Some Rigidity Properties of von Neumann Algebras

Amine Marrakchi

NNT : 2018SACLS132

THÈSE DE DOCTORAT

de

L'UNIVERSITÉ PARIS-SACLAY

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Université Paris-Sud

Laboratoire d'accueil : Laboratoire de mathématiques d'Orsay, UMR 8628 CNRS

Spécialité de doctorat : Mathématiques fondamentales

Amine MARRAKCHI

Quelques propriétés de rigidité des algèbres de von Neumann

Date de soutenance : 18 Juin 2018

Après avis des rapporteurs :
GEORGES SKANDALIS (Université Paris 7)
NARUTAKA OZAWA (RIMS, Kyoto)

Jury de soutenance :

CLAIRE ANANTHARAMAN	(Université d'Orléans)	Examinateur
CYRIL HOUDAYER	(Université Paris-Sud)	Directeur de thèse
FRÉDÉRIC PAULIN	(Université Paris-Sud)	Examinateur
SORIN POPA	(UCLA)	Président du Jury
GEORGES SKANDALIS	(Université Paris 7)	Rapporteur
STEFAAN VAES	(K.U. Leuven)	Examinateur

Remerciements

Tout d'abord, je tiens à exprimer ma plus profonde gratitude à mon directeur de thèse, Cyril Houdayer. Durant toutes ces années, il a été d'une très grande disponibilité. Nos discussions mathématiques ont toujours été passionnantes et j'ai beaucoup appris à ses côtés. Je suis très reconnaissant envers Cyril pour la liberté qu'il m'a laissée dans le choix des problèmes sur lesquels je travaillais. Il a su m'écouter, m'aider et m'encourager même lorsque mes idées étaient confuses ou farfelues. Enfin, grâce à son caractère généreux et bon vivant, Cyril est devenu, pour moi, un ami plus qu'un directeur de thèse.

Je remercie très chaleureusement Narutaka Ozawa et Georges Skandalis pour avoir accepté de rapporter cette thèse. Je remercie aussi Claire Anantharaman, Frédéric Paulin, Sorin Popa et Stefaan Vaes qui ont accepté de faire partie de mon jury de soutenance. J'en suis très honoré.

Je veux aussi remercier tous ceux qui m'ont invité à donner des exposés durant ma thèse. Grâce à eux j'ai pu rencontrer de nombreux chercheurs un peu partout sur la planète. Je ne peux tous les citer mais je tiens à remercier tout particulièrement Yusuke Isono et tous nos autres collègues japonais qui m'ont, à deux reprises, si bien accueilli. Kyoto est une ville merveilleuse et j'ai hâte d'y retourner.

Mais ces années de thèse auraient été bien ternes sans les très bons moments passés avec mes amis. Je pense bien sûr à Aymeric, Jacko, Jaouad, Rémi et à tous les autres, de l'ENS, de l'Echiquier Fou ou d'ailleurs, trop nombreux pour tous les citer ici !

Enfin, je remercie ma famille, à qui je dois tout. Du fait de la distance, mes parents et ma petite sœur me manquent toujours terriblement, mais leur soutien compte beaucoup. Je remercie également tous les Bensaid qui ont été, ici en France, comme une deuxième famille pour moi.

Table des matières

Première partie
Introduction

La théorie des algèbres de von Neumann tire ses origines de la physique. En effet, en mécanique quantique, l'algèbre a priori commutative des observables d'un système doit en fait être remplacée par une algèbre d'observables non-commutatives. Pour modéliser ces nouvelles observables, on utilise des opérateurs sur un espace de Hilbert. C'est en cherchant à élaborer un cadre mathématique adéquat pour la mécanique quantique que Murray et von Neumann introduisent et développent à partir de 1936 la théorie des algèbres d'opérateurs qu'on appelle aujourd'hui *algèbres de von Neumann*.

Avec le temps, de plus en plus de liens sont découverts entre la théorie des algèbres de von Neumann et d'autres branches des mathématiques telles que la théorie ergodique, la théorie des groupes et de leurs représentations, les probabilités libres ou encore, de façon plus surprenante, la théorie des nœuds et la topologie en basse dimension grâce à la découverte remarquable du polynôme de Jones.

Durant les années 70, la théorie des algèbres de von Neumann vit une petite révolution. Le développement de la théorie modulaire par Tomita et Takesaki, puis les travaux remarquables de Connes sur les facteurs injectifs vont permettre de donner une classification complète des algèbres de von Neumann *moyennables*, une propriété étroitement liée à la notion de moyennabilité pour les groupes. Ce développement s'accompagne de progrès similaires et tout aussi remarquables en théorie ergodique.

Aujourd'hui, la plupart des problèmes relatifs aux algèbres de von Neumann moyennables sont résolus. En revanche, le cas non-moyennable demeure largement incompris. Ainsi, le problème d'isomorphisme des facteurs de groupes libres est toujours ouvert. Des développements récents, grâce notamment à la théorie de la déformation/rigidité de Popa, ont toutefois permis de démontrer des résultats de rigidité impressionnants. Ainsi, certaines actions ergodiques peuvent être entièrement reconstruites à partir de leur algèbre de von Neumann. Il y a là un contraste frappant avec le cas moyennable, où au contraire, toute l'information est oubliée.

1 Définition des algèbres de von Neumann

Dans cette section nous définissons les algèbres de von Neumann et énonçons leurs premières propriétés.

L'exemple fondamental d'algèbre de von Neumann sera $\mathbf{B}(H)$, l'algèbre des opérateurs bornés sur un espace de Hilbert H. Cette algèbre est munie d'une opération particulière : l'opération d'*adjonction* $T \mapsto T^*$. C'est donc un exemple particulier d'$*$-*algèbre* au sens de la définition suivante :

Définition 1.1. Une $*$-*algèbre* est une $\mathbb{C}$-algèbre A (avec unité) munie d'une opération $a \mapsto a^*$ qui est :

(i) involutive : $(a^*)^* = a$.

(ii) anti-linéaire : $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$.

(iii) anti-multiplicative : $(ab)^* = b^* a^*$.

Par analogie avec le cas de $\mathbf{B}(H)$, on dit qu'un élément a d'une $*$-algèbre A est *normal* lorsque $a^*a = aa^*$, *auto-adjoint* lorsque $a^* = a$ et *positif* lorsqu'il existe $b \in A$ tel que $a = b^*b$.

Définition 1.2. Soit M une $*$-algèbre. On dit que M est une *algèbre de von Neumann* lorsqu'il existe une norme $|| \cdot ||$ sur M telle que :

(i) Pour tout $x, y \in M$, $\|xy\| \leq \|x\|\|y\|$.

(ii) Pour tout $x \in M$, $\|x^*x\| = \|x\|^2$.

(iii) $(M, \| \cdot \|)$ est un espace de Banach dual, i.e. il existe un espace de Banach $(E, \| \cdot \|_E)$ tel que $(M, \| \cdot \|) = (E, \| \cdot \|_E)^*$.

Cette définition peut sembler étrange au premier abord car on demande seulement l'existence de la norme et de l'espace de Banach E sans les inclure dans la structure de M. En réalité, et c'est une des premières propriétés remarquables des algèbres de von Neumann, la norme et l'espace de Banach E sont nécessairement *uniques*. Autrement dit, ils sont déterminés par la structure algébrique de M. Plus précisément, on a le théorème suivant.

Théorème 1.3. *Soit M une algèbre de von Neumann. Alors il existe une unique norme $\| \cdot \|$ sur M qui vérifie les conditions de la définition précédente.*

De plus, si $(E_i, \| \cdot \|_{E_i})$, $i = 1, 2$ sont deux espaces de Banach tels que $(E_i, \| \cdot \|_{E_i})^* = (M, \| \cdot \|)$, alors il existe un et un seul isomorphisme $\theta : (E_1, \| \cdot \|_{E_1}) \to (E_2, \| \cdot \|_{E_2})$ dont l'application duale vérifie $\theta^* = \mathrm{id}_M$.

En raison de cette propriété, on notera à partir de maintenant M_* l'espace de Banach qui vérifie $(M_*)^* = M$. On l'appellera le *prédual* de M. On préfère voir les éléments de M_* comme des formes linéaires sur M grâce à l'inclusion canonique de M_* dans son bidual $(M_*)^{**} = M^*$. En général, M^* est beaucoup plus gros que M_*. Notons qu'une algèbre de von Neumann admet donc, en plus de la topologie issue de la norme $\| \cdot \|$, une autre topologie naturelle : la *topologie faible-$*$* issue de la dualité avec M_*.

Nous nous intéressons maintenant à l'exemple le plus fondamental d'algèbre de von Neumann.

Proposition 1.4. *Soit H un espace de Hilbert. Alors l'$*$-algèbre $\mathbf{B}(H)$ est une algèbre de von Neumann.*

La norme de $\mathbf{B}(H)$ n'est pas difficile à deviner. Il s'agit de la *norme d'opérateur* :

$$||T|| = \sup_{||\xi||=1} ||T\xi||.$$

Le prédual $\mathbf{B}(H)_*$ est quant à lui lié à la notion d'*état* d'un système en mécanique quantique. En effet, les physiciens définissent l'état d'un système quantique par un vecteur normalisé $\xi \in H$ considéré *à une phase près*. Autrement dit, ils ne s'intéressent qu'à la forme linéaire

$$\omega_\xi : T \mapsto \langle T\xi, \xi \rangle.$$

Pour toute observable du système, représentée par un opérateur $T \in \mathbf{B}(H)$, la quantité $\omega_\xi(T)$ donne la *valeur moyenne* de cette observable lorsqu'elle est mesurée dans l'état ξ.

Proposition 1.5. *Soit H un espace de Hilbert. Alors le prédual $\mathbf{B}(H)_* \subset \mathbf{B}(H)^*$ est le sous-espace fermé de $\mathbf{B}(H)^*$ engendré par les ω_ξ pour $\xi \in H$.*

L'exemple de l'algèbre de von Neumann $\mathbf{B}(H)$ est fondamental en raison du théorème suivant. Il montre que toute algèbre de von Neumann peut être représentée comme une algèbre d'opérateurs sur un espace de Hilbert (mais de façon non canonique).

Définition 1.6. Soit M une algèbre de von Neumann. Soit N une sous-$*$-algèbre de M. On dit que N est une *sous-algèbre de von Neumann* de M lorsque N est fermée pour la topologie faible-$*$ de M. Dans ce cas, N est une algèbre de von Neumann, la norme de N est la restriction de la norme de M et le prédual N_* est le quotient de M_* par le sous-espace

$$N^\perp = \{\omega \in M_* \mid \omega_{|N} = 0\}$$

Théorème 1.7. *Toute algèbre de von Neumann est isomorphe à une sous-algèbre de von Neumann de $\mathbf{B}(H)$ pour un certain espace de Hilbert H.*

En réalité, c'est essentiellement ainsi, c'est-à-dire comme des sous-$*$-algèbres de $\mathbf{B}(H)$ qui sont fermées pour la topologie faible-$*$, que von Neumann a historiquement défini les algèbres qui portent son nom. Il a en outre démontré le célèbre *théorème du bicommutant* qui donne encore un exemple frappant d'interaction entre structure algébrique et structure topologique.

Théorème 1.8. *Soit M une sous-$*$-algèbre de $\mathbf{B}(H)$. Alors les deux propriétés suivantes sont équivalentes.*

(i) *M est une sous-algèbre de von Neumann de $\mathbf{B}(H)$, i.e. M est fermée pour la topologie faible-$*$ de $\mathbf{B}(H)$.*

(ii) *$M = (M')'$ où, pour toute partie $A \subset \mathbf{B}(H)$, on note*

$$A' = \{T \in \mathbf{B}(H) \mid \forall S \in A,\ TS = ST\}.$$

2 Algèbres de von Neumann commutatives

Dans cette section, nous allons voir qu'il y a une correspondance parfaite entre algèbres de von Neumann commutatives et espaces mesurables. Pour cela, nous introduisons la catégorie suivante qui est *la* bonne catégorie pour faire de la théorie de la mesure. La notion d'espace localisable généralise la notion d'espace σ-fini.

Définition 2.1. La *catégorie des espaces mesurables* est la catégorie définie de la façon suivante.

(i) Les objets sont les espaces mesurés $(X, \mathcal{B}, \mu)$ qui sont *localisables*, i.e. qui peuvent s'écrire comme une somme directe (non nécessairement dénombrable) d'espaces mesurés de mesure finie.

(ii) Les morphismes entre deux objets $(X, \mathcal{B}, \mu)$ et $(Y, \mathcal{C}, \nu)$ sont les classes, modulo égalité presque partout, de fonctions mesurables $\theta : X \to Y$ définies presque partout et telles que $\theta^{-1}(C)$ est de mesure nulle pour tout $C \in \mathcal{C}$ de mesure nulle.

Donnons une description différente de cette catégorie.

Définition 2.2. Une *algèbre de Boole* est un ensemble ordonné $\mathbb{A}$ tel que :

(i) Toute partie de $\mathbb{A}$ admet une borne inférieure et une borne supérieure (notées avec les symboles $\bigwedge$ et $\bigvee$ respectivement, on note 0 et 1 le plus petit élément et le plus grand élément de $\mathbb{A}$).

(ii) Les opérations $\bigwedge$ et $\bigvee$ sont distributives l'une par rapport à l'autre.

(iii) Pour tout $a \in \mathbb{A}$, il existe un élément $a^c \in \mathbb{A}$ tel que $a \vee a^c = 1$ et $a \wedge a^c = 0$.

Un morphisme entre deux algèbres de Boole est une application qui préserve les opérations $\bigwedge$ et $\bigvee$.

Une *mesure* sur une algèbre de Boole $\mathbb{A}$ est une application $\mu : \mathbb{A} \to \mathbb{R}_+ \cup \{+\infty\}$ telle que pour tout $a \in \mathbb{A}$ et toute partition $(a_i)_{i\in I}$ de a (même infinie non dénombrable) on a

$$\mu(a) = \sum_{i\in I} \mu(a_i).$$

L'algèbre de Boole $\mathbb{A}$ est dite *mesurable* si pour tout $a \in \mathbb{A}$ avec $a \neq 0$, il existe une mesure μ sur $\mathbb{A}$ telle que $0 < \mu(a) < +\infty$.

Théorème 2.3. *Soit $(X, \mathcal{B}, \mu)$ un objet de la catégorie des espaces mesurables. Alors le quotient $\mathcal{B}^\mu = \mathcal{B}/\{A \in \mathcal{B} \mid \mu(A) = 0\}$ est une algèbre de Boole mesurable. Pour tout morphisme $\theta : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$, l'application réciproque θ^{-1} induit un morphisme d'algèbres de Boole de $\mathcal{C}^\nu$ vers $\mathcal{B}^\mu$. Le foncteur ainsi défini est une équivalence de catégorie.*

À partir de maintenant, on appellera *espace mesurable* et on notera simplement X un objet abstrait de la catégorie des espaces mesurables. Un morphisme entre deux objets $f : X \to Y$ est une *application mesurable.* On note $\mathfrak{P}(X)$ l'algèbre de Boole associée à X par le théorème ci-dessus. Les éléments de $\mathfrak{P}(X)$ seront appelées *parties* de X et on utilisera les notations ensemblistes usuelles $\emptyset$, X, $\bigcup$, $\bigcap$ pour désigner respectivement, son plus petit élément, son plus grand élément, le supremum, l'infinimum etc. Enfin, par *mesure* sur X on entend une mesure sur $\mathfrak{P}(X)$.

On associe à chaque espace mesurable X, l'$*$-algèbre $\mathrm{L}^\infty(X)$ des classes de fonctions mesurables bornées à valeurs dans $\mathbb{C}$. L'involution $*$ est donnée par la conjugaison complexe.

Théorème 2.4. *Soit X un espace mesurable. Alors $\mathrm{L}^\infty(X)$ est une algèbre de von Neumann commutative. Réciproquement, toute algèbre de von Neumann commutative est de cette forme.*

En effet, $\mathrm{L}^\infty(X)$ est une algèbre de von Neumann car $\mathrm{L}^\infty(X)$, muni de la norme usuelle $\|\cdot\|_\infty$, est l'espace de Banach dual de $\mathcal{M}(X)$, l'espace des mesures complexes sur X.

Réciproquement, si M est une algèbre de von Neumann commutative alors l'ensemble des projections auto-adjointes $\mathcal{P}(M) = \{p \in M \mid p = p^* = p^2\}$ est une algèbre de Boole mesurable. On peut donc l'identifier à $\mathfrak{P}(X)$ pour un espace mesurable X, puis montrer que M s'identifie à $\mathrm{L}^\infty(X)$.

Enfin, mentionnons que les applications mesurables de X vers Y s'identifient précisément aux morphismes d'algèbres de $\mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(X)$ qui préservent l'involution et qui sont continus pour la topologie faible-$*$.

En raison de cette équivalence entre espaces mesurables et algèbres de von Neumann commutatives, la théorie des algèbres de von Neumann est souvent considérée comme une généralisation non-commutative de la théorie de la mesure. Les projections continuent à jouer un rôle crucial dans l'analyse des algèbres de von Neumann non-commutatives et il est souvent utile de les voir comme les parties d'un prétendu "espace mesurable non-commutatif". De même pour la notion suivante de *poids*, qui généralise la notion de mesure.

Définition 2.5. Soit M une algèbre de von Neumann. Soit $M^+ = \{x \in M \mid \exists y \in M,\ x = y^*y\}$ le *cône positif* de M.

Un *poids* sur M est une application $\varphi : M^+ \to \mathbb{R}_+ \cup \{\infty\}$ qui vérifie :

(i) $\varphi(0) = 0$ et $\varphi(\lambda x + \mu y) = \lambda\varphi(x) + \mu\varphi(y)$ pour tout $\lambda, \mu > 0$ et $x, y \in M^+$.

(ii) $\varphi(x) = \sup_i \varphi(x_i)$ pour toute suite $(x_i)_{i\in I} \in M^+$ qui croît vers $x \in M^+$.

On dit que φ est *semi-fini* lorsque $\{x \in M^+ \mid \varphi(x) < +\infty\}$ engendre un sous-espace $*$-faiblement dense dans M. On dit que φ est *fidèle* lorsque $\varphi(x) = 0$ implique que $x = 0$ pour tout $x \in M^+$.

Lorsque $M = \mathrm{L}^\infty(X)$ pour un espace mesurable X, il y a une correspondance parfaite entre les poids sur M et les mesures sur X grâce à la théorie de l'intégration.

$$\forall f \geq 0, \quad \varphi(f) = \int_X f d\mu.$$

La propriété (ii) est la traduction de la propriété de convergence monotone.

Lorsque φ est un poids *fini*, i.e. $\varphi(1) < +\infty$, il s'étend naturellement en une forme linéaire $\varphi \in M_*$. Lorsque $\varphi(1) = 1$, on dit que φ est un *état*. Une paire (M, φ) constituée d'une algèbre de von Neumann et d'un état φ sur M est l'analogue non-commutatif d'un espace de probabilité.

3 Facteurs et classification en types

Une fois que l'on a introduit les algèbres de von Neumann, la question qui se pose naturellement est celle de leur classification à isomorphisme près. Afin de s'attaquer à ce problème, Murray et von Neumann s'intéressent à une classe très spéciale d'algèbres de von Neumann, qui sont les moins commutatives possible au sens suivant.

Définition 3.1. Soit M une algèbre de von Neumann. On dit que M est un *facteur* lorsque le centre de M, i.e.

$$\mathcal{Z}(M) = \{a \in M \mid \forall b \in M,\ ab = ba\}$$

est réduit aux scalaires, i.e. $\mathcal{Z}(M) = \mathbb{C}$.

L'étude des facteurs est fondamentale pour la raison suivante. Si M est une algèbre de von Neumann quelconque, alors $\mathcal{Z}(M)$ est une algèbre de von Neumann commutative et s'écrit donc $\mathcal{Z}(M) = \mathrm{L}^\infty(X)$ pour un certain espace mesurable X. Murray et von Neumann montrent alors que M se désintègre de façon unique en un *champ de facteurs* au-dessus de X. La conclusion est que pour classifier toutes les algèbres de von Neumann il suffit de classifier les facteurs.

Pour analyser la structure des facteurs, Murray et von Neumann posent la définition suivante.

Définition 3.2. Soit M une algèbre de von Neumann. On dit que deux projections $p, q \in \mathcal{P}(M)$ sont équivalentes et on note alors $p \sim q$, lorsqu'il existe $u \in M$ tel que

$$p = u^*u \text{ et } q = uu^*.$$

Bien sûr, cette définition est sans intérêt dans le cas commutatif. Elle devient fondamentale lorsque M est un facteur en raison du théorème suivant.

Théorème 3.3. *Soit M un facteur (σ-fini). Il existe un poids τ sur M, unique à multiplication par un scalaire $\lambda \in \mathbb{R}_+^*$ près, tel que :*

$$\forall p, q \in \mathcal{P}(M), \quad p \sim q \iff \tau(p) = \tau(q)$$

De plus, l'ensemble des valeurs que prend le poids τ sur les projections de M coïncide, à une renormalisation près, avec l'un des ensemble suivants :

$\{0, 1, 2, \ldots, n\}$, $n \in \mathbb{N}$,	et on dit alors que M est de type	I_n
$\{0, 1, 2, \ldots, \infty\}$		I_∞
$[0, 1]$		II_1
$[0, +\infty]$		II_∞
$\{0, +\infty\}$		III

Les facteurs de type I_n, $n \in \mathbb{N} \cup \{\infty\}$ sont bien connus. En effet, on montre que ce sont exactement les algèbres de la forme $\mathbf{B}(H)$ pour H un espace de Hilbert (de dimension n). Le poids τ du théorème est alors donné (à un scalaire près) par la *trace* définie par

$$\mathrm{tr}(T) = \sum_{i \in I} \langle T\xi_i, \xi_i \rangle \text{ pour tout } T \in \mathbf{B}(H)^+$$

où $(\xi_i)_{i \in I}$ est une base orthonormée quelconque de H. La trace d'une projection est égale à son *rang* et deux projections sont équivalentes si et seulement si elles ont le même rang.

Pour les facteurs de type II, tout se passe comme si le rang des projections pouvait varier continûment au lieu de ne prendre que des valeurs entières discrètes. Par analogie avec le cas des facteurs de type I, le poids τ est toujours appelé *trace*. Quand le facteur M est de type II_1, la trace τ est finie et on peut donc la normaliser de sorte que $\tau(1) = 1$. Elle s'étend alors en une forme linéaire $\tau \in M_*$ qui vérifie $\tau(xy) = \tau(yx)$ pour tout $x, y \in M$.

Reste le cas des facteurs de type III. Ce sont les seuls facteurs pour lesquels le poids du théorème 3.3 est purement infini et toutes les projections non nulles sont donc équivalentes. Pour cette raison, von Neumann considéra les facteurs de type III comme étant pathologiques et douta même pendant un moment de leur existence. En réalité, les facteurs de type III sont très intéressants mais la richesse de leur structure ne peut être révélée que grâce à *la théorie modulaire* que nous verrons plus loin.

4 Exemples

Dans cette partie, on donne des exemples de constructions intéressantes d'algèbres de von Neumann en lien avec la théorie des groupes et la théorie ergodique.

Définition 4.1. Soit Γ un groupe discret. Considérons l'espace de Hilbert $\ell^2(\Gamma)$. Pour tout $g \in \Gamma$, définissons un opérateur unitaire $u_g \in \mathbf{B}(\ell^2(\Gamma))$ par

$$(u_g \cdot \xi)(h) = \xi(g^{-1}h) \text{ pour tout } \xi \in \ell^2(\Gamma).$$

L'algèbre de von Neumann de Γ, notée $\mathcal{L}(\Gamma)$, est la sous-algèbre de von Neumann de $\mathbf{B}(\ell^2(\Gamma))$ engendrée par les unitaires $\{u_g \mid g \in \Gamma\}$.

Un fait remarquable est que l'algèbre de von Neumann $\mathcal{L}(\Gamma)$ possède une trace canonique.

Proposition 4.2. *Soit Γ un groupe discret. Soit $\delta_e \in \ell^2(\Gamma)$ la suite qui vaut 1 sur l'élément neutre $e \in \Gamma$ et 0 ailleurs. Pour tout $x \in \mathcal{L}(\Gamma)$, posons*

$$\tau(x) = \langle x\delta_e, \delta_e \rangle.$$

Alors τ est un état tracial fidèle sur $\mathcal{L}(\Gamma)$.

Lorsque Γ est commutatif, $\mathcal{L}(\Gamma)$ est aussi commutative et doit donc être de la forme $\mathrm{L}^\infty(X)$ pour un certain espace mesurable X. En fait, $\mathcal{L}(\Gamma)$ est canoniquement isomorphe à $\mathrm{L}^\infty(\hat{\Gamma})$ où $\hat{\Gamma}$ est le *dual de Pontryagin* de Γ, i.e. le groupe compact formé par les caractères $\hat{\Gamma} = \hom(\Gamma, \mathbb{U})$ équipé de sa mesure de Haar (qui s'identifie à la trace τ sur $\mathcal{L}(\Gamma)$ précédemment décrite).

Lorsque Γ n'est pas commutatif, la dualité de Pontryagin classique ne fonctionne plus et $\hat{\Gamma}$ est en général trop petit. Si Γ est suffisamment non-commutatif, $\mathcal{L}(\Gamma)$ devient un facteur. Plus précisément, on a la caractérisation suivante.

Proposition 4.3. *Soit Γ un groupe discret. Alors $\mathcal{L}(\Gamma)$ est un facteur si et seulement si Γ est à classes de conjugaison infinies, i.e. pour tout $g \in \Gamma$, $g \neq e$, l'ensemble $\{hgh^{-1} \mid h \in \Gamma\}$ est infini.*

Notons que lorsque $\mathcal{L}(\Gamma)$ est un facteur, il est nécessairement de type II_1 et la trace du théorème 3.3 est précisément celle donnée par la proposition 4.2. Des exemples de groupes à classes de conjugaison infinies sont donnés par les groupes libres $\mathbb{F}_n$, $n \geq 2$, ou bien le groupe S_∞ des permutations de $\mathbb{N}$ à support fini.

Maintenant, nous allons voir une autre construction d'algèbres de von Neumann à partir d'une action d'un groupe discret sur un espace mesurable, qui nous donnera en particulier des exemples de facteurs de type III.

Définition 4.4. Une *action* d'un groupe discret Γ sur un espace mesurable X est un morphisme de groupes $\sigma : \Gamma \to \mathrm{Aut}(X)$ et on note $\sigma : \Gamma \curvearrowright X$.

On dit que l'action σ est *libre* lorsque pour tout $g \in \Gamma$ et tout $A \subset X$, on a

$$\sigma(g)|_A = \mathrm{id}_A \quad \Longleftrightarrow \quad g = e \text{ ou } A = \emptyset.$$

On dit que l'action σ est *ergodique* lorsque les seules parties de X qui sont invariantes par σ sont $\emptyset$ et X.

À partir d'une action σ d'un groupe discret Γ sur un espace mesurable X, on obtient une action, qu'on note encore σ, de Γ sur $\mathrm{L}^\infty(X)$ définie par

$$\sigma_g(f) = f \circ \sigma_g^{-1}.$$

On peut alors à partir de cette donnée construire une algèbre de von Neumann $\mathrm{L}^\infty(X) \rtimes_\sigma \Gamma$ qui est le *produit croisé* de Γ et de $\mathrm{L}^\infty(X)$. Plutôt que de détailler la construction de $\mathrm{L}^\infty(X) \rtimes_\sigma \Gamma$, contentons-nous simplement de dire qu'elle est engendrée par une copie de $\mathrm{L}^\infty(X)$ et une copie de $\mathcal{L}(\Gamma)$ qui sont liées par les relations

$$\forall g \in \Gamma,\ \forall f \in \mathrm{L}^\infty(X), \quad u_g f u_g^* = \sigma_g(f),$$

où les $(u_g)_{g\in\Gamma}$ sont les unitaires canonique de $\mathcal{L}(\Gamma)$.

Proposition 4.5. *Si l'action $\sigma : \Gamma \curvearrowright X$ est libre et ergodique alors $\mathrm{L}^\infty(X) \rtimes_\sigma \Gamma$ est un facteur.*

Maintenant, nous allons déterminer le type du facteur $\mathrm{L}^\infty(X) \rtimes_\sigma \Gamma$ pour une action libre et ergodique σ. Ce type sera lié à l'existence de mesures sur X qui sont invariantes sous l'action σ. L'ergodicité de l'action assure que si une telle mesure invariante non triviale (i.e. qui n'est ni la mesure nulle ni la mesure purement infinie) existe, alors elle est nécessairement unique. Ceci se démontre grâce au théorème de Radon-Nikodym. Une telle mesure s'étend alors naturellement en un poids tracial sur $\mathrm{L}^\infty(X) \rtimes_\sigma \Gamma$. On a alors la classification suivante :

Théorème 4.6. *Soit $\sigma : \Gamma \curvearrowright X$ une action libre et ergodique. Le facteur $\mathrm{L}^\infty(X) \rtimes_\sigma \Gamma$ est :*

— *de type* I *si et seulement si X est discret. Dans ce cas, il est de type* I_n *si Γ est fini de cardinal n et de type* I_∞ *sinon. La mesure de comptage est une mesure invariante non triviale.*

— *de type* II *si et seulement si X est diffus et admet une mesure invariante non triviale μ. Dans ce cas, il est de type* II_1 *si μ est finie et de type* II_∞ *sinon.*

— *de type* III *si et seulement si X ne possède pas de mesure invariante non triviale.*

Ce théorème fournit de nombreux exemples de facteurs de type III puisqu'une action générique n'admet pas de mesure invariante non triviale.

Enfin, pour terminer cette section, mentionnons que pour une action *libre* $\sigma : \Gamma \curvearrowright X$, le produit croisé $\mathrm{L}^\infty(X) \rtimes_\sigma \Gamma$ ne dépend en fait que de la *relation d'équivalence orbitale*

$$\mathcal{R} = \{(x, g \cdot x) \mid x \in X,\ g \in \Gamma\} \subset X \times X.$$

Ainsi, la construction du produit croisé ne retient que la partition de X en orbites et oublie a priori une grande partie de l'information sur l'action σ.

5 Théorie modulaire

La *théorie modulaire* développée par Tomita, Takesaki et Connes [Co72, Ta73, CT76] est la clé qui a permis de comprendre la structure des facteurs de type III.

Le point de départ de la théorie est une découverte de Tomita. Il a montré qu'on pouvait associer canoniquement, à chaque poids semi-fini fidèle φ sur une algèbre de von Neumann M, un groupe à un paramètre d'automorphismes

$$\sigma^\varphi : \mathbb{R} \ni t \mapsto \sigma_t^\varphi \in \mathrm{Aut}(M)$$

qu'on appelle le *flot modulaire* de φ. Ce flot mesure le défaut de tracialité de φ. En particulier, il est trivial si et seulement si φ est une trace.

Plus tard, durant sa thèse, Connes montre une version non-commutative du théorème de Radon-Nikodym qui permet de comprendre comment le flot modulaire σ^φ dépend du choix du poids φ. Il montre, en particulier, que si ψ est un autre poids semi-fini fidèle sur M, les flots modulaires σ^φ et σ^ψ sont égaux modulo le groupe $\mathrm{Inn}(M)$ des automorphismes intérieurs de M. Autrement dit, le groupe à un paramètre

$$\delta_M : \mathbb{R} \ni t \mapsto [\sigma_t^\varphi] \in \mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$$

ne dépend *pas* du choix de φ.

Ces résultats, combinés à la théorie de la dualité de Takesaki pour les produits croisés, ont permis d'aboutir au théorème suivant, qui est le Graal de la théorie modulaire.

Théorème 5.1 ([Ta73, CT76])**.** *Soit M une algèbre de von Neumann. Alors, il existe un quadruplé (N, τ, θ, ι) tel que :*

(i) *N est une algèbre de von Neumann.*

(ii) *τ est une trace semi-finie fidèle sur N*

(iii) *$\theta : \mathbb{R}_+^* \curvearrowright N$ est une action continue qui vérifie $\tau \circ \theta_\lambda = \lambda^{-1}\tau$ pour tout $\lambda \in \mathbb{R}_+^*$.*

(iv) *$\iota : M \to N^\theta$ est un isomorphisme de M sur l'algèbre des points fixes*

$$N^\theta = \{x \in N \mid \forall \lambda \in \mathbb{R}_+^*,\ \theta_\lambda(x) = x\}.$$

De plus, si $(N', \tau', \theta', \iota')$ est un autre quadruplet qui vérifie les mêmes conditions alors il existe un unique isomorphisme $\pi : N \to N'$ qui conjugue (τ, θ, ι) et (τ', θ', ι').

L'algèbre de von Neumann N est appelée le *cœur* de M et est notée $c(M)$. Lorsque M est commutative, i.e. de la forme $\mathrm{L}^\infty(X)$ pour un certain espace mesurable X, alors $c(M)$ est aussi commutative. Notons $c(X)$ l'espace mesurable défini par $\mathrm{L}^\infty(c(X)) = c(\mathrm{L}^\infty(X))$. L'espace $c(X)$ s'identifie alors au *fibré de Radon-Nikodym* de X. C'est le fibré au-dessus de X, de groupe structural $\mathbb{R}_+^*$, et dont les sections correspondent aux mesures semi-finies fidèles sur X. Le théorème de Radon-Nikodym affirme que ce fibré $c(X)$ est *principal*. De plus, $c(X)$ est équipé d'une mesure semi-finie fidèle τ canonique qui est dilatée par l'action de $\mathbb{R}_+^*$. L'intérêt de cette construction est qu'elle est complètement canonique. Ainsi, toute action $\sigma : \Gamma \curvearrowright X$ d'un groupe discret Γ induit naturellement une action $c(\sigma) : \Gamma \curvearrowright c(X)$ qui a l'avantage de préserver la mesure τ. Cette action induite, qu'on appelle *extension de Maharam*, est fondamentale pour comprendre les actions qui n'admettent pas de mesure invariante.

Un phénomène similaire se produit pour une algèbre de von Neumann M quelconque. Le cœur $c(M)$, qui est donc l'analogue non-commutatif du fibré de Radon-Nikodym, est tracial et on peut retrouver M à partir de $c(M)$ en prenant l'algèbre des points fixes $c(M)^\theta$. Ainsi, en théorie, pour classifier les algèbres de von Neumann quelconques, il suffit de classifier les actions de $\mathbb{R}_+^*$ qui dilatent la trace sur des algèbres de von Neumann traciales (N, τ).

Enfin, notons que même lorsque M est un facteur, le cœur $c(M)$ n'est pas nécessairement un facteur. Cela permet de définir un nouvel invariant.

Définition 5.2. Soit M un facteur. Le *flot des poids* de M est la restriction de l'action θ au centre $\mathcal{Z}(c(M))$ de $c(M)$.

Comme $\mathcal{Z}(c(M))$ est commutative, on peut toujours voir le flot des poids comme une action ergodique de $\mathbb{R}_+^*$ sur un espace mesurable. On montre que M est de type I ou II si et seulement si son flot des poids est isomorphe à l'action de $\mathbb{R}_+^*$ sur lui-même par multiplication. En revanche, toute autre action ergodique de $\mathbb{R}_+^*$ peut-être réalisée comme le flot des poids d'un facteur de type III. On dit ainsi qu'un facteur M est :

— de type III_1 lorsque son flot des poids est trivial (i.e. action triviale sur un espace à un point), ou autrement dit, lorsque $c(M)$ est un facteur.

— de type III_λ, $\lambda \in]0,1[$ lorsque son flot des poids est périodique de période λ, c'est-à-dire isomorphe à l'action de $\mathbb{R}_+^*$ sur le groupe quotient $\mathbb{R}_+^*/\lambda^{\mathbb{Z}}$.

— de type III_0 dans tous les autres cas, c'est-à-dire quand son flot des poids n'est pas *transitif*.

6 Classification des facteurs

Nous avons déjà vu que les facteurs de type I sont complètement compris et classifiés. Murray et von Neumann s'intéressent ensuite logiquement aux facteurs de type II_1. Ils démontrent le théorème suivant.

Théorème 6.1. *À isomorphisme près, il n'existe qu'un seul facteur R de type II_1 qui vérifie la propriété suivante : il existe une suite croissante de sous-$*$-algèbres de dimension finie $Q_n \subset R$, $n \in \mathbb{N}$ telle que $\bigcup_{n\in\mathbb{N}} Q_n$ est dense dans R pour la topologie faible-$*$.*

On dit d'un facteur qui vérifie cette propriété d'approximation qu'il est *hyperfini*. Ainsi, il n'existe, à isomorphisme près, qu'un seul facteur hyperfini de type II_1. Il peut être réalisé, par exemple, comme l'algèbre de von Neumann $\mathcal{L}(S_\infty)$ du groupe S_∞ des permutations à support fini de $\mathbb{N}$.

La question qui se pose alors est : existe-t-il des facteurs II_1 qui ne soient pas hyperfinis ? Murray et von Neumann répondent à cette question en montrant que le facteur $\mathcal{L}(\mathbb{F}_2)$ n'est *pas* hyperfini. Ils montrent ainsi qu'il existe au moins deux facteurs II_1 non-isomorphes.

Un progrès majeur survient avec les travaux de Connes dans les années 1970. Il démontre qu'un facteur de type II_1 est hyperfini si et seulement s'il est *moyennable*, une propriété en apparence beaucoup plus faible et analogue à la notion de moyennabilité pour les groupes. Il en déduit en particulier que le facteur $\mathcal{L}(\Gamma)$ d'un groupe Γ à classes de conjugaisons infinies est hyperfini si et seulement si Γ est moyennable.

De plus, ce théorème de Connes, combiné à la théorie modulaire expliquée précédemment, a permis de donner une classification complète des facteurs moyennables de type quelconque. Le cas le plus difficile, celui des facteurs de type III_1, a été complété par Haagerup.

Théorème 6.2 ([Co75b, Ha85])**.** *Il existe, à isomorphisme près, un unique facteur moyennable (à prédual séparable) de chacun des types suivants : I_n, $n \in \mathbb{N}$, I_∞, II_1, II_∞, et III_λ, $\lambda \in]0,1]$.*

Les classes d'isomorphisme de facteurs moyennables de type III_0 sont entièrement classifiés par leur flot des poids, et donc en correspondance bijective avec les flots ergodiques non transitifs de $\mathbb{R}_+^*$.

Ce résultat de classification clôt donc tout un chapitre de la théorie et on peut aujourd'hui dire que le monde des algèbres de von Neumann moyennables est totalement compris et finalement assez pauvre puisque tous les facteurs de type II_1 provenant de groupes moyennables ou d'actions de groupes moyennables sur des espaces mesurés sont isomorphes.

En revanche, la situation est radicalement différente dans le cas non moyennable. D'abord parce que l'on sait très peu de choses comme le montre la question suivante de Murray et von Neumann qui reste encore ouverte de nos jours.

Question 1. Les facteurs de groupes libres $\mathcal{L}(\mathbb{F}_n)$, $n \geq 2$ sont-ils tous isomorphes ?

Mais aussi parce que, contrairement au cas moyennable, on voit apparaître des phénomènes de *rigidité* spectaculaires. Ainsi, certaines familles d'actions peuvent être entièrement reconstruites à partir de leur algèbre de von Neumann. Ce genre de résultat est obtenu grâce à la théorie de la déformation/rigidité de Popa. Citons par exemple le théorème suivant de Ioana.

Théorème 6.3 ([I11])**.** *Soit Γ un groupe à classes de conjugaisons infinies qui a la propriété de Kazhdan. Soit $(X, \mu) = (X_0, \mu_0)^\Gamma$ le processus de Bernoulli indexé par Γ sur un espace de probabilité non trivial (X_0, μ_0). Considérons l'action $\Gamma \curvearrowright X$ par décalage de Bernoulli. Si une action libre ergodique $\Lambda \curvearrowright Y$ vérifie*

$$\mathrm{L}^\infty(X) \rtimes \Gamma \cong \mathrm{L}^\infty(Y) \rtimes \Lambda$$

alors $\Gamma \cong \Lambda$ et les actions $\Gamma \curvearrowright X$ et $\Lambda \curvearrowright Y$ sont conjuguées.

Une autre conjecture de rigidité, énoncée par Connes, demeure largement ouverte :

Conjecture 6.1. *Soit Γ un groupe à classes de conjugaisons infinies qui a la propriété de Kazhdan. Alors pour tout groupe Λ, on a $\mathcal{L}(\Lambda) \cong \mathcal{L}(\Gamma)$ si et seulement si $\Lambda \cong \Gamma$.*

7 Aperçu des principaux résultats

Mes travaux de thèses ont portés sur diverses propriétés de rigidités des algèbres de von Neumann, ainsi que des relations d'équivalences mesurées.

Solidité des produits croisés Bernoulli de type III

Dans [Oz03], Ozawa a découvert une propriété de rigidité remarquable des algèbres de von Neumann qu'il a appelée *solidité*. L'intérêt principal de cette notion est qu'un facteur solide non moyennable est nécessairement *premier* au sens suivant :

Définition 7.1. Un facteur M est dit *premier* s'il n'est pas de type I et s'il n'admet pas de décomposition de la forme $M = P \mathbin{\overline{\otimes}} Q$ avec P et Q des facteurs qui ne sont pas de type I.

Le célèbre théorème d'Ozawa, démontré dans [Oz03], affirme que l'algèbre de von Neumann $\mathcal{L}(\Gamma)$ d'un groupe hyperbolique quelconque Γ est solide. En particulier, les facteurs de groupes libres $\mathcal{L}(\mathbb{F}_n)$, $n \geq 2$ sont premiers.

Dans la deuxième partie de ce mémoire, reposant sur l'article [Ma16a], je démontre un résultat de *solidité relative* (voir Section 8) pour des produits croisés Bernoulli de type III qui généralise un théorème de Chifan et Ioana [CI10] en type II. Plus précisément, si l'on se donne une algèbre de von Neumann A_0 munie d'un état normal fidèle φ_0, alors pour tout groupe discret Γ, on a une action naturelle de Γ sur le produit tensoriel infini $(A_0, \varphi_0)^{\overline{\otimes}\Gamma}$ qui généralise l'action usuelle par décalage de Bernoulli de $\mathbb{Z}$ sur $(X_0, \mu_0)^{\overline{\otimes}\mathbb{Z}}$, où (X_0, μ_0) est un espace de probabilité. Le produit croisé correspondant $M = (A_0, \varphi_0)^{\overline{\otimes}\Gamma} \rtimes \Gamma$ est appelé *le produit croisé Bernoulli de* (A_0, φ_0) *par* Γ.

Théorème 7.2. *Soit A_0 une algèbre de von Neumann moyennable munie d'un état normal fidèle φ_0. Alors, pour tout groupe discret Γ, le produit croisé Bernoulli $M = (A_0, \varphi_0)^{\overline{\otimes}\Gamma} \rtimes \Gamma$ est solide relativement à $\mathcal{L}(\Gamma)$. En particulier, M est un facteur premier dès que Γ est non-moyennable et $A_0 \neq \mathbb{C}$*

Ce résultat est dû à Chifan et Ioana lorsque φ_0 est une trace. Je l'étends au cas où φ_0 est un état fidèle normal quelconque et où A_0 est donc possiblement de type III. J'obtiens ainsi de nombreux nouveaux exemples de facteurs premiers de type III ainsi que les premiers exemples de relations d'équivalences ergodiques de type III solides et non-moyennables. La preuve est une adaptation de la preuve originale de Chifan et Ioana. Elle repose sur une version non-traciale d'un argument de déformation/rigidité de Popa [Po08]. Elle utilise aussi la version non-traciale de sa théorie de l'entrelacement obtenue dans [HI15] (voir aussi l'appendice).

Trou spectral et facteurs pleins

La troisième partie de ce mémoire repose essentiellement sur [Ma16b], [HMV16] et [Ma17a]. Elle traite de la propriété de trou spectral et de la notion de facteur plein. Un facteur M est dit *plein* [Co74] lorsque l'application

$$\mathrm{Ad} : \mathcal{U}(M) \to \mathrm{Aut}(M),$$

qui à un unitaire $u \in \mathcal{U}(M)$ associe l'automorphisme intérieur $\mathrm{Ad}(u) : x \mapsto uxu^*$, est *ouverte sur son image*. Cela signifie que pour toute suite généralisée d'unitaires $(u_i)_{i \in I}$ on a $\mathrm{Ad}(u_i) \to \mathrm{id}$ si et seulement s'il existe une suite de scalaires $z_i \in \mathbb{U} = \ker \mathrm{Ad}$ telle que $u_i - z_i \to 0$ quand $i \to \infty$. Lorsque M est plein, le sous-groupe $\mathrm{Inn}(M) = \{\mathrm{Ad}(u) \mid u \in \mathcal{U}(M)\} \subset \mathrm{Aut}(M)$ est fermé et le groupe quotient $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ est séparé pour la topologie quotient.

La notion de facteur plein a permi à Murray et von Neumann de donner le premier exemple de facteur de type II_1 qui ne soit pas hyperfini. En effet, ils ont démontré que le facteur $\mathcal{L}(\mathbb{F}_2)$ est plein alors qu'un facteur hyperfini ne l'est jamais. Il s'agit donc d'une notion importante qu'on peut voir comme une propriété de rigidité plus forte que la non moyennabilité. Elle est intimement liée à la notion d'ergodicité forte en théorie ergodique. Par exemple, un facteur produit croisé $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_2$ issu d'une action libre et ergodique $\mathbb{F}_2 \curvearrowright X$ est plein si et seulement si l'action est *fortement ergodique* (toute partie de X presque invariante par l'action est presque triviale).

Le résultat principal de cette partie est une caractérisation de type *trou spectral* des facteurs pleins qui généralise un résultat de Connes dans le cas II_1 [Co75b, Théorème 2.1] aux facteurs de type quelconque. Le trou spectral permet de mieux exploiter la rigidité des facteur pleins et d'étudier plus finement la topologie de

$\operatorname{Out}(M)$. La preuve du trou spectral que l'on donne ici est grandement simplifiée par rapport à la preuve originale de [Ma16b] et elle repose sur un résultat de trou spectral local plus général [Ma17a]. Voici les principales applications.

La première application donne une condition nécessaire et suffisante, facile à vérifier, pour que le cœur d'un facteur de type III_1 soit plein.

Théorème 7.3 ([Ma16b])**.** *Soit M un facteur de type III_1. Alors $c(M)$ est plein si et seulement si M est plein et le morphisme canonique $\delta_M : \mathbb{R} \to \operatorname{Out}(M)$ est un homéomorphisme sur son image.*

Grâce à un renforcement du trou spectral, obtenu conjointement avec Cyril Houdayer et Peter Verraedt, nous obtenons aussi le résultat suivant qui n'était connu auparavant que pour les facteurs semi-finis [Co75b, Corollary 2.3].

Théorème 7.4 ([HMV17])**.** *Soient M et N deux facteurs pleins. Alors $M \mathbin{\overline{\otimes}} N$ est aussi un facteur plein.*

Enfin, la troisième application est un résultat d'*unique décomposition McDuff* qui généralise un résultat similaire de Popa dans le cas tracial [Po06, Theorem 5.1].

Théorème 7.5 ([HMV17])**.** *Soient M et N deux facteurs pleins infinis, et P et Q deux facteurs moyennables infinis. Supposons qu'il existe un isomorphisme $\Psi : M \mathbin{\overline{\otimes}} P \to N \mathbin{\overline{\otimes}} Q$. Alors il existe un unitaire $u \in N \mathbin{\overline{\otimes}} Q$ tel que $\Psi(M) = uNu^*$ et $\Psi(P) = uQu^*$.*

Relations d'équivalences stables et facteurs McDuff

Soit R le facteur hyperfini de Murray et von Neumann. Alors, il est facile de voir que $R \mathbin{\overline{\otimes}} R$ est aussi hyperfini et donc $R \cong R \mathbin{\overline{\otimes}} R$. Un facteur M de type II_1 à prédual séparable est dit *McDuff* [McD69] lorsque $M \cong M \mathbin{\overline{\otimes}} R$. La propriété de ne *pas* être McDuff est donc une propriété de rigidité plus forte que la non moyennabilité. De façon analogue, on dit qu'une relation d'équivalence ergodique $\mathcal{R}$ de type II_1 est *stable* [JS87] lorsque $\mathcal{R} \cong \mathcal{R} \otimes \mathcal{R}_0$ où $\mathcal{R}_0$ est l'unique relation d'équivalence ergodique hyperfinie de type II_1. Par exemple, pour toute action $\mathbb{F}_2 \curvearrowright X$ libre et ergodique préservant une mesure de probabilité, la relation d'équivalence orbitale associée n'est pas stable et le produit croisé $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_2$ n'est pas McDuff.

Le théorème principal de cette partie donne une caractérisation locale de type "trou spectral" de la non-stabilité.

Théorème 7.6. *Soit $\mathcal{R}$ une relation ergodique préservant la mesure sur un espace de probabilité (X, μ). Alors $\mathcal{R}$ n'est pas stable si et seulement s'il existe une partie finie K du pseudo-groupe plein $[[\mathcal{R}]]$ et une constante $\kappa > 0$ telles que*

$$\|vp - pv\|_2 \leq \kappa \sum_{u \in K} \|uv - vu\|_2 + \|pu - up\|_2$$

pour tout $v \in [[\mathcal{R}]]$ et $p \in \mathfrak{P}(X)$.

Cette caractérisation renforce la caractérisation obtenue dans [JS87] qui montre seulement que le membre de gauche de cette inégalité tend vers 0 dès que le membre de droite tend vers 0. La démonstration repose sur un argument de maximalité.

Grâce à cette caractérisation renforcée, il devient possible de démontrer le résultat suivant.

Théorème 7.7 ([Ma17b]). *Soit $\mathcal{R}$ et $\mathcal{S}$ deux relations d'équivalences ergodiques de type* II_1*. Alors $\mathcal{R} \otimes \mathcal{S}$ est stable si et seulement si $\mathcal{R}$ est stable ou $\mathcal{S}$ est stable.*

Notons qu'il est aussi possible de démontrer un analogue du Théorème 7.6 pour les facteurs II_1 non McDuff. Mais cette caractérisation s'avère insuffisante pour démontrer la conjecture suivante, analogue au Théorème 7.7.

Conjecture 7.1. *Soit M et N deux facteurs de type* II_1*. Alors $M \mathbin{\overline{\otimes}} N$ est McDuff si et seulement si M est McDuff ou N est McDuff.*

Cependant, je donne quelques résultats partiels qui permettent de démontrer la conjecture pour tous les exemples concrets connus.

Notations and conventions

Let M be a von Neumann algebra. We never assume that the predual M_* of M is separable unless explicitly stated. We use the following notations.

— $\mathcal{P}(M)$ is the set of all self-adjoint projections of M,

— $\mathcal{U}(M)$ is the unitary group of M,

— $\mathrm{Ball}(M)$ is the unit ball of M,

— M^+, M_*^+ are the positive cones of M and M_* respectively and $\overline{M^+}$, $\overline{M_*^+}$ are the extended positive cones [Ha77a].

By *weak* topology* on M, we mean the weak* topology with respect to its predual M_*. For us, the *strong topology* is the topology induced by the seminorms $\|x\|_\varphi = \varphi(x^*x)^{1/2}$ for $\varphi \in M_*^+$. The $*$*-strong topology* is the topology induced by the seminorms $\|x\|_\varphi + \|x^*\|_\varphi$ for $\varphi \in M_*^+$. We never use the σ-weak/ultraweak, σ-strong/ultrastrong terminology.

We denote by $(c(M), \theta, \tau)$ the *non-commutative flow of weights* of M and we identify M with the fixed point subalgebra $c(M)^\theta$ [Ta73, CT76]. There is a canonical order preserving bijection from $\overline{M_*^+}$ to the set $\{h \in \overline{c(M)^+} \mid \forall \lambda > 0,\ \theta_\lambda(h) = \lambda^{-1}h\}$ which sends every normal weight $\varphi \in \overline{M_*^+}$ to the unique $h \in \overline{c(M)^+}$ such that $\tau(h\cdot)$ is the dual weight of φ. The operator h will simply be denoted by φ. In other words, we view $\overline{M_*^+}$ as a subset of $\overline{c(M)^+}$. When φ is faithful and semifinite, $t \mapsto \varphi^{it}$ is a one parameter group of unitaries in $\mathcal{U}(c(M))$ and we have $\sigma_t^\varphi(x) = \varphi^{it}x\varphi^{-it}$ for all $x \in M$.

Let $\mathcal{A}$ be the $*$-algebra of all τ-measurable operators affiliated with $c(M)$. Then $M_*^+ \subset \mathcal{A}$ and its linear span is

$$\mathrm{L}^1(M) = \{\omega \in \mathcal{A} \mid \forall \lambda > 0,\ \theta_\lambda(\omega) = \lambda^{-1}\omega\}$$

which is naturally identified with the predual M_*. For every $\omega \in \mathrm{L}^1(M) = M_*$, we let $\langle \omega \rangle = \omega(1)$.

The space of *half-densities*

$$\mathrm{L}^2(M) = \{\xi \in \mathcal{A} \mid \forall \lambda > 0,\ \theta_\lambda(\xi) = \lambda^{-1/2}\xi\}$$

is a Hilbert space for the inner product $\langle \xi, \eta \rangle = \langle \xi\eta^* \rangle$ where $\xi\eta^* \in M_*$. The Hilbert space $\mathrm{L}^2(M)$, equipped with the left and right action of $M \subset \mathcal{A}$ by multiplication, the antilinear involution $J : \xi \mapsto \xi^*$, and the positive cone $\mathrm{L}^2(M)^+ = \mathrm{L}^2(M) \cap \mathcal{A}^+$, is identified with the standard form of M [Ha73]. If $\varphi \in M_*^+$, then $\varphi^{1/2} \in \mathrm{L}^2(M)^+$. Conversely, if $\xi \in \mathrm{L}^2(M)$, then $\xi^*\xi \in M_*^+$. We say that ξ is *left φ-bounded* when there is a constant $\kappa > 0$ such that $\xi^*\xi \leq \kappa\varphi$. This is equivalent to $\xi \in M\varphi^{1/2}$. We say that ξ is *φ-bounded* if both ξ and ξ^* are left φ-bounded.

A *topological group* is a group G equipped with a topology (not necessarily Hausdorff) making the map $(g, h) \in G \times G \mapsto gh^{-1}$ continuous. A topological group G is said to be *complete* if it is complete with respect to the uniform structure generated by the following sets

$$U_\mathcal{V} = \{(g, h) \in G \times G \mid gh^{-1} \in \mathcal{V} \text{ and } g^{-1}h \in \mathcal{V}\},$$

where $\mathcal{V}$ runs over the neighborhoods of 1 in G. If H is a subgroup of G, then it is a topological group for the induced topology. If H is normal in G then G/H is also a topological group with respect to the quotient topology. If G is complete and H is closed in G, then H and G/H are complete.

Let M be a von Neumann algebra. Then the restriction of the weak* topology, the strong topology and the $*$-strong topology all coincide on $\mathcal{U}(M)$ and they turn $\mathcal{U}(M)$ into a complete topological group. If moreover M_* has separable predual, then $\mathcal{U}(M)$ is Polish.

The group $\mathrm{Aut}(M)$ of all $*$-automorphisms of M acts on M_* by $\theta(\varphi) = \varphi \circ \theta^{-1}$ for all $\theta \in \mathrm{Aut}(M)$ and all $\varphi \in M_*$. Following [Co74, Ha73], $\mathrm{Aut}(M)$ is equipped with the *u-topology*. This is the topology of pointwise norm convergence on M_*, meaning that a net $(\theta_i)_{i \in I}$ in $\mathrm{Aut}(M)$ converges to the identity id_M in the u-topology if and only if for all $\varphi \in M_*$ we have $\|\theta_i(\varphi) - \varphi\| \to 0$ as $i \to \infty$. This turns $\mathrm{Aut}(M)$ into a complete topological group. When M_* is separable, $\mathrm{Aut}(M)$ is Polish. The group $\mathrm{Aut}(M)$ also acts naturally on the non-commutative flow of weights $(c(M), \theta, \tau)$, hence also on $\mathrm{L}^2(M)$. We have $\theta(\varphi^{1/2}) = \theta(\varphi)^{1/2}$ for every $\varphi \in M_*^+$. Then, by the Powers-Størmer inequality, the u-topology is also the topology of pointwise norm convergence on $\mathrm{L}^2(M)$.

We denote by $\mathrm{Ad} : \mathcal{U}(M) \to \mathrm{Aut}(M)$ the continuous homomorphism which sends a unitary u to the corresponding inner automorphism $\mathrm{Ad}(u)$. We denote by $\mathrm{Inn}(M)$ the image of Ad. We denote by $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$ the quotient group and by $\pi_M : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ the quotient map. The group $\mathrm{Inn}(M)$ is equipped with the restriction of the u-topology while $\mathrm{Out}(M)$ is equipped with the quotient topology. Finally, we denote by $\overline{\mathrm{Inn}}(M)$ the closure of $\mathrm{Inn}(M)$ in $\mathrm{Aut}(M)$.

Part II

Solidity of type III Bernoulli crossed products

In [Oz03], Ozawa discovered a remarkable rigidity property of von Neumann algebras that he called *solidity*. The main interest of this notion is that any solid non-amenable factor is *prime*, i.e. not a tensor product of two non type I factors. Ozawa's celebrated result states that the group von Neumann algebra $\mathcal{L}(\Gamma)$ of any hyperbolic group Γ (e.g. free groups) is solid.

In this chapter, which is based on [Ma16a], we generalize a theorem of Chifan and Ioana [CI10] by proving that for any, possibly type III, amenable von Neumann algebra $A_0 \neq \mathbb{C}$, any faithful normal state φ_0 and any discrete group Γ, the associated Bernoulli crossed product von Neumann algebra $M = (A_0, \varphi_0)^{\overline{\otimes}\Gamma} \rtimes \Gamma$ is solid *relatively* to $\mathcal{L}(\Gamma)$. In particular, M is solid if and only if $\mathcal{L}(\Gamma)$ is solid and M is prime if and only if Γ is non-amenable. This gives many new examples of solid or prime type III factors as well as the first examples of solid non-amenable type III equivalence relations.

The proof is based on an adaptation of a deformation/rigidity argument to the type III context and follows the lines of Chifan and Ioana's original proof. We also use the type III generalization of Popa's intertwining theory [HI15] which is exposed in details in the Appendix.

8 Relative solidity

In [Oz03], N. Ozawa introduced the notion of *solid* von Neumann algebras. It is easy to check that in the II_1 case, the definition of solidity given in [Oz03] is equivalent to the following one (see Proposition 8.4).

Definition 8.1. Let M be a von Neumann algebra. We say that M is *solid* if every properly non-amenable subalgebra with expectation $Q \subset 1_Q M 1_Q$ has discrete center.

Equivalently, M is *solid* if and only if every subalgebra with expectation $Q \subset M$ is a direct sum of an amenable von Neumann algebra and a family (possibly empty) of non-amenable factors. This clearly shows the analogy with the notion of *solid ergodicity* for equivalence relation discovered in [CI10].

In this section, we are interested in a *relative* version of solidity. We refer to the Appendix for Popa's intertwining theory and the explanation of the symbol $\prec_M$.

Definition 8.2. Let M be a von Neumann algebra and $N \subset M$ a subalgebra with expectation. We say that M is *solid relatively to* N if every properly non-amenable subalgebra with expectation $Q \subset 1_Q M 1_Q$ such that $Q \not\prec_M N$ has discrete center.

Clearly, a von Neumann algebra M is solid if and only if it is solid relatively to $\mathbb{C}$. The following property justifies the terminology.

Proposition 8.3. *Let $P \subset N \subset M$ be inclusions of von Neumann algebras with expectations. If M is solid relatively to N and N is solid relatively to P then M is solid relatively to P. In particular, if M is solid relatively to N and N is solid then M is solid.*

Proof. Take a properly non-amenable subalgebra with expectation $Q \subset 1_Q M 1_Q$ such that Q has diffuse center. We have to show that $Q \prec_M P$. Since M is solid relatively to N, we already know that $Q \prec_M N$. Take a nonzero partial isometry $v \in \mathcal{I}^l_M(Q, N)$ and $\pi_v : pQp \to qNq$ the associated embedding. Then $D = \pi_v(pQp) \subset qNq$ is again a subalgebra with expectation which is properly non-amenable and it has diffuse center. Since N is solid relatively to P, this implies that $D \prec_N P$. Take a nonzero $w \in \mathcal{D}^l_N(D, P)$. Since q is the N-support of v, we have $vw \neq 0$. And we have $vw \in \mathcal{D}^l_M(Q, D)\mathcal{D}^l_N(D, P) \subset \mathcal{D}^l_M(Q, P)$. Hence $Q \prec_M P$. □

Next we present other possible formulations of relative solidity.

Proposition 8.4. *Let M be a von Neumann algebra and $N \subset M$ a subalgebra with expectation. The following are equivalent:*

1. *M is solid relatively to N.*
2. *For every diffuse subalgebra with expectation $Q \subset 1_Q M 1_Q$ such that $Q' \cap 1_Q M 1_Q$ is non-amenable we have $Q' \cap 1_Q M 1_Q \prec_M N$.*
3. *For every non-amenable subalgebra with expectation $Q \subset 1_Q M 1_Q$ such that $Q' \cap 1_Q M 1_Q$ is diffuse we have $Q \prec_M N$.*

Proof. (1) ⇒ (2). Suppose that $Q' \cap 1_Q M 1_Q \not\prec_M N$. Take $A \subset Q$ a diffuse abelian subalgebra with expectation. Then $P = A \vee (Q' \cap 1_Q M 1_Q)$ has diffuse center and $P \not\prec_M N$. Hence P is amenable by (1). Therefore $Q' \cap 1_Q M 1_Q$ is also amenable.

(2) $\Rightarrow$ (3). Let $P = Q' \cap 1_Q M 1_Q$. Then P is diffuse and since Q is non-amenable and $Q \subset P' \cap 1_Q M 1_Q$ we have that $P' \cap 1_Q M 1_Q$ is non-amenable. Hence $P' \cap 1_Q M 1_Q \prec_M N$ by (2). Therefore $Q \prec_M N$.

(3) $\Rightarrow$ (1). Let $Q \subset 1_Q M 1_Q$ be a properly non-amenable subalgebra with expectation such that Q has diffuse center. Then $Q' \cap 1_Q M 1_Q$ is also diffuse. Hence by (3), we have $Q \not\prec_M N$. Therefore M is solid relatively to N. $\square$

Note that [CI10, Theorem 2] as well as [Bo12, Theorem B] and [Bo12, Theorem C] are examples of relative solidity results. Relative solidity can be used to prove primeness or fullness even when true solidity fails. This can happen for example when it is combined with the following absorption property.

Definition 8.5. Let M be a von Neumann algebra and $N \subset M$ a subalgebra with expectation. We say that $N \subset M$ is *absorbing* if for every diffuse subalgebra with expectation $Q \subset 1_Q N 1_Q$, we have $Q' \cap 1_Q M 1_Q \subset 1_Q N 1_Q$.

Example 1 ([Po04, Section 3]). Let (A, φ) be a von Neumann algebra with a faithful normal state φ and $\sigma : \Gamma \to \mathrm{Aut}(A, \varphi)$ a φ-preserving action of a discrete group Γ, let $M = A \rtimes_\sigma \Gamma$ be the crossed product von Neumann algebra. Suppose that the action σ is *mixing*, i.e.

$$\forall a, b \in A, \ \lim_{g \to \infty} \varphi(a \sigma_g(b)) = \varphi(a)\varphi(b).$$

Then the inclusion $\mathcal{L}(\Gamma) \subset M$ is absorbing.

Other examples of this absorption phenomenon come from some group inclusions as well as free products [Po83] and amalgamated free products [IPP08, Theorem 1.1].

Lemma 8.6. *Let $N \subset M$ be an absorbing inclusion. Let $Q \subset 1_Q M 1_Q$ be a diffuse subalgebra with expectation. If $Q \prec_M N$ then there exists a nonzero partial isometry $v \in M$ such that $vv^* \in Q \vee (Q' \cap 1_Q M 1_Q)$, $v^* v \in N$ and $v^*(Q \vee (Q' \cap 1_Q M 1_Q))v \subset N$. In particular, we have $Q \vee (Q' \cap 1_Q M 1_Q) \prec_M N$.*

Proof. Take a nonzero partial isometry $v \in \mathcal{I}^l_M(Q, N)$ and let $\pi_v : pQp \to qNq$ be the associated embedding. We then have $vv^* \in (pQp)' \cap pMp$, $v^* v \in \pi_v(pQp)' \cap qMq$ and

$$v^*(pQp \vee ((pQp)' \cap pMp))v \subset \pi_v(pQp) \vee (\pi_v(pQp)' \cap qMq)$$

Since $\pi_v(pQp)$ is diffuse, and N is absorbing, then $\pi_v(pQp)' \cap qMq = \pi_v(pQp)' \cap qNq \subset qNq$. Moreover, we have $pQp \vee ((pQp)' \cap pMp) = p(Q \vee (Q' \cap 1_Q M 1_Q))p$ so that $vv^* \in Q \vee (Q' \cap 1_Q M 1_Q)$ and $v^*(Q \vee (Q' \cap 1_Q M 1_Q))v \subset qNq$ as we wanted. $\square$

Proposition 8.7. *Let M be a von Neumann algebra and $N \subset M$ a subalgebra with expectation. Suppose that M is solid relatively to N and the inclusion $N \subset M$ is absorbing. Let $P \subset 1_P M 1_P$ be any non-amenable factor with expectation such that $P \not\prec_M N$. Then P is prime. If P has moreover a separable predual then it is full.*

Proof. Suppose that $P = P_1 \overline{\otimes} P_2$ where P_1 and P_2 are two diffuse factors. Since P is non-amenable then one of them, say P_1, is non-amenable. Since M is solid relatively to N, by Proposition 8.4, we must have $P_2 \prec_M N$. Since $N \subset M$ is absorbing, this implies that $P = P_2 \vee (P_2' \cap P) \prec_M N$ by Lemma 8.6. Contradiction.

Now we suppose that P has separable predual and we show that P is full. On the contrary, suppose that P is not full. Then, by [HU15, Theorem 3.1], there exists a decreasing sequence of diffuse abelian subalgebras $Q_i \subset P$ with expectation such that $P = \bigvee_{i\in\mathbb{N}}(Q_i' \cap P)$. Suppose that for some i, we have $Q_i' \cap 1_P M 1_P \prec_M N$. Note that $Q_i \subset Q_i' \cap 1_P M 1_P$. So by Lemma 8.6, we know that there exists a nonzero partial isometry $v \in M$ such that $e = v^*v \in Q_i' \cap 1_P M 1_P$, $f = vv^* \in N$ and $v(Q_i' \cap qMq)v^* \subset fNf$ with expectation. Note that for all $j \geq i$, we have $Q_j \subset Q_i \subset Q_i' \cap 1_P M 1_P$ with expectation and Q_j is diffuse. Hence, since N is absorbing we have $v(Q_j' \cap 1_P M 1_P)v^* \subset (vQ_jv^*)' \cap fMf \subset fNf$. Therefore for all $j \geq i$, we have $v(Q_j' \cap 1_P M 1_P)v^* \subset fNf$. Thus $v(\bigvee_{i\in\mathbb{N}}(Q_i' \cap 1_P M 1_P))v^* \subset fNf$. In particular $\bigvee_{i\in\mathbb{N}}(Q_i' \cap 1_P M 1_P) \prec_M N$. Since $P \subset \bigvee_{i\in\mathbb{N}}(Q_i' \cap 1_P M 1_P)$ with expectation we get $P \prec_M N$. But this is not possible by assumption on P. Hence we must have $Q_i' \cap 1_P M 1_P \not\prec_M N$ for all i. By relative solidity and using Proposition 8.4, this implies that $Q_i' \cap 1_P M 1_P$ is amenable for all i. In particular, $Q_i' \cap P$ is amenable for all i. Hence $P = \bigvee_{i\in\mathbb{N}}(Q_i' \cap P)$ is also amenable. From this contradiction we conclude that P is full. $\square$

9 Coarse inclusions

Let $M \subset N$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation $\mathrm{E}_M : N \to M$. Then E_M gives rise to an inclusion of M-M-bimodules $\mathrm{L}^2(M) \subset \mathrm{L}^2(N)$. We say that the inclusion $M \subset N$ is *coarse*[1] if, for some choice of a faithful normal conditional expectation $\mathrm{E}_M : N \to M$, the M-M-bimodule

$$\mathrm{L}^2(N) \ominus \mathrm{L}^2(M) = \{\xi \in \mathrm{L}^2(N) \mid \xi \perp \mathrm{L}^2(M)\}$$

is weakly contained in the coarse M-M-bimodule $\mathrm{L}^2(M) \mathbin{\overline{\otimes}} \mathrm{L}^2(M)$.

The following key lemma is an abstract non-tracial version of an argument used in [Po08, Lemma 5.1]. See also [HI15b, Theorem 4.1].

Lemma 9.1. *Let $M \subset N$ be an inclusion of σ-finite von Neumann algebras with expectation. Suppose that the inclusion $M \subset N$ is coarse. Let $P \subset M$ be a subalgebra with expectation and suppose that P is properly non-amenable. Let ω be any free ultrafilter on $\mathbb{N}$. Then we have $P' \cap N^\omega \subset M^\omega$.*

Proof. Let $\mathrm{E}_M : N \to M$ be a faithful normal conditional expectation as in the definition of a coarse inclusion. Let $E^\omega : N^\omega \to N$ the canonical conditional expectation and $\mathrm{E}_{M^\omega} : N^\omega \to M^\omega$ the conditional expectation induced by E_M.

Now, suppose, by contradiction, that there is $Y \in P' \cap N^\omega$ with $Y \neq 0$ and such that $\mathrm{E}_{M^\omega}(Y) = 0$. We have $\mathrm{E}_M(Y^*Y) \in P' \cap M$. Let $c \in P' \cap M$ be an element such that $q = \mathrm{E}_M(Y^*Y)^{1/2}c$ is a nonzero projection in $P' \cap M$. Then $Yc \in P' \cap N^\omega$ and $\mathrm{E}_{M^\omega}(Yc) = 0$ and we have $\mathrm{E}_M((Yc)^*(Yc)) = q$. So, without loss of generality, we can directly suppose that $q = \mathrm{E}_M(Y^*Y) \in P' \cap M$ is a nonzero projection. We will show that the P-P-bimodule $q\mathrm{L}^2(M)$ is weakly contained in the P-P-bimodule $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)$. Let $V : \mathrm{L}^2(M) \to \mathrm{L}^2(N)$ and $W : \mathrm{L}^2(N) \ominus \mathrm{L}^2(M) \to \mathrm{L}^2(N)$ be the inclusion of bimodules. Note that $VV^* = 1 - WW^* = e_M$. Pick a sequence $(y_n)_{n\in\mathbb{N}}$ representing Y and define a completely positive map

$$\Phi : \mathbf{B}(\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)) \to \mathbf{B}(\mathrm{L}^2(M))$$

1. We thank R. Boutonnet for suggesting this name.

$$T \mapsto \lim_{n\to\omega} (V^* y_n^* W T W^* y_n V) \text{ in the weak}^* \text{ topology.}$$

We have

$$\Phi(1) = \lim_{n\to\omega} (V^* y_n^* (1 - e_M) y_n V) = \mathrm{E}_M(Y^*Y) - \mathrm{E}_M(Y^*)\mathrm{E}_M(Y) = q$$

Hence Φ takes its values in $q\mathbf{B}(\mathrm{L}^2(M)q \simeq \mathbf{B}(q\mathrm{L}^2(M))$ which means that we can view Φ as a unital completely positive map from $\mathbf{B}(\mathrm{L}^2(N) \ominus \mathrm{L}^2(M))$ to $\mathbf{B}(q\mathrm{L}^2(M))$. And since $Y \in P' \cap N^\omega$ we see that Φ preserves the P-P-bimodule representations. Hence the P-P- bimodule $q\mathrm{L}^2(M)$ is weakly contained in $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)$. Since P is with expectation in M, we have an inclusion of P-P-bimodules $\mathrm{L}^2(P) \subset \mathrm{L}^2(M)$. Hence the P-P-bimodule $q\mathrm{L}^2(P)$ is weakly contained in $\mathrm{L}^2(N) \ominus \mathrm{L}^2(M)$. By the coarse inclusion property, this implies in particular that $q\mathrm{L}^2(P)$ is weakly contained in the coarse P-P-bimodule. We conclude easily that qP is amenable and this contradicts the assumption that P is properly non-amenable. □

10 Spectral gap rigidity for non-tracial von Neumann algebras

In this section, we prove an abstract non-tracial version of Popa's spectral gap rigidity principle [Po08, Lemma 5.2]. We start by recalling the notion of symmetric malleable deformations, or *s-malleable* deformations [Po04]. Let M be a von Neumann algebra. A *malleable deformation* of M is a pair $(\widetilde{M}, \theta)$ where $M \subset \widetilde{M}$ is an inclusion with expectation and $\theta : \mathbb{R} \to \mathrm{Aut}(\widetilde{M})$ is a continuous action of $\mathbb{R}$. The deformation $(\widetilde{M}, \theta)$ is said to be *symmetric* if there exists $\beta \in \mathrm{Aut}(\widetilde{M})$ such that $\beta_{|M} = \mathrm{Id}$ and for all $t \in \mathbb{R}$, $\beta \circ \theta_t \circ \beta = \theta_{-t}$. We will say that a subalgebra $Q \subset M$ with expectation is *rigid relatively to* the deformation $(\widetilde{M}, \theta)$ if θ converges uniformly on the unit ball of Q: for every $*$-strong neighborhood $\mathcal{V}$ of 0 in $\widetilde{M}$ there exists $t_0 > 0$ such that

$$\forall t \in [-t_0, t_0], \forall x \in (Q)_1,\ \theta_t(x) - x \in \mathcal{V}.$$

Now we can state the main theorem of this section.

Theorem 10.1. *Let M be a σ-finite von Neumann algebra, $(\widetilde{M}, \theta)$ a symmetric malleable deformation of M and $Q \subset M$ a subalgebra with expectation. Suppose that*

(i) *The inclusion $M \subset \widetilde{M}$ is coarse.*

(ii) *Q is finite.*

(iii) *$Q' \cap 1_Q M 1_Q$ is properly non-amenable.*

Then Q is rigid relatively to the deformation $(\widetilde{M}, \theta)$ and $Q \prec_{\widetilde{M}} \theta_1(Q)$.

Proof. We proceed in 4 steps following the lines of Popa's original argument. In fact, only step 1 is different from the tracial case.

Step 1 - *The subalgebra Q is rigid relatively to $(\widetilde{M}, \theta)$.*

Suppose that Q is not relatively rigid. Then we can find a $*$-strong neighborhood of 0 in $\widetilde{M}$ denoted by $\mathcal{V}$, a sequence $x_n \in (Q)_1$ and a sequence of reals $t_n \to 0$ such that $\theta_{t_n}(x_n) - x_n \notin \mathcal{V}$ for all n. Let ω be any free ultrafilter on $\mathbb{N}$. Since Q is finite the sequence $(x_n)_{n\in\mathbb{N}}$ defines an element x in the ultraproduct $Q^\omega \subset M^\omega \subset \widetilde{M}^\omega$. Also, since $t_n \to 0$ we have $||\varphi \circ \theta_{t_n} - \varphi|| \to 0$ for all $\varphi \in \widetilde{M}_*$. Using this, we obtain an automorphism of $\widetilde{M}^\omega$ defined by

$$\Theta((y_n)^\omega) = (\theta_{\frac{t_n}{2}}(y_n))^\omega.$$

Note that the choice of x_n and t_n we made implies that $\Theta^2(x) \neq x$. Now, observe that $\Theta(y) = y$ for all $y \in \widetilde{M}$ because $t_n \to 0$. In particular, if we let $P = Q' \cap 1_Q M 1_Q$ then we have $\Theta(P) = P$ and therefore $\Theta(P' \cap 1_Q \widetilde{M}^\omega 1_Q) = P' \cap 1_Q \widetilde{M}^\omega 1_Q$. Since $x \in P' \cap 1_Q \widetilde{M}^\omega 1_Q$, we get $\Theta(x) \in P' \cap 1_Q \widetilde{M}^\omega 1_Q$. Lemma 9.1 applies to the coarse inclusion $1_Q M 1_Q \subset 1_Q \widetilde{M} 1_Q$ and shows that

$$P' \cap 1_Q \widetilde{M}^\omega 1_Q \subset 1_Q M^\omega 1_Q.$$

Therefore we get $\Theta(x) \in M^\omega$. Now, choose a symmetry β for $(\widetilde{M}, \theta)$ and extend it naturally to an automorphism $\beta \in \operatorname{Aut}(\widetilde{M}^\omega)$. Then β fixes M^ω and we have $(\beta \circ \Theta \circ \beta)(x) = \Theta^{-1}(x)$. Since $x \in M^\omega$ and $\Theta(x) \in M^\omega$, we conclude that $\Theta(x) = \Theta^{-1}(x)$. And this contradicts the fact that $\Theta^2(x) \neq x$. Therefore Q is rigid relatively to the deformation $(\widetilde{M}, \theta)$.

STEP 2 - *For sufficiently small t there exists a nonzero $\theta_t(Q)$-Q-intertwiner.*

Since Q is finite and with expectation, we can take ψ a faithful normal state on $\widetilde{M}$ such that Q is in the centralizer $\widetilde{M}^\psi$. By Step 1, Q is rigid relatively to $(\widetilde{M}, \theta)$. Hence we can find t_0 small enough so that for all $|t| \leq t_0$ we have

$$\forall u \in \mathcal{U}(Q),\ \mathfrak{Re}(\psi(\theta_t(u)u^*)) \geq \frac{\psi(1_Q)}{2} > 0.$$

Now take $\mathcal{C} \subset (\widetilde{M})_1$ the weak* closed convex hull of $\{\theta_t(u)u^* \mid u \in \mathcal{U}(Q)\}$ and let $w_t \in \mathcal{C}$ the unique element which minimizes $||w_t||_\psi$. Then, we have $w_t \in \theta_t(1_Q)\widetilde{M}1_Q$ and by the above inequality $w_t \neq 0$. Since $Q \subset \widetilde{M}^\psi$ we have $||\theta_t(u)w_t u^*||_\psi = ||w_t||_\psi$ for all $u \in \mathcal{U}(Q)$. Therefore, by the uniqueness of w_t we have

$$\forall x \in Q,\ \theta_t(x)w_t = w_t x.$$

So w_t is indeed a nonzero $\theta_t(Q)$-Q-intertwiner.

STEP 3 - *If there exists a nonzero $\theta_t(Q)$-Q-intertwiner then there exists a nonzero $\theta_{2t}(Q)$-Q-intertwiner.*

Take a nonzero $\theta_t(Q)$-Q-intertwiner $w_t \in \theta_t(1_Q)\widetilde{M}1_Q$ so that

$$\forall x \in Q,\ \theta_t(x)w_t = w_t x.$$

Take a symmetry $\beta \in \operatorname{Aut}(\widetilde{M})$ for $(\widetilde{M}, \theta)$. Let $P = Q' \cap 1_Q M 1_Q$. Note that for all $d \in P$, the element $w_t d\beta(w_t^*)$ is a $\theta_t(Q)$-$\theta_{-t}(Q)$-intertwiner. Indeed, for all $x \in Q$, we have

$$\theta_t(x)w_t d\beta(w_t^*) = w_t dx\beta(w_t^*) = w_t d\beta(xw_t^*) = w_t d\beta(w_t^*\theta_t(x)) = w_t d\beta(w_t^*)\theta_{-t}(x).$$

Hence, $\theta_t(w_t d\beta(w_t^*))$ is $\theta_{2t}(Q)$-Q-intertwiner. Therefore, we just need to find d such that $w_t d\beta(w_t^*) \neq 0$. Suppose that $w_t d\beta(w_t^*) = 0$ for all $d \in P$. Let $q \in \widetilde{M}$ be the unique projection such that $\widetilde{M}q$ is the weak* closure of the left ideal $\widetilde{M}w_t P$. Then we have $q\beta(q) = 0$. However, note that $q \in P' \cap 1_Q \widetilde{M} 1_Q$ (because $\widetilde{M}q$ is invariant by the right multiplication by elements of P so that $qx = qxq$ for all $x \in P$). Hence by Lemma 9.1, we get that $q \in M$. Thus, we have $q = q\beta(q) = 0$. Since $w_t \in \widetilde{M}q$, this contradicts the fact that $w_t \neq 0$ and we are done.

Step 4 - *Conclusion.*

Take $t = \frac{1}{2^n}$ sufficiently small and choose a nonzero $\theta_t(Q)$-Q-intertwiner. Then build recursively nonzero $\theta_{2^k t}(Q)$-Q-intertwiners until $k = n$. This gives a nonzero $\theta_1(Q)$-Q-intertwiner as we wanted.

□

11 Bernoulli crossed products

In this section, we prove our main results using the spectral gap rigidity principle of the previous section.

Fix (A_0, φ_0) any von Neumann algebra with a faithful normal state φ_0. Let $(A, \varphi) = (A_0, \varphi_0)^{\overline{\otimes}\Gamma}$ be the infinite tensor product indexed by Γ where Γ is any infinite discrete group. Let $\sigma : \Gamma \to \mathrm{Aut}(A)$ be the *Bernoulli action* of Γ on A obtained by shifting the tensors. The crossed product von Neumann algebra $M = A \rtimes_\sigma \Gamma$ is called the *non-commutative Bernoulli crossed product* of Γ with base (A_0, φ_0). If $A_0 \neq \mathbb{C}$ and Γ is infinite, it is well known that the Bernoulli action σ is *ergodic* (i.e. the fixed point algebra A^σ is trivial) and *properly outer* (i.e. if $\sigma_g(x)v = vx$ for all $x \in A$ and some nonzero $v \in A$ then $g = 1$) and so, in this case, $M = A \rtimes_\sigma \Gamma$ is a factor. There is a canonical faithful normal conditional expectation $\mathrm{E}_A : M \to A$ allowing us to extend φ to a faithful normal state φ on M by the formula $\varphi \circ \mathrm{E}_A = \varphi$. The action σ preserves the state φ so that $\mathcal{L}(\Gamma)$ is contained in the centralizer M^φ. An important fact for us is that the Bernoulli action is *mixing*:

$$\forall a, b \in A,\ \varphi(a\sigma_g(b)) \to \varphi(a)\varphi(b) \text{ when } g \to \infty.$$

Therefore, the inclusion $\mathcal{L}(\Gamma) \subset M$ is *absorbing* (Example 1).

Now, following [CI10], we define a symmetric malleable deformation of M. Let $(\widetilde{A}_0, \psi_0) = (A_0, \varphi_0) * (\mathcal{L}(\mathbb{Z}), \tau)$ be the free product von Neumann algebra where τ is the Haar trace of $\mathcal{L}(\mathbb{Z})$. As before, let $(\widetilde{A}, \psi) = (\widetilde{A}_0, \psi_0)^{\overline{\otimes}\Gamma}$ be the infinite tensor product and $\widetilde{M} = \widetilde{A} \rtimes \Gamma$ the crossed product with respect to the Bernoulli action. The von Neumann algebra $\widetilde{M}$ contains M with a normal conditional expectation $\mathrm{E}_M : \widetilde{M} \to M$ such that $\psi = \varphi \circ \mathrm{E}_M$. Now, take v the canonical Haar unitary generating $\mathcal{L}(\mathbb{Z})$ and let $h \in \mathcal{L}(\mathbb{Z})$ be a self-adjoint element such $v = e^{ih}$. For every $t \in \mathbb{R}$, let $v_t = e^{ith}$ and define an automorphism $\theta_t^0 \in \mathrm{Aut}(\widetilde{A}_0)$ by $\theta_t^0(x) = v_t x v_t^*$. Then θ_t^0 induces an automorphism θ_t of $\widetilde{M}$ that fixes the elements of $\mathcal{L}(\Gamma)$ and preserves ψ. Moreover, the action $\theta : t \mapsto \theta_t \in \mathrm{Aut}(\widetilde{M})$ is continuous. Let $\beta_0 \in \mathrm{Aut}(\widetilde{A}_0)$ be the automorphism defined by $\beta_0(a) = a$ for all $a \in A_0$ and $\beta_0(v) = v^*$. Then β_0 induces naturally an automorphism β of $\widetilde{M}$ such that $\beta \circ \theta_t \circ \beta = \theta_{-t}$. Therefore $(\widetilde{M}, \theta)$ is indeed a symmetric malleable deformation of M. In fact, it is malleable *over* $\mathcal{L}(\Gamma)$ meaning that we have the following *commuting square relation* for the ψ-preserving conditional expectations:

$$\mathrm{E}_M \circ \mathrm{E}_{\theta_1(M)} = \mathrm{E}_{\theta_1(M)} \circ \mathrm{E}_M = \mathrm{E}_{\mathcal{L}(\Gamma)}.$$

This fact is important since in many cases it allows one to obtain an intertwining relation in M from an intertwining relation in $\widetilde{M}$. In our specific case we get the following dichotomy which was obtained in the tracial case by S. Popa [Po04, Theorem 4.1 and 4.4] and completed by A. Ioana [I07, Theorem 3.3 and 3.6] (see also [Hou11, Theorem 7.3, step 3]).

Lemma 11.1. *Let $Q \subset 1_Q M 1_Q$ be a finite subalgebra with expectation. If $Q \prec_{\widetilde{M}} \theta_1(M)$ then one of the following holds:*

(i) *$Q \prec_M \mathcal{L}(\Gamma)$.*

(ii) *$Q \prec_M \overline{\otimes}_F A_0$ for some finite subset $F \subset \Gamma$.*

Proof. Suppose that $Q \not\prec_M \mathcal{L}(\Gamma)$ and $Q \not\prec_M \overline{\otimes}_F A_0$ for every finite subset $F \subset \Gamma$. Then by Theorem 23.11, we can find a net $u_i \in \mathcal{U}(Q)$ such that

$$\forall x, y \in M, \ \mathrm{E}_{\mathcal{L}(\Gamma)}(x u_i y^*) \to 0$$

$$\forall F \subset \Gamma \text{ finite }, \forall x, y \in M, \ \mathrm{E}_{\overline{\otimes}_F A_0}(x u_i y^*) \to 0$$

in the $*$-strong topology. We will contradict the assumption that $Q \prec_{\widetilde{M}} \theta_1(M)$ by showing that

$$\forall x, y \in \widetilde{M}, \ \mathrm{E}_{\theta_1(M)}(x u_i y^*) \to 0.$$

First, we can suppose that $x, y \in \widetilde{A}$ since $\mathcal{L}(\Gamma) \subset \theta_1(M)$ and $\widetilde{M} = \widetilde{A} \rtimes \Gamma$. We can suppose that $x = \otimes_{g \in K} x_g$ and $y = \otimes_{g \in K'} y_g$ where $K, K' \subset \Gamma$ are finite subsets and $x_g, y_g \in \widetilde{A_0}$. Also the result is obvious if both $x, y \in \theta_1(A)A$ since $\mathrm{E}_{\theta_1(M)} \circ \mathrm{E}_M = \mathrm{E}_{\mathcal{L}(\Gamma)}$. So we can suppose that x is orthogonal to $\theta_1(A)A$ for example. Now, note that we have the following relation for all $g \in \Gamma$

$$\mathrm{E}_{\theta_1(A)}(\mathrm{E}_{\theta_1(M)}(x u_i y^*) u_g^*) = \mathrm{E}_{\theta_1(A)}(x \mathrm{E}_{\overline{\otimes}_{K \cup gK'} A_0}(u_i u_g^*) \sigma_g(y^*)).$$

This shows first that if g is outside some finite set $F \subset \Gamma$, so that the support of x and $\sigma_g(y)$ are disjoint, then

$$\mathrm{E}_{\theta_1(A)}(\mathrm{E}_{\theta_1(M)}(x u_i y^*) u_g^*) = 0$$

because x is orthogonal to $\theta_1(A)A$. Hence we have a finite sum

$$\mathrm{E}_{\theta_1(M)}(x u_i y^*) = \sum_{g \in F} \mathrm{E}_{\theta_1(A)}(\mathrm{E}_{\theta_1(M)}(x u_i y^*) u_g^*) u_g$$

and each term of this sum converges to 0 because $\mathrm{E}_{\overline{\otimes}_{K \cup gK'} A_0}(u_i u_g^*) \to 0$. Therefore we have $\mathrm{E}_{\theta_1(M)}(x u_i y^*) \to 0$ for all $x, y \in \widetilde{M}$ as we wanted. □

In order to apply our deformation/rigidity principle, we still need to show, as in the original proof of Chifan and Ioana, that the inclusion $M \subset \tilde{M}$ is coarse, i.e. that the M-M-bimodule $\mathrm{L}^2(\tilde{M}) \ominus \mathrm{L}^2(M)$ is weakly contained in the coarse M-M-bimodule (see Section 4).

Theorem 11.2. *Suppose that A_0 is amenable. Then the inclusion $M \subset \widetilde{M}$ is coarse.*

Proof. We will compute the bimodule $\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M)$ following [CI10, Lemma 5]. The computations still hold even though ψ is not a trace.

Let $\mathcal{A}_0 \subseteq A_0$ be a φ_0-orthonormal base of A_0 with $1 \in \mathcal{A}_0$. In $\widetilde{A}_0 = A_0 * \mathcal{L}(\mathbb{Z})$ consider the set

$$\widetilde{\mathcal{A}}_0 = \{ v^{n_0} a_1 \cdots v^{n_{k-1}} a_k v^{n_k} \mid k \geq 0, \quad n_i \in \mathbb{Z} \setminus \{0\}, \quad a_i \in \mathcal{A}_0 \setminus \{1\} \}.$$

Then it is easy to check that the subspaces $A_0\tilde{a}_0A_0$ are pairwise ψ_0-orthogonal for $\tilde{a}_0 \in \widetilde{\mathcal{A}}_0$. Thus, we have

$$\mathrm{L}^2(\widetilde{A}_0) \ominus \mathrm{L}^2(A_0) = \bigoplus_{\tilde{a}_0 \in \widetilde{\mathcal{A}}_0} \overline{A_0\tilde{a}_0A_0\psi_0^{1/2}}.$$

Now, let $\widetilde{\mathcal{A}}$ be the set of elements of $\widetilde{A}$ of the form

$$\tilde{a} = \otimes_{g\in\Gamma}\tilde{a}_g$$

where $\tilde{a}_g \in \widetilde{\mathcal{A}}_0$ for finitely many (and at least one) g and $\tilde{a}_g = 1$ otherwise. Then we have

$$\mathrm{L}^2(\widetilde{A}) \ominus \mathrm{L}^2(A) = \bigoplus_{\tilde{a}\in\widetilde{\mathcal{A}}} \overline{A\tilde{a}A\psi^{1/2}}.$$

Now, focus on the M-M-bimodules $H_{\tilde{a}} = \overline{M\tilde{a}M\psi^{1/2}} \subset \mathrm{L}^2(\widetilde{M})$ for $\tilde{a} \in \widetilde{\mathcal{A}}$. We note that $H_{\tilde{a}} = H_{\sigma_g(\tilde{a})}$ for all $g \in \Gamma$ while $H_{\tilde{a}}$ and $H_{\tilde{a}'}$ are orthogonal when $\tilde{a}$ and $\tilde{a}'$ are not in the same Γ-orbit. So let Ω be the set of Γ-orbits of $\widetilde{\mathcal{A}}$ and for every $\pi \in \Omega$ define

$$H_\pi = H_{\tilde{a}}$$

where $\tilde{a}$ is any element of the orbit π. Then we have an M-M-bimodules decomposition

$$\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M) = \bigoplus_{\pi\in\Omega} H_\pi.$$

In order to conclude, it suffices to show that for each $\pi \in \Omega$, H_π is weakly contained in $\mathrm{L}^2(M) \mathbin{\overline{\otimes}} \mathrm{L}^2(M)$. So let π be such an orbit, represented by $\tilde{a} \in \widetilde{\mathcal{A}}$. Write $\tilde{a} = \otimes_{g\in\Gamma}\tilde{a}_g$ and let F the non-empty finite set of elements $g \in \Gamma$ such that $\tilde{a}_g \neq 1$. The stabilizer of $\tilde{a}$ inside Γ is denoted by

$$S = \{g \in \Gamma \mid \sigma_g(\tilde{a}) = \tilde{a}\}.$$

It is a finite subgroup since it must leave the support F invariant. Now let

$$K = A_0^{\overline{\otimes}\Gamma\setminus F} \rtimes S \subset M.$$

The von Neumann algebra K is globally invariant by the modular flow of φ. So there exists a unique normal conditional expectation E_K from M to K that preserves φ. We will show that there is an isomorphism of M-M-bimodules

$$H_\pi = H_{\tilde{a}} \simeq \mathrm{L}^2(\langle M, K\rangle)$$

where $\langle M, K\rangle = (J_MKJ_M)' \subset \mathbf{B}(\mathrm{L}^2(M))$ is the basic construction. Let $e_K \in \langle M, K\rangle$ be the Jones projection associated to E_K. It is known that the $*$-subalgebra Me_KM is dense in $\langle M, K\rangle$. Let $\hat{\varphi}$ be the unique normal faithful semi-finite weight on $\langle M, K\rangle$ which satisfies

$$\forall x, y \in M, \hat{\varphi}(xe_Ky) = \varphi(xy).$$

We have

$$\mathrm{L}^2(\langle M, K\rangle) = \overline{Me_KM\hat{\varphi}^{1/2}}.$$

Denote by $U : H_{\tilde{a}} \to \mathrm{L}^2(\langle M, K\rangle)$ the linear map densely defined by

$$U(x\tilde{a}y\psi^{1/2}) = xe_Ky\hat{\varphi}^{1/2}$$

for all $x, y \in M$ which are φ-analytic. We claim that U extends to a unitary map which is an M-M-bimodule isomorphism. In fact U clearly commutes with the left action, and it also commutes with the right action because for $z \in M$ analytic we have

$$U(x\tilde{a}y\psi^{1/2}z) = U(x\tilde{a}yz'\psi^{1/2}) = xe_K yz'\hat{\varphi}^{1/2} = xe_K y\hat{\varphi}^{1/2}z = U(x\tilde{a}y\psi^{1/2})z.$$

where $z' = \sigma^{\varphi}_{\frac{-i}{2}}(z) = \sigma^{\psi}_{\frac{-i}{2}}(z) = \sigma^{\hat{\varphi}}_{\frac{-i}{2}}(z)$ (Note that the modular flows of ψ and $\hat{\varphi}$ coincide on M with the modular flow of φ). So it only remains to check that U defines indeed a unitary, i.e. that

$$\psi(y_1^*\tilde{a}^*x_1^*x_2\tilde{a}y_2) = \hat{\varphi}(y_1^*e_K x_1^*x_2e_K y_2)$$

for all $x_1, x_2, y_1, y_2 \in M$ which are φ-analytic. Once again, since the modular flow of $\hat{\varphi}$ and ψ coincide on M, we can pass y_1^* to the other side

$$\psi(y_1^*\tilde{a}^*x_1^*x_2\tilde{a}y_2) = \psi(\tilde{a}^*x_1^*x_2\tilde{a}y_2\sigma^{\varphi}_{-i}(y_1^*))$$

$$\hat{\varphi}(y_1^*e_K x_1^*x_2e_K y_2) = \hat{\varphi}(e_K x_1^*x_2e_K y_2\sigma^{\varphi}_{-i}(y_1^*))$$

and so we just need to check that

$$\forall x, y \in M, \psi(\tilde{a}^*x\tilde{a}y) = \hat{\varphi}(e_K xe_K y) = \varphi(\mathrm{E}_K(x)y).$$

In order to prove this, we can suppose, by density, that x and y are of the form

$$x = (\otimes_g x_g)u_\gamma$$

$$y = (\otimes_g y_g)u_\delta$$

with $\gamma, \delta \in \Gamma$, $x_g, y_g \in A_0$ for all g and $x_g = y_g = 1$ except for finitely many g. We also have $\tilde{a} = \otimes_g \tilde{a}_g$ with $\tilde{a}_g \in \widetilde{\mathcal{A}}_0$ for finitely many (not zero) g and $\tilde{a}_g = 1$ otherwise. Recall that $\widetilde{\mathcal{A}}_0$ is our orthonormal base.

Now we compute $\psi(\tilde{a}^*x\tilde{a}y)$. First, if $\delta\gamma \neq 1$ then $\psi(\tilde{a}^*x\tilde{a}y) = 0$. So suppose that $\delta = \gamma^{-1}$. Then we have

$$\psi(\tilde{a}^*x\tilde{a}y) = \psi(\otimes_g(\tilde{a}_g^*x_g\tilde{a}_{\gamma^{-1}g}y_{\gamma^{-1}g})) = \prod_{g\in\Gamma}\psi_0(\tilde{a}_g^*x_g\tilde{a}_{\gamma^{-1}g}y_{\gamma^{-1}g}).$$

For this product to be nonzero, we must have $\tilde{a}_g = \tilde{a}_{\gamma^{-1}g}$ for all g. This means that $\sigma_\gamma(\tilde{a}) = \tilde{a}$, i.e. $\gamma \in S$. In this case, the usual computation of free probability gives

$$\psi(\tilde{a}^*x\tilde{a}y) = \prod_{g\in F}\varphi_0(x_g)\varphi_0(y_{\gamma^{-1}g})\prod_{g\in\Gamma\setminus F}\varphi_0(x_g y_{\gamma^{-1}g}).$$

So we have shown that $\psi(\tilde{a}^*x\tilde{a}y)$ is given by the formula above when $\delta = \gamma^{-1} \in S$ and is equal to 0 otherwise.

Now, in order to compute $\varphi(\mathrm{E}_K(x)y)$, we just need to check the formula

$$\mathrm{E}_K(x) = \mathrm{E}_K((\otimes_g x_g)u_\gamma) = 1_S(\gamma)\prod_{g\in F}\varphi_0(x_g)(\otimes_{g\in\Gamma\setminus F}x_g)u_\gamma$$

and we conclude that the equality

$$\psi(\tilde{a}^*x\tilde{a}y) = \varphi(\mathrm{E}_K(x)y)$$

is true in all cases.

Hence, we have shown that there is an isomorphism of M-M-bimodules

$$H_\pi = H_{\tilde{a}} \simeq \mathrm{L}^2(\langle M, K\rangle).$$

Finally, since $K = A_0^{\overline{\otimes}\Gamma\setminus F} \rtimes S$ is the crossed product of an amenable von Neumann algebra by a finite group S then K is also amenable. Therefore, its commutant $\langle M, K\rangle$ is also amenable. In particular, this implies that H_π is weakly contained in $\mathrm{L}^2(M) \mathbin{\overline{\otimes}} \mathrm{L}^2(M)$ as an M-M-bimodule. Since this is true for all $\pi \in \Omega$, we conclude that the M-M-bimodule

$$\mathrm{L}^2(\widetilde{M}) \ominus \mathrm{L}^2(M) = \bigoplus_{\pi\in\Omega} H_\pi$$

is weakly contained in $\mathrm{L}^2(M) \mathbin{\overline{\otimes}} \mathrm{L}^2(M)$ as we wanted. □

Theorem 11.3. *Let $A_0 \neq \mathbb{C}$ be an amenable von Neumann algebra, φ_0 a faithful normal state on A_0 and Γ any discrete group. Let $M = (A_0, \varphi_0)^{\overline{\otimes}\Gamma} \rtimes \Gamma$ be the associated Bernoulli crossed product von Neumann algebra. Then M is solid relatively to Γ. In particular, M is prime if and only if Γ is non-amenable and M is solid if and only if $\mathcal{L}(\Gamma)$ is solid.*

Proof. We first show that M is solid relatively to $\mathcal{L}(\Gamma)$. Suppose we have a properly non-amenable subalgebra with expectation $Q \subset 1_Q M 1_Q$ such that $Z = \mathcal{Z}(Q)$ is diffuse. We have to show that $Q \prec_M \mathcal{L}(\Gamma)$. Since $\mathcal{L}(\Gamma) \subset M$ is absorbing and $Q \subset Z' \cap 1_Q M 1_Q$, then by using Lemma 8.6, we see that it is enough to show that $Z \prec_M \mathcal{L}(\Gamma)$. By Theorem 10.1 we know that $Z \prec_{\widetilde{M}} \theta_1(M)$. Hence by Lemma 11.1 we must have $Z \prec_M \mathcal{L}(\Gamma)$ or $Z \prec_M \overline{\otimes}_F A_0$ for some finite subset $F \subset \Gamma$. So we just need to show that the case where $Z \prec_M \overline{\otimes}_F A_0$ leads to a contradiction. Take $v \in \mathcal{I}^l_M(Z, \overline{\otimes}_F A_0)$ and let $\pi_v : Zp \to q(\overline{\otimes}_F A_0)q$. Let $C = \pi_v(Zp)$ and denote $q = 1_C$. We have $v^*(Z' \cap 1_Q M 1_Q)v \subset v^*v(C' \cap 1_C M 1_C)v^*v$ hence $Z' \cap 1_Q M 1_Q \prec_M C' \cap 1_C M 1_C$. Thus, we get $Q \prec_M C' \cap 1_C M 1_C$ because $Q \subset Z' \cap 1_Q M 1_Q$. Recall that, by assumption, Q is properly non-amenable. Therefore, we just need to show that $C' \cap 1_C M 1_C$ is amenable in order to get a contradiction. Let $x \in C' \cap 1_C M 1_C$. We claim that $x_g = \mathrm{E}_A(x u_g^*) = 0$ for $g \notin FF^{-1}$. In fact, since C is abelian and diffuse (because Z is abelian and diffuse), there exists a net of unitaries $u_i \in \mathcal{U}(C)$ which tends weakly to 0. Take $g \notin FF^{-1}$. Then we have $\sigma_g(u_i) \in \overline{\otimes}_{\Gamma\setminus F} A_0$. Hence

$$\forall a, b \in A,\ \mathrm{E}_{\overline{\otimes}_F A_0}(a \sigma_g(u_i) b) \to 0$$

in the $*$-strong topology. Since $u_i x_g = x_g \sigma_g(u_i)$ we get

$$u_i \mathrm{E}_{\overline{\otimes}_F A_0}(x_g x_g^*) = \mathrm{E}_{\overline{\otimes}_F A_0}(x_g \sigma_g(u_i) x_g^*) \to 0$$

in the $*$-strong topology. Thus $x_g = 0$. Therefore we have shown that

$$C' \cap 1_C M 1_C \subset \sum_{g\in FF^{-1}} A u_g$$

We will show that this implies that $C' \cap 1_C M 1_C$ is amenable. Let $D = C' \cap 1_C M 1_C \oplus (1 - 1_C)\mathbb{C}$. Let $\mathrm{E}_D : M \to D$ be a faithful normal conditional expectation. Since A

is amenable, there is a conditional expectation $\Psi : \mathbf{B}(\mathrm{L}^2(A)) \to A$. Define a map $\Phi : \mathbf{B}(\mathrm{L}^2(M)) \to D$ by

$$\Phi(T) = \sum_{g \in FF^{-1}} \mathrm{E}_D(\Psi(e_A T u_g^* e_A) u_g)$$

where $e_A : \mathrm{L}^2(M) \to \mathrm{L}^2(A)$ is the Jones projection. Since $D \subset \sum_{g \in FF^{-1}} A u_g$, a little computation shows that $\Phi(x) = x$ for all $x \in D$. Moreover Φ is completely bounded (compositions and finite sums of completely bounded maps are still completely bounded). Therefore, using [Pi93, Corollaire 5], we have that D is amenable which means that $C' \cap 1_C M 1_C$ is amenable as we wanted.

For the second part of the theorem we can apply Proposition 8.3 and Proposition 8.7 to get the desired conclusion. Note that when Γ is non-amenable and $A_0 \neq \mathbb{C}$ then M is a non-amenable factor and $M \not\prec_M \mathcal{L}(\Gamma)$. Indeed, since $A_0 \neq \mathbb{C}$ then A is diffuse and we can find a diffuse abelian subalgebra with expectation $B \subset A$. Let $u_i \in B$ a net of unitaries which tends weakly to 0. Then we have $\mathrm{E}_{\mathcal{L}(\Gamma)}(x u_i y) \to 0$ in the $*$-strong topology for all $x, y \in M$ (by density, it suffices to check it for x, y of the form $a u_g$ with $g \in \Gamma$ and $a \in A$). Therefore, by Theorem 23.11, we know that $B \not\prec_M \mathcal{L}(\Gamma)$. A fortiori, we get $M \not\prec_M \mathcal{L}(\Gamma)$. $\square$

We obtain many new examples of solid type III factors. Note that there was previously no known example of a non-amenable solid type III factor with a Cartan subalgebra.

Corollary 11.4. *For every countable subgroup $\Lambda \subset \mathbb{R}^*_+$, there exists a solid non-amenable type* III *factor with separable predual and with a Cartan subalgebra such that its* Sd *invariant is Λ. For any topology τ_0 on $\mathbb{R}$ induced by an injective continuous separable unitary representation of $\mathbb{R}$, there exists a solid non-amenable type* III_1 *factor with separable predual and with a Cartan subalgebra such that its τ invariant is τ_0.*

Proof. See ([Co74], Proposition 3.9) and the constructions in ([Co74], Corollary 4.4) for the first part and ([Co74], Theorem 5.2) for the second part. In both cases, the examples are obtained by taking Bernoulli crossed products $M = (A_0, \varphi_0) \rtimes \mathbb{F}_2$ where $\mathbb{F}_2$ is the free group on 2 generators and A_0 is some non-trivial amenable algebra with separable predual. Since $\mathcal{L}(\mathbb{F}_2)$ is solid and non-amenable [Oz03] then by Theorem 11.3 we know that M is solid and non-amenable. If B_0 is a Cartan subalgebra in the centralizer of A_0, then it is not hard to check (using the fact that the Bernoulli action is properly outer) that $\overline{\otimes}_{\mathbb{F}_2} B_0$ is again a Cartan subalgebra of M. $\square$

Using Theorem 11.3 and Proposition 8.7, we can also generalize [CI10, Theorem 7] by removing the assumption that the base equivalence relation is measure preserving and hence we also obtain the first examples of non-amenable solid type III equivalence relations.

Corollary 11.5. *Let (X_0, μ_0) be a standard probability space, $\mathcal{R}_0$ an arbitrary amenable equivalence relation (not necessarily measure preserving) on X_0 and Γ any countable group. Let $\mathcal{R} = \mathcal{R}_0 \wr \Gamma$ be the* wreath product equivalence relation *on $(X_0, \mu_0)^\Gamma$. Then $\mathcal{R}$ is solid, i.e. for any subequivalence relation $\mathcal{S} \subset \mathcal{R}$ there exists a countable partition of $(X_0, \mu_0)^\Gamma$ into $\mathcal{S}$-invariant components $Z_n, n \in \mathbb{N}$ such that*

— $\mathcal{S}_{|Z_0}$ *is amenable.*

— $\mathcal{S}_{|Z_n}$ *is strongly ergodic and prime for all* $n \geq 1$.

Proof. We can suppose that X_0 is not a point and that Γ is infinite (otherwise the result is obvious). Let $X = (X_0, \mu_0)^\Gamma$. Let $A_0 = \mathcal{L}(\mathcal{R}_0)$ and let φ_0 be the faithful normal state on A_0 induced by μ_0. Let $B_0 = \mathrm{L}^\infty(X_0) \subset A_0$ be the canonical Cartan subalgebra. Since $A_0 \neq \mathbb{C}$ and Γ is infinite, the Bernoulli action of Γ on $(A_0, \varphi_0)^{\overline{\otimes}\Gamma}$ is properly outer. Using this, it is not hard to check that $(B_0, \varphi_0)^{\overline{\otimes}\Gamma} \subset M = (A_0, \varphi_0)^{\overline{\otimes}\Gamma} \rtimes \Gamma$ is again a Cartan subalgebra and that we can identify canonically the Cartan pair $(M, (B_0, \varphi_0)^{\overline{\otimes}\Gamma})$ with the Cartan pair $(\mathcal{L}(\mathcal{R}), \mathrm{L}^\infty(X))$.

Let $\mathcal{S} \subset \mathcal{R}$ be a subequivalence relation. Then with the preceding identification we have that $(B_0, \varphi_0)^{\overline{\otimes}\Gamma} \subset \mathcal{L}(\mathcal{S}) \subset M$ with expectations. Since the subalgebra $(B_0, \varphi_0)^{\overline{\otimes}\Gamma} \subset (A_0, \varphi_0)^{\overline{\otimes}\Gamma}$ is diffuse, it is easy to check that $(B_0, \varphi_0)^{\overline{\otimes}\Gamma} \not\prec_M \mathcal{L}(\Gamma)$. Therefore $\mathcal{L}(\mathcal{S}) \not\prec_M \mathcal{L}(\Gamma)$. Hence, we get a sequence of projections $z_n \in \mathcal{Z}(\mathcal{L}(\mathcal{S}))$ with $\sum_n z_n = 1$ such that $\mathcal{L}(\mathcal{S})z_0$ is amenable and $\mathcal{L}(\mathcal{S})z_n$ is a non-amenable factor for all $n \geq 1$. By Proposition 8.7, $\mathcal{L}(\mathcal{S})z_n$ is a full prime factor for all $n \geq 1$. By identifying the projections z_n with $\mathcal{S}$-invariant measurable subsets $Z_n \subset X$ we get the desired conclusion since fullness and primeness of $\mathcal{L}(\mathcal{S}_{|Z_n}) = \mathcal{L}(\mathcal{S})z_n$ imply strong ergodicity and primeness of $\mathcal{S}_{|Z_n}$. $\square$

Part III

Spectral gap and full factors

This chapter is based on [Ma16b], [HMV16] and [Ma17a]. The main result is a spectral gap characterization of full type III factors which is similar to Connes' spectral gap characterization of full type II_1 factors. We give a proof based on a more general local spectral gap principle which is considerably simpler than the original proof of [Ma16b] and which has a wider range of applications.

The main consequences are the following. If M and N are full factors (possibly of type III) then $M \mathbin{\overline{\otimes}} N$ is also a full factor. Moreover, the map $\mathrm{Out}(M) \times \mathrm{Out}(N) \to \mathrm{Out}(M \mathbin{\overline{\otimes}} N)$ is a homeomorphism on its range. In particular, the τ-invariant $\tau(M \mathbin{\overline{\otimes}} N)$ can be computed easily from $\tau(M)$ and $\tau(N)$. We also generalize a theorem of Jones on fullness of crossed products to factors of arbitrary type. We deduce that the core $c(M)$ of a type III_1 factor M is full if and only if M is full and $\tau(M)$ is the usual topology. Finally, we generalize a theorem of Popa on unique McDuff decompositions to factors of arbitrary type.

12 A general local spectral gap theorem

We fix M a von Neumann algebra and $\varphi \in M_*$ a normal state. We de not assume that φ is faithful. We let $\Sigma_\varphi \subset \mathrm{L}^2(M)$ denote the set of all φ-bounded vectors, i.e. $\Sigma_\varphi = \varphi^{1/2} M \cap M \varphi^{1/2}$. If φ is a faithful normal *trace*, then Σ_φ can be identified with M.

We are interested in the following spectral gap property.

Definition 12.1. Let $N \subset M$ a von Neumann subalgebra and $\Sigma \subset \Sigma_\varphi$. We say that N has a *spectral gap* with respect to (Σ, φ) if there exists a constant $\kappa > 0$ and a finite subset $S \subset \Sigma$ such that for all $x \in N$, we have

$$\|x - \varphi(x)\|_\varphi \leq \kappa \sum_{\xi \in S} \|x\xi - \xi x\|.$$

It is clear that if N has spectral gap with respect to (Σ, φ), then any *bounded* net $(x_i)_{i \in I}$ in N which is Σ-central, meaning that $\lim_i \|x_i \xi - \xi x_i\| = 0$ for all $\xi \in \Sigma$, must be trivial, meaning that $\lim_i \|x_i - \varphi(x_i)\|_\varphi = 0$. The converse is not true in general. In fact, any strongly ergodic probability measure preserving action $\Gamma \curvearrowright X$ which does not have spectral gap provides a counter example by taking $N = \mathrm{L}^\infty(X)$ and $\Sigma = \{u_g \mid g \in \Gamma\}$ inside $M = \mathrm{L}^\infty(X) \rtimes \Gamma$. Another interesting counter example is obtained by taking $N = M = \mathcal{L}(\Gamma)$ and $\Sigma = \{u_g \mid g \in \Gamma\}$ where Γ is an inner amenable i.c.c. group Γ such that the II_1 factor $\mathcal{L}(\Gamma)$ is full (such a group exists by [Va09]).

Nevertheless, the main theorem of this section provides a partial converse in the form of a *local spectral gap* property.

Theorem 12.2 ([Ma17a]). *Let N be a von Neumann subalgebra of M and take $\Sigma = \Sigma^* \subset \Sigma_\varphi$. Suppose that every bounded Σ-central net $(x_i)_{i \in I}$ in N satisfies $\lim_i \|x_i - \varphi(x_i)\|_\varphi = 0$. Then for any $\varepsilon > 0$, we can find a projection $p \in N$ with $\varphi(p) > 1 - \varepsilon$ such that pNp has a spectral gap with respect to $(p\Sigma p, \frac{1}{\varphi(p)} p\varphi p)$.*

The proof is elementary and is based on a maximality argument. But we first need the following inequalities which are inspired by Namioka's trick. Item (i) goes back to [CS78]. Our new input is item (ii).

Lemma 12.3. *Let M be any von Neumann algebra. For $t \geq 0$ and $x \in M$, we use the notation $u_t(x) = u 1_{[t, +\infty)}(|x|)$ where $x = u|x|$ is the polar decomposition of x.*

(i) *Let $x, y \in M$ and $\xi \in \mathrm{L}^2(M)$. If x and y are self-adjoint or if $x = y$, we have*

$$\frac{1}{2}\|x\xi - \xi y\|^2 \leq \int_0^\infty \|u_{t^{1/2}}(x)\xi - \xi u_{t^{1/2}}(y)\|^2 \, \mathrm{d}t \leq 4\|x\xi - \xi y\|(\|x\xi\| + \|\xi y\|).$$

(ii) *Let $x \in M^+$ and let $\varphi \in M_*$ be any state. We have*

$$\|x - \varphi(x)\|_\varphi^2 \leq \int_0^\infty \varphi(u_{t^{1/2}}(x))\varphi\left(1 - u_{t^{1/2}}(x)\right) \mathrm{d}a$$

Proof. For item (i), see the proof of [CS78, Theorem 2]. With the notations used there, the case where x and y are self-adjoint follows from the following inequality for all $h, k \in [0, +\infty[$

$$\frac{1}{2}|h - k|^2 \leq \int_0^{+\infty} |F_{t^{1/2}}(h) - F_{t^{1/2}}(k)|^2 \, \mathrm{d}t \leq 4|h - k|(|h| + |k|).$$

The case where $x = y$ follows from the first case by applying it to

$$x = y = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \text{ and } \xi = \begin{pmatrix} \xi & 0 \\ 0 & \xi^* \end{pmatrix}.$$

We will now prove (ii). First, note that $u_{t^{1/2}}(x) = u_t(x^2)$. Moreover, we have $\|x - \varphi(x)\|_\varphi^2 = \varphi(x^2) - \varphi(x)^2$ and

$$\varphi(x^2) = \int_0^\infty \varphi(u_t(x^2)) \,\mathrm{d}t$$

Therefore, we only have to show that

$$\int_0^\infty \varphi(u_t(x^2))^2 \,\mathrm{d}t \leq \varphi(x)^2$$

In $M \mathbin{\overline{\otimes}} M$, we have $u_t(x^2) \otimes u_t(x^2) \leq u_t(x \otimes x)$. Hence, by appying $\varphi \otimes \varphi$ we get

$$\varphi(u_t(x^2))^2 \leq (\varphi \otimes \varphi)(u_t(x \otimes x))$$

and thus, after integrating over t, we finally get

$$\int_0^\infty \varphi(u_t(x^2))^2 \,\mathrm{d}u \leq (\varphi \otimes \varphi)(x \otimes x) = \varphi(x)^2$$

□

We deduce from the previous inequalities the following criterion which shows that it is enough to check the spectral gap inequality on projections.

Lemma 12.4 ([Ma17a])**.** *Let N be a von Neumann subalgebra of M and take $\Sigma = \Sigma^* \subset \Sigma_\varphi$. Then N has a spectral gap with respect to (Σ, φ) if and only if there exists a constant $\kappa > 0$ and a finite subset $S \subset \Sigma$ such that for any projection $p \in N$ we have*

$$\varphi(p)\varphi(1-p) \leq \kappa \sum_{\xi \in S} \|p\xi - \xi p\|^2.$$

Proof. Assume that there exists a constant $\kappa > 0$ and a finite subset $S \subset \Sigma$ such that for any projection $p \in N$ we have

$$\varphi(p)\varphi(1-p) \leq \kappa \sum_{\xi \in S} \|p\xi - \xi p\|^2.$$

We can assume that S is self-adjoint. A direct application of Lemma 12.3 shows that for every element $x \in N^+$, we have

$$\|x - \varphi(x)\|_\varphi^2 \leq 4\kappa \sum_{\xi \in S} \|x\xi + \xi x\|(\|x\xi\| + \|\xi x\|)$$

Since $S \subset \Sigma_\varphi$, we can find a constant $C > 0$ such that for all $x \in N^+$ and $\xi \in S$, we have $\|x\xi\| + \|\xi x\| \leq C\|x\|_\varphi$. Hence, we get

$$\forall x \in N^+, \quad \|x - \varphi(x)\|_\varphi^2 \leq 4C\kappa \|x\|_\varphi \sum_{\xi \in S} \|x\xi - \xi x\|$$

Now, for every $x = x^* \in N$ with $\varphi(x) = 0$, write $x = x_+ - x_-$ where $x_+, x_- \in N^+$ and $x_+ x_- = 0$. Then we have

$$\|x\|_\varphi^2 \leq 2(\|x_+ - \varphi(x_+)\|_\varphi^2 + \|x_- - \varphi(x_-)\|_\varphi^2) \leq 8C\kappa \|x\|_\varphi \sum_{\xi \in S} (\|x_+\xi - \xi x_+\| + \|x_-\xi - \xi x_-\|)$$

and since $\|x_\pm \xi - \xi x_\pm\| \leq \|x\xi - \xi x\|$, we obtain

$$\|x\|_\varphi \leq 16C\kappa \sum_{\xi \in S} \|x\xi - \xi x\|$$

By applying this inequality to $x - \varphi(x)$, we obtain the desired spectral gap inequality for every self-adjoint $x \in N$. Finally, since S is self-adjoint, it is easy to obtain the spectral gap inequality for every element $x \in N$ by decomposing it into its real and imaginary part. $\square$

Proof of Theorem 12.2. Fix $0 < \varepsilon < \frac{1}{4}$. Since every bounded Σ-central net is trivial, there must exist a finite self-adjoint subset $S \subset \Sigma$ and some $\eta > 0$ such that for every projection $p \in N$ we have

$$\sum_{\xi \in S} \|p\xi - \xi p\|^2 \leq \eta \implies \min(\varphi(p), \varphi(1-p)) \leq \varepsilon$$

Consider the set Λ of all projections e in N such that $\varphi(e) \leq \varepsilon$ and

$$\sum_{\xi \in S} \|e\xi - \xi e\|^2 \leq \eta \varphi(e)$$

Then Λ is closed for the strong topology, hence it is an inductive set. Therefore, by Zorn's lemma, we can choose e a maximal element of Λ. Let $p = 1 - e$. Take a projection $f \in pNp$. Suppose that

$$\varphi(f)\varphi(p-f) > \frac{1}{\eta} \sum_{\xi \in S} \|f(p\xi p) - (p\xi p)f\|^2$$

Then up to replacing f by $p - f$, we can suppose that $\varphi(f) \leq \frac{1}{2}$. Now let $q = e + f$. Then we can check that

$$\sum_{\xi \in S} \|q\xi - \xi q\|^2 \leq \sum_{\xi \in S} \|e\xi - \xi e\|^2 + \sum_{\xi \in S} \|f(p\xi p) - (p\xi p)f\|^2 \leq \eta\varphi(e) + \eta\varphi(f) = \eta\varphi(q)$$

But, since e is maximal in Λ, we know that $q \notin \Lambda$. Hence we must have $\varphi(q) > \varepsilon$. Therefore, by the choice of S and η, we must have $\varphi(q) \geq 1 - \varepsilon$. But $\varphi(q) = \varphi(e) + \varphi(f) \leq \varepsilon + \frac{1}{2}$. Since $\varepsilon < \frac{1}{4}$, this is a contradiction. Hence, for all projections $f \in pNp$, we have

$$\varphi(f)\varphi(p-f) \leq \frac{1}{\eta} \sum_{\xi \in S} \|f(p\xi p) - (p\xi p)f\|^2$$

Finally, we can use Lemma 12.4 to conclude that pNp has a spectral gap with respect to $(p\Sigma p, \frac{1}{\varphi(p)} p\varphi p)$. $\square$

Let us mention the following direct application of Theorem 12.2. It shows that the local spectral gap property introduced in [BISG15] is actually equivalent to strong ergodicity.

Theorem 12.5 ([Ma17a]). *Let $\Gamma \curvearrowright (X,\mu)$ be an ergodic measure-preserving action of a discrete group Γ. Then $\Gamma \curvearrowright X$ is strongly ergodic if and only if there exists $B \subset X$ with $0 < \mu(B) < +\infty$, a finite set $S \subset \Gamma$ and a constant $\kappa > 0$ such that*

$$\forall f \in \mathrm{L}^\infty(X), \quad \|f - \mu_B(f)\|_{\mu_B} \leq \kappa \sum_{g \in S} \|g(f) - f\|_{\mu_B}$$

where μ_B is the probability measure on X defined by $\mu_B(A) = \frac{1}{\mu(B)}\mu(B \cap A)$ for all $A \subset X$.

We finish this section with a lemma which will be useful in applications.

Lemma 12.6. *Let N be a von Neumann subalgebra of M and take $\Sigma \subset \Sigma_\varphi$. Suppose that $\varphi^{1/2} \in \Sigma$. Then the following holds:*

(i) *N has a spectral gap with respect to (Σ, φ) if and only if pNp has a spectral gap with respect to (Σ, φ) where $p = \mathrm{supp}(\varphi)$.*

(ii) *If N has a spectral gap with respect to (Σ, φ) and ψ is a normal state which satisfies $\lambda^{-1}\varphi \leq \psi \leq \lambda\varphi$ for some $\lambda \geq 1$ then N has a spectral gap with respect to (Σ, ψ).*

(iii) *If N has a spectral gap with respect to (Σ, φ) and $v \in M$ is a partial isometry such that $v^*v = \mathrm{supp}(\varphi)$, then N has a spectral gap with respect to $(v\Sigma v^*, v\varphi v^*)$ and also with respect to $(\Sigma \cup v\Sigma v^* \cup \{v\varphi^{1/2}\}, t\varphi + (1-t)v\varphi v^*)$ for all $t \in]0,1[$.*

Proof. (i) This follows from the inequalities

$$\|x - \varphi(x)\|_\varphi \leq \|pxp - \varphi(pxp)\|_\varphi + \|x\varphi^{1/2} - \varphi^{1/2}x\|$$

and

$$\|(pxp)\xi - \xi(pxp)\| \leq \|x\xi - \xi x\|$$

for all $x \in N$ and $\xi \in \Sigma$.

(ii) Since $\varphi \leq \lambda\psi$, we have $\Sigma_\varphi \subset \Sigma_\psi$ and since $\psi \leq \lambda\varphi$, for every $x \in M$ we have

$$\|x - \psi(x)\|_\psi \leq \|x - \varphi(x)\|_\psi \leq \lambda^{1/2}\|x - \varphi(x)\|_\varphi.$$

(iii) Let $e = vv^*$ and $\phi = v\varphi v^*$. Then for all $x \in eMe$, we have

$$\|x - \phi(x)\|_\phi = \|v^*xv - \varphi(v^*xv)\|_\varphi.$$

Moreover, for every $\xi \in \Sigma$, we have

$$\|v^*xv\xi - \xi v^*xv\| = \|x(v\xi v^*) - (v\xi v^*)x\|.$$

This shows that eNe has spectral gap with respect to $(v\Sigma v^*, \phi)$, hence N has also spectral gap with respect to $(v\Sigma v^*, \phi)$ by (i). Now let $\psi = t\varphi + (1-t)\phi$ with $t \in]0,1[$. Observe that $\Sigma \cup v\Sigma v^* \cup \{v\varphi^{1/2}\} \subset \Sigma_\psi$. Now we check that for all $x \in M$, we have

$$\|x - \psi(x)\|_\psi \leq \|x - \varphi(x)\|_\varphi + \|x - \phi(x)\|_\phi + 2\|x(v\varphi^{1/2}) - (v\varphi^{1/2})x\|.$$

□

13 Full factors

Definition 13.1. [Co74] We say that a factor M is *full* if every bounded net $(x_i)_{i\in I}$ in M that is *centralizing*, meaning that $\lim_i \|x_i\varphi - \varphi x_i\| = 0$ for all $\varphi \in M_*$, must be *trivial*, meaning that there exists a bounded net $(\lambda_i)_{i\in I}$ in $\mathbb{C}$ such that $x_i - \lambda_i 1 \to 0$ strongly as $i \to \infty$.

Theorem 13.2 ([Ma16b])**.** *Let M be a full factor. Then there exists a normal state φ, a finite set $S \subset \Sigma_\varphi$ and a constant $\kappa > 0$ such that for all $x \in M$, we have*

$$\|x - \varphi(x)\|_\varphi \leq \kappa \sum_{\xi \in S} \|x\xi - \xi x\|.$$

If M is semifinite, we can choose φ to be the trace of pMp for any nonzero finite projection $p \in M$.

If M if of type III_λ, $\lambda \in]0,1[$, we can choose φ to be any given normal state.

If M is σ-finite and of type III_1, we can choose φ to be faithful.

Proof. We have to show that M has a spectral gap with respect to $(\Sigma_\varphi, \varphi)$ for some state $\varphi \in M_*$. Let ψ be a normal state on M. Let $e = \mathrm{supp}(\psi)$. Since eMe is full, every Σ_ψ-central bounded net in eMe must be trivial. Hence a direct application of Theorem 12.2 shows that there exists a nonzero projection $p \in eMe$ such that eMe has spectral gap with respect to $(p\Sigma_\psi p, \frac{1}{\psi(p)} p\psi p)$. Let $\varphi = \frac{1}{\psi(p)} p\psi p$. Observe that $p\Sigma_\psi p \subset \Sigma_\varphi$. Hence eMe has spectral gap with respect to $(\Sigma_\varphi, \varphi)$. Finally, by Lemma 12.6.(i), M also has a spectral gap with respect to $(\Sigma_\varphi, \varphi)$.

Now, suppose that M is semifinite. Let $p \in M$ be any nonzero finite projection. Let τ_p be the unique trace of pMp. Then the first part applied to $\psi = \tau_p$ shows that M has a spectral gap with respect to $(\Sigma_{\tau_q}, \tau_q)$ for some projection $q \leq p$. Moreover, we can take q such that $\tau_p(q) > \frac{1}{2}$. Let $r \in pMp$ be any projection which is equivalent to q and such that $p \leq q + r$. Then by Lemma 12.6.(iii), we know that M has a spectral gap with respect to (Σ_ψ, ψ) where $\psi = \frac{1}{2}(\tau_q + \tau_r)$. Since $\frac{1}{2}\tau_p \leq \psi \leq \tau_p$, then by Lemma 12.6.(ii), we conclude that M has a spectral gap with respect to $(\Sigma_{\tau_p}, \tau_p)$.

Suppose M is of type III_λ with $\lambda \in]0,1[$. The first part shows that M has a spectral gap with respect to $(\Sigma_\varphi, \varphi)$ for some normal state φ. Let ψ be any other normal state on M. Since M is of type III_λ, we can find a partial isometry $v \in M$ such that $vv^* = \mathrm{supp}(\varphi)$, $v^*v = \mathrm{supp}(\psi)$ and

$$\lambda\varphi \leq v\psi v^* \leq \lambda^{-1}\varphi.$$

Then we conclude that M has spectral gap with respect to (Σ_ψ, ψ) thanks to Lemma 12.6.

Finally, the last part follows from Lemma 12.6.(iii) and the fact that all nonzero projections are equivalent in a σ-finite type III factor. □

Definition 13.3. A normal state as in Theorem 13.2 will be called a *spectral gap state.*

The next theorem was obtained in [HMV16]. In the case where M or N is semifinite, it follows easily from Theorem 13.2. But the general case requires more work. This can be intuitively understood from the following observation. Let $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ be two bounded sequences in M and N respectively and suppose that $(a_i \otimes b_i)_{i\in I}$ is *centralizing* in $M \mathbin{\overline{\otimes}} N$. Then it is easy to see that $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$

must be *central* in M and N respectively. However, they may not be *centralizing*. Indeed, if φ and ϕ are two normal states on M and N respectively, then one might have for example $\|a_i\varphi - \lambda\varphi a_i\| \to 0$ and $\|b_i\phi - \lambda^{-1}\phi b_i\| \to 0$ for some $\lambda > 0$. In fact, this phenomenon turns out to be the only serious issue and we are able to overcome it by enhancing the spectral gap inequality of Theorem 13.2 (see Lemma 13.6).

Theorem 13.4 ([HMV16])**.** *Let M and N be two full factors. Then $M \mathbin{\overline{\otimes}} N$ is a full factor.*

More generally, if M is a full factor and N is any von Neumann algebra, then $M' \cap (M \mathbin{\overline{\otimes}} N)^\omega = N^\omega$ for any cofinal ultrafilter ω on any directed set I.

Lemma 13.5. *Let M be any von Neumann algebra and let $x \in M$ and $\xi, \eta \in \mathrm{L}^2(M)$, then we have*

$$\| \, |x|\xi - \xi|x| \, \|^2 + \| \, |x^*|\eta - \eta|x^*| \, \|^2 \leq \|x\xi - \eta x\|^2 + \|x^*\eta - \xi x^*\|^2.$$

Proof. First we prove the inequality when x is self-adjoint and $\xi = \eta$. In that case, we just have to show that

$$\| \, |x|\xi - \xi|x| \, \|^2 \leq \|x\xi - \xi x\|^2.$$

Since x is self-adjoint, the operators $\lambda(x)$ and $\rho(x)$ generate a commutative von Neumann algebra in $\mathbf{B}(\mathrm{L}^2(M))$. Hence by the classical triangle inequality we have $| \, |\lambda(x)| - |\rho(x)| \, |^2 \leq |\lambda(x) - \rho(x)|^2$. Hence by applying the positive linear form $\langle \cdot \xi, \xi \rangle$ we get $\|(|\lambda(x)| - |\rho(x)|)\xi\|^2 \leq \|(\lambda(x) - \rho(x))\xi\|^2$ which means that $\| \, |x|\xi - \xi|x| \, \|^2 \leq \|x\xi - \xi x\|^2$ as we wanted.

Now, back to the general case, we work in $N = M_2(\mathbb{C}) \otimes M = M_2(M)$ and we consider

$$y = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \in N \text{ and } \alpha = \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \in \mathrm{L}^2(N)$$

Then we have

$$y\alpha - \alpha y = \begin{pmatrix} 0 & x^*\eta - \xi x^* \\ x\xi - \eta x & 0 \end{pmatrix}$$

so that $\|y\alpha - \alpha y\|^2 = \|x\xi - \eta x\|^2 + \|x^*\eta - \xi x^*\|^2$. We also have

$$|y| = \begin{pmatrix} |x| & 0 \\ 0 & |x^*| \end{pmatrix}$$

which means that

$$|y|\alpha - \alpha|y| = \begin{pmatrix} |x|\xi - \xi|x| & 0 \\ 0 & |x^*|\eta - \eta|x^*| \end{pmatrix}$$

so that $\||y|\alpha - \alpha|y|\|^2 = \| \, |x|\xi - \xi|x| \, \|^2 + \| \, |x^*|\eta - \eta|x^*| \, \|^2$. Since y is self-adjoint, the conclusion now follows from the first case.

□

Lemma 13.6. *Let M be a full factor and let $\varphi \in M_*$ be a spectral gap state. Then we can find a finite set $F \subset M$, with $a\varphi^{1/2} = \varphi^{1/2}a^*$ for all $a \in F$, and a constant $\kappa > 0$ such that for all $x \in M$ and all $\lambda > 0$ we have*

$$\|x - \varphi(x)\|_\varphi^2 \leq \kappa \left(\sum_{a \in F} \|ax - xa\|_\varphi^2 + \|x\varphi^{1/2} - \lambda\varphi^{1/2}x\|^2 \right). \tag{13.1}$$

Proof. Since φ is a spectral gap state, we can find a finite set $S \subset \Sigma_\varphi$ and a constant $\kappa > 0$ such that for all $x \in M$, we have

$$\|x - \varphi(x)\|_\varphi \leq \kappa \sum_{\xi \in S} \|x\xi - \xi x\|. \tag{13.2}$$

We can always assume that all the elements of S are self-adjoint. Write $S = \{\xi_1, \dots, \xi_k\}$. Let $\xi_\varphi = \varphi^{1/2}$. For all $j \in \{1, \dots, k\}$, let $a_j = \xi_j \varphi^{-1/2}$ so that $\xi_j = a_j \xi_\varphi = \xi_\varphi a_j^*$. We will show that $F = \{a_1, \dots, a_k\}$ satisfies the desired inequality.

By homogeneity, it is sufficient to prove that for all sequences $x_n \in M$ and $\lambda_n \in \mathbb{R}_+$ ($n \in \mathbb{N}$) such that $\lim_n \|x_n a_j - a_j x_n\|_\varphi = 0$ for every $j \in \{1, \dots, k\}$ and $\lim_n \|x_n \xi_\varphi - \lambda_n \xi_\varphi x_n\| = 0$, we have that $\lim_n \|x_n - \varphi(x_n)1\|_\varphi = 0$.

Put $\mu_n = \|x_n\|_\varphi \in \mathbb{R}_+$ for every $n \in \mathbb{N}$. Observe that the sequence $(\mu_n)_{n\in\mathbb{N}}$ may be unbounded. Since $\lim_n \|x_n\xi_\varphi - \lambda_n \xi_\varphi x_n\| = 0$, we have that $\lim_n |\mu_n - \lambda_n \|x_n^*\|_\varphi| = 0$. For every $n \in \mathbb{N}$, write $x_n = u_n|x_n| = u_n|x_n|u_n^* u_n = |x_n^*|u_n$ for the polar decomposition of $x_n \in M$. For every $j \in \{1, \dots, k\}$ and every $n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_n\xi_j - \lambda_n \xi_j x_n\| &= \|(x_n a_j - a_j x_n)\xi_\varphi + a_j(x_n\xi_\varphi - \lambda_n\xi_\varphi x_n)\| \\ &\leq \|x_n a_j - a_j x_n\|_\varphi + \|x_n\xi_\varphi - \lambda_n \xi_\varphi x_n\|.\end{aligned}$$

Thus, we obtain $\lim_n \|x_n\xi_j - \lambda_n\xi_j x_n\| = 0$ for every $j \in \{1, \dots, k\}$. By Lemma 13.5 applied to "$\xi = \xi_j$" and "$\eta = \lambda_n \xi_j$", and since $\xi_j = \xi_j^*$, we also have

$$2\|x_n\xi_j - \lambda_n\xi_j x_n\|^2 \geq \||x_n|\xi_j - \xi_j|x_n|\|^2 + \|\lambda_n(|x_n^*|\xi_j - \xi_j|x_n^*|)\|^2.$$

Hence we obtain $\lim_n \||x_n|\xi_j - \xi_j|x_n|\| = 0$ and $\lim_n \lambda_n \||x_n^*|\xi_j - \xi_j|x_n^*|\| = 0$ for every $j \in \{1, \dots, k\}$. Then (13.2) yields $\lim_n \||x_n| - \varphi(|x_n|)1\|_\varphi = 0$ and $\lim_n \lambda_n\||x_n^*| - \varphi(|x_n^*|)\|_\varphi = 0$.

Since $\||x_n|\|_\varphi = \|x_n\|_\varphi = \mu_n$ for every $n \in \mathbb{N}$ and since $\lim_n \||x_n| - \varphi(|x_n|)\|_\varphi = 0$, we have $\lim_n |\mu_n - \varphi(|x_n|)| = 0$. Hence we obtain $\lim_n \||x_n| - \mu_n\|_\varphi = 0$ and by multiplying by u_n on the left, we obtain $\lim_n \|x_n - \mu_n u_n\|_\varphi = 0$. Since $\xi_j \in \Sigma_\varphi$, this yields $\lim_n \|x_n\xi_j - \mu_n u_n \xi_j\| = 0$ for all j.

Likewise, since $\lim_n |\mu_n - \lambda_n\|x_n^*\|_\varphi| = 0$ and since $\lim_n \|\lambda_n|x_n^*| - \lambda_n\varphi(|x_n^*|)1\|_\varphi = 0$, we have $\lim_n |\mu_n - \lambda_n\varphi(|x_n^*|)| = 0$ and hence we obtain $\lim_n \|\lambda_n|x_n^*| - \mu_n\|_\varphi = 0$. By multiplying by u_n^* on the right, we obtain $\lim_n \|\lambda_n x_n^* - \mu_n u_n^*\|_\varphi = 0$. Since $\xi_j \in \Sigma_\varphi$, this yields $\lim_n \|\lambda_n \xi_j x_n - \mu_n \xi_j u_n\| = \lim_n \|\lambda_n x_n^* \xi_j - \mu_n u_n^* \xi_j\| = 0$ for all j.

Finally, since $\lim_n \|x_n\xi_j - \lambda_n\xi_j x_n\| = 0$, we obtain $\lim_n \mu_n\|u_n\xi_j - \xi_j u_n\| = 0$ for all $j \in \{1, \dots, k\}$. Then (13.2) implies that $\lim_n \|\mu_n u_n - \varphi(\mu_n u_n)1\|_\varphi = 0$. Since $\lim_n \|x_n - \mu_n u_n\|_\varphi = 0$, Cauchy-Schwarz inequality yields $\lim_n |\varphi(x_n - \mu_n u_n)| = 0$. Therefore, we finally obtain $\lim_n \|x_n - \varphi(x_n)1\|_\varphi = 0$. $\square$

Lemma 13.7. *Let H be any Hilbert space and $S : \mathcal{D}(S) \to H$ any closed positive densily defined operator. Let $A \in \mathbf{B}(H)$ be any selfadjoint bounded operator which satisfies*

$$\forall \xi \in \mathcal{D}(S),\ \forall \lambda > 0, \quad \langle A\xi, \xi\rangle \leq \|(\lambda S - 1)\xi\|^2.$$

Then for every Hilbert space K and every closed positive densily defined operator $T : \mathcal{D}(T) \to K$, we have

$$\forall \eta \in \mathcal{D}(S \otimes T), \quad \langle (A \otimes 1)\eta, \eta\rangle \leq \|(S \otimes T - 1)\eta\|^2.$$

Proof. Write $A_0 = \{T^{\mathrm{i}t} : t \in \mathbb{R}\}$ for the von Neumann subalgebra of $\mathbf{B}(K)$ generated by T. Choose an intermediate maximal abelian subalgebra $A_0 \subset A \subset \mathbf{B}(K)$. Then we can identify $(A \subset \mathbf{B}(K)) = \left(\mathrm{L}^\infty(X, \mu) \subset \mathbf{B}(\mathrm{L}^2(X, \mu))\right)$ for some measure space (X, μ). Since T is affiliated with $A = \mathrm{L}^\infty(X, \mu)$, the spectral theorem for unbounded operators (see e.g. [KR97, Theorem 5.6.4]) shows that we can assume that $T = M_f$ acts by multiplication on $\mathrm{L}^2(X, \mu)$ by some measurable function $f : (X, \mu) \to \mathbb{R}_+$. Then

$$\mathcal{D}(T) = \left\{ g \in \mathrm{L}^2(X, \mu) : \int_X |fg|^2 \,\mathrm{d}\mu < +\infty \right\}.$$

Let $\eta = \sum_{i=1}^p \xi_i \otimes g_i$ be any element in the algebraic tensor product $\mathcal{D}(S) \odot \mathcal{D}(T)$. Identify $H \otimes K = \mathrm{L}^2(X, \mu, H)$ and regard η as a function from X to $\mathcal{D}(S)$. Then a simple computation shows that

$$\langle (A \otimes 1)\eta, \eta \rangle = \int_{x \in X} \langle A\eta(x), \eta(x) \rangle \,\mathrm{d}\mu(x)$$

and

$$\|(S \odot T - 1)\eta\|^2 = \int_{x \in X} \|(f(x)S - 1)\eta(x)\|^2 \,\mathrm{d}\mu(x).$$

Hence, we have

$$\forall \eta \in \mathcal{D}(S) \odot \mathcal{D}(T), \quad \langle (A \otimes 1)\eta, \eta \rangle \leq \|(S \odot T - 1)\eta\|^2.$$

Since $S \otimes T$ is the closure of $S \odot T$, for every $\eta \in \mathcal{D}(S \otimes T)$, we can find a sequence $\eta_n \in \mathcal{D}(S) \odot \mathcal{D}(T)$ such that $\eta_n \to \eta$ and $(S \odot T)\eta_n \to (S \otimes T)\eta$ as $n \to \infty$. For every $n \in \mathbb{N}$, we have

$$\langle (A \otimes 1)\eta_n, \eta_n \rangle \leq \|(S \odot T - 1)\eta_n\|^2.$$

Therefore we obtain

$$\langle (A \otimes 1)\eta, \eta \rangle \leq \|(S \otimes T - 1)\eta\|^2$$

as we wanted. $\square$

The proof of Theorem 13.4 follows from the following inequality.

Lemma 13.8. *Let M be a full factor. Let $\varphi \in M_*$ be any spectral gap state. Then there exists a finite subset $F \subset M$, with $a\varphi^{1/2} = \varphi^{1/2}a^*$ for all $a \in F$, and a constant $\kappa > 0$ such that for any von Neumann algebra N, for any $\phi, \psi \in N_*^+$ and any $z \in M \mathbin{\overline{\otimes}} N$ we have*

$$\|z - \mathrm{E}_\varphi(z)\|^2_{\varphi \otimes \psi} \leq \kappa \left(\sum_{a \in F} \|za - az\|^2_{\varphi \otimes \psi} + \|z(\varphi \otimes \psi)^{1/2} - (\varphi \otimes \phi)^{1/2} z\|^2 \right) \quad (13.3)$$

where $\mathrm{E}_\varphi = \varphi \otimes \mathrm{id}_N : M \mathbin{\overline{\otimes}} N \to N$ is the normal conditional expectation induced by φ.

Proof. We take $F \subset M$ and $\kappa > 0$ as in Lemma 13.6. Let $S = \Delta_\varphi^{1/2}$ and let p be the support of φ. Let $P \in \mathbf{B}(\mathrm{L}^2(M)$ the projection on $\mathbb{C}\varphi^{1/2}$. Then (13.1) implies that

$$\forall \xi \in \mathcal{D}(S),\ \forall \lambda > 0, \quad \langle A\xi, \xi \rangle \leq \|(\lambda S - 1)\xi\|^2,$$

where

$$A = \frac{1}{\kappa}(JpJ - P) - \sum_{a \in F} |a - JaJ|^2.$$

Indeed, it is enough to check this inequality for ξ of the form $x\varphi^{1/2}$ with $x \in Mp$.

Now, let N be any von Neumann algebra and let $\psi, \phi \in N_*^+$. Let $T = \Delta_{\phi,\psi}^{1/2}$. We apply Lemma 13.7 to S and T and we obtain

$$\forall \eta \in \mathcal{D}(S \otimes T), \quad \langle (A \otimes 1)\eta, \eta \rangle \leq \|(S \otimes T - 1)\eta\|^2.$$

Since $S \otimes T = \Delta_{\varphi\otimes\phi,\varphi\otimes\psi}^{1/2}$ and since the orthogonal projection $P \otimes 1 : \mathrm{L}^2(M \mathbin{\overline{\otimes}} N) \to \mathbb{C}\varphi^{1/2} \otimes \mathrm{L}^2(N)$ satisfies $P(z(\varphi \otimes \phi)^{1/2}) = \mathrm{E}_\varphi(z)(\varphi \otimes \phi)^{1/2}$ for all $z \in M \mathbin{\overline{\otimes}} N$, we finally obtain (13.3). $\square$

Proof of Theorem 13.4. Let $\varphi \in M_*$ be a spectral gap state. Let $(x_i)_{i\in I}$ be a centralizing net in $\mathrm{Ball}(M \mathbin{\overline{\otimes}} N)$. Let $y_i = \mathrm{E}_\varphi(x_i)$ for every $i \in I$. Observe that $(y_i)_{i\in I}$ is a centralizing net in N. Moreover, (13.3) shows that $\lim_i \|x_i - y_i\|_{\varphi\otimes\psi} = 0$ for any state $\psi \in N_*$. This means that $(x_i - y_i)p \to 0$ strongly as $i \to \infty$ where $p \in M$ is the support of φ. Since $(x_i)_{i\in I}$ is asymptotically central, this shows that $(x_i - y_i)q \to 0$ strongly as $i \to \infty$ for any projection $q \in M$ which is equivalent to p. Since M is a factor, we conclude that $x_i - y_i \to 0$ strongly as $i \to \infty$.

Let $\varphi \in M_*$ be a spectral gap state and $\psi \in N_*$ be any state. Denote by $\mathrm{E}_\varphi : M \mathbin{\overline{\otimes}} N \to N$ the normal conditional expectation induced by φ and let $e = p \otimes q$ be the support of $\varphi \otimes \psi$ in $M \mathbin{\overline{\otimes}} N$. We will show that $eN^\omega e = e(M' \cap (M \mathbin{\overline{\otimes}} N)^\omega)e$. Assume by contradiction that $eN^\omega e \subsetneq e(M' \cap (M \mathbin{\overline{\otimes}} N)^\omega)e$. Let S be the restriction of $\Delta_{(\varphi\otimes\psi)^\omega}^{1/2}$ to the nonzero Hilbert space $\mathrm{L}^2(e(M' \cap (M \mathbin{\overline{\otimes}} N)^\omega)e) \ominus \mathrm{L}^2(eN^\omega e)$. Then S is a closed positived definite operator. Take some $\lambda > 0$ in the spectrum of S. Then, for any $\varepsilon > 0$, we can find $Z \in e(M' \cap (M \mathbin{\overline{\otimes}} N)^\omega)e$ such that $\|Z\|_{(\varphi\otimes\psi)^\omega} = 1$, $\mathrm{E}_\varphi^\omega(Z) = 0$ and $\|((\varphi \otimes \psi)^\omega)^{1/2} Z - \lambda Z((\varphi \otimes \psi)^\omega)^{1/2}\| < \varepsilon$. This means that for any finite set $F \subset M$, we can find $z \in M \mathbin{\overline{\otimes}} N$ such that $\|z\|_{\varphi\otimes\psi} = 1$, $\mathrm{E}_\varphi(z) = 0$, $\|z(\varphi \otimes \psi)^{1/2} - \lambda^{-1}(\varphi \otimes \psi)^{1/2} z\| \leq 2\lambda^{-1}\varepsilon$ and $\|za - az\|_{\varphi\otimes\psi} \leq \varepsilon$ for every $a \in F$. This however contradicts Lemma 13.4 if $\varepsilon > 0$ is small enough (we apply it with $\phi = \lambda^{-2}\psi$). Therefore, we must have $e(M' \cap (M \mathbin{\overline{\otimes}} N)^\omega)e = eN^\omega e$. Since this holds for every state $\psi \in N_*$, we get $p(M' \cap (M \mathbin{\overline{\otimes}} N)^\omega) = pN^\omega$ where p is the support of φ. Since M is a factor, this finally yields $M' \cap (M \mathbin{\overline{\otimes}} N)^\omega = N^\omega$. $\square$

14 Outer automorphism groups of full factors

Note that a net of unitaries $(u_i)_{i\in I}$ in $\mathcal{U}(M)$ is centralizing in M if and only if $\mathrm{Ad}(u_i) \to \mathrm{id}_M$ as $i \to \infty$. This observation implies the following characterization of fullness.

Proposition 14.1 ([Co74]). *Let M be a factor. Then M is full if and only if the map*

$$\mathrm{Ad} : \mathcal{U}(M) \to \mathrm{Aut}(M)$$

is open on its range. In that case, $\mathrm{Inn}(M)$ *is closed in* $\mathrm{Aut}(M)$ *and* $\mathrm{Out}(M)$ *is Hausdorff.*

The following theorem is the key result for all this section. It is a generalization of a result of Jones in the II_1 case [Jo81].

Theorem 14.2 ([Ma16b]). *Let M be a full factor with a spectral gap state $\varphi \in M_*$. Let $\mathcal{V}$ be a neighborhood of the identity in* $\mathrm{Out}(M)$*. Then there exists a finite set*

$S \subset \Sigma_\varphi$ and a constant $\kappa > 0$ such that for all $x \in M$ and all $\theta \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$ we have

$$\|x\|_\varphi \leq \kappa \sum_{\xi \in S} \|x\xi - \theta(\xi)x\|. \tag{14.1}$$

Before we prove Theorem 14.2, let us mention the following closely related result of Connes. While Connes' criterion is very general and does not require any assumption on M, Theorem 14.2 is more powerful in the full factor case since it gives a *uniform* inequality with the same state φ, the same set S and the same constant κ for all automorphisms outside some neighborhood of the identity. This will be crucial for our applications.

Theorem 14.3 ([Co85, Theorem III.1])**.** *Let M be a factor and let $\theta \in \mathrm{Aut}(M) \setminus \overline{\mathrm{Inn}}(M)$. Then there exists a state $\varphi \in M_*$, a finite set $S \subset \Sigma_\varphi$ and a constant $\kappa > 0$ such that for all $x \in M$ we have*

$$\|x\|_\varphi \leq \kappa \sum_{\xi \in S} \|x\xi - \theta(\xi)x\|.$$

For the proof of Theorem 14.2, we first need two technical lemmas. The first one strengthens slightly [DM71][Proposition 1].

Lemma 14.4. *Let M be any von Neumann algebra. Then the invertible elements are $*$-strongly dense in the unit ball of M.*

Proof. Let $Q = \{x \in M \mid \|x\| \leq 1 \text{ and } x \text{ is invertible}\}$. Let $\overline{Q}$ be the $*$-strong closure of Q. It is easy to see by the functional calculus that any normal element in the unit ball of M is in $\overline{Q}$. Hence, since $\overline{Q}$ is stable under multiplication, it is enough, thanks to the polar decomposition, to show that any partial isometry $u \in M$ belongs to $\overline{Q}$. By [DM71][Lemma 1], we can find an isometry $v \in M$ and a coisometry $w \in M$ such that $u = vw(u^*u)$. Therefore, since $\overline{Q}$ contains all projections and is stable by adjunction and multiplication, it is enough to show that any isometry $v \in M$ is in $\overline{Q}$. By the proof of [DM71][Lemma 2], we can find a sequence of projections $p_n \in M$ converging to 1 strongly (hence $*$-strongly) and partial isometries $w_n \in M$ such that $vp_n + w_n$ is a unitary for all n. Then let $x_n = vp_n + \frac{1}{n}w_n \in Q$. We have $x_n \to v$ in the $*$-strong topology. Hence $v \in \overline{Q}$. $\square$

The second lemma uses the factoriality in order to transform a convergence on a corner into a convergence modulo inner automorphisms. Recall that $\pi_M : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ is the quotient map.

Lemma 14.5. *Let M be a factor. Let $p \in M$ be a nonzero projection. Let $\theta_i \in \mathrm{Aut}(M)$, $i \in I$ be a net of automorphisms such that $\theta_i(\xi) \to \xi$ for all $\xi \in p\mathrm{L}^2(M)p$. Then we have $\pi_M(\theta_i) \to 1$.*

Proof. After replacing p by a smaller projection if necessary, we can suppose that p is part of a system of matrix units $(e_{kl})_{k,l \in J}$ in M with $e_{00} = p$. Let $x_i = \sum_k \theta_i(e_{k0})e_{0k}$ where the sum is $*$-strongly convergent. Then for all $\xi \in \mathrm{L}^2(M)$ we have $\theta_i(\xi)x_i - x_i\xi \to 0$. Indeed we have

$$\theta_i(\xi)x_i = \sum_{k,l} \theta_i(e_{k0})\theta_i(e_{0k}\xi e_{l0})e_{0l}$$

and

$$x_i\xi = \sum_{k,l} \theta_i(e_{k0})(e_{0k}\xi e_{l0})e_{0l}$$

Thus we get $\theta_i(\xi)x_i - x_i\xi \to 0$ since by assumption we know that $\theta_i(e_{0k}\xi e_{l0}) \to e_{0k}\xi e_{l0}$ for all k, l.

We also note that $x_i^* x_i \leq 1$ and $x_i^* x_i \to 1$ strongly hence $|x_i| \to 1$ strongly and therefore we have $|x_i|\xi - \xi \to 0$ for all ξ. Similarly, we have $\theta_i^{-1}(|x_i^*|) \to 1$ strongly and therefore $\theta_i(\xi)|x_i^*| - \theta_i(\xi) \to 0$ for all ξ. By Lemma 14.4, we can find a net of invertible elements y_i in the unit ball of M such that $y_i - x_i \to 0$ and $\theta_i^{-1}(y_i) - \theta_i^{-1}(x_i) \to 0$ in the $*$-strong topology. Then we still have $\theta_i(\xi)y_i - y_i\xi \to 0$, $|y_i|\xi - \xi \to 0$ and $\theta_i(\xi)|y_i^*| - \theta_i(\xi) \to 0$. Hence, by writing the polar decomposition $y_i = u_i|y_i| = |y_i^*|u_i$ with unitaries u_i, we get $\theta_i(\xi)u_i - u_i\xi \to 0$ for all $\xi \in \mathrm{L}^2(M)$. This shows that $\mathrm{Ad}(u_i)^{-1} \circ \theta_i \to 1$ in $\mathrm{Aut}(M)$ hence $\pi_M(\theta_i) \to 1$ in $\mathrm{Out}(M)$. ☐

Proof of Theorem 14.2. If the conclusion of the theorem does not holds, then we can find a net $x_i \in M, i \in I$ and a net $\theta_i \notin \pi_M^{-1}(\mathcal{N})$ such that $\|x_i\|_\varphi = 1$ and $x_i\xi - \theta_i(\xi)x_i \to 0$ for all $\xi \in \Sigma_\varphi$. Thanks to Lemma 14.4, we can assume that every x_i is invertible. Note that the net $(x_i)_{i\in I}$ is not necessarily bounded. Let $x_i = u_i|x_i| = |x_i^*|u_i$, $u_i \in \mathcal{U}(M)$ be the polar decomposition of x_i.

For all $\xi \in \Sigma_\varphi$, we have $x_i\xi - \theta_i(\xi)x_i \to 0$ and therefore $|x_i|\xi - \xi|x_i| \to 0$ by Lemma 13.5. Since φ is a spectral gap state, we get $\|\,|x_i| - \varphi(|x_i|)\,\|_\varphi \to 0$. Since $\|x_i\|_\varphi = 1$ this means that $\|\,|x_i| - 1\|_\varphi \to 0$. Therefore, we get $(x_i - u_i)\xi = u_i(|x_i| - 1)\xi \to 0$ for all $\xi \in \Sigma_\varphi$.

Similarly, for all $\xi \in \Sigma_\varphi$, we have $|x_i^*|\theta_i(\xi) - \theta_i(\xi)|x_i^*| \to 0$ by Lemma 13.5. Hence, if we let $y_i = \theta_i^{-1}(x_i^*)$ then we have $|y_i|\xi - \xi|y_i| \to 0$ for all $\xi \in \Sigma_\varphi$ and thus $\|\,|y_i| - 1\|_\varphi \to 0$ for the same reason as before. Therefore, we get $(y_i - \theta_i^{-1}(u_i^*))\xi = \theta_i^{-1}(u_i^*)(|y_i| - 1)\xi \to 0$ for all $\xi \in \Sigma_\varphi$. By taking the adjoint and composing with θ_i, we obtain $\theta_i(\xi)(x_i - u_i) \to 0$ for all $\xi \in \Sigma_\varphi$.

By combining this with the fact that $x_i\xi - \theta_i(\xi)x_i \to 0$, we get $u_i\xi - \theta_i(\xi)u_i \to 0$ for all $\xi \in \Sigma_\varphi$. Let p be the support of φ. Then Σ_φ is dense in $p\mathrm{L}^2(M)p$. Hence, we really have $(\mathrm{Ad}(u_i^*) \circ \theta_i)(\xi) \to \xi$ for all $\xi \in p\mathrm{L}^2(M)p$. By Lemma 14.5, we get $\pi_M(\theta_i) = \pi_M(\mathrm{Ad}(u_i^*) \circ \theta_i) \to 1$ in $\mathrm{Out}(M)$ which is a contradiction since $\pi_M(\theta_i) \notin \mathcal{V}$ for all $i \in I$. ☐

Let M and N be two von Neumann algebras. Then there exist a unique homomorphism $\iota : \mathrm{Out}(M) \times \mathrm{Out}(N) \to \mathrm{Out}(M \mathbin{\overline{\otimes}} N)$ such that $\iota(\pi_M(\alpha), \pi_N(\beta)) = \pi_{M\overline{\otimes}N}(\alpha \otimes \beta)$ for all $(\alpha, \beta) \in \mathrm{Aut}(M) \times \mathrm{Aut}(N)$. This homomorphism is always continuous and injective. When M is a full factor, we obtain the following theorem.

Theorem 14.6 ([HMV16])**.** *Let M be a full factor and N any von Neumann algebra. Then the group homomorphism $\iota : \mathrm{Out}(M) \times \mathrm{Out}(N) \to \mathrm{Out}(M \mathbin{\overline{\otimes}} N)$ is a homeomorphism onto its range.*

Before we prove this theorem, let us point out that it allows us in particular to compute the τ-*invariant* of tensor products of full factors. Recall that if M is a von Neumann algebra then the homomorphism $\delta_M = \pi_M \circ \sigma^\phi : \mathbb{R} \to \mathrm{Out}(M)$ does not depend on the choice of the faithful normal semifinite weight ϕ on M [Co72]. The τ-*invariant* of M, denoted by $\tau(M)$, is the weakest topology on $\mathbb{R}$ which makes the map δ_M continuous [Co74]. The following is a direct application of Theorem 14.6.

Corollary 14.7 ([HMV16]). *Let M be a full factor and N any von Neumann algebra. Then for every net $(t_i)_{i\in I}$ in $\mathbb{R}$, we have*

$$t_i \to 0 \text{ w.r.t. } \tau(M \mathbin{\overline{\otimes}} N) \quad \text{if and only if} \quad t_i \to 0 \text{ w.r.t. } \tau(M) \text{ and } \tau(N).$$

Theorem 14.6 follows easily from Theorem 14.2 when M or N is finite. However in the general case, one encounters the same difficulty as for Theorem 13.4. Thus the proof will follow the same pattern.

Lemma 14.8. *Let M be a full factor and let $\varphi \in M_*$ be any spectral gap state. Let $\mathcal{V}$ be a neighborhood of the identity in $\mathrm{Out}(M)$. Then we can find a finite set $F \subset M$, with $a\varphi^{1/2} = \varphi^{1/2}a^*$ for all $a \in F$, and a constant $\kappa > 0$ such that for all $x \in M$, all $\theta \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$ and all $\lambda > 0$ we have*

$$\|x\|_\varphi^2 \leq \kappa \left(\sum_{a\in F} \|xa - \theta(a)x\|_\varphi^2 + \|x\varphi^{1/2} - \lambda\theta(\varphi)^{1/2}x\|^2 \right). \tag{14.2}$$

Proof. By Theorem 13.2 and Theorem 14.2, we can find a finite set $S \subset \Sigma_\varphi$ and a constant $\kappa > 0$ such that for all $x \in M$ and all $\theta \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$ we have

$$\|x - \varphi(x)\|_\varphi \leq \kappa \sum_{\xi \in S} \|x\xi - \xi x\|. \tag{14.3}$$

and

$$\|x\|_\varphi \leq \kappa \sum_{\xi\in S} \|x\xi - \theta(\xi)x\|. \tag{14.4}$$

We can always assume that all the elements of S are self-adjoint. Write $S = \{\xi_1, \dots, \xi_k\}$. Let $\xi_\varphi = \varphi^{1/2}$. For all $j \in \{1, \dots, k\}$, let $a_j = \xi_j\varphi^{-1/2}$ so that $\xi_j = a_j\xi_\varphi = \xi_\varphi a_j^*$. We will show that $F = \{a_1, \dots, a_k\}$ satisfies the desired inequality. By homogeneity, it is sufficient to prove that for all sequences $x_n \in M$, $\lambda_n \in \mathbb{R}_+$ and $\theta_n \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$ ($n \in \mathbb{N}$) such that $\lim_n \|x_n a_j - \theta_n(a_j)x_n\|_\varphi = 0$ for every $j \in \{1, \dots, k\}$ and $\lim_n \|x_n\varphi^{1/2} - \lambda_n\theta_n(\varphi^{1/2})x_n\| = 0$, we have that $\lim_n \|x_n\|_\varphi = 0$.

Put $\mu_n = \|x_n\|_\varphi \in \mathbb{R}_+$ for every $n \in \mathbb{N}$. Since $\lim_n \|x_n\varphi^{1/2} - \lambda_n\theta_n(\varphi^{1/2})x_n\| = 0$, we have that $\lim_n |\mu_n - \lambda_n\|\theta_n^{-1}(x_n^*)\|_\varphi| = \lim_n |\mu_n - \lambda_n\|\theta_n(\xi_\varphi)x_n\|| = 0$. For every $n \in \mathbb{N}$, write $x_n = u_n|x_n| = u_n|x_n|u_n^* u_n = |x_n^*|u_n$ for the polar decomposition of $x_n \in M$. For every $j \in \{1, \dots, k\}$ and every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|x_n\xi_j - \lambda_n\theta_n(\xi_j)x_n\| &= \|(x_n a_j - \theta_n(a_j)x_n)\varphi^{1/2} + \theta_n(a_j)(x_n\varphi^{1/2} - \lambda_n\theta_n(\varphi^{1/2})x_n)\| \\ &\leq \|x_n a_j - \theta_n(a_j)x_n\|_\varphi + \|x_n\varphi^{1/2} - \lambda_n\theta_n(\varphi^{1/2})x_n\|. \end{aligned}$$

Thus, we obtain $\lim_n \|x_n\xi_j - \lambda_n\theta_n(\xi_j)x_n\| = 0$ for every $j \in \{1, \dots, k\}$. By Lemma 13.5 applied to "$\xi = \xi_j$" and "$\eta = \lambda_n\theta_n(\xi_j)$", we get

$$2\|x_n\xi_j - \lambda_n\theta_n(\xi_j)x_n\|^2 \geq \||x_n|\xi_j - \xi_j|x_n|\|^2 + \|\lambda_n(|x_n^*|\theta_n(\xi_j) - \theta_n(\xi_j)|x_n^*|)\|^2.$$

Hence we obtain $\lim_n \||x_n|\xi_j - \xi_j|x_n|\| = 0$ and $\lim_n \lambda_n\|\,\theta_n^{-1}(|x_n^*|)\xi_j - \xi_j\theta_n^{-1}(|x_n^*|)\| = 0$ for every $j \in \{1, \dots, k\}$. Then (14.3) implies that $\lim_n \||x_n| - \varphi(|x_n|)1\|_\varphi = 0$ and $\lim_n \lambda_n\|\theta_n^{-1}(|x_n^*|) - \varphi(\theta_n^{-1}(|x_n^*|))1\|_\varphi = 0$.

Since $\||x_n|\|_\varphi = \|x_n\|_\varphi = \mu_n$ for every $n \in \mathbb{N}$ and since $\lim_n \||x_n| - \varphi(|x_n|)1\|_\varphi = 0$, we have $\lim_n |\mu_n - \varphi(|x_n|)| = 0$ and hence we obtain $\lim_n \||x_n| - \mu_n\|_\varphi = 0$. Therefore, by multiplying by u_n on the left, we obtain $\lim_n \|x_n - \mu_n u_n\|_\varphi = 0$ and since $\xi_j \in \Sigma_\varphi$, we get $\lim_n \|x_n\xi_j - \mu_n u_n\xi_j\| = 0$ for all $j \in \{1, \dots, k\}$.

Likewise, since $\lim_n |\mu_n - \lambda_n \|\theta_n^{-1}(x_n^*)\|_\varphi| = 0$ and since $\lim_n \|\lambda_n \theta_n^{-1}(|x_n^*|) - \lambda_n \varphi(\theta_n^{-1}(|x_n^*|))\|_\varphi = 0$, we have $\lim_n |\mu_n - \lambda_n \varphi(\theta_n^{-1}(|x_n^*|))| = 0$. This in turn implies that $\lim_n \|\lambda_n \theta_n^{-1}(|x_n^*|) - \mu_n\|_\varphi = 0$. By multiplying by $\theta_n^{-1}(u_n^*)$ on the left, we obtain $\lim_n \|\lambda_n \theta_n^{-1}(x_n^*) - \mu_n \theta_n^{-1}(u_n^*)\|_\varphi = 0$ and since $\xi_j \in \Sigma_\varphi$, we get $\lim_n \|\lambda_n \theta_n(\xi_j) x_n - \theta_n(\xi_j)\mu_n u_n\| = \|\lambda_n \theta_n^{-1}(x_n^*)\xi_j - \mu_n \theta_n^{-1}(u_n^*)\xi_j\| = 0$ for all $j \in \{1, \dots, k\}$.

Since $\lim_n \|x_n \xi_j - \lambda_n \theta_n(\xi_j) x_n\| = 0$, we obtain $\lim_n \mu_n \|u_n \xi_j - \theta_n(\xi_j) u_n\| = 0$ for all $j \in \{1, \dots, k\}$. Then (14.4) implies that $\lim_n \|\mu_n u_n\|_\varphi = 0$. Since $\lim_n \|x_n - \mu_n u_n\|_\varphi = 0$, we obtain $\lim_n \|x_n\|_\varphi = 0$ as we wanted. $\square$

Lemma 14.9. *Let M be a full factor and $\varphi \in M_*$ any spectral gap state. Let $\mathcal{V}$ be a neighborhood of the identity in $\mathrm{Out}(M)$. Then there exists a finite subset $F \subset M$, with $a\varphi^{1/2} = \varphi^{1/2}a^*$ for all $a \in F$, and a constant $\kappa > 0$ such that for any von Neumann algebra N, any $\phi, \psi \in N_*^+$, any $z \in M \mathbin{\overline{\otimes}} N$ and any $\theta \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$ we have*

$$\|z\|_{\varphi\otimes\psi}^2 \leq \kappa \left(\sum_{a\in F} \|za - \theta(a)z\|_{\varphi\otimes\psi}^2 + \|z(\varphi\otimes\psi)^{1/2} - (\theta(\varphi)\otimes\phi)^{1/2}z\|^2 \right). \tag{14.5}$$

Proof. We take $F \subset M$ and $\kappa > 0$ as in Lemma 14.8. Take $\theta \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$. Let $S = \Delta_{\theta(\varphi),\varphi}^{1/2}$ and let p be the support of φ. Then (14.2) implies that

$$\forall \xi \in \mathcal{D}(S),\ \forall \lambda > 0, \quad \langle A\xi, \xi\rangle \leq \|(\lambda S - 1)\xi\|^2,$$

where

$$A = \frac{1}{\kappa} JpJ - \sum_{a \in F} |\theta(a) - JaJ|^2.$$

Indeed, it is enough to check this inequality for ξ of the form $x\varphi^{1/2}$ with $x \in Mp$.

Now, let N be any von Neumann algebra and let $\psi, \phi \in N_*^+$. Let $T = \Delta_{\phi,\psi}^{1/2}$. We apply Lemma 13.7 to S and T and we obtain

$$\forall \eta \in \mathcal{D}(S \otimes T), \quad \langle (A \otimes 1)\eta, \eta \rangle \leq \|(S \otimes T - 1)\eta\|^2.$$

Since $S \otimes T = \Delta_{\theta(\varphi)\otimes\phi, \varphi\otimes\psi}^{1/2}$, we finally obtain (14.5). $\square$

Proof of Theorem 14.6. We have to show that for any net $(\alpha_i \otimes \beta_i)_{i\in I}$ in $\mathrm{Aut}(M) \times \mathrm{Aut}(N)$ such that $\pi_{M\overline{\otimes}N}(\alpha_i \otimes \beta_i) \to 1$ in $\mathrm{Out}(M \mathbin{\overline{\otimes}} N)$ as $i \to \infty$ we have $\pi_M(\alpha_i) \to 1$ in $\mathrm{Out}(M)$ and $\pi_N(\beta_i) \to 1$ in $\mathrm{Out}(N)$ as $i \to \infty$. We can always assume that I is large enough so that there exists a decreasing net $(\mathcal{W}_i)_{i\in I}$ of $*$-strong neighborhoods of 0 in N which is cofinal, meaning that for every $*$-strong neighborhood $\mathcal{W}$ of 0 in N there exists $i \in I$ such that $\mathcal{W}_i \subset \mathcal{W}$.

First, we prove that $\pi_M(\alpha_i) \to 1$ in $\mathrm{Out}(M)$ as $i \to \infty$. Assume by contradiction that this is not the case. Then there exist an open neighborhood $\mathcal{V}$ of 1 in $\mathrm{Out}(M)$ and a subnet $(\alpha_j)_{j\in J}$ such that $\pi_M(\alpha_j) \notin \mathcal{V}$ for every $j \in J$. Let $\varphi \in M_*$ be a spectral gap state and let $\psi \in N_*$ be any state. Since $\pi_{M\overline{\otimes}N}(\alpha_j \otimes \beta_j) \to 1$ in $\mathrm{Out}(M \mathbin{\overline{\otimes}} N)$ as $j \to \infty$, there exists a net $(u_j)_{j\in J}$ in $\mathcal{U}(M \mathbin{\overline{\otimes}} N)$ such that $\mathrm{Ad}(u_j)^{-1} \circ (\alpha_j \otimes \beta_j) \to \mathrm{id}_{M\overline{\otimes}N}$ in $\mathrm{Aut}(M \mathbin{\overline{\otimes}} N)$ as $j \to \infty$. In particular, we have $\lim_j \|u_j a - \alpha_j(a)u_j\|_{\varphi\otimes\psi} = 0$ for every $a \in M$ and $\lim_j \|u_j(\varphi\otimes\psi)^{1/2} - (\alpha_j(\varphi) \otimes \beta_j(\psi))^{1/2}u_j\| = 0$. Since $\|u_j\|_{\varphi\otimes\psi} = 1$ for every $j \in J$, this contradicts Lemma 14.9.

Secondly, we prove that $\pi_N(\beta_i) \to 1$ in $\mathrm{Out}(N)$ as $i \to \infty$. Since we do not assume that N is a full factor, this require a bit more work then the first part.

Since $\pi_M(\alpha_i) \to 1$ in $\mathrm{Out}(M)$ we have that $\pi_{M \overline{\otimes} N}(\alpha_i \otimes \mathrm{id}_N) \to 1$ in $\mathrm{Out}(M \mathbin{\overline{\otimes}} N)$ as $i \to \infty$. By assumption, $\pi_{M \overline{\otimes} N}(\alpha_i \otimes \beta_i) \to 1$ in $\mathrm{Out}(M \mathbin{\overline{\otimes}} N)$. Thus, we also have $\pi_{M \overline{\otimes} N}(\mathrm{id}_M \otimes \beta_i) \to 1$ in $\mathrm{Out}(M \mathbin{\overline{\otimes}} N)$ as $i \to \infty$. Hence there exists a net $(v_i)_{i \in I}$ in $\mathcal{U}(M \mathbin{\overline{\otimes}} N)$ such that $\mathrm{Ad}(v_i)^{-1} \circ (\mathrm{id}_M \otimes \beta_i) \to \mathrm{id}_{M \overline{\otimes} N}$ in $\mathrm{Aut}(M \mathbin{\overline{\otimes}} N)$ as $i \to \infty$. Write $y_i = \mathrm{E}_\varphi(v_i) \in \mathrm{Ball}(N)$ for every $i \in I$. By using Lemma 14.4, we can find for every $i \in I$ an invertible element $x_i \in \mathrm{Ball}(N)$ such that $y_i - x_i \in \mathcal{W}_i$ and $\beta_i^{-1}(y_i) - \beta_i^{-1}(x_i) \in \mathcal{W}_i$. Then we have $x_i - y_i \to 0$ and $\beta_i^{-1}(y_i) - \beta_i^{-1}(x_i) \to 0$ in the $*$-strong topology as $i \to \infty$. Let $x_i = u_i|x_i|$ be the polar decomposition of x_i with $u_i \in \mathcal{U}(N)$. We will show that $\mathrm{Ad}(u_i)^{-1} \circ \beta_i \to \mathrm{id}_N$ in $\mathrm{Aut}(N)$ as $i \to \infty$. For this, it is enough to show that $\lim_i \|u_i\psi^{1/2} - \beta_i(\psi)^{1/2}u_i\| = 0$ for any state $\psi \in N_*$.

First, since $\mathrm{Ad}(v_i)^{-1} \circ (\mathrm{id}_M \otimes \beta_i) \to \mathrm{id}_{M \overline{\otimes} N}$ in $\mathrm{Aut}(M \mathbin{\overline{\otimes}} N)$ as $i \to \infty$, we have $\lim_i \|v_i a - a v_i\|_{\varphi \otimes \psi} = 0$ for every $a \in M$ and $\lim_i \|v_i(\varphi \otimes \psi)^{1/2} - (\varphi \otimes \beta_i(\psi))^{1/2} v_i\| = 0$. Then Lemma 13.8 implies that $\lim_i \|v_i - y_i\|_{\varphi \otimes \psi} = 0$. Since v_i is a unitary, this means that $\lim_i \|x_i\|_\psi = \lim_i \|y_i\|_\psi = 1$. Since $\lim_i \|v_i(\varphi \otimes \psi)^{1/2} - (\varphi \otimes \beta_i(\psi))^{1/2} v_i\| = 0$, by applying the orthogonal projection $P \otimes 1 : \mathrm{L}^2(M \mathbin{\overline{\otimes}} N) \to \mathbb{C}\varphi^{1/2} \otimes \mathrm{L}^2(N)$, we also have $\lim_i \|y_i\psi^{1/2} - \beta_i(\psi)^{1/2} y_i\| = 0$ and thus $\lim_i \|x_i\psi^{1/2} - \beta_i(\psi)^{1/2} x_i\| = 0$. In particular, we obtain $\lim_i \|\beta_i^{-1}(x_i^*)\|_\psi = \lim_i \|x_i\|_\psi = 1$. Since $x_i \in \mathrm{Ball}(N)$, this easily implies that $\lim_i \|x_i - u_i\|_\psi = \lim_i \|\beta_i^{-1}(x_i^* - u_i^*)\|_\psi = 0$. This, combined with $\lim_i \|x_i\psi^{1/2} - \beta_i(\psi)^{1/2} x_i\| = 0$, gives $\lim_i \|u_i\psi^{1/2} - \beta_i(\psi)^{1/2} u_i\| = 0$ as we wanted. This finally shows that $\pi_N(\beta_i) \to 1$ in $\mathrm{Out}(N)$ as $i \to \infty$. $\square$

Remark 14.10. Fix M a semifinite factor and p a nonzero finite projection in M and let φ be the trace of pMp.

Suppose that M is full and let $\mathcal{V}$ be a neighborhood of the identity in $\mathrm{Out}(M)$. Then Lemma 14.8 shows that we can find a finite set $F \subset pMp$ and a constant $\kappa > 0$ such that for all $\theta \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$ and all $x \in M$ we have

$$\|x\|_\varphi \leq \kappa \sum_{a \in F} \|xa - \theta(a)x\|_\varphi.$$

Indeed, this follows from (14.2) applied with $\lambda = \mathrm{mod}(\theta)^{-1/2}$.

We point out a striking difference with Theorem 14.3. Indeed, when M is not full, it is not true in general that for any $\theta \in \mathrm{Aut}(M) \setminus \overline{\mathrm{Inn}}(M)$, we can find a finite set $F \subset pMp$ and a constant $\kappa > 0$ such that for all $x \in M$ we have

$$\|x\|_\varphi \leq \kappa \sum_{a \in F} \|xa - \theta(a)x\|_\varphi.$$

A counter-example can be obtained by taking M the hyperfinite II_∞ factor and θ an automorphism of M with $\mathrm{mod}(\theta) \neq 1$. Then one can find a nonzero partial isometry $v \in M^\omega$ such that $va = \theta(a)v$ for all $a \in M$.

15 Fullness of crossed products

In this section, we investigate fullness of crossed product factors. We will need the following notion.

Definition 15.1. Let M be a factor. Let $\sigma : G \to \mathrm{Aut}(M)$ be a continuous action of a locally compact group G. We say that σ is

(i) *Outer* if the induced map $\pi_M \circ \sigma : G \to \mathrm{Out}(M)$ is injective.

(ii) *Fully outer* if the induced map $\pi_M \circ \sigma : G \to \mathrm{Out}(M)$ is a homeomorphism on its range.

Theorem 15.2 ([Jo81, Ma16b])**.** *Let M be a full factor and let $\sigma : \Gamma \to \mathrm{Aut}(M)$ be a fully outer action of a discrete group Γ. Then $M \rtimes_\sigma \Gamma$ is a full factor.*

Proof. Since σ is outer, then $M \rtimes_\sigma \Gamma$ is a factor. Since σ is fully outer, we can find a neighborhood of the identity $\mathcal{V} \subset \mathrm{Out}(M)$ such that $\pi_M(g) \notin \mathcal{V}$ for all $g \in \Gamma \setminus \{1\}$. Choose a spectral gap state $\varphi \in M_*$. Let $\mathrm{E} : M \rtimes_\sigma \Gamma \to M$ be the canonical faithful normal conditional expectation and use it to lift φ to a state on $M \rtimes_\sigma \Gamma$. We will show that φ is a spectral gap state of $M \rtimes_\sigma \Gamma$.

By Theorem 13.2 and Theorem 14.2, we can find a finite set $S \subset \Sigma_\varphi$ and a constant $\kappa > 0$ such that for all $x \in M$ and all $\theta \in \mathrm{Aut}(M) \setminus \pi_M^{-1}(\mathcal{V})$, we have

$$\|x - \varphi(x)\|_\varphi^2 \leq \kappa \sum_{\xi \in S} \|x\xi - \xi x\|^2$$

and

$$\|x\|_\varphi^2 \leq \kappa \sum_{\xi \in S} \|x\xi - \theta(\xi)\xi\|^2.$$

Taje $x \in M \rtimes_\sigma G$. Let $x^g = \mathrm{E}(u_g^* x) \in M$ for all $g \in \Gamma$. Then we have

$$\|\mathrm{E}(x) - \varphi(x)\|_\varphi^2 \leq \kappa \sum_{\xi \in S} \|\mathrm{E}(x)\xi - \xi \mathrm{E}(x)\|^2$$

and

$$\forall g \in \Gamma \setminus \{1\}, \quad \|x^g\|_\varphi^2 \leq \kappa \sum_{\xi \in S} \|x^g \xi - \sigma_{g^{-1}}(\xi) x^g\|^2$$

Now, since for all $\xi \in S$, we have

$$\|x\xi - \xi x\|^2 = \|\mathrm{E}(x)\xi - \xi \mathrm{E}(x)\|^2 + \sum_{g \in \Gamma \setminus \{1\}} \|x^g \xi - \sigma_{g^{-1}}(\xi) x^g\|^2,$$

and since

$$\|x - \varphi(x)\|_\varphi^2 = \|\mathrm{E}(x) - \varphi(x)\|_\varphi^2 + \|x - \mathrm{E}(x)\|_\varphi^2 = \|\mathrm{E}(x) - \varphi(x)\|_\varphi^2 + \sum_{g \in \Gamma \setminus \{1\}} \|x^g\|_\varphi^2,$$

we conclude that for all $x \in M \rtimes_\sigma \Gamma$, we have

$$\|x - \varphi(x)\|_\varphi^2 \leq \kappa \sum_{\xi \in S} \|x\xi - \xi x\|^2.$$

This shows that φ is a spectral gap state for $M \rtimes_\sigma G$. Hence $M \rtimes_\sigma G$ is full. $\square$

Theorem 15.3. *Let M be a factor and $\sigma : \Gamma \curvearrowright M$ an action of a discrete group Γ. Suppose that M is not full and that Γ is amenable. Then we can find a nontrivial bounded centralizing net $(x_i)_{i \in I}$ in M such that $\sigma_g(x_i) - x_i \to 0$ strongly for all $g \in \Gamma$.*

Lemma 15.4. *Let M be a II_1 factor and let $\sigma : \Gamma \curvearrowright M$ be an action of a discrete group Γ. Suppose that every bounded central net $(x_i)_{i \in I}$ in M which satisfies $\lim_i \|\sigma_g(x_i) - x_i\|_2 = 0$ for all $g \in \Gamma$ is trivial. Then there exists a finite set of unitaries $S \subset \mathcal{U}(M)$, a finite set $K \subset \Gamma$, and a constant $\kappa > 0$ such that for all $x \in M$ we have*

$$\|x - \tau(x)\|_2 \leq \kappa \left(\sum_{u \in S} \|ux - xu\|_2 + \sum_{g \in K} \|\sigma_g(x) - x\|_2 \right)$$

Proof. Let $N = M \rtimes_\sigma \Gamma$. Let $\Sigma = \mathcal{U}(M) \cup \{u_g \mid g \in \Gamma\} \subset N$. A direct application of Theorem 12.2 shows that we can find a projection $p \in M$ with $\tau(p) > \frac{1}{2}$, finite sets $S \subset \mathcal{U}(M)$ and $K \subset \Gamma$, and a constant $\kappa > 0$ such that for all $x \in pMp$ with $\tau(x) = 0$ we have

$$\|x\|_2 \leq \kappa \left(\sum_{u \in S} \|p(ux - xu)p\|_2 + \sum_{g \in K} \|p(u_g x - x u_g)p\|_2 \right)$$

Let $v = 2p - 1 \in \mathcal{U}(M)$ and let $w \in \mathcal{U}(M)$ be any unitary which satisfies $w(1-p)w^* \leq p$. Let $S' = S \cup \{v, w\}$. Then it is not hard to check that there exists some constant $\kappa' > 0$ such that for all $x \in M$ we have

$$\|x - \tau(x)\|_2 \leq \kappa' \left(\sum_{u \in S'} \|ux - xu\|_2 + \sum_{g \in K} \|\sigma_g(x) - x\|_2 \right)$$

□

Proof of Theorem 15.3. First we deal with the case where M is a II_1 factor. Since M is not full, we can find a net $(p_i)_{i \in I}$ of nonzero projections $p_i \in M$ such that $\lim_i p_i = 0$ and $\lim_i \frac{1}{\tau(p_i)} \|up_i - p_i u\|_1 = 0$ for all $u \in \mathcal{U}(M)$. Fix $\varepsilon > 0$ and $K \subset \Gamma$ a finite subset. Then, since Γ is amenable, we can find $F \subset \Gamma$ such that $|gF \triangle F| < \varepsilon |F|$ for all $g \in K$. Let

$$x_i = \left(\frac{1}{|F|\tau(p_i)} \sum_{g \in F} \sigma_g(p_i) \right)^{1/2} \in M$$

Note that $\|x_i\|_2 = 1$ for all i and $\lim_i \tau(x_i) = 0$. We also have

$$\|ux_iu^* - x_i\|_2^2 \leq \|ux_i^2u^* - x_i^2\|_1 \leq \frac{1}{|F|\tau(p_i)} \sum_{g \in F} \|\sigma_g^{-1}(u)p_i - p_i\sigma_g^{-1}(u)\|_1 \to 0$$

Moreover, for all $g \in K$, we have

$$\|\sigma_g(x_i) - x_i\|_2^2 \leq \|\sigma_g(x_i^2) - x_i^2\|_1 \leq \frac{|gF \triangle F|}{|F|} \leq \varepsilon$$

This contradicts the conclusion of Lemma 15.4. Hence, the theorem is proved when M is a II_1 factor.

Now, we deal with the general case. Since M is not full, we can find a cofinal ultrafilter ω on some directed set I such that M_ω is diffuse. Let τ be the canonical trace of M_ω and let $\sigma_\omega : \Gamma \curvearrowright M_\omega$ be the trace preserving action of Γ induced by σ. To prove the theorem, it is enough to construct a nontrivial bounded asymptotically Γ-invariant net in M_ω. If the restricted probability measure preserving action $\mathcal{Z}(\sigma_\omega) : \Gamma \curvearrowright \mathcal{Z}(M_\omega)$ is not strongly ergodic, then we are done. On the other hand, since Γ is amenable, if the action $\mathcal{Z}(\sigma_\omega)$ is strongly ergodic, then it must be conjugate to a transitive action of Γ on some finite set. Let e be a minimal projection in $\mathcal{Z}(\sigma_\omega)$ and consider the restricted action $\pi : H \curvearrowright e(M_\omega)e$ where $H < \Gamma$ is the stablizer of e (which is of finite index in Γ). Since $e(M_\omega)e$ is a II_1 factor, the first step of the proof applies and we can find a nontrivial bounded net $(x_j)_{j \in J}$ in $e(M_\omega)e$ which is asympotically H-invariant. Now, take a finite set $F \subset \Gamma$ such that the cosets gH, $g \in F$ form a partition of Γ. Then $\sigma_g(e), g \in F$ is partition of unity in $\mathcal{Z}(M_\omega)$. Define $y_j = \sum_{g \in F} \sigma_g(x_j)$ for all $j \in J$. One checks easily that $(y_j)_{j \in J}$ is a nontrivial bounded asymptotically Γ-invariant net in M_ω. □

Theorem 15.5. *Let M be a factor and let $\sigma : \Gamma \curvearrowright M$ be an outer action of a discrete group Γ. Assume that Γ is abelian. Then $M \rtimes_\sigma \Gamma$ is full if and only if M is full and σ is fully outer.*

Proof. The if direction is precisely Theorem 15.2. Let us prove the other direction. Since Γ is abelian hence amenable, it follows from Theorem 15.3 that M must be full. Let us show that σ is fully outer. Assume this is not the case. Then there exists a net $(\gamma_i)_{i\in I}$ in $\Gamma \setminus \{1\}$ such that $\pi_M(\sigma_{\gamma_i}) \to 1$ in $\mathrm{Out}(M)$ as $i \to \infty$. Then there exists a net of unitaries $(u_i)_{i\in I}$ in $\mathcal{U}(M)$ such that $\mathrm{Ad}(u_i) \circ \sigma_{\gamma_i} \to \mathrm{id}_M$ in $\mathrm{Aut}(M)$ as $i \to \infty$. Since Γ is abelian, for every $\gamma \in \Gamma$, we have still have $\mathrm{Ad}(\sigma_\gamma(u_i)) \circ \sigma_{\gamma_i} \to \mathrm{id}_M$ in $\mathrm{Aut}(M)$ as $i \to \infty$. This further implies that for every $\gamma \in \Gamma$, we have $\mathrm{Ad}(u_i\sigma_\gamma(u_i^*)) \to \mathrm{id}_M$ in $\mathrm{Aut}(M)$ as $i \to \infty$.

Fix a cofinal ultrafilter ω on I. Since M is full and hence $M_\omega = \mathbb{C}$, there exists a mapping $g : \Gamma \to \mathbf{T}$ such that for every $\gamma \in \Gamma$, we have that $u_i\sigma_\gamma(u_i^*) \to g(\gamma)$ strongly as $i \to \omega$. It is easy to see that $g : \Gamma \to \mathbf{T}$ is a group homomorphism which means exactly that g is an element of the dual compact group $G = \widehat{\Gamma}$.

Let $N = M \rtimes_\sigma \Gamma$. For every $\gamma \in \Gamma$, denote by v_γ the implementing unitary in N. We denote by $\widehat{\sigma} : G \curvearrowright N$ the dual action of σ characterized by $\widehat{\sigma}_g(x) = x$ for all $x \in M$ and $\widehat{\sigma}_g(v_\gamma) = \langle g, \gamma\rangle v_\gamma$ for all $\gamma \in \Gamma$. Let us show that $g = 1_G$ and $\mathrm{Ad}(u_i v_{\gamma_i}) \to \mathrm{id}_N$ in $\mathrm{Aut}(N)$ as $i \to \omega$.

Firstly, we have that $\mathrm{Ad}(u_i)(v_\gamma) = u_i v_\gamma u_i^* \to \langle g, \gamma\rangle\, v_\gamma = \widehat{\sigma}_g(v_\gamma)$ strongly as $i \to \omega$ for every $\gamma \in \Gamma$. Since Γ is abelian, this further implies that $\mathrm{Ad}(u_i v_{\gamma_i})(v_\gamma) \to \widehat{\sigma}_g(v_\gamma)$ strongly as $i \to \omega$ for every $\gamma \in \Gamma$. Since $\mathrm{Ad}(u_i) \circ \sigma_{\gamma_i} \to \mathrm{id}_N$ in $\mathrm{Aut}(M)$ as $i \to \infty$, we have that $\mathrm{Ad}(u_i v_{\gamma_i})(x) \to x$ strongly as $i \to \infty$ for every $x \in N$. This altogether implies that $\mathrm{Ad}(u_i v_{\gamma_i}) \to \widehat{\sigma}_g$ in $\mathrm{Aut}(N)$ as $i \to \infty$. Since N is full, $\overline{\mathrm{Inn}}(N)$ is closed in $\mathrm{Aut}(N)$ and this implies that there exists $u \in \mathcal{U}(N)$ such that $\widehat{\sigma}_g = \mathrm{Ad}(u)$. For every $x \in N$, since $\widehat{\sigma}_g(x) = x$, we have $uxu^* = x$ and hence $u \in M' \cap N$. But $M' \cap N = \mathbb{C}$ because σ is outer. Thus, $u \in \mathbf{T}$ and hence $\widehat{\theta}_g = \mathrm{id}_N$ which implies that $g = 1_G$. We finally have that $\mathrm{Ad}(u_i v_{\gamma_i}) \to \mathrm{id}_N$ in $\mathrm{Aut}(N)$ as $i \to \omega$. Hence $(u_i v_{\gamma_i})^\omega \in N_\omega$. Moreover $\mathrm{E}_M(u_i v_{\gamma_i}) = 0$ for all $i \in I$ because $\gamma_i \in \Gamma \setminus \{1\}$. This shows that $N_\omega \neq \mathbb{C}$ contradicting the fullness of N. □

Corollary 15.6 ([Ma16b])**.** *Let M be a factor of type III_1. Then $c(M)$ is full if and only if M is full and $\tau(M)$ is the usual topology on $\mathbb{R}$.*

Proof. First note that an action of $\mathbb{R}$ on a full factor M is fully outer if and only if its restriction to $\mathbb{Z}$ is fully outer. Let $\theta : \mathbb{R}^*_+ \curvearrowright c(M)$ be the trace scaling action. Let ψ be any faithful normal weight on M and let $\sigma^\psi : \mathbb{R} \curvearrowright M$ be its modular flow. Fix $\lambda > 0$ and let $T = \frac{2\pi}{\log(\lambda)}$. It follows from Takesaki's duality that $M \rtimes_{\sigma^\psi_T} \mathbb{Z}$ is isomorphic to $c(M) \rtimes_{\theta_\lambda} \mathbb{Z}$. By applying Theorem 15.5, we know that $c(M) \rtimes_{\theta_\lambda} \mathbb{Z}$ is full if and only if $c(M)$ is full and θ is fully outer, while $M \rtimes_{\sigma^\psi_T} \mathbb{Z}$ is full if and only if M is full and σ^ψ is fully outer. But θ is always fully outer and σ^ψ is fully outer precisely when $\tau(M)$ is the usual topology. Hence, we conclude that $c(M)$ is full if and only if M is full and $\tau(M)$ is the usual topology. □

16 Unique McDuff decomposition

In this section, we give an application of Theorem 13.4 to obtain a *unique McDuff decomposition* result.

Following [McD69, Co75a], we say that a factor $\mathcal{M}$ with separable predual is *McDuff* if it absorbs tensorially the hyperfinite type II_1 factor R, that is, $\mathcal{M} \cong \mathcal{M} \mathbin{\overline{\otimes}} R$. We introduce the following terminology.

Definition 16.1. Let $\mathcal{M}$ be any McDuff factor with separable predual.

(i) We say that $\mathcal{M}$ admits a *McDuff decomposition* if there exist a non-McDuff factor M and a non-type I amenable factor P such that $\mathcal{M} = M \mathbin{\overline{\otimes}} P$.

(i) We say that $\mathcal{M}$ has a *unique* McDuff decomposition if the above decomposition $\mathcal{M} = M \mathbin{\overline{\otimes}} P$ is unique up to *stable unitary conjugacy* (see Definition 26.1).

Note that there exists McDuff factors with separable predual which do not admit any McDuff decomposition as Corollary 20.7 shows.

The class of McDuff factors with separable predual that admit a unique McDuff decomposition is well understood in the type II_1 case. Indeed, by a theorem of Popa (see [Po06, Theorem 5.1]), if M is a full factor of type II_1 with separable predual and R is the hyperfinite type II_1 factor, then the tensor product factor $M \mathbin{\overline{\otimes}} R$ has a unique McDuff decomposition. Conversely, by a theorem of Hoff (see [Ho15, Theorem B]), if $\mathcal{M}$ is a McDuff factor of type II_1 with separable predual and with a unique McDuff decomposition $\mathcal{M} = M \mathbin{\overline{\otimes}} R$, then its non-McDuff part M must be full.

We extend this characterization of McDuff factors with a unique McDuff decomposition to type III factors.

Theorem 16.2 ([HMV16]). *Let $\mathcal{M}$ be any McDuff factor with separable predual. The following conditions are equivalent:*

(i) *$\mathcal{M} = M \mathbin{\overline{\otimes}} P$, where M is a full factor and P is a non-type* I *amenable factor.*

(ii) *$\mathcal{M}$ has a unique McDuff decomposition.*

Proposition 16.3. *Let $\mathcal{M}$ be any McDuff factor with separable predual and with two McDuff decompositions $\mathcal{M} = M_1 \mathbin{\overline{\otimes}} P_1 = M_2 \mathbin{\overline{\otimes}} P_2$. The following conditions are equivalent:*

(i) *$M_1 \prec_{\mathcal{M}} M_2$.*

(ii) *(M_1, P_1) and (M_2, P_2) are stably unitarily conjugate.*

Proof. (i) $\Rightarrow$ (ii) By Proposition 26.3 (v), we know that there exists a tensor product decomposition $M_2 = C \mathbin{\overline{\otimes}} D$ such that $(M_1, P_1) \sim_\infty (C, D \mathbin{\overline{\otimes}} P_2)$. Hence P_1 is stably isomorphic to $D \mathbin{\overline{\otimes}} P_2$ and therefore D is amenable. But $M_2 = C \mathbin{\overline{\otimes}} D$ is not McDuff, hence D must be type I. Therefore, by Proposition 26.3 (ii), we have $(C, D \mathbin{\overline{\otimes}} P_2) \sim_\infty (C \mathbin{\overline{\otimes}} D, P_2) = (M_2, P_2)$. Finally, by Proposition 26.3 (i), we conclude that $(M_1, P_1) \sim_\infty (M_2, P_2)$.

(ii) $\Rightarrow$ (i) follows from Proposition 26.3 (v). $\square$

Proof of Theorem 16.2. (i) $\Rightarrow$ (ii) Assume that $\mathcal{M}$ has a McDuff decomposition $\mathcal{M} = M \mathbin{\overline{\otimes}} P$ where M is a full factor and P is a non-type I amenable factor. Fix a faithful normal conditional expectation $\mathrm{E}_P : \mathcal{M} \to P$. Let $\mathcal{M} = M_0 \mathbin{\overline{\otimes}} P_0$ be another McDuff decomposition. We want to show that these two decompositions are stably unitarily conjugate. By Proposition 16.3, it suffices to show that $M \prec_{\mathcal{M}} M_0$.

Assume first that P_0 is of type III_1. Then $P_0 \cong R_\infty$ is the unique Araki–Woods factor of type III_1 [Co75b, Co85, Ha85] and hence we can write P_0 as an infinite tensor product

$$(P_0, \varphi) = \overline{\bigotimes}_{n\in\mathbb{N}}(\mathbf{M}_2(\mathbb{C}) \otimes \mathbf{M}_2(\mathbb{C}), \omega_{\lambda_1} \otimes \omega_{\lambda_2}) \tag{16.1}$$

where $0 < \lambda_1, \lambda_2 < 1$, $\frac{\log \lambda_1}{\log \lambda_2} \notin \mathbb{Q}$ and $\omega_{\lambda_i} = \frac{1}{1+\lambda_i}\mathrm{Tr}_{\mathbf{M}_2(\mathbb{C})}(\,\cdot\,\mathrm{diag}(1, \lambda_i))$ for every $i \in \{1, 2\}$. Then $\varphi \in (P_0)_*$ is an almost periodic faithful state satisfying $((P_0)_\varphi)' \cap P_0 = \mathbb{C}1$. Indeed, (16.1) implies that $(P_0, \varphi) \cong (R_{\lambda_1}, \varphi_{\lambda_1}) \mathbin{\overline{\otimes}} (R_{\lambda_2}, \varphi_{\lambda_2})$ where R_{λ_i} is the Powers factor of type III_{λ_i} for every $i \in \{1, 2\}$. Since $((R_{\lambda_i})_{\varphi_{\lambda_i}})' \cap R_{\lambda_i} = \mathbb{C}1$ for every $i \in \{1, 2\}$ and since $(R_{\lambda_1})_{\varphi_{\lambda_1}} \mathbin{\overline{\otimes}} (R_{\lambda_2})_{\varphi_{\lambda_2}} \subset (P_0)_\varphi$, we indeed have $((P_0)_\varphi)' \cap P_0 = \mathbb{C}1$ (see also [Ha85, Example 1.6]). For every $n \in \mathbb{N}$, define the finite dimensional unital $\ast$-subalgebra $Q_n \subset R_\infty$ by $Q_n = \overline{\bigotimes}_{k=0}^{n}(\mathbf{M}_2(\mathbb{C}) \otimes \mathbf{M}_2(\mathbb{C}), \omega_{\lambda_1} \otimes \omega_{\lambda_2})$. We show that there exists $n \in \mathbb{N}$ such that $(Q_n' \cap P_0)_\varphi \prec_{\mathcal{M}} P$.

Assume by contradiction that $(Q_n' \cap P_0)_\varphi \nprec_{\mathcal{M}} P$ for every $n \in \mathbb{N}$. By [HI15, Theorem 4.3 (5)], we can find a sequence $u_n \in \mathcal{U}((Q_n' \cap P_0)_\varphi)$ such that $\mathrm{E}_P(u_n) \to 0$ strongly as $n \to \infty$. Fix a nonprincipal ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$. By construction, we have $(u_n)^\omega \in \mathcal{M}_\omega$. Hence we must have $(u_n)^\omega = (\mathrm{E}_P(u_n))^\omega$ by Theorem 13.4. This contradicts the fact that $(\mathrm{E}_P(u_n))^\omega = 0$. From this contradiction, we deduce that there exists $n \in \mathbb{N}$ such that $(Q_n' \cap P_0)_\varphi \prec_{\mathcal{M}} P$. Since $(Q_n' \cap P_0, \varphi|_{Q_n' \cap P_0}) \cong (P_0, \varphi)$, we have $((Q_n' \cap P_0)_\varphi)' \cap \mathcal{M} = M_0 \mathbin{\overline{\otimes}} Q_n$. Then [HI15, Lemma 4.9] implies that $M \prec_{\mathcal{M}} M_0 \mathbin{\overline{\otimes}} Q_n$. Since Q_n is a type I factor, [HI15, Remark 4.2 (2) and Remark 4.5] imply that $M \prec_{\mathcal{M}} M_0$. Assume now that P_0 is any non-type I amenable factor. Then we can consider $\mathcal{M} \mathbin{\overline{\otimes}} R_\infty = M \mathbin{\overline{\otimes}} (P \mathbin{\overline{\otimes}} R_\infty) = M_0 \mathbin{\overline{\otimes}} (P_0 \mathbin{\overline{\otimes}} R_\infty)$. Since $P_0 \mathbin{\overline{\otimes}} R_\infty$ is of type III_1 by [CT76, Corollary 6.8] and amenable, we can apply the result obtained in the previous paragraph and we obtain $M \otimes \mathbb{C}1_{R_\infty} \prec_{\mathcal{M} \overline{\otimes} R_\infty} M_0 \otimes \mathbb{C}1_{R_\infty}$. Hence, by Lemma 26.2, we finally obtain $M \prec_{\mathcal{M}} M_0$.

(ii) $\Rightarrow$ (i) Assume that $\mathcal{M}$ is a McDuff factor with separable predual and with a McDuff decomposition $\mathcal{M} = M \mathbin{\overline{\otimes}} P$ where M is not McDuff and P is any non-type I amenable factor. Assuming that M is not full and following the proof of [Ho15, Theorem B], we show the existence of $\Psi \in \mathrm{Aut}(\mathcal{M})$ such that $\Psi(P) \nprec_{\mathcal{M}} P$. By Proposition 16.3 and Proposition 26.3, this will prove that $\mathcal{M}$ has two McDuff decompositions that are not stably unitarily conjugate.

Since M is not full, [HU15, Theorem 3.1] shows there exist a diffuse abelian subalgebra $B \subset M$ with faithful normal conditional expectation $\mathrm{E}_B : M \to B$ and an M-central sequence of mutually commuting projections $(q_n)_{n\in\mathbb{N}}$ in B such that $q_n \to \frac{1}{2}$ weakly as $n \to \infty$. Choose a faithful state $\phi \in M_*$ such that $\phi \circ \mathrm{E}_B = \phi$. Since P is a non-type I amenable factor, P is McDuff (by combining results in [Co72, Co75b, CT76, Co85, Ha85]) and we may write $P = P_0 \mathbin{\overline{\otimes}} R$ where R is the hyperfinite type II_1 factor and $P_0 \cong P$. Observe that $\mathcal{M} = M \mathbin{\overline{\otimes}} P_0 \mathbin{\overline{\otimes}} R$. Choose any faithful state $\phi_0 \in (P_0)_*$ and put $\varphi = \phi \otimes \phi_0 \otimes \tau \in \mathcal{M}_*$.

Write $(R, \tau) = \overline{\bigotimes}_{\mathbb{N}}(\mathbf{M}_2(\mathbb{C}), \mathrm{Tr}_{\mathbf{M}_2(\mathbb{C})})$. Let $\pi_n : \mathbf{M}_2(\mathbb{C}) \to R$ be the trace preserving embedding at the n-th position and put

$$p_n = \pi_n\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \qquad v_n = \pi_n\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right), \qquad w_n = 1 - 2p_n q_n.$$

By construction, $(w_n)_{n\in\mathbb{N}}$ is a sequence of mutually commuting unitaries in $\mathcal{U}(\mathcal{M}_\varphi)$. Moreover, since $w_n^* = w_n$ for every $n \in \mathbb{N}$ and since $(w_n)_{n\in\mathbb{N}}$ is a centralizing sequence, we have that $\mathrm{Ad}(w_n) \to \mathrm{id}_{\mathcal{M}}$ in $\mathrm{Aut}(\mathcal{M})$ as $n \to \infty$. Since $\mathcal{M}$ has separable predual, $\operatorname{Aut}(\mathcal{M})$ is a Polish group. Fix a complete metric d on $\operatorname{Aut}(\mathcal{M})$ compatible with the u-topology. We may choose inductively a subsequence $(w_{n_k})_{k\in\mathbb{N}}$ such that $w_{n_0} = w_0$ and

$$\mathrm{d}(\operatorname{Ad}(w_{n_0}\cdots w_{n_k}), \operatorname{Ad}(w_{n_0}\cdots w_{n_{k+1}})) < 2^{-(k+1)}$$

for every $k \in \mathbb{N}$. Put $W_k = w_{n_0}\cdots w_{n_k} \in \mathcal{U}(\mathcal{M}_\varphi)$ for every $k \in \mathbb{N}$. The triangle inequality yields $\mathrm{d}(\operatorname{Ad}(W_p), \operatorname{Ad}(W_q)) < 2^{-p}$ for all $q \geq p$ and hence $(\operatorname{Ad}(W_p))_{p\in\mathbb{N}}$ is a Cauchy sequence in $\operatorname{Aut}(\mathcal{M})$. Since the metric d is complete on $\operatorname{Aut}(\mathcal{M})$, the sequence $(\operatorname{Ad}(W_p))_{p\in\mathbb{N}}$ converges to $\Psi \in \operatorname{Aut}(\mathcal{M})$ with respect to the u-topology.

Observe that we now have a new McDuff decomposition $\mathcal{M} = \Psi(\mathcal{M}) = \Psi(M)\mathbin{\overline{\otimes}} \Psi(P)$. It remains to prove that $\Psi(P) \not\prec_{\mathcal{M}} P$. Note that $W_\ell v_{n_k} W_\ell = (1-2p_{n_k}q_{n_k})v_{n_k}(1-2p_{n_k}q_{n_k}) = (1-2q_{n_k})v_{n_k}$ for all $\ell \geq k$ and hence $\Psi(v_{n_k}) = (1-2q_{n_k})v_{n_k}$ for every $k \in \mathbb{N}$. In particular, for every $x \in M$, we have

$$\begin{aligned}
\|\mathrm{E}_P(x\Psi(v_{n_k}))\|_\varphi &= \|\mathrm{E}_P(x(1-2q_{n_k})v_{n_k})\|_\varphi \\
&= \|\mathrm{E}_P(x(1-2q_{n_k}))v_{n_k}\|_\varphi \quad (\text{since } v_{n_k} \in \mathcal{U}(P)) \\
&= \|\mathrm{E}_P(x(1-2q_{n_k}))\|_\varphi \qquad (\text{since } v_{n_k} \in \mathcal{U}(\mathcal{M}_\varphi)) \\
&= |\varphi(x(1-2q_{n_k}))|.
\end{aligned}$$

Since $q_{n_k} \to \frac{1}{2}$ weakly as $k \to \infty$, we have that $\lim_k \varphi(xq_{n_k}) = \frac{1}{2}\varphi(x)$ and hence

$$\lim_{k\to\infty} \|\mathrm{E}_P(x\Psi(v_{n_k}))\|_\varphi = \lim_{k\to\infty} |\varphi(x) - 2\varphi(xq_{n_k}))| = 0.$$

Since $(\Psi(v_{n_k}))_{k\in\mathbb{N}}$ is moreover a centralizing sequence in $\mathcal{M}$, we obtain that

$$\lim_k \|\mathrm{E}_P(x^*\Psi(v_{n_k})y)\|_\varphi = \lim_k \|\mathrm{E}_P(x^*y\Psi(v_{n_k}))\|_\varphi = 0$$

for all $x, y \in \operatorname{span}(M \cdot P)$. By Theorem 23.11, we obtain that $\Psi(R) \not\prec_{\mathcal{M}} P$. Finally, since $\Psi(R) \subset \Psi(P)$, we conclude that $\Psi(P) \not\prec_{\mathcal{M}} P$. □

Part IV

Stable equivalence relations and McDuff factors

This chapter is based on the article [Ma17b]. It is motivated by the following conjecture.

Conjecture 16.1. *Let M and N be type II_1 factors with separable predual. Suppose that $M \mathbin{\overline{\otimes}} N \cong M \mathbin{\overline{\otimes}} N \mathbin{\overline{\otimes}} R$ where R is the hyperfinite type II_1 factor. Then we have $M \cong M \mathbin{\overline{\otimes}} R$ or $N \cong N \mathbin{\overline{\otimes}} R$.*

The type II_1 factors with separable predual satisfying $M \cong M \mathbin{\overline{\otimes}} R$ are called *McDuff* factors [McD69]. By analogy, Jones and Schmidt called an ergodic type II_1 equivalence relation $\mathcal{R}$ *stable* if it is isomorphic to $\mathcal{R}\otimes\mathcal{R}_0$ where $\mathcal{R}_0$ is the hyperfinite ergodic type II_1 equivalence relation [JS87].

Our main results are new "spectral gap like" characterizations of McDuff factors and stable equivalence relations. As an application, we solve the analog of conjecture 16.1 for equivalence relations. We also give some partial results in the von Neumann algebraic case.

17 A local characterization of stable equivalence relations

The notion of stable equivalence relation was introduced and studied by Jones and Schmidt in [JS87] by analogy with McDuff's work [McD69]. In particular, they obtained the following characterization.

Theorem 17.1 ([JS87]). *Let $\mathcal{R}$ be an ergodic measure preserving equivalence relation on a probability space (X, μ). Then $\mathcal{R}$ is stable if and only if for every finite set $K \subset [[\mathcal{R}]]$ and every $\varepsilon > 0$, there exists $v \in [[\mathcal{R}]]$ such that*

$$v^2 = 0,$$

$$vv^* + v^*v = 1,$$

$$\forall u \in K, \quad \|vu - uv\|_2 < \varepsilon.$$

The conditions $v^2 = 0$ and $vv^* + v^*v = 1$ just mean that v generates a copy of the unique transitive equivalence relation $\mathcal{S}_2$ on $\{0, 1\}$. The condition $\forall u \in K, \quad \|vu - uv\|_2 < \varepsilon$ means that this copy must be almost central.

The main theorem of this section strengthens this characterization by removing the condition $vv^* + v^*v = 1$, thus allowing v to be arbitrarily small. The proof is based on a maximality argument inspired by [Co75b, Theorem 2.1] and [Co85, Theorem 2].

Theorem 17.2 ([Ma17b]). *Let $\mathcal{R}$ be an ergodic measure preserving equivalence relation on a probability space (X, μ). Then $\mathcal{R}$ is stable if and only if for every finite set $K \subset [[\mathcal{R}]]$ and every $\varepsilon > 0$, there exists $v \in [[\mathcal{R}]]$ such that*

$$v^2 = 0,$$

$$\forall u \in K, \quad \|vu - uv\|_2 < \varepsilon \|v\|_2.$$

Proof. Suppose that $\mathcal{R}$ satisfies this condition. First, we claim that every corner of $\mathcal{R}$ also satisfies this condition. Suppose this were not the case. Then we can find a finite set $K \subset p[[\mathcal{R}]]p$ and a constant $\kappa > 0$ such that for all $v \in p[[\mathcal{R}]]p$ with $v^2 = 0$, we have

$$\|v\|_2^2 \leq \kappa \sum_{u \in K} \|vu - uv\|_2^2.$$

Since $\mathcal{R}$ is ergodic, we can find a finite set $S \subset [[\mathcal{R}]]$ such that $\sum_{w \in S} w^*w = p^\perp$ and $ww^* \leq p$ for all $w \in S$. Then, for every $v \in [[\mathcal{R}]]$, we have

$$\|v\|_2^2 = \|pv\|_2^2 + \sum_{w \in S} \|wv\|_2^2$$

and for all $w \in S$, we have

$$\|wv\|_2^2 2(|wv - vw\|_2^2 + \|vw\|_2^2) \leq 2(\|wv - vw\|_2^2 + \|vp\|_2^2).$$

Hence, we obtain

$$\|v\|_2^2 \leq 2|S|(\|vp\|_2^2 + \|pv\|_2^2) + 2 \sum_{w \in S} \|wv - vw\|_2^2.$$

Moreover, we have

$$\|pv\|_2^2 + \|vp\|_2^2 = \|pv - vp\|_2^2 + 2\|pvp\|_2^2,$$

hence

$$\|v\|_2^2 \leq 2|S|\|pv - vp\|_2^2 + 4|S|\|pvp\|_2^2 + \sum_{w \in S} \|wv - vw\|_2^2.$$

Since $v^2 = 0$, we also have $(pvp)^2 = 0$. Thus, by assumption, we have

$$\|pvp\|_2^2 \leq \kappa \sum_{u \in K} \|(pvp)u - u(pvp)\|_2^2 \leq \kappa \sum_{u \in K} \|vu - uv\|_2^2.$$

Therefore, we finally obtain

$$\|v\|_2^2 \leq 2|S|\|pv - vp\|_2^2 + 4|S|\kappa \sum_{u \in K} \|vu - uv\|_2^2 + 2 \sum_{w \in S} \|wv - vw\|_2^2.$$

This shows that $\mathcal{R}$ does not satisfy the condition of the theorem. Hence the claim is proved.

Now, we use a maximality argument to show that our condition implies the condition of Theorem 17.1. Let $u_1, \ldots, u_n \in [[\mathcal{R}]]$ be a finite family and let $\varepsilon > 0$ and $\delta = 4\varepsilon$. Consider the set Λ of all $(v, U_1, \ldots, U_n) \in [[\mathcal{R}]]^{n+1}$ such that

— $v^2 = 0$.

— $[U_k, vv^* + v^*v] = 0$ for all $k = 1, \ldots, n$.

— $\|vU_k - U_kv\|_2 \leq \varepsilon\|v\|_2$ for all $k = 1, \ldots, n$.

— $\|U_k - u_k\|_2 \leq \delta\|v\|_2$.

Recall that $d(v, w) = \|v - w\|_2^2$ is a distance on $[[\mathcal{R}]]$. On Λ put the order relation given by

$$(v, U_1, \ldots, U_n) \leq (v', U_1', \ldots, U_n')$$

if and only if $v \leq v'$ and $\|U_k' - U_k\|_2^2 \leq \delta^2(\|v'\|_2^2 - \|v\|_2^2)$ for all $k = 1, \ldots, n$. Then Λ is an inductive set because of the completeness of $([[\mathcal{R}]], d)$. Let $v \in \Lambda$ be a maximal element. Suppose that $q = vv^* + v^*v \neq 1$. Thanks to the claim we proved, we can find a nonzero element $w \in q^\perp[[\mathcal{R}]]q^\perp$, with $w^2 = 0$ such that

$$\|wU_k - U_kw\|_2 \leq \varepsilon\|w\|_2$$

$$\|wU_k^* - U_k^*w\|_2 \leq \varepsilon\|w\|_2$$

for all $k = 1, \ldots, n$.

Now, let

— $p = ww^* + w^*w$

— $U_k' = pU_kp + p^\perp U_k p^\perp$

— $v' = v + w$

— $q' = v'(v')^* + (v')^*v' = q + p$

Note that $(v')^2 = 0$ and $[U_k', q'] = 0$ for all k. We also have

$$\|v'U_k' - U_k'v'\|_2^2 \leq \|vU_k - U_kv\|_2^2 + \|wU_k - U_kw\|_2^2 \leq \varepsilon^2\|v\|_2^2 + \varepsilon^2\|w\|_2^2 = \varepsilon^2\|v'\|_2^2.$$

Moreover, we have

$$\begin{aligned}\|U_k' - U_k\|_2 &= \|[U_k, p]\|_2 \\ &\le \|[U_k, ww^*]\|_2 + \|[U_k, w^*w]\|_2 \\ &= \|[U_k, w]w^* + w[U_k, w^*]\|_2 + \|[U_k, w^*]w + w^*[U_k, w]\|_2 \\ &\le 2(\|[U_k, w]\|_2 + \|[U_k, w^*]\|_2) \\ &\le 4\varepsilon\|w\|_2\end{aligned}$$

Since $\|v'\|_2^2 = \|v\|_2^2 + \|w\|_2^2$, this implies that

$$\|U_k' - U_k\|_2^2 \le \delta^2(\|v'\|_2^2 - \|v\|_2^2)$$

and using the fact that d is a distance, we also get

$$\|U_k' - u_k\|_2^2 \le \|U_k' - U_k\|_2^2 + \|U_k - u_k\|_2^2 \le \delta^2\|v'\|_2^2.$$

Therefore $v' \in \Lambda$ and $v \le v'$. This contradicts the maximality of v. Hence we must have $v^*v + vv^* = q = 1$. Finally, since

$$\|vu_k - u_kv\|_2 \le \|vU_k - U_kv\|_2 + 2\|U_k - u_k\|_2,$$

$$\|vU_k - U_kv\|_2 \le \varepsilon$$

and

$$\|U_k - u_k\|_2 \le \delta = 4\varepsilon,$$

we conclude that

$$\|vu_k - u_kv\|_2 \le 9\varepsilon.$$

Since such a v exists for every $\varepsilon > 0$ and every finite family $u_1, \dots, u_n$, we have proved that $\mathcal{R}$ satisfies the condition of Theorem 17.1, hence it is stable. $\square$

Theorem 17.2 can be stated in a form which looks like a spectral gap.

Corollary 17.3. *Let $\mathcal{R}$ be an ergodic measure preserving equivalence relation on a probability space (X, μ). Then $\mathcal{R}$ is not stable if and only if there exists a finite set $K \subset [[\mathcal{R}]]$ and a constant $\kappa > 0$ such that*

$$\|vp - pv\|_2 \le \kappa \sum_{u \in K} \|uv - vu\|_2 + \|up - pu\|_2$$

for all $v \in [[\mathcal{R}]]$ and all $p \in \mathfrak{P}(X)$.

Proof. Suppose that there exists a a finite set $K \subset [\mathcal{R}]$ and a constant $\kappa > 0$ such that

$$\|vp - pv\|_2 \le \kappa \sum_{u \in K} \|uv - vu\|_2 + \|up - pu\|_2$$

for all $v \in [[\mathcal{R}]]$ and all $p \in \mathfrak{P}(X)$. Let $v \in [[\mathcal{R}]]$ such that $v^2 = 0$ and take $p = v^*v$. Then we get

$$\|v\|_2 = \|vp - pv\|_2 \le \kappa \sum_{u \in K} \|uv - vu\|_2 + \|up - pu\|_2$$

and since $\|up - pu\|_2 \leq \|vu - uv\|_2 + \|vu^* - u^*v\|_2$ for all $u \in K$, this shows that the criterion of Theorem 17.2 is not satisfied, hence $\mathcal{R}$ is not stable.

Conversely, if $\mathcal{R}$ is not stable then by Theorem 17.2, there exists $\kappa > 0$ and a finite set $K \subset [[\mathcal{R}]]$ such that

$$\|v\|_2 \leq \kappa \sum_{u \in K} \|vu - uv\|_2$$

for all $v \in [[\mathcal{R}]]$ with $v^2 = 0$. Now take any $v \in [[\mathcal{R}]]$ and any $p \in \mathfrak{P}(X)$. Since $(pvp^\perp)^2 = (p^\perp vp)^2 = 0$, then we get

$$\begin{aligned}
\|pv - vp\|_2 &\leq \|pvp^\perp\|_2 + \|p^\perp vp\|_2 \\
&\leq \kappa \sum_{u \in K} \|(pvp^\perp)u - u(pvp^\perp)\|_2 + \|(p^\perp vp)u - u(p^\perp vp)\|_2 \\
&\leq \kappa \sum_{u \in K} \|vu - uv\|_2 + 2\|pu - up\|_2.
\end{aligned}$$

This is what we wanted. □

18 Stable product equivalence relations

The following theorem is the main application of the previous characterizations of stable equivalence relations.

Theorem 18.1. *Let $\mathcal{R}$ and $\mathcal{S}$ be two ergodic p.m.p. equivalence relations. Then the product equivalence relation $\mathcal{R} \otimes \mathcal{S}$ is stable if and only if $\mathcal{R}$ is stable or $\mathcal{S}$ is stable.*

For the proof, we need to explain the *currying* procedure which allows us to decompose any element of $[[\mathcal{R} \otimes \mathcal{S}]]$ as a function from $\mathcal{R}$ to $[[\mathcal{S}]]$.

First, recall that for any projection $p \in \mathfrak{P}(X \otimes Y)$, there exists a unique measurable function $p_X : X \to \mathfrak{P}(Y)$ such that $p(x, y) = (p_X(x))(y)$ for a.e. $(x, y) \in X \otimes Y$. This function p_X is called the *curried form of p over X*.

Now, let $\mathcal{R}$ be a p.m.p equivalence relation on a probability space (X, μ). We denote by $\tilde{\mu}$ the canonical faithful semifinite measure on $\mathcal{R}$ induced by μ. Then $\mathrm{L}^2(\mathcal{R}) = \mathrm{L}^2(\mathcal{R}, \tilde{\mu})$ can be identified with the canonical L^2-space of $\mathcal{L}(\mathcal{R})$. For every $x \in \mathcal{L}(\mathcal{R})$, we denote by $\widehat{x}$ the corresponding vector in $\mathrm{L}^2(\mathcal{R})$. If $v \in [[\mathcal{R}]]$, then $\widehat{v}$ is just the indicator function of the graph of v. Let $\mathcal{S}$ be a second p.m.p. equivalence relation on (Y, ν). Then, for any $v \in [[\mathcal{R} \otimes \mathcal{S}]]$, there exists a unique measurable function $v_\mathcal{R} : \mathcal{R} \to [[\mathcal{S}]]$ which satisfies

$$\widehat{v}(x, x', y, y') = \widehat{v_\mathcal{R}(x, x')}(y, y')$$

for a.e. $(x, x', y, y') \in \mathcal{R} \otimes \mathcal{S}$. The function $v_\mathcal{R}$ is called the *curried form of v over $\mathcal{R}$*.

For all $v, w \in [[\mathcal{R} \otimes \mathcal{S}]]$ we have the following formulas

$$(vw)_\mathcal{R}(x, x'') = \sum_{x' \sim_\mathcal{R} x} v_\mathcal{R}(x, x') w_\mathcal{R}(x', x'')$$

for a.e. $(x, x'') \in \mathcal{R}$.

This applies in particular to any projection $p \in \mathfrak{P}(X \otimes Y) \subset [[\mathcal{R} \otimes \mathcal{S}]]$. In that case, the curried form $p_\mathcal{R}$ of p over $\mathcal{R}$ is supported on the diagonal and is given by $p_\mathcal{R}(x, x) = p_X(x)$ for a.e. $x \in X$.

Proof of Theorem 18.1. Clearly, if $\mathcal{R}$ or $\mathcal{S}$ is stable then $\mathcal{R}\otimes\mathcal{S}$ is also stable. Now, suppose that $\mathcal{R}$ and $\mathcal{S}$ are not stable. Then, by Corollary 17.3, we can find a constant $\kappa_1 > 0$ and a finite set $K_1 \subset [\mathcal{R}]$ such that for all $v \in [[\mathcal{R}]]$ and all $p \in \mathfrak{P}(X)$ we have

$$\|vp - pv\|_2^2 \leq \kappa_1 \sum_{u\in K_1} \|vu - uv\|_2^2 + \|pu - up\|_2^2.$$

Similarly, we can find a constant $\kappa_2 > 0$ and a finite set $K_2 \subset [\mathcal{S}]$ such that for all $v \in [[\mathcal{S}]]$ and all $f \in \mathfrak{P}(Y)$ we have

$$\|vp - pv\|_2^2 \leq \kappa_2 \sum_{u\in K_2} \|vu - uv\|_2^2 + \|pu - up\|_2^2.$$

Now, let $v \in [[\mathcal{R}\otimes\mathcal{S}]]$ and let $p \in \mathfrak{P}(X\otimes Y)$. Let $v_1 : \mathcal{S} \to [[\mathcal{R}]]$ be the curried form of v over $\mathcal{S}$, and $p_1 : Y \to \mathfrak{P}(X)$ be the curried form of p over Y and define $\xi_1 : \mathcal{S} \to \mathcal{L}(\mathcal{R})$ by

$$\xi_1(y, y') = v_1(y, y')p_1(y') - p_1(y')v_1(y, y'), \quad \text{for a.e. } (y, y') \in \mathcal{S}.$$

Similarly, let $v_2 : \mathcal{R} \to [[\mathcal{S}]]$ be the curried form of v over $\mathcal{R}$, and $p_2 : X \to \mathfrak{P}(Y)$ be the curried form of p over X and define $\xi_2 : \mathcal{R} \to \mathcal{L}(\mathcal{S})$ by

$$\xi_2(x, x') = v_2(x, x')p_2(x) - p_2(x)v_2(x, x'), \quad \text{for a.e. } (x, x') \in \mathcal{R}.$$

Then, for a.e. $(x, x', y, y') \in \mathcal{R}\otimes\mathcal{S}$, we have

$$\begin{aligned}
\xi_1\widehat{(y,y')}(x,x') &= \widehat{v}(x,x'y,y')p(x',y') - p(x,y')\widehat{v}(x,x',y,y')\\
\xi_2\widehat{(x,x')}(y,y') &= \widehat{v}(x,x'y,y')p(x,y') - p(x,y)\widehat{v}(x,x',y,y')\\
\widehat{[v,p]}(x,x',y,y') &= \widehat{v}(x,x'y,y')p(x',y') - p(x,y)\widehat{v}(x,x',y,y')\\
&= \xi_1\widehat{(y,y')}(x,x') + \xi_2\widehat{(x,x')}(y,y')
\end{aligned}$$

hence we obtain

$$\|vp - pv\|_2 \leq \left(\int_{\mathcal{S}} \|\xi_1(y,y')\|_2^2 \,\mathrm{d}\tilde{\nu}(y,y')\right)^{1/2} + \left(\int_{\mathcal{R}} \|\xi_2(x,x')\|_2^2 \,\mathrm{d}\tilde{\mu}(x,x')\right)^{1/2}.$$

But, we also have

$$\begin{aligned}
\int_{\mathcal{S}} \|\xi_1(y,y')\|_2^2 \,\mathrm{d}\tilde{\nu}(y,y') &= \int_{\mathcal{S}} \|v_1(y,y')p_1(y') - p_1(y')v_1(y,y')\|_2^2 \,\mathrm{d}\tilde{\nu}(y,y')\\
&\leq \kappa_1 \sum_{u\in K_1} \int_{\mathcal{S}} \|v_1(y,y')u - uv_1(y,y')\|_2^2 + \|up_1(y') - p_1(y')u\|_2^2 \,\mathrm{d}\tilde{\nu}(y,y')\\
&= \kappa_1 \sum_{u\in K_1} \|v(u\otimes 1) - (u\otimes 1)v\|_2^2 + \|p(u\otimes 1) - (u\otimes 1)p\|_2^2,
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_{\mathcal{R}} \|\xi_2(x,x')\|_2^2 \,\mathrm{d}\tilde{\mu}(x,x') &= \int_{\mathcal{R}} \|v_2(x,x')p_2(x) - p_2(x)v_2(x,x')\|_2^2 \,\mathrm{d}\tilde{\mu}(x,x')\\
&\leq \kappa_2 \sum_{u\in K_2} \int_{\mathcal{R}} \|v_2(x,x')u - uv_2(x,x')\|_2^2 + \|up_2(x) - p_2(x)u\|_2^2 \,\mathrm{d}\tilde{\mu}(x,x')\\
&= \kappa_2 \sum_{u\in K_2} \|v(1\otimes u) - (1\otimes u)v\|_2^2 + \|p(1\otimes u) - (1\otimes u)p\|_2^2.
\end{aligned}$$

Therefore, we obtain

$$\|vp - pv\|_2^2 \leq \kappa \sum_{u \in K} \|vu - uv\|_2^2 + \|pu - up\|_2^2$$

with $\kappa = 2(\kappa_1 + \kappa_2)$ and $K = (K_1 \otimes 1) \cup (1 \otimes K_2)$. By Corollary 17.3, we conclude that $\mathcal{R} \otimes \mathcal{S}$ is not stable. $\square$

19 A local characterization of McDuff factors

In this section, we establish the following analog of Theorem 17.2. However, the proof is more involved in this context. Even if one is only interested in item (iii), one still needs first to prove that it is equivalent to (iv)$'$.

Theorem 19.1. *Let M be a factor of type II_1 with separable predual. Then the following are equivalent:*

(i) *M is McDuff.*

(ii) *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists a partial isometry $v \in M$ such that*

$$vv^* + v^*v = 1,$$
$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon.$$

(iii) *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists a partial isometry $v \in M$ such that*

$$v^2 = 0,$$
$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon \|v\|_2.$$

(iii)$'$ *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists $x \in M$ such that*

$$x^2 = 0,$$

$$\forall a \in F, \quad \|xa - ax\|_2 < \varepsilon \|x\|_2.$$

(iv) *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists a partial isometry $v \in M$ such that*

$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon \|vv^* - v^*v\|_2.$$

(iv)$'$ *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists $x \in M$ such that*

$$\forall a \in F, \quad \|x\|_2 \cdot \|xa - ax\|_2 < \varepsilon \||x| - |x^*|\|_2^2.$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is proved in [McD69]. Our goal is to show that (iii) $\Rightarrow$ (ii) but first, we will show that (iii) $\Leftrightarrow$ (iii)$'$ $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iv)$'$. For this, we will prove the following implications (iii) $\Rightarrow$ (iv)$'$ $\Rightarrow$ (iv) $\Rightarrow$ (iii)$'$ $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv)$'$. If v satisfies (iii) then $x = v$ also satisfies (iv)$'$ since $\||x| - |x^*|\|_2 = \sqrt{2}\|x\|_2$.

(iv)$'$ $\Rightarrow$ (iv). Suppose, by contradiction, that there exists a finite set $F \subset M$ and a constant $\kappa > 0$ such that for all partial isometries $v \in M$ we have

$$\|vv^* - v^*v\|_2^2 \leq \kappa \sum_{a \in F} \|va - av\|_2^2.$$

Let $x \in M$. Then, the above inequality applied to $v = u_t(x)$ yields

$$\|u_t(|x^*|) - u_t(|x|)\|_2^2 \leq \kappa \sum_{a \in F} \|u_t(x)a - au_t(x)\|_2^2$$

for all $t \geq 0$. By Lemma 13.5, we then have

$$\frac{1}{2}\||x^*|-|x|\|_2^2 \leq \int_0^\infty \|u_{t^{1/2}}(|x^*|)-u_{t^{1/2}}(|x|)\|_2^2 \,\mathrm{d}t \leq \kappa \sum_{a \in F} \int_0^\infty \|u_{t^{1/2}}(x)a-au_{t^{1/2}}(x)\|_2^2 \,\mathrm{d}t.$$

For every $a \in F$, Lemma 13.5 also gives

$$\int_0^\infty \|u_{t^{1/2}}(x)a-au_{t^{1/2}}(x)\|_2^2 \,\mathrm{d}t \leq 4\|xa-ax\|_2(\|xa\|_2+\|ax\|_2) \leq 8\|a\|_\infty\|x\|_2\|xa-ax\|_2,$$

Hence, we obtain

$$\||x^*| - |x|\|_2^2 \leq 16\kappa \left(\max_{a \in F} \|a\|_\infty\right) \|x\|_2 \sum_{a \in F} \|xa - ax\|_2$$

and this contradicts (iv)$'$.

(iv) $\Rightarrow$ (iii)$'$. Let $F \subset M$ be a finite self-adjoint set and $\varepsilon > 0$. Pick $v \in M$ a partial isometry such that

$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon\|vv^* - v^*v\|_2.$$

Let $x_1 = (1 - v^*v)v$ and $x_2 = v(1 - vv^*)$. Note that $x_1^2 = x_2^2 = 0$. Let $x = x_1$ if $\|x_1\| \geq \|x_2\|$ and $x = x_2$ otherwise. Then we have

$$\|vv^* - v^*v\|_2^2 = \|x_1\|_2^2 + \|x_2\|_2^2 \leq 2\|x\|_2^2.$$

Moreover, since F is self-adjoint, we have

$$\forall a \in F, \quad \|xa - ax\|_2 \leq 3\|va - av\|_2.$$

Therefore, we obtain

$$\forall a \in F, \quad \|xa - ax\|_2 < 3\sqrt{2}\varepsilon\|x\|_2.$$

(iii)$'$ $\Rightarrow$ (iii). Suppose, by contradiction, that there exists a finite set $F \subset M$ and a constant $\kappa > 0$ such that for all partial isometries $v \in M$ with $v^2 = 0$ we have

$$\|v\|_2^2 \leq \kappa \sum_{a \in F} \|va - av\|_2^2.$$

We can assume that $F \subset M^+$. Let $x \in M$ such that $x^2 = 0$. Then for every $t > 0$, we have $u_t(x)^2 = 0$ hence

$$\|x\|_2^2 = \int_0^\infty \|u_{t^{1/2}}(x)\|_2^2 \,\mathrm{d}t \leq \kappa \sum_{a \in F} \int_0^\infty \|u_{t^{1/2}}(x)a - au_{t^{1/2}}(x)\|_2^2 \,\mathrm{d}t.$$

For every $a \in F$, Lemma 13.5 gives

$$\int_0^\infty \|u_{t^{1/2}}(x)a-au_{t^{1/2}}(x)\|_2^2 \,\mathrm{d}t \leq 4\|xa-ax\|_2(\|xa\|_2+\|ax\|_2) \leq 8\|a\|_\infty\|x\|_2\|xa-ax\|_2.$$

Therefore, we get

$$\|x\|_2 \leq 8\left(\max_{a\in F}\|a\|_\infty\right)\kappa\sum_{a\in F}\|xa-ax\|_2$$

and this contradicts (iii)$'$.

We have now finished the proof of the equivalences (iii) $\Leftrightarrow$ (iii)$'$ $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iv)$'$. Next, we will prove that if M satisfies (iii) then pMp also satisfies (iii) for every nonzero projection $p\in M$. Suppose, by contradiction, pMp does not satisfy (iii). Then pMp does not satisfy (iv)$'$. Hence we can find a constant $\kappa>0$ and a finite set $F\subset pMp$ such that

$$\forall x\in pMp,\quad \||x|-|x^*|\|_2^2 \leq \kappa\|x\|_2\sum_{a\in F}\|ax-xa\|_2.$$

Take $S\subset M$ a finite set of partial isometries such that $\sum_{w\in S} w^*w = p^\perp$ and $ww^*\leq p$ for all $w\in S$. Now, take a partial isometry $v\in M$ with $v^2=0$ and let $x=pvp$. Then we have

$$\|v\|_2^2 = \|pv\|_2^2 + \sum_{w\in S}\|wv\|_2^2$$

and for all $w\in S$, we have

$$\|wv\|_2^2 \leq 2(\|wv-vw\|_2^2+\|vw\|_2^2) \leq 2(\|wv-vw\|_2^2+\|vp\|_2^2).$$

Hence, we obtain

$$\|v\|_2^2 \leq 2|S|(\|vp\|_2^2+\|pv\|_2^2) + 2\sum_{w\in S}\|wv-vw\|_2^2.$$

Moreover, we have

$$\|pv\|_2^2+\|vp\|_2^2 = 2\|x\|_2^2+\|pv-vp\|_2^2,$$

hence

$$\|v\|_2^2 \leq 4|S|\|x\|_2^2 + 2|S|\|pv-vp\|_2^2 + 2\sum_{w\in S}\|wv-vw\|_2^2.$$

Now, by assumption, we have

$$\||x|-|x^*|\|_2^2 \leq \kappa\|x\|_2\sum_{a\in F}\|ax-xa\|_2.$$

Moreover, we have

$$\||x|-|pv|\|_2 \leq \|x-pv\|_2 \leq \|vp-pv\|_2$$

and

$$\||x^*|-|vp|\|_2 \leq \|x^*-vp\|_2 \leq \|vp-pv\|_2.$$

Hence, by using the fact that $v^2=0$, we get

$$\|x\|_2 \leq \|pv\|_2 \leq \||pv|-|vp|\|_2 \leq \||x|-|x^*|\|_2 + 2\|vp-pv\|_2$$

which implies that

$$\|x\|_2^2 \leq 2\kappa\|x\|_2\sum_{a\in F}\|ax-xa\|_2 + 8\|pv-vp\|_2^2.$$

Therefore we obtain

$$\|v\|_2^2 \leq 8|S|\kappa\|x\|_2 \sum_{a\in F} \|ax - xa\|_2 + 34|S|\|pv - vp\|_2^2 + 2\sum_{w\in S} \|wv - vw\|_2^2.$$

which implies that

$$\|x\|_2^2 \leq 2\kappa\|x\|_2 \sum_{a\in F} \|ax - xa\|_2 + 8\|pv - vp\|_2^2.$$

Therefore we obtain

$$\|v\|_2^2 \leq 8|S|\kappa\|x\|_2 \sum_{a\in F} \|ax - xa\|_2 + 34|S|\|pv - vp\|_2^2 + 2\sum_{w\in S} \|wv - vw\|_2^2.$$

Finally, using the fact that

$$\|x\|_2 \sum_{a\in F} \|ax - xa\|_2 \leq \|v\|_2 \sum_{a\in F} \|av - va\|_2,$$

$$\|pv - vp\|_2^2 \leq 2\|v\|_2\|pv - vp\|_2$$

and

$$\|wv - vw\|_2^2 \leq 2\|v\|_2\|wv - vw\|_2,$$

we can conclude that

$$\|v\|_2 \leq \kappa' \sum_{a\in F'} \|av - va\|_2.$$

for some $\kappa' > 0$, some finite set $F' \subset M$ and all partial isometries $v \in M$ with $v^2 = 0$. This shows that M does not satisfy (iii) as we wanted.

Finally, we show (iii) $\Rightarrow$ (ii) by using the same maximality argument that we used in the proof of Theorem 17.2. The only difference now is that the function $(v, w) \mapsto \|v - w\|_2^2$ is no longer a distance. This is why we replace it by $d(v, w) = \|v - w\|_1$. Let $a_1, \dots, a_n \in \mathrm{Ball}(M)$ be a finite family and let $\varepsilon > 0$ and $\delta = 8\varepsilon$. Consider the set Λ of all $(v, A_1, \dots, A_n) \in \mathrm{Ball}(M)^{n+1}$ such that

— v is a partial isometry and $v^2 = 0$.

— $[A_k, vv^* + v^*v] = 0$ for all $k = 1, \dots, n$.

— $\|vA_k - A_kv\|_2 \leq \varepsilon\|v\|_2$ for all $k = 1, \dots, n$.

— $\|A_k - a_k\|_1 \leq \delta\|v\|_1$.

On Λ put the order relation given by

$$(v, A_1, \dots, A_n) \leq (v', A_1', \dots, A_n')$$

if and only if $v \leq v'$ and $\|A_k' - A_k\|_1 \leq \delta(\|v'\|_1 - \|v\|_1)$ for all $k = 1, \dots, n$. Then Λ is an inductive set (because $\mathrm{Ball}(M)$ is complete for the distance given by $\|\cdot\|_1$). By Zorn's lemma, let $v \in \Lambda$ be a maximal element. Suppose that $q = vv^* + v^*v \neq 1$. Since, by the previous step, all corners of M also satisfy (iii), we can find a nonzero partial isometry $w \in q^\perp M q^\perp$, with $w^2 = 0$ such that

$$\|wA_k - A_kw\|_2 \leq \varepsilon\|w\|_2$$

$$\|wA_k^* - A_k^*w\|_2 \leq \varepsilon\|w\|_2$$

for all $k = 1, \dots, n$.

Now, let

— $p = ww^* + w^*w$
— $A'_k = pA_kp + p^\perp A_k p^\perp$
— $v' = v + w$
— $q' = v'(v')^* + (v')^*v' = q + p$

Note that $(v')^2 = 0$ and $[A'_k, q'] = 0$ for all k. We also have

$$\|v'A'_k - A'_kv'\|_2^2 \leq \|vA_k - A_kv\|_2^2 + \|wA_k - A_kw\|_2^2 \leq \varepsilon^2\|v\|_2^2 + \varepsilon^2\|w\|_2^2 = \varepsilon^2\|v'\|_2^2.$$

Moreover, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|A'_k - A_k\|_1 &\leq \|pA_kp^\perp\|_1 + \|p^\perp A_kp\|_1 \\ &\leq \|p\|_2(\|pA_kp^\perp\|_2 + \|p^\perp A_kp\|_2) \\ &\leq \sqrt{2}\|p\|_2\|[A_k, p]\|_2 \\ &\leq 2\sqrt{2}\|p\|_2(\|[A_k, w]\|_2 + \|[A_k, w^*]\|_2) \\ &\leq 4\sqrt{2}\varepsilon\|p\|_2\|w\|_2 \\ &= 8\varepsilon\|w\|_2^2 \\ &= \delta\|w\|_1.\end{aligned}$$

Since $\|v'\|_1 = \|v\|_1 + \|w\|_1$, this implies that

$$\|A'_k - A_k\|_1 \leq \delta(\|v'\|_1 - \|v\|_1)$$

and

$$\|A'_k - a_k\|_1 \leq \|A'_k - A_k\|_1 + \|A_k - a_k\|_1 \leq \delta\|v'\|_1.$$

Therefore $v' \in \Lambda$ and $v \leq v'$. This contradicts the maximality of v. Hence we must have $v^*v + vv^* = q = 1$. Moreover, since

$$\|va_k - a_kv\|_2 \leq \|vA_k - A_kv\|_2 + 2\|A_k - a_k\|_2,$$

$$\|vA_k - A_kv\|_2 \leq \varepsilon$$

and

$$\|A_k - a_k\|_2^2 \leq 2\|A_k - a_k\|_1 \leq 2\delta = 16\varepsilon,$$

we conclude that

$$\|va_k - a_kv\|_2 \leq \varepsilon + 8\sqrt{\varepsilon}.$$

Since such a v exists for every $\varepsilon > 0$ and every finite family $a_1, \ldots, a_n \in \mathrm{Ball}(M)$, we have proved (ii). □

20 McDuff tensor product factors

Unfortunately, one cannot prove Conjecture 16.1 by following the same pattern as in Theorem 18.1. The reason is that there is no analog, in the von Neumann algebraic context, of the currying procedure we used in the proof of Theorem 18.1. Nevertheless, in this section, we present some partial solutions to Conjecture 16.1 by using a different technique.

For the first result, we point out that all concrete examples of non-McDuff factors M in the literature do satisfy the assumption that there exists an abelian subalgebra $A \subset M$ such that $M_\omega \subset A^\omega$ (in fact, this is how we show that they are not McDuff). For these kind of factors, the problem is solved.

Theorem 20.1. *Let M be a II_1 factor with separable predual. Suppose that there exists an abelian subalgebra $A \subset M$ such that $M_\omega \subset A^\omega$ (in particular M is not McDuff). Then for every II_1 factor N with separable predual we have that $M \mathbin{\overline{\otimes}} N$ is McDuff if and only if N is McDuff.*

Fix once and for all a free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$. Recall that a type II_1 factor M with separable predual is McDuff if and only if M_ω is not abelian [McD69]. Keeping this in mind, our second result solves Conjecture 16.1 under the additional assumption that $(M \mathbin{\overline{\otimes}} N)_\omega$ is a factor.

Theorem 20.2. *Let M and N be type II_1 factors with separable predual and suppose that $(M \mathbin{\overline{\otimes}} N)_\omega$ is a factor. Then both M_ω and N_ω are factors. In particular, if $M \mathbin{\overline{\otimes}} N$ is McDuff, then M is McDuff or N is McDuff.*

For the proof of these theorems, we will use the following lemma which is extracted from [IV15]. It is inspired by a trick used in [Ha84]. Recall that if M is a von Neumann algebra, then $\mathrm{L}^2(M^\omega)$ is in general much smaller than the ultraproduct Hilbert space $\mathrm{L}^2(M)^\omega$ (see [Co75b, Proposition 1.3.1]).

Lemma 20.3. *Let M and N be finite von Neumann algebras with separable predual. Fix a tracial state τ on M and pick an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of (M,τ). Let $A = \mathrm{L}^\infty(\mathbb{T}^\mathbb{N}) = \mathrm{L}^\infty(\mathbb{T})^{\overline{\otimes}\mathbb{N}}$ and for each $n \in \mathbb{N}$, let $u_n \in \mathcal{U}(A)$ be the canonical generator of the nth copy of $\mathrm{L}^\infty(\mathbb{T})$. Let $V : \mathrm{L}^2(M) \to \mathrm{L}^2(A)$ be the unique (non-surjective) isometry which sends e_n to u_n for every $n \in \mathbb{N}$.*

Then the naturally defined ultraproduct isometry

$$(V \otimes 1)^\omega : \mathrm{L}^2(M \mathbin{\overline{\otimes}} N)^\omega \to \mathrm{L}^2(A \mathbin{\overline{\otimes}} N)^\omega$$

sends $\mathrm{L}^2((M \mathbin{\overline{\otimes}} N)^\omega)$ into $\mathrm{L}^2((A \mathbin{\overline{\otimes}} N)^\omega)$.

Lemma 20.3 is useful because it allows us to reduce many problems on sequences in tensor products $M \mathbin{\overline{\otimes}} N$ to the case where M is abelian. We now present two applications of this principle.

The first one slightly generalizes [IV15, Corollary]. We will need it for Theorem 20.1.

Proposition 20.4. *Let M and N be finite von Neumann algebras with separable predual. For any von Neumann subalgebras $Q, P \subset N$ such that $Q' \cap N^\omega \subset P^\omega$, we have*

$$(1 \otimes Q)' \cap (M \mathbin{\overline{\otimes}} N)^\omega \subset (M \mathbin{\overline{\otimes}} P)^\omega.$$

Proof. First, we deal with the case where M is abelian, i.e. $M = \mathrm{L}^\infty(T,\mu)$ for some probability space (T,μ). Take $(x_n)^\omega$ in the unit ball of $(1 \mathbin{\overline{\otimes}} Q)' \cap (M \mathbin{\overline{\otimes}} N)^\omega$ and write $x_n = (t \mapsto x_n(t)) \in M \mathbin{\overline{\otimes}} N = \mathrm{L}^\infty(T,\mu,N)$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose a finite set $F \subset Q$ and $\delta > 0$ such that for every x in the unit ball of N we have

$$(\forall a \in F,\ \|[x,a]\|_2 \leq \delta) \Longrightarrow \|x - \mathrm{E}_P(x)\|_2 \leq \varepsilon.$$

Since $(x_n)^\omega \in (1 \mathbin{\overline{\otimes}} Q)' \cap (M \mathbin{\overline{\otimes}} N)^\omega$, we have

$$\lim_{n\to\omega} \mu\left(\{t \in T \mid \forall a \in F,\ \|[x_n(t),a]\|_2 \leq \delta\}\right) = 1.$$

Hence, we have

$$\lim_{n\to\omega} \mu\left(\{t \in T \mid \|x_n(t) - \mathrm{E}_P(x_n(t))\|_2 \leq \varepsilon\}\right) = 1.$$

This means that

$$\lim_{n\to\omega} \|x_n - \mathrm{E}_{M\overline{\otimes}P}(x_n)\|_2 \leq \varepsilon$$

and since this holds for every $\varepsilon > 0$, we conclude that $(x_n)^\omega \in (M \mathbin{\overline{\otimes}} P)^\omega$.

Now, we extend to the general case where M is not necessarily abelian. Let $\xi \in \mathrm{L}^2((M \mathbin{\overline{\otimes}} N)^\omega)$ be an Q-central vector. We want to show that $\xi \in \mathrm{L}^2((M \mathbin{\overline{\otimes}} P)^\omega)$. By Lemma 20.3, we know that $\eta = (V \otimes 1)^\omega(\xi) \in \mathrm{L}^2((A \mathbin{\overline{\otimes}} N)^\omega)$. Since $(V \otimes 1)^\omega$ is N-bimodular, we know that η is Q-central. Hence, by the abelian case, we obtain that $\eta \in \mathrm{L}^2((A \mathbin{\overline{\otimes}} P)^\omega)$. But this clearly implies that $\xi \in \mathrm{L}^2((M \mathbin{\overline{\otimes}} P)^\omega)$. □

Proof of Theorem 20.1. Suppose that the factor $M \mathbin{\overline{\otimes}} N$ is McDuff, i.e. $(M \mathbin{\overline{\otimes}} N)_\omega$ is non-commutative. By Proposition 20.4, we know that $(M \mathbin{\overline{\otimes}} N)_\omega \subset (A \mathbin{\overline{\otimes}} N)_\omega$ so that $(A \mathbin{\overline{\otimes}} N)_\omega$ is also non-commutative. Thus, we can find $x = (x_n)^\omega$ and $y = (y_n)^\omega$ in $(A \mathbin{\overline{\otimes}} N)_\omega$ with $\|x_n\|_\infty, \|y_n\|_\infty \leq 1$ for all n, such that $\|[x, y]\|_2 = \delta > 0$. Let $A = \mathrm{L}^\infty(T, \mu)$ with (T, μ) a probability space. Write $x_n = (t \mapsto x_n(t)) \in A \mathbin{\overline{\otimes}} N = \mathrm{L}^\infty(T, \mu, N)$ with $\|x_n(t)\|_\infty \leq 1$ for all n and t. Similarly, let $y_n = (t \mapsto y_n(t))$. Fix $F \subset N$ a finite subset and $\varepsilon > 0$. Since $x, y \in (A \mathbin{\overline{\otimes}} N)_\omega$, we know that

$$\lim_{n\to\omega} \mu(\{t \in T \mid \forall a \in F,\ \|[x_n(t), a]\|_2 \leq \varepsilon\}) = 1$$

and

$$\lim_{n\to\omega} \mu(\{t \in T \mid \forall a \in F,\ \|[y_n(t), a]\|_2 \leq \varepsilon\}) = 1.$$

Moreover, since $\|[x, y]\|_2 = \delta > 0$, we have

$$\lim_{n\to\omega} \mu(\{t \in T \mid \|[x_n(t), y_n(t)]\|_2 \geq \delta/2\}) > 0.$$

Hence, for n large enough, the intersection of these three sets is non-empty, i.e. there exists t such that

$$\begin{aligned}
\forall a \in F,\ \|[x_n(t), a]\|_2 \leq \varepsilon,\\
\forall a \in F,\ \|[y_n(t), a]\|_2 \leq \varepsilon,\\
\|[x_n(t), y_n(t)]\|_2 \geq \delta/2.
\end{aligned}$$

Hence, by iterating this procedure, we can extract a sequence $a_k = x_{n_k}(t_k)$, $k \in \mathbb{N}$ and $b_k = y_{n_k}(t_k)$, $k \in \mathbb{N}$ such that $a = (a_k)^\omega$ and $b = (b_k)^\omega$ are in N_ω and $\|[a, b]\|_2 \geq \delta/2$. Thus N_ω is not commutative, i.e. N is McDuff as we wanted. □

The second application is the following lemma which we will need in the proof of Theorem 20.2.

Lemma 20.5. *Let M and N be finite von Neumann algebras with separable predual. Then we have*

$$\mathcal{Z}(N_\omega) \subset \mathcal{Z}(N' \cap (M \mathbin{\overline{\otimes}} N)^\omega).$$

Proof. First, we treat the case where M is abelian, i.e. $M = \mathrm{L}^\infty(T, \mu)$ for some probability space (T, μ). Let $(a_k)_{k\in\mathbb{N}}$ be a $\|\cdot\|_2$-dense sequence in $(N)_1$ and let

$$N_k = \{x \in (N)_1 \mid \forall r \leq k,\ \|[x, a_r]\|_2 \leq 1/k\}.$$

Let $y = (y_n)^\omega \in \mathcal{Z}(N_\omega)$ with $\|y_n\|_\infty$ for all n. By [McD69, Lemma 10], there exists a sequence of sets $U_k \in \omega$, $k \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N},\ \forall x \in N_k,\ \forall n \in U_k, \quad \|[y_n, x]\|_2 \leq 1/k.$$

Let $x = (x_n)^\omega \in (1 \otimes N)' \cap (M \mathbin{\overline{\otimes}} N)^\omega$ with $\|x_n\|_\infty \leq 1$ for all $n \in \mathbb{N}$. We want to show that $x(1 \otimes y) = (1 \otimes y)x$. Write $x_n = (t \mapsto x_n(t)) \in M \mathbin{\overline{\otimes}} N = \mathrm{L}^\infty(T, \mu, N)$ with $\|x_n(t)\|_\infty \leq 1$ for all t and all $n \in \mathbb{N}$. Since $x \in (1 \otimes N)' \cap (M \mathbin{\overline{\otimes}} N)^\omega$, there exists a sequence of sets $V_k \in \omega$ such that

$$\mu(\{t \in T \mid x_n(t) \in N_k\}) \geq 1 - 1/k^2$$

for all $n \in V_k$.

Therefore, for all $n \in U_k \cap V_k$, we have

$$\mu(\{t \in T \mid \|[y_n, x_n(t)]\|_2 \leq 1/k\}) \geq 1 - 1/k^2$$

which implies that

$$\|[1 \otimes y_n, x_n]\|_2^2 = \int_T \|[y_n, x_n(t)]\|_2^2 \, \mathrm{d}\mu(t) \leq 5/k^2.$$

Since $U_k \cap V_k \in \omega$ for all $k \in \mathbb{N}$, we conclude that $\lim_{n \to \omega} \|[1 \otimes y_n, x_n]\|_2 = 0$ as we wanted.

Finally, we extend to the general case where M is not necessarily abelian. Let $\xi \in \mathrm{L}^2((M \mathbin{\overline{\otimes}} N)^\omega)$ be an N-central vector. We want to show that ξ is $\mathcal{Z}(N_\omega)$-central. By Lemma 20.3, we know that $\eta = (V \otimes 1)^\omega(\xi) \in \mathrm{L}^2((A \mathbin{\overline{\otimes}} N)^\omega)$. Since $(V \otimes 1)^\omega$ is N-bimodular, we know that η is N-central. Hence, by the abelian case, we obtain that η is $\mathcal{Z}(N_\omega)$-central. Since $(V \otimes 1)^\omega$ is N^ω-bimodular, we conclude that ξ is also $\mathcal{Z}(N_\omega)$-central. □

Proof of Theorem 20.2. By Lemma 20.5, we know that $\mathcal{Z}(N_\omega)$ is contained in the center of $N' \cap (M \mathbin{\overline{\otimes}} N)^\omega$ hence it is also contained in the center of $(M \mathbin{\overline{\otimes}} N)_\omega$. Since $(M \mathbin{\overline{\otimes}} N)_\omega$ is a factor, this implies that N_ω is also a factor, and the same argument shows that M_ω is a factor.

Now suppose that $M \mathbin{\overline{\otimes}} N$ is McDuff. Then $(M \mathbin{\overline{\otimes}} N)_\omega$ is nontrivial. Thus M_ω or N_ω is also nontrivial (use [Co75b, Corollary 2.2] or Proposition 20.4). This means that M_ω or N_ω is a nontrivial factor. In particular, M or N is McDuff. □

Examples of factors with factorial asymptotic centralizers are obtained by taking infinite tensor products of full II_1 factors. We provide a proof if this fact for the reader's convenience.

Proposition 20.6. *Let $M = \overline{\bigotimes}_{n \in \mathbb{N}} M_n$ be an infinite tensor product of full type II_1 factors M_n, $n \in \mathbb{N}$ with separable predual. Then M_ω is a factor.*

Proof. For every $n \in \mathbb{N}$, we let

$$Q_n = M_0 \mathbin{\overline{\otimes}} M_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} M_n \otimes 1 \otimes 1 \otimes \cdots \subset M.$$

Suppose that $(x_k)_{k \in \mathbb{N}}$ is a nontrivial central sequence in M with $\|x_k\|_2 = 1$ and $\tau(x_k) = 0$ for all $k \in \mathbb{N}$. Then, since Q_n is full, we know by [Co75b, Theorem 2.1] that $\lim_k \|x_k - \mathrm{E}_{Q_n' \cap M}(x_k)\|_2 = 0$ Hence we can find a sequence $(n_k)_{k \in \mathbb{N}}$ with

$n_k \to \infty$ such that $\|\mathrm{E}_{Q'_{n_k} \cap M}(x_k)\|_2 \geq \frac{1}{2}$ for all $k \in \mathbb{N}$. Let $y_k = \mathrm{E}_{Q'_{n_k} \cap M}(x_k)$. Since $Q'_{n_k} \cap M$ is a finite factor, we know that $0 = \tau(x_k) = \tau(y_k)$ is in the weakly closed convex hull of

$$\{u y_k u^* \mid u \in \mathcal{U}(Q'_{n_k} \cap M)\}.$$

Hence, there must exist some unitary $u_k \in \mathcal{U}(Q'_{n_k} \cap M)$ such that

$$\|u_k y_k u_k^* - y_k\|_2 \geq \frac{1}{2}\|y_k\|_2 \geq \frac{1}{4}$$

which yields $\|[u_k, x_k]\|_2 \geq \frac{1}{4}$. But, by construction, $(u_k)_{k\in\mathbb{N}}$ is a central sequence in M. This shows that $(x_k)_{k\in\mathbb{N}}$ is not in the center of M_ω. Therefore M_ω is a factor. $\square$

Infinite tensor products of full factors were studied in [Po10]. By combining [Po10, Theorem 4.1] and Theorem 20.2, we obtain the following Corollary which is not related to Conjecture 16.1. It provides the first example of a McDuff II_1 factor that does not admit any *McDuff decomposition* (see Definition 16.1).

Corollary 20.7. *Let $M = \overline{\bigotimes}_{n\in\mathbb{N}} M_n$ be an infinite tensor product of full type II_1 factors M_n, $n \in \mathbb{N}$ with separable predual. Suppose that $M \cong N \mathbin{\overline{\otimes}} R$ for some factor N. Then $M \cong N$.*

Proof. By Proposition 20.6, we know that M_ω is a factor. By Theorem 20.2, we then know that N_ω is also a factor. Moreover, N is not full because of [Po10, Theorem 4.1]. This means that N_ω is non-commutative or equivalently that N is McDuff. Thus $N \cong N \mathbin{\overline{\otimes}} R \cong M$ as we wanted. $\square$

Part V

Appendix : Discrete correspondences and Popa's intertwining theory

In [Po04, Po6], Sorin Popa has developed his *intertwining by bimodules* theory. It is a powerful tool to obtain unitary conjugacy between subalgebras of a given von Neumann algebra and for this reason it is a key ingredient in every deformation/rigidity based argument. It has been extended to non-tracial von Neumann algebras in [HI15].

In this appendix, we introduce the notion of *discrete correspondence* between arbitrary von Neumann algebras and we use it to give a unified approach to Popa's intertwining theory as well as a better understanding of quasinormalizers, discrete inclusions and quasiregular inclusions. More precisely, to each pair of von Neumann subalgebras $A, B \subset M$, one can associate three closed subspaces of M denoted

$$\mathcal{D}^l_M(A,B), \quad \mathcal{D}^r_M(A,B) = \mathcal{D}^l_M(B,A)^* \text{ and } \mathcal{D}_M(A,B) = \mathcal{D}^l_M(A,B) \cap \mathcal{D}^r_M(A,B)$$

whose elements are respectively called left discrete, right discrete and (both left and right) discrete from A to B. When A and B are abelian, this has been studied in [PS03]. These subspaces are, in some sense, generated by intertwiners so that $A \prec_M B$ if and only if $\mathcal{D}^l_M(A,B) \neq 0$. But unlike intertwiners, they are particularly well-behaved and stable under many operations such as reduction to corners, passing to subalgebras etc. They also satisfy the following multiplicativity relation

$$\mathcal{D}^l_M(A,B)\mathcal{D}^l_M(B,C) = \mathcal{D}^l_M(A,C).$$

In particular, $\mathcal{D}_M(A) = \mathcal{D}_M(A,A)$ is a von Neumann subalgebra of M. We have $M = \mathcal{D}_M(A)$ if and only if the inclusion $A \subset \langle M, A\rangle$ is with expectation. When A is finite $\mathcal{D}_M(A)$ is precisely the quasinormalizer of A inside M. In general, for subalgebras A and B of arbitrary type, the space $\mathcal{D}_M(A,B)$ measures the possibility to conjugate A and B inside M *up to finite index.*

21 Inclusions with expectation and compatible projections

We will use a lot the following theorem of Haagerup.

Theorem 21.1 ([Ha77b, Theorem 6.6])**.** *Let $N \subset M$ be an inclusion of von Neumann algebras. Then the following conditions are equivalent:*

(i) *There exists a faithful family of normal conditional expectations $\mathrm{E}_i : M \to N$, $i \in I$.*

(ii) *There exists a faithful semifinite normal operator valued weight $T : M \to N$ such that its restriction $T_{|N' \cap M} : N' \cap M \to \mathcal{Z}(N)$ is also semifinite.*

(iii) *For every semifinite normal operator valued weight $T : M \to N$, its restriction $T_{|N' \cap M} : N' \cap M \to \mathcal{Z}(N)$ is also semifinite.*

When these properties hold, the map $T \mapsto T_{|N' \cap M}$ is a bijection from the set of operator valued weights from M to N to the set of operator valued weights from $N' \cap M$ to $\mathcal{Z}(N)$.

We will say that N is *with expectation* inside M if it satisfies the conditions above. Note that there always exists a largest projection $p \in N' \cap M$ such that $Np \subset pMp$ is with expectation. When $N' \cap M$ is σ-finite, N is with expectation in M if and only if it admits a faithful normal conditional expectation.

We introduce the following notion of compatible projections because they are particularly well-behaved.

Definition 21.2. Let $N \subset M$ be an inclusion of von Neumann algebras. We say that a projection $p \in M$ is *compatible* with N when there exists two projections $e \in N$ and $f \in N' \cap M$ such that $p = ef$.

We gather the basic properties of compatible projections in the following proposition.

Proposition 21.3. . *Let N be a subalgebra of a von Neumann algebra M and p an N-compatible projection. Then we have :*

(i) *pNp is a von Neumann subalgebra of pMp.*

(ii) *$p(N' \cap M)p = (pNp)' \cap pMp$.*

(iii) *$p(N \vee N' \cap M)p = pNp \vee p(N' \cap M)p$.*

(iv) *If a projection q in pMp is pNp-compatible then it is also N-compatible.*

(v) *If $N \subset M$ is with expectation then $pNp \subset pMp$ is with expectation.*

We now state the main technical result of this section. The assumption that N is with expectation in M is essential (a counter example is given by taking a diffuse abelian subalgebra of a type I factor). Recall that the relation $p \precsim q$ between two projections $p, q \in M$ means that there exists a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* \leq q$.

Proposition 21.4. *Let $N \subset M$ an inclusion of von Neumann algebras with expectation. Then for any nonzero projection $p \in M$ there exists a nonzero N-compatible projection $q \in M$ such that $q \precsim p$.*

The assumption that the inclusion is with expectation is necessary. A counter example is obtained by taking $N = A$ a diffuse masa in $M = \mathbf{B}(H)$ and taking p a minimal projection in M.

In order to prove Proposition 21.4, we need a few lemmas.

Lemma 21.5. *Let $N \subset M$ an inclusion of von Neumann algebras with expectation and suppose that M is semifinite. Then there exists a nonzero finite projection $e \in M$ which is N-compatible.*

Proof. Since N is with expectation then it is also semi-finite. Thus, up to cutting down by a nonzero finite projection in N, we may suppose that N is finite. Pick τ_0 a normal faithful finite trace on N and τ a normal faithful semifinite trace on M. Let $T : M \to N$ be the unique n.f.s operator valued weight such that $\tau = \tau_0 \circ T$. Since N is with expectation, we know by Theorem 21.1 that $T_{|N' \cap M} : N' \cap M \to \mathcal{Z}(N)$ is semifinite. This implies that $\tau_{|N' \cap M}$ is semifinite. Therefore we can find a nonzero projection $e \in N' \cap M$ which is finite in M and we are done. $\square$

Lemma 21.6. *Let $N \subset M$ be an inclusion of von Neumann algebras with expectation. Then there exists a nonzero σ-finite projection $e \in M$ which is N-compatible.*

Proof. Let φ be a normal state on N and $\mathrm{E} : M \to N$ a normal conditional expectation. Then $\varphi \circ \mathrm{E}$ is a normal state on M. Its support is pq where $p \in N$ is the support of φ and $q \in N' \cap M$ is the support of E. Hence pq is nonzero, σ-finite in M and N-compatible. $\square$

Lemma 21.7. *Let M be a finite von Neumann algebra with central valued trace* ctr *and let $N \subset M$ be a von Neumann subalgebra. Suppose that $N \not\subset \mathcal{Z}(M)$. Then there exists a nonzero N-compatible projection $q \in M$ such that* $\mathrm{ctr}(q) \leq \frac{1}{2}$.

Proof. Let $q \in N$ a projection such that $q \notin \mathcal{Z}(M)$. Then $\mathrm{ctr}(q)$ is not a projection (otherwise we would have $q = \mathrm{ctr}(q) \in \mathcal{Z}(M)$). Thus, replacing q by $1 - q$ if necessary, we may suppose that the spectral projection $z = 1_{[0,\frac{1}{2}]}(\mathrm{ctr}(q))$ is nonzero. Then zq is a nonzero N-compatible projection such that $\mathrm{ctr}(zq) = z\mathrm{ctr}(q) \leq \frac{1}{2}$. $\square$

Proof of Proposition 21.4. We deal first with the case where M is finite. Let $\mathrm{ctr} : M \to \mathcal{Z}(M)$ be the central valued trace of M. Up to cutting down by a central projection (which is of course N-compatible) we can suppose that $\mathrm{ctr}(p) \geq \varepsilon$ for some $\varepsilon > 0$. In particular p has central support 1. If there exists a nonzero N-compatible projection $q \in M$ such that $qNq \subset \mathcal{Z}(M)q$, then we have $q(N' \cap M)q = qMq$. So in this case we are done because p has central support 1 which means that we can find a nonzero projection $e \leq q$ such that $e \precsim p$. The projection e will be N-compatible because $e \in q(N' \cap M)q$. So now suppose that there does not exist a nonzero N-compatible projection $q \in M$ such that $qNq \subset \mathcal{Z}(M)q$. Then by applying lemma 21.7 recursively we can obtain a decreasing sequence of nonzero N-compatible projections q_n such that $\mathrm{ctr}(q_n) \leq 2^{-n}$. Hence, for n big enough we will have $\mathrm{ctr}(q_n) \leq \varepsilon \leq \mathrm{ctr}(p)$ which means that $q_n \precsim p$.

Now we deal with the general case where M is not necessarily finite. Let $z \in \mathcal{Z}(M)$ be the projection on the type III part of M. Suppose first that $pz \neq 0$ and let f be the central support of pz. By Lemma 21.6, we can find a nonzero σ-finite projection $e \in Mf$ which is compatible with Nf, hence with N. Since Mf is of type III, we know that p is properly infinite in Mf, hence $e \precsim pz \leq p$. Now suppose that $pz = 0$. Then Mf is semifinite where f is the central support of p. By lemma 21.5, we can find a nonzero finite projection $e \in Mf$ which is N-compatible. Since f is the central support of p we can find a nonzero projection $r \leq e$ such that $r \precsim p$. Now we apply the first part of the proof to the inclusion of finite von Neumann algebras $eNe \subset eMe$ and we get a nonzero eNe-compatible projection q such that $q \precsim r$. In particular, q is N-compatible and $q \precsim p$. $\square$

22 Discrete correspondences

Let E and F be two sets. Then a *correspondence* from E to F is simply a relation between the elements of E and F, which can be defined by its graph $\mathcal{C} \subset E \times F$. The notion of correspondence can be generalized to arbitrary measurable spaces. Indeed, the category of measurable spaces (which is the opposite of the category of commutative von Neumann algebras) has all products. In particular, if X and Y are two measurable spaces, we let $X \times_u Y$ denote their categorical product. It is always much larger then the usual spatial product unless X or Y is discrete. We call $X \times_u Y$ the *universal product* of X and Y. A *correspondence from X to Y* is then defined simply as a measurable subset $\mathcal{C} \subset X \times_u Y$. For example the usual spatial product $X \otimes X$ is the *coarse* correspondence. When X is diffuse, it is disjoint from the *identity* correspondence which is represented by the diagonal $\Delta_X \subset X \times_u X$.

The notion of correspondences has been further generalized to all von Neumann algebras by Connes. If M and N are two von Neumann algebras, we denote by $M \mathbin{\overline{\otimes}_u} N$ their *universal tensor product*. It is a von Neumann algebra equipped with a binormal $*$-homomorphism $\iota : M \odot N \to M \mathbin{\overline{\otimes}_u} N$ which is universal for this property, i.e. any binormal $*$-homomorphism from $M \odot N$ into some von Neumann algebra factors uniquely through ι. The usual spatial tensor product $M \mathbin{\overline{\otimes}} N$ is a quotient of $M \mathbin{\overline{\otimes}_u} N$ and they are equal if and only if M or N is discrete. We then define a *correspondence from M to N* as a central projection $Z \in \mathcal{P}(\mathcal{Z}(M \mathbin{\overline{\otimes}_u} N^{\mathrm{op}}))$.

A *concrete correspondence from M to N* is a normal representation of $M \mathbin{\overline{\otimes}_u} N^{\mathrm{op}}$ on a Hilbert space H or equivalently a binormal representation of $M \odot N^{\mathrm{op}}$ on H also called a *M-N-bimodule*. The support of this representation is a central projection in $M \mathbin{\overline{\otimes}_u} N^{\mathrm{op}}$ which is an abstract correspondence. Therefore, since the representation theory of von Neumann algebras is not very subtle, there is often no danger to abuse language and forget the distinction between abstract and concrete correspondence. We refer the reader to [Po86, Ri74, Pa73] for details and different point of views on correspondences. In particular, to every representation of M on a self-dual right N-module, one can associate an M-N-bimodule $X \otimes_N \mathrm{L}^2(N)$. The inverse operation associates to every M-N-bimodule H a representation of M on a self-dual N-module denoted $H/\mathrm{L}^2(N)$ and defined as the set of all bounded operators $T : \mathrm{L}^2(N) \to H$ which commute with the right action of N. Finally, for every normal completely positive map $\Phi : M \to N$, one can perform the GNS construction to obtain the unique representation π of M on a self-dual N-module X_Φ together with a cyclic vector $\xi_\Phi \in X_\Phi$ such that $\Phi(x) = \langle \pi(x)\xi_\Phi, \xi_\Phi \rangle$ for all $x \in M$. We also let $H_\Phi = X_\Phi \otimes_N \mathrm{L}^2(N)$.

Here is the main definition of this appendix.

Definition 22.1. Let M and N be two von Neumann algebras. Let H be an M-N-bimodule and let $\pi : M \odot N^{\mathrm{op}} \to \mathbf{B}(H)$ be the associated binormal representation, we say that H is *left discrete* if the inclusion $\pi(M) \subset \pi(N^{\mathrm{op}})'$ is with expectation.

We say that H is *right discrete* if the opposite N-M-bimodule $\overline{H}$ is left discrete. We say that H is *discrete* if it is both left discrete and right discrete.

Left discreteness passes to submodules and is stable under direct sums. It is a property of the underlying abstract correspondence and does not depend on the concrete bimodule representation. For any M-N-bimodule H with the associated binormal representation $\pi : M \odot N^{\mathrm{op}} \to \mathbf{B}(H)$, there exists a largest projection $p \in \pi(M)' \cap \pi(N^{\mathrm{op}})'$ such that $\pi(M)p \subset p\pi(N^{\mathrm{op}})'p$ is with expectation. The range of p is the largest left discrete sub-M-N-bimodule of H. We call it the *left discrete part* of H. We define in the obvious way the *right discrete part* and the *discrete part* of H.

We gather some other basic properties in the following proposition.

Proposition 22.2. *The following properties are satisfied:*

(i) *If a A-B-bimodule H is left (resp. right) discrete, then so is the pAp-B-bimodule pH for any projections $p \in A$. Conversely, if pH is left (resp. discrete), then so is the A-B-bimodule zH where $z \in \mathcal{Z}(A)$ is the central support of p.*

(ii) *Let H be a A-B-bimodule and suppose that A is discrete. Then H is always left discrete, and H is right discrete if and only if B is discrete.*

(iii) *Let H be an A-B-bimodule where A and B are two finite von Neumann algebras. Then H is left discrete if and only if H decomposes as a direct sum of sub-A-B-bimodules which are finitely generated as right B-modules.*

(iv) *Let H be a A-B-bimodule and K a C-D-bimodule with left discrete parts H_d and K_d respectively. Then the left discrete part of the $A \mathbin{\overline{\otimes}} C$-$B \mathbin{\overline{\otimes}} D$-bimodule $H \otimes K$ is $H_d \otimes K_d$.*

(v) *Suppose that H is a left discrete A-B-bimodule and K is a left discrete B-C-bimodule. Then $H \otimes_B K$ is a left discrete A-C-bimodule.*

Proof. We only prove (iii) and (v). We leave the others to the reader.

(iii). Suppose that H is left discrete. Let $\pi : A \odot B^{\mathrm{op}} \to \mathbf{B}(H)$ be the binormal representation associated to the bimodule H. It is enough to prove the result for Hz for every σ-finite projection z in the center of B. Hence we can assume that B is σ-finite. Let τ be a faithful tracial state on $\pi(B^{\mathrm{op}})$ and τ' the corresponding dual semifinite trace on $\pi(B^{\mathrm{op}})'$. Since $\pi(A)$ is finite and with expectation in $\pi(B^{\mathrm{op}})'$, we know that τ' is semifinite on $\pi(A)' \cap \pi(B^{\mathrm{op}})'$ (see the proof of Lemma 21.5). This means that we can find a partition of unity $(p_i)_{i \in I}$ in $\pi(A)' \cap \pi(B^{\mathrm{op}})'$ such that $\tau'(p_i) < +\infty$ for all $i \in I$. Then, if we let H_i be the range of p_i for all $i \in I$, we have $H = \bigoplus_{i \in I} H_i$ and the H_i are A-B-bimodules with finite τ-dimension over B. Finally, for each i, since H_i has finite τ-dimension over B, we can find a partition of unity $(z_j)_{j \in J}$ in the center of B, such that for all j, the A-B-bimodule $H_i z_j$ is finitely generated as a right B-module and we are done since the other direction is obvious.

For (v), it is better to use the self-dual module point of view. We prove first the following.

Claim. Let M and N be two von Neumann algebras. Let X be any right M-module and let (Y, π) be an N-representation of M. Let $\rho : \mathcal{L}_M(X) \to \mathcal{L}_N(X \otimes_M Y)$ be the natural normal homomorphism defined by $\rho(T)(\xi \otimes \eta) = (T\xi) \otimes \eta$. If $\pi(M)$ is with expectation in $\mathcal{L}_N(Y)$, then $\rho(\mathcal{L}_M(X))$ is with expectation in $\mathcal{L}_N(X \otimes_M Y)$.

Proof of the claim. We can view X as a sub-M-module of $M^I = \bigoplus_{i\in I} M$ for some set I. Then $X \otimes_M Y$ is then identified with a sub-N-module of Y^I. Let $p \in \mathcal{L}_M(M^I)$ be the projection on X so that $\mathcal{L}_M(X) = p\mathcal{L}_M(M^I)p$. Then ρ is just the restriction to $\mathcal{L}_M(X)$ of the normal homomorphism

$$\mathrm{id} \otimes \pi : \mathcal{L}_M(M^I) = \mathbf{B}(\ell^2(I)) \mathbin{\overline{\otimes}} M \longrightarrow \mathcal{L}_N(Y^I) = \mathbf{B}(\ell^2(I)) \mathbin{\overline{\otimes}} \mathcal{L}_N(Y).$$

Since $\pi(M)$ is with expectation in $\mathcal{L}_N(Y)$, then $(\mathrm{id}\otimes\pi)(\mathcal{L}_M(M^I))$ is with expectation in $\mathcal{L}_N(Y^I)$, hence $\rho(\mathcal{L}_M(X)) = (\mathrm{id}\otimes\pi)(p\mathcal{L}_M(M^I)p)$ is with expectation in $\mathcal{L}_N(X\otimes_M Y) = (\mathrm{id}\otimes\pi)(p)\mathcal{L}_N(Y^I)(\mathrm{id}\otimes\pi)(p)$. This finishes the proof of the claim. □

Now, back to (v), let (X, π_1) be the B-representation of A associated to H and (Y, π_2) be the C-representation of B associated to K. Then we have a natural C-representation of A on $X \otimes_B Y$ given by $\rho \circ \pi_1 : A \to \mathcal{L}_C(X \otimes_B Y)$ where $\rho : \mathcal{L}_B(X) \to \mathcal{L}_C(X \otimes_N Y)$ is the natural homomorphism induced by π_2. Since by assumption $\pi_1(A)$ is with expectation in $\mathcal{L}_B(X)$ and by the claim, $\rho(\mathcal{L}_B(X))$ is with expectation in $\mathcal{L}_C(X \otimes_B Y)$, we conclude that $\rho(\pi_1(A))$ is with expectation in $\mathcal{L}_C(X \otimes_B Y)$. This means that $H \otimes_B K \cong X \otimes_B Y \otimes_C \mathrm{L}^2(C)$ is left discrete. □

Definition 22.3. We say that a normal completely positive map $\Phi : M \to N$ is left (resp. right) discrete if the associated bimodule H_Φ is left (resp. right) discrete.

When M and N are abelian von Neumann algebras, our notion of discrete normal completely positive map Φ is equivalent to the one used in [PS03]. Like in the abelian case we have the following property

Proposition 22.4. *If $\Phi : M \to N$ and $\Psi : N \to P$ are left (resp. right) discrete normal completely positive maps, then so is $\Psi \circ \Phi$.*

Proof. This follows from Proposition 22.2.(v) and the fact that $X_{\Psi\circ\Phi} \subset X_\Phi \otimes_N X_\Psi$. □

23 Popa's intertwining theory

If $A \subset M$ is a von Neumann subalgebra, we let $A^\vee = A \vee A' \cap M$. If A is with expectation, then there exists a unique faithful normal conditional expectation from M to $A^\vee$ and we can use it to view $\mathrm{L}^2(A^\vee)$ as a subspace of $\mathrm{L}^2(M)$ in a canonical way.

Proposition 23.1. *Let M be a von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be two von Neumann subalgebras. Let $x \in 1_A M 1_B$ and define a normal completely positive map $\Phi : A \to 1_B M 1_B$ by $\Phi(a) = x^* a x$ for all $a \in M$. Suppose that $B \subset 1_B M 1_B$ is with expectation. Then the following are equivalent:*

(i) *For every normal conditional expectation $\mathrm{E} : 1_B M 1_B \to B$, the completely positive map $\mathrm{E} \circ \Phi : A \to B$ is left (resp. right) discrete.*

(ii) *For some faithful family of normal conditional expectations $\mathrm{E}_i : 1_B M 1_B \to B$, the completely positive maps $\mathrm{E}_i \circ \Phi : A \to B$ are left (resp. right) discrete.*

(iii) *The A-B-bimodule $\overline{A^\vee x \mathrm{L}^2(B^\vee)}$ is left (resp. right) discrete.*

Proof. Let H be the left (resp. right) discrete part of the A-B-bimodule $1_A\mathrm{L}^2(M)1_B$. Let $\mathrm{E} : 1_BM1_B \to B$ be any normal conditional expectation. Use it to view $\mathrm{L}^2(B)$ as a sub-B-B-bimodule of $\mathrm{L}^2(B^\vee)$. Then, the A-B-bimodule $H_{\mathrm{E}\circ\Phi}$ associated to $\mathrm{E}\circ\Phi$ is isomorphic to $\overline{Ax\mathrm{L}^2(B)}$. Indeed, the following map

$$ax\eta \mapsto a\xi_{\mathrm{E}\circ\Phi} \otimes_B \eta$$

for $a \in A$ and $\eta \in \mathrm{L}^2(B)$ extends to a well-defined A-B-bimodular unitary $U : \overline{Ax\mathrm{L}^2(B)} \to H_{\mathrm{E}\circ\Phi}$.

In particular, $\mathrm{E}\circ\Phi$ is left (resp. right) discrete if and only if $\overline{Ax\mathrm{L}^2(B)} \subset H$. Since H is a $A^\vee$-$B^\vee$-module, this is equivalent to

$$\overline{A^\vee xp\mathrm{L}^2(B^\vee)} = \overline{A^\vee Ax\mathrm{L}^2(B)B^\vee} \subset H$$

where $p \in B' \cap 1_BM1_B$ is the support of E. It is now clear that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). $\square$

Definition 23.2. Let M be a von Neumann algebra. Let $A \subset 1_AM1_A$ and $B \subset 1_BM1_B$ be two von Neumann subalgebras with expectation. We say that an element $x \in 1_AM1_B$ is *left (resp. right) discrete from A to B* if it satisfies the equivalent conditions above.

We denote the set of all left (resp. right) discrete elements from A to B by $\mathcal{D}^l_M(A,B)$ (resp. $\mathcal{D}^r_M(A,B)$). We let $\mathcal{D}_M(A,B) = \mathcal{D}^l_M(A,B) \cap \mathcal{D}^r_M(A,B)$.

The following proposition is not obvious from the definition.

Proposition 23.3. *Let M be a von Neumann algebra. Let $A \subset 1_AM1_A$ and $B \subset 1_BM1_B$ be two von Neumann subalgebras with expectation. We have $\mathcal{D}^l_M(A,B)^* = \mathcal{D}^r_M(B,A)$. In particular, we have $\mathcal{D}_M(A,B) = \mathcal{D}_M(B,A)^* = \mathcal{D}^l_M(A,B) \cap \mathcal{D}^l_M(B,A)^*$.*

Proof. Let $x \in 1_AM1_B$. We have to show that the A-B-bimodule $\overline{A^\vee x\mathrm{L}^2(B^\vee)}$ is left (resp. right) discrete if and only if the A-B-bimodule $\overline{\mathrm{L}^2(A^\vee)xB^\vee} = J\overline{B^\vee x^*\mathrm{L}^2(A^\vee)}$ is left (resp. right) discrete.

Let $H \subset 1_A\mathrm{L}^2(M)1_B$ be the left (resp. right) discrete part of the A-B-bimodule $1_A\mathrm{L}^2(M)1_B$. Let μ be any normal faithful semifinite weight on $A^\vee$ such that A is globally invariant under σ^μ and lift μ to a n.f.s. weight on 1_AM1_A still denoted μ by using the unique f.n. conditional expectation on $A^\vee$. Similarly, let ν be any normal faithful semifinite weight on $B^\vee$ such that B is globally invariant under σ^ν and lift ν to a n.f.s. weight on 1_BM1_B. Consider the unbounded positive operator $\Delta_{\mu,\nu}$ on $1_A\mathrm{L}^2(M)1_B$ whose graph is the closure of

$$\{(y\nu^{1/2}, \mu^{1/2}y) \mid y \in 1_AM1_B,\ \nu(y^*y) < +\infty \text{ and } \mu(yy^*) < +\infty\}.$$

Note that for all $t \in \mathbb{R}$, the unitary $U = \Delta^{\mathrm{i}t}_{\mu,\nu}$ satisfies $U(a\xi b) = \sigma^{\mathrm{i}t}_\mu(a)(U\xi)\sigma^{-\mathrm{i}t}_\nu(b)$ for all $\xi \in 1_A\mathrm{L}^2(M)1_B$. This implies easily that UH is a left (resp. right) discrete A-B-bimodule. Therefore H is an invariant subspace for U and hence for $\Delta_{\mu,\nu}$ since t is arbitrary. Now, let $a \in A^\vee$ such that $\mu(aa^*) < +\infty$ and $b \in B^\vee$ such that $\nu(b^*b) < +\infty$. Then $(axb\nu^{1/2}, \mu^{1/2}axb)$ is in the graph of $\Delta_{\mu,\nu}$. Since H is an invariant subspace of $\Delta_{\mu,\nu}$, this shows that $axb\nu^{1/2} \in H$ if and only if $\mu^{1/2}axb \in H$. By density, we conclude that $\overline{A^\vee x\mathrm{L}^2(B^\vee)} \subset H$ if and only if $\overline{\mathrm{L}^2(A^\vee)xB^\vee} \subset H$. $\square$

Remark 23.4. Note that the subspaces $\mathcal{D}^l_M(A,B)$, $\mathcal{D}^r_M(A,B)$ and $\mathcal{D}_M(A,B)$ are strongly closed in M, hence also weakly closed by [Ta03a, Theorem 2.1]. They are all $A^\vee$-$B^\vee$-bimodules. For our investigations on this subspaces, it is enough, by Proposition 23.3, to focus on $\mathcal{D}^l_M(A,B)$. If $p \in 1_A M 1_A$ is an A-compatible projection and $q \in 1_B M 1_B$ is a B-compatible projection, then we have the following useful relation

$$p\mathcal{D}^l_M(A,B)q = \mathcal{D}^l_M(pAp, qBq).$$

If $A_0 \subset A$ and $B \subset B_0$ are inclusions with expectation then

$$\mathcal{D}^l_M(A,B) \subset \mathcal{D}^l_M(A_0,B_0).$$

We also note that, inside $M_2(M) = M_2(\mathbb{C}) \otimes M$, we have the following relation

$$\mathcal{D}^l_M(A,B) \otimes e_{12} = \mathcal{D}^l_{M_2(M)}(A \otimes e_{11}, B \otimes e_{22}) = (1_A \otimes e_{11})\mathcal{D}^l_{1_C M_2(M) 1_C}(C,C)(1_B \otimes e_{22})$$

where $C = (A \otimes e_{11}) \oplus (B \otimes e_{22})$ and $1_C = (1_A \otimes e_{11}) + 1_B \otimes e_{22})$ is the unit of C. This allows to reduce almost all questions on general $\mathcal{D}^l_M(A,B)$ to the case where $A = B$ and $A \subset M$ is unital. In that case, we will use the notation $\mathcal{D}^l_M(A) = \mathcal{D}^l_M(A,A)$.

Definition 23.5. Let M be a von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be two von Neumann subalgebras. An element $x \in 1_A M 1_B$ is called a *left A-B-intertwiner* if there exists two projections $p \in A$, $q \in B$ with $pxq = x$ and a morphism $\pi : pAp \to qBq$ such that $ax = x\pi(a)$ for all $a \in pAp$.

Let x be a left A-B-intertwiner. Note that if we take p and q to be the smallest projections in A and B respectively such that $pxq = x$, then there exists a *unique* morphism $\pi_x : pAp \to qBq$ such that $ax = x\pi_x(a)$ for all $a \in pAp$. In that case, we always have $xx^* \in (A' \cap 1_A M 1_A)p$ and $x^*x \in \pi_x(pAp)' \cap qMq$. Moreover, if v is the partial isometry in the polar decomposition $x = v|x|$ then we still have $v \in \mathcal{I}^l_M(A,B)$ and $\pi_v = \pi_x$.

We denote by $\mathcal{I}^l_M(A,B)$ the set of all left A-B-intertwiners. We let $\mathcal{I}^r_M(A,B) = \mathcal{I}^l_M(B,A)^*$ and $\mathcal{I}_M(A,B) = \mathcal{I}^l_M(A,B) \cap \mathcal{I}^r_M(A,B)$. Note that if $x \in \mathcal{I}_M(A,B)$, then $\pi_x : pAp \to qBq$ is an isomorphism and $\pi_x^{-1} = \pi_{x^*}$.

Proposition 23.6. *Let M be a von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be two von Neumann subalgebras with expectation. Let $x \in \mathcal{I}^l_M(A,B)$ and let $\pi_x : pAp \to qBq$ be the morphism it implements. Then $\pi_x(pAp)$ is with expectation in qMq and $x \in \mathcal{D}^l_M(A,B)$.*

Proof. Let $\pi = \pi_x$. Let $\mathrm{E}_i : 1_A M 1_A \to A$ be a faithful family of normal conditional expectations. Then we can define a faithful family of normal conditional expectations $\mathrm{E}^x_i : qMq \to \pi(pAp)$ by

$$\mathrm{E}^x_i(y) = \pi\left(\mathrm{E}_i(xx^*)^{-1}\mathrm{E}_i(xyx^*)\right).$$

Therefore $\pi(A)$ is with expectation in qMq. Now let $\Phi : A \ni a \mapsto x^*ax \in 1_B M 1_B$ and let $\mathrm{E} : 1_B M 1_B \to B$ be a normal conditional expectation. We want to show that $H_{\mathrm{E}\circ\Phi}$ is left discrete. Recall that it is generated by the vectors of the form $a\xi_{\mathrm{E}\circ\Phi} \otimes_B \eta$ with $a \in A$ and $\eta \in \mathrm{L}^2(B)$. Observe that $p\xi_{\mathrm{E}\circ\Phi} = \xi_{\mathrm{E}\circ\Phi}$. Hence if z is the central support of p in A, then $zH_{\mathrm{E}\circ\Phi} = H_{\mathrm{E}\circ\Phi}$, and in order to show that the

A-B-bimodule $H_{\mathrm{E}\circ\Phi}$ is left discrete, it is enough to show that the pAp-B-bimodule $pH_{\mathrm{E}\circ\Phi}$ is left discrete. Now, observe that the map

$$pap\xi_{\mathrm{E}\circ\Phi} \otimes_B \eta \mapsto \pi(pap)\mathrm{E}(x^*x)\eta$$

extends to an isomorphism of pAp-B-bimodule from $pH_{\mathrm{E}\circ\Phi}$ to $q\mathrm{L}^2(B)$ where $q\mathrm{L}^2(B)$ is equipped with the obvious left pAp-action induced by π. Since $\pi(pAp)$ is with expectation in qMq and in particular in qBq, then the pAp-B-bimodule $pH_{\mathrm{E}\circ\Phi} \cong q\mathrm{L}^2(B)$ is left discrete. Therefore $\mathrm{E} \circ \Phi$ is left discrete for every normal conditional expectation $\mathrm{E} : 1_B M 1_B \to B$. This shows that $x \in \mathcal{D}^l_M(A, B)$ as we wanted. □

Theorem 23.7. *Let M be a von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be two von Neumann subalgebras with expectation. Suppose that $\mathrm{E}_B : 1_B M 1_B \to B$ is a faithful normal conditional expectation and use it to view $\mathrm{L}^2(B)$ as a sub-B-B-bimodule of $\mathrm{L}^2(1_B M 1_B)$. Then every left discrete sub-A-B-bimodule of $1_A \mathrm{L}^2(M) 1_B$ is a direct sum of sub-A-B-bimodules of the form $\overline{Ax\mathrm{L}^2(B)}$ with $x \in \mathcal{I}^l(A, B)$.*

Proof. Without loss of generality, we assume that $1_A = 1_B = 1_M$. By a straightforward maximality argument, it is enough to show that any nonzero left discrete A-B-bimodule $H \subset \mathrm{L}^2(M)$ contains a nonzero A-B-bimodule of the desired form. Let $e_B \in \langle M, B\rangle$ be the Jones projection associated to E_B and let $\widehat{\mathrm{E}}_B : \langle M, B\rangle \to M$ be the dual operator valued weight. Let $e \in A' \cap \langle M, B\rangle$ be the projection on H. Then $Ae \subset e\langle M, B\rangle e$ is with expectation. This means that $\widehat{\mathrm{E}}_B$ is semifinite on $(Ae)' \cap e\langle M, B\rangle e = e(A' \cap \langle M, B\rangle)e$. Hence we can find a nonzero projection $e_1 \in A' \cap \langle M, B\rangle$ such that $\widehat{\mathrm{E}}_B(e_1) < +\infty$. Since e_B has central support 1 in $\langle M, B\rangle$, we can find a nonzero partial isometry $u \in e_1\langle M, B\rangle e_B$. Since $Ae_1 \subset e_1\langle M, B\rangle e_1$ is with expectation, then by Proposition 21.4, we can find a nonzero partial isometry $v \in e_1\langle M, B\rangle e_1$ such that vv^* is compatible with Ae_1 and $v^*v \leq uu^*$. Since vv^* is compatible with Ae_1, we can find a projection $e_2 \in e_1(A' \cap \langle M, B\rangle)e_1$ and a projection $p \in A$ such that $vv^* = pe_2$. Let $w = vu$. Since $w^*w \leq e_B$ and $e_B\langle M, B\rangle e_B = Be_B$, we can write $w^*w = qe_B$ with $q \in B$. Moreover, there exists a unique $*$-isomorphism $\rho : pe_2\langle M, B\rangle pe_2 \to qBq$ such that $xw = w\rho(x)$ for all $x \in pe_2\langle M, B\rangle pe_2$.

Now, let $K \subset \mathrm{L}^2(M)$ be the range of e_2. Since $e_2 \in e(A' \cap \langle M, B\rangle)e$, then K is a nonzero sub-A-B-bimodule of H. Since $\widehat{\mathrm{E}}_B(e_2) \leq \widehat{\mathrm{E}}_B(e_1) < +\infty$, we know by *Popa's push down lemma* that $w = xe_B$ where $x = \widehat{\mathrm{E}}_B(w) \in M$ (see [PP84, Lemma 1.2] and [ILP96, Proposition 2.2]). This means that

$$K \supset pK = \mathrm{ran}(w) = \mathrm{ran}(xe_B) = \overline{x\mathrm{L}^2(B)}.$$

Since K is an A-module, we get $K \supset \overline{Ax\mathrm{L}^2(B)}$. Moreover, we have $x = pxq$ and for all $a \in pAp$, we have

$$ax = a\widehat{\mathrm{E}}_B(w) = \widehat{\mathrm{E}}_B(aw) = \widehat{\mathrm{E}}_B(w\pi(ae_2)) = x\rho(ae_2) = x\pi(a)$$

where $\pi : pAp \to qBq$ is the morphism defined by $\pi(a) = \rho(ae_2)$. This shows that $x \in \mathcal{I}^l_M(A, B)$ as we wanted.

□

Corollary 23.8. *Let M be a von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be two von Neumann subalgebras with expectation. Then the left discrete part of the A-B-bimodule $1_A \mathrm{L}^2(M) 1_B$ is given by*

$$\overline{\mathcal{D}^l_M(A, B)\mathrm{L}^2(B^\vee)} = \overline{\mathrm{L}^2(A^\vee)\mathcal{D}^l_M(A, B)}.$$

The following corollary is very useful. Despite the appearances, it does not follow *directly* from Proposition 22.4.

Corollary 23.9. *Let M be a von Neumann algebra. Let $A \subset 1_A M 1_A$, $B \subset 1_B M 1_B$ and $C \subset 1_C M 1_C$ be three von Neumann subalgebras with expectation. Then we have*

$$\mathcal{D}^l_M(A,B) \cdot \mathcal{D}^l_M(B,C) \subset \mathcal{D}^l_M(A,C).$$

In particular, $\mathcal{D}^l_M(A)$ is a (non self-adjoint) closed subalgebra of $1_A M 1_A$ and $\mathcal{D}_M(A)$ is a von Neumann subalgebra of $1_A M 1_A$.

Proof. Let us first show that

$$\mathcal{I}^l_M(A,B) \cdot \mathcal{D}^l_M(B,C) \subset \mathcal{D}^l_M(A,C).$$

Take $x \in \mathcal{I}^l_M(A,B)$ and $y \in \mathcal{D}^l_M(B,C)$. Let $\pi_x : pAp \to qBq$ be the morphism implemented by x. Let $Q = \pi_x(pAp) \subset qBq$. Note that $qy \in \mathcal{D}^l_M(qBq, C) \subset \mathcal{D}^l_M(Q,C)$. Also, we have $|x| \in Q' \cap qMq$. Hence $z = |x|y \in \mathcal{D}^l_M(Q,C)$. Let $\mathrm{E} : 1_C M 1_C \to C$ be a normal conditional expectation. Then for all $a \in pAp$, we have

$$\mathrm{E}(y^* x^* a x y) = \mathrm{E}(y^* x^* x \pi_x(a) y^*) = \mathrm{E}(z^* \pi_x(a) z).$$

The map $\pi_x : pAp \to Q$ is an isomorphism, hence it is left discrete as a completely positive map. The map $m \mapsto \mathrm{E}(z^* m z)$ is left discrete from Q to C because $z \in \mathcal{D}^l_M(Q,C)$. Hence, by composition, the map $a \mapsto \mathrm{E}(y^* x^* a x y)$ is left discrete from pAp to C. This shows that $xy \in \mathcal{D}^l_M(pAp, C) \subset \mathcal{D}^l_M(A,C)$ as we wanted.

Now let X be the linear span of $\mathcal{I}^l_M(A,B)$. By Theorem 23.7, we have

$$\overline{\mathcal{D}^l_M(A,B)\mathrm{L}^2(B^\vee)} = \overline{X\mathrm{L}^2(B^\vee)}.$$

Therefore, we compute

$$\begin{aligned}
\overline{\mathcal{D}^l_M(A,B)\mathcal{D}^l_M(B,C)\mathrm{L}^2(C^\vee)} &= \overline{\mathcal{D}^l_M(A,B)\mathrm{L}^2(B^\vee)\mathcal{D}^l_M(B,C)} \\
&= \overline{X\mathrm{L}^2(B^\vee)\mathcal{D}^l_M(B,C)} \\
&= \overline{X\mathcal{D}^l_M(B,C)\mathrm{L}^2(C^\vee)} \\
&\subset \overline{\mathcal{D}^l_M(A,C)\mathrm{L}^2(C^\vee)}
\end{aligned}$$

This shows that

$$\mathcal{D}^l_M(A,B)\mathcal{D}^l_M(B,C) \subset \mathcal{D}^l_M(A,C)$$

as we wanted. □

Finally, in the finite case, Popa obtained a very useful criterion to show the existence of left discrete elements. This criterion has been generalized to the case where the ambient von Neumann algebra M is of arbitrary type by Houdayer and Isono in [HI15].

Lemma 23.10. *Let M be a von Neumann algebra. Let $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be two von Neumann subalgebras with expectation. Choose a normal conditional expectation $\mathrm{E}_B : M \to B$ and use it to view $\mathrm{L}^2(B) \subset \mathrm{L}^2(M)$. Take $x \in 1_A M 1_B$ and assume that A is finite. Then the following are equivalent:*

(i) *The left discrete part of the A-B-bimodule $\overline{Ax\mathrm{L}^2(B)}$ is zero.*

(ii) *There exists a net of unitaries* $(u_i)_{i\in I}$ *in* A *such that* $\mathrm{E}_B(x^*au_ibx) \to 0$ *strongly when* $i \to \infty$ *for all* $a, b \in A$.

Proof. For simplicity, we assume that $1_A = 1_B = 1$ and that A admits a faithful normal conditional expectation $\mathrm{E}_A : M \to A$ and a faithful normal trace τ. We also assume that E_B is faithful. The proof can easily be adapted to the general case.

(i) $\Rightarrow$ (ii). Suppose that (ii) does not hold. Then we can find φ a normal state on M with $\varphi \circ \mathrm{E}_B = \varphi$, $\varepsilon > 0$, a finite set $F \subset A$ such that

$$\sum_{a,b\in F} \|\mathrm{E}_B(x^*b^*uax)\|_\varphi^2 \geq \varepsilon$$

for all $u \in \mathcal{U}(A)$. Let $d_0 = \sum_{a\in F} axe_Bx^*a^*$ where $e_B \in \langle M, B\rangle$ is the Jones projection. Then $\widehat{\mathrm{E}}_B(d_0) = \sum_{a\in F} axx^*a^*$ is bounded where $\widehat{\mathrm{E}}_B : \langle M, B\rangle \to M$ is the dual operator valued weight. Let C be the weak* closed convex hull of $\{ud_0u^* \mid u \in \mathcal{U}(A)\}$ in $\langle M, B\rangle$. Observe that $\|\widehat{\mathrm{E}}_B(d)\|_\infty \leq \|\widehat{\mathrm{E}}_B(d_0)\|_\infty$ for all $d \in C$. Define a faithful semifinite weight $\psi = \tau \circ \mathrm{E}_A \circ \widehat{\mathrm{E}}_B$ on $\langle M, B\rangle$. Let $d_1 \in C$ be the unique element of minimal $\|\cdot\|_\psi$-norm. Then d_1 must be fixed by $\mathcal{U}(A)$. Hence $d_1 \in A' \cap \langle M, B\rangle$.

Let us show that $d_1 \neq 0$. Let $\varphi^{\frac{1}{2}} \in \mathrm{L}^2(M)$ be the cyclic vector associated to φ. Then for every $u \in \mathcal{U}(A)$ we have

$$\sum_{a\in F} \langle u^*d_0uax\varphi^{\frac{1}{2}}, ax\varphi^{\frac{1}{2}}\rangle = \sum_{a,b\in F} \langle e_Bx^*b^*uax\varphi^{\frac{1}{2}}, x^*b^*uax\varphi^{\frac{1}{2}}\rangle = \sum_{a,b\in F} \|\mathrm{E}_B(x^*b^*uax)\|_\varphi^2 \geq \varepsilon.$$

Therefore we also have

$$\sum_{a\in F} \langle d_1ax\varphi^{\frac{1}{2}}, ax\varphi^{\frac{1}{2}}\rangle \geq \varepsilon$$

so that $d_1 \neq 0$.

Now, by construction $(\mathrm{E}_A \circ \widehat{\mathrm{E}}_B)(d_1)$ is bounded. Hence if we let $e \in A' \cap \langle M, B\rangle$ be the support projection of d_1, then $Ae \subset e\langle M, B\rangle e$ is with expectation, which means precisely that the range of e is a left discrete A-B-bimodule. Moreover, the range of e is contained in $\overline{Ax\mathrm{L}^2(B)}$ by construction. Hence (i) does not hold. $\square$

Theorem 23.11. *Let* M *be a von Neumann algebra. Let* $A \subset 1_AM1_A$ *and* $B \subset 1_BM1_B$ *be two von Neumann subalgebras with expectation. Then the following are equivalent:*

(i) $A \prec_M B$, *i.e.* $\mathcal{I}^l_M(A, B) \neq \{0\}$.

(ii) $\mathcal{D}^l_M(A, B) \neq \{0\}$.

(iii) $1_A\mathrm{L}^2(M)1_B$ *contains a nonzero left discrete* A-B-*bimodule.*

If A *is finite, then all these properties are equivalent to a fourth one:*

(iv) *There does not exist a net of unitaries* $u_i \in A$, $i \in I$ *such that* $\mathrm{E}(y^*u_ix) \to 0$ *strongly for all* $x, y \in 1_AM1_B$ *and every (or some faithful family of) normal conditional expectation* $\mathrm{E} : 1_BM1_B \to B$.

Remark 23.12. We point out the fact that the condition $\mathcal{D}^l_M(A, B) \neq 0$ is often easier to use than the condition $A \prec_M B$ because the subspaces $\mathcal{D}^l_M(A, B)$ have good stability properties (see Remark 23.4). We also recall the fact that the relation $\prec_M$ is *not* transitive. When using $\mathcal{D}^l_M(A, B)$, this issue appears in the following way: let p be the projection in M defined by the weakly closed left ideal

$$Mp^\perp = \{x \in M \mid x\mathcal{D}^l_M(A, B) = \{0\}\}.$$

We call p the left support of $\mathcal{D}^l_M(A,B)$. We defined similarly the right support of $\mathcal{D}^l_M(A,B)$. Now, if we know that $A \prec_M B$ and $B \prec_M C$ and we want to show that $A \prec_M C$, i.e. that $\mathcal{D}^l_M(A,C) \neq 0$, then it is enough to show that the right support of $\mathcal{D}^l_M(A,B)$ intersects the left support of $\mathcal{D}^l_M(B,C)$. Indeed, this would imply that $\mathcal{D}^l_M(A,B)\mathcal{D}^l_M(B,C) \neq \{0\}$. Note that these supports are always contained in $\mathcal{Z}(\mathcal{D}_M(B))$ thanks to Corollary 23.9.

24 Discrete inclusions

The following notion of discrete inclusions is different from the notion of discreteness of [ILP96]. We believe that the inclusions studied in [ILP96] should be called *quasiregular* instead.

Definition 24.1. We call an inclusion of von Neumann algebras $N \subset M$ *discrete* if it satisfies the following equivalent conditions:

(i) The N-M-bimodule ${}_N\mathrm{L}^2(M)_M$ is discrete.

(ii) For some (hence any) spatial realization $M \subset \mathbf{B}(H)$, the inclusions $N \subset M$ and $M' \subset N'$ are both with expectation.

(iii) There exists a faithful family of normal conditional expectations $\mathrm{E}_i : M \to N$ which are left discrete.

(iv) N is with expectation in M and $1 \in \mathcal{D}^l_M(M,N)$.

For a von Neumann algebra M, it is well-known that the inclusion $\mathbb{C} \subset M$ is discrete if and only if M is discrete, i.e. M is generated by its minimal projections.

We will prove a relative version of this property for an arbitrary discrete inclusion $N \subset M$. However, in the relative case, the minimal projections are replaced by projections $p \in N' \cap M$ such that $Np \subset pMp$ is *irreducible with finite index* in the following sense. This definition is not standard but it is very natural in our context.

Definition 24.2. Let $N \subset M$ be an inclusion of von Neumann algebras. We say that $N \subset M$ is *irreducible* if $N' \cap M = \mathcal{Z}(N) = \mathcal{Z}(M)$. In that case, we define the *index* of $N \subset M$ by

$$[M : N] = \widehat{\mathrm{E}}(1) \in \overline{\mathcal{Z}(M)^+}$$

where $\mathrm{E} : M \to N$ is the unique normal conditional expectation and $\widehat{\mathrm{E}} : \langle M, N\rangle \to M$ is its dual operator valued weight.

More generally, if $N \subset M$ is irreducible and z is the largest projection in $\mathcal{Z}(M)$ such that $Nz \subset Mz$ is with expectation, we let $[M : N] = [Mz : Nz]z + \infty \cdot z^{\perp}$. When $[M : N]$ is bounded, we say that $N \subset M$ has *finite index*.

Theorem 24.3. *Let $N \subset M$ be an inclusion of von Neumann algebras. Then $N \subset M$ is discrete if and only if there exists a partition of unity $(p_i)_{i\in I}$ in $N' \cap M$ such that $Np_i \subset p_iMp_i$ is irreducible with finite index for all $i \in I$.*

Proof. Suppose first that $N \subset M$ is an irreducible inclusion with finite index. Let $\mathrm{E} : M \to N$ be the unique normal conditional expectation. Then $\widehat{\mathrm{E}}(1) = [M : N] \in \mathcal{Z}(M)^+$. Hence $\widehat{\mathrm{E}} : \langle M, N\rangle \to M$ is a bounded faithful normal operator valued weight. Therefore, the inclusion $M \subset \langle M, N\rangle$ is with expectation which means that $N \subset M$ is discrete.

Now, suppose that there exists a partition of unity $(p_i)_{i\in I}$ in $N'\cap M$ such that $Np_i \subset p_iMp_i$ is irreducible with finite index for all $i \in I$. Then $Np_i \subset p_iMp_i$ is with expectation for all i. Hence $N \subset M$ is with expectation. Moreover, since $Np_i \subset p_iMp_i$ is discrete, then $p_i \in \mathcal{D}^l_{p_iMp_i}(p_iMp_i, Np_i) \subset \mathcal{D}^l_M(M,N)$ for all i. Hence $1 = \sum_{i\in I} p_i \in \mathcal{D}^l_M(M,N)$ and $N \subset M$ is discrete. This proves the if direction.

For the only if direction, it is sufficient, by a standard maximality argument, to find a nonzero projection $p \in N'\cap M$ such that $Np \subset pMp$ is irreducible with finite index. Let $N^\vee = N \vee (N'\cap M)$ Since $N \subset M$ is discrete and $N^\vee$ is with expectation in M, then $N \subset N^\vee$ is also discrete. Hence $N^\vee \subset \langle N^\vee, N\rangle$ is with expectation. Thus the inclusion $N'\cap M = N'\cap N^\vee \subset N'\cap \langle N^\vee, N\rangle = \langle N'\cap M, \mathcal{Z}(N)\rangle$ is also with expectation. This shows that the inclusion $\mathcal{Z}(N) \subset N'\cap M$ is discrete. In particular, we have $N'\cap M \prec_{N'\cap M} \mathcal{Z}(N)$. Since $\mathcal{Z}(N)$ is in the center of $N'\cap M$, this really means that we can find a nonzero projection $p \in N'\cap M$ such that $p(N'\cap M)p = \mathcal{Z}(N)p$. Note that the inclusion $M'p \subset pN'p$ is also discrete. Hence, by repeating the same argument, we can find a nonzero $q \in (M'p)'\cap pN'p = p(N'\cap M)p$ such that $q(N'\cap M)q = \mathcal{Z}(M'p)q = \mathcal{Z}(M)q$. Since, we already have $p(N'\cap M)p = \mathcal{Z}(N)p$, we really have $q(N'\cap M)q = \mathcal{Z}(M)q = \mathcal{Z}(N)q$, or equivalently, the inclusion $Nq \subset qMq$ is irreducible. Let $\mathrm{E} : qMq \to Nq$ be the unique normal conditional expectation. Since $Nq \subset qMq$ is discrete, then $qMq \subset \langle M,N\rangle = p\langle M,N\rangle q$ is with expectation. Hence the restriction of $\widehat{\mathrm{E}}$ to

$$(qMq)'\cap \langle qMq, Nq\rangle = (M'\cap \langle M,N\rangle)q = \mathcal{Z}(qMq)$$

is semifinite. Equivalently, this means that $[qMq : Nq] = \widehat{\mathrm{E}}(1)$ is a semifinite element of $\overline{\mathcal{Z}(qMq)^+}$. Let $e \in \mathcal{Z}(qMq)$ be a nonzero projection such that $[qMq : Nq]e$ is bounded. Then $Ne \subset eMe$ is irreducible with finite index $[eMe : Ne] = [qMq : Nq]e$ and we are done. $\square$

Let A and B be two von Neumann algebras. We say that an A-B-bimodule H is irreducible if $\pi(A)'\cap\pi(B^{\mathrm{op}})' = \mathcal{Z}(\pi(A)) = \mathcal{Z}(\pi(B^{\mathrm{op}}))$ where $\pi : A\odot B^{\mathrm{op}} \to \mathbf{B}(H)$ is the associated representation. Note that H is irreducible if and only if the inclusion $\pi(A) \subset \pi(B^{\mathrm{op}})'$ (or equivalently $\pi(B^{\mathrm{op}}) \subset \pi(A)'$) is irreducible. In that case we define the *index of* H as the index of the inclusion $\pi(A) \subset \pi(B^{\mathrm{op}})'$ (or equivalently the index of $\pi(B^{\mathrm{op}}) \subset \pi(A)'$) which is an element of the extended positive cone of $\mathcal{Z}(\pi(A)) = \mathcal{Z}(\pi(B^{\mathrm{op}}))$.

Finally, observe that H is discrete if and only if the inclusion $\pi(A) \subset \pi(B^{\mathrm{op}})'$ is discrete. Hence, we obtain the following.

Corollary 24.4. *Let A and B be two von Neumann algebras. Then a A-B-bimodule is discrete if and only if it is a direct sum of irreducible bimodules with finite index.*

Note that an inclusion of type I von Neumann algebras is irreducible if and only if it is a trivial inclusion. Hence, a bimodule between two type I von Neumann algebras A and B is discrete if and only if it is a direct sum of irreducible bimodules and the irreducible A-B-bimodules are the bimodules of the form $\mathrm{L}^2(\theta)$ where $\theta : Ae \to Bf$ is an isomorphism and e and f are two central projections in A and B respectively.

When A and B are infinite factors, an irreducible A-B-bimodule with finite index is just a bimodule of the form $\mathrm{L}^2(\rho)$ where $\rho : A \to B$ is a morphism such that the inclusion $\rho(A) \subset B$ is irreducible with finite index.

In general, irreducible bimodules with finite index should be considered as generalized symmetries which serve as building blocks for all discrete correspondences.

25 Quasiregular inclusions

The following notion of quasiregular inclusions coincides, in the case of factors, with the notion of discrete inclusions of [ILP96].

Definition 25.1. Let $N \subset M$ be an inclusion of von Neumann algebras with expectation. We call $\mathcal{D}_M(N)$ the *quasinormalizer of N inside M.*

We say that the inclusion $N \subset M$ is *quasiregular* if it satisfies the following equivalent conditions:

(i) $\mathcal{D}_M(N) = M$.

(ii) The inclusion $N \subset \langle M, N\rangle$ is with expectation.

(iii) The inclusion $N \subset \langle M, N\rangle$ is discrete.

(iv) The N-N-bimodule ${}_N\mathrm{L}^2(M)_N$ is discrete.

The terminology is justified by the next theorem which gives an explicit description of the von Neumann algebra $\mathcal{D}_M(N)$ for arbitrary inclusions $N \subset M$ and also under more restrictive assumptions. It shows, for example, that a maximal abelian subalgebra is quasiregular if and only if it is a Cartan subalgebra. Also all inclusions coming from crossed products are quasiregular.

Theorem 25.2. *Let $N \subset M$ be an inclusion of von Neumann algebras with expectation. The following properties hold.*

(i) *$\mathcal{D}_M(N)$ is the von Neumann algebra generated by the set of all partial isometries $v \in M$ such that there exists a morphism $\pi : pNp \to qNq$ with $pvq = v$ and $av = v\pi(a)$ for all $a \in N$ and $\pi(pNp) \subset qNq$ is irreducible with finite index.*

(ii) *If N is finite then $\mathcal{D}_M(N)$ is the von Neumann algebra generated by the set of all $x \in M$ for which there exists a finite set $F \subset M$ such that*

$$xN \subset \sum_{y \in F} Ny \quad \textit{and} \quad Nx \subset \sum_{y\in F} yN.$$

(iii) *If N is abelian then $\mathcal{D}_M(N)$ is the von Neumann algebra generated by the set of all partial isometries $v \in M$ such that*

$$\{vv^*, v^*v\} \subset N' \cap M \quad \textit{and} \quad Nv = vN.$$

Proof. For simplicity, we assume that N admits a faithful normal conditional expectation $\mathrm{E} : M \to N$ and we use it to view $\mathrm{L}^2(N) \subset \mathrm{L}^2(M)$. The general case can be deduced easily from this case by cutting down with σ-finite projections in $N' \cap M$.

(i). Let Q be the von Neumann algebra generated by the set of all $x \in \mathcal{I}^l_M(N, N)$ such that the inclusion $\pi_x(pNp) \subset qNq$ is irreducible with finite index. For all such x, we have $x \in \mathcal{D}_M(N, \pi_x(pNp))$. And since $\pi_x(pNp)$ is irreducible with finite index in qNq, we have $q \in \mathcal{D}_M(\pi_x(pNp), qNq)$. Hence $x = xq \in \mathcal{D}_M(N, N)$. This shows that $Q \subset \mathcal{D}_M(N)$. Conversely, observe that Q contains N and $N' \cap M$, hence $Q' \cap M \subset Q$ and there exists a canonical faithful normal conditional expectation from M to Q, use it to view $\mathrm{L}^2(Q) \subset \mathrm{L}^2(M)$. Suppose there exists a nonzero discrete sub-N-N-bimodule $H \subset \mathrm{L}^2(M) \ominus \mathrm{L}^2(Q)$. Thanks to 24.4, we can assume that H is irreducible with finite index. By Theorem 23.8, it contains a sub-bimodue of the form

$\overline{N x \mathrm{L}^2(N)}$ for some nonzero $x \in \mathcal{I}_M^l(N,N)$. Let $\pi_x : pNp \to qNq$ be the morphism it implements. The pNp-qNq bimodule $p\overline{N x \mathrm{L}^2(N)}q$ is still irreducible with finite index and it contains $\mathrm{L}^2(\pi_x)$. This means that $\pi_x(pNp) \subset qNq$ is irreducible with finite index. Take the polar decomposition $x = |x^*|v$, then we get $v \in Q$. Since Q contains $N \vee N' \cap M$, we also have $|x^*| \in p(N' \cap M)p \subset Q$. Hence $x \in Q$ and therefore $H \subset \mathrm{L}^2(Q)$ as we wanted.

(ii). Take $x \in M$ for which there exists a finite set $F \in M$ such that

$$xN \subset \sum_{y \in F} Ny \quad \text{and} \quad Nx \subset \sum_{y \in F} yN.$$

Let $\mathrm{E} : M \to N$ be a normal conditional expectation and use it to view $\mathrm{L}^2(N)$ as subspace of $\mathrm{L}^2(M)$. Then $\overline{N x \mathrm{L}^2(N)} \subset \overline{F \mathrm{L}^2(N)}$ is finitely generated as a right N-module, hence it is left discrete, or equivalently, x is left discrete. Similarly, we show that x is right discrete. Thus we have that $x \subset \mathcal{D}_M(N)$.

Now, let Q be the von Neumann algebra generated by the set of all such x. Suppose that the discrete N-N-bimodule $\mathrm{L}^2(\mathcal{D}_M(N)) \ominus \mathrm{L}^2(Q)$ is nonzero. Then it contains a nonzero sub-N-N-bimodule H which is finitely generated both as a left and right N-module. By Theorem 23.7, we can find a nonzero $x \in M$ such that $\overline{N x \mathrm{L}^2(N)} \subset H$. Then by the Gram-Schmidt algorithm, we can find a finite family $y_1, \dots, y_q \in NxN$ which generate $\overline{N x \mathrm{L}^2(N)}$ as a right N-module and such that $\mathrm{E}(y_i^* y_j) = \delta_{i,j}$ for all $1 \leq i,j \leq q$. Then for all $a \in Nx$, we have $a = \sum_{i=1}^q y_i \mathrm{E}(y_i^* a)$. Hence $Nx \subset \sum_{i=1}^q y_i N$. Observe that $\overline{\mathrm{L}^2(N) x N}$ is isomorphic to $\overline{N x \mathrm{L}^2(N)}$ as a N-N-bimodule, hence $\overline{\mathrm{L}^2(N) x N}$ is also finitely generated as a left N-module. Hence by repeating the same argument, we can find another finite family $y_{q+1}, \dots, y_r$ such that $xN \subset \sum_{i=q+1}^r N y_i$. By taking $F = \{y_1, \dots, y_r\}$, we conclude that $x \in Q$. A contradiction. Hence, we must have $\mathrm{L}^2(\mathcal{D}_M(N)) = \mathrm{L}^2(Q)$ and therefore $\mathcal{D}_M(N) = Q$ as we wanted.

(iii) This follows from (i). Indeed, for a partial isometry $v \in \mathcal{I}_M^l(N,N)$, the inclusion of abelian von Neumann algebras $\pi_v(pNp) \subset \pi_v(qNq)$ is irreducible if and only if $\pi_v(pNp) = qNq$, so that π_v is a partial isomorphism of N. This happens exactly when

$$\{vv^*, v^*v\} \subset N' \cap M \quad \text{and} \quad Nv = vN.$$

□

Note that one can deduce similar descriptions of $\mathcal{D}_M(A,B)$ for any pair of von Neumann subalgebras A and B by using Remark 23.4. In particular, we have the following criterion.

Corollary 25.3. *Let M be a von Neumann algebra, $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ two von Neumann subalgebras with expectation. Then the following are equivalent:*

(i) *There exist a morphism $\pi : pAp \to qBq$ and a non-zero partial isometry $v \in pMq$ such that $av = v\pi(a)$ for all $a \in pAp$ and $\pi(pNp) \subset qNq$ is irreducible with finite index.*

(ii) $\mathcal{D}_M(A,B) \neq \{0\}$

(iii) *The discrete part of the A-B-bimodule $1_A \mathrm{L}^2(M) 1_B$ is nonzero.*

Finally, one can recover all the results of [PS03] and generalize them to the non-tracial situation. Indeed, our $\mathcal{D}_M(A,B)$ is a generalization of what is denoted $N_q^w(A,B)$ in [PS03] for abelian A and B.

26 Stable unitary conjugacy

In this section, we study the notion of *stable unitary conjugacy* for tensor product decompositions [HMV16].

Definition 26.1. Let M be any σ-finite factor. A *tensor product decomposition* of M is a pair (A_1, A_2) of subalgebras of M such that $M = A_1 \mathbin{\overline{\otimes}} A_2$. Two decompositions (A_1, A_2) and (B_1, B_2) are said to be:

(i) *unitarily conjugate* if there exists $u \in \mathcal{U}(M)$ such that $uA_iu^* = B_i$ for all $i \in \{1, 2\}$. We then write $(A_1, A_2) \sim (B_1, B_2)$.

(ii) *stably unitarily conjugate* if there exist type I factors with separable predual F_1 and F_2 such that the two tensor product decompositions $(A_1 \mathbin{\overline{\otimes}} F_1, A_2 \mathbin{\overline{\otimes}} F_2)$ and $(B_1 \mathbin{\overline{\otimes}} F_1, B_2 \mathbin{\overline{\otimes}} F_2)$ are unitarily conjugate in $M \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} F_2$. We then write $(A_1, A_2) \sim_\infty (B_1, B_2)$.

We will need the following lemma.

Lemma 26.2. *Let M and N be any von Neumann algebras and $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ be any von Neumann subalgebras with expectation.*

The following conditions are equivalent:

(i) $A \prec_M B$.

(ii) $A \otimes \mathbb{C} \prec_{M \overline{\otimes} N} B \otimes \mathbb{C}$.

(iii) $A \otimes \mathbb{C} \prec_{M \overline{\otimes} N} B \mathbin{\overline{\otimes}} N$.

Proof. This follows from Proposition 22.2. $\square$

We record in Proposition 26.3 below several useful properties of the notion of stable unitary conjugacy.

Proposition 26.3. *Let M be any σ-finite factor. The following properties hold true:*

(i) *The relations $\sim$ and $\sim_\infty$ are equivalence relations.*

(ii) *If $M = A_1 \mathbin{\overline{\otimes}} A_2$ is a tensor product decomposition of M and if F is a type* I *factor with separable predual then $(A_1 \mathbin{\overline{\otimes}} F, A_2) \sim_\infty (A_1, F \mathbin{\overline{\otimes}} A_2)$ in $M \mathbin{\overline{\otimes}} F$.*

(iii) *If $M = A \mathbin{\overline{\otimes}} F = B \mathbin{\overline{\otimes}} G$ are two tensor product decompositions of M where F and G are infinite type* I *factors with separable predual then we can find a nonzero partial isometry $v \in M$ with $v^*v = p \in A$ and $vv^* = q \in B$ such that $v\,pAp\,v^* = qBq$ and $v\,pFp\,v^* = qGq$. If both A and B are infinite then we can choose $p = q = 1$ so that $(A, F) \sim (B, G)$.*

(iv) *If $M = A_1 \mathbin{\overline{\otimes}} A_2 = B_1 \mathbin{\overline{\otimes}} B_2$ are two tensor product decompositions of M then we have $(A_1, A_2) \sim_\infty (B_1, B_2)$ if and only if there exist nonzero projections $p_i \in A_i$ and $q_i \in B_i$ and a partial isometry $v \in M$ with $v^*v = p = p_1p_2$ and $vv^* = q = q_1q_2$ such that $v\,pA_1p\,v^* = qB_1q$ and $v\,pA_2p\,v^* = qB_2q$. If A_i and B_i are infinite for every $i \in \{1, 2\}$ then we can choose $p = q = 1$ so that $(A_1, A_2) \sim (B_1, B_2)$.*

(v) *If $M = A_1 \mathbin{\overline{\otimes}} A_2 = B_1 \mathbin{\overline{\otimes}} B_2$ are two tensor product decompositions of M then we have $A_1 \prec_M B_1$ if and only if there exists a tensor product decomposition $B_1 = C \mathbin{\overline{\otimes}} D$ such that $(A_1, A_2) \sim_\infty (C, D \mathbin{\overline{\otimes}} B_2)$.*

Proof. (i) It is obvious that $\sim$ is an equivalence relation and that $\sim_\infty$ is symmetric and reflexive. Let us prove that $\sim_\infty$ is transitive. Suppose that (A_1, A_2), (B_1, B_2) and (C_1, C_2) are three tensor product decompositions of M such that $(A_1, A_2) \sim_\infty (B_1, B_2)$ and $(B_1, B_2) \sim_\infty (C_1, C_2)$. Then we can find type I factors with separable predual F_i and G_i for $i \in \{1, 2\}$ such that $(A_1 \mathbin{\overline{\otimes}} F_1, A_2 \mathbin{\overline{\otimes}} F_2) \sim (B_1 \mathbin{\overline{\otimes}} F_1, B_2 \mathbin{\overline{\otimes}} F_2)$ in $M \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} F_2$ and $(B_1 \mathbin{\overline{\otimes}} G_1, B_2 \mathbin{\overline{\otimes}} G_2) \sim (C_1 \mathbin{\overline{\otimes}} G_1, C_2 \mathbin{\overline{\otimes}} G_2)$ in $M \mathbin{\overline{\otimes}} G_1 \mathbin{\overline{\otimes}} G_2$. Then we naturally have $(A_1 \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} G_1, A_2 \mathbin{\overline{\otimes}} F_2 \mathbin{\overline{\otimes}} G_2) \sim (C_1 \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} G_1, C_2 \mathbin{\overline{\otimes}} F_2 \mathbin{\overline{\otimes}} G_2)$ in $M \mathbin{\overline{\otimes}} (F_1 \mathbin{\overline{\otimes}} G_1) \mathbin{\overline{\otimes}} (F_2 \mathbin{\overline{\otimes}} G_2)$. Hence $(A_1, A_2) \sim_\infty (C_1, C_2)$ as we wanted.

(ii) Let G_1, G_2 be two infinite type I factors with separable predual. Then G_1 is isomorphic to $G_1 \mathbin{\overline{\otimes}} F$ and $F \mathbin{\overline{\otimes}} G_2$ is isomorphic to G_2. Hence we can find an automorphism ϕ of $G_1 \mathbin{\overline{\otimes}} F \mathbin{\overline{\otimes}} G_2$ such that $\phi(G_1) = G_1 \mathbin{\overline{\otimes}} F$ and $\phi(F \mathbin{\overline{\otimes}} G_2) = G_2$. Since $G_1 \mathbin{\overline{\otimes}} F \mathbin{\overline{\otimes}} G_2$ is a type I factor, ϕ is inner. Hence $(G_1, F \mathbin{\overline{\otimes}} G_2) \sim (G_1 \mathbin{\overline{\otimes}} F, G_2)$ in $G_1 \mathbin{\overline{\otimes}} F \mathbin{\overline{\otimes}} G_2$. Therefore $(A_1 \mathbin{\overline{\otimes}} G_1, A_2 \mathbin{\overline{\otimes}} F \mathbin{\overline{\otimes}} G_2) \sim (A_1 \mathbin{\overline{\otimes}} F \mathbin{\overline{\otimes}} G_1, A_2 \mathbin{\overline{\otimes}} G_2)$ which means exactly that $(A_1, F \mathbin{\overline{\otimes}} A_2) \sim_\infty (A_1 \mathbin{\overline{\otimes}} F, A_2)$ as we wanted.

(iii) Take $(e_{k,l})_{k,l \in \mathbb{N}}$ and $(f_{k,l})_{k,l \in \mathbb{N}}$ two systems of matrix units for F and G respectively. Choose p and q two nonzero projections in A and B respectively such that pe_{00} and qf_{00} are equivalent in M. If both A and B are infinite, then we can take $p = q = 1$. Take $v \in M$ such that $v^*v = pe_{00}$ and $vv^* = qf_{00}$ and define $V = \sum_{k \in \mathbb{N}} f_{k0} v e_{0k}$. Then we have $V^*V = p$, $VV^* = q$ and $V\, pAp\, V^* = qBq$ and $V\, pFp\, V^* = qGq$.

(iv) Firstly, we prove the "if" direction. Take nonzero projections $p_i \in A_i$ and $q_i \in B_i$ and a partial isometry $v \in M$ with $v^*v = p = p_1p_2$ and $vv^* = q = q_1q_2$ such that $v\, pA_1p\, v^* = qB_1q$ and $v\, pA_2p\, v^* = qB_2q$. Take F_1, F_2 two infinite type I factors with separable predual. Then there exist isometries $r_i \in A_i \overline{\otimes} F_i$ and $s_i \in B_i \overline{\otimes} F_i$ such that $r_ir_i^* = p_i$ and $s_is_i^* = q_i$. Then $U = (s_1s_2)^* v (r_1r_2)$ is a unitary in $M \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} F_2$ such that $U(A_i \mathbin{\overline{\otimes}} F_i)U^* = B_i \mathbin{\overline{\otimes}} F_i$.

Secondly, we prove the "only if" direction. Assume that $(A_1, A_2) \sim_\infty (B_1, B_2)$. Then we can find two type I factors with separable predual F_1, F_2 and a unitary $u \in \mathcal{U}(M \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} F_2)$ such that $u(A_i \mathbin{\overline{\otimes}} F_i)u^* = B_i \mathbin{\overline{\otimes}} F_i$ for every $i \in \{1, 2\}$. Now, by applying (iii) to the two tensor product decompositions of $N_i = uA_iu^* \mathbin{\overline{\otimes}} uF_iu^* = B_i \mathbin{\overline{\otimes}} F_i$, we see that we can find a nonzero partial isometry $v_i \in N_i$ with $v_i^*v_i = up_iu^* \in uA_iu^*$ and $v_iv_i^* = q_i \in B_i$ (and $p_i = q_i = 1$ if A_i and B_i are infinite) such that $(v_iu)\, p_iA_ip_i\, (v_iu)^* = q_iB_iq_i$ and $(v_iu)\, p_iF_ip_i\, (v_iu)^* = q_iF_iq_i$ for every $i \in \{1, 2\}$. Then there is a unique automorphism ϕ_i of F_i such that $\phi(x)v_iu = v_iux$ for all $x \in F_i$ and ϕ_i must be inner because F_i is of type I. Thus, up to replacing v_i by w_iv_i for some unitary $w_i \in F_i$, we may assume that v_iu commutes with F_i. Finally, if we let $V = v_1v_2u \in M \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} F_2 = N_1 \mathbin{\overline{\otimes}} N_2$, we see that V commutes with $F_1 \mathbin{\overline{\otimes}} F_2$ which means that $V \in M$ and we have $V^*V = p = p_1p_2$, $VV^* = q = q_1q_2$ and $V\, pA_ip\, V^* = q_iB_iq$ for every $i \in \{1, 2\}$. Moreover, if all A_i and B_i are infinite then we can choose $p = q = 1$ and hence $(A_1, A_2) \sim (B_1, B_2)$.

(v) Suppose that $A_1 \prec_M B_1$. Take a nonzero partial isometry $v \in \mathcal{I}^l_M(A_1, B_1)$. Let $\pi_v : p_1A_1p_1 \to q_1B_1q_1$ be the morphism it implements. By reducing q_1 if necessary, and reducing v accordingly, we may assume that q_1 is a minimal projection in some type I subfactor K of B_1. Write $vv^* = p = p_1p_2$ with $p_2 \in A_2$ and $v^*v = q = q_1q_2$ with $q_2 \in B_2$. Let $P = v^*A_1v \subset qB_1q$ and $Q = P' \cap qB_1q$. Since $v^*(A_2)v$ must be equal to $P' \cap qMq$ and $A_1 and A_2$ generate M, we know that P and $P' \cap qMq$ must also generate qMq. Hence P and Q must generate qB_1q, i.e. we have $qB_1q = P \mathbin{\overline{\otimes}} Q$. Now, pick an isomorphism $\phi : q_1B_1q_1 \mathbin{\overline{\otimes}} K \to B_1$ such that $\phi(x \otimes q_1) = x$ for all $x \in q_1B_1q_1$ and let $C = \phi(P \mathbin{\overline{\otimes}} K)$ and $D = \phi(Q)$.

Then we have $B_1 = C \mathbin{\overline{\otimes}} D$, $p_i \in A_i$, $q_1 \in C$, $q_2 \in D \mathbin{\overline{\otimes}} B_2$ and $v\, pA_1p\, v^* = qCq$ and $v\, pA_2p\, v^* = q(D \mathbin{\overline{\otimes}} B_2)q$. Hence by (iv), we obtain $(A_1, A_2) \sim_\infty (C, D \mathbin{\overline{\otimes}} B_2)$. Conversely, if $(A_1, A_2) \sim_\infty (C, D \mathbin{\overline{\otimes}} B_2)$ with $B_1 = C \mathbin{\overline{\otimes}} D$, then there exist type I factors F_1 and F_2 such that $A_1 \mathbin{\overline{\otimes}} F_1$ is unitarily conjugate to $C \mathbin{\overline{\otimes}} F_1$ in $M \mathbin{\overline{\otimes}} F_1 \mathbin{\overline{\otimes}} F_2$ which implies that $A_1 \prec_M C$ by Lemma 26.2 and therefore $A_1 \prec_M B_1$. $\square$

References

[Aa68] J. F. Aarnes, *On the Mackey-topology for a von Neumann algebra.* Math. Scan. **22** (1968), 87–107.

[Ar70] H. Araki, *Asymptotic Ratio Set and Property L'_λ.* Publ. Res. Inst. Math. Sci. **6** (1970), 443–460.

[AH12] H. Ando, U. Haagerup, *Ultraproducts of von Neumann algebras.* J. Funct. Anal. **266** (2014), 6842–6913.

[BB16] R. Boutonnet, A. Brothier, *Crossed-products by locally compact groups: Intermediate subfactors.* To appear in J. Operator Theory. `arXiv:1611.10121`

[BISG15] R. Boutonnet, A. Ioana, A. Salehi Golsefidy, *Local spectral gap in simple Lie groups and applications.* To appear in Invent. Math. `arXiv:1503.06473`

[Bo12] R. Boutonnet, *On solid ergodicity for Gaussian actions.* J. Funct. Anal., **263** (2012) 1040—1063.

[CI10] I. Chifan, A. Ioana *Ergodic subequivalence relations induced by a Bernoulli action.* Geom. and Funct. Anal. **20** (2010), 53–67.

[CP12] V. Capraro, L. Paunescu, *Product between ultrafilters and applications to Connes embedding problem*, J. Operator Theory **68** (2012), 165–172.

[Co72] A. Connes, *Une classification des facteurs de type* III. Ann. Sci. École Norm. Sup. **6** (1973), 133–252.

[Co74] A. Connes, *Almost periodic states and factors of type* III_1, J. Funct. Anal. **16** (1974), 415–445.

[Co75a] A. Connes, *Outer conjugacy classes of automorphisms of factors.* Ann. Sci. École Norm. Sup. **8** (1975), 383–419.

[Co75b] A. Connes, *Classification of injective factors. Cases* II_1*,* II_∞*,* III_λ*,* $\lambda \neq 1$*.* Ann. of Math. **74** (1976), 73–115.

[Co85] A. Connes, *Factors of type* III_1*, property* L'_λ*, and closure of inner automorphisms.* J. Operator Theory **14** (1985), 189–211.

[CS78] A. Connes, E. Størmer, *Homogeneity of the state space of factors of type* III_1, J. Funct. Anal. **28** (1978), 187–196.

[CT76] A. Connes, M. Takesaki, *The flow of weights of factors of type* III. Tohoku Math. Journ. **29** (1977), 473–575.

[DM71] J. Dixmier, O. Marechal, *Vecteurs totalisateurs d'une alg'ebre de von Neumann.* Commun. Math. Phys. **22** (1971), 44–50.

[FM75] J. Feldman, C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras.* I, II. Trans. Amer. Math. Soc. **234** (1977), 289–324, 325–359.

[Go75] V. J. Golodets, *Spectral properties of modular operators and the asymptotic ratio set.* (in Russian) Izv. Akad. Nauk SSSR Math. Ser. Mat. Tom **39** (1975), English translation in Math. USSR Izv. **9** (1975), 599–619.

[Ha73] U. Haagerup, *The standard form of von Neumann algebras.* Math. Scand. **37** (1975), 271–283.

[Ha77a] U. Haagerup, *Operator valued weights in von Neumann algebras*, I. J. Funct. Anal. **32** (1979), 175–206.

[Ha77b] U. Haagerup, *Operator valued weights in von Neumann algebras*, II. J. Funct. Anal. **33** (1979), 339–361.

[Ha84] U. Haagerup, *A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space.* J. Funct. Anal. **62** (1985), 160–201.

[Ha85] U. Haagerup, *Connes' bicentralizer problem and uniqueness of the injective factor of type* III_1. Acta Math. **69** (1986), 95–148.

[HI15] C. Houdayer, Y. Isono, *Unique prime factorization and bicentralizer problem for a class of type* III *factors.* Adv. Math. **305** (2017), 402–455.

[HI15b] C. Houdayer, Y. Isono, *Bi-exact groups, strongly ergodic actions and group measure space type* III *factors with no central sequence.* Comm. Math. Phys. **348** (2016), 991–1015.

[HMV16] C. Houdayer, A. Marrakchi, P. Verraedt, *Fullness and Connes' τ invariant of type* III *tensor product factors.* To appear in J. Math. Pures Appl. `arXiv:1611.07914`

[HMV17] C. Houdayer, A. Marrakchi, P. Verraedt, *Strongly ergodic equivalence relations: spectral gap and type* III *invariants.* To appear in Ergodic Theory Dynam. Systems.

[Hou11] C. Houdayer, *Invariant percolation and measured theory of non-amenable groups.* Séminaire Bourbaki, 2011.

[Ho15] D.J. Hoff, *Von Neumann algebras of equivalence relations with nontrivial one-cohomology.* J. Funct. Anal. **270** (2016), 1501–1536.

[HP17] C. Houdayer, S. Popa, *Singular masas in type* III *factors and Connes' bicentralizer property.* To appear in Proceedings of the 9th MSJ-SI "Operator Algebras and Mathematical Physics". `arXiv:1704.07255`

[HR14] C. Houdayer, S. Raum, *Asymptotic structure of free Araki-Woods factors.* Math. Ann. **363** (2015), 237–267.

[HU15] C. Houdayer, Y. Ueda, *Rigidity of free product von Neumann algebras..* Compos. Math. **152** (2016), 2461–2492.

[I07] A. Ioana, *Rigidity results for wreath product* II_1 *factors.* J. Func. Analysis, **252** (2007), 763—791

[I11] A. Ioana, *W*-superrigidity for Bernoulli actions of property (T) groups.* J. Amer. Math. Soc. **24** (2011), 1175–1226

[ILP96] M. Izumi, R. Longo, S. Popa, *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras.* J. Funct. Anal. **155** (1998), 25–63.

[IPP08] A. Ioana, J. Peterson, S. Popa, *Amalgamated free products of w-rigid factors and calculation of their symmetry groups.* Acta Math. **200** (2008), 85–153

[IV15] A. Ioana, S. Vaes, *Spectral gap for inclusions of von Neumann algebras.* Appendix to the article *Cartan subalgebras of amalgamated free product* II_1 *factors* by A. Ioana in Ann. Sci. École Norm. Sup. **48** (2015), 71–130.

[Jo81] V.F.R. Jones, *Central sequences in crossed products of full factors.* Duke Math. J. **49** (1982), 29–33.

[Jo82] V.F.R. Jones, *Index for subfactors.* Invent. Math. **72** (1983), 1–25.

[JS87] V.F.R. Jones, K. Schmidt, *Asymptotically Invariant Sequences and Approximate Finiteness.* American. J. Math. **109** (1987), 91–114.

[Ka67] R.V. Kadison, *Problems on von Neumann algebras.* Baton Rouge Conference, 1967 (unpublished).

[Ko85] H. Kosaki, *Extension of Jones' theory on index to arbitrary factors.* J. Funct. Anal. **66** (1986), 123–140.

[KR97] R.V. Kadison, J.R. Ringrose, *Fundamentals of the theory of operator algebras.* I. *Elementary theory.* Reprint of the 1983 original. Graduate Studies in Mathematics, **15**. Amer. Math. Soc., Providence, RI, 1997. xvi+398 pp.

[Ma16a] A. Marrakchi, *Solidity of type* III *Bernoulli crossed croducts.* Comm. Math. Phys. **350** (2016), 897–916.

[Ma16b] A. Marrakchi, *Spectral gap characterization of full type* III *factors.* To appear in J. Reine Angew. Math. 2016, `arXiv:1605.09613.`

[Ma17a] A. Marrakchi, *Strongly ergodic actions have local spectral gap.* To appear in Proc. Amer. Math. Soc. 2017, `arXiv:1707.00438.`

[Ma17b] A. Marrakchi, *Stability of products of equivalence relations.* To appear in Compos. Math. 2017, `arXiv:1709.00357.`

[McD69] D. McDuff, *Central sequences and the hyperfinite factor.* Proc. London Math. Soc. **21** (1970), 443–461.

[Oz03] N. Ozawa, *Solid von Neumann algebras.* Acta Math. **192** (2004), 111–117.

[Pa73] W. L. Paschke, *Inner product modules over B^*-algebras.* Trans. Amer. Math. Soc. **182** (1973), 443–468.

[Pi93] G. Pisier, *Espace de Hilbert d'opérateurs et interpolation complexe.* Comptes Rendus Acad. Sci. Paris Série I, **316** (1993), 47—52.

[PP84] M. Pimsner, S. Popa, *Entropy and index for subfactors.* Ann. Sci. École Norm. Sup. **19** (1986), 57–106.

[Po81] S. Popa, *On a problem of R.V. Kadison on maximal abelian $*$-subalgebras in factors.* Invent. Math. **65** (1981), 269–281.

[Po83] S. Popa, *Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras.* J. Operator Theory **9** (1983), 253–268.

[Po86] S. Popa, *Correspondences* Preprint INCREST (1986).

[Po95] S. Popa, *Classification of subfactors and their endomorphisms.* CBMS Regional Conference Series in Mathematics, **86**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1995. x+110 pp.

[Po98] S. Popa, *On the relative Dixmier property for inclusions of* C^**-algebras.* J. Funct. Anal. **171** (2000), 139–154.

[Po04] S. Popa, *Strong rigidity of* II_1 *factors arising from malleable actions of w-rigid groups*, I. Inventiones Mathematicae. **165** (2004), 369–408.

[Po6] S. Popa, *On a class of type* II_1 *factors with Betti numbers invariants.* Ann. of Math. **163** (2006), 809–899.

[Po06] S. Popa, *On Ozawa's property for free group factors.* Int. Math. Res. Not. IMRN **2007**, no. 11, Art. ID rnm036, 10 pp.

[Po08] S. Popa, *On the superrigidity of malleable actions with spectral gap.* J. Amer. Math. Soc. **21** (2008), 981—1000.

[Po10] S. Popa, *On spectral gap rigidity and Connes' invariant* $\chi(M)$*.* Proc. Amer. Math. Soc. **138** (2010), 3531–3539.

[PS03] S. Popa, D. Shlyakhtenko, *Cartan subalgebras and bimodule decompositions of* II_1 *factors.* Mathematica Scandinavica **92** (2003), 93–102

[Ri74] M .A. Rieffel, *Morita equivalence for C*-algebras and W*-algebras.* J. Pure Appl. Alg. **5** (1974), 51–96.

[Sc63] J. Schwartz, *Two finite, non-hyperfinite, non-isomorphic factors.* Comm. Pure Appl. Math. **16** (1963), 19–26.

[Ta73] M. Takesaki, *Duality for crossed products and structure of von Neumann algebras of type* III. Acta Math. **131** (1973), 249–310.

[Ta03a] M. Takesaki, *Theory of operator algebras.* I. Encyclopaedia of Mathematical Sciences, **124**. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003. xix+415 pp.

[Ta03b] M. Takesaki, *Theory of operator algebras.* II. Encyclopaedia of Mathematical Sciences, **125**. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003. xxii+518 pp.

[Ta03c] M. Takesaki, *Theory of operator algebras.* III. Encyclopaedia of Mathematical Sciences, **127**. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003. xxii+548 pp.

[Va09] S. Vaes, *An inner amenable group whose von Neumann algebra does not have property Gamma.* Acta Math. **208** (2009), 389–394.

Titre : Quelques propriétés de rigidité des algèbres de von Neumann

Mots Clefs : algèbres de von Neumann, facteurs de type III, facteurs pleins, trou spectral, solidité, relations d'équivalences stables.

Résumé : Dans cette thèse, je m'intéresse à diverses propriétés de rigidité des algèbres de von Neumann. Je démontre d'abord la *solidité relative* des produits croisés issus d'actions Bernoulli de type quelconque. Ce résultat généralise un théorème de Chifan et Ioana en type II. Comme conséquence, dès que le groupe qui agit est non-moyennable, ces produits croisés sont *premiers*. Ensuite, je m'interesse aux facteurs *pleins*. Je montre notamment que tout facteur plein de type III vérifie une propriété de trou spectral similaire à celle obtenue par Connes dans le cas II_1. Grâce à cela, je généralise un théorème de Jones en donnant une condition suffisante pour qu'un produit croisé soit plein. Ceci permet de caractériser complètement les facteurs de type III_1 dont le cœur est plein. Dans un travail en collaboration avec C. Houdayer et P. Verraedt, nous montrons également qu'un produit tensoriel de deux facteurs pleins est encore plein et nous calculons ses invariants de Connes. Enfin, je m'intéresse aux facteurs *McDuff* ainsi qu'à leurs analogues en théorie ergodique, les relations d'équivalences *stables*. Je donne notamment une nouvelle caractérisation de type "trou spectral" de cette propriété de stabilité qui permet de démontrer le résultat de rigidité suivant : un produit direct de deux relations d'équivalences est stable si et seulement si l'une des deux est stable.

Title : Some rigidity properties of von Neumann algebras

Keys words : von Neumann algebras, type III factors, full factors, spectral gap, solidity, stable equivalence relations.

Abstract : In this dissertation, we study several rigidity properties of von Neumann algebras. We first prove the *relative solidity* of Bernoulli crossed products of arbitrary type. This result generalizes a theorem of Chifan and Ioana in the tracial case. As a consequence, when the acting group is non-amenable, the crossed product is *prime*. Next, we study *full* factors. The main result is a spectral gap characterization of full type III factors which is similar to Connes' characterization in the tracial case. Thanks to this new characterization, we generalize a theorem of Jones by giving a sufficient condition for a crossed product to be full. In particular, we obtain a complete characterization of the type III_1 factors whose core is full. In a joint work with C. Houdayer and P. Verraedt, we also show that a tensor product of two full factors is still full and we compute its Connes invariants. Finally, we study *McDuff* factors as well as their counterpart in ergodic theory, the so-called *stable* equivalence relations. We obtain a new "spectral gap like" characterization of this property which allows us to prove the following rigidity result: a direct product of two stable equivalence relations is stable if and only if one of them is already stable.