École doctorale IAEM Lorraine

Estimation d'état, estimation paramétrique et identifiabilité des modèles quasi-LPV

THÈSE

présentée et soutenue publiquement le 28 juin, 2018

pour l'obtention du

Doctorat de l'Université de Lorraine

(mention Automatique)

par

Krishnan SRINIVASARENGAN

Composition du jury

Président :	Eric LEVRAT	Professeur, Université de Lorraine
Rapporteurs :	Marcin WITCZAK	Professeur, University of Zielona Góra, Poland
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	Didier MAQUIN	Professeur, Université de Lorraine
Invité :	José RAGOT	Professeur, Université de Lorraine

Centre de Recherche en Automatique de Nancy — UMR 7039

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Acronyms

AHU Air Handling Unit

AI Algebraic Identifiability

ARX Autoregressive Exogenous

BRL Bounded Real Lemma

CAS Computer Algebra Systems

CSPI Combined State and Parameter Identifiability

DAISY Differential Algebra for Identifiability of SYstems

EiT Energy in Time

FAC Fault Adaptive Control

FCU Fan Control Unit

FDI Fault Detection and Isolation

FMO Finite Memory Observer

genSSI Generating Series for testing Structural Identifiability

HVAC Heating, Ventillation and Air Conditioning

I-O-P Input-Output-Parameter

LFT Linear Fractional Transformation

LMI Linear Matrix Inequality

LPV Linear Parameter Varying

LPV-SS Linear Parameter Varying State-Space

LSI Local Structural Identifiability

LTV Linear Time-Varying

MIMO Multiple Input Multiple Output

MMPC Multiple Model Predictive Control

PCA Principal Component Analysis

PI Proportional Integral

PID Proportional Integral and Derivative

PLC Programmanle Logic Controller

PMI Proportional Multiple Integral

SISO Single Input Single Output

SNL Sector Nonlinearity

T-S Takagi-Sugeno

UL Université de Lorraine

VAV Variable Air Volume

WP Work Package

Introduction et motivation

Dans ce chapitre, un résumé étendu de la motivation et de la résolution des problèmes traités dans cette thèse est donné en Français.

Introduction

La climatisation des bâtiments constitue une grande partie de la consommation d'énergie globale dans un pays. Selon le rapport du département d'énergie (DoE) des États-Unis, les bâtiments résidentiels et commerciaux représentaient près de 40% de l'énergie totale consommée aux États-Unis en 2010. On estime que la phase d'exploitation d'un immeuble représente 80% de l'énergie totale de son cycle de vie, cela étant la conséquence de l'utilisation des systèmes de chauffage, ventilation et climatisation, de l'éclairage et de la consommation d'énergie des divers autres appareils. Par conséquent, les mesures d'économie d'énergie et de coûts d'utilisation qui ciblent la phase opérationnelle d'un bâtiment ont un impact majeur tant au niveau local pour les propriétaires de bâtiments, qu'au niveau mondial pour la l'approvisionnement énergétique du pays.

L'objectif du projet 'Energy in Time (EiT)' est de développer une approche intégrée de contrôle. Celle-ci combinera des techniques de modélisation de pointe avec le développement de méthodes de commande basées sur la simulation pour automatiser la génération de plans de fonctionnement optimaux adaptés aux besoins réels du bâtiment et des utilisateurs. Cela devrait réduire les inefficacités du système et améliorer l'efficacité énergétique des bâtiments tout en maintenant le confort des occupants. Le projet EiT cible les bâtiments non résidentiels existants pour lesquels les marges d'améliorations dont élevées et peuvent avoir un impact important. Notons que les stratégies développées peuvent également être adaptées à de nouveaux bâtiments dès leur leur mise en service initiale.

Contributions au projet européen

L'équipe de l'Université de Lorraine a été impliquée dans les 'workpackages' concernant le développement de méthodes à bases de modèles pour la surveillance et le contrôle. Afin de tester des approches à base d'observateurs, un benchmark de simulation été développé en utilisant le logiciel MATLAB et la boîte à outils SIMBAD. Ce simulateur comporte les éléments clés impliqués dans les systèmes de chauffage, ventilation et climatisation et un modèle représentatif des zones d'occupation multiples (deux à six zones). Dans cette contribution, trois fautes au niveau du système, ont été considérées. La première est un encrassement qui réduit l'efficacité d'un échangeur de chaleur. La seconde correspond à une pompe ou une vanne bloquée, ce qui affecte la vitesse de l'eau chaude. Enfin, la troisième concerne une dégradation du fonctionnement des ventilateurs ou des registres d'air ce qui affecte la délivrance des flux d'air aux différentes zones. La détection de ces défauts est effectuée à l'aide d'un filtre de Kalman étendu. Deux types d'adaptation pour des défauts sont aussi développé : modification du point de consigne (pour le défaut de l'actionneur) et l'approche de capteur virtuel (pour le défaut de capteur).

Spécification du problème

Considérons par exemple le blocage d'une vanne du circuit d'eau chaude dans une unité de climatisation. La position de la vanne (et donc le débit massique de l'eau) est l'entrée de commande et une vanne bloquée correspond à un défaut d'actionneur. Les échangeurs de chaleur sont souvent affectés par le dépôt de matériaux pendant son fonctionnement, appelé encrassement. Il en résulte une décroissance lente ou une dégradation de son efficacité affectant le transfert de chaleur efficace entre l'air et l'eau. Un entretien périodique est généralement prévu pour nettoyer les tubes internes de l'échangeur de chaleur sur la base d'une estimation du coefficient d'encrassement. Un module de détection et d'estimation de défaut de la vanne doit pouvoir fonctionner même si l'échangeur est encrassé (variation paramétrique du modèle considéré).

Considérons un autre exemple : une grande salle comme un hall dans un aéroport ou un restaurant dans l'immeuble d'une entreprise. La climatisation est contrôlée par plusieurs boîtes VAV (Variable Air Volume), qui à leur tour fournissent l'air chaud/froid à travers plusieurs gaines d'aération. Un défaut bien connu est une obturation complète ou partielle du conduite d'air, soit en raison de blocages dans les gaines d'aération, soit parce que l'amortisseur VAV est bloqué dans une position indésirable. La détection d'une défaut et son estimation dans de tels scénarios doivent être accomplie en utilisant divers signaux de contrôle et de mesure de la température de l'air à divers points de la salle. La température de l'air dans la pièce est également affectée par des facteurs tels que l'état d'une porte/fenêtre ouverte ou des fuites dans l'isolation. Ces facteurs apparaissent comme des paramètres constants ou lentement variables dans le temps lorsqu'ils sont modélisés par les premiers principes. Dans ces scénarios, le système de détection et d'estimation des défauts doit estimer simultanément un ou plusieurs paramètres inconnus.

Un observateur basé sur un modèle dans un tel cas doit estimer à la fois les états et certains paramètres inconnus. Un problème connexe est la capacité d'estimer les états et les paramètres en utilisant une certaine forme d'observateurs. L'étude de l'identifiabilité des paramètres apparaît donc nécessaire. Ce dernier problème est également intéressant en raison de sa pertinence dans le problème de placement des capteurs pour l'estimation des paramètres, la détectabilité des défauts, etc.

Principaux problèmes traités dans cette thèse:

- **Conception d'observateurs pour l'estimation des états et de quelques paramètres inconnus.**
- **Analyse de l'identifiabilité des paramètres inconnus.**

Observateurs pour l'estimation de l'état et des paramètres

La conception d'observateurs pour l'estimation de l'état et des paramètres ainsi que l'identifiabilité des paramètres sont des problèmes très génériques. Les travaux développés dans cette thèse se sont focalisés sur des modèles non linéaires décrits dans l'espace d'état pouvant être ré-écrit sous forme quasi-LPV (Linear Parameter Varying).

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) &= C(\rho(t))x(t) \end{aligned} \tag{1}$$

où $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$ et $u \in \mathbb{R}^{n_u}$ représentent les états, les sorties et les entrées, et $\rho \in \mathbb{R}^{n_p}$ représente les variables d'ordonnancement ou de prémisse.

La paramétrisation du modèle de système considérée est affine. Le modèle dans (1) peut être de la forme:

$$\begin{aligned} \dot{x}(t) &= A(\rho(t),\theta)x(t) + B(\rho(t),\theta)u(t) + F(\rho(t),\theta) \\ y(t) &= C(\rho(t),\theta)x(t) \end{aligned} \tag{2}$$

où

$$X(\rho(t),\theta) = X_0(\rho(t)) + \sum_{j=1}^{n_\theta} \theta_j \bar{X}_j(\rho(t)) \tag{3}$$

avec X représentant l'une des quatre matrices système possibles dans (2).

Le modèle d'intérêt: modèles quasi-LPV avec paramétrisation affine.

De façon générale, les observateurs sont utilisés pour générer des résidus qui permettent la détection de défauts. La variation de la valeur de certains paramètres du modèle peut être révélatrice de l'apparition de défauts de système. Il est donc utile de concevoir des observateurs capables non seulement d'estimer l'état du système mais également la valeur de certains paramètres. Il existe deux types d'approches pour la conception de tels observateurs. Dans la première, les paramètres inconnus sont ajoutés au vecteur d'état pour construire un nouveau vecteur d'état étendu et donc un nouveau modèle. Ce vecteur d'état est alors estimé en utilisant le plus fréquemment un filtre de Kalman. Cette approche ne permet cependant pas d'obtenir des garanties de convergence de l'erreur d'estimation. La deuxième approche conduit à ce que l'on appelle communément les observateurs adaptatifs qui estiment conjointement les états et es paramètres. Nous avons choisi ici cette seconde approche.

Problème 1: Conception d'observateurs adaptatifs pour les modèles MIMO quasi-LPV avec paramétrisation affine

Placement de capteurs pour l'estimation des paramètres

La procédure de conception d'un observateur suppose que les mesures disponibles permettent d'effectuer l'estimation des états (et/ou des paramètres du modèle). Cette propriété structurelle s'appelle observabilité lorsqu'elle est relative à l'état et identifiabilité lorsqu'elle concerne les paramètres. L'évaluation de cette propriété est de première importance si l'on souhaite développer des stratégies optimales de placement des capteurs.

Le placement de capteurs dans de grandes infrastructures comme les bâtiments est contraint par divers facteurs : contrainte d'espace due à l'espace ouvert important, contraintes de coût dues au nombre de capteurs requis pour surveiller de vastes espaces, etc. La détection et la correction des défauts dans une telle grande infrastructure doivent faire partie d'une action de maintenance planifiée où l'analyse des données se combine avec des connaissances d'experts. De manière similaire, la dégradation des performances (par exemple, l'encrassement d'un échangeur de chaleur, le blocage du filtre à air) est également évaluée par l'intermédiaire d'une heuristique et d'une compréhension experte. Au-delà de la maintenance planifiée, toute détection de défaillance ou analyse de dégradation de performance est effectuée en s'appuyant sur des mesures de variables clés à différents points de l'infrastructure. Pour équiper ces technologies du point de vue de la maintenance du système, des approches adéquates de placement des capteurs pour l'observation de l'état, l'identifiabilité des paramètres, la détectabilité des défauts et l'isolabilité sont nécessaires. L'estimation d'état et de paramètres repose sur l'identifiabilité des paramètres du modèle du système sous-jacent : c'est le second problème d'intérêt de cette thèse.

Problème 2: Méthodes d'analyse de l'identifiabilité des paramètres pour les modèles quasi-LPV avec paramétrisation affine

Plan de la thèse

La figure 1 présente l'organisation de la thèse avec la description du contenu des chapitres et leurs interactions. Le premier chapitre donne un bref aperçu du projet, qui constitue la motivation de l'application pour la thèse. Le projet s'inscrit dans le contexte du diagnostic des défaillances basé sur un modèle ainsi que problème de contrôle sous-jacent. Le chapitre 2 présente tout d'abord l'approche polytopique ou de Takagi-Sugeno (TS) pour la modélisation des systèmes et la conception d'observateurs. S'ensuit une étude bibliographique de la conception des observateurs TS appliqués dans le domaine des systèmes énergétiques du bâtiment. Cette analyse a conduit à développer plus particulièrement des observateurs permettant l'estimation conjointe des états et des paramètres des modèles considérés. La conception de tels observateurs sur la base de modèles quasi-LPV décrits en temps continu est présentée lors du chapitre 3. La dynamique d'estimation des paramètres est issue d'une analyse de stabilité à l'aide d'une fonction de Lyapunov.

Une méthode plus classique (observateur de type proportionnel intégral) a été utilisée dans le cas des systèmes décrits en temps discret. Celle-ci est décrite au chapitre 4. Les cas de paramètres constants et variant dans le temps sont analysés.

Une stratégie différente est exposée au chapitre 5. Il s'agit de découpler l'estimation d'état de l'estimation paramétrique. Une approche de type espace de parité est tout d'abord utilisée afin d'éliminer, des équations du modèle, le vecteur d'état. Les équations résultantes permettent d'estimer les paramètres du modèle. Ceux-ci ayant été déterminés, un observateur à mémoire finie (Finite Memory Observer – FMO) permet ensuite d'estimer les états.

Cette approche a fait apparaître plusieurs difficultés en particulier celle relative à l'identifiabilité paramétrique. Le chapitre 6 a donc été consacré à l'analyse de cette identifiabilité pour les modèles sous forme quasi-LPV. Quelques résultats initiaux sur l'utilisation du calcul de l'espace nul symbolique et numérique dans l'analyse d'identifiabilité sont donnés.

La thèse se termine par le chapitre 7 qui résume les contributions de la thèse et présente un large éventail de perspectives tirées des contributions des chapitres 3 à 6.

Figure 1: Esquisse de la thèse

Chapter 1

Introduction and motivation

The origins of this thesis and the research performed in the last few years lie in the European commission project, Energy in Time (EiT). The project gave the initial impetus to the directions to take for the research and most importantly the financial support. In this chapter, a very brief outline of the project and the work as part of the project are given. This is followed by a discussion into different problems to explore and then finalizing the problems of interest in this thesis. The chapter ends with an outline to the organization of the thesis.

Contents

Buildings constitute a large part of the overall energy consumption in a country. According to the United States' Department of Energy (DoE) report, residential and commercial buildings accounted for nearly 40% of the total energy consumed in the US for 2010 (see Fig. 1.1). It is estimated that operational stage represents 80% of a building's life cycle cost[1]. Out of this about 50% is the consequence of the energy use. Further, up to 90% of the buildings' life cycle carbon emission occur during the operational phase, mainly as a consequence of the HVAC (Heating, Ventilation and Air Conditioning) systems, lighting and other appliances' energy use. A typical break-up of such energy consumption is given in Fig. 1.2 for 2010[2]. Hence, energy and cost saving measures that target the operational phase of a building will have a major impact both at a local level for the building owners, and at the global level for the energy safety of the country.

The Energy in Time (EiT) project[3] aims to address this by going beyond building control techniques, developing an integrated control and operations approach. This will combine the state of the art modeling techniques with the development of a simulation based control technique

[1] Details based on an internal document, Annex-I - Description of Work

[2] The graphs were created based on open data available at www.eia.gov

[3] www.energyintime.eu

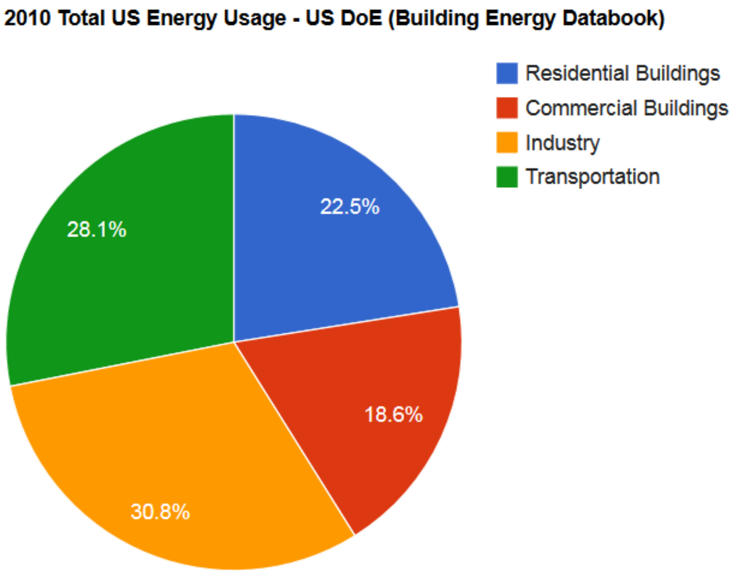

Figure 1.1: Total Energy consumption split for the US in 2010

to automate the generation of optimal operation plans tailored to the actual building and user requirements. This is expected to reduce system inefficiencies and hence improve building energy efficiency at the same time maintaining occupant comfort. The EiT project targets existing non-residential buildings, which are the types that apparently guarantees higher impact and a lot of room for improvement because of the wide variety of facilities and equipments covered. Further, a methodology to enhance the implementation of such strategies for new buildings from the time of their initial commissioning, is also envisioned.

1.1 EiT project description

To realize the above motivation, the project was split into multiple work packages that would split various tasks associated. In this section, an abridged summary of this is given. It is to be noted that this is not the exact breakup or organization of the project, but a description that is useful to illustrate the components associated with the works relevant to the thesis. This avoids sharing confidential details but at the same time provides the context necessary to connect to the thesis. A simplified representation of the blocks involved in the project implementation is given in Fig. 1.3. The blocks are further split into multiple actionable tasks that are termed work packages (WPs), that track and illustrate the effectiveness of the EiT architecture. The project involves 9 work packages and could be accessed from the project website[4]. In the following, a brief summary of the architecture and the information flow along with an abridged version of the relevant work packages are given:

The WP1 covers collating requirements and formulating the EiT system architecture. This includes the data acquisition module which should define the sensor, energy and weather data to be collected for the demonstration sites. The *WP2: Simulation Reference Model* deals with

[4]http://energyintime.eu/work-packages/

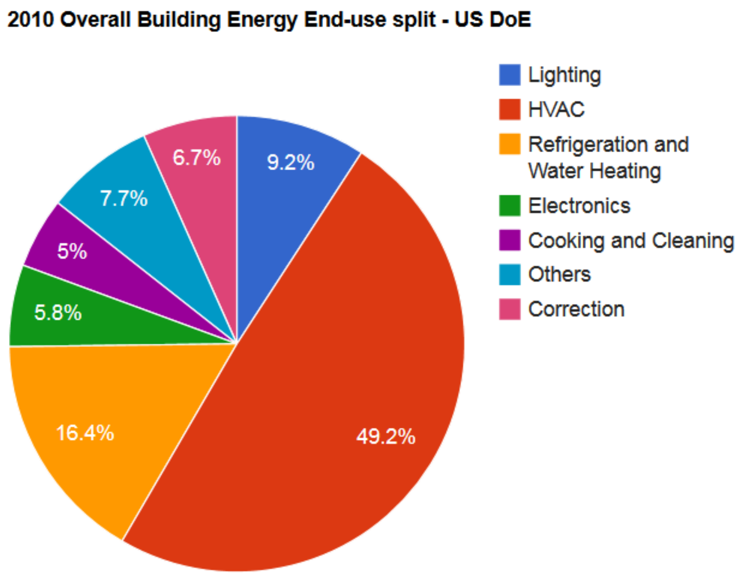

Figure 1.2: Building Energy consumption split for the US in 2010

characterizing the building using energy usage, occupancy data, HVAC equipments, weather loads, control loops etc. This would also be used as a forecast tool in the EiT system architecture. The relevant modules for this thesis in the form of building control and maintenance are realised through the work packages, *WP3: Whole building Intelligent Control System* and *WP4: Diagnosis and Continuous Commissioning*. The building intelligent control is responsible for key tasks such as preparing an operational plan for the energy system's working, a dynamic model on demand control as well as a fault adaptive module for the entire system. The building maintenance module includes tasks such as continuous commissioning, fault diagnosis and operational maintenance of equipments.

Out of these, the relevant tasks related to the contents of this part of the thesis are: fault diagnosis and fault adaptive control. A brief of the contributions to the project is given in Sec. 1.2. The other work packages relate to data analysis, validation of the EiT architecture at the laboratory scale as well as in the demonstration sites. More details could be obtained from the project website or from the deliverable documents for partners.

Project partners and demonstration sites

The EiT project involves a number of partners bringing various skill sets to the table. Further, partners include those who manage the four demonstration sites and a laboratory scale set-up where the parts of EiT architecture could be evaluated. The project partners are:

- Acciona Infraestructuras S.A (ACCIONA), Spain
- ANA - Aeroportos de Portugal, S.A (ANA), Portugal
- Fundacion Circe Centro de Investigacion de Recursos y Consumos Energeticos (CIRCE), Spain

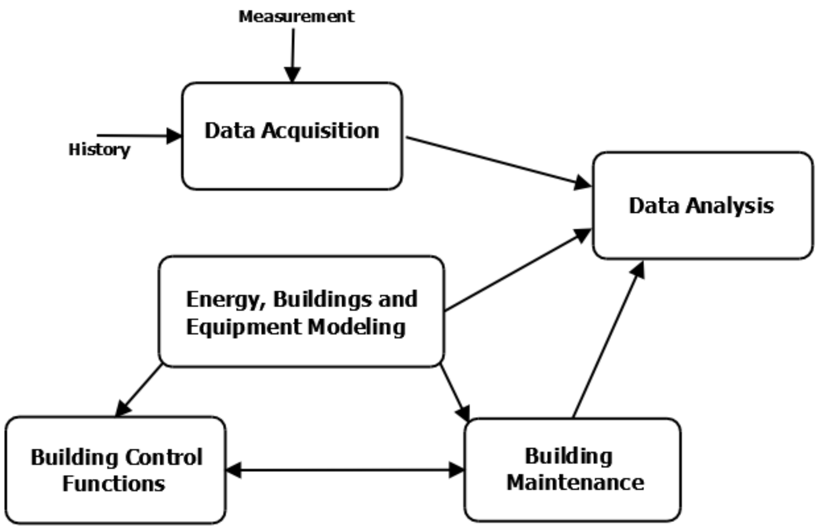

Figure 1.3: An abridged version of the project organization

- Cork Institute of Technology (CIT), Ireland
- Université de Lorraine (UL), France
- Centre Scientifique et Technique du Bâtiment (CSTB), France
- Fundacion Universitaria Iberoamericana (Funiber), Spain
- Institutule de Cercetari Electrotehnice (ICPE), Romania
- Integrated Environmental Solutions Limited (IES), United Kingdom
- STAM SRL (STAM), Italy
- Universidad de Granada (UGR), Spain
- United Technologies Research Centre Ireland, Limited (UTRCI), Ireland
- YIT Kiinteistötekniikka Oy (YIT), Finland

The demonstration sites provide a diversity in its characterization and could be briefly summarized as follows:

FARO Airport, Portugal This is an airport building with an area of about 41000m^2 which was refurbished in 2001. The airport contains large open spaces with large flow of people at certain times. A part of this airport building is available for demonstration purposes.

ICPE, Bucharest, Romania This is an office and test labs facility of an area of 17384m^2 built in 1982. The space inside the building are closed and distributed with a constant flow of people with a scheduled occupancy.

Levi-Lapland, Finland A large 42500m^2 area hotel built in 2010. The building has three distinct spaces for occupancy. The occupancy is seasonal and has high variability.

Panorama, Helsinki, Finland A commercial and office building of 38160m^2 in size and built in 1999. The occupancy spaces in this building are open and distributed. The occupancy has a varied flow, but with clear scheduling.

Specific details about the demonstration sites, from energy usage to occupancy patterns, from energy sources to HVAC equipments are confidential. Further, only a part of these sites are available for demonstration purposes. Given below are some relevant technical characteristics of the sites. They are specific enough to be useful to understand the context of this part of the thesis, but general enough and not localized to the sites so as not to affect any confidentiality aspects.

Energy sources Apart from the electricity supply from the grid, which is used in some split air conditioning and lighting, the heating and cooling of the buildings are accomplished through a variety of energy sources, including,

- District heating
- Solar water heating
- Heat pump
- Boilers
- Ground thermal heating
- Water condensing machines
- Chillers
- Ice banks

Transmission system The hot or cold water is supplied to the buildings in different ways. In some buildings, radiators and convection apparatus let the space to be heated or cooled directly. For buildings that use district heating or a centralized heat source, a network of heat exchangers are employed to regulate the temperature of the water being supplied to various loads.

Air handling In the buildings where the heating/cooling is not accomplished through direct convection, an air handling set-up is used. Typically they are of two categories:

- Heat exchangers (water-air) with a VAV (Variable Air Volume) box
- Fan coil unit (FCU)

Air Handling Units (AHU) also incorporate ways to save energy by recapturing heat from exhaust air through mixers.

Spaces served A variety of different occupancy spaces are served by the HVAC systems in the building. This includes,

- Office spaces
- Logistics/warehouses

- Common areas, stairs
- Restaurants
- Kitchen
- Shopping spaces
- Auditoriums
- Guest rooms

Control infrastructure These buildings have a variety of control infrastructures installed in monitoring and controlling the energy usage. This ranges from programmable logical controllers (PLCs) based set-up to latest building automation systems technologies that have fine granular understanding of the energy usage.

Based on the literature survey and taking into account the various objectives of the project, this section summarizes the ideal requirement specifications to realize those objectives.

1.2 Project contributions: a summary

As discussed in Sec. 1.1, the project is split into multiple work packages. The main contribution from the team at the Université de Lorraine (UL) was in the work packages *WP3: Whole building intelligent control system* and *WP4: Diagnosis and Continuous Commissioning.* Under these work packages, team UL was involved in the fault adaptive control part in WP3 and the diagnosis part in WP4. There were three possible scenarios to evaluate algorithms to be developed, which in-turn influenced the approach to follow:

- Demonstration sites
- Laboratory scale set-up
- Simulation benchmark

The access to demonstration sites proved to be restrictive, partly because of the lack of sufficient instrumentation and partly because of the requirements for the fault diagnosis and adaptive control modules (data collection before, during and after fault). This brought the focus to the laboratory scale set-up where all the necessary instrumentation for purposes such as fault realization was available. However, this also turned futile and hence it was agreed that the project partner responsible for the laboratory scale set-up evaluation would develop fault diagnosis and adaptation methods using data-based techniques and UL would focus on model based methods.

Simulation benchmark

The SIMBAD toolbox [5] of the MATLAB software package was the chosen platform to develop a benchmark on which the algorithms would be evaluated. SIMBAD (SIMulator of Building And Devices) is a MATLAB/Simulink component library dedicated to the modelling and dynamic simulation of fully equipped buildings developed by CSTB (Centre Scientifique et Technique du Bâtiment). The simulation benchmark model has the key elements involved in the AHU and a

[5] `http://www.simbad-cstb.fr/index.html`

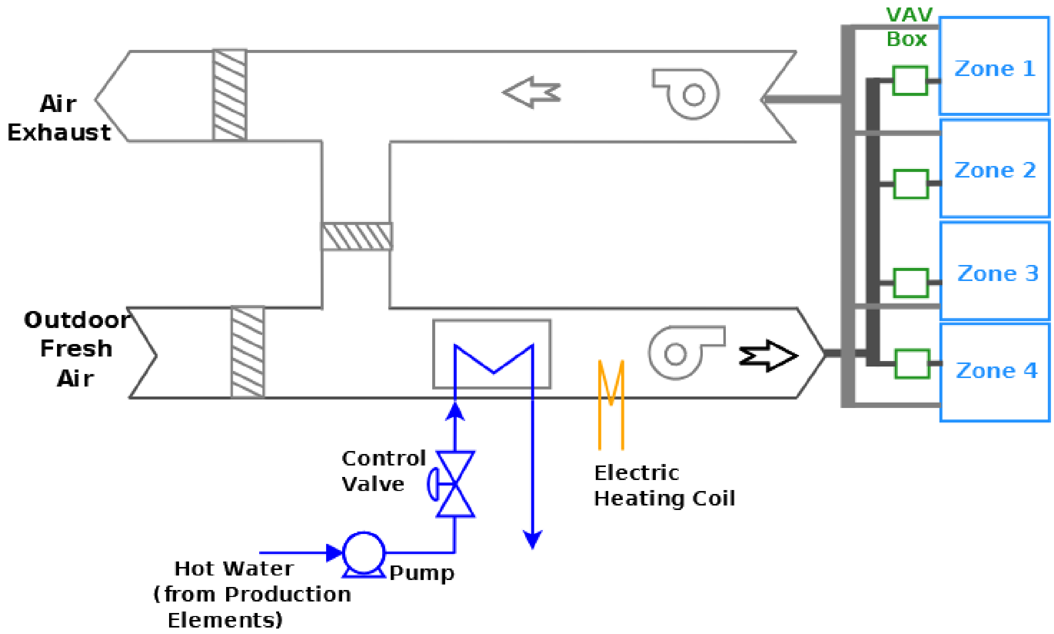

Figure 1.4: Schematic of the simulation benchmark

representative model of occupancy areas with multiple zones (two or six zones). A general outline of the benchmark is given in Fig. 1.4. A VAV based system is used to control the temperature of the room. The idea is to use the faults that can occur in these elements and illustrate the energy loss and comfort index degradation. It would also be used to test the model based observers developed for Fault detection and Diagnosis. The benchmark used a mix of SIMBAD library components and Simulink library components in conjunction with other MATLAB elements. Modifications were made to incorporate aspects that are useful in illustration, such as:

- More complicated dynamics: e.g., dynamics of heat exchanger were enhanced to display the nonlinear characteristics
- Incorporation of degradations: e.g., fouling in heat exchanger
- Incorporation of faults: e.g., stuck damper in the VAV box

The spectrum of work that team UL contributed within the work packages of WP3 and WP4 are summarized below:

- *D3.3* Fault-adaptive control algorithms
- *D4.1* System and equipment level fault detection module
- *D4.2* Building operational fault detection module
- *D4.3* Sensor diagnostics module
- A host of predictive maintenance and continuous commissioning modules

The predictive maintenance modules were executed by another team within UL and is out of scope for this thesis. For the other 4 modules, team UL provided some strategies and approaches that were envisaged by the lead project partner of those modules. A select set of these modules is discussed in the next two sections.

1.2.1 Observers for fault diagnosis

Observers form the core of the model based fault detection and diagnosis strategies. In this contribution, three faults, all at the system level, were considered [6].

- **Fouling**: Fouling is the process of deposition or accumulation of unwanted materials such as scale, algae, suspended solids and insoluble salts on the internal or external surfaces of heat exchangers. The efficiency of the heat exchanger could be severely affected by fouling. When fouling accumulates in a heat exchanger the resistance to heat transfer increases, which decreases the overall heat transfer coefficient (referred to as UA) and hence will increase the energy cost of the heat exchanger.

- **Pump or valve stuck**: Mass flow rate of water is one of the parameter that is used as a control input for heat exchangers, though in the benchmark simulator, the focus has been on the temperature. Water flow rate in a heat exchanger system could be influenced by many factors. A malfunctioning pump or a valve and can lead to heat exchanger efficiency issues and comfort index degradation. Hence estimation of mass flow rate changes is crucial in the effective understanding of heat exchanger operation.

- **Fan or air damper stuck**: Mass flow rate of air is a crucial parameter in the function of the heat exchanger. In a VAV system, the air temperature is kept constant and the VAV adjusts the flow of air into the room using the damper. During the operation, to avoid pressure build up in the ducts, the air flow rate may be adjusted through the fan. If the fan gets stuck or is not able to operate at certain speeds, it could cause a difference in the air temperature and hence the comfort. Estimating the mass flow rate and comparing it with the command to fan (through appropriate conversion), one could detect the fault and estimate the same.

These could be classified as process fault (fouling) and actuator faults (the other two). A discretized nonlinear model of a heat exchanger was used and an Extended Kalman filter was designed for each of the fault detection scenario and the results were illustrated on the benchmark simulator.

1.2.2 Fault adaptive control strategies

A brief outline of the contribution in the fault adaptive control (FAC) module is discussed here. First, the proposed performance indices to better understand the effect of fault adaptive control strategies are given. This is followed by two strategies proposed for fault adaptation and their corresponding results are given.

Performance indices

Many internal subsystems in a HVAC system have local control loops that can mask faults occurring in the equipments in these subsystems. On one hand, this is useful to avoid drastic impact of the fault, this is a problem from the perspective of optimizing the balance between the overall comfort of the occupants and the energy usage. To visualize this lack of balance, the

[6] Other faults at subsystem level such as VAV damper stuck and temperature sensor fault were considered for the project contribution but were executed by other members of the team and hence not outlined in this thesis.

following two indices are proposed to evaluate the performances of the comfort and the energy:

$$C_I = \frac{100}{T_{SP} \times t_{total}} \times \left(\sum_{|T_z - T_{SP}| > T_{th}} (\|T_z - T_{SP}\| - T_{th}) \times t_s \right) \tag{1.1}$$

and

$$E_I = \frac{E_n - E_{ref}}{E_{ref}} \times 100 \tag{1.2}$$

with the summation is performed only when the zone temperature exceeds from the limit of $\pm T_{th}$ from the set point T_{SP}. where

C_I and E_I	-	comfort and energy indices (no units)
T_z and T_{SP}	-	zone temperature measured and its set point respectively ($°C$)
T_{th}	-	threshold (above or below T_{SP}) indicating loss of comfort ($°C$)
t_s	-	the sampling interval (second)
t_{total}	-	the total time period for which the index is computed (second)
E_n, E_{ref}	-	the measured and reference energy (in J) respectively

The energy is computed based on the following formula,

$$E = \sum_{t_{total}} (C_{p-air} \times \dot{m}_{air}(k) \times \Delta T_{air}(k) \times t_s)$$

Where, $\Delta T_{air}(k)$ corresponds to the change in the air temperature from the input to the output of the heat exchanger, $\dot{m}_{air}$ is the mass flow rate (kg/s), and C_{p-air} is the specific heat capacity of the air (in J/(kg-K)).

Fault adaptation strategies

For the benchmark model, system-level analysis comprises of the AHU with the heat exchanger and the associated components. Most of the system level equipment faults are managed by the robustness of the internal loops in the AHU. However, two interesting cases of faults, one at the system-level and another at subsystem level with adaptation at the system level could be illustrated:

- System level: hot water mass flow rate fault (pump or valve stuck)
- Subsystem level: damper stuck fault

The fault adaptive control proposed strategies assumed the availability of a perfectly working fault detection and isolation module. With this, the strategy proposed involves the following (respectively for the two scenarios above):

- Controller reconfiguration: Detection of the fault would lead to a new controller in the loop
- Actuator effort distribution through reference management: Adjustment of reference at the system level to account for fault at the subsystem level.

Pump or valve fault

The mass flow rate of water is normally used to control the temperature of air that is supplied to the VAV boxes. This flow rate is controlled either using a control valve (if the hot water source is separate from AHU) or a pump (if source is local). An obstruction in the mass flow rate could be caused due to either a pump fault or a valve stuck at a position. Such faults lead to either an increase or decrease in the air temperature to the VAV and can lead to a reduction in comfort index. If the mass flow rate of water is stuck, a redundant loop to control the temperature of the water could be activated. Fig. 1.5 illustrates this strategy. As pointed out in the schematic, this fault adaptation strategy involves a redundant controller which was implemented using the Model Predictive Control (MPC) strategy.

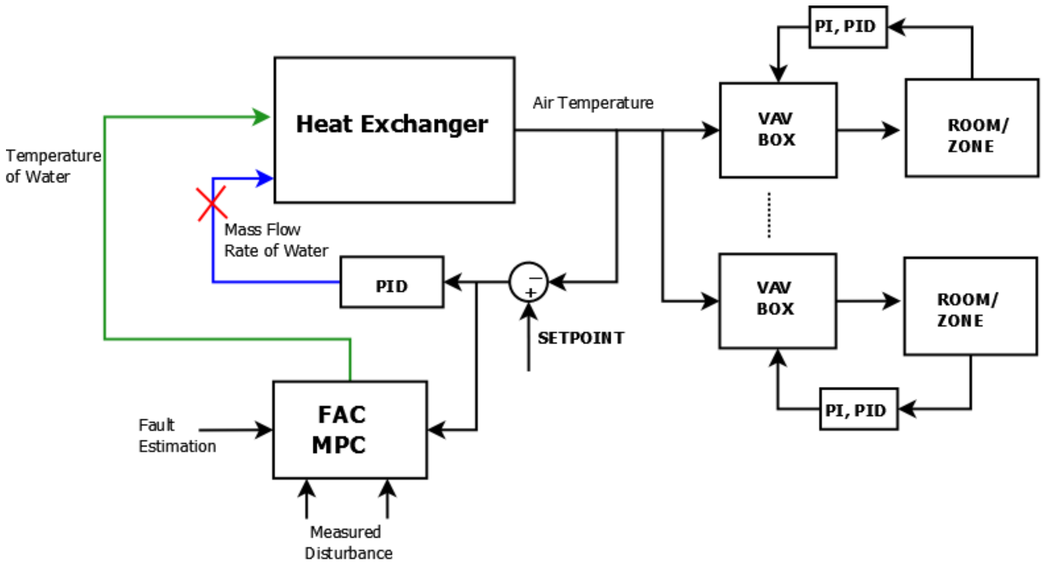

Figure 1.5: Schematic of fault adaptation strategy for the water mass flow rate fault

Damper stuck fault

One insight obtained during the simulations for the controller reconfiguration based fault adaptation was on the distribution of actuator effort. It was observed that even if the AHU cannot deliver the required amount of energy in the air, VAV based systems, with their internal control loop, may adjust to allow for a correction. This inspires the idea to handle faults at the subsystem level using changes at the system level. This is also particularly useful in big buildings where zones are tightly interconnected without any walls between them. For example, a waiting hall in FARO airport where there would be multiple VAVs supplying heated/cooled air.

A schematic of one possible way to implement the proposed strategy is given in Fig. 1.6. A reference management technique is implemented using a PID controller. As simulation was just to illustrate the idea, the implementation was done without prolonged tuning of the PID controller. This approach can be implemented with a more sophisticated MPC based control whereupon all the constraints of different actuators could be taken into account and the comfort index could be improved further as well as better use of energy. The effectiveness of the strategy was illustrated for two simulation cases: one fault in each zone leading either to heating or

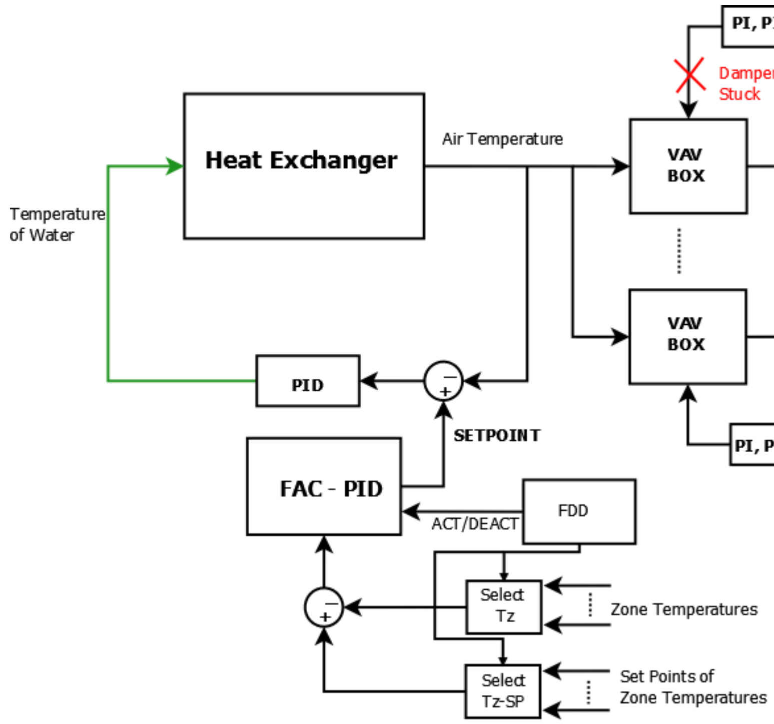

Figure 1.6: Schematic of fault adaptation strategy for the damper stuck fault

cooling the zone. It was also illustrated that with this strategy, there is little change in the comfort index of the zone which is not affected by the damper stuck fault.

1.3 Research inspirations

In this section, an overview of the research problems that are tackled in the thesis is given. To motivate the choice of the problems, especially in the context of the project work, the challenges encountered in the project are discussed first in Sec. 1.3.1. This is followed by a brief about the control problem that is taken out of the project. A clear specification of the problems handled in this thesis concludes this section.

1.3.1 Project work challenges

The Sec. 1.2 illustrated the contribution to the project in terms of simulations. The choice of simulation allowed simplified assumptions on available models, data and duration of data, etc. During the project, significant limitations came up during the discussions on the data collection. First, fault diagnosis module was a small part of the overall architecture. Second, apart from the sensor data, there was a need to conceive and induce faults in the demonstration sites and then collect measurement data accordingly. These factors were bottlenecks to capture sufficient data before and after the occurrence of faults. The concerns regarding the difficulty to simulate faults in demonstration sites were raised early. This lead to the choices of the type of faults simulated for the contributions in Sec. 1.2, which focused on the load side and AHU components. However, fault data for these appliances were also not available until towards the end of the project. The delay in obtaining the data, their focus only on a single demonstration scenario made it difficult to use this data as a central part of the thesis work.

The model based approach was the choice that the simulations took. This relied heavily on the availability of accurate description of the demonstration sites in the form of dynamical models. However, there were several challenges in this regard. First, the modeling part of the project, carried out by another partner, focused on the static characteristics of the building. Their primary goal was to capture the energy performance of the building without any dynamic characteristics, which are useful in capturing abnormal behaviours such as faults. Second, the team at the Université de Lorraine was not associated in any of the work packages that deals with modeling. This made it difficult to get access to as well as drive the modeling approach to suit to our needs.

The challenges described above turned the focus towards problems at a more fundamental level. These are described in the next section.

1.3.2 Research directions

Fault detection, isolation (FDI) and estimation in large and complex dynamical systems have several challenges. One is the presence of unknown parameters in the system model. These unknown parameters could be due to performance degradation in equipment manifesting as modeling uncertainties, changes in model parameters due to changes in operation characteristics etc. Common remedy for this challenge is the use of robust design methods to reduce the influence of such parameters on the fault detection modules [1]. In some circumstances, these parameters need to be estimated simultaneously with the states of the system as well the faults.

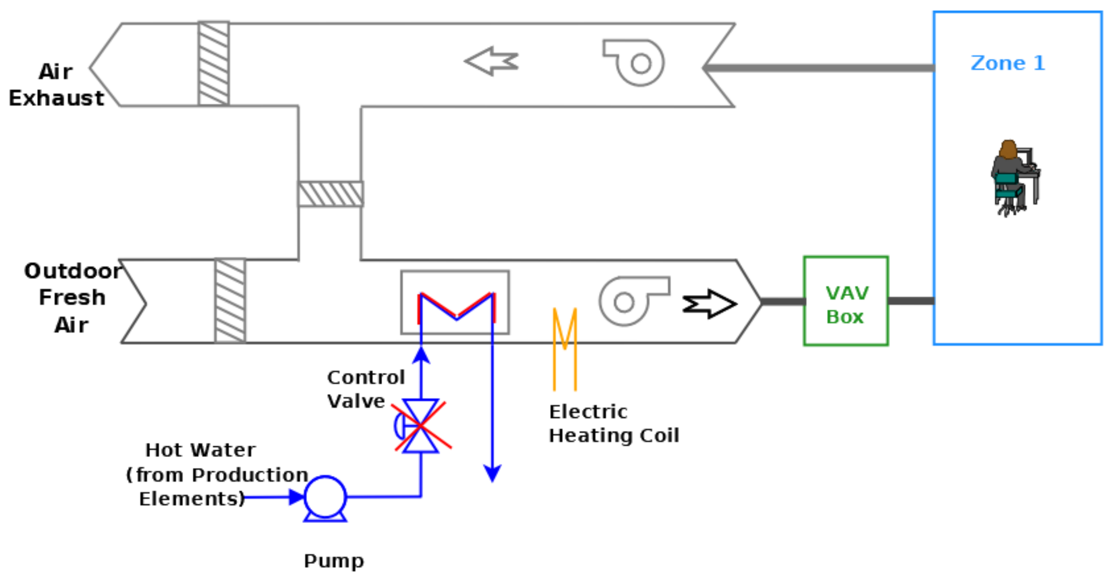

Figure 1.7: Schematic of a simple air handling unit (AHU)

Take the example of valve stuck fault detection and estimation in an AHU (see Fig. 1.7). The valve is on the water path that supplies hot water to the heat exchanger. The hot water is supplied from a remote location, such as the district heating. The temperature of the air on the secondary (2° in the figure) side of the heat exchanger is the controlled output (heat exchanger control loop). The valve position (and hence the mass flow rate of water) is the control input and a stuck valve indicates an actuator fault. Further, the input air flow rate and its temperature are assumed to be known. A practical heat exchanger is affected by deposition of materials during its operation, termed *fouling* (as discussed in Sec. 1.2.1). This results in slow decay or degradation of its efficiency affecting the effective heat transfer between the air and the water. Periodic maintenance is scheduled to clean the internals of the heat exchanger based on an estimation of the fouling coefficient. A fault detection and estimation module for the valve stuck fault should work under fouling whose coefficients also need to be estimated.

Consider another example in the building energy systems of a large hall, such as in an airport or a restaurant in an office building. A hall is climate controlled by multiple VAV boxes, which in-turn supply the hot/cold air through multiple vents. A well-known fault in such a set-up is blockage in the air path, either due to blockages in the vents or the VAV damper being stuck in an undesirable position. FDI and estimation in such scenarios are to be accomplished using various control signals and measurement of air temperature at various points in the hall. The air temperature in the room would also be affected by factors such as the state of an open door/window or leakages in the insulation. These factors appear as constant or slowly time-varying parameters when modeled through first principles.

In these scenarios, the fault detection and estimation problem has to simultaneously estimate some unknown parameter(s). A model based observer in such a case should estimate both the states and some unknown parameters. A related problem is the ability to estimate the states and the parameters using some form of observers. For the state, the well-studied property of observability plays the role, whereas identifiability forms the basis for the parameters. The study of parameter identifiability in relation to estimation of the parameters through an observer is of interest. The latter problem is of interest also due to its relevance in sensor placement for parameter estimation, fault detectability etc., which is further discussed later in this section. At the outset, the problems of interest could be stated as follows, more specific version of this and their motivations are discussed subsequently.

Thesis problems of interest:

- **Designing observers for the estimation of states and some unknown parameters.**
- **Analyzing identifiability of unknown parameters**

1.3.3 Specific research problems

Observer design for state and parameter estimation and parameter identifiability is a very generic problem. To narrow this down, a couple of specifics that are general to both the problem of interest, model type, and parametrization type are described here. This thesis would consider state space models, nonlinear in nature and can be converted into a quasi-LPV (Linear Parameter

Varying) form. These models have a general form,

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) &= C(\rho(t))x(t) \end{aligned} \tag{1.3}$$

where $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$ and $u \in \mathbb{R}^{n_u}$ represent the states, outputs and the inputs, and $\rho \in \mathbb{R}^{n_p}$ represents the scheduling or premise variables which appears as a scalar function in the matrix entries and the matrices having appropriate dimensions. The system is LPV if ρ is an external parameter and quasi-LPV if it is one of the system variables such as the states, inputs or outputs. Three scenarios that are possible:

- Only known and measured premise variables: y and u. This case is by default considered in the entirety of the thesis.
- Some unmeasured premise variables. When this is considered, it is explicitly mentioned.
- The premise variables contain feedback control inputs. This option is not considered this work as it is difficult to clearly articulate the conservativeness of the quasi-LPV form.

In some special cases, an affine term F on top of the A and B terms could also be present.

The system model parametrization considered is affine. While affine parametrization is not a significant part of the possible parametrizations, they are however important in engineering systems. The model in (1.3) could be of the form:

$$\begin{aligned} \dot{x}(t) &= A(\rho(t), \theta)x(t) + B(\rho(t), \theta)u(t) + F(\rho(t), \theta) \\ y(t) &= C(\rho(t), \theta)x(t) \end{aligned} \tag{1.4}$$

where

$$X(\rho(t), \theta) = X_0(\rho(t)) + \sum_{j=1}^{n_\theta} \theta_j \bar{X}_j(\rho(t)) \tag{1.5}$$

with X representing one of the four possible system matrices in (1.4).

The model of interest: quasi-LPV models with affine parametrization.

Observers for state and parameter estimation

One form of model-based fault detection methods has a stable, state observer design at its core. The observers would be used to obtain residuals, which along with a threshold would help detect faults in the system [2]. In the scenario under discussion, these observers would provide a state estimation corrected for the unknown parameters, which are also estimated. There are two broad categories to design observers that perform a joint estimation of states and parameters. In the first category, the unknown parameters are augmented to the state vector to construct a new extended state vector and hence a new model. Consider the linear model with an affine unknown parameter,

$$\begin{aligned} \dot{x}(t) &= (A_0 + \bar{A}\theta)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

the extended model is given by,

$$\dot{x}_e(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A_0 + \bar{A}\theta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \theta \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \theta \end{bmatrix}$$

A state observer designed to estimate $\hat{x}_e$ will simultaneously estimate the states and the parameters of the model. In the literature, Kalman filter based approaches are popular for this type of approach (see for e.g., [3]). For nonlinear models, however, there is no guarantee for convergence with the extended Kalman filter.

The second category of such observers is commonly referred to as *adaptive observers* in the literature [4]. The parameter estimation is either through a choice of observer structure that is rooted in intuition (see [5], [6]) or carefully designed structure based on the underlying dynamic model of the system [7].

This thesis would focus on the second category of observer design, that is, adaptive observers. For a class of single output nonlinear system models with affine parametrization, there have been well-established studies on adaptive observer design strategies that admit a quadratic Lyapunov function [8]. However, analytical understanding for MIMO systems have been limited. Design strategies such as in [7] and some fundamental insights provided in [9] could be used to develop the Takagi-Sugeno (T-S) polytopic version of adaptive observers for quasi-LPV models.

Problem 1: Design of adaptive observers for MIMO quasi-LPV models with affine parametrization

From an application point of view, the observer design strategy shall be extended to fault detection, isolation and estimation scenarios, such as that discussed in [10].

Sensor placement for parameter estimation

An observer design procedure assumes that the available measurements guarantee the states (and the parameters) to be estimated, which more formally refers to state observability (and parameter identifiability). State-space models arise naturally when the first principles modeling approach is used to capture the characteristics of a system. Models so obtained have their structural properties such as state observability, parameter identifiability and from an application point of view, fault detectability, isolability, etc., governed by the available measurements. This makes the problem of evaluating these structural properties and developing optimal sensor placement strategies to mitigate any lack of these structural properties closely linked to the observer design problem.

Sensor placement in large infrastructures like buildings is constrained by various factors: space constraint due to the large open space, cost constraints due to the number of sensors required to monitor the vast space etc. This is further exaggerated in the case of analysing performance degradation and faults in those sensors itself. Consider the scenario that is illustrated in Fig. 1.8. If one of the vents in the large open space has a block, even if the fault is detected, the isolation of the fault to the specific vent is severely constrained, partly because the sensors are usually placed in locations where the air is sufficiently mixed.

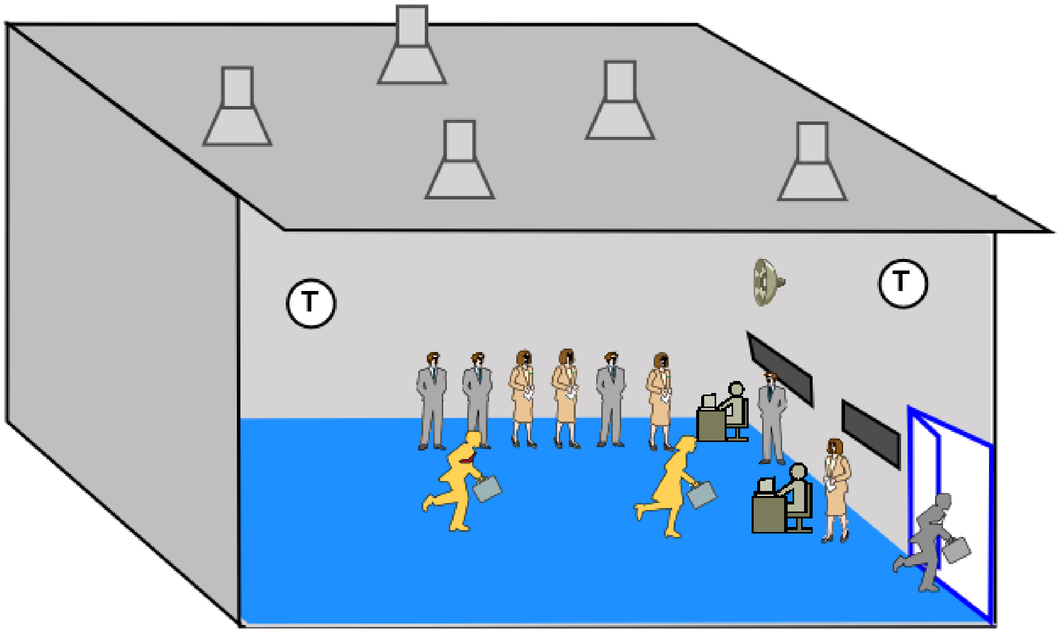

Figure 1.8: Schematic of a large room with multiple VAVs/vents

Fault detection and correction in such large infrastructure is hence part of a scheduled maintenance where data analysis combines with expert knowledge [11], [12]. Similarly, degradation of performances (e.g., fouling in heat exchanger, blocking of air filter, opening and closing positions of valves/dampers) are also evaluated through heuristics and expert understanding. Beyond the scheduled maintenance, any urgent fault detection or performance degradation analysis is performed through manual measurement of key variables at various points in the infrastructure (in this case, building). Only recently semi-automated tools are being developed for continuous-condition based maintenance platforms for buildings (see for instance [13]). One interesting direction is the emergence of new sensor capabilities which could mitigate the problem of evaluating faults and performance degradation in a timely manner. Terrestrial [14] and airborne [15] robots are already coming up in management of logistics and could follow suit for applications such as building energy management.

To equip these technologies from a system maintenance perspective, adequate approaches to sensor placement for state observability, parameter identifiability, fault detectability and isolability would be required. For state and parameter estimation, parameter identifiability of the given state-space model of an underlying system is a key factor. This would enable to obtain a trajectory for human-held or robot-mounted sensors to traverse through to collect measurement data that would lead to the estimation of the parameters. This would hence be the second problem of interest in this thesis.

Problem 2: Methods to verify Parameter Identifiability for quasi-LPV models with affine parametrization

With these problems in place, the thesis is outlined in the following section.

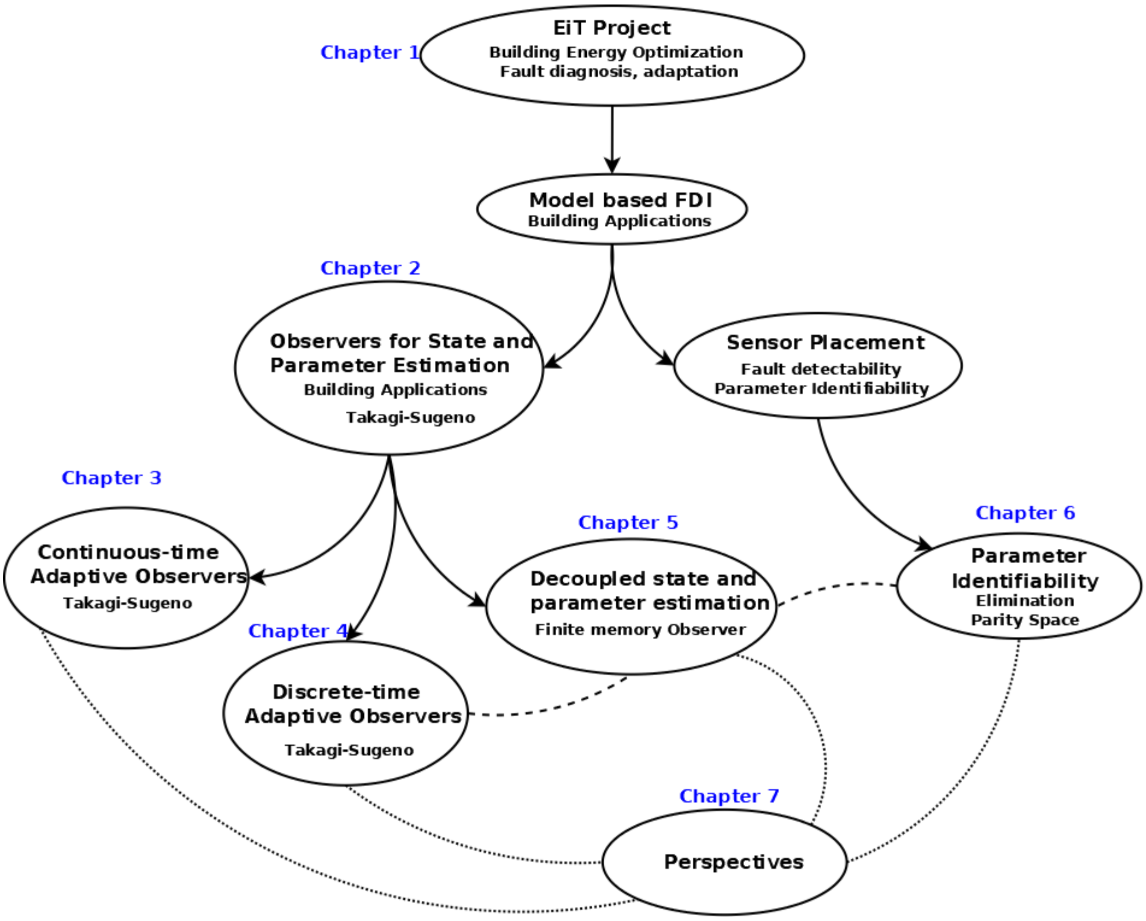

Figure 1.9: Thesis outline

1.4 Thesis outline

With the research inspiration and the problems clarified, the thesis organization is briefly outlined in this section. A schematic of the outline of the thesis is given in Fig. 1.9, where the contents of the chapters and their relationships to others are outlined. This is further elaborated in this section, where the key contribution in each of the chapter is presented as well as how they connect with each other.

The present chapter gave a brief overview of the project, which forms the application motivation for the thesis. The project also directed towards the broad area of *model based fault diagnosis* as the underlying control problem. In Chapter 2, initial explorations in this regard are given. The T-S polytopic modeling and design approach that is used for the observer design in this thesis is outlined first. Following this, relevant existing literature in T-S observer design are applied to some problems in the domain of building energy systems.

The explorations in Chapter 2 revealed possible theoretical contributions in the observer design for state and parameter estimation. The primary among them being the reduction in the complexity of the LMI (Linear Matrix Inequality) conditions for the case when the parameters are constant. The design approach obtained to do it, for continuous-time quasi-LPV models are described in Chapter 3. Here, the adaptive observer design follows a control Lyapunov function-like strategy to obtain the parameter estimation dynamics providing an asymptotic estimation guarantees against the much conservative $\mathbb{L}_2$ bounded guarantees in Chapter 2.

The design approach developed in Chapter 3 couldn't be extended to discrete-time because the control Lyapunov function strategy doesn't transfer well into the discrete-time case. Hence, the Chapter 4 develops the discrete-time version of the design approach that was used in the Chapter 2 for building energy applications. Both the constant and time-varying parameters cases are analyzed. The main challenge in this chapter is to overcome several of the bottlenecks that had to be overcome to show the convergence. However, the results obtained were significantly conservative.

The theoretical challenges and the complex LMI conditions obtained for the discrete-time adaptive observer in Chapter 4 turned the attention towards following a different strategy. One attempt in that direction that provided some insights are discussed in Chapter 5, where a parity space-like approach [16] is used to decouple the estimation of the parameters and the states. In the first step, the states are eliminated to estimate the parameters, and subsequently, the states are estimated. This is set up as a finite memory observer (FMO) to estimate the states and the parameters. This approach provides exact estimates under no modeling errors without noise since it performs a sort of system-inversion.

Several problems arose in the realization of the decoupled state and parameter estimation work. This paved way to treat the problem of parameter identifiability. Incidentally, this is also central to the problem of sensor placement, which did not get significant attention during the major part of the research work. This convergence led to a study of the parameter identifiability methods for nonlinear models and in particular to those that could be represented in quasi-LPV form. These are summarized in Chapter 6. Some initial results on the use of symbolic and numerical null space computation in identifiability analysis are given.

Finally, in Chapter 7, the thesis contributions are summarized and a wide range of perspectives that came out of the contributions in Chapters 3 through 6 are discussed.

Chapter 2

State and parameter estimation in building energy systems

This chapter provides a prelude to the two subsequent chapters on the observer design for state and parameter estimation using the Takagi-Sugeno (T-S) approach. To start with, an introduction to the T-S modeling approach and the observer design process is given. This is followed by the implementation of an existing result on joint state and parameter estimation with the T-S approach for two building energy applications. The chapter ends with a brief on the limitations faced during the implementation and the directions for the subsequent contributions.

Contents

2.1 Modeling and observer design with Takagi-Sugeno approach

Observers use the input and output signals of a system, together with a system model to generate an estimate of the system's state, which is then deployed in control, monitoring, fault detection, etc. The origins of observers date back to the seminal work by Luenberger in [17] for linear systems. For nonlinear systems, one of the popular observer design approaches is using the Takagi-Sugeno models in polytopic formulation [18]. This is made use of in several of the contributions in this thesis. In this section, a brief overview of Takagi-Sugeno models, the observer design methods and the key techniques involved in it are discussed.

2.1.1 Origins of Takagi-Sugeno models

The origins of Takagi-Sugeno (T-S) models go back to the article [19] by Tomohiro Takagi and Michio Sugeno where the approach was proposed to model of fuzzy systems. It consists of several *if-then* rules to capture the characteristics of a system. For example, the ith rule of a T-S model is given as,

$$\begin{aligned}&\text{Model rule } i:\\&\text{If } z_1 \text{ is } Z_1^i \text{ and } Z_2 \text{ is } Z_2^i \text{ and } \cdots\ z_n \text{ is } Z_n^i \text{ then } y = g_i(z)\end{aligned}$$

The vector z contains the premise or scheduling variables with each z_j belonging to a fuzzy set Z_j^i for the $i$the model. The dependence of the region of scheduling variables on the choice of the g_i function is captured through weights $w_i(z)$ to obtain the overall output as,

$$y = \sum_{i=1}^{m} w_i(z) g_i(z) \tag{2.1}$$

The consequent $g_i(z)$ requires to be a static function, but there is no specific restriction on what y is. Authors have taken liberty to to use *linguistic variables* $\dot{x}$ and y to consider state-space models of the form,

$$\begin{aligned}&\text{Model rule } i:\\&\text{If } z_1 \text{ is } Z_1^i \text{ and } z_2 \text{ is } Z_2^i \text{ and } \cdots\ z_n \text{ is } Z_n^i \text{ then,}\\&\qquad\qquad \dot{x} = A_i x + B_i u\\&\qquad\qquad y = C_i x + D_i u\end{aligned} \tag{2.2}$$

It is to be noted that the $\dot{x}$ one one side of the equation is not related to the x on the other side. However, this notation is not wrong within the realms of linguistic variables. This allows further the following fuzzy model,

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{n} w_i(z)(A_i x + B_i u)\\ y &= \sum_{i=1}^{n} w_i(z)(C_i x + D_i u)\end{aligned} \tag{2.3}$$

We would refer to this representation as T-S fuzzy model. For more details on this formulation, refer to [18], [20].

2.1.2 Takagi-Sugeno polytopic model

As was hinted above, there is an abuse of notation in the T-S fuzzy modeling literature and this extends to the T-S polytopic representation. To clarify this, first the method to obtain a T-S polytopic model from a nonlinear model is discussed. Given a nonlinear model,

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t))\\ y(t) &= g(x(t), u(t))\end{aligned}$$

which can be rewritten[7] into a quasi-LPV form,

$$\begin{aligned}\dot{x}(t) &= A(x(t), u(t))x(t) + B(x(t), u(t))u(t)\\ y(t) &= C(x(t), u(t))x(t) + D(x(t), u(t))u(t)\end{aligned} \tag{2.4}$$

[7] if feasible, using nonlinear embedding techniques to be discussed shortly

where $x \in \mathbb{R}^{n_x}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the states, inputs and the output variables. $A(.)$, $B(.)$, $C(.)$, and $D(.)$ are smooth nonlinear matrix functions of appropriate dimensions. The quasi-LPV model can be further turned in a polytopic form with the use of the sector nonlinearity approach [21] which embeds the nonlinearity into a weighting function $\mu_i(.)$ that depends on system variables to obtain a polytopic model of the form,

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{m} \mu_i(x(t), u(t))(A_i x(t) + B_i u(t)) \\ y(t) &= \sum_{i=1}^{m} \mu_i(x(t), u(t))(C_i x(t) + D_i u(t)) \end{aligned} \tag{2.5}$$

Notice that the representations in (2.3) and (2.5) are exactly same except for the deliberate usage of different notations for the weighting functions. If we consider z to contain the system variables, the representations are the same. However, this notational abuse is sometimes confusing as unlike in the fuzzy representation, the scheduling/premise variables are system variables and hence do not have a fuzzy nature as in (2.2). In this thesis, representation would be referred to as T-S polytopic model.

To obtain a T-S polytopic model from a nonlinear model, there are two major approaches: linearisation and nonlinear embedding. Linearisation method uses a two-step process to arrive at the T-S model: First, to decide on the scheduling variables on which the weighting functions will depend on and second choosing sufficient number of linearisation points to obtain the system matrices. Linearisation method provides an approximation to the original nonlinear model. One example of a linearisation method is [22]. In this thesis, the focus is on methods that can provide exact representation of the nonlinear model within a compact set and so linearisation method is not discussed any further.

Nonlinear embedding Obtaining a T-S polytopic model involves a two-step process. In the first step, the nonlinear model is converted into an equivalent quasi-LPV form. Depending upon the choice of the premise variables and how the entries of the matrices in (2.4) are constructed, different quasi-LPV models can be obtained for the same nonlinear model. For a simple nonlinear model, this can be done manually and for large and complex models, formal steps are required. Once the quasi-LPV model is obtained, the nonlinearities have to be embedded into the weighting functions so that the system matrices $A(.)$ etc. are replaced with constant matrices A_i. This step involves finding polytopic bounds for the nonlinear matrix function $A(.)$ formed by the vertices A_i. Depending upon the choice of the vertices, the polytopic model provides an exact characterization of the nonlinear model within the polytope.

A popular way to obtain a T-S polytopic model is to manually choose the premise variables to obtain the quasi-LPV form in (2.4) and then apply the sector nonlinearity approach in [21]. The approach obtains a T-S polytopic model of the form,

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2^{n_p}} \mu_i(z(t))(A_i x(t) + B_i u(t)) \\ y(t) &= \sum_{i=1}^{2^{n_p}} \mu_i(z(t))(C_i x(t) + D_i u(t)) \end{aligned} \tag{2.6}$$

where $z(t)$ refers to the n_p premise variables leading to 2^{n_p} submodels. The weighting functions (μ_is) absorb the nonlinearity in the model and they also satisfy the convex sum property,

$$\sum_{i=1}^{r} \mu_i(z(t)) = 1 \text{ and } 0 \leq \mu_i(z(t)) \leq 1, \quad \forall t, \quad \forall i \in \{1, 2, ...n_p\}$$

The sector nonlinearity approach allows to define the weighting functions such that the resultant T-S model exactly represents the original nonlinear behaviour within the sector. For a given premise variable z_1, enclosed within the sector of $[z_1^{\min}, z_1^{\max}]$, the membership functions are given by,

$$\tilde{\mu}_1^1(z(t)) = \frac{z(t) - z_1^{\min}}{z_1^{\max} - z_1^{\min}}, \text{ and } \tilde{\mu}_1^2(z(t)) = \frac{z_1^{\max} - z(t)}{z_1^{\max} - z_1^{\min}} \tag{2.7}$$

For practical systems the extremum values of the premise variables, $z_1^{\min}$ and $z_1^{\max}$ can be obtained from experience. For a general scenario, the models could be simulated to find the range within which the model max/min value would be satisfactory. The weighting functions $\mu_i(z(t))$ are then obtained by normalizing the products of the membership functions of individual premise variables. For instance, for a system with two premise variables, there are four submodels/weighting functions given by (the dependence $z(t)$ is omitted for simplified notation),

$$\mu_1 = \frac{\tilde{\mu}_1^1\tilde{\mu}_2^1}{\sum_{ij}\tilde{\mu}_1^i\tilde{\mu}_2^j}, \qquad \mu_2 = \frac{\tilde{\mu}_1^1\tilde{\mu}_2^2}{\sum_{ij}\tilde{\mu}_1^i\tilde{\mu}_2^j}, \qquad \mu_1 = \frac{\tilde{\mu}_1^2\tilde{\mu}_2^1}{\sum_{ij}\tilde{\mu}_1^i\tilde{\mu}_2^j}, \qquad \mu_4 = \frac{\tilde{\mu}_1^2\tilde{\mu}_2^2}{\sum_{ij}\tilde{\mu}_1^i\tilde{\mu}_2^j}$$

The chosen premise variables would replace the corresponding functions of x and u in the matrices. By replacing these premise variables with their appropriate extremum values, the constant matrices A_i, B_i, C_i, D_i in (2.6) are obtained. Examples for the applications are in the following chapters as well as can be referred to in [21], [18], [20].

The sector nonlinear approach forms a polytope around the nonlinear model trajectories that does not take into account the relationship that may exist between the premise variables. This leads to conservative models which, while still offering an exact representation within the polytope, can be restrictive when this model is used for applications such as control, fault diagnosis etc. In the works [23], [24], the authors develop an approach to find a balance between accuracy and conservatism of the final model with the use of principal component analysis (PCA). Their approach to find a balance can be summarized as follows:

- Generate *typical* trajectories of scheduling signals
- Compute parameter trajectories generated by these scheduling signal trajectories and normalize this data to form a matrix of normalized parameter data.
- Apply PCA to find a coordinate transform that yields a tighter parameter set
- Apply the normalization and the mapping derived from the data to the parameter functions to determine the desired representation

Other approaches to reduce conservativeness could be referred to such as in [25], and can eventually make use of linear programming techniques such as in [26].

Instead of the manual approach to obtain a quasi-LPV model from a nonlinear model, it is possible to use systematic procedures to do so. In [27], the authors propose a method to obtain a quasi-LPV model for control-affine SISO models. The method exploits the feedback linearisation technique by constructing a diffeomorphism which forms the tranformation where the final quasi-LPV model has only measured premise variables. The authors in [23] also provide a systematic approach to obtain which consists of the following four steps:

- Rewrite the original model in standard form
- Classify the terms
- Assign terms for the state-space matrices
- Combine the terms

This systematic procedure is combined with the conservativeness reduction procedure discussed above to obtain appropriate LPV models for control system problems in application.

2.1.3 Observer design using T-S polytopic approach

One of the contributions in this thesis relates to the design of observers to estimate the states and parameters for quasi-LPV model. To achieve this, the state observer design for T-S polytopic models is used as the foundation. In general, T-S polytopic models are used in control design due to the ability to formulate the design problem to that of solving a set of LMIs. Typically these problems could be classified into one of: LMI feasibility problem and LMI optimization problem (eigenvalue problem or generalized eigenvalue problem). More details on these formulations could be referred to from [28], [29]. Given the availability of a host of LMI solvers that use semidefinite programming techniques (for example LMILab from MATLAB, SeDuMi [30]) the design problems can be easily evaluated with the modern computers.

At the core of the observer design using the T-S models is the Lyapunov's direct method. This makes the approach only a sufficient condition, that is, if the LMI solvers cannot return a solution, it does not provide any definitive answer on the stability of the T-S model. The most popular of the Lyapunov function used for stability analysis is the quadratic Lyapunov function, that is (the time dependence (t) is dropped for simplifying notation),

$$V(x) = x^T P x \tag{2.8}$$

where $P = P^T > 0$ is a symmetric positive definite matrix. An autonomous T-S model $(u = 0)$ of the form (2.6) is quadratically stable if the Lyapunov function (2.8) decreases and asymptotically goes to zero for all the trajectories permitted by the system model. That is (the time dependence (t) is dropped for simplifying notation),

$$\dot{V} = \left(\sum_{i=1}^{2^{n_p}} \mu_i(z) A_i x \right)^T P x + x^T P \left(\sum_{i=1}^{2^{n_p}} \mu_i(z) A_i x \right) = \left(\sum_{i=1}^{2^{n_p}} \mu_i(z) x^T (P A_i + A_i^T P) x \right) \tag{2.9}$$

Since the weighting functions satisfy convex sum properties, the following results follow:

Theorem 1. *[31] The autonomous model*

$$\dot{x} = \sum_{i=1}^{2^{n_p}} \mu_i(z) A_i x$$

is globally asymptotically stable if there exists a matrix $P = P^T > 0$ such that the following LMI feasibility problem has a solution,

$$PA_i + A_i^T P < 0 \tag{2.10}$$

for $i = 1, 2, \cdots, 2^{n_p}$

It is important to note that the Lyapunov matrix P has to be common to all the different vertices of the polytope (the matrices $A_i,\ \forall\ i$). This is the reason why conservativeness of the T-S model is an important issue as a very conservative model may end up having widely varying A_is and hence would make it very difficult to find a common P which satisfies (2.10). Pole placement approach could be used to make the quadratic stability analysis to have arbitrary speed of convergence and is termed D-stability. This is realized by adding extra LMI constraints, more details could be referred to from [32]. There are non-quadratic Lyapunov functions that are also possible to be used with continuous-time T-S models and more details could be referred to in [33], [34]. For discrete-time models, special Lyapunov functions which verify the reduction in the Lyapunov function dynamics over a few sampling intervals have been proposed in [35]. For the purpose of this thesis, the focus would mostly be on quadratic Lyapunov functions.

For a non-autonomous case, especially when there are unknown or unmeasured disturbances that exist in the equation (2.6), the analysis is carried out to minimize the impact of the disturbances on the stability. These could be achieved through the bounded real lemma (BRL) discussed in the following chapter. To illustrate the relevance of the stability analysis for observer design, consider a simple case of continuous-time T-S model in (2.6) where the premise variables z are composed of either known or measured variables. For this system, the following observer is designed,

$$\begin{aligned}
\dot{\hat{x}}(t) &= \sum_{i=1}^{2^{n_p}} \mu_i(z(t))(A_i\hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\\
\hat{y}(t) &= \sum_{i=1}^{2^{n_p}} \mu_i(z(t))(C_i\hat{x}(t) + D_i u(t))
\end{aligned} \tag{2.11}$$

The error dynamics obtained will be,

$$\dot{e}(t) = \sum_{i=1}^{2^{n_p}} \sum_{j=1}^{2^{n_p}} \mu_i(z(t))\mu_j(z(t))(A_i - L_iC_j)e(t) \tag{2.12}$$

This is now an autonomous T-S model and the results in Theorem 1 can be extended to,

Theorem 2. *[31] The estimation error dynamics associated with the observer design in (2.12) is asymptotically stable, if there exists $P = P^T > 0$, and $L_i,\ i = 1, \cdots, 2^{n_p}$ such that,*

$$P(A_i - L_iC_i) + (A_i - L_iC_i)^T P < 0 \tag{2.13}$$

$$P(A_i - L_iC_j + A_j - L_jC_i) + (A_i - L_iC_j + A_j - L_jC_i)^T P < 0 \tag{2.14}$$

for $i = 1, 2, \cdots, 2^{n_p}$ and $j = i + 1, i + 2, ..., 2^{n_p}$ assuming that the two submodels are active simultaneously or the corresponding weighting function is non-zero.

Note that the two conditions (2.13)-(2.14) are not LMIs due to the cross terms of two unknowns L_i and P. However, a simple change of variables would be sufficient to cast them as an LMI. That is, consider, $R_i = PL_i,\ i = 1, \cdots, 2^{n_p}$, then the conditions are reformulated as,

$$PA_i - R_iC_i + A_i^T P - C_i^T R_i^T < 0 \tag{2.15}$$

$$PA_i - R_iC_j + PA_j - R_jC_i + A_i^T P - C_j^T R_i^T + A_j^T P - C_i^T R_j^T < 0 \tag{2.16}$$

The observer gain L_i is then obtained by $L_i = P^{-1}R_i$. These two theorems form the basis of the results in this chapter as well as the subsequent chapters where observers for state and parameter estimation are investigated and proposed.

2.1.4 Some useful results

Casting a given control design problem into an LMI problem involves a number of properties of matrix inequalities. Some of those which are used later in the thesis are given below:

Lemma 1. *[36] Consider two matrices X and Y with appropriate dimensions, a time varying matrix $\Delta(k)$ and a positive scalar λ. The following property is verified:*

$$X^T\Delta^T(k)Y + Y^T\Delta(k)X \leq \lambda X^TX + \lambda^{-1}Y^TY \tag{2.17}$$

for $\Delta^T(k)\Delta(k) \leq I$

Lemma 2. *[37] The following LMIs, for appropriately suitable dimensions, are equivalent:*

$$A^TPA - Q < 0,\ P > 0 \tag{2.18}$$

$$\begin{bmatrix} -Q & A^TP \\ PA & -P \end{bmatrix} < 0,\ P > 0 \tag{2.19}$$

Lemma 3. ***(Bounded Real Lemma**[28]) Assume that A is stable, (A, B, C) is minimal, and $D^TD < 0$. Then, the following are equivalent, for the $\mathbb{L}_2$ gain Γ:*

1. *The system*

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du, \qquad x(0) = 0$$

   is nonexpansive, i.e., satisfies

$$\int_0^T y(t)^Ty(t)dt \leq \int_0^T u(t)^T\Gamma u(t)dt$$

   for all u and $T \geq 0$

2. *The LMI*

$$P > 0, \qquad \begin{bmatrix} A^TP + PA + C^TC & PB + C^TD \\ B^TP + D^TC & D^TD - \Gamma \end{bmatrix} \leq 0 \tag{2.20}$$

   in the variable $P = P^T$ is feasible. This corresponds to the existence of a quadratic Lyapunov function $V(x) = x^TPx$ that satisfies

$$\dot{V} + y^Ty - u^T\Gamma u \leq 0 \tag{2.21}$$

Lemma 4. ***(Discrete-time Bounded Real Lemma*** *[38]) For a discrete-time system, of the form,*

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{2.22}$$

the Bounded Real Lemma equivalent LMI condition for stability with an $\mathbb{L}_2$ *gain* Γ *is,*

$$P > 0, \begin{bmatrix} A^T PA - P + C^T C & A^T PB + C^T D \\ B^T PA + D^T C & D^T D + B^T PB - \Gamma \end{bmatrix} \leq 0 \tag{2.23}$$

where this corresponds to existence of a quadratic Lyapunov function $V(x_k) = x_k^T P x_k$ *such that*

$$V(x_{k+1}) - V(x_k) + y_k^T y_k - u_k^T \Gamma u_k \leq 0 \tag{2.24}$$

Lemma 5. ***(Schur's Complement)*** *For a symmetric matrix* M*, given by,*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

if C *is invertible, then the following properties hold:*

1. $M > 0$ *iff* $C > 0$ *and* $A - BC^{-1}B^T > 0$
2. *if* $C > 0$*, then* $M \geq 0$ *iff* $A - BC^{-1}B^T \geq 0$.

2.2 State and parameter estimation in building energy systems

In Sec. 1.3.2, some examples from building energy systems were stated to motivate the problem of joint state and parameter estimation. In the current section, a result from existing literature is customized and applied for models of some building energy modules. The focus is on quasi-LPV models and the analysis is through the polytopic Takagi-Sugeno modeling approach.

To give the context, a brief review of the use of Takagi-Sugeno models in control and estimation problem in building energy systems is given. The following works give a flavour of those that use a data-based model building and use in control applications. In [39], the authors use a multiple model predictive control (MMPC) for the temperature control in the AHU of a HVAC system. A divide-and-conquer strategy by which the nonlinear characteristics is split into a set of T-S models at a lower level and fuzzy integrated LPV model at the global level is proposed. This allows them to develop a hierarchical MMPC strategy using parallel distribution compensation method. The weights corresponding to the various T-S submodels are learned through a fuzzy satisfactory clustering (FSC) algorithm. This approach is illustrated in an HVAC system with three chillers, three zones with three AHUs. The authors in [40] develop a Fuzzy MPC to the temperature control of an industrial scale cross-flow heat exchanger. To train the model, the local linear model tree algorithm (LOLIMOT) is applied. The authors in [41] choose a T-S modeling approach to model a parallel flow heat exchanger which is part of a thermal plant identified through an iterative fuzzy clustering algorithm.

Very limited number of works deploy a T-S model obtained through first principles. In [42], the authors consider a tubular heat exchanger of counter flow type. The heat exchanger model is obtained according to the direct lumping procedure and the dynamical model that's considered has four states and two outputs in the quasi-LPV form. The aim of the paper is to estimate fouling and the modeling approach renders the fouling parameter to appear in two different unknown parameters. The authors rewrite the model such that the difference between the nominal value of the unknown parameters are bunched together as two unknown inputs. These unknown inputs are considered as polynomials of degree 2, and an extended state vector with the unknown inputs and their non-zero derivatives is formed. A T-S observer is designed for this extended model that estimates both the state and the unknown inputs. To improve performance, a D-stability approach is used. One assumption that makes this approach work for fouling detection is that the temperatures on the primary and secondary sides of the heat exchanger for the two counter flowing liquids are the same. This assumption, however is very difficult to achieve in practice even if the two liquids are the same. In order to test the proposed approach, the authors use input/output data generated by simulation using a refined model of the considered heat exchanger in the ANSYS Fluent CFD software. In [43], the authors follow a similar modeling approach for the heat exchanger as in [42] and the detection of fouling is attached to two different unknown parameter detection. Unlike in the previous work, the authors consider these parameters as constant and add them to the state vector. The observer design considers the model with uncertain terms and follows a sum of squares (SOS) approach.

2.2.1 State and parameter estimation using a T-S design approach

One key observation in the models of several building energy components is that the unknown parameters appear affinely when represented in the LPV state-space (LPV-SS) form. For instance, this enabled the authors in [42] to rewrite the fouling detection problem as an additive unknown input observer problem. That is, the models are of the form,

$$\begin{aligned} \dot{x}(t) &= \left(A_0(z(t)) + \sum_{i=1}^{n_\theta} \bar{A}_i(z(t))\theta_i \right) x(t) + \left(B_0(z(t)) + \sum_{i=1}^{n_\theta} \bar{B}_i(z(t))\theta_i \right) u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2.25}$$

where $z(t)$ refers to the premise or scheduling variables. θ_i is one of the n_θ unknown parameters. The unknown parameters can be time-varying to represent effects such a degradation, faults etc. Further, since these models are obtained through transformation from nonlinear counterparts, the premise or scheduling variables shall consist of the system variables such as inputs, outputs or one of the unmeasured states, making the models quasi-LPV. The requirements lead to use of the joint state and parameter estimation approach proposed and developed in papers such as [44] and [6]. They use the Takagi-Sugeno polytopic approach. The motivation for the choice could be summarized as:

- the unknown parameters, θ_i, be time-varying
- the premise variables, z, be unmeasured/unknown
- no specific restriction on the dynamics of the unknown parameters

The fundamental idea behind the parameter estimation approach proposed in [44] is to rewrite a given unknown parameter using the sector nonlinearity (SNL) transformation [21]. The main assumption required for this construction is that the maximum and minimum values within which the parameter varies are known. That is,

$$\theta(t) = \mu_1(\theta)\theta^1 + \mu_2(\theta)\theta^2 \tag{2.26}$$

where θ^1 and θ^2 represent the minimum and maximum values that the parameter can attain and the membership functions $\mu_i(\theta)$ are given by,

$$\mu_1(\theta) = \frac{\theta^2 - \theta(t)}{\theta^2 - \theta^1}, \qquad \mu_2(\theta) = \frac{\theta(t) - \theta^1}{\theta^2 - \theta^1}$$

When this approach is extended to a model in quasi-LPV state-space form, SNL transformation is applied to both the premise variables and the unknown parameters. First, the application of SNL transformation to an LPV/quasi-LPV model (without any unknown parameters) would lead to Takagi-Sugeno models of the form,

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2^{n_p}} \mu_i^z(z(t))\left(A_i x(t) + B_i u(t)\right) \\ y(t) &= Cx(t) \end{aligned} \tag{2.27}$$

where there are n_p premise variables leading to 2^{n_p} submodels. Further details about the model could be referred to in Sec. 2.1. For the model in (2.25), while considering the parameters as unknown and time-varying, the SNL transformation would lead to,

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{2^{n_p}} \sum_{j=1}^{2^{n_\theta}} \mu_i^z(z(t))\mu_j^\theta(\theta(t))(A_{ij}x(t) + B_{ij}u(t)) \\ y(t) &= Cx(t) \end{aligned} \tag{2.28}$$

with,

$$A_{ij} = \breve{A}_i + \sum_{j=1}^{n_\theta} \theta_j^k \bar{A}_j, \qquad B_{ij} = \breve{B}_i + \sum_{j=1}^{n_\theta} \theta_j^k \bar{B}_j \tag{2.29}$$

where $\theta_j \in [\theta_j^1, \theta_j^2]$ and the weighting functions are obtained using the bounds assumed for the corresponding premise variables and the parameters. The terms $\breve{A}_i$ (and $\breve{B}_i$) refers to the components of the matrix A_{ij} (and B_{ij}) that does not depend on the parameters θ. The reader is referred to the original reference [45] for more details about the SNL transformation and modeling. For a system defined by (2.28)-(2.29), the authors in the original reference propose an observer with a structure given by

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{2^{n_p}} \sum_{j=1}^{2^{n_\theta}} \mu_i^z(\hat{z}(t))\mu_j^\theta(\hat{\theta}(t))[A_{ij}\hat{x} + B_{ij}u + L_{ij}(y(t) - \hat{y}(t))] \\ \dot{\hat{\theta}}(t) &= \sum_{i=1}^{2^{n_p}} \sum_{j=1}^{2^{n_\theta}} \mu_i^z(\hat{z}(t))\mu_j^\theta(\hat{\theta}(t))[K_{ij}(y(t) - \hat{y}(t)) - \eta_{ij}\hat{\theta}(t)] \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{2.30}$$

where L_{ij}, K_{ij} and η_{ij} have to be adjusted to estimate the states and the parameters. Note the use of $\hat{z}(t)$ in the weighting function, which indicates that the premise variable can be considered unmeasured and that an estimate of the premise based on the state estimates go into it. Also of interest to note is that the parameter estimation component in the observer has a proportional-integral (PI) structure.

The analysis of the convergence of the estimation requires the computation of the error between the original and estimated states. However, the weighting functions in the observer model depend on estimated premise variables ($\mu_j^\theta(\hat{\theta})$, $\mu_i^z(\hat{z}(t))$) which makes the comparison of the system equations (2.28) and the observer equations (2.30) difficult. This is avoided by representing the original system in an uncertain-like form as,

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \mu_i^z(\hat{z}(t))\mu_j^\theta(\hat{\theta}(t))\left[(A_{ij}+\Delta A(t))x(t)+(B_{ij}+\Delta B(t))u(t)\right]\\ y(t) &= Cx(t)\end{aligned} \tag{2.31}$$

where $\Delta A(t)$ and $\Delta B(t)$ are bounded time varying factors and a function of the differences between the weighting functions given by

$$\begin{aligned}\Delta A(t) &= \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \left[\mu_i^z(z(t))\mu_j^\theta(\theta(t)) - \mu_i^z(\hat{z}(t))\mu_j^\theta(\hat{\theta}(t))\right] A_{ij} = \mathcal{A}\Sigma_A(t)E_A\\ \Delta B(t) &= \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \left[\mu_i^z(z(t))\mu_j^\theta(\theta(t)) - \mu_i^z(\hat{z}(t))\mu_j^\theta(\hat{\theta}(t))\right] B_{ij} = \mathcal{B}\Sigma_B(t)E_B\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}\mathcal{A} &= \begin{bmatrix}A_{11} & \dots & A_{2^{n_p}2^{n_\theta}}\end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix}B_{11} & \dots & B_{2^{n_p}2^{n_\theta}}\end{bmatrix}\\ E_A &= \begin{bmatrix}I_{n_x} & \dots & I_{n_x}\end{bmatrix}^T, \qquad E_B = \begin{bmatrix}I_{n_u} & \dots & I_{n_u}\end{bmatrix}^T\\ \Sigma_A(t) &= \mathrm{diag}\left(\delta_{11}I_{n_x}, \cdots, \delta_{2^{n_p}2^{n_\theta}}I_{n_x}\right), \qquad \Sigma_B(t) = \mathrm{diag}\left(\delta_{2^{n_p}2^{n_\theta}}I_{n_u}, \cdots, \delta_{2^{n_p}2^{n_\theta}}I_{n_u}\right)\\ \text{with} \qquad \delta_{ij} &= \mu_i^z(z(t))\mu_j^\theta(\theta(t)) - \mu_i^z(\hat{z}(t))\mu_j^\theta(\hat{\theta}(t))\end{aligned}$$

The matrix representation $\mathcal{A}\Sigma_A(t)E_A$ for $\Delta A(t)$ (as well as that for $\Delta B(t)$) is useful later when deriving LMI conditions. This representation helps to obtain the observer error dynamics of the form:

$$\dot{e}_a(t) = \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \mu_i^z(\hat{z}(t))\mu_j^\theta(\hat{\theta}(t))[\Phi_{ij}e_a(t) + \Psi_{ij}(t)\tilde{u}(t)] \tag{2.33}$$

where $e_a(t) \triangleq [e_x(t)\ e_\theta(t)]^T$ combines the error dynamics of both the states and the parameter, with $\tilde{u} \triangleq \begin{bmatrix}x(t) & \theta(t) & \dot{\theta}(t) & u(t)\end{bmatrix}^T$ and,

$$\Phi_{ij} = \begin{bmatrix}A_{ij} - L_{ij}C & 0\\ -K_{ij}C & -\eta_{ij}\end{bmatrix}, \qquad \Psi_{ij}(t) = \begin{bmatrix}\Delta A(t) & 0 & 0 & \Delta B(t)\\ 0 & \eta_{ij} & I & 0\end{bmatrix} \tag{2.34}$$

The problem is now reduced to finding observer gains such that the error will decay and the effect from the external inputs $\tilde{u}$ is minimized with guaranteed bounds. Considering a quadratic Lyapunov function, $V(t) = e_a^T(t)Pe_a(t)$, $P = P^T > 0$, the following inequality needs to hold,

$$\dot{V}(t) + e_a^T(t)e_a(t) - \tilde{u}^T(t)\Gamma_2\tilde{u}(t) < 0 \tag{2.35}$$

where Γ_2 corresponds to the matrix with block entries corresponding component in $\tilde{u}$. This can be expanded taking into account the dynamics in (2.33) to obtain

$$\sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \mu_i(z(t))\mu_j(\hat{\theta}(t)) \begin{bmatrix}e_a(t)\\ \tilde{u}(t)\end{bmatrix}^T \begin{bmatrix}\Phi_{ij}^TP + P\Phi_{ij} + I & P\Psi_{ij}(t)\\ \Psi_{ij}^T(t)P & -\Gamma_2\end{bmatrix} \begin{bmatrix}e_a(t)\\ \tilde{u}(t)\end{bmatrix} < 0 \tag{2.36}$$

To obtain the conditions to guarantee stability and bounded influence of the elements in $\tilde{u}(t)$ on the convergence, BRL is a candidate of application. The uncertain terms inside $\Psi_{ij}(t)$ in (2.34) makes it a time-varying factor and application of BRL requires the error dynamics to have constant matrices. The uncertain terms in $\Psi_{ij}(t)$ in (2.34) depend on weighting functions which are time varying but follow the convex sum property which is exploited by the authors. The authors use the Lemma 1 to obtain an upper bound for the uncertain terms, which depend on the weighting functions. The resulting matrix inequalities are not linear, however and require the following two steps to obtain LMIs:

- The Lyapunov matrix P has a block diagonal structure, that is, $P = \text{diag}(P_0, P_1)$ of the sizes corresponding to n_x and n_θ. This makes it easy to reduce the nonlinear matrix inequalities to LMIs.
- Change of variables for bilinear terms: $R_{ij} = P_0 L_{ij}$, $F_{ij} = P_1 K_{ij}$ and $\bar{\eta}_{ij} = P_1 \eta_{ij}$.

This leads to LMI conditions which are summarized in the following theorem.

Theorem 3. *[6] There exists a robust state and parameter observer for the polytopic system with a time varying parameter with a bounded $\mathcal{L}_2$ gain β of the transfer from $\tilde{u}(t)$ to $e_a(t)$ ($\beta > 0$) if there exists $P_0 = P_0^T > 0$, $P_1 = P_1^T > 0$, $\beta > 0$, λ_1, λ_2, Γ_2^0, Γ_2^1, Γ_2^2, Γ_2^3, η_{ij}, F_{ij} and R_{ij} solution for the optimization problem* (2.37) *under the LMI constraints* (2.39) *(for $i = 1, 2, ..., 2^{n_p}$ $j = 1, ..., 2^{n_\theta}$):*

$$\underset{P_0,P_1,R_{ij},F_{ij},\bar{\eta}_{ij},\lambda_1,\lambda_2,\Gamma_2^k}{\textit{minimize}} \quad \beta \tag{2.37}$$

$$\Gamma_2^k < \beta I \textit{ for } k = 0, 1, 2, 3, 4 \tag{2.38}$$

$$\begin{bmatrix} Q_{ij}^{11} & -C^T F_{ij}^T & 0 & 0 & 0 & 0 & P_0\mathcal{A} & P_0\mathcal{B} \\ * & Q_{ij}^{22} & 0 & \bar{\eta}_{ij} & P_1 & 0 & 0 & 0 \\ * & * & Q^{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 & 0 & 0 \\ * & * & * & * & * & Q^{66} & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & -\lambda_1 I_{n_a} & 0 \\ * & * & * & * & * & * & 0 & -\lambda_2 I_{n_b} \end{bmatrix} < 0 \tag{2.39}$$

with,

$$\begin{aligned} Q_{ij}^{11} &= P_0 A_{ij} + A_{ij}^T P_0 - R_{ij} C - C^T R_{ij}^T + I_{n_x} \\ Q_{ij}^{22} &= -\bar{\eta}_{ij} - \bar{\eta}_{ij}^T + I_{n_\theta} \\ Q^{33} &= -\Gamma_2^0 + \lambda_1 E_A^T E_A \\ Q^{66} &= -\Gamma_2^3 + \lambda_1 E_B^T E_B \end{aligned}$$

and the entry '$*$' refers to the transpose of the corresponding element due to the symmetric nature. The indices for the identity matrices are given by: $n_a = n_x \times 2^{n_p+n_\theta}$ and $n_b = n_u \times 2^{n_p+n_\theta}$. The solution to this LMI would then be used to obtain the observer gains by:

$$\begin{cases} L_{ij} &= P_0^{-1} R_{ij} \\ K_{ij} &= P_1^{-1} F_{ij} \\ \eta_{ij} &= P_1^{-1} \bar{\eta}_{ij} \end{cases} \tag{2.40}$$

2.2.2 Customizations for implementation

The results in [6] need some adaptation when applying to specific problems, such as the following:

- The models contain known or measured premise variables $\Rightarrow$ A joint state and parameter estimation result with known premise variables is envisaged.

- An $\mathbb{L}_2$ bound for noise influence on the estimation error. The authors analyze this for the linear case in [44], but extension to quasi-LPV is pending.

- Reducing the number of variables to be determined:

  - The gains Γ_2^k in (2.35) are chosen. This follows the Remark 3 in [6]. However, choosing all the Γ_2^k parameters transforms the constrained optimization problem with LMI constraint in Theorem 3 to an LMI feasibility problem.
  - The gains η_{ij} in (2.34) are chosen. The presence of a non-zero integral gain allows for the matrix Φ_{ij} in (2.34) to have non-zero eigenvalues and hence choosing them simplifies the problem.

- Numerical constraints:

  - To obtain the LMIs from nonlinear matrix inequalities, change of variables is employed. This leads to the final observer gain computation as given in (2.40) which require the computation of P_0^{-1} and P_1^{-1}. If the computed P_0 and P_1 are close to 0, the inverse computation will be numerically intractable. To avoid this, additional conditions are added such that $P_0 > P_{0init}$ and $P_1 > P_{1init}$ for sufficiently large P_{0init} and P_{1init}.
  - It was observed during simulations that there is a need for between the gains K_{ij} and η and an additional LMI constraint is considered as,

$$F_{ij} > \rho P_1 \eta \tag{2.41}$$

    where ρ is chosen such that the integral gain would not let the estimation vanish to zero due to the effect of η. ρ shall be a diagonal matrix of different values depending upon the scale of each the unknown parameter θ. This aspect requires further analysis for cases such as multiple θ.

For the purpose of illustration, the integral gain η_{ij} is considered the same across different submodels and denoted as η_0. The chosen $\mathbb{L}_2$ gains for the component of $\tilde{u}$ is given by $\Gamma = \begin{bmatrix}\Gamma_x & \Gamma_\theta & \Gamma_{\dot{\theta}} & \Gamma_u & \Gamma_\nu\end{bmatrix}^T$, where the last element corresponds to the measurement noise $\nu(t)$. The system model considered for the implementation is given by,

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{2^{n_p}} \sum_{j=1}^{2^{n_\theta}} \mu_i^z(z(t))\mu_j^\theta(\theta(t))(A_{ij}x(t) + B_{ij}u(t)) \\ y(t) &= Cx(t) + H\nu(t)\end{aligned} \tag{2.42}$$

with the noise $\nu(t)$ affecting the measurement through the transmission matrix H. The observer structure is chosen to be

$$\begin{aligned}
\dot{\hat{x}}(t) &= \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \mu_i^z(z(t))\mu_j^\theta(\hat{\theta}(t))[A_{ij}\hat{x} + B_{ij}u + L_{ij}(y(t) - \hat{y}(t))] \\
\dot{\hat{\theta}}(t) &= \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \mu_i^z(z(t))\mu_j^\theta(\hat{\theta}(t))[K_{ij}(y(t) - \hat{y}(t)) - \eta_0\hat{\theta}(t)] \\
\hat{y}(t) &= C\hat{x}(t)
\end{aligned} \tag{2.43}$$

Theorem 4. *There exists a robust state and parameter observer* (2.43) *for the Takagi-Sugeno model with time-varying parameters* (2.42) *with a chosen bounded gain of* Γ *from* $\tilde{u}(t)$ *to* $e_a(t)$*, if there exists* $P_0 = P_0^T > 0$*,* $P_1 = P_1^T > 0$*,* λ_1*,* $\lambda_2 > 0$*,* F_{ij}*,* R_{ij} *such that (for* $i = 1, 2, ..., 2^{n_p}$ *and* $j = 1, 2, ..., 2^{n_\theta}$*), the following LMIs are satisfied*

$$\begin{bmatrix}
Q_{ij}^{11} & -C^TF_{ij}^T & 0 & 0 & 0 & 0 & -R_{ij}H & P_0\mathcal{A} & P_0\mathcal{B} \\
* & Q_{ij}^{22} & 0 & P_1\eta_0 & P_1 & 0 & -F_{ij}H & 0 & 0 \\
* & * & Q^{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\Gamma_\theta & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\Gamma_{\dot{\theta}} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & Q^{66} & 0 & 0 & 0 \\
* & * & * & * & * & * & -\Gamma_\nu & 0 & 0 \\
* & * & * & * & * & * & * & -\lambda_1 I_{n_a} & 0 \\
* & * & * & * & * & * & * & * & -\lambda_2 I_{n_b}
\end{bmatrix} < 0 \tag{2.44}$$

$$F_{ij} > \rho P_1\eta_0 I_{n_\theta} \tag{2.45}$$

where Q_{ij}^{11} *refers to that in* (2.39) *and*

$$\begin{aligned}
Q^{22} &= -\eta_0 - \eta_0^T + P_1 + I_{n_\theta} \\
Q^{33} &= -\Gamma_x + \lambda_1 E_A^T E_A \\
Q^{66} &= -\Gamma_u + \lambda_2 E_B^T E_B
\end{aligned}$$

The observer gains are given by:

$$L_{ij} = P_0^{-1}R_{ij} \text{ and } K_{ij} = P_1^{-1}F_{ij} \tag{2.46}$$

Proof. Though the premise variables are considered measured, the proof follows the steps in [6]. The central crux of the convergence analysis is to handle the difference between the actual and the estimated weighting functions. This means that whether the premise variables or known or not, the conservativeness of the proposed results remains the same. In the modified scenario with known premise variable, this difference continues to remain. This is because the weighting function is product of the membership functions corresponding to the premise variables and the unknown parameters and the latter remains *estimated*. That is, the uncertain-like terms $\Delta A(t)$ and $\Delta B(t)$ in (2.32) are modified, but have the same structure and characteristic as given by

$$\begin{aligned}
\Delta A(t) &= \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \mu_i^z(z(t))\left[\mu_j^\theta(\theta(t)) - \mu_j^\theta(\hat{\theta}(t))\right]A_{ij} \\
\Delta B(t) &= \sum_{i=1}^{2^{n_p}}\sum_{j=1}^{2^{n_\theta}} \mu_i^z(z(t))\left[\mu_j^\theta(\theta(t)) - \mu_j^\theta(\hat{\theta}(t))\right]B_{ij}
\end{aligned} \tag{2.47}$$

This means that whether the premise variables are known or not, the conservativeness of the proposed result remains the same. Further, since there is an additional measurement noise considered, the analysis leads to the vector $\tilde{u}(t)$ to be extended to include the measurement noise such that,

$$\tilde{u}(t) = \begin{bmatrix} x(t) & \theta(t) & \dot{\theta}(t) & u(t) & \nu(t) \end{bmatrix}^T$$

with the $\Psi(t)$ matrix in (2.34) having the form,

$$\Psi_{ij}(t) = \begin{bmatrix} \Delta A(t) & 0 & 0 & \Delta B(t) & -L_{ij}H \\ 0 & \eta_0 & I & 0 & -K_{ij}H \end{bmatrix}$$

To accommodate this, the $\mathbb{L}_2$ gain matrix adds another block Γ_ν and corresponding modification in the LMI condition with the addition of a row and column all following the steps in [44]. □

In the subsequent sections, this result will be applied to several examples from building energy systems.

2.2.3 Application: heat exchanger

Heat exchanger is a recurring component in different types of HVAC systems. One way to model heat exchanger is from the first principles considering the energy balance between the heat exchanging fluids, the interaction with the intervening material and the external environment. In this sense, the model used in [42] provides an ideal starting point. One way to simplify the model in [42] is to lump the variables at the ends of the heat exchanger. Further, specific choice of the fluids are made with the water being supplied to the heat exchanger to supply heat to (or take away heat from) the air to be fed to the occupancy zones. This is represented by the nonlinear differential equations

$$\frac{dT_{ao}(t)}{dt} = \frac{q_a(t)}{M_a}(T_{ai}(t) - T_{ao}(t)) + \frac{U_A(t)}{2C_{pa}M_a}\Delta T(t) \tag{2.48}$$

$$\frac{dT_{wo}(t)}{dt} = \frac{q_w(t)}{M_w}(T_{wi}(t) - T_{wo}(t)) - \frac{U_A(t)}{2C_{pw}M_w}\Delta T(t) \tag{2.49}$$

where

T_{ao}, T_{ai}	: Output/input air temperature (K)
T_{wo}, T_{wi}	: Output/input water temperature (K)
$q_a,\ q_w$	: Air and water mass flow rates (g/s)
$C_{pa},\ C_{pw}$	: Specific heat capacities (J/g.K)
U_A	: Overall heat transfer coefficient (J/K)

The following are assumed for the model:

- The mass flow rates of air and water do no undergo any change inside the heat exchanger.
- The temperature variation is considered lumped at the ends of the heat exchanger.
- There is no heat loss in the transfer through the metal conductor.
- The term ΔT is chosen as $T_{wo} + T_{wi} - T_{ao} - T_{ai}$ as discussed in [42] which is considered a better approximation.

- $U_A(t)$, the heat transfer coefficient is a time varying factor. However, it changes very slowly (over weeks) and hence could be considered constant for problems where time duration is small.

By making the following definitions for the states and inputs of the system:

$$x_1(t) \triangleq T_{ao}(t), \qquad x_2(t) \triangleq T_{wo}(t), \qquad d(t) \triangleq T_{ai}(t), \qquad u_1(t) \triangleq T_{wi}(t)$$

and defining the constants as

$$\alpha_1 \triangleq \frac{1}{M_a}, \qquad \alpha_{2a} \triangleq \frac{1}{2C_{pa}M_a}, \qquad \alpha_{2w} \triangleq \frac{1}{2C_{pw}M_w}, \alpha_3 = \frac{1}{M_w}$$

the model could be represented as

$$\dot{x}_1(t) = \alpha_1 q_a(t)(d(t) - x_1(t)) + \alpha_{2a}U_A(t)(x_2(t) + u_1(t) - x_1(t) - d(t)) \tag{2.50}$$
$$\dot{x}_2(t) = \alpha_3 q_w(t)(u_1(t) - x_2(t)) - \alpha_{2w}U_A(t)(x_2(t) + u_1(t) - x_1(t) - d(t)) \tag{2.51}$$

This can further be represented in a matrix form as

$$\dot{x}(t) = A(x,u)x(t) + B(x,u)u(t) \tag{2.52}$$

where

$$A(x,u) = \begin{bmatrix} -\alpha_1 q_a(t) - \alpha_{2a}U_A(t) & \alpha_{2a}U_A(t) \\ \alpha_{2w}U_A(t) & -\alpha_3 q_w(t) - \alpha_{2w}U_A(t) \end{bmatrix}$$
$$B(x,u) = \begin{bmatrix} \alpha_1 q_a(t) - \alpha_{2a}U_A(t) & \alpha_{2a}U_A(t) \\ \alpha_{2w}U_A(t) & \alpha_3 q_w(t) - \alpha_{2w}U_A(t) \end{bmatrix}$$

The input air temperature is considered as a measured disturbance since the external weather is not controlled, but measured.

To represent the heat exchanger model, the premise variables considered are the mass flow rates $z(t) = \begin{bmatrix} q_a(t) & q_w(t) \end{bmatrix}$. Thus the time varying system matrices are given by,

$$A_i(t) = \begin{bmatrix} -\alpha_1 z_1^i - \alpha_{2a}U_A(t) & \alpha_{2a}U_A(t) \\ \alpha_{2w}U_A(t) & -\alpha_3 z_2^i - \alpha_{2w}U_A(t) \end{bmatrix}$$
$$B_i(t) = \begin{bmatrix} \alpha_1 z_1^i - \alpha_{2a}U_A(t) & \alpha_{2a}U_A(t) \\ \alpha_{2w}U_A(t) & \alpha_3 z_2^i - \alpha_{2w}U_A(t) \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{2.53}$$

for $i = 1, 2, 3, 4$. The superscript i on the premise variable refers to the corresponding maximum or minimum value of the premise variable of the ith submodel. That is, $i = 1$ shall correspond to the minimum values of both the premise variables, and hence z_1^1 and z_1^2 are the minimum values of the two premise variables. The four possible combinations are represented as four submodels. As is evident, the model assumes the temperature of air at the secondary side of the heat exchanger as measured.

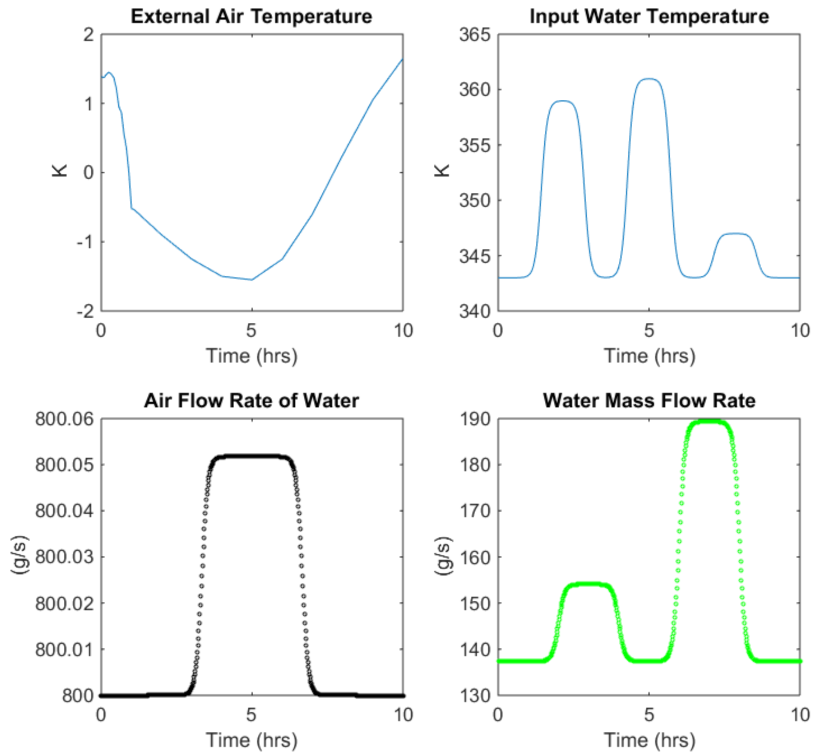

Figure 2.1: Typical inputs and premise variables for the model in (2.48)-(2.48)

In the following, three example cases of joint state and parameter estimation using the heat exchanger model (2.48)-(2.49) are given. A typical set of inputs and premise variables are shown in Fig. 2.1. The models were simulated over a 10 hour period with a sampling interval of 1 minute. It is to be noted that over a 10 hour period, it can be assumed that there would be no noticeable change in the heat transfer coefficient $U_A(t)$. Hence for the illustration of the estimation of $U_A(t)$, it has been shown to vary over this short time period. The results are expected to hold with appropriate time scaling. All simulations were carried out in MATLAB with the Yalmip toolbox in conjunction with the LMILab solver from Mathworks. As discussed in the previous section, the parameters whose values were fixed are given in table 2.1. For

Table 2.1: Simulation parameters comparing Examples 1, 2 and 3

Parameters	Example 1	Example 2	Example 3
η_0	10^{-4}	5×10^{-5}	10^{-7}
ρ	10^5	10^6	10^7
Γ_θ	0.01	100	100
$\Gamma_{\dot{\theta}}$	0.01	10	10
Γ_x, Γ_u	0.01	1	1
Γ_ν	0.01	1	1

the simulation, the measurement noise added was a zero mean Gaussian noise with a standard deviation of $1K$. The sector bounds considered for the premise variables and the time varying parameters for the simulation are given by: $U_A(t) \in [27,\ 324]\ J/K$, $q_w(t) \in [80,\ 1200]\ g/s$ and $q_a(t) \in [200,\ 1500]\ g/s$.

Example 1. *(Fouling estimation) The heat exchanger surfaces suffer deposition of unwanted materials over time. This phenomenon, known as fouling, leads to reduction of effectiveness of the heat exchanger. Due to this deposition, the heat transfer coefficient $U_A(t)$ changes over a long period of time. This can be analyzed by constructing a parameter estimator for $\theta(t) \triangleq U_A(t)$. With the premise variables as $z_1(t) \triangleq q_a(t)$ and $z_2(t) \triangleq q_w(t)$, the system matrices in* (2.28) *and* (2.29) *for this problem can be given by,*

$$\breve{A}_i = \begin{bmatrix} -\alpha_1 z_1^i & 0 \\ 0 & -\alpha_3 z_2^i \end{bmatrix} \quad \bar{A} = \begin{bmatrix} -\alpha_{2a} & \alpha_{2a} \\ \alpha_{2w} & -\alpha_{2w} \end{bmatrix}$$
$$\breve{B}_i = \begin{bmatrix} \alpha_1 z_1^i & 0 \\ 0 & \alpha_3 z_2^i \end{bmatrix} \quad \bar{B} = \begin{bmatrix} -\alpha_{2a} & \alpha_{2a} \\ \alpha_{2w} & -\alpha_{2w} \end{bmatrix} \tag{2.54}$$

with $i = 1, 2, 3, 4$.

For this example, the estimated states and parameters using the observer in (4) is given in Fig. 2.2. For the given simulation, the error characteristics are summarized in table 2.2. The mean and standard deviation values given are for the absolute error over the entire simulation period. The characteristics of fouling was simulated by representing it as a part of a cosine wave as suggested in [46].

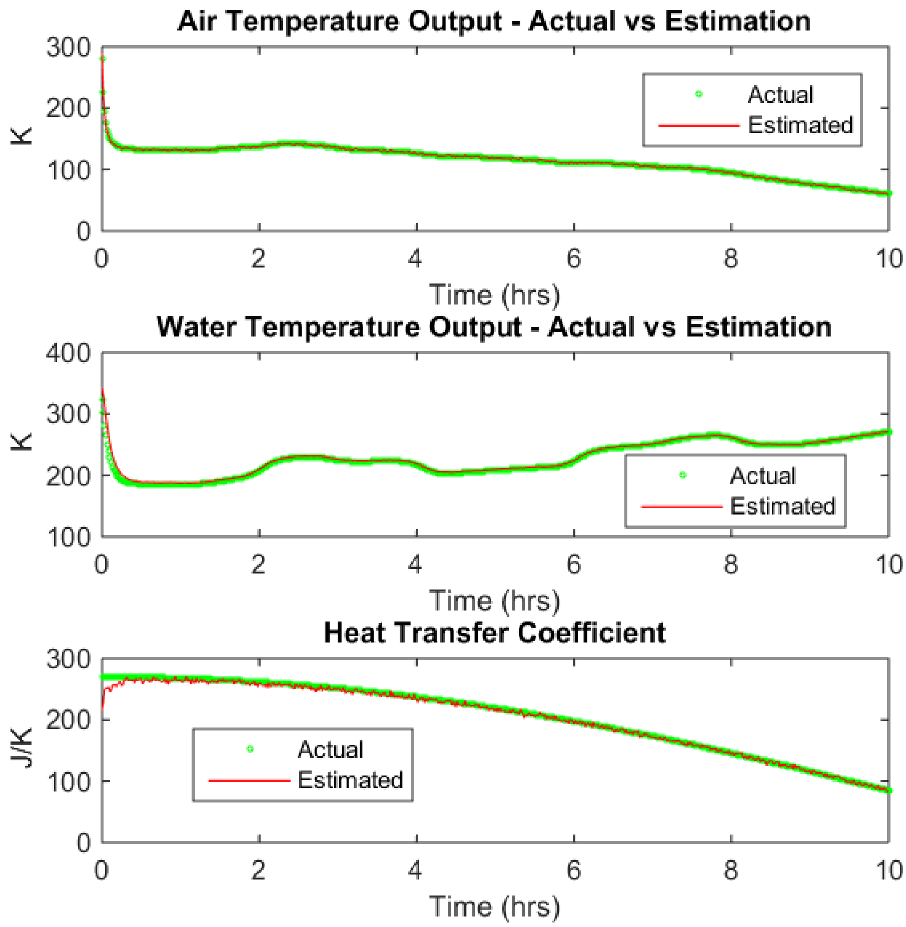

Figure 2.2: State and parameter estimates: U_A estimation (Example 1)

Table 2.2: Results: $U_A(t)$ Estimation

Error	**Mean (%)**	**Standard Deviation (%)**
$\lvert e_{x_1}\rvert$	2.4	2
$\lvert e_{x_2}\rvert$	1.4	0.4
$\lvert e_\theta\rvert$	1.1	1.2

Example 2. *(Air mass flow rate estimation) The mass flow rate of air is maintained by the fan speed as well as the pressure balance in the air network. An estimation of the air mass flow rate as a parameter can potentially point out to malfunctioning of the fan or problems in the duct. With $z(t) \triangleq q_w(t)$ and $\theta(t) \triangleq q_a(t)$, the system matrices for the parameter estimation become,*

$$\begin{aligned} \breve{A}_i &= \begin{bmatrix} -\alpha_{2au} & \alpha_{2au} \\ \alpha_{2wu} & -\alpha_3 z^i - \alpha_{2wu} \end{bmatrix} \quad \bar{A} = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & 0 \end{bmatrix} \\ \breve{B}_i &= \begin{bmatrix} -\alpha_{2au} & \alpha_{2au} \\ \alpha_{2wu} & \alpha_3 z^i - \alpha_{2wu} \end{bmatrix} \quad \bar{B} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.55}$$

with $i = 1, 2$.

The air flow rate estimation results along with the corresponding state estimation are given in Fig 2.3. The estimation errors' mean and standard deviation are given in table 2.3. As is evident from the figure, the estimation of the air mass flow rate follows the actual value.

Table 2.3: Results: $q_a(t)$ Estimation

Error	**Mean (%)**	**Standard Deviation (%)**
$\lvert e_{x_1}\rvert$	0.7	0.32
$\lvert e_{x_2}\rvert$	0.57	0.52
$\lvert e_\theta\rvert$	1.3	1.5

Example 3. *(Water mass flow rate estimation) The mass flow rate of the water is one of the control inputs used in the industry to regulate and maintain the output air temperature of the heat exchanger. The water mass flow rate could be affected by a number of reasons: a stuck or malfunctioning valve, a malfunctioning pump, etc. Since the time scale over which this occurs is considerably small, the heat transfer coefficient is considered constant: $\alpha_{2wu} \triangleq \alpha_{2w} U_A$ and $\alpha_{2au} \triangleq \alpha_{2a} U_A$. With $z(t) \triangleq q_a(t)$ and $\theta(t) \triangleq q_w(t)$, the system matrices for the parameter estimation become,*

$$\begin{aligned} \breve{A}_i &= \begin{bmatrix} -\alpha_1 z^i - \alpha_{2au} & \alpha_{2au} \\ \alpha_{2wu} & -\alpha_{2wu} \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha_3 \end{bmatrix} \\ \breve{B}_i &= \begin{bmatrix} \alpha_1 z^i - \alpha_{2au} & \alpha_{2au} \\ \alpha_{2wu} & -\alpha_{2wu} \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_3 \end{bmatrix} \end{aligned} \tag{2.56}$$

with $i = 1, 2$.

The estimated water mass flow rate along with the estimated states are given in Fig 2.4. The mean and standard deviation of the estimation errors for the entire simulation duration is given in table 2.4. It can be seen that there is considerable error in the parameter estimation in this case. Part of the reason is due to the initial period of oscillations, but also due to model characteristics, further details of which follow in the discussions.

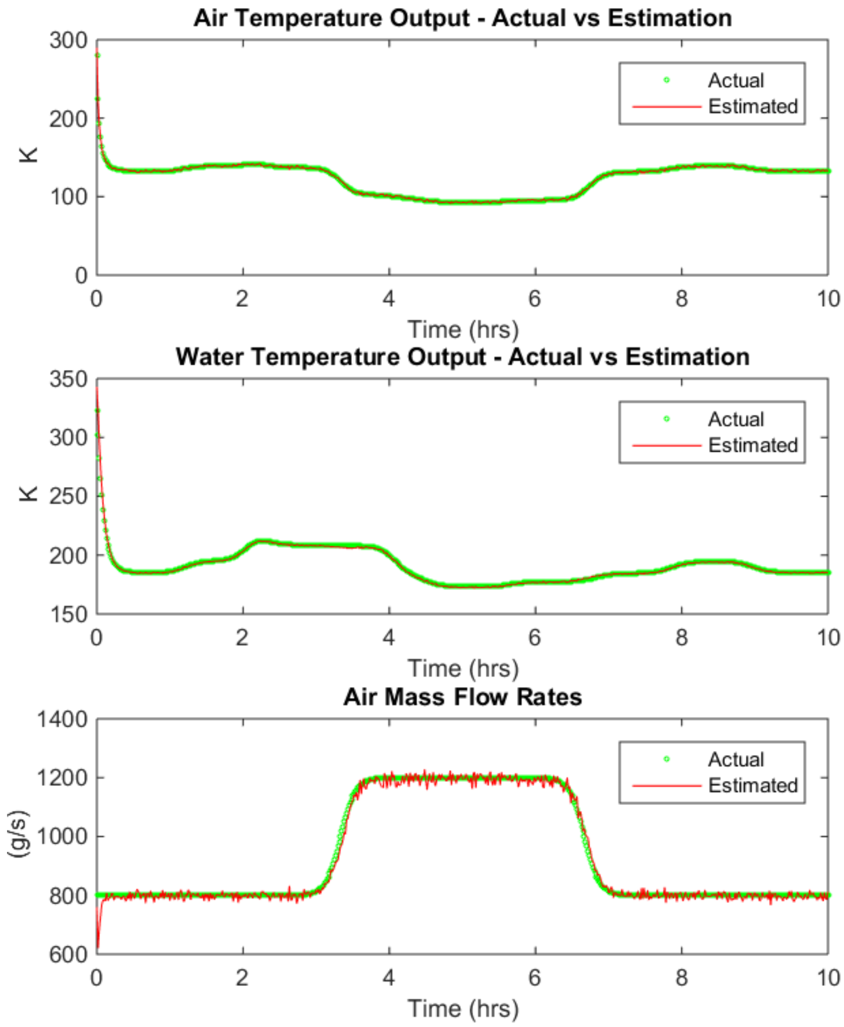

Figure 2.3: State and parameter estimates: q_a estimation (Example 2)

Remark 1. *During the simulation, it was observed that the condition $K_{ij}/\eta_{ij} > \rho$ realized using the LMI condition 2.45 is not sufficient. During the initial simulation period or during sudden changes in the parameter to be estimated, the error $y(t) - \hat{y}(t)$ is significantly high and hence a high value of K_{ij} leads to significant oscillations. This oscillations could be damped too slowly for a meaningful response if the ratio (and hence K_{ij}) is too high. This phenomenon could be seen partly in the simulation results in Fig. 2.3 and Fig. 2.4. Given the time varying nature of the residue, an appropriate observer will have a K_{ij} value that would be adapted with time and residue amplitude.*

Remark 2. *The simulations were performed with the unit of gram for mass since kg introduced scaling issues leading to stiffness in the mass flow rate estimation models.*

Remark 3. *It is apparent from the results that the estimate of $q_w(t)$ is the least convincing of the three. This could be understood by a simple local algebraic observability analysis (see for example, [47]). From* (2.50)-(2.51)*, the $U_A(t)$ and $q_a(t)$ estimates can be written as (dropping*

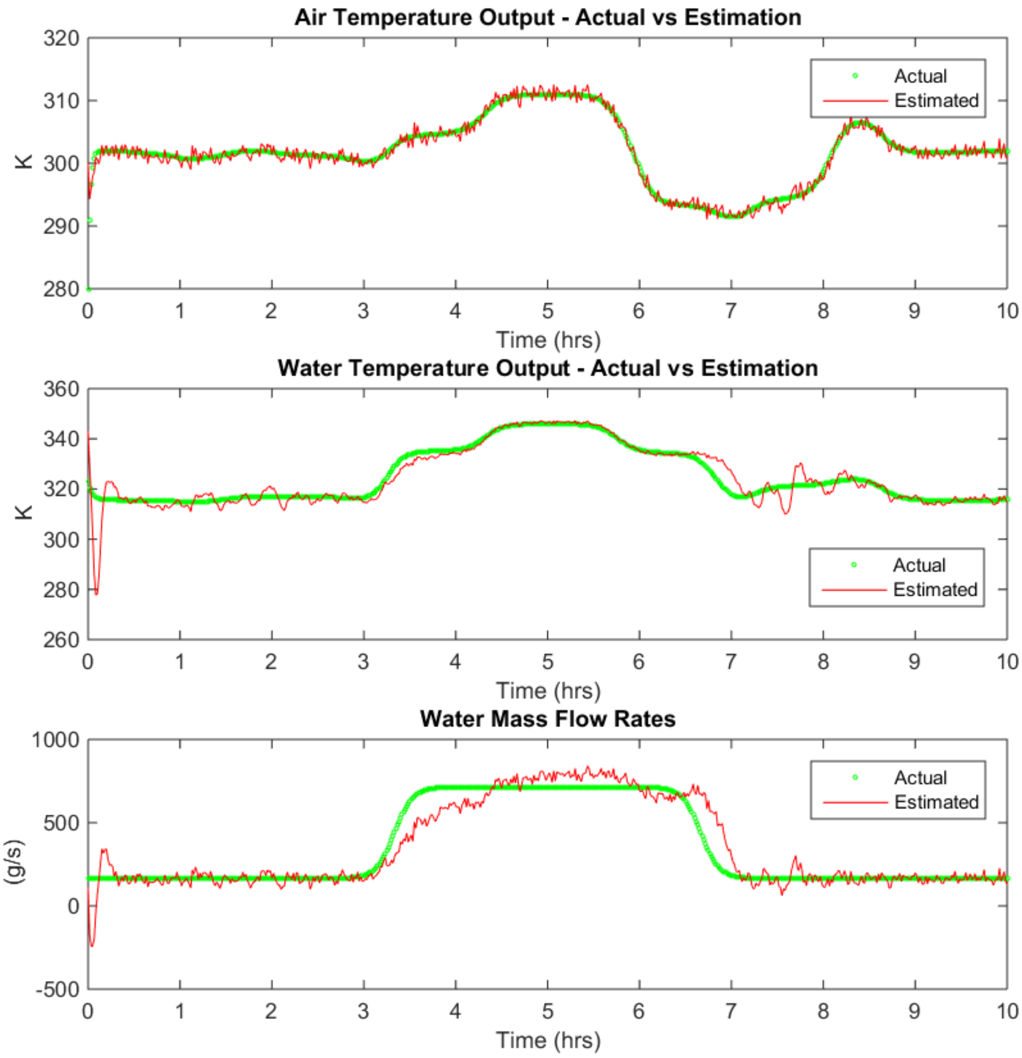

Figure 2.4: State and parameter estimates: q_w estimation (Example 3)

(t) for simplicity and considering $y = x_1$),

$$U_A = \frac{\dot{y} - \alpha_1 q_a(d-y)}{\alpha_{2a}(u_1 + x_2 - d - y)}$$
$$q_a = \frac{\dot{y} - \alpha_{2a} U_A (u_1 + x_2 - d - y)}{\alpha_1(d-y)}$$

They depend only on $\dot{y}$, and the other unknown x_2 can be obtained from (2.51) without the need for $\ddot{y}$. However, q_w is given by,

$$q_w = \frac{\dot{x}_2 + \alpha_{2w} U_A (u_1 + x_2 - d - y)}{\alpha_3(u_1 - x_2)} \tag{2.57}$$

The dependence on $\dot{x}_2$ directly indicates the need for $\ddot{y}$ to compute q_w and hence explains the observation of noise and delay in the estimate of q_w. One possible approach to avoid this could be to design the parameter estimation observer with a filtering mechanism inbuilt to minimize the effect of the measurement noise.

Table 2.4: Results: $q_w(t)$ Estimation

Error	Mean (%)	Standard Deviation (%)
$\|e_{x_1}\|$	0.22	0.64
$\|e_{x_2}\|$	0.32	1.04
$\|e_\theta\|$	17	25

2.2.4 Application: air handling unit with a VAV box

In the previous section, the three examples considered cases with measured or known premise variables. In this section, an application example is formulated where the premise variable is unmeasured. The system considered is an AHU with a heat exchanger at its core and the air to be distributed to two zones through a VAV box. A schematic of the system considered is given in Fig. 2.5. The following are the points to be noted for the model:

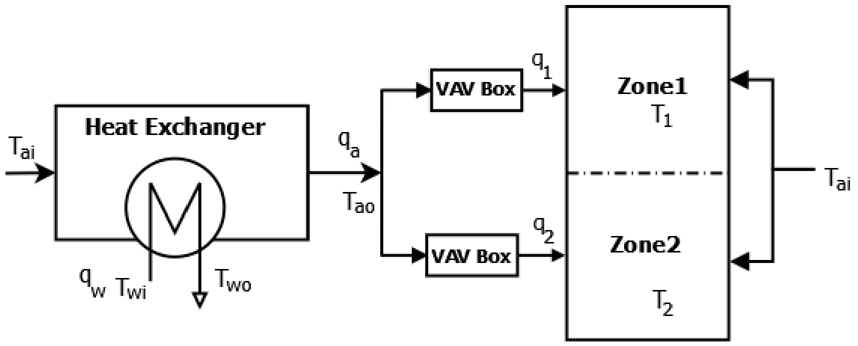

Figure 2.5: Schematic of the system under consideration

- Only two zones are considered for the system, though this could be easily extended to any number of zones supplied by a single AHU.
- The VAV boxes have an internal control loop that adjusts the damper position based on the deviation of zone temperature from the set point.
- The two zone temperatures are measured along with the air mass flow rate q_a.
- The mass flow rate of water (q_w) is the manipulated variable. The water temperature is considered constant and known.
- Further the overall heat transfer coefficient of the heat exchanger, $U_A(t)$, is assumed constant, since fouling takes place over a very long period of time.
- The heat exchanger is assumed extract air from the environment, which is also the same ambient temperature outside the walls of the zone.

The heat exchanger (main component of the AHU) model structure is the same as that used in the Sec. 2.2.3. The zone models were derived based on an energy balance approach and inspired by the thermal modeling (see for e.g., [48], [49]) and given by:

$$\begin{aligned}
C_1\frac{dT_1(t)}{dt} &= q_1(t)C_{pa}(T_{ao}(t)-T_1(t)) + K_{12}(T_2(t)-T_1(t)) + K_{1amb}(T_{ai}(t)-T_1(t)) + K_{d_2}d_2(t)\\
C_2\frac{dT_2(t)}{dt} &= q_2(t)C_{pa}(T_{ao}(t)-T_2(t)) + K_{21}(T_1(t)-T_2(t)) + K_{2amb}(T_{ai}(t)-T_2(t)) + K_{d_3}d_3(t)
\end{aligned} \tag{2.58}$$

where q_1, q_2 are the air mass flow rate into each zones. The known, measured or forecast disturbances like occupancy, solar radiation are combined and represented as d_2 and d_3. C_1, C_2 are the corresponding zone capacitance. The gains K_{12}, K_{21}, K_{1amb}, K_{2amb} correspond to the heat transfer coefficient offered by a wall or a window that remains between the zone temperature and the other zone and the zone and the external environment respectively. K_{d2} and K_{d3} represent the general resistance offered for the exchange of thermal energy between objects in the room with the air. The values are computed following the approach in [48].

The modeling of the VAV boxes and other components on the air flow rate path is simplified by the following assumptions:

- The control loop in VAV terminal boxes have a negligible time constant compared to that of the AHU and the zones and the actuator actions are instantaneous.
- Let β_1 and β_2 be ratios of the air flow rate into each zone. The air mass balance gives,

$$\begin{gathered} q_1(t) = \beta_1 q_a(t), \qquad q_2(t) = \beta_2 q_a(t) \\ \beta_1 + \beta_2 = 1 \end{gathered} \tag{2.59}$$

- The fan dynamics are ignored, while the fan speed and hence $q_a(t)$, is measured.

With these assumptions in place, the overall model of the AHU-VAV-Zone combination could be given as (time is dropped for simplicity),

$$\dot{x}_1 = \alpha_1 q_a(d_1 - x_1) + \alpha_{2au}(T_{wi} + x_2 - d_1 - x_1) \tag{2.60}$$
$$\dot{x}_2 = \alpha_3 u(T_{wi} - x_2) - \alpha_{2wu}(T_{wi} + x_2 - d_1 - x_1) \tag{2.61}$$
$$\dot{x}_3 = \alpha_4 \beta_1 q_a(x_1 - x_3) + \alpha_5(x_4 - x_3) + \alpha_6(d_1 - x_3) + \alpha_7 d_2 \tag{2.62}$$
$$\dot{x}_4 = \alpha_8(1 - \beta_1) q_a(x_1 - x_4) + \alpha_9(x_3 - x_4) + \alpha_{10}(d_1 - x_4) + \alpha_{11} d_3 \tag{2.63}$$

where the state and the input variables are defined as,

$$x_1 \triangleq T_{ao},\ x_2 \triangleq T_{wo},\ x_3 \triangleq T_1,\ x_4 \triangleq T_2,\ u \triangleq q_w,\ d_1 \triangleq T_{ai}$$

and the constants as,

$$\begin{aligned} &\alpha_1 \triangleq \frac{1}{M_a}, \alpha_3 \triangleq \frac{1}{M_w}, \alpha_{2au} \triangleq \frac{U_A}{2C_{pa}M_a}, \alpha_{2wu} \triangleq \frac{U_A}{2C_{pw}M_w} \\ &\alpha_4 \triangleq \frac{C_{pa}}{C_1},\ \alpha_5 \triangleq \frac{K_{12}}{C_1},\ \alpha_6 \triangleq \frac{K_{1amb}}{C_1},\ \alpha_7 \triangleq \frac{K_{d1}}{C_1}, \\ &\alpha_8 \triangleq \frac{C_{pa}}{C_2},\ \alpha_9 \triangleq \frac{K_{21}}{C_2},\ \alpha_{10} \triangleq \frac{K_{2amb}}{C_2},\ \alpha_{11} \triangleq \frac{K_{d3}}{C_2} \end{aligned}$$

To obtain the T-S equivalent model of the AHU-VAV-Zones system in the polytopic form, the equations (2.60)-(2.63) are written in state space form as,

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_u(t)u(t) + B_d(t)d(t) + B_T T_{wi} \\ y(t) &= Cx(t) + H\nu(t) \end{aligned} \tag{2.64}$$

where the premise variables to be considered for this problem are $z_1(t) \triangleq q_a(t)$ and $z_2(t) \triangleq x_2(t)$. These premise variables are assumed to be within a sector, $z_j \in [z_j^{min}\ z_j^{max}]$. $\nu(t)$

is the measurement noise and H its distribution matrix. Given that the AHU water output temperature is not measured, this is an unmeasured premise variable. The matrices in (2.64) are given by,

$$A(t) = \begin{bmatrix} -\alpha_1 z_1(t) - \alpha_{2au} & \alpha_{2au} & 0 & 0 \\ \alpha_{2wu} & -\alpha_{2wu} & 0 & 0 \\ \alpha_4\beta_1 z_1(t) & 0 & -\alpha_4\beta_1 z_1(t) - \alpha_5 - \alpha_6 & \alpha_5 \\ \alpha_8(1-\beta_1)z_1(t) & 0 & \alpha_9 & -\alpha_8(1-\beta_1)z_1(t) - \alpha_9 - \alpha_{10} \end{bmatrix}$$

$$B_u(t) = \begin{bmatrix} 0 \\ \alpha_3(T_{wi} - z_2(t)) \\ 0 \\ 0 \end{bmatrix}, \; B_d(t) = \begin{bmatrix} \alpha_1 z_1(t) - \alpha_{2au} & 0 & 0 \\ \alpha_{2wu} & 0 & 0 \\ \alpha_6 & \alpha_7 & 0 \\ \alpha_{10} & 0 & \alpha_{11} \end{bmatrix}$$

$$B_T = \begin{bmatrix} \alpha_{2au} \\ -\alpha_{2wu} \\ 0 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.65}$$

To obtain the model of the form (2.27), the matrices' entries of $z_1(t)$ and $z_2(t)$ are replaced with the corresponding sector extremum values corresponding to the submodel i, i.e., z_j^{min} or z_j^{max}. The membership functions are obtained by,

$$\mu_i^1(z_j(t)) = \frac{z_j(t) - z_j^{min}}{z_j^{max} - z_j^{min}}, \; \mu_i^2(z_j(t)) = \frac{z_j^{max} - z_j(t)}{z_j^{max} - z_j^{min}}$$

and the weighting functions are obtained through the product of these membership functions for the corresponding extremum values.

For the application problem scenario, there is one unknown parameter $\beta_1 \in [\beta_1^1,\ \beta_1^2]$ which affects only the state matrix. Hence, the corresponding representation for (2.29) would be, $A_{ij} = \breve{A}_i + \beta_1^j \bar{A}_j$, where $j = 1, 2$ and β_1^j corresponds to either the minimum of maximum value of the parameter. The system matrices corresponding to (2.28) are given by,

$$\breve{A}_i = \begin{bmatrix} -\alpha_1 z_1^i - \alpha_{2au} & \alpha_{2au} & 0 & 0 \\ \alpha_{2wu} & -\alpha_{2wu} & 0 & 0 \\ 0 & 0 & -\alpha_5 - \alpha_6 & \alpha_5 \\ \alpha_8 z_1^i & 0 & \alpha_9 & -\alpha_8 z_1^i - \alpha_9 - \alpha_{10} \end{bmatrix}$$

$$\bar{A}_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_4 z_1^i & 0 & -\alpha_4 z_1^i & 0 \\ -\alpha_8 z_1^i & 0 & 0 & \alpha_8 z_1^i \end{bmatrix}$$

$$\breve{B}_{ui} = \begin{bmatrix} 0 \\ \alpha_3(T_{wi} - z_2^i) \\ 0 \\ 0 \end{bmatrix}, \qquad \breve{B}_{di} = \begin{bmatrix} \alpha_1 z_1^i - \alpha_{2au} & 0 & 0 \\ \alpha_{2wu} & 0 & 0 \\ \alpha_6 & \alpha_7 & 0 \\ \alpha_{10} & 0 & \alpha_{11} \end{bmatrix}$$

and $\bar{B}_{ui} = 0_{4\times 1}$, $\bar{B}_{di} = 0_{4\times 3}$ with B_T and C remaining unchanged from (2.65). z_1^i and z_2^i refer to the values of the premise variables corresponding to the ith submodel.

To illustrate the methodology, a simulation of the AHU-VAV-Zones system combination was executed in MATLAB. The yalmip [50] LMI parser was used along with the lmilab solver in the LMI toolbox. The system model parameters were derived to be reasonably representative and not accurate. The heat exchanger component was designed based on steady state analysis of the model after fixing some of the variables. The zone models were obtained considering zones of dimensions $3 \times 5 \times 4$ and $3 \times 4 \times 4$ (m). The sector maximum and minimum vales of premise variables and the unknown time varying parameters are given in the Table 2.5. It is to be noted that the parameter β_1 was scaled by 100 to allow for a reasonably close scale for the parameters and states of the observer. The simulation parameters that were chosen for the LMI feasibility problem are given in the Table 2.6.

Table 2.5: Sector minimum and maximum values of model parameters

Parameter	**Min**	**Max**
z_1	0.16 kg/s	1.6 kg/s
z_2	293 K	368 K
β_1	0	100

Table 2.6: Simulation and Model parameters

Parameters	**Values**
η	10^{-4}
ρ	10^5
Γ_2^{θ}	0.1
$\Gamma_2^{\dot{\theta}}$	0.1
Γ_2^{x}	$0.1 I_4$
Γ_2^{u}	0.1
Γ_2^{ν}	0.1

The results are shown in Figures 2.6 and 2.7. The inputs used to generate these results are shown in Fig 2.8. The error statistics of the observer estimates are summarized in Table 2.7.

Table 2.7: Simulation Results: $\beta_1(t)$ Estimation

Error	**Mean (%)**	**Standard Deviation (%)**
$\lvert e_{x_1} \rvert$	0.04	0.4
$\lvert e_{x_2} \rvert$	0.07	0.55
$\lvert e_{x_3} \rvert$	0.03	0.3
$\lvert e_{x_4} \rvert$	0.03	0.3
$\lvert e_{\theta} \rvert$	10.9	18.34

Remark 4. *It is clear that the main error deviation on the estimated parameter is at the initial periods of the simulation. It was observed that by dropping the first few estimates, the standard deviation came down by* 5%*. The other significant behaviour is the noisy estimate. This could be partially mitigated through filtering the estimate.*

The above results were communicated in [51] and [52].

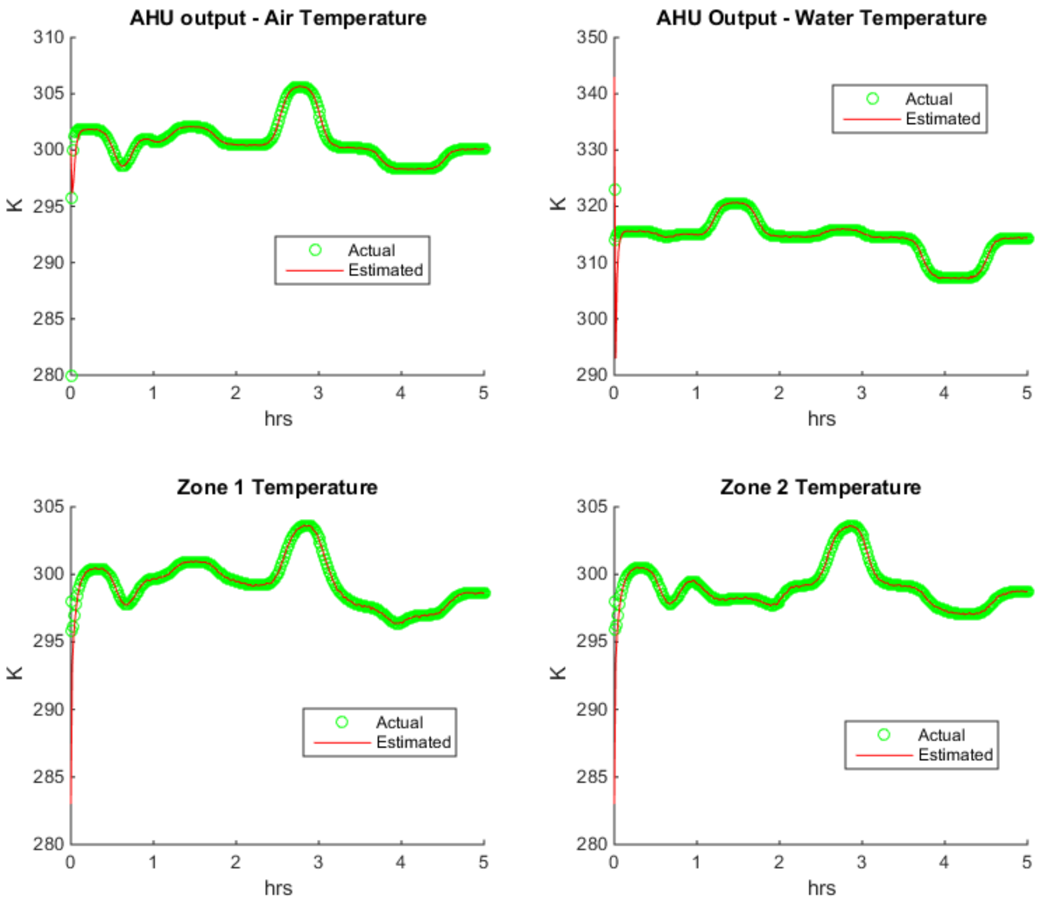

Figure 2.6: Estimated and actual states

2.3 Perspectives for the thesis

In the previous chapter, the Takagi-Sugeno modeling framework and its use to design observers for joint state and parameters were discussed. Further, the approach developed in [6] was customized and illustrated for some application examples in building energy system models. This exercise provided necessary insights to envisage some characteristics to avoid and some to expect as an improvement. They are briefly summarized below.

Choice of model structure The work in [6] as well as other works that used T-S approach for heat exchanger in [42] and [43] adapted observer model structures from the linear systems. The complications in the derivations to obtain the sufficient conditions in [6] were a direct result of fixing the model structure at the start. Hence a desired characteristics of an observer design for joint state and parameter estimation is to have an observer structure that reflects better the underlying nonlinear model structure.

Complexity of the obtained LMIs The observer design process in Sec. 2.2 results in LMIs whose dimensions and cardinality grow exponentially with that of the system model and the

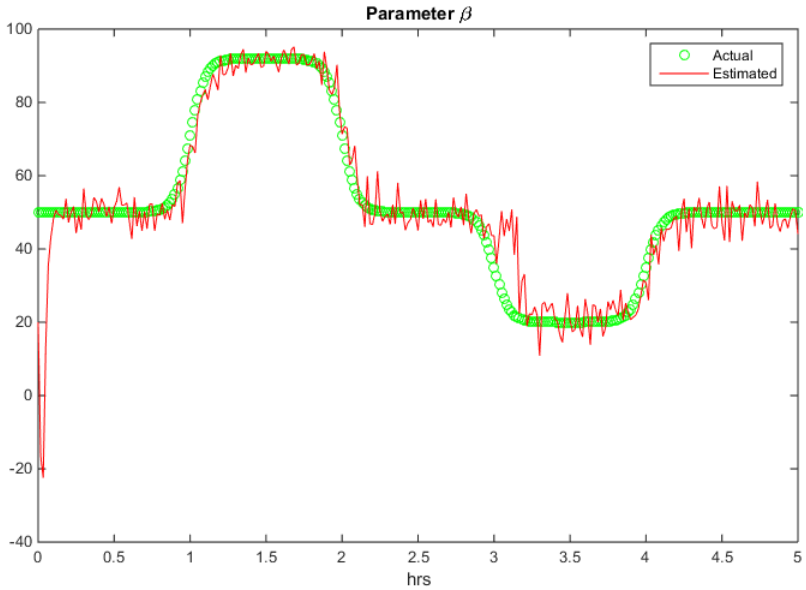

Figure 2.7: Estimated and actual parameter β

number of premise variables (or nonlinearity). The dimension of the LMIs in (2.44) is given by:

$$2n_x + 3n_\theta + n_u + n_y + (n_x + n_u)(2^{n_p+n_\theta})$$

Further, the number of LMIs that the feasibility problem should solve is $2^{n_p+n_\theta}$. Hence design solutions to problems where system models are large is difficult to obtain. For each of the example in Sec. 2.2, the dimension of the LMIs is given in table 2.8. A desirable characteristics of the observer design process could be to reduce the dimension of the LMIs.

Table 2.8: Dimension of LMIs for application examples in Sec. 2.2

Example	Dimension
Example 1 in Sec. 2.2.3	42
Example 2 in Sec. 2.2.3	18
Example 3 in Sec. 2.2.3	18
Example in Sec. 2.2.4	63

Remark 5. *The complexity of the LMIs are due to the generalized scenario handled by the observer design process, namely unmeasured premise variables and time-varying parameters. Hence, the concern on complexity is not a drawback of the design process itself, but a desirable quality for the application scenarios we are looking for.*

Insights into structural characteristics The observer design process in Sec. 2.2 relies on state observability and parameter identifiability assumptions without explicitly stating them. Further, the significance of the choice of the affine parametrization is not discussed. This can make it difficult to precisely identify the reasons for not finding a solution to the LMI problem: whether it is because of the sufficient nature of the conditions (either numerical issues or that the quadratic Lyapunov function is insufficient) or is it due to structural issues associated with the model itself.

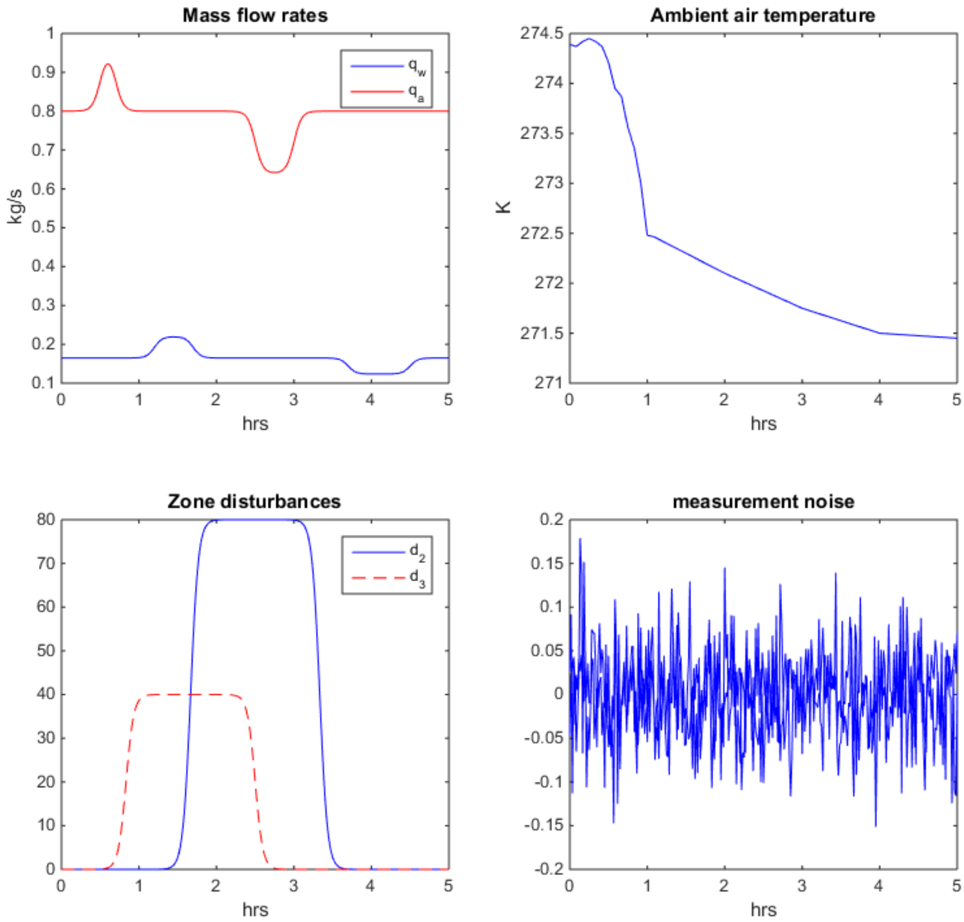

Figure 2.8: Various inputs used for the simulation

The observer design procedures use Lyapunov approach, which for nonlinear system models, can only give sufficient conditions and hence the first ambiguity is difficult to mitigate. However, an observer design approach can also provide better insights into the necessary structural requirements of the underlying system model so that a joint state and parameter estimation exists. This is not necessarily a global necessary condition for all nonlinear system models with unknown parameters, but for those models for which the proposed design procedure will work. This would be another desirable feature of the observer design.

Case of measured premise variables The observer design approach in Sec. 2.2 considered a general case where the premise variable is unmeasured. However, the complexities of the final LMI conditions do not reduce when the premise variables are measured/known, as illustrated in Theorem 4. This is a limitation of the design approach. It is envisaged to develop an approach that will have progressively complex conditions/steps as the problem moves from measured to unmeasured premise variable case. This would ensure that whenever simpler scenarios are available, the design process would be simpler.

The next chapter addresses some of these through a contribution of an observer design process.

Chapter 3

Adaptive observers for continuous-time models

Adaptive observer refers to those observers that provide a joint estimation of states and unknown parameters of the system model. Towards the end of Chapter 2, a number of avenues for improvement were suggested. This chapter attempts to realize some of them. The chapter begins by outlining the existing literature on adaptive observer design for nonlinear models in general as well as those for T-S polytopic models. The proposed adaptive observer design follows a control-Lyapunov function like strategy, that is, it derives the parameter estimation part of the observer based on the choice of the Lyapunov function. Some insights into the design process follow with an illustrative example.

Contents

3.1 Review of literature

In the adaptive systems paradigm [4], adaptive control refers to the control of partially known systems, where some or all of the model parameters are not accurately known. To realize this paradigm of control, observer based control are employed. These observers, which estimate the parameters in addition to the states of the model, are referred to as adaptive observers. This paradigm is adopted in this literature as the adaptive observers are destined to be used for applications such as fault diagnosis under unknown parameters.

The choice of developing adaptive observers over a more popular Extended Kalman filter (EKF) approaches stems from the difficulty to perform stability analysis for the latter. This is one of the main motivation for early ventures into adaptive observer design such as [53].

These approaches exploited the feedback linearization idea and transformed a given nonlinear model with linearizable dynamics through state feedback. For system with unknown parameters, their existence and the design for single output system have been worked out in [54] and then with exponential convergence in [8]. These approaches however had limited reach as they do not transfer well beyond the narrow class of single output nonlinear models with linearizable dynamics.

One of the first formal adaptive observer design process for MIMO nonlinear systems was proposed in [7]. The authors consider linear output equation and assume that the output of the system and the parameters have a positive real connection. By employing suitable persistence of excitation conditions and the use of pseudo inverse of the output matrix, the authors provide a systematic design approach that provides the observer gains by solving LMIs. A systematic study of the design approach proposed by [7] was performed by [9], where the authors introduce an adaptive observer form to characterize the models for which the design process in the former is feasible. A number of works have descended based on the design approach proposed in these two works. A related, but different MIMO design was proposed in [55] where the authors achieve an exponential convergence under noise-free conditions. The work is proposed for time-varying models, though the authors claim it can work, under certain assumptions, for quasi-LPV models with measured premise variable, a class of model considered in this chapter. The state estimation part uses a Luenberger form whereas a two-stage filter structure is deployed for the parameter estimation part. The design of observer gains, however is not clear and is one reason for not adopting this approach for this chapter. The application aim of developing adaptive observers for this thesis is to deploy it for fault diagnosis. In this respect one can refer to the use of adaptive observers in fault diagnosis in [56].

For T-S observers, the literature of adaptive observers could be viewed in two broad categories: those that consider an additive unknown parameter in the form of unknown inputs and those that consider unknown parameters affecting any system matrices. In the former case, a number of works that could be considered relevant including, [57], [58], [59], [60], [61]. A general characterization of these works could be that they use a Luenberger form for the state observer and a Proportional Integral (PI) or Proportional Multiple Integral (PMI) form for the parameter estimation part. They focus on specific design nuances such as the use of decoupling class approach in [57], a robust observer design in [59] or considering unmeasured premise variables in [58]. The use of PMI form for parameter estimation is driven by the assumption that the parameter is not constant, but its derivative is zero beyond a certain order of differentiation. Two application examples for the T-S adaptive observer design could be the two works described in Chapter 2: [42] and [43].

For adaptive observers considering multiplicative unknown parameters, different observer structures are considered. In [20], an adaptive observer is designed for the estimation of unmodeled dynamics in a T-S system. The authors propose an observer structure inspired from that in [7]. These results have been applied to estimate the uncertainties in the state matrices of a two-degrees-of-freedom robot arm model in [62]. The estimates are then used to control the robot arm. In this chapter, the design process is motivated from [7] and improves upon the results in [20].

3.2 Adaptive observer design

In this section, the main contributions of the chapter are given. The continuous-time T-S adaptive observer design problem is formulated and then the proposed results are given with an illustrative example.

3.2.1 Problem formulation

The system models considered in this chapter follows that in Chapter 2. That is, LPV and quasi-LPV models that have unknown parameters appearing affinely and any nonlinear models that could be transformed to this form. This could be written in a more extended form as

$$\begin{aligned} \dot{x}(t) &= A(z(t),\theta)x(t) + B(z(t),\theta)u(t) + F(z(t),\theta) \\ y(t) &= Cx(t) \end{aligned} \tag{3.1}$$

where $z(t)$ is the premise or scheduling variable and θ is the unknown parameter. Note that θ is represented as a constant, which is a requirement for the analysis used in the proposed approach. However, as would be shown in the example, this is not a limitation in practice. The system matrices are given by the structure,

$$X(z(t),\theta) = X_0(z(t)) + \sum_{j=1}^{n_\theta} \bar{X}_j \theta_j$$

with X used as a place holder for the system matrices A, B, and F. For this model, the observer design is envisaged to have the following structural characteristics:

- A Luenberger-like form for the state estimation part
- A nonlinear structure for the parameter estimation part that will be obtained from the error convergence analysis

That is,

$$\begin{aligned} \dot{\hat{x}}(t) &= A(z(t),\hat{\theta})\hat{x}(t) + B(z(t),\hat{\theta})u(t) + F(z(t),\hat{\theta}) + L(z(t))(y(t) - \hat{y}(t)) \\ \dot{\hat{\theta}}_j(t) &= f_j(\hat{\theta}_i(t), \hat{x}(t), y(t)), \qquad \forall j = 1, ..., n_\theta \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned} \tag{3.2}$$

Remark 6. *As can be noted in* (3.2)*, the scheduling or premise variables are considered known/ measured. The unmeasured premise variable case is not treated in this chapter.*

Consider the nonlinear system of the form used in [7] ,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \phi(x(t),u(t)) + bf(x(t),u(t))\theta \\ y(t) &= Cx(t) \end{aligned} \tag{3.3}$$

where b is a transmission vector. For this model, it has been shown that a state and parameter observer with asymptotically vanishing error is possible under certain conditions. This model structure will lead to a LPV/quasi-LPV model of the form (3.1) (with possibly unmeasured premise variables) when put through the procedure of, say [23]. The choice of the observer structure in this chapter is motivated from these works.

The Takagi-Sugeno polytopic approach would be the tool used for this design. Hence the model (3.1) can be transformed, through SNL transformation as

$$
\begin{aligned}
\dot{x} &= \sum_{i=1}^{2^{n_p}} \mu_i(z) \left((A_i + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij})x + (B_i + \sum_{j=1}^{n_\theta} \theta_j \bar{B}_{ij})u + (F_i + \sum_{j=1}^{n_\theta} \theta_j \bar{F}_{ij}) \right) \\
y &= Cx
\end{aligned} \tag{3.4}
$$

Note that the time factor (t) has been dropped for notational convenience. While the unknown parameters are shown to affect all the matrices, the corresponding transmission matrices $(\bar{A}_{ij}, \bar{B}_{ij}, \bar{F}_{ij})$ could be zero, if a particular unknown parameter θ_j does not affect it. Thus observer structure is given by,

$$
\begin{aligned}
\dot{\hat{x}} &= \sum_{i=1}^{2^{n_p}} \mu_i(z) \left((A_i + \sum_{j=1}^{n_\theta} \hat{\theta}_j \bar{A}_{ij})\hat{x} + (B_i + \sum_{j=1}^{n_\theta} \hat{\theta}_j \bar{B}_{ij})u + (F_i + \sum_{j=1}^{n_\theta} \hat{\theta}_j \bar{F}_{ij}) + L_i(y - \hat{y}) \right) \\
\dot{\hat{\theta}}_j &= f_j(\hat{\theta}_j, \hat{x}, y), \ \forall j = 1, ..., n_\theta \\
\hat{y} &= C\hat{x}
\end{aligned} \tag{3.5}
$$

where $f_j(.)$ is a nonlinear function to be obtained through the design process. Note that the estimated parameter $\hat{\theta}$ is not constant and the aim is to have $\hat{\theta} \to \theta$ as $t \to \infty$.

3.2.2 Assumptions and initial steps

The observer design for the T-S system of the form (3.4) is given under the following assumptions:

1. There exists a $\bar{\theta}$ such that $|\theta_j| < \bar{\theta}$, $\forall j$. This value is assumed to be known. However, the maximum value allowed by the design process could be determined as described later.
2. All the submodels are sufficiently excited, illustrated by the variation in the weighting functions of each submodels, so that the system is under a persistence of excitation.
3. $\dot{\theta} = 0$. The proof for the main result uses this condition to assume that the unknown parameters are constant. However, it is shown in the examples that, on a practical point of view, this approach will work for slowly varying parameters as well.

Define the state estimation error between the system (3.4) with that of the Luenberger observer structure in (3.5) and that of the parameter estimation errors as

$$
e_x(t) \triangleq x(t) - \hat{x}(t), \qquad e_{\theta_j}(t) \triangleq \theta_j - \hat{\theta}_j(t)
$$

The dynamics based on the model and observer equations are given by,

$$
\dot{e}_x(t) = \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left((A_i - L_i C)e_x(t) + \sum_{j=1}^{n_\theta} (\theta_j \bar{A}_{ij} x(t) - \hat{\theta}_j \bar{A}_{ij} \hat{x}(t)) + (\bar{B}_{ij} u(t) + \bar{F}_{ij}) e_{\theta_j}(t) \right)
$$

By adding and subtracting $\sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij} \hat{x}$, the error dynamics becomes,

$$
\dot{e}_x(t) = \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left(\left[A_i - L_i C + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij} \right] e_x(t) + \left[\sum_{j=1}^{n_\theta} \bar{A}_{ij} \hat{x}(t) + \bar{B}_{ij} u(t) + \bar{F}_{ij} \right] e_{\theta_j}(t) \right)
$$

To analyze stability, consider the following Lyapunov function

$$V(e_x, e_\theta) = e_x^T(t)Pe_x(t) + \sum_{j=1}^{n_\theta} e_{\theta_j}(t)\rho_j e_{\theta_j}(t) \tag{3.6}$$

where $P = P^T > 0$ is a positive definite matrix and $\rho_j > 0,\ \forall j$. Its derivative is then given by

$$\dot{V}(t) = \dot{e}_x^T(t)Pe_x(t) + e_x^T(t)P\dot{e}_x(t) + 2\sum_{j=1}^{n_\theta} \rho_j \dot{e}_{\theta_j}(t)e_{\theta_j}(t)$$

Considering,

$$G_{ij} \triangleq P(A_i - L_iC + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij}) + (A_i - L_iC + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij})^T P$$

and since $\dot{\theta} = 0$,

$$2\sum_{j=1}^{n_\theta} \rho_j \dot{e}_{\theta_j}(t)e_{\theta_j}(t) = -2\sum_{j=1}^{n_\theta} \rho_j \dot{\hat{\theta}}_j(t)e_{\theta_j}(t) \tag{3.7}$$

leads to,

$$\begin{aligned}\dot{V}(t) = \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) &\left(e_x^T(t)G_{ij}e_x(t) + 2\sum_{j=1}^{n_\theta} e_{\theta_j}(t)(\bar{A}_{ij}\hat{x}(t) + \bar{B}_{ij}u(t) + \bar{F}_{ij})^T Pe_x(t) \right) \\ &- 2\sum_{j=1}^{n_\theta} \rho_j \dot{\hat{\theta}}_j(t)e_{\theta_j}(t)\end{aligned} \tag{3.8}$$

To ensure $\dot{V}(t) < 0$, each term on the right hand side (RHS) of (3.8) is analyzed. The first contains a quadratic term with the unknown parameter in G_{ij}. Following the Assumption 1, the result of robust stability analysis for error dynamics as in [63] (and [64]) is applied. This translates to,

$$P = P^T,\ P > 0,\ Q = Q^T,\ Q > 0 \tag{3.9}$$

$$P(A_i - L_iC) + (A_i - L_iC)^T P < -Q \tag{3.10}$$

$$n_\theta \bar{a}\bar{\theta} \leq \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)} \tag{3.11}$$

where $\bar{a}$ is the maximum of the norms of the different matrices $\bar{A}_{ij}$, that is

$$\bar{a} = \max_{ij}\|\bar{A}_{ij}\|,\ \forall\ i,j \tag{3.12}$$

The equivalent LMI condition for (3.10) is given by:

$$PA_i + A_i^T P - M_iC - C^T M_i^T < -Q \tag{3.13}$$

with the observer gain obtained as, $L_i = P^{-1}M_i$. The condition (3.11) will be satisfied if the following LMI condition is met,

$$\begin{bmatrix} Q - \gamma I & P \\ P & I \end{bmatrix} > 0 \tag{3.14}$$

where

$$\gamma = (n_\theta \bar{a}\bar{\theta})^2,\ \forall i,j \tag{3.15}$$

The second and third terms in the RHS of (3.8) relate to the coefficients of e_θ and one way to manage the Lyapunov function is to annihilate the coefficients of each error e_{θ_j},

$$\sum_{i=1}^{2^{n_p}} \mu_i(z(t))(\bar{A}_{ij}\hat{x}(t) + \bar{B}_{ij}u(t) + \bar{F}_{ij})^T P e_x(t) - \rho_j \dot{\hat{\theta}}_j(t) = 0, \ \forall j \tag{3.16}$$

This would lead to the condition,

$$\dot{\hat{\theta}}_j(t) = \frac{1}{\rho_j} \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left(\bar{A}_{ij}\hat{x}(t) + \bar{B}_{ij}u(t) + \bar{F}_{ij}\right)^T P e_x(t), \quad \forall \ j \tag{3.17}$$

Since $e_x(t)$ is not available, construction of $e_x(t)$ from $e_y(t) = y(t) - \hat{y}(t)$ for the conditions specified is to be explored. This could be resolved in multiple ways and discussed as follows.

3.2.3 Main results

In this section, the open question above is resolved in several ways based on both existing literature and original results. To start with, the proposed observer for the model (3.4) is given as

$$\begin{aligned}
\dot{\hat{x}}(t) &= \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left(\breve{A}_{ij}\hat{x}(t) + \breve{B}_{ij}u(t) + \breve{F}_{ij} + L_i(y(t) - \hat{y}(t))\right) \\
\dot{\hat{\theta}}_j(t) &= \frac{1}{\rho_j} \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left(\bar{A}_{ij}\hat{x}(t) + \bar{B}_{ij}u(t) + \bar{F}_{ij}\right)^T P C^{\dagger}(y(t) - \hat{y}(t)), \quad \forall \ j \\
\hat{y}(t) &= C\hat{x}(t)
\end{aligned} \tag{3.18}$$

where $C^{\dagger}$ is the pseudo inverse of C, and

$$\breve{A}_{ij} = A_i + \sum_{j=1}^{n_\theta} \hat{\theta}_j(t)\bar{A}_{ij}, \qquad \breve{B}_{ij} = B_i + \sum_{j=1}^{n_\theta} \hat{\theta}_j(t)\bar{B}_{ij}, \qquad \breve{F}_{ij} = F_i + \sum_{j=1}^{n_\theta} \hat{\theta}_j(t)\bar{F}_{ij}$$

This is an adaptive observer for the system (3.4) if one of the following theorems are satisfied, which resolve the problem in (3.17), that is, to obtain the parameter estimation component as function of measured/estimated parameters.

Theorem 5. *(adapted from [20]) The observer* (3.18) *for the system* (3.4) *provides an asymptotically converging estimates for the states and the parameters, if*

1. *The conditions* (3.9), (3.13) *and* (3.14) *are satisfied*
2. N_i *is of full column rank and*

$$rank(CN_i) = rank(N_i), \ \forall i \tag{3.19}$$

where N_i *is a pre-multiplying matrix that is common for all* $\bar{A}_{ij}, \bar{B}_{ij}, \bar{F}_{ij} (\forall j)$ *such that,*

$$\bar{A}_{ij} = N_i\tilde{A}_{ij}, \ \bar{B}_{ij} = N_i\tilde{B}_{ij}, \ \bar{F}_{ij} = N_i\tilde{F}_{ij}, \ \forall j \tag{3.20}$$

Alternately this could be stated in the form of $\bar{A}_{ij}, \bar{B}_{ij}, \bar{F}_{ij} (\forall i, j)$ are of full column rank, and

$$\begin{aligned} rank(C\bar{A}_{ij}) &= rank(\bar{A}_{ij}) \\ rank(C\bar{B}_{ij}) &= rank(\bar{B}_{ij}) \\ rank(C\bar{F}_{ij}) &= rank(\bar{F}_{ij}) \end{aligned} \tag{3.21}$$

Proof. Given the rank conditions, there exists, an R_i such that, $R_i C = N_i^T P$ which could be used in (3.17) to give

$$\dot{\hat{\theta}}_j = \frac{1}{\rho_j} \sum_{i=1}^{2^{n_p}} \mu_i(z) \left(\tilde{A}_{ij}\hat{x} + \tilde{B}_{ij}u + \tilde{F}_{ij} \right)^T R_i e_y, \quad \forall\ j \tag{3.22}$$

Considering $R_i = N_i^T P C^\dagger$, the parameter estimation part of the observer would become

$$\dot{\hat{\theta}}_j = \frac{1}{\rho_j} \sum_{i=1}^{2^{n_p}} \mu_i(z) \left(\bar{A}_{ij}\hat{x} + \bar{B}_{ij}u + \bar{F}_{ij} \right)^T P C^\dagger e_y, \quad \forall\ j \tag{3.23}$$

Hence the proof. □

Remark 7. *This result is considerably restrictive due to the rank conditions on the system matrices of model chosen for this work. Apart from the structural constraints to be satisfied, there is no standard procedure that connects choice of R_i with that of the Lyapunov matrix P. To mitigate these problems, another approach is considered and given in the following theorem.*

Theorem 6. *The system* (3.5) *forms an observer for the system* (3.4)*, if*

1. *The conditions* (3.9), (3.13) *and* (3.14) *are satisfied*
2. *For every θ_j,*

$$\begin{aligned} \bar{A}_{ij}^T P H &= 0, \ \forall i \\ \bar{B}_{ij}^T P H &= 0, \ \forall i \\ \bar{F}_{ij}^T P H &= 0, \ \forall i \end{aligned} \tag{3.24}$$

where $H \triangleq I - C^\dagger C$ with I being the identity matrix of appropriate dimensions.

Proof. Consider the following lemma,

Lemma 6. *([65]) Let $A \in \mathbb{R}^{m \times n}$ and $A^\dagger$ be any generalized inverse of A. Then a general solution of a consistent nonhomogeneous equation $A\mathbf{x} = \mathbf{y}$ is*

$$\mathbf{x} = A^\dagger \mathbf{y} + H\omega \tag{3.25}$$

where ω is an arbitrary vector and $H = I - C^\dagger C$. The necessary and sufficient condition that $A\mathbf{x} = \mathbf{y}$ is consistent is,

$$AA^\dagger \mathbf{y} = \mathbf{y} \tag{3.26}$$

The nonhomogeneous equation in the problem under focus is, $e_y = Ce_x$, and based on this lemma,

$$e_x = C^\dagger e_y + H\omega \tag{3.27}$$

for some arbitrary ω. Applying this to (3.17) leads to,

$$\begin{aligned}\hat{x}^T\bar{A}_{ij}^T P e_x &= \hat{x}^T\bar{A}_{ij}^T P(C^\dagger e_y + H\omega),\ \forall i\\ u^T\bar{B}_{ij}^T P e_x &= u^T\bar{B}_{ij}^T P(C^\dagger e_y + H\omega),\ \forall i\\ \bar{F}_{ij}^T P e_x &= \bar{F}_{ij}^T P(C^\dagger e_y + H\omega),\ \forall i\end{aligned} \tag{3.28}$$

The second term on the right hand side of the equation would lead to zero, if the conditions in (3.24) are satisfied. Hence, the proof. □

Convergence of $\hat{\theta}$ For a given parameter $\theta_j,\ \ j = 1, 2, \cdots, n_\theta$, the above proof of convergence for the parameter only guarantees that,

$$\sum_{i=1}^{2^{n_p}} \mu_i(z)\left(\theta_j\bar{A}_{ij}x - \hat{\theta}_j\bar{A}_{ij}\hat{x} + \bar{B}_{ij}u + \bar{F}_{ij}\right)e_{\theta_j} \to 0$$

Given $(\hat{x} - x) \to 0$ as $t \to \infty$, this can be written as,

$$\sum_{i=1}^{2^{n_p}} \mu_i(z)\left(\bar{A}_{ij}x + \bar{B}_{ij}u + \bar{F}_{ij}\right)e_{\theta_j} \to 0$$

and hence the convergence of the parameter θ_j is guaranteed under the following persistence of excitation condition. $\exists\ \alpha, \beta, \delta$ such that,

$$\alpha I \geq \int_{t_0}^{t_0+\delta}\left(\sum_{i=1}^{2^{n_p}}\left[\bar{A}_{ij}x + \bar{B}_{ij}u + \bar{F}_{ij}\right]\left[\bar{A}_{ij}x + \bar{B}_{ij}u + \bar{F}_{ij}\right]^T d\tau\right) \geq \beta I \qquad \forall\ t_0 \tag{3.29}$$

Corollary 1. *If the value of $\bar{\theta}$ as in the Assumption 1 is not known, the maximum $\bar{\theta}$ allowed by a particular design could be obtained by rewriting the above results as an optimization problem considering γ as an objective to be maximized. That is,*

$$\underset{P,Q,L_i}{\mathit{maximize}}\quad \gamma \tag{3.30}$$

$\forall i$, such that, the conditions (3.9)*,* (3.13) *and* (3.14) *are satisfied.*

One concern that may arise in Theorem 6 is that on the ability to satisfy equality conditions using numerical approaches under the simulation tools. To mitigate this, the following theorem rewrites the equality conditions as an objective to minimize.

Theorem 7. *The observer* (3.18) *for the system* (3.4)*, could be designed if the following optimization problem has a solution,*

$$\min_{P,M_i} \sum_{j=1}^{n_\theta}\beta_j \tag{3.31}$$

under the constraints of (3.9)*,* (3.13)*,* (3.14)*. Here,*

$$\beta_j = \sum_{i=1}^{2^{n_p}}\|\bar{A}_{ij}^T PH\| + \|\bar{B}_{ij}^T PH\| + \|\bar{F}_{ij}^T PH\| \tag{3.32}$$

It is useful to note that the Lyapunov analysis performed in the previous section is valid for the Theorem 6 and not for that in Theorem 7. Theorem 7 is only a procedure to evaluate it when the equality conditions are not satisfied.

Remark 8. *It is to be noted that the parameter ρ_j is not considered in the design process. It is manually tuned to improve the dynamics of the parameter estimation and the values would depend on the scale of the corresponding θ_j.*

3.2.4 Simulation example

Example 4. *Consider a nonlinear system of the form*

$$\begin{aligned}
\dot{x}_1 &= -0.7x_1^2 - x_2 + x_3 + (1 - 0.8x_1)\theta \\
\dot{x}_2 &= -x_1x_3 - 2x_2 + (x_2 + u)\theta \\
\dot{x}_3 &= 0.5x_1 - 2x_3 + u \\
y_1 &= x_1 + x_2 \\
y_2 &= x_2
\end{aligned} \tag{3.33}$$

Using the sector nonlinearity approach, this system could be transformed into a system of the form (3.4)*.* $z \triangleq x_1$ *is the premise variable and is assumed to be in the sector of* $[0,\ 2]$*. The weighting functions are given by* $\mu_1 = \frac{z_1}{2}$ *and* $\mu_2 = 1 - \mu_1$*. The following are the system matrices obtained after applying the sector nonlinearity approach.*

$$A_1 = \begin{bmatrix} -1.4 & -1 & 1 \\ 0 & -2 & -2 \\ 0.5 & 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 0 \\ 0.5 & 0 & -2 \end{bmatrix}$$

$$\bar{A}_{11} = \bar{A}_{21} = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \bar{B}_{11} = \bar{B}_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

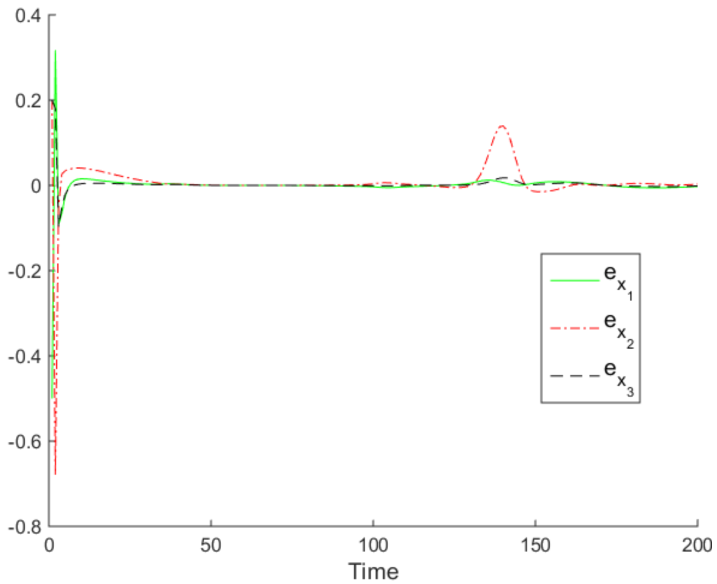

Figure 3.1: State errors' evolution over time

The simulation of this example was carried out on MATLAB with the *Yalmip* modeling interface [50] and using the *SeDuMi* solver [30]. As could be seen, the conditions in (3.24) are

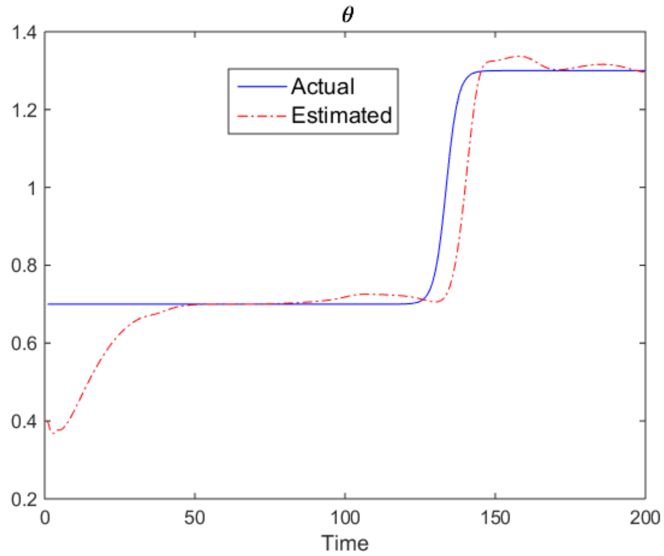

Figure 3.2: Unknown parameter and its estimate

satisfied for this example only if P is constrained to be block diagonal. By using Theorem 7, the results show that this constraint is not required.

The state estimation errors are shown in the Fig. 3.1. As could be seen, the estimation results are fairly accurate. The parameter estimation tracking is shown in the Fig. 3.2, which shows a good tracking even when the unknown parameter changes ($\rho_1 = 1$ is used for this simulation). The input used for the simulation is shown in the Fig. 3.3 and the weighting functions in Fig. 3.4 illustrate the sufficient excitation of both the submodels. The Lyapunov matrix obtained through the optimization problem was,

$$\left[\begin{array}{cc|c} 1.169 & 0.657 & \multirow{2}{*}{-4.7×10^{-14}} \\ 0.657 & 1.153 & \\ \hline -4.7\times10^{-14} & -3.3\times10^{-14} & 1.365 \end{array}\right] \tag{3.34}$$

and the value of the $\beta_1 = 1.31 \times 10^{-13}$.

3.2.5 Structural connotations

It is interesting to note the correlation between the structural constraints that arise out of this result with that of the nonlinear adaptive observer form as proposed in [9]. In [9], the state equations are split into those that are measured and unmeasured. Further, the unknown parameter θ is allowed to appear only on the dynamics of the measured states. In [7], this constraint is given, for the system in (3.3), as $b^T P$ should be in the space spanned by C. This is seen from the constraints in (3.24),

$$\bar{A}_{ij}^T P = (\bar{A}_{ij}^T P C^\dagger) C \Rightarrow \bar{A}_{ij}^T P \in \text{span}(C) \tag{3.35}$$

However, the design procedure in [7] and the T-S equivalent approach in [20] are ambiguous on how integrate the two components: obtain a P that will simultaneously satisfy the structural requirement (the need for a path between the parameter and the output) and the LMI conditions. The approach proposed in this thesis provides a procedure where the two aspects are

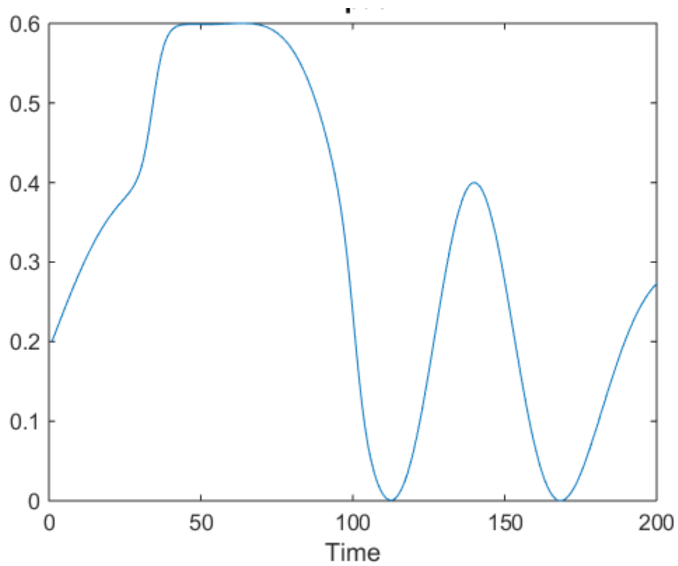

Figure 3.3: Input used for the illustration

connected. And to ease the restrictive nature of the equality constraint an optimization problem was formulated.

Further insights could be obtained by the observation on the connection between the structures of C and $\bar{A}_{ij}, \bar{B}_{ij}, \bar{F}_{ij}$ with that of the Lyapunov matrix P . Consider the structure of C of the form,

$$C = \begin{bmatrix} X_{n_y \times n_y} & 0_{n_y \times (n_x - n_y)} \end{bmatrix}, \tag{3.36}$$

with $X \in \mathbb{R}^{n_y \times n_y}$ is a regular full rank matrix. This follows the following assumptions:

- The C matrix is of full row rank. This is reasonable, for the redundant measurements, if exist, could be dropped.
- There are some states that are not directly measured.

Given C is full row rank, $C^\dagger$ could be computed as

$$C^\dagger = C^T (CC^T)^{-1} = \begin{bmatrix} X^T \\ 0 \end{bmatrix} \begin{bmatrix} XX^T \end{bmatrix}^{-1} \tag{3.37}$$

This would lead to the matrix $H = I - C^\dagger C$,

$$H = I_{n_x} - \begin{bmatrix} X^T \\ 0 \end{bmatrix} \begin{bmatrix} XX^T \end{bmatrix}^{-1} \begin{bmatrix} X & 0 \end{bmatrix} = \begin{bmatrix} I_{n_y} & 0 \\ 0 & I_{n_x - n_y} \end{bmatrix} - \begin{bmatrix} X^T (XX^T)^{-1} X & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & I_{n_x - n_y} \end{bmatrix}$$

With this structure of H, some insights could be obtained for the structure of P that can satisfy the LMI conditions. Consider,

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{3.38}$$

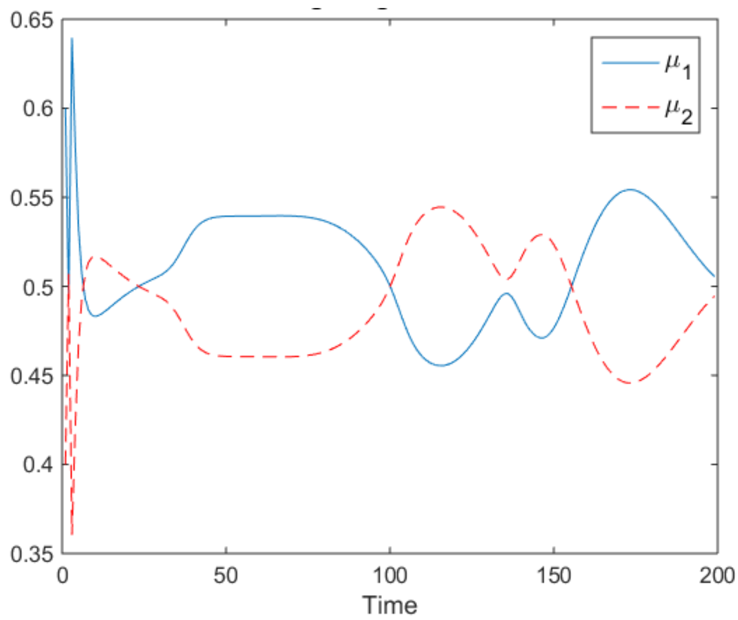

Figure 3.4: Weighting function evolution for the simulation

with $P_1 = P_1^T > 0 \in \mathbb{R}^{n_y \times n_y}$ and $P_2 > 0$ is a diagonal matrix of dimension $(n_x - n_y)$. If the transmission matrices $(\bar{A}_{ij}, \bar{B}_{ij}, \bar{F}_{ij})$ have a structure such that the unknown parameters affect only the measured states, then a Lyapunov matrix with the above structure would guarantee the equality conditions in (3.24). However, there are no standard procedures to enforce such a structure on P. Fortunately, the optimization procedure in Theorem 7 facilitates this process without an explicit choice of the structure. Hence it could be asserted that the algorithm facilitates an integration of connecting the stability requirements of the state estimation and the structural requirements of the parameter estimation. This could be seen by the value of P obtained in the Example 4 in (3.34), the procedure numerically tends to a structure of P with a full rank for the first $n_y = 2$ block and then towards a diagonal for the $n_x - n_y = 1$ block.

The results discussed above were communicated in [66].

3.3 Extensions

In this section, some extensions of the observer design procedure to other scenarios are explored, such as the presence of measurement noise. Often in a practical scenario, the system models would be affected by noise. In this extension, we consider a model which is affected by a measurement noise. To illustrate the design approach, we consider the following model

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{2^{n_p}} \mu_i(z) \left((A_i + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij})x + (B_i + \sum_{j=1}^{n_\theta} \theta_j \bar{B}_{ij})u + (F_i + \sum_{j=1}^{n_\theta} \theta_j \bar{F}_{ij}) \right) \\ y &= Cx + E\nu(t) \end{aligned} \tag{3.39}$$

Here, $\nu(t) \in \mathbb{R}^{n_\nu}$ is the measurement noise affecting the measurement through the transmission matrix $E \in \mathbb{R}^{n_y \times n_\nu}$. For this model, the observer proposed is the same as the previous case, that is,

$$\dot{\hat{x}}(t) = \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left(\breve{A}_{ij}\hat{x}(t) + \breve{B}_{ij}u(t) + \breve{F}_{ij} + L_i(y(t) - \hat{y}(t)) \right)$$

$$\dot{\hat{\theta}}_j(t) = \frac{1}{\rho_j} \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left(\bar{A}_{ij}\hat{x}(t) + \bar{B}_{ij}u(t) + \bar{F}_{ij} \right)^T PC^{\dagger}(y(t) - \hat{y}(t)), \quad \forall\, j$$

$$\hat{y}(t) = C\hat{x}(t) \tag{3.40}$$

Theorem 8. *The system* (3.40) *is an observer for the system* (3.39) *has an* $\mathbb{L}_2$ *gain bound of* Γ_2^x *for the state error in case, if a solution to the following optimization problem is found,*

$$\min_{P, M_i} \sum_{j=1}^{n_\theta} \beta_j \tag{3.41}$$

under the constraints,

$$P = P^T > 0, \; Q = Q^T > 0 \tag{3.42}$$

$$\begin{bmatrix} PA_i + A_i^T P - K_i C - C^T K_i^T + W_x & K_i E \\ E^T K_i^T & -\Gamma^x \end{bmatrix} < -Q, \; \forall i \tag{3.43}$$

$$\begin{bmatrix} Q - \gamma I & P \\ P & I \end{bmatrix} > 0 \tag{3.44}$$

where γ *is given in* (3.15) *and*

$$\beta_j = \sum_{i=1}^{2^{n_p}} \|\bar{A}_{ij}^T PH\| + \|\bar{B}_{ij}^T PH\| + \|\bar{F}_{ij}^T PH\| \tag{3.45}$$

Proof. The error dynamics is given by,

$$\dot{e}_x(t) = \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left((A_i - L_i C)e_x(t) + \sum_{j=1}^{n_\theta} (\theta_j \bar{A}_{ij} x(t) - \hat{\theta}_j \bar{A}_{ij}\hat{x}(t)) + (\bar{B}_{ij}u(t) + \bar{F}_{ij})e_{\theta_j}(t) + L_i E\nu(t) \right)$$

By adding and subtracting $\sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij}\hat{x}$, the error dynamics becomes,

$$\dot{e}_x(t) = \sum_{i=1}^{2^{n_p}} \mu_i(z(t)) \left(\left[A_i - L_i C + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij} \right] e_x(t) + \left[\sum_{j=1}^{n_\theta} \bar{A}_{ij}\hat{x}(t) + \bar{B}_{ij}u(t) + \bar{F}_{ij} \right] e_{\theta_j}(t) + L_i E\nu(t) \right)$$

To analyze stability, consider the following Lyapunov function

$$V(e_x, e_\theta) = e_x^T(t) P e_x(t) + \sum_{j=1}^{n_\theta} e_{\theta_j}(t) \rho_j e_{\theta_j}(t) \tag{3.46}$$

where $P = P^T > 0$ is a positive definite matrix and $\rho_j > 0$, $\forall j$ are positive scalars. The Bounded Real Lemma (Lemma 3) offers a way to integrate the reduction of influence of perturbations (such as $\nu(t)$) on the estimates by considering the following equation for the derivative of the Lyapunov condition to satisfy:

$$\dot{V}(t) + e_x(t)^T W_x e_x(t) - \nu^T(t)\Gamma_2^x \nu(t) < 0 \tag{3.47}$$

The matrix W_x could be chosen and it could be an identity matrix of appropriate dimensions. Following the steps in the main results and considering

$$G_{ij} \triangleq P(A_i - L_iC + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij}) + (A_i - L_iC + \sum_{j=1}^{n_\theta} \theta_j \bar{A}_{ij})^T P$$

and further making use of the observer equation for the parameter estimation dynamics would lead to a simplification in the matrix form as,

$$\sum_{i=1}^{2^{n_p}} \mu_i(z) \begin{bmatrix} e_x(t) \\ \nu(t) \end{bmatrix}^T \begin{bmatrix} PG_i + G_i^T P + W_x & PL_iE \\ E^T L_i^T P & -\Gamma_2^x \end{bmatrix} \begin{bmatrix} e_x(t) \\ \nu(t) \end{bmatrix} < 0 \tag{3.48}$$

The term G_i contains the parameter θ and it could be bounded using the same strategy followed in the previous result. Further doing the change of variable $K_i = PL_i$, we have the LMI conditions as in (3.44). Hence the proof. □

Remark 9. *The design approach in Theorem 8 ignores the effect of measurement noise on the parameter estimation error. This omission is due to the inability to bring upon a bound on this influence. When the effects of ν on both e_x and e_θ was included in* (3.47)*, an approach that mimics that in [6], that is,*

$$\dot{V}(t) + e_a(t)^T e_a(t) - \nu^T(t)\Gamma_2^x \nu(t) < 0, \qquad \textit{where,} \quad e_a = \begin{bmatrix} e_x \\ e_\theta \end{bmatrix}$$

the LMI condition corresponding to (3.43) *becomes,*

$$\begin{bmatrix} PA_i + A_i^T P - K_iC - C^T K_i^T + I & 0 & K_iE \\ 0 & I & 0 \\ E^T K_i^T & 0 & -\Gamma_2^x \end{bmatrix} < -Q, \ \forall i \tag{3.49}$$

which is an impossible condition to satisfy. The reason for this limitation arises in the manner in which the observer structures was chosen. The observer state equation structure was fixed a priori, whereas, the parameter structure was obtained through the design process. This makes it difficult to integrate the two errors in (3.47)*.*

The above remark hints at a way to suppress the effect of ν on e_θ indirectly. That is, if e_θ is a perturbation in (3.47) and its effect on e_x is suppressed, then it could indirectly mean that the effect of ν on e_θ is also suppressed. This means that the derivative of Lyapunov function should satisfy,

$$\dot{V}(t) + e_x(t)^T W_x e_x(t) - \tilde{u}^T(t)\Gamma_2 \tilde{u}(t) < 0, \qquad \text{where,} \quad \tilde{u} = \begin{bmatrix} e_\theta \\ \nu \end{bmatrix}, \quad \text{and} \quad \Gamma_2 = \begin{bmatrix} \Gamma_2^\theta & 0 \\ 0 & \Gamma_2^\nu \end{bmatrix}$$

which would then lead to

$$\sum_{i=1}^{2^{n_p}} \mu_i(z) \begin{bmatrix} e_x(t) \\ \tilde{u}(t) \end{bmatrix}^T \begin{bmatrix} PG_i + G_i^T P + W_x & PL_iE \\ E^T L_i^T P & -\Gamma_2 \end{bmatrix} \begin{bmatrix} e_x(t) \\ \tilde{u}(t) \end{bmatrix} < 0$$

The result could then be summarized as the following corollary:

Corollary 2. *The system* (3.39) *is an observer for the system* (3.40) *with an asymptotically vanishing error in case of no noise and an* $\mathbb{L}_2$ *gain bound of* $\Gamma_2 = \begin{bmatrix} \Gamma_2^\theta & 0 \\ 0 & \Gamma_2^\nu \end{bmatrix}$ *for the state error in case, if the following LMI conditions replace* (3.43)*,*

$$\begin{bmatrix} PA_i + A_i^T P - K_i C - C^T K_i^T + W_x & 0 & K_i E \\ 0 & -\Gamma_2^\theta & 0 \\ E^T K_i^T & 0 & -\Gamma_2^\nu \end{bmatrix} < -Q, \ \forall i \tag{3.50}$$

A careful choice of Γ_2 would enable the conditions to be satisfied.

Chapter 4

Adaptive observers for discrete-time models

The results in this chapter borne out of the explorations towards developing a discrete-time version of joint state and parameter estimation observer. After a brief review of the literature, a state and time-varying parameter estimation method that follows the existing continuous-time results discussed in Chapter 2 are given. A Lyapunov approach with quadratic Lyapunov function is used. The result is also customized for a case with constant parameters. Simulation examples illustrate the obtained results.

Contents

4.1 Review of literature

As discussed in Chapter 3, a systematic approach for adaptive observer design for nonlinear system was proposed in [7]. However, the extension of this approach to discrete-time models is not straightforward. The design in [7] derives the parameter estimation component of the observer equation using the continuous-time trajectories of the Lyapunov function dynamics. This does not transfer well into the discrete-time. But discrete-time adaptive observer literature still has interesting works, some focusing on specific applications.

An approach that combines a diagnostic observer with an adaptive uncertainty estimation is proposed in [67]. The observer assumes the uncertainties have a parametric model. Further, all the states are assumed to be measured, which allows to consider an innovation term $e_{x,k+1} -$

$(A - K_o)e_{x,k}$, where A and K_o are system and state observer gain matrices respectively with $e_{x,k}$ represents the error. This innovation allows for a cancellation of terms that complicates the stability analysis as it uses the same form as in [7]. The fault detection then depends on the particular choice of thresholds that make the use of the estimated parameter. This approach is restricted to specific circumstances of adaptive observer, and the conditions on the eigenvalues of the Lyapunov and gain matrices makes the design process cumbersome to generalize.

In [68], a fault detection method for discrete-time nonlinear system is proposed. The considered system model contains nonlinearities and model uncertainties on both the state and the output equation. Further, the authors also consider additive faults on the actuator and the sensor affecting both the equations. The uncertainties are assumed to be bounded with a known constant. The observer uses a Luenberger form for the state estimation component. The fault components are represented by a general class of what is termed, online function approximators in discrete-time (OLAD). The OLAD allows for a mechanism such that the observer updates its parameter only after the fault occurs. In order to avoid false alarms due to unmodeled dynamics, the proposed fault detection scheme utilizes a dead zone to improve the robustness whose values depend on the bounds of unmodeled dynamics. This tight interconnection of the design makes the approach effective for fault diagnosis but difficult to adapt for general adaptive observers.

One way to develop observers for general nonlinear systems could be using equivalent forms. In the above works, a recurring theme was boundedness of the transition matrices and inputs. If we further add the condition that the states too are bounded, the quasi-LPV equivalent forms could be obtained. With the systematic procedure proposed in [23], it is possible to obtain a quasi-LPV form for a given nonlinear system. For the discrete-time version of the adaptive observer form given in [7],

$$\begin{aligned} x_{k+1} &= Hx_k + \phi(x_k, u_k) + bf(x_k, u_k)\theta \\ y_k &= Cx_k \end{aligned} \tag{4.1}$$

the quasi-LPV form would be,

$$\begin{aligned} x_{k+1} &= A(x_k, u_k)x_k\theta + B(x_k, u_k)u_k\theta \\ y_k &= Cx_k \end{aligned} \tag{4.2}$$

and possibly an affine term.

Two possible directions exist for the observer design for such quasi-LPV forms: observer design for Linear Time Varying (LTV) models and Tagaki-Sugeno (T-S) polytopic models. The reason for considering LTV systems in inspired by the assertion in [69] that the observer design strategy for the LTV systems of the form,

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{aligned} \tag{4.3}$$

covers the design problems for systems of the form,

$$\begin{aligned} x_{k+1} &= A_k(y_k, u_k)x_k + B_k(y_k, u_k) \\ y_k &= C_k(u_k)x_k \end{aligned} \tag{4.4}$$

which is intuitive under some assumptions such as the sampling interval is larger than measurement delay, for instance. This structure covers majority of the models obtained based on the adaptive observer form characterized in [9].

For discrete-time LTV systems, the adaptive observers proposed in [5], which takes inspiration from its continuous-time counterpart [55] is notable. The authors consider a stochastic MIMO time-varying system with bounded zero mean noise affecting the state and output equations as well as unknown parameters (w_k, v_k, e_k respectively).

$$\begin{aligned} \theta_{k+1} &= \theta_k + e_k \\ x_{k+1} &= A_k x_k + B_k u_k + \Psi_k \theta_k + w_k \\ y_k &= C_k x_k + v_k \end{aligned} \tag{4.5}$$

For this system, the proposed observer with the gain K_k is of the form,

$$\begin{aligned} \Upsilon_{k+1} &= (A_k - K_k C_k)\Upsilon_k + \Psi_k \\ \hat{\theta}_{k+1} &= \hat{\theta}_k + \mu_k \Upsilon_k^T C_k^T (y_k - C_k \hat{x}_k) \\ \hat{x}_{k+1} &= A_k \hat{x}_k + B_k u_k + \Psi_k \hat{\theta}_k + K_k (y_k - C_k \hat{x}_k) + \Upsilon_{k+1}(\hat{\theta}_{k+1} - \hat{\theta}_k) \end{aligned}$$

where Υ_k is a matrix sequence obtained by linearly filtering Ψ_k. The exponential convergence of the observer is proven based on boundedness of the available matrices and the boundedness of factors by the choice of the scalar μ_k. For a noisy case, the authors assert that with bounded and zero mean noises, the estimated values of the state and parameter error will be bounded and tend to zero. The two main drawbacks of this approach are: lack of a clear procedure to compute the gains of the matrix (specifically μ_k), and the need to check and guarantee the boundedness conditions at every sampling instant.

These criticisms are reiterated in [70] and claimed it is hence not effective in controlling the decay rate of the estimation error ($x_k - \hat{x}_k$). To this end, the authors in [70] propose an exponential forgetting factor based approach adopted from [69] and implement an interconnected forgetting factor design - one for the state and one for the parameters. This approach mimics a Kalman filter, with update steps at measurements as well as propagation step between measurements. The complete uniform observability of the system and the invertibility of A_k, $\forall k$ are the main assumptions. The decay rates of the state and parameter estimation errors are independently tuned through separate factors. However, as remarked earlier, some parts of nonlinear systems cannot be handled by this type of approach.

4.2 Problem formulation for adaptive observer design

The existing literature covers a broad class of adaptive observers for quasi-LPV models if we assume that the observers designed for LTV systems in (4.3) also works well for quasi-LPV models in (4.4). Beyond this, there is a class of system for which the existing literature does not cater to. That is the class of systems where the premise variables are not measured and hence the adaptation of results from LTV systems does not apply. This is the interest in this chapter and the corresponding problem is formulated in this section.

Representing a time-varying parameter using SNL

The idea of the estimation of time-varying parameter lies in representing it using the SNL transformation. This is a direct adaptation of the method proposed in [6] The SNL assumes that the parameter is bounded and its boundary values are known. For a scalar parameter $\theta_k \in [\theta^1, \theta^2]$, we could write,

$$\theta_k = \mu^1(\theta_k)\theta^1 + \mu^2(\theta_k)\theta^2 \tag{4.6}$$

where

$$\mu^1(\theta_k) = \frac{\theta^2 - \theta_k}{\theta^2 - \theta^1}, \qquad \mu^2(\theta_k) = \frac{\theta_k - \theta^1}{\theta^2 - \theta^1} \tag{4.7}$$

The membership functions satisfy the convex sum property, that is,

$$\sum_i \mu^i(.) = 1, \qquad 0 \leq \mu^i(.) \leq 1, \ \forall i \tag{4.8}$$

Hence each parameter could be represented by a weighted sum of two elements. For a vector case, or for T-S systems with unknown parameters, the membership functions can be manipulated to obtain weighting functions that depend on the same membership functions. To illustrate this, take the case of two unknown parameters, $\theta_{1,k} \in [\theta_1^1, \theta_1^2]$ and $\theta_{2,k} \in [\theta_2^1, \theta_2^2]$ and represented as,

$$\begin{aligned} \theta_{1,k} &= \mu_1^1(\theta_{1,k})\theta_1^1 + \mu_1^2(\theta_{1,k})\theta_1^2 \\ \theta_{2,k} &= \mu_2^1(\theta_{2,k})\theta_2^1 + \mu_2^2(\theta_{2,k})\theta_2^2 \end{aligned} \tag{4.9}$$

We can now create a new formulation for each unknown parameter by multiplying the summation of the weighting function of the other parameter. Since this summation equals to 1 due to the convex sum property, this does not change anything but to obtain a representation with the same weighting functions. That is,

$$\begin{aligned} \theta_{1,k} &= \Big(\mu_2^1(\theta_{2,k}) + \mu_2^2(\theta_{2,k})\Big)\,\theta_{1,k} \\ \theta_{2,k} &= \Big(\mu_1^1(\theta_{1,k}) + \mu_1^2(\theta_{1,k})\Big)\,\theta_{2,k} \end{aligned} \tag{4.10}$$

By rewriting the parameters in (4.9) with this alternative representation, we obtain the membership functions that depend on the same, but 4 weighting functions, which are the products of the membership functions of the original representation. In general, this approach would lead to 2^{n_θ} submodels, where n_θ is the number of parameters. A detailed treatment of this representation could be obtained from [71].

System model structure

Consider the following quasi-LPV model with unknown time-varying parameters:

$$\begin{aligned} x_{k+1} &= A(z_k, \Theta_k)x_k + B(z_k, \Theta_k)u_k \\ y_k &= Cx_k \end{aligned} \tag{4.11}$$

where $\Theta_k \in \mathbb{R}^{n_\theta}$ is the vector of the time-varying parameter $\theta_{i,k}$, $\forall i \in [1, \ldots, n_\theta]$, k being the time index and $z_k \in \mathbb{R}^{n_p}$ is the vector of premise or scheduling variables which are either measured or unmeasured. Further, $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^{n_y}$. The differentiation between z_k and Θ_k comes from two aspects:

- The scheduling or premise variables z_k that are not system variables, that is, exogenous inputs, are assumed measured. The unmeasured variables have to be one of the states of the systems.
- The unknown parameters Θ_k can be considered as unmeasured scheduling variables. However, the extra assumption is that the parametrization is affine in the quasi-LPV form, that is,

$$A(z_k, \Theta_k) = A_0(z_k) + \sum_{i=1}^{n_\theta} \theta_{i,k}\bar{A}_i(z_k), \qquad B(z_k, \Theta_k) = B_0(z_k) + \sum_{i=1}^{n_\theta} \theta_{i,k}\bar{B}_i(z_k) \tag{4.12}$$

To this model, the SNL transformation is applied in two steps, first to obtain a T-S model that has unknown parameters,

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{2^{n_p}} \mu_i(z_k) \left((A_0 + \sum_{j=1}^{n_\theta} \theta_{j,k}\bar{A}_{ij})x_k + (B_0 + \sum_{j=1}^{n_\theta} \theta_{j,k}\bar{B}_{ij})u_k \right) \\ y_k &= Cx_k \end{aligned}$$

In the second step, the SNL transformation is applied to the unknown parameters to obtain,

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{2^{n_p}} \sum_{j=1}^{2^{n_\theta}} \mu_i(z_k)\mu_j(\Theta_k) \left(A_{ij}x_k + B_{ij}u_k\right) \\ y_k &= Cx_k \end{aligned}$$

To simplify the notations, the two different indices i and j to represent the submodels corresponding to that of premise variables and unknown parameters could be combined together such that,

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} h_i(z_k, \Theta_k) \left(A_i x_k + B_i u_k\right) \\ y_k &= Cx_k \end{aligned} \tag{4.13}$$

where $r = 2^{n_p+n_\theta}$ and $h_i(z_k, \Theta_k)$ refers to the product of the two weighting functions corresponding to z_k and Θ_k. A_i and B_i are constant matrices obtained with appropriate replacement of the premise variables and the unknown parameters for their maximum and minimum values (for the appropriate submodel). For the model in (4.13), we propose an observer of the form,

$$\begin{aligned} \hat{x}_{k+1} &= \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k)(A_i\hat{x}_k + B_i u_k + L_i(y_k - \hat{y}_k)) \\ \hat{\Theta}_{k+1} &= \hat{\Theta}_k + \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k)(K_{y,i}(y_k - \hat{y}_k) - K_\theta\hat{\Theta}_k) \\ \hat{y}_k &= C\hat{x}_k \end{aligned} \tag{4.14}$$

The gains $L_i \in \mathbb{R}^{n_x \times n_y}$ and $K_{y,i} \in \mathbb{R}^{n_\theta \times n_y}$ are to be estimated while the gain $K_\theta \in \mathbb{R}^{n_\theta \times n_\theta}$ is chosen. The choice of K_θ shall typically be in the form of a diagonal matrix. In the initial work [45], this was introduced to avoid a marginal stability condition for the error dynamics. As discussed in Sec. 2.2.2, choosing this reduces the number of variables in the final LMI to solve and hence allows for a computationally tractable problem. Further, in the discrete-time case, K_θ as a variable leads to unresolvable nonlinear terms in the matrix inequalities.

Error dynamics

Let the state estimation error be $e_{x,k} = x_k - \hat{x}_k$. The analysis of the dynamics of the errors based on the system and observer models in (4.13) and (4.14) would involve comparing systems weighted by functions that depend on mismatched variables (i.e., z_k, Θ_k vs $\hat{z}_k, \hat{\Theta}_k$). This is a typical problem in observer design for T-S systems with unmeasured premise variables. To mitigate this, the original system model can be rewritten as,

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k)(A_i x_k + B_i u_k) + \Delta_k \\ y_k &= C x_k \end{aligned} \tag{4.15}$$

with,

$$\Delta_k = \sum_{i=1}^{r} (h_i(z_k, \Theta_k) - h_i(\hat{z}_k, \hat{\Theta}_k))(A_i x_k + B_i u_k)$$

This leads to the error dynamics as

$$e_{x,k+1} = \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k)(A_i - L_i C) e_{x,k} + \Delta_k \tag{4.16}$$

The challenge is to obtain the conditions that are necessary to show the convergence of the estimates or the vanishing of the error (4.16). The approach followed is exploiting the uncertain-like model representation explored in [58] and used for joint state and parameter estimation for continuous-time models in [45]. This approach leads to finding bounds for Δ_k exploiting the convex sum property satisfied by the weighting functions and hence to obtain the bounds for the estimation error e_k. The next sections discuss these results.

4.3 Proposed observer design method

In this section, the approach used in Sec. 2.2.2 following the works of [45] and [58] would be used to analyze the error dynamics arising out of the observer model.

Uncertain-like representation

We split the term Δ_k in (4.15) into $\Delta_k = \Delta A_k + \Delta B_k$ with

$$\begin{aligned} \Delta A_k &= \sum_{i=1}^{r} (h_i(z_k, \Theta_k) - h_i(\hat{z}_k, \hat{\Theta}_k)) A_i = \mathcal{A} \Sigma_{A,k} E_A \\ \Delta B_k &= \sum_{i=1}^{r} (h_i(z_k, \Theta_k) - h_i(\hat{z}_k, \hat{\Theta}_k)) B_i = \mathcal{B} \Sigma_{B,k} E_B \end{aligned} \tag{4.17}$$

where

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_r \end{bmatrix} \in \mathbb{R}^{n_x \times n_x . r}, \qquad E_A = \begin{bmatrix} I_{n_x} & I_{n_x} & \dots & I_{n_x} \end{bmatrix}^T \in \mathbb{R}^{n_x . r \times n_x}$$

$$\Sigma_{A,k} = \begin{bmatrix} \Big(h_1(z_k, \Theta_k) - h_1(\hat{z}_k, \hat{\Theta}_k)\Big) I_{n_x} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Big(h_r(z_k, \Theta_k) - h_r(\hat{z}_k, \hat{\Theta}_k)\Big) I_{n_x} \end{bmatrix} \tag{4.18}$$

and

$$\mathcal{B} = \begin{bmatrix} B_1 & B_2 & \dots & B_r \end{bmatrix} \in \mathbb{R}^{n_u \times n_u . r}, \qquad E_B = \begin{bmatrix} I_{n_u} & I_{n_u} & \dots & I_{n_u} \end{bmatrix}^T \in \mathbb{R}^{n_u . r \times n_x}$$

$$\Sigma_{B,k} = \begin{bmatrix} \left(h_1(z_k,\Theta_k) - h_1(\hat{z}_k,\hat{\Theta}_k)\right) I_{n_u} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(h_r(z_k,\Theta_k) - h_r(\hat{z}_k,\hat{\Theta}_k)\right) I_{n_u} \end{bmatrix} \tag{4.19}$$

Noting that $-1 \leq h_i(z_k,\Theta_k) - h_i(\hat{z}_k,\hat{\Theta}_k) \leq 1$, the matrices $\Sigma_{A,k} \in \mathbb{R}^{n_x r \times n_x r}$, $\Sigma_{B,k} \in \mathbb{R}^{n_u r \times n_u r}$ have the useful property,

$$\Sigma_{A,k}^T \Sigma_{A,k} \leq I, \qquad \Sigma_{B,k}^T \Sigma_{B,k} \leq I \tag{4.20}$$

which will later be used to bound the time-varying difference between the known and estimated weighting functions. The system model could now be rewritten as,

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} h_i(\hat{z}_k,\hat{\Theta}_k)[(A_i + \Delta A_k)x_k + (B_i + \Delta B_k)u_k] \\ y_k &= C x_k \end{aligned} \tag{4.21}$$

Now, the main result is expressed in the following theorem,

Theorem 9. *Given the system model of the form* (4.13) *an observer of the form* (4.14) *exists, if there exists* P_0, P_1, R_i, F_i, λ_1, λ_2, λ_3, λ_4, Γ_2^j $(\forall i \in [1,r], \forall j \in \{x,u,\theta,\Delta\theta\})$*, such that,*

$$\begin{aligned} & P_0 = P_0^T > 0, \; P_1 = P_1^T > 0 \\ & \lambda_m > 0, \; \forall m \in \{1,2,3,4\}, \; \Gamma_2^j > 0, \forall j \end{aligned} \tag{4.22}$$

$$\left[\begin{array}{cccc|cccc} -P+I & Q_{A,i} & \Phi_i^T P & 0 & P_{0,i}\mathcal{A} & P_{0,i}\mathcal{B} & 0 & 0 \\ * & T_{22} & 0 & Q_B & 0 & 0 & 0 & 0 \\ * & * & -P & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 & P_0\mathcal{A} & P_0\mathcal{B} \\ \hline & & & & -\lambda_1 I & 0 & 0 & 0 \\ & & * & & * & -\lambda_3 I & 0 & 0 \\ & & & & * & * & -\lambda_2 I & 0 \\ & & & & * & * & * & -\lambda_4 I \end{array}\right] \tag{4.23}$$

Here,

$$P_{0,i} = A_i^T P_0 - C^T R_i^T, \qquad P = \mathrm{diag}(P_0, P_1), \qquad \Phi_i^T P = \begin{bmatrix} P_{0,i} & -C^T F_i^T \\ 0 & -K_\theta^T P_1 \end{bmatrix}$$

$$Q_{A,i} = \begin{bmatrix} 0 & 0 & -C^T F_i^T (I+K_\theta) & -C^T F_i^T \\ 0 & 0 & -K_\theta^T P_1 (I+K_\theta) & -K_\theta^T P_1 \end{bmatrix}, \qquad Q_B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & (I+K_\theta)^T P_1 \\ 0 & P_1 \end{bmatrix}$$

$$T_{22} = \begin{bmatrix} T_{22}^{11} & 0 & 0 & 0 \\ 0 & T_{22}^{22} & 0 & 0 \\ 0 & 0 & -\Gamma_2^\theta & 0 \\ 0 & 0 & 0 & -\Gamma_2^{\Delta\theta} \end{bmatrix} \tag{4.24}$$

where $T_{22}^{11} = -\Gamma_2^x + (\lambda_1 + \lambda_2) E_A^T E_A$ *and* $T_{22}^{22} = -\Gamma_2^u + (\lambda_3 + \lambda_4) E_B^T E_B$.

The observer gains are given by,

$$K_{y,i} = P_1^{-1} F_i, \qquad L_i = P_0^{-1} R_i, \qquad \forall\ i \tag{4.25}$$

Proof. Defining errors of the form,

$$e_{x,k} = x_k - \hat{x}_k, \quad e_{\Theta,k} = \Theta_k - \hat{\Theta}_k$$

and comparing the uncertain-like representation of the system in (4.21) with the observer (4.14),

$$\begin{aligned} e_{x,k+1} &= \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k) \left((A_i - L_i C) e_{x,k} + \Delta A_k x_k + \Delta B_k u_k \right) \\ e_{\Theta,k+1} &= \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k) \left(\Delta\Theta_k + (I + K_\theta)\Theta_k - K_{y,i} C e_{x,k} - K_\theta e_{\Theta,k} \right) \end{aligned} \tag{4.26}$$

where $\Delta\Theta_k = \Theta_{k+1} - \Theta_k$. This error dynamics can be represented as

$$\begin{bmatrix} e_{x,k+1} \\ \\ e_{\Theta,k+1} \end{bmatrix} = \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k) \left(\Phi_i \begin{bmatrix} e_{x,k} \\ \\ e_{\Theta,k} \end{bmatrix} + \Psi_{i,k} \begin{bmatrix} x_k \\ u_k \\ \Theta_k \\ \Delta\Theta_k \end{bmatrix} \right)$$

where

$$\Phi_i = \begin{bmatrix} A_i - L_i C & 0 \\ -K_{y,i} C & -K_\theta \end{bmatrix}, \qquad \Psi_{i,k} = \begin{bmatrix} \Delta A_k & \Delta B_k & 0 & 0 \\ 0 & 0 & I + K_\theta & I \end{bmatrix} \tag{4.27}$$

Considering,

$$e_{a,k} = \begin{bmatrix} e_{x,k}^T & e_{\Theta,k}^T \end{bmatrix}^T, \ \tilde{u}_k = \begin{bmatrix} x_k^T & u_k^T & \Theta_k^T & \Delta\Theta_k^T \end{bmatrix}^T$$

the error dynamics can be written by,

$$e_{a,k+1} = \sum_{i=1}^{r} h_i(\hat{z}_k, \hat{\Theta}_k) \left(\Phi_i e_{a,k} + \Psi_{i,k} \tilde{u}_k \right) \tag{4.28}$$

The aim here is the vanishing of the error and the minimization of the effect of $\tilde{u}_k$ on the error. It is to be noted that Φ_i has constant entries, but $\Psi_{i,k}$ has time varying entries, while both contain the design parameters. To analyze the stability of (4.28), the following Lyapunov candidate is considered,

$$V_k = e_{a,k}^T P e_{a,k} \tag{4.29}$$

Since there are time-varying perturbations that affect the error $e_{a,k}$ in (4.28), the sufficient condition for stability could be enforced using

$$V_{k+1} < V_k - (e_{a,k}^T e_{a,k} - \tilde{u}_k^T \Gamma_2 \tilde{u}_k) \tag{4.30}$$

where Γ_2 is a block diagonal matrix with the entries

$$\Gamma_2 = \mathrm{diag}(\Gamma_2^x, \Gamma_2^u, \Gamma_2^\theta, \Gamma_2^{\Delta\theta}) \tag{4.31}$$

that represent the $\mathbb{L}_2$-gains of the effect of the elements in $\tilde{u}$ on the error $e - a$, respectively. By applying the discrete-time version of the bounded real lemma (BRL) in Lemma 4 to (4.28),

$$\begin{bmatrix} \Phi_i^T P \Phi_l - P + I & \Phi_i^T P \Psi_{l,k} \\ * & \Psi_{i,k}^T P \Psi_{l,k} - \Gamma_2 \end{bmatrix} < 0, \qquad \forall\ i, l \tag{4.32}$$

is obtained. Note the introduction of the index l. This is to indicate the cross terms that would emerge when applying BRL (because of the quadratic term in the Lyapunov function as well as the term $e_{a,k}^T e_{a,k}$ in (4.30)). This however would make the final LMI significantly complex. To avoid this, a more conservative, but less complicated conditions can be obtained if the submodels considered are vertices corresponding to $l = i$. This assumption works well for discrete-time models as discussed in Theorem 17 in [33]. This hence makes the condition in (4.32) as,

$$\begin{bmatrix} \Phi_i^T P \Phi_i - P + I & \Phi_i^T P \Psi_{i,k} \\ * & \Psi_{i,k}^T P \Psi_{i,k} - \Gamma_2 \end{bmatrix} < 0, \qquad \forall\ i \tag{4.33}$$

Further, the condition in (4.33) are not LMIs and following changes are required to convert them:

- Linearization of nonlinear terms ($\Phi_i^T P \Phi_i$, $\Phi_i^T P \Psi_{i,k}$ and their transposes)
- Obtain bounds for the time-varying terms (in $\Phi_i^T P \Psi_{i,k}$ and $\Psi_{i,k}^T P \Psi_{i,k}$)

Linearization The quadratic terms associated with Φ_i and $\Psi_{i,k}$ could be reduced to linear terms by using the Schur complements (Lemma 5) for the nonlinear terms, the matrix terms in (4.32) could be reduced to,

$$\begin{bmatrix} -P + I & \Phi_i^T P \Psi_{i,k} & \Phi_i^T P & 0 \\ * & -\Gamma_2 & 0 & \Psi_{i,k}^T P \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix} < 0, \qquad \forall\ i \tag{4.34}$$

This has not resolved all the nonlinear entries, though, and the residual factors are in the form of unresolvable terms inside $\Phi_i^T P$ and $\Phi_i^T P \Psi_{i,k}$. This is because as in (4.27), Φ_i has two variables L_i and $K_{y,i}$, as part of the matrix split into n_x and n_θ blocks. This issue is alleviated in two steps:

- Consider a diagonal structure for the Lyapunov matrix $P = \begin{bmatrix} P_0 & 0 \\ 0 & P_1 \end{bmatrix}$
- This Lyapunov structure would lead to terms $P_0 L_i$ and $P_1 K_{y,i}$. These quadratic terms are eliminated by introducing new variables,

$$R_i = P_0 L_i, \qquad F_i = P_1 K_{y,i} \tag{4.35}$$

These steps would reduce the nonlinear matrix entries in (4.34) to linear terms. First, to simplify notations, consider

$$P_{0,i} = A_i^T P_0 - C^T R_i^T \tag{4.36}$$

From the definition of Φ_i in (4.27), the term $\Phi_i^T P$ would reduce to,

$$\Phi_i^T P = \begin{bmatrix} P_{0,i} & -C^T F_i^T \\ 0 & -K_\theta^T P_1 \end{bmatrix}$$

Further, the linearized time-varying matrices are split into those with constant entries and time-varying terms,

$$\begin{aligned} \Phi_i^T P \Psi_{i,k} &= Q_{A,i} + \mathcal{L}_{U,i,k} \\ \Psi_{i,k}^T P &= Q_B + \mathcal{L}_{L,k}^T \end{aligned} \tag{4.37}$$

where

$$Q_{A,i} = \begin{bmatrix} 0 & 0 & -C^T F_i^T (I + K_\theta) & -C^T F_i^T \\ \\ 0 & 0 & -K_\theta^T P_1 (I + K_\theta) & -K_\theta^T P_1 \end{bmatrix}, \quad Q_B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & (I + K_\theta)^T P_1 \\ 0 & P_1 \end{bmatrix}$$

$$\mathcal{L}_{U,i,k} = \begin{bmatrix} P_{0,i}\Delta A_k & P_{0,i}\Delta B_k & 0 & 0 \\ \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}_{L,k} = \begin{bmatrix} P_0 \Delta A_k & P_0 \Delta B_k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.38}$$

Bounds for time-varying terms The linearized version of the inequality in (4.34) can now be split into terms with and without time-varying terms and their corresponding transposes,

$$Q_i + \mathcal{L}_{i,k} + \mathcal{L}_{i,k}^T < 0 \tag{4.39}$$

where

$$Q_i = \begin{bmatrix} -P + I & Q_{A,i} & \Phi_i^T P & 0 \\ * & -\Gamma_2 & 0 & Q_B \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix} \tag{4.40}$$

with $Q_{A,i}$ and Q_B given in (4.24). The time-varying terms are gathered as below,

$$\mathcal{L}_{i,k} = \begin{bmatrix} 0 & \mathcal{L}_{U,i,k} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}_{L,k} & 0 & 0 \end{bmatrix} \tag{4.41}$$

and its transpose with the terms defined in (4.38).

There are 4 time-varying terms and their transposes in (4.39). A bound for each of these terms could be obtained using the matrix representation idea shown in (4.17) and the fact that these matrices have interesting bounds that could be exploited (see (4.20)). Similarly each of the uncertain-like term in $\mathcal{L}_{i,k}$ could be split. Let the four terms as part of individual matrices be $\mathcal{L}_{A1,k}$, $\mathcal{L}_{A2,k}$, $\mathcal{L}_{B1,k}$, $\mathcal{L}_{B2,k}$ corresponding to the time-varying factors, $P_{0,i}\Delta A_k$, $P_0\Delta A_k$, $P_{0,i}\Delta B_k$, $P_0\Delta B_k$ respectively. That is,

$$\mathcal{L}_{A1,k} = \begin{bmatrix} 0 & \begin{bmatrix} P_{0,i}\Delta A_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{L}_{B2,k} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 0 & P_0\Delta B_k & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 & 0 \end{bmatrix} \tag{4.42}$$

and similarly for $\mathcal{L}_{B1,k}$ and $\mathcal{L}_{A2,k}$, so that,

$$\mathcal{L}_{i,k} = \mathcal{L}_{A1,k} + \mathcal{L}_{A2,k} + \mathcal{L}_{B1,k} + \mathcal{L}_{B2,k} \tag{4.43}$$

It is to be noted that the 0 entries in the matrices have appropriate dimension and are usually grouped together to make the representation easier. By taking cue from the representation in (4.17),

$$\mathcal{L}_{A1,k} = \begin{bmatrix} \begin{bmatrix} P_{0,i}\mathcal{A} \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Sigma_{A,k} \begin{bmatrix} 0 & \begin{bmatrix} E_A & 0 & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}, \; \mathcal{L}_{A2,k} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \begin{bmatrix} P_0\mathcal{A} \\ 0 \end{bmatrix} \end{bmatrix} \Sigma_{A,k} \begin{bmatrix} 0 & \begin{bmatrix} E_A & 0 & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

$$\mathcal{L}_{B1,k} = \begin{bmatrix} \begin{bmatrix} P_{0,i}\mathcal{B} \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Sigma_{B,k} \begin{bmatrix} 0 & \begin{bmatrix} 0 & E_B & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}, \; \mathcal{L}_{B2,k} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \begin{bmatrix} P_0\mathcal{B} \\ 0 \end{bmatrix} \end{bmatrix} \Sigma_{B,k} \begin{bmatrix} 0 & \begin{bmatrix} 0 & E_B & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

The Lemma 1 is now applied on the sum of these terms and their transposes.

$$\mathcal{L}_{A1,k} + \mathcal{L}_{A1,k}^T \leq \lambda_1^{-1} \begin{bmatrix} \begin{bmatrix} P_{0,i}\mathcal{A} \\ 0 \end{bmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \mathcal{A}^T P_{0,i} & 0 \end{bmatrix} & 0 & 0 & 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 0 \\ \begin{bmatrix} E_A^T \\ 0 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} E_A & 0 & 0 \end{bmatrix} & 0 & 0 \end{bmatrix}$$

$$\leq \begin{pmatrix} \begin{bmatrix} \lambda_1^{-1} P_{0,i}\mathcal{A}\mathcal{A}^T P_{0,i} & 0 \\ 0 & 0 \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 0 & \lambda_1 E_A^T E_A & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for some scalar λ_1. Similarly,

$$\mathcal{L}_{A2,k} + \mathcal{L}_{A2,k}^T \leq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} \lambda_2 E_A^T E_A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{bmatrix} \lambda_2^{-1} P_0\mathcal{A}\mathcal{A}^T P_0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix}$$

$$\mathcal{L}_{B1,k} + \mathcal{L}_{B1,k}^T \leq \begin{pmatrix} \begin{bmatrix} \lambda_3^{-1} P_{0,i} \mathcal{B}\mathcal{B}^T P_{0,i} & 0 \\ 0 & 0 \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_3 E_B^T E_B & 0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{L}_{B2,k} + \mathcal{L}_{B2,k}^T \leq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_4 E_B^T E_B & 0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{bmatrix} \lambda_4^{-1} P_0 \mathcal{B}\mathcal{B}^T P_0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix}$$

Adding them all up gives,

$$\mathcal{L}_{i,k} + \mathcal{L}_{i,k}^T \leq \begin{pmatrix} \begin{bmatrix} \mathcal{L}_i^1 & 0 \\ 0 & 0 \end{bmatrix} & 0 & 0 & 0 \\ 0 & \mathcal{L}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{bmatrix} \mathcal{L}^3 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix}$$

with,

$$\begin{aligned} \mathcal{L}_i^1 &= \lambda_1^{-1} P_{0,i} \mathcal{A}\mathcal{A}^T P_{0,i} + \lambda_3^{-1} P_{0,i} \mathcal{B}\mathcal{B}^T P_{0,i}, \\ \mathcal{L}^2 &= \begin{bmatrix} (\lambda_1 + \lambda_2) E_A^T E_A & 0 & 0 & 0 \\ 0 & (\lambda_3 + \lambda_4) E_B^T E_B & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \mathcal{L}^3 &= \lambda_2^{-1} P_0 \mathcal{A}\mathcal{A}^T P_0 + \lambda_4^{-1} P_0 \mathcal{B}\mathcal{B}^T P_0 \end{aligned} \tag{4.44}$$

This would lead to the inequality in (4.39) to,

$$\begin{pmatrix} \mathcal{L}_i^{11} & Q_{A,i} & \Phi_i^T P & 0 \\ * & -\Gamma_2 + \mathcal{L}^2 & 0 & Q_B \\ * & * & -P & 0 \\ * & * & * & \mathcal{L}^{44} \end{pmatrix} < 0 \tag{4.45}$$

where

$$\mathcal{L}_i^{11} = \begin{bmatrix} -P_0 + I + \mathcal{L}_i^1 & 0 \\ 0 & P_1 \end{bmatrix}, \qquad \mathcal{L}^{44} = \begin{bmatrix} -P_0 + I + \mathcal{L}^3 & 0 \\ 0 & P_1 \end{bmatrix} \tag{4.46}$$

These terms have quadratic entries ($\mathcal{L}_i^1$ and $\mathcal{L}^3$) that could be handled by applying Schur's complement. In this way, we could consider,

$$\mathcal{L}_i^{11} < 0 \Leftrightarrow \begin{bmatrix} -P_0+I & 0 & P_{0,i}\mathcal{A} & P_{0,i}\mathcal{B} \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & -\lambda_1 I & 0 \\ 0 & 0 & 0 & -\lambda_3 I \end{bmatrix} < 0,$$
$$\mathcal{L}_i^{44} < 0 \Leftrightarrow \begin{bmatrix} -P_0+I & 0 & P_{0,i}\mathcal{A} & P_{0,i}\mathcal{B} \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & -\lambda_2 I & 0 \\ 0 & 0 & 0 & -\lambda_4 I \end{bmatrix} < 0 \tag{4.47}$$

By putting them together and rearranging, we get,

$$\left[\begin{array}{c|c} \begin{pmatrix} \begin{bmatrix} -P_0+I & 0 \\ 0 & -P_1+I \end{bmatrix} & Q_{A,i} & \Phi_i^T P & 0 \\ Q_{A,i}^T & T_{22} & 0 & Q_B \\ P\Phi_{ij} & 0 & \begin{bmatrix} -P_0 & 0 \\ 0 & -P_1 \end{bmatrix} & 0 \\ 0 & Q_B^T & 0 & \begin{bmatrix} -P_0 & 0 \\ 0 & -P_1 \end{bmatrix} \end{pmatrix} & \begin{pmatrix} P_{0,i}\mathcal{A} & P_{0,i}\mathcal{B} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P_0\mathcal{A} & P_0\mathcal{B} \end{pmatrix} \\ \hline \begin{pmatrix} \mathcal{A}^T P_{0,i} & 0 & 0 & 0 \\ \mathcal{B}^T P_{0,i} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{A}^T P_0 \\ 0 & 0 & 0 & \mathcal{B}^T P_0 \end{pmatrix} & \begin{pmatrix} -\lambda_1 I & 0 & 0 & 0 \\ 0 & -\lambda_3 I & 0 & 0 \\ 0 & 0 & -\lambda_2 I & 0 \\ 0 & 0 & 0 & -\lambda_4 I \end{pmatrix} \end{array}\right] \tag{4.48}$$

which in a simplified form can be represented as,

$$\begin{bmatrix} -P+I & Q_{A,i} & \Phi_i^T P & 0 & P_{0,i}\mathcal{A} & P_{0,i}\mathcal{B} & 0 & 0 \\ * & T_{22} & 0 & Q_B & 0 & 0 & 0 & 0 \\ * & * & -P & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 & P_0\mathcal{A} & P_0\mathcal{B} \\ * & * & * & * & -\lambda_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda_3 I & 0 & 0 \\ * & * & * & * & * & * & -\lambda_2 I & 0 \\ * & * & * & * & * & * & * & -\lambda_4 I \end{bmatrix} < 0 \tag{4.49}$$

which is the same as the LMI condition (4.23). Hence the proof. □

Corollary 3. *The observer design problem can be formulated as an optimization problem with the objective to minimize the* $\mathbb{L}_2$*-gain between the perturbation factors* $\tilde{u}_k$ *and the errors* $e_{a,k}$ *in* (4.28)*. This could be achieved by minimizing a scalar* β*, such that,*

$$\min_{P_0,P_1,F_i,R_i,\Gamma_2^j,\lambda_m} \beta \tag{4.50}$$

$\forall i \in [1,r]$, $\forall j \in \{x,u,\theta,\Delta\theta\}$, $\forall m \in \{1,2,3,4\}$ *such that, the LMIs in* (4.23) *are satisfied along with,*

$$\beta I > \Gamma_2^j, \quad \forall j \in \{x,u,\theta,\Delta\theta\} \tag{4.51}$$

There is an inherent assumption that the $\mathbb{L}_2$*-gains of various perturbations are scaled appropriately so that using a single* β *makes sense. This could otherwise be achieved by using appropriate scaling factor instead of* I *on the left hand side of*(4.51)*.*

Remark 10. *It is to be noted that since there are only two time-varying terms ΔA_k and ΔB_k in* (4.41)*, we could split the time-varying terms into only two additive factors and hence apply the Lemma 1 twice. However, the resulting matrix inequality is nonlinear with crossover terms making it impossible to resolve. Hence four additive factors were used.*

Remark 11. *In the continuous-time version in [45], the factor K_θ allowed to avoid numerical issues in the LMI conditions. This has been followed through in the discrete-time case presented here. The value of K_θ however, is also important because it may lead to the effect of the innovation term $K_{y,i}(y_k - \hat{y}_k)$ to become negligible due to relative scaling between K_θ and $K_{y,i}$ as discussed in Sec. 2.2.2. This could be done by adding an extra condition. For example, for a scalar parameter estimation case, let k_θ be the scalar value of the observer gain K_θ, which would lead to the condition,*

$$\frac{1}{k_\theta} K_{y,i} > \rho \tag{4.52}$$

where $\rho > 1$ is a constant chosen depending upon the relative scaling between θ and $y_k - \hat{y}_k$. Further conditions along with the LMIs in (4.23) *allows mitigating this problem. For instance, consider a scalar parameter case,*

$$F_i > \rho P_1 k_\theta \tag{4.53}$$

Remark 12. *The term $-P + I$ in* (4.23) *has its origins in the Lyapunov function of the form $e_{a,k}^T I e_{a,k}$. This could provide a numerical limitation while solving the LMI feasibility problem. This could be modified by choosing the Lyapunov function as $e_{a,k}^T Q_e e_{a,k}$, where $Q_e > 0$ that could be chosen such that a solution exists.*

Remark 13. *Some problem specific conditions could be added to obtain an optimum solution to the problem. For example, pole placement for the state observers $A_i - L_i C$ could be added as a separate LMI condition so as to achieve favourable rate of convergence. Further, some minimum value for the gain corresponding to parameter estimation, $K_i, \forall i$ could be imposed so that the innovation term is useful in augmenting the estimated θ (due to the relative scaling between the values of $\hat{\theta}$ and $y - \hat{y}$).*

4.4 Extensions

In this section, a couple of extensions to the proposed observer design method are discussed.

4.4.1 Measurement noise

If the system in (4.13) has measurement noise, this would lead to

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} h_i(z_k, \Theta_k) \left(A_i x_k + B_i u_k \right) \\ y_k &= C x_k + H \nu_k \end{aligned} \tag{4.54}$$

where $\nu_k \in \mathbb{R}^{n_y}$ is the measurement noise with $H \in \mathbb{R}^{n_y \times n_\nu}$ the transmission matrix. This leads to the perturbation variable to become $\tilde{u}_k = \begin{bmatrix} x_k & u_k & \Theta_k & \Delta\Theta_k & \nu_k \end{bmatrix}^T$ and the matrix $\Psi_{i,k}$ in (4.27) as,

$$\Psi_{i,k} = \begin{bmatrix} \Delta A_k & \Delta B_k & 0 & 0 & -L_i H \\ 0 & 0 & I + K_\theta & I & -K_{y,i} H \end{bmatrix} \tag{4.55}$$

which would then lead to the same LMIs as in (4.63) with the modifications in the following components in (4.24) as,

$$Q_{A,i} = \begin{bmatrix} 0 & 0 & -C^T F_i^T (I + K_\theta) & -C^T F_i^T & -R_i H \\ & & & & \\ 0 & 0 & -K_\theta^T P_1 (I + K_\theta) & -K_\theta^T P_1 & -F_i H \end{bmatrix}$$
$$T_{22} = \begin{bmatrix} T_{22}^{11} & 0 & 0 & 0 & 0 \\ 0 & T_{22}^{22} & 0 & 0 & 0 \\ 0 & 0 & -\Gamma_2^\theta & 0 & 0 \\ 0 & 0 & 0 & -\Gamma_2^{\Delta\theta} & 0 \\ 0 & 0 & 0 & 0 & -\Gamma_2^\nu \end{bmatrix} \tag{4.56}$$

where Γ_2^ν is the $\mathbb{L}_2$-gain between the noise ν and the error $e_{a,k}$. It is to be noted that Γ_2^ν will also be added as a diagonal block in the matrix Γ_2.

4.4.2 Constant parameters

If we consider the parameters θ to be constant, some simplification could be attempted. Our aim is to design an adaptive observer assuming that we know a range of values $[\theta_i^1, \theta_i^2]$ in which the true value of each of the $\theta_i, \forall i \in 1, .., n_\theta$ lies. This would then transform the problem in the same lines of the bounds of a time-varying parameter considered in the previous section. This would mean, we could represent the system model as

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{s} h_i(z_k, \Theta)(A_i x_k + B_i u_k) \\ y_k &= C x_k \end{aligned} \tag{4.57}$$

where $s = 2^{n_p + n_\theta}$ and $h_i(z_k, \Theta)$ is the weighting function obtained by normalizing the product of membership functions associated with the premise variables z_k and the parameters Θ. For this type of system, we propose an observer of the form,

$$\begin{aligned} \hat{x}_{k+1} &= \sum_{i=1}^{s} h_i(\hat{z}_k, \hat{\Theta}_k) \left[A_i \hat{x} + B_i u_k + L_i (y_k - \hat{y}_k) \right] \\ \hat{\Theta}_{k+1} &= \hat{\Theta}_k + \sum_{i=1}^{s} h_i(\hat{z}_k, \hat{\Theta}_k) K_{y,i} (y_k - \hat{y}_k) \\ \hat{y}_k &= C x_k \end{aligned} \tag{4.58}$$

As could be noted, the K_θ gain term has been dropped. One main reason is the simplification this offers. Further, in the continuous-time case, the condition $K_\theta = 0$ lead to unsolvable LMIs but not in discrete-time. To compute the state and parameter error, we follow the uncertain-like model approach, the augmented error dynamics is given by,

$$e_{a,k+1} = \begin{bmatrix} A_i - L_i C & 0 \\ -K_{y,i} C & 0 \end{bmatrix} e_{a,k} + \begin{bmatrix} \Delta A_k & \Delta B_k \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

As could be seen, the number of perturbations has reduced and hence the dynamic matrices simplified. By applying discrete-time BRL and following it up with the application of LMI equivalence using Schur's complement (Lemma 5), then splitting the Lyapunov matrix to be of the form $P = \begin{bmatrix} P_0 & 0 \\ 0 & P_1 \end{bmatrix}$, and applying the variable transformations $R_i = P_0 L_i$ and $F_i = P_1 K_{y,i}$, we get,

$$\begin{bmatrix} -P+I & \Phi_i^T P \Psi_{i,k} & \Phi_i^T P & 0 \\ * & -\Gamma_2 & 0 & \Psi_{i,k}^T P \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix} < 0 \tag{4.59}$$

with

$$\Phi_i^T P \Psi_{i,k} = \begin{bmatrix} P_{0,i}\Delta A_k & P_{0,i}\Delta B_k \\ 0 & 0 \end{bmatrix}, \; \Phi_i^T P = \begin{bmatrix} P_{0,i} & -C^T F_i^T \\ 0 & 0 \end{bmatrix}, \; \Psi_{i,k}^T P = \begin{bmatrix} \Delta A_k^T P_0 & \Delta B_k^T P_0 \\ 0 & 0 \end{bmatrix} \tag{4.60}$$

Further following the same steps described in the proof of Theorem 9, we can summarize the results as follows,

Theorem 10. *Given a discrete-time T-S model of the form* (4.57)*, an observer of the form* (4.58) *exists, if there exists* P_0, P_1, R_i, F_i, λ_1, λ_2, λ_3, λ_4, Γ_2^j *(*$\forall i \in [1, r], \forall j \in \{x, u\}$*), such that,*

$$P_0 = P_0^T > 0, \; P_1 = P_1^T > 0 \tag{4.61}$$

$$\lambda_m > 0, \; \forall m \in 1,2,3,4, \; \Gamma_2^j > 0, \forall j \tag{4.62}$$

$$\begin{bmatrix} -P+I & 0 & \Phi_i^T P & 0 & P_{0,i}\mathcal{A} & P_{0,i}\mathcal{B} & 0 & 0 \\ * & T_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -P & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 & P_0\mathcal{A} & P_0\mathcal{B} \\ * & * & * & * & -\lambda_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -\lambda_3 I & 0 & 0 \\ * & * & * & * & * & * & -\lambda_2 I & 0 \\ * & * & * & * & * & * & * & -\lambda_4 I \end{bmatrix} < 0 \tag{4.63}$$

where $\Phi_i^T P$ *is given in* (4.60)*, and there are some structural changes to accommodate the changes in the number of zero rows due to the change of* T_{22} *to,*

$$T_{22} = \begin{bmatrix} T_{22}^{11} & 0 \\ 0 & T_{22}^{22} \end{bmatrix}$$

where $T_{22}^{11} = -\Gamma_2^x + (\lambda_1 + \lambda_2)E_A^T E_A$ *and* $T_{22}^{11} = -\Gamma_2^u + (\lambda_3 + \lambda_4)E_B^T E_B$.

Proof. The proof follows that of Theorem 9 with the changes in the matrix block entries discussed above. □

4.5 Simulation examples

4.5.1 Time-varying parameter

Example 5. *Consider the second order nonlinear system,*

$$\begin{aligned} x_{1,k+1} &= -0.5x_{1,k}^2 + 0.5x_{2,k} + x_{1,k}\theta_k \\ x_{2,k+1} &= 0.8x_{1,k} + (1 - 0.5\theta_k)u_k \\ y_k &= x_{1,k} \end{aligned} \tag{4.64}$$

A T-S model of the nonlinear model could be obtained by considering that the state $x_{1,k} \in [0,1]$, $\theta_k \in [-1,1]$ and $z \triangleq x_1$ as the premise variables. The system matrices for this is given by,

$$A_{11} = \begin{bmatrix} -1 & 0.5 \\ 0.8 & 0 \end{bmatrix} \; A_{12} = \begin{bmatrix} 1 & 0.5 \\ 0.8 & 0 \end{bmatrix} \; A_{21} = \begin{bmatrix} -1.5 & 0.5 \\ 0.8 & 0 \end{bmatrix} \; A_{22} = \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0 \end{bmatrix} \tag{4.65}$$

and with

$$B_{11} = B_{13} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad B_{21} = B_{22} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \tag{4.66}$$

The weighting functions are given by,

$$\mu_1(\hat{x}_1) = 1 - \hat{x}_1, \; \mu_2(\hat{x}_1) = \hat{x}_1, \; \tilde{\mu}_1(\hat{\theta}) = \frac{1-\hat{\theta}}{2}, \; \tilde{\mu}_2(\hat{\theta}) = \frac{\hat{\theta}+1}{2} \tag{4.67}$$

For developing the observer, the aspects were considered:

- The value of η was fixed at -0.995. Apart from avoiding the nonlinear matrix term, this would also avoid Φ_{ij} to become marginal stable in (4.27).
- A condition to ensure that the value of K_{ij} are sufficiently larger than that of η as discussed in Sec. 2.2.2, the following LMI condition was introduced

$$F_{ij} > P_1 \eta$$

- The chosen values for $Q_e = diag(Q_{e0}, Q_{e1})$ (See Remark 12) are given by

$$Q_{e0} = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \; Q_{e1} = 0.001$$

The resultant observer gains were as follows:

$$L_{11} = \begin{bmatrix} -1 \\ 0.8 \end{bmatrix} \; L_{12} = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \; L_{21} = \begin{bmatrix} -1.5 \\ 0.8 \end{bmatrix} \; L_{22} = \begin{bmatrix} 0.5 \\ 0.8 \end{bmatrix} \quad \text{and} \quad K_{ij} = 1.014 \; \forall i,j \tag{4.68}$$

The state and the parameter estimation results are shown in the Fig. 4.1. The input used for the simulation as well as the variation of the four weighting functions during the simulation are shown in the Fig.4.2.

4.5.2 Constant parameter

Example 6. *We consider the discrete-time version of the simplified waste water treatment plant from [45]. The simplification concerns reducing the 10-state system to a 2-state model, given by,*

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + T_s \left[\frac{a x_{1,k}}{x_{2,k}+b} x_{2,k} - x_{1,k} u_k \right] \\ x_{2,k+1} &= x_{2,k} + T_s \left[-\frac{c a x_{1,k}}{x_{2,k}+b} x_{2,k} + (d - x_{2,k}) u_k \right] \\ y_k &= x_{1,k} \end{aligned} \tag{4.69}$$

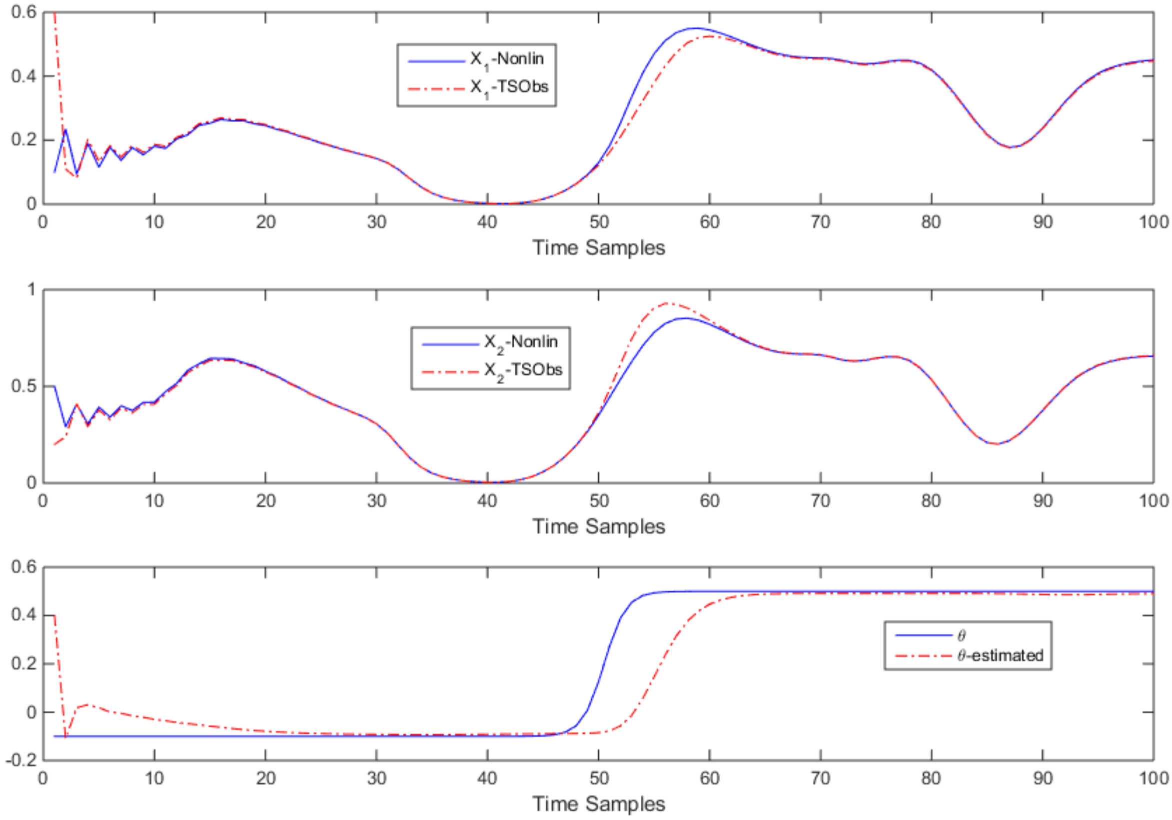

Figure 4.1: States and the parameter of the system and their estimates for Example 5

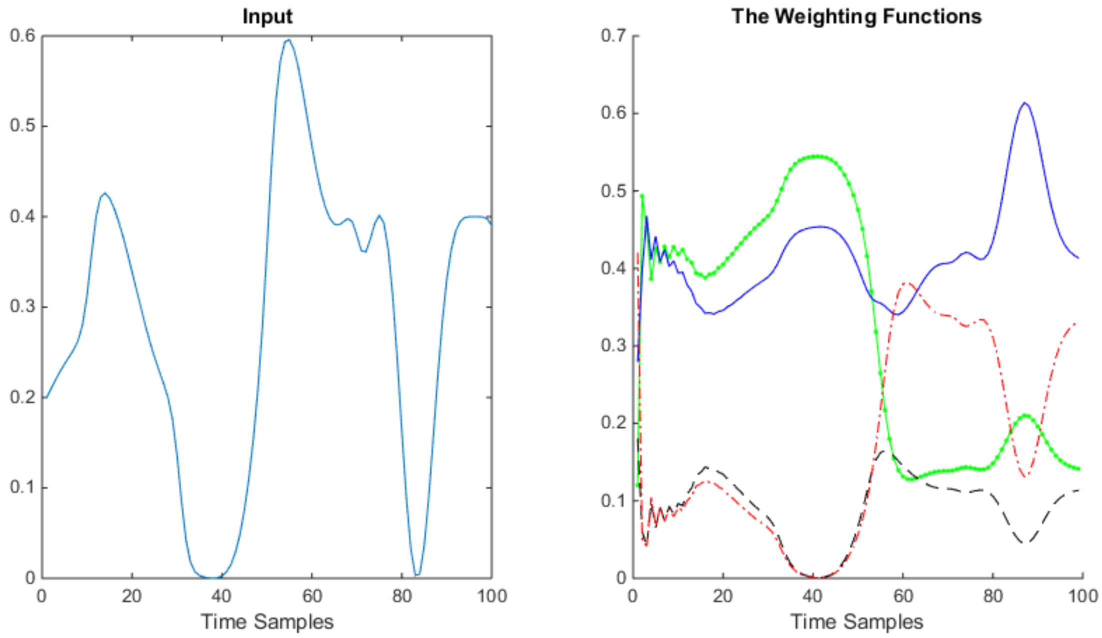

Figure 4.2: Input used and the weighting function trajectory generated for Example 5

Table 4.1: Model Parameters

Parameter	**Value**
a_0	0.5
b	0.4
c	0.4
d	2
T_s	1

In this model, we consider an uncertainty in the parameter a, that is,

$$a = a_0 + \theta \tag{4.70}$$

The parameters of the model used are given in Table 4.1. The unknown parameter θ is constant; however, for the observer design purposes we assume it to be known and in the range,

$$\theta \in [-0.3, 0.3] \tag{4.71}$$

We choose the premise variables,

$$z_{1,k} = u_k, \qquad \text{and,} \qquad z_{2,k} = \frac{x_{1,k}}{x_{2,k} + b} \tag{4.72}$$

It is evident that $z_{2,k}$ depends on the unmeasured state $x_{2,k}$ making it unmeasured. Assuming a range of values for $u_k \in [0, 0.4]$, $x_{1,k} \in [0.01, 6]$, and $x_{2,k} \in [0.01, 3]$, we get the range of the premise variables as,

$$z_{1,k} \in [0, 0.4], \qquad z_{2,k} \in [0.003, 14.63] \tag{4.73}$$

With these parameters, we get the model

$$x_{k+1} = \sum_{i=1}^{8} h_i(z_k, \theta_k) \left[A_i x_k + B_i u_k\right] \tag{4.74}$$

where $h_i(z_k, \theta)$ is obtained from the product of membership functions of $z_{1,k}$, $z_{2,k}$ and θ corresponding to the submodel i. The system matrices are given by,

$$A_1 = \begin{bmatrix} 1 & 5.9\times10^{-4} \\ 0 & 0.99 \end{bmatrix} A_2 = \begin{bmatrix} 1 & 0.0024 \\ 0 & 0.99 \end{bmatrix} A_3 = \begin{bmatrix} 1 & 2.92 \\ 0 & -0.17 \end{bmatrix} A_4 = \begin{bmatrix} 1 & 11.7 \\ 0 & -3.68 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0.6 & 5.9\times10^{-4} \\ 0 & 0.6 \end{bmatrix} A_6 = \begin{bmatrix} 0.6 & 0.0024 \\ 0 & 0.6 \end{bmatrix} A_7 = \begin{bmatrix} 0.6 & 2.93 \\ 0 & -0.57 \end{bmatrix} A_8 = \begin{bmatrix} 0.6 & 11.7 \\ 0 & -4.1 \end{bmatrix}$$

$$\text{and,} \qquad B_i = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ \forall i$$

MATLAB with Yalmip [50] interface and LMIlab toolbox were used to solve the LMI conditions in (4.63). Further, as noted in Remark 12, the value of Q_e was chosen as

$$Q_e = \begin{bmatrix} 0.001 I_{n_x} & 0 \\ 0 & 0.1 I_{n_\theta} \end{bmatrix}$$

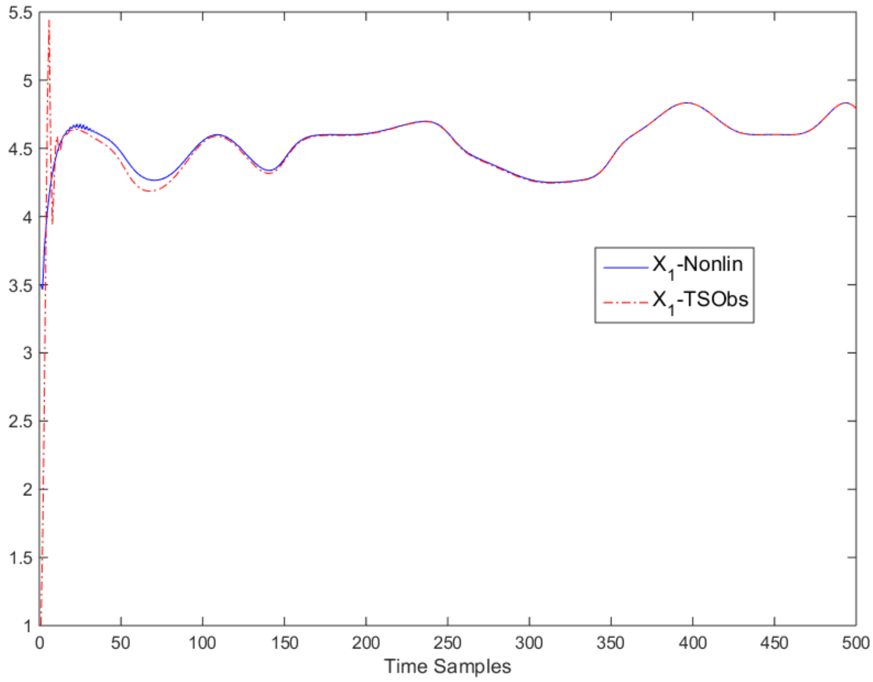

Figure 4.3: Estimation of $x_{1,k}$ (Example 6)

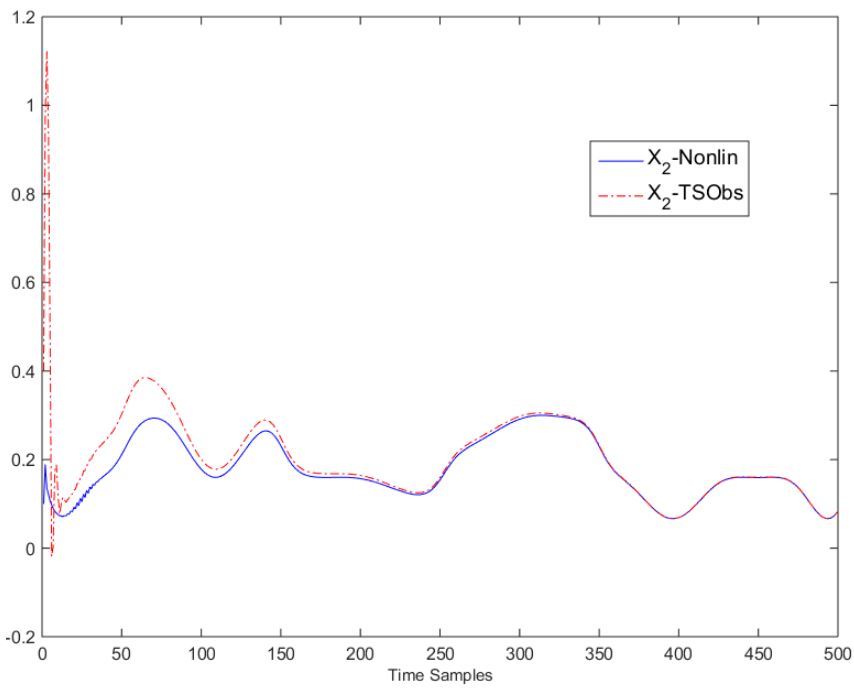

Figure 4.4: Estimation of $x_{2,k}$ (Example 6)

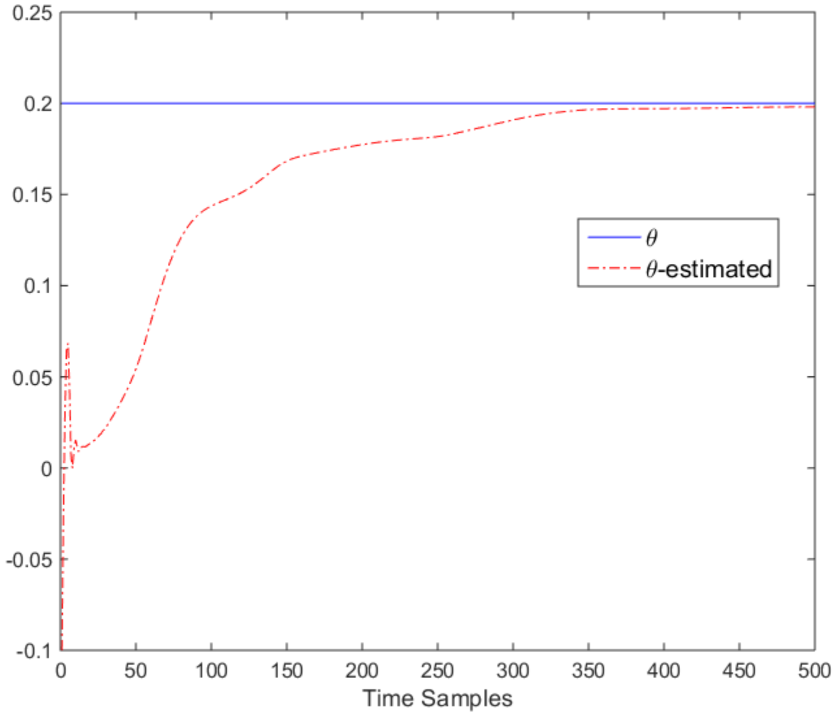

Figure 4.5: Estimation of θ_k (Example 6)

It is to be noted that there are a number of variables to be determined by the LMI solver. This could be reduced by fixing some of the parameters as was done in Sec. 2.2.2. For this example, the values for $\Gamma_2^x = \Gamma_2^u = 0.1$ and $\lambda_i = 0.001$, $\forall i = 1, 2, 3, 4$ were chosen. This significantly reduces the computational complexity of the problem. With the above simplifications and added conditions, the following observer gain values were obtained,

$$L_1 = \begin{bmatrix} 0.23 \\ 0.24 \end{bmatrix} L_2 = \begin{bmatrix} 0.23 \\ 0.24 \end{bmatrix} L_3 = \begin{bmatrix} 0.31 \\ 0.21 \end{bmatrix} L_4 = \begin{bmatrix} 0.61 \\ 0.09 \end{bmatrix}$$
$$L_5 = \begin{bmatrix} -0.41 \\ 0.30 \end{bmatrix} L_6 = \begin{bmatrix} -0.41 \\ 0.30 \end{bmatrix} L_7 = \begin{bmatrix} -0.41 \\ 0.30 \end{bmatrix} L_8 = \begin{bmatrix} -0.27 \\ 0.24 \end{bmatrix}$$

and $K_i = 0.03$, $\forall i$. The state estimation results are shown in the Fig 4.3 and 4.4 (here 'Nonlin' and 'TSObs' refer to the results from nonlinear system T-S observer respectively). The estimation of the unknown parameter is given in Fig 4.5. Further, the input used for the simulation is shown in the Fig 4.6. To illustrate the nonlinearity of the model, the weighting function trajectories over the simulation period $h_i(\hat{x}, \hat{\theta})$ is given in the Fig 4.7.

Remark 14. *The transient response characteristics of polytopic observers have some known issues. This concerns not explicitly taking into account the known bound for the states in the observer design. This could cause the transient response of estimated states (and hence the parameters) to go out of the bounds and lead to jerks in the response as is seen in Fig 4.5. This could be partially mitigated through approaches such as that described in [72].*

The results presented in this chapter were published as [73] and [74]

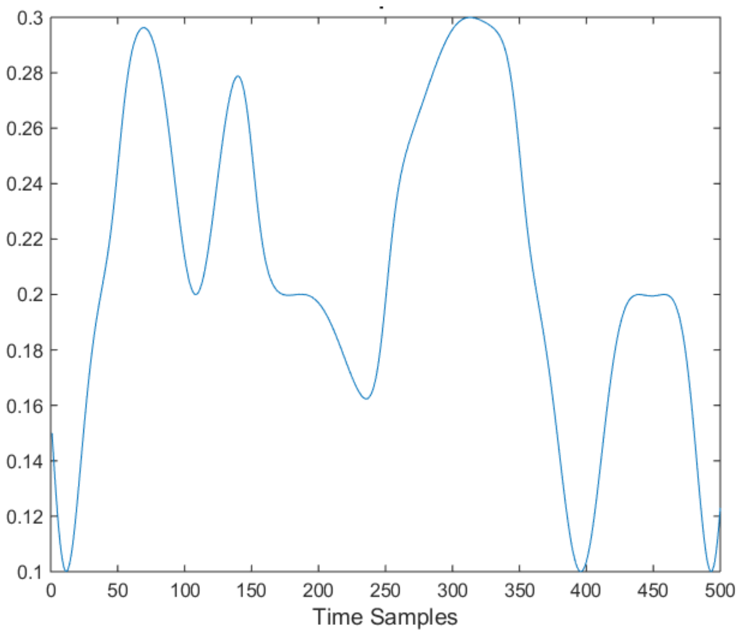

Figure 4.6: Input used for simulation (Example 6)

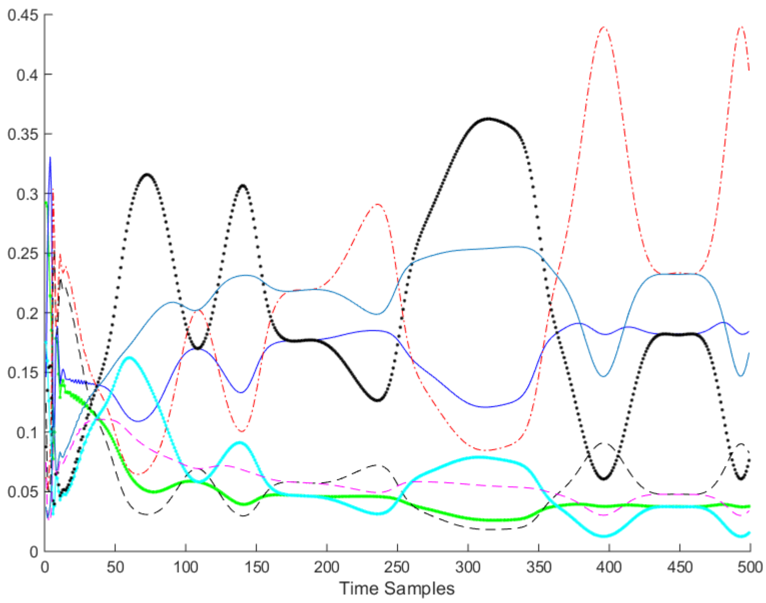

Figure 4.7: Weighting functions of the submodels (Example 6)

Chapter 5

A Finite Memory Observer for decoupled state and parameter estimation

This short chapter provides a bridge between the preceding chapters and the following ones. The work in this chapter stems from the attempt to decouple the state and parameter estimation problem, that is, first to eliminate the states and estimate the parameters, and then use the estimated parameters for state estimation. In this chapter, an outline of this decoupling realized using a parity-space technique using a Finite Memory Observer (FMO) strategy is given. The discussion is brief because the open questions and challenges during this attempt lead to a more detailed work in the parameter identifiability problem in the next chapter.

Contents

5.1 Introduction and motivation

In the preceding chapters, it was observed that the choice of the observer structure plays a significant role in the overall complexity of the design process. In particular, the choice of the observer structure for the parameter estimation was tricky and plays a role in the ability to show the convergence of the estimation. This is further complicated in the discrete-time models. One direction to mitigate this is to consider decoupling the estimation of states and the parameters. This would make it easier to prove the convergence. For example, the recent work on state and parameter estimation for switching systems in [75] decouples the LMI conditions for the design of states and parameter estimation.

Algebraic approaches offer an alternative option to realize the decoupling. Algebraic observers have been proposed in continuous-time case [76] which make use of the ability to obtain stable higher order derivatives. For the discrete-time case, the notion of FMO provides a more streamlined way to handle observer design in the algebraic framework. The idea of using a fixed memory for estimation rather than the infinite past stems from the possibility that aspects such as faults and related phenomenon are more visible in recent data. One of the early works in this direction is in [77] where the authors propose a continuous finite memory observer as an alternative to Luenberger and Kalman filter approaches. They were then applied in fault diagnosis applications in subseauent works such as in [78] and [79]. In the recent times, this has also been extended to fault detection applications in linear time varying [80] and polytopic models [81]. In [80], a detailed study on the use of FMO for fault diagnosis is given, including the case when there is noise. One recent work that follows this idea is proposed in [82] for polytopic LPV models. The parameters are estimated using an iterative constrained optimization in the first step and then the states are estimated using a robust polytopic observer.

Parity-space approach (or Chow-Wilsky approach) introduced in [16] was directed at fault detection in the context of robustness to model uncertainties. Analytical redundancy relations of the underlying system are derived and used to design residual generation methods which in-turn could aid in minimizing the effects of the uncertainties. Parity-space approach can be realized in several ways and polynomial null-space basis computation is one of them (see [83], [84]). The combination of parity-space and null-space could also be used in case of parameter estimation and is attempted in this chapter.

5.2 Problem formulation

The quasi-LPV models of interest are of the form,

$$\begin{aligned} x_{k+1} &= A(\rho_k, \theta)x_k + B(\rho_k, \theta)u_k \\ y_k &= Cx_k \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$, $\theta \in \mathbb{R}^q$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and ρ_k is the premise variable composed of one of the system variables or an extraneous signal. The following are the assumptions associated with this model:

Assumption 1. *The output equations are linear*

The discussion in this chapter considers the output equations to be linear. However, the results are easily extended to nonlinear outputs as long as they satisfy the other assumptions.

Assumption 2. *The premise variables ρ_k are composed of only measured or known variables*

That is, if the premise variable is one of the system variables (or a function of system variables), it has to be either known or measured or a combination of these variables.

Assumption 3. *The parameters θ are constant*

The underlying procedure assumes that the parameters are constants. However, as would be illustrated, the FMO strategy can be tuned to allow for time-varying parameters to be also estimated using the same framework.

Assumption 4. *The parametrization is affine in the system matrices. That is, the system matrices are of the form,*

$$A(\rho_k, \theta) = A_0(\rho_k) + \sum_{i=1}^{q} \bar{A}_i(\rho_k)\theta_i, \qquad B(\rho_k, \theta) = B_0(\rho_k) + \sum_{i=1}^{q} \bar{B}_i(\rho_k)\theta_i \tag{5.2}$$

Further, the proposed approach also works for cases where the parameters appear affine in the state equation. That is, the state equation could be of the form,

$$x_{k+1} = A(\rho_k, \theta)x_k + B(\rho_k, \theta)u_k + F(\rho_k)\theta$$

However, this is not explicitly discussed.

Assumption 5. *The system model is noise-free*

This assumption stems from the system inversion performed to eliminate states and estimate the parameters. However, as FMO strategy could be equipped to handle noise, it is possible to develop strategies to handle noise, though they are not rigorously discussed here. With these assumptions in place, the aim here is the estimation of states and parameters in the following steps:

1. Eliminate states using the parity-space approach realized through null-space computation of appropriate matrices.
2. Estimate the parameters.
3. Estimate the states.

The contribution in this chapter focuses on the first two steps of eliminating the states and estimating the parameters. The state estimation step could be realized using a number of strategies, including a polytopic observer design as in [82]. In this chapter, a least-squares solution for the state estimates is given. It can be noted that, if feasible, analytical redundancy equations that rely only on inputs and outputs can be obtained to extend this approach to fault detection applications.

5.3 The elimination-estimation process

In this section, the steps involved in the decoupled estimation of the states and parameters are discussed. To make the visualization simple, the system matrices in (5.1) are simplified by the following notations:

$$A_k^\theta \triangleq A(\rho_k, \theta), \qquad B_k^\theta \triangleq B(\rho_k, \theta)$$

Formulation of algebraic equations Given the system model of the form (5.1), the following well-known algebraic equations in the matrix form could be written,

$$\mathbb{Y}_k = \mathbb{O}_k^\theta x_k + \mathbb{G}_k^\theta \mathbb{U}_k \tag{5.3}$$

where

$$\mathbb{Y}_k = \begin{bmatrix} y_k \\ y_{k+1} \\ y_{k+2} \\ \vdots \\ y_{k+w} \end{bmatrix}, \qquad \mathbb{U}_k = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+w-1} \end{bmatrix}, \qquad \mathbb{O}_k^\theta = \begin{bmatrix} C \\ CA_k^\theta \\ CA_{k+1}^\theta A_k^\theta \\ \vdots \\ CA_{k+w-1}^\theta \cdots A_k^\theta \end{bmatrix},$$

$$\mathbb{G}_k^\theta = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ CB_k^\theta & \mathbf{0} & \cdots & \mathbf{0} \\ CA_{k+1}^\theta B_k^\theta & CB_{k+1}^\theta & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ CA_{k+w-1}^\theta B_k^\theta & CA_{k+w-2}^\theta \cdots A_{k+2}^\theta B_{k+1}^\theta & \cdots & CB_{k+w-1}^\theta \end{bmatrix} \tag{5.4}$$

The index w is indicative of the size of the window $(w+1)$ to be chosen in formulating the algebraic equations. In the FMO formulation, this indicates the size of the finite memory.

Null-space computation To eliminate the states in (5.3), the left null-space of $\mathbb{O}_k^\theta$ should be computed. That is, to find Ω_k^θ such that,

$$(\Omega_k^\theta)^T \mathbb{O}_k^\theta = \mathbf{0} \tag{5.5}$$

The null-space computation can be realized in two ways:

- Symbolic computation: Symbolic computation can be realized through packages such as the symbolic computation toolbox in MATLAB. This would help in obtaining the null-space and the subsequent steps through symbolic values for known/measured variables. The consequence is that, all the steps preceding the parameter estimation step would be performed only once.

- Numerical computation: The entries of the matrix $\mathbb{O}_k^\theta$ are composed of known/measured variables (except the parameters), and hence it is possible to compute the null-space numerically. To achieve this, the problem is posed as the computation of the null-space basis of a polynomial matrix. The matrix $\mathbb{O}_k^\theta$ is a polynomial in θ because of the affine parametrization (Assumption 4). This can be better explained with a scalar parameter θ_1 and the system as follows:

$$\begin{aligned} x_{k+1} &= \left(A_0(\rho_k) + \theta_1 \bar{A}_1(\rho_k)\right) x_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

or in a simplified representation,

$$\begin{aligned} x_{k+1} &= \left(A_k + \theta_1 \bar{A}_k\right) x_k + Bu_k \\ y_k &= Cx_k \end{aligned}$$

For this system model with for example $w = 2$, the matrix is given by,

$$\begin{aligned} \mathbb{O}_k^\theta &= \begin{bmatrix} C \\ C(A_k + \theta_1 \bar{A}_k) \\ C(A_{k+1} + \theta_1 \bar{A}_{k+1})(A_k + \theta_1 \bar{A}_k) \end{bmatrix} \\ &= \begin{bmatrix} C \\ CA_k \\ CA_{k+1}A_k \end{bmatrix} + \theta_1 \begin{bmatrix} \mathbf{0} \\ C\bar{A}_k \\ C(A_{k+1}\bar{A}_k + \bar{A}_{k+1}A_k) \end{bmatrix} + \theta_1^2 \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ C\bar{A}_{k+1}\bar{A}_k \end{bmatrix} \\ &= \mathbb{O}_{0,k} + \theta_1 \mathbb{O}_{1,k} + \theta_1^2 \mathbb{O}_{2,k} \end{aligned}$$

That is, for a length of window size w, the observability matrix $\mathbb{O}_k^\theta$ is parametrized as:

$$\mathbb{O}_k^\theta = \mathbb{O}_{0,k} + \theta_1 \mathbb{O}_{1,k} + \cdots + \theta_1^{w-1} \mathbb{O}_{w-1,k} \tag{5.6}$$

The null-space can also be parametrized, for some s as,

$$\Omega_k^\theta = \Omega_{0,k} + \theta_1 \Omega_{1,k} + \theta_1^2 \Omega_{2,k} + \cdots + \theta_1^s \Omega_{s,k} \tag{5.7}$$

For the condition in (5.5), the coefficient of θ_1 on either side of the equation can be evaluated. These equations can be gathered into a matrix form as,

$$\begin{bmatrix} \Omega_{0,k}^T & \Omega_{1,k}^T & \Omega_{2,k}^T & \cdots & \Omega_{s,k}^T \end{bmatrix} \mathbb{P} = 0 \tag{5.8}$$

where,

$$\mathbb{P} = \begin{bmatrix} \mathbb{O}_{0,k} & \mathbb{O}_{1,k} & \mathbb{O}_{2,k} & \cdots & \mathbb{O}_{w-1,k} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbb{O}_{0,k} & \mathbb{O}_{1,k} & \cdots & \mathbb{O}_{w-2,k} & \mathbb{O}_{w-1,k} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbb{O}_{0,k} & \cdots & \mathbb{O}_{w-3,k} & \mathbb{O}_{w-2,k} & \mathbb{O}_{w-1,k} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

The left null-space of the matrix $\mathbb{P}$ would give the desired null-space. The number of row blocks in $\mathbb{P}$ would be equal to $s+1$. One aspect that is not clear here is the choice of s. It is not possible to predict *a priori* the degree of the polynomial of the null-space as it will depend entirely on the specific model. An iterative approach followed in [85], [86] solves this problem and computes the polynomial basis for the null-space. It is important to note that, this polynomial null-space computation however, has not been extended to multivariate case. Hence their applicability to the present problem is limited.

Obtaining a set of Input-Output-Parameter (I-O-P) equations Once the null-space is computed, the states are eliminated to obtain,

$$(\Omega_k^\theta)^T \left(\mathbb{Y}_k - \mathbb{G}_k^\theta \mathbb{U}_k \right) = 0 \tag{5.9}$$

which are a set of equations in the known and measured variables along with the parameter θ such that,

$$\Psi(y_k, y_{k+1}, \cdots, y_{k+w}, u_k, \cdots, u_{k+w}, \theta) = 0 \tag{5.10}$$

The dimension of $\Psi(.)$ would depend on the chosen size of the finite memory, w and the observability properties of the model.

Estimating the parameters Depending upon the structure of I-O-P equations, $\Psi(.)$, it can be analytically solved or an optimization solution could be found.

- Solving for the parameters: If the I-O-P equations have a polynomial structure in θ, then the solution could be obtained by either directly solving through algebraic operations or more formally, by computing the Gröbner basis using Buchberger algorithm [87]. That is, given a set of polynomial equations in θ,

  $$\Psi = \{\psi_1, \psi_2, \cdots, \psi_r\}$$

  where each ψ_i represents an equation of the form,

  $$\psi_i(y_k, \cdots, y_{k+w}, u_k, \cdots, u_{k+w}, \theta) = 0$$

  the Gröbner basis of these equations would be the set of all the solutions (if they exist). If the solution is not unique (but finitely many), specific conditions on the parameter values could be put to choose one of the solutions. It is to be noted that the Gröbner basis approach assumes that the I-O-P equations have a polynomial structure.

- Nonlinear optimization: Formulating a nonlinear optimization problem is an alternative when the direct solving approach fails. While this approach is complex to solve and prone to local solutions, additional information about parameters could be introduced as constraints to obtain the right solution. This option has not been evaluated during this investigation.

Estimating the states Once an estimate of the parameter $\hat{\theta}$ is obtained, the next step is to obtain an estimate for the states of the system. Considering the least squares estimation of the states based on (5.3) would give,

$$\hat{x}_k = \left((\mathbb{O}_k^{\hat{\theta}})^T \mathbb{O}_k^{\hat{\theta}} \right)^{-1} (\mathbb{O}_k^{\hat{\theta}})^T \left(\mathbb{Y}_k - \mathbb{G}_k^{\hat{\theta}} \mathbb{U}_k \right) \tag{5.11}$$

assuming that the inverse exists, that is, the matrix $\mathbb{O}_k^{\hat{\theta}}$ has full row rank. In this context, if there is noise in the state or measurement equation, the estimation strategy followed in [80] could be adapted.

5.4 The FMO strategy

A finite memory observer strategy typically falls into two categories [80]:

- Filtering FMO: if the measurement window is from $k - w_1$ to k and the attempt is to estimate the states x_k. The term filtering refers to the fact that past values are used to predict/refine the current estimates.

- Predicting FMO: if the measurement window is from $k - w_1$ and up to $k + w_2$ where $w_2 > 0$, then, the observer is in prediction form. That is, for $w_2 = 1$, the observer is a 1-step prediction. The idea is that in real-time, k corresponds to the current instant and estimating x_{k+w_2} corresponds to future values that are to be predicted.

These terminologies effectively stem from the choice of the start of the measurement window. In this chapter, there is an abuse of notation in this regard as well as the strategy followed and outlined as follows:

- The measurement/input data used is from the sample instants k to $k + w$

- The estimation of the parameters and states corresponding to that of k is achieved with a delay of w samples

- The estimation happens over a sliding window of measurements

Aligning the proposed method to the formal categories is possible, though is not attempted. This FMO strategy would be useful to detect both constant parameters and cases where the parameters model an abrupt fault scenario, that is, the parameter remains constant and changes abruptly and then remains constant again. This is illustrated in Example 9 in the next section.

Choice of the window size w All the preceding discussions assume that the window size w will guarantee that the parameters are estimated within this window. This considers two aspects:

- The parameters can be estimated with measurement data from k to $k+w$: this relates to the parameter identifiability property.
- The window size w can be determined *a priori*: closely related to the observability index of the system model in (5.1).

These are discussed further in the next section and in depth in the next chapter. To paraphrase, the choice of w is equal to the system dimension n for rational and control-affine analytical models. However, for other systems there are no proof to show a relationship between the choice of w and the dimensions of system variables.

5.5 Illustrative examples

In this section, a set of simple examples is given to illustrate the steps discussed in the previous sections.

Example 7. *Consider the following model with the scheduling variables as $\rho_{1,k} \triangleq y_k$ and $\rho_{2,k} \triangleq u_k$,*

$$x_{k+1} = \begin{bmatrix} \rho_{1,k} & -0.5 \\ 0.5\theta & \rho_{2,k} \end{bmatrix} x_k + \begin{bmatrix} \theta \\ 0 \end{bmatrix} u_k, \qquad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

The matrices in (5.4) *are given (with $w=3$) as,*

$$\mathbb{O}_k^\theta = \begin{bmatrix} 1 & 0 \\ y_k & -0.5 \\ y_k y_{k+1} - 0.25\theta & -0.5(u_k + y_{k+1}) \end{bmatrix}, \qquad \mathbb{G}_k^\theta = \begin{bmatrix} 0 & 0 \\ \theta & 0 \\ \theta y_{k+1} & \theta \end{bmatrix}$$

And the corresponding null-space and the I-O-P equations are given by,

$$\Omega_k^\theta = \begin{bmatrix} 0.25\theta + u_k y_k \\ -u_k - y_{k+1} \\ 1 \end{bmatrix}$$

$$y_{k+2} - y_{k+1}(u_k + y_{k+1}) - \theta u_{k+1} + y_k(0.25\theta + u_k y_k) + \theta u_k^2 = 0$$

In this example, given that there is a single parameter to be estimated, the I-O-P equation is rewritten to obtain $\hat{\theta}$ as a function of known and measured variables. For some multi-variable cases the solution could be found symbolically to represent all the parameters as a function of known and measured variables (e.g., *solve* in MATLAB). The parameter thus estimated $\hat{\theta}$ is assigned to the system matrices in (5.3) to obtain an estimate for the state,

$$\hat{x}_k = \left((\mathbb{O}_k^{\hat{\theta}})^T (\mathbb{O}_k^{\hat{\theta}}) \right)^{-1} (\mathbb{O}_k^{\hat{\theta}})^T \left(Y_k - \mathbb{G}_k^{\hat{\theta}} U_k \right) \tag{5.12}$$

The existence of the solution for $\hat{\theta}$ and the inverse $\left((\mathbb{O}_k^{\hat{\theta}})^T (\mathbb{O}_k^{\hat{\theta}}) \right)^{-1}$ depend on the numerical values of the inputs and outputs and is equivalent to the persistence of excitation condition used in adaptive observer design.

Example 8. *This example considers multiple unknown parameters with multiple outputs.*

$$\begin{aligned} x_{1,k+1} &= 0.2x_{1,k} + \theta_2 x_{4,k} + u_k, & y_{1,k} &= x_{1,k} \\ x_{2,k+1} &= \theta_1 x_{2,k} - 0.5x_{3,k}, & y_{2,k} &= x_{2,k} \\ x_{3,k+1} &= u_k x_{2,k} + 0.2x_{4,k} \\ x_{4,k+1} &= 0.5x_{3,k} + 0.8x_{4,k} \end{aligned}$$

A quasi-LPV model with the scheduling variable $\rho_k \triangleq u_k$ is,

$$x_{k+1} = \begin{bmatrix} 0.2 & 0 & 0 & \theta_2 \\ 0 & \theta_1 & -0.5 & 0 \\ 0 & \rho_k & 0 & 0.2 \\ 0 & 0 & 0.5 & 0.8 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_k$$
$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_k$$

For $w = 3$ the null-space obtained symbolically is given by,

$$\Omega_k^\theta = \begin{bmatrix} 0.16 & -0.02/theta_2 \\ -\theta_1\theta_2 & 0.5u_k \\ -1 & 0.1/\theta_2 \\ \theta_2 & -\theta_1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The I-O-P equations are given by,

$$0.8u_k - u_{k+1} + 0.16y_{1,k} - y_{1,k+1} + y_{1,k+2} + \theta_2 y_{2,k+1} - \theta_1\theta_2 y_{2,k} = 0$$
$$y_{2,k+2} - \theta_1 y_{2,k+1} + 0.5u_k y_{2,k} - \frac{y_{1,k}}{50\theta_2} - 0.1\frac{u_k - y_{1,k+1}}{\theta_2} = 0$$

The parameter estimate $\hat{\theta}$ can be obtained by solving the two equations.

Example 9. *In this example, a case where the parameter only appears in the input matrix in the LPV form is given. If the parameter is considered time-varying in this case, this models a multiplicative actuator fault scenario. Considering the same scheduling variables as $\rho_{1,k} \triangleq y_k$ and $\rho_{2,k} \triangleq u_k$ we have,*

$$x_{k+1} = \begin{bmatrix} 0.2 & -0.5 & 0 \\ 0 & \rho_{1,k} & 0.5 \\ 0 & 0.5\rho_{2,k} & -0.5 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1+\theta \\ 0 \end{bmatrix} u_k$$
$$y_k = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x_k$$

Given that the state transmission matrix A doesn't have θ appearing, the null-space would not contain θ. Symbolic computation leads to the following null-space and the I-O-P equation,

$$\Omega_k^\theta = \begin{bmatrix} 0.05u_k + 0.1y_k \\ 0.2y_{k+1} - 0.5y_k - 0.25u_k - 0.1 \\ 0.3 - y_{k+1} \\ 1 \end{bmatrix}$$

$$y_{k+3} + \frac{1+\theta}{4}(u_k + 2u_{k+1}) - y_{k+1}(\frac{u_k}{4} + \frac{y_k}{2} - \frac{y_{k+1}}{5} + 0.1) - y_{k+2}(y_{k+1} - 0.3) + y_k(\frac{u_k}{20} + 0.1y_k) = 0$$

For this example, we show the results of the estimation of both the parameter and the states. MATLAB computing environment was used to implement with the aid of the symbolic computation toolbox. In Fig. 5.1, the parameter estimation is illustrated for a case where the parameter varies in a waytypical in abtrupt fault scenarios. There is a delay because it takes

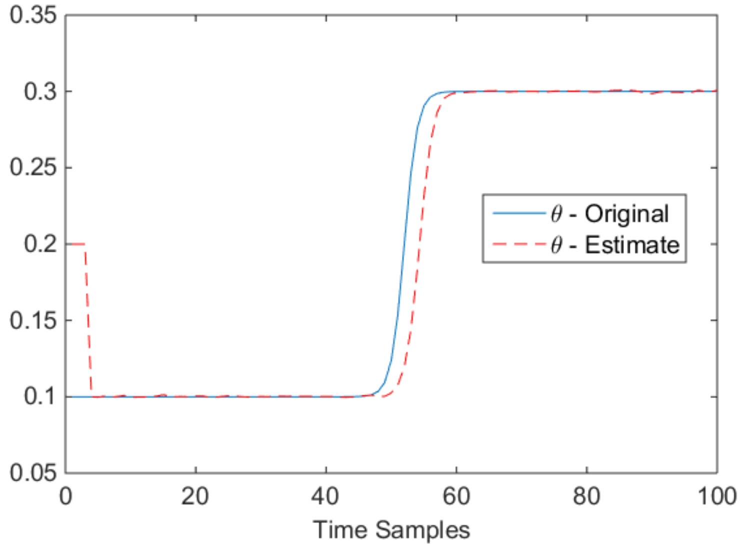

Figure 5.1: True and estimated values of the parameter

$w = 4$ measurements to be available before an accurate estimation can be achieved. In Fig. 5.2, the errors of state estimation are illustrated as percentage of their values. We accounted for the delay to calculate these errors since the states have a dynamic nature. The errors on the states x_1 and x_2 are less than 1% and hence appear to be almost zero. The spike in the state estimation error for x_3 coincides with the transition on the parameter θ.

These results were communicated in [88].

5.6 Discussion

The development of the proposed FMO for decoupled state and parameter estimation was hampered by several factors. The efforts to solve these bottlenecks revealed the opportunities for algorithms to evaluate parameter identifiability (more in Chapter 6), and diverted efforts away from FMO. The key factors responsible for this action are outlined in this section.

Choice of window size w The choice of window size w plays an important role in any FMO design, more so for this chapter, because it should guarantee that θ can be estimated. The decoupling strategy eliminates the states and hence the window size is directly related to the observability index of the system model. That is, if the window size w is equal to the observability index of the model, then $\mathbb{Y}_k$ in (5.3) has sufficient data to guarantee that the null-space Ω_k^θ exists. For some specific types of nonlinear models, the system dimension n forms the upper bound for the observability index. For MIMO system, this upper bound is highly likely to be conservative as each output's observability index would add up to a lot more than n in general. So the window size needs to be determined by checking the definition (see [89] or Chapter 6 for more details).

Parameter identifiability and obtaining I-O-P equations The estimation of parameters assume that the unknown parameters are identifiable. Further, to estimate these parameters, it

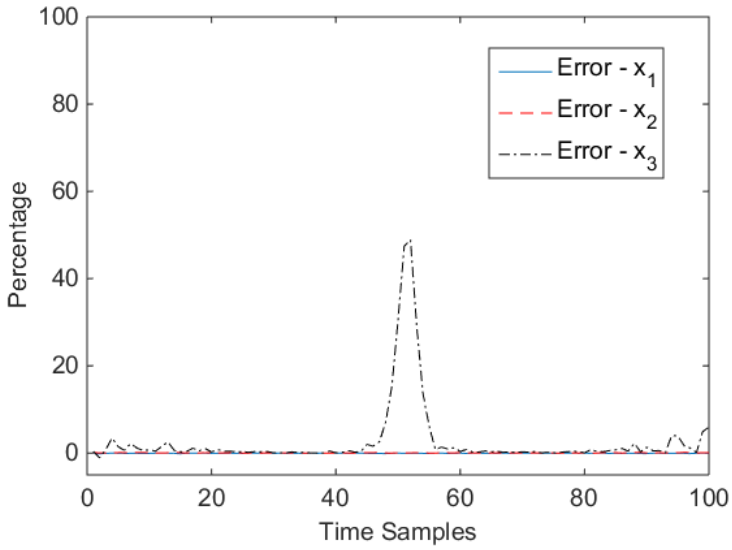

Figure 5.2: State estimation errors

should also be possible to write them in I-O-P form so that subsequent steps of Gröbner basis or optimization problem can be formulated and solved. Search for answers to this question lead to the works discussed in Chapter 6.

Solving for the parameters The parameter estimation step is computationally expensive (except for some simple cases) for real-time implementation. The illustrative examples used MATLAB's *solve* command which do not scale well. The Gröbner basis approach is computationally expensive to apply symbolically, though a one-time analysis to obtain θ as a function of the input-output values is possible (if the I-O-P equations are lower degree polynomials). Numerical assignment of input-output values leads to a much simpler set of polynomials to solve using Gröbner basis. But performing this for every sliding window is computationally expensive. A completely different take on this could be to use Ritt's algorithm as given in [90] to obtain θ as a function of the inputs and outputs[8]. For system models where Gröbner basis is unsuitable, an optimization problem is suggested to be formulated. This again is computationally expensive.

Time-varying parameter estimation The FMO strategy, under certain circumstances allows one to estimate the unknown parameters that are time-varying. The main assumptions here are:

- The parameters are constant for a period of time (at least w samples)
- There is no noise on the measurements

A brief of this strategy could be described as follows: Based on the assumption, the parameter is constant from k to $k+w$. This would mean that, for this finite memory window, there is only one unknown, θ_k. Once θ_k it is estimated, for the next sliding window (from $k+1$ to $k+w+1$), there will be at most one unknown parameter, θ_{k+w+1}. This is because $\theta_{k+i} = \theta_k, \ \forall \ i = 0, \cdots, w$. This process could be continued to estimate any time-varying parameter as long as it varies slower than the sampling rate.

[8]The discrete-time validity of the results in [90] can be referred to in [91]

To summarize the ambiguity in the choice of window size w and the computational effort required for real-time estimation, directed the effort towards identifiability analysis, which is an *a priori* analysis. However, the above discussion contains hindsights arrived at after the work in Chapter 6 and hence opportunities exists to extend the works in this chapter and discussed in Chapter 7.

Chapter 6

Parameter identifiability

Parameter identifiability is an implicit condition for the design of adaptive observers. Further, the design problem of sensor placement for fault detection, degradation estimation, etc., rely on the analysis of parameter identifiability of the model. In this chapter, a parity-space [16] based procedure for the parameter identifiability analysis of quasi-LPV models is proposed. The parameters are assumed to be constant and the parametrization affine.

Contents

6.1 Motivation

Broadly, the study of parameter identifiability is motivated by the need to have a well-posed problem in several applications. For instance, when the problem of parameter estimation is transformed into an optimization problem of minimizing a cost function (e.g., [82]), there is an expectation that there is a unique set of parameters which satisfy the input, output values available from the experiment. If multiple parameter sets satisfy the input-output data, the next interest will be if at least locally, the parameter set is unique. It is in this respect the distinguishability property of parameters is defined and forms the basis of parameter identifiability analysis. Consider the general nonlinear system model of the form,

$$\Sigma_\theta : \begin{cases} \dot{x}(t) = f(x(t), u(t), \theta) & (6.1a) \\ y(t) = h(x(t), u(t), \theta) & (6.1b) \end{cases}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and the constant parameters $\theta \in \mathbb{R}^q$. Distinguishability property of a model structure refers to the following [90],

$$\tilde{y}(t|\theta') \equiv \tilde{y}(t|\theta'') \Rightarrow \theta' = \theta'' \tag{6.2}$$

where, $\tilde{y}$ refers to a set of measurements of the output (6.1b) and the presence of θ'/θ'' indicate the parameter vector that generated the output. In the following, the importance of the distinguishability property in the context of the research problems of interest in Sec. 1.3.2 is outlined. It is important to note that a priori parameter identifiability assumes error-free model and noise-free data and thus is a necessary and not a sufficient condition for the existence of a solution.

Consider the problem of sensor placement for maintenance activities (such as degradation estimation, fault estimation, etc.). Maintenance activities are rare and typically involve gathering data by deploying temporary sensors (e.g., hand-held monitors). Consequently, a finite amount of data is available to estimate relevant parameters that are indicative of the underlying phenomena. This estimation procedure requires that the available input-output data admit a unique solution for the set of unknown parameters, at least within a known range of parameter values. This constraint is satisfied by local parameter identifiability of the model.

The second situation is when measurements from permanently installed sensors are used to perform the same task. Adaptive observers (as discussed in earlier chapters), or Kalman filter like observers provide an alternative. In the latter case, an extended model is formed by augmenting the unknown parameters to the state vector. To understand the effect of identifiability properties on the combined state and parameter estimation, we refer to the preliminary analysis in [92]. The authors conceive a definition of combined state and parameter identifiability (CSPI) that is valid for autonomous single-input-single-output (SISO) linear systems. Then, they provide necessary and sufficient conditions for non-identifiability of the models of different orders. Consider a linear model of second order,

$$\dot{x}(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \qquad x(0) = x_0$$

For this autonomous model, CSPI is defined assuming as if the system is subjected by an impulse input through a input transmission matrix $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. CSPI refers to the ability to estimate the parameters (a_{ij}) and two additional parameters (b_1, b_2) of the impulse input matrix. The authors perform this analysis by comparing the transfer function obtained from the state-space model to a generic SISO transfer function of 2nd order. We reproduce here three of the five conditions [9] given by the authors for non-identifiability of the above model:

[9]The other two conditions are important, but not relevant for this discussion.

1. $a_{12} = 0$
2. $a_{12} \neq 0$, $a_{21} = 0$ and a_{12} is unknown
3. $a_{12} \neq 0$, $a_{21} \neq 0$, and a_{12} and a_{21} are unknown

To understand the relationship between these results and parameter identifiability of the original model, consider an example of a second order linear model:

$$\dot{x}(t) = \begin{bmatrix} 1 & 1+\theta \\ 0 & 1 \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{6.3}$$

which satisfies the condition 2 as given above, implying that the model is unidentifiable in the CSPI sense.

This example could also be analyzed for parameter identifiability using some existing methods in the literature[10]. For example, consider the extended observability analysis approach proposed in [93] and that is valid locally for general nonlinear models of type (6.1). As the name refers to, extended observability analysis considers parameters as states without dynamics and augments them to the state vector. The identifiability is verified if the extended observability matrix (referred to as the observability-identifiability matrix [94]), obtained by computing the $n+q-1$ derivatives of the output has rank $n+q$. For the above example, the observability-Identifiability matrix of the extended state vector (where $\tilde{x} = \begin{bmatrix} x_1 & x_2 & \theta \end{bmatrix}^T$) is given by,

$$\frac{\partial \Phi}{\partial \tilde{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1+\theta & x_2 \\ 1 & 2(1+\theta) & 2x_2 \end{bmatrix}$$

which has a (structural) rank of 2, which is less that $n+q=3$ required for θ to be identifiable. So θ is not identifiable.

The motivation behind this illustration is to show the role of parameter identifiability in the joint estimation of states and parameters. This aspect is not generally discussed explicitly in the literature where they are masked by a host of other structural conditions[11]. However, it is not possible to say beyond the fact that parameter identifiability appears to be a necessary condition for the joint state and parameter identifiability. This is because, in [92] the authors provide two examples (II.2 and II.3) that are not CSPI. However, by the observability-identifiability matrix approach above, the example II.2 can be verified to be parameter identifiable (locally) whereas II.3 is not.

Having established the motivation to perform parameter identifiability, the chapter is arranged as follows to tackle the problem for quasi-LPV models: a select review of the relevant identifiability works is given in the Sec. 6.2. The identifiability problem is formulated in Sec. 6.3 where the relevant definitions and the assumptions are given. The components of the proposed method for continuous-time models is given and illustrated with examples in Sec. 6.4 while the discrete-time version is summarized with examples in Sec. 6.5. To realize the methods into a systematic algorithm, the challenges are discussed in Sec. 6.6 where systematic algorithms for specific scenarios are provided. A discussion in Sec. 6.7 covers an ensemble of details and limitations related to the proposed approach.

[10] A more detailed discussion of these methods follows in the next section

[11] For e.g., in [7] it appears as the positive real connection between output and the parameter. In [9], it is the existence of certain transformation.

6.2 Review of literature

As discussed earlier, parameter identifiability primarily drives to show the distinguishability property of the underlying model. Given the wide range of models encompassed by (6.1), an array of different identifiability definitions exists in the literature. These definitions and the corresponding characterizations vary due to several factors, including,

- The characteristics of the functions f and h (e.g., analytic, homogeneous, meromorphic)
- The characteristics of the inputs (sufficiently continuous/differentiable, piecewise continuous etc.)
- Initial (state) conditions assumptions (known/unknown/partial)
- Neighbourhood of identifiability (local/global, parameter/state+parameter)

There are also nuances associated with strong and weak notions of the identifiability. For example, in [95], the authors note that their identifiability definitions are weaker compared to that in [96] because the distinguishability in the latter work is for 'all the admissible inputs within an open dense subset', whereas it is just 'at least one input' as a sufficient condition in their own work[12]. For the purpose of this review, the following definition is considered (adapted from [97]). For the system model Σ_θ in (6.1), $\Sigma_{\theta_1} = \Sigma_{\theta_1 *}$ refers to the models indistinguishable by their outputs.

6.2.1 Identifiability of continuous-time models

The problem of structural identifiability was initially investigated in [98]. Several works have since been developed to characterize parameter identifiability. The basic premise is to validate the distinguishability property (6.2) of a given model. In a broad sense, the methods could be classified as those that perform:

1. Analysis of observables
2. Analysis of the system map

The term *observables* has been borrowed from [97] and roughly refers to the outputs and the parameter information embedded in them. That is, the first classification refers to verifying directly, whether the outputs (and inputs) provide a way to validate the distinguishability property in (6.2). Methods such as Taylor series approach and generating series approaches fall into this category. The second class of methods look at some specific properties of the system model to check for identifiability. Isomorphism based approaches or approaches that consider identifiability as an extended observability property belong to the second class. This classification is not strict as several methods cross over. For example, the implicit function theorem based approach and the differential algebraic tools based approach exploit the system model properties to eliminate the latent (state) variables and then analyze the observables.

[12] But at the same time, the authors in [95] consider piece-wise continuous inputs which are more representative in biological system identification.

Taylor series and generating series approach

Taylor series approaches were one of the first methods to be proposed for identifiability analysis in [99]. The idea is to consider the output as an analytic function of time and hence their derivatives should hold all the possible information about the unknown parameters θ. And the uniqueness of the Taylor series expansion of this function should be an indication of the identifiability of the system. With the coefficients obtained for the Taylor's series expansion, it can now be verified whether the they admit only one or multiple possible parameter values. If the test fails, however, it does not mean identifiability. More coefficients are to be computed and verified again.

The generating series approach [100] is conceptually similar to the Taylor series approach and is applicable on control affine models of the form,

$$\begin{aligned} \dot{x} &= f(x,\theta) + g(x,\theta)u, \qquad x(t_0) = x(\theta) \\ y &= h(x,\theta) \end{aligned} \tag{6.4}$$

Instead of using derivatives, Lie derivative expansion of the output functions along the vector fields f and g are computed. The coefficients of the output functions and their Lie derivatives are termed *exhaustive summary*, and their uniqueness validates the structural global identifiability of the model.

Both these approaches have the drawback of the lack of knowledge on the number of derivatives over which the identifiability can be verified. That is, the results provided by these methods are sufficient only. To mitigate some of these issues, iterative approaches have been suggested. The *identifiability tableau* proposed in [101] applied it for Taylor series approach. It simplifies the procedure to understand the identifiable parameters given the outputs and their derivatives. This iterative procedure was also paired with the generating series method in [97] to develop the genSSI MATLAB toolbox.

Isomorphism based approach

The isomorphism based approaches answer the distinguishability question by analyzing the relationship between state-space realizations. These methods exploit the fact that, under certain conditions, indistinguishable state space models have locally isomorphic state spaces. So the identifiability analysis works to show that state isomorphisms must have certain properties within the class of state space systems considered. This helps to parametrize indistinguishable state space models and if the isomorphism can be shown to be identity, then global identifiability is also verified.

One of the earliest works to analyze the identifiability property through the state-space realization theory is in [102]. For nonlinear systems, local state isomorphism based works were proposed in [103] which were followed up by works such as [104] for uncontrolled systems, [105] for homogeneous systems, and [95] for polynomial and rational models with provision for piece-wise continuous inputs. These approaches assume the system model is minimal, however minimality is not necessary for identifiability. One critique of this approach is the lack of systematic approach to verify identifiability. While this was mitigated to some extent in the systematic solution proposed in [106], this continues to remain an issue to use this method for complex nonlinear models [97]. For specific types of nonlinear models, realization theory associated with those models are used to turn the identifiability problem to verifying the rank of a certain matrix. For example, the identifiability results in [95] depend on the realization theory developed for rational and polynomial systems in [107]. Several results, including [106], use the local state isomorphism theorem from [108].

Differential algebraic approach

The potential of the differential algebraic tools in identifiability was discussed in the seminal paper [90]. The authors deploy the Ritt's algorithm to find the characteristic set of the polynomial ideal generated by the system model (which is assumed to be either polynomial or those models that could be written in such a form). Given the system model Σ_θ in (6.1b), the idea is to rewrite the state-space model as a set of polynomials,

$$g_i(\frac{d}{dt}, u, y, \theta) = 0 \tag{6.5}$$

along with $\dot{\theta} = 0$. Here $i = 1, 2, , \cdots$ and $\frac{d}{dt}$ stands for all the higher order derivatives of the inputs and outputs. Then after a careful choice of ranking of the variables, the characteristic set of the ideal generated by the set of polynomials (6.5) is obtained through Ritt's algorithm. The characteristic set has the following form,

$$\begin{gathered}\mathcal{A}_1(u, y), \cdots, \mathcal{A}_p(u, y), \mathcal{B}_1(u, y, \theta_1), \\ \mathcal{B}_2(u, y, \theta_1, \theta_2), \cdots, \mathcal{B}_q(u, y, \theta_1, \cdots, \theta_q), \\ \mathcal{C}_1(u, y, \theta, x), \cdots, \mathcal{C}_n(u, y, \theta, x)\end{gathered} \tag{6.6}$$

which leads to the following conclusions,

- Non identifiable: For some i one has $\mathcal{B}_i = \dot{\theta}_i$
- Globally identifiable: All $\mathcal{B}_i$ are of order 0 and degree 1 in θ_i
- Locally identifiable: All $\mathcal{B}_i$ are of order 0 and degree 1 in θ_i, some $\mathcal{B}_j$ is of degree greater than 1 in θ_j

The authors also show that the identifiability and the estimation of the parameters are guaranteed when for each parameter, a linear regression form,

$$P_i(\frac{d}{dt}, u, y) + \theta_i Q_i(\frac{d}{dt}, u, y) = 0 \tag{6.7}$$

is obtained. Implementation of differential algebra approach has different flavour. One approach, proposed in [109] uses a differential ring that does not consider θ, a strategy framed in [110]. For biological systems, this proves useful, as they have a huge number of parameters and hence if the differential ring includes them, would lead to enormous computational effort. This is elaborated in [111] which develops identifiability tools for biological systems.

Differential geometric approach

In [93], the authors discuss several identifiability definitions. These definitions are characterized as an extended observability problem, where the parameters are added to the state vector and the observability of the new model is evaluated. These results are local in nature, but have an intuitive appeal to it that it lead to the development of the toolbox STRIKE-GOLDD [112].

In [96], the implicit function theorem is employed as a means to derive local identifiability results. The authors consider geometric identifiability, which is a structural identifiability construct that is local both around the states and the parameters. Local structural identifiability is formulated as algebraic identifiability. While the relationship between the various local identifiability characterizations are clearly given, the actual computation steps to validate identifiability is slightly ambiguous and the example provided also seems *ad hoc*. The results in this chapter mitigate this to an extent.

6.2.2 Discrete-time identifiability

For the discrete-time case, the identifiability results are limited. In [91], the authors formulated the cryptographic key's ability to be cracked as an identifiability problem and then reused the continuous-time results in the discrete-time context. The authors in [113] develop a discrete-time version of the local state isomorphism theorem and use it to establish identifiability results for discrete-time systems with polynomial nonlinearities.

The authors in [114] develop discrete-time local identifiability results using the implicit function theorem similar to that of continuous-time in [96]. These results, however, don't provide any specific systematic procedure and the examples provided are too simple to provide insight into the procedure.

For identifiability of LPV models, all the works consider models with static dependence on the scheduling variables. The authors in [115] derive some perspectives on those models that could be represented using the linear fractional transformation (LFT) approach. They provide an identifiability characterization of such models using the existence of similarity transformation between two realizations. In [116], the authors deal with the dual problems of identifiability and informativity that concerns the parameter estimation. The models considered are the input-output models with LPV-ARX structure in contrast to the state-space models of interest in this chapter. Discrete-time affine LPV model identifiability is discussed in [117], where the authors use the realization theory developed in [118]. The authors provide a systematic procedure that culminates in a rank condition that would verify the presence of an isomorphism between the realizations. The results are necessary and sufficient for local structural identifiability and sufficient for global structural identifiability.

6.2.3 Software packages for identifiability evaluation

While there are several systematic approaches to validate identifiability for different systems, implementation of each of those methods for comparison purposes would be difficult. In this respect, the existing software packages which were used to validate the results obtained from the approach proposed in the chapter were: DAISY ([119]), genSSI ([97]) and STRIKE-GOLDD ([112]).

DAISY is a package developed under the REDUCE platform and implements the ideas that originated in [109] and elaborated in [111]. The package uses the Ritt's algorithm to eliminate the states of the system and compute the characteristic set associated with the differential ideal generated by the system differential equations. The differential ring used is $\mathbb{R}[x, y, u]$ (instead of the $\mathbb{R}[x, y, u, \theta]$ as in [90]) and hence a normalized input-output relation is obtained from

Table 6.1: Comparison between the available identifiability software packages

Software	**Model scope**	**Neighbourhood scope**	**Applicability scope**
DAISY [119]	Models of the form (6.1) with analytic functions	Local, Global	Necessary and Sufficient
genSSI [97]	Control affine models (6.4) with analytic functions	Local, Global	Sufficient
STRIKE-GOLDD [112]	Models of the form (6.1) with $h(.)$ independent of inputs	Local	Necessary and Sufficient

the characteristic set. The exhaustive summary [100] is extracted from the normalized input-output relations by gathering the functions of parameters that appear as coefficients. Further, the authors assign random numerical values to the parameters and subject it to the Buchberger algorithm to compute their Gröbner basis. Depending upon the number of solutions it admits, the original system is globally, locally or non-identifiable.

The genSSI package is a toolbox under the MATLAB computing platform. This package evaluates the identifiability of models in the control affine form using the generating series approach from [100]. The package requires the maximum number of derivatives to be computed as an input. Hence it provides a sufficient condition for the identifiability results. The identifiability tableau introduced in [101] is included as an aid to understand the identifiable and unidentifiable parameters.

The STRIKE-GOLDD toolbox also works on the MATLAB computing platform. It is based on the extended observability approach of [93] and computes $n+q$ derivatives of the output and then evaluates the Jacobian of the resulting equation with respect to the extended state vector that includes the unknown parameters. Given the use of observability-identifiability-condition (OIC), the results are local in nature.

A comparison of the capabilities of the three software toolboxes, in terms of the class of models considered and the locality/globality of the results obtained is given in Table 6.1.

6.2.4 Motivation for the chapter

There is a wealth of literature available for nonlinear model identifiability analysis. For the objective of adapting or developing tools to perform identifiability analysis for quasi-LPV models, it is useful to note that quasi-LPV models can represent different types of nonlinear models, including polynomial, rational and transcendental. Hence, developing a unified method for quasi-LPV models is a difficult challenge. Elimination strategies seem appropriate in this context as it can work, to some extent, agnostic to the underlying model. One of the underlying themes in the literature of elimination techniques is to arrive at exhaustive summary. In [100], it is through a generating series, whereas in [111], it is through differential algebra (Ritt's algorithm). In this chapter, a parity-space based approach is used to eliminate the system states and arrive at the exhaustive summary. The results from the proposed method are compared with that of DAISY [119], as the strategies are similar.

6.3 Problem formulation

In this section, the problem of interest is specified precisely.

To start with, a reiteration of the model structure for which identifiability is envisaged is given,

Assumption 6 (Model structure). *The models of interest are those nonlinear models that could be written in the quasi-LPV form with affine parametrization. That is,*

$$\begin{aligned} \dot{x}(t) &= A(\rho(t),\theta)x(t) + B(\rho(t),\theta)u(t) \\ \dot{y}(t) &= C(\rho(t),\theta)x(t) + D(\rho(t),\theta)u(t) \end{aligned} \tag{6.8}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $\rho \in \mathbb{R}^\xi$ $\theta \in \mathbb{R}^q$ with the appropriate dimensions for the system matrices (A, B, C, D) which are of the form,

$$X(\rho(t),\theta) = X_0(\rho(t)) + \sum_{j=1}^{q} \theta_j \bar{X}_j(\rho(t)) \tag{6.9}$$

The scheduling or premise variable, $\rho(t)$, is either composed of external variables with static dependence (in which case the model is LPV) or that of system variables such as an inputs, states and outputs (in which case the model is quasi-LPV).

Assumption 7 (Premise variables). *The premise variables of the quasi-LPV model are known or measured.*

When writing nonlinear models in quasi-LPV form, one of the unmeasured states can appear as a premise variable. However, this chapter considers only those models where the premise variables are composed of known or measured variables. This assumption also puts some constraints on the structure of $C(.)$ and $D(.)$ matrices (or $h(.)$ in Σ_θ) such that they are restricted to one or more of the following:

- The premise variables are functions of states that are directly measured. For example when the output equation is of the form,

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)$$

- The premise variables are functions of states that can be resolved algebraically using the output equations. For example,

$$y_1(t) = x_1(t)x_2(t), \qquad y_2(t) = x_1(t)u(t)$$

  or in quasi-LPV form,

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & x_1(t) \\ u(t) & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

  with two premise variables $\rho_1(t) = x_1(t)$ and $\rho_2(t) = u(t)$

- The measured nonlinearity appears as a whole as premise variables. For instance, for a model of the form,

$$\begin{aligned}\dot{x}_1(t) &= x_1^2(t)x_2(t) + \theta_1 x_2(t)\\ \dot{x}_2(t) &= \theta_2 x_1(t) + \theta_3 x_2(t)\\ y(t) &= x_1(t)x_2(t)\end{aligned}$$

  a quasi-LPV model can be obtained with the premise variable $\rho(t) = x_1(t)x_2(t)$ such that

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}\rho(t) & \theta_1\\ \theta_2 & \theta_3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\\ y(t) &= \rho(t)\end{aligned}$$

Remark 15. *The nonlinear models of the form* (6.1) *can be rewritten into quasi-LPV forms using several of the existing embedding techniques (see for example, [21], [23], [27]). The quasi-LPV representation is not unique, and only those models are of interest in this thesis which result in known or measured premise variables. Further, it is assumed that the methods do not change the identifiability properties of the original model during the embedding from nonlinear model to quasi-LPV model.*

Definition 1. *Identifiability*

A parameter θ_i, $i \in \{1, \cdots, q\}$, is structurally globally (or uniquely) identifiable if for almost any $\theta_i^* \in \Theta$,

$$\Sigma_{\theta_i} = \Sigma_{\theta_i^*} \Rightarrow \theta_i = \theta_i^*$$

A parameter θ_i, $i \in \{1, \cdots, q\}$, is structurally locally identifiable if for almost any $\theta_i^* \in \Theta$, there exists a neighbourhood $\eta(\theta^*)$ such that

$$\Sigma_{\theta_i} = \Sigma_{\theta_i^*} \Rightarrow \theta_i = \theta_i^*$$

Consequently, a parameter θ_i, $i \in \{1, \cdots, q\}$, is structurally non-identifiable if for almost any $\theta_i^* \in \Theta$, there exists no neighbourhood $\eta(\theta^*)$ such that

$$\Sigma_{\theta_i} = \Sigma_{\theta_i^*} \Rightarrow \theta_i = \theta_i^*$$

The above definition formalizes the distinguishability property through structural identifiability. To *a priori* verify identifiability using standard mathematical tools, more tangible definitions are required. In this respect two approaches of interest are discussed below, namely, structural identifiability [111] and algebraic identifiability [96], which are shown to be related shortly. This characterization requires the following notation and terminology:

- **Exhaustive summary** ([100]) of an experiment is a set of functions, $\Pi(\theta)$, if it contains only, but all the information about θ that can be extracted from the knowledge of u and y. That is, they embody the parameter dependence of the input-output model completely. These are also referred to as the observational parameter vector in [120]. Some authors use a slightly different terminology, for example, in [111], the authors refer to the set of equations,

$$\Pi(\theta) = \tilde{\theta} \tag{6.10}$$

as exhaustive summary, where $\tilde{\theta}$ refers to the specific instance of θ used to verify if $\Pi(\theta)$ admits only on solution, that is, $\tilde{\theta}$. In this thesis, we use *exhaustive summary* to refer to the generic set of equations denoted by $\Pi(\theta)$ whereas (6.10) would be referred to as *exhaustive summary evaluation.*

- $\Phi(.)$ **Set of identifiability equations**: These are q equations which are functions of only the known and measured variables and their derivatives and the unknown parameters and are of the form:
$$\Phi(y, \dot{y}, \ddot{y}, \cdots, u, \dot{u}, \cdots, \theta) = 0$$

Given a system model (6.1), it is possible to obtain its exhaustive summary through various methods and then validate the number of solutions admitted by it. This characterization is formalized as follows. Note that the use of $y(\Pi(\theta), t)$ is in reference to the experiment which provides a set of outputs which depend on the exhaustive summary.

Definition 2 (Structural identifiability ([111]))**.** *A parameter θ_i is,*

globally (or uniquely) identifiable if and only if, for almost any $\tilde{\theta}$, the following system has only one solution, $\theta_i = \tilde{\theta}_i$:

$$y(\Pi(\theta), t) = y(\tilde{\Pi}(\tilde{\theta}), t) \tag{6.11}$$

locally (nonuniquely) identifiable (LSI) if and only if, for almost any $\tilde{\theta}$, the system (6.11) *has (for θ_i) more than one, but a finite number of solutions.*

non-identifiable if and only if, for almost any $\tilde{\theta}$, the system (6.11) *has (for θ_i) infinite number of solutions.*

The second definition of interest is the algebraic identifiability in [96].

Definition 3 (Algebraic identifiability (AI) ([96]))**.** *The system model Σ_θ is said to be algebraically identifiable if there exist a $T > 0$, a positive integer k and a meromorphic function $\Phi : \mathbb{R}^q \times \mathbb{R}^{(k+1)m} \times \mathbb{R}^{(k+1)p} \to \mathbb{R}^q$ such that*

$$det \frac{\partial \Phi}{\partial \theta} \neq 0 \tag{6.12}$$

and

$$\Phi(\theta, y, \dot{y}, ..., y^{(k)}, u, \dot{u}, ..., u^{(k)}) = 0 \tag{6.13}$$

hold, on $[0, T]$, for all $(\theta, u, ...u^{(k)}, y, \dot{y}, ..., y^{(k)})$ where (θ, x_0, u) belong to an open and dense subset.

The relationship between algebraic identifiability and the local structural identifiability is clarified in the following proposition. This is done by reiterating the characterization of the two definitions to illustrate the equivalence.

Proposition 1 (AI vs LSI)**.** *A system model of type Σ_θ is algebraically identifiable (AI) if and only if it is locally structurally identifiable (LSI)*[13]*.*

[13] Generically, for almost all cases

Proof To validate AI, we need to find $\Phi = \{\phi_1, \cdots \phi_q\}$, a set of q equations, such that,

$$\text{rank}\frac{\partial \Phi}{\partial \theta} = q, \qquad \text{and} \qquad \phi_i(y, \dot{y}, \cdots, u, \dot{u}, \cdots, \theta) = 0, \text{ for } i = 1, \cdots, q$$

To validate LSI, we need to arrive at Γ, a set of equations in the inputs, outputs, their derivatives and the parameter so as to obtain the exhaustive summary. For a given experiment, the cardinality would be equal to the number of outputs p. In the DAISY software package, these are referred to as the 'normalized input-output equations', such that,

$$\Gamma = \{\gamma_1, \cdots, \gamma_p\}, \qquad \text{with,} \qquad \gamma_i(y, \dot{y}, \cdots, u, \dot{u}, \cdots, \theta) = 0 \tag{6.14}$$

The exhaustive summary $\Pi(\theta)$ is obtained from Γ by extracting the coefficients containing θ considering it as a set of polynomials in inputs, outputs and their derivatives. If the model is LSI, then, the exhaustive summary $\Pi(\theta)$ admits finitely many solutions, or a unique solution locally around a certain value of the vector θ. The equivalence for the two identifiability definitions is achieved if one can show,

$$\Phi \Leftrightarrow \Gamma, \qquad \text{for} \qquad q = p, \quad q < p, \quad q > p \tag{6.15}$$

(i) LSI $\Rightarrow$ AI: For case $q = p$, this is trivial as $\Phi = \Gamma$ and given LSI, rank$\frac{\partial \Gamma}{\partial \theta} = q$.

For $q < p$, then $\Phi \subset \Gamma$ and hence it is a matter of finding the right set of equations such that rank$\frac{\partial \Phi}{\partial \theta} = q$. For example, one can compute the Jacobian $\frac{\partial \Gamma}{\partial \theta}$ and pick the relevant rows and columns.

For $q > p$, the set Φ can be constructed by augmenting Γ with the derivatives of the polynomials that is contained in it. That is,

$$\Phi = \{\Gamma, \gamma_1^{(1)}, \cdots \gamma_1^{(g_1)}, \cdots \gamma_p^{(1)}, \cdots \gamma_p^{(g_p)}\}, \qquad \text{where,} \qquad g_1 + \cdots + g_p = q - p$$

(ii) AI $\Rightarrow$ LSI: Given that AI is satisfied, it is sufficient to pick the exhaustive summary (Π) from Φ which will permit a unique local solution. More on this could be referred to in [120].

Remark 16. *The proposition illustrates what appears to be a fairly straightforward result. However, the discussion also emphasizes the equivalence of the Jacobian computation and Gröbner basis evaluation when the problem is local. The importance of this would be evident during the later discussions.*

Assumption 8 (Characteristics of f and h)**.** *The state and the output functions, f and h respectively, are assumed to be meromorphic. Further,*

$$\text{rank}\left(\frac{\partial h(x, \theta, u)}{\partial x}\right) = p \tag{6.16}$$

The assumptions on the model functions are a superset to the assumptions given in [111] and [96] to complete the definitions 2 and 3. In terms of the quasi-LPV model, this condition requires the following

- The nonlinearities that appear in the matrices $A(.)$, $B(.)$, $C(.)$, and $D(.)$ are meromorphic.
- The rows of the matrix $C(.)x + D(.)u$ are locally independent, that is,

$$\text{rank}\left(\frac{\partial}{\partial x}(C(.)x + D(.)u)\right) = p$$

Assumption 9 (Initial conditions)**.** *The state initial conditions are arbitrary, but do not contain special values that invalidate identifiability result.*

The initial conditions play a role in the identifiability analysis in the sense that it is possible that for some specific initial conditions, the model loses identifiability for some parameters. These specific initial conditions form a very thin (invariant) set [121], but have the power to render the identifiability analysis, which assumes arbitrary initial conditions, void. See, for instance, Example 4 in [94], Example 2 in [122]. To illustrate the idea, consider the three compartment model (Example 1 in [121]),

$$\begin{aligned} \dot{x}_1 &= \theta_{13}x_3 + \theta_{12}x_2 - \theta_{21}x_1 + u, && x_1(0) = x_{10}, \\ \dot{x}_2 &= -\theta_{12}x_2 + \theta_{21}x_1, && x_2(0) = x_{20}, \\ \dot{x}_3 &= -\theta_{13}x_3, && x_3(0) = 0, \\ y &= x_2 \end{aligned}$$

For arbitrary initial values of x_1 and x_2 and the specific initial value of $x_3(0) = 0$, it is very straightforward to see that θ_{13} would never appear on the output and their successive derivatives. It is always multiplied by x_3 and $x_3 = 0$ is an invariant set. The above model, for arbitrary initial conditions, is identifiable. However, for the specific initial condition $x_3(0) = 0$, the parameter θ_3 is not identifiable[14]. In general terms, this relates to a form of reachability of the given model as discussed in [121]. The corresponding analysis for the parity-space approach is a future work.

Assumption 10 (Inputs)**.** *The higher order derivatives of the inputs are defined and are known*

This assumption is required at least up to the order required for identifiability analysis so that it is possible to formulate $\Phi(.)$ in (6.13) or Γ in (6.14).

Remark 17 (Discrete-time case)**.** *The discussion in this section has focused on continuous-time models, though it holds for the discrete-time case with the exchange of shift in discrete-time for derivatives in continuous-time as commented in [91].*

6.4 Parameter identifiability for continuous-time models

In this section, an overview of the proposed parity-space based identifiability analysis method is given. A set of examples illustrate the steps involved as well as comparing the results with that obtained from DAISY.

6.4.1 Steps involved

The procedure for the identifiability analysis proposed is inspired by the parity-space approach in [16] as a means to eliminate the states of the system so as to obtain a set of equations of the form Φ in (6.13) or of the form Γ in (6.14). The analysis of the Jacobian of these equations or their exhaustive summary would then allow to determine the identifiability. The procedure could be summarized as follows:

[14] It is to be noted that this linear model has a pole-zero cancellation in its transfer function leading to unobservability of the state x_3. However, similar problems could be seen in other nonlinear examples in the literature

Step 1: Formulation of algebraic equations The quasi-LPV model in (6.8) is rewritten for illustration purposes as follows,

$$\begin{aligned} x^{(1)} &= A^{(0)}x^{(0)} + B^{(0)}u^{(0)} \\ y^{(0)} &= C^{(0)}x^{(0)} + D^{(0)}u^{(0)} \end{aligned} \tag{6.17}$$

where the superscript refers to the order of derivatives, that is,

$$A^{(j)} = \frac{d^j \left(A(\rho(t), \theta)\right)}{dt^j}$$

with the dependence on time, premise variables and the parameters is omitted for the sake of simplicity. To illustrate the gathering of algebraic equations, consider that there are up to 2nd order derivatives of the output, it is possible to write the following set of algebraic equations,

$$\begin{aligned} y^{(0)} &= C^{(0)}x^{(0)} + D^{(0)}u^{(0)} \\ x^{(1)} &= A^{(0)}x^{(0)} + B^{(0)}u^{(0)} \\ y^{(1)} &= C^{(1)}x^{(0)} + C^{(0)}x^{(1)} + D^{(1)}u^{(0)} + D^{(0)}u^{(1)} \\ x^{(2)} &= A^{(1)}x^{(0)} + A^{(0)}x^{(1)} + B^{(1)}u^{(0)} + B^{(0)}u^{(1)} \\ y^{(2)} &= C^{(2)}x^{(0)} + 2C^{(1)}x^{(1)} + C^{(0)}x^{(2)} + D^{(2)}u^{(0)} + 2D^{(1)}u^{(1)} + D^{(0)}u^{(2)} \end{aligned} \tag{6.18}$$

which could then be rewritten in matrix form, but taking all the known and measured components to the left hand side as,

$$\begin{bmatrix} y^{(0)} - D^{(0)}u^{(0)} \\ B^{(0)}u^{(0)} \\ y^{(1)} - D^{(1)}u^{(0)} - D^{(0)}u^{(1)} \\ B^{(1)}u^{(0)} + B^{(0)}u^{(1)} \\ y^{(2)} - D^{(2)}u^{(0)} - 2D^{(1)}u^{(1)} - D^{(0)}u^{(2)} \end{bmatrix} = \begin{bmatrix} C^{(0)} & \mathbf{0} & \mathbf{0} \\ -A^{(0)} & I_n & \mathbf{0} \\ C^{(1)} & C^{(0)} & \mathbf{0} \\ -A^{(1)} & -A^{(0)} & I_n \\ C^{(2)} & 2C^{(1)} & C^{(0)} \end{bmatrix} \begin{bmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{bmatrix} \tag{6.19}$$

which could be rearranged to give,

$$\begin{bmatrix} y^{(0)} - D^{(0)}u^{(0)} \\ y^{(1)} - D^{(1)}u^{(0)} - D^{(0)}u^{(1)} \\ y^{(2)} - D^{(2)}u^{(0)} - 2D^{(1)}u^{(1)} - D^{(0)}u^{(2)} \\ B^{(0)}u^{(0)} \\ B^{(1)}u^{(0)} + B^{(0)}u^{(1)} \end{bmatrix} = \begin{bmatrix} C^{(0)} & \mathbf{0} & \mathbf{0} \\ C^{(1)} & C^{(0)} & \mathbf{0} \\ C^{(2)} & 2C^{(1)} & C^{(0)} \\ -A^{(0)} & I_n & \mathbf{0} \\ -A^{(1)} & -A^{(0)} & I_n \end{bmatrix} \begin{bmatrix} x^{(0)} \\ x^{(1)} \\ x^{(2)} \end{bmatrix} \tag{6.20}$$

More generally for up to an order w,

$$\begin{bmatrix} \mathbb{Y} \\ \mathbf{0}_{w \times n} \end{bmatrix} + \begin{bmatrix} -\mathbb{D}(\theta) \\ \mathbb{B}(\theta) \end{bmatrix} \mathbb{U} = \begin{bmatrix} \mathbb{C}(\theta) \\ \mathbb{A}(\theta) \end{bmatrix} \mathbb{X} \tag{6.21}$$

with the left hand side containing known and measured terms. The indication (θ) indicates the explicit appearance of the parameter in the matrices. Notice, however that, all the elements except $\mathbb{U}$ are indirectly dependent on θ. Here,

$$\begin{aligned} \mathbb{Y} &= \begin{bmatrix} (y^{(0)})^T & (y^{(1)})^T & (y^{(2)})^T & \cdots & (y^{(w)})^T \end{bmatrix}^T \\ \mathbb{U} &= \begin{bmatrix} (u^{(0)})^T & (u^{(1)})^T & (u^{(2)})^T & \cdots & (u^{(w)})^T \end{bmatrix}^T \\ \mathbb{X} &= \begin{bmatrix} (x^{(0)})^T & (x^{(1)})^T & (x^{(2)})^T & \cdots & (x^{(w)})^T \end{bmatrix}^T \end{aligned} \tag{6.22}$$

and,

$$\mathbb{B}(\theta) = \begin{bmatrix} B^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ B^{(1)} & B^{(0)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ B^{(2)} & 2B^{(1)} & B^{(0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ B^{(3)} & 3B^{(2)} & 3B^{(1)} & B^{(0)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ B^{(w-1)} & \binom{w}{2}B^{(w-2)} & \binom{w}{3}B^{(w-3)} & \binom{w}{4}B^{(w-4)} & \cdots & B^{(0)} & \mathbf{0} \end{bmatrix} \tag{6.23}$$

$$\mathbb{D}(\theta) = \begin{bmatrix} D^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ D^{(1)} & D^{(0)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ D^{(2)} & 2D^{(1)} & D^{(0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ D^{(3)} & 3D^{(2)} & 3D^{(1)} & D^{(0)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ D^{(w)} & \binom{w+1}{2}D^{(w-1)} & \binom{w+1}{3}D^{(w-2)} & \binom{w+1}{4}D^{(w-3)} & \cdots & 2D^{(1)} & D^{(0)} \end{bmatrix} \tag{6.24}$$

$$\mathbb{C}(\theta) = \begin{bmatrix} C^{(0)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ C^{(1)} & C^{(0)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ C^{(2)} & 2C^{(1)} & C^{(0)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ C^{(3)} & 3C^{(2)} & 3C^{(1)} & C^{(0)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ C^{(w)} & \binom{w+1}{2}C^{(w-1)} & \binom{w+1}{3}C^{(w-2)} & \binom{w+1}{4}C^{(w-3)} & \cdots & 2C^{(1)} & C^{(0)} \end{bmatrix} \tag{6.25}$$

$$\mathbb{A}(\theta) = \begin{bmatrix} -A^{(0)} & I_n & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ -A^{(1)} & -A^{(0)} & I_n & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ -A^{(2)} & -2A^{(1)} & -A^{(0)} & I_n & \cdots & \mathbf{0} & \mathbf{0} \\ -A^{(4)} & -3A^{(2)} & -3A^{(1)} & -A^{(0)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ -A^{(w-1)} & -\binom{w}{2}A^{(w-2)} & -\binom{w}{3}A^{(w-3)} & -\binom{w}{4}A^{(w-4)} & \cdots & -A^{(0)} & I_n \end{bmatrix} \tag{6.26}$$

As is apparent, each of these matrices form a Pascal's triangle with the increasing order of derivatives. This aides in easier representation to automate the algorithms. The representation in (6.21) is further simplified to indicate the dependence on the unknown parameter as,

$$\mathbb{Y}_0 + \mathbb{G}(\theta)\mathbb{U} = \mathbb{O}(\theta)\mathbb{X} \tag{6.27}$$

Notice that the matrix $\mathbb{O}(\theta)$ has a dimension of $(wp + (w-1)n) \times ((w-1)n)$.

Remark 18 (Choice of the set of algebraic equations)**.** *The choice of the particular form of algebraic equations in* (6.19) *has some specific advantages. It is possible to consider the more common observability matrix to obtain the set of algebraic equations, that is,*

$$\begin{bmatrix} y^{(0)} \\ y^{(1)} \end{bmatrix} = \begin{bmatrix} C^{(0)} \\ C^{(1)} + C^{(0)}A^{(0)} \end{bmatrix} x^{(0)} + \begin{bmatrix} D^{(0)} & 0 \\ C^{(0)}B^{(0)} + D^{(1)} & D^{(0)} \end{bmatrix} \begin{bmatrix} u^{(0)} \\ u^{(1)} \end{bmatrix} \tag{6.28}$$

However, the form chosen in the parity-space approach provides the following advantages:

- *The observability matrices have several matrix multiplications leading to a significant complexity in the computation. This is already visible when consider only the* 1*st order derivatives in* (6.28)*.*

- *Given that we consider affine parametrization, the degree of θ in the form* (6.19) *or its condensed notation* (6.21) *is* 1*, that is $\mathbb{O}(\theta)$ is affine in θ. In the form* (6.28)*, this will definitely be higher due to the matrix multiplications. This plays a significant role in the null-space computation step. In a numerical approach to be discussed later, this would mean the computation of the null-space for a degree* 1 *polynomial.*

Step 2: Computation of null-space Once a set of algebraic equations are formulated, the next step is to eliminate the state variables and their derivatives. This is achieved if the left null-space of $\mathbb{O}(\theta)$ is computed, that is, to find a matrix $\Omega(\theta)$, such that,

$$\Omega^T(\theta)\mathbb{O}(\theta) = \mathbf{0}$$

For a given $\mathbb{O}(\theta)$, the null-space $\Omega(\theta)$, if it exists, can be computed using symbolic computations such as under the symbolic computation toolbox under MATLAB. The existence of the null-space is directly related to the output and the state matrices which populate $\mathbb{O}(\theta)$

Step 3: Formulation of the Input-Output-Parameter (I-O-P) equations Once the null-space has been obtained, then one can compute,

$$\Omega^T(\theta)\left(\mathbb{Y}_0 + \mathbb{G}(\theta)\right) = \mathbf{0} \tag{6.29}$$

which can alternatively represented as,

$$\Psi(\theta, y, \cdots, y^{(w)}, u, \cdots, u^{(w)}) = \mathbf{0} \tag{6.30}$$

where $\Psi(.)$ is termed as Input-Output-Parameter (I-O-P) equations to signify its dependence on inputs, outputs and parameters, and their derivatives.

Step 4: Identifiability evaluation Once the I-O-P equations are obtained, the verification of identifiability is achieved through one of the following approaches:

- Following the final step in the DAISY package [119]:
  - Extract the coefficients of $\Psi(.)$ considering it as polynomials in inputs, outputs, and their derivatives. Those coefficients that depend on the parameters θ form the exhaustive summary $\Pi(\theta)$
  - Assign symbolic values for each of the parameters $\{\theta_1, \cdots, \theta_q\}$ and evaluate the exhaustive summary to obtain $\tilde{\Pi}(\tilde{\theta})$. For large scale problems, symbolic values can be replaced with numerical values.
  - Apply Buchberger algorithm on $\tilde{\Pi}(\tilde{\theta})$ to obtain all the solutions. Depending upon the number of solutions $\tilde{\Pi}(\tilde{\theta})$ admits, identifiability can be evaluated using Definition 2. In case of numerical approach, the last two steps are repeated several times. Since the results are *generic*, that is, valid for almost all numerical values except for a set of measure zero. This repetition would help to avoid reaching conclusions based on possibly choosing a numerical value of this set of measure zero.

- Compute the Jacobian of $\Psi(.)$ with respect to the parameters θ, that is,

$$\text{rank}\left(\frac{\partial\Psi}{\partial\theta}\right) = q$$

  If the rank is q, then local identifiability is verified.

6.4.2 Illustrative examples

In this section, several examples are given to show the steps and the effectiveness of the parity-space approach. The results obtained from the parity-space approach is validated by comparing it with that of DAISY, STRIKE-GOLDD and genSSI software packages. Further, the exhaustive summary obtained using the parity-space method is compared with that obtained using DAISY.

Example 10. *This example is used to show all the steps of the proposed approach. Further, it also concerns a model which is not identifiable. Consider the second order nonlinear model,*

$$\begin{aligned}\dot{x}_1 &= \theta_1 x_1 + \theta_2 x_2 u,\\ \dot{x}_2 &= \theta_3 x_1 - x_2,\\ y &= x_1 u,\end{aligned}$$

A quasi-LPV equivalent form with $x = [x_1 \;\; x_2]^T$ and $\rho(t) = u(t)$ is,

$$\dot{x} = \begin{bmatrix}\theta_1 & \theta_2 u\\ \theta_3 & -1\end{bmatrix} x \qquad \text{and} \qquad y = \begin{bmatrix}u & 0\end{bmatrix} x$$

Based on the specifications of the model structure required, the genSSI software cannot handle this example. Using the parity-space approach with output up to $y^{(2)}$, we obtain the following representation to that in (6.27),

$$\begin{bmatrix}y\\ \dot{y}\\ \ddot{y}\\ 0\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}u & 0 & 0 & 0 & 0 & 0\\ \dot{u} & 0 & u & 0 & 0 & 0\\ \ddot{u} & 0 & 2\dot{u} & 0 & u & 0\\ -\theta_1 & -\theta_2 u & 1 & 0 & 0 & 0\\ -\theta_3 & 1 & 0 & 1 & 0 & 0\\ 0 & -\theta_2\dot{u} & -\theta_1 & -\theta_2 u & 1 & 0\\ 0 & 0 & -\theta_3 & 1 & 0 & 1\end{bmatrix} \begin{bmatrix}x_1\\ x_2\\ \dot{x}_1\\ \dot{x}_2\\ \ddot{x}_1\\ \ddot{x}_2\end{bmatrix}$$

The left null-space of the matrix $\mathbb{O}(\theta)$ is given by,

$$\Omega^T(\theta) = \begin{bmatrix}(u\dot{u} + u\ddot{u} + \theta_1 u^2 - 3\dot{u}^2 + \theta_2\theta_3 u^3 - 2\theta_1 u\dot{u})/u^3\\ (3\dot{u} - u + \theta_1 u0)/u^2\\ -1/u\\ (u - \dot{u})/u\\ \theta_2 u\\ 1\\ 0\end{bmatrix}^T$$

which leads to the I-O-P equation as follows,

$$\Psi(.) = \theta_1 u^2 y - 3\dot{u}^2 y - u^2\ddot{y} - u^2\dot{y} + \theta_1 u^2\dot{y} + u\dot{u}y + 3u\dot{u}\dot{y} + u\ddot{u}y - 2\theta_1 u\dot{u}y + \theta_2\theta_3 u^3 y$$

And the exhaustive summary obtained by extracting the coefficients considering $\Psi(.)$ as a polynomial in inputs, outputs and their derivatives. Considering only those coefficients that depend on θ, the following is obtained,

$$\Pi(\theta) = \{1 - 2\theta_1,\; \theta_1 - 1,\; \theta_1,\; \theta_2\theta_3\} \tag{6.31}$$

To verify the number of solutions admitted by this exhaustive summary, a Gröbner basis analysis is performed. One strategy is to assign symbolic values for for each of the parameter ($\tilde{\theta}_1 = a$, $\tilde{\theta}_2 = b$, $\tilde{\theta}_3 = c$) and evaluate the exhaustive summary to obtain the specific exhaustive summary,

$$\{2\theta_1 - 2a,\ \theta_1 - a,\ \theta_1 - a,\ \theta_2\theta_3 - bc\}$$

For this simple example, it is straightforward to see that only θ_1 is identifiable as it admits a unique solution and the other two parameters can have several solutions. Hence the model is not identifiable. To formally verify this, these polynomial equations were given as input to the Buchberger algorithm implemented in MuPAD CAS under MATLAB. The Gröbner basis for this set is given by,

$$\{\theta_1 - a,\ \theta_2\theta_3 - bc\}$$

which, if the equations admit a unique solution should have returned $\theta_i = \tilde{\theta}_i$ for $i = 1, 2, 3$.

Comparison with DAISY The normalized input-output equation obtained through the DAISY package is given by,

$$\Gamma = \ddot{u}u^4y - 3\dot{u}^2u^3y + 3\dot{u}\dot{y}u^4 + \dot{u}u^4y(-2\theta_1 + 1) - \ddot{y}u^5 + \dot{y}u^5(\theta_1 - 1) + u^6y\theta_2\theta_3 + u^5y\theta_1$$

which has the same set of exhaustive summary as given in (6.31). The DAISY package results also verify those inferred above.

Example 11. *In this example, the case of local identifiability is illustrated.*

$$\begin{aligned}\dot{x}_1 &= \theta_1x_1 + \theta_2x_2u,\\ \dot{x}_2 &= \theta_2x_1 - \theta_3x_2,\\ y &= x_1u,\end{aligned}$$

Using parity-space approach, the I-O-P equation obtained is

$$\Psi(.) = \theta_1u^2\dot{y} - u^2\ddot{y} - 3\dot{u}^2y - \theta_3u^2\dot{y} + \theta_2^2u^3y + 3u\dot{u}\dot{y} + u\ddot{u}y - 2\theta_1u\dot{u}y + \theta_3u\dot{u}y + \theta_1\theta_3u^2y$$

which has the exhaustive summary of,

$$\Pi(\theta) = \{\theta_3 - 2\theta_1,\ \theta_1 - \theta_3,\ \theta_1\theta_3,\ \theta_2^2\}$$

The Gröbner for this summary with the symbolic assignment of $\tilde{\theta}_1 = a$, $\tilde{\theta}_2 = b$ and $\tilde{\theta}_3 = c$ was obtained as,

$$\{\theta_1 - a,\ \theta_3 - c,\ \theta_2^2 - b^2\}$$

which indicates that while θ_1 and θ_3 are identifiable, θ_2 is only locally identifiable. The results and the exhaustive summary compares with that obtained from DAISY.

Example 12 (Air Handling Unit)**.** *Consider a simple model of a heat exchanger that was discussed in Sec. 2.2.3. The model has been simplified by considering that the inlet air temperature and water temperature are known and constant. A quasi-LPV representation of that model is given by,*

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-\theta_1u_1 - \theta_2 & \theta_2\\ \theta_4 & -\theta_3u_2 - \theta_4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\theta_1 & 0\\ 0 & 5\theta_3\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}\\ y &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned}$$

The exhaustive summary obtained for this model is

$$\Pi(\theta) = \{3 - \theta_4 - \theta_2,\ \theta_2 + \theta_4 - 2,\ \theta_1,\ -\theta_1,\ 5\theta_2\theta_3,\ -\theta_3,\\ \theta_3 - \theta_2\theta_3,\ \theta_1\theta_4, -\theta_1,\ 2\theta_1 - \theta_1\theta_4,\ \theta_1\theta_3,\ -\theta_1\theta_3\}$$

By choosing the numerical values, $\tilde{\theta}_1 = 1$, $\tilde{\theta}_2 = 2$, $\tilde{\theta}_3 = 3$, $\tilde{\theta}_4 = 5$, the specific instance of the exhaustive summary was obtained and the Gröbner basis obtained is

$$\{\theta_2 - 2,\ \theta_4 - 5,\ \theta_3 - 3,\ \theta_1 - 1\}$$

indicating that the model is globally identifiable. And these results verify with those obtained by DAISY both for the exhaustive summary and the eventual identifiability interpretation. Because a numerical value was used, the results are not representative, though, as suggested in [119], several set of numerical values be chosen to gain confidence on the obtained results.

6.5 Parameter identifiability for discrete-time models

In practical scenarios, parameter estimation involves discrete-time models. Hence it is vital to consider the identifiability of system models in discrete-time. In this section, a brief outline to extend the parity-space method to discrete-time quasi-LPV models is given. It is to be noted that the effect of discretization on the identifiability is not treated here. Consider a discrete-time quasi-LPV model of the form,

$$\begin{aligned} x_{k+1} &= A(\rho_k, \theta)x_k + B(\rho_k, \theta) \\ y_k &= C(\rho_k, \theta)u_k + D(\rho_k, \theta) \end{aligned} \tag{6.32}$$

For this type of model, the procedure for parameter identifiability follows a similar trajectory. The key difference is in the first step where the set of algebraic equations are obtained in a different way. For the sake of simplicity, in the following, A_k would be used in place of $A(\rho_k, \theta)$ and similarly for other matrices.

For the discrete-time case, the algebraic equations take a far simpler structure compared to that in the continuous-time case. The continuous-time algebraic equations in (6.21) is rewritten for the discrete-time case as:

$$\begin{bmatrix} \mathbb{Y}_k \\ \mathbf{0}_{w\times n} \end{bmatrix} + \begin{bmatrix} -\mathbb{D}_k \\ \mathbb{B}_k \end{bmatrix} \mathbb{U}_k = \begin{bmatrix} \mathbb{C}_k \\ \mathbb{A}_k \end{bmatrix} \mathbb{X}_k \tag{6.33}$$

where,

$$\begin{aligned} \mathbb{Y}_k &= \begin{bmatrix} y_k^T & y_{k+1}^T & \cdots & y_{k+w}^T \end{bmatrix}^T \\ \mathbb{U}_k &= \begin{bmatrix} u_k^T & u_{k+1}^T & \cdots & u_{k+w}^T \end{bmatrix}^T \\ \mathbb{X}_k &= \begin{bmatrix} x_k^T & x_{k+1}^T & \cdots & x_{k+w}^T \end{bmatrix}^T \end{aligned} \tag{6.34}$$

and

$$\mathbb{B}_k = \begin{bmatrix} B_k & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_{k+1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_{k+2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_{k+w-1} & \mathbf{0} \end{bmatrix} \tag{6.35}$$

$$\mathbb{D}_k = \begin{bmatrix} D_k & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{k+1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_{k+2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & D_{k+w} \end{bmatrix} \tag{6.36}$$

$$\mathbb{A}_k = \begin{bmatrix} -A_k & I_n & \mathbf{0} & & \mathbf{0}\cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -A_{k+1} & I_n & & \mathbf{0}\cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -A_{k+2} & I_n & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{k+w-1} & I_n \end{bmatrix} \tag{6.37}$$

$$\mathbb{C}_k = \begin{bmatrix} C_k & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{k+1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{k+2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \ddots & & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_{k+w} \end{bmatrix} \tag{6.38}$$

The other three steps in this case follow that of the continuous time approach with the derivatives replaced with time shifts.

Illustrative examples

All the software packages in the literature are available only for continuous-time models. Hence, comparison of results is not feasible. The illustrative examples here compare the results obtained by parity-space methods with the results in the work from which the example is adapted.

Example 13. *This is an example of the Henon map adapted from [91]*

$$\begin{aligned} x_{1,k+1} &= \theta_1 x_{1,k}^2 + \theta_2 x_{2,k} + u_k \\ x_{2,k+1} &= \theta_3 x_{1,k} + \theta_4 u_k \\ y_k &= x_{1,k} \end{aligned}$$

The exhaustive summary obtained using the parity-space approach is:

$$\Pi(\theta) = \{\theta_1,\ \theta_2\theta_4,\ \theta_2\theta_3\}$$

The identifiability results verify with that from [91] that only the parameter θ_1 is identifiable.

Example 14. *This is also an example from [91] of Burgers map*

$$\begin{aligned} x_{1,k+1} &= (1+\theta_1)x_{1,k} + x_{1,k}x_{2,k} + u_k \\ x_{2,k+1} &= (1-\theta_2)x_{2,k} - x_{1,k}^2 u_k \\ y_k &= x_{1,k} \end{aligned}$$

The exhaustive summary obtained for this example is:

$$\Pi(\theta) = \{\theta_1, \theta_2, \theta_1 - \theta_2 - \theta_1\theta_2\}$$

It is easy to see that the model is identifiable and agrees with the results in [91].

6.6 Systematic formulation of the proposed method

The next step in the identifiability analysis is to obtain a systematic algorithm using the proposed steps. In this section, some details that would aid to develop a systematic algorithm for identifiability analysis based on the parity-space approach is given. Further, algorithmic steps for sample scenarios (including those for the illustrative example in the previous section) are given. Even though these algorithms are yet to be completely realized, they provide the necessary directions for the future implementation. These formulations are for both continuous-time and discrete-time models with appropriate modifications, though the discussion focuses on continuous-time models.

6.6.1 The choice on the number of derivatives

The discussion in the preceding sections did not explicitly talk about w, the number of derivatives (or shifts in discrete-time) for which the null-space $\Omega^T(\theta)$ exists and hence the I-O-P equations and the exhaustive summary that follow. This corresponds to the observability index of the system model. A detailed discussion on observability index of a nonlinear system could be referred to in [89] though a brief idea is given below. Consider a SISO system of the form (6.1) and the observability index of this model is defined as $w > 0$, if in the neighbourhood of x_0,

$$\begin{aligned} &\operatorname{rank}\left[L_f^{w-1}h, L_f^{w-2}h, \cdots, L_f^0 h\right] = w \\ \text{and} \qquad &\operatorname{rank}\left[L_f^{w}h, L_f^{w-1}h, \cdots, L_f^0 h\right] = w \end{aligned}$$

where, $L_f h$ corresponds to the Lie derivative of h over f, that is,

$$L_f h \triangleq \frac{\partial h(x,u,\theta)}{\partial x} f(x,u,\theta)$$

and $L_f^i h$ refers to ith successive application of the Lie derivative. Essentially it means that, locally, the dimension of the space spanned by the model does not grow after $(w-1)$ derivatives. For MIMO systems, the observability index is defined for each output. A discrete-time version of this is briefly discussed in Chapter 5 of [123].

This means that for a SISO system, using w derivatives would guarantee that $\Omega^T(\theta)$ exists and hence would provide the I-O-P equation corresponding to the output. The next question is to know whether the observability index has been connected to the system dimensions theoretically. That is, is it possible to obtain the index without verifying the rank condition. For linear systems, this is equal to the number of states that is easy to verify using Cayley-Hamilton theorem. For nonlinear systems, for the following two classes:

- models of the form (6.1) where the functions are rational
- models in a control affine form (6.4) with analytical functions

it has been shown that, locally, the observability index has an upper bound, equal to the number of states in the system n. That is n derivatives of outputs are sufficient to guarantee that the null-space $\Omega^T(\theta)$ exists. For a more detailed discussion, see [124]. This is an upper bound because: one, the model may not be minimal and has unobservable spaces and hence the observability index is less than n. Second, for MIMO systems, each output would have different observability indices and hence the total number of derivatives required to span the entire observability space can be less than n.

Consequently, for single output systems, n derivatives of output would guarantee that the null-space $\Omega^T(\theta)$ exists. Hence $w = n$ for SISO systems. For MIMO systems, this is further complicated. Each output's observability index has an upper bound of n, but is more likely to be lower than n. A systematic approach to handle this scenario is discussed later in this section.

6.6.2 Higher order derivatives of system matrices

One aspect of the algorithm that is not discussed is obtaining the matrices $A^{(n)},\ n > 0$, that is symbolically computing the higher order derivatives of the elements in a matrix. In the examples implemented, these were computed manually. This process is simple partly because the derivatives are computed with respect to an independent parameter t and hence is easier to perform (in the examples, it is about introducing new symbols). Even for large models, this step would not be cumbersome, though a systematic implementation is pending. The simplicity of this step relies on the nonlinear embedding procedure that performs the task of finding an appropriate quasi-LPV model. The systematic procedure for generating quasi-LPV models such as that proposed in works like [23], [27] should be included if the procedure has to evaluate identifiability of nonlinear models. Further, for the case when the quasi-LPV model has unmeasured premise variables, this step needs to be amended.

6.6.3 Algorithm for parameter identifiability

The algorithm that was used for the analysis of identifiability for the illustrative examples discussed in the previous section is summarized in Algo. 1. The implementation was done on the MATLAB computing environment with the use of symbolic computation toolbox and MuPAD computer algebra systems (CAS). Once the set of I-O-P equations are obtained and the exhaustive summary extracted, the Gröbner basis evaluation is performed through the MuPAD CAS scripts. Hence, at this moment, there are components of the algorithms that require manual intervention. The first step in the algorithm chooses the upper bound on the number of derivatives to be n. While this is applicable for the models chosen for illustrative example, this is not in general other than the models specified before. Further, for MIMO systems, since the observability index depend on individual outputs, a step by step analysis starting from 0 derivatives is considered. Further optimization is envisaged in this respect.

Analyzing outputs independently One of the assumptions that is part of the problem specifications (and adopted from [96]) is,

$$\text{rank}\left(\frac{\partial h(x,u,\theta)}{\partial x}\right) = p$$

That is, the outputs are at least locally independent. This provides an opening to develop local structural identifiability analysis methods that can provide the following advantages:

- Obtain the local identifiability results through Jacobian analysis instead of the Buchberger algorithm to obtain the Gröbner basis.
- As suggested in [119], there are p normalized input-output equations. By considering one input at a time, the exit criterion for the algorithm could be set as one I-O-P equation per output by considering the system with one output at a time.

This is realized as an algorithm in Algo. 2.

Algorithm 1 An algorithm for parameter identifiability

The upper bound on the number of derivatives $w = n$
Evaluate the matrices and their higher order derivatives (element-wise)
for w = 0 to n **do**
  Formulate $\mathbb{Y}_0 + \mathbb{G}(\theta)\mathbb{U} = \mathbb{O}(\theta)\mathbb{X}$ as in (6.27)
  Compute $\Omega^T(\theta)$, the left null-space of $\mathbb{O}(\theta)$ using symbolic computation
  Obtain the I-O-P equations $\Psi(.)$ and extract the coefficients to obtain the exhaustive summary $\Pi(\theta)$.
  for j = 1 to NbIter **do**
    Choose random values for the parameters $\theta_1, \cdots, \theta_q$
    Evaluate Gröbner basis and verify the number of solutions admitted by the exhaustive summary
  end for
  if Global or Local identifiability verified **then**
    END
  end if
end for

A numerical approach The STRIKE-GOLDD toolbox [112] offers to evaluate identifiability either numerically or symbolically. In the numerical approach, a random set of initial conditions are chosen for the states and random values are associated for inputs and their derivatives (these random values are chosen as prime numbers to avoid undesirable cancellations). This significantly reduces the computational effort required to compute the Jacobian. It is to be noted that numerical approach here is not completely numerical. It still requires computing symbolically the Lie derivatives to set up the Observability-Identifiability matrix.

A prominent work in the semi-numerical analysis was proposed in [125], where a polynomial-time algorithm to test local algebraic observability is proposed. The theoretical concept underlying the approach is the differential geometric approach in [93] and used by STRIKE-GOLDD. To avoid symbolic effort, the authors choose to assign random integer values to the initial states as discussed above. Further, they approximate the Lie derivative computation by computing the power series expansion of the partial derivatives with respect to the initial states $x(0)$ and parameters θ. This method was used in [126] to implement a structural identifiability analysis method for large scale models.

The same set of ideas could be incorporated in the Jacobian computation step of the parity-space approach. However, this requires further scrutiny and understanding and left as perspectives. The realization of this semi-numerical scenario with the parity-space approach is illustrated in Algo. 3 which is essentially the same as Algo. 2 except for

- the initial assignments step
- the null-space computation step
- the Jacobian computation step

The numerical approach reduces the symbolic effort both at the null-space computation step and the Jacobian step. In practical scenarios, an approximate value of the initial conditions of states

Algorithm 2 An algorithm for local structural identifiability

```
The maximum value for observability index for each output (w_1, ..., w_p) is n.
Evaluate the matrices and their higher order derivatives (element-wise)
for i = 1 to p (for each output) do
    for w = 0 to n do
        Formulate Y_0 + G(θ)U = O(θ)X as in (6.27)
        Compute Ω^T(θ), the left null-space of O(θ) using symbolic computation
        Obtain the I-O-P equation ψ_i(.)
        if 1 I-O-P equation obtained then
            Calculation ends for output i
        end if
    end for
    Add the I-O-P for output to the overall I-O-P, Ψ(.) = {Ψ(.), ψ_i(.)}
end for
Evaluate ∂Ψ/∂θ and compute the rank
if rank ∂Ψ/∂θ = q then
    Model is Locally structurally identifiable
else
    Not Identifiable
end if
```

are known. Further, an approximate range over which the parameter values exist is also known. Hence, for practical scenarios, this approach simplifies the structural identifiability verification. Using numerical values also allows to completely forego the symbolic null-space computation in favour of the polynomial null-space computation approach proposed in [85] which can realize the null-space computation step through stable numerical techniques. The polynomial null-space computation idea is briefly outlined in the next section.

6.7 Discussion

In this section, some relevant concepts and details of the proposed method omitted in the previous sections are discussed.

Rigorous analysis of the algorithm steps

In the previous sections, the steps of the algorithms and some formalizing them were presented. To make a case for the algorithm, some rigorous analysis of the algorithmic steps and the mathematical underpinnings is required. It is known that the I-O-P equations are not unique because the null-space $\Omega^T(\theta)$ is only a basis to the null-space. In the DAISY software, a normalization of the characteristic set is required before obtaining the Input-Output equations from which the exhaustive summary is extracted. In this context, the following question arises:

- What are the conditions under which the I-O-P equations obtained from parity-space are equivalent to the normalized Input-Output equations obtained by DAISY?

That is, conditions for $\Psi = \Gamma$ (or $\Psi = \Phi$)[15]. The subsequent and a more relevant question follows:

[15] The equality sign represents a comparison between the two sets.

Algorithm 3 An algorithm for local structural identifiability semi-numerically

```
Assign random prime numbers (or random integers) to initial states and the inputs and their higher order derivatives.
The maximum value for observability index for each output (w_1, ..., w_p) is n.
Evaluate the matrices and their higher order derivatives (element-wise). Assign chosen numerical values where appropriate.
for i = 1 to p (for each output) do
  for w_i = 0 to n do
    Formulate Y_0 + G(θ)U = O(θ)X as in (6.27)
    Compute null-space Ω^T(θ) using a symbolic approach or a numerical approach.
    Obtain the I-O-P equation ψ_i(.)
    if 1 I-O-P equation obtained then
      Calculation ends for output i
    end if
  end for
  Add the I-O-P for output to the overall I-O-P, Ψ(.) = {Ψ(.), ψ_i(.)}
end for
Evaluate ∂Ψ/∂θ and compute the rank
if rank ∂Ψ/∂θ = q then
  Model is Locally structurally identifiable
else
  Model is not Identifiable
end if
```

- What are the conditions that guarantee that the exhaustive summary obtained by the parity-space approach and that from DAISY would be the same.

This has been observed in the examples and has intuitive sense that this should be true. However, it still remains to be shown rigorously. For instance, in [109], the authors note that to guarantee uniqueness of the characteristic set, the controllability of the model is assumed. This is clearly pointed out in [121] and is attributed to special initial conditions. While this has been addressed in Assumption 9, it would still be interesting to investigate the scenarios which results in the parity-space approach resulting in non-unique I-O-P equations and hence some effect on the exhaustive summary.

Polynomial null-space computation

In the Algorithm 3, a numerical approach for local structural identifiability is proposed which uses a polynomial null-space computation approach as an alternative. Polynomial null-space computation through a matrix pencil approach was initiated in [127]. Further, non-pencil approaches such as those exploiting the Toeplitz structure ([85], [86]) or those using the generalized resultant approach ([128]) have also been developed in the last few decades. To illustrate the connection between the null-space computation requirement in the thesis and that of the polynomial null-space computation, consider the discrete-time system model with scalar parameter,

$$
\begin{aligned}
x_{k+1} &= (A_{0,k} + \theta \bar{A}_k)x_k + (B_{0,k} + \theta \bar{A}_k)u_k \\
y_k &= (C_{0,k} + \theta \bar{C}_k)x_k
\end{aligned}
$$

The matrix $\mathbb{O}(\theta)$ in (6.27) can be written as,

$$\mathbb{O}(\theta) = \mathbb{O}_0 + \theta\mathbb{O}_1 \tag{6.39}$$

and for $w = 2$,

$$\mathbb{O}_0 = \begin{bmatrix} C_{0,k} & 0 & 0 \\ 0 & C_{0,k+1} & 0 \\ -A_{0,k} & I & 0 \\ 0 & -A_{0,k+1} & I \end{bmatrix}, \qquad \mathbb{O}_1 = \begin{bmatrix} \bar{C}_k & 0 & 0 \\ 0 & \bar{C}_{k+1} & 0 \\ -\bar{A}_k & 0 & 0 \\ 0 & -\bar{A}_{k+1} & 0 \end{bmatrix}$$

The null-space $\Omega(\theta)$ can be parametrized as a matrix polynomial in θ of degree s as,

$$\Omega(\theta) = \Omega_0 + \theta\Omega_1 + \cdots + \theta^s\Omega_s \tag{6.40}$$

Considering the null-space definition, $\Omega^T(\theta)\mathbb{O}(\theta) = \mathbf{0}$ and comparing the coefficient matrices of θ^j, $j = 0, \cdots, s$ on either side would give,

$$\begin{bmatrix} \Omega_0^T & \Omega_1^T & \cdots & \Omega_s^T \end{bmatrix} \mathbb{P} = 0 \tag{6.41}$$

where,

$$\mathbb{P} = \begin{bmatrix} \mathbb{O}_0 & \mathbb{O}_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbb{O}_0 & \mathbb{O}_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mathbb{O}_0 & \mathbb{O}_1 & 0 & \cdots & 0 \\ \cdots & \ddots & & \cdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbb{O}_0 & \mathbb{O}_1 \end{bmatrix}$$

where the matrix $\mathbb{P}$ has s block rows. And the polynomial null-space computation problem is now converted to finding the left null-space of the matrix $\mathbb{P}$. This has a Toeplitz matrix structure and as shown in [85] and [86], the minimal null-space basis for the original polynomial matrix can be obtained through an iterative procedure. The iterative nature of the procedure is due to the lack of prior knowledge of the degree s of the null-space polynomial.

Hence the polynomial null-space computation is suitable for the problem of interest. However, polynomial null-space computation results concern only univariate polynomials. However, the extension to multi-variate polynomial is possible and an outline of the direction is discussed in the perspectives.

Computational complexity and efficiency

The computational complexity analysis of the proposed algorithm would be a topic of future interest once the implementation is realized completely in a computational environment like MATLAB. This would include analysing the relative efficiencies of deploying Buchberger algorithm for Gröbner basis computation versus the Jacobian evaluation for local structural identifiability.

Another related interest is the computational comparison with other methods. The DAISY software package envisioned to reduce the computational complexity of the approach in [90] by a choice of differential ring that does not contain θ. This works well for biological systems with a number of parameters and a small number of states. However, in engineering systems (for instance, the building energy system), one often encounters a model with large number of states and a relatively few parameters. An application such as DAISY suffers from the same type of computational overhead as [90] had for biological systems. It is to be analyzed whether the parity-space approach can bring in any specific advantages. Similarly to analyze the same type of models for same characteristics, the parity-space approach should also be compared with genSSI and STRIKE-GOLDD toolboxes.

Chapter 7

Perspectives

The thesis presented contributions to quasi-LPV models in two broad categories: estimation of states and parameters, and parameter identifiability. For the estimation part, adaptive observers were proposed using Takagi-Sugeno polytopic techniques to jointly estimate the states and the parameters. This contribution took two different paths for the continuous-time and discrete-time models, but both used the quadratic Lyapunov functions. While the continuous-time case could achieve asymptotic convergence when there was no noise, the discrete-time case provided a bound on the estimation error. For a class of discrete-time quasi-LPV models, a brief of a decoupled state and parameter estimation strategy was also proposed. This investigation was limited by several factors one of which was expanded into an independent contribution: parameter identifiability of quasi-LPV models. Steps towards developing a systematic procedure to verify parameter identifiability of quasi-LPV models was given. The parity-space strategy deploying null-space computation formed the core of this approach applicable to models with measured or known premise variables.

In this chapter, an outline of the open ends from the contributions that are of immediate future interests. These perspectives are split into three parts, that mimics the thesis contributions.

Observer design

In Chapter 3, the fundamental assumption in the proposed continuous-time T-S observer design is that the parameters are constant. Nevertheless, it was shown that, for practical purposes, slowly time-varying parameters can also be estimated. However, this capability has not been rigorously proven. In the unknown input estimation literature, the following strategy is often followed: assume that the parameter θ is a polynomial function of time of degree $(d_\theta - 1)$. That is, $\dot{\theta} \neq 0$ and the higher order derivatives up to $d_\theta - 1$ are non-zero such that $\theta^{(d_\theta)} = 0$. With this, the authors in the unknown input estimation literature use a proportional multiple integral (PMI) structure for parameter estimation equation which fits well into the Lyapunov analysis. This, however, does not directly translate to the results in Chapter 3 for one key reason: the structure for parameter estimation equation is not a priori fixed, but obtained as a consequence of the choice of the Lyapunov function. This makes it difficult to adopt the choice of PMI structure for parameter estimation as in literature difficult. Initial investigation to formally handle time-varying θ through similar and alternative means did not yield useful results. This is an open end for future investigation.

An important class of quasi-LPV models consists of those which have unmeasured premise variables. The approach proposed in Chapter 3 works only for models with measured premise variables. The case for unmeasured premise variable would bring about a discrepancy between the weighting functions $\mu_i(z)$ of the system model and that of $\mu_i(\hat{z})$ of the observer model and hence their differences in the analysis of the error dynamics. A straightforward method to handle unmeasured premise variables is the Lipschitz approach, that is to consider that the extra terms arising out of the discrepancy in weighting functions as bounded by a known Lipschitz constant. The approach is used in several works, for example, see Sec. 7.4 in [20], where it is applied for adaptive observers. The Lipschitz approach is restrictive in the sense that the Lipschitz constant has to be known a priori. Another approach is that used in the Chapter 2 for continuous-time models and in Chapter 4 for discrete-time models. The idea here is to bound the extra terms arising out of the weighting functions' discrepancy by exploiting the convex sum property of the weighting functions. This strategy, as was discussed in the earlier chapters leads to complex LMIs and conservative conditions. There are other approaches to handle unmeasured premise variables as discussed in [129], where the idea of immersion is introduced for this task. This approach is yet to be extended to cases when the system model has unknown parameters. Further, the impact of the immersion procedure on the observability and the parameter identifiability also needs to be investigated. This approach has been extended to the discrete-time case in [130] and can be explored as an alternative to the complex LMI conditions obtained in Chapter 4. These are avenues for future investigations.

Decoupled state and parameter estimation

Chapter 5 provided a brief on decoupling the state and parameter estimation through the use of parity-space. However, this was not sufficiently developed due to the focus on one of the requirements for the decoupled state and parameter estimation, that is, parameter identifiability. The proposed approach can be, in practice, be used for less complex models where the parameters can be represented as a function of inputs and outputs. This is performed a priori using null-space computation in this thesis. The parity-space approach does not guarantee that the parameter can be represented as a function of inputs and outputs. A differential algebra based approach using Ritt's algorithm, as proposed in [90] can be deployed to arrive at the I-O-P equation which has a linear regression structure. This would make the step of representing parameters as a function of inputs, outputs and their derivatives a trivial affair. A brief illustration of this for discrete-time models can be referred to in [91].

An FMO observer strategy was used in Chapter 5 for state estimation. This could, however, be replaced with a polytopic observer for the state estimation as postulated in [82]. These steps need to be formalized and this would be an immediate future work in this direction.

As noted in Sec. 5.6, the FMO strategy is capable of providing an estimate for parameters which are time-varying under certain circumstances. This needs to be developed further from an implementation point of view. As discussed in the works such as [80], the effect of noise in the measurement can be suppressed for the state estimation through well-known methods. This is not straightforward for the parameter estimation case. A simple strategy could be to consider a larger size of the finite memory beyond the minimum required to guarantee parameter estimation (that is, larger than the observability index for applicable models). The parameter shall be estimated several times within this larger memory and an average of these estimates used as the estimated parameter value. This needs to be studied further and formalized.

Parameter identifiability

Parameter identifiability results in Chapter 6 are only the starting point for a complete identifiability analysis algorithm for quasi-LPV models. In this regard, several extensions to the works in this thesis can be envisaged.

Developing a systematic implementation

A systematic and complete implementation of all the steps discussed in Chapter 6 is an immediate future work. The realization could be in the form of a toolbox in MATLAB similar to those such as [97] or [112]. That is a detailed strategy for input methods, options for different methods to evaluate the identifiability and the final display of the results and relevant information. In this regard, an evaluation of the computational complexity and comparison with other methods is also due.

This systematic development also should look to include some methods that have been used in other identifiability analysis tools. For example, *identifiability tableau* of [101] has been integrated in the genSSI toolbox [97]. This would provide better insights into understanding the unidentifiable parameters.

Another notable work to consider is the numerical Jacobian computation approach proposed in [125]. In this work, the authors develop a numerical approximation to compute the Jacobian of a large matrix with careful choice of numerical values and characteristics of the inputs. The authors use this algorithm to verify state observability and parameter identifiability of a nonlinear model. This has been implemented for large-scale models in works such as [126].

A related open problem, but in a different domain, is the multivariate polynomial null-space computation which should extend the works such as in [128], [85] to multivariate case.

Extending results to newer class of system models

The results in Chapter 6 left out some of the extensions possible. It was noted that the parity-space approach would work well for polynomial parametrization as well. This should be formally extended. The initial conditions in Chapter 6 were considered arbitrary. However, there are cases where such assumption can be detrimental as pointed earlier, such as Example 4 in [94] and Example 2 in [122]. These cases need to be carefully handled in the implementation to cover a larger spectrum of models and initial conditions. Further, the case of known initial conditions and partially known initial conditions shall also be handled in this extension.

The results were restricted to measured or known premise variables. The first step in the accommodation of some models with unmeasured premise variables could be through the idea of immersion discussed above, using the ideas outlined in [129]. This would require rigorous analysis of the effect of the immersion procedure on the identifiability properties of the original model.

Sensor placement and optimization

One of the motivations taken out of the project deals with placing sensors for maintenance activities such as fault detection, degradation estimation etc. In this context, the identifiability verification procedure in Chapter 6 can be envisaged as a component to resolve the problem of sensor placement. Graph based approaches are quite popular in the sensor placement optimization problems in linear systems. In [131] the authors provide a procedure to place sensors in chemical plants based on a fault diagnostics observability criteria. Similar strategies have also been developed for parameter identification [132]. Relevant recent works could be referred to in, [133], [134]. However, these works focus on fault diagnosis and hence do not generalize for parameter estimation/identification. An interesting recent work in this direction is in [135] where the authors provide a graph based characterization of parameter identifiability. It is still an open work to use this characterization for optimizing the sensor placement following the works of [133]. In this context, the question that comes up is, whether the insight obtained in parameter identifiability characterization in Chapter 6 can help to extend sensor optimization results for some LPV and quasi-LPV models. For a starting point, consider quasi-LPV models with constant output matrix. Here, the structure associated with the matrix $\mathbb{O}(\theta)$ in (6.21) is

$$\mathbb{O}(\theta) = \begin{bmatrix} \mathbb{C} \\ \mathbb{A} \end{bmatrix}$$

Given that the matrix $\mathbb{A}$ would remain the same with sensor addition, it is interesting to explore if the effect of modifying the measurements could be understood better by analysing the changes in the rank of the matrix $\mathbb{O}(\theta)$ (and hence the null-space and the I-O-P equations). This can help towards establishing constraints for an optimization procedure leading to optimising sensor placement.

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Résumé

Dans cette thèse, deux problèmes liés aux approches basées sur des modèles pour le diagnostic de défauts et l'estimation du niveau de dégradation des équipements dans un bâtiment sont étudiés: la conception d'observateurs adaptatifs pour l'estimation de l'état et des paramètres, et l'analyse de l'identifiabilité des paramètres. La classe des modèles considérés est celle des modèles quasi-linéaires à paramètres variants dans le temps (quasi-LPV) avec paramétrisation affine des matrices d'état. Utilisant l'approche polytopique de Takagi-Sugeno (T-S), deux types d'observateurs sont proposés, un pour des systèmes en temps continu et l'autre pour des systèmes en temps discret. La structure de Luenberger (correction de la dynamique à l'aide de l'erreur d'estimation de la sortie) est choisie pour la partie d'estimation d'état de l'observateur pour les deux et leur conception s'appuie sur l'approche de Lyapunov. Pour la partie d'estimation des paramètres, une structure originale est proposée en temps continu et une structure proportionnelle-intégrale (PI) est utilisée en temps discret. La troisième contribution présente succinctement une méthode d'estimation d'état et des paramètres de façon découplée. Elle utilise conjointement l'approche de l'espace de parité et un observateur à mémoire finie. Pour la quatrième contribution relative à l'identifiabilité des paramètres, les états du système sont tout d'abord éliminés en utilisant une approche de type espace de parité. Cela permet d'extraire le 'résumé exhaustif' du modèle qui aide à établir l'identifiabilité du modèle. Tous les résultats sont illustrés à l'aide d'exemples.

Mots-clés: modèle de Takagi-Sugeno, observateurs polytopiques, estimation conjointe des états et des paramètres, identifiabilité paramétrique, espace de parité

Abstract

Two problems relevant to the model-based approaches to fault diagnosis and degradation estimation in commissioned buildings are investigated in this thesis: adaptive observers for state and parameter estimation, and parameter identifiability. The system models considered are the quasi-LPV models with affine parameterization. Using the Takagi-Sugeno (T-S) polytopic approach, two observer designs, one for continuous-time models and another for discrete-time models are provided. Both models use a Luenberger structure for the state estimation part and deploy the Lyapunov design approach. An innovative non-linear estimation model is obtained through the design process for the continuous-time parameter estimation whereas a proportional-integral (PI) structure is used for discrete-time. A brief third contribution is a decoupled state and parameter estimation that makes use of the parity-space approach and realized using a finite memory observer strategy. For the fourth contribution of parameter identifiability, a parity-space formulation using null-space computation is used for the elimination of states of the model from which the exhaustive summary of the model is extracted and the identifiability of the model verified. All the results are illustrated using examples.

Keywords: Takagi-Sugeno modeling, Polytopic observers, Joint state and parameter estimation, Parameter Identifiability, Parity-space