01769598* (2018).

[31] J. Brasseur, J. Coville, F. Hamel & E. Valdinoci, “Liouville type results for a nonlocal obstacle problem”, *Preprint. hal-01672149* (2017).

[32] J. Brasseur & S. Dipierro, “Some monotonicity results for general systems of nonlinear elliptic PDEs”, *J. Diff. Equations* **261** (2016), no. 5, p. 2854-2880.

[33] H. Brezis, “How to recognize constant functions. connections with Sobolev spaces”, *Uspekhi Mat. Nauk. (A volume in honour of M. Visik)* **57** (2002), p. 59-74.

[34] M. Bricchi, *Complements and results on h-sets*, Function Spaces, Differential Operators and Nonlinear Analysis (The Hans Triebel Anniversary Volume), p. 219-229, Birkäuser, 2003.

[35] ———, “Tailored Besov spaces and h-sets”, *Math. Nachr.* **263-264** (2004), p. 36-52.

[36] A. M. Bruckner & E. Ostrow, “Some function classes related to the class of convex functions”, *Pacific J. Math.* **12** (1962), no. 4, p. 1203-1215.

[37] X. Cabré & E. Cinti, “Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian”, *Disc. Cont. Dyn. Syst.* **28** (2010), no. 3, p. 1179-1206.

[38] ———, "Sharp energy estimates for nonlinear fractional diffusion equations", *Calc. Var. Partial Differential Equations* **49** (2014), no. 1-2, p. 233-269.

[39] A. M. Caetano & S. D. Moura, "Local growth envelopes of spaces of generalized smoothness: the critical case", *Math. Inequal. Appl.* **7** (2004), no. 4, p. 573-606.

[40] ———, "Local growth envelopes of spaces of generalized smoothness: the subcritical case", *Math. Nachr.* **273** (2004), no. 1, p. 43-57.

[41] L. Caffarelli, J.-M. Roquejoffre & O. Savin, "Nonlocal minimal surfaces", *Comm. Pure Appl. Math.* **63** (2010), no. 9, p. 1111-1144.

[42] L. Caffarelli, O. Savin & E. Valdinoci, "Minimization of a fractional perimeter-Dirichlet integral functional", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **32** (2015), no. 4, p. 901-924.

[43] L. Caffarelli & L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32** (2007), no. 7-9, p. 1245-1260.

[44] L. Cafferelli, S. Dipierro & E. Valdinoci, "A logistic equation with nonlocal interactions", *Kinet. Relat. Models* **10** (2017), no. 1, p. 141-170, Special issue - dedicated to the 60th birthday of Peter Markowich.

[45] X. Chen, "Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations", *Adv. Diff. Equations* **2** (1997), p. 125-160.

[46] D. Chiron, "On the definitions of Sobolev and BV spaces into singular spaces and the trace problem", *Commun. Contemp. Math.* **7** (2007), no. 4, p. 473-513.

[47] P. Clément & G. Sweers, "Existence and multiplicity results for a semilinear elliptic eigenvalue problem", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **14** (1987), p. 97-121.

[48] F. Cobos, L. M. Fernández-Cabrera & H. Triebel, "Abstract and concrete logarithmic interpolation spaces", *J. Lond. Math. Soc.* **70** (2004), no. 2, p. 231-243.

[49] C. Cosner, J. Dávila & S. Martínez, "Evolutionary stability of ideal free nonlocal dispersal", *J. Biol. Dyn.* **6** (2012), p. 395-405.

[50] J. Coville, "Travelling fronts in asymmetric nonlocal reaction diffusion equations: the bistable and ignition cases", *Preprint: hal-00696208* (2007).

[51] ———, "On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators", *J. Diff. Equations* **249** (2010), p. 2921-2953.

[52] J. Coville & L. Dupaigne, "On a non-local reaction diffusion equation arising in population dynamics", *Proc. Math. Roy. Soc. of Edinburgh Sect. A* **137** (2007), no. 4, p. 1-29.

[53] J. Coville, F. Li & X. Wang, "On eigenvalue problems arising from nonlocal diffusion models", *Disc. Cont. Dyn. Syst. A* **37** (2017), p. 879-903.

[54] J. Dávila, "On an open question about functions of bounded variations", *Calc. Var. Partial Differential Equations* **15** (2002), p. 519-527.

[55] S. Dipierro, "Geometric inequalities and symmetry results for elliptic systems", *Disc. Cont. Dyn. Syst.* **33** (2013), no. 8, p. 3473-3496.

[56] S. Dipierro & A. Pinamonti, "A geometric inequality and a symmetry result for elliptic systems involving the fractional Laplacian", *J. Diff. Equations* **255** (2013), no. 1, p. 85-119.

[57] S. Dipierro, O. Savin & E. Valdinoci, "A nonlocal free boundary problem", *SIAM J. Math. Anal.* **47** (2015), no. 6, p. 4559-4605.

[58] ———, "All functions are locally s-harmonic up to a small error", *J. EMS* **19** (2017), p. 957-966.

[59] ———, "Local approximation of arbitrary functions by solutions of nonlocal equations", *Preprint. arXiv:1609.04438* (2017).

[60] D. Edmunds & H. Triebel, "Spectral theory for isotropic fractal drums", *C. R. Acad. Sci. Paris* **326** (1998), no. 11, p. 1269-1274.

[61] ———, "Eigenfrequencies of isotropic fractal drums", *Oper. Theory Adv. Appl.* **110** (1999), p. 81-102.

[62] A. Farina, "Some symmetry results for entire solutions of an elliptic system arising in phase separation", *Disc. Cont. Dyn. Syst.* **34** (2014), no. 6, p. 2505-2511.

[63] A. Farina, B. Sciunzi & E. Valdinoci, "Bernstein and De Giorgi type problems: new results via a geometric approach", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **7** (2008), no. 4, p. 741-791.

[64] A. Farina & N. Soave, "Monotonicity and 1-dimensional symmetry for solutions of an elliptic system arising in Bose-Einstein condensation", *Arch. Ration. Mech. Anal.* **213** (2014), no. 1, p. 287-326.

[65] W. Farkas & H. G. Leopold, "Characterization of function spaces of generalized smoothness", *Ann. Mat. Pura Appl.* **185** (2006), no. 1, p. 1-62.

[66] M. Fazly & N. Ghoussoub, "De Giorgi type results for elliptic systems", *Calc. Var. Partial Differential Equations* **47** (2013), no. 3-4, p. 809-823.

[67] M. Fazly & Y. Sire, "Symmetry results for fractional elliptic systems and related problems", *Comm. Partial Differential Equations* **40** (2015), no. 6, p. 1070-1095.

[68] R. Ferreira, C. Kreisbeck & A. M. Ribeiro, "Characterization of polynomials and higher-order Sobolev spaces in terms of functionals involving difference quotients", *Nonlinear Anal.* **112** (2015), p. 199-214.

[69] P. C. Fife, "An integrodifferential analog of semilinear parabolic PDEs", *Lecture Notes in Pure and Appl. Math.* **177** (1996), p. 137-145.

[70] P. C. Fife & J. B. McLeod, "The approach of solutions of non-linear diffusion equations to traveling front solutions", *Arch. Ration. Mech. Anal.* **65** (1977), p. 335-361.

[71] D. G. de Figueiredo, E. M. dos Santos & O. H. Miyagaki, "Sobolev spaces of symmetric functions and applications", *J. Funct. Anal.* **261** (2011), no. 4.

[72] A. Fiscella, R. Servadei & E. Valdinoci, "Density properties for fractional Sobolev spaces", *Ann. Acad. Sci. Fenn. Math.* **40** (2015), p. 235-253.

[73] R. A. Fisher, "The wave of advance of advantageous genes", *Ann. Eugenics* **7** (1937), p. 353-369.

[74] P. Garbaczewski, "Fractional laplacians and Lévy flights in bounded domains", *Preprint. arXiv:1802.09853* (2018).

[75] J. García-Melián & J. Rossi, "On the principal eigenvalue of some nonlocal diffusions problem", *J. Diff. Equations* **246** (2009), p. 21-38.

[76] G. Gilboa & S. Osher, "Nonlocal linear image regularization and supervised segmentation", *Multiscale Model. Simul.* **6** (2007), no. 2, p. 595-630.

[77] ———, "Nonlocal operators with applications to image processing", *Multiscale Model. Simul.* **7** (2008), no. 3, p. 1005-1028.

[78] P. Gurka & B. Opic, "Sharp embeddings of Besov spaces with logarithmic smoothness", *Rev. Mat. Complutense* **18** (2005), p. 81-110.

[79] ———, "Sharp embeddings of Besov-type spaces", *J. Comput. Appl. Math.* **208** (2007), p. 235-269.

[80] O. Hallatschek & D. S. Fisher, "Acceleration of evolutionary spread by long-range dispersal", *PNAS* **111** (2014), no. 46, p. E4911-E4919.

[81] D. Haroske & S. D. Moura, "Continuity envelopes of spaces of generalised smoothness, entropy and approximation numbers", *J. Approx. Theory* **128** (2004), p. 151-174.

[82] N. E. Humphries, N. Queiroz, J. R. M. Dyer, N. G. Pade, M. K. Musyl, K. M. Schaefer, D. W. Fuller, J. M. Brunnschweiler, T. K. Doyle, J. D. R. Houghton, G. C. Hays, C. S. Jones, L. R. Noble, V. J. Wearmouth, E. J. Southall & D. W. Sims, "Environmental context explains Lévy and Brownian movement patterns of marine predators", *Nature* **465** (2010), p. 1066-1069.

[83] V. Hutson, S. Martinez, K. Mischaikow & G. T. Vickers, "The evolution of dispersal", *J. Math. Biol.* **47** (2003), p. 483-517.

[84] M. Izuki, Y. Sawano & H. Tanaka, "Weighted Besov-Morrey spaces and Triebel-Lizorkin spaces", *Harmonic analysis and nonlinear partial differential equations, 21C60, RIMS Kôkyûroku Bessatsu, B22, Res. Inst. Math. Sci. (RIMS), Kyoto* (2010).

[85] N. Jacob, *Pseudo differential operators & Markov processes. Volume III: Markov processes and applications*, Imperial College Press, 2005.

[86] S. Jaffard, "Théorèmes de trace et "dimensions négatives"", *C. R. Acad. Sci. Paris* **320** (1995), no. 4, p. 409-413.

[87] N. J. Kalton, N. T. Peck & J. W. Roberts, *An F-space sampler*, Cambridge Univ. Press, Lecture Notes, no. 89, London Math. Soc., 1985.

[88] G. A. Kalyabin & P. I. Lizorkin, "Spaces of functions of generalized smoothness", *Math. Nachr.* **133** (1987), p. 7-32.

[89] G. E. Karadzhov, M. Milman & J. Xiao, "Limits of higher-order Besov spaces and sharp reiteration theorems", *J. Funct. Anal.* **221** (2005), p. 323-339.

[90] J. Karamata, "Théorèmes sur la sommabilité exponentielle et d'autres sommabilités s'y rattachant", *Mathematica (Cluj)* **9** (1935), p. 164-178.

[91] ———, "Über einen Konvergenzsatz des Herrn Knopp", *Math. Zeit.* **40** (1935), no. 1, p. 421-425.

[92] V. Knopova & M. Zähle, "Spaces of generalized smoothness on h-sets and related Dirichlet forms", *Studia Math.* **174** (2006), p. 277-308.

[93] A. N. Kolmogorov, I. G. Petrovskii & N. S. Piskunov, "A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem", *Selected Works of A. N. Kolmogorov I, V. M. Tikhomirov, editor, Kluwer* (1991), p. 248-270, Translated by V. M. Volosov from Bull. Moscow Univ., *Math. Mech.* **1**, 125, (1937).

[94] V. I. Kolyada & A. K. Lerner, "On limiting embeddings of Besov spaces", *Studia Math.* **171** (2005), no. 1, p. 1-13.

[95] J. K. Kowalski, "A method of approximation of Besov spaces", *Studia Math.* **96** (1990), no. 2, p. 183-193.

[96] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, Edited by A. Gilányi, Birkhäuser, 2000.

[97] L. D. Kudryavtsev & S. M. Nikol'skii, *Spaces of differentiable functions of several variables and imbedding theorem. Analysis III, Spaces of differentiable functions*, Encycl. of Math. Sciences, no. 26, Heidelberg: Springer, p. 4-140, 1990.

[98] X. Lamy & P. Mironescu, "Characterization of function spaces via low regularity mollifiers", *Disc. Cont. Dyn. Syst.* **35** (2015), no. 12, p. 6015-6030.

[99] H.-G. Leopold, "On Besov spaces of variable order of differentiation", *Z. Anal. Anwendungen* **8** (1989), no. 1, p. 69-82.

[100] ———, *Embeddings and entropy numbers in Besov spaces of generalized smoothness*, in Function Spaces: the fifth conference, Lecture notes in pure and applied math., no. 213 (H. Hudzig and L. Skrzypcazk) Marcel Dekker eds., p. 323-336, 2000.

[101] H.-G. Leopold & E. Schroh, "Trace theorems for Sobolev spaces of variable order of differentiation", *Math. Nachr.* **179** (1996), p. 223-245.

[102] C. D. Levermore & J. X. Xin, "Multidimensional stability of traveling waves in a bistable reaction-diffusion equation", *Comm. Part. Diff. Eq.* **17** (1992), p. 1901-1924.

[103] P. I. Lizorkin, *Spaces of generalized smoothness*, Mir, Moscow, p. 381-415, 1986, Appendix to Russian ed. of H. Triebel. Theory of function spaces. Birkhäuser, Basel, (1983).

[104] M. A. Lomholt, K. Tal, R. Metzle & K. Joseph, "Lévy strategies in intermittent search processes are advantageous", *Proc. Natl. Acad. Sci. USA* **105** (2008), no. 32, p. 11055-11059.

[105] V. Maz'ya & T. Shaposhnikova, "On the Bourgain, Brezis and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces", *J. Funct. Anal.* **195** (2002), p. 230-238.

[106] A. McNabb, "Strong comparison theorems for elliptic equations of second order", *J. Math. Mech.* **10** (1961), p. 431-440.

[107] P. Mironescu, "Personal communication", (2015).

[108] P. Mironescu, E. Russ & Y. Sire, "Lifting in Besov spaces", *Preprint. hal-01517735* (2017).

[109] D. S. Mitrinović, *Analytic inequalities*, no. 165, Springer-Verlag Berlin Heidelberg, 1970.

[110] S. D. Moura, J. S. Neves & C. Schneider, "Spaces of generalized smoothness in the critical case: optimal embeddings, continuity envelopes and approximation numbers", *J. Approx. Theory* **187** (2014), p. 82-117.

[111] S. Moura, "Function spaces of generalized smoothness", *Dissertationes Math.* **398** (2001), p. 1-88.

[112] S. Nakamura, T. Noi & Y. Sawano, "Generalized Morrey spaces and trace operator", *Science China Mathematics* **59** (2016), no. 2, p. 281-336.

[113] E. D. Nezza, G. Palatucci & E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces", *Bull. Sci. math.* **136** (2012), no. 5, p. 521-573.

[114] S. M. Nikol'skii, *Approximation of functions of several variables and embedding theorems*, Springer-Verlag (New York-Heidelberg), 1975.

[115] B. Noris, H. Tavares, S. Terracini & G. Verzini, "Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition", *Comm. Pure Appl. Math.* **63** (2010), no. 3, p. 267-302.

[116] L. E. Payne & H. F. Weinberger, "An optimal Poincaré inequality for convex domains", *Arch. Rational Mech. Anal.* **5** (1960), p. 286-292.

[117] J. Peetre, "Espaces d'interpolation et théorème de Soboleff", *Ann. Inst. Fourier* **16** (1966), no. 1, p. 279-317.

[118] A. Ponce, “An estimate in the spirit of Poincaré's inequality”, *J. Eur. Math. Soc.* **6** (2004), p. 1-15.

[119] ———, “A new approach to Sobolev spaces and connections to Gamma-convergence”, *Calc. Var. Partial Differential Equation* **19** (2004), no. 3, p. 229-255.

[120] P. Pucci & J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications, no. 73, Birkhäuser Verlag, 2007, 431-440 pages.

[121] A. M. Reynolds, “Scale-free animal movement patterns: Lévy walks outperform fractional Brownian motions and fractional Lévy motions in random search scenarios”, *J. Phys. A: Math. Theor.* **42** (2009), no. 43, p. 434006-434017.

[122] A. M. Reynolds & C. J. Rhodes, “The Lévy flight paradigm: random search patterns and mechanisms”, *Ecology* **90** (2009), no. 4, p. 877-887.

[123] L. Roques, *Modèles de réaction-diffusion pour l'écologie spatiale*, Ed. Quae, 2013.

[124] O. Savin & E. Valdinoci, “Regularity of nonlocal minimal cones in dimension 2”, *Calc. Var. Partial Differential Equations* **48** (2013), no. 1-2, p. 33-39.

[125] ———, “Some monotonicity results for minimizers in the calculus of variations”, *J. Funct. Anal.* **264** (2013), no. 10, p. 2469-2496.

[126] R. Schneider, O. Reichmann & C. Schwab, “Wavelet solution of variable order pseudodifferential equations”, *C. Calcolo* **47** (2010), no. 2, p. 65-101.

[127] J. Serrin, “A symmetry problem in potential theory”, *Arch. Ration. Mech. Anal.* **43** (1971), p. 304-318.

[128] L. Seuront & H. E. Stanley, “Anomalous diffusion and multifractality enhance mating encounters in the ocean”, *Proc. Natl. Acad. Sci. USA* **111** (2014), no. 6, p. 2206-2211.

[129] J. Simon, “Sobolev, Besov and Nikolskii fractional spaces: imbeddings and comparisons for vector valued spaces on an interval”, *Ann. Mat. Pura Appl.* **157** (1990), no. 1, p. 117-148.

[130] D. W. Sims & N. E. Humphries, “Lévy flight search patterns of marine predators not questioned: a reply to Edwards et al.”, *arXiv:1210.2288* (2012).

[131] D. Spector, “Characterization of Sobolev and BV spaces”, *Dissertation.* **78** (2011).

[132] S. A. Stasyuk & S. Y. Yanchenko, “Approximation of functions from Nikol’skii-Besov type”, *Anal. Math.* **41** (2015), p. 311-334.

[133] S. Terracini, G. Verzini & A. Zilio, “Uniform Hölder regularity with small exponent in competition-fractional diffusion systems”, *Disc. Cont. Dyn. Syst.* **34** (2014), no. 6, p. 2669-2691.

[134] ———, “Uniform Hölder bounds for strongly competing systems involving the square root of the Laplacian”, *J. Eur. Math. Soc.* **18** (2016), no. 12, p. 2865-2924.

[135] J. Trageser, “Local and nonlocal models in thin-plate and bridge dynamics”, *Dissertation, Theses and Student Research Paper in Mathematics* **64** (2015).

[136] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, no. 78, Birkhäuser, 1983.

[137] ———, *Fractals and spectra related to Fourier analysis and function spaces*, Monographs in Mathematics, no. 91, Birkhäuser, 1997.

[138] ———, *Interpolation theory, function spaces, differential operators*, Monographs in Mathematics, Wiley-VCH; 2nd Revised and Enlarged Edition edition, 1998.

[139] ———, *The structure of functions*, Monographs in Mathematics, no. 97, Birkhäuser, 2001.

[140] ———, *Theory of function spaces III*, Monographs in Mathematics, no. 100, Birkhäuser, 2006.

[141] ———, *Bases in function spaces, sampling, discrepancy, numerical integration*, EMS Publ. House, Zürich, 2010.

[142] ———, “Limits of Besov norms”, *Arch. Math.* **96** (2011), p. 169-175.

[143] N. Trudinger, “Hölder gradient estimates for fully nonlinear elliptic equations”, *Proc. Roy. Soc. Edinburgh Sect. A* **108** (1988), p. 57-65.

[144] P. Turchin, *Quantitative analysis of movement: measuring and modeling population redistribution in animals and plants*, Sinauer Associates, Sunderland, MA, 1998.

[145] E. Valdinoci, “From the long jump random walk to the fractional Laplacian”, *Bol. Soc. Esp. Mat. Apl.* **49** (2009), p. 33-44.

[146] P. F. Verhulst, “Recherches mathématiques sur la loi d’accroissement de la population”, *Nouveaux Mémoires de l’Académie Royale des Sciences et Belles-Lettres de Bruxelles* **18** (1845), p. 1-42.

[147] G. Verzini & A. Zilio, “Strong competition versus fractional diffusion: the case of Lotka-Volterra interaction”, *Comm. Partial Differential Equations* **39** (2014), no. 1-2, p. 2284-2313.

[148] V. Volpert & S. Petrovskii, “Reaction-diffusion waves in biology”, *Phys. Life Rev.* **6** (2009), p. 267-310.

[149] N. N. Vorobiev, *Fibonacci numbers*, Probus Publishing Co., 1983.

[150] K. Wang, “On the De Giorgi type conjecture for an elliptic system modeling phase separation”, *Comm. Partial Differential Equations* **39** (2014), no. 4, p. 696-739.

[151] ———, “Harmonic approximation and improvement of flatness in a singular perturbation problem”, *Manuscripta Math.* **146** (2015), no. 1-2, p. 281-298.

[152] K. Wang & J. Wei, “On the uniqueness of solutions of a nonlocal elliptic system”, *Math. Ann.* **365** (2016), no. 1-2, p. 105-153.

[153] H. Weinberger, “Long-time behavior of a class of biological models”, *SIAM J. Math. Anal.* **13** (1982), p. 353-396.

[154] ———, “On spreading speed and travelling waves for growth and migration models in a periodic habitat”, *J. Math. Biol.* **45** (2002), p. 511-548.

[155] H. Yagisita, “Existence of traveling wave solutions for a nonlocal bistable equation: an abstract approach”, *Publ. Res. Inst. Math. Sci.* **45** (2009), no. 4, p. 955-979.