Hypoelliptic Laplacian and twisted trace formula

Bingxiao Liu

NNT : 2018SACLS165

THÈSE DE DOCTORAT

de

l'Université Paris-Saclay

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Université Paris-Sud

Laboratoire d'accueil : Laboratoire de mathématiques d'Orsay, UMR 8628 CNRS

Spécialité de doctorat : Mathématiques fondamentales

Bingxiao LIU

Laplacien hypoelliptique et formule des traces tordue

Date de soutenance : 15 Juin 2018

Après avis des rapporteurs :
George MARINESCU (Universität zu Köln)
Werner MÜLLER (Mathematisches Institut der Universität Bonn)

Jury de soutenance :

Jean-Michel BISMUT (Université Paris-Sud)	Directeur de thèse
Jochen BRÜNING (Humboldt-Universität zu Berlin)	Examinateur
Laurent CLOZEL (Université Paris-Sud)	Examinateur
Gérard FREIXAS I MONTPLET (C.N.R.S.- I.M.J.)	Examinateur
Colin GUILLARMOU (C.N.R.S.- L.M.O.)	Président du jury
George MARINESCU (Universität zu Köln)	Rapporteur
Werner MÜLLER (Mathematisches Institut der Universität Bonn)	Rapporteur

Thèse préparée au
Département de Mathématiques d'Orsay
Laboratoire de Mathématiques d'Orsay (UMR 8628), Bâtiment 307
Université Paris-Sud
91405 Orsay Cedex
France

A mes parents et ma sœur

Remerciements

Tout d'abord, je voudrais exprimer ma plus profonde gratitude à mon directeur de thèse, Jean-Michel BISMUT. Avec sa gentillesse et sa grande expérience, il m'a bien guidé et apporté son soutien tout au long de la préparation de la thèse. Les nombreuses discussions que nous avons eues ainsi que ses conseils sont pour beaucoup dans le résultat final de ce travail, et ses nombreuses relectures et corrections de cette thèse ont été très appréciables. Je le remercie vivement pour tout le temps qu'il m'a consacré et pour m'avoir appris le métier de chercheur.

Je tiens à remercier vivement George MARINESCU et Werner MÜLLER d'avoir accepté de rapporter sur ma thèse et de faire partie du jury.

Je remercie également Jochen BRÜNING, Laurent CLOZEL, Gérard FREIXAS I MONTPLET et Colin GUILLARMOU d'avoir accepté d'être membres du jury.

Je souhaite exprimer ma reconnaissance à Xinan MA, qui m'a présenté le grand univers de l'analyse géométrique pendant mes études universitaires et m'a suggéré de venir à Paris. Je remercie aussi la Fondation Sciences Mathématiques de Paris et l'Université Paris Diderot pour me donner la chance de suivre le programme de master à Paris.

Mes vifs remerciements vont également à Xiaonan MA pour le soutien et les encouragements constants qu'il m'a prodigués. Il m'a beaucoup aidé non seulement en mathématiques, mais aussi dans la vie quotidienne depuis mon arrivée en France.

Cette thèse a été effectuée au sein du Laboratoire de Mathématiques d'Orsay dont je voudrais remercier tous les membres et particulièrement sa directrice Elisabeth GASSIAT, ses secrétaires et ses informaticiens pour leur aide et leur accueil chaleureux. Je voudrais exprimer ma gratitude à Stéphane NONNENMACHER et Frédéric PAULIN pour leur soutien administratif.

Merci à Wei Guo FOO, Louis IOOS, Quang Huy NGUYEN, Shu SHEN, Yeping ZHANG pour des discussions toujours fructueuses. Merci à Junyan CAO, Yang CAO, Xianglong DUAN, Siarhei FINSKI, Weikun HE, Dinh Tuan HUYNH, Bo LIU, Sai MA, Zicheng QIAN, Chaoyu QUAN, Guokuan SHAO, Rui SHI, Ruoci SUN, Wen SUN, Xiaozong WANG, Qilong WENG, Bo XIA, Cong XUE, Hui ZHU, ... pour avoir partagé leurs connaissances. Merci également à tous les amis que je me suis faits en France. Leur soutien constant a été d'une valeur inestimable.

À titre plus personnel, je remercie chaleureusement ma chère Yuyao qui a su me soutenir, me supporter, m'encourager. . . pendant la durée de ma thèse et plus particulièrement durant les derniers mois de rédaction qui n'ont pas toujours été faciles. Je lui suis très reconnaissant de sa compagnie.

Enfin, je tiens à remercier mes parents et ma sœur pour leur soutien tout au long de mes études en France.

Laplacien hypoelliptique et formule des traces tordue

Résumé

Dans cette thèse, on donne une formule géométrique explicite pour les intégrales orbitales semisimples tordues du noyau de la chaleur sur un espace symétrique, en utilisant la méthode du laplacien hypoelliptique développée par Bismut. On montre que nos résultats sont compatible avec les résultats classiques de la théorie de l'indice équivariant local sur les espaces localement symétriques compacts.

On utilise notre formule explicite pour évaluer le terme dominant dans l'asymptotique quand $d \to +\infty$ de la torsion analytique équivariante de Ray-Singer associée à une famille de fibrés vectoriels plats F_d sur un espace localement symétrique compact. On montre que le terme dominant peut être calculé à l'aide de W-invariants au sens de Bismut-Ma-Zhang.

Mots clefs : laplacian hypoelliptique, intégrale orbitale tordue, formule des traces tordue, torsion analytique équivariante.

Hypoelliptic Laplacian and twisted trace formula

Abstract

In this thesis, we give an explicit geometric formula for the twisted semisimple orbital integrals associated with the heat kernel on symmetric spaces. For that purpose, we use the method of the hypoelliptic Laplacian developed by Bismut. We show that our results are compatible with classical results in local equivariant index theory.

We also use this formula to evaluate the leading term of the asymptotics as $d \to +\infty$ of the equivariant Ray-Singer analytic torsion associated with a family of flat vector bundles F_d on a compact locally symmetric space. We show that the leading term can be evaluated in terms of the W-invariants constructed by Bismut-Ma-Zhang.

Keywords : Hypoelliptic Laplacian, twisted orbital integral, twisted trace formula, equivariant analytic torsion.

Contents

Introduction (en français)

L'objet de cette thèse est de donner une formule géométrique explicite pour les intégrales orbitales semisimples tordues associées au noyau de la chaleur, en utilisant la méthode du laplacien hypoelliptique développée dans [B11].

On utilise notre formule explicite pour évaluer le terme dominant dans l'asymptotique quand $d \to +\infty$ de la torsion analytique équivariante de Ray-Singer associée à une famille de fibrés vectoriels plats F_d sur un espace localement symétrique compact. On montre que le terme dominant peut être calculé à l'aide de W-invariants au sens de [BMZ17].

Table des matières

i. **Un groupe réductif réel.** Soit G un groupe réductif réel connexe d'algèbre de Lie $\mathfrak{g}$, et soit $\theta \in \mathrm{Aut}(G)$ une involution de Cartan de G. Soit K l'ensemble des points fixes de θ dans G. Alors K est un sous-groupe maximal compact de G. Soit $\mathfrak{k}$ l'algèbre de Lie de K, et soit $\mathfrak{p} \subset \mathfrak{g}$ l'espace propre de l'action de θ associé à la valeur propre -1. La décomposition de Cartan de $\mathfrak{g}$ est donnée par

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{i-1}$$

Alors on a

$$[\mathfrak{p}, \mathfrak{p}],\ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{i-2}$$

Soit B une forme bilinéaire symétrique non-dégénérée qui est invariante par G et θ telle que B soit positive sur $\mathfrak{p}$ et négative sur $\mathfrak{k}$.

On pose $m = \dim \mathfrak{p}$, $n = \dim \mathfrak{k}$.

Soit $X = G/K$ l'espace symétrique associé. On note $p : G \to X$ la projection canonique, donc il est un K-fibré principal sur X. Le scindage (i-1) induit une forme de connexion sur ce K-fibré principal.

Le groupe K agit sur $\mathfrak{p}$ par l'action adjointe, on a

$$TX = G \times_K \mathfrak{p}. \tag{i-3}$$

Alors B induit une métrique riemannienne sur X telle que la forme de connexion sur $p : G \to X$ induit la connexion de Levi-Civita ∇^{TX}.

On a que $X \simeq \mathbb{R}^m$ est de courbure sectionnelle nonpositive. On note $d(\cdot, \cdot)$ la distance riemannienne sur X.

ii. **Laplacien hypoelliptique et espaces symétriques.** Soit $\rho^E : K \to \mathrm{Aut}(E)$ une representation unitaire de dimension finie de K, et soit $F = G \times_K E$ le fibré vectoriel associé sur X avec une connexion unitaire ∇^F. En particulier, $\mathfrak{k}$ induit un fibré vectoriel N sur X.

Le fibré vectoriel $TX \oplus N$ est canoniquement trivial sur X. Soit $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ l'espace total de $TX \oplus N$. On a $\widehat{\mathcal{X}} \simeq X \times \mathfrak{g}$.

Soit $U\mathfrak{g}$ l'algèbre enveloppante de $\mathfrak{g}$, et soit $C^{\mathfrak{g}} \in U\mathfrak{g}$ l'opérateur de Casimir associé à B, qui est dans le centre de $U\mathfrak{g}$. Si $e_1, \cdots, e_{m+n}$ est une base de $\mathfrak{g}$, et si $e_1^*, \cdots, e_{m+n}^*$ est la base duale de $\mathfrak{g}$ relativement à B, alors

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m+n} e_i^* e_i. \tag{ii-1}$$

Le Casimir $C^{\mathfrak{g}}$ induit un opérateur elliptique $C^{\mathfrak{g},X}$ agissant sur $C^\infty(X, F)$. Soit $\mathcal{L}^X$ l'opérateur qui diffère par une constante explicite de l'action de $\frac{1}{2} C^{\mathfrak{g},X}$ sur $C^\infty(X, F)$. Pour $t > 0$, on note $\exp(-t\mathcal{L}^X)$ l'opérateur de la chaleur associé.

Par [B11, Sections 0.1, 0.3 et 0.6], le laplacien hypoelliptique $\mathcal{L}_b^X$ est une déformation de $\mathcal{L}^X$, de sorte que si $b \to 0$, $\mathcal{L}_b^X$ converge dans le sens adéquat vers $\mathcal{L}^X$. On rappelle la construction de $\mathcal{L}_b^X$ en abrégé.

Soit $\widehat{D}^{\mathfrak{g},X}$ l'opérateur de Dirac de Kostant [Kos97] associé à $(\mathfrak{g}, B)$. Alors $\widehat{D}^{\mathfrak{g},X}$ agit sur $C^\infty(X, F)$, et son carré est égal à $-2\mathcal{L}^X$. Dans [B11, Chapitre 2], l'auteur a défini un opérateur de Dirac généralisé $\mathfrak{D}_b^X$, $b > 0$ agissant sur $C^\infty(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\cdot(T^*X \oplus N^*) \otimes F))$ en utilisant $\widehat{D}^{\mathfrak{g},X}$ et une version de l'opérateur de Dirac sur la fibre $TX \oplus N$.

Dans [B11, Section 2.13], le laplacien hypoelliptique $\mathcal{L}_b^X$ sur $\widehat{\mathcal{X}}$ est défini par

$$\mathcal{L}_b^X = -\frac{1}{2}\widehat{D}^{\mathfrak{g},X,2} + \frac{1}{2}\mathfrak{D}_b^{X,2}. \tag{ii-2}$$

Par [B11, Proposition 2.15.1], on a

$$[\mathfrak{D}_b^X, \mathcal{L}_b^X] = 0. \tag{ii-3}$$

Soit $\Delta^{TX\oplus N}$ le Laplace usuel le long des fibres $TX \oplus N$. La formule explicite suivante de $\mathcal{L}_b^X$ est établie dans [B11, Section 2.13],

$$\begin{aligned}\mathcal{L}_b^X = \frac{1}{2}|[Y^N, Y^{TX}]|^2 + \frac{1}{2b^2}(-\Delta^{TX\oplus N} + |Y|^2 - m - n) + \frac{N^{\Lambda^\cdot(T^*X\oplus N^*)}}{b^2} \\ + \frac{1}{b}\Big(\nabla^{C^\infty(TX\oplus N, \widehat{\pi}^*(\Lambda^\cdot(T^*X\oplus N^*)\otimes F))}_{Y^{TX}} + \widehat{c}(\mathrm{ad}(Y^{TX})) \\ - c(\mathrm{ad}(Y^{TX}) + i\theta\mathrm{ad}(Y^N)) - i\rho^E(Y^N)\Big).\end{aligned} \tag{ii-4}$$

Par un résultat de Hömander [Hör67], $\mathcal{L}_b^X$ est un opérateur hypoelliptique. La structure de $\mathcal{L}_b^X$ est proche de la structure du laplacien hypoelliptique étudiée dans le travail de Bismut [B05] et de Bismut-Lebeau [BL08]. En fait, étant donnée une variété riemannienne M, la théorie générale du laplacian hypoelliptique [B05] peut donner une famille d'opérateur $L_b|_{b>0}$ sur TM interpolant le laplacien elliptique sur

M (lorsque $b \to 0$) et le flot géodésique sur TM (lorsque $b \to +\infty$). Alors on peut espérer que l'opérateur $\mathcal{L}_b^X$ a des propriétés similaires.

Des méthodes d'analyse ont été développées par Bismut pour obtenir les propriétés convenables de la résolvante de $\mathcal{L}_b^X$. Pour $t > 0$, soit $\exp(-t\mathcal{L}_b^X)$ l'opérateur de la chaleur associé à $\mathcal{L}_b^X$. Dans [B11], on a démontré que $\exp(-t\mathcal{L}_b^X)$ possède un noyau de la chaleur lisse $q_{b,t}^X$, et que le noyau $q_{b,t}^X$ converge dans le sens adéquat vers le noyau de $\exp(-t\mathcal{L}^X)$ lorsque $b \to 0$.

Dans la section 3 de la présente thèse, nous rappelons la construction de $\mathcal{L}_b^X$ sur $\widehat{\mathcal{X}}$ en plus de détail. Dans la sous-section 3.7, nous rappelons aussi des résultats sur $q_{b,t}^X$ établis dans [B11, Chapitres 4 et 11].

iii. **Intégrales orbitales semisimples.** Soit $\mathrm{Isom}(X)$ le groupe de Lie d'isométries de X, et soit $\mathrm{Isom}(X)^0$ la composante connexe de l'identité. Nous avons l'homomorphisme évident $G \to \mathrm{Isom}(X)^0$.

Si $\phi \in \mathrm{Isom}(X)$, soit d_ϕ la fonction de déplacement sur X associée à ϕ. Alors d_ϕ est une fonction convexe. Comme dans [E96], ϕ est dit semisimple si d_ϕ atteint sa valeur infimum m_ϕ dans X, et ϕ est dit elliptique si ϕ a des points fixes dans X. Si ϕ est semisimple, soit $X(\phi) \subset X$ l'ensemble minimisant de d_ϕ, qui est une sous-variété convexe de X.

Dans [B11, Chapitres 3 et 4], on a donné une interprétation géométrique pour les intégrales orbitales associées à un élément semisimple $\gamma \in G$. Ainsi que $X(\gamma)$ est un espace symétrique associé au centralisateur $Z(\gamma)$ de γ. Alors l'espace total du fibré normal $N_{X(\gamma)/X}$ peut être identifié avec X. Etant donné un opérateur dont le noyau de Schwartz a une propriété de décroissance gaussienne appropriée, son intégrale orbitale associée à γ peut être écrite comme intégration le long de la fibre $N_{X(\gamma)/X}$. En particulier, les intégrales orbitales $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$, $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)]$ sont bien définies. Ces intégrales orbitales sont dites elliptiques et hypoelliptiques.

Dans [B11, Chapitre 4], on a montré que les intégrales orbitales $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$, $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)]$ coïncident pour $t > 0$, $b > 0$. En utilisant ce fait, dans [B11, Chapitre 6], on a donné une formule géométrique explicite pour $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$, qui est obtenue en calculant la limite de $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)]$ lorsque $b \to +\infty$.

En utilisant ce résultat, Shu Shen [S18] a donné une démonstration de la conjecture de Fried pour des espaces compacts localement symétriques, complétant le travail de Moscovici et Stanton dans [MS91].

iv. **Intégrales orbitales tordues.** Soit Σ le sous-groupe compact de $\mathrm{Aut}(G)$ qui se compose des automorphismes de (G, B, θ). Si $\sigma \in \Sigma$, soit Σ^σ le sous-groupe fermé de Σ engendré par σ. On pose

$$G^\sigma = G \times \Sigma^\sigma,\ K^\sigma = K \rtimes \Sigma^\sigma. \tag{iv-1}$$

Si $\sigma \in \Sigma$, on définit la conjugation tordue C^σ sur G telle que si $\gamma, h \in G$,

$$C^\sigma(h)\gamma = h\gamma\sigma(h^{-1}). \tag{iv-2}$$

Alors C^σ définit une action de G sur G. Si $\gamma \in G$, on dénote $[\gamma]_\sigma$ l'orbite de γ. Soit $Z(\gamma\sigma)$ le stabilisateur de γ sous l'action de G par C^σ. Alors on a

$$Z(\gamma\sigma) = \{g \in G \ : \ \gamma\sigma g = g\gamma\sigma \in G^\sigma\}. \tag{iv-3}$$

On a l'identification

$$[\gamma]_\sigma \simeq Z(\gamma\sigma)\backslash G. \tag{iv-4}$$

Comme l'action de σ préserve K et B, G^σ agit sur X isométriquement. Soit $\gamma \in G$ tel que $\gamma\sigma$ soit semisimple, et soit $X(\gamma\sigma) \subset X$ l'ensemble minimisant de $d_{\gamma\sigma}$. On montre que $X(\gamma\sigma)$ est aussi un espace symétrique et que $[\gamma]_\sigma$ est un sous-ensemble fermé dans G. Dans la sous-section 1.5, nous étendons les constructions géométriques de [B11, Chapitre 3] à notre cas.

Nous supposons également que E s'étend comme représentation unitaire de K^σ : la question de l'existence de tels relèvements sera examinée plus en détail dans la section 2. L'action de G^σ sur X se relève à F.

Soit $\mathcal{Q}^\sigma$ une algèbre d'opérateurs agissant sur $C^b(X,F)$ qui commutent avec G^σ et qui ont une propriété de décroissance gaussienne appropriée.

Dans la section 4, nous montrons que si $\gamma\sigma$ est semisimple, si $Q \in \mathcal{Q}^\sigma$ a pour le noyau $q \in C(G, \mathrm{End}(E))$, on peut définit une intégrale $\mathrm{Tr}^{[\gamma\sigma]}[Q]$ par la formule

$$\mathrm{Tr}^{[\gamma\sigma]}[Q] = \int_{Z(\gamma\sigma)\backslash G} \mathrm{Tr}^E[\rho^E(\sigma)q(g^{-1}\gamma\sigma(g))]dg. \tag{iv-5}$$

Comme indiqué par la notation, $\mathrm{Tr}^{[\gamma\sigma]}[Q]$ ne dépend que de la classe de conjugaison $[\gamma\sigma]$ de $\gamma\sigma$ dans G^σ. On les appelle intégrales orbitales tordues [L80, Fli82, C84, ArC89, Lip15, BeLi17]. Dans la sous-section 4.2, nous donnons aussi une description géométrique pour $\mathrm{Tr}^{[\gamma\sigma]}[Q]$.

Les opérateurs $\mathcal{L}^X$, $\mathcal{L}^X_b$ commutent avec l'action de G^σ. Donc $\exp(-t\mathcal{L}^X)$ est dans $\mathcal{Q}^\sigma$, alors on a l'intégrale orbitale tordue correspondante $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)]$. Dans la sous-section 4.3, nous étendons la définition des intégrales orbitales tordues aux intégrales orbitales tordues hypoelliptiques $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_b)]$.

Soit Γ un sous-groupe discret cocompact de G tel que $\sigma(\Gamma) \subset \Gamma$. Pour simplifier, nous supposons que Γ est sans torsion, de sorte que $Z = \Gamma\backslash X$ est une variété lisse compacte équipée d'une action de Σ^σ.

Le fibré vectoriel F descend en un fibré vectoriel sur Z que nous notons encore F. L'action de Σ^σ sur Z se relève au fibré F. Si $Q \in \mathcal{Q}^\sigma$, alors Q descend en un opérateur Q^Z agissant sur $C^\infty(Z,F)$.

Dans la sous-section 1.8, nous montrons que si $\gamma \in \Gamma$, $\gamma\sigma$ est semisimple, de telle sorte que $\mathrm{Tr}^{[\gamma\sigma]}[Q]$ est bien définie. De plus, $\Gamma \cap Z(\gamma\sigma)$ est un sous-groupe discret cocompact de $Z(\gamma\sigma)$, de telle sorte que $\Gamma \cap Z(\gamma\sigma)\backslash X(\gamma\sigma)$ est compact.

Soit $\underline{C}$ l'ensemble des classes de conjugaison tordues de Γ définies à la Definition 1.8.2. Dans la sous-section 4.5, d'après Langlands [L80], Flicker [Fli82] et Bergeron-Lipnowski [BeLi17], nous récupérons une version tordue de la formule des traces de Selberg [Sel56],

$$\mathrm{Tr}[\sigma^Z Q^Z] = \sum_{\underline{[\gamma]_\sigma} \in \underline{C}} \mathrm{Vol}(\Gamma \cap Z(\gamma\sigma)\backslash X(\gamma\sigma))\mathrm{Tr}^{[\gamma\sigma]}[Q]. \tag{iv-6}$$

Dans la suite, on considère le cas où $Q = \exp(-t\mathcal{L}^X)$, $t > 0$.

v. **Résultats de la thèse.** Dans la section 4, nous établissons une identité fondamentale qui dit que, pour $b > 0$, $t > 0$,

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_b^X)]. \tag{v-1}$$

En fait, en utilisant (ii-3), on montre que la dérivée du côté droit de (v-1) par rapport à $b > 0$ est nulle, alors le côté droit ne dépend pas de b. Donc (v-1) est une conséquence du fait que le noyau $q_{b,t}^X$ converge dans le sens adéquat vers le noyau de $\exp(-t\mathcal{L}^X)$ lorsque $b \to 0$.

Nous faisons alors $b \to +\infty$ dans (v-1). L'évaluation de la limite du côté droit se concentre autour $X(\gamma\sigma)$, où la description géométrique des intégrales orbitales tordues joue un rôle important.

Décrivons plus en détail notre résultat principal. Si $\gamma\sigma$ est semisimple, après conjugaison, on peut supposer que

$$\gamma = e^a k^{-1},\ a \in \mathfrak{p},\ k \in K,\ \mathrm{Ad}(k^{-1})\sigma a = a. \tag{v-2}$$

On pose

$$K(\gamma\sigma) = Z(\gamma\sigma) \cap K. \tag{v-3}$$

Soit $\mathfrak{z}(\gamma\sigma)$, $\mathfrak{k}(\gamma\sigma)$ algèbres de Lie de $Z(\gamma\sigma)$, $K(\gamma\sigma)$. On a le scindage

$$\mathfrak{z}(\gamma\sigma) = \mathfrak{p}(\gamma\sigma) \oplus \mathfrak{k}(\gamma\sigma), \tag{v-4}$$

où $\mathfrak{p}(\gamma\sigma)$ est l'intersection de $\mathfrak{z}(\gamma\sigma)$ et $\mathfrak{p}$.

On pose $\mathfrak{z}_0 = \ker \mathrm{ad}(a)$. Alors $\mathfrak{z}(\gamma\sigma) \subset \mathfrak{z}_0$. Soit $\mathfrak{z}_0^\perp$ l'orthogonal à $\mathfrak{z}_0$ dans $\mathfrak{g}$. Soit $\mathfrak{z}_0^\perp(\gamma\sigma)$ l'orthogonal à $\mathfrak{z}(\gamma\sigma)$ dans $\mathfrak{z}_0$, alors on a le scindage

$$\mathfrak{z}_0^\perp(\gamma\sigma) = \mathfrak{p}_0^\perp(\gamma\sigma) \oplus \mathfrak{k}_0^\perp(\gamma\sigma). \tag{v-5}$$

Dans la sous-section 5.1, pour $Y_0^\mathfrak{k} \in \mathfrak{k}(\gamma\sigma)$, nous définissons une fonction analytique $J_{\gamma\sigma}$ sur $\mathfrak{k}(\gamma\sigma)$ par la formule

$$\begin{aligned} J_{\gamma\sigma}(Y_0^\mathfrak{k}) = & \frac{1}{|\det(1-\mathrm{Ad}(\gamma\sigma))|_{\mathfrak{z}_0^\perp}|^{1/2}} \frac{\widehat{A}(i\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{p}(\gamma\sigma)})}{\widehat{A}(i\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{k}(\gamma\sigma)})} \\ & \left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}} \frac{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}_0^\perp(\gamma\sigma)}}{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}_0^\perp(\gamma\sigma)}}\right]^{1/2}. \end{aligned} \tag{v-6}$$

Le résultat essentiel de cette thèse est le suivant.

Théorème 1. *Pour $t > 0$, on a l'identité suivante :*

$$\begin{aligned} \mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = & \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \\ & \int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}(Y_0^\mathfrak{k}) \mathrm{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y_0^\mathfrak{k}))] \\ & \exp(-|Y_0^\mathfrak{k}|^2/2t) \frac{dY_0^\mathfrak{k}}{(2\pi t)^{q/2}}. \end{aligned} \tag{v-7}$$

Si $\sigma = \mathrm{Id}_G$, nous récupérons la formule obtenue dans [B11, Théorème 6.1.1]. En utilisant (v-7), on peut aussi obtenir des formules pour les intégrales orbitales tordues plus générales.

Notez que les fonctions $\widehat{A}$ sur $\mathfrak{p}$ et $\mathfrak{k}$ (avec des rôles différents) apparaissent dans la fonction $J_{\gamma\sigma}(Y_0^{\mathfrak{k}})$. Le caractère de (E, ρ^E) apparaît aussi naturellement dans (v-7). La formule (v-7) présent des analogies avec la formule du point fixe de Lefschetz d'Atiyah-Bott [AB67, AB68], où on a le $\widehat{A}$-genre équivariant et le caractère de Chern équivariant.

Soit A un endomorphisme auto adjoint de E qui commute avec l'action de K^σ. On considère A comme une section parallèle de $\mathrm{End}(F)$ commutant avec l'action de G^σ. On pose

$$\mathcal{L}_A^X = \mathcal{L}^X + A. \tag{v-8}$$

Le Théorème 1 s'étend au cas de $\exp(-t\mathcal{L}_A^X)$.

vi. **Intégrales orbitales tordues et théorème de l'indice local.** L'opérateur $\mathcal{L}^X$ descend en un laplacien $\mathcal{L}^Z$ sur Z. Soit D^Z l'opérateur de Dirac sur Z. Par [B11, Sections 7.2 et 7.3], à une constante près, $D^{Z,2}$ coïncide avec $2\mathcal{L}^Z$. Le nombre de Lefschetz $\chi_\sigma(F)$ est donné par

$$\chi_\sigma(F) = \mathrm{Tr_s}[\sigma^Z \exp(-tD^{Z,2}/2)]. \tag{vi-1}$$

Soit ${}^\sigma Z$ l'ensemble des points fixes de σ dans Z. Alors $\chi_\sigma(F)$ peut être calculé par le théorème du point fixe de Lefschetz de Atiyah-Bott [AB67, AB68], de telle sorte que

$$\chi_\sigma(F) = \int_{{}^\sigma Z} \widehat{A}^\sigma(TZ|_{{}^\sigma Z}, \nabla^{TZ|_{{}^\sigma Z}})\mathrm{ch}^\sigma(F, \nabla^F). \tag{vi-2}$$

Dans le Lemme 1.8.7, on montre que ${}^\sigma Z$ est l'union de $\Gamma \cap Z(\gamma\sigma)\backslash X(\gamma\sigma) \subset Z$ avec elliptique $\gamma\sigma$, $\gamma \in \Gamma$. Dans la section 7, nous vérifions que si nous évaluons le côté droit de (vi-1) en utilisant (iv-6), (v-7), nous récupérons l'équation (vi-2). Pour ce faire, nous devons explorer en détail la théorie de la représentation du groupe K^σ.

Dans la section 2, à la suite de [L80, C84, Bou87, DK00, BeLi17], nous donnons une classification des représentations de K^σ à l'aide des racines de K, pour nous permettre d'évaluer le caractère de K^σ dans la partie droite de (v-7). Plus précisément, on construit un élément $\tau \in \mathrm{Aut}(K)$ d'ordre fini, de telle sorte que les représentations de K^σ puissent être transformées en représentations de K^τ. Alors τ agit sur l'ensemble des poids dominants P_{++}.

Soient $\mathrm{Irr}(K^\sigma)$, $\mathrm{Irr}(\Sigma^\sigma)$ les ensembles des classes d'équivalence des représentations unitaires irréductibles de K^σ, Σ^σ. Dans la sous-section 2.4, nous montrons que

$$\mathrm{Irr}(\Sigma^\sigma)\backslash\mathrm{Irr}(K^\sigma) \simeq \left\{\begin{matrix}\text{orbites dans } P_{++} \text{ sous l'action du} \\ \text{groupe fini engendré par } \tau\end{matrix}\right\}. \tag{vi-3}$$

vii. **Torsion analytique équivariante de Ray-Singer sur** Z**.** Si E est une représentation de G^σ, alors $F = G\times_K E$ est un fibré vectoriel plat sur X, et F descend en un fibré vectoriel plat sur Z. Soit $\nabla^{F,f}$ la connexion plat canonique sur F, et soit $(\Omega^\cdot(Z,F), d^{Z,F})$ le complexe de de Rham associé. Soit $d^{Z,F,*}$ l'adjoint formal de $d^{Z,F}$. On pose

$$\mathbf{D}^{Z,F} = d^{Z,F} + d^{Z,F,*}. \tag{vii-1}$$

À un endomorphisme auto adjoint près, l'opérateur $2\mathcal{L}^Z$ coïncide avec le laplacien de Hodge $\mathbf{D}^{Z,F,2}$. Soit $N^{\Lambda^\cdot(T^*Z)}$ l'opérateur de nombre sur $\Omega^\cdot(Z,F)$. Soit $P^\perp$ la projection orthogonale sur $(\ker \mathbf{D}^{Z,F})^\perp$, l'espace orthogonal à $\ker \mathbf{D}^{Z,F}$ dans $\Omega^\cdot(Z,F)$, et soit $[\mathbf{D}^{Z,F,2}]^{-1}$ l'inverse de $\mathbf{D}^{Z,F,2}$ agissant sur $(\ker \mathbf{D}^{Z,F})^\perp$.

Pour $s\in\mathbb{C}$ et $\mathrm{Re}(s)$ assez grand, on pose

$$\vartheta_\sigma(g^{TZ},\nabla^{F,f},g^F)(s) = -\mathrm{Tr_s}\left[N^{\Lambda^\cdot(T^*Z)}\sigma[\mathbf{D}^{Z,F,2}]^{-s}P^\perp\right]. \tag{vii-2}$$

Alors $\vartheta_\sigma(g^{TZ},\nabla^{F,f},g^F)(s)$ s'étend en une fonction méromorphe de $s\in\mathbb{C}$, qui est holomorphe en $s=0$.

On définit la torsion analytique équivariante de Ray-Singer par la formule

$$\mathcal{T}_\sigma(g^{TZ},\nabla^{F,f},g^F) = \frac{1}{2}\frac{\partial\vartheta_\sigma(g^{TZ},\nabla^{F,f},g^F)}{\partial s}(0). \tag{vii-3}$$

Si $\sigma=\mathrm{Id}_G$, il s'agit simplement de la torsion analytique ordinaire de Ray-Singer [RS71, RS73], on la note par $\mathcal{T}(g^{TZ},\nabla^{F,f},g^F)$.

Dans la sous-section 7.8, comme dans [BMZ17, Section 8], nous obtenons une formule géométrique pour les intégrales orbitales tordues pour le noyau de la chaleur qui apparaîssent dans l'évaluation de la torsion analytique équivariante de Ray-Singer. On obtient alors des résultats sur $\mathcal{T}_\sigma(g^{TZ},\nabla^{F,f},g^F)$.

Si γ est sous la forme dans (v-2), on pose

$$\epsilon(\gamma\sigma) = \mathrm{rk}_\mathbb{C}(Z(\gamma\sigma)) - \mathrm{rk}_\mathbb{C}(K(\gamma\sigma)) \in \mathbb{N}. \tag{vii-4}$$

L'entier $\epsilon(\gamma\sigma)$ ne dépend que de la class $[\gamma]_\sigma$.

Proposition 1. *Si une des trois hypothèses est vérifiée :*

(1) m est pair et σ preserve l'orientation de $\mathfrak{p}$;

(2) m est impair et σ ne preserve pas l'orientation de $\mathfrak{p}$;

(3) Pour $\gamma\in\Gamma$, $\epsilon(\gamma\sigma)\neq 1$,

alors on a

$$\mathcal{T}_\sigma(g^{TZ},\nabla^{F,f},g^F) = 0 \tag{vii-5}$$

La Proposition 1 étend des résultats dans [MS91, Corollaire 2.2], [Lot94, Proposition 9], [BL95, Théorème 3.26], [B11, Section 7.9], [BMZ17, Théorème 8.6].

viii. **Asymptotique de la torsion équivariante de Ray-Singer.** Dans la section 9, nous utilisons le Théorème 1 pour obtenir une version locale de l'asymptotique de la torsion analytique équivariante de Ray-Singer lorsque le fibré vectoriel F tend vers l'infini en un sens adéquat.

Bergeron et Venkatesh [BeV13] ont considéré le comportement asymptotique de la torsion analytique d'espaces localement symétriques par revêtement fini. Müller [Mül12] a initié l'étude de la torsion analytique de Ray-Singer pour les puissances symétriques d'un fibré vectoriel plat donné sur les variétés hyperboliques. Bismut-Ma-Zhang [BMZ17] et Müller-Pfaff [MüP13] ont également étudié la suite de fibrés vectoriels plats associés aux multiples d'un poids dominant qui induisent les représentations correspondantes de la forme compacte U de G. Ici, nous nous intéressons à l'asymptotique de la torsion analytique équivariante de Ray-Singer pour un espace localement symétrique compact Z. Ce problème a déjà été considéré par Ksenia Fedosova [Fed15] par des méthodes d'analyse harmonique sur le groupe réductif G. Ici, comme dans [BMZ17], nous allons utiliser la formule explicite du Théorème 1.

Nous supposons que l'action de σ sur G s'étend en un automorphisme de U. On pose

$$U^{\sigma} = U \rtimes \Sigma^{\sigma}. \tag{viii-1}$$

Dans la suite, nous supposons que (E, ρ^E) est une représentation unitaire de U^{σ}. En utilisant l'astuce unitaire de Weyl, cette représentation s'étend en une représentation de G^{σ}, alors on obtient un fibré vectoriel plat F sur X ou Z équipé d'une action de Σ^{σ}. On considère principalement la torsion analytique équivariante associée à l'action de σ.

Dans la sous-section 8.2, en conséquence de (v-7) et (vi-3), nous nous ramenons au cas où E est U-irréductible, et où le poids λ de E est fixé par σ. D'abord, on peut construire une famille de représentations E_d de U associée à λ, en remplaçant λ par $d\lambda$, $d \in \mathbb{N}$. Ensuite, par (vi-3), on peut étendre chaque E_d en une représentation de U^{σ}, mais en générale, l'extension n'est pas unique. On utilise l'idée de [BMZ17] pour donner une façon canonique de construire les extensions.

Soit M_{λ} la variété de drapeaux associée à λ, de telle sorte que U^{σ} agit holomorphiquement sur M_{λ}, et que cette action se relève au fibré en droite canonique associé $L_{\lambda} \to M_{\lambda}$. Alors pour $d \in \mathbb{N}$, U^{σ} agit sur $H^{(0,0)}(M_{\lambda}, L_{\lambda}^d)$. Nous obtenons une famille de représentations irréductibles (E_d, ρ^{E_d}) de U^{σ} donnée par $H^{(0,0)}(M_{\lambda}, L_{\lambda}^d)$, $d \in \mathbb{N}$.

Soit F_d le fibré vectoriel plat associé à l'action de G^{σ} sur E_d, et soit $\mathbf{D}^{Z,F_d}$ l'opérateur défini dans (vii-1) pour le fibré F_d. Dans [BMZ17], on a introduit une condition de non-dégénérescence, et on a montré que si cette condition est vérifiée, il y a des constantes $c > 0$, $C > 0$ telles que pour $d \in \mathbb{N}$,

$$\mathbf{D}^{Z,F_d} \geq cd^2 - C. \tag{viii-2}$$

Dans [BMZ17], un résultat important est la construction du W-invariant, où on a montré que sous ladite condition de non-dégénérescence, le terme dominant de l'asymptotique de $\mathcal{T}(g^{TZ}, \nabla^{F_d,f}, g^{F_d})$ lorsque $d \to +\infty$ est donné par le W-invariant, qui peut être calculé localement.

Dans la dernière partie de la présente thèse, on se consacre à l'extension de ce résultat en cas de la torsion équivariante. Dans la sous-section 9.3, on montre que sous la même condition de non-dégénérescence, quand $d \to +\infty$, l'asymptotique de $\mathcal{T}_\sigma(g^{TZ}, \nabla^{F_d,f}, g^{F_d})$ peut être évaluée à l'aide de formes différentielles explicites $W_{\gamma\sigma}$ associées à des éléments elliptiques $\gamma\sigma$, $\gamma \in \Gamma$, qui sont exactement les W-invariants au sens de [BMZ17, Sections 2 et 8]. Les invariants $W_{\gamma\sigma}$ forment un ensemble fini d'invariants sur les points fixes de σ sur Z.

Une différence entre notre résultats et les résultats de [BMZ17, Section 8] est la présence de facteurs oscillants de la forme $\exp(c_{\gamma\sigma}d\sqrt{-1})$, $c_{\gamma\sigma} \in \mathbb{R}$. En fait, en calculant le terme dominant de l'asymptotique de $\mathcal{T}_\sigma(g^{TZ}, \nabla^{F_d,f}, g^{F_d})$ par (v-7), il faut évaluer l'asymptotique de $\mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma\exp(iy/d))]$ avec $k \in K$, $y \in \mathfrak{k}(\gamma\sigma)$ lorsque $d \to +\infty$. Quand on utilise le théorème du point fixe de Berline-Vergne [BV85] pour ce faire, on voir que le terme dominant de $\mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma\exp(iy/d))]$ est une somme finie des intégrales de Duistermaat-Heckman [DH82, DH83] associées les points fixes de $k^{-1}\sigma$ sur M_λ. Si $z \in M_\lambda$ est fixé par $k^{-1}\sigma$, l'action de $k^{-1}\sigma$ sur $L_{\lambda,z}$ est représentée par un nombre $h \in \mathbb{S}^1$, alors pour $d \in \mathbb{N}$, l'action de $k^{-1}\sigma$ sur $L^d_{\lambda,z}$ est représentée par h^d, qui est justement un facteur oscillant susdit.

Dans la Proposition 9.3.1, en utilisant (viii-2), nous montrons que la contribution des éléments non-elliptiques $\gamma\sigma$, $\gamma \in \Gamma$ à l'asymptotique de la torsion analytique équivariante de Ray-Singer est exponentiellement petite.

Nos résultats sont compatibles avec les résultats de Ksenia Fedosova [Fed15]. Dans [Fed15], on a considéré l'asymptotique de la torsion analytique de Ray-Singer pour des orbifolds hyperboliques compacts. En utilisant la formule des traces de Selberg, elle a montré que les éléments elliptiques de Γ contribuaient à l'asymptotique de la torsion analytique de Ray-Singer par un pseudo-polynôme en d contenant également des facteurs oscillants, et que la contribution des éléments non-elliptiques dans Γ est exponentiellement petite.

ix. **Structure de la thèse.** La thèse est structurée de la manière suivante. Dans la section 1, nous introduisons une extension $\widetilde{G}$ de G par un groupe compact d'automorphismes Σ, et nous établissons les constructions géométriques associées à l'action des éléments semisimples de $\widetilde{G}$ sur X.

Dans la section 2, nous classifions les représentations irréductibles de K^σ, et nous donnons une formule de caractère de Weyl pour K^σ.

Dans la section 3, nous rappelons la construction du laplacien hypoelliptique associé à (G, K), et les propriétés de son noyau de la chaleur dans [B11].

Dans la section 4, nous définissons les intégrales orbitales tordues associées à un élément semisimple $\gamma\sigma$. Dans la sous-section 4.5, nous dérivons la version tordue de la formule des traces de Selberg pour les espaces localements symétriques compacts.

Dans la section 5, nous montrons le résultat essentiel de la présente thèse, et nous donnons quelques extensions.

Dans la section 6, nous rappelons la formule explicite du noyau de la chaleur hypoelliptique sur l'espace vectoriel euclidien, et nous montrons que notre formule est compatible avec les calculs de [B11, Section 10.6].

Dans la section 7, nous montrons la compatibilité de notre formule avec les résultats de la théorie de l'indice équivariant local. Dans la sous-section 7.8, nous obtenons des résultats sur l'évaluation de la torsion analytique équivariante de Ray-Singer sur les espaces localement symétriques compacts.

Dans la section 8, nous construisons une suite de représentations E_d de G^σ et une famille de fibrés vectoriels plats F_d sur Z.

Enfin, dans la section 9, nous calculons l'asymptotique de la torsion analytique équivariante de Ray-Singer lorsque $d \to +\infty$.

Dans tout la thèse, si $E = E_+ \oplus E_-$ est un espace vectoriel $\mathbb{Z}_2$-gradué, et si $\tau = \pm 1$ définit cette structure $\mathbb{Z}_2$-graduée sur E, si $A \in \mathrm{End}(E)$, on définit la supertrace de A par

$$\mathrm{Tr_s}[A] = \mathrm{Tr}^E[\tau A]. \tag{ix-1}$$

Si $\mathcal{A}$ est une algèbre $\mathbb{Z}_2$-graduée, si $a, b \in \mathcal{A}$, on note $[a, b]$ le supercommutateur de a, b, de tell sorte que

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba. \tag{ix-2}$$

Si $\mathcal{B}$ est une autre algèbre $\mathbb{Z}_2$-graduée, on note $\mathcal{A}\widehat{\otimes}\mathcal{B}$ le produit tensoriel $\mathbb{Z}_2$-gradué de $\mathcal{A}$ et $\mathcal{B}$.

Introduction

The purpose of this thesis is to give an explicit geometric formula for the twisted semisimple orbital integrals associated with the heat kernel. For that purpose, we use the method of the hypoelliptic Laplacian developed in [B11].

We also show that our results are compatible with classical results in the local equivariant index theory for compact locally symmetric spaces. In the last part of the present thesis, we use this formula to evaluate the leading term in the asymptotic expansion as $d \to +\infty$ of the equivariant Ray-Singer analytic torsion associated with a family of flat vector bundles F_d on a compact locally symmetric space.

0.1. **A real reductive group.** Let G be a connected real reductive group with Lie algebra $\mathfrak{g}$, and let $\theta \in \mathrm{Aut}(G)$ be a Cartan involution. Let K be the fixed point set of θ in G. Then K is a maximal compact subgroup of G. Let $\mathfrak{k}$ be its Lie algebra, and let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace of θ associated with the eigenvalue -1. The Cartan decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{0.1.1}$$

And we have

$$[\mathfrak{p}, \mathfrak{p}],\ [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{0.1.2}$$

Let B be a G and θ-invariant nondegenerate bilinear form on $\mathfrak{g}$, which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Put $m = \dim \mathfrak{p}$, $n = \dim \mathfrak{k}$.

One main geometric object in this thesis is the symmetric space $X = G/K$. The form B induces a Riemannian metric on X, so that $X \simeq \mathfrak{p}$ with nonpositive sectional curvature. Let $d(\cdot, \cdot)$ denote the Riemannian distance on X.

0.2. **Hypoelliptic Laplacian and symmetric spaces.** If (E, ρ^E) is a unitary representation of K of finite dimension, then $F = G \times_K E$ is a Hermitian vector bundle on X. In particular, $\mathfrak{p}$, $\mathfrak{k}$ descends to the vector bundles TX, N on X. Then $TX \oplus N$ is canonically trivial on X. Let $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ be the total space of $TX \oplus N$, so that $\widehat{\mathcal{X}} = X \times \mathfrak{g}$.

Let $U\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$, and let $C^{\mathfrak{g}} \in U\mathfrak{g}$ be the Casimir operator associated with B. Then $C^{\mathfrak{g}}$ lies in the center of $U\mathfrak{g}$. Also $C^{\mathfrak{g}}$ descends to an elliptic operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X, F)$. Let $\mathcal{L}^X$ be the operator which differs by an explicit constant from the action of $\frac{1}{2}C^{\mathfrak{g},X}$ on $C^\infty(X, F)$. For $t > 0$, let $\exp(-t\mathcal{L}^X)$ be the associated heat operator.

As explained by Bismut in [B11, Sections 0.1, 0.3 and 0.6], the hypoelliptic Laplacian $\mathcal{L}^X_b$ is considered to be a deformation of $\mathcal{L}^X$, so that as $b \to 0$, $\mathcal{L}^X_b$ converges in the proper sense to $\mathcal{L}^X$. We introduce briefly the construction of $\mathcal{L}^X_b|_{b>0}$.

Let $\widehat{D}^{\mathfrak{g},X}$ be the Dirac operator of Kostant [Kos97] associated with $(\mathfrak{g}, B)$, whose square coincides with $-2\mathcal{L}^X$ acting on $C^\infty(X, F)$. In [B11, Section 2.12], the author defined a generalized Dirac operator $\mathfrak{D}^X_b$, $b > 0$ acting on $C^\infty(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\cdot(T^*X \oplus N^*) \otimes F))$ by combining $\widehat{D}^{\mathfrak{g},X}$ and a version of Dirac operator along the fiber $TX \oplus N$.

The hypoelliptic Laplacian $\mathcal{L}_b^X$ on $\widehat{\mathcal{X}}$ is defined as

$$\mathcal{L}_b^X = -\frac{1}{2}\widehat{D}^{\mathfrak{g},X,2} + \frac{1}{2}\mathfrak{D}_b^{X,2}. \tag{0.2.1}$$

Let $\Delta^{TX\oplus N}$ be the standard Laplace along the fiber $TX \oplus N$. The following explicit formula of $\mathcal{L}_b^X$ is established in [B11, Section 2.13],

$$\begin{aligned}\mathcal{L}_b^X = \frac{1}{2}|[Y^N, Y^{TX}]|^2 + \frac{1}{2b^2}(-\Delta^{TX\oplus N} + |Y|^2 - m - n) + \frac{N^{\Lambda^\cdot(T^*X\oplus N^*)}}{b^2}\\ +\frac{1}{b}\Big(\nabla_{Y^{TX}}^{C^\infty(TX\oplus N,\widehat{\pi}^*(\Lambda^\cdot(T^*X\oplus N^*)\otimes F))} + \widehat{c}(\mathrm{ad}(Y^{TX}))\\ -c(\mathrm{ad}(Y^{TX}) + i\theta\mathrm{ad}(Y^N)) - i\rho^E(Y^N)\Big).\end{aligned} \tag{0.2.2}$$

The structure of $\mathcal{L}_b^X$ is very closed to the structure of the hypoelliptic Laplacian studied in the work of Bismut [B05] and Bismut-Lebeau [BL08].

In [B11], the proper functional analytic machinery was developed in order to obtain the analytic properties of the resolvent and of the heat kernel of $\mathcal{L}_b^X$.

Let $\exp(-t\mathcal{L}_b^X)$ be the heat operator associated with $\mathcal{L}_b^X$. In [B11], Bismut proved that there is a smooth heat kernel $q_{b,t}^X$ associated with $\exp(-t\mathcal{L}_b^X)$, and that as $b \to 0$, the kernel $q_{b,t}^X$ converges in the proper sense to the kernel of $\exp(-t\mathcal{L}^X)$.

In section 3 of the present thesis, we recall the construction of $\mathcal{L}_b^X|_{b>0}$ on $\widehat{\mathcal{X}}$ with more details, and in subsection 3.7, we also recall some results on $q_{b,t}^X$ established in [B11, Chapters 4 and 11].

0.3. **Semisimple orbital integrals.** Let $\mathrm{Isom}(X)$ be the Lie group of isometries of X, and let $\mathrm{Isom}(X)^0$ be the connected component containing the identity. We have the obvious homomorphism of Lie groups $G \to \mathrm{Isom}(X)^0$.

If $\phi \in \mathrm{Isom}(X)$, let $d_\phi(x) = d(x, \phi(x))$ be the displacement function associated with ϕ. As in [E96], ϕ is called semisimple if d_ϕ reaches its infimum value m_ϕ in X, and ϕ is called elliptic if ϕ has fixed points in X. If ϕ is semisimple, let $X(\phi) \subset X$ be the minimizing set of d_ϕ, which is a convex submanifold of X.

In [B11, Chapters 3], given a semisimple element $\gamma \in G$, Bismut showed that $X(\gamma)$ is a symmetric space associated with the centralizer $Z(\gamma)$ of γ, then he constructed a normal coordinate system for X based on $X(\gamma)$. Based on this, Bismut gave a geometric interpretation for the associated orbital integrals, so that it can be written as an integration along the fiber of normal bundle $N_{X(\gamma)/X}$ of $X(\gamma)$. In particular, the orbital integrals $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$, $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)]$ are well-defined. These orbital integrals are said to be respectively elliptic and hypoelliptic.

In [B11, Section 4.6], Bismut showed that for $t > 0$, $b > 0$, $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$, $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)]$ coincide. Then by making $b \to +\infty$ in $\mathrm{Tr_s}^{[\gamma]}[\exp(-t\mathcal{L}_b^X)]$, Bismut obtained an explicit geometric formula for $\mathrm{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X)]$.

Using this formula, Shu Shen [S18] gave a full proof of the Fried conjecture for compact locally symmetric spaces, completing the work of Moscovici and Stanton [MS91].

0.4. **Twisted orbital integrals.** Let Σ be the compact subgroup of $\mathrm{Aut}(G)$ consisting of the automorphisms of (G, B, θ). If $\sigma \in \Sigma$, let Σ^σ be the closed subgroup of Σ generated by σ. Put

$$G^\sigma = G \rtimes \Sigma^\sigma,\ K^\sigma = K \rtimes \Sigma^\sigma. \tag{0.4.1}$$

If $\sigma \in \Sigma$, we define the σ-twisted conjugation C^σ on G such that if $h, \gamma \in G$,

$$C^\sigma(h)\gamma = h\gamma\sigma(h^{-1}). \tag{0.4.2}$$

Then C^σ gives an action of G on itself. Let $Z(\gamma\sigma) \subset G$ be the twisted centralizer of $\gamma \in G$, and let $[\gamma]_\sigma$ be the orbit of $\gamma \in G$ by the action C^σ. We have

$$[\gamma]_\sigma \simeq Z(\gamma\sigma)\backslash G. \tag{0.4.3}$$

Then the twisted orbital integrals [L80, Fli82, C84, ArC89, Lip15, BeLi17] are referred to certain integrals on $Z(\gamma\sigma)\backslash G$.

The group G^σ acts on X isometrically. Let $\gamma \in G$ be such that $\gamma\sigma$ is semisimple, and let $X(\gamma\sigma) \subset X$ be the minimizing set of $d_{\gamma\sigma}$. In subsection 1.5, we extend the geometric constructions in [B11, Chapter 3] to our case, so that $X(\gamma\sigma)$ is the symmetric space associated with $Z(\gamma\sigma)$.

We also assume that E extends as a unitary representation of K^σ, the question of the existence of such lifts will be revisited in more detail in section 2. Then the action of G^σ on X lifts to F. Let $\mathcal{Q}^\sigma$ be an algebra of operators which commute with G^σ and have the proper Gaussian decay.

In section 4, for $Q \in \mathcal{Q}^\sigma$, one has a geometric formulation for the twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[Q]$. In particular, the elliptic heat kernel has well-defined twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)]$. In subsection 4.3, we extend the definition of twisted orbital integrals to the hypoelliptic orbital integrals $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_b)]$.

Let Γ be a cocompact discrete subgroup of G such that $\sigma(\Gamma) \subset \Gamma$. For simplicity, we assume that Γ is torsion free, so that $Z = \Gamma\backslash X$ is a compact smooth manifold equipped with an action of Σ^σ.

The vector bundle F descends to a vector bundle on Z, which we still denote it by F. The action of Σ^σ on Z lifts to F. If $Q \in \mathrm{Q}^\sigma$, then Q descends to an operator Q^Z acting on $C^\infty(Z, F)$.

In subsection 1.8, we show that if $\gamma \in \Gamma$, $\gamma\sigma$ is semisimple. Let $\underline{C}$ be the twisted conjugacy classes of Γ defined in Definition 1.8.2. In subsection 4.5, following Langlands [L80], Flicker [Fli82] and Bergeron-Lipnowski [BeLi17], we rederive a twisted version of Selberg's trace formula [Sel56],

$$\mathrm{Tr}[\sigma^Z Q^Z] = \sum_{\underline{[\gamma]_\sigma} \in \underline{C}} V(\gamma\sigma)\mathrm{Tr}^{[\gamma\sigma]}[Q], \tag{0.4.4}$$

where the factor $V(\gamma\sigma)$ is a volume term only depending on the class $\underline{[\gamma]_\sigma}$.

0.5. **The results of this thesis.** In subsection 4.4, we establish the fundamental identity which says that, for $b > 0$, $t > 0$,

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_b)]. \tag{0.5.1}$$

We then make $b \to +\infty$ in (0.5.1), and the right-hand side can be localized near $X(\gamma\sigma)$. The geometric formulation of the twisted orbital integrals established in section 4 plays an essential role here.

If $\gamma\sigma$ is semisimple, after conjugation, we may and we will assume that $\gamma = e^a k^{-1}$, where $a \in \mathfrak{p}, k \in K$ and $\mathrm{Ad}(k)a = \sigma a$. Put $K(\gamma\sigma) = Z(\gamma\sigma) \cap K$, and let $\mathfrak{k}(\gamma\sigma)$ be its Lie algebra. In subsection 5.1, we define an analytic function $J_{\gamma\sigma}(Y_0^{\mathfrak{k}})$ in $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$ by an explicit formula.

Our main result of this thesis is as follows.

Theorem 0.5.1. *For $t > 0$, the following identity holds:*

$$\begin{aligned}\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] =& \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \\ & \int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) \mathrm{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y_0^{\mathfrak{k}}))] \\ & \exp(-|Y_0^{\mathfrak{k}}|^2/2t) \frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.\end{aligned} \tag{0.5.2}$$

If $\sigma = \mathrm{Id}_G$, we recover the formula obtained in [B11].

0.6. **Connections with equivariant local index theory.** The operator $\mathcal{L}^X$ descends to a Laplacian $\mathcal{L}^Z$ on Z. Let D^Z be the classical Dirac operator on Z. By [B11, Sections 7.2 and 7.3], $D^{Z,2}$ coincides with $2\mathcal{L}^Z$ up to an explicit constant. The Lefschetz number is given by

$$\chi_\sigma(F) = \mathrm{Tr_s}[\sigma^Z \exp(-tD^{Z,2}/2)]. \tag{0.6.1}$$

Let ${}^\sigma Z$ be the fixed point set of σ in Z. Then $\chi_\sigma(F)$ can be computed by the Lefschetz fixed point theorem of Atiyah-Bott [AB67, AB68], so that

$$\chi_\sigma(F) = \int_{{}^\sigma Z} \widehat{A}^\sigma(TZ|_{{}^\sigma Z}, \nabla^{TZ|_{{}^\sigma Z}}) \mathrm{ch}^\sigma(F, \nabla^F). \tag{0.6.2}$$

In section 7, we verify that when evaluating the right-hand side of (0.6.1) using (0.4.4), (0.5.2), we recover equation (0.6.2). To do this, we have to explore in more detail the representation theory of K^σ.

In section 2, following [L80, C84, Bou87, DK00, BeLi17], we give a classification of representations of K^σ in terms of a root data of K, so that we can evaluate the character of K^σ in the right-hand side of (0.5.2). More precisely, we construct an element $\tau \in \mathrm{Aut}(K)$ of finite order, so that the representations of K^σ can be transformed to representations of K^τ. Also τ acts on the set of dominant weights P_{++}.

Let $\mathrm{Irr}(K^\sigma)$, $\mathrm{Irr}(\Sigma^\sigma)$ be the sets of the equivalence classes of the irreducible unitary representations of K^σ, Σ^σ respectively. In subsection 2.4, we show

$$\mathrm{Irr}(\Sigma^\sigma)\backslash\mathrm{Irr}(K^\sigma) \simeq \left\{\begin{matrix}\text{orbits of } P_{++} \text{ under the action of} \\ \text{the finite group generated by } \tau\end{matrix}\right\}. \tag{0.6.3}$$

0.7. **Equivariant Ray-Singer analytic torsion of Z.** If E is a representation of G^σ, then $F = G \times_K E$ is a flat vector bundle on X, and F descends to a flat vector bundle on Z with a canonical flat connection $\nabla^{F,f}$. In this case, the operator $2\mathcal{L}^Z$ is just the Hodge Laplacian $\mathbf{D}^{Z,F,2}$ up to a known self-adjoint endomorphism.

Let $N^{\Lambda^\cdot(T^*Z)}$ denote the number operator on $\Omega^\cdot(Z,F)$. Let $[\mathbf{D}^{Z,F,2}]^{-1}$ be the inverse of $\mathbf{D}^{Z,F,2}$ acting on the orthogonal space of $\ker \mathbf{D}^{Z,F}$ in $\Omega^\cdot(Z,F)$.

For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ large enough, set

$$\vartheta_\sigma(g^{TZ}, \nabla^{F,f}, g^F)(s) = -\mathrm{Tr_s}\left[N^{\Lambda^\cdot(T^*Z)}\sigma[\mathbf{D}^{Z,F,2}]^{-s}\right]. \tag{0.7.1}$$

By standard heat equation methods, $\vartheta_\sigma(g^{TZ}, \nabla^{F,f}, g^F)(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic near $s = 0$.

Put

$$\mathcal{T}_\sigma(g^{TZ}, \nabla^{F,f}, g^F) = \frac{1}{2}\frac{\partial \vartheta_\sigma(g^{TZ}, \nabla^{F,f}, g^F)}{\partial s}(0). \tag{0.7.2}$$

The quantity (0.7.2) is called the equivariant Ray-Singer analytic torsion of the de Rham complex $(\Omega^\cdot(Z,F), d^{Z,F})$. If σ is the identity map, this is just the ordinary Ray-Singer analytic torsion [RS71, RS73].

In subsection 7.8, as in [BMZ17, Section 8], we obtain an explicit formula for the twisted orbital integrals for the heat kernel that appears in the evaluation of $\mathcal{T}_\sigma(g^{TZ}, \nabla^{F,f}, g^F)$. Then we get some nontrivial algebraic conditions on $\mathfrak{p}$ and σ such that $\mathcal{T}_\sigma(g^{TZ}, \nabla^{F,f}, g^F)$ vanishes.

0.8. **Asymptotics of equivariant Ray-Singer analytic torsions.** In section 9, we will use our explicit formula to obtain an explicit local version of the asymptotics of the equivariant Ray-Singer analytic torsion when the vector bundle F tends to infinity in the proper sense.

Bergeron and Venkatesh [BeV13] has considered the asymptotic behaviour of analytic torsion of locally symmetric spaces under finite coverings. Müller [Mül12] initiated the study of Ray-Singer analytic torsion for symmetric powers of a given flat vector bundle on hyperbolic manifolds. Also Bismut-Ma-Zhang [BMZ17] and Müller-Pfaff [MüP13] studied the case where one considers a sequence of flat vector bundles associated with multiples of a given highest weight defining a representation of the compact form U of G. Here, we will be concerned with the asymptotics of the equivariant Ray-Singer analytic torsion for a compact locally symmetric space Z. This problem has already been considered by Fedosova [Fed15] using methods of harmonic analysis on the reductive group G. Here, as in [BMZ17], we will exploit instead the explicit formula of Theorem 0.5.1.

We assume that the action of σ on G extends to U. Put $U^\sigma = U \rtimes \Sigma^\sigma$. In the sequel, we assume that (E, ρ^E) is a unitary representation of U^σ. This representation extends to a representation of G^σ.

In subsection 8.2, as a consequence of (0.5.2) and (0.6.3), we show that we may assume that E is also U-irreducible, so that the highest weight λ of E is fixed by σ. Let M_λ be the flag manifold associated with λ. We show that U^σ acts holomorphically on M_λ and that this action lifts to the associated canonical line bundle $L_\lambda \to M_\lambda$. Then for $d \in \mathbb{N}$, U^σ acts on $H^{(0,0)}(M_\lambda, L_\lambda^d)$. We get a family of irreducible representations (E_d, ρ^{E_d}) of U^σ given by $H^{(0,0)}(M_\lambda, L_\lambda^d)$. The flat vector bundles F_d are the ones associated with the action of G^σ on E_d.

In subsection 9.3, we show that under nondegeneracy condition, as $d \to +\infty$, the asymptotics of $\mathcal{T}_\sigma(g^{TZ}, \nabla^{F_d,f}, g^{F_d})$ can be evaluated in terms of explicit forms $W_{\gamma\sigma}$ associated with elliptic elements $\gamma\sigma$, $\gamma \in \Gamma$, which are the W-invariant defined in [BMZ17]. A difference from the result in [BMZ17, Section 8] is that the coefficients of $W_{\gamma\sigma}$ have oscillating factors of the form $\exp(c_{\gamma\sigma} d\sqrt{-1})$, $c_{\gamma\sigma} \in \mathbb{R}$.

Also, in Proposition 9.3.1, we show that under the above nondegeneracy condition, the contribution of non-elliptic elements $\gamma\sigma$, $\gamma \in \Gamma$ to the asymptotic equivariant Ray-Singer analytic torsion is exponentially small.

Our results are compatible with the results of Ksenia Fedosova [Fed15], where she considered the asymptotics of Ray-Singer analytic torsions for compact hyperbolic orbifolds. Using Selberg's trace formula, she showed that the elliptic elements of Γ contributed to the asymptotic Ray-Singer analytic torsion by a so-called pseudo-polynomial in d containing the oscillating factors in the same way, and that the contribution of non-elliptic elliptic elements in Γ is exponentially small.

0.9. **The organization of the thesis.** The thesis is organized as follows. In section 1, we introduce an extension $\widetilde{G}$ of G by a compact group of automorphism Σ, and we establish the associated geometric structures on the symmetric space X.

In section 2, we classify the irreducible representation of K^σ in terms of root data of K, and we give a Weyl character formula.

In section 3, we recall the construction of the hypoelliptic Laplacian associated with (G, K) and the properties of its heat kernel proved in [B11].

In section 4, we define the twisted orbital integrals associated with $\gamma\sigma$. In subsection 4.5, we rederive a twisted version of Selberg trace formula for the locally symmetric space.

In section 5, we prove the main result of the present thesis, and give some extensions.

In section 6, we recall the explicit formula for the hypoelliptic heat kernel on the Euclidean vector space, and we show that in this case, our formula is compatible with the computations in [B11, Section 10.6].

In section 7, we show the compatibility of our formula for twisted orbital integrals to the results in local equivariant index theory.

In section 8, we construct a sequence of representations E_d of G^σ and an associated sequence of flat vector bundles F_d on Z.

Finally, in section 9, we compute the asymptotics of equivariant Ray-Singer analytic torsions as $d \to +\infty$.

Notation: if $E = E_+ \oplus E_-$ is a $\mathbb{Z}_2$-graded vector space, and if $\tau = \pm 1$ defines the $\mathbb{Z}_2$-grading, if $A \in \mathrm{End}(E)$, we denote by $\mathrm{Tr_s}[A]$ the supertrace of A.

If $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra, if $a, b \in \mathcal{A}$, $[a, b]$ will be our notation for the supercommutator of a, b, so that

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba. \tag{0.9.1}$$

If $\mathcal{B}$ is another $\mathbb{Z}_2$-graded algebra, we denote by $\mathcal{A} \widehat{\otimes} \mathcal{B}$ the $\mathbb{Z}_2$-graded tensor product of $\mathcal{A}$ and $\mathcal{B}$.

1. The symmetric space $X = G/K$ and semisimple isometries

This section is to introduce a compact subgroup Σ of $\mathrm{Aut}(G)$ and establish the geometric structures of X associated with semisimple elements of the semidirect product of G and Σ.

This section is organized as follows. In Subsection 1.1, we introduce the real reductive group G and the symmetric space X. We describe the semisimple isometries of X.

In subsection 1.2, we introduce a compact subgroup Σ of $\mathrm{Aut}(G)$ and the semidirect product $\widetilde{G}$ of G and Σ. We show that X is a quotient space of $\widetilde{G}$.

In subsection 1.3, we describe the semisimple elements in $\widetilde{G}$ and their centralizers.

In subsection 1.4, if $\gamma\sigma$, $\gamma \in G$, and if $\sigma \in \Sigma$ is semisimple, we get a representation of $X(\gamma\sigma)$ in a global geodesic coordinate system.

In subsection 1.5, we describe the normal bundle $N_{X(\gamma\sigma)/X}$. We get a normal coordinate system based on $X(\gamma\sigma)$ that identifies X to the total space of $N_{X(\gamma\sigma)/X}$.

In subsection 1.6, we introduce the return map along the geodesics in $X(\gamma\sigma)$ that connects x, $\gamma\sigma(x)$, $x \in X(\gamma\sigma)$. We also interpret $X(\gamma\sigma)$ as the fixed point set of a symplectic diffeomorphism of the total space of the cotangent bundle of X.

In subsection 1.7, we recall a pseudodistance on $\mathcal{X}$, $\widehat{\mathcal{X}}$, and we extend the estimates obtained in [B11, Section 3.9] to our case.

Finally, in subsection 1.8, we introduce a cocompact discrete subgroup Γ of G preserved by σ, and we show that if $\gamma \in \Gamma$, $\gamma\sigma$ is semisimple. We also introduce the locally symmetric space Z, and describe the fixed point set of σ in Z.

If H be a Lie group of finite dimension, let H^0 be the connected component of H containing the identity element of H. In the sequel, we will call H^0 the identity component of H.

1.1. **Symmetric space and displacement function.**

Let G be a connected real reductive group [K02, §7.2], and let θ be a Cartan involution of G whose fixed point set K is a compact maximal subgroup of G. Then K is connected. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{k}$ be the Lie algebra of K. The Cartan decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{1.1.1}$$

The vector spaces $\mathfrak{p}$,$\mathfrak{k}$ are the eigenspaces corresponding to the eigenvalues -1, 1 of θ acting on $\mathfrak{g}$. Then we have

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}], [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{1.1.2}$$

Put

$$m = \dim \mathfrak{p}, \qquad n = \dim \mathfrak{k}. \tag{1.1.3}$$

Then $\dim \mathfrak{g} = m + n$.

Let B be a nondegenerate θ and G invariant bilinear symmetric form on $\mathfrak{g}$ which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Let $\langle \cdot, \cdot \rangle$ be the scalar product on $\mathfrak{g}$ defined by $-B(\cdot, \theta \cdot)$.

For $g, g' \in G$, put

$$C(g)g' = gg'g^{-1} \in G. \tag{1.1.4}$$

Let $\mathrm{Ad}(\cdot)$, $\mathrm{ad}(\cdot)$ denote respectively the adjoint actions of G, $\mathfrak{g}$ on $\mathfrak{g}$. We also use $\mathrm{Ad}(g)$ abusively to denote the conjugation $C(g)$ on G.

Let X be the symmetric space G/K. The tangent bundle TX is given by

$$TX = G \times_K \mathfrak{p}. \tag{1.1.5}$$

The scalar product $B|_{\mathfrak{p}}$ induces a Riemannian metric g^{TX} on TX. Then G and θ act on X isometrically. In the following, let $d(\cdot,\cdot)$ denote the Riemann distance on X.

Let $\omega^{\mathfrak{g}}$ be the canonical left-invariant 1-form on G with values in $\mathfrak{g}$, then

$$d\omega^{\mathfrak{g}} = -\frac{1}{2}[\omega^{\mathfrak{g}}, \omega^{\mathfrak{g}}]. \tag{1.1.6}$$

Let $\omega^{\mathfrak{k}}, \omega^{\mathfrak{p}}$ be the $\mathfrak{k}, \mathfrak{p}$ components of $\omega^{\mathfrak{g}}$ with respect to (1.1.1). Then

$$\omega^{\mathfrak{g}} = \omega^{\mathfrak{k}} + \omega^{\mathfrak{p}}. \tag{1.1.7}$$

By (1.1.1), (1.1.2), equation (1.1.6) splits as

$$d\omega^{\mathfrak{p}} = -[\omega^{\mathfrak{k}}, \omega^{\mathfrak{p}}], \quad d\omega^{\mathfrak{k}} = -\frac{1}{2}[\omega^{\mathfrak{k}}, \omega^{\mathfrak{k}}] - \frac{1}{2}[\omega^{\mathfrak{p}}, \omega^{\mathfrak{p}}]. \tag{1.1.8}$$

The projection $p : G \to G/K$ defines a $K-$principal bundle on X, and the connection form corresponding to the splitting (1.1.1) is just $\omega^{\mathfrak{k}}$. Let Ω be the associated curvature, then by (1.1.8),

$$\Omega = -\frac{1}{2}[\omega^{\mathfrak{p}}, \omega^{\mathfrak{p}}] \in \Lambda^2(\mathfrak{p}^*) \otimes \mathfrak{k}. \tag{1.1.9}$$

By (1.1.5), $\omega^{\mathfrak{k}}$ induces an Euclidean connection ∇^{TX} on TX. By the first identity in (1.1.8), ∇^{TX} is the Levi-Civita connection of (TX, g^{TX}). Let R^{TX} be its curvature. If $a, b, c \in \mathfrak{p}$, by (1.1.5), (1.1.9), R^{TX} is just the equivariant representation of the map $a, b, c \in \mathfrak{p} \to -[[a, b], c] \in \mathfrak{p}$. If $a, b \in \mathfrak{p}$,

$$\langle -[[a, b], b], a \rangle = -\langle [a, b], [a, b] \rangle. \tag{1.1.10}$$

By (1.1.10), we deduce that X has nonpositive sectional curvature. Given a point $x \in X$, the exponential map $T_xX \to X$ is a covering. Since X is simply connected, then this map one to one. In particular, if $x = p1 \in X$, then the exponential map $\exp_x : \mathfrak{p} \to X$ given by $Y^{\mathfrak{p}} \in \mathfrak{p} \to \exp_x(Y^{\mathfrak{p}}) = \exp(Y^{\mathfrak{p}}) \cdot x$ is a diffeomorphism between $\mathfrak{p}$ and X.

If (E, ρ^E) is an orthogonal (or a unitary) representation of K on an Euclidean (or a Hermitian) space E of finite dimension, then $F = G \times_K E$ is an Euclidean (or a Hermitian) vector bundle on X. The connection form $\omega^{\mathfrak{k}}$ induces an Euclidean (or a Hermitian) connection ∇^F on F.

The action of G on X lifts to an action on F, so that if $g, h \in G$,$x = ph \in X$, $f \in E$, then

$$g : F_x \to F_{gx} \quad (h, f) \to (gh, f). \tag{1.1.11}$$

Let $C^\infty(G,E)$ be the set of smooth functions on G valued in E. The right multiplication of K on G induces an action of K on $C^\infty(G,E)$, such that for $k\in K$, $s\in C^\infty(G,E)$,

$$(k.s)(g)=\rho^E(k)s(gk). \tag{1.1.12}$$

Let $C^\infty_K(G,E)$ be the subspace of $C^\infty(G,E)$ of the sections fixed by K. Let $C^\infty(X,F)$ be the vector space of the smooth sections of F over X. Then we have

$$C^\infty(X,F)=C^\infty_K(G,E). \tag{1.1.13}$$

Also the left action of G on itself induces an action of G on $C^\infty(X,F)$ such that if $s\in C^\infty_K(G,E)$, if $g,h\in G$, then

$$(hs)(g)=s(h^{-1}g). \tag{1.1.14}$$

Moreover, ∇^F is G-invariant.

Put

$$N=G\times_K\mathfrak{k}. \tag{1.1.15}$$

We call N the normal bundle on X. Let ∇^N be the connection on N associated with $\omega^{\mathfrak{k}}$.

By (1.1.5), (1.1.15), we have

$$TX\oplus N=G\times_K\mathfrak{g}. \tag{1.1.16}$$

Let $\nabla^{TX\oplus N}$ be the connection on $TX\oplus N$ associated with $\omega^{\mathfrak{k}}$, equivalently, $\nabla^{TX\oplus N}=\nabla^{TX}\oplus\nabla^N$. As in [B11, Section 2.2], the map $[g,a]\in G\times_K\mathfrak{g}\to(pg,\mathrm{Ad}(g)a)\in X\times\mathfrak{g}$ gives an identification of vector bundles

$$TX\oplus N\simeq X\times\mathfrak{g}. \tag{1.1.17}$$

In the whole thesis, let $\pi:\mathcal{X}\to X$ be the total space of TX to X, and let $\widehat{\pi}:\widehat{\mathcal{X}}\to X$ be the total space of $TX\oplus N$ to X. We also denote by $\underline{\pi}:\widehat{\mathcal{X}}\to\mathcal{X}$ the obvious projection.

Let $U\mathfrak{k}$ be the enveloping algebra of $\mathfrak{k}$. Let $v_1,\cdots,v_n$ be a orthonormal basis of $\mathfrak{k}$ with respect to $-B|_{\mathfrak{k}}$, then the Casimir operator $C^{\mathfrak{k}}\in U\mathfrak{k}$ of K with respect to $B|_{\mathfrak{k}}$ is given by

$$C^{\mathfrak{k}}=\sum_{i=1}^n v_i^2. \tag{1.1.18}$$

Then $C^{\mathfrak{k}}$ lies in the center of $U\mathfrak{k}$.

We denote by $C^{\mathfrak{k},E}\in\mathrm{End}(E)$ the corresponding Casimir operator acting on E, so that

$$C^{\mathfrak{k},E}=\sum_{i=1}^n\rho^{E,2}(v_i). \tag{1.1.19}$$

In particular, let $C^{\mathfrak{k},\mathfrak{k}}\in\mathrm{End}(\mathfrak{k})$, $C^{\mathfrak{k},\mathfrak{p}}\in\mathrm{End}(\mathfrak{p})$ be the Casimir operators associated with the adjoint actions of K on $\mathfrak{k}$, $\mathfrak{p}$ respectively. Moreover, we can regard $C^{\mathfrak{k},\mathfrak{p}}$ as a section of $\mathrm{End}(TX)$.

Let Ric^X be the Ricci tensor of X, let S^X be its scalar curvature. Then by [B11, (2.6.8)], we have

$$\mathrm{Ric}^X = C^{\mathfrak{k},\mathfrak{p}}, \quad S^X = \mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]. \tag{1.1.20}$$

Let $\mathrm{Isom}(X)$ be the Lie group of isometries of X. Then we have a group homomorphism $G \to \mathrm{Isom}(X)$.

Definition 1.1.1. If $\phi \in \mathrm{Isom}(X)$, the displacement function d_ϕ of ϕ is the function on X defined as

$$d_\phi(x) = d(x, \phi x)\ ,\ x \in X. \tag{1.1.21}$$

Put $m_\phi = \inf_{x\in X} d_\phi(x)$.

Since X has nonpositive sectional curvature, by [E96, Chapter 1, Example 1.6.6], d_ϕ is a continuous nonnegative convex function and d_ϕ^2 is a smooth convex function.

Definition 1.1.2. We say $\phi \in \mathrm{Isom}(X)$ is semisimple if $d_\phi(x)$ reaches its infimum m_ϕ in X. A semisimple isometry ϕ is called elliptic if it has fixed points in X, i.e. $m_\phi = 0$. If ϕ is semisimple, put $X(\phi) = \{x \in X \mid d_\phi(x) = m_\phi\}$.

Remark 1.1.3. If ϕ is semisimple, $X(\phi)$ is just the set of all critical points of d_ϕ^2, which is a convex subset of X. If ϕ is elliptic, then $X(\phi)$ is the set of fixed points of ϕ.

If $x(s)$, $s \in [0,1]$ is a smooth path in X, let $\dot{x}(s)$ denote its tangent vector at $x(s)$. If $f \in C^\infty(X)$, let ∇f denote the gradient of f with respect to g^{TX}.

Lemma 1.1.4. *Take $\phi \in \mathrm{Isom}(X)$ and $x \in X$ such that $d_\phi(x) > 0$. Let $x(s), s \in [0,1]$ be the unique geodesic in X joining x and $\phi(x)$ with constant speed. Then*

$$\nabla d_\phi(x) = \frac{1}{d_\phi(x)}((\phi^{-1})_*\dot{x}(1) - \dot{x}(0)). \tag{1.1.22}$$

Proof. At first, we have

$$\nabla d_\phi(x) = \frac{1}{2d_\phi(x)}\nabla d_\phi^2(x). \tag{1.1.23}$$

Then the calculus on the length of the geodesic shows the identity in (1.1.22). This completes the proof of our lemma. □

As a consequence, if ϕ is a semisimple isometry of X with $m_\phi > 0$. Fix a point $x \in X$, let $x(s), s \in [0,1]$ be the unique geodesic in X joining x and $\phi(x)$ with constant speed. Then $x \in X(\phi)$ if and only if $\phi_*\dot{x}(0) = \dot{x}(1)$. In this case m_ϕ is just the length of the path $x(\cdot)$.

1.2. **A compact subgroup of** $\mathrm{Aut}(G)$**.** Let $\mathrm{Aut}(G)$ be the Lie group of automorphism of G [Hoc52, Theorem 2], and let $\mathfrak{aut}(G)$ be its Lie algebra. Let 1 denote the unit element of G, and let $\mathbb{1}_G$ be the identity automorphism of G. We have the injective group morphism,

$$\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\mathfrak{g}). \tag{1.2.1}$$

Definition 1.2.1. The semidirect product of G and $\mathrm{Aut}(G)$ is given by

$$G \rtimes \mathrm{Aut}(G) := \{(g,\phi) \mid g \in G, \phi \in \mathrm{Aut}(G)\}, \tag{1.2.2}$$

with the group multiplication:

$$(g_1,\phi_1)\cdot(g_2,\phi_2) = (g_1\phi_1(g_2), \phi_1\phi_2). \tag{1.2.3}$$

The unit element is $(1,\mathbb{1}_G)$. Also $(g,\phi)^{-1} = (\phi^{-1}(g^{-1}),\phi^{-1})$.

We can view G and $\mathrm{Aut}(G)$ as Lie subgroups of $G \rtimes \mathrm{Aut}(G)$. In particular, G is a normal subgroup of $G \rtimes \mathrm{Aut}(G)$. We have the exact sequence of Lie groups,

$$1 \to G \to G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G) \to 1. \tag{1.2.4}$$

We have the corresponding exact sequence of Lie algebras,

$$0 \to \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{aut}(G) \to \mathfrak{aut}(G) \to 0. \tag{1.2.5}$$

Then $\mathfrak{g}$ is an ideal of $\mathfrak{g} \oplus \mathfrak{aut}(G)$.

We will often use the notation $g\phi$ instead of (g,ϕ). Let $C(\theta)$ be the inner automorphism of $G \rtimes \mathrm{Aut}(G)$ associated with θ. Then $C(\theta)$ is an involution.

Definition 1.2.2. Put

$$\Sigma := \{\phi \in \mathrm{Aut}(G) \ : \ \phi\theta = \theta\phi, \ \phi \text{ preserves the bilinear form } B\}. \tag{1.2.6}$$

Then Σ is a compact Lie subgroup of $\mathrm{Aut}(G)$, and let $\mathfrak{e}$ be its Lie algebra. The action of Σ on $\mathfrak{g}$ preserves the splitting (1.1.1) and the scalar product of $\mathfrak{g}$. In particular, Σ contains all the inner automorphisms defined by elements in K.

Let $\widetilde{G}$ be the preimage of Σ under the projection $G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$. Then $\widetilde{G} = G \rtimes \Sigma$. Let $\tilde{\mathfrak{g}}$ be its Lie algebra, then

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{e}, \tag{1.2.7}$$

Moreover, the adjoint action of $\mathfrak{e}$ on $\mathfrak{g}$ preserves the splitting (1.1.1).

Remark 1.2.3. In general, the group $\widetilde{G}$ is not necessary to be reductive. An example is the Euclidean space $\mathbb{R}^n$. In this case $\widetilde{G} = \mathbb{R}^n \rtimes \mathrm{O}(n)$ and the corresponding Lie algebra $\tilde{\mathfrak{g}} = \mathbb{R}^n \oplus \mathfrak{so}(n)$ with a twisted Lie bracket. One can show that $\tilde{\mathfrak{g}}$ is not a reductive Lie algebra. We will return to this case in section 6.

The group automorphism $C(\theta)$ maps $\widetilde{G}$ into itself, i.e., if $\phi \in \Sigma$, $g \in G$,

$$C(\theta)(g\phi) = \theta(g)\phi. \tag{1.2.8}$$

Let $\widetilde{K}$ be the fixed set of $C(\theta)$ in $\widetilde{G}$. Then

$$\widetilde{K} = K \rtimes \Sigma. \tag{1.2.9}$$

Let $\tilde{\mathfrak{k}}$ be the Lie algebra of $\widetilde{K}$, by (1.2.9),

$$\tilde{\mathfrak{k}} = \mathfrak{k} \oplus \mathfrak{e}. \tag{1.2.10}$$

We have a splitting of $\tilde{g}$ associated with $C(\theta)$,

$$\tilde{\mathfrak{g}} = \mathfrak{p} \oplus \tilde{\mathfrak{k}}, \tag{1.2.11}$$

where $\mathfrak{p}, \tilde{\mathfrak{k}}$ are the eigenspaces of $C(\theta)$ in $\tilde{\mathfrak{g}}$ corresponding to eigenvalues -1, 1 respectively.

If $\sigma \in \Sigma$, let Σ^σ be the closed subgroup of Σ generated by σ. Let G^σ be the closed subgroup of $\widetilde{G}$ generated by G and σ, and let K^σ be the closed subgroup $\widetilde{K}$ generated by K and σ. Then we have

$$G^\sigma = G \rtimes \Sigma^\sigma,\ K^\sigma = K \rtimes \Sigma^\sigma. \tag{1.2.12}$$

If σ is chosen and fixed, let $\mathfrak{g}^\sigma$, $\mathfrak{k}^\sigma$ be the Lie algebras of G^σ, K^σ. Then the analogues of (1.2.7) - (1.2.11) for the groups Σ^σ, G^σ, K^σ hold.

If $\tilde{g} \in \widetilde{G}$, let $\widetilde{Z}(\tilde{g})$ be the centralizer of $\tilde{g} \in \widetilde{G}$ in $\widetilde{G}$. If $\sigma \in \Sigma$, put

$$\begin{aligned} Z(\tilde{g}) &= \widetilde{Z}(\tilde{g}) \cap G, \\ Z^\sigma(\tilde{g}) &= \widetilde{Z}(\tilde{g}) \cap G^\sigma. \end{aligned} \tag{1.2.13}$$

In particular, if $\tilde{g} \in G$ (resp. G^σ), $Z(\tilde{g})$ (resp. $Z^\sigma(\tilde{g})$) is just the centralizer group of $\tilde{g}$ in G (resp. G^σ). We denote respectively by $\widetilde{Z}^0(\tilde{g})$, $Z^0(\tilde{g})$, $Z^{\sigma,0}(\tilde{g})$ the identity components of $\widetilde{Z}(\tilde{g})$, $Z(\tilde{g})$, $Z^\sigma(\tilde{g})$, and we denote respectively by $\tilde{\mathfrak{z}}(\tilde{g})$, $\mathfrak{z}(\tilde{g})$, $\mathfrak{z}^\sigma(\tilde{g})$ their Lie algebras. Then,

$$\begin{aligned} \mathfrak{z}(\tilde{g}) &= \tilde{\mathfrak{z}}(\tilde{g}) \cap \mathfrak{g}, \\ \mathfrak{z}^\sigma(\tilde{g}) &= \tilde{\mathfrak{z}}(\tilde{g}) \cap \mathfrak{g}^\sigma. \end{aligned} \tag{1.2.14}$$

Given $\sigma \in \Sigma$, the map $g \in G \to \sigma(g) \in G$ descends to a diffeomorphism of X: $x \in X \to \sigma(x) \in X$. By (1.1.5), (1.2.6), the tangent map of σ is given by the map $(g, f) \to (\sigma(g), \sigma(f))$ with $g \in G, f \in \mathfrak{p}$. Then $\sigma \in \mathrm{Isom}(X)$.

Recall that the left actions of G on X are also isometries. Then $\widetilde{G}$ acts on X isometrically.

Proposition 1.2.4. *We have the identification of manifolds,*

$$X = \widetilde{G}/\widetilde{K}. \tag{1.2.15}$$

Proof. We know that $\widetilde{G}$ acts on X transitively. Put $x = p1 \in X$, and let $\widetilde{G}_x \subset \widetilde{G}$ be the centralizer of x. If $g\phi \in \widetilde{G}$ is such that $g\phi(x) = x$ then $gx = x$, this is equivalent to $g \in K$, so that $\widetilde{G}_x = \widetilde{K}$. Then we have $X = \widetilde{G}/\widetilde{K}$. This completes the proof of our proposition. □

Remark 1.2.5. The group injection $G \to \widetilde{G}$ induces the identification between G/K and $\widetilde{G}/\widetilde{K}$ described above and its inverse is given by the canonical projection $\widetilde{G} \to G$.

A consequence of Proposition 1.2.4 is that if $\tilde{g} \in \widetilde{G}$, there exist unique $f \in \mathfrak{p}$, $\tilde{k} \in \widetilde{K}$ such that

$$\tilde{g} = e^f \tilde{k}. \tag{1.2.16}$$

If $\sigma \in \Sigma$, using the same arguments as in the proof of Proposition 1.2.4, we get

$$X = G^\sigma / K^\sigma. \tag{1.2.17}$$

We always use the identifications in (1.2.15), (1.2.17) without specific mention. We also use p denote both the projections $\widetilde{G} \to X$ and $G^\sigma \to X$.

By (1.2.15), (1.2.17), we get

$$TX = \widetilde{G} \times_{\widetilde{K}} \mathfrak{p} = G^\sigma \times_{K^\sigma} \mathfrak{p}. \tag{1.2.18}$$

Remark 1.2.6. By [H79, Chapter 4, §3, Remark 2], the group actions of $\widetilde{G}$ on X give a closed Lie subgroup of $\mathrm{Isom}(X)$. The kernel of this group morphism $\widetilde{G} \to \mathrm{Isom}(X)$ is given by $\{\tilde{k} \in \widetilde{K} \ : \ \mathrm{Ad}(\tilde{k})|_{\mathfrak{p}} = \mathbb{1}_{\mathfrak{p}}\} = \ker(\mathrm{Ad} : \widetilde{K} \to O(\mathfrak{p}))$.

It also follows from the Cartan fixed point theorem and the same arguments as in [E96, Proposition 1.13.14] that $\widetilde{K}$ is maximal compact subgroup of $\widetilde{G}$.

If the representation $\rho^E : K \to \mathrm{Aut}(E)$ lifts to a representation of $\widetilde{K}$, which is still denoted by ρ^E, then we have

$$F = \widetilde{G} \times_{\widetilde{K}} E. \tag{1.2.19}$$

As in (1.1.11), the action of $\mu \in \Sigma$ on F is given by $\mu(g, f) \to (\mu(g), \rho^E(\mu) f)$.

As in (1.1.13), we have

$$C^\infty(X, F) = C^\infty_{\widetilde{K}}(\widetilde{G}, E). \tag{1.2.20}$$

Then $\widetilde{G}$ acts on $C^\infty(X, F)$. If $s \in C^\infty(X, F)$ is represented by a section in $C^\infty_K(G, E)$, then by (1.1.14), if $\mu \in \Sigma^\sigma$, $g \in G$,

$$(\mu s)(g) = \rho^E(\mu) s(\mu^{-1}(g)). \tag{1.2.21}$$

Also ∇^F is invariant under the action of $\widetilde{G}$.

Lemma 1.2.7. *Let $C^{\mathfrak{k},E}$ be the Casimir operator defined in* (1.1.19)*. If (E, ρ^E) is a repesentation of $\widetilde{K}$, if $\mu \in \Sigma$, then*

$$\rho^E(\mu) C^{\mathfrak{k},E} = C^{\mathfrak{k},E} \rho^E(\mu). \tag{1.2.22}$$

The endomorphism $C^{\mathfrak{k},E}$ descends to a parallel section of $\mathrm{End}(F)$ *over X which commutes with Σ.*

Proof. If $v \in \mathfrak{k}$, we have

$$\rho^E(\sigma) \rho^E(v) = \rho^E(\sigma(v)) \rho^E(\sigma). \tag{1.2.23}$$

Then using the fact that σ acting on $\mathfrak{k}$ preserves B and (1.1.19), (1.2.23), we get (1.2.22).

By (1.2.21), (1.2.22), the second part of our lemma is clear. This completes the proof of our lemma. □

Remark 1.2.8. If (E,ρ^E) lifts to a representation of K^σ, then the analogues of (1.2.19) - (1.2.22) still hold for the pair (G^σ, K^σ).

If we take $E=\mathfrak{k}$ with the adjoint action, then by (1.1.15), we get

$$N=\widetilde{G}\times_{\widetilde{K}}\mathfrak{k}=G^\sigma\times_{K^\sigma}\mathfrak{k}. \tag{1.2.24}$$

1.3. The decomposition of semisimple elements in $\widetilde{G}$.

Definition 1.3.1. We say an element $\tilde{g}\in\widetilde{G}$ to be semisimple(resp. elliptic) if its isometric action on X is semisimple(resp. elliptic).

If $\tilde{g}\in\widetilde{G}$, then $d_{\tilde{g}}$ is invariant by the action of $\widetilde{Z}(\tilde{g})$. Recall that if $\tilde{g}$ is semisimple, $X(\tilde{g})$ is the minimizing set of $d_{\tilde{g}}$.

We can extend the results in [B11, Theorem 3.1.2] to our case.

Theorem 1.3.2. *We assume that $\gamma\in\widetilde{G}$ is semisimple. If $g\in\widetilde{G}$, $x=p(g)\in X$, then $x\in X(\gamma)$ if and only if there exist $a\in\mathfrak{p}, k\in\widetilde{K}$ such that $\mathrm{Ad}(k)a=a$ and $\gamma=C(g)(e^ak^{-1})$. In this case, $m_\gamma=|a|$.*

Proof. Using Lemma 1.1.4 and by (1.2.6), (1.2.15), the proof of our theorem is just a modification of the proof of [B11, Theorem 3.1.2]. □

By Theorem 1.3.2, $\gamma\in\widetilde{G}$ is elliptic if and only if it is conjugate in $\widetilde{G}$ to an element of $\widetilde{K}$. An element $\gamma\in\widetilde{G}$ is said to be hyperbolic if it is conjugate in $\widetilde{G}$ to $e^a, a\in\mathfrak{p}$.

Remark 1.3.3. Since σ preserves the splitting in (1.1.1), the conjugation of σ on $\widetilde{G}$ preserves semisimple elements in $\widetilde{G}$. Moreover, it preserves elliptic elements and hyperbolic elements in $\widetilde{G}$.

If $a\in\mathfrak{g}$, put

$$\widetilde{Z}(a)=\{g\in\widetilde{G}\ :\ \mathrm{Ad}(g)a=a\}. \tag{1.3.1}$$

If $a\in\mathfrak{p}$, by [B11, Proposition 3.2.8], and using the uniqueness of Cartan decomposition in (1.2.16), we get

$$\widetilde{Z}(e^a)=\widetilde{Z}(a). \tag{1.3.2}$$

Remark 1.3.4. In general, a modification of [B11, Proofs of Theorem 3.2.6 and Proposition 3.2.8] shows that (1.3.2) holds for $a\in\mathfrak{g}$.

The Lie algebra of $\widetilde{Z}(a)$ is given by

$$\tilde{\mathfrak{z}}(a)=\{f\in\tilde{\mathfrak{g}}\ :\ [f,a]=0\}. \tag{1.3.3}$$

Let $Z(a)$ be the centralizer of a in G and let $\mathfrak{z}(a)$ be its Lie algebra. Then

$$Z(a)=\widetilde{Z}(a)\cap G,\ \mathfrak{z}(a)=\ker(\mathrm{ad}(a)|_{\mathfrak{g}})=\tilde{\mathfrak{z}}(a)\cap\mathfrak{g}. \tag{1.3.4}$$

We now assume that $\gamma=e^ak^{-1}\in\widetilde{G}$ is such that

$$a\in\mathfrak{p},\ k\in\widetilde{K},\ \mathrm{Ad}(k)a=a. \tag{1.3.5}$$

By Theorem 1.3.2, γ is semisimple, and $x = p1 \in X(\gamma)$.

If $h \in \widetilde{Z}(\gamma)$, then $h(X(\gamma)) = X(\gamma)$. As in [E96, Theorem 2.19.23] and [B11, eq.(3.1.7)], we have the result as follows.

Proposition 1.3.5. *We have*

$$\widetilde{Z}(\gamma) = \widetilde{Z}(e^a) \cap \widetilde{Z}(k). \tag{1.3.6}$$

Proof. It is clear that $\widetilde{Z}(e^a) \cap \widetilde{Z}(k) \subset \widetilde{Z}(\gamma)$, we only need to prove the reverse direction. We adapt the proofs of [B11, Theorem 3.2.6 and Proposition 3.2.8] to get this conclusion.

Take $h \in \widetilde{Z}(\gamma)$. Let $f \in \mathfrak{p}$ and $k' \in \widetilde{K}$ be such that $h = e^f k'$ as in (1.2.16). Then $hx = pe^f \in X(\gamma)$. Put $y = \gamma x = pe^a \in X(\gamma)$, then $h\gamma x = \gamma hx \in X(\gamma)$.

Put $y_s = pe^{sa}, s \in [0,1]$ the unique geodesic in X joining x and y and $x_t = pe^{tf}, t \in [0,1]$ the unique geodesic connecting x and hx. Since $X(\gamma)$ is geodesically convex, then the paths $y_\cdot, x_\cdot$ lie in $X(\gamma)$. Also we have two other geodesics $\gamma x_\cdot, hy_\cdot$ in $X(\gamma)$. These four geodesics form a geodesic rectangle in $X(\gamma)$ with the vertexes $x, y, hx, \gamma hx = h\gamma x$.

Let $c_t(s), 0 \le s \le 1$ be the geodesic connecting x_t and γx_t for all t. In particular, if $s, t \in [0,1]$,

$$c_0(s) = y_s,\ c_1(s) = hy_s,\ c_t(0) = x_t,\ c_t(1) = \gamma x_t. \tag{1.3.7}$$

If $t \in [0,1]$, let $E_f(t)$ be the energy function associated with the geodesics $c_t(\cdot)$, i.e.,

$$E_f(t) = \frac{1}{2} d_\gamma^2(x_t). \tag{1.3.8}$$

In particular, $E_f(t)$ is a constant function in t, so that

$$E''_f(0) = 0. \tag{1.3.9}$$

Put $J_s = \frac{\partial}{\partial t}|_{t=0} c_t(s)$ the Jacobi field along y_s. By (1.3.7), in the trivialization given by parallel transport,

$$\begin{aligned} &\ddot{J}_s - \mathrm{ad}^2(a) J_s = 0, \\ &J_0 = f, \quad J_1 = \mathrm{Ad}(k^{-1}) f, \end{aligned} \tag{1.3.10}$$

where the differentials $\dot{J}, \ddot{J}$ are taken with respect to the Levi-Civita connection along $y_\cdot$.

Also we have

$$E''_f(0) = \int_0^1 \left(|\dot{J}_s|^2 + |[a, J_s]|^2 \right) ds. \tag{1.3.11}$$

By (1.3.9), (1.3.10), (1.3.11), we get

$$f \in \mathfrak{z}(a) \cap \mathfrak{p}, \quad \mathrm{Ad}(k) f = f. \tag{1.3.12}$$

Applying (1.3.12) to $h = e^f k'$, $h\gamma = \gamma h$, we obtain

$$e^{\mathrm{Ad}(k')a} k' k^{-1} = e^a k^{-1} k'. \tag{1.3.13}$$

Using the uniqueness of Cartan decomposition in (1.2.16), we get

$$\mathrm{Ad}(k')a = a,\ k'k^{-1} = k^{-1}k'. \tag{1.3.14}$$

By (1.3.12), (1.3.14), we get $h \in \widetilde{Z}(e^a) \cap \widetilde{Z}(k)$. This completes the proof of our proposition. □

In general, if $\gamma \in \widetilde{G}$ is semisimple, then by Theorem 1.3.2, there exist $g \in \widetilde{G}$, $a \in \mathfrak{p}$, $k \in \widetilde{K}$ such that

$$\gamma = ge^a k^{-1} g^{-1},\ \mathrm{Ad}(k)a = a. \tag{1.3.15}$$

Put

$$\gamma_h = ge^a g^{-1},\ \ \gamma_e = gk^{-1}g^{-1}. \tag{1.3.16}$$

The element γ_h (resp. γ_e) is called the hyperbolic (resp. elliptic) part of γ. Then $\gamma = \gamma_h\gamma_e = \gamma_e\gamma_h$. By Proposition 1.3.5,

$$\widetilde{Z}(\gamma) = \widetilde{Z}(\gamma_e) \cap \widetilde{Z}(\gamma_h). \tag{1.3.17}$$

Theorem 1.3.6. *Let $\gamma = \gamma_e\gamma_h = \gamma_h\gamma_e$ be the semisimple element given in* (1.3.15), (1.3.16)*. If there exist $g' \in \widetilde{G}$, $a' \in \mathfrak{p}$, $k' \in \widetilde{K}$ such that*

$$\mathrm{Ad}(k')a' = a',\ \gamma = g'e^{a'}(k')^{-1}(g')^{-1}. \tag{1.3.18}$$

Then

$$\gamma_e = g'e^{a'}(g')^{-1}, \gamma_h = g'(k')^{-1}(g')^{-1}. \tag{1.3.19}$$

Proof. We can rewrite the identities in (1.3.15), (1.3.18) as follows,

$$\gamma = ge^a k^{-1} g^{-1} = g'e^{a'}(k')^{-1}(g')^{-1}. \tag{1.3.20}$$

Put $h = g^{-1}g'$, by (1.3.20),

$$he^{a'}(k')^{-1}h^{-1} = e^a k^{-1}. \tag{1.3.21}$$

We only need to prove that

$$he^{a'}h^{-1} = e^a\ ,\ h(k')^{-1}h^{-1} = k^{-1}. \tag{1.3.22}$$

Put $\gamma' = e^a k^{-1}$.

There is unique $f \in \mathfrak{p}$ and $k'' \in \widetilde{K}$ such that $h = e^f k''$. Put $x = p1, y = ph$, then by Theorem 1.3.2, $x, y \in X(\gamma')$, and $\gamma' x, \gamma' y \in X(\gamma')$.

Put $x(s) = pe^{sf}, s \in [0,1]$ the geodesic connecting x and y. If $t \in [0,1]$, put $\tilde{l}(t) = pe^{ta}, l(t) = hpe^{ta'}$. Then $\tilde{l}(\cdot)$ is the unique geodesic joining x and $\gamma' x$, and $l(\cdot)$ is the unique geodesic joining y and $\gamma' y$.

Furthermore, we have the fourth geodesic given by $\gamma' x(\cdot)$ joining $\gamma' x$ and $\gamma' y$. All the vertexes and geodesics lie in $X(\gamma')$. Then they form a geodesic rectangle in $X(\gamma')$, so that the same arguments using the Jacobi field and energy function as in the proof of Proposition 1.3.5 show that

$$e^f \in \widetilde{Z}(a) \cap \widetilde{Z}(k). \tag{1.3.23}$$

By (1.3.21), (1.3.23),

$$e^{\mathrm{Ad}(k'')a'}k''(k')^{-1}(k'')^{-1} = e^a k^{-1}. \tag{1.3.24}$$

Then

$$\mathrm{Ad}(k'')a' = a\ ,\ k''(k')^{-1}(k'')^{-1} = k^{-1}. \tag{1.3.25}$$

It follows from (1.3.25) that

$$k''e^{a'}(k'')^{-1} = e^a\ ,\ k''(k')^{-1}(k'')^{-1} = k^{-1}. \tag{1.3.26}$$

Let $C(e^f)$ act on both sides of identities in (1.3.26), we get (1.3.22). This completes the proof of our theorem. □

1.4. **The minimizing set** $X(\gamma\sigma)$**.** In this subsection, we fix $\gamma \in G,\ \sigma \in \Sigma$ such that $\gamma\sigma$ is semisimple in $\widetilde{G}$. Recall that $X(\gamma\sigma)$ is the minimizing set of $d_{\gamma\sigma}$, so that $X(\gamma\sigma)$ is a $\widetilde{Z}(\gamma\sigma)$- invariant closed convex subset of X. Recall that the group G^σ is the closed subgroup of $\widetilde{G}$ generated by G and σ.

Since $\theta\sigma = \sigma\theta$, if $x \in X$,

$$d_{\theta(\gamma)\sigma}(\theta x) = d_{\gamma\sigma}(x). \tag{1.4.1}$$

If $\gamma \in K$, then $X(\gamma\sigma)$ is preserved by θ.

If $g \in G$,

$$C(g)(\gamma\sigma) = g\gamma\sigma(g^{-1})\sigma \in G^\sigma. \tag{1.4.2}$$

Let $C^\sigma : G \to G$ be such that if $g, h \in G$,

$$C^\sigma(g)h = gh\sigma(g^{-1}) \in G. \tag{1.4.3}$$

Fix $g_0 \in G$ such that $x_0 = p(g_0) \in X(\gamma\sigma)$. By Theorem 1.3.2, there exists $a \in \mathfrak{p}$, $k \in K$ such that

$$\mathrm{Ad}(k)a = \sigma a, \quad \gamma = C^\sigma(g_0)(e^a k^{-1}). \tag{1.4.4}$$

As in (1.3.15), put $\tilde{\gamma}_h = g_0 e^a g_0^{-1}, \tilde{\gamma}_e = C^\sigma(g_0)(k^{-1})\sigma$, then

$$\gamma\sigma = \tilde{\gamma}_h\tilde{\gamma}_e = \tilde{\gamma}_e\tilde{\gamma}_h. \tag{1.4.5}$$

By (1.3.17) and using the fact that $g_0 \in G$, we get

$$Z(\gamma\sigma) = Z(\tilde{\gamma}_h) \cap Z(\tilde{\gamma}_e) = C(g_0)(Z(e^a) \cap Z(k^{-1}\sigma)). \tag{1.4.6}$$

Let $\mathfrak{z}(k^{-1}\sigma)$ be the Lie algebra of $Z(k^{-1}\sigma)$. Then

$$\mathfrak{z}(k^{-1}\sigma) = \{f \in \mathfrak{g} \mid \mathrm{Ad}(k)f = \sigma f\}. \tag{1.4.7}$$

By (1.3.4), (1.4.6), we get

$$\mathfrak{z}(\gamma\sigma) = \mathrm{Ad}(g_0)(\mathfrak{z}(a) \cap \mathfrak{z}(k^{-1}\sigma)). \tag{1.4.8}$$

Proposition 1.4.1. *As submanifolds of X, we have*

$$X(\gamma\sigma) = g_0(X(e^a) \cap X(k^{-1}\sigma)) \subset X. \tag{1.4.9}$$

Proof. If $y = pg \in X(\gamma\sigma)$, by Theorem 1.3.2, there exists $a' \in \mathfrak{p}, k' \in K$ such that $\gamma\sigma = C(g)(e^{a'}(k')^{-1}\sigma)$. Also by Theorem 1.3.2, Proposition 1.3.6,

$$p(g_0^{-1}g) = g_0^{-1}y \in X(e^a) \cap X(k^{-1}\sigma). \tag{1.4.10}$$

Then

$$X(\gamma\sigma) \subset g_0(X(e^a) \cap X(k^{-1}\sigma)). \tag{1.4.11}$$

If $y = pg \in X(e^a) \cap X(k^{-1}\sigma)$. By Theorem 1.3.2, there exist $a' \in \mathfrak{p}$, $k_1, k_2 \in K$ such that

$$e^a = C(g)(e^{a'}k_1^{-1})\ ,\ \mathrm{Ad}(k_1)a' = a'\ ,\ k^{-1} = C^\sigma(g)(k_2^{-1}). \tag{1.4.12}$$

By (1.3.17), (1.4.4), we have $k_2^{-1}\sigma \in C(g^{-1})\widetilde{Z}(e^a) = \widetilde{Z}(a') \cap \widetilde{Z}(k_1)$. Put $k' = k_2k_1 \in K$, then $e^ak^{-1}\sigma = ge^{a'}(k')^{-1}\sigma g^{-1}$ with $\mathrm{Ad}(k')a' = \sigma a'$. Thus $y = pg \in X(e^ak^{-1}\sigma)$ and $g_0y \in X(\gamma\sigma)$. This completes the proof of our proposition. □

We can use $x_0 = pg_0$ as the base point to get a global geodesic coordinate for X. Indeed, we have a diffeomorphism,

$$\Phi_{g_0} : \mathrm{Ad}(g_0)\mathfrak{p} \to X,\ \ y \mapsto \exp(y)x_0. \tag{1.4.13}$$

In the case when $g_0 = 1$, this coordinate system is just $(\exp_x, \mathfrak{p})$ defined in subsection 1.1.

Proposition 1.4.2. *In the coordinate system defined by Φ_{g_0}, we have,*

(1) $g_0X(e^a) = \mathrm{Ad}(g_0)(\mathfrak{z}(a) \cap \mathfrak{p})$;
(2) $g_0X(k^{-1}\sigma) = \mathrm{Ad}(g_0)(\mathfrak{z}(k^{-1}\sigma) \cap \mathfrak{p})$.

Proof. The first identification is proved in [B11, Theorem 3.2.6]. We only prove the second one. Clearly, $\mathrm{Ad}(g_0)(\mathfrak{z}(k^{-1}\sigma) \cap \mathfrak{p}) \subset g_0X(k^{-1}\sigma)$.

If $b \in \mathfrak{p}$ is such that $\Phi_{g_0}(\mathrm{Ad}(g_0)b) \in g_0X(k^{-1}\sigma)$, then there exists $k' \in K$ such that

$$k^{-1}\exp(\sigma(b)) = \exp(b)k'. \tag{1.4.14}$$

We can rewrite (1.4.14) as

$$\exp(\mathrm{Ad}(k^{-1})\sigma b)k^{-1} = \exp(b)k'. \tag{1.4.15}$$

Then we get

$$\mathrm{Ad}(k^{-1}\sigma)b = b\ ,\ k' = k^{-1}. \tag{1.4.16}$$

From (1.4.16), $b \in \mathfrak{z}(k^{-1}\sigma) \cap \mathfrak{p}$. This completes the proof of our proposition. □

Theorem 1.4.3. *In the coordinate system defined by Φ_{g_0}, we have*

$$X(\gamma\sigma) = \mathfrak{z}(\gamma\sigma) \cap \mathrm{Ad}(g_0)\mathfrak{p}. \tag{1.4.17}$$

Proof. This is just a consequence of (1.4.8) and Propositions 1.4.1, 1.4.2. □

Remark 1.4.4. Since $\mathfrak{z}(\gamma\sigma) = \tilde{\mathfrak{z}}(\gamma\sigma) \cap \mathfrak{g}$, we can rewrite (1.4.17) as

$$X(\gamma\sigma) = \tilde{\mathfrak{z}}(\gamma\sigma) \cap \mathrm{Ad}(g_0)\mathfrak{p}. \tag{1.4.18}$$

We have another Cartan decomposition of $\mathfrak{g}$ associated with g_0,

$$\mathfrak{g} = \mathrm{Ad}(g_0)\mathfrak{p} \oplus \mathrm{Ad}(g_0)\mathfrak{k}. \tag{1.4.19}$$

Put $\mathfrak{p}_{g_0}(\gamma\sigma) := \mathfrak{z}(\gamma\sigma) \cap \mathrm{Ad}(g_0)\mathfrak{p}$ and $\mathfrak{k}_{g_0}(\gamma\sigma) := \mathfrak{z}(\gamma\sigma) \cap \mathrm{Ad}(g_0)\mathfrak{k}$. Since σ preserves the splitting (1.1.1), by (1.4.7),(1.4.8), we get

$$\mathfrak{z}(\gamma\sigma) = \mathfrak{p}_{g_0}(\gamma\sigma) \oplus \mathfrak{k}_{g_0}(\gamma\sigma). \tag{1.4.20}$$

Then in the coordinate $(\Phi_{g_0}, \mathrm{Ad}(g_0)\mathfrak{p})$, (1.4.17) is equivalent to

$$X(\gamma\sigma) = \mathfrak{p}_{g_0}(\gamma\sigma). \tag{1.4.21}$$

Recall that $\widetilde{Z}(\gamma\sigma)$ acts on $X(\gamma\sigma)$ isometrically.

Definition 1.4.5. We define a map $\tilde{p} : \widetilde{Z}(\gamma\sigma) \to X(\gamma\sigma)$ by

$$\tilde{p}(\tilde{g}) = \tilde{g}x_0. \tag{1.4.22}$$

Note that if $g_0 = 1$, the map $\tilde{p}$ in (1.4.22) is just the restriction of p to $\widetilde{Z}(\gamma\sigma)$.

Lemma 1.4.6. *The action of $Z(\gamma\sigma)$ on $X(\gamma\sigma)$ is transitive, and the stabilizing subgroup of x_0 is given by $Z(\gamma\sigma) \cap C(g_0)K$. Moreover, $Z^0(\gamma\sigma)$ acts on $X(\gamma\sigma)$ transitively.*

Proof. Let $g \in G$ be such that $x = p(g) \in X(\gamma\sigma)$, by Theorem 1.4.3, there exists $y \in \mathfrak{p}_{g_0}(\gamma\sigma)$ such that $p(g) = \exp(y)x_0$. Clearly, $\exp(y) \in Z^0(\gamma\sigma)$, so that $\tilde{p}(\exp(y)) = x$. Then $Z^0(\gamma\sigma)$ acts on $X(\gamma\sigma)$ transitively, so does $Z(\gamma\sigma)$.

If $g \in Z(\gamma\sigma)$ fixes x_0, then

$$C(g_0^{-1})g \in K, \tag{1.4.23}$$

this is equivalent to that $g \in Z(\gamma\sigma)\cap C(g_0)K$. The proof of our lemma is completed. □

In the sequel, we put

$$\begin{aligned} K_{g_0}(\gamma\sigma) &= Z(\gamma\sigma) \cap C(g_0)K, \\ K^\sigma_{g_0}(\gamma\sigma) &= Z^\sigma(\gamma\sigma) \cap C(g_0)K^\sigma, \\ \widetilde{K}_{g_0}(\gamma\sigma) &= \widetilde{Z}(\gamma\sigma) \cap C(g_0)\widetilde{K}. \end{aligned} \tag{1.4.24}$$

Theorem 1.4.7. *We have the identification of $Z(\gamma\sigma)$-manifolds,*

$$\begin{aligned} X(\gamma\sigma) &\simeq Z(\gamma\sigma)/K_{g_0}(\gamma\sigma) \\ &\simeq \widetilde{Z}(\gamma\sigma)/\widetilde{K}_{g_0}(\gamma\sigma). \end{aligned} \tag{1.4.25}$$

As submanifolds of X, we have

$$\begin{aligned} X(\gamma\sigma) &= Z(\gamma\sigma)/K_{g_0}(\gamma\sigma) \cdot x_0 \\ &= \widetilde{Z}(\gamma\sigma)/\widetilde{K}_{g_0}(\gamma\sigma) \cdot x_0 \subset X. \end{aligned} \tag{1.4.26}$$

Proof. This is a consequence of Lemma 1.4.6. □

Corollary 1.4.8. *Induced by the map $\tilde{p}$, we also have the identification of $Z^0(\gamma\sigma)$-manifolds,*

$$X(\gamma\sigma) \simeq Z^0(\gamma\sigma)/(Z^0(\gamma\sigma) \cap C(g_0)K) \\ \simeq \widetilde{Z}^0(\gamma\sigma)/(\widetilde{Z}^0(\gamma\sigma) \cap C(g_0)\widetilde{K}). \tag{1.4.27}$$

Moreover, the groups $Z^0(\gamma\sigma) \cap C(g_0)K$, $\widetilde{Z}^0(\gamma\sigma) \cap C(g_0)\widetilde{K}$ coincide with the identity components $K^0_{g_0}(\gamma\sigma)$, $\widetilde{K}^0_{g_0}(\gamma\sigma)$ of $K_{g_0}(\gamma\sigma)$, $\widetilde{K}_{g_0}(\gamma\sigma)$ respectively.

The group embeddings $K_{g_0}(\gamma\sigma) \to Z(\gamma\sigma)$ and $\widetilde{K}_{g_0}(\gamma\sigma) \to \widetilde{Z}(\gamma\sigma)$ induce respectively the isomorphisms of finite groups,

$$K^0_{g_0}(\gamma\sigma)\backslash K_{g_0}(\gamma\sigma) \simeq Z^0(\gamma\sigma)\backslash Z(\gamma\sigma), \\ \widetilde{K}^0_{g_0}(\gamma\sigma)\backslash \widetilde{K}_{g_0}(\gamma\sigma) \simeq \widetilde{Z}^0(\gamma\sigma)\backslash \widetilde{Z}(\gamma\sigma). \tag{1.4.28}$$

Proof. The identifications (1.4.27) is clear.

Using the fact that $X(\gamma\sigma)$ is contractible, we get that $Z^0(\gamma\sigma) \cap C(g_0)K$ and $\widetilde{Z}^0(\gamma\sigma) \cap C(g_0)\widetilde{K}$ are connected. Then

$$K^0_{g_0}(\gamma\sigma) = Z^0(\gamma\sigma) \cap C(g_0)K, \\ \widetilde{K}^0_{g_0}(\gamma\sigma) = \widetilde{Z}^0(\gamma\sigma) \cap C(g_0)\widetilde{K}. \tag{1.4.29}$$

Since K and $\widetilde{K}$ both are compact, the groups in (1.4.28) are finite. By (1.4.25), (1.4.27), (1.4.29), we get (1.4.28). The proof of this corollary is completed. □

Remark 1.4.9. In (1.4.25), (1.4.26), (1.4.27), (1.4.28), (1.4.29), we can replace $\widetilde{Z}(\gamma\sigma)$, $\widetilde{K}_{g_0}(\gamma\sigma)$ together with their identity components by $Z^\sigma(\gamma\sigma)$, $K^\sigma_{g_0}(\gamma\sigma)$ and their identity components. In particular, we have

$$K^{\sigma,0}_{g_0}(\gamma\sigma) = Z^{\sigma,0}(\gamma\sigma) \cap C(g_0)K^\sigma. \tag{1.4.30}$$

Remark 1.4.10. Note that the representation of $X(\gamma\sigma)$ in (1.4.26) does not depend on the choice of the base point x_0.

We also can choose a representative $\tilde{g}_0 \in \widetilde{G}$ for the point $x_0 \in X(\gamma\sigma)$, and the analogues of the above results with respect to $\tilde{g}_0$ can be obtained immediately.

Using Propositions 1.4.1 and 1.4.2, Theorem 1.4.3 and by (1.4.19), if we use the Cartan decomposition (1.4.19) instead of (1.1.1) and we use the left transition L_{g_0} to identify subsets of X, we can reduce our assumption of γ in (1.4.4) to the simple case where $g_0 = 1$.

Then we can rewrite (1.4.4) as follows,

$$\gamma = e^a k^{-1}, \qquad \mathrm{Ad}(k)a = \sigma a, \\ a \in \mathfrak{p}, \qquad k \in K. \tag{1.4.31}$$

In the following sections, we will only consider this simple case, and we will drop the subscript g_0 in all the associated notation.

1.5. **The normal coordinate system on X based at $X(\gamma\sigma)$.** In this subsection, we always assume that $\gamma\sigma \in \widetilde{G}$ is of the form given in (1.4.31). Then $x = p1 \in X(\gamma\sigma)$.

By (1.4.8) with $g_0 = 1$, a lies in the center of $\mathfrak{z}(\gamma\sigma)$. Let $\mathfrak{z}^{a,\perp}(\gamma\sigma)$ be the orthogonal subspace to a in $\mathfrak{z}(\gamma\sigma)$, let $\mathfrak{p}^{a,\perp}(\gamma\sigma)$ be the orthogonal subspace to a in $\mathfrak{p}$. Then we have

$$\mathfrak{z}^{a,\perp}(\gamma\sigma) = \mathfrak{p}^{a,\perp}(\gamma\sigma) \oplus \mathfrak{k}(\gamma\sigma). \tag{1.5.1}$$

Moreover, $\mathfrak{z}^{a,\perp}(\gamma\sigma)$ is an ideal of $\mathfrak{z}(\gamma\sigma)$.

Let $Z^{a,\perp,0}(\gamma\sigma)$ be the connected Lie subgroup of $Z^0(\gamma\sigma)$ that corresponds the Lie algebra $\mathfrak{z}^{a,\perp}(\gamma\sigma)$. Note that if $a \neq 0$, we have

$$Z^0(\gamma\sigma) \simeq Z^{a,\perp,0}(\gamma\sigma) \times \mathbb{R}, \tag{1.5.2}$$

where e^{ta} maps into $t|a| \in \mathbb{R}$.

As in [B11, Theorem 3.3.1], let $X^{a,\perp}(\gamma\sigma)$ be the image of $Z^{a,\perp,0}(\gamma\sigma)$ by the projection p, which is a convex submanifold of $X(\gamma\sigma)$. Then we have

$$X^{a,\perp}(\gamma\sigma) = Z^{a,\perp,0}(\gamma\sigma)/K^0(\gamma\sigma). \tag{1.5.3}$$

If $a \neq 0$, by (1.5.2), (1.5.3), we have the identification of Riemannian $Z^0(\gamma\sigma)$-manifolds,

$$X(\gamma\sigma) \simeq X^{a,\perp}(\gamma\sigma) \times \mathbb{R}, \tag{1.5.4}$$

so that the action of e^{ta} on $X(\gamma\sigma)$ corresponds to the translation by $t|a|$ on $\mathbb{R}$, and the action of $\gamma\sigma$ on $X(\gamma\sigma)$ is just the translation by $|a|$.

Let $\mathfrak{z}^{\perp}(\gamma\sigma)$ be the orthogonal subspace of $\mathfrak{z}(\gamma\sigma)$ in $\mathfrak{g}$ with respect to B. Put

$$\mathfrak{p}^{\perp}(\gamma\sigma) = \mathfrak{z}^{\perp}(\gamma\sigma) \cap \mathfrak{p}, \ \mathfrak{k}^{\perp}(\gamma\sigma) = \mathfrak{z}^{\perp}(\gamma\sigma) \cap \mathfrak{k}. \tag{1.5.5}$$

Then the splitting (1.4.20) with $g_0 = 1$ shows that

$$\mathfrak{z}^{\perp}(\gamma\sigma) = \mathfrak{p}^{\perp}(\gamma\sigma) \oplus \mathfrak{k}^{\perp}(\gamma\sigma). \tag{1.5.6}$$

The normal bundle of $X(\gamma\sigma)$ in X is given by

$$\begin{aligned} N_{X(\gamma\sigma)/X} &= Z(\gamma\sigma) \times_{K(\gamma\sigma)} \mathfrak{p}^{\perp}(\gamma\sigma), \\ &= \widetilde{Z}(\gamma\sigma) \times_{\widetilde{K}(\gamma\sigma)} \mathfrak{p}^{\perp}(\gamma\sigma). \end{aligned} \tag{1.5.7}$$

Let $\mathcal{N}_{X(\gamma\sigma)/X}$ be the total space of $N_{X(\gamma\sigma)/X} \to X(\gamma\sigma)$. By (1.5.7), a point in $\mathcal{N}_{X(\gamma\sigma)/X}$ is represented by a pair (g, f) with $g \in Z(\gamma\sigma)$ or $\widetilde{Z}(\gamma\sigma)$ and $f \in \mathfrak{p}^{\perp}(\gamma\sigma)$.

Let $P_{\gamma\sigma} : X \to X(\gamma\sigma)$ be the orthogonal projection from X into $X(\gamma\sigma)$. As in [B11, Theorems 3.4.1 and 3.4.3], we can define a normal coordinate system on X based at $X(\gamma\sigma)$ as follows.

Theorem 1.5.1. *We have the diffeomorphism of $\widetilde{Z}(\gamma\sigma)$-manifolds,*

$$\rho_{\gamma\sigma} : \mathcal{N}_{X(\gamma\sigma)/X} \longrightarrow X, \tag{1.5.8}$$

defined by

$$\rho_{\gamma\sigma}(g,f)=p(g\exp(f))\in X. \tag{1.5.9}$$

Under this diffeomorphism, the action of $\gamma\sigma$ on X is represented by the map $(g,f)\mapsto(\exp(a)g,\mathrm{Ad}(k^{-1})\sigma f)$, and the projection $P_{\gamma\sigma}$ is given by $P_{\gamma\sigma}(g,f)=(g,0)$.

Proof. The proof of this theorem is the same as the first part of the proof of [B11, Theorem 3.4.1]. □

Proposition 1.5.2. *If $(g,f)\in\mathcal{N}_{X(\gamma\sigma)/X}$, then*

$$d_{\gamma\sigma}(\rho_{\gamma\sigma}(g,f))=d_{\gamma\sigma}(\rho_{\gamma\sigma}(1,f)), \tag{1.5.10}$$

Moreover, there exists a constant $c_{\gamma\sigma}>0$, for $f\in\mathfrak{p}^\perp(\gamma\sigma)$ with $|f|\geq 1$, such that

$$d_{\gamma\sigma}(\rho_{\gamma\sigma}(1,f))\geq|a|+c_{\gamma\sigma}|f|. \tag{1.5.11}$$

There exist $C'_{\gamma\sigma}>0$, $C''_{\gamma\sigma}>0$ such that, for $f\in\mathfrak{p}^\perp(\gamma\sigma)$, if $|f|\geq 1$,

$$|\nabla d_{\gamma\sigma}(\rho_{\gamma\sigma}(1,f))|\geq C'_{\gamma\sigma}, \tag{1.5.12}$$

and if $|f|\leq 1$,

$$|\nabla d^2_{\gamma\sigma}(\rho_{\gamma\sigma}(1,f))/2|\geq C''_{\gamma\sigma}|f|. \tag{1.5.13}$$

In particular, the function $d^2_{\gamma\sigma}/2$ is a Morse-Bott function, whose critical set is $X(\gamma\sigma)$, and its Hessian on $X(\gamma\sigma)$ is given by the symmetric positive endomorphism on $\mathfrak{p}^\perp(\gamma\sigma)$,

$$\nabla^{TX}\nabla d^2_{\gamma\sigma}/2=\frac{\mathrm{ad}(a)}{\sinh(\mathrm{ad}(a))}\left(2\cosh(\mathrm{ad}(a))-(\mathrm{Ad}(k^{-1})\sigma+\sigma^{-1}\mathrm{Ad}(k))\right). \tag{1.5.14}$$

Proof. As we have seen, the geometric structures of X associated with $\gamma\sigma$ are the same as the ones considered in [B11, Chapter 3], we can adapt the proof of [B11, Theorem 3.4.1] to prove this proposition. We here only give the detail of the geometric part of the proof.

The equality in (1.5.10) comes from the fact $g\in\widetilde{Z}(\gamma\sigma)$.

For $f\in\mathfrak{p}^\perp(\gamma\sigma)$, if $t\in\mathbb{R}$, set

$$\varphi_f(t)=d_{\gamma\sigma}(pe^{tf}). \tag{1.5.15}$$

It is a convex function from $t\in\mathbb{R}$ to $\mathbb{R}_{\geq 0}$.

First we assume that $\gamma\sigma$ is elliptic, i.e. $a=0$, $\gamma\sigma=\tilde{\gamma}_e$. Since $1-\mathrm{Ad}(\tilde{\gamma}_e)$ is invertible on $\mathfrak{p}^\perp(\gamma\sigma)$, there is $c_{\gamma\sigma}>0$ such that

$$|(1-\mathrm{Ad}(\tilde{\gamma}_e)f|\geq c_{\gamma\sigma}|f|. \tag{1.5.16}$$

Use the results of [E96, Proposition 1.4.1], we have

$$d_{\gamma\sigma}(\rho_{\gamma\sigma}(1,f))=d(\rho_{\gamma\sigma}(1,f),\rho_{\gamma\sigma}(1,\mathrm{Ad}(\tilde{\gamma}_e)f))\geq|(1-\mathrm{Ad}(\tilde{\gamma}_e)f|. \tag{1.5.17}$$

Then we get (1.5.11) for this case. Using the convexity of $\varphi_f(t)$, we can also get (1.5.12), (1.5.13) when $a=0$.

Let us assume that $\gamma\sigma$ is non-elliptic, i.e. $a \neq 0$. Let $f \in \mathfrak{p}^{\perp}(\gamma\sigma)$ be such that $|f| = 1$. By (1.5.15), $\varphi_f(0) = |a| > 0$. Then $\varphi_f(t)$ is a smooth convex function.

The curves $pe^{tf}, \gamma\sigma pe^{tf}$ for $t \in \mathbb{R}$ are two geodesics in X orthogonal to $X(\gamma\sigma)$. Let $c_t(s), s \in [0,1]$ be the unique geodesic connecting pe^{tf} and $\gamma\sigma pe^{tf}$. We have $c_0(s) = pe^{sa}$.

Put $J_{f,s} = \frac{\partial}{\partial t}|_{t=0} c_t(s)$ the Jacobi field along $c_0(s)$. As in (1.3.10), in the trivialization given by parallel transport, we have

$$\begin{aligned} &\ddot{J}_{f,s} - \mathrm{ad}^2(a) J_{f,s} = 0, \\ &J_{f,0} = f\ ,\ J_{f,1} = \mathrm{Ad}(k^{-1}\sigma) f = \mathrm{Ad}(\tilde{\gamma}_e) f. \end{aligned} \tag{1.5.18}$$

The unique solution of (1.5.18) is given by

$$J_{f,s} = \cosh(s\mathrm{ad}(a)) f + \frac{\sinh(s\mathrm{ad}(a))}{\sinh(\mathrm{ad}(a))} (\mathrm{Ad}(k^{-1})\sigma - \cosh(\mathrm{ad}(a))) f. \tag{1.5.19}$$

As in (1.3.8), set $E_f(t) = \frac{1}{2}\varphi_f^2(t)$. Then we have

$$E_f''(0) = |a| \varphi_f''(0). \tag{1.5.20}$$

Also we have,

$$E_f''(0) = \int_0^1 (|\dot{J}_{f,s}|^2 + |[a, J_{f,s}]|^2) ds. \tag{1.5.21}$$

Thus $E_f''(0)$ continuously depends on $f \in \mathfrak{p}^{\perp}(\gamma\sigma)$. By (1.5.19), there exists $C > 0$ such that if $f \in \mathfrak{p}^{\perp}(\gamma\sigma)$, $|f| = 1$, then $E_f''(0) \geq C$.

Now we can proceed the proof using the same arguments as in [B11, eq.(3.4.22) - eq.(3.4.28)], when replacing k^{-1} by $k^{-1}\sigma$, we get (1.5.11) - (1.5.13).

The identity (1.5.14) follows from (1.5.19) and [B11, eq.(3.4.29)].

The proof of our proposition is completed. □

Remark 1.5.3. If $x \in X(\gamma\sigma)$, under the identification in Theorem 1.5.1, by (1.5.11) the displacement function $d_{\gamma\sigma}$ is increasing at least linearly along the normal fiber at x. This property will be used in the geometric interpretation of the twisted orbital integrals in section 4.

The group $K(\gamma\sigma)$ (resp. $\widetilde{K}(\gamma\sigma)$) acts on the left on K (resp. $\widetilde{K}$). We define a vector bundle $\mathfrak{p}^{\perp}(\gamma\sigma)_{K(\gamma\sigma)} \times K$ (resp. $\mathfrak{p}^{\perp}(\gamma\sigma)_{\widetilde{K}(\gamma\sigma)} \times \widetilde{K}$) on $K(\gamma\sigma)\backslash K$ (resp. $\widetilde{K}(\gamma\sigma)\backslash\widetilde{K}$) by the relation, for $f \in \mathfrak{p}^{\perp}(\gamma\sigma)$, $k \in K$ (resp. $\widetilde{K}$) and $h \in K(\gamma\sigma)$ (resp. $\widetilde{K}(\gamma\sigma)$),

$$(f, k) \sim (\mathrm{Ad}(h) f, hk). \tag{1.5.22}$$

We also define the right action of $k \in K$ (resp. $\widetilde{K}$) on $\mathfrak{p}^{\perp}(\gamma\sigma) \times K$ (resp. $\mathfrak{p}^{\perp}(\gamma\sigma) \times \widetilde{K}$) is the multiplication on K (resp. $\widetilde{K}$) from the right side by k.

By Theorem 1.5.1, we can define two maps as follows,

$$\begin{aligned} &\varrho_{\gamma\sigma} : (g, f, k) \in Z(\gamma\sigma) \times_{K(\gamma\sigma)} (\mathfrak{p}^{\perp}(\gamma\sigma) \times K) \to ge^f k \in G, \\ &\tilde{\varrho}_{\gamma\sigma} : (g, f, k) \in \widetilde{Z}(\gamma\sigma) \times_{\widetilde{K}(\gamma\sigma)} (\mathfrak{p}^{\perp}(\gamma\sigma) \times \widetilde{K}) \to ge^f k \in \widetilde{G}. \end{aligned} \tag{1.5.23}$$

Theorem 1.5.4. *The map $\varrho_{\gamma\sigma}$ in (1.5.23) a diffeomorphism of left $Z(\gamma\sigma)$-spaces and of right K-spaces, and $\tilde{\varrho}_{\gamma\sigma}$ is a diffeomorphism of left $\widetilde{Z}(\gamma\sigma)$-spaces and of right $\widetilde{K}$-spaces. The projection p is represented by $(g,f,k) \mapsto (g,f)$. Moreover, under this diffeomorphism, we have*

$$\begin{aligned}\mathfrak{p}^{\perp}(\gamma\sigma)_{K(\gamma\sigma)} \times K &= Z(\gamma\sigma)\backslash G,\\ \mathfrak{p}^{\perp}(\gamma\sigma)_{\widetilde{K}(\gamma\sigma)} \times \widetilde{K} &= \widetilde{Z}(\gamma\sigma)\backslash \widetilde{G}.\end{aligned} \tag{1.5.24}$$

Proof. The first two statements in our theorem follow from Theorem (1.5.1). The identifications in (1.5.24) are just consequences of the diffeomorphisms in (1.5.23). □

Remark 1.5.5. In Theorems 1.5.1 and 1.5.4 and in Proposition 1.5.2, we also can replace $Z(\gamma\sigma), \widetilde{Z}(\gamma\sigma), K(\gamma\sigma), \widetilde{K}(\gamma\sigma)$ by their identity components. These results are also true for $Z^{\sigma}(\gamma\sigma), K^{\sigma}(\gamma\sigma)$ and their identity components. In particular, we have

$$\begin{aligned}\mathfrak{p}^{\perp}(\gamma\sigma)_{K^0(\gamma\sigma)} \times K &= Z^0(\gamma\sigma)\backslash G,\\ \mathfrak{p}^{\perp}(\gamma\sigma)_{\widetilde{K}^0(\gamma\sigma)} \times \widetilde{K} &= \widetilde{Z}^0(\gamma\sigma)\backslash \widetilde{G},\\ \mathfrak{p}^{\perp}(\gamma\sigma)_{K^{\sigma,0}(\gamma\sigma)} \times K^{\sigma} &= Z^{\sigma,0}(\gamma\sigma)\backslash G^{\sigma}.\end{aligned} \tag{1.5.25}$$

If $\tilde{\gamma} \in \widetilde{G}$, let $[\tilde{\gamma}] \subset \widetilde{G}$ be the conjugacy class of $\tilde{\gamma}$ in $\widetilde{G}$, i.e.,

$$[\tilde{\gamma}] = \{C(\tilde{g})\tilde{\gamma} \ : \ \tilde{g} \in \widetilde{G}\}. \tag{1.5.26}$$

Proposition 1.5.6. *If $\tilde{\gamma} \in \widetilde{G}$ is semisimple, then the conjugacy class of $\tilde{\gamma}$ in $\widetilde{G}$ is a closed subset.*

Proof. We can assume that $\tilde{\gamma} = \gamma\sigma$ given by (1.4.31). Then the above geometric constructions are applicable.

We suppose that $\{\tilde{\gamma}_i\}_{i\in\mathbb{N}} \subset [\gamma\sigma]$ is a Cauchy sequence in $\widetilde{G}$ with the limit $\tilde{g}_0 \in \widetilde{G}$. In particular, we have, as $i \to +\infty$,

$$d(p\tilde{\gamma}_i, p\tilde{g}_0) \to 0. \tag{1.5.27}$$

By (1.5.24), for $i \in \mathbb{N}$, there exists $g_i = e^{f_i}k_i$, $f_i \in \mathfrak{p}^{\perp}(\gamma\sigma)$, $k_i \in \widetilde{K}$ such that

$$\tilde{\gamma}_i = g_i^{-1}\gamma\sigma g_i. \tag{1.5.28}$$

By (1.5.27), (1.5.28), we get, as $i \to +\infty$,

$$d(\gamma\sigma p e^{f_i}, g_i p\tilde{g}_0) \to 0. \tag{1.5.29}$$

Use the triangle inequality for the distance d on X, by (1.5.29), we get, as $i \to +\infty$,

$$d(\gamma\sigma p e^{f_i}, p e^{f_i}) \to d(p1, p\tilde{g}_0). \tag{1.5.30}$$

Using (1.5.11), (1.5.30), we get the set $\{f_i\}_{i\in\mathbb{N}}$ is a bounded set in $\mathfrak{p}^\perp(\gamma\sigma)$. Then we can assume that there exist $f'\in\mathfrak{p}^\perp(\gamma\sigma)$, $k'\in\widetilde{K}$ such that, as $i\to+\infty$,

$$f_i\to f',\quad k_i\to k'. \tag{1.5.31}$$

Put $g'=e^{f'}k'\in\widetilde{G}$, then as $i\to+\infty$,

$$g_i\to g'. \tag{1.5.32}$$

By (1.5.28), we get

$$\tilde{g}_0=(g')^{-1}\gamma\sigma g'\in[\gamma\sigma]. \tag{1.5.33}$$

This completes the proof of our proposition. □

Let dx be the volume element on X induced by the Riemannian metric. Recall that $Y^\mathfrak{p}\in\mathfrak{p}\to\exp_{p1}(Y^\mathfrak{p})\in X$ defines a global geodesic coordinate of X. Let $dY^\mathfrak{p}$ be the volume element on the Euclidean space $\mathfrak{p}$. Then there is a positive smooth function η on $\mathfrak{p}$ such that $\eta(0)=1$ and, under the identification of manifolds,

$$dx=\eta(Y^\mathfrak{p})dY^\mathfrak{p}. \tag{1.5.34}$$

By [B11, eq. (4.1.12)], there exist $c>0, C>0$ such that

$$\eta(Y^\mathfrak{p})\leq c\exp(C|Y^\mathfrak{p}|). \tag{1.5.35}$$

Let dk be the normalized Haar measure of K. Put

$$dg=dxdk. \tag{1.5.36}$$

Then dg is a left-invariant Haar measure on G. Since G is unimodular, dg is also right-invariant.

Let dy be the volume element on $X(\gamma\sigma)$ induced by Riemannian metric, let df be the volume element on the Euclidean space $\mathfrak{p}^\perp(\gamma\sigma)$. Then $dydf$ is a volume element on $Z^0(\gamma\sigma)\times_{K^0(\gamma\sigma)}\mathfrak{p}^\perp(\gamma\sigma)$ which is $Z^0(\gamma\sigma)$-invariant. By Theorem 1.5.1, there is a smooth positive $K^0(\gamma\sigma)$-invariant function $r(f)$ on $\mathfrak{p}^\perp(\gamma\sigma)$ such that we have the identity of volume elements on X,

$$dx=r(f)dydf, \tag{1.5.37}$$

with $r(0)=1$. Moreover, by [B11, (3.4.36)], there exist $C>0,\ C'>0$ such that for $f\in\mathfrak{p}^\perp(\gamma\sigma)$,

$$r(f)\leq C\exp(C'|f|). \tag{1.5.38}$$

Let dk'^0 be the Haar measure on $K^0(\gamma\sigma)$ that gives volume 1 to $K^0(\gamma\sigma)$, and let du^0 be the $K-$invariant volume form on $K^0(\gamma\sigma)\backslash K$, so that

$$dk=dk'^0du^0. \tag{1.5.39}$$

Set

$$dz^0=dydk'^0. \tag{1.5.40}$$

Then dz^0 is a left invariant Haar measure on $Z^0(\gamma\sigma)$. Combining (1.5.37) - (1.5.40), we get

$$dg=r(f)dz^0dfdu^0. \tag{1.5.41}$$

Also by (1.5.25), $r(f)dfdu^0$ can be viewed as a measure on $Z^0(\gamma\sigma)\backslash G$ such that

$$dg = dz^0 \cdot r(f)dfdu^0. \tag{1.5.42}$$

Then $dv^0 = r(f)dfdu^0$ is exactly the measure on $Z^0(\gamma\sigma)\backslash G$ that is canonically associated with dg and dz^0.

When replacing $Z^0(\gamma\sigma)$, $K^0(\gamma\sigma)$ by $Z(\gamma\sigma)$, $K(\gamma\sigma)$, one can define measures dk', du, dz, dv on $K(\gamma\sigma)$, $K(\gamma\sigma)\backslash K$, $Z(\gamma\sigma)$, $Z(\gamma\sigma)\backslash G$ such that the analogues of (1.5.39) - (1.5.42) still hold.

Let dn be the normalized Haar measure on $K^0(\gamma\sigma)\backslash K(\gamma\sigma)$ such that

$$dk' = dk'^0 dn. \tag{1.5.43}$$

By the normalization of dk'^0, dk', we have

$$\int_{K^0(\gamma\sigma)\backslash K(\gamma\sigma)} dn = \mathrm{Vol}(K(\gamma\sigma)) = 1. \tag{1.5.44}$$

Moreover, using (1.4.28) for the groups $K^\sigma(\gamma\sigma)$ and $Z^\sigma(\gamma\sigma)$, we get

$$dz = dz^0 dn. \tag{1.5.45}$$

Using the canonical projection $Z^0(\gamma\sigma)\backslash G \to Z(\gamma\sigma)\backslash G$, and by (1.4.28), we get

$$dv^0 = dndv. \tag{1.5.46}$$

Let $d\mu$ be the normalized Haar measure of Σ^σ. Put

$$d\tilde{g} = dgd\mu. \tag{1.5.47}$$

This defines a bi-invariant Haar measure on G^σ. If $d\tilde{k}$ is the normalized Haar measure on K^σ, then $d\tilde{k} = dkd\mu$.

Let $d\tilde{k}^\sigma$ be the normalized Haar measure on $K^\sigma(\gamma\sigma)$, let $d\tilde{u}^\sigma$ be the $K^\sigma-$invariant measure on $K^\sigma(\gamma\sigma)\backslash K^\sigma$ such that

$$d\tilde{k} = d\tilde{k}^\sigma d\tilde{u}^\sigma. \tag{1.5.48}$$

Set

$$d\tilde{z}^\sigma = dyd\tilde{k}^\sigma. \tag{1.5.49}$$

Then $d\tilde{z}^\sigma$ is a left invariant Haar measure on $Z^\sigma(\gamma\sigma)$. Furthermore,

$$d\tilde{g} = dxd\tilde{k} = r(f)dydfd\tilde{k}^\sigma d\tilde{u}^\sigma = d\tilde{z}^\sigma \cdot r(f)dfd\tilde{u}^\sigma. \tag{1.5.50}$$

Then $d\tilde{v}^\sigma = r(f)dfd\tilde{u}^\sigma$ is a measure on $Z^\sigma(\gamma\sigma)\backslash G^\sigma$.

Also the analogues of (1.5.47) - (1.5.50) can be formulated for the groups $\widetilde{G}$, $\widetilde{K}$, $\widetilde{\Sigma}$, $\widetilde{Z}(\gamma\sigma)$ and the orbit space $\widetilde{Z}(\gamma\sigma)\backslash\widetilde{G}$.

In the sequel, we will always use these measures for the associated integrations.

1.6. **The return map along the minimizing geodesic in** $X(\gamma\sigma)$**.** In this subsection, we still assume that $\gamma\sigma \in \widetilde{G}$ is of the form given in (1.4.31).

Recall that $\pi : \mathcal{X} \to X$ is the total space of the tangent bundle TX to X. Let $\mathcal{X}^*$ be the total space of the cotangent bundle T^*X. We still use π denote the canonical projection from T^*X to X. Let p be the generic element of T^*X, and then $\vartheta = \pi^* p$ is a smooth 1-form on $\mathcal{X}^*$. Put $\omega = d^{\mathcal{X}^*}\vartheta$, which is the canonical symplectic form on $\mathcal{X}^*$. The identification of the fibres TX and T^*X by the metric g^{TX} identifies the manifolds $\mathcal{X}$ and $\mathcal{X}^*$.

Put $\mathcal{H}(x,p) = \frac{1}{2}|p|^2$ the Hamiltonian on $\mathcal{X}^*$. Let V be the Hamiltonian vector field associated with $\mathcal{H}$. Then V is the generator of geodesic flow. Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the corresponding $1-$parameter subgroup of diffeomorphisms of $\mathcal{X}^*$, which preserves the symplectic form. When identifying $\mathcal{X}$ and $\mathcal{X}^*$, we may consider φ_t as a flow of symplectic diffeomorphisms of $\mathcal{X}$. If $(x, Y^{TX}) \in \mathcal{X}$, if $(x_t, Y^{TX}_t) = \varphi_t(x, Y^{TX})$, then $t \in \mathbb{R} \to x_t \in X$ is the unique geodesic in X such that $x_0 = x, \dot{x}_0 = Y^{TX}$.

The action of $\gamma\sigma$ lifts to $\mathcal{X}$ and $\mathcal{X}^*$. Since $\gamma\sigma$ is isometry, these actions correspond by the identification through the metric g^{TX}. Then $\gamma\sigma$ preserves the symplectic form of $\mathcal{X}$ or $\mathcal{X}^*$. Now we study the symplectic diffeomorphism $(\gamma\sigma)^{-1}\varphi_1$ of $\mathcal{X}$.

Set

$$\mathcal{F}_{\gamma\sigma} = \{z \in \mathcal{X} \ : \ (\gamma\sigma)^{-1}\varphi_1(z) = z\}. \tag{1.6.1}$$

The element $a \in \mathfrak{p}$ defines a constant section of $X \times \mathfrak{g}$. By (1.1.17), we can view a as a smooth section of $TX \oplus N$. Let a^{TX}, a^N the corresponding parts of this section in TX, N respectively. Recall that we have a global geodesic coordinate system centered at $x = p1$ which identifies $\mathfrak{p}$ with X by $Y^{\mathfrak{p}} \in \mathfrak{p} \to \exp(Y^{\mathfrak{p}})x$. By [B11, Proposition 3.2.4], we have

$$a^{TX}(Y^{\mathfrak{p}}) = \cosh(\mathrm{ad}(Y^{\mathfrak{p}}))a, \qquad a^N(Y^{\mathfrak{p}}) = -\sinh(\mathrm{ad}(Y^{\mathfrak{p}}))a. \tag{1.6.2}$$

Definition 1.6.1. Let $i_a : X \to \mathcal{X}$ be the embedding

$$x \in X \to (x, a^{TX}) \in \mathcal{X}. \tag{1.6.3}$$

We get the extension of [B11, Proposition 3.5.1] to our case.

Lemma 1.6.2. *we have*

$$\mathcal{F}_{\gamma\sigma} = i_a X(\gamma\sigma). \tag{1.6.4}$$

Proof. For $x \in X(\gamma\sigma)$, let $g \in Z^0(\gamma\sigma)$ be such that $pg = x$. Then $a^{TX}(x)$ is given by $[g, (\mathrm{Ad}(g^{-1})a)^{\mathfrak{p}}]$. By (1.4.6), we get $a^{TX}(x) = [g, a]$. Then

$$\varphi_1((x, a^{TX})) = [ge^a, a] = \gamma\sigma(x, a^{TX}(x)) \in \mathcal{X}. \tag{1.6.5}$$

We get $i_a X(\gamma\sigma) \subset \mathcal{F}(\gamma\sigma)$.

If $(x, Y^{TX}) \in \mathcal{F}(\gamma\sigma)$, then x_t is a geodesic connecting x and $\gamma\sigma(x)$ such that $(\gamma\sigma)_* Y^{TX}_0 = Y^{TX}_1$. Since $\dot{x}_t = Y^{TX}_t$, we get that x is a critical point of $d^2_{\gamma\sigma}$. By Remark 1.1.3, we get $x \in X(\gamma\sigma)$. Furthermore, $Y^{TX} = a^{TX}(x)$. This completes the proof of our lemma. □

Let $\mathfrak{z}_0 = \mathfrak{z}(a)$, and put

$$\mathfrak{p}_0 = \ker \mathrm{ad}(a) \cap \mathfrak{p}, \quad \mathfrak{k}_0 = \ker \mathrm{ad}(a) \cap \mathfrak{k}. \tag{1.6.6}$$

Let $\mathfrak{z}_0^\perp$, $\mathfrak{p}_0^\perp$, $\mathfrak{k}_0^\perp$ be the orthogonal spaces to $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$ in $\mathfrak{g}, \mathfrak{p}, \mathfrak{k}$ with respect to B, then

$$\mathfrak{z}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0, \quad \mathfrak{z}_0^\perp = \mathfrak{p}_0^\perp \oplus \mathfrak{k}_0^\perp. \tag{1.6.7}$$

By (1.4.8), the space $\mathfrak{z}(\gamma\sigma)$ is a Lie subalgebra of $\mathfrak{z}_0$. Also $\mathfrak{p}(\gamma\sigma)$, $\mathfrak{k}(\gamma\sigma)$ are subspaces of $\mathfrak{p}_0$, $\mathfrak{k}_0$ respectively. Let $\mathfrak{z}_0^\perp(\gamma\sigma)$, $\mathfrak{p}_0^\perp(\gamma\sigma)$, $\mathfrak{k}_0^\perp(\gamma\sigma)$ be the orthogonal spaces to $\mathfrak{z}(\gamma\sigma)$, $\mathfrak{p}(\gamma\sigma)$, $\mathfrak{k}(\gamma\sigma)$ in $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$. Then

$$\mathfrak{z}_0^\perp(\gamma\sigma) = \mathfrak{p}_0^\perp(\gamma\sigma) \oplus \mathfrak{k}_0^\perp(\gamma\sigma). \tag{1.6.8}$$

Moreover, the action $\mathrm{ad}(a)$ gives an isomorphism between $\mathfrak{p}_0^\perp$ and $\mathfrak{k}_0^\perp$. Let ρ be the isomorphism from $\mathfrak{p}_0^\perp \oplus \mathfrak{k}_0^\perp$ to $\mathfrak{p}_0^\perp \oplus \mathfrak{p}_0^\perp$ given by

$$\rho(e, f) = (e, -\mathrm{ad}(a)f). \tag{1.6.9}$$

The connection ∇^{TX} on TX induces a splitting

$$T\mathcal{X} \simeq \pi^*(TX \oplus TX). \tag{1.6.10}$$

In (1.6.10), the first copy is identified with its horizontal lift, and the second copy is the tangent bundle along the fibre, which is the kernel of $d\pi : T\mathcal{X} \to TX$.

If $x \in X(\gamma\sigma)$, then $(x, a^{TX}) \in \mathcal{F}_{\gamma\sigma}$. The differential $d((\gamma\sigma)^{-1}\varphi_1)$ is an automorphism of $T_{(x,a^{TX})}\mathcal{X}$. Let $g \in Z(\gamma\sigma)$ be such that $x = pg$. We identify $T_{(x,a^{TX})}\mathcal{X}$ with the vector space $\mathfrak{p} \oplus \mathfrak{p}$ by the left action g on $T_{(p1,a)}\mathcal{X}$.

We also have the extension of [B11, Theorem 3.5.2].

Proposition 1.6.3. *The following identity holds at $(x, a^{TX}) \in \mathcal{F}_{\gamma\sigma}$,*

$$d((\gamma\sigma)^{-1}\varphi_1)|_{\mathfrak{p}\oplus\mathfrak{p}} = \begin{bmatrix} \sigma^{-1}\mathrm{Ad}(k) & 0 \\ 0 & \sigma^{-1}\mathrm{Ad}(k) \end{bmatrix} \exp\left(\begin{bmatrix} 0 & 1 \\ \mathrm{ad}^2(a) & 0 \end{bmatrix} \right). \tag{1.6.11}$$

In particular, we have

$$\begin{aligned} d((\gamma\sigma)^{-1}\varphi_1)|_{\mathfrak{p}_0\oplus\mathfrak{p}_0} &= \begin{bmatrix} \sigma^{-1}\mathrm{Ad}(k)|_{\mathfrak{p}_0} & \sigma^{-1}\mathrm{Ad}(k)|_{\mathfrak{p}_0} \\ 0 & \sigma^{-1}\mathrm{Ad}(k)|_{\mathfrak{p}_0} \end{bmatrix} \\ d((\gamma\sigma)^{-1}\varphi_1)|_{\mathfrak{p}_0^\perp\oplus\mathfrak{p}_0^\perp} &= \rho \circ \left(\sigma^{-1}\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp} \circ \rho^{-1}. \end{aligned} \tag{1.6.12}$$

The eigenspace of $d((\gamma\sigma)^{-1}\varphi_1)$ associated with the eigenvalue 1 is just $T\mathcal{F}_{\gamma\sigma} \simeq \mathfrak{p}(\gamma\sigma) \oplus \{0\} \subset \mathfrak{p}_0 \oplus \mathfrak{p}_0$.

Proof. We adapt the proof of [B11, Theorem 3.5.2] to prove our proposition.

Let $x_s, s \in [0,1]$ be the unique geodesic connecting x and $\gamma\sigma(x)$ with constant speed. Let $J_s \in T_{x_s}X$ be a Jacobi field along this geodesic which satisfies

$$\ddot{J} + R^{TX}_{x_s}(J, \dot{x})\dot{x} = 0, \tag{1.6.13}$$

Using the splitting (1.6.10) of $T\mathcal{X}$, the differential $d\varphi_1$ at (x, a^{TX}) is given by the linear map $(J_0, \dot{J}_0) \to (J_1, \dot{J}_1)$.

We trivialize the tangent space along x_s to the vector space T_xX by the parallel transport with respect to the Levi-Civita connection. Then (1.6.13) becomes the differential equation for $J_s \in \mathfrak{p}$

$$\ddot{J} - \mathrm{ad}^2(a)J = 0, \tag{1.6.14}$$

this is equivalent to

$$\frac{\partial}{\partial s}\begin{bmatrix} J_s \\ \dot{J}_s \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mathrm{ad}^2(a) & 0 \end{bmatrix}\begin{bmatrix} J_s \\ \dot{J}_s \end{bmatrix}. \tag{1.6.15}$$

Then we have

$$\begin{bmatrix} J_1 \\ \dot{J}_1 \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & 1 \\ \mathrm{ad}^2(a) & 0 \end{bmatrix}\right)\begin{bmatrix} J_0 \\ \dot{J}_0 \end{bmatrix}. \tag{1.6.16}$$

By (1.6.16), we get (1.6.11). If we take $(J_0, \dot{J}_0) \in \mathfrak{p}_0 \oplus \mathfrak{p}_0$, we get the first identity in (1.6.12).

If $(J_0, \dot{J}_0) \in \mathfrak{p}_0^\perp \oplus \mathfrak{p}_0^\perp$, then $(J_s, \dot{J}_s) \in \mathfrak{p}_0^\perp \oplus \mathfrak{p}_0^\perp$. Put

$$H_s = \rho^{-1}(J_s, \dot{J}_s) \in \mathfrak{p}_0^\perp \oplus \mathfrak{k}_0^\perp \tag{1.6.17}$$

Then (1.6.14) is equivalent to

$$\dot{H} + [a, H] = 0, \tag{1.6.18}$$

so that

$$H_1 = \mathrm{Ad}(e^{-a})H_0. \tag{1.6.19}$$

By (1.6.17), (1.6.19), we get the second identity in (1.6.12).

The second identity of (1.6.12) shows that the kernel of $d((\gamma\sigma)^{-1}\varphi_1) - 1$ in $\mathfrak{p} \oplus \mathfrak{p}$ is just the kernel of $d((\gamma\sigma)^{-1}\varphi_1) - 1$ in $\mathfrak{p}_0 \oplus \mathfrak{p}_0$. Then, by (1.4.8) for $g_0 = 1$ and the first identity in (1.6.12), we get that this kernel coincides with $\mathfrak{p}(\gamma\sigma) \oplus \{0\}$. Since $\mathcal{F}_{\gamma\sigma}$ is the fixed point set of $(\gamma\sigma)^{-1}\varphi_1$, then the kernel of $d((\gamma\sigma)^{-1}\varphi_1) - 1$ is just $T\mathcal{F}_{\gamma\sigma}$. This completes the proof of our proposition. □

Recall that $\widehat{\mathcal{X}}$ is the total space of $TX \oplus N$ and $\underline{\pi} : \widehat{\mathcal{X}} \to \mathcal{X}$ denote the natural projection. The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ lifts to a flow of diffeomorphisms of $\widehat{\mathcal{X}}$. If $(x, Y^{TX}, Y^N) \in \widehat{\mathcal{X}}$, set

$$(x_t, Y_t^{TX}, Y_t^N) = \varphi_t(x, Y^{TX}, Y^N), \tag{1.6.20}$$

then x_t is just the geodesic starting at x with speed Y_t^{TX}, and Y_t^N is the parallel transport of Y^N along x_t.

Recall that $\mathfrak{k}(k^{-1}\sigma)$ is the eigenspace of $\mathfrak{k}$ corresponding to the eigenvalue 1 of $\mathrm{Ad}(k^{-1}\sigma)$. Clearly $K^0(\gamma\sigma)$ acts on $\mathfrak{k}(k^{-1}\sigma)$. Put

$$N(k^{-1}\sigma) = Z^0(\gamma\sigma) \times_{K^0(\gamma\sigma)} \mathfrak{k}(k^{-1}\sigma). \tag{1.6.21}$$

Then $N(k^{-1}\sigma)$ is a subbundle of $N|_{X(\gamma\sigma)}$. Let $\mathcal{N}(k^{-1}\sigma)$ be the total space of $N(k^{-1}\sigma)$. Let $\widehat{i}_a$ be the embedding $(x, Y^N) \in \mathcal{N}(k^{-1}\sigma) \to (x, a^{TX}, Y^N) \in \widehat{\mathcal{X}}$.

Set

$$\widehat{\mathcal{F}}_{\gamma\sigma} = \{z \in \widehat{\mathcal{X}}, (\gamma\sigma)^{-1}\varphi_1 z = z\}. \tag{1.6.22}$$

As in [B11, Section 3.6] and Lemma 1.6.2, we have

Proposition 1.6.4. *We have the following identities,*

$$\begin{aligned} N(k^{-1}\sigma) &= \{Y^N \in N|_{X(\gamma\sigma)} \mid \mathrm{Ad}(k^{-1}\sigma)Y^N = Y^N\},\\ \underline{\pi}\widehat{\mathcal{F}}_{\gamma\sigma} &= \mathcal{F}_{\gamma\sigma},\\ \widehat{\mathcal{F}}_{\gamma\sigma} &= \widehat{i}_a\mathcal{N}(k^{-1}\sigma). \end{aligned} \tag{1.6.23}$$

Proof. The first identity in (1.6.23) follows from (1.6.21), and the second one follows from (1.6.1), (1.6.20), (1.6.22). Using the first two identities in (1.6.23) and by Lemma 1.6.2, we get the third identity. □

1.7. **A pseudodistance on $\mathcal{X}$.** If $x, x' \in X$, let $\tau_x^{x'}$ be the parallel transport from $T_{x'}X$ into T_xX with respect to ∇^{TX} along the unique geodesic joining x' et x. We recall a definition in [B11, Section 3.8] as follows.

Definition 1.7.1. If $(x,f),(x',f') \in \mathcal{X}$, set

$$\delta((x,f),(x',f')) = d(x,x') + |\tau_x^{x'}f' - f|. \tag{1.7.1}$$

We call it a pseudodistance on $\mathcal{X}$.

If $x_0 = p1 \in X$, put

$$d_{x_0}((x,f),(x',f')) = d(x,x') + |\tau_{x_0}^{x}f - \tau_{x_0}^{x'}f'|. \tag{1.7.2}$$

Then d_{x_0} is a distance on $\mathcal{X}$. By [B11, eqs. (3.8.9), (3.8.10)], there exists $C > 0$ such that

$$|\delta((x,f),(x',f')) - d_{x_0}((x,f),(x',f'))| \leq Cd(x,x'). \tag{1.7.3}$$

By Lemma 1.6.2, if $x \notin X(\gamma\sigma)$, for any $t \in \mathbb{R}$,

$$(\gamma\sigma)^{-1}\varphi_t(x,Y^{TX}) \neq (x,Y^{TX}). \tag{1.7.4}$$

If $(x,Y^{TX}) \in \mathcal{X}$ with $x \notin X(\gamma\sigma)$, $|Y^{TX}| = 1$, set

$$(x',Y^{TX\prime}) = \gamma\sigma(x,Y^{TX}). \tag{1.7.5}$$

Then for $t \in \mathbb{R}$, we have

$$\varphi_t(x,Y^{TX}) \neq \varphi_{-t}(x',Y^{TX\prime}). \tag{1.7.6}$$

Proposition 1.7.2. *Given $\beta > 0$, there exists $C_{\gamma\sigma,\beta} > 0$ such that if $x \in X$ is such that $d(x,X(\gamma\sigma)) \geq \beta$, if $Y^{TX} \in T_xX, |Y^{TX}| = 1$, for $t > 0$, then*

$$\delta(\varphi_t(x,Y^{TX}),\varphi_{-t}\gamma\sigma(x,Y^{TX})) \geq C_{\gamma\sigma,\beta}. \tag{1.7.7}$$

Proof. Using Lemma 1.1.4 and Proposition 1.5.2, an easy modification of the proof of [B11, Theorems 3.9.1] gives a proof of our proposition.

□

Similarly, following the same arguments of the proofs of [B11, Theorems 3.9.2 - 3.9.4], we also have their analogues as follows.

Proposition 1.7.3. *Given $\beta, M>0$, there exists $C_{\gamma\sigma,\beta,M}>0$ such that if $x\in X$ is such that $d(x,X(\gamma\sigma))\geq\beta$, if $Y^{TX}\in T_xX$, for $0\leq t\leq M$,*

$$\delta(\varphi_t(x,Y^{TX}),\varphi_{-t}\gamma\sigma(x,Y^{TX}))\geq C_{\gamma\sigma,\beta,M}. \tag{1.7.8}$$

Proposition 1.7.4. *Given $\beta>0,\mu>0$, there exists $C_{\gamma\sigma,\beta,\mu}>0$ such that if $f\in\mathfrak{p}^{\perp}(\gamma\sigma)$, $|f|\leq\beta$, $x=\rho_{\gamma\sigma}(1,f)$, $|Y^{TX}-a^{TX}|\geq\mu$,*

$$\delta(\varphi_{1/2}(x,Y^{TX}),\varphi_{-1/2}\gamma\sigma(x,Y^{TX}))\geq C_{\gamma\sigma,\beta,\mu}. \tag{1.7.9}$$

Given $\nu>0$, there exists $C_\nu>0$ such that if $f\in\mathfrak{p}^{\perp}(\gamma\sigma)$, $|f|\leq 1$, $x=\rho_{\gamma\sigma}(1,f)$, $Y^{TX}\in T_xX$, $|Y^{TX}|\leq\nu$, then

$$\delta(\varphi_{1/2}(x,Y^{TX}),\varphi_{-1/2}\gamma\sigma(x,Y^{TX}))\geq C_\nu(|f|+|Y^{TX}-a^{TX}|). \tag{1.7.10}$$

For $x,x'\in X$, we still denote by $\tau_x^{x'}$ the parallel transport from $N_{x'}$ into N_x along the unique geodesic connecting x' to x with respect to ∇^N.

Take $(x,Y)\in\widehat{\mathcal{X}}$. Set

$$(x',Y')=\gamma\sigma(x,Y). \tag{1.7.11}$$

Put

$$\underline{x}_t=\widehat{\pi}\varphi_t(x,Y),\ \underline{x}'_t=\widehat{\pi}\varphi_{-t}(x',Y'). \tag{1.7.12}$$

Let $\underline{Y}^N_{\cdot}$, $\underline{Y}^{N\prime}_{\cdot}$ be the parallel transports of Y^N, $Y^{N\prime}$ along $\underline{x}_t$, $\underline{x}'_t$ with respect to ∇^N.

The same as in [B11, Theorem 3.9.5], there exists $c_{\gamma\sigma}>0$ such that $f\in\mathfrak{p}^{\perp}(\gamma\sigma)$, $|f|\leq 1$, $x=\rho_{\gamma\sigma}(1,f)$, if $|Y^{TX}-a^{TX}|\leq 1$, then

$$\left|\tau_{\underline{x}_{1/2}}^{\underline{x}'_{1/2}}\underline{Y}^{N\prime}_{1/2}-\underline{Y}^N_{1/2}\right|\geq\left|(\mathrm{Ad}(k^{-1}\sigma)-1)Y^N\right|-c_{\gamma\sigma}(|f|+|Y^{TX}-a^{TX}|)|Y^N|. \tag{1.7.13}$$

We can extend δ to a pseudodistance on $\widehat{\mathcal{X}}$. Combining Proposition 1.7.4 and (1.7.13), an estimate can be established for this pseudodistance on $\widehat{\mathcal{X}}$.

1.8. **The locally symmetric space Z.** Let Γ be a cocompact discrete subgroup of G.

Lemma 1.8.1. *If Γ is a cocompact discrete subgroup of G, then any $\gamma\in\Gamma$ is semisimple, and $\Gamma\cap Z(\gamma)$ is a cocompact discrete subgroup of $Z(\gamma)$. More generally, if $\sigma\in\Sigma$, $\sigma(\Gamma)\subset\Gamma$, if $\gamma\in\Gamma$, then $\gamma\sigma\in\widetilde{G}$ is also semisimple, and $\Gamma\cap Z(\gamma\sigma)$ is a cocompact discrete subgroup of $Z(\gamma\sigma)$.*

Proof. The first part of this lemma was proved in [M17, Proposition 3.9]. Also if $\gamma\in\Gamma$, $\sigma(\Gamma)\subset\Gamma$, by [Sel60, Lemmas 1,2], $\Gamma\cap Z(\gamma\sigma)$ is a discrete cocompact subgroup of $Z(\gamma\sigma)$. We only need to prove that $\gamma\sigma$ is semisimple. We will adapt the proof of [M17, Proposition 3.9] to obtain this conclusion.

Recall that $p:G\to X$ is a proper projection. Since Γ is cocompact, we choose and fix a compact fundamental domain $U\subset G$ for $\Gamma\backslash G$. Then we have

$$G=\Gamma\cdot U. \tag{1.8.1}$$

There is a sequence $\{g_i\}_{i\in\mathbb{N}} \subset G$ such that

$$d_{\gamma\sigma}(pg_i) \to m_{\gamma\sigma} \text{ as } i \to +\infty. \tag{1.8.2}$$

Using (1.8.1), there exists a sequence $\{\gamma_i\}_{i\in\mathbb{N}} \subset \Gamma$ such that $h_i = \gamma_i g_i \in U$. Then the convergence becomes

$$d(ph_i, \gamma_i\gamma\sigma(\gamma_i^{-1})p\sigma(h_i)) \to m_{\gamma\sigma}. \tag{1.8.3}$$

Since U is compact, we may and we will assume that $\{h_i\}_{i\in\mathbb{N}}$ is a convergent sequence with limit $h \in U$.

As in [M17, eq.(3.47)], we have

$$\begin{aligned} d(ph, \gamma_i\gamma\sigma(\gamma_i^{-1})p\sigma(h)) \leq\ & d(ph, ph_i) + d(ph_i, \gamma_i\gamma\sigma(\gamma_i^{-1})p\sigma(h_i)) \\ & + d(\gamma_i\gamma\sigma(\gamma_i^{-1})p\sigma(h_i), \gamma_i\gamma\sigma(\gamma_i^{-1})p\sigma(h)) \\ =\ & 2d(ph, ph_i) + d(pg_i, \gamma\sigma p g_i). \end{aligned} \tag{1.8.4}$$

By (1.8.2), the right-hand side of (1.8.4) converges to $m_{\gamma\sigma}$ as $i \to +\infty$, then the set $\{d(ph, \gamma_i\gamma\sigma(\gamma_i^{-1})p\sigma(h))\}_{i\in\mathbb{N}}$ is bounded. Note that $\gamma_i\gamma\sigma(\gamma_i^{-1}) \in \Gamma$, since Γ is discrete, the set of such $\gamma_i\gamma\sigma(\gamma_i^{-1})$ is finite. This implies that there exist infinite γ_{m_i} such that $\gamma_{m_i}\gamma\sigma(\gamma_{m_i}^{-1}) = \gamma' \in \Gamma$. Then $m_{\gamma'\sigma} = m_{\gamma\sigma}$, and we have

$$m_{\gamma\sigma} = d(ph, \gamma' p\sigma(h)) = d(p(\gamma_{m_i}h), \gamma\sigma p(\gamma_{m_i}h)). \tag{1.8.5}$$

Therefore, $\gamma\sigma$ is semisimple. □

Definition 1.8.2. If $\gamma_1, \gamma_2 \in \Gamma$, we say that $\gamma_1 \sim \gamma_2$ if there exists $\gamma \in \Gamma$ such that

$$\gamma_2 = C^\sigma(\gamma)\gamma_1. \tag{1.8.6}$$

which is the same as,

$$\gamma_2\sigma = C(\gamma)(\gamma_1\sigma). \tag{1.8.7}$$

By (1.8.7), one verifies that $\sim$ is an equivalence relation. We denote by $\underline{C}$ the set of equivalence classes in Γ. Let $\underline{[\gamma]}_\sigma$ be the equivalence class of $\gamma \in \Gamma$. If $\gamma\sigma$ is elliptic, we say that $\underline{[\gamma]}_\sigma$ is an elliptic class. Let $\underline{E}$ be the set of elliptic classes in $\underline{C}$.

The map $\gamma' \in \Gamma \mapsto (\gamma')^{-1}\gamma\sigma(\gamma') \in \underline{[\gamma]}_\sigma$ induces the identification

$$\underline{[\gamma]}_\sigma \simeq \Gamma \cap Z(\gamma\sigma)\backslash\Gamma. \tag{1.8.8}$$

Lemma 1.8.3. *The set $\underline{E}$ is finite.*

Proof. Let $U \subset G$ be the compact fundamental domain for $\Gamma\backslash G$ as in the proof of Lemma 1.8.1.

Put

$$V = p^{-1}(p(U)) = U \cdot K. \tag{1.8.9}$$

Then V is a compact subset of G. We denote by V^{-1} the set of the inverses of elements in V. Then V^{-1} and $V \cdot \sigma(V^{-1})$ are compact in G.

For any $\underline{[\gamma]}_\sigma \in \underline{E}$, there exists $\gamma' \in \underline{[\gamma]}_\sigma$ such that $\gamma'\sigma$ has fixed points in $p(V) = p(U)$. Let $g_\gamma \in U$ such that pg_γ is fixed by $\gamma'\sigma$. Then we get

$$\gamma' \in UK\sigma(U^{-1}) \cap \Gamma \subset V \cdot \sigma(V^{-1}) \cap \Gamma. \tag{1.8.10}$$

Since $V \cdot \sigma(V^{-1})$ is compact and Γ is discrete, $V \cdot \sigma(V^{-1}) \cap \Gamma$ is a finite set. This completes the proof of our lemma. □

Remark 1.8.4. If we take $\sigma = \mathbb{1}_G$, then the set $\underline{E}$ is just the set of conjugacy classes of elliptic elements in Γ, which is a finite set.

Proposition 1.8.5. *We have*

$$\inf_{\underline{[\gamma]}_\sigma \in \underline{C} \backslash \underline{E}} m_{\gamma\sigma} > 0. \tag{1.8.11}$$

Proof. Suppose that we have a sequence of $\underline{[\gamma_i]}_\sigma \in \underline{C}\backslash\underline{E}$, $i \in \mathbb{N}$ such that $m_{\gamma_i\sigma} \to 0$ as $i \to +\infty$.

Let $U \subset G$ be the compact fundamental domain in the proof of Lemma 1.8.3. Then for each class $\underline{[\gamma_i]}_\sigma$, there exists $\gamma_i' \in \underline{[\gamma_i]}_\sigma$, $x_i \in p(U)$ such that

$$d_{\gamma_i'\sigma}(x_i) = m_{\gamma_i\sigma}. \tag{1.8.12}$$

Since U is compact, we may and we will assume that $\{x_i\}_{i\in\mathbb{N}}$ is a convergent sequence with the limit $x \in p(U)$.

The triangle inequality shows

$$d(x, \gamma_i'\sigma(x)) \leq d(x, x_i) + d(x_i, \gamma_i'\sigma(x_i)) + d(\gamma_i'\sigma(x_i), \gamma_i'\sigma(x)). \tag{1.8.13}$$

By the assumption, there exists $i_0 \in \mathbb{N}$ such that if $i \geq i_0$, then

$$d(x, \gamma_i'\sigma(x)) \leq 1/2. \tag{1.8.14}$$

Since Γ is discrete, there exists only finite number of γ_i' such that (1.8.14) holds, then this contradicts the assumption that $m_{\gamma_i\sigma} \to 0$ as $i \to +\infty$. This completes the proof of our proposition. □

Set

$$c_{\Gamma,\sigma} = \inf_{\underline{[\gamma]}_\sigma \in \underline{C} \backslash \underline{E}} m_{\gamma\sigma}. \tag{1.8.15}$$

By Proposition 1.8.5, we have

$$c_{\Gamma,\sigma} > 0. \tag{1.8.16}$$

Lemma 1.8.6. *There exist $c > 0$, $C > 0$ such that for $R > 0$, $x \in X$, we have*

$$\sharp\{\gamma\sigma \text{ non-elliptic} \ : \ \gamma \in \Gamma,\ d_{\gamma\sigma}(x) \leq R\} \leq C\exp(cR). \tag{1.8.17}$$

Proof. If $x \in X$, $R > 0$, let $B_R(x)$ be the metric ball centred at x of radius R. Then by (1.5.34), (1.5.35), there exists $c' > 0$, $C' > 0$ such that, for $x \in X$, $R > 0$,

$$\mathrm{Vol}(B_R(x)) \leq C' \exp(c'R). \tag{1.8.18}$$

If Γ is torsion free, then using the same arguments as in the proof of [MüP13, Proposition 3.2], we get (1.8.17). Note that it also a special case of [MM15, eq.(3.19)].

If Γ is not torsion free, let $E(\Gamma) \subset \Gamma$ be the set of elliptic elements in Γ. By Remark 1.8.4, $E(\Gamma)$ is a disjoint union of finite conjugacy classes in Γ. Then by Proposition 1.5.2, there exists $c_0 > 0$ such that if $\gamma \in E(\Gamma)$, $x \in X$, then

$$c_0 d(x, X(\gamma)) \leq d_\gamma(x). \tag{1.8.19}$$

Put

$$c_\Gamma = c_{\Gamma, \mathbb{1}_G} > 0. \tag{1.8.20}$$

Let ε be such that

$$0 < \varepsilon < \frac{1}{4} \min(c_\Gamma, c_{\Gamma,\sigma}). \tag{1.8.21}$$

By (1.8.15), (1.8.20), (1.8.21), if $\gamma, \gamma' \in \Gamma$, $\gamma\sigma$, $\gamma'\sigma$ are non-elliptic, and if $\gamma^{-1}\gamma'$ is non-elliptic, then if $x \in X$, we have

$$\gamma\sigma B_\varepsilon(x) \cap \gamma'\sigma B_\varepsilon(x) = \emptyset. \tag{1.8.22}$$

If $\gamma\sigma B_\varepsilon(x) \cap \gamma'\sigma B_\varepsilon(x) \neq \emptyset$, then $\gamma^{-1}\gamma'$ is elliptic, and there exists $x' \in \sigma B_\varepsilon(x)$ such that

$$d(x', \gamma^{-1}\gamma' x') \leq 2\varepsilon. \tag{1.8.23}$$

Put $r = (\frac{2}{c_0} + 1)\varepsilon + \frac{1}{8}$. By (1.8.19), $\gamma^{-1}\gamma'$ has fixed points in $\sigma B_r(x)$. We can fix ε small enough such that $r < 1$.

Let $U \subset G$ be a compact fundamental domain for $\Gamma\backslash G$, and let V_1 be the closed 1-tube neighbourhood of $p(U)$ in X. The same arguments in the proof of Lemma 1.8.3 show that

$$l(U) = \sharp\{\gamma \in \Gamma \ : \ \gamma \text{ has fixed points in } V_1 \subset X\} \tag{1.8.24}$$

is finite. If $\gamma \in \Gamma$, then $l(U) = l(\gamma U)$.

Fix $x \in X$, $R > 0$. If $\gamma \in \Gamma$ is such that

$$d(x, \gamma\sigma(x)) \leq R. \tag{1.8.25}$$

Then

$$\gamma\sigma B_\varepsilon(x) \subset B_{R+\varepsilon}(x). \tag{1.8.26}$$

There exists $\gamma_0 \in \Gamma$ such that

$$\sigma B_r(x) \subset p(\gamma_0 U). \tag{1.8.27}$$

Let $\gamma \in \Gamma$ be such that $\gamma\sigma$ is not elliptic. Set

$$I(\gamma\sigma) = \{\gamma' \in \Gamma : \gamma'\sigma \text{ non-elliptic}, \gamma\sigma B_\varepsilon(x) \cap \gamma'\sigma B_\varepsilon(x) \neq \emptyset\}. \tag{1.8.28}$$

By the arguments (1.8.22) - (1.8.24), we get

$$\sharp I(\gamma\sigma) \leq l(U). \tag{1.8.29}$$

By (1.8.26), (1.8.28), (1.8.29), we get

$$\sharp\{\gamma\sigma \text{ non-elliptic } : \gamma\in\Gamma,\ d_{\gamma\sigma}(x)\le R\} \le l(U)\mathrm{Vol}(B_{R+\varepsilon}(x))/\mathrm{Vol}(B_\varepsilon(x)). \tag{1.8.30}$$

By (1.8.18), (1.8.30), we get (1.8.17). □

Put $Z=\Gamma\backslash X=\Gamma\backslash G/K$. By [ALR07, Example 1.20], Z is a compact orbifold. Recall that the vector bundle F on X is defined by a K-representation (E,ρ^E) in subsection 1.1. Then F descends to an orbifold vector bundle F on Z. In particular, the tangent bundle TX descends to the orbifold tangent bundle TZ, and N also descends to a orbifold bundle, which we still denote it by N.

We now assume that Γ is torsion free, so that Z is a smooth compact manifold. Let $\sigma\in\Sigma$ be such that $\sigma(\Gamma)=\Gamma$. Then the action of Σ^σ preserves Γ, and Σ^σ acts isometrically on Z.

Let ${}^\sigma Z\subset Z$ is the fixed point set of σ in Z. If $g\in G$, we denote by $[g]_X$ (resp. $[g]_Z$) the corresponding point in X (resp. Z). If $A\subset X$, we denote by $[A]_Z\subset Z$ the image of $A\subset X$ under the canonical projection $X\to Z$.

Lemma 1.8.7. *Then $[g]_Z\in {}^\sigma Z$ if and only if there is an elliptic element $\gamma\sigma$, $\gamma\in\Gamma$ such that $[g]_X\in X(\gamma\sigma)\subset X$. If $\gamma_1,\gamma_2\in\Gamma$ are in the same class in $\underline{C}$, then we have*

$$[X(\gamma_1\sigma)]_Z=[X(\gamma_2\sigma)]_Z\subset Z. \tag{1.8.31}$$

If $\gamma_1,\ \gamma_2$ are not in the same class in $\underline{E}$, then we have

$$[X(\gamma_1\sigma)]_Z\cap[X(\gamma_2\sigma)]_Z=\emptyset. \tag{1.8.32}$$

Proof. For any $g\in G$, if $[g]_Z\in{}^\sigma Z$, then there are $\gamma_0\in\Gamma$ and $k_0\in K$ such that

$$\sigma(g)=\gamma_0 g k_0. \tag{1.8.33}$$

Then $\gamma_0^{-1}\sigma(g)=gk_0$, this implies that $[g]_X\in X$ is a fixed point of $\gamma_0^{-1}\sigma$, so that $\gamma_0^{-1}\sigma$ is elliptic.

If $x\in X$ and $\gamma\sigma(x)=x$, then $[x]_Z=[\sigma(x)]_Z\in Z$. This completes the proof of the first part of our lemma.

If $\gamma_1,\ \gamma_2$ are in the same class in $\underline{C}$, then by (1.8.7), there is $\gamma\in\Gamma$ such that

$$\gamma_1\sigma=\gamma\gamma_2\sigma\gamma^{-1}. \tag{1.8.34}$$

Then we have

$$X(\gamma_1\sigma)=\gamma X(\gamma_2\sigma)\subset X, \tag{1.8.35}$$

so that (1.8.31) holds.

Suppose that $\underline{[\gamma_1]}_\sigma$, $\underline{[\gamma_2]}_\sigma$ are in $\underline{E}$. If $[X(\gamma_1\sigma)]_Z\cap[X(\gamma_2\sigma)]_Z\neq\emptyset$ in Z, since $\gamma_1\sigma,\gamma_2\sigma$ are elliptic, we can find $\gamma\in\Gamma$ and $x\in X$ such that

$$\gamma^{-1}\gamma_1\sigma(\gamma)\sigma(x)=\gamma_2\sigma(x)=x. \tag{1.8.36}$$

Then $\gamma_2^{-1}\gamma^{-1}\gamma_1\sigma(\gamma)\sigma(x)=\sigma(x)$. Since Γ is torsion free, then $\gamma_2=\gamma^{-1}\gamma_1\sigma(\gamma)$, which says that $\underline{[\gamma_1]}_\sigma=\underline{[\gamma_2]}_\sigma$. Then we get (1.8.32). □

Using Lemma 1.8.7, we get that

$$ {}^{\sigma}Z = \cup_{\underline{[\gamma]}_{\sigma}\in \underline{E}}[X(\gamma\sigma)]_Z. \tag{1.8.37} $$

Moreover, the right-hand side in (1.8.37) is a disjoint union.

By Lemma 1.8.1, $\Gamma\cap Z(\gamma\sigma)$ is a cocompact discrete subgroup of $Z(\gamma\sigma)$. Moreover, since Γ is torsion free, $\Gamma\cap Z(\gamma\sigma)$ is also torsion free. Then $\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma)$ is a compact smooth manifold

Take $\underline{[\gamma]}_{\sigma}\in \underline{E}$, let $\gamma\in\Gamma$ be one representative of $\underline{[\gamma]}_{\sigma}$. If $x\in X(\gamma\sigma)$, if $\gamma_0\in\Gamma$ is such that $\gamma_0 x\in X(\gamma\sigma)$, then a similar argument like (1.8.36) gives that $\gamma_0\in Z(\gamma\sigma)$. Thus the projection $X\to Z$ induces an identification between $\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma)$ and $[X(\gamma\sigma)]_Z\subset Z$. Then (1.8.37) can be rewritten as

$$ {}^{\sigma}Z = \cup_{\underline{[\gamma]}_{\sigma}\in \underline{E}}\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma), \tag{1.8.38} $$

Let $C(Z,F)$ be the vector space of continuous sections of F on Z. We can identify this vector space with the subspace of $C(X,F)$ consisting of continuous sections over X which are left $\Gamma-$invariant, i.e.,

$$ C(Z,F) = C(X,F)^{\Gamma}. \tag{1.8.39} $$

By (1.2.20), (1.8.39), we obtain

$$ C(Z,F) = C_K(G,E)^{\Gamma}. \tag{1.8.40} $$

We now assume that (E,ρ^E) lifts to a representation of K^{σ}. If $s\in C_K(G,E)^{\Gamma}, \mu\in \Sigma^{\sigma}$, then $\mu s\in C_K^b(G,E)$ is given by (1.2.21). If $\gamma\in\Gamma, g\in G$, then

$$ \begin{aligned} (\mu s)(\gamma g) &= \rho^E(\mu)s(\mu^{-1}(\gamma g)) \\ &= \rho^E(\mu)s(\mu^{-1}(\gamma)\mu^{-1}(g)) \\ &= \rho^E(\mu)s(\mu^{-1}(g)) = (\mu s)(g). \end{aligned} \tag{1.8.41} $$

Then $\mu s\in C_K(G,E)^{\Gamma}$. The action of $\mu\in\Sigma^{\sigma}$ descends to an action μ^Z on $C(Z,F)$.

Proposition 1.8.8. *Take $\underline{[\gamma]}_{\sigma}\in \underline{E}$ with the representative $\gamma\in\Gamma$. Under the identification in* (1.8.38)*, the action of σ on the bundles over ${}^{\sigma}Z$ restricted to $[X(\gamma\sigma)]_Z$ is given by the action of $\gamma\sigma$ on the corresponding vector bundles over $\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma)$.*

Proof. Take $x_0 = p(g_0)\in X(\gamma\sigma)$. There is $k\in K$ such that

$$ \gamma = C^{\sigma}(g_0)(k^{-1}). \tag{1.8.42} $$

By Proposition 1.4.1 and (1.8.42), we have

$$ X(\gamma\sigma) = g_0(X(k^{-1}\sigma)). \tag{1.8.43} $$

By (1.2.19), (1.8.43), we have the identification of vector bundles corresponding to the identification in (1.8.38),

$$ F|_{[X(\gamma\sigma)]_Z} \simeq \Gamma\cap Z(\gamma\sigma)\backslash\, g_0\big(Z(k^{-1}\sigma)\times_{K(k^{-1}\sigma)} E\big). \tag{1.8.44} $$

If $g\in Z(k^{-1}\sigma)$, by (1.8.42), we get

$$ \sigma(g_0 g) = \gamma^{-1} g_0 g k^{-1}. \tag{1.8.45} $$

Put $x = p(g_0 g) \in X(\gamma\sigma)$ and $z = [g_0 g]_Z \in [X(\gamma\sigma)]_Z$. The computation on an small neighbourhood of z is equivalent to do the computation on a neighbourhood of x. If $v \in F_z \simeq E$, then

$$\begin{aligned}\sigma(z, v) &= (\sigma(z), \sigma v) \\ &= [(\sigma(g_0 g), \rho^E(\sigma)v)]_Z \\ &= [(g_0 g, \rho^E(k^{-1}\sigma)v)]_Z \in F_{\sigma(z)}.\end{aligned} \tag{1.8.46}$$

Take the lift of $[(g_0 g, \rho^E(k^{-1}\sigma)v)]_Z$ around x, we have

$$[(g_0 g, \rho^E(k^{-1}\sigma^E)v)]_Z = g_0 k^{-1}\sigma g_0^{-1}(x, v) = \gamma\sigma(x, v). \tag{1.8.47}$$

This completes the proof of our proposition. □

Remark 1.8.9. If Γ is not torsion free, then Z is a compact orbifold. In this case, (1.8.37) still holds, but the union in the right-hand side of (1.8.37) is not a disjoint union any more.

For γ_1, $\gamma_2 \in \Gamma$, if $\gamma_1\sigma$, $\gamma_2\sigma$ are elliptic, if $\gamma_2^{-1}\gamma_1$ is elliptic, then we have

$$X(\gamma_1\sigma) \cap X(\gamma_2\sigma) = \sigma^{-1}(X(\gamma_2^{-1}\gamma_1)). \tag{1.8.48}$$

If $\gamma_2^{-1}\gamma_1$ is not elliptic, then we have

$$X(\gamma_1\sigma) \cap X(\gamma_2\sigma) = \emptyset. \tag{1.8.49}$$

These identities are compatible with the corresponding results in the proof of Lemma 1.8.6.

2. A classification of representations of K^σ

This section is devoted to a proper classification of the irreducible representations of K^σ, and the question of lifting representations of K to representations of K^σ.

This section is organized as follows. In subsection 2.1, we reduce the classification of representations of K^σ to the classification of representations of a subgroup $K^{\sigma\prime}$ of $K \rtimes \mathrm{Aut}(K)$.

In subsection 2.2, we recall a Weyl character formula for non-connected compact Lie group obtained in [DK00].

In subsection 2.3, we give a classification of the irreducible unitary representations of a finite extension K^τ of K by the orbits in the set of dominants weights.

Finally, in subsection 2.4, we give a constructive correspondence between representations of $K^{\sigma\prime}$ and representations of K^τ, so that a classification of representations of K^σ is established. In the last part, we give a criterion for the extension of a K-representation to a K^σ-representation to exist.

In this section, we use the same notation as in subsections 1.1 and 1.2.

2.1. **Irreducible representations of K^σ.** Fix $\sigma \in \Sigma$, let Σ^σ be the closed subgroup of $\mathrm{Aut}(G)$ generated by σ. Recall that $\widetilde{K} = K \rtimes \Sigma$ and $K^\sigma = K \rtimes \Sigma^\sigma$.

Since σ preserve the group K, we have the natural homomorphism of Lie groups:

$$f : \Sigma^\sigma \to \mathrm{Aut}(K). \tag{2.1.1}$$

Put $H = \ker f$. Let $\Sigma^{\sigma\prime} \subset \mathrm{Aut}(K)$ be the image of f. The group $\Sigma^{\sigma\prime}$ is a compact subgroup of $\mathrm{Aut}(K)$ generated by $f(\sigma)$.

Definition 2.1.1. Let $K^{\sigma\prime}$ be the closed subgroup of $K \rtimes \mathrm{Aut}(K)$ which is generated by K and $f(\sigma)$.

Then

$$K^{\sigma\prime} = K \rtimes \Sigma^{\sigma\prime}. \tag{2.1.2}$$

The homomorphism f extends trivially to a homomorphism from K^σ onto $K^{\sigma\prime}$, which we still denote by f.

We regard H as a closed Lie subgroup of K^σ, then it lies in the center of K^σ. Then

$$K^{\sigma\prime} = K^\sigma / H. \tag{2.1.3}$$

If (E, ρ^E) is an irreducible unitary representation of $K^{\sigma\prime}$, then using f, one can get a corresponding irreducible unitary representation of K^σ. Conversely, if $(E, \widetilde{\rho}^E)$ is an irreducible unitary representation of K^σ, then Schur's lemma says that for $\tilde{h} \in H$, $\widetilde{\rho}^E(\tilde{h})$ is a scalar operator in $\mathrm{Aut}(E)$. Let $\mathbb{1}_E$ be the identity map of E.

Proposition 2.1.2. *For an irreducible unitary representation $(E, \widetilde{\rho}^E)$ of K^σ, there exists an 1-dimensional representation $(L, \widetilde{\rho}^L)$ of Σ^σ such that the representations $(E \otimes L, \widetilde{\rho}^{E\otimes L})$ is an irreducible representation of K^σ satisfying that if $\tilde{h} \in H$, $\rho^{E\otimes L}(\tilde{h}) = \mathbb{1}_E$.*

Proof. The restriction of $\widetilde{\rho}^E$ to H can be regarded as a 1-dimensional representation of H over $\mathbb{C}$. Let $(L, \widetilde{\rho}^L)$ be its dual representation. We can extend $(L, \widetilde{\rho}^L)$ to the group Σ^σ. One can verify that $(L, \widetilde{\rho}^L)$ is just the representation we want. This completes the proof of our proposition. □

Let χ^E, χ^L, $\chi^{E\otimes L}$ be the characters of these representations of K^σ appeared above, then, on K^σ, we have

$$\chi^{E\otimes L} = \chi^L \cdot \chi^E. \tag{2.1.4}$$

In particular, the values of χ^L depend only on the factor in Σ^σ, i.e., if $\tilde{g} \in K^\sigma$, $k \in K$, then

$$\chi^E(\tilde{g}k) = (\chi^L(\tilde{g}))^{-1}\chi^{E\otimes L}(\tilde{g}k). \tag{2.1.5}$$

By (2.1.3), the representation $E \otimes L$ in Proposition 2.1.2 can be regarded as an irreducible unitary representation of $K^{\sigma\prime}$. Then in most cases where we need to deal with the characters of representations, it is enough to work on representations of $K^{\sigma\prime}$ instead of representations of K^σ.

2.2. **Finite extension of K and a Weyl character formula.** Let $\widehat{K}$ be a compact Lie group such that the identity component $\widehat{K}^0 = K$. Then $\widehat{K}/K$ is a finite group.

Let $T \subset K$ be a maximal torus of K with Lie algebra $\mathfrak{t}$. Let $W(K,T)$ be the associated Weyl group. Let $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$ be the complexification of $\mathfrak{k}$, $\mathfrak{t}$. Let $R(\mathfrak{k},\mathfrak{t}) \subset \mathfrak{t}^*$ be the associated (real) root system. Let $R_+(\mathfrak{k},\mathfrak{t}) \subset R(\mathfrak{k},\mathfrak{t})$ be a system of positive roots with the simple root system $\Phi(\mathfrak{k},\mathfrak{t})$. If there is no risk of confusion, we use the notation W, R, R_+, Φ instead of $W(K,T)$, $R(\mathfrak{k},\mathfrak{t})$, $R_+(\mathfrak{k},\mathfrak{t})$, $\Phi(\mathfrak{k},\mathfrak{t})$.

Note that if $\omega \in W$

$$\det(\omega) = (-1)^{|R_+\backslash\omega\cdot R_+|}. \tag{2.2.1}$$

Let $\mathfrak{c}$ be the Weyl chamber defining the positive root system R_+. Let $P_{++} \subset \mathfrak{t}^*$ be the set of dominate weights with respect to R_+. As in [DK00, eq. (3.15.2)], put

$$N_{\widehat{K}}(\mathfrak{c}) = \{\widehat{g} \in \widehat{K} \mid \mathrm{Ad}(\widehat{g})(\mathfrak{c}) = \mathfrak{c}\}. \tag{2.2.2}$$

Then $N_{\widehat{K}}(\mathfrak{c})$ is a closed Lie subgroup of $\widehat{K}$ with Lie algebra $\mathfrak{t}$. Let $N_{\widehat{K}}(\mathfrak{c})^0$ be the identity component of $N_{\widehat{K}}(\mathfrak{c})$. Then $N_{\widehat{K}}(\mathfrak{c})^0 = T$. Moreover, if $u \in N_{\widehat{K}}(\mathfrak{c})$, the action $\mathrm{Ad}(u)$ on $\mathfrak{t}^*$ preserves R_+.

By [DK00, Proposition (3.15.1)], the injections $N_{\widehat{K}}(\mathfrak{c}) \to \widehat{K}$ and $T \to K$ induces an isomorphism of finite groups:

$$N_{\widehat{K}}(\mathfrak{c})/T \to \widehat{K}/K \tag{2.2.3}$$

In particular, $N_{\widehat{K}}(\mathfrak{c}) \cap K = T$.

As in (1.3.1), if $v \in \mathfrak{k}$, set

$$\widehat{Z}(v) = \{\widehat{g} \in \widehat{K} \ : \ \mathrm{Ad}(\widehat{g})v = v\}. \tag{2.2.4}$$

By [DK00, Proposition (3.15.2)], there exists $v \in \mathfrak{c}$ such that

$$\widehat{Z}(v) = N_{\widehat{K}}(\mathfrak{c}). \tag{2.2.5}$$

Using [DK00, Lemma (3.15.3)], if $u \in N_{\widehat{K}}(\mathfrak{c})$, then there exists $z \in N_{\widehat{K}}(\mathfrak{c})$ which is arbitrary close to u such that $S = K^0(z)$ is a torus in K, i.e., z is a regular element in $\widehat{K}$, and z commutes with u. One can verify S is a subtorus of T. Moreover, if $\widehat{g}$ and z are in the same connected component of $\widehat{K}$, then $\widehat{g}$ is conjugate by an element of K to an element of zS. Then a consequence of (2.2.3) is that the character of a representation of $\widehat{K}$ is determined by its restriction to the subgroup $N_{\widehat{K}}(\mathfrak{c})$.

Set

$$\rho_{\mathfrak{k}} = \frac{1}{2} \sum_{\alpha \in R_+} \alpha. \tag{2.2.6}$$

Then if $u \in N_{\widehat{K}}(\mathfrak{c})$,

$$\mathrm{Ad}(u)\rho_{\mathfrak{k}} = \rho_{\mathfrak{k}}. \tag{2.2.7}$$

For a subset $Q \subset R$, put

$$\mathfrak{k}_Q = \oplus_{\alpha \in Q} \mathfrak{k}_\alpha. \tag{2.2.8}$$

In particular, set

$$\mathfrak{n} = \mathfrak{k}_{R_+}. \tag{2.2.9}$$

If $u \in N_{\widehat{K}}(\mathfrak{c})$, set

$$W(uT) = \{\omega \in W \mid \mathrm{Ad}(u)|_{\mathfrak{t}} \text{ commutes with } \omega\}. \tag{2.2.10}$$

If $\omega \in W(uT)$, then $\mathrm{Ad}(u)$ preserves the subspace $\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}$.

Let the function δ in $u \in N_{\widehat{K}}(\mathfrak{c})$ be given by

$$\begin{aligned} \delta(u) &= \det(1 - \mathrm{Ad}(u^{-1}))_{\mathfrak{n}} \\ &= \sum_{\omega \in W(uT)} \det(\omega) \det(\mathrm{Ad}(u^{-1}))|_{\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}}. \end{aligned} \tag{2.2.11}$$

Note that δ is just the function defined in [DK00, eq.(3.15.11)], and that the second equation in (2.2.11) follows from [DK00, eq.(4.13.32)].

Lemma 2.2.1. *If $u \in N_{\widehat{K}}(\mathfrak{c})$, $t \in T$, then*

$$\delta(tu) = \sum_{\omega \in W(uT)} \det(\omega) \det(\mathrm{Ad}(u^{-1}))|_{\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}} e^{2\pi i \omega \cdot \rho_{\mathfrak{k}} - 2\pi i \rho_{\mathfrak{k}}}(t). \tag{2.2.12}$$

Proof. Since the actions $\mathrm{Ad}(u^{-1})$, $\mathrm{Ad}(t^{-1})$ and $\mathrm{Ad}(t^{-1}u^{-1})$ on $\mathfrak{k}$ preserve the subspace $\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}$, then

$$\begin{aligned} &\det(\mathrm{Ad}(t^{-1}u^{-1}))|_{\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}} \\ &\qquad = \det(\mathrm{Ad}(t^{-1}))|_{\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}} \det(\mathrm{Ad}(u^{-1}))|_{\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}}. \end{aligned} \tag{2.2.13}$$

Also we have

$$\det(\mathrm{Ad}(t^{-1}))|_{\mathfrak{k}_{R_+ \backslash \omega \cdot R_+}} = e^{-2\pi i \sum_{\alpha \in R_+ \backslash \omega \cdot R_+} \alpha}(t). \tag{2.2.14}$$

Then it is enough to show that

$$\sum_{\alpha\in R_+\backslash\omega\cdot R_+}\alpha=\rho_{\mathfrak{k}}-\omega\cdot\rho_{\mathfrak{k}}, \tag{2.2.15}$$

which is a classical result of the root system theory of Lie groups [DK00, Section 4.9]. □

Let (E,ρ^E) be an irreducible unitary representation of $\widehat{K}$. Set

$$E(\mathfrak{n})=\{e\in E\ :\ \rho^E(v)e=0 \text{ for all } v\in\mathfrak{n}\}. \tag{2.2.16}$$

Similarly, if $\omega\in W(uT)$, set

$$E(\omega\cdot\mathfrak{n})=\{e\in E\ :\ \rho^E(v)e=0 \text{ for all } v\in\mathfrak{k}_{\omega\cdot R_+}\}. \tag{2.2.17}$$

By (2.2.2), (2.2.10), the subspaces $E(\mathfrak{n})$, $E(\omega\cdot\mathfrak{n})$ are invariant by the action of $N_{\widehat{K}}(\mathfrak{c})$.

When we regard (E,ρ^E) as a unitary representation of K, each irreducible component corresponds a highest weight $\lambda\in P_{++}$. Let $\Omega(E,\rho^E)$ be the set of these highest weights. Since $N_{\widehat{K}}(\mathfrak{c})$ preserves R_+, $N_{\widehat{K}}(\mathfrak{c})$ also preserves $\Omega(E,\rho^E)$. By the discussion in [DK00, pp. 307], since (E,ρ^E) is $\widehat{K}$-irreducible, the set $\Omega(E,\rho^E)$ consists of one single orbit under the adjoint action of $N_{\widehat{K}}(\mathfrak{c})$.

Lemma 2.2.2. *If (E,ρ^E) is an irreducible unitary representation of $\widehat{K}$ and also an irreducible representation of K. Let $\lambda\in P_{++}$ be its highest weight with respect to R_+. Then λ is fixed by the adjoint action of $N_{\widehat{K}}(\mathfrak{c})$.*

Proof. In this case,

$$\Omega(E,\rho^E)=\{\lambda\}. \tag{2.2.18}$$

Then our lemma follows from that if $u\in N_{\widehat{K}}(\mathfrak{c})$, $\mathrm{Ad}(u)|_{\mathfrak{t}}$ preserves the set $\Omega(E,\rho^E)$. □

If $\lambda\in\Omega(E,\rho^E)$, let E_λ be the subspace of E associated with the weight λ. By [DK00, Corollary (4.13.2)]

$$E(\mathfrak{n})=\oplus_{\lambda\in\Omega(E,\rho^E)}E_\lambda. \tag{2.2.19}$$

Let χ^E be the character of (E,ρ^E). By [DK00, eq.(4.13.34)], if $u\in N_{\widehat{K}}(\mathfrak{c})$, then

$$\delta(u)\chi^E(u)=\sum_{\omega\in W(uT)}\det(\omega)\det(\mathrm{Ad}(u^{-1})|_{\mathfrak{k}_{R_+\backslash\omega\cdot R_+}}\mathrm{Tr}[\rho^E(u)|_{E(\omega\cdot\mathfrak{n})}]. \tag{2.2.20}$$

Now we suppose that (E,ρ^E) is an irreducible unitary representation of $\widehat{K}$ and also an irreducible representation of K with the highest weight λ. By Lemma 2.2.2 and (2.2.19), we have

$$E(\mathfrak{n})=E_\lambda. \tag{2.2.21}$$

If $\omega \in W(uT)$, then $E(\omega \cdot \mathfrak{n}) = E_{\omega\cdot\lambda}$. If $u \in N_{\widehat{K}}(\mathfrak{c})$, $t \in T$, by (2.2.12), (2.2.20), (2.2.21), we get

$$\begin{aligned} &\delta(tu)\chi^E(tu) \\ &= \sum_{\omega\in W(uT)} \det(\omega)\det(\mathrm{Ad}(u^{-1}))|_{\mathfrak{k}_{R_+\setminus\omega\cdot R_+}} \\ &\qquad \times \mathrm{Tr}[\rho^E(u)|_{E_{\omega\cdot\lambda}}]e^{2\pi i(\omega\cdot\rho_{\mathfrak{k}}+\omega\cdot\lambda-\rho_{\mathfrak{k}})}(t). \end{aligned} \tag{2.2.22}$$

If $u = 1$, (2.2.22) becomes the classical Weyl character formula for K.

2.3. **Representations of the principal extension of K.** We assume that K is compact, semisimple, connected and simply connected. Then the center $Z(K)$ of K is a finite Abelian group.

The identity component of $\mathrm{Aut}(K)$ is just the group of inner automorphism $\mathrm{Inn}(K)$. The outer automorphism group of K is defined as

$$\mathrm{Out}(K) = \mathrm{Aut}(K)/\mathrm{Inn}(K). \tag{2.3.1}$$

Choosing a basis of Chevalley, any automorphism τ of associated Dynkin diagram could be lift to an automorphism of K canonically, which we still denote by τ. Then we get an embedding $\mathrm{Out}(K) \hookrightarrow \mathrm{Aut}(K)$, so that we can identify $\mathrm{Out}(K)$ with its image in $\mathrm{Aut}(K)$, which is a finite group and acts on $\mathrm{Inn}(K)$ canonically. By the results in [Bo04, Chapter VIII, §4.4 and Chapter IX, §4.10], we have the group isomorphism

$$\mathrm{Aut}(K) = \mathrm{Inn}(K) \rtimes \mathrm{Out}(K). \tag{2.3.2}$$

As in (1.2.4), we get an exact sequence of Lie groups from (2.3.2),

$$1 \to \mathrm{Inn}(K) \to \mathrm{Aut}(K) \to \mathrm{Out}(K) \to 1. \tag{2.3.3}$$

Moreover, the group $\mathrm{Out}(K)$ acts on K. Note that the decomposition in (2.3.2) depends on the choice of maximal torus T and the root system (R, R_+).

Set

$$\widehat{K} = K \rtimes \mathrm{Out}(K), \tag{2.3.4}$$

which is so-called the principal extension of K. We can regard $\widehat{K}$ as a model group for the group $K^{\sigma\prime}$.

Remark 2.3.1. In fact, we can drop the assumption that K is simply connected. If $\alpha \in R$, let $V(\alpha)$ denote the 2−dimensional real vector subspace of $\mathfrak{k}$ such that its complexification is just $\mathfrak{k}_\alpha + \mathfrak{k}_{-\alpha}$. If $\alpha \in \Phi$, we fix a $v_\alpha \in V(\alpha)$ such that $B(v_\alpha, v_\alpha) = -1$. The pair $(\Phi, (v_\alpha)_{\alpha\in\Phi})$ is called a framing of (K, T). Let O be the subgroup of $\mathrm{Aut}(K)$ that leave $(\Phi, (v_\alpha)_{\alpha\in\Phi})$ stable. Then $\mathrm{Aut}(K)$ is the semi-direct product of $\mathrm{Inn}(K)$ and O as in (2.3.2). In particular, O is isomorphic to $\mathrm{Out}(K)$. We refer to [Bo04, pp. 324] for more detail.

Let $\tau \in \mathrm{Out}(K)$ with order N_0. We denote by $\langle\tau\rangle$ the finite cyclic group generated by τ in $\mathrm{Out}(K)$. Let K^τ be the closed subgroup of $\widehat{K}$ generated by K and τ. Then

$$K^\tau = K \rtimes \langle\tau\rangle. \tag{2.3.5}$$

In the sequel, we classify the irreducible complex representations of K^τ by explicit constructions.

Let $\mathrm{Irr}(K^\tau)$ be the set of the equivalent classes of the irreducible complex representations of K^τ. Let C_{N_0} be the set of all N_0^{th} roots of 1.

Let (E, ρ^E) be an irreducible unitary representations of K^τ, and let χ^E be its character. We can decompose it as a direct sum of irreducible unitary representations of K, written as:

$$E = \oplus_{i=1}^d E_i. \tag{2.3.6}$$

Let $\lambda_i \in P_{++}$ be the highest weight of E_i.

In the following, let $(E_\lambda, \rho^{E_\lambda})$ be an irreducible unitary representation of K with the highest weight $\lambda \in P_{++}$ and let χ_λ be the corresponding character. Then (2.3.6) can be rewritten as

$$E = \oplus_{i=1}^d E_{\lambda_i}. \tag{2.3.7}$$

When restricting to K, we have

$$\chi^E = \sum_{i=1}^d \chi_{\lambda_i}. \tag{2.3.8}$$

Lemma 2.3.2. *(1) If $k \in K$, put $\rho^{E_\lambda,\tau^{-1}}(k) = \rho^{E_\lambda}(\tau^{-1}(k)) \in \mathrm{Aut}(E)$. Then $(E_\lambda, \rho^{E_\lambda,\tau^{-1}})$ is an irreducible representation with the highest weight $\tau\lambda \in P_{++}$.*

(2) If (E, ρ^E) is a representation of K^τ, if $W \subset E$ is a K-invariant subspace which is an irreducible representation of K with the highest weight λ, then $\rho^E(\tau)W$ is also a K-invariant subspace of E, and it is an irreducible representation of K with the highest weight $\tau\lambda$.

Proof. These two statements follow from that if $t \in T$, $v \in E_\lambda$, then

$$\rho^E(\tau^{-1}(t))v = e^{2\pi i\lambda}(\tau^{-1}(t))v = e^{2\pi i\tau\lambda}(t)v. \tag{2.3.9}$$

□

Lemma 2.3.3. *Let (E, ρ^E) be a finite-dimensional unitary representation of K^τ such that it can be written as a direct sum of m copies of E_λ as K-representation. If there exists $d \in \mathbb{N}_{>0}$ such that $\tau^d \cdot \lambda = \lambda$, then there exists a K-invariant subspace W of E such that (W, ρ^E) is irreducible representation of K with the highest weight λ, and W is invariant under the action of $\rho^E(\tau^d)$*

Proof. Since λ is fixed by the adjoint action of τ^d, then the representation $(E_\lambda, \rho^{E_\lambda})$ is isomorphic to $(E_\lambda, \rho^{E_\lambda,\tau^d})$, there is $J \in \mathrm{GL}(E_\lambda)$ such that $J \circ \rho^{E_\lambda}(k) = \rho^{E_\lambda}(\tau^d(k)) \circ J$. By the Schur's lemma, the map J is unique up to a non-zero constant multiplication. Then

$$\mathrm{Hom}_K((E_\lambda, \rho^{E_\lambda}), (E_\lambda, \rho^{E_\lambda,\tau^d})) = \mathbb{C}J \subset \mathrm{End}(E_\lambda). \tag{2.3.10}$$

By the assumption in the lemma, we have $E=E_\lambda^{\oplus m}$. Then for $k\in K$

$$\rho^E(k)=\begin{bmatrix}\rho^{E_\lambda}(k) & 0 & \cdots & 0\\ 0 & \rho^{E_\lambda}(k) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \rho^{E_\lambda}(k)\end{bmatrix} \tag{2.3.11}$$

We also have

$$\rho^E(\tau^d)\rho^E(k)=\rho^E(\tau^d(k))\rho^E(\tau^d). \tag{2.3.12}$$

Let $\tau_{ij}\in \mathrm{End}(E_\lambda)$ be the (i,j) block of $\rho^E(\tau^d)$ under this decomposition. By (2.3.12), we get

$$\tau_{ij}\in \mathrm{Hom}_K((E_\lambda,\rho^{E_\lambda}),(E_\lambda,\rho^{E_\lambda,\tau^d})). \tag{2.3.13}$$

Then by (2.3.10), there exists $a_{ij}\in\mathbb{C}$ such that $\tau_{ij}=a_{ij}J$. Put $A_{\tau^d}=(a_{ij})\in M_{m\times m}(\mathbb{C})$, which is a non-zero complex matrix.

Since J is an isomorphism, there exists a non-zero vector $v\in E_\lambda$ and $a\in\mathbb{C}^*$ such that $Jv=av$. Also there exists a non-zero $w=(w_1,\cdots,w_m)\in\mathbb{C}^m$ and $b\in\mathbb{C}$ such that $A_{\tau^d}w=bw$. Put $\widetilde{v}=(w_1v,\cdots,w_mv)\in E$. Then $\widetilde{v}\neq 0$, and we have

$$\rho^E(\tau^d)\widetilde{v}=ab\widetilde{v}. \tag{2.3.14}$$

Since $\rho^E(\tau^d)$ is invertible, then $ab\neq 0$. Let W be the smallest subspace of E invariant by K-actions containing $\widetilde{v}$. Then W is a representation of K, and it is stable by the action $\rho^E(\tau^d)$.

Now we show that W as $K-$representation is isomorphic to $(E_\lambda,\rho^{E_\lambda})$.

Suppose that $w_1\neq 0$. Let P_1 be the projection from E to the first copy of E_λ. Then for any $k\in K$,

$$P_1\rho^E(k)=\rho^{E_\lambda}(k)P_1. \tag{2.3.15}$$

Put $P_W=P_1|_W: W\to E_\lambda$. Then (2.3.15) implies that P_W is a morphism between these two $K-$representations. Since $P_W\widetilde{v}=w_1v\neq 0\in E_\lambda$, we get that P_W is surjective. We only need to show that P_W is injective.

If $u\in W$, there are some $c_k\in\mathbb{C}$ for finite numbers of $k\in K$ such that $u=\sum_k c_k\rho^E(k)\widetilde{v}$. Then

$$\begin{aligned}P_W\rho^E(\tau^d)u&=abw_1\sum_k c_k\rho^{E_\lambda}(\tau^d(k))v\\ &=bw_1\sum_k c_k\rho^{E_\lambda}(\tau^d(k))Jv\\ &=bJ\sum_k c_k\rho^{E_\lambda}(k)w_1v\\ &=bJP_Wu.\end{aligned} \tag{2.3.16}$$

If $P_Wu=0$, then

$$P_W\rho^E(\tau^d)u=0. \tag{2.3.17}$$

By (2.3.16), we get

$$bw_1 J \sum_k c_k \rho^{E_\lambda}(k)v = 0. \tag{2.3.18}$$

Since $bw_1 \neq 0$, we get

$$\sum_k c_k \rho^{E_\lambda}(k)v = 0. \tag{2.3.19}$$

In the same time, the i^{th}-component of $\rho^E(\tau^d)u$ is given by

$$\sum_j a_{ij} w_j J \sum_k c_k \rho^{E_\lambda}(k)v. \tag{2.3.20}$$

Using the identity $\sum_j a_{ij}w_j = bw_i$, (2.3.19) is equivalent to $u = 0$, so that P_W is injective. Then (W, ρ^E) is isomorphic to E_λ as K-representations. This completes the proof of our lemma. □

As an analogue of the results in [DK00, pp. 307], we have the following result.

Proposition 2.3.4. *These λ_i in (2.3.7) are distinct and they form an orbit of length d in P_{++} under the action of τ. In particular, we have $d\,|N_0$.*

Proof. This proposition follows from Lemmas 2.3.2 and 2.3.3. □

Set $\tau^E = \rho^E(\tau) \in \mathrm{Aut}(E)$. If $c \in C_{N_0}$, then we define a new irreducible unitary representation of K^τ with the same vector space E and the following actions,

$$\begin{aligned} &\rho^E_c(\tau) = c\tau^E, \\ &\rho^E_c(k) = \rho^E(k), \text{ if } k \in K. \end{aligned} \tag{2.3.21}$$

By (2.3.7), (2.3.21) and Proposition 2.3.4, the representation (E, ρ^E_c) has the same orbit in P_{++} as (E, ρ^E).

We can define an action of C_{N} on $\mathrm{Irr}(K^\tau)$ by the map $c : \rho^E \to \rho^E_c$.

Proposition 2.3.5. *Let Λ be the map which sends the irreducible unitary representation of K^τ to its corresponding orbit in P_{++}, then Λ induces an $1-1$ correspondence Λ' of two orbit spaces*

$$\Lambda' : C_{\mathrm{N}}\backslash \mathrm{Irr}(K^\tau) \simeq \langle\tau\rangle\backslash P_{++}. \tag{2.3.22}$$

Proof. We prove that Λ' defined in (2.3.22) is a bijection. Firstly, we prove the surjectivity of Λ'.

If $\lambda \in P_{++}$. let $[\lambda]$ denote the orbit of λ under the action of $\langle\tau\rangle$. Let d be the length of $[\lambda]$, then $d|N_0$. Then

$$[\lambda] = \{\lambda, \tau\lambda, \cdots, \tau^{d-1}\lambda\} \subset P_{++}. \tag{2.3.23}$$

In particular, $\tau^d\lambda = \lambda$.

Let (E_0, h^{E_0}, ρ_0) is an irreducible unitary representation of K with the highest weight λ, where h^{E_0} is an invariant Hermitian metric on E_0. If $i = 1, \cdots, d-1$, put

$$E_i = E_0, \quad \rho_i = \rho_0^{\tau^{-i}}, \quad h^{E_i} = h^{E_0}. \tag{2.3.24}$$

Then (E_i, h^{E_i}, ρ_i) is an irreducible unitary representation of K with the highest weight $\tau^i\lambda \in [\tau]$. Since $\tau^d\lambda = \lambda$, there exists $J_0 \in \mathrm{Aut}(E_0)$ such that if $k \in K$, then

$$J_0\rho_0(k) = \rho_0(\tau^d(k))J_0. \tag{2.3.25}$$

Since $\tau^{N_0} = 1$ and using the Schur's lemma, we get that $J_0^{N_0/d} \in \mathrm{Aut}(E_0)$ is a scalar operator. Then after a rescaling by a number, we assume that

$$J_0^{N_0/d} = \mathrm{Id}_{E_0}. \tag{2.3.26}$$

Furthermore, $J_0 \in U(E_0, h^{E_0})$.

Put $(E, \rho^E) = \oplus_{i=0}^d (E_i, \rho_i)$ with $h^E = \oplus_i h^{E_i}$. Note that as vector spaces, $E = E_0^{\oplus d}$.

Let τ^E be an automorphism of E in the following form,

$$\tau^E = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & J_0 \\ \mathrm{Id}_{E_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Id}_{E_0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{Id}_{E_0} & 0 \end{bmatrix} \in U(E, h^E). \tag{2.3.27}$$

We have

$$\tau^{E,d} = \mathrm{diag}\{J_0, \cdots, J_0\}. \tag{2.3.28}$$

Then

$$\tau^{E,N_0} = \mathrm{Id}_E. \tag{2.3.29}$$

Moreover, one can verify that if $k \in K$, then

$$\tau^E\rho^E(k) = \rho^E(\tau(k))\tau^E. \tag{2.3.30}$$

Set $\rho^E(\tau) = \tau^E$, then (E, ρ^E) becomes a unitary representation of K^τ. In fact, this representation is irreducible (see (2.3.36) in Remark 2.3.6). By our construction, we have $\Lambda(E, \rho^E) = [\lambda]$.

Next we prove the injectivity of Λ'. Suppose that (F, ρ^F) is an irreducible unitary representations of K^τ that has the same orbit $[\lambda]$. We still denote by (E, ρ^E) the representation constructed above. Then after an isomorphism of $K-$representations, we can assume that $F = E$, $\rho^E|_K = \rho^F|_K$. Put

$$\tau^F = \rho^F(\tau). \tag{2.3.31}$$

Under the decomposition $E = E_0^{\oplus d}$, by Lemma 2.3.2 and the Schur's lemma, we get the automorphism τ^F must have a matrix representation as follows,

$$\tau^F = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & c_d J_0 \\ c_1 \mathrm{Id}_{E_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & c_2 \mathrm{Id}_{E_0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{d-1} \mathrm{Id}_{E_0} & 0 \end{bmatrix}, \tag{2.3.32}$$

where $c_1, \cdots, c_d \in \mathbb{C}^*$ are constant. Similar to (2.3.28), we have

$$\tau^{F,d} = c_1 \cdots c_d \operatorname{diag}\{J_0, \cdots, J_0\}, \tag{2.3.33}$$

and

$$(c_1 \cdots c_d)^{N_0/d} = 1. \tag{2.3.34}$$

Let $c \in C_{N_0}$ be such that $c^d = c_1 \cdots c_d$. Then $(c\tau^E)^d = \tau^{F,d}$. We see that the character of the representation (E, ρ^E_c) is the same as the character of (F, ρ^F) on K^τ, then $(F, \rho^F) \simeq (E, \rho^E_c)$ as the representations of K^τ. This completes the proof of our proposition. □

Remark 2.3.6. Let $(E = E_0^{\oplus d}, \rho^E)$ be the irreducible representation constructed in the proof of Proposition 2.3.5 for $[\lambda] \subset P_{++}$. Let J_0 be the automorphism given in the above proof, then we can write down a formula of the character χ^E of (E, ρ^E): if $k \in K$,

$$\begin{aligned} &\chi^E(k) = \sum_{i=0}^{d-1} \chi^{\tau^i \lambda}(k); \\ &\chi^E(k\tau^i) = 0, \text{ if } \ d \nmid i \ ; \\ &\chi^E(k\tau^{jd}) = \sum_{i=0}^{d-1} \mathrm{Tr}^{E_0}[\rho^{E_0}(\tau^{-i}(k)) J_0^j], \text{ for } \ j = 1, 2, \cdots, N_0/d \ . \end{aligned} \tag{2.3.35}$$

The equations in (2.3.35) are compatible with (2.2.20). Also if dk^τ is the normalized Haar measure of K^τ, a direct calculation shows that

$$\int_{K^\tau} |\chi^E|^2 dk^\tau = 1. \tag{2.3.36}$$

2.4. **Irreducible unitary representations of $K^{\sigma\prime}$.** This section is devoted to classify the irreducible representations of $K^{\sigma\prime}$. We still use the same notation associated with K as in subsections 2.2 and 2.3. If there is no risk of confusion, if $k \in K$, we will denote by $\mathrm{Ad}(k)$ both the conjugation of K by k and the adjoint action of k on the Lie algebras.

Recall that $\Sigma^{\sigma\prime}$ is the compact Abelian group generated by $f(\sigma) \in \mathrm{Aut}(K)$. We also use σ abusively instead of $f(\sigma)$ in this section. Let $\mathrm{Irr}(\Sigma^{\sigma\prime})$ be the set of irreducible unitary representations of $\Sigma^{\sigma\prime}$. Note that the representations in $\mathrm{Irr}(\Sigma^{\sigma\prime})$ are 1-dimensional. It is well-known that $\mathrm{Irr}(\Sigma^{\sigma\prime})$ can be realized as a discrete group.

Remark 2.4.1. Since $\Sigma^{\sigma\prime}$ is a product of a torus and an Abelian finite group, we get that $\mathrm{Irr}(\Sigma^{\sigma\prime})$ is isomorphic to the product of some $\mathbb{Z}^k$ and a finite subgroup of $\mathbb{S}^1 \subset \mathbb{C}^*$.

We define a left group action of $\mathrm{Irr}(\Sigma^{\sigma\prime})$ on $\mathrm{Irr}(K^{\sigma\prime})$. For $(L, \rho^L) \in \mathrm{Irr}(\Sigma^{\sigma\prime})$, we can regard (L, ρ^L) as a representation of $K^{\sigma\prime}$ through the group projection $K^{\sigma\prime} \to \Sigma^{\sigma\prime}$. Then the group action of (L, ρ^L) on $\mathrm{Irr}(K^{\sigma\prime})$ is defined for $(E, \rho^E) \in \mathrm{Irr}(K^{\sigma\prime})$ by

$$(L, \rho^L) \cdot (E, \rho^E) = (L \otimes E, \rho^L \otimes \rho^E). \tag{2.4.1}$$

It is clear that $(L \otimes E, \rho^L \otimes \rho^E)$ is also an irreducible unitary representation of $K^{\sigma\prime}$. Let $\mathrm{Irr}(\Sigma^{\sigma\prime})\backslash\mathrm{Irr}(K^{\sigma\prime})$ be the orbit space of $\mathrm{Irr}(K^{\sigma\prime})$ under the action of $\mathrm{Irr}(\Sigma^{\sigma\prime})$.

Let τ be the image of $f(\sigma)$ under the group projection $\mathrm{Aut}(K) \to \mathrm{Out}(K)$, which is uniquely determined by $f(\sigma)$.

As in subsection 2.3, after choosing a maximal torus T and the positive root system R_+, we have the identification of groups in (2.3.2). Then τ can be identified with an element in $\mathrm{Aut}(K)$, which is still denoted by τ. By (2.3.2), there exists $k^* \in K$ such that

$$\sigma = \mathrm{Ad}(k^*) \circ \tau \in \mathrm{Aut}(K). \tag{2.4.2}$$

In general, k^* can be differed by any element in $Z(K)$. We just fix one choice of k^* in the sequel.

Proposition 2.4.2. *Let $(E, \rho^E) \in \mathrm{Irr}(K^{\sigma\prime})$. There exists $c_\tau \in \mathbb{C}$ such that the formulas*

$$\begin{aligned} \widetilde{\rho}^E(\tau) &= c_\tau \rho^E((k^*)^{-1})\rho^E(\sigma), \\ \widetilde{\rho}^E(k) &= \rho^E(k), \text{ if } k \in K, \end{aligned} \tag{2.4.3}$$

define an irreducible representation $(E, \widetilde{\rho}^E)$ of K^τ.

Proof. Put

$$A = \rho^E((k^*)^{-1})\rho^E(\sigma) \in \mathrm{Aut}(E). \tag{2.4.4}$$

Then if $k \in K$,

$$A\rho^E(k) = \rho^E(\tau(k))A. \tag{2.4.5}$$

Set

$$\begin{aligned} \widehat{k} &= k^*\tau(k^*) \cdots \tau^{N_0-1}(k^*) \\ &= \sigma^{N_0-1}(k^*) \cdots \sigma(k^*)k^* \in K. \end{aligned} \tag{2.4.6}$$

Then

$$\begin{aligned} \sigma(\widehat{k}) &= \widehat{k} \in K, \\ \sigma^{N_0} &= \mathrm{Ad}(\widehat{k}) \in \mathrm{Aut}(K). \end{aligned} \tag{2.4.7}$$

In the same time, we have

$$A^{N_0} = \rho^E(\widehat{k}^{-1})\rho^E(\sigma^{N_0}). \tag{2.4.8}$$

Then A^{N_0} commutes with $\rho^E(\sigma)$. If $k \in K$, then

$$A^{N_0}\rho^E(k) = \rho^E(\widehat{k}^{-1})\rho^E(\sigma^{N_0})\rho^E(k) = \rho^E(k)A^{N_0}. \tag{2.4.9}$$

Since E is irreducible as $K^{\sigma\prime}-$representation, A^{N_0} is a scalar operator, so that there exists $c_\tau \in \mathbb{C}$ such that

$$c_\tau^{N_0} A^{N_0} = \mathrm{Id}_E. \tag{2.4.10}$$

Then we can set

$$\widetilde{\rho}^E(\tau) = c_\tau A, \tag{2.4.11}$$

so that (2.4.3) defines a representation of K^τ.

If E has a proper subspace invariant under K^τ, then this subspace must be invariant under $K^{\sigma\prime}$. We conclude that the representation $(E, \widetilde{\rho}^E)$ of K^τ is irreducible. □

Remark 2.4.3. The number c_τ is determined by (2.4.10), therefore the different choices of this number are the multiplications of a fixed c_τ by numbers in C_{N_0}.

Proposition 2.4.4. *The construction of representations of K^τ from representations of $K^{\sigma\prime}$ in Proposition 2.4.2 induces an injection*

$$\phi : \mathrm{Irr}(\Sigma^{\sigma\prime})\backslash \mathrm{Irr}(K^{\sigma\prime}) \to C_N\backslash \mathrm{Irr}(K^\tau). \tag{2.4.12}$$

Moreover, ϕ is independent of the choice of k^* and the choice of c_τ in Proposition 2.4.2.

Proof. If $(L, \rho^L) \in \mathrm{Irr}(\Sigma^{\sigma\prime})$, the representation $(L, \rho^L) \cdot (E, \rho^E)$ is isomorphic to (E, ρ^E) as representations of K. Then their associated representations of K^τ constructed in Proposition 2.4.2 correspond to the same orbit in P_{++}. We get that the map ϕ above is well-defined. In particular, the different choices of k^* and $c_\tau$do not change the orbit of E in P_{++} as $K-$representations. By Proposition 2.3.5, we see that ϕ is independent of the choices of k^* and c_τ.

Now we prove the injectivity of ϕ. Suppose that (E_1, ρ_1) and (E_2, ρ_2) are two irreducible representations of $K^{\sigma\prime}$ which have the same image under the map ϕ. Let $(E_1, \widetilde{\rho}_1), (E_2, \widetilde{\rho}_2)$ be the corresponding irreducible representations of K^τ defined in Proposition 2.4.2 with suitable choices of c_τ. Then by Remark 2.4.3, we could and we will assume that $E = E_1 = E_2, \widetilde{\rho}^E = \widetilde{\rho}_1 = \widetilde{\rho}_2$.

Let c_1, c_2 be the two numbers chosen for defining $\widetilde{\rho}_1(\tau)$ and $\widetilde{\rho}_2(\tau)$, i.e.,

$$c_1\widetilde{\rho}^E((k^*)^{-1})\rho_1(\sigma) = c_2\widetilde{\rho}^E((k^*)^{-1})\rho_2(\sigma), \tag{2.4.13}$$

and

$$c_1^{N_0}\widetilde{\rho}^E(\widehat{k}^{-1})\rho_1(\sigma^{N_0}) = \mathrm{Id}_E = c_2^{N_0}\widetilde{\rho}^E(\widehat{k}^{-1})\rho_2(\sigma^{N_0}). \tag{2.4.14}$$

Let a be a non-zero eigenvalue for $\rho_2(\sigma)$ with an eigenvector $v \in E$. Put $L = \mathbb{C}v \subset E$. The equality above shows that v is also an eigenvector of $\rho_2(\sigma)$ for the eigenvalue $\dfrac{c_2}{c_1}a$. Then the complex line L, with the restriction of ρ_1 (resp. ρ_2) to the Abelian group $\Sigma^{\sigma\prime}$, becomes a representation of $\Sigma^{\sigma\prime}$, we denote it by (L_1, ρ_1^L) (resp.

$(L_2, \rho_2^L))$. These constructions imply that the representation $(E_1 \otimes L_1, \rho_1 \otimes \rho_1^L)$ of $K^{\sigma\prime}$ is isomorphic to the representation $(E_2 \otimes L_2, \rho_2 \otimes \rho_2^L)$. This is equivalent to that (E_1, ρ_1) and (E_2, ρ_2) lie in the same orbit in $\mathrm{Irr}(\Sigma^{\sigma\prime})\backslash\mathrm{Irr}(K^{\sigma\prime})$. This proves the injectivity of ϕ, so that the proof of our proposition is completed. □

Theorem 2.4.5. *We have a canonical bijection between the two orbit spaces:*

$$\mathrm{Irr}(\Sigma^{\sigma\prime})\backslash\mathrm{Irr}(K^{\sigma\prime}) \simeq C_N\backslash\mathrm{Irr}(K^{\tau}). \tag{2.4.15}$$

Proof. To prove (2.4.15), we only need to prove the surjectivity of ϕ defined in Proposition 2.4.4. Take an irreducible representation $(E_\lambda, \rho^{E_\lambda})$ of K associated with $\lambda \in P_{++}$. Since K is a closed subgroup of $K^{\sigma\prime}$, we have the induced $K^{\sigma\prime}$-representation $\mathrm{Ind}_K^{K^{\sigma\prime}}(E_\lambda)$. Now take any $K^{\sigma\prime}$-irreducible component V of $\mathrm{Ind}_K^{K^{\sigma\prime}}(E_\lambda)$, which is always of finite dimension.

By the Frobenius reciprocity [DK00, Theorem (4.7.1)], we have

$$\mathrm{Hom}_{K^{\sigma\prime}}(V, \mathrm{Ind}_K^{K^{\sigma\prime}}(E_\lambda)) \simeq \mathrm{Hom}_K(\mathrm{Res}_K^{K^{\sigma\prime}} V, E_\lambda). \tag{2.4.16}$$

The left-hand side of (2.4.16) is non-empty, then the restriction of V to K has a K-irreducible component corresponding to λ. Then the $K^{\sigma\prime}$-representation V is sent to the orbit $[\lambda]$ by ϕ. This completes the proof of our theorem. □

Remark 2.4.6. In fact, we have the group identification $\mathrm{Irr}(\langle\tau\rangle) = C_{N_0}$. Then we can rewrite (2.4.15) in an uniform way

$$\mathrm{Irr}(\Sigma^{\sigma\prime})\backslash\mathrm{Irr}(K^{\sigma\prime}) \simeq \mathrm{Irr}(\langle\tau\rangle)\backslash\mathrm{Irr}(K^{\tau}). \tag{2.4.17}$$

Combining Proposition 2.1.2 and Theorem 2.4.5, we get a bijection,

$$\mathrm{Irr}(\Sigma^{\sigma})\backslash\mathrm{Irr}(K^{\sigma}) \simeq C_{N_0}\backslash\mathrm{Irr}(K^{\tau}). \tag{2.4.18}$$

One important observation to (2.4.17) is that we can get a version of Weyl character formula for K^σ from the Weyl character formula given in subsection 2.2 for K^τ, which is in terms of the root data of K. We will use this observation in subsection 7.3.

We still use the root data of K fixed in subsection 2.2 and the group identification (2.3.2). Recall that τ is the projection of σ in $\mathrm{Out}(K)$. Recall that if (E, ρ^E) is a finite-dimensional unitary representation of K, $\Omega(E, \rho^E) \subset P_{++}$ denote the set of the highest weights associated with the K-irreducible components of E.

Using the correspondence in (2.4.18), we get a criterion for the extension of a K-representation to a K^σ-representation to exist.

Proposition 2.4.7. *If (E, ρ^E) is a finite-dimensional unitary representation of K, then we can extend it to a representation of K^σ if and only if the following conditions are satisfied:*

(1) If $\lambda \in \Omega(E, \rho^E)$, then $\tau\lambda \in \Omega(E, \rho^E)$, i.e., $\Omega(E, \rho^E)$ is a disjoint union of τ-orbits in P_{++};

(2) If $\lambda \in \Omega(E, \rho^E)$, then the multiplicity of E_λ in E is equal to the multiplicity of $E_{\tau\lambda}$ in E.

Moreover, the representation E can be extended to an irreducible representation of K^σ if and only if $\Omega(E, \rho^E)$ has only one τ-orbit and the multiplicity of any K-irreducible component is 1*. In this case, the extension is unique up to a* 1*-dimensional unitary representation of Σ^σ.*

Proof. This is just a consequence of Propositions 2.1.2, 2.3.5 and 2.4.2 and Theorem 2.4.5. □

3. The hypoelliptic Laplacian on X

The purpose of this section is to recall the construction of the hypoelliptic Laplacian $\mathcal{L}_b^X$, $b>0$ on $\widehat{\mathcal{X}}$ in [B11, Chapter 2]. The constructions involve Clifford algebras and the Dirac operator of Kostant [Kos97].

This section is organized as follows. In subsection 3.1, we introduce the general Clifford algebras.

In subsection 3.2, we recall the construction of the flat connections on vector bundle $\Lambda^\cdot(T^*X\oplus N^*)$ on X.

In subsection 3.3, we introduce the harmonic oscillator on an Euclidean space, and the corresponding K-invariant operator on $\mathfrak{g}$ with respect to the bilinear form B.

In subsection 3.4, we consider the Casimir operator associated with $\mathfrak{g}$ and the Dirac operator of Kostant.

In subsection 3.5, we introduce the Dirac operator $\mathfrak{D}_b$, $b>0$ acting on $C^\infty(G\times\mathfrak{g},\Lambda^\cdot(\mathfrak{g}^*))$ and a key formula of its square obtained in [B11, Section 2.11]. We explain how the operator $\mathfrak{D}_b$ descends to an operator $\mathfrak{D}_b^X$ acting on $C^\infty(\widehat{\mathcal{X}},\widehat{\pi}^*(\Lambda^\cdot(T^*X\oplus N^*)\otimes F))$.

Finally, in subsection 3.6, we introduce the hypoelliptic Laplacian $\mathcal{L}_b^X$ defined in [B11, Section 2.13] and the associated Bianchi identity in [B11, Section 2.15].

In subsection 3.7, we introduce results on the heat kernel of $\mathcal{L}_b^X$ obtained in [B11, Chapters 9 and 11].

3.1. **Clifford algebras.** Let V be a real vector space of dimension m equipped with a real-valued symmetric bilinear form B. The Clifford algebra $c(V)$ of V with respect to B is an associative algebra generated by 1 and $a\in V$ with the relations, if $a,b\in V$,

$$ab+ba=-2B(a,b). \tag{3.1.1}$$

We will denote by $\widehat{c}(V)$ the Clifford algebra associated with $-B$.

Then $c(V)$, $\widehat{c}(V)$ are filtered by length, their associated $\mathbf{Gr}^\cdot$ is just $\Lambda^\cdot(V)$. Also they are $\mathbb{Z}_2$-graded algebras, we can write

$$c(V)=c_+(V)\oplus c_-(V),\;\; \widehat{c}(V)=\widehat{c}_+(V)\oplus\widehat{c}_-(V). \tag{3.1.2}$$

Now we assume that B is nondegenerate. Then B induces an isomorphism φ between V and V^*, i.e., if $a,b\in V$, then $\varphi(a)\in V^*$ is given by

$$\langle\varphi a,b\rangle=B(a,b). \tag{3.1.3}$$

Let B^* be the corresponding bilinear form on V^*, i.e., if $\alpha,\beta\in V^*$, then

$$B^*(\alpha,\beta)=B(\varphi^{-1}\alpha,\varphi^{-1}\beta)=\langle\alpha,\varphi^{-1}\beta\rangle. \tag{3.1.4}$$

Then B^* induces a nondegenerate symmetric bilinear form on $\Lambda^\cdot(V^*)$, which we still denote by B^*.

If $a\in V$, let $c(a)$, $\widehat{c}(a)\in\operatorname{End}(\Lambda^\cdot(V^*))$ be given by

$$c(a)=\varphi(a)\wedge-i_a,\;\widehat{c}(a)=\varphi(a)\wedge+i_a. \tag{3.1.5}$$

Then $c(a)$, $\widehat{c}(a)$ are odd operators, which are respectively antisymmetric, symmetric with respect to B^*.

If $a, b \in V$, then

$$[c(a), c(b)] = -2B(a,b),\ [\widehat{c}(a), \widehat{c}(b)] = 2B(a,b),\ [c(a), \widehat{c}(b)] = 0. \tag{3.1.6}$$

By (3.1.6), $\Lambda^{\cdot}(V^*)$ is a $c(V)\widehat{\otimes}\widehat{c}(V)$ module. If $D \in c(V)$ or $\widehat{c}(V)$, then we denote by $c(D)$ or $\widehat{c}(D)$ the corresponding actions on $\Lambda^{\cdot}(V^*)$ defined in (3.1.5).

Definition 3.1.1. The symbol map $\boldsymbol{\sigma} : c(V) \to \Lambda^{\cdot}(V^*)$ is such that if $D \in c(V)$, then

$$\boldsymbol{\sigma}(D) = c(D)1 \in \Lambda^{\cdot}(V^*). \tag{3.1.7}$$

The map $\boldsymbol{\sigma}$ identifies $c(V)$ with $\Lambda^{\cdot}(V^*)$ as vector spaces. Similarly, we have a symbol map for $\widehat{c}(V)$.

Let $e_1, \cdots, e_m$ be a basis of V, and let $e_1^*, \cdots, e_m^*$ be the dual basis of V with respect to B, so that $B(e_i, e_j^*) = \delta_{ij}$. If $a \in V$, then

$$a = \sum_{i=1}^{m} B(a, e_i^*) e_i. \tag{3.1.8}$$

Let $e^1, \cdots, e^m$ be the basis of V^* which is dual to the basis $e_1, \cdots, e_m$. Then $e^i = \varphi(e_i^*)$.

If $\alpha \in \Lambda^p(V^*)$, then the inverse map of $\boldsymbol{\sigma}$ is given by

$$c(\alpha) = \frac{1}{p!} \sum_{1 \le i_1, \cdots, i_p \le m} \alpha(e_{i_1}^*, \cdots, e_{i_p}^*) c(e_{i_1}) \cdots c(e_{i_p}) \in c(V). \tag{3.1.9}$$

Let $\mathcal{A}(V)$ be the Lie algebra of endomorphisms of V that are antisymmetric with respect to B. Then $\mathcal{A}(V)$ embeds as a Lie algebra in $c(V)$. If $A \in \mathcal{A}(V)$, the image $c(A)$ of A in $c(V)$ is given by

$$c(A) = \frac{1}{4} \sum_{i,j} B(Ae_i^*, e_j^*) c(e_i) c(e_j). \tag{3.1.10}$$

If $a \in V$, then

$$[c(A), c(a)] = c(Aa). \tag{3.1.11}$$

Note that $A \in \mathcal{A}(V)$ defines naturally an element $\alpha \in \Lambda^2(V^*)$ by

$$\alpha = \frac{1}{2} \sum_{i,j} B(Ae_i, e_j) e^i \wedge e^j. \tag{3.1.12}$$

Then $c(\alpha) = 2c(A) \in c(V)$.

When replacing B by $-B$, e_i^* is changed to $-e_i^*$, and $c(e_i)$ is changed to $-\widehat{c}(e_i)$. If $A \in \mathcal{A}(V)$, then it is also antisymmetric with respect to $-B$. We denote by $\widehat{c}(A)$ the corresponding element in $\widehat{c}(V)$. By (3.1.10), we get

$$\widehat{c}(A) = -\frac{1}{4} \sum_{i,j} B(e_i^*, e_j^*) \widehat{c}(e_i) \widehat{c}(e_j). \tag{3.1.13}$$

Instead of (3.1.11), we have

$$[\widehat{c}(A), \widehat{c}(a)] = \widehat{c}(Aa). \tag{3.1.14}$$

If $A \in \mathrm{End}(V)$, then A induces an action on $\Lambda^{\cdot}(V^*)$, and this action is given by

$$A|_{\Lambda^{\cdot}(V^*)} = -\langle Ae_i, e^j \rangle e^i i_{e_j}. \tag{3.1.15}$$

By [B11, eq.(1.1.14)], if $A \in \mathcal{A}(V)$, then

$$A|_{\Lambda^{\cdot}(V^*)} = c(A) + \widehat{c}(A). \tag{3.1.16}$$

Definition 3.1.2. The number operator $N^{\Lambda^{\cdot}(V^*)}$ on $\Lambda^{\cdot}(V^*)$ is such that, if $\alpha \in \Lambda^p(V^*)$, then

$$N^{\Lambda^{\cdot}(V^*)}\alpha = p\alpha. \tag{3.1.17}$$

By [B11, eq.(1.1.15)], we have

$$N^{\Lambda^{\cdot}(V^*)} - \frac{m}{2} = \frac{1}{2}c(e_i^*)\widehat{c}(e_i). \tag{3.1.18}$$

If (V', B') is another pair like (V, B), then $(V \oplus V', B \oplus B')$ is still another such a pair. We have the identifications of Clifford algebras,

$$c(V \oplus V') = c(V)\widehat{\otimes}c(V'),\ \widehat{c}(V \oplus V') = \widehat{c}(V)\widehat{\otimes}\widehat{c}(V'). \tag{3.1.19}$$

We refer to [LM89], [BGV04, Chapter 3], [B11, Chapter 1] for more detailed discussions on Clifford algebras.

3.2. **The flat connections on $\Lambda^{\cdot}(T^*X \oplus N^*)$.** Recall that the map $(g, f) \in G \times \mathfrak{g} \to \mathrm{Ad}(g)f \in \mathfrak{g}$ identifies the vector bundle $TX \oplus N$ with the trivial vector bundle $\mathfrak{g}$ on X. Recall that the connection $\nabla^{TX\oplus N} = \nabla^{TX} \oplus \nabla^N$ is the Euclidean connection of $TX \oplus N$ induced by the connection form $\omega^{\mathfrak{k}}$. Let $\nabla^{TX\oplus N,f}$ denote the flat connection on $\mathfrak{g}$, i.e. the connection associated with the connection form $\omega^{\mathfrak{g}}$.

By (1.1.7), we get

$$\nabla^{TX\oplus N,f} = \nabla^{TX\oplus N} + \mathrm{ad}(\omega^{\mathfrak{p}}). \tag{3.2.1}$$

Let $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f}$ be the connection on $\Lambda^{\cdot}(T^*X \oplus N^*)$ induced by $\nabla^{TX\oplus N,f}$. Then by (3.1.16), (3.2.1),

$$\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f} = \nabla^{\Lambda^{\cdot}(T^*X\oplus N^*)} + c(\mathrm{ad}(\cdot)) + \widehat{c}(\mathrm{ad}(\cdot)). \tag{3.2.2}$$

Let $\nabla^{TX\oplus N,f,*}$ be the dual connection of $\nabla^{TX\oplus N,f}$ with respect to the metric on $TX \oplus N$, then

$$\nabla^{TX\oplus N,f,*} = \nabla^{TX\oplus N} - \mathrm{ad}(\omega^{\mathfrak{p}}). \tag{3.2.3}$$

Let $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f,*}$ be the associated connection on $\Lambda^{\cdot}(T^*X \oplus N^*)$. As in (3.2.2), we get

$$\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f,*} = \nabla^{\Lambda^{\cdot}(T^*X\oplus N^*)} - c(\mathrm{ad}(\cdot)) - \widehat{c}(\mathrm{ad}(\cdot)). \tag{3.2.4}$$

Moreover, both $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f}$, $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f,*}$ preserve the degree of $\Lambda^{\cdot}(T^*X \oplus N^*)$.

We recall another connection on $\Lambda^{\cdot}(T^*X \oplus N^*)$ defined in [B11, Definition 2.4.1], *Definition* 3.2.1. Put

$$\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f*,\widehat{f}} = \nabla^{\Lambda^{\cdot}(T^*X\oplus N^*)} - c(\mathrm{ad}(\cdot)) + \widehat{c}(\mathrm{ad}(\cdot)). \tag{3.2.5}$$

By [B11, Proposition 2.4.2], $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f*,\widehat{f}}$ is a flat connection on $\Lambda^{\cdot}(T^*X\oplus N^*)$. Also the connection $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f*,\widehat{f}}_{\cdot}$ does not preserve the degree of $\Lambda^{\cdot}(T^*X\oplus N^*)$.

3.3. **The harmonic oscillator on an Euclidean space V.** Let V be an Euclidean space with scalar product g^V, let Δ^V denote the associated Euclidean Laplacian on V, and let $c(V)$, $\widehat{c}(V)$ be the corresponding Clifford algebras. Let $e_1, \cdots, e_m$ be an orthonormal basis of (V, g^V).

Let $Y \in V$ denote the tautological section of V, and let Y^* denote the metric dual of Y in V^*. If $v \in V$, let ∇_v denote the differential operator on V along the vector v.

Let d^V be de Rham operator on V, and let $d^{V,*}$ be its formal adjoint. Set

$$\overline{d} = \exp(-|Y|^2/2)d^V \exp(|Y|^2/2), \tag{3.3.1}$$

i.e., $\overline{d}$ is the Witten twist [Wit82] of d^V associated with the function $|Y|^2/2$. We have

$$\overline{d} = d^V + Y^* \wedge . \tag{3.3.2}$$

Let $\overline{d}^*$ be the formal adjoint of $\overline{d}$. Then by [B11, eq.(1.6.7)],

$$\overline{d}^* = d^{V,*} + i_Y. \tag{3.3.3}$$

Then the corresponding Hodge Laplacian is given by

$$[\overline{d}, \overline{d}^*] = -\Delta^V + |Y|^2 - m + 2N^{\Lambda^{\cdot}(V^*)}. \tag{3.3.4}$$

Set

$$\begin{aligned} \mathcal{D}^V &= \sum_{j=1}^m c(e_j)\nabla_{e_j}, \quad \mathcal{E}^V = \widehat{c}(Y), \\ \mathcal{D}^{V\prime} &= \sum_{j=1}^m \widehat{c}(e_j)\nabla_{e_j}, \quad \mathcal{E}^{V\prime} = c(Y). \end{aligned} \tag{3.3.5}$$

Then $\mathcal{D}^V$, $\mathcal{E}^V$, $\mathcal{D}^{V\prime}$, $\mathcal{E}^{V\prime}$ are linear differential operators acting on $C^\infty(V)\otimes\Lambda^{\cdot}(V^*)$. In particular, $\mathcal{D}^V$ is a classical Dirac operator.

By [B11, eq.(1.6.2)], we have

$$\begin{aligned} \overline{d} + \overline{d}^* &= \mathcal{D}^V + \mathcal{E}^V, \\ \overline{d} - \overline{d}^* &= \mathcal{D}^{V\prime} + \mathcal{E}^{V\prime}. \end{aligned} \tag{3.3.6}$$

Then

$$[\overline{d}, \overline{d}^*] = (\mathcal{D}^V + \mathcal{E}^V)^2 = -(\mathcal{D}^{V\prime} + \mathcal{E}^{V\prime})^2. \tag{3.3.7}$$

The kernel of the unbounded operator $[\overline{d}, \overline{d}^*]$ is an one-dimensional line spanned by the function $\exp(-|Y|^2/2)/\pi^{m/4}$.

Let $c(\mathfrak{g})$, $\widehat{c}(\mathfrak{g})$ be the Clifford algebras associated with $(\mathfrak{g}, B)$, $(\mathfrak{g}, -B)$. Then G acts by automorphisms of $c(\mathfrak{g})$, $\widehat{c}(\mathfrak{g})$, so that if $g \in G$, $e \in \mathfrak{g}$, then

$$g \cdot c(e) = c(\mathrm{Ad}(g)e), \; g \cdot \widehat{c}(e) = \widehat{c}(\mathrm{Ad}(g)e). \tag{3.3.8}$$

By restricting B to $\mathfrak{p}$, $\mathfrak{k}$, we get the Clifford algebras $c(\mathfrak{p})$, $\widehat{c}(\mathfrak{p})$, $c(\mathfrak{k})$, $\widehat{c}(\mathfrak{k})$. By (1.1.1), (3.1.19), we have

$$c(\mathfrak{g}) = c(\mathfrak{p})\widehat{\otimes}c(\mathfrak{k}), \qquad \widehat{c}(\mathfrak{g}) = \widehat{c}(\mathfrak{p})\widehat{\otimes}\widehat{c}(\mathfrak{k}). \tag{3.3.9}$$

The scalar product $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$ is given by $-B(\cdot, \theta\cdot)$. Let $e_1, \cdots, e_m$ be an orthonormal basis of $\mathfrak{p}$, and let $e_{m+1}, \cdots, e_{m+n}$ be an orthonormal basis of $\mathfrak{k}$. Let $e_1^*, \cdots, e_{m+n}^*$ be the dual basis of $\mathfrak{g}$ with respect to B, then

$$\begin{aligned} &e_j^* = e_j \text{ for } 1 \le j \le m; \\ &e_j^* = -e_j \text{ for } m+1 \le j \le m+n. \end{aligned} \tag{3.3.10}$$

If $Y \in \mathfrak{g}$, we split Y in the form

$$Y = Y^{\mathfrak{p}} + Y^{\mathfrak{k}}, \tag{3.3.11}$$

with $Y^{\mathfrak{p}} \in \mathfrak{p}$, $Y^{\mathfrak{k}} \in \mathfrak{k}$. As in (3.3.5), set

$$\mathcal{D}^{\mathfrak{p}} = \sum_{j=1}^{m} c(e_j)\nabla_{e_j}, \;\; \mathcal{E}^{\mathfrak{p}} = \widehat{c}(Y^{\mathfrak{p}}). \tag{3.3.12}$$

By (3.3.4), (3.3.7), we have

$$\frac{1}{2}(\mathcal{D}^{\mathfrak{p}} + \mathcal{E}^{\mathfrak{p}})^2 = \frac{1}{2}\big(- \Delta^{\mathfrak{p}} + |Y^{\mathfrak{p}}|^2 - m\big) + N^{\Lambda^{\cdot}(\mathfrak{p}^*)}. \tag{3.3.13}$$

Note that B is negative on $\mathfrak{k}$. We define the operators $\mathcal{D}^{\mathfrak{k}}$, $\mathcal{E}^{\mathfrak{k}}$ by the formulas,

$$\mathcal{D}^{\mathfrak{k}} = \sum_{j=m+1}^{m+n} c(e_j^*)\nabla_{e_j}, \;\; \mathcal{E}^{\mathfrak{k}} = \widehat{c}(Y^{\mathfrak{k}}). \tag{3.3.14}$$

Let $\mathcal{D}^{\mathfrak{k}\prime}$, $\mathcal{E}^{\mathfrak{k}\prime}$ be the operators defined in (3.3.5) on the Euclidean space $(\mathfrak{k}, -B|_{\mathfrak{k}})$. By [B11, eq.(2.8.10)], we have

$$\mathcal{D}^{\mathfrak{k}} = \mathcal{D}^{\mathfrak{k}\prime}, \;\; \mathcal{E}^{\mathfrak{k}} = -\mathcal{E}^{\mathfrak{k}\prime}. \tag{3.3.15}$$

By (3.3.4), (3.3.7), (3.3.15), we have

$$\frac{1}{2}(-i\mathcal{D}^{\mathfrak{k}} + i\mathcal{E}^{\mathfrak{k}})^2 = \frac{1}{2}\big(- \Delta^{\mathfrak{k}} + |Y^{\mathfrak{k}}|^2 - n\big) + N^{\Lambda^{\cdot}(\mathfrak{k}^*)}. \tag{3.3.16}$$

Since K preserves the scalar products on $\mathfrak{p}$ and $\mathfrak{k}$, the above constructions are K-equivariant.

Then $\mathcal{D}^{\mathfrak{p}}$, $\mathcal{E}^{\mathfrak{p}}$, $\mathcal{D}^{\mathfrak{k}}$, $\mathcal{E}^{\mathfrak{k}}$ are linear differential operators acting on $\Lambda^{\cdot}(\mathfrak{g}^*) \otimes C^{\infty}(\mathfrak{g})$. Moreover,

$$[\mathcal{D}^{\mathfrak{p}} + \mathcal{E}^{\mathfrak{p}}, -i\mathcal{D}^{\mathfrak{k}} + i\mathcal{E}^{\mathfrak{k}}] = 0. \tag{3.3.17}$$

Let $\Delta^{\mathfrak{g}}$ be the Euclidean Laplacian of $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$. By (3.3.13), (3.3.16), (3.3.17), we get

$$\frac{1}{2}\big(\mathcal{D}^{\mathfrak{p}} + \mathcal{E}^{\mathfrak{p}} - i\mathcal{D}^{\mathfrak{k}} + i\mathcal{E}^{\mathfrak{k}}\big)^2 = \frac{1}{2}\big(- \Delta^{\mathfrak{g}} + |Y|^2 - (m+n)\big) + N^{\Lambda^{\cdot}(\mathfrak{g}^*)}. \tag{3.3.18}$$

3.4. **The Casimir operator and the Dirac operator of Kostant.** Let $U\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. If we identify $\mathfrak{g}$ to the vector space of left-invariant vector fields on G, then the enveloping algebra $U\mathfrak{g}$ is identified with the algebra of left-invariant differential operators on G. Moreover, the adjoint action of $\widetilde{G}$ on $\mathfrak{g}$ induces a corresponding action on $U\mathfrak{g}$.

Let $C^{\mathfrak{g}} \in U\mathfrak{g}$ be the Casimir element of G associated with the bilinear form B. If $e_1, \cdots, e_{m+n}$ is a basis of $\mathfrak{g}$ and if $e_1^*, \cdots, e_{m+n}^*$ is the dual basis of $\mathfrak{g}$ with respect to B, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m+n} e_i^* e_i. \tag{3.4.1}$$

Also $C^{\mathfrak{g}}$ lies in the center of $U\mathfrak{g}$. Following Lemma 1.2.7, $C^{\mathfrak{g}}$ commutes with $\widetilde{G}$.

If $e_1, \cdots, e_m$ is an orthonormal basis of $\mathfrak{p}$, and if $e_{m+1}, \cdots, e_{m+n}$ is an orthonormal basis of $\mathfrak{k}$, by (3.3.10), (3.4.1), we have

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m} e_i^2 + \sum_{i=m+1}^{m+n} e_i^2. \tag{3.4.2}$$

Set

$$C^{\mathfrak{g},H} = -\sum_{i=1}^{m} e_i^2. \tag{3.4.3}$$

Recall that the Casimir operator $C^{\mathfrak{k}}$ of K was defined in (1.1.18), then

$$C^{\mathfrak{g}} = C^{\mathfrak{g},H} + C^{\mathfrak{k}}. \tag{3.4.4}$$

Put $\kappa^{\mathfrak{g}} \in \Lambda^3(\mathfrak{g}^*)$ such that if $a, b, c \in \mathfrak{g}$,

$$\kappa^{\mathfrak{g}}(a,b,c) = B([a,b],c). \tag{3.4.5}$$

Since the action of $\widetilde{g} \in \widetilde{G}$ preserves B, we have

$$\mathrm{Ad}(\widetilde{g})\kappa^{\mathfrak{g}} = \kappa^{\mathfrak{g}}. \tag{3.4.6}$$

We can view $\kappa^{\mathfrak{g}}$ as a closed left and right invariant 3-form on G.

Let B^* be the bilinear form on $\Lambda^{\cdot}(\mathfrak{g}^*)$ given by (3.1.4). By [B11, eqs.(2.6.4) and (2.6.11)], we have

$$\begin{aligned} B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}) &= \frac{1}{6}\sum_{i,j=1}^{m+n} B([e_i, e_j], [e_i^*, e_j^*]) \\ &= \frac{1}{2}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{1}{6}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \end{aligned} \tag{3.4.7}$$

Let $\kappa^{\mathfrak{k}} \in \Lambda^3(\mathfrak{k}^*)$ be the element defined by the same formula as in (3.4.5) with respect to $(\mathfrak{k}, B|_{\mathfrak{k}})$. Then by (3.4.7), we get

$$B^*(\kappa^{\mathfrak{k}}, \kappa^{\mathfrak{k}}) = \frac{1}{6}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}]. \tag{3.4.8}$$

Recall that the Clifford elements $c(\kappa^{\mathfrak{g}})$, $\widehat{c}(-\kappa^{\mathfrak{g}})$, $c(\kappa^{\mathfrak{k}})$, $\widehat{c}(-\kappa^{\mathfrak{k}})$ are given as in (3.1.9). If $e \in \mathfrak{k}$, let $\mathrm{ad}(e)|_{\mathfrak{p}}$ be the restriction of $\mathrm{ad}(e)$ to $\mathfrak{p}$. Then $\widehat{c}(\mathrm{ad}(e)|_{\mathfrak{p}}) \in \widehat{c}(\mathfrak{p})$. By [B11, eq.(2.7.4)], we have

$$\widehat{c}(-\kappa^{\mathfrak{g}}) = -2 \sum_{i=m+1}^{m+n} \widehat{c}(e_i)\widehat{c}(\mathrm{ad}(e)|_{\mathfrak{p}}) + \widehat{c}(-\kappa^{\mathfrak{k}}). \tag{3.4.9}$$

Definition 3.4.1. Let $D^{\mathfrak{g}} \in c(\mathfrak{g}) \otimes U\mathfrak{g}$, $\widehat{D}^{\mathfrak{g}} \in \widehat{c}(\mathfrak{g}) \otimes U\mathfrak{g}$ be the Dirac operators,

$$\begin{aligned} D^{\mathfrak{g}} &= \sum_{i=1}^{m+n} c(e_i^*)e_i + \frac{1}{2}c(\kappa^{\mathfrak{g}}), \\ \widehat{D}^{\mathfrak{g}} &= \sum_{i=1}^{m+n} \widehat{c}(e_i^*)e_i + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{g}}). \end{aligned} \tag{3.4.10}$$

The operators $D^{\mathfrak{g}}, \widehat{D}^{\mathfrak{g}}$ are the Dirac operators of Kostant [Kos97].

Set

$$\begin{aligned} \widehat{D}^{\mathfrak{g}}_H &= \sum_{j=1}^{m} \widehat{c}(e_j)e_j, \\ \widehat{D}^{\mathfrak{g}}_V &= -\sum_{j=m+1}^{m+n} \widehat{c}(e_j)(e_j + \widehat{c}(\mathrm{ad}(e_j)|_{\mathfrak{p}})) + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{k}}). \end{aligned} \tag{3.4.11}$$

By [B11, eq.(2.7.6)], we have

$$\widehat{D}^{\mathfrak{g}} = \widehat{D}^{\mathfrak{g}}_H + \widehat{D}^{\mathfrak{g}}_V. \tag{3.4.12}$$

By [B11, Theorem 2.7.2], we have

$$[\widehat{D}^{\mathfrak{g}}_H, \widehat{D}^{\mathfrak{g}}_V] = 0, \quad \widehat{D}^{\mathfrak{g},2} = -C^{\mathfrak{g}} - \frac{1}{4}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}). \tag{3.4.13}$$

We have the analogues of (3.4.11) - (3.4.13) for $D^{\mathfrak{g}}$. In particular, we have

$$D^{\mathfrak{g},2} = C^{\mathfrak{g}} + \frac{1}{4}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}). \tag{3.4.14}$$

3.5. **The operator $\mathfrak{D}^X_b$.** As we saw in subsection 3.4, $\widehat{D}^{\mathfrak{g}}$ is a first order differential operator acting on $C^\infty(G, \Lambda^\cdot(\mathfrak{g}^*))$. Recall that $\mathcal{D}^{\mathfrak{p}} + \mathcal{E}^{\mathfrak{p}} - i\mathcal{D}^{\mathfrak{k}} + i\mathcal{E}^{\mathfrak{k}}$ is a differential operator acting on $\Lambda^\cdot(\mathfrak{g}^*) \otimes C^\infty(\mathfrak{g})$. We have

$$C^\infty(G, \Lambda^\cdot(\mathfrak{g}^*) \otimes C^\infty(\mathfrak{g})) = C^\infty(G \times \mathfrak{g}, \Lambda^\cdot(\mathfrak{g}^*)). \tag{3.5.1}$$

Definition 3.5.1. For $b > 0$, let $\mathfrak{D}_b$ on $C^\infty(G \times \mathfrak{g}, \Lambda^\cdot(\mathfrak{g}^*))$ be the differential operator given by,

$$\mathfrak{D}_b = \widehat{D}^{\mathfrak{g}} + ic([Y^{\mathfrak{k}}, Y^{\mathfrak{p}}]) + \frac{1}{b}(\mathcal{D}^{\mathfrak{p}} + \mathcal{E}^{\mathfrak{p}} - i\mathcal{D}^{\mathfrak{k}} + i\mathcal{E}^{\mathfrak{k}}). \tag{3.5.2}$$

If $Y \in \mathfrak{g}$, let $\underline{Y^{\mathfrak{p}}}$, $\underline{Y^{\mathfrak{k}}}$ denote the tangent vector fields on G associated with $Y^{\mathfrak{p}}$, $Y^{\mathfrak{k}} \in \mathfrak{g}$. The following identity is obtained in [B11, Section 2.11].

Theorem 3.5.2 (Bismut). *We have the following formula for $\mathfrak{D}_b^2$,*

$$\begin{aligned}\frac{\mathfrak{D}_b^2}{2} = \frac{\widehat{D}^{\mathfrak{g},2}}{2} + \frac{1}{2}|[Y^{\mathfrak{k}}, Y^{\mathfrak{p}}]|^2 + \frac{1}{2b^2}(-\Delta^{\mathfrak{p}\oplus\mathfrak{k}} + |Y|^2 - m - n) + \frac{N^{\Lambda^{\cdot}(\mathfrak{g}^*)}}{b^2} \\ + \frac{1}{b}\bigg(\underline{Y^{\mathfrak{p}}} + i\underline{Y^{\mathfrak{k}}} - i\nabla^{\mathfrak{g}}_{[Y^{\mathfrak{k}},Y^{\mathfrak{p}}]} + \widehat{c}(ad(Y^{\mathfrak{p}} + iY^{\mathfrak{k}})) \\ + 2ic(ad(Y^{\mathfrak{k}})|_{\mathfrak{p}}) - c(ad(Y^{\mathfrak{p}}))\bigg).\end{aligned} \tag{3.5.3}$$

As we saw before, the kernel $H \subset \Lambda^{\cdot}(\mathfrak{g}^*)\otimes L_2(\mathfrak{g})$ of the operator $\mathcal{D}^{\mathfrak{p}}+\mathcal{E}^{\mathfrak{p}}-i\mathcal{D}^{\mathfrak{k}}+i\mathcal{E}^{\mathfrak{k}}$ is one-dimensional and spanned by $\exp(-|Y|^2/2)$. Let P be the orthogonal projection operator on H, then by [B11, eq.(2.10.2)], we have

$$P\bigg(\widehat{D}^{\mathfrak{g}} + ic([Y^{\mathfrak{k}}, Y^{\mathfrak{p}}])\bigg)P = 0. \tag{3.5.4}$$

Recall that (E, ρ^E) is a unitary representation of K^{σ}. If $s \in C^{\infty}(G\times\mathfrak{g}, \Lambda^{\cdot}(\mathfrak{g}^*)\otimes E)$, as in (1.1.12), the action of $k \in K$ is given by,

$$(k.s)(g, Y) = \rho^{\Lambda^{\cdot}(\mathfrak{g}^*)\otimes E}(k)s(gk, \mathrm{Ad}(k^{-1})Y). \tag{3.5.5}$$

Let $C_K^{\infty}(G \times \mathfrak{g}, \Lambda^{\cdot}(\mathfrak{g}^*) \otimes E)$ denote the set of K-invariant sections.

Recall that $\widehat{\pi} : \widehat{\mathcal{X}} \to X$ is the total space of $TX \oplus N$. Let $Y = Y^{TX} + Y^N$, $Y^{TX} \in TX$, $Y^N \in N$ be the tautological section of $\widehat{\pi}^*(TX \oplus N)$ over $\widehat{\mathcal{X}}$.

Definition 3.5.3. Let $\mathcal{H}$ be the vector space of smooth sections over X of the vector bundle $C^{\infty}(TX \oplus N, \widehat{\pi}^*(\Lambda^{\cdot}(T^*X \oplus N^*) \otimes F))$.

We can identify $\mathcal{H}$ with $C^{\infty}(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^{\cdot}(T^*X\oplus N^*)\otimes F))$. Let $\nabla^{\mathcal{H}}$ be the connection induced by the connection form $\omega^{\mathfrak{k}}$ on X. We can identify the element of $\mathfrak{p}$ and TX to the corresponding horizontal lift in $T\widehat{\mathcal{X}}$ by the connection $\nabla^{TX} \oplus \nabla^N$. The Bochner Laplacian $\Delta^{H,X}$ acting on $\mathcal{H}$ is given by

$$\Delta^{H,X} = \sum_{j=1}^{m} \nabla^{\mathcal{H}}_{e_j}. \tag{3.5.6}$$

By (3.4.3), (3.5.6), we have the identity of operators acting on $\mathcal{H}$,

$$C^{\mathfrak{g},H} = -\Delta^{H,X}. \tag{3.5.7}$$

Let $e \in \mathfrak{k}$, $[e, Y]$ on $\mathfrak{g}$ is a Killing vector field. Let $L^V_{[e,Y]}$ be the Lie derivative acting on $C^{\infty}(\mathfrak{g}, \Lambda^{\cdot}(\mathfrak{g}^*))$, then by [B11, eq.(2.12.4)],

$$L^V_{[e,Y]} = \nabla_{[e,Y]} - (c + \widehat{c})(\mathrm{ad}(e)). \tag{3.5.8}$$

By [B11, eq.(2.12.16)], we have the identity of operators acting on $\mathcal{H}$,

$$C^{\mathfrak{k}} = \sum_{j=m+1}^{m+n} (L^V_{[e_j,Y]} - \rho^E(e_j))^2. \tag{3.5.9}$$

The operators $\widehat{D}^{\mathfrak{g}}$, $\widehat{D}^{\mathfrak{g}}_H$, $\widehat{D}^{\mathfrak{g}}_V$ are $K-$invariant. Let $\widehat{D}^{\mathfrak{g},X}$, $\widehat{D}^{\mathfrak{g},X}_H$, $\widehat{D}^{\mathfrak{g},X}_V$ be the corresponding differential operators on the smooth sections of $\mathcal{H}$. We still use the notation $C^{\mathfrak{g}}$ to denote the Casimir operator on $\widehat{\mathcal{X}}$. By [B11, Theorem 2.12.2], we have the following identities of operators,

$$
\begin{aligned}
&C^{\mathfrak{g}} = C^{\mathfrak{g},H} + C^{\mathfrak{k}},\\
&\widehat{D}^{\mathfrak{g},X} = \widehat{D}^{\mathfrak{g},X}_H + \widehat{D}^{\mathfrak{g},X}_V, \;\; [\widehat{D}^{\mathfrak{g},X}_H, \widehat{D}^{\mathfrak{g},X}_V] = 0,\\
&\widehat{D}^{\mathfrak{g},X}_H = \sum_{j=1}^{m} \widehat{c}(e_i)\nabla^{\mathcal{H}}_{e_i},\\
&\widehat{D}^{\mathfrak{g},X}_V = -\sum_{j=m+1}^{m+n} \widehat{c}(e_j)(L^V_{[e_j,Y]} + \widehat{c}(\mathrm{ad}(e_j)|_{\mathfrak{p}}) - \rho^E(e_j)) + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{k}}),\\
&\widehat{D}^{\mathfrak{g},X,2} = -C^{\mathfrak{g}} - \frac{1}{4}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}).
\end{aligned}
\tag{3.5.10}
$$

Let $\mathcal{D}^{TX}$, $\mathcal{E}^{TX}$, $\mathcal{D}^N$, $\mathcal{E}^N$ be the differential operator on $\widehat{\pi}^*(\Lambda^{\cdot}(T^*X\oplus N^*)\otimes F)$ along the fiber $\widehat{\mathcal{X}}$ induced by $\mathcal{D}^{\mathfrak{p}}$, $\mathcal{E}^{\mathfrak{p}}$, $\mathcal{D}^{\mathfrak{k}}$, $\mathcal{E}^{\mathfrak{k}}$. Then the operator $\mathfrak{D}_b$ defined in (3.5.2) induces an operator $\mathfrak{D}^X_b$ on $C^\infty(TX\oplus N, \widehat{\pi}^*(\Lambda^{\cdot}(T^*X\oplus N^*)\otimes F))$. By (3.5.2), we get

$$
\mathfrak{D}^X_b = \widehat{D}^{\mathfrak{g},X} + ic([Y^N, Y^{TX}]) + \frac{1}{b}(\mathcal{D}^{TX} + \mathcal{E}^{TX} - i\mathcal{D}^N + i\mathcal{E}^N).
\tag{3.5.11}
$$

By [B11, Theorem 2.12.5], we have

$$
\begin{aligned}
\frac{1}{2}\mathfrak{D}^{X,2}_b =& \frac{1}{2}\widehat{D}^{\mathfrak{g},X,2} + \frac{1}{2}|[Y^N, Y^{TX}]|^2\\
&+ \frac{1}{2b^2}(-\Delta^{TX\oplus N} + |Y|^2 - m - n) + \frac{N^{\Lambda^{\cdot}(T^*X\oplus N^*)}}{b^2}\\
&+ \frac{1}{b}\bigg(\nabla^{\mathcal{H}}_{Y^{TX}} + \widehat{c}(\mathrm{ad}(Y^{TX}))\\
&- c(\mathrm{ad}(Y^{TX}) + i\theta\mathrm{ad}(Y^N)) - i\rho^E(Y^N)\bigg).
\end{aligned}
\tag{3.5.12}
$$

The connection $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f*,\widehat{f}}$ is defined by (3.2.5). Let $\nabla^{\mathcal{H},f*,\widehat{f}}$ be the connection on $\mathcal{H}$ that is induced by $\nabla^{\Lambda^{\cdot}(T^*X\oplus N^*),f*,\widehat{f}}$, ∇^F. By (3.2.5), (3.5.12), we get

$$
\begin{aligned}
\frac{1}{2}\mathfrak{D}^{X,2}_b =& \frac{1}{2}\widehat{D}^{\mathfrak{g},X,2} + \frac{1}{2}|[Y^N, Y^{TX}]|^2\\
&+ \frac{1}{2b^2}(-\Delta^{TX\oplus N} + |Y|^2 - m - n) + \frac{N^{\Lambda^{\cdot}(T^*X\oplus N^*)}}{b^2}\\
&+ \frac{1}{b}\big(\nabla^{\mathcal{H},f*,\widehat{f}}_{Y^{TX}} - c(i\theta\mathrm{ad}(Y^N)) - i\rho^E(Y^N)\big).
\end{aligned}
\tag{3.5.13}
$$

3.6. **The hypoelliptic Laplacian.** Let $C^{\mathfrak{g},X}$ be the operator acting on $C^\infty(X,F)$ induced by $C^{\mathfrak{g}}$, and we still denote by $\Delta^{H,X}$ the Bochner Laplacian acting on $C^\infty(X,F)$. Then $C^{\mathfrak{g},H}$ descends to $-\Delta^{H,X}$.

By (1.1.18), (1.1.19), $C^{\mathfrak{k}}$ induces an endomorphism $C^{\mathfrak{k},E}$ of E, it descends to $C^{\mathfrak{k},F}$ acting $C^{\infty}(X,F)$. Then

$$C^{\mathfrak{g},X} = -\Delta^{H,X} + C^{\mathfrak{k},F}. \tag{3.6.1}$$

We now recall the definition of the elliptic operator $\mathcal{L}^X$ in [B11, Definition 2.13.1].

Definition 3.6.1. Let $\mathcal{L}^X$ be the operator acting on $C^{\infty}(X,F)$,

$$\mathcal{L}^X = \frac{1}{2}C^{\mathfrak{g},X} + \frac{1}{8}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}). \tag{3.6.2}$$

Then $\mathcal{L}^X$ commutes with G^{σ}.

Let $(\cdot,\cdot)$ denote the Hermitian metric on $\Lambda^{\cdot}(T^*X \oplus N^*) \otimes F$ associated with B^* and g^F. The Cartan involution θ acts on $\widehat{\mathcal{X}}$, so that

$$\theta(Y^{TX} + Y^N) = -Y^{TX} + Y^N. \tag{3.6.3}$$

Let $dv_{\widehat{\mathcal{X}}}$ be the volume form on $\widehat{\mathcal{X}}$ coming from the Riemann metric on X and the Euclidean scalar product on $TX \oplus N$.

Let $\eta(\cdot,\cdot)$ be the Hermitian form on the space of smooth compactly supported sections of $\widehat{\pi}^*(\Lambda^{\cdot}(T^*X \oplus N^*) \otimes F)$ over $\widehat{\mathcal{X}}$,

$$\eta(s,s') = \int_{\widehat{\mathcal{X}}} (s\circ\theta, s')dv_{\widehat{\mathcal{X}}}. \tag{3.6.4}$$

As in [B11, Sections 2.12 and 2.13], we put

$$\mathcal{L}^X_b = -\frac{1}{2}\widehat{D}^{\mathfrak{g},X,2} + \frac{1}{2}\mathfrak{D}^{X,2}_b. \tag{3.6.5}$$

The operator $\mathcal{L}^X_b$ acts on $C^{\infty}(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^{\cdot}(T^*X \oplus N^*) \otimes F))$.

The following result is taken from [B11, Theorem 2.13.2].

Theorem 3.6.2. *The operator $\mathcal{L}^X_b$ is formally self-adjoint with respect to $\eta(\cdot,\cdot)$. Moreover, $\frac{\partial}{\partial t} + \mathcal{L}^X_b$ is hypoelliptic.*

The operator $\mathcal{L}^X_b$ is called the hypoelliptic Laplacian associated with (G,K). By [B11, equation (2.13.5)], for $b>0$, we have

$$\begin{aligned}\mathcal{L}^X_b = {}& \frac{1}{2}|[Y^N, Y^{TX}]|^2 + \frac{1}{2b^2}(-\Delta^{TX\oplus N} + |Y|^2 - m - n) + \frac{N^{\Lambda^{\cdot}(T^*X\oplus N^*)}}{b^2}\\ &+\frac{1}{b}\bigg(\nabla^{\mathcal{H}}_{Y^{TX}} + \widehat{c}(\mathrm{ad}(Y^{TX})) - c(\mathrm{ad}(Y^{TX}) + i\theta\mathrm{ad}(Y^N))\\ &-i\rho^E(Y^N)\bigg).\end{aligned} \tag{3.6.6}$$

By [B11, Proposition 2.15.1], we have the identity

$$[\mathfrak{D}^X_b, \mathcal{L}^X_b] = 0. \tag{3.6.7}$$

As in (1.1.14), the left action of G^{σ} on itself induces actions of G^{σ} on $C^{\infty}(X,F)$ and $C^{\infty}(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^{\cdot}(T^*X \oplus N^*) \otimes F))$. Since σ preserves the bilinear form B and the

Cartan decomposition (1.1.1), we find that G^σ commutes with $\widehat{D}^{\mathfrak{g},X}$ and $\mathfrak{D}_b^X$, so that $\mathcal{L}_b^X$ commutes with G^σ.

3.7. **The hypoelliptic heat kernel.** Let A be a self-adjoint element of $\mathrm{End}(E)$ which commutes with the action of K^σ. Then A descends to a self-adjoint parallel section of $\mathrm{End}(F)$ which commutes with the left action of G^σ.

Definition 3.7.1. Let $\mathcal{L}_A^X$ be the operator acting on $C^\infty(X,F)$,

$$\mathcal{L}_A^X = \mathcal{L}^X + A. \tag{3.7.1}$$

From [B11, Section 4.4], for $t>0$, the operator $\exp(-t\mathcal{L}_A^X)$ has a smooth kernel $p_t^X(x,x')$ with respect to the volume element dx on X.

The section A lifts to $\widehat{\mathcal{X}}$. As in [B11, eq.(4.5.1)], let $\mathcal{L}_{A,b}^X$ be the differential operator acting on $C^\infty(\widehat{\mathcal{X}}, \hat{\pi}^*(\Lambda^\cdot(T^*X\oplus N^*)\otimes F))$ given by

$$\mathcal{L}_{A,b}^X = \mathcal{L}_b^X + A. \tag{3.7.2}$$

In [B11, Sections 4.5 and 11.8], Bismut showed that the heat operator $\exp(-t\mathcal{L}_{A,b}^X)$ is well-defined for $b>0, t>0$ with a smooth kernel $q_{b,t}^X((x,Y),(x',Y'))$. By [B11, Section 11.8], given $b>0$, $t>0$, $q_{b,t}^X((x,Y),(x',Y'))$ is rapidly decreasing together with its derivatives in the variables (x',Y'), the decay in the variable x' is measured via $d(x,x')$.

Moreover, using the same argument as in [BGV04, Theorem 2.48], we can get a Duhamel's formula for $q_{b,t}^X((x,Y),(x',Y'))$,

$$\begin{split}\frac{\partial}{\partial b} q_{b,t}^X((x,Y),(x',Y')) = \\ -\int_0^t \Bigg(\int_{(x'',Y'')\in\widehat{\mathcal{X}}} q_{b,t-s}^X((x,Y),(x'',Y'')) \\ (\frac{\partial \mathcal{L}_{A,b}^X}{\partial b})_{(x'',Y'')} q_{b,s}^X((x'',Y''),(x',Y'))dx''dY''\Bigg) ds.\end{split} \tag{3.7.3}$$

Equivalently, we also have this Duhamel's formula written in operator form,

$$\frac{\partial}{\partial b}\exp(-t\mathcal{L}_{A,b}^X) = -\int_0^t \exp(-(t-s)\mathcal{L}_{A,b}^X)\frac{\partial \mathcal{L}_{A,b}^X}{\partial b}\exp(-s\mathcal{L}_{A,b}^X)ds. \tag{3.7.4}$$

As in [B11, Sections 4.5], let $\mathbf{P}$ be the projection from $\Lambda^\cdot(T^*X\oplus E^*)\otimes F$ on $\Lambda^0(T^*X\oplus E^*)\otimes F$. For $t>0$ and $(x,Y),(x',Y')\in\widehat{\mathcal{X}}$, put

$$q_{0,t}^X((x,Y),(x',Y')) = \mathbf{P}p_t^X(x,x')\pi^{-(m+n)/2}\exp(-\frac{1}{2}\big(|Y|^2+|Y'|^2\big))\mathbf{P}. \tag{3.7.5}$$

We recall a result established in [B11, Theorem 4.5.2 and Chapter 14].

Theorem 3.7.2. *Given $M\geq\epsilon>0$, there exist $C,C'>0$ such that for $0<b\leq M, \epsilon\leq t\leq M, (x,Y),(x',Y')\in\widehat{\mathcal{X}}$,*

$$\big|q_{b,t}^X((x,Y),(x',Y'))\big| \leq C\exp\big(-C'(d^2(x,x')+|Y|^2+|Y'|^2)\big). \tag{3.7.6}$$

Moreover, as $b \to 0$, we have the convergence in any C^k-norm on any compact subset,

$$q^X_{b,t}((x,Y),(x',Y')) \to q^X_{0,t}((x,Y),(x',Y')). \tag{3.7.7}$$

4. The twisted orbital integrals

This section is devoted to give a geometric interpretation for the twisted orbital integrals associated with a semisimple element in $\widetilde{G}$.

Recall that if $\sigma \in \Sigma$, Σ^σ is the closed subgroup of Σ generated by σ, and that

$$G^\sigma = G \rtimes \Sigma^\sigma, \quad K^\sigma = K \rtimes \Sigma^\sigma. \tag{4.0.1}$$

Let $\gamma \in G$ be such that $\gamma\sigma$ is a semisimple element in G^σ. By subsection 1.4, we may and we will suppose that $\gamma\sigma$ is such that

$$\begin{aligned} &\gamma = e^a k^{-1}, \qquad && \mathrm{Ad}(k)a = \sigma a, \\ &a \in \mathfrak{p}, && k \in K. \end{aligned} \tag{4.0.2}$$

In this section, we always assume that (E, ρ^E) is a K^σ-representation.

This section is organized as follows. In subsection 4.1, we introduce an algebra $\mathcal{Q}^\sigma$ of invariant kernels on X.

In subsection 4.2, we introduce a geometric formalism of the twisted orbital integrals associated with $\gamma\sigma$. We show that they vanish on commutators.

In subsection 4.3, when replacing F by $\Lambda^\cdot(T^*X \oplus N^*) \otimes C^{\infty,b}(TX \oplus N, \mathbb{R}) \otimes F$, we introduce the associated algebra $\mathfrak{Q}^\sigma$ of invariant kernels, and we obtain the corresponding twisted orbital integrals.

In subsection 4.4, we introduce the twisted orbital integrals for elliptic heat kernel and hypoelliptic heat kernel. We show that they coincide.

Finally, in subsection 4.5, we rederive a twisted version of Selberg trace formula for the locally symmetric space Z.

4.1. An algebra of invariant kernels on X.

In [B11, Chapter 4], a vector space $\mathcal{Q}$ of continuous invariant kernels was defined. We recall its definition and some properties as follows.

Definition 4.1.1. Let $\mathcal{Q}$ be the vector space of continuous kernels $q \in C(G, \mathrm{End}(E))$ satisfying the following two properties:

— There exist $C, C' > 0$, such that

$$|q(g)| \leq C \exp(-C' d^2(p1, pg)), \ \forall\, g \in G. \tag{4.1.1}$$

— For $k, k' \in K$, we have

$$q(kgk') = \rho^E(k) q(g) \rho^E(k'). \tag{4.1.2}$$

Recall that dk is the normalized Haar measure on K and that $dg = dxdk$ is a bi-invariant Haar measure on G.

For $q \in \mathcal{Q}$ and $g, g' \in G$, put

$$q(g, g') = q(g^{-1}g') \in \mathrm{End}(E). \tag{4.1.3}$$

Let $C^b(G, E)$ be the set of bounded contiuous sections of E on G, and let $C^b_K(G, E)$ be the set of K-invariant sections in $C^b(G, E)$. For $s \in C^b_K(G, E)$, put

$$(Qs)(g) = \int_G q(g, g') s(g') dg'. \tag{4.1.4}$$

By (1.5.34), (1.5.35), (1.5.36) and the condition in (4.1.1), the integral (4.1.4) is well-defined. Moreover, the conditions (4.1.1) and (4.1.2) guarantee that $Qs \in C^b_K(G,E)$. Then $q \in \mathcal{Q}$ defines an integral operator Q acting on $C^b(X,F)$ commuting with the action of G on F. Let $q(x,x') \in \mathrm{Hom}(F_{x'},F_x)$ be the corresponding continuous kernel on $X \times X$, which is just the descent of $q(g,g')$ to $X \times X$.

On $\mathcal{Q}$, the composition of two kernels is given by

$$q * q'(g) = \int_G q(g')q'((g')^{-1}g)dg', \tag{4.1.5}$$

which defines the operator QQ'. Then $(\mathcal{Q},*)$ becomes an associative algebra.

Put $\sigma^E = \rho^E(\sigma) \in \mathrm{Aut}(E)$.

Definition 4.1.2. Let $\mathcal{Q}^\sigma$ be the vector subspace of the $q \in \mathcal{Q}$ such that

$$q(\sigma(g)) = \sigma^E q(g)(\sigma^E)^{-1} \in \mathrm{End}(E). \tag{4.1.6}$$

Equivalently, for any $x,x' \in X$,

$$q^X(\sigma(x),\sigma(x')) = \sigma q^X(x,x')\sigma^{-1} \in \mathrm{Hom}(F_{\sigma(x')},F_{\sigma(x)}). \tag{4.1.7}$$

Then $\mathcal{Q}^\sigma$ is the subalgebra of $\mathcal{Q}$ consisting all the kernels commuting with the action of σ on $C^\infty(X,F)$.

Also we can extend $q \in \mathcal{Q}^\sigma$ to a continuous map $\tilde{q} \in C(G^\sigma,\mathrm{End}(E))$ by,

$$\tilde{q}(g\mu) = q(g)\rho^E(\mu) \in \mathrm{End}(E)\ ,\ g \in G, \mu \in \Sigma^\sigma. \tag{4.1.8}$$

Then we lift it to a continuous kernel defined on $G^\sigma \times G^\sigma$,

$$\tilde{q}(g\mu,h\mu') = \tilde{q}((g\mu)^{-1}h\mu') \in \mathrm{End}(E). \tag{4.1.9}$$

Then by (4.1.1), (4.1.2), for $\tilde{g} \in G^\sigma$, $\tilde{k} \in K^\sigma$, we have

$$\begin{aligned}&|\tilde{q}(\tilde{g})| \leq C\exp(-C'd^2(p1,p\tilde{g})),\\ &\tilde{q}(\tilde{k}\tilde{g}) = \rho^E(\tilde{k})\tilde{q}(\tilde{g}),\ \tilde{q}(\tilde{g}\tilde{k}) = \tilde{q}(\tilde{g})\rho^E(\tilde{k}).\end{aligned} \tag{4.1.10}$$

Recall that $d\mu$ is the normalized Haar measure of Σ^σ and that $d\tilde{g} = dgd\mu$ is a bi-invariant Haar measure on G^σ. Then the operator Q defined above is also defined by the kernel $\tilde{q}$, i.e., if $s \in C^b_{K^\sigma}(G^\sigma,E)$,

$$(Qs)(\tilde{g}) = \int_{G^\sigma} \tilde{q}(\tilde{g},\tilde{g})s(\tilde{g}')d\tilde{g}'. \tag{4.1.11}$$

4.2. **Twisted orbital integrals.** If $q \in \mathcal{Q}^\sigma$, and if $x \in X$, then $\gamma\sigma q(x,\gamma\sigma(x)) \in \mathrm{End}(F_{\gamma\sigma(x)})$, so that $\mathrm{Tr}^F[\gamma\sigma q(x,\gamma\sigma(x))]$ is well-defined.

Let $h(y)$ be a compactly supported bounded measurable function on $X(\gamma\sigma)$. Then we have an analogue of [B11, Theorem 4.2.1] as follows,

Proposition 4.2.1. *The function* $\mathrm{Tr}^F[\gamma\sigma q(x,\gamma\sigma(x))]h(p_{\gamma\sigma}x)$ *is integrable on* X. *We have the identity,*

$$\begin{aligned}&\int_X \mathrm{Tr}^F[\gamma\sigma q(x,\gamma\sigma(x))]h(p_{\gamma\sigma}x)dx\\ &= \int_{\mathfrak{p}^\perp(\gamma\sigma)} \mathrm{Tr}^E[\sigma^E q(e^{-f}\gamma e^{\sigma f})]r(f)df \int_{X(\gamma\sigma)} h(y)dy.\end{aligned} \tag{4.2.1}$$

Proof. We adapt the proof of [B11, Theorem 4.2.1] to prove our proposition. By Proposition 1.5.2 and (4.1.1), we have

$$\begin{aligned} |\sigma^E q(e^{-f}\gamma e^{\sigma f})| &= |\tilde{q}(e^f, \gamma\sigma e^f)| \\ &\leq C\exp(-C' d^2(pe^f, p\gamma\sigma e^f)) \\ &= C\exp(-C' d^2(pe^f, p\gamma\sigma e^f)) \\ &= C\exp(-C' d^2_{\gamma\sigma}(\rho_{\gamma\sigma}(1,f))) \\ &\leq C\exp(-C'(|a| + c_{\gamma\sigma}|f|)^2). \end{aligned} \tag{4.2.2}$$

By (1.5.38), (4.2.2), the function $\mathrm{Tr}^E[\sigma^E q(e^{-f}\gamma e^{\sigma f})]r(f)$ is integrable in $f \in \mathfrak{p}^\perp(\gamma\sigma)$. Using Fubini's theorem, we get (4.2.1). This completes the proof of our proposition. □

By (1.5.24), (1.5.25), and using the fact that the Haar measures of K, $K^0(\gamma\sigma)$, $K(\gamma\sigma)$ have volume 1, we have

$$\begin{aligned} \int_{\mathfrak{p}^\perp(\gamma\sigma)} \mathrm{Tr}^E[\sigma^E q(e^{-f}\gamma e^{\sigma f})]r(f)df &= \int_{Z^0(\gamma\sigma)\backslash G} \mathrm{Tr}^E[\sigma^E q(v^{-1}\gamma\sigma(v))]dv^0 \\ &= \int_{Z(\gamma\sigma)\backslash G} \mathrm{Tr}^E[\sigma^E q(v^{-1}\gamma\sigma(v))]dv. \end{aligned} \tag{4.2.3}$$

If we use the kernel $\tilde{q}$ on G^σ and the related measures defined in subsection 1.5, we also have,

$$\int_{\mathfrak{p}^\perp(\gamma\sigma)} \mathrm{Tr}^E[\sigma^E q(e^{-f}\gamma e^{\sigma f})]r(f)df = \int_{Z^\sigma(\gamma\sigma)\backslash G^\sigma} \mathrm{Tr}^E[\tilde{q}(\tilde{v}^{-1}\gamma\sigma\tilde{v}]d\tilde{v}^\sigma. \tag{4.2.4}$$

Remark 4.2.2. If E is a representation of $\widetilde{K}$ and if q commutes with the action of Σ on $C^\infty(X,F)$, then we can extend $\tilde{q}$ to $\widetilde{G}$. If $d\tilde{v}$ is the corresponding measure on $\widetilde{Z}(\gamma\sigma)\backslash\widetilde{G}$, then we get

$$\int_{\mathfrak{p}^\perp(\gamma\sigma)} \mathrm{Tr}^E[\sigma^E q(e^{-f}\gamma e^{\sigma f})]r(f)df = \int_{\widetilde{Z}(\gamma\sigma)\backslash\widetilde{G}} \mathrm{Tr}^E[\tilde{q}(\tilde{v}^{-1}\gamma\sigma\tilde{v}]d\tilde{v}. \tag{4.2.5}$$

Let $[\gamma\sigma]$ denote the conjugation class of $\gamma\sigma$ in G^σ.

Definition 4.2.3. We define the orbital integral associated with $\gamma\sigma$ for $q \in \mathcal{Q}^\sigma$ by the formula,

$$\begin{aligned} \mathrm{Tr}^{[\gamma\sigma]}[Q] &= \int_{Z^0(\gamma\sigma)\backslash G} \mathrm{Tr}^E[\sigma^E q(v^{-1}\gamma\sigma(v))]dv^0 \\ &= \int_{\mathfrak{p}^\perp(\gamma\sigma)} \mathrm{Tr}^E[\sigma^E q(e^{-f}\gamma e^{\sigma f})]r(f)df. \end{aligned} \tag{4.2.6}$$

Integrals like (4.2.3) - (4.2.6) are called twisted orbital integrals.

The same arguments in [B11, Page 80] show that $\mathrm{Tr}^{[\gamma\sigma]}[Q]$ only depends on the conjugacy class of $\gamma\sigma$ in G^σ. Indeed, if $\tilde{h} \in G^\sigma$, the map $g \in G^\sigma \to C(\tilde{h})g \in G^\sigma$

induces a map $Z^\sigma(\gamma\sigma)\backslash G^\sigma \to Z^\sigma(C(\tilde{h})(\gamma\sigma))\backslash G^\sigma$. Since $d\tilde{g}$ is bi-invariant on G^σ, $C(\tilde{h})$ maps the volume element $d\tilde{v}^\sigma$ of $Z^\sigma(\gamma\sigma)\backslash G^\sigma$ to the corresponding volume element on $Z^\sigma(C(\tilde{h})(\gamma\sigma))\backslash G^\sigma$. Then the integral in the right-hand side of (4.2.4) remains the same if we replace $\gamma\sigma$ by $C(\tilde{h})(\gamma\sigma)$.

Remark 4.2.4. Following Remark 4.2.2, if E is a representation of $\widetilde{K}$ and q commutes with the action of Σ on $C^\infty(X,F)$, then $\mathrm{Tr}^{[\gamma\sigma]}[Q]$ only depends on the conjugacy class of $\gamma\sigma$ in $\widetilde{G}$.

The following proposition extends [B11, Theorem 4.2.3].

Proposition 4.2.5. *For $Q, Q' \in \mathcal{Q}^\sigma$, we have*

$$\mathrm{Tr}^{[\gamma\sigma]}[[Q,Q']] = 0. \tag{4.2.7}$$

Equivalently, $\mathrm{Tr}^{[\gamma\sigma]}[\cdot]$ *is a trace on the algebra $\mathcal{Q}^\sigma$.*

Proof. Using the formalism in (4.2.4) and Definition 4.2.3, we can adapt the proof of [B11, Theorem 4.2.3] to prove our proposition.

Let $\delta_{\gamma\sigma}$ be the current on G^σ so that

$$\int_{G^\sigma} f\delta_{\gamma\sigma} = \int_{Z^\sigma(\gamma\sigma)\backslash G^\sigma} f((\tilde{v})^{-1}\gamma\sigma\tilde{v})d\tilde{v}^\sigma. \tag{4.2.8}$$

Since $d\tilde{v}^\sigma$ is invariant under the right-action of G^σ on $Z^\sigma(\gamma\sigma)\backslash G^\sigma$, $\delta_{\gamma\sigma}$ is invariant by conjugation. If $q \in \mathcal{Q}^\sigma$, let $\tilde{q}$ be the function on G^σ given in (4.1.8). Then by (4.2.4), (4.2.6),

$$\mathrm{Tr}^{[\gamma\sigma]}[Q] = \int_{G^\sigma} \mathrm{Tr}^E[\tilde{q}]\delta_{\gamma\sigma} = \mathrm{Tr}^E[\tilde{q} * \delta_{\gamma\sigma}(1)]. \tag{4.2.9}$$

Also we have

$$QR_{(\gamma\sigma)^{-1}} = R_{(\gamma\sigma)^{-1}}Q. \tag{4.2.10}$$

As in (4.1.8) - (4.1.11), the current $\delta_{(\gamma\sigma)^{-1}}$ on G^σ defines an operator $R_{(\gamma\sigma)^{-1}}$. Then we can rewrite (4.2.9) as

$$\mathrm{Tr}^{[\gamma\sigma]}[Q] = \mathrm{Tr}^{[1]}[QR_{(\gamma\sigma)^{-1}}]. \tag{4.2.11}$$

By (4.2.10), (4.2.11), we get

$$\mathrm{Tr}^{[\gamma\sigma]}[[Q,Q']] = \mathrm{Tr}^{[1]}[[Q,Q']R_{(\gamma\sigma)^{-1}}] = \mathrm{Tr}^{[1]}[[Q,Q'R_{(\gamma\sigma)^{-1}}]] = 0. \tag{4.2.12}$$

This completes the proof of our proposition. □

4.3. **Infinite dimensional orbital integrals.** Recall that $\widehat{\pi}: \widehat{\mathcal{X}} \to X$ is the total space of $TX \oplus N$. In the sequel, if V is a real vector space and if E is a complex vector space, we will denote by $V \otimes E$ the complex vector space $V \otimes_{\mathbb{R}} E$. We use the same convention for the tensor product of vector bundles.

Let $dY^{\mathfrak{p}}$, $dY^{\mathfrak{k}}$ be the volume elements on the Euclidean vector spaces $\mathfrak{p}$, $\mathfrak{k}$. Since K^σ acts isometrically on $\mathfrak{p}$ and $\mathfrak{k}$, these volume elements are $K^\sigma-$invariant. Then $dY = dY^{\mathfrak{p}} dY^{\mathfrak{k}}$ is a volume element on $\mathfrak{g}$ which is $G^\sigma-$invariant. Let dY^{TX}, dY^N, dY be the corresponding volume elements on the fibres of $TX, N, TX \oplus N$ over X.

Let $C^{\infty,b}(\mathfrak{g}, \mathbb{R})$ be the vector space of real valued smooth bounded functions on $\mathfrak{g}$. We replace the finite-dimensional vector space by the infinite dimensional space $\mathcal{E} = \Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes C^{\infty,b}(\mathfrak{g}, \mathbb{R}) \otimes E$ with the natural group action of K^σ. Then the vector bundle F on X is replaced by

$$\mathcal{F} = \Lambda^\cdot(T^*X \oplus N^*) \otimes C^{\infty,b}(TX \oplus N, \mathbb{R}) \otimes F. \tag{4.3.1}$$

Let $C^b(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\cdot(T^*X \oplus N^*) \otimes F))$ be the vector space of continuous bounded sections of $\widehat{\pi}^*(\Lambda^\cdot(T^*X \oplus N^*) \otimes F)$ over $\widehat{\mathcal{X}}$.

The group K^σ acts on $C^b(G^\sigma \times \mathfrak{g}, \Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$, so that if $s \in C^b(G^\sigma \times \mathfrak{g}, \Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$ then for $\tilde{k} \in K^\sigma$

$$(\tilde{k}s)(\tilde{g}, Y) = \rho^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E}(\tilde{k}) s(\tilde{g}\tilde{k}, \mathrm{Ad}(\tilde{k}^{-1})Y). \tag{4.3.2}$$

Let $C^b_{K^\sigma}(G^\sigma \times \mathfrak{g}, \Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$ be the vector space of K^σ-invariant continuous bounded function on $G^\sigma \times \mathfrak{g}$ with values in $\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E$. Then we have

$$\begin{aligned} C^b(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\cdot(T^*X \oplus N^*) \otimes F)) &= C^b_{K^\sigma}(G^\sigma \times \mathfrak{g}, \Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E) \\ &= C^b_K(G \times \mathfrak{g}, \Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E). \end{aligned} \tag{4.3.3}$$

Definition 4.3.1. Let $\mathfrak{Q}^\sigma$ be the vector space of continuous kernels $q(g, Y, Y')$ defined on $G \times \mathfrak{g} \times \mathfrak{g}$ with values in $\mathrm{End}(\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$ such that

— If $g \in G, k, k' \in K, Y, Y' \in \mathfrak{g}$, then

$$\begin{aligned} &q(kgk', Y, Y') \\ &= \rho^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E}(k) q(g, \mathrm{Ad}(k^{-1})Y, \mathrm{Ad}(k')Y') \rho^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E}(k'). \end{aligned} \tag{4.3.4}$$

— If $\sigma^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E} = \rho^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E}(\sigma) \in \mathrm{Aut}(\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E)$, then

$$q(\sigma(g), \sigma Y, \sigma Y') = \sigma^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E} q(g, Y, Y') (\sigma^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E})^{-1}. \tag{4.3.5}$$

— There exist $C, C' > 0$ such that

$$|q(g, Y, Y')| \leq C \exp(-C'(d^2(p1, pg) + |Y|^2 + |Y'|^2)). \tag{4.3.6}$$

We will denote $\mathfrak{Q}^{\sigma,\infty}$ the subspace of $\mathfrak{Q}^\sigma$ consisting of smooth kernels.

Since $\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E$ is a representation of K^σ, as in (4.1.8), we can extend q to a kernel $\tilde{q}$ defined on $G^\sigma \times \mathfrak{g} \times \mathfrak{g}$,

$$\tilde{q}(g\mu, Y, Y') = q(g, Y, \mu Y') \rho^{\Lambda^\cdot(\mathfrak{p}^* \oplus \mathfrak{k}^*) \otimes E}(\mu), \ Y, Y' \in \mathfrak{g}, \ \mu \in \Sigma^\sigma. \tag{4.3.7}$$

If $q \in \mathfrak{Q}^\sigma$, put $q((g,Y),(g',Y')) = q(g^{-1}g', Y, Y')$. If $s \in C^b_K(G\times\mathfrak{g}, \Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E)$, put

$$(Qs)(g,Y) = \int_{G\times\mathfrak{g}} q((g,Y),(g',Y'))s(g',Y')dY'dg'. \tag{4.3.8}$$

By (4.3.4), (4.3.6), Q is an operator acting on $C^b_K(G\times\mathfrak{g}, \Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E)$. Recall that the action of σ is given by

$$(\sigma s)(g,Y) = \sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} s(\sigma^{-1}(g), \sigma^{-1}Y). \tag{4.3.9}$$

Then (4.3.5) is equivalent to $Q\sigma = \sigma Q$.

Equivalently, the operator Q acts on $C^b(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda^\cdot(T^*X\oplus N^*)\otimes F))$ with kernel $q((x,Y),(x',Y'))$.

By [B11, Proposition 4.3.2] and using the fact that σ preserves $dxdY$, $\mathfrak{Q}^\sigma$ is an associative algebra with respect to the composition of operators. Let $[\cdot,\cdot]$ be the supercommutator on $\mathfrak{Q}^\sigma$ defined by the $\mathbb{Z}_2-$graded structure of $\mathrm{End}(\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E)$, and let $\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\cdot]$ be the supertrace on $\mathrm{End}(\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E)$.

If $g\in G$, let $q(g)$ be the operator on $\mathcal{E}$ defined by the kernel $q(g,Y,Y')$. Let $\sigma^{\mathcal{E}} \in \mathrm{End}(\mathcal{E})$ denote the action of σ on $\mathcal{E}$.

Then for $g\in G$, $\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))$ acting on $\mathcal{E}$ is given by the continuous kernel $\sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y')$ on $\mathfrak{g}\times\mathfrak{g}$. When restricting to the diagonal, this kernel is also continuous. By (4.3.6), $\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y)]$ is integrable on $Y\in\mathfrak{g}$.

If $\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))$ is trace class, with the decay condition and by [Duf72, Proposition 3.1.1], we get

$$\begin{aligned}\mathrm{Tr_s}^{\mathcal{E}}[\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))] = \\ \int_{\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y)]dY.\end{aligned} \tag{4.3.10}$$

Remark 4.3.2. A sufficient condition for our operator to be a trace class is that the kernel together with its derivatives in Y, Y' of arbitrary orders lie in the Schwartz space of $\mathfrak{g}\times\mathfrak{g}$.

Using (4.3.6), there exists $C_{\gamma\sigma}>0$ such that

$$\begin{aligned}\left|\int_{\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(g^{-1}\gamma\sigma(g), \sigma^{-1}Y, Y)]dY\right| \\ \leq C_{\gamma\sigma}\exp(-C'd^2(pg, \gamma\sigma pg)).\end{aligned} \tag{4.3.11}$$

By Proposition 1.5.2, along the normal fiber of $X(\gamma\sigma)$, the displacement function $d_{\gamma\sigma}$ is increasing at least linearly with respect to the norm of normal vectors, the same arguments in Proposition 4.2.1 show that the left-hand side of (4.3.11) is integrable on $\mathfrak{p}^\perp(\gamma\sigma)$. Then we have the analogue of (4.2.1), if $h(y)$ is a compactly supported bounded measurable function on $X(\gamma\sigma)$,

$$\begin{aligned}
&\int_{\widehat{\mathcal{X}}} \mathrm{Tr}^F[\gamma\sigma q((x,Y),\gamma\sigma(x,Y))]h(p_{\gamma\sigma}x)dxdY\\
&= \int_{X(\gamma\sigma)} h(y)dy\\
&\times \int_{\mathfrak{p}^\perp(\gamma\sigma)\times\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(e^{-f}\gamma e^{\sigma f},Y,\sigma Y)]r(f)dfdY.
\end{aligned} \tag{4.3.12}$$

Definition 4.3.3. We define $\mathrm{Tr_s}^{[\gamma\sigma]}[Q]$ as in (4.2.6) for $Q \in \mathfrak{Q}^\sigma$, i.e.,

$$\begin{aligned}
&\mathrm{Tr_s}^{[\gamma\sigma]}[Q]\\
&= \int_{(Z^0(\gamma\sigma)\backslash G)\times\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(v^{-1}\gamma\sigma(v),Y,\sigma Y)]dvdY\\
&= \int_{\mathfrak{p}^\perp(\gamma\sigma)\times\mathfrak{g}} \mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E}[\sigma^{\Lambda^\cdot(\mathfrak{p}^*\oplus\mathfrak{k}^*)\otimes E} q(e^{-f}\gamma e^{\sigma f},Y,\sigma Y)]r(f)dfdY.
\end{aligned} \tag{4.3.13}$$

Expressions such as (4.3.13) are called twisted orbital supertraces.

If $\sigma^{\mathcal{E}} q(g^{-1}\gamma\sigma(g))$ is trace class for any $g \in G$, using (4.3.10), we can rewrite (4.3.13) as

$$\begin{aligned}
\mathrm{Tr_s}^{[\gamma\sigma]}[Q] &= \int_{Z^0(\gamma\sigma)\backslash G} \mathrm{Tr_s}^{\mathcal{E}}[\sigma^{\mathcal{E}} q(v^{-1}\gamma\sigma(v))]dv\\
&= \int_{\mathfrak{p}^\perp(\gamma\sigma)} \mathrm{Tr}^{\mathcal{E}}[\sigma^{\mathcal{E}} q(e^{-f}\gamma e^{\sigma f})]r(f)df.
\end{aligned} \tag{4.3.14}$$

Proposition 4.3.4. *If $Q, Q' \in \mathfrak{Q}^\sigma$, then*

$$\mathrm{Tr_s}^{[\gamma\sigma]}[[Q,Q']] = 0. \tag{4.3.15}$$

Proof. By the above constructions, the proof of our proposition is just an easy modification of the proof of Proposition 4.2.5. This extends [B11, Theorem 4.3.4]. □

4.4. **A fundamental identity.** Recall that the operators $\mathcal{L}_A^X$, $\mathcal{L}_{A,b}^X$ are defined in subsection 3.6.

Proposition 4.4.1. *For any $t > 0$, $p_t^X \in \mathcal{Q}^\sigma$.*

Proof. This follows from [B11, Proposition 4.4.2] and from the fact that $\mathcal{L}^X$ commutes with the left action of σ. □

It follows from subsection 4.2 and Proposition 4.4.1 that for $t > 0$, the twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^X)]$ is well-defined.

Using (3.7.6) and the fact that $\mathcal{L}_{A,b}^X$ commutes with σ. If $b > 0$, $t > 0$, then $q_{b,t}^X \in \mathfrak{Q}^{\sigma,\infty}$. By subsection 4.3, $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_{A,b}^X)]$ is well-defined.

As an extension of [B11, Theorem 4.6.1], we have a fundamental identity as follows.

Theorem 4.4.2. *For any $b > 0, t > 0$, the following identity holds,*

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = \mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_A)]. \tag{4.4.1}$$

Proof. By (3.7.3), (4.3.13) and using Proposition 4.3.4, we get

$$\frac{\partial}{\partial b}\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = -t\mathrm{Tr_s}^{[\gamma\sigma]}[\frac{\partial}{\partial b}\mathcal{L}^X_{A,b}\exp(-t\mathcal{L}^X_{A,b})]. \tag{4.4.2}$$

As in [B11, eq. (4.6.4)-(4.6.7)], we have

$$\frac{\partial}{\partial b}\mathcal{L}^X_{A,b} = \frac{1}{2}[\mathfrak{D}^X_b, \frac{\partial}{\partial b}\mathfrak{D}^X_b], \ \ [\mathfrak{D}^X_b, \mathcal{L}^X_{A,b}] = 0. \tag{4.4.3}$$

By (4.4.2), we have

$$\begin{aligned}\frac{\partial}{\partial b}\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] &= -\frac{t}{2}\mathrm{Tr_s}^{[\gamma\sigma]}\big[[\mathfrak{D}^X_b, \frac{\partial}{\partial b}\mathfrak{D}^X_b]\exp(-t\mathcal{L}^X_{A,b})\big] \\ &= -\frac{t}{2}\mathrm{Tr_s}^{[\gamma\sigma]}\big[[\mathfrak{D}^X_b, \frac{\partial}{\partial b}\mathfrak{D}^X_b\exp(-t\mathcal{L}^X_{A,b})]\big].\end{aligned} \tag{4.4.4}$$

By Proposition 4.3.4 and (4.4.2) - (4.4.4), we get

$$\frac{\partial}{\partial b}\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = 0. \tag{4.4.5}$$

It is now enough to prove that

$$\lim_{b\to 0}\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = \mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_A)]. \tag{4.4.6}$$

By (1.5.11) and Theorem 3.7.2, given $t > 0$, there exist $C, C' > 0$ such that for $0 < b \le 1, f \in \mathfrak{p}^\perp(\gamma\sigma), Y \in (TX \oplus N)_{e^f p1}$,

$$\left|q^X_{b,t}((e^f p1, Y), \gamma\sigma(e^f p1, Y))\right| \le C\exp\big(-C'(|f|^2 + |Y|^2)\big) \tag{4.4.7}$$

Using (3.7.7), (4.2.6), (4.3.13) and dominated convergence, we get (4.4.6). The proof of our theorem is completed. □

4.5. **A twisted trace formula for Z.** Let Γ be a cocompact discrete subgroup of G. Let $\sigma \in \Sigma$ be such that $\sigma(\Gamma) = \Gamma$. We still assume that the vector bundle F is given by a finite-dimensional representation (E, ρ^E) of K^σ.

Put $Z = \Gamma\backslash X = \Gamma\backslash G/K$. We use the notation in subsection 1.8. Recall that the vector bundles TX, N, F over X descend to the orbifold vector bundles TZ, N, F over Z. Moreover, Σ^σ acts isometrically on Z and its action lifts to an action on the bundles TZ, N, F.

For simplicity, we assume that Γ is torsion free, then Z is a compact smooth manifold. Let $\widehat{\pi} : \widehat{\mathcal{Z}} \to Z$ be the total space of $TZ \oplus N$. Let dz be the volume element of Z induced by the Riemannian metric. We still denote by dg the volume element on $\Gamma\backslash G$ induced by dg.

Let $Q \in \mathcal{Q}^\sigma$ with the associated kernel q. The operator Q descends to an operator Q^Z acting on $C(Z,F)$. Let $q^Z(z,z')$, $z,z' \in Z$ be the continuous kernel of Q^Z over Z. Then by [B11, eq.(4.8.6)],

$$q^Z(z,z') = \sum_{\gamma\in\Gamma} \gamma q^X(\gamma^{-1}z, z') = \sum_{\gamma\in\Gamma} q^X(z,\gamma z')\gamma. \tag{4.5.1}$$

Recall that σ^Z is the induced action of σ on $C^\infty(Z,F)$ as in (1.8.41). Then σQ descends to the operator $\sigma^Z Q^Z$. We also denote by z,z' their arbitrary lifts in X. By (4.5.1), the kernel of $\sigma^Z Q^Z$ is given by

$$(\sigma^Z Q^Z)(z,z') = \sum_{\gamma\in\Gamma} \sigma\gamma q^X(\gamma^{-1}\sigma^{-1}(z), z') = \sum_{\gamma\in\Gamma} \sigma q^X(\sigma^{-1}(z), \gamma z')\gamma. \tag{4.5.2}$$

If $Q \in \mathfrak{Q}^\sigma$ as in subsection 4.3, the analogues of (4.5.1), (4.5.2) still hold.

Since the operators $\mathcal{L}^X_A$, $\mathcal{L}^X_{A,b}$ commutes with G, they descend to operators $\mathcal{L}^Z_A$, $\mathcal{L}^Z_{A,b}$ on Z, $\widehat{\mathcal{Z}}$ respectively. Also for $t>0, b>0$, the operators $\exp(-t\mathcal{L}^Z_A)$, $\exp(-t\mathcal{L}^Z_{A,b})$ are trace class. By (4.4.3), we have the following analogue of [B11, Theorem 4.8.1],

Theorem 4.5.1. *For any $t>0, b>0$,*

$$\mathrm{Tr_s}[\sigma^Z \exp(-t\mathcal{L}^Z_{A,b})] = \mathrm{Tr}[\sigma^Z \exp(-t\mathcal{L}^Z_A)]. \tag{4.5.3}$$

Proof. The differential operator $\mathfrak{D}^X_b$ descends to a differential operator $\mathfrak{D}^Z_b$, so that the analogue of (3.6.7) still holds.

When replacing the twisted orbital supertraces by the standard supertraces, we can establish the analogues of (4.4.2) - (4.4.5), so that

$$\frac{\partial}{\partial b}\mathrm{Tr_s}[\sigma^Z \exp(-t\mathcal{L}^Z_{A,b})] = 0. \tag{4.5.4}$$

Since Z is compact, by Theorem 3.7.2, as $b\to 0$, $\mathrm{Tr_s}[\sigma^Z \exp(-t\mathcal{L}^Z_{A,b})]$ converges to $\mathrm{Tr}[\sigma^Z \exp(-t\mathcal{L}^Z_A)]$. This completes the proof of our theorem. □

By (4.1.7) and (4.5.2), we have the identity,

$$(\sigma^Z Q^Z)(z,z') = \sum_{\gamma\in\Gamma} q^X(z,\gamma\sigma(z'))\gamma\sigma. \tag{4.5.5}$$

By (1.8.40), the kernel $(\sigma^Z Q^Z)(z,z')$ lifts to $G\times G$, so that

$$(\sigma^Z Q^Z)(g,g') = \sum_{\gamma\in\Gamma} q(g^{-1}\gamma\sigma(g'))\sigma^E \in \mathrm{End}(E). \tag{4.5.6}$$

If $\sigma^Z Q^Z$ is of trace-class, then

$$\begin{aligned}\mathrm{Tr}[\sigma^Z Q^Z] &= \int_Z \mathrm{Tr}^F[(\sigma^Z Q^Z)(z,z)]dz \\ &= \int_{\Gamma\backslash G} \mathrm{Tr}^E[(\sigma^Z Q^Z)(g,g)]dg.\end{aligned} \tag{4.5.7}$$

Recall that $\underline{C}$ is the set of twisted conjugacy classes defined in Definition 1.8.2. If $\underline{[\gamma_0]}_\sigma \in \underline{C}$, set

$$q^{X,\underline{[\gamma_0]}_\sigma}(g,g') = \sum_{\gamma\in\underline{[\gamma_0]}_\sigma} q(g^{-1}\gamma\sigma(g'))\sigma^E. \tag{4.5.8}$$

Then (4.5.6) can be rewritten as

$$(\sigma^Z Q^Z)(g,g') = \sum_{\underline{[\gamma]}_\sigma\in\underline{C}} q^{X,\underline{[\gamma]}_\sigma}(g,g'). \tag{4.5.9}$$

The function $q^{X,\underline{[\gamma_0]}_\sigma}(g,g)$ is left $\Gamma-$invariant. Put

$$\mathrm{Tr}[Q^{Z,\underline{[\gamma]}_\sigma}] = \int_Z \mathrm{Tr}[q^{X,\underline{[\gamma_0]}_\sigma}(z,z)]dz. \tag{4.5.10}$$

Then

$$\mathrm{Tr}[\sigma^Z Q^Z] = \sum_{\underline{[\gamma]}_\sigma\in\underline{C}} \mathrm{Tr}[Q^{Z,\underline{[\gamma]}_\sigma}]. \tag{4.5.11}$$

We have

$$\mathrm{Tr}[Q^{Z,\underline{[\gamma]}_\sigma}] = \int_{\Gamma\backslash G} \mathrm{Tr}[q^{X,\underline{[\gamma_0]}_\sigma}(g,g)]dg. \tag{4.5.12}$$

By (1.8.8), we get

$$\mathrm{Tr}[Q^{Z,\underline{[\gamma]}_\sigma}] = \int_{\Gamma\cap Z(\gamma\sigma)\backslash G} \mathrm{Tr}^E[\sigma^E q(g^{-1}\gamma\sigma(g))]dg. \tag{4.5.13}$$

We use the notation in subsection 4.2. By (4.5.13), we obtain

$$\mathrm{Tr}[Q^{Z,\underline{[\gamma]}_\sigma}] = \mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash Z(\gamma\sigma)) \int_{Z(\gamma\sigma)\backslash G} \mathrm{Tr}^E[\sigma^E q(v^{-1}\gamma\sigma(v))]dv \tag{4.5.14}$$

Since Γ is torsion free, $\Gamma\cap Z(\gamma\sigma)$ acts freely on $X(\gamma\sigma)$. By Lemma 1.8.1, $\Gamma\cap Z(\gamma\sigma)\backslash Z(\gamma\sigma)$ is a smooth compact manifold. The compact group $K(\gamma\sigma)$ acts freely on the right on $\Gamma\cap Z(\gamma\sigma)\backslash Z(\gamma\sigma)$, so that

$$\mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash Z(\gamma\sigma)) = \mathrm{Vol}(K(\gamma\sigma))\mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma)) \tag{4.5.15}$$

By (1.5.46) and (4.2.6), we have

$$\frac{\mathrm{Vol}(K(\gamma\sigma))}{\mathrm{Vol}(K)} \int_{Z(\gamma\sigma)\backslash G} \mathrm{Tr}^E[\sigma^E q(v^{-1}\gamma\sigma(v))]dv = \mathrm{Tr}^{[\gamma\sigma]}[Q]. \tag{4.5.16}$$

Then using $\mathrm{Vol}(K)=1$, (4.5.14) can be rewritten as

$$\mathrm{Tr}[Q^{Z,\underline{[\gamma]}_\sigma}] = \mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma))\mathrm{Tr}^{[\gamma\sigma]}[Q]. \tag{4.5.17}$$

Remark 4.5.2. The identity (4.5.3) is compatible with Theorem 4.4.2, (4.5.11) and (4.5.17).

Let $\underline{e}$ be an elliptic class in $\underline{E}$, let $[\underline{e}]$ be all the elliptic classes in $\underline{E}$ which are $C^\sigma-$conjugate to $\underline{e}$ by elements of G. Thus there is $k \in K$ such that for each $\underline{[\gamma]}_\sigma \in [\underline{e}]$, we have $g_\gamma \in G$ satisfying

$$k = C^\sigma(g_\gamma^{-1})\gamma. \tag{4.5.18}$$

Without the risk of confusion, if $g \in G$, we now denote by $[g]$ the corresponding point in $\Gamma\backslash G$. We define a right action of $k\sigma$ on $\Gamma\backslash G$ by

$$R(k\sigma)[g] = [\sigma^{-1}(gk)]. \tag{4.5.19}$$

Let $(\Gamma\backslash G)_{k\sigma}$ be the fixed points set of $R(k\sigma)$ in $\Gamma\backslash G$. Then we have the following identify,

$$(\Gamma\backslash G)_{k\sigma} = \bigcup_{\underline{[\gamma]}_\sigma \in [\underline{e}]} \big(\Gamma \cap Z(\gamma\sigma)\backslash Z(\gamma\sigma)\big) \cdot g_\gamma. \tag{4.5.20}$$

This union is a disjoint union. One can verify that (4.5.20) is a refined version of (1.8.38).

By (4.5.14), we have

$$\sum_{\underline{[\gamma]}_\sigma \in [\underline{e}]} \operatorname{Tr}[Q^{Z,\underline{[\gamma]}_\sigma}] = \operatorname{Vol}((\Gamma\backslash G)_{k\sigma}) \int_{Z(k\sigma)\backslash G} \operatorname{Tr}[\sigma^E q(v^{-1}k\sigma(v))]dv. \tag{4.5.21}$$

A direct computation shows that $Z(k\sigma)$ acts on $(\Gamma\backslash G)_{k\sigma}$ on the right. Recall that $K(k\sigma) = K \cap Z(k\sigma)$. Let $\Delta(k\sigma)$ be the subgroup of $K(k\sigma)$ of the elements that act like identity on $(\Gamma\backslash G)_{k\sigma}$. It is a finite group of $\Gamma \cap K(k\sigma)$. Then we get

$$\operatorname{Vol}((\Gamma\backslash G)_{k\sigma}) = \frac{\operatorname{Vol}(K(k\sigma))}{|\Delta(k\sigma)|}\operatorname{Vol}((\Gamma\backslash G)_{k\sigma}/K(k\sigma)). \tag{4.5.22}$$

Equation (4.5.22) is of special interest in connection with the equivariant index formulas for orbifolds [V96].

5. A formula for semisimple twisted orbital integrals

The purpose of this section is devoted to give a proof of Theorem 0.5.1. The geometric constructions of sections 1 and 4 play an important role in the proof. The proof is partly composed from [B11, Chapter 9].

This section is organized as follows. In subsection 5.1, we introduce an explicit function $J_{\gamma\sigma}$ on $\mathfrak{k}(\gamma\sigma)$.

In subsection 5.2, we give a proof of our geometric formula.

Finally, in subsection 5.3, we extend our formula to general twisted orbital integrals.

5.1. **The function $J_{\gamma\sigma}(Y_0^{\mathfrak{k}})$ on $\mathfrak{k}(\gamma\sigma)$.** Recall that the function $\widehat{A}(x)$ is given by

$$\widehat{A}(x)=\frac{x/2}{\sinh(x/2)}. \tag{5.1.1}$$

Let H be a finite-dimensional Hermitian vector space. If $B\in\operatorname{End}(H)$ is self-adjoint, then $\dfrac{B/2}{\sinh(B/2)}$ is a self-adjoint positive endomorphism. Put

$$\widehat{A}(B)=\det{}^{1/2}\left[\frac{B/2}{\sinh(B/2)}\right]. \tag{5.1.2}$$

Recall that $\widetilde{G}=G\rtimes\Sigma$, $\widetilde{K}=K\rtimes\Sigma$. Let $\widetilde{\gamma}\in\widetilde{G}$ be a semisimple element of the form $\widetilde{\gamma}=e^a\widetilde{k}^{-1}$ with $a\in\mathfrak{p},\widetilde{k}\in\widetilde{K}$ and $\operatorname{Ad}(\widetilde{k})a=a$. We can write $\widetilde{k}^{-1}=k^{-1}\sigma$, $k\in K$, $\sigma\in\Sigma$. Put $\gamma=e^ak^{-1}\in G$ so that $\widetilde{\gamma}=\gamma\sigma$. Recall that Σ^σ is the closed subgroup of Σ generated by σ, and that

$$G^\sigma=G\rtimes\Sigma^\sigma,\quad K^\sigma=K\rtimes\Sigma^\sigma. \tag{5.1.3}$$

We now recall the notation in subsection 1.6. Let $\mathfrak{z}_0=\mathfrak{z}(a)$. Put

$$\mathfrak{p}_0=\ker\operatorname{ad}(a)\cap\mathfrak{p},\quad \mathfrak{k}_0=\ker\operatorname{ad}(a)\cap\mathfrak{k}. \tag{5.1.4}$$

Recall that $\mathfrak{z}_0^\perp$, $\mathfrak{p}_0^\perp$, $\mathfrak{k}_0^\perp$ are the orthogonal spaces to $\mathfrak{z}_0$, $\mathfrak{p}_0$, $\mathfrak{k}_0$ in $\mathfrak{g},\mathfrak{p},\mathfrak{k}$ with respect to B, so that

$$\mathfrak{z}_0=\mathfrak{p}_0\oplus\mathfrak{k}_0,\quad \mathfrak{z}_0^\perp=\mathfrak{p}_0^\perp\oplus\mathfrak{k}_0^\perp. \tag{5.1.5}$$

Also $\mathfrak{z}(\gamma\sigma)$ is a Lie subalgebra of $\mathfrak{z}_0$, and $\mathfrak{p}(\gamma\sigma)$, $\mathfrak{k}(\gamma\sigma)$ are subspaces of $\mathfrak{p}_0$, $\mathfrak{k}_0$ respectively. Recall that $\mathfrak{z}_0^\perp(\gamma\sigma)$, $\mathfrak{p}_0^\perp(\gamma\sigma)$, $\mathfrak{k}_0^\perp(\gamma\sigma)$ are the orthogonal spaces to $\mathfrak{z}(\gamma\sigma)$, $\mathfrak{p}(\gamma\sigma)$, $\mathfrak{k}(\gamma\sigma)$ in $\mathfrak{z}_0,\mathfrak{p}_0,\mathfrak{k}_0$. Then

$$\mathfrak{z}_0^\perp(\gamma\sigma)=\mathfrak{p}_0^\perp(\gamma\sigma)\oplus\mathfrak{k}_0^\perp(\gamma\sigma). \tag{5.1.6}$$

For $Y_0^{\mathfrak{k}}\in\mathfrak{k}(\gamma\sigma)$, $\operatorname{ad}(Y_0^{\mathfrak{k}})$ preserves $\mathfrak{p}(\gamma\sigma),\mathfrak{k}(\gamma\sigma),\mathfrak{p}_0^\perp(\gamma\sigma),\mathfrak{k}_0^\perp(\gamma\sigma)$, and it is an antisymmetric endomorphism with respect to the scalar product.

If $Y_0^{\mathfrak{k}}\in\mathfrak{k}(\gamma\sigma)$, as explained in [B11, pp. 105], the following function in $Y_0^{\mathfrak{k}}$ has a natural square root, which depends analytically on $Y_0^{\mathfrak{k}}$,

$$\det(1-\exp(-i\theta\operatorname{ad}(Y_0^{\mathfrak{k}}))\operatorname{ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}\det(1-\operatorname{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}. \tag{5.1.7}$$

Remark 5.1.1. If $\dim \mathfrak{z}_0^\perp(\gamma\sigma) = 1$, then $\mathrm{Ad}(k^{-1}\sigma)|_{\mathfrak{z}_0^\perp(\gamma\sigma)} = -1$ and $\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{z}_0^\perp(\gamma\sigma)} = 0$, the square root is 2. If $\mathfrak{z}_0^\perp(\gamma\sigma)$ is of dimension 2, if $\mathrm{Ad}(k^{-1})\sigma$ is a rotation of angle ϕ and $\theta\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{z}_0^\perp(\gamma\sigma)}$ acts by an infinitesimal rotation of angle ϕ', such a square root is given by

$$4\sin(\frac{\phi}{2})\sin(\frac{\phi-\sqrt{-1}\phi'}{2}). \tag{5.1.8}$$

We will denote the above square root by

$$\Big[\det(1-\exp(-i\theta\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}\Big]^{1/2}. \tag{5.1.9}$$

If $Y_0^\mathfrak{k} = 0$, then this square root has the value $\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}$.

In (5.1.9), we may as well replace $\mathfrak{z}_0^\perp(\gamma\sigma)$ by $\mathfrak{p}_0^\perp(\gamma\sigma)$ or $\mathfrak{k}_0^\perp(\gamma\sigma)$, where θ acts as -1 or 1. Then the following function $A(Y_0^\mathfrak{k})$ has a natural square root that is analytic in $Y_0^\mathfrak{k} \in \mathfrak{k}(\gamma\sigma)$,

$$\begin{aligned} A(Y_0^\mathfrak{k}) =& \frac{1}{\det(1-\mathrm{ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}} \\ &\cdot \frac{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{ad}(k^{-1}\sigma))|_{\mathfrak{k}_0^\perp(\gamma\sigma)}}{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{ad}(k^{-1}\sigma))|_{\mathfrak{p}_0^\perp(\gamma\sigma)}} \end{aligned} \tag{5.1.10}$$

Its square root is denoted by

$$\begin{aligned} A^{1/2}(Y_0^\mathfrak{k}) =& \Bigg[\frac{1}{\det(1-\mathrm{ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}} \\ &\cdot \frac{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{ad}(k^{-1}\sigma))|_{\mathfrak{k}_0^\perp(\gamma\sigma)}}{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{ad}(k^{-1}\sigma))|_{\mathfrak{p}_0^\perp(\gamma\sigma)}}\Bigg]^{1/2} \end{aligned} \tag{5.1.11}$$

Definition 5.1.2. Let $J_{\gamma\sigma}(Y_0^\mathfrak{k})$ be the analytic function of $Y_0^\mathfrak{k} \in \mathfrak{k}(\gamma\sigma)$ given by

$$\begin{aligned} J_{\gamma\sigma}(Y_0^\mathfrak{k}) =& \frac{1}{|\det(1-\mathrm{Ad}(\gamma\sigma))|_{\mathfrak{z}_0^\perp}|^{1/2}} \frac{\widehat{A}(i\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{p}(\gamma\sigma)})}{\widehat{A}(i\mathrm{ad}(Y_0^\mathfrak{k})|_{\mathfrak{k}(\gamma\sigma)})} \\ &\Bigg[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_0^\perp(\gamma\sigma)}} \frac{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}_0^\perp(\gamma\sigma)}}{\det(1-\exp(-i\mathrm{ad}(Y_0^\mathfrak{k}))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}_0^\perp(\gamma\sigma)}}\Bigg]^{1/2}. \end{aligned} \tag{5.1.12}$$

If $\sigma = \mathbb{1}_G$, then the function $J_\gamma(Y_0^\mathfrak{k})$ given by (5.1.12) is exactly the same function defined in [B11, eq. (5.5.5)]

By (5.1.12), there exist $c_{\gamma\sigma},\ C_{\gamma\sigma} > 0$ such that,

$$|J_{\gamma\sigma}(Y_0^\mathfrak{k})| \leq c_{\gamma\sigma}\exp(C_{\gamma\sigma}|Y_0^\mathfrak{k}|). \tag{5.1.13}$$

Example 5.1.3. If $G = K$, $X = G/K$ is reduced to a point. Let $\gamma = k^{-1} \in K, \sigma \in \mathrm{Aut}(K)$. In this case, we have for $Y_0^{\mathfrak{k}} \in \mathfrak{k}(k^{-1}\sigma)$,

$$
\begin{aligned}
J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) =& \frac{1}{\widehat{A}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(\gamma\sigma)})} \\
& \left[\frac{\det(1-\exp(-i\mathrm{ad}(Y_0^{\mathfrak{k}}))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}^{\perp}(\gamma\sigma)}}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}^{\perp}(\gamma\sigma)}}\right]^{1/2}.
\end{aligned}
\tag{5.1.14}
$$

Put $p = \dim \mathfrak{p}(\gamma\sigma)$, $q = \dim \mathfrak{k}(\gamma\sigma)$, then $r = \dim \mathfrak{z}(\gamma\sigma) = p + q$. Let $e_1, \cdots, e_p$ be an orthonormal basis of $\mathfrak{p}(\gamma\sigma)$, and let $e_{p+1}, \cdots, e_r$ be an orthonormal basis of $\mathfrak{k}(\gamma\sigma)$. Let $e^1, \cdots, e^r$ be the corresponding dual basis of $\mathfrak{z}(\gamma\sigma)^*$. Let $\underline{\mathfrak{z}}(\gamma\sigma)$, $\underline{\mathfrak{z}}(\gamma\sigma)^*$ be another copies of $\mathfrak{z}(\gamma\sigma)$, $\mathfrak{z}(\gamma\sigma)^*$. We underline the obvious objects associated with $\underline{\mathfrak{z}}(\gamma\sigma)$, $\underline{\mathfrak{z}}(\gamma\sigma)^*$.

Put

$$
\alpha = \sum_{i=1}^{r} c(e_i)\underline{e}^i \in c(\mathfrak{z}(\gamma\sigma))\widehat{\otimes}\Lambda^{\cdot}(\underline{\mathfrak{z}}^*(\gamma\sigma)).
\tag{5.1.15}
$$

By the splitting (1.1.1) of $\mathfrak{g}$, we have

$$
\mathfrak{p} \times \mathfrak{g} = \mathfrak{p} \times (\mathfrak{p} \oplus \mathfrak{k}).
\tag{5.1.16}
$$

We denote by y the tautological section of the first copy of $\mathfrak{p}$ in the right-hand side of (5.1.16), and by $Y^{\mathfrak{g}} = Y^{\mathfrak{p}} + Y^{\mathfrak{k}}$ the tautological section of $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We also denote by dy, $dY^{\mathfrak{g}} = dY^{\mathfrak{p}}dY^{\mathfrak{k}}$ the volume forms on $\mathfrak{p}$, $\mathfrak{g}$ respectively. Recall that $\Delta^{\mathfrak{p}\oplus\mathfrak{k}} = \Delta^{\mathfrak{p}} + \Delta^{\mathfrak{k}}$ is the standard Laplacian on $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, i.e., the second factor in the right-hand side of (5.1.16). Let ∇^H denote differentiation in the variable $y \in \mathfrak{p}$, and let ∇^V denote the differentiation in the variable $Y^{\mathfrak{g}} \in \mathfrak{g}$.

As an analogue in [B11, Section 5.1], let $\mathcal{P}_{a,Y_0^{\mathfrak{k}}}$ be the differential operator acting on $C^{\infty}(\mathfrak{p} \times \mathfrak{g}, \Lambda^{\cdot}(\mathfrak{g}^*)\widehat{\otimes}\Lambda^{\cdot}(\underline{\mathfrak{z}}^*(\gamma\sigma)))$ defined as follows. If $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$, set

$$
\begin{aligned}
\mathcal{P}_{a,Y_0^{\mathfrak{k}}} =& \frac{1}{2}|[Y^{\mathfrak{k}}, a] + [Y_0^{\mathfrak{k}}, Y^{\mathfrak{p}}]|^2 - \frac{1}{2}\Delta^{\mathfrak{p}\oplus\mathfrak{k}} + \alpha - \nabla^H_{Y^{\mathfrak{p}}} \\
& - \nabla^V_{[a+Y_0^{\mathfrak{k}},[a,y]]} - \widehat{c}(\mathrm{ad}(a)) + c(\mathrm{ad}(a) + i\theta\mathrm{ad}(Y_0^{\mathfrak{k}})).
\end{aligned}
\tag{5.1.17}
$$

By Hörmander [Hör67], the operator $\frac{\partial}{\partial t} + \mathcal{P}_{a,Y_0^{\mathfrak{k}}}$ is hypoelliptic.

Let $R_{Y_0^{\mathfrak{k}}}$ be the smooth kernel of $\exp(-\mathcal{P}_{a,Y_0^{\mathfrak{k}}})$ with respect to the volume $dydY^{\mathfrak{g}}$ on $\mathfrak{p} \times \mathfrak{g}$. Then for $(y, Y^{\mathfrak{g}}), (y', Y^{\mathfrak{g}\prime}) \in \mathfrak{p} \times \mathfrak{g}$,

$$
R_{Y_0^{\mathfrak{k}}}((y, Y^{\mathfrak{g}}), (y', Y^{\mathfrak{g}\prime})) \in \mathrm{End}(\Lambda^{\cdot}(\mathfrak{z}^{\perp}(\gamma\sigma)^*))\widehat{\otimes}c(\mathfrak{z}(\gamma\sigma))\widehat{\otimes}\Lambda^{\cdot}(\underline{\mathfrak{z}}^*(\gamma\sigma)).
\tag{5.1.18}
$$

Definition 5.1.4. Let $\widehat{\mathrm{Tr}_{\mathrm{s}}}$ be the supertrace functional on $c(\mathfrak{z}(\gamma\sigma))\widehat{\otimes}\Lambda^{\cdot}(\underline{\mathfrak{z}}^*(\gamma\sigma))$ such that it vanishes on monomials of nonmaximal length, and gives the value $(-1)^r$ to $c(e_1)\underline{e}^1 \cdots c(e_r)\underline{e}^r$. We extend it to a supertrace functional on the vector space $\mathrm{End}(\Lambda^{\cdot}(\mathfrak{z}^{\perp}(\gamma\sigma)^*))\widehat{\otimes}c(\mathfrak{z}(\gamma\sigma))\widehat{\otimes}\Lambda^{\cdot}(\underline{\mathfrak{z}}^*(\gamma\sigma))$ by the supertrace on $\mathrm{End}(\Lambda^{\cdot}(\mathfrak{z}^{\perp}(\gamma\sigma)^*))$. We still denote it by $\widehat{\mathrm{Tr}_{\mathrm{s}}}$.

Now we give an important result established in [B11, Theorem 5.5.1].

Proposition 5.1.5. *For $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$, we have*

$$J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) = (2\pi)^{r/2} \int_{\mathfrak{p}^{\perp}(\gamma\sigma)\times(\mathfrak{p}\oplus\mathfrak{k}^{\perp}(\gamma\sigma))} \widehat{\mathrm{Tr_s}}\Big[\mathrm{Ad}(k^{-1}\sigma) R_{Y_0^{\mathfrak{k}}}((y,Y^{\mathfrak{g}}),\mathrm{Ad}(k^{-1}\sigma)(y,Y^{\mathfrak{g}}))\Big] dydY^{\mathfrak{g}}. \tag{5.1.19}$$

Proof. Note that the operator $\mathcal{P}_{a,Y_0^{\mathfrak{k}}}$ has the same expression as the operator defined in [B11, Definition 5.1.2].

If $\sigma = \mathbb{1}_G$, then (5.1.19) is just the result of [B11, Theorem 5.5.1]. In the proof of [B11, Theorem 5.5.1], the computations of the supertrace functional in the right-hand side of (5.1.19) only depend on the adjoint actions of γ, k^{-1} and a and the fact that they commute with each other.

In general, when replacing γ, k^{-1} by $\gamma\sigma$, $k^{-1}\sigma$, the computations in [B11, Chapter 5] still hold. Then the result of [B11, Theorem 5.5.1] still hold. This completes the proof of our proposition. □

Remark 5.1.6. If $t > 0$, if we replace B by B/t, the function $J_{\gamma\sigma}$ is unchanged.

5.2. **A formula for the twisted orbital integrals for the heat kernel.** We assume that (E,ρ^E) is a unitary finite-dimensional representation of K^{σ}. Recall that $A \in \mathrm{End}(E)$ commutes with the action of K^{σ}, and that $[\gamma\sigma]$ is the conjugacy class of $\gamma\sigma$ in G^{σ}.

Theorem 5.2.1. *For any $t > 0$, the following identity holds:*

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^X)] = \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \times \int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\mathrm{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y_0^{\mathfrak{k}}) - tA)]e^{-|Y_0^{\mathfrak{k}}|^2/2t}\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}. \tag{5.2.1}$$

Proof. For $b > 0$, $s(x,Y) \in C^{\infty}\big(\widehat{X}, \widehat{\pi}^*(\Lambda^{\cdot}(T^*X \oplus N^*) \otimes F)\big)$, set

$$F_b s(x,Y) = s(x,bY). \tag{5.2.2}$$

For $t > 0$, we denote with an extra subscript t the hypoelliptic Laplacian defined in subsection 3.6 associated with the bilinear form B/t. Then by [B11, eq.(2.14.4)], we have

$$F_{\sqrt{t}} t^{N^{\Lambda^{\cdot}(T^*X\oplus N^*)}/2} \mathcal{L}_{b,t}^X t^{-N^{\Lambda^{\cdot}(T^*X\oplus N^*)}/2} F_{\sqrt{t}}^{-1} = t\mathcal{L}_{\sqrt{t}b}^X. \tag{5.2.3}$$

Using Remark (5.1.6) and by (5.2.3), it is enough to prove (5.2.1) with $t = 1$. Then by (4.4.1), we only need to make $b \to +\infty$ in $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-\mathcal{L}_{A,b}^X)]$.

Since all the analytic and geometric constructions of [B11] only depend on the fact that G acts on X as a group of isometries, replacing G by G^{σ} does not change anything from that point of view. This is why we will freely use the arguments in [B11, Chapter 9].

Using Propositions 1.7.2 - 1.7.4 and (1.7.13), and following the arguments in [B11, Section 15.7], we extend the estimates on hypoelliptic heat kernels of [B11, Theorem 9.1.1]to our case when replacing γ, k^{-1} by $\gamma\sigma$, $k^{-1}\sigma$. Then using Theorem 1.5.1, and by (4.3.13), as $b \to +\infty$, $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-\mathcal{L}^X_{A,b})]$ is localized to an integral near $\mathcal{F}_{\gamma\sigma} \subset \mathcal{X}$.

Using the rescaling techniques in [B11, Sections 9.2 - 9.5] to the above integral, we get

$$\begin{aligned}
&\lim_{b\to+\infty} \mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-\mathcal{L}^X_{A,b})] \\
&= \exp(-|a|^2/2) \int_{\substack{(y,Y^{\mathfrak{g}},Y^{\mathfrak{k}}_0)\\ \in \mathfrak{p}^\perp(\gamma\sigma)\times(\mathfrak{p}\oplus\mathfrak{k}^\perp(\gamma\sigma))\times\mathfrak{k}(\gamma\sigma)}} \\
&\widehat{\mathrm{Tr_s}}\left[\mathrm{Ad}(k^{-1}\sigma) R_{Y^{\mathfrak{k}}_0}((f,Y^{\mathfrak{g}}),\mathrm{Ad}(k^{-1}\sigma)(f,Y^{\mathfrak{g}}))\right] \\
&\mathrm{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y^{\mathfrak{k}}_0)-A)]\exp(-|Y^{\mathfrak{k}}_0|^2/2)dydY^{\mathfrak{g}}dY^{\mathfrak{k}}_0.
\end{aligned} \tag{5.2.4}$$

By (5.1.19), (5.2.4), we get (5.2.1). This completes the proof of our theorem. □

Remark 5.2.2. Let (E, ρ^E) be a representation of K. If K has trivial center, and if $k_0 \in K$, $\sigma = C(k_0) \in \Sigma$, put

$$\rho^E(\sigma) = \rho^E(k_0). \tag{5.2.5}$$

Then using (5.2.5), we extend (E, ρ^E) to a representation of K^σ. In this case, we have the identity of orbital integrals,

$$\mathrm{Tr}^{[\sigma]}[\exp(-t\mathcal{L}^X_A)] = \mathrm{Tr}^{[k_0]}[\exp(-t\mathcal{L}^X_A)]. \tag{5.2.6}$$

5.3. **A formula for general twisted orbital integrals.** Let $\Delta^{\mathfrak{z}(\gamma\sigma)}$ be the standard Laplacian on $\mathfrak{z}(\gamma\sigma)$ with respect to the scalar product induced by the scalar product of $\mathfrak{g}$. For $t > 0$, let $\exp(t\Delta^{\mathfrak{z}(\gamma\sigma)}/2)$ be the corresponding heat operator with the Gaussian heat kernel denoted by $\exp(t\Delta^{\mathfrak{z}(\gamma\sigma)}/2)((y, Y^{\mathfrak{k}}_0), (y', Y^{\mathfrak{k}\prime}_0))$. Here the heat kernel is computed with respect to the volume element on $\mathfrak{z}(\gamma\sigma)$ induced by the scalar product. Let y, $Y^{\mathfrak{k}}_0$ denote the elements in $\mathfrak{p}(\gamma\sigma)$, $\mathfrak{k}(\gamma\sigma)$ respectively.

Then

$$\begin{aligned}
&\exp(t\Delta^{\mathfrak{z}(\gamma\sigma)}/2)((y, Y^{\mathfrak{k}}_0), (y', Y^{\mathfrak{k}\prime}_0)) = \\
&\qquad \frac{1}{(2\pi t)^{(p+q)/2}} \exp(-|y-y'|^2/2t - |Y^{\mathfrak{k}}_0 - Y^{\mathfrak{k}\prime}_0|^2/2t).
\end{aligned} \tag{5.3.1}$$

Let $\delta_{y=a}$ be a distribution on $\mathfrak{z}(\gamma\sigma) = \mathfrak{p}(\gamma\sigma) \oplus \mathfrak{k}(\gamma\sigma)$ associated with the subspace $\{y = a\}$. Then $J_{\gamma\sigma}(Y^{\mathfrak{k}}_0)\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y^{\mathfrak{k}}_0))\delta_{y=a}$ is a distribution on $\mathfrak{z}(\gamma\sigma)$ with values in $\mathrm{End}(E)$. Applying the heat operator $\exp(t\Delta^{\mathfrak{z}(\gamma\sigma)}/2 - tA)$ to this distribution, we get a smooth function over $\mathfrak{z}(\gamma\sigma)$ with values in $\mathrm{End}(E)$. This function can be evaluated at $0 \in \mathfrak{z}(\gamma\sigma)$. Then Theorem 5.2.1 can be rewritten as follows,

$$\begin{aligned}
\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_A)] = \mathrm{Tr}^E\Big[&\exp(t\Delta^{\mathfrak{z}(\gamma\sigma)}/2 - tA) \\
&[J_{\gamma\sigma}(Y^{\mathfrak{k}}_0)\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y^{\mathfrak{k}}_0))\delta_{y=a}]\Big](0)
\end{aligned} \tag{5.3.2}$$

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$, let $\mathcal{S}^{\mathrm{even}}(\mathbb{R})$ be the space of even functions in $\mathcal{S}(\mathbb{R})$. The Fourier transform of $h \in \mathcal{S}(\mathbb{R})$ is given by

$$\widehat{h}(y) = \int_{\mathbb{R}} e^{-2i\pi yx} h(x) dx. \tag{5.3.3}$$

Take $\mu \in \mathcal{S}^{\mathrm{even}}(\mathbb{R})$, then $\widehat{\mu} \in \mathcal{S}^{\mathrm{even}}(\mathbb{R})$. We now assume that there exists $C > 0$ such that for any $k \in \mathbb{N}$, there exists $c_k > 0$ such that

$$|\widehat{\mu}^{(k)}(y)| \leq c_k \exp(-C|y|^2). \tag{5.3.4}$$

Then $\mu(\sqrt{\mathcal{L}^X + A})$ is a self-adjoint operator with a smooth kernel, which we denote $\mu(\sqrt{\mathcal{L}^X + A})(x, x') \in \mathrm{Hom}(F_{x'}, F_x)$, $x, x' \in X$. As explained in [B11, pp. 115], we have

$$\mu(\sqrt{\mathcal{L}^X + A}) \in \mathcal{Q}. \tag{5.3.5}$$

Since σ commutes with $\mathcal{L}^X + A$, we can get $\mu(\sqrt{\mathcal{L}^X + A}) \in \mathcal{Q}^\sigma$. Then the corresponding twisted orbital integral $\mathrm{Tr}^{[\gamma\sigma]}[\mu(\sqrt{\mathcal{L}^X + A})]$ is well-defined. From (5.3.4), the kernel of $\mu(\sqrt{-\Delta^{\mathfrak{z}(\gamma\sigma)}/2 + A})$ on $\mathfrak{z}(\gamma\sigma)$ has a Gaussian-like decay.

Theorem 5.3.1. *The following identity holds:*

$$\begin{aligned}\mathrm{Tr}^{[\gamma\sigma]}\big[\mu(\sqrt{\mathcal{L}^X + A})\big] =& \mathrm{Tr}^E\Big[\mu(\sqrt{-\Delta^{\mathfrak{z}(\gamma\sigma)}/2 + A}) J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) \\ & \rho^E(k^{-1}\sigma) \exp(-i\rho^E(Y_0^{\mathfrak{k}}))\delta_{y=a}\Big](0).\end{aligned} \tag{5.3.6}$$

Proof. This is just an analogue of [B11, Theorem 6.2.2]. Using Theorem 5.2.1 and by (5.1.13), (5.3.2), (5.3.4), an easy modification of the proof of [B11, Theorem 6.2.2] proves our theorem. □

6. The case of Euclidean vector space

The purpose of this section is to compute explicitly the twisted orbital integrals and twisted orbital supertraces for the heat kernels in the case of Euclidean vector space. As in [B11, Section 10.6], we will show that the formulas fit with our formula in Theorem 5.2.1.

This section is organized as follows. In subsection 6.1, we recall the explicit formula for the hypoelliptic heat kernel in the case of an Euclidean vector space.

In subsection 6.2, using the explicit formula of hypoelliptic heat kernel, we compute the associated twisted orbital supertraces, and we show that these computations are compatible with Theorem 5.2.1.

6.1. **An Euclidean vector space.** Let E be an Euclidean vector space of dimension m. We will consider the case where $G = E$.

The Cartan involution is given by $\theta(x) = -x$ for $x \in E$, so that $K = \{0\}$. The Lie algebra of G is given by

$$\mathfrak{g} = E, \tag{6.1.1}$$

so that

$$\mathfrak{p} = E, \qquad \mathfrak{k} = 0. \tag{6.1.2}$$

The bilinear form B is just the scalar product of E.

Let $\mathrm{O}(E)$ be the orthogonal group of E, let $\mathrm{I}(E)$ be the group of isometries of E. Then

$$\mathrm{I}(E) = E \rtimes \mathrm{O}(E). \tag{6.1.3}$$

In this case, we have

$$\mathrm{Aut}(G) = \mathrm{GL}(E). \tag{6.1.4}$$

By the definition of Σ in (1.2.6), we get

$$\Sigma = \mathrm{O}(E). \tag{6.1.5}$$

Then

$$\widetilde{G} = G \rtimes \Sigma = \mathrm{I}(E), \ \widetilde{K} = \mathrm{O}(E). \tag{6.1.6}$$

Moreover, the adjoint action of Σ on $\mathfrak{g}$ is just given by the matrix action on E.

If $\gamma \in E, \sigma \in \mathrm{O}(E)$, then $\gamma\sigma \in \widetilde{G}$, then is

$$\begin{aligned} Z(\gamma\sigma) &= \ker(1-\sigma), \\ \mathfrak{z}(\gamma\sigma) &= \ker(1-\sigma). \end{aligned} \tag{6.1.7}$$

Moreover, we have the orthogonal splitting,

$$E = \ker(1-\sigma) \oplus \mathrm{Im}(1-\sigma). \tag{6.1.8}$$

Let $\mathfrak{z}^{\perp}(\gamma\sigma)$ be the orthogonal space of $\mathfrak{z}(\gamma\sigma)$ in E. Then

$$\mathfrak{z}^{\perp}(\gamma\sigma) = \mathrm{Im}(1-\sigma). \tag{6.1.9}$$

In our case, $X = E$. For any $\gamma \in E, \sigma \in \mathrm{O}(E)$, $x \in X$, the action of $\gamma\sigma$ on X is given by $\gamma\sigma(x) = \sigma x + \gamma \in X$. Moreover, $TX = E$, $N = 0$. The Euclidean connection and the flat connection on TX coincide. Also the displacement function associated with $\gamma\sigma$ is

$$d_{\gamma\sigma}(x) = |\gamma + \sigma(x) - x| \tag{6.1.10}$$

Lemma 6.1.1. *Any element in* $\mathrm{I}(E)$ *is semisimple. The element* $\gamma\sigma$ *is elliptic if and only if* $\gamma \in \mathrm{Im}(1-\sigma)$.

Proof. Let $\gamma = \gamma_1 + \gamma_2$ be the orthogonal decomposition of γ by (6.1.8), where $\gamma_1 \in \ker(1-\sigma), \gamma_2 \in \mathrm{Im}(1-\sigma)$.

By the formula (6.1.10), we see that if $x \in E$, then

$$d_{\gamma\sigma}(x) \geq |\gamma_1|. \tag{6.1.11}$$

In particular, we have

$$d_{\gamma\sigma}((1-\sigma)^{-1}\gamma_2) = |\gamma_1|. \tag{6.1.12}$$

By Definition 1.1.2, $\gamma\sigma$ is semisimple, and is elliptic if and only if $\gamma_1 = 0$, which is equivalent to that $\gamma \in \mathrm{Im}(1-\sigma)$. □

A similar argument also shows that $\gamma\sigma$ is conjugate to $\gamma_1\sigma$. This result is just a version of Theorem 1.3.2 in this case. The minimizing set associate with $\gamma\sigma$ is

$$X(\gamma\sigma) = (1-\sigma)^{-1}\gamma_2 + \ker(1-\sigma). \tag{6.1.13}$$

The normal bundle $N_{X(\gamma\sigma)\backslash X}$ of $X(\gamma\sigma)$ in X is just $\mathfrak{z}^{\perp}(\gamma\sigma)$. The decomposition in (6.1.8) is the normal coordinate system defined in Theorem 1.5.1.

We consider the trivial vector bundle $F = \mathbb{R}$ over X. Now, we recall some results about the hypoelliptic heat kernel obtained in [B11, Chapter 10]

Let $\Delta^{E,H}$ be the scalar Laplacian on E. Then the operator $\mathcal{L}^X$ defined by (3.6.2) over E is given by

$$\mathcal{L}^X = -\frac{1}{2}\Delta^{E,H}, \tag{6.1.14}$$

If $x = (x^1, \cdots, x^m)$ is the canonical coordinate of E, then

$$\Delta^{E,H} = \sum_{j=1}^{m} \frac{\partial^2}{\partial (x^j)^2}. \tag{6.1.15}$$

For $t > 0$, let $p_t(x, x')$ be the smooth heat kernel on E associated with $\exp(t\Delta^{E,H}/2)$. Then

$$p_t(x,x') = \frac{1}{(2\pi t)^{m/2}} \exp(-\frac{1}{2t}|x - x'|^2). \tag{6.1.16}$$

The bundle $TX \oplus N$ is just E and $\widehat{\mathcal{X}} = \mathcal{X} = E \times E$. The first copy of E is identified with X, and the second copy with TX. We use (x, Y) denote the generic element of $E \times E$. The operator $\mathcal{L}^X_b$ for $b > 0$ defined in (3.6.6) acts on $C^\infty(E \times E, \Lambda^\cdot(E^*))$. Let $\Delta^{E,V}$ be the Laplacian along the second copy of E and ∇^H be the derivative along X. Let $N^{\Lambda^\cdot(E^*)}$ be the number operator of $\Lambda^\cdot(E^*)$. The hypoelliptic Laplacian $\mathcal{L}_b^X$ defined in (3.6.6) is given by,

$$\mathcal{L}_b^X = \frac{1}{2b^2}(-\Delta^{E,V} + |Y|^2 - m) + \frac{N^{\Lambda^\cdot(E^*)}}{b^2} + \frac{1}{b}\nabla_Y^H. \tag{6.1.17}$$

For $t > 0, b > 0$, let $q_{b,t}^X((x,Y),(x',Y'))$ denote the smooth kernel associated with $\exp(-t\mathcal{L}_b^X)$.

Put

$$\begin{aligned} H_{b,t}((x,Y),(x',Y')) = \frac{b^2}{2}\left(\tanh(t/2b^2)\big(|Y|^2 + |Y'|^2\big) + \frac{|Y-Y'|^2}{\sinh(t/b^2)}\right) \\ + \frac{1}{2(t - 2b^2\tanh(t/2b^2))}\left|x' - x - b^2\tanh(t/2b^2)(Y+Y')\right|^2. \end{aligned} \tag{6.1.18}$$

and

$$\begin{aligned} K_{b,t}((x,Y),(x',Y')) = \frac{b^2}{2\sinh(t/b^2)}\left|e^{-t/2b^2}Y - e^{t/2b^2}Y'\right|^2 \\ + \frac{1}{2(t - 2b^2\tanh(t/2b^2))}\left|x' - x - b^2\tanh(t/2b^2)(Y+Y')\right|^2. \end{aligned} \tag{6.1.19}$$

Set

$$\begin{aligned} h_{b,t}^E((x,Y),(x',Y')) =& \left[\frac{b^2 e^{t/b^2}}{4\pi^2\sinh(t/b^2)(t - 2b^2\tanh(t/2b^2))}\right]^{m/2} \\ & \exp\big(-H_{b,t}((x,Y),(x',Y'))\big), \\ k_{b,t}^E((x,Y),(x',Y')) =& \left[\frac{b^2 e^{t/b^2}}{4\pi^2\sinh(t/b^2)(t - 2b^2\tanh(t/2b^2))}\right]^{m/2} \\ & \exp\big(-K_{b,t}((x,Y),(x',Y'))\big). \end{aligned} \tag{6.1.20}$$

By [B11, eq. (10.5.3)], we have the identity

$$k_{b,t}^E((x,Y),(x',Y')) = \exp\left(\frac{b^2}{2}(|Y|^2 - |Y'|^2)\right)h_{b,t}^E((x,Y),(x',Y')). \tag{6.1.21}$$

An explicit formula for the kernel $q_{b,t}^X((x,Y),(x',Y')) \in \mathrm{End}(\Lambda^\cdot(E^*))$ is given in [B11, Proposition 10.6.1].

Proposition 6.1.2. *For $b > 0, t > 0$, the following identity holds:*

$$\begin{aligned} & q_{b,t}^X((x,Y),(x',Y')) = \\ & \qquad b^{-m}h_{b,t}^E((x,-Y/b),(x',-Y'/b))\exp(-tN^{\Lambda^\cdot(E^*)}/b^2). \end{aligned} \tag{6.1.22}$$

As explained in [B11, Remark 10.5.2], we can see the estimate as in (3.7.6) from the explicit formula in (6.1.22): given $M \geq \epsilon > 0$, there exist $C_{\epsilon,M}, C'_{\epsilon,M} > 0$ such that for $0 < b \leq M, \epsilon \leq t \leq M, (x,Y),(x',Y') \in \widehat{\mathcal{X}}$,

$$\begin{aligned} & \left|q_{b,t}^X((x,Y),(x',Y'))\right| \leq \\ & \qquad C_{\epsilon,M}\exp\big(-C'_{\epsilon,M}(|x-x'|^2 + |Y|^2 + |Y'|^2)\big). \end{aligned} \tag{6.1.23}$$

In this case, if we make $b \to 0$, we get

$$H_{b,t}((x,-Y/b),(x',-Y'/b)) \to \frac{1}{2}\big(|Y|^2+|Y'|^2\big)+\frac{1}{2t}\big|x'-x\big|^2. \tag{6.1.24}$$

By (6.1.20), (6.1.22) and (6.1.24), the convergence of the hypoelliptic heat kernels to the elliptic heat kernel in (3.7.7) is clear in this case.

Remark 6.1.3. The hypoelliptic Laplacian $\mathcal{L}^X_{b,t}$ on the Euclidean space serves as a model operator for general versions, and the heat kernel $q^X_{b,t}((x,Y),(x',Y'))$ given by (6.1.22) is a step stone for the estimates (3.7.6) of the general hypoelliptic heat kernel. We refer to [B11, Sections 13.2, 13.3 and 15.1] for more details.

6.2. **Twisted orbital integrals on an Euclidean vector space.** Since we have the explicit formulas for the elliptic heat kernel $p_t(x,x')$ and the hypoelliptic heat kernel $q^X_{b,t}((x,Y),(x',Y'))$, we can calculate their orbital integrals by the definitions, i.e., (4.2.6), (4.3.13).

Now we fix $\gamma\in E,\sigma\in \mathrm{O}(E)$ such that $\gamma\in\ker(1-\sigma)$. We put $\widetilde{\gamma}=\gamma\sigma\in \mathrm{I}(E)$. By (6.1.7), the centralizer of $\widetilde{\gamma}$ in E is

$$Z(\widetilde{\gamma})=Z(\sigma)=\ker(1-\sigma). \tag{6.2.1}$$

Then we have

$$\mathfrak{p}(\widetilde{\gamma})=\mathfrak{z}(\widetilde{\gamma})=\mathfrak{z}(\sigma),\;\; \mathfrak{p}^\perp(\widetilde{\gamma})=\mathfrak{z}^\perp(\widetilde{\gamma})=\mathfrak{z}^\perp(\sigma),\;\; \mathfrak{k}(\widetilde{\gamma})=0. \tag{6.2.2}$$

By (6.1.13), the minimizing set is

$$X(\widetilde{\gamma})=\ker(1-\sigma) \tag{6.2.3}$$

Proposition 6.2.1. *We have the following identity:*

$$\mathrm{Tr}^{[\widetilde{\gamma}]}[\exp(t\Delta^{E,H}/2)]=\frac{\exp(-|\gamma|^2/2t)}{(2\pi t)^{p/2}}\frac{1}{\det(1-\sigma)|_{\mathrm{Im}(1-\sigma)}}. \tag{6.2.4}$$

Proof. The kernel $p_t(x,x')$ is given explicitly in (6.1.16), by Definition 4.2.3, we have

$$\begin{aligned}\mathrm{Tr}^{[\widetilde{\gamma}]}[\exp(t\Delta^{E,H}/2)]=&\int_{\mathfrak{z}^\perp(\widetilde{\gamma})}p_t(f,\gamma+\sigma f)df\\ =&\frac{\exp(-|\gamma|^2/2t)}{(2\pi t)^{m/2}}\int_{\mathfrak{z}^\perp(\widetilde{\gamma})}\exp(-\frac{1}{2t}|(1-\sigma)f|^2)df\\ =&\frac{\exp(-|\gamma|^2/2t)}{(2\pi t)^{p/2}}\frac{1}{\det(1-\sigma)|_{\mathfrak{z}^\perp(\widetilde{\gamma})}}.\end{aligned} \tag{6.2.5}$$

By (6.1.9), (6.2.5), we get (6.2.4). □

Proposition 6.2.2. *For $b>0, t>0$, the following identities hold:*

$$\begin{aligned}\mathrm{Tr_s}^{[\widetilde{\gamma}]}[\exp\big(-t\mathcal{L}_b^X\big)] =& \det(1-e^{-t/b^2}\sigma)|_E \\ &\times \int_{(f,Y)\in\mathfrak{z}^\perp(\widetilde{\gamma})\times E} k^E_{b,t}((f,Y),(\gamma+\sigma f,\sigma Y))df dY \\ =& \frac{\exp(-|\gamma|^2/2t)}{(2\pi t)^{p/2}}\frac{1}{\det(1-\sigma)|_{\mathrm{Im}(1-\sigma)}}.\end{aligned} \tag{6.2.6}$$

Proof. Since $\sigma\in \mathrm{O}(E)$, its transpose $\sigma^T=\sigma^{-1}$, then

$$\mathrm{Tr_s}^{\Lambda^\cdot(E^*)}[\sigma\exp(-tN^{\Lambda^\cdot(E^*)}/b^2)] = \det(1-e^{-t/b^2}\sigma)|_E. \tag{6.2.7}$$

Using the fact that $|Y|=|\sigma Y|$, (6.1.21) and Proposition 6.1.2, we get the first equality.

The kernel function $k^E_{b,t}((x,Y),(x',Y'))$ is given by (6.1.20). We rewrite the splitting of E in (6.1.8),

$$E=\mathfrak{z}(\widetilde{\gamma})\oplus\mathfrak{z}^\perp(\widetilde{\gamma}). \tag{6.2.8}$$

If $Y\in E$, let $Y=Y_1+Y_2$ be the corresponding orthogonal decomposition. Then $dY=dY_1dY_2$. And σ acting on E preserves this decomposition. We recall that $\gamma\in\mathfrak{z}(\widetilde{\gamma})$.

By (6.1.20), we get

$$\begin{aligned}&\int_{(f,Y)\in\mathfrak{z}^\perp(\tilde{\gamma})\times E} k^E_{b,t}((f,Y),(\gamma+\sigma f,\sigma Y))df dY \\ &= \left[\frac{b^2e^{t/b^2}}{4\pi^2\sinh(t/b^2)(t-2b^2\tanh(t/2b^2))}\right]^{m/2} \\ &\times\int_{\mathfrak{z}^\perp(\tilde{\gamma})\times E}\exp(-K_{b,t}((f,Y),(\gamma+\sigma f,\sigma Y)))df dY.\end{aligned} \tag{6.2.9}$$

By (6.1.19), (6.2.8), if $f\in\mathfrak{z}^\perp(\tilde{\gamma})$, we have

$$\begin{aligned}&K_{b,t}((f,Y),(\gamma+\sigma f,\sigma Y)) \\ &= \frac{1}{2(t-2b^2\tanh(t/2b^2))}\big|(1-\sigma)f+b^2\tanh(t/2b^2)(1+\sigma)Y_2\big|^2 \\ &\quad+\frac{tb^2\tanh(t/2b^2)}{t-2b^2\tanh(t/2b^2)}\big|Y_1-\frac{\gamma}{t}\big|^2+\frac{1}{2t}|\gamma|^2 \\ &\quad\quad+\frac{b^2e^{-t/b^2}}{2\sinh(t/b^2)}|(1-e^{t/b^2}\sigma)Y_2|^2.\end{aligned} \tag{6.2.10}$$

We can separate the integration in (6.2.9) to the product of three integrals with respect to df, dY_1, dY_2. Then we get

$$\begin{aligned}&\int_{(f,Y)\in\mathfrak{z}^{\perp}(\widetilde{\gamma})\times E} k^E_{b,t}((f,Y),(\gamma+\sigma f,\sigma Y))dfdY\\ &=\frac{\exp(-|\gamma|^2/2t)}{(2\pi t)^{p/2}}\frac{1}{\det(1-\sigma)|_{\mathfrak{z}^{\perp}(\widetilde{\gamma})}}\frac{(e^{t/b^2})^{m-p}}{(1-e^{-t/b^2})^p}\frac{1}{\big|\det(1-e^{t/b^2}\sigma)|_{\mathfrak{z}^{\perp}(\widetilde{\gamma})}\big|}.\end{aligned}\tag{6.2.11}$$

Since $\sigma\in \mathrm{O}(E)$, the following identities hold:

$$\begin{aligned}\big|\det(1-e^{t/b^2}\sigma)|_{\mathfrak{z}^{\perp}(\widetilde{\gamma})}\big| &= \big(e^{t/b^2}\big)^{m-p}\det(1-e^{-t/b^2}\sigma)|_{\mathfrak{z}^{\perp}(\widetilde{\gamma})},\\ \det(1-e^{-t/b^2}\sigma)|_E &= (1-e^{-t/b^2})^p\det(1-e^{-t/b^2}\sigma)|_{\mathfrak{z}^{\perp}(\widetilde{\gamma})}.\end{aligned}\tag{6.2.12}$$

Combining (6.1.9), (6.2.11) and (6.2.12), we get the second identity in (6.2.6). □

Remark 6.2.3. The identities (6.2.4) and (6.2.6) are compatible with the identities in [B11, Propositions 10.6.2 and 10.6.3].

The last equation in (6.2.6) shows that the twisted orbital supertrace $\mathrm{Tr_s}^{[\widetilde{\gamma}]}[\exp\big(-t\mathcal{L}^X_b\big)]$ does not depend on $b>0$, and it is equal to $\mathrm{Tr}^{[\widetilde{\gamma}]}[\exp(t\Delta^{E,H}/2)]$, which is a consequence of Theorem 4.4.2. Now we verify that these results are compatible with our formula in (5.2.1) for semisimple orbital integrals.

Use the notation in subsection 5.1, we have

$$\mathfrak{z}_0=\mathfrak{p}_0=E,\quad \mathfrak{k}_0=0.\tag{6.2.13}$$

And

$$\mathfrak{z}_0^{\perp}(\widetilde{\gamma})=\mathfrak{p}_0^{\perp}(\widetilde{\gamma})=\mathfrak{z}^{\perp}(\widetilde{\gamma}),\quad \mathfrak{k}_0^{\perp}(\widetilde{\gamma})=\mathfrak{z}_0^{\perp}=0.\tag{6.2.14}$$

Put $p=\dim\mathfrak{z}(\widetilde{\gamma})$.

Since $\mathfrak{k}(\widetilde{\gamma})=0$, the function defined in (5.1.12) is just

$$J_{\widetilde{\gamma}}(0)=\frac{1}{\det(1-\sigma)|_{\mathrm{Im}(1-\sigma)}}.\tag{6.2.15}$$

Recall that the representation E here is just the trivial representation on $\mathbb{R}$. Then the right-hand side of (5.2.1) reduces to the same number in (6.2.4):

$$\frac{\exp(-|\gamma|^2/2t)}{(2\pi t)^{p/2}}\frac{1}{\det(1-\sigma)|_{\mathrm{Im}(1-\sigma)}}.\tag{6.2.16}$$

7. Connections with local equivariant index theory

This section is devoted to verify the compatibility of our formula in Theorem 5.2.1 for the twisted orbital integrals of heat kernels to the Lefschetz fixed point theorem of Atiyah-Bott [AB67, AB68] on locally symmetric spaces. Recall that the McKean-Singer formula [McS67] expresses the equivariant index of a Dirac operator D as a supertrace involving the heat kernel for D^2.

This section is organized as follows. In subsection 7.1, we construct the Dirac operator D^X acting on the twisted spinors over X. We show that under a proper assumption of K^σ, D^X is invariant by the action of Σ^σ. We show that if $\gamma\sigma$ is semisimple but non-elliptic, $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)]$ vanishes.

In subsection 7.2, we introduce the equivariant characteristic forms of TX and of N. We state a formula for $\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)]$ in terms of equivariant characteristic forms when $\gamma\sigma$ is elliptic.

In subsection 7.3, we establish the main result of subsection 7.2.

In subsection 7.4, we prove the compatibility of our formula to the Lefschetz formulas for the action of Σ^σ on $Z = \Gamma\backslash X$.

In subsection 7.5, we consider the case of de Rham operator of X.

In subsection 7.6, we consider the case where $G = K$. Then we get an identity of characters of K^σ.

In subsection 7.7, we consider the de Rham operator associated with a flat bundle obtained via a representation of G^σ.

Finally, in subsection 7.8, we apply Theorem 5.2.1 to the evaluation of the equivariant Ray-Singer analytic torsions over Z.

7.1. **The classical Dirac operator on X.**

Here we will assume $\mathfrak{p}$ to be even dimensional and oriented, and K to be semisimple, connected and simply connected. Recall that $\dim \mathfrak{p} = m$.

Let $\mathrm{Spin}(\mathfrak{p})$ be the Spin group of $\mathfrak{p}$. We have the exact sequence of Lie groups,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(\mathfrak{p}) \longrightarrow \mathrm{SO}(\mathfrak{p}) \longrightarrow 1. \tag{7.1.1}$$

If $m \geq 4$, $\mathrm{Spin}(\mathfrak{p})$ is just the universal cover of $\mathrm{SO}(\mathfrak{p})$. Since K is connected and simply connected, the adjoint representation $K \to \mathrm{SO}(\mathfrak{p})$ lifts to a homomorphism $K \to \mathrm{Spin}(\mathfrak{p})$.

To avoid confusion with the notation in subsection 3.1, let $\bar{c}(\mathfrak{p})$ denote the Clifford algebra of $(\mathfrak{p}, B|_{\mathfrak{p}})$, and let $S^{\mathfrak{p}} = S^{\mathfrak{p}}_+ \oplus S^{\mathfrak{p}}_-$ be the $\mathbb{Z}_2$-graded complex Hermitian vector space of $\mathfrak{p}$-spinors. Then we have the classical identification of $\mathbb{Z}_2$ graded algebras by [ABS64, Part I: §5],

$$\bar{c}(\mathfrak{p}) \otimes \mathbb{C} \simeq \mathrm{End}(S^{\mathfrak{p}}). \tag{7.1.2}$$

Moreover, $\mathrm{Spin}(\mathfrak{p})$ embeds in $\bar{c}_+(\mathfrak{p})$. Then $\mathrm{Spin}(\mathfrak{p})$ acts unitarily on $S^{\mathfrak{p}}$ and preserves the $\mathbb{Z}_2$-grading. Therefore, K acts on $S^{\mathfrak{p}}$ via a representation $\rho^{S^{\mathfrak{p}}}$ induced by the action of $\mathrm{Spin}(\mathfrak{p})$. In particular, the action of K preserves $S^{\mathfrak{p}}_\pm$.

By (3.1.10), if $f \in \mathfrak{k}$, we have

$$\rho^{S^{\mathfrak{p}}}(f) = \bar{c}(\mathrm{ad}(f)|_{\mathfrak{p}}). \tag{7.1.3}$$

The group K acts on $\mathrm{SO}(\mathfrak{p})$, $\mathrm{Spin}(\mathfrak{p})$ by conjugation. Set

$$P_{\mathrm{SO}}(X) = G \times_K \mathrm{SO}(\mathfrak{p}),\ P_{\mathrm{Spin}}(X) = G \times_K \mathrm{Spin}(\mathfrak{p}). \tag{7.1.4}$$

Then the projection in (7.1.1) induces a double cover of principal bundles $P_{\mathrm{Spin}}(X) \to P_{\mathrm{SO}}(X)$. This gives a spin structure on X. Moreover, $S^{\mathfrak{p}}$ descends to the Hermitian vector bundle $S^{TX} = S^{TX}_+ \oplus S^{TX}_-$ of (TX, g^{TX})-spinors. Let $\nabla^{S^{TX}}$ denote the induced connection on S^{TX} by the connection form $\omega^{\mathfrak{k}}$.

We fix $\sigma \in \Sigma$, and we assume that its action on $\mathfrak{p}$ preserves the orientation. Recall that $K^\sigma = K \rtimes \Sigma^\sigma$. Then K^σ acts naturally on $P_{\mathrm{SO}}(X)$.

We will assume that the homomorphism $K \to \mathrm{Spin}(\mathfrak{p})$ can be extended to a homomorphism $K^\sigma \to \mathrm{Spin}(\mathfrak{p})$. Then the action of K^σ on $P_{\mathrm{SO}}(X)$ lifts to an action on the bundle $P_{\mathrm{Spin}}(X)$. By [LM89, Definition 14.10 in Chapter 3], this is equivalent to say that the action of K^σ preserves the spin structure.

If $e \in \mathfrak{p}$, then

$$\rho^{S^{\mathfrak{p}}}(\sigma)\bar{c}(e)\rho^{S^{\mathfrak{p}}}(\sigma^{-1}) = \bar{c}(\sigma e) \in \bar{c}(\mathfrak{p}). \tag{7.1.5}$$

In particular, we have the unitary representation

$$\rho^{S^{\mathfrak{p}}} : K^\sigma \to \mathrm{Aut}^{\mathrm{even}}(S^{\mathfrak{p}}). \tag{7.1.6}$$

We also assume that the representations (E, ρ^E) of K satisfy the conditions in Proposition 2.4.7. This representation extends to a representation of K^σ, which is still denoted by ρ^E. In general, this extension is not unique, we just fix one choice. Now G^σ acts on sections of $S^{TX} \otimes F$ over X, and this action is compatible with its action on X. Recall that ∇^F is a unitary connection on F with the curvature R^F, and that ∇^F is invariant under the action of G^σ.

Let D^X be the classical Dirac operator acting on $C^\infty(X, S^{TX} \otimes F)$. If $e_1, \cdots, e_m$ is an orthogonal basis of TX, then

$$D^X = \sum_{i=1}^m \bar{c}(e_i)\nabla^{S^{TX}\otimes F}_{e_i}. \tag{7.1.7}$$

We can write D^X in matrix form with respect to the $\mathbb{Z}_2$-splitting of $C^\infty(X, S^{TX} \otimes F)$, so that

$$D^X = \begin{bmatrix} 0 & D^X_- \\ D^X_+ & 0 \end{bmatrix}. \tag{7.1.8}$$

Let $\Delta^{X,H}$ be the Bochner Laplacian acting on $C^\infty(X, S^{TX} \otimes F)$. Recall that S^X is the scalar curvature of X, which is a constant given by (1.1.20). By a formula of Lichnerowicz [Lic62], we have

$$D^{X,2} = -\Delta^{X,H} + \frac{S^X}{4} + \frac{1}{2}\sum_{1\leq i,j\leq m} \bar{c}(e_i)\bar{c}(e_j)R^F(e_i, e_j). \tag{7.1.9}$$

Let $\mathcal{L}^X$ be the operator defined in (3.6.2), with E replaced by $S^{\mathfrak{p}} \otimes E$. Then by [B11, Theorem 7.2.1], we have

$$\frac{D^{X,2}}{2} = \mathcal{L}^X - \frac{1}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] - \frac{1}{2}C^{\mathfrak{k},E}. \tag{7.1.10}$$

Put

$$\mathcal{A} = -\frac{1}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] - \frac{1}{2}C^{\mathfrak{k},E}. \tag{7.1.11}$$

By Lemma 1.2.7, $\mathcal{A}$ commutes with K^σ. We get

$$\frac{D^{X,2}}{2} = \mathcal{L}_{\mathcal{A}}^X \in \mathcal{Q}^\sigma. \tag{7.1.12}$$

Let $\gamma \in G$ be such that $\gamma\sigma$ is semisimple. We still assume that

$$\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k)a = \sigma a. \tag{7.1.13}$$

Theorem 7.1.1. *If $\gamma\sigma$ is nonelliptic, i.e., if $a \neq 0$, for $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$,*

$$\mathrm{Tr_s}^{S^{\mathfrak{p}}}[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma)\exp(-i\rho^{S^{\mathfrak{p}}}(Y_0^{\mathfrak{k}}))] = 0. \tag{7.1.14}$$

For any $t > 0$, we have

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] = 0. \tag{7.1.15}$$

Proof. By [BGV04, Proposition 3.23], we have

$$\begin{aligned}(-1)^{m/2}\big(\mathrm{Tr_s}^{S^{\mathfrak{p}}}[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma)\exp(-i\rho^{S^{\mathfrak{p}}}(Y_0^{\mathfrak{k}}))]\big)^2 =\\ \det(1 - \mathrm{Ad}(k^{-1}\sigma)\exp(-i\mathrm{ad}(Y_0^{\mathfrak{k}})))|_{\mathfrak{p}}.\end{aligned} \tag{7.1.16}$$

If $a \neq 0$, then a is an eigenvector in $\mathfrak{p}$ of $\mathrm{Ad}(k^{-1}\sigma)\exp(-i\mathrm{ad}(Y_0^{\mathfrak{k}}))$ associated with the eigenvalue 1, so that (7.1.14) holds.

To prove (7.1.15), we use the formula in Theorem 5.2.1. Inside the integral in (5.2.1), we have

$$\begin{aligned}\mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}[\rho^{S^{\mathfrak{p}}\otimes E}(k^{-1}\sigma)\exp(-i\rho^{S^{\mathfrak{p}}\otimes E}(Y_0^{\mathfrak{k}}) - t\mathcal{A})] =\\ \mathrm{Tr_s}^{S^{\mathfrak{p}}}[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma)\exp(-i\rho^{S^{\mathfrak{p}}}(Y_0^{\mathfrak{k}}))]\\ \times \mathrm{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-i\rho^{\otimes E}(Y_0^{\mathfrak{k}}) - t\mathcal{A})]\end{aligned} \tag{7.1.17}$$

By (5.2.1), (7.1.14), (7.1.17), we get (7.1.15). □

7.2. **The elliptic case.** We still use the same assumptions as in subsection 7.1. We can apply the results of section 2.

Let $\gamma\sigma \in G^\sigma$ be an elliptic element. We may and we will assume that $\gamma = k^{-1}, k \in K$. Then $X(\gamma\sigma) \subset X$ is just the fixed point set of $\gamma\sigma$. Recall that $X(\gamma\sigma)$ is a totally geodesic submanifold of X and $p1 \in X(\gamma\sigma)$. Recall that

$$p = \dim \mathfrak{p}(\gamma\sigma). \tag{7.2.1}$$

On $X(\gamma\sigma)$, let $N_{X(\gamma\sigma)/X}$ denote the normal of $X(\gamma\sigma)$ in X. Then

$$TX|_{X(\gamma\sigma)} = TX(\gamma\sigma) \oplus N_{X(\gamma\sigma)/X}. \tag{7.2.2}$$

Note that $\gamma\sigma$ acts isometrically on $TX|_{X(\gamma\sigma)}$ and preserves (7.2.2). We have

$$\dim TX(\gamma\sigma) = p,\ \dim N_{X(\gamma\sigma)/X} = m - p. \tag{7.2.3}$$

In particular, p and $m - p$ are even.

Let $\pm\theta_1, \cdots, \pm\theta_s, 0 < \theta_i \leq \pi$ be the distinct nonzero angles of this action on $N_{X(\gamma\sigma)/X}$, which correspond to the distinct angles of the action of $\mathrm{Ad}(k^{-1})\sigma$ on $\mathfrak{p}^{\perp}(\gamma\sigma)$. Let $N_{X(\gamma\sigma)/X,\theta_i}, 1 \leq i \leq s$ be the part of $N_{X(\gamma\sigma)/X}$ on which $\gamma\sigma$ acts by a rotation of angle θ_i.

The action of $\gamma\sigma$ on $TX|_{X(\gamma\sigma)}$ is parallel, so that ∇^{TX} induces metric connections on the above subbundles of $TX|_{X(\gamma\sigma)}$. Let $R^{TX(\gamma\sigma)}$, $R^{N_{X(\gamma\sigma/X),\theta_i}}$, $1 \leq i \leq s$ be the their curvatures.

If $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$, set

$$\widehat{A}^{\theta}(x) = \frac{1}{2\sinh(\frac{x+i\theta}{2})} \tag{7.2.4}$$

Given θ_i, let $\widehat{A}^{\theta_i}(N_{X(\gamma\sigma/X),\theta_i}, \nabla^{N_{X(\gamma\sigma/X),\theta_i}})$ be the corresponding multiplicative genus. The equivariant $\widehat{A}$-form of $(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$ is given by

$$\begin{aligned} &\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}}) \\ &= \widehat{A}(-\frac{R^{TX(\gamma\sigma)}}{2\pi i})\prod_{i=1}^{s}\widehat{A}^{\theta_i}(N_{X(\gamma\sigma/X),\theta_i}, \nabla^{N_{X(\gamma\sigma/X),\theta_i}}) \in \Omega^{\cdot}(X(\gamma\sigma)). \end{aligned} \tag{7.2.5}$$

We have a similar formula for the closed form $\widehat{A}^{\gamma\sigma}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}})$.

Note that there are questions of signs to be taken care of, because of the need to distinguish between θ_i and $-\theta_i$, especially for the case where $\theta_i = \pi$. We refer to [AB67, AB68] and also [LM89, Theorem 14.11 in Chapter 3], [BGV04, Chapter 6] for more detail.

Let $o(TX(\gamma\sigma))$, $o(N_{X(\gamma\sigma\backslash X)})$ be the orientation lines of $TX(\gamma\sigma)$, $N_{X(\gamma\sigma)\backslash X}$ respectively. Because of the ±1 sign ambiguity in (7.2.5) explained as above, the differential form $\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$ can be regarded as a section of $\Lambda^{\cdot}(T^*X(\gamma\sigma)) \otimes o(N_{X(\gamma\sigma)/X})$. Since the orientation of TX is equivalent to the orientation of $\mathfrak{p}$, then $\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$ can be identified naturally to a section of $\Lambda^{\cdot}(T^*X(\gamma\sigma)) \otimes o(TX(\gamma\sigma))$.

The equivariant Chern character form of the bundle (F, ∇^F) is given by

$$\mathrm{ch}^{\gamma\sigma}(F|_{X(\gamma\sigma)}, \nabla^{F|_{X(\gamma\sigma)}}) = \mathrm{Tr}[\rho^E(k^{-1}\sigma)\exp(-\frac{R^F|_{X(\gamma\sigma)}}{2\pi i})]. \tag{7.2.6}$$

The closed forms in (7.2.5), (7.2.6) on $X(\gamma\sigma)$ are exactly the ones that appear in the Lefschetz fixed point formula of Atiyah-Bott [AB67, AB68].

Let the function $\widehat{A}^{\gamma\sigma|_{\mathfrak{p}}}(0)$ on $X(\gamma\sigma)$ be the component of degree 0 of the form $\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})$, and let the function $\widehat{A}^{\gamma\sigma|_{\mathfrak{k}}}(0)$ be the component of degree 0 of $\widehat{A}^{\gamma\sigma}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}})$. These are constants on $X(\gamma\sigma)$. Put

$$\widehat{A}^{\gamma\sigma}(0) = \widehat{A}^{\gamma\sigma|_{\mathfrak{p}}}(0)\widehat{A}^{\gamma\sigma|_{\mathfrak{k}}}(0). \tag{7.2.7}$$

Using the same arguments as in the proof of [B11, Proposition 7.1.1] and (1.1.9), one can prove the following identities of differential forms on $X(\gamma\sigma)$,

$$\begin{aligned}\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})\widehat{A}^{\gamma\sigma}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}}) = \widehat{A}^{\gamma\sigma}(0).\\ \mathrm{ch}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}}) + \mathrm{ch}^{\gamma\sigma}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}})\\ = \mathrm{Tr}^{\mathfrak{g}}[\mathrm{ad}(k^{-1}\sigma)].\end{aligned} \tag{7.2.8}$$

Let Ψ be the canonical section of norm 1 in $\Lambda^p(\mathfrak{p}(\gamma\sigma)^*) \otimes o(\mathfrak{p}(\gamma\sigma))$ (respectively $\Lambda^p(T^*X(\gamma\sigma)) \otimes o(TX(\gamma\sigma))$). For $\alpha \in \Lambda^{\cdot}(\mathfrak{p}(\gamma\sigma)^*) \otimes o(\mathfrak{p}(\gamma\sigma))$ (respectively $\Lambda^{\cdot}(T^*X(\gamma\sigma)) \otimes o(TX(\gamma\sigma))$), for $0 \le l \le p$, let $\alpha^{(l)}$ be the component of α of degree l. We define $\alpha^{\max} \in \mathbb{R}$ by

$$\alpha^{(p)} = \alpha^{\max}\Psi. \tag{7.2.9}$$

Theorem 7.2.1. *If $\gamma\sigma = k^{-1}\sigma, k \in K$, for any $t > 0$,*

$$\begin{aligned}&\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)]\\ &= \frac{1}{(2\pi t)^{p/2}} \int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\mathrm{Tr_s}^{S^{\mathfrak{p}}\otimes E}[\rho^{S^{\mathfrak{p}}\otimes E}(k^{-1}\sigma)\exp(-i\rho^{S^{\mathfrak{p}}\otimes E}(Y_0^{\mathfrak{k}}) - t\mathcal{A})]\\ &\qquad \exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}\\ &= [\widehat{A}^{\gamma\sigma}(TX|_{X(\gamma\sigma)}, \nabla^{TX|_{X(\gamma\sigma)}})\mathrm{ch}^{\gamma\sigma}(F, \nabla^F)]^{\max}.\end{aligned} \tag{7.2.10}$$

Proof. The first identity in (7.2.10) follows from Theorem 5.2.1. The next section is devoted to the proof of the second identity in (7.2.10). □

7.3. **Proof of the second identity in** (7.2.10)**.** Recall that $f(\sigma) \in \mathrm{Aut}(K)$ is the restriction of σ to K, and that $K^{\sigma\prime}$ is the closed subgroup of $K \rtimes \mathrm{Aut}(K)$ generated by K and $f(\sigma)$.

Since K is simply connected, by [DK00, Corollary (3.15.5)], if $\mu \in \mathrm{Aut}(K)$, $K(\mu)$ is a connected Lie subgroup of K.

Let $\mathfrak{k}^{\mathrm{reg}}$ be the set of regular elements in $\mathfrak{k}$. By [DK00, Lemma (3.15.4)] and since $\mathfrak{k}$ is semisimple, there exists $\hat{\tau}' \in \mathrm{Aut}(K)$ such that $\hat{\tau}'$ lies in the same connected component of $\mathrm{Aut}(K)$ as $f(\sigma)$ and $S = K(\hat{\tau}')$ is a torus of K, i.e., $\hat{\tau}'$ is regular in $\mathrm{Aut}(K)$. Let $\mathfrak{s} = \mathfrak{k}(\hat{\tau}')$ be the Lie algebra of S.

By [DK00, Lemma (3.15.4)], there exists $v \in \mathfrak{s} \cap \mathfrak{k}^{\mathrm{reg}}$. If $\mathfrak{t} = \mathfrak{k}(v) \subset \mathfrak{k}$, then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Let $T \subset K$ be the corresponding maximal torus of K, and let W be the associated Weyl group. Let $\mathfrak{c} \subset \mathfrak{t}$ be the Weyl chamber that contains v. Let R be the root system associated with $(\mathfrak{k}, \mathfrak{t})$, and let $R_+ \subset R$ be the positive root system associated with $\mathfrak{c}$.

Since $\hat{\tau}'$ fixs v, $\hat{\tau}'$ preserves $\mathfrak{t}$ and $\mathfrak{c}$, so it preserves T and R_+. Also $\mathfrak{s} \subset \mathfrak{t}$, so that S is a subtorus of T. In particular,

$$S = T(\hat{\tau}'),\ \mathfrak{s} = \mathfrak{t}(\hat{\tau}'). \tag{7.3.1}$$

As explained in subsection 2.3, using the above root data of $\mathfrak{k}$, we can construct a group inclusion

$$\mathrm{Out}(K) \hookrightarrow \mathrm{Aut}(K), \tag{7.3.2}$$

so that

$$\mathrm{Aut}(K) \simeq \mathrm{Inn}(K) \rtimes \mathrm{Out}(K). \tag{7.3.3}$$

By (7.3.2), (7.3.3), we identify $\mathrm{Out}(K)$ with a finite subgroup of $\mathrm{Aut}(K)$ that acts on K and preserves T, R_+.

Let $\tau \in \mathrm{Out}(K)$ is the image of $f(\sigma) \in \mathrm{Aut}(K)$ under the projection $\mathrm{Aut}(K) \to \mathrm{Out}(K)$. By (7.3.3), we identify τ with an element in $\mathrm{Aut}(K)$. Recall that

$$K^\tau = K \rtimes \langle \tau \rangle. \tag{7.3.4}$$

Here $\langle \tau \rangle$ is the finite cyclic group generated by τ in $\mathrm{Out}(K)$.

There exists $k' \in K$ (not unique in general) such that

$$\mathrm{Ad}(k') \circ \tau = \hat{\tau}' \in \mathrm{Aut}(K). \tag{7.3.5}$$

Put $\hat{\tau} = k'\tau \in K^\tau$. By (7.3.1), we have

$$S = K(\hat{\tau}),\ \mathfrak{s} = \mathfrak{k}(\hat{\tau}),\ k' \in T. \tag{7.3.6}$$

There exists $k_0 \in K$ such that

$$\mathrm{Ad}(k^{-1}) \circ f(\sigma) = \mathrm{Ad}(k_0) \circ \mathrm{Ad}(\hat{\tau}) \in \mathrm{Aut}(K). \tag{7.3.7}$$

By (2.4.2), (7.3.5), (7.3.7), we can put

$$k^* = kk_0k' \in K \tag{7.3.8}$$

so that

$$f(\sigma) = \mathrm{Ad}(k^*) \circ \tau \in \mathrm{Aut}(K). \tag{7.3.9}$$

By [Seg68, Proposition I.4] and [BtD85, Proposition 4.3], there exists $k_1 \in K$ and $s_0 \in S$ such that

$$k_0 = k_1 s_0 \mathrm{Ad}(\hat{\tau})(k_1^{-1}). \tag{7.3.10}$$

Then

$$K(k^{-1}\sigma) = \mathrm{Ad}(k_1)(K(s_0\hat{\tau})) \subset K. \tag{7.3.11}$$

Moreover,

$$S \subset K(s_0\hat{\tau}). \tag{7.3.12}$$

Lemma 7.3.1. *The torus S is a maximal torus of $K(s_0\hat{\tau})$.*

Proof. Since S is a torus in $K(s_0\hat{\tau})$, if S' is a torus in $K(s_0\hat{\tau})$ containing S, then S' is fixed by $\hat{\tau}'$, so that $S = S'$. □

By (7.3.11) and by Lemma 7.3.1, $\mathrm{Ad}(k_1)S$ is a maximal torus of $K(k^{-1}\sigma)$.

Recall that $\rho_{\mathfrak{k}}$ is defined in (2.2.6). Since the adjoint actions $\hat{\tau}$, τ preserve R_+, we have

$$\mathrm{Ad}(\hat{\tau})\rho_{\mathfrak{k}} = \tau\rho_{\mathfrak{k}} = \rho_{\mathfrak{k}} \in \mathfrak{t}^*. \tag{7.3.13}$$

Using the scalar product of $\mathfrak{k}$ restricting to $\mathfrak{s}$, $\mathfrak{t}$, we identify $\mathfrak{s}$, $\mathfrak{t}$ with $\mathfrak{s}^*$, $\mathfrak{t}^*$, so that we can regard $\mathfrak{s}^*$ as a subspace of $\mathfrak{t}^*$. By (7.3.6), we have

$$\mathfrak{s}^* = \mathfrak{t}^*(\hat{\tau}) = \mathfrak{t}^*(\tau). \tag{7.3.14}$$

By (7.3.13), we get

$$\rho_{\mathfrak{k}} \in \mathfrak{s}^*. \tag{7.3.15}$$

By [Kos99, Proposition 1.84] and [B11, Proposition 7.5.1], we have

$$4\pi^2|\rho_{\mathfrak{k}}|^2 = -\frac{1}{24}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] = -\frac{1}{4}B^*(\kappa^{\mathfrak{k}},\kappa^{\mathfrak{k}}). \tag{7.3.16}$$

Let $\pi_{\mathfrak{k}} : \mathfrak{t} \to \mathbb{C}$ be the polynomial function

$$\pi_{\mathfrak{k}}(t) = \prod_{\alpha\in R_+}\langle 2i\pi\alpha, t\rangle. \tag{7.3.17}$$

Let $\Delta^{\mathfrak{k}}$, $\Delta^{\mathfrak{t}}$ be the standard Laplacians in $\mathfrak{k}$, $\mathfrak{t}$. When acting on Ad-invariant functions on $\mathfrak{k}$, we have the identity of differential operators

$$\Delta^{\mathfrak{k}} = \frac{1}{\pi_{\mathfrak{k}}}\Delta^{\mathfrak{t}}\,\pi_{\mathfrak{k}}. \tag{7.3.18}$$

Recall that $r_+ = |R_+|$ and that R_+ defines a natural orientation on $\mathfrak{k}/\mathfrak{t}$. Let $\varrho_{\mathfrak{k}} : \mathfrak{t} \to \mathbb{C}$ be the denominator in Weyl's character formula,

$$\begin{aligned}\varrho_{\mathfrak{k}}(t) &= \prod_{\alpha\in R_+}\left(\exp(i\pi\langle\alpha,t\rangle) - \exp(-i\pi\langle\alpha,t\rangle)\right)\\ &= (-i)^{r_+}\det{}^{1/2}(1-\mathrm{Ad}(e^{-t}))|_{\mathfrak{k}/\mathfrak{t}}.\end{aligned} \tag{7.3.19}$$

The function $\varrho_{\mathfrak{k}}(t)$ can be extended to a function on T.

As in (2.2.2), set

$$N_{K^\tau}(\mathfrak{c}) = \{\widehat{g} \in K^\tau \mid \mathrm{Ad}(\widehat{g})(\mathfrak{c}) = \mathfrak{c}\}. \tag{7.3.20}$$

Then $N_{K^\tau}(\mathfrak{c})$ is a Lie subgroup of K^τ, and

$$\hat{\tau} \in N_{K^\tau}(\mathfrak{c}). \tag{7.3.21}$$

In fact, one can verify

$$N_{K^\tau}(\mathfrak{c}) = T \rtimes \langle\tau\rangle. \tag{7.3.22}$$

Recall that the function δ is defined on $N_{K^\tau}(\mathfrak{c})$ by (2.2.11). By (2.2.6), (7.3.19), if $t \in T$, then

$$\delta(t) = e^{-2\pi i\rho_{\mathfrak{k}}}(t)\varrho_{\mathfrak{k}}(t). \tag{7.3.23}$$

Lemma 7.3.2. *There exists $c(s_0\hat{\tau}) \in \mathbb{S}^1$ such that if $t \in \mathfrak{t}$, then*

$$\det{}^{1/2}(1-\exp(-\mathrm{ad}(t))\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{k}/\mathfrak{t}} = c(s_0\hat{\tau})e^{-2\pi i\langle\rho_{\mathfrak{k}},t\rangle}\delta(e^{-t}s_0\hat{\tau}). \tag{7.3.24}$$

Proof. Let $\mathfrak{r} \subset \mathfrak{k}$ be the orthogonal space of $\mathfrak{t}$. Let $c(\mathfrak{r})$ denote the associated Clifford algebra. The adjoint actions of $s_0\hat{\tau}$ and T preserves $\mathfrak{r}$ and its scalar product.

By (2.2.9), we get

$$\mathfrak{r} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{n} \oplus \bar{\mathfrak{n}}. \tag{7.3.25}$$

Since $s_0\hat{\tau}$ preserves R_+, it preserves the splitting in (7.3.25). Moreover, if $t \in \mathfrak{s}$, then the adjoint action of $\exp(-t) \in S$ also preserves the splitting in (7.3.25).

Let $\lambda_1, \cdots, \lambda_{r_+}$ be its eigenvalues of $\mathrm{Ad}(s_0\hat{\tau})$ on $\mathfrak{n}$ with corresponding eigenvectors $v_1, \cdots, v_{r_+} \in \mathfrak{n}$, which form a $\mathbb{C}-$basis of $\mathfrak{n}$. Then

$$\mathrm{Ad}(s_0\hat{\tau})\bar{v}_j = \bar{\lambda}_j \bar{v}_j, \qquad j = 1, \cdots, r_+. \tag{7.3.26}$$

where $\bar{v}_1, \cdots, \bar{v}_{r_+}$ is a basis of $\bar{\mathfrak{n}}$.

Take $\theta_j \in [0, 2\pi)$ such that $\lambda_j = e^{\sqrt{-1}\theta_j}$. Put

$$f_j = v_j + \bar{v}_j \in \mathfrak{r}, \qquad e_j = \sqrt{-1}v_j - \sqrt{-1}\bar{v}_j \in \mathfrak{r}. \tag{7.3.27}$$

Then f_1, e_1, $\cdots$, f_{r_+}, e_{r_+} form a $\mathbb{R}$-basis of $\mathfrak{r}$, and each subspace $\mathfrak{r}_j$ spanned by f_j, e_j is invariant by $\mathrm{Ad}(s_0\hat{\tau})$. Moreover, under the oriented basis f_j, e_j, $\mathrm{Ad}(s_0\hat{\tau})$ acts on $\mathfrak{r}_j$ by the matrix,

$$\begin{bmatrix} \cos(\theta_j) & -\sin(\theta_j) \\ \sin(\theta_j) & \cos(\theta_j) \end{bmatrix} \tag{7.3.28}$$

Put

$$A = \begin{bmatrix} 0 & -\theta_1 & \cdots & 0 & 0 \\ \theta_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\theta_d \\ 0 & 0 & \cdots & \theta_d & 0 \end{bmatrix} \tag{7.3.29}$$

Then $A \in \mathfrak{so}(\mathfrak{r})$ and $\mathrm{Ad}(s_0\hat{\tau}) = e^A \in \mathrm{SO}(\mathfrak{r})$. Moreover, A preserves the splitting in (7.3.25) and

$$Av_j = \sqrt{-1}\theta_j v_j. \tag{7.3.30}$$

Put $S^{\mathfrak{r}} = \Lambda^{\cdot}(\mathfrak{n})$. By [BGV04, Proposition 3.19], $S^{\mathfrak{r}}$ is just the spinor space associated with $\mathfrak{r}$. Let $\rho^{S^{\mathfrak{r}}}$ denote the action of $c(\mathfrak{r})$ on $S^{\mathfrak{r}}$. Using the identification of Lie algebras between $\mathfrak{so}(\mathfrak{r})$ and $c^2(\mathfrak{r})$ in [BGV04, Proposition 3.7], $\mathfrak{so}(\mathfrak{r})$ acts on $S^{\mathfrak{r}}$ by $\rho^{S^{\mathfrak{r}}}$. In the same time, $\mathfrak{so}(\mathfrak{r})$ acts on $S^{\mathfrak{r}}$ by its action on $\mathfrak{n}$, which we denote by λ.

By [BGV04, Lemma 3.29], we have

$$\rho^{S^{\mathfrak{r}}}(A) = \lambda(A) - \frac{1}{2}\mathrm{Tr}^{\mathfrak{n}}[A]. \tag{7.3.31}$$

Put

$$\widetilde{g} = e^{\rho^{S^{\mathfrak{r}}}(A)} \in \mathrm{Spin}(\mathfrak{r}). \tag{7.3.32}$$

Then $\widetilde{g}$ is a lift of $\mathrm{Ad}(s_0\hat{\tau}) \in \mathrm{SO}(\mathfrak{r})$. Another lift is $-\widetilde{g}$.

If $t \in \mathfrak{t}$, then as in (7.3.31), we have

$$\rho^{S^{\mathfrak{r}}}(-\mathrm{ad}(t)) = \lambda(-\mathrm{ad}(t)) + \frac{1}{2}\mathrm{Tr}^{\mathfrak{n}}[\mathrm{ad}(t)]. \tag{7.3.33}$$

By (2.2.6), (2.2.9), we have

$$\frac{1}{2}\mathrm{Tr}^{\mathfrak{n}}[\mathrm{ad}(t)] = 2\pi i\langle\rho_{\mathfrak{k}}, t\rangle. \tag{7.3.34}$$

Note that if $g \in \mathrm{U}(\mathfrak{n})$, then

$$\mathrm{Tr_s}^{\Lambda^{\cdot}\mathfrak{n}}[\lambda(g)] = \det(1-g)|_{\mathfrak{n}}. \tag{7.3.35}$$

By [BGV04, Proposition 3.24], we have

$$\begin{aligned}
&\det{}^{1/2}(1-\exp(-\mathrm{ad}(t))\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{r}}\\
&= \pm i^{r_+}\mathrm{Tr}_{\mathrm{s}}^{\Lambda^{\cdot}\mathfrak{n}}[\exp(-\rho^{S^{\mathfrak{r}}}(\mathrm{ad}(t))\widetilde{g}]\\
&= \pm i^{r_+}e^{-\frac{1}{2}\mathrm{Tr}^{\mathfrak{n}}[A]}\det(1-e^{-t}\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{n}}\exp(2\pi i\langle\rho_{\mathfrak{k}}, t\rangle).
\end{aligned} \tag{7.3.36}$$

By (2.2.11), we get

$$\begin{aligned}
&\det(1-e^{-t}\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{n}}\\
&= (-1)^{r_+}\det\mathrm{Ad}(s_0\hat{\tau})|_{\mathfrak{n}}\exp(-4\pi i\langle\rho_{\mathfrak{k}}, t\rangle)\delta(e^{-t}s_0\hat{\tau}).
\end{aligned} \tag{7.3.37}$$

Set

$$c(s_0\hat{\tau}) = \pm(-i)^{r_+}e^{-\frac{1}{2}\mathrm{Tr}^{\mathfrak{n}}[A]}\det\mathrm{Ad}(s_0\hat{\tau})|_{\mathfrak{n}} \in \mathbb{S}^1. \tag{7.3.38}$$

By (7.3.36), (7.3.37), (7.3.38), we get (7.3.24).

□

Remark 7.3.3. Lemma 7.3.2 is an extension of [Bou87, Lemmas 2.3.3 and 3.6.3].

Using (7.3.9) and the fact that $\sigma \in \mathrm{Aut}(G)$, we can extend the action of τ on K to an automorphism of G, which we still denote by τ, i.e.,

$$\tau = \mathrm{Ad}((k^*)^{-1}) \circ \sigma \in \mathrm{Aut}(G). \tag{7.3.39}$$

Note that τ as an automorphism of G is no longer of finite order. Moreover, $\tau \in \Sigma$, and then we can regard $\hat{\tau}$ as an element in $\widetilde{K}$.

As in (7.3.11), we get,

$$Z(k^{-1}\sigma) = \mathrm{Ad}(k_1)Z(s_0\hat{\tau}). \tag{7.3.40}$$

Also

$$\begin{aligned}
\mathfrak{z}(k^{-1}\sigma) &= \mathrm{Ad}(k_1)\mathfrak{z}(s_0\hat{\tau}),\\
\mathfrak{k}(k^{-1}\sigma) &= \mathrm{Ad}(k_1)\mathfrak{k}(s_0\hat{\tau}),\\
\mathfrak{p}(k^{-1}\sigma) &= \mathrm{Ad}(k_1)\mathfrak{p}(s_0\hat{\tau}).
\end{aligned} \tag{7.3.41}$$

From Proposition 2.1.2 and (2.1.5), we may and we will assume that (E, ρ^E) is an irreducible unitary representation of $K^{\sigma\prime}$.

By Proposition 2.4.2 and (7.3.9), there is an irreducible unitary representation $(E, \widetilde{\rho}^E)$ of K^τ and a constant $c_\tau \in \mathbb{S}^1$ such that

$$\begin{aligned}
\widetilde{\rho}^E(\tau) &= c_\tau\rho^E((k^*)^{-1})\rho^E(f(\sigma)),\\
\widetilde{\rho}^E(k) &= \rho^E(k).
\end{aligned} \tag{7.3.42}$$

We will denote by χ^E the character of ρ^E on $K^{\sigma\prime}$, and denote by $\widetilde{\chi}^E$ the character of $\widetilde{\rho}^E$ on K^τ.

If $h \in K$, by (7.3.8), (7.3.10), (7.3.42), we get

$$\widetilde{\chi}^E(k_1^{-1}hk_1s_0\hat{\tau}) = c_\tau \chi^E(hk^{-1}f(\sigma)). \tag{7.3.43}$$

Let P_{++} be the system of dominant weights with respect to the root data R_+ of (K,T). Let $\lambda \in P_{++}$ be the highest weight of an irreducible component of E as a representation of K, then by [B11, eq.(7.5.7)], when restricting to this irreducible component, we have

$$C^{\mathfrak{k},E} = -4\pi^2\big(|\rho_{\mathfrak{k}}+\lambda|^2 - |\rho_{\mathfrak{k}}|^2\big). \tag{7.3.44}$$

Since (E,ρ^E) is $K^{\sigma\prime}$-irreducible, by Theorem 2.4.5, the set of highest weights associated with the different K-irreducible components of E is a τ-orbit in P_{++}. By (7.3.13), the identity in (7.3.44) holds for all K-irreducible components of E. Then by (7.1.11), (7.3.16), the following identity in $\mathrm{End}(E)$ holds,

$$\mathcal{A} = 2\pi^2|\rho_{\mathfrak{k}}+\lambda|^2. \tag{7.3.45}$$

Recall that Ψ is the unit volume form on $\mathfrak{p}(\gamma\sigma)$ with values in $o(\mathfrak{p}(\gamma\sigma))$. Let $\mathrm{Pf}[\cdot]$ be the Pfaffian on $\mathfrak{so}(\mathfrak{p}(\gamma\sigma))$ defined by Ψ.

Proof of the second identity in (7.2.10). If the restricting of (E,ρ^E) to K is not irreducible, by (2.2.20), (2.3.35), (7.3.42), both sides of the second identity in (7.2.10) vanish. So we may as well assume that (E,ρ^E) is an irreducible representation of K.

Let $\lambda \in P_{++}$ be the highest weight of (E,ρ^E). By Lemma 2.2.2, we have,

$$\tau\cdot\lambda = \lambda. \tag{7.3.46}$$

Moreover, by (7.3.14), (7.3.15), (7.3.46), we have

$$\lambda + \rho_{\mathfrak{k}} \in \mathfrak{s}^*. \tag{7.3.47}$$

As in [B11, (7.7.7)], if $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$,

$$\begin{aligned}&\mathrm{Tr_s}^{S^{\mathfrak{p}}}[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma)\exp(-ic(\mathrm{ad}(Y_0^{\mathfrak{k}})))]\\ &= \mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}]\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)})\big(\widehat{A}^{\sigma^{-1}ke^{iY_0^{\mathfrak{k}}}|_{\mathfrak{p}^\perp(\gamma\sigma)}}(0)\big)^{-1}.\end{aligned} \tag{7.3.48}$$

By (5.1.12) and (7.3.48), we have

$$\begin{aligned}&J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\mathrm{Tr_s}^{S^{\mathfrak{p}}}[\rho^{S^{\mathfrak{p}}}(k^{-1}\sigma)\exp(-ic(\mathrm{ad}(Y_0^{\mathfrak{k}})))]\\ &= (-1)^{\dim\mathfrak{p}^\perp(\gamma\sigma)/2}\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}]\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(\gamma\sigma)})\\ &\qquad \widehat{A}^{\sigma^{-1}k|_{\mathfrak{p}^\perp(\gamma\sigma)}}(0)\Big\{\frac{\det(1-\exp(-i\mathrm{ad}(Y_{\mathfrak{k}}^0))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}^\perp(\gamma\sigma)}}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}^\perp(\gamma\sigma)}}\Big\}^{1/2}\end{aligned} \tag{7.3.49}$$

Using (7.2.5), (7.3.39), (7.3.40), (7.3.41), if we replace $\gamma\sigma$ by $s_0\hat{\tau}$ and replace $Y_0^{\mathfrak{k}}$ by $\mathrm{Ad}(k_1^{-1})Y_0^{\mathfrak{k}} \in \mathfrak{k}(s_0\hat{\tau})$ in the right-hand side of (7.3.49), the identity in (7.3.49) still holds.

Combining (5.2.1), (7.3.42),(7.3.45) and (7.3.49), we get

$$\begin{aligned}&\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)]\\&=\frac{(-1)^{\dim \mathfrak{p}^{\perp}(s_0\hat{\tau})/2}c_{\tau}^{-1}}{(2\pi t)^{p/2}}e^{-2\pi^2 t|\lambda+\rho_{\mathfrak{k}}|^2}\\&\int_{\mathfrak{k}(s_0\hat{\tau})}\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(s_0\hat{\tau})}]\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(s_0\hat{\tau})})\\&\qquad\widehat{A}^{\hat{\tau}^{-1}s_0^{-1}|_{\mathfrak{p}^{\perp}(s_0\hat{\tau})}}(0)\Big\{\frac{\det(1-\exp(-i\mathrm{ad}(Y_{\mathfrak{k}}^0))\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})}}{\det(1-\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})}}\Big\}^{1/2}\\&\qquad\qquad\mathrm{Tr}^E[\widetilde{\rho}^E(s_0\hat{\tau})\exp(-i\widetilde{\rho}^E(Y_0^{\mathfrak{k}}))]\exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.\end{aligned}\tag{7.3.50}$$

Let $\Omega^{\mathfrak{z}(s_0\hat{\tau})}$ be the curvature form associated with $Z^0(s_0\hat{\tau})\to X(s_0\hat{\tau})$ as an analogue of Ω in (1.1.9), when replacing $\mathfrak{g}$ by $\mathfrak{z}(s_0\hat{\tau})$. In particular,

$$\Omega^{\mathfrak{z}(s_0\hat{\tau})}\in\Lambda^2(\mathfrak{p}(s_0\hat{\tau})^*)\otimes\mathfrak{k}(s_0\hat{\tau}).\tag{7.3.51}$$

If $\alpha,\beta\in\Lambda^{\cdot}(\mathfrak{p}(s_0\hat{\tau})^*)$, $a,b\in\mathfrak{k}(s_0\hat{\tau})$, we define

$$\langle\alpha\otimes a,\beta\times b\rangle'=\alpha\wedge\beta\langle a,b\rangle\in\Lambda^{\cdot}(\mathfrak{p}(s_0\hat{\tau})^*).\tag{7.3.52}$$

Also we put

$$|\alpha\otimes a|^{\prime,2}=\langle\alpha\otimes a,\alpha\otimes a\rangle'.\tag{7.3.53}$$

By [B11, eq. (7.5.17)], we have

$$\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(s_0\hat{\tau})}]=[\exp(-\langle Y_0^{\mathfrak{k}},\Omega^{\mathfrak{z}(s_0\hat{\tau})}\rangle')]^{\max}.\tag{7.3.54}$$

As in [B11, eq. (7.5.19)], an explicit calculation shows,

$$|\Omega^{\mathfrak{z}(s_0\hat{\tau})}|^{\prime,2}=0.\tag{7.3.55}$$

Then we can rewrite (7.3.50) as follows,

$$\begin{aligned}&\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)]\\&=\frac{(-1)^{\dim \mathfrak{p}^{\perp}(s_0\hat{\tau})/2}c_{\tau}^{-1}}{(2\pi t)^{p/2}}e^{-2\pi^2 t|\lambda+\rho_{\mathfrak{k}}|^2}\\&\times\Bigg[\int_{\mathfrak{k}(s_0\hat{\tau})}\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(s_0\hat{\tau})})\widehat{A}^{\hat{\tau}^{-1}s_0^{-1}|_{\mathfrak{p}^{\perp}(s_0\hat{\tau})}}(0)\\&\Big\{\frac{\det(1-\exp(-i\mathrm{ad}(Y_{\mathfrak{k}}^0))\mathrm{ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})}}{\det(1-\mathrm{ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})}}\Big\}^{1/2}\\&\mathrm{Tr}^E[\widetilde{\rho}^E(s_0\hat{\tau})\exp(-i\widetilde{\rho}^E(Y_0^{\mathfrak{k}}))]\exp(-|Y_0^{\mathfrak{k}}+t\Omega^{\mathfrak{z}(s_0\hat{\tau})}|^{\prime,2}/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}\Bigg]^{\max}.\end{aligned}\tag{7.3.56}$$

Set

$$L_t = \Big[\int_{\mathfrak{k}(s_0\hat{\tau})} \widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(s_0\hat{\tau})}) \det{}^{1/2}(1-\exp(-i\mathrm{ad}(Y_{\mathfrak{k}}^0))\mathrm{ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})} \mathrm{Tr}^E[\widetilde{\rho}(s_0\hat{\tau})\exp(-i\widetilde{\rho}^E(Y_0^{\mathfrak{k}}))]\exp(-|Y_0^{\mathfrak{k}}+t\Omega^{\mathfrak{z}(s_0\hat{\tau})}|^{\prime,2}/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}\Big]^{\max}. \tag{7.3.57}$$

Let $\Delta^{\mathfrak{k}(s_0\hat{\tau})}$ and $\Delta^{\mathfrak{s}}$ be the standard Laplacian in $\mathfrak{k}(s_0\hat{\tau})$ and $\mathfrak{s}$ respectively. Then we can rewrite (7.3.57) as

$$L_t = \Big[\exp(t\Delta^{\mathfrak{k}(s_0\hat{\tau})}/2)\Big(\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(s_0\hat{\tau})}) \det{}^{1/2}(1-\exp(-i\mathrm{ad}(Y_{\mathfrak{k}}^0))\mathrm{ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})} \widetilde{\chi}^E(s_0\hat{\tau}\exp(-iY_0^{\mathfrak{k}}))\Big)(-t\Omega^{\mathfrak{z}(s_0\hat{\tau})})\Big]^{\max}. \tag{7.3.58}$$

Let R' be the root system of $(\mathfrak{k}(s_0\hat{\tau}),\mathfrak{s})$ and let R'_+ be a positive root system in R'. Let $\pi_{\mathfrak{k}(s_0\hat{\tau})}(y)$, $\varrho_{\mathfrak{k}(s_0\hat{\tau})}(y)$, $y\in\mathfrak{s}$ be the functions defined as in (7.3.17), (7.3.19) with respect to $(\mathfrak{k}(s_0\hat{\tau}),\mathfrak{s})$. Put $r'_+ = |R'_+|$.

The function

$$\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{k}(s_0\hat{\tau})}) \det{}^{1/2}(1-\exp(-i\mathrm{ad}(Y_{\mathfrak{k}}^0))\mathrm{ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})} \widetilde{\chi}^E(s_0\hat{\tau}\exp(-iY_0^{\mathfrak{k}})) \tag{7.3.59}$$

is invariant by adjoint action of $K(s_0\hat{\tau})$. By (7.3.18), we get

$$L_t = \Big[\frac{1}{\pi_{\mathfrak{k}(s_0\hat{\tau})}}\exp(t\Delta^{\mathfrak{s}}/2)\Big(\pi_{\mathfrak{k}(s_0\hat{\tau})}(y)\widehat{A}^{-1}(i\mathrm{ad}(y)|_{\mathfrak{k}(s_0\hat{\tau})}) \det{}^{1/2}(1-\exp(-i\mathrm{ad}(y))\mathrm{ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})} \widetilde{\chi}^E(s_0\hat{\tau}\exp(-iy))\Big)(-t\Omega^{\mathfrak{z}(s_0\hat{\tau})})\Big]^{\max}. \tag{7.3.60}$$

The function appearing in the right-hand side of (7.3.60) is viewed as a function of $y\in\mathfrak{s}$, which is invariant by the Weyl group $W(K^0(s_0\hat{\tau}),S)$, and lifts to a central function on $\mathfrak{k}(s_0\hat{\tau})$. This guarantees that the function can be evaluated at $-t\Omega^{\mathfrak{k}(s_0\hat{\tau})}$.

If $y\in\mathfrak{s}$, then

$$\widehat{A}^{-1}(\mathrm{ad}(y)|_{\mathfrak{k}(s_0\hat{\tau})}) = \frac{\varrho_{\mathfrak{k}(s_0\hat{\tau})}(y)}{\pi_{\mathfrak{k}(s_0\hat{\tau})}(y)}. \tag{7.3.61}$$

By (7.3.17), (7.3.19), (7.3.61), we get

$$\begin{aligned}\pi_{\mathfrak{k}(s_0\hat{\tau})}(y)\widehat{A}^{-1}(i\mathrm{ad}(y)|_{\mathfrak{k}(s_0\hat{\tau})}) &= (-i)^{r'_+}\varrho_{\mathfrak{k}(s_0\hat{\tau})}(iy)\\ &= (-1)^{r'_+}\det{}^{1/2}(1-\exp(-i\mathrm{ad}(y)))|_{\mathfrak{k}(s_0\hat{\tau})/\mathfrak{s}}.\end{aligned} \tag{7.3.62}$$

Using the decomposition $\mathfrak{k}/\mathfrak{s} = \mathfrak{k}/\mathfrak{t} \oplus \mathfrak{t}/\mathfrak{s}$ and (7.3.24), we get

$$\begin{aligned}&\pi_{\mathfrak{k}(s_0\hat{\tau})}(y)\widehat{A}^{-1}(i\mathrm{ad}(y)|_{\mathfrak{k}(s_0\hat{\tau})}) \det{}^{1/2}(1-\exp(-i\mathrm{ad}(y))\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})}\\&= (-1)^{r'_+}c(s_0\hat{\tau})\det{}^{1/2}(1-\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{t}/\mathfrak{s}}e^{2\pi\langle\rho_{\mathfrak{k}},y\rangle}\delta(s_0\hat{\tau}e^{-iy}).\end{aligned} \tag{7.3.63}$$

Using (2.2.22) for the representation $(E, \widetilde{\rho}^E)$ with $u = s_0\hat{\tau}$ and $t = \exp(-iy), y \in \mathfrak{s}$, we have

$$\begin{aligned}&e^{2\pi\langle\rho_{\mathfrak{k}},y\rangle}\delta(s_0\hat{\tau}e^{-iy})\widetilde{\chi}^E(s_0\hat{\tau}\exp(-iy))\\&= \sum_{s\in W(s_0\hat{\tau})}\det(s)\det(\mathrm{Ad}((s_0\hat{\tau})^{-1}))|_{\mathfrak{k}_{R_+\backslash s\cdot R_+}}\mathrm{Tr}[\rho^E(s_0\hat{\tau})|_{E_{s\cdot\lambda}}]e^{2\pi\langle s\cdot\rho_{\mathfrak{k}}+s\cdot\lambda,y\rangle}\end{aligned} \tag{7.3.64}$$

Then the right-hand side in (7.3.64) implies that

$$\begin{aligned}L_t =&e^{2\pi^2t|\rho_{\mathfrak{k}}+\lambda|^2}\Big[\widehat{A}^{-1}(i\mathrm{ad}(-t\Omega^{\mathfrak{z}(s_0\hat{\tau})})|_{\mathfrak{k}(s_0\hat{\tau})})\\&\det{}^{1/2}(1-\exp(it\mathrm{ad}(\Omega^{\mathfrak{z}(s_0\hat{\tau})}))\mathrm{ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})}\\&\widetilde{\chi}^E(s_0\hat{\tau}\exp(it\Omega^{\mathfrak{z}(s_0\hat{\tau})}))\Big]^{\max}.\end{aligned} \tag{7.3.65}$$

Then using (7.2.5), (7.2.7), we get

$$\begin{aligned}&\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)]\\&= \frac{c_\tau^{-1}}{(2\pi t)^{p/2}}\Big[i^{\dim\mathfrak{k}^{\perp}(s_0\hat{\tau})/2}\hat{A}^{s_0\hat{\tau}}(0)\hat{A}^{-1}(i\mathrm{ad}(t\Omega^{\mathfrak{z}(s_0\hat{\tau})})|_{\mathfrak{k}(s_0\hat{\tau})})\\&\det{}^{1/2}(1-\exp(it\mathrm{ad}(\Omega^{\mathfrak{z}(s_0\hat{\tau})}))\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})}\widetilde{\chi}^E(s_0\tau\exp(it\Omega^{\mathfrak{z}(s_0\hat{\tau})}))\Big]^{\max}\\&= \frac{c_\tau^{-1}}{(2\pi t)^{p/2}}\Big[\widehat{A}^{s_0\hat{\tau}}(0)(\widehat{A}^{s_0\hat{\tau}})^{-1}(i\mathrm{ad}(t\Omega^{\mathfrak{z}(s_0\hat{\tau})})|_{\mathfrak{k}})\widetilde{\chi}^E(s_0\hat{\tau}\exp(it\Omega^{\mathfrak{z}(s_0\hat{\tau})}))\Big]^{\max}.\end{aligned} \tag{7.3.66}$$

Also the parameter t is killed automatically in the right-hand side of (7.3.66).

Note that the curvature $R^{TX|_{X(\gamma\sigma)}}$ is given by the adjoint action of the connection form $\Omega^{\mathfrak{z}(\gamma\sigma)}$ associated with $Z^0(\gamma\sigma) \to X(\gamma\sigma)$, and that $R^F|_{X(\gamma\sigma)} = \rho^E(\Omega^{\mathfrak{z}(\gamma\sigma)})$. By (7.3.41), (7.3.42), (7.3.43), the last identity in (7.3.66) is just

$$\Big[\widehat{A}^{\gamma\sigma}(0)(\widehat{A}^{\gamma\sigma})^{-1}(N|_{X(\gamma\sigma)}, \nabla^{N|_{X(\gamma\sigma)}})\chi^E(\rho^E(k^{-1}\sigma\exp(-\frac{R^F}{2\pi i})))\Big]^{\max} \tag{7.3.67}$$

Then by (7.2.6), (7.2.8), (7.3.67), we get the second identity in (7.2.10). □

Remark 7.3.4. We make a useful observation here. If $E = \mathbb{C}$ is the trivial representation of $K^{\sigma\prime}$, then the highest weight $\lambda = 0$, by (7.3.63), (7.3.64), the function on $\mathfrak{s}$

$$y \to \pi_{\mathfrak{k}(s_0\hat{\tau})}(y)\widehat{A}^{-1}(i\mathrm{ad}(y)|_{\mathfrak{k}(s_0\hat{\tau})})\det{}^{1/2}(1-\exp(-i\mathrm{ad}(y))\mathrm{Ad}(s_0\hat{\tau}))|_{\mathfrak{k}^{\perp}(s_0\hat{\tau})} \tag{7.3.68}$$

is an eigenfunction of $\Delta^{\mathfrak{s}}$ associated with the eigenvalue $4\pi^2|\rho_{\mathfrak{k}}|^2$.

By (7.3.16), (7.3.41), the function on $\mathfrak{k}(\gamma\sigma)$ given by

$$y \to \pi_{\mathfrak{k}(\gamma\sigma)}(y)\widehat{A}^{-1}(i\mathrm{ad}(y)|_{\mathfrak{k}(\gamma\sigma)})\det{}^{1/2}(1-\exp(-i\mathrm{ad}(y))\mathrm{Ad}(\gamma\sigma))|_{\mathfrak{k}^{\perp}(\gamma\sigma)} \tag{7.3.69}$$

is an eigenfunction of $\Delta^{\mathfrak{t}(\gamma\sigma)}$ associated with the eigenvalue $-\frac{1}{4}B^*(\kappa^{\mathfrak{k}},\kappa^{\mathfrak{k}})$.

7.4. **The local equivariant index theorem on Z.** We make the same assumptions as in subsections 1.8, 4.5, and we use the corresponding notation. In particular, we assume $\sigma(\Gamma)=\Gamma$.

Recall that $Z=\Gamma\backslash X$ is a compact orbifold, and the Abelian group Σ^σ acts on Z. Also the bundle of TX-spinors S^{TX} descends to the bundle of TZ-spinors S^{TZ}. The assumptions in subsection 7.1 make S^{TZ} an equivariant Clifford module over Z with respect to the action of Σ^σ. Moreover, the Clifford connection $\nabla^{S^{TZ}\otimes F}$ is Σ^σ-invariant.

The operator D^X descends to the classical Dirac operator D^Z on Z, which acts on $C^\infty(Z,S^{TZ}\otimes F)$ and commutes with Σ^σ. Similarly, the operator $\mathcal{L}^X_{\mathcal{A}}$ descends to an operator $\mathcal{L}^Z_{\mathcal{A}}$. By (7.1.12), we have

$$\frac{1}{2}D^{Z,2}=\mathcal{L}^Z_{\mathcal{A}}. \tag{7.4.1}$$

Let D^Z_+ be the corresponding component of D^Z with respect to the decomposition in (7.1.8). Then D^Z_+ is a Fredholm operator.

Let $\ker D^Z$ be the kernel of D^Z in $C^\infty(Z,S^{TZ}\otimes F)$, which is naturally a finite-dimensional representation of Σ^σ. The equivariant index of D^Z (or Lefschetz number) associated with σ is defined by

$$\mathrm{Ind}_{\Sigma^\sigma}(\sigma,D^Z)=\mathrm{Tr_s}^{\ker D^Z}[\sigma]. \tag{7.4.2}$$

We now assume that Γ is torsion free. Then Z is a compact smooth manifold. Recall that ${}^\sigma Z\subset Z$ is the fixed point set of σ, which is a finite disjoint union of $[X(\gamma\sigma)]$, $\underline{[\gamma]}_\sigma\in\underline{E}$ by (1.8.37). Let $\widehat{A}^\sigma(TZ|_{{}^\sigma Z},\nabla^{TZ|_{{}^\sigma Z}})$, $\mathrm{ch}^\sigma(F,\nabla^F)$ be the closed differential forms on ${}^\sigma Z$ defined by (7.2.5), (7.2.6).

By [AB67, AB68] and [LM89, Theorem 14.11 in Chapter 3], $\mathrm{Ind}_{\Sigma^\sigma}(\sigma,D^Z)$ can be computed by the Lefschetz fixed point formula of Atiyah-Bott, so that

$$\begin{aligned}\mathrm{Ind}_{\Sigma^\sigma}(\sigma,D^Z)&=\mathrm{Tr_s}[\sigma^Z\exp(-tD^{Z,2}/2)]\\&=\int_{{}^\sigma Z}\widehat{A}^\sigma(TZ|_{{}^\sigma Z},\nabla^{TZ|_{{}^\sigma Z}})\mathrm{ch}^\sigma(F,\nabla^F).\end{aligned} \tag{7.4.3}$$

By Proposition 1.8.8, if $\underline{[\gamma]}_\sigma\in\underline{E}$, the action of σ on $S^{TZ}\otimes F|_{[X(\gamma\sigma)]}$ is equivalent to the action of $k^{-1}\sigma$ on the corresponding vector bundle $S^{TX}\otimes F$ over $\Gamma\cap Z(k^{-1}\sigma)\backslash X(k^{-1}\sigma)$. Then on each component $[X(\gamma\sigma)]$ of ${}^\sigma Z$, the following function is constant,

$$\left[\widehat{A}^\sigma(TZ|_{{}^\sigma Z},\nabla^{TZ|_{{}^\sigma Z}})\mathrm{ch}^\sigma(F,\nabla^F)\right]^{\max} \tag{7.4.4}$$

and it is equal to

$$\left[\widehat{A}^{k^{-1}\sigma}(TX|_{X(k^{-1}\sigma)},\nabla^{TX|_{X(k^{-1}\sigma)}})\mathrm{ch}^{k^{-1}\sigma}(F,\nabla^F)\right]^{\max}. \tag{7.4.5}$$

Then by (4.5.11), (4.5.17) and using Theorem 7.1.1, Theorem 7.2.1, we get

$$\begin{aligned}\mathrm{Tr_s}[\sigma^Z e^{-tD^{Z,2}/2}] &= \sum_{\underline{[\gamma]_\sigma}\in \underline{E}} \mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma)) \\ &\qquad \times \left[\widehat{A}^\sigma(TZ|_{\sigma Z},\nabla^{TZ|_{\sigma Z}})\mathrm{ch}^\sigma(F,\nabla^F)\right]^{\max} \\ &= \sum_{\underline{[\gamma]_\sigma}\in \underline{E}} \int_{[X(\gamma\sigma)]} \widehat{A}^\sigma(TZ|_{\sigma Z},\nabla^{TZ|_{\sigma Z}})\mathrm{ch}^\sigma(F,\nabla^F).\end{aligned} \tag{7.4.6}$$

By (1.8.38), (7.4.6) is equivalent to the second identity in (7.4.3).

7.5. **The de Rham operator.** In this subsection, we no longer assume that $\dim \mathfrak{p}$ is of even dimension or that K is simply connected. We assume that G has compact center. Recall our notation $m = \dim \mathfrak{p}$.

Let $(\Omega^{\cdot}_{\mathrm{c}}(X), d^X)$ be the de Rham complex of smooth forms on X with compact support. Let d^{X*} be the formal adjoint with respect to the L_2 product induced by the Riemannian structure on X.

Put

$$D^X = d^X + d^{X*}. \tag{7.5.1}$$

Then $D^{X,2} = [d^X, d^{X*}]$ is the Hodge Laplacian of X.

Let $\mathcal{L}^X$ be the operator defined in (3.6.2) with $E = \Lambda^{\cdot}(\mathfrak{p}^*)$. By [B11, Proposition 7.8.1], we have,

$$\frac{D^{X,2}}{2} = \mathcal{L}^X - \frac{1}{8}B^*(\kappa^{\mathfrak{k}},\kappa^{\mathfrak{k}}) - \frac{1}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]. \tag{7.5.2}$$

Set

$$\beta = -\frac{1}{8}B^*(\kappa^{\mathfrak{k}},\kappa^{\mathfrak{k}}) - \frac{1}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]. \tag{7.5.3}$$

By (3.4.7), (3.4.8), we also have

$$\beta = -\frac{1}{8}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}). \tag{7.5.4}$$

It is a scalar operator on $\Omega_{\mathrm{c}}^{\cdot}(X)$. By (7.5.2), (7.5.3), we can write

$$\frac{1}{2}D^{X,2} = \mathcal{L}^X_\beta. \tag{7.5.5}$$

Recall that the Casimir operator $C^{\mathfrak{g}}$ descends to the operator $C^{\mathfrak{g},X}$ acting on $\Omega^{\cdot}_c(X)$. By (3.6.2), (7.5.4), (7.5.5), we get

$$D^{X,2} = C^{\mathfrak{g},X}. \tag{7.5.6}$$

Let $e(TX,\nabla^{TX})$ be the Euler form of TX that is associated with the Euclidean connection ∇^{TX}. If $\dim X$ is even-dimensional, then

$$e(TX,\nabla^{TX}) = \mathrm{Pf}\left[\frac{R^{TX}}{2\pi}\right]. \tag{7.5.7}$$

If $\dim X$ is odd-dimensional, then $e(TX,\nabla^{TX})$ vanishes identically.

The semisimple element $\gamma\sigma$ is assumed to be in the form in (7.1.13). Let $T(\gamma\sigma) \subset K(\gamma\sigma)$ be a maximal torus with Lie algebra $\mathfrak{t}(\gamma\sigma) \subset \mathfrak{k}$. As in [B17, eq.(8.9)], set

$$\mathfrak{b}(\gamma\sigma) = \{e \in \mathfrak{p}(k^{-1}\sigma) \mid [e, \mathfrak{t}(\gamma\sigma)] = 0\}. \tag{7.5.8}$$

Then

$$a \in \mathfrak{b}(\gamma\sigma). \tag{7.5.9}$$

If $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$, there exists $k_0 \in K(\gamma\sigma)$ such that $\mathrm{Ad}(k_0)Y_0^{\mathfrak{k}} \in \mathfrak{t}(\gamma\sigma)$. Then if $v \in \mathfrak{b}(\gamma\sigma)$, we have

$$\exp(\mathrm{ad}(Y_0^{\mathfrak{k}}))k^{-1}\sigma \mathrm{Ad}(k_0^{-1})v = v. \tag{7.5.10}$$

Theorem 7.5.1. *If* $\dim \mathfrak{b}(\gamma\sigma) \geq 1$*, if* $t > 0$*,*

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] = 0. \tag{7.5.11}$$

In particular, if $\gamma\sigma$ *is nonelliptic, then* (7.5.11) *holds.*

Proof. Note if $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$, then

$$\begin{aligned} &\mathrm{Tr_s}^{\Lambda^{\cdot}(\mathfrak{p}^*)}[\exp(-i\rho^{\Lambda^{\cdot}(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}))\rho^{\Lambda^{\cdot}(\mathfrak{p}^*)}(k^{-1}\sigma)] \\ &\qquad = \det(1 - \exp(i\mathrm{ad}(Y_0^{\mathfrak{k}}))\mathrm{Ad}(\sigma^{-1}k))|_{\mathfrak{p}}. \end{aligned} \tag{7.5.12}$$

If $\dim \mathfrak{b}(\gamma\sigma) \geq 1$, then by (7.5.10), the right-hand side in (7.5.12) vanishes identically. Using the formula in (5.2.1), we get (7.5.11). If $\gamma\sigma$ is nonelliptic, then $a \neq 0$. By (7.5.9), we get $\dim \mathfrak{b}(\gamma\sigma) \geq 1$.

□

Theorem 7.5.2. *If* $\gamma\sigma$ *is elliptic, for* $t > 0$*,*

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] = \left[e(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)})\right]^{\max}. \tag{7.5.13}$$

Proof. Now $\gamma = k^{-1} \in K$. Then

$$\mathfrak{b}(\gamma\sigma) \subset \mathfrak{p}(\gamma\sigma). \tag{7.5.14}$$

Moreover, by [K86, pp. 129], $\mathfrak{b}(\gamma\sigma) \oplus \mathfrak{t}(\gamma\sigma)$ is a Cartan subalgebra of $\mathfrak{z}(\gamma\sigma)$.

Case 1: if m is odd and σ preserves the orientation of $\mathfrak{p}$, or if m is even and σ does not preserve the orientation of $\mathfrak{p}$, then the right-hand side of (7.5.12) vanishes identically so that the left-hand side of (7.5.13) vanishes. Also $\dim \mathfrak{p}(\gamma\sigma)$ is odd, so that the right-hand side of (7.5.13) vanishes.

Case 2: if $\dim \mathfrak{b}(\gamma\sigma) \geq 1$, then by Theorem 7.5.1, the left-hand side in (7.5.13) vanishes. Let $\omega^{\mathfrak{z}(\gamma\sigma)} = \omega^{\mathfrak{k}(\gamma\sigma)} + \omega^{\mathfrak{p}(\gamma\sigma)}$ be the left-invariant 1-form on $Z^0(\gamma\sigma)$ with values in $\mathfrak{z}(\gamma\sigma)$. Recall that $\Omega^{\mathfrak{z}(\gamma\sigma)}$ is the curvature of the connection form $Z^0(\gamma\sigma) \to X(\gamma\sigma)$, i.e.,

$$\Omega^{\mathfrak{z}(\gamma\sigma)} = -\frac{1}{2}[\omega^{\mathfrak{p}(\gamma\sigma)}, \omega^{\mathfrak{p}(\gamma\sigma)}] \in \Lambda^2(\mathfrak{p}(\gamma\sigma)^*) \otimes \mathfrak{k}(\gamma\sigma). \tag{7.5.15}$$

By (7.5.10), if $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$, then

$$\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})] = 0. \tag{7.5.16}$$

By (7.5.7), (7.5.15), (7.5.16), we get $e(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)}) = 0$.

Case 3: if $\mathfrak{b}(\gamma\sigma) = \{0\}$, then $\mathfrak{t}(\gamma\sigma)$ is a Cartan subalgebra of $\mathfrak{z}(\gamma\sigma)$. We may and we will assume that either m is even and σ preserves the orientation of $\mathfrak{p}$, or m is odd and σ does not preserve the orientation of $\mathfrak{p}$. Then $\dim \mathfrak{p}(\gamma\sigma)$ is even.

By (7.5.12), we get

$$\begin{aligned} &\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}[\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}))\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)] \\ &= (-1)^{\dim \mathfrak{p}(\gamma\sigma)/2}\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}]^2 \widehat{A}^{-2}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}) \\ &\qquad \det(1 - e^{-i\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{p}^\perp(\gamma\sigma)}. \end{aligned} \tag{7.5.17}$$

By (5.1.12), we get

$$\begin{aligned} &J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}[\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}))\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)] \\ &= (-1)^{\dim \mathfrak{p}(\gamma\sigma)/2}\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}]^2 \widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{z}(\gamma\sigma)}) \\ &\qquad \left[\frac{\det(1 - e^{-i\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}{\det(1 - \mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}\right]^{1/2}. \end{aligned} \tag{7.5.18}$$

We use the arguments as in (7.3.51) - (7.3.58). Then

$$\begin{aligned} &\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] = \frac{(-1)^{\dim \mathfrak{p}(\gamma\sigma)/2}}{(2\pi t)^{p/2}} e^{-t\beta} \\ &\times \Bigg[\exp(t\Delta^{\mathfrak{k}(\gamma\sigma)}/2)\Bigg(\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}]\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{z}(\gamma\sigma)}) \\ &\qquad [\frac{\det(1 - e^{-i\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}{\det(1 - \mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}]^{1/2}\Bigg)(-t\Omega^{\mathfrak{z}(\gamma\sigma)})\Bigg]^{\max} \end{aligned} \tag{7.5.19}$$

Let $\pi_{\mathfrak{k}(\gamma\sigma)}(Y_0^{\mathfrak{k}})$ be the corresponding function on $\mathfrak{t}(\gamma\sigma)$ as in (7.3.17). By (7.3.18), we get

$$\begin{aligned} &\exp(t\Delta^{\mathfrak{k}(\gamma\sigma)}/2)\Bigg(\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}]\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{z}(\gamma\sigma)}) \\ &\qquad [\frac{\det(1 - e^{-i\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}{\det(1 - \mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}]^{1/2}\Bigg) \\ &= \frac{1}{\pi_{\mathfrak{k}(\gamma\sigma)}(Y_0^{\mathfrak{k}})}\exp(t\Delta^{\mathfrak{t}(\gamma\sigma)}/2)\Bigg(\pi_{\mathfrak{k}(\gamma\sigma)}(Y_0^{\mathfrak{k}})\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}] \\ &\qquad \widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{z}(\gamma\sigma)})[\frac{\det(1 - e^{-i\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}{\det(1 - \mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^\perp(\gamma\sigma)}}]^{1/2}\Bigg). \end{aligned} \tag{7.5.20}$$

If we compare the right-hand side of (7.5.20) and the right-hand side of (7.3.60), we may see that if we replace G by its compact form, then we can apply the same arguments as in (7.3.61) - (7.3.66) to evaluate (7.5.20). More precisely, put

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}, \qquad \mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}. \tag{7.5.21}$$

The Lie algebra $\mathfrak{u}$ is called the compact form of $\mathfrak{g}$. By (7.5.21), we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}. \tag{7.5.22}$$

Also B extends a negative definite bilinear form on $\mathfrak{u}$ and a $\mathbb{C}$-bilinear form on $\mathfrak{g}_{\mathbb{C}}$. Let $G_{\mathbb{C}}$ and U be the connected group of complex matrices associated with the Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{u}$. Since G has compact center, by [K86, Proposition 5.3], U is a compact Lie group.

Let $\kappa^{\mathfrak{u}} \in \Lambda^3(\mathfrak{u}^*)$ be defined by (3.4.5). One can verify

$$B^*(\kappa^{\mathfrak{u}},\kappa^{\mathfrak{u}}) = B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}). \tag{7.5.23}$$

Since σ preserves the splitting (1.1.1), then σ is an automorphism of Lie algebra $\mathfrak{u}$. We will assume temporarily that σ lifts to an automorphism σ of U.

Let $U(\gamma\sigma)$ be the centralizer of $\gamma\sigma$ in U with Lie algebra $\mathfrak{u}(\gamma\sigma) \subset \mathfrak{u}$. Then

$$\mathfrak{u}(\gamma\sigma) = \sqrt{-1}\mathfrak{p}(\gamma\sigma) \oplus \mathfrak{k}(\gamma\sigma). \tag{7.5.24}$$

Moreover, $\mathfrak{t}(\gamma\sigma)$ is a Cartan subalgebra of $\mathfrak{u}(\gamma\sigma)$. We still use $T(\gamma\sigma) \subset U$ denote the corresponding maximal torus. Similarly, we have

$$\mathfrak{u}^{\perp}(\gamma\sigma) = \sqrt{-1}\mathfrak{p}^{\perp}(\gamma\sigma) \oplus \mathfrak{k}^{\perp}(\gamma\sigma). \tag{7.5.25}$$

The root system $R(\mathfrak{k}(\gamma\sigma),\mathfrak{t}(\gamma\sigma))$ can be extend to a root system $R(\mathfrak{u}(\gamma\sigma),\mathfrak{t}(\gamma\sigma))$. Also the positive root system $R^+(\mathfrak{k}(\gamma\sigma),\mathfrak{t}(\gamma\sigma))$ can be extended to a positive root system $R^+(\mathfrak{u}(\gamma\sigma),\mathfrak{t}(\gamma\sigma))$.

Let $\pi_{\mathfrak{u}(\gamma\sigma)}(Y_0^{\mathfrak{k}})$ be the function on $\mathfrak{t}(\gamma\sigma)$ defined in (7.3.17) with respect to $\mathfrak{u}(\gamma\sigma)$. Then

$$\pi_{\mathfrak{u}(\gamma\sigma)}(Y_0^{\mathfrak{k}}) = \pm i^{\dim\mathfrak{p}(\gamma\sigma)/2}\mathrm{Pf}[\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{p}(\gamma\sigma)}]\pi_{\mathfrak{k}(\gamma\sigma)}(Y_0^{\mathfrak{k}}). \tag{7.5.26}$$

The right-hand side in (7.5.20) can be rewritten as

$$\begin{aligned}&\frac{\pm(-i)^{\dim\mathfrak{p}(\gamma\sigma)/2}}{\pi_{\mathfrak{u}(\gamma\sigma)}(Y_0^{\mathfrak{k}})}\exp(t\Delta^{\mathfrak{t}(\gamma\sigma)}/2)\bigg(\pi_{\mathfrak{u}(\gamma\sigma)}(Y_0^{\mathfrak{k}})\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{u}(\gamma\sigma)})\\&\qquad[\frac{\det(1-e^{-i\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{u}^{\perp}(\gamma\sigma)}}{\det(1-\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{u}^{\perp}(\gamma\sigma)}}]^{1/2}\bigg)\end{aligned} \tag{7.5.27}$$

By (7.3.69) in Remark 7.3.4, the function in $Y_0^{\mathfrak{k}} \in \mathfrak{t}(\gamma\sigma)$,

$$\pi_{\mathfrak{u}(\gamma\sigma)}(Y_0^{\mathfrak{k}})\widehat{A}^{-1}(i\mathrm{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{u}(\gamma\sigma)})[\frac{\det(1-e^{-i\mathrm{ad}(Y_0^{\mathfrak{k}})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{u}^{\perp}(\gamma\sigma)}}{\det(1-\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{u}^{\perp}(\gamma\sigma)}}]^{1/2} \tag{7.5.28}$$

is an eigenfunction of $\Delta^{\mathfrak{t}(\gamma\sigma)}$ associated with the eigenvalue $-\frac{1}{4}B^*(\kappa^{\mathfrak{u}},\kappa^{\mathfrak{u}})$, which is equal to $-\frac{1}{4}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}})$.

By (7.5.4), (7.5.19), we get

$$\begin{aligned}\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] = \frac{(-1)^{\dim\mathfrak{p}(\gamma\sigma)/2}}{(2\pi t)^{p/2}}\Big[&\mathrm{Pf}[\mathrm{ad}(-t\Omega^{\mathfrak{z}(\gamma\sigma)})|_{\mathfrak{p}(\gamma\sigma)}]\\ \widehat{A}^{-1}(i\mathrm{ad}(-t\Omega^{\mathfrak{z}(\gamma\sigma)})|_{\mathfrak{z}(\gamma\sigma)})[\frac{\det(1-e^{-i\mathrm{ad}(-t\Omega^{\mathfrak{z}(\gamma\sigma)})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^{\perp}(\gamma\sigma)}}{\det(1-\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^{\perp}(\gamma\sigma)}}]^{1/2}&\Big]^{\max}\end{aligned} \tag{7.5.29}$$

By (7.5.15) and using the same arguments as in the proof of [B11, Proposition 7.1.1], one can prove that

$$\widehat{A}^{-1}(i\mathrm{ad}(-t\Omega^{\mathfrak{z}(\gamma\sigma)})|_{\mathfrak{z}(\gamma\sigma)})\Big[\frac{\det(1-e^{-i\mathrm{ad}(-t\Omega^{\mathfrak{z}(\gamma\sigma)})}\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^{\perp}(\gamma\sigma)}}{\det(1-\mathrm{Ad}(k^{-1}\sigma))_{\mathfrak{z}^{\perp}(\gamma\sigma)}}\Big]^{1/2}=1. \tag{7.5.30}$$

By (7.5.29), (7.5.30), we get

$$\begin{aligned}\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-tD^{X,2}/2)] &= \frac{(-1)^{\dim\mathfrak{p}(\gamma\sigma)/2}}{(2\pi t)^{p/2}}\Big[\mathrm{Pf}[\mathrm{ad}(-t\Omega^{\mathfrak{z}(\gamma\sigma)})|_{\mathfrak{p}(\gamma\sigma)}]\Big]^{\max}\\ &= \Big[\mathrm{Pf}\big[\frac{\mathrm{ad}(\Omega^{\mathfrak{z}(\gamma\sigma)})|_{\mathfrak{p}(\gamma\sigma)}}{2\pi}\big]\Big]^{\max}.\end{aligned} \tag{7.5.31}$$

By (7.5.7), we get (7.5.13).

In general, the adjoint action of σ on $\mathfrak{u}$ does not lift to an automorphism of U. If $\mathfrak{u}$ is semisimple, then by [K86, Theorem 4.26], there exists a finite cover group U' of U which is simply connected so that σ lifts to an automorphism of U'.

If $\mathfrak{u}$ is not semisimple, let $\mathfrak{z}(\mathfrak{u})$ be the center of $\mathfrak{u}$, then

$$\mathfrak{u}=\mathfrak{z}(\mathfrak{u})\oplus[\mathfrak{u},\mathfrak{u}]. \tag{7.5.32}$$

Since G has compact center, we have

$$\mathfrak{z}(\mathfrak{u})\subset\mathfrak{k}. \tag{7.5.33}$$

Also the action of σ on $\mathfrak{u}$ preserves the splitting in (7.5.32). Let $Z^0(U)$ be the identity component of the center of U and let U^{ss} be the analytic subgroup of U with the Lie algebra $[\mathfrak{u},\mathfrak{u}]$. Let $\widetilde{U}^{\mathrm{ss}}$ be the compact universal cover group of U^{ss}. Then $U'=Z^0(U)\times\widetilde{U}^{\mathrm{ss}}$ is a compact finite cover group of U. The action σ on $\mathfrak{u}$ lifts to an automorphism of U'.

Let $K'\subset U'$ be the analytic subgroup associated with the Lie subalgebra $\mathfrak{k}$. Then K' is a finite cover group of K. If $k\in K$, then $\mathrm{Ad}(k)$ on $\mathfrak{u}$ can be replaced by $\mathrm{Ad}(k')$ with some $k'\in K'$.

We use U' and $k'\in K'$ instead of U and $k\in K$, the arguments (7.5.26) - (7.5.31) still hold. This completes the proof of (7.5.13) in full generality. □

Let $N^{\Lambda^{\cdot}(\mathfrak{p}^*)}$, $N^{\Lambda^{\cdot}(T^*X)}$ be the number operators of $\Lambda^{\cdot}(\mathfrak{p}^*)$, $\Lambda^{\cdot}(T^*X)$ respectively. If g is an isometry of $\mathfrak{p}$, then

$$\mathrm{Tr_s}^{\Lambda^{\cdot}(\mathfrak{p}^*)}\big[N^{\Lambda^{\cdot}(\mathfrak{p}^*)}g\big]=\frac{\partial}{\partial b}|_{b=0}\det(1-g^{-1}e^{b}). \tag{7.5.34}$$

If the eigenspace associated with the eigenvalue 1 is of dimension ≥ 2, then the quantity in (7.5.34) vanishes. If m is even and g preserves the orientation of $\mathfrak{p}$, or if m is odd and g does not preserve the orientation of $\mathfrak{p}$, then by [B11, eq.(7.9.2)], we have

$$\mathrm{Tr_s}^{\Lambda^{\cdot}(\mathfrak{p}^*)}\big[(N^{\Lambda^{\cdot}(\mathfrak{p}^*)}-\frac{m}{2})g\big]=0. \tag{7.5.35}$$

By Theorem 5.2.1, we obtain

$$\begin{aligned}(7.5.36)\qquad &\mathrm{Tr_s}^{[\gamma\sigma]}[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)\exp(-tD^{X,2}/2)]\\ &=\frac{1}{(2\pi t)^{p/2}}\exp(-t\beta-|a|^2/2t)\int_{\mathfrak{k}(\gamma\sigma)}J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\\ &\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)}-\frac{m}{2}\big)\exp(-i\mathrm{ad}(Y_0^{\mathfrak{k}}))\mathrm{ad}(k^{-1}\sigma)\big]\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.\end{aligned}$$

Now an extension of [B11, Theorem 7.9.1] can be established.

Theorem 7.5.3. *If one of the following three assumptions is verified:*

(1) m is even and σ preserves the orientation of $\mathfrak{p}$;

(2) m is odd and σ does not preserve the orientation of $\mathfrak{p}$;

(3) $\dim\mathfrak{b}(\gamma\sigma)\geq 2$,

then for $t>0$, we have

$$(7.5.37)\qquad \mathrm{Tr_s}^{[\gamma\sigma]}[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)\exp(-tD^{X,2}/2)]=0.$$

Proof. The first two cases follows from (7.5.35) and (7.5.36). The third case follows from (7.5.10), (7.5.12), (7.5.29) and (7.5.36). □

7.6. **The case $G=K$.** We now assume $G=K$. Then $\mathfrak{g}=\mathfrak{k}$, $\mathfrak{p}=0$. The space $X=G/K$ is reduced to one point. Then by (1.2.6),

$$(7.6.1)\qquad \Sigma=\mathrm{Aut}(K)$$

We now take $\sigma\in\mathrm{Aut}(K)$. Recall that K^σ is the compact group generated by K and σ in $K\rtimes\mathrm{Aut}(K)$. Let (E,ρ^E) be a finite-dimensional unitary representation of K^σ. Put $\widetilde{\gamma}=k^{-1}\sigma$, $k\in K$.

Let $\mathcal{A}$ be the endomorphism of E defined in (7.1.11). Then

$$(7.6.2)\qquad \mathcal{L}^X_{\mathcal{A}}=0.$$

The kernel $\widetilde{q}(k)$ on K^σ associated with $\exp(-t\mathcal{L}^X_{\mathcal{A}})$ is given by

$$(7.6.3)\qquad \widetilde{q}(k)=\rho^E(k)\in\mathrm{Aut}(E).$$

Recall that the function $J_{\widetilde{\gamma}}(Y_0^{\mathfrak{k}})$ is given by (5.1.14).

Theorem 7.6.1. *If $t>0$, then*

$$\begin{aligned}(7.6.4)\qquad &\int_{\mathfrak{k}(\widetilde{\gamma})}J_{\widetilde{\gamma}}(Y_0^{\mathfrak{k}})\mathrm{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y_0^{\mathfrak{k}})-t\mathcal{A})]\\ &\qquad\exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}=\mathrm{Tr}^E[\rho^E(k^{-1}\sigma)].\end{aligned}$$

Proof. By (4.2.4), (4.2.6), (5.2.1), we get (7.6.4). This is a special case of Theorem 7.2.1. □

7.7. **The de Rham operator associated with a flat bundle.** We still assume that G has compact center. Let (E, ρ^E) be a representation of G^σ. We use the same notation ρ^E for the restrictions of this representation to G, to K and to K^σ.

Recall $\mathfrak{g}_\mathbb{C}$, $\mathfrak{u}$ are given in (7.5.21). Let $U\mathfrak{u}$, $U\mathfrak{g}_\mathbb{C}$ be the enveloping algebras of $\mathfrak{u}$, $\mathfrak{g}_\mathbb{C}$ respectively. Recall that U, $G_\mathbb{C}$ are the connected groups of complex matrices associated with $\mathfrak{u}$, $\mathfrak{g}_\mathbb{C}$. Then $U\mathfrak{g}_\mathbb{C}$ can be identified with the left-invariant holomorphic differential operators on $G_\mathbb{C}$. By [K86, Proposition 5.6], $G_\mathbb{C}$ is still reductive, and G, U are closed subgroups of $G_\mathbb{C}$. In particular, U is a maximal compact subgroup of $G_\mathbb{C}$.

Let $C^\mathfrak{u}$ be the Casimir operator of U associated with B, by (1.1.18), (3.4.2), we have

$$C^\mathfrak{u} = C^\mathfrak{g} \in U\mathfrak{g} \cap U\mathfrak{u}. \tag{7.7.1}$$

The representation (E, ρ^E) can be regarded as a representation of $\mathfrak{u}$, or a $\mathbb{C}$-linear representation of $\mathfrak{g}_\mathbb{C}$. By Weyl's unitary trick [K86, Proposition 5.7], if U is simply connected, then it is equivalent to consider representations of G, of U on E, or holomorphic representations of $G_\mathbb{C}$ on E. Also by the arguments in Case 3 of the proof of Theorem 7.5.2, when replacing U by a finite cover group of U, we can always assume that σ extends to an automorphism of U and that the representation of $\mathfrak{u}$ on E can be extended to a representation of U. Then by (7.7.1), we have

$$C^{\mathfrak{u},E} = C^{\mathfrak{g},E} \in \mathrm{End}(E). \tag{7.7.2}$$

Let T' be a maximal torus of U with Lie algebra $\mathfrak{t}' \subset \mathfrak{u}$. Let $R(\mathfrak{u}, \mathfrak{t}')$ be the associated root system with the positive roots system $R_+(\mathfrak{u}, \mathfrak{t}')$. Recall that $\rho_\mathfrak{u}$ is defined as in (2.2.6). If (E, ρ^E) is an irreducible unitary representation of U with the highest weight $\lambda' \in \mathfrak{t}'^*$, then by (7.3.44), (7.7.2), we get

$$C^{\mathfrak{g},E} = -4\pi^2(|\rho_\mathfrak{u} + \lambda'|^2 - |\rho_\mathfrak{u}|^2). \tag{7.7.3}$$

Also by (7.3.16), (7.5.23), we have

$$-\frac{1}{4}B^*(\kappa^\mathfrak{g}, \kappa^\mathfrak{g}) = 4\pi^2|\rho_\mathfrak{u}|^2. \tag{7.7.4}$$

The group Σ^σ now embeds in $\mathrm{Aut}(U)$. Put

$$U^\sigma = U \rtimes \Sigma^\sigma. \tag{7.7.5}$$

Then (E, ρ^E) is a representation of U^σ. We equip E with a Hermitian metric h^E invariant by U^σ, then h^E is also invariant by the action of K^σ and ρ^E maps $\mathfrak{p}$ to self-adjoint elements in $\mathrm{End}(E)$.

Put $F = G \times_K E$. Let ∇^F be the Hermitian connection induced by the connection form $\omega^\mathfrak{k}$. Then the map $(g, v) \in G \times_K E \to \rho^E(g)v \in E$ gives the canonical identification of vector bundles on X,

$$G \times_K E = X \times E. \tag{7.7.6}$$

Then F is equipped with a canonical flat connection $\nabla^{F,f}$ so that

$$\nabla^{F,f} = \nabla^F + \rho^E(\omega^\mathfrak{p}). \tag{7.7.7}$$

Recall that R^F is the curvature of ∇^F,

$$R^F = -\frac{1}{2}\rho^E([\omega^{\mathfrak{p}}, \omega^{\mathfrak{p}}]). \tag{7.7.8}$$

As in (7.2.8), we claim that if $k \in K$, then if $x \in X(k^{-1}\sigma)$

$$\mathrm{Tr}^F_x[\rho^F(k^{-1}\sigma)\exp(-\frac{R^F}{2\pi i})] = \mathrm{Tr}^E[\rho^E(k^{-1}\sigma)]. \tag{7.7.9}$$

Indeed, using (7.7.8), one gets at $x = p1 \in X(k^{-1}\sigma)$, $s \in \mathbb{R}$,

$$\begin{aligned}&\frac{\partial}{\partial s}\mathrm{Tr}^E[\rho^{\Lambda^{\cdot}(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma)\exp(-\frac{sR^F}{2\pi i})] \\ &= \frac{1}{4\pi i}\mathrm{Tr}^E\big[[\rho^E(\omega^{\mathfrak{p}}), \rho^E(\omega^{\mathfrak{p}})\rho^{\Lambda^{\cdot}(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma)\exp(\frac{s\rho^E([\omega^{\mathfrak{p}},\omega^{\mathfrak{p}}])}{4\pi i})]\big] \\ &= 0.\end{aligned} \tag{7.7.10}$$

When taking $s = 0$ and $s = 1$ in $\mathrm{Tr}^E[\rho^{\Lambda^{\cdot}(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma)\exp(-\frac{sR^F}{2\pi i})]$, we get (7.7.9).

Let $(\Omega^{\cdot}_c(X, F), d^{X,F})$ be the de Rham complex associated with the flat vector bundle $(F, \nabla^{F,f})$. Let $d^{X,F,*}$ be the adjoint operator of $d^{X,F}$ with respect to the L_2 metric on $\Omega^{\cdot}_c(X, F)$. The Dirac operator $\mathbf{D}^{X,F}$ of this de Rham complex is given by

$$\mathbf{D}^{X,F} = d^{X,F} + d^{X,F,*}. \tag{7.7.11}$$

Recall that $c(\mathfrak{p})$, $\widehat{c}(\mathfrak{p})$ act on $\Lambda^{\cdot}(\mathfrak{p}^*)$ by (3.1.5). Similarly, $c(TX)$, $\widehat{c}(TX)$ act on $\Lambda^{\cdot}(T^*X)$. We still use $e_1, \cdots, e_m$ to denote an orthonormal basis of $\mathfrak{p}$ or TX, and let $e^1, \cdots, e^m$ be the corresponding dual basis of $\mathfrak{p}^*$ or T^*X.

Let $\nabla^{\Lambda^{\cdot}(T^*X)\otimes F,u}$ be the connection on $\Lambda^{\cdot}(T^*X) \otimes F$ induced by ∇^{TX} and ∇^F. Then the standard Dirac operator is given by

$$D^{X,F} = \sum_{j=1}^m c(e_j)\nabla^{\Lambda^{\cdot}(T^*X)\otimes F,u}_{e_j}. \tag{7.7.12}$$

By [BMZ17, eq.(8.42)], we have

$$\mathbf{D}^{X,F} = D^{X,F} + \sum_{j=1}^m \widehat{c}(e_j)\rho^E(e_j). \tag{7.7.13}$$

The Casimir operator $C^{\mathfrak{g}}$ descends to an elliptic differential operator $C^{\mathfrak{g},X}$ acting on $C^\infty(X, \Lambda^{\cdot}(T^*X) \otimes F)$, and recall that $C^{\mathfrak{g},E}$ defines a smooth section of endomorphism of F. As in (3.6.2), set

$$\mathcal{L}^{X,F} = \frac{1}{2}C^{\mathfrak{g},X} + \frac{1}{8}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}). \tag{7.7.14}$$

By [BMZ17, Proposition 8.4] and (3.4.8), (7.5.4), we have

$$\frac{\mathbf{D}^{X,F,2}}{2} = \mathcal{L}^{X,F} - \frac{1}{2}C^{\mathfrak{g},E} - \frac{1}{8}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}). \tag{7.7.15}$$

In particular, $\mathbf{D}^{X,F,2}$ commutes with the action of G^σ.

We still assume that $\gamma\sigma$ is a semisimple element given by (7.1.13). Recall that $\mathfrak{b}(\gamma\sigma) \subset \mathfrak{p}(k^{-1}\sigma)$ is given in (7.5.8) and that the notation $[\cdot]^{\max}$ refers to the forms on $X(\gamma\sigma)$.

Theorem 7.7.1. *For $t > 0$, the following identity holds:*

$$\begin{aligned}
&\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathbf{D}^{X,F,2}/2)] \\
&= \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \exp\Big(\frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] + \frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]\Big) \int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) \\
&\qquad \mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}\Big[\rho^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma)\exp\Big(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}(Y_0^{\mathfrak{k}}) + \frac{t}{2}C^{\mathfrak{g},E}\Big)\Big] \\
&\qquad\qquad \exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.
\end{aligned} \tag{7.7.16}$$

If $\dim \mathfrak{b}(\gamma\sigma) \geq 1$, *then*

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathbf{D}^{X,F,2}/2)] = 0. \tag{7.7.17}$$

If $\gamma\sigma$ is elliptic, then

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathbf{D}^{X,F,2}/2)] = \big[e(TX(\gamma\sigma), \nabla^{TX(\gamma\sigma)})\big]^{\max}\mathrm{Tr}^E[\rho^E(k^{-1}\sigma)]. \tag{7.7.18}$$

Proof. The identity in (7.7.16) follows from (5.2.1), (7.5.3), (7.7.15).

As in (7.1.17), one can write

$$\begin{aligned}
&\mathrm{Tr_s}^{\Lambda^\cdot(p^*)\otimes E}\Big[\rho^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma)\exp\Big(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}(Y_0^{\mathfrak{k}}) + \frac{t}{2}C^{\mathfrak{g},E}\Big)\Big] \\
&= \mathrm{Tr_s}^{\Lambda^\cdot(p^*)}[\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}))] \\
&\qquad \times \mathrm{Tr}^E\Big[\rho^E(k^{-1}\sigma)\exp\Big(-i\rho^E(Y_0^{\mathfrak{k}}) + \frac{t}{2}C^{\mathfrak{g},E}\Big)\Big]
\end{aligned} \tag{7.7.19}$$

By (7.7.19), the proof of (7.7.17) is exactly the same as the proof of Theorem 7.5.1.

The proof of (7.7.18) is a combination of the proofs of Theorem 7.2.1 and of Theorem 7.5.2. We still use the same notation as in the proof of Theorem 7.5.2.

The arguments in Case 1 and Case 2 of the proof of Theorem 7.5.2 are still applicable. Then we reduce the proof of (7.7.18) to Case 3 where $\dim \mathfrak{b}(\gamma\sigma) = 0$. Then $\mathfrak{t}(\gamma\sigma)$ is the Cartan subalgebra of $\mathfrak{k}(\gamma\sigma)$ and of $\mathfrak{u}(\gamma\sigma)$ in the same time.

Using the arguments (7.5.17) - (7.5.27), we get

$$
\begin{aligned}
&\operatorname{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathbf{D}^{X,F,2}/2)] = \frac{\pm(-i)^{\dim \mathfrak{p}(\gamma\sigma)/2}}{(2\pi t)^{p/2}} \exp(-t\beta) \\
&\Bigg[\exp(t\Delta^{\mathfrak{u}(\gamma\sigma)}/2)\bigg(\widehat{A}^{-1}(i\operatorname{ad}(Y_0^{\mathfrak{k}})|_{\mathfrak{u}(\gamma\sigma)}) \\
&\qquad [\frac{\det(1-e^{-i\operatorname{ad}(Y_0^{\mathfrak{k}})}\operatorname{Ad}(k^{-1}\sigma))_{\mathfrak{u}^\perp(\gamma\sigma)}}{\det(1-\operatorname{Ad}(k^{-1}\sigma))_{\mathfrak{u}^\perp(\gamma\sigma)}}]^{1/2} \\
&\qquad\quad \operatorname{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-i\rho^E(Y_0^{\mathfrak{k}}) + \frac{t}{2}C^{\mathfrak{g},E})]\bigg)(-t\Omega^{\mathfrak{u}(\gamma\sigma)})\Bigg]^{\max}.
\end{aligned}
\tag{7.7.20}
$$

We may and we will assume that (E, ρ^E) is an irreducible unitary representation of U^σ. Now we can proceed the arguments in subsection 7.3 to (7.7.20). Using the corresponding character formula of U as in (7.3.64) and by (7.5.30), (7.7.3), (7.7.4), (7.7.9), we get (7.7.18). □

Remark 7.7.2. If we take $E = \mathbb{C}$ with the trivial representation, we get Theorem 7.5.1 and Theorem 7.5.2 as consequences of Theorem 7.7.1.

If we take $G = K$, (7.7.18) reduces to (7.6.4).

Theorem 7.7.3. *If $t > 0$, the following identity holds:*

$$
\begin{aligned}
&\operatorname{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F,2}/2)\Big] \\
&= \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \exp(\frac{t}{48}\operatorname{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] + \frac{t}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]) \int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}}) \\
&\qquad \operatorname{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}\Big[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)} - \frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}(k^{-1}\sigma) \\
&\qquad\quad \exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)\otimes E}(Y_0^{\mathfrak{k}}) + \frac{t}{2}C^{\mathfrak{g},E})\Big]\exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.
\end{aligned}
\tag{7.7.21}
$$

If m is even and σ acting on $\mathfrak{p}$ preserves the orientation, or m is odd and σ does not preserve the orientation of $\mathfrak{p}$, or if $\dim \mathfrak{b}(\gamma\sigma) \geq 2$, *then* (7.7.21) *vanishes.*

Proof. The proof of (7.7.21) follows from (5.2.1), (7.5.3), (7.7.15). The proof of the rest part is the same as the proof of Theorem 7.5.3. □

Corollary 7.7.4. *If $\gamma\sigma$ is elliptic, i.e., $\gamma = k^{-1} \in K$, if* $\dim \mathfrak{b}(\gamma\sigma) = 0$, *then*

$$
\operatorname{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F,2}/2)\Big] = 0.
\tag{7.7.22}
$$

Proof. As in the proof of Theorem 7.5.2, when $\gamma\sigma$ is elliptic, $\mathfrak{b}(\gamma\sigma) \oplus \mathfrak{t}(\gamma\sigma)$ is a Cartan subalgebra of $\mathfrak{z}(\gamma\sigma)$.

If $\dim \mathfrak{b}(\gamma\sigma) = 0$, then $\dim \mathfrak{p}(\gamma\sigma)$ is even. If γ preserves the orientation of $\mathfrak{p}$, then $\dim \mathfrak{p}^{\perp}(\gamma\sigma)$ is even. If γ does not preserves the orientation of $\mathfrak{p}$, then $\dim \mathfrak{p}^{\perp}(\gamma\sigma)$ is odd. By Theorem 7.7.3, we get (7.7.22). □

7.8. **Equivariant Ray-Singer analytic torsions on Z.** Let Z be the compact smooth manifold considered in subsection 7.4. The group Σ^{σ} acts on Z isometrically.

The flat vector bundle F defined in subsection 7.7 descends to a flat vector bundle on Z, which we still denote by F on which Σ^{σ} also acts. Also the operator $\mathbf{D}^{X,F}$ descends to the corresponding operator $\mathbf{D}^{Z,F}$ so that

$$\mathbf{D}^{Z,F} = d^{Z,F} + d^{Z,F,*}. \tag{7.8.1}$$

Then $\mathbf{D}^{Z,F}$ commutes with Σ^{σ}.

Let $H^{\cdot}(Z,F)$ be the cohomology of $(\Omega^{\cdot}(Z,F), d^{Z,F})$. By Hodge theory,

$$\ker \mathbf{D}^{Z,F} \simeq H^{\cdot}(Z,F). \tag{7.8.2}$$

Recall that the equivariant index of $D^{Z,F}$ is defined in (7.4.2). In this case, we will change the notation to

$$\chi_{\sigma}(F) = \mathrm{Tr_s}^{H^{\cdot}(Z,F)}[\sigma]. \tag{7.8.3}$$

Let $N^{\Lambda^{\cdot}(T^*Z)}$ denote the number operator on $\Omega^{\cdot}(Z,F)$. By standard heat equation methods, there exists l with $2l \in \mathbb{N}_{>0}$ such that as $t \to 0$, for $k \in \mathbb{N}$,

$$\begin{aligned} \mathrm{Tr_s}\big[N^{\Lambda^{\cdot}(T^*Z)}\sigma^{Z}\exp(-t\mathbf{D}^{Z,F,2})\big] = \frac{a_l}{t^l} + \frac{a_{l-1/2}}{t^{l-1/2}} + \cdots + a_0 + a_{1/2}t^{1/2} + \\ \cdots + a_{k-1/2}t^{k-1/2} + a_k t^k + o(t^k). \end{aligned} \tag{7.8.4}$$

Let $(\mathbf{D}^{Z,F,2})^{-1}$ be the inverse of $\mathbf{D}^{Z,F,2}$ acting on the orthogonal space of $\ker \mathbf{D}^{Z,F}$ in $\Omega^{\cdot}(Z,F)$.

Definition 7.8.1. For $s \in \mathbb{C}$, $\mathrm{Re}(s) > l$, set

$$\vartheta_{\sigma}(g^{TZ}, \nabla^{F,f}, g^F)(s) = -\mathrm{Tr_s}\big[N^{\Lambda^{\cdot}(T^*Z)}\sigma^{Z}(\mathbf{D}^{Z,F,2})^{-s}\big]. \tag{7.8.5}$$

By [See67], $\vartheta_{\sigma}(g^{TZ}, \nabla^{F,f}, g^F)(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic near $s = 0$.

Definition 7.8.2. Put

$$\mathcal{T}_{\sigma}(g^{TZ}, \nabla^{F,f}, g^F) = \frac{1}{2}\frac{\partial \vartheta_{\sigma}(g^{TZ}, \nabla^{F,f}, g^F)}{\partial s}(0). \tag{7.8.6}$$

Then (7.8.6) is called the equivariant Ray-Singer analytic torsion of the de Rham complex $(\Omega^{\cdot}(Z,F), d^{Z,F})$[RS71, RS73, BG04, BL08].

For $t>0$, as in [BL08, eq.(1.8.5)], put

$$\begin{aligned} b_t(F,g^F) &= \frac{1}{2}\mathrm{Tr_s}\Big[\big(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\big)\sigma^Z(1-t\mathbf{D}^{Z,F,2}/2)\exp(-t\mathbf{D}^{Z,F,2}/4)\Big] \\ &= \frac{1}{2}(1+2t\frac{\partial}{\partial t})\mathrm{Tr_s}\Big[\big(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\big)\sigma^Z\exp(-t\mathbf{D}^{Z,F,2}/4)\Big]. \end{aligned} \tag{7.8.7}$$

Then $b_t(F,g^F)$ is a smooth function in $t>0$.

Put

$$\chi'_\sigma(F)=\sum_{j=0}^{m}(-1)^j j\,\mathrm{Tr}^{H^j(Z,F)}[\sigma]. \tag{7.8.8}$$

By [BL08, eqs.(1.8.7),(1.8.8)], as $t\to 0$,

$$b_t(F,g^F)=\mathcal{O}(\sqrt{t}). \tag{7.8.9}$$

As $t\to+\infty$,

$$b_t(F,g^F)=\frac{1}{2}\chi'_\sigma(F)-\frac{m}{4}\chi_\sigma(F)+\mathcal{O}(1/\sqrt{t}). \tag{7.8.10}$$

Set

$$b_\infty(F,g^F)=\frac{1}{2}\chi'_\sigma(F)-\frac{m}{4}\chi_\sigma(F). \tag{7.8.11}$$

Let $\Gamma(s)$ be the Gamma function. By [BL08, eq.(1.8.11)], we have

$$\begin{aligned} \mathcal{T}_\sigma(g^{TZ},\nabla^{F,f},g^F) = -\int_0^1 b_t(F,g^F)\frac{dt}{t} &- \int_1^{+\infty}(b_t(F,g^F)-b_\infty(F,g^F))\frac{dt}{t} \\ &-(\Gamma'(1)+2(\log(2)-1))b_\infty(F,g^F). \end{aligned} \tag{7.8.12}$$

By (4.5.11), (4.5.17), we get, for $t>0$,

$$\begin{aligned} &\mathrm{Tr_s}\Big[\big(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\big)\sigma^Z\exp(-t\mathbf{D}^{Z,F,2}/4)\Big] \\ &=\sum_{\underline{[\gamma]}_\sigma\in\underline{C}}\mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma)) \\ &\qquad \mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F,2}/4)\Big]. \end{aligned} \tag{7.8.13}$$

For $\gamma\in\Gamma$, if $\gamma\sigma$ is conjugate to an element $e^a k^{-1}\sigma$ as in (7.1.13), put

$$\epsilon(\gamma\sigma)=\dim\mathfrak{b}(e^a k^{-1}\sigma). \tag{7.8.14}$$

Then $\epsilon(\gamma\sigma)$ is an integer which depends only the class $\underline{[\gamma]}_\sigma\in\underline{C}$. We also put

$$\epsilon(\underline{[\gamma]}_\sigma)=\epsilon(\gamma\sigma). \tag{7.8.15}$$

Proposition 7.8.3. *If one of the following three assumptions is verified:*

(1) m is even and σ preserves the orientation of $\mathfrak{p}$;

(2) m is odd and σ does not preserve the orientation of $\mathfrak{p}$;

(3) For $\gamma\in\Gamma$, $\epsilon(\gamma\sigma)\neq 1$,

then we have

$$\mathcal{T}_\sigma(g^{TZ}, \nabla^{F,f}, g^F) = 0 \tag{7.8.16}$$

Proof. If m and σ verify one assumption of the first two cases in Proposition 7.8.3, then by Theorem 7.7.3 and (7.8.13), we get, for $t > 0$,

$$\mathrm{Tr_s}[\big(N^{\Lambda^\cdot(T^*Z)} - \frac{m}{2}\big)\sigma^Z \exp(-t\mathbf{D}^{Z,F,2}/4)] = 0. \tag{7.8.17}$$

By (7.8.7), (7.8.17), we get the function $b_t(F, g^F)$ vanishes identically for $t > 0$. In particular,

$$b_\infty(F, g^F) = 0. \tag{7.8.18}$$

By (7.8.12), we get (7.8.16).

Note that if $\gamma \in \Gamma$ is such that $\gamma\sigma$ is non-elliptic, then $\epsilon(\gamma\sigma) \geq 1$. If the third assumptions is verified, then if $\gamma \in \Gamma$, by Theorem 7.7.3, Corollary 7.7.4, the identity (7.8.17) still holds. Then the same arguments above shows that (7.8.16) holds. This completes the proof of our proposition. □

Note that Proposition 7.8.3 is just an analogue of some classical results of the analytic torsion as in [MS91, Corollary 2.2], [Lot94, Proposition 9], [BL95, Theorem 3.26], [B11, Section 7.9], [BMZ17, Theorem 8.6].

8. A Kirillov formula and the W-invariant

The purpose of this section is to construct a sequence of representations associated with a given representation of U^σ. Also we prove a Kirillov formula to compute the asymptotic behaviour of associated characters.

Also in we recall the construction of the W-invariant of locally symmetric space in [BMZ17], which will be applied to the commutator $Z^0(\gamma\sigma)$.

This section is organized as follows. In subsection 8.1, we recall a fixed point formula of Berline and Vergne [BV85] for U^σ.

In subsection 8.2, when E is a representation of U^σ with a fixed highest weight λ by σ, we construct a sequence of representations of U^σ using the geometry of the flag manifold M_λ, eventually by replacing λ by $d\lambda$.

In subsection 8.3, we recall the constructions of the form W on X associated with the group G. This construction will be applied to the symmetric space $X(\gamma\sigma)$ associated with $Z^0(\gamma\sigma)$ with $\gamma \in K$.

In subsection 8.4, we show that the nondegeneracy condition associated with G implies the nondegeneracy condition associated with $Z^0(\gamma\sigma)$ in the case of the coadjoint orbit of λ.

8.1. **A fixed point formula of Berline and Vergne.** We use the same notation as in subsections 7.5, 7.7. Recall that U is the compact form of G, so that K is a closed subgroup of U. We assume that σ acts on U as an automorphism. Then Σ^σ acts on U. Recall that

$$U^\sigma = U \rtimes \Sigma^\sigma. \tag{8.1.1}$$

Let M be a compact complex manifold equipped with a holomorphic action of U^σ. We denote by TM the holomorphic tangent bundle of M. Let g^{TM} be a U^σ-invariant Hermitian metric on TM.

Let L be a holomorphic line bundle on M, let g^L be a Hermitian metric on L, and let ∇^L denote the corresponding Chern connection. If r^L is the curvature of ∇^L, then

$$c_1(L, g^L) = -\frac{r^L}{2\pi i}. \tag{8.1.2}$$

In the sequel, we assume that $c_1(L, g^L)$ is a positive $(1,1)$-form, i.e., if $B \in TM$, then $-ic_1(L, g^L)(B, \overline{B})$ defines a Hermitian metric on TM. We also assume that the holomorphic action of U^σ on M lifts to a holomorphic unitary action on L.

If $y \in \mathfrak{u}$, let y^M be the associated real vector field on M, and let L^L_y denote the natural action of y on the smooth sections of L, which lifts y^M to L. Then $y^{M(1,0)}$ is a holomorphic section of TM. Let $\mu : M \to \mathfrak{u}^*$ be the map such that

$$\mathrm{L}^L_y = \nabla^L_{y^M} - 2\pi i \langle \mu, y\rangle. \tag{8.1.3}$$

We call μ the moment map associated with the action of U on L.

If y_1, $y_2 \in \mathfrak{u}$, then by [BMZ17, eq.(3.8)], we have

$$\langle \mu, [y_1, y_2]\rangle = c_1(L, g^L)(y_1^M, y_2^M). \tag{8.1.4}$$

Let ∇^{TM} be the Chern connection on TM, and let R^{TM} be its curvature. If $y \in \mathfrak{u}$, let L_y^{TM} be the natural action of y on the smooth sections of TM. Let $\nu^{TM}(y)$ be the map given by

$$2\pi i \nu^{TM}(y) = \nabla_{y^M}^{TM} - \mathrm{L}_y^{TM}. \tag{8.1.5}$$

Set

$$\xi = \det TM. \tag{8.1.6}$$

Then ξ is a holomorphic line bundle on M.

The metric g^{TM} induces a Hermitian metric g^{ξ} on ξ, and U^{σ} acts holomorphically and isometrically on ξ. Also the analogues of (8.1.2) - (8.1.4) hold. Let ∇^{ξ} denote the corresponding Chern connection on ξ, and let r^{ξ} be the curvature of ∇^{ξ}. Then

$$r^{\xi} = \mathrm{Tr}[R^{TM}]. \tag{8.1.7}$$

By (8.1.2), we have

$$c_1(\xi, g^{\xi}) = -\frac{1}{2\pi i}\mathrm{Tr}[R^{TM}]. \tag{8.1.8}$$

Let $\nu : M \to \mathfrak{u}^*$ be moment map associated with the action of U on M and $c_1(\xi, g^{\xi})$. Then by (8.1.3),

$$\mathrm{L}_y^{\xi} = \nabla_{y^M}^{\xi} - 2\pi i \langle \nu, y \rangle. \tag{8.1.9}$$

By (8.1.5), (8.1.9), we have

$$\mathrm{Tr}[\nu^{TM}(y)] = \langle \nu, y \rangle. \tag{8.1.10}$$

Put $L^d = L^{\otimes d}$. If $i \geq 0$, $H^{(0,i)}(M, L^d)$ is a finite-dimensional representation of U^{σ}. By Kodaira's vanishing theorem, for $d \in \mathbb{N}$ such that $c_1(L^d \otimes \xi, g^{L^d \otimes \xi}) > 0$, if $i > 0$, $H^{(0,i)}(M, L^d)$ vanishes.

If B is a complex (q, q) matrix, put

$$|B| = \sup_{s \in \mathrm{Sp}(B)} |s|. \tag{8.1.11}$$

If B is such that $|B| < 2\pi$, set

$$\mathrm{Td}(B) = \det \frac{B}{1 - e^{-B}}. \tag{8.1.12}$$

If B is Hermitian, no condition on B is necessary to define $\mathrm{Td}(B)$.

Set

$$e(B) = \det B. \tag{8.1.13}$$

We fix $u_0 \in U^{\sigma}$. Put

$$\underline{Z} = U^{\sigma}(u_0) \cap U. \tag{8.1.14}$$

Let $\underline{Z}^0$ be the connected component of $\underline{Z}$ containing the identity, and let $\mathfrak{z} \subset \mathfrak{u}$ be the Lie algebra of $\underline{Z}$.

Let ${}^{u_0}M$ be the fixed point set of M. Then ${}^{u_0}M$ is a complex submanifold of M, and the group $\underline{Z}$ acts holomorphically and isometrically on ${}^{u_0}M$.

If $x \in {}^{u_0}M$, let $e^{i\theta_1}, \cdots, e^{i\theta_l}$, $0 \leq \theta_1, \cdots, \theta_l < 2\pi$ be the distinct eigenvalues of u_0 acting on T_xM. Since u_0 is parallel, these eigenvalues are locally constant on ${}^{u_0}M$. Then $TX|_{{}^{u_0}M}$ splits holomorphically as an orthogonal sum of the subbundles TM^{θ_j}. The Chern connection $\nabla^{TM|_{{}^{u_0}M}}$ on $TM|_{{}^{u_0}M}$ splits as the sum of the Chern connection on TM^{θ_j}. Let R^{θ_j} denote the corresponding curvature.

The equivariant Todd genus is given by

$$\mathrm{Td}^{u_0}(TM|_{{}^{u_0}M}, g^{TM|_{{}^{u_0}M}}) = \mathrm{Td}(-\frac{R^0}{2\pi i}) \prod_{\theta_j \neq 0} \big(\frac{\mathrm{Td}}{e}\big)\big(-\frac{R^{\theta_j}}{2\pi i} + i\theta_j\big). \tag{8.1.15}$$

If $y \in \underline{\mathfrak{z}}$, let $\nu^{TM|_{{}^{u_0}M}}(y)$ be the restriction of $\nu^{TM}(y)$ to ${}^{u_0}M$, which is given by the same formula as in (8.1.5) with respect to the action of Z on $TM|_{{}^{u_0}M}$.

The action of $\nu^{TM|_{{}^{u_0}M}}(y)$ preserves the splitting of $TM|_{{}^{u_0}M}$. Then the equivariant Todd genus $\mathrm{Td}^{u_0}_y(TM|_{{}^{u_0}M}, g^{TM|_{{}^{u_0}M}})$ is given by the same formula in (8.1.15) when replacing $-\frac{R^{\theta_j}}{2\pi i}$ by the equivariant curvature $-\frac{R^{\theta_j}}{2\pi i} + \nu^{TM^{\theta_j}}(y)$. Also, we denote by $\mathrm{Td}^{u_0}_y(TM|_{{}^{u_0}M})$ the equivariant cohomology class of $\mathrm{Td}^{u_0}_y(TM|_{{}^{u_0}M}, g^{TM|_{{}^{u_0}M}})$. We refer to [BV85], [BGV04, Chapter 7] for more details.

If $x \in {}^{u_0}M$, u_0 acts on L_x by a complex number $\rho^L(u_0)$ of modulo 1. The equivariant Chern character form of $L|_{{}^{u_0}M}$ is given by

$$\mathrm{ch}^{u_0}_y(L|_{{}^{u_0}M}, g^{L|_{{}^{u_0}M}}) = \rho^L(u_0) \exp\big(2\pi i\langle \mu, y\rangle + c_1(L|_{{}^{u_0}M}, g^{L|_{{}^{u_0}M}})\big). \tag{8.1.16}$$

If $d \in \mathbb{N}$, then

$$\mathrm{ch}^{u_0}_y(L^d|_{{}^{u_0}M}, g^{L^d|_{{}^{u_0}M}}) = \rho^L(u_0)^d \exp\big(2\pi i d\langle \mu, y\rangle + dc_1(L|_{{}^{u_0}M}, g^{L|_{{}^{u_0}M}})\big). \tag{8.1.17}$$

For $u \in U^\sigma$, set

$$\chi_{L,d}(u) = \mathrm{Tr_s}^{H^{(0,\cdot)}(M,L^d)}[u]. \tag{8.1.18}$$

By [BV85, Theorem 3.23], if y is in a small neighbourhood of $\underline{\mathfrak{z}}$, we have

$$\begin{aligned}\chi_{L,d}(u_0e^y) = \int_{{}^{u_0}M} & \mathrm{Td}^{u_0}_y(TM|_{{}^{u_0}M})\rho^L(u_0)^d \\ & \exp\big(2\pi i d\langle \mu, y\rangle + dc_1(L|_{{}^{u_0}M}, g^{L|_{{}^{u_0}M}})\big).\end{aligned} \tag{8.1.19}$$

8.2. **A sequence of unitary representations of U^σ.** Let $\mathfrak{u}^{\mathrm{reg}}$ be the set of regular elements in $\mathfrak{u}$.

Lemma 8.2.1. *We have*

$$\mathfrak{u}(\sigma) \cap \mathfrak{u}^{\mathrm{reg}} \neq \emptyset. \tag{8.2.1}$$

Proof. If $\mathfrak{u}$ is Abelian, then $0 \in \mathfrak{u}^{\mathrm{reg}}$. If $\mathfrak{u}$ is not Abelian, as in (7.5.32), if $\mathfrak{z}(\mathfrak{u})$ is the center of $\mathfrak{u}$, then

$$\mathfrak{u} = \mathfrak{z}(\mathfrak{u}) \oplus [\mathfrak{u}, \mathfrak{u}]. \tag{8.2.2}$$

Since $[\mathfrak{u}, \mathfrak{u}]$ is semisimple, it is isomorphic to the Lie algebra of $\mathrm{Aut}(U)$. In particular, if $\sigma \in \mathrm{Aut}(U)$, by [DK00, Lemma (3.15.4)], $[\mathfrak{u}, \mathfrak{u}](\sigma)$ contains regular elements in $[\mathfrak{u},\mathfrak{u}]$. By (8.2.2), a regular element in $[\mathfrak{u},\mathfrak{u}]$ is also regular in $\mathfrak{u}$. This completes the proof. □

We fix $v \in \mathfrak{u}(\sigma)\cap\mathfrak{u}^{\mathrm{reg}}$. If $\mathfrak{t}' = \mathfrak{u}(v)$, then $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{u}$. Let $T' \subset U$ be the corresponding maximal torus. Let R_U be the associated root system, and let W_U be the associated Weyl group. Let $\mathfrak{c} \subset \mathfrak{t}'$ be the Weyl chamber containing v, and let $P_{++}(\mathfrak{c})$ be set of the dominant weights on $\mathfrak{u}$ with respect to $\mathfrak{c}$.

Put

$$\mathfrak{a} = \mathfrak{t}' \cap \mathfrak{u}(\sigma). \tag{8.2.3}$$

Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{u}(\sigma)$. Let $A \subset U^0(\sigma)$ be the corresponding maximal torus.

Let (E,ρ^E) be an irreducible unitary representation of U^σ, and let χ^E be its character. Since σ preserves R_{U+}, then by Theorem 2.4.5, $\rho^E(\sigma)$ permutes the different U-irreducible components in E. If (E,ρ^E) is not irreducible as a U-representation, then if $u \in U$, $\chi^E(u\sigma) = 0$.

We may and we will assume that (E,ρ^E) is also irreducible as a U-representation. Let $\lambda \in P_{++}(\mathfrak{c})$ be the highest weight of (E,ρ^E). Then

$$\sigma \cdot \lambda = \lambda. \tag{8.2.4}$$

Then $\lambda \in \mathfrak{a}^*$.

Set

$$U^\sigma(\lambda) = \{u \in U^\sigma \ : \ u \cdot \lambda = \lambda\}. \tag{8.2.5}$$

Put

$$U(\lambda) = U^\sigma(\lambda) \cap U. \tag{8.2.6}$$

Then

$$U^\sigma(\lambda) = U(\lambda) \rtimes \Sigma^\sigma. \tag{8.2.7}$$

Recall that the group $N_{U^\sigma}(\mathfrak{c})$ is defined as in (7.3.20) for the group U^σ, then

$$N_{U^\sigma}(\mathfrak{c}) \subset U^\sigma(\lambda). \tag{8.2.8}$$

By [W73, Lemma 6.2.2], $U(\lambda)$ is a connected compact subgroup of U. Moreover, by [P09, Propositions 1.2.20 and 1.2.22], there exists a torus

$$T_1 \subset A \tag{8.2.9}$$

such that

$$U(\lambda) = Z_U(T_1). \tag{8.2.10}$$

It is clear that

$$T' \subset U(\lambda). \tag{8.2.11}$$

If λ is regular, we have $U(\lambda) = T'$ and $T_1 = A$.

Let $\mathfrak{u}(\lambda)$ be the Lie algebra of $U(\lambda)$, and let $\mathfrak{t}_1$ be the Lie algebra of T_1. Then

$$\mathfrak{t}_1 \subset \mathfrak{t}' \subset \mathfrak{u}(\lambda). \tag{8.2.12}$$

Let R_1 denote the root system of $(\mathfrak{u}(\lambda), \mathfrak{t}')$. Then

$$R_1 \subset R_U. \tag{8.2.13}$$

Definition 8.2.2. A Weyl chamber $\mathfrak{c}'$ relative to $(\mathfrak{u}, \mathfrak{t}')$ is called T_1-admissible if there exists a Weyl chamber $\mathfrak{c}_1$ of $(\mathfrak{u}(\lambda), \mathfrak{t}')$ such that if R_{U+}, $R_{1,+}$ are the positive roots systems of R_U, R_1 associated with $\mathfrak{c}'$, $\mathfrak{c}_1$, then

(1) $R_{1,+} = R_{U+} \cap R_1$;
(2) If $\alpha \in R_{U+} \backslash R_{1,+}$, $\alpha' \in R_1$, if $\alpha + \alpha' \in R_U$, then $\alpha + \alpha' \in R_{U+} \backslash R_{1,+}$.

By [W73, Lemma 6.2.9], there always exists T_1-admissible Weyl chamber $\mathfrak{c}'$ of $(\mathfrak{u}, \mathfrak{t}')$. Put

$$\mathfrak{b}_+ = \sum_{\alpha \in R_{U+} \backslash R_{1,+}} \mathfrak{u}_\alpha. \tag{8.2.14}$$

One can verify that

$$[\mathfrak{u}(\lambda), \mathfrak{b}_+] \subset \mathfrak{b}_+, \quad [\mathfrak{b}_+, \mathfrak{b}_+] \subset \mathfrak{b}_+. \tag{8.2.15}$$

Set

$$M_\lambda = U/U(\lambda). \tag{8.2.16}$$

Then by [W73, Lemma 6.2.13], M_λ is a complex manifold with

$$TM_\lambda = U \times_{U(\lambda)} \mathfrak{b}_+. \tag{8.2.17}$$

Moreover, U acts holomorphically and isometrically on M_λ. Put n_λ the complex dimension of M_λ.

In fact, if λ is regular, then we can take $\mathfrak{c}' = \mathfrak{c}$, M_λ is a complex manifold which does not depend on λ. Put

$$M = U/T'. \tag{8.2.18}$$

We have a holomorphic projection

$$p_\lambda : M \to M_\lambda. \tag{8.2.19}$$

Moreover, p_λ is U-equivariant.

Since σ preserves $U(\lambda)$, the group U^σ acts on M_λ. We have the identification of homogeneous spaces,

$$M_\lambda = U^\sigma / U^\sigma(\lambda). \tag{8.2.20}$$

Use the arguments in [W73, Proof of Lemma 6.2.9], there always exists a T_1-admissible Weyl chamber $\mathfrak{c}'$ such that R_{U+} is preserved by σ, and that λ is a dominant weight with respect to $\mathfrak{c}'$. Then the action of σ preserves $\mathfrak{b}_+$, and the holomorphic action of U on M_λ extends to a holomorphic action of U^σ on M_λ.

By (8.2.17), (8.2.20), we get

$$TM_\lambda = U^\sigma \times_{U^\sigma(\lambda)} \mathfrak{b}_+. \tag{8.2.21}$$

Let $E^{\mathfrak{b}_+} \subset E$ be the vector space

$$E^{\mathfrak{b}_+} = \{w \in E \,:\, \text{if } v \in \mathfrak{b}_+, \text{ then } \rho^E(v)w = 0\} \tag{8.2.22}$$

Then $E^{\mathfrak{b}_+}$ is preserved by $U^\sigma(\lambda)$. Recall that E_λ is the highest weight line of E.

Lemma 8.2.3. *We have*

$$E^{\mathfrak{b}+} = E_\lambda. \tag{8.2.23}$$

Moreover, the differential of the representation E_λ of $U(\lambda)$ at $1 \in U(\lambda)$ is given by $2\pi i\lambda : \mathfrak{u}(\lambda) \to \mathbb{C}$.

Proof. Clearly, $E_\lambda \subset E^{\mathfrak{b}+}$. We only need to prove that $\dim_{\mathbb{C}} E^{\mathfrak{b}+} = 1$.

We claim that $E^{\mathfrak{b}+}$ is an irreducible unitary representation of $U(\lambda)$ with highest weight λ. Indeed, if there are two linearly independent, non-zero highest weight vectors in $E^{\mathfrak{b}+}$ as a $U(\lambda)$ representation, then these two vectors are also the highest weight vectors with respect to the action of U on E, so that this is a contradiction with the assumption that E is U-irreducible.

If $\alpha \in R_{1,+}$, then $\langle \alpha, \lambda \rangle = 0$. Let $\rho_{\mathfrak{u}(\lambda)}$ be the element defined in (2.2.6) for $R_{1,+}$. Then by the dimension formula [BtD85, Chapter VI, Theorem (1.7)], we have

$$\dim_{\mathbb{C}} E^{\mathfrak{b}+} = \prod_{\alpha \in R_{1,+}} \frac{\langle \alpha, \lambda + \rho_{\mathfrak{u}(\lambda)} \rangle}{\langle \alpha, \rho_{\mathfrak{u}(\lambda)} \rangle} = 1. \tag{8.2.24}$$

□

Put

$$L_\lambda = U^\sigma \times_{U^\sigma(\lambda)} E_\lambda. \tag{8.2.25}$$

Then L_λ is a holomorphic line bundle over M_λ with the U^σ-invariant Hermitian metric g^{L_λ}, and $c_1(L_\lambda, g^{L_\lambda})$ is a closed symplectic $(1,1)$-form. The action of U^σ on M_λ lifts to a holomorphic unitary action on L_λ.

By [W73, Theorem 6.3.7], $H^{(0,0)}(M_\lambda, L_\lambda)$ is a unitary representation of U^σ isomorphic to (E, ρ^E). If $d \in \mathbb{N}_{>0}$, put

$$E_d = H^{(0,0)}(M_\lambda, L_\lambda^d). \tag{8.2.26}$$

Then (E_d, ρ^{E_d}) is an irreducible unitary representation of U^σ associated with the highest weight $d\lambda \in P_{++}(\mathfrak{c}')$. Let χ_d be the character of U^σ associated with (E_d, ρ^{E_d}).

By the results in subsection 8.1, the character χ_d is given by (8.1.19). In the sequel, if $u \in U^\sigma$, we will give an explicit description for the fixed point set of u in M_λ.

Put

$$N_U(T')(\sigma) = \{u \in N_U(T') \ : \ \mathrm{Ad}(u)|_{\mathfrak{t}'} \text{ commutes with } \sigma|_{\mathfrak{t}'}\}. \tag{8.2.27}$$

Then $N_U(T')(\sigma)$ is a closed subgroup of $N_U(T')$. Let $N_U(A)$ be the normalizer of A in U, then one can verify that

$$N_U(T')(\sigma) = N_U(A). \tag{8.2.28}$$

If $u \in N_U(T')(\sigma)$, then

$$u \cdot \lambda \in \mathfrak{a}^*. \tag{8.2.29}$$

If $\gamma \in U$, put $u_0 = \gamma\sigma \in U^\sigma$, then by [Seg68, Proposition I.4], there exists $u \in U$, $t_0 \in A$ such that

$$u_0 = ut_0\sigma(u^{-1})\sigma. \tag{8.2.30}$$

Remark 8.2.4. Since in general σ is not of finite order, (8.2.30) is not a direct consequence of [Seg68, Proposition I.4]. But a slight modification of the proof of [Seg68, Proposition I.4] will extend its conclusion to our case.

As in (8.1.14), put $\underline{Z} = U(u_0)$, and let $\underline{\mathfrak{z}}$ be its Lie algebra. Put

$$a_0 = t_0\sigma \in A^\sigma. \tag{8.2.31}$$

Let $U(a_0)$ be the centralizer of a_0 in U, and let $U^0(a_0)$ be its identity connected component. Moreover,

$$\mathfrak{t}' \cap \mathfrak{u}(a_0) = \mathfrak{a}. \tag{8.2.32}$$

Then A is also a maximal torus of $U^0(a_0)$.

By (8.2.30), we get

$$\underline{Z} = uU(a_0)u^{-1}, \;\; \underline{Z}^0 = uU^0(a_0)u^{-1}. \tag{8.2.33}$$

Then $\mathrm{Ad}(u)(A)$ is a maximal torus of $\underline{Z}$.

Let ${}^{u_0}M$ be the fixed point set of u_0 in M, let ${}^{u_0}M_\lambda$ be the fixed point set of u_0 in M_λ. If $u' \in U$, let $[u']_\lambda$, $[u']$ denote, respectively, the corresponding points in M_λ, M.

Lemma 8.2.5. *We have*

$${}^{u_0}M_\lambda = \underline{Z}^0 uN_U(T')(\sigma)U(\lambda)/U(\lambda) \subset M_\lambda. \tag{8.2.34}$$

Moreover, ${}^{u_0}M_\lambda$ has finite connected components. If $u' \in uN_U(T')(\sigma)$, then the connected component of $[u']_\lambda$ is isomorphic to the flag manifold $\underline{Z}^0/\underline{Z}^0(u' \cdot \lambda)$ as complex manifolds.

Proof. If λ is regular, then $M_\lambda = M$, and (8.2.34) is just an equivalent version of the results in [DHV84, I.2 : Lemme (7)] and [Bou87, Lemme 6.1.1]. In general, (8.2.34) can be regarded as a consequence of [Bou87, Lemma 7.2.2], where the author use a different formulation. Here we give a proof of our lemma using our notation.

One can verify that the left-hand side of (8.2.34) does not depend on the choice u and t_0 satisfying (8.2.30). Let ${}^{a_0}M_\lambda$ be the fixed point set of a_0, by (8.2.30), we have

$${}^{u_0}M_\lambda = u \cdot {}^{a_0}M_\lambda. \tag{8.2.35}$$

We claim that

$$p_\lambda({}^{u_0}M) = {}^{u_0}M_\lambda. \tag{8.2.36}$$

Indeed, the first set in (8.2.36) is included in the second set. If $u \in U$ is such that $[u]_\lambda \in {}^{u_0}M_\lambda$, then

$$u^{-1}u_0u \in U^\sigma(\lambda). \tag{8.2.37}$$

As in (8.2.30), there exists $g \in U(\lambda)$ such that

$$g^{-1}u^{-1}u_0 ug \in A^{\sigma}. \tag{8.2.38}$$

Then $[ug] \in {}^{u_0}M$, and $p_\lambda([ug]) = [u]_\lambda$.

The same arguments in the proof of [Bou87, Lemme 6.1.1] shows that

$${}^{a_0}M = U^0(a_0)N_U(T')(\sigma)/T' \subset M. \tag{8.2.39}$$

By(8.2.33), (8.2.35), (8.2.36), (8.2.39), we get (8.2.34).

Recall that $N_U(T')(\sigma)/T'$ is finite, by (8.2.11), $N_U(T')(\sigma)U(\lambda)/U(\lambda)$ is a finite set. Then ${}^{u_0}M_\lambda$ is a finite union of $\underline{Z}^0$-orbits in M_λ.

Fix $u' \in uN_U(T')(\sigma)$ and $x = [u']_\lambda \in {}^{u_0}M_\lambda$. Then the centralizer of x in $\underline{Z}^0$ is just the subgroup $\underline{Z}^0(u' \cdot \lambda)$.

Let $t \in N_{U^\sigma}(\mathfrak{c}') \subset U^\sigma(\lambda)$ be such that

$$u_0 u' = u't. \tag{8.2.40}$$

Then by (8.2.21), the action of u_0 on $T_x M_\lambda$ is identified with the adjoint action of t on $\mathfrak{b}_+$. Then

$$T_x {}^{u_0}M_\lambda = \mathfrak{b}_+(t). \tag{8.2.41}$$

It is clear that $\mathrm{Ad}(u')\mathfrak{b}_+(t) \subset \underline{\mathfrak{z}}_{\mathbb{C}}$, and then we can take the complex structure on $\underline{Z}^0/\underline{Z}^0(u'\cdot\lambda)$ such that $\mathrm{Ad}(u')\mathfrak{b}_+(t)$ is identified with the holomorphic tangent bundle of $\underline{Z}^0/\underline{Z}^0(u' \cdot \lambda)$. This completes the proof of our lemma.

□

Remark 8.2.6. If $\lambda \in \mathfrak{a}^*$, let $\mathcal{O}_\lambda \subset \mathfrak{u}^*$ denote the coadjoint orbit of λ in $\mathfrak{u}^*$. Then

$$\mathcal{O}_\lambda \simeq M_\lambda. \tag{8.2.42}$$

Moreover, the moment map μ on M_λ is just given by the inclusion $\mathrm{i} : \mathcal{O}_\lambda \hookrightarrow \mathfrak{u}^*$.

By [BMZ17, eq.(8.123)], we have

$$\mathcal{O}_\lambda \cap (\mathfrak{t}')^* = W_U \cdot \lambda. \tag{8.2.43}$$

The fixed points set of u_0 in $\mathcal{O}_\lambda$ is given by $\mathcal{O}_\lambda \cap \underline{\mathfrak{z}}^*$. Then by (8.2.42), we get

$${}^{u_0}M \simeq \mathcal{O}_\lambda \cap \underline{\mathfrak{z}}^*. \tag{8.2.44}$$

Since the set

$$u \cdot \big(\mathcal{O}_\lambda \cap \mathfrak{a}^*\big) = \mathcal{O}_\lambda \cap \mathrm{Ad}(u)(\mathfrak{a}^*) \tag{8.2.45}$$

is finite, we get that $\mathcal{O}_\lambda \cap \underline{\mathfrak{z}}^*$ is a finite union of $\underline{Z}^0$-orbits. From this formalism, we get another proof of Lemma 8.2.5.

By Lemma 8.2.5, let ${}^{u_0}M_\lambda^j$, $j \in \mathcal{J}$ be distinct connected components of ${}^{u_0}M_\lambda$. In particular, $\mathcal{J}$ is a finite set. Take $u^j \in uN_U(T')(\sigma)$ such that

$$x_j = [u^j]_\lambda \in {}^{u_0}M_\lambda^j. \tag{8.2.46}$$

Then by Lemma 8.2.5, we have the identification

$${}^{u_0}M_\lambda^j \simeq \underline{Z}^0/\underline{Z}^0(u^j \cdot \lambda). \tag{8.2.47}$$

We will denote

$$h^j = (u^j)^{-1} u_0 u^j \in N_{U^\sigma}(\mathfrak{c}') \subset U^\sigma(\lambda). \tag{8.2.48}$$

If $j \in \mathcal{J}$, set

$$n_j = \dim_{\mathbb{C}} \underline{Z}^0 / \underline{Z}^0(u^j \cdot \lambda). \tag{8.2.49}$$

Since $\underline{Z}$ acts on ${}^{u_0}M$, then we can divide ${}^{u_0}M$ into different $\underline{Z}$-orbits, in particular, $\underline{Z}$ will permute the different connected components. If $j \in \mathcal{J}$, let $[j]$ denote the set of indexes j' in $\mathcal{J}$ such that ${}^{u_0}M_\lambda^{j'}$ and ${}^{u_0}M_\lambda^j$ lie in the same $\underline{Z}$-orbits.

Let $C\mathcal{J}$ be the set of classes $[j]$ in $\mathcal{J}$. Suppose that

$$C\mathcal{J} = \{[j_1], \cdots, [j_s]\}. \tag{8.2.50}$$

We will denote the class $[j_\ell]$ by κ_ℓ, $\ell = 1, \cdots, s$. Moreover, the dimension n_j only depends on the class $[j]$, we will denote $n_{\kappa_\ell} = n_{j_\ell}$.

If $x \in {}^{u_0}M_\lambda$, there exist $j \in \mathcal{J}$, $z \in \underline{Z}$ such that

$$x = z x_j. \tag{8.2.51}$$

Then $zu^j \in U$ is a representative of the point x. By (8.2.48), we have

$$u_0 z u^j = z u^j h^j. \tag{8.2.52}$$

Recall that $\rho^{L_\lambda}(u_0) : {}^{u_0}M_\lambda \to \mathbb{S}^1$ represents the action of u_0 on $L_\lambda|_{{}^{u_0}M_\lambda}$. Then by (8.2.52), we get at x,

$$\rho^{L_\lambda}(u_0) = \rho^{E_\lambda}(h^j). \tag{8.2.53}$$

By (8.2.53), $\rho^{L_\lambda}(u_0)$ is constant on each $\underline{Z}$-orbits, it is a locally constant function on ${}^{u_0}M_\lambda$.

Let $\underline{\mathfrak{z}}(u^j \cdot \lambda)$ be the Lie algebra of $\underline{Z}^0(u^j \cdot \lambda)$, and let $\underline{\mathfrak{z}}^\perp(u^j \cdot \lambda)$ be the orthogonal of $\underline{\mathfrak{z}}(u^j \cdot \lambda)$ in $\underline{\mathfrak{z}}$. Put

$$\underline{\mathfrak{q}}(u^j \cdot \lambda) = \underline{\mathfrak{q}} \cap \mathfrak{u}(u^j \cdot \lambda). \tag{8.2.54}$$

Let $\underline{\mathfrak{q}}^\perp(u^j \cdot \lambda)$ be the orthogonal of $\underline{\mathfrak{q}}(u^j \cdot \lambda)$ in $\underline{\mathfrak{q}}$. Then

$$\mathfrak{u}(u^j \cdot \lambda) = \underline{\mathfrak{z}}(u^j \cdot \lambda) \oplus \underline{\mathfrak{q}}(u^j \cdot \lambda). \tag{8.2.55}$$

Let $\mathrm{Td}^{u_0}(TM_\lambda|_{{}^{u_0}M_\lambda}, g^{TM_\lambda|_{{}^{u_0}M_\lambda}})$ be the equivariant Todd genus on ${}^{u_0}M_\lambda$ defined in subsection 8.1, and let the function φ_{u_0} on ${}^{u_0}M_\lambda$ denote the component of degree zero of $\mathrm{Td}^{u_0}(TM_\lambda|_{{}^{u_0}M_\lambda}, g^{TM_\lambda|_{{}^{u_0}M_\lambda}})$.

By (7.2.5), (8.1.12), (8.1.15), if $x = [u]_\lambda \in {}^{u_0}M_\lambda$, then

$$\varphi_{u_0}(x) = \frac{1}{\det{}^{1/2}(\mathrm{Ad}(u_0)|_{\underline{\mathfrak{q}}^\perp(u\cdot\lambda)})} \widehat{A}^{u_0|_{\underline{\mathfrak{q}}^\perp(u\cdot\lambda)}}(0). \tag{8.2.56}$$

It is clear that φ_{u_0} is a locally constant function on ${}^{u_0}M_\lambda$. In particular, if $x \in {}^{u_0}M_\lambda^j$, then

$$\varphi_{u_0}(x) = \varphi_{u_0}(x_j). \tag{8.2.57}$$

Moreover, $\varphi_{u_0}(x_j)$ only depends on the class $[j] \in C\mathcal{J}$.

Definition 8.2.7. Put

$$n(u_0) = \max\{n_j \ : \ j \in \mathcal{J}\}. \tag{8.2.58}$$

Let $C\mathcal{J}(u_0) \subset C\mathcal{J}$ be the set of κ_ℓ such that $n_{\kappa_\ell} = n(u_0)$.

Recall that $\mu : M_\lambda \to \mathfrak{u}^*$ is the moment map associated with the action of U on $L_\lambda \to M_\lambda$.

Definition 8.2.8. If $y \in \underline{\mathfrak{z}}$, $j \in \mathcal{J}$, set

$$R^j_{u_0,\lambda}(y) = \int_{{}^{u_0}M^j_\lambda} \exp\big(2\pi i\langle \mu, y\rangle + c_1(L_\lambda|_{{}^{u_0}M}, g^{L_\lambda|_{{}^{u_0}M}})\big). \tag{8.2.59}$$

If $\ell \in \{1, \cdots, s\}$, set

$$R^{\kappa_\ell}_{u_0,\lambda}(y) = \varphi_{u_0}(x_{j_\ell}) \sum_{j\in\kappa_\ell} R^j_{u_0,\lambda}(y). \tag{8.2.60}$$

Note that $R^j_{u_0,\lambda}(y)$ is just of the same type as the functions defined in [BMZ17, Section 1.4]. We can verify that $R^j_{u_0,\lambda}$ is a $\underline{Z}^0$-invariant function on $\underline{\mathfrak{z}}$, and that $R^{\kappa_\ell}_{u_0,\lambda}$ is a $\underline{Z}$-invariant function on $\underline{\mathfrak{z}}$. Also $R^j_{u_0,\lambda}(y)$ can be computed by the localization formulas in [BGV04, Chapter 7], [DH82, DH83].

Let $\Delta^{\underline{\mathfrak{z}}}$ be the standard Laplace on $\underline{\mathfrak{z}}$, then by [BMZ17, eq.(8.146)], we have

$$\Delta^{\underline{\mathfrak{z}}} R^j_{u_0,\lambda} = -4\pi^2|\lambda|^2 R^j_{u_0,\lambda}. \tag{8.2.61}$$

Proposition 8.2.9. *If $y \in \underline{\mathfrak{z}}$, as $d \to +\infty$, then*

$$\chi_d(e^{y/d}u_0) = d^{n(u_0)} \sum_{\kappa_\ell \in C\mathcal{J}(u_0)} \rho^{E_\lambda}(h^{j_\ell})^d R^{\kappa_\ell}_{u_0,\lambda}(y) + \mathcal{O}(d^{n(u_0)-1}). \tag{8.2.62}$$

Proof. By (8.1.19), as $d \to +\infty$, the leading term of $\chi_d(e^{y/d}u_0)$ is given by the integrals over the connected components of ${}^{u_0}M_\lambda$ with maximal dimension $n(u_0)$. Then using (8.1.19), (8.2.53), (8.2.56), (8.2.59), we get (8.2.62). □

8.3. **The forms e_t, d_t and the W-invariant.** In this subsection, we recall the construction of the forms e_t, d_t introduced in [BMZ17].

Let $S\mathfrak{g}$ be the symmetric algebra of $\mathfrak{g}$, which can be identified with the algebra of real differential operators with constant coefficients on $\mathfrak{g}$. Let $\boldsymbol{\sigma} : U\mathfrak{g} \to S\mathfrak{g}$ be the symbol map of $U\mathfrak{g}$, which is also an isomorphism of vector spaces. For instance, if $u, v \in \mathfrak{g}$,

$$\boldsymbol{\sigma}(uv) = \frac{1}{2}(uv + vu) + \frac{1}{2}[u, v]. \tag{8.3.1}$$

Let $\widehat{\mathfrak{p}}$ be another copy of $\mathfrak{p}$. Recall that the symbol map of Clifford algebras is defined in Definition 3.1.1, then we get a symbol map

$$\boldsymbol{\sigma} : \widehat{c}(\widehat{\mathfrak{p}}) \otimes U\mathfrak{g} \to \Lambda^\cdot(\widehat{\mathfrak{p}}^*) \otimes S\mathfrak{g}, \tag{8.3.2}$$

which is an identification of filtered $\mathbb{Z}_2$-graded vector spaces.

Recall that $\omega^{\mathfrak{p}}$ is the left-invariant 1-form on G with values in $\mathfrak{p}$. Let $e_1, \cdots, e_m$ be an orthonormal basis of $\mathfrak{p}$, then $\widehat{e}_1, \cdots, \widehat{e}_m$ is a basis of $\widehat{\mathfrak{p}}$, and let $\widehat{e}^1, \cdots, \widehat{e}^m$ be the corresponding dual basis of $\widehat{\mathfrak{p}}^*$. Put

$$\beta = \sum_{i=1}^{m} \widehat{e}^i e_i \in \widehat{\mathfrak{p}}^* \otimes \mathfrak{g}. \tag{8.3.3}$$

By [BMZ17, eq.(1.8)], $\beta^2 \in \Lambda^2(\widehat{\mathfrak{p}}^*) \otimes \mathfrak{k}$ is given by

$$\beta^2 = \frac{1}{2}[\beta, \beta] = \frac{1}{2}\widehat{e}^i \widehat{e}^j [e_i, e_j]. \tag{8.3.4}$$

Let $\underline{\beta}$ be the corresponding element of β in $\Lambda^\cdot(\widehat{\mathfrak{p}}^*) \otimes U\mathfrak{g}$. Then $\underline{\beta}^2 \in \Lambda^2(\widehat{\mathfrak{p}}^*) \otimes U\mathfrak{g}$ coincides with β^2 in (8.3.4).

Let $|\beta|^2 \in S\mathfrak{g}$ be given by

$$|\beta|^2 = \sum_{i=1}^{m} e_i^2. \tag{8.3.5}$$

Let $\Delta^{\mathfrak{p}}$ be the Laplacian of Euclidean space $\mathfrak{p}$, then

$$|\beta|^2 = \Delta^{\mathfrak{p}}. \tag{8.3.6}$$

By [BMZ17, eq.(1.10)], we have

$$|\beta|^2 \in S^2\mathfrak{g} \cap S^2\mathfrak{u}, \; |\beta|^2 = -|i\beta|^2 \in S^2\mathfrak{g}_{\mathbb{C}}. \tag{8.3.7}$$

Set

$$|\underline{\beta}|^2 = \sum_{i=1}^{p} \underline{\beta}(\widehat{e}_i)^2 \in U\mathfrak{g}. \tag{8.3.8}$$

By [BMZ17, eq.(1.14)], we have

$$|\underline{\beta}|^2 \in U\mathfrak{g} \cap U\mathfrak{u}, \; |\underline{\beta}|^2 = -|i\underline{\beta}|^2 \in U\mathfrak{g}_{\mathbb{C}}. \tag{8.3.9}$$

Then

$$\boldsymbol{\sigma}(|\underline{\beta}|^2) = |\beta|^2. \tag{8.3.10}$$

Set

$$\widehat{c}(\underline{\beta}) = \sum_{i=1}^{m} \widehat{c}(\widehat{e}_i)\underline{\beta}(\widehat{e}_i) \in \widehat{c}(\widehat{\mathfrak{p}}) \otimes U\mathfrak{g}. \tag{8.3.11}$$

Then we have

$$\boldsymbol{\sigma}(\widehat{c}(\underline{\beta})) = \beta. \tag{8.3.12}$$

Recall that $TX \oplus N = G \times_K \mathfrak{g}$. Note that the Lie bracket of $\mathfrak{g}$ lifts to a Lie bracket on the fiber of $TX \oplus N$. In the sequel, let $\mathfrak{g}_r$ be a copy of $TX \oplus N$ equipped with the Lie bracket on the fiber, so that $\mathfrak{g}_r$ is a family of Lie algebras on X. Also we get the bundle of enveloping algebras

$$U\mathfrak{g}_r = G \times_K U\mathfrak{g}. \tag{8.3.13}$$

Similarly, put

$$S\mathfrak{g}_r = G \times_K S\mathfrak{g}. \tag{8.3.14}$$

Let $\widehat{TX}$ (resp. $\widehat{T^*X}$) be another copies of TX (resp. T^*X) on X. Recall that ∇^{TX} is the Levi-Civita connection of TX.

Let $\nabla^{\mathfrak{g}_r,\widehat{u}}$ be the connections on $\widehat{T^*X} \otimes \mathfrak{g}_r$ induced by the connection form $\omega^{\mathfrak{k}}$, and let $\nabla^{U\mathfrak{g}_r,\widehat{u}}$ be the connections on $\widehat{T^*X} \otimes U\mathfrak{g}_r$ induced by $\omega^{\mathfrak{k}}$. We still denote by $\nabla^{U\mathfrak{g}_r,\widehat{u}}$ the corresponding connection on $\widehat{c}(\widehat{TX}) \otimes U\mathfrak{g}_r$.

Then $\omega^{\mathfrak{p}}$ can be considered as a section of $T^*X \otimes \mathfrak{g}_r$, and β, $\underline{\beta}$ can be considered as a section of $\widehat{T^*X} \otimes \mathfrak{g}_r$, $\widehat{T^*X} \otimes U\mathfrak{g}_r$ respectively. By [BMZ17, eq.(1.41)], we have

$$\nabla^{\mathfrak{g}_r,\widehat{u}}\beta = 0,\ \nabla^{U\mathfrak{g}_r,\widehat{u}}\underline{\beta} = 0. \tag{8.3.15}$$

Definition 8.3.1. For $t \geq 0$, let $\mathcal{A}_t$ be the superconnection

$$\mathcal{A}_t = \nabla^{U\mathfrak{g}_r,\widehat{u}} + \sqrt{t}\widehat{c}(\underline{\beta}). \tag{8.3.16}$$

As in [BMZ17, Definition 1.2], $\mathcal{A}_t^2$ is a smooth section of $[\Lambda^{\cdot}(T^*X)\widehat{\otimes}\widehat{c}(\widehat{TX})]^{\mathrm{even}} \otimes U\mathfrak{g}_r$, and $\boldsymbol{\sigma}(\mathcal{A}_t^2)$ is a smooth section of $[\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(\widehat{T^*X})]^{\mathrm{even}} \otimes S\mathfrak{g}_r$.

Recall that the product $\langle\cdot,\cdot\rangle'$ is defined in (7.3.52). By [BMZ17, Theorem 1.3 and eq.(8.70)], we have

$$\begin{aligned} \mathcal{A}_t^2 = {} & \frac{1}{4}\langle\widehat{e}_i, R^{\widehat{TX}}\widehat{e}_j\rangle\widehat{c}(\widehat{e}_i)\widehat{c}(\widehat{e}_j) - \omega^{\mathfrak{p},2} \\ & + t|\underline{\beta}|^2 + \frac{t}{2}\widehat{c}(\widehat{e}_i)\widehat{c}(\widehat{e}_j)\beta^2(\widehat{e}_i,\widehat{e}_j), \\ \boldsymbol{\sigma}(\mathcal{A}_t^2) = {} & -\frac{1}{2}\langle\omega^{\mathfrak{p},2},\beta^2\rangle' - \omega^{\mathfrak{p},2} + t|\underline{\beta}|^2 + t\beta^2. \end{aligned} \tag{8.3.17}$$

Let N be a compact complex manifold, and let η^N be a smooth real closed nondegenerate $(1,1)$-form on N. We assume that U acts holomorphically on N and preserves the form η^N. Let $\mu : N \to \mathfrak{u}^*$ be the moment map associated with the action of U and η^N.

If $y \in \mathfrak{u}$, set

$$\widetilde{R}(y) = \int_N \exp(2\pi i\langle\mu,y\rangle + \eta^N). \tag{8.3.18}$$

Then $\widetilde{R}$ is U-invariant function, we can extend it to a holomorphic function $\mathfrak{u}_{\mathbb{C}} \to \mathbb{C}$. If $y \in \mathfrak{u}_{\mathbb{C}}$, let $\mathrm{Im}(y)$ denote the component of y in $i\mathfrak{u}$.

The algebra $S\mathfrak{u}$ acts on $\widetilde{R}(y)$. Then by [BMZ17, eq.(1.24)],

$$\exp(-t|\beta|^2)\widetilde{R}(y) = \int_N \exp(-4\pi^2 t|\langle\mu,i\beta\rangle|^2 + 2\pi i\langle\mu,y\rangle + \eta_N). \tag{8.3.19}$$

We regard $\mathfrak{k}^*$ as a subspace of $\mathfrak{u}^*$ by the metric dual of $\mathfrak{k} \subset \mathfrak{u}$.

Definition 8.3.2. We say that (N,μ) is nondegenerate (with respect to $\omega^{\mathfrak{p}}$) if

$$\mu(N) \cap \mathfrak{k}^* = \emptyset. \tag{8.3.20}$$

Equivalently, there exists $c > 0$ such that

$$|\langle\mu,i\beta\rangle|^2 \geq c. \tag{8.3.21}$$

If there is no confusion, we also that say μ is nondegenerate (with respect to $\omega^{\mathfrak{p}}$).

By [BMZ17, eq.(1.27)], if (N,μ) is nondegenerate, there exists $C_0>0$, $C_1>0$ such that, if $y\in\mathfrak{u}_{\mathbb{C}}$,

$$|\exp(-t|\beta|^2)\widetilde{R}(y)|\leq C_0\exp(-tc+C_1|\mathrm{Im}(y)|). \tag{8.3.22}$$

Definition 8.3.3. The Berezin integral $\int^{\widehat{B}}$ is the linear map from $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(\widehat{T^*X})$ into $\Lambda^{\cdot}(T^*X)$ such that, if $\alpha\in\Lambda^{\cdot}(T^*X)$, $\alpha'\in\Lambda^{\cdot}(\widehat{T^*X})$,

$$\begin{aligned}&\int^{\widehat{B}}\alpha\alpha'=0,\ \text{if}\ \deg\alpha'<m;\\&\int^{\widehat{B}}\alpha\widehat{e}^1\wedge\cdots\wedge\widehat{e}^m=\frac{(-1)^{m(m+1)/2}}{\pi^{m/2}}\alpha.\end{aligned} \tag{8.3.23}$$

More generally, let $o(\widehat{\mathfrak{p}})$ be the orientation line of $\widehat{\mathfrak{p}}$, which can be identified with $o(\mathfrak{p})$. Then $\int^{\widehat{B}}$ defines a map from $\Lambda^{\cdot}(T^*X)\widehat{\otimes}\Lambda^{\cdot}(\widehat{T^*X})$ into $\Lambda^{\cdot}(T^*X)\widehat{\otimes}o(\widehat{\mathfrak{p}})$.

If B is an antisymmetric endomorphism of $\widehat{TX}$, let $\omega_B\in\Lambda^2(\widehat{T^*X})$ be the form given by $v_1,v_2\in\widehat{TX}\to\langle v_1,Bv_2\rangle$. By [BMZ17, eq.(1.30)], we have

$$\int^{\widehat{B}}\exp(-\omega_B/2)=\mathrm{Pf}[\frac{B}{2\pi}]. \tag{8.3.24}$$

Set

$$L=\sum_{i=1}^m e^i\wedge\widehat{e}^i. \tag{8.3.25}$$

Let ψ be the endomorphism of $\Lambda^{\cdot}(T^*X)\otimes_{\mathbb{R}}\mathbb{C}$ which maps $\alpha\in\Lambda^k(T^*X)\otimes_{\mathbb{R}}\mathbb{C}$ into $(2\pi i)^{-k/2}\alpha$.

Definition 8.3.4. For $t\geq 0$, set

$$\begin{aligned}d_t&=-(2\pi i)^{m/2}\psi\int^{\widehat{B}}\sqrt{t}\frac{\omega^{\mathfrak{p}}\wedge\beta}{2}\exp(-\boldsymbol{\sigma}(\mathcal{A}_t^2))\widetilde{R}(0),\\e_t&=(2\pi i)^{m/2}\psi\int^{\widehat{B}}\frac{L}{4\sqrt{t}}\exp(-\boldsymbol{\sigma}(\mathcal{A}_t^2))\widetilde{R}(0).\end{aligned} \tag{8.3.26}$$

Then d_t, e_t are smooth real forms on X.

Let $[\cdot]^{\max}$ be defined as in (7.2.9) for X. Then $[d_t]^{\max}$, $[e_t]^{\max}$ are constant function on X. By [BMZ17, Theorem 2.10], we have

$$(1+2t\frac{\partial}{\partial t})[e_t]^{\max}=[d_t]^{\max}. \tag{8.3.27}$$

Also if (N,μ) is nondegenerate, and if H is a compact subset of X, there exists $c_H>0$ such that, on H, as $t\to+\infty$,

$$d_t=\mathcal{O}(e^{-c_Ht}),\ \ e_t=\mathcal{O}(e^{-c_Ht}). \tag{8.3.28}$$

Definition 8.3.5. If (N,μ) is nondegenerate, set

$$W=-\int_0^{+\infty} d_t\frac{dt}{t}. \tag{8.3.29}$$

Then W is a smooth form on X with values in $o(TX)$. We call it the W-invariant.

As shown in [BMZ17], W appears naturally as the leading term in the asymptotics of analytic torsions.

Note that the above constructions are universal, so that we can apply to any reductive group. If $\gamma\sigma\in G^\sigma$ satisfies (7.1.13), then $Z^0(\gamma\sigma)$ is real reductive group. We can define the forms e_t, d_t associated with $Z^0(\gamma\sigma)$. In particular, the form $\omega^{\mathfrak{p}}$ is replaced by $\omega^{\mathfrak{p}}(\gamma\sigma)$, and we still have a nondegeneracy condition as in Definition 8.3.2 for this case.

If the nondegeneracy condition is verified, we will denote the form defined in (8.3.29) by $W_{\gamma\sigma}$ to indicate its relation with $\gamma\sigma$.

8.4. **A nondegeneracy condition.** Recall that M_λ is the complex manifold defined in subsection 8.2. As in Remark 8.2.6, we can always identify M_λ with the coadjoint orbit $\mathcal{O}_\lambda$ in $\mathfrak{u}^*$. In [BMZ17, Proposition 8.12], the authors gave an equivalent condition for the nondegeneracy of (M_λ,μ) with respect to $\omega^{\mathfrak{p}}$ using the Weyl group of U.

Recall that

$$\mathfrak{u}=\sqrt{-1}\mathfrak{p}\oplus\mathfrak{k}. \tag{8.4.1}$$

Using the orthogonal relations, we have

$$\mathfrak{u}^*=\sqrt{-1}\mathfrak{p}^*\oplus\mathfrak{k}^*. \tag{8.4.2}$$

Then the nondegeneracy condition of $\omega^{\mathfrak{p}}$ is also equivalent to that if $v\in\mathcal{O}_\lambda$, v always has a nonzero $\sqrt{-1}\mathfrak{p}^*$-part in the splitting (8.4.2).

Let $\gamma\in K$, then $\gamma\sigma$ is an elliptic element in G^σ. We can also consider it as an element in U^σ.

Recall that the fixed point set of $\gamma\sigma$ in M_λ is given by

$${}^{\gamma\sigma}M_\lambda=\cup_{j\in\mathcal{J}}{}^{\gamma\sigma}M^j_\lambda, \tag{8.4.3}$$

and that each connected component ${}^{\gamma\sigma}M^j_\lambda$ is complex submanifold of M_λ equipped with a holomorphic action of $U^0(\gamma\sigma)$.

Note that the function $R^j_{\gamma\sigma,\lambda}$ defined in (8.2.59) is just the function (8.3.18) associated with the group $Z^0(\gamma\sigma)$ and the complex manifold ${}^{\gamma\sigma}M^j_\lambda$.

Recall that $R^j_{\gamma\sigma,\lambda}$, $R^{\kappa_\ell}_{\gamma\sigma,\lambda}$ are the functions defined in Definition 8.2.8. For $t>0$, let the forms $e^{\kappa_\ell}_t$, $d^{\kappa_\ell}_t$ on $X(\gamma\sigma)$ be defined in Definition 8.3.4 with respect to the function $R^{\kappa_\ell}_{\gamma\sigma,\lambda}$.

As in Remark (8.2.6), given $j\in\mathcal{J}$, the moment map associated with the action of $U^0(\gamma\sigma)$ on ${}^{\gamma\sigma}M^j_\lambda$ is just the restriction of μ to ${}^{\gamma\sigma}M^j_\lambda$, which we will denote by μ^j.

Given κ^ℓ, if for $j\in\kappa^\ell$, μ^j is nondegenerate with respect to the action of $U^0(\gamma\sigma)$ on ${}^{\gamma\sigma}M^j_\lambda$ and $\omega^{\mathfrak{p}(\gamma\sigma)}$, then by (8.3.28), if H is a compact subset of $X(\gamma\sigma)$, there exists

$c_H > 0$ such that, on H and for $\ell \in \{1, \cdots, s\}$, as $t \to +\infty$,

$$d_t^{\kappa_\ell} = \mathcal{O}(e^{-c_H t}), \quad e_t^{\kappa_\ell} = \mathcal{O}(e^{-c_H t}). \tag{8.4.4}$$

Put

$$W_{\gamma\sigma}^{\kappa_\ell} = -\int_0^{+\infty} d_t^{\kappa_\ell} \frac{dt}{t}. \tag{8.4.5}$$

Then $W_{\gamma\sigma}^{\kappa_\ell}$ is a smooth differential form on $X(\gamma\sigma)$ valued in $o(TX(\gamma\sigma))$. Since $\dim \mathfrak{p}(\gamma\sigma)$ is odd, by (8.3.17), (8.3.26), (8.4.5), the degree of $W_{\gamma\sigma}^{\kappa_\ell}$ is odd.

Proposition 8.4.1. *Let $\gamma \in K$ be as above. If (M_λ, μ) is nondegenerate with respect to $\omega^{\mathfrak{p}}$, then for $j \in \mathcal{J}$, $({}^{\gamma\sigma}M_\lambda^j, \mu^j)$ is nondegenerate with respect to $\omega^{\mathfrak{p}(\gamma\sigma)}$.*

Proof. As in Remark 8.2.6, we get

$${}^{\gamma\sigma}M \simeq \mathcal{O}_\lambda \cap \mathfrak{u}(\gamma\sigma)^*. \tag{8.4.6}$$

The splitting (8.4.2) induces a splitting of $\mathfrak{u}(\gamma\sigma)^*$,

$$\mathfrak{u}(\gamma\sigma)^* = \sqrt{-1}\mathfrak{p}(\gamma\sigma)^* \oplus \mathfrak{k}(\gamma\sigma)^*. \tag{8.4.7}$$

By Definition 8.3.2, if (M_λ, μ) is nondegenerate, then $\mu(M_\lambda) \cap \mathfrak{k}^* = \emptyset$, so that $\mu^j({}^{\gamma\sigma}M_\lambda^j) \cap \mathfrak{k}(\gamma\sigma)^* = \emptyset$, which says that $({}^{\gamma\sigma}M_\lambda^j, \mu^j)$ is nondegenerate with respect to $\omega^{\mathfrak{p}(\gamma\sigma)}$. This completes the proof of our proposition. $\square$

9. The asymptotics of the equivariant Ray-Singer analytic torsion

In this section, we compute the asymptotics of the equivariant Ray-Singer analytic torsion associated with a family of flat vector bundles defined by the representations in subsection 8.2. We extend the results of [Mül12], [BMZ17, Section 8] and [Fed15].

In subsection 9.1, we recall some results on the spectral gap of Hodge Laplacian obtained in [BMZ17, Section 4] under nondegeneracy condition. Also we establish estimates on the elliptic heat kernel on X, which allows us to evaluate the contributions of non-elliptic twisted orbital integrals when t is small.

In subsection 9.2, using the formula of Proposition 8.2.9, we compute the asymptotics of the elliptic twisted orbital integrals when $\dim \mathfrak{b}(\gamma\sigma) = 1$.

In subsection 9.3, we compute the leading term of the equivariant Ray-Singer analytic torsion $\mathcal{T}_\sigma(g^{TZ}, \nabla^{F_d,f}, g^{F_d})$ as $d \to +\infty$ using the twisted trace formula on Z established in subsection 4.5. We show that only the elliptic twisted orbital integrals contribute to the leading terms. Finally, we describe the asymptotics of the equivariant Ray-Singer analytic torsion in terms of the W-invariants associated with $Z^0(\gamma\sigma)$, $\gamma \in \Gamma$.

9.1. A lower bound for the Hodge Laplacian on X.

Let $e_1, \cdots, e_m$ be the orthogonal basis of TX or $\mathfrak{p}$. Recall that $C^{\mathfrak{g},H}$ is defined in (3.4.3). Let $C^{\mathfrak{g},H,E}$ be its action on E. Then

$$C^{\mathfrak{g},E} = C^{\mathfrak{g},H,E} + C^{\mathfrak{k},E}. \tag{9.1.1}$$

Let $\Delta^{H,X}$ be the Bochner-Laplace operator on bundle $\Lambda^\cdot(T^*X) \otimes F$.

By [BMZ17, eq.(8.39)], we have

$$\begin{aligned}\mathbf{D}^{X,F,2} = -\Delta^{H,X} + \frac{S^X}{4} - \frac{1}{8}\langle R^{TX}(e_i,e_j)e_k, e_\ell\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell) \\ -C^{\mathfrak{g},H,E} + \frac{1}{2}\big(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\big)R^F(e_i,e_j).\end{aligned} \tag{9.1.2}$$

Put

$$\begin{aligned}\Theta(E) = \frac{S^X}{4} - \frac{1}{8}\langle R^{TX}(e_i,e_j)e_k, e_\ell\rangle c(e_i)c(e_j)\widehat{c}(e_k)\widehat{c}(e_\ell) \\ -C^{\mathfrak{g},H,E} + \frac{1}{2}\big(c(e_i)c(e_j) - \widehat{c}(e_i)\widehat{c}(e_j)\big)R^F(e_i,e_j).\end{aligned} \tag{9.1.3}$$

Then $\Theta(E)$ is a self-adjoint section of $\mathrm{End}(\Lambda^\cdot(T^*X) \otimes F)$, which is parallel with respect to $\nabla^{\Lambda^\cdot(T^*X)\otimes F}$. Then we rewrite (9.1.2) as

$$\mathbf{D}^{X,F,2} = -\Delta^{H,X} + \Theta(E). \tag{9.1.4}$$

Let $\langle\cdot,\cdot\rangle_{L_2}$ be the L_2 scalar product of $\Omega^\cdot_c(X,F)$. By (9.1.4), if $s \in \Omega^\cdot_c(X,F)$, then

$$\langle \mathbf{D}^{X,F,2}s, s\rangle_{L_2} \geq \langle \Theta(E)s, s\rangle_{L_2}. \tag{9.1.5}$$

Let $\Delta^{H,X,i}$ denote the Bochner-Laplace operator acting on $\Omega^i(X,F)$, and let $p^{H,i}_t(x,x')$ be the kernel of $\exp(t\Delta^{H,X,i}/2)$ on X with respect to dx'. We will denote by $p^{H,i}_t(g) \in \mathrm{End}(\Lambda^i(\mathfrak{p}^*) \otimes E)$ its lift to G explained in subsection 4.1. Let Δ^X_0 be the scalar Laplacian on X with the heat kernel $p^{X,0}_t$.

Let $||p_t^{H,i}(g)||$ be the operator norm of $p_t^{H,i}(g)$ in $\mathrm{End}(\Lambda^i(\mathfrak{p}^*)\otimes E)$. By [MüP13, Proposition 3.1], we have, for $g\in G$,

$$||p_t^{H,i}(g)||\leq p_t^{X,0}(g). \tag{9.1.6}$$

Let p_t^H be the kernel of $\exp(t\Delta^{H,X}/2)$, then

$$p_t^H=\oplus_{i=1}^p p_t^{H,i}. \tag{9.1.7}$$

Let $q_t^{X,F}$ be the heat kernel associated with $D^{X,F,2}/2$, then by (9.1.4), for $x,x'\in X$,

$$q_t^{X,F}(x,x')=\exp(-t\Theta(E)/2)p_t^H(x,x'). \tag{9.1.8}$$

We use the same notation as in subsection 8.2. Recall that $\omega^{\mathfrak{p}}$ is given in (1.1.7), and that $\mu: M_\lambda\to\mathfrak{u}^*$ is the moment map associated with the action of U on $L_\lambda\to M_\lambda$.

Let E_d, $d\in\mathbb{N}$ be the sequence of representations constructed in subsection 8.2. If μ is nondegenerate with respect to $\omega^{\mathfrak{p}}$, by [BMZ17, Theorem 4.4 and Remark 4.5], there exist $c>0$, $C>0$ such that, for $d\in\mathbb{N}$,

$$\Theta(E_d)\geq cd^2-C. \tag{9.1.9}$$

By (9.1.4), (9.1.5), (9.1.9), we get

$$\mathbf{D}^{X,F_d,2}\geq cd^2-C. \tag{9.1.10}$$

Lemma 9.1.1. *If μ is nondegenerate with respect to $\omega^{\mathfrak{p}}$, there exists $d_0\in\mathbb{N}$ and $c_0>0$ such that if $d\geq d_0$, $x,x'\in X$*

$$||q_t^{X,F_d}(x,x')||\leq e^{-c_0d^2t}p_t^{X,0}(x,x'). \tag{9.1.11}$$

Proof. By (9.1.9), there exist $d_0\in\mathbb{N}$, $c'>0$ such that if $d\geq d_0$,

$$\Theta(E_d)\geq c'd^2. \tag{9.1.12}$$

Then if $t>0$,

$$||\exp(-t\Theta(E_d)/2)||\leq e^{-c'd^2t/2}. \tag{9.1.13}$$

By (9.1.6), (9.1.7), (9.1.8), (9.1.13), we get (9.1.11). This completes the proof of our lemma. □

Let Γ be a discrete cocompact subgroup of G such that $\sigma(\Gamma)=\Gamma$. Recall that $\underline{C}$ is the set of twisted conjugacy classes in Γ defined in Definition 1.8.2. Recall that $\underline{E}$ is the set of elliptic classes in $\underline{C}$. Note that by Lemma 1.8.3, $\underline{E}$ is a finite set. Recall that by Proposition 1.8.5,

$$c_{\Gamma,\sigma}=\inf_{\underline{[\gamma]_\sigma}\in\underline{C}\setminus\underline{E}}m_{\gamma\sigma}>0. \tag{9.1.14}$$

Let q_t^{X,E_d} be the heat kernel associated with $D^{X,F_d,2}/2$. If $x\in X$, $\gamma\in\Gamma$, set

$$\begin{aligned}&v_t(E_d,\gamma\sigma,x)\\&=\frac{1}{2}\mathrm{Tr_s}^{\Lambda^\cdot(T^*X)\otimes F_d}\Big[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)q_{t/2}^{X,E_d}(x,\gamma\sigma(x))\gamma\sigma\Big].\end{aligned} \tag{9.1.15}$$

Lemma 9.1.2. *If μ is nondegenerate with respect to $\omega^{\mathfrak{p}}$, there exists $C_0 > 0$, $c_0 > 0$ such that if d is large enough, for $t > 0$, $x \in X$, $\gamma \in \Gamma$,*

$$|v_t(E_d, \gamma\sigma, x)| \leq C_0 \dim(E_d) e^{-c_0 d^2 t} p_t^{X,0}(x, \gamma\sigma(x)). \tag{9.1.16}$$

Proof. Using Lemma 9.1.1, by (9.1.15), we get (9.1.16). This completes the proof of our lemma.

□

Remark 9.1.3. This lemma is an analogue of the estimate in [MüP13, Proposition 5.3].

Proposition 9.1.4. *There exist constants $C > 0$, $c > 0$ such that if $x \in X$, $t \in]0, 1]$, then*

$$\sum_{\gamma \in \Gamma, \gamma\sigma \text{ non-elliptic}} p_t^{X,0}(x, \gamma\sigma(x)) \leq C \exp(-c/t). \tag{9.1.17}$$

Proof. By [Don79, Theorem 3.3], there exists $C_0 > 0$ such that when $0 < t \leq 1$, one has

$$p_t^{X,0}(x, x') \leq C_0 t^{-m/2} \exp(-\frac{d^2(x, x')}{4t}). \tag{9.1.18}$$

By Lemma 1.8.6, (9.1.14), (9.1.18), and using the same arguments as in the proof of [MüP13, Proposition 3.2], we get (9.1.17). □

9.2. **Asymptotics of the elliptic twisted orbital integrals.** In this subsection, we always assume that $\gamma = k^{-1} \in K$. As we saw in Theorem 7.7.3, Corollary 7.7.4, if $\gamma\sigma$ is elliptic, the orbital integral in (7.7.21) vanishes except $\dim \mathfrak{b}(\gamma\sigma) = 1$. In the sequel, we will concentrate on this case, so that $\dim \mathfrak{p}(\gamma\sigma)$ is odd.

As in (8.2.1), there exists

$$v' \in \mathfrak{k}(\gamma\sigma) \cap \mathfrak{k}^{\mathrm{reg}}. \tag{9.2.1}$$

If

$$\mathfrak{t} = \mathfrak{k}(v'), \tag{9.2.2}$$

then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Let T be the corresponding maximal torus of K.

Put

$$\mathfrak{t}(\gamma\sigma) = \mathfrak{t} \cap \mathfrak{k}(\gamma\sigma). \tag{9.2.3}$$

Then $\mathfrak{t}(\gamma\sigma)$ is a Cartan subalgebra of $\mathfrak{k}(\gamma\sigma)$. Let $\mathfrak{b}(\gamma\sigma) \subset \mathfrak{p}(\gamma\sigma)$ be the subspace defined in (7.5.8). Then $\mathfrak{b}(\gamma\sigma) \oplus \mathfrak{k}(\gamma\sigma)$ is a Cartan subalgebra of $\mathfrak{z}(\gamma\sigma)$.

We can also regard $\gamma\sigma$ as an element of U^σ. Then

$$\mathfrak{u}(\gamma\sigma) = \sqrt{-1}\mathfrak{p}(\gamma\sigma) \oplus \mathfrak{k}(\gamma\sigma). \tag{9.2.4}$$

Put

$$\mathfrak{h}(\gamma\sigma) = \sqrt{-1}\mathfrak{b}(\gamma\sigma) \oplus \mathfrak{t}(\gamma\sigma). \tag{9.2.5}$$

Then $\mathfrak{h}(\gamma\sigma)$ is a Cartan subalgebra of $\mathfrak{u}(\gamma\sigma)$.

Set $u_0 = \gamma\sigma \in U^\sigma$, if we use the notation in subsection 8.2, then $\underline{Z} = U(\gamma\sigma)$ and $\underline{\mathfrak{z}} = \mathfrak{u}(\gamma\sigma)$. We will use the associated notation in subsection 8.2 by replacing u_0 by $\gamma\sigma$.

Let (E_d, ρ^{E_d}), $d \in \mathbb{N}$ be the family of irreducible unitary representations of U^σ defined in (8.2.26). Then we extend this family to a family of representations of G^σ. Set

$$F_d = G \times_K E_d. \tag{9.2.6}$$

Now we can establish an extension of [BMZ17, Theorem 8.14].

Theorem 9.2.1. *Suppose that* $\dim \mathfrak{b}(\gamma\sigma) = 1$*. For* $t > 0$*, as* $d \to +\infty$*,*

$$\begin{aligned}
&d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big]\\
&= 2\sum_{\kappa_\ell \in C\mathcal{J}(\gamma\sigma)} \rho^{E_\lambda}(h^{j_\ell})^d [e^{\kappa_\ell}_{t/2}]^{\max} + \mathcal{O}(d^{-1}),\\
&d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)(1-\frac{t\mathbf{D}^{X,F_d,2}}{d^2})\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big]\\
&= 2\sum_{\kappa_\ell \in C\mathcal{J}(\gamma\sigma)} \rho^{E_\lambda}(h^{j_\ell})^d [d^{\kappa_\ell}_{t/2}]^{\max} + \mathcal{O}(d^{-1}).
\end{aligned} \tag{9.2.7}$$

Proof. To prove (9.2.7), we will adapt the proof of [BMZ17, Theorem 8.14] .

By (7.7.19), (7.7.21), for $d \in \mathbb{N}_{>0}$, we get

$$\begin{aligned}
&\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big]\\
&= \frac{d^p}{(2\pi t)^{p/2}}\exp\Big(\frac{t}{48d^2}\mathrm{Tr}^{\mathfrak{k}}[C^{\mathfrak{k},\mathfrak{k}}] + \frac{t}{16d^2}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]\Big)\int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}}/d)\\
&\times \mathrm{Tr}_s^{\Lambda^\cdot(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)} - \frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}/d))\big]\\
&\times \mathrm{Tr}^{E_d}\big[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(Y_0^{\mathfrak{k}}/d) + \frac{t}{2d^2}C^{\mathfrak{g},E_d})\big]\exp\Big(-\frac{|Y_0^{\mathfrak{k}}|^2}{2t}\Big)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.
\end{aligned} \tag{9.2.8}$$

By (5.1.12), as $d \to +\infty$,

$$J_{\gamma\sigma}(Y_0^{\mathfrak{k}}/d) = \frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^\perp(\gamma\sigma)}} + \mathcal{O}(d^{-1}). \tag{9.2.9}$$

Let $\rho_{\mathfrak{u}}$ be the half of the sum of the positive roots given by the Weyl chamber $\mathfrak{c}'$. By (7.7.2), (7.7.3), we have

$$C^{\mathfrak{g},E_d} = -4\pi^2(|d\lambda + \rho_{\mathfrak{u}}|^2 - |\rho_{\mathfrak{u}}|^2). \tag{9.2.10}$$

By (9.2.10), as $d \to +\infty$,

$$\frac{C^{\mathfrak{g},E_d}}{d^2} \to -4\pi^2|\lambda|^2. \tag{9.2.11}$$

By (8.2.62) in Proposition 8.2.9, as $d \to +\infty$,

$$\begin{aligned} d^{-n(\gamma\sigma)}\mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(Y_0^{\mathfrak{k}}/d)] \\ = \sum_{\kappa_\ell \in C\mathcal{J}(\gamma\sigma)} \rho^{E_\lambda}(h^{j_\ell})^d R^{\kappa_\ell}_{\gamma\sigma,\lambda}(-iY_0^{\mathfrak{k}}) + \mathcal{O}(d^{-1}). \end{aligned} \tag{9.2.12}$$

Let $\mathfrak{b}^\perp(\gamma\sigma) \subset \mathfrak{p}(\gamma\sigma)$ be the space orthogonal to the one-dimensional line $\mathfrak{b}(\gamma\sigma)$ in $\mathfrak{p}(\gamma\sigma)$. Take $Y_0^{\mathfrak{k}} \in \mathfrak{t}(\gamma\sigma)$, then by [BMZ17, eq.(8.133)],

$$\begin{aligned} \mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)} - \frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}/d))] \\ = -\det(1-\exp(-i\mathrm{ad}(Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{b}^\perp(\gamma\sigma)} \\ \det(1-\mathrm{Ad}(k^{-1}\sigma)\exp(-i\mathrm{ad}(Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{p}^\perp(\gamma\sigma)} \end{aligned} \tag{9.2.13}$$

By [BMZ17, eq.(8.134)] and (9.2.13), as $d \to +\infty$,

$$\begin{aligned} d^{p-1}\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)} - \frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}/d))] \\ = -\det(i\mathrm{ad}(Y_0^{\mathfrak{k}}))|_{\mathfrak{b}^\perp(\gamma\sigma)}\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^\perp(\gamma\sigma)} + \mathcal{O}(d^{-1}). \end{aligned} \tag{9.2.14}$$

Recall that $\Omega^{\mathfrak{z}(\gamma\sigma)}$ is curvature associated with $Z^0(\gamma\sigma) \to X(\gamma\sigma)$. Let $\widehat{\Omega}^{\mathfrak{z}(\gamma\sigma)} \in \Lambda^2(\widehat{\mathfrak{p}}(\gamma\sigma)^*) \otimes \mathfrak{k}(\gamma\sigma)$ be a copy of $\Omega^{\mathfrak{z}(\gamma\sigma)}$. Now let L and the Berezin integral be the ones as in (8.3.15) and (8.3.23) associated with $\mathfrak{p}(\gamma\sigma)$. Then by (8.3.24), we have

$$\pi^{-p/2}\det(i\mathrm{ad}(Y_0^{\mathfrak{k}}))|_{\mathfrak{b}^\perp(\gamma\sigma)} = -\bigg[\int^{\widehat{B}} L\exp(\langle Y_0^{\mathfrak{k}}, \Omega^{\mathfrak{z}(\gamma\sigma)} + \widehat{\Omega}^{\mathfrak{z}(\gamma\sigma)}\rangle)\bigg]^{\max}. \tag{9.2.15}$$

By (9.2.14), (9.2.15), we get, as $d \to +\infty$,

$$\begin{aligned} \pi^{-p/2}d^{p-1}\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)} - \frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(Y_0^{\mathfrak{k}}/d))] \\ = \bigg[\int^{\widehat{B}} L\exp(\langle Y_0^{\mathfrak{k}}, \Omega^{\mathfrak{z}(\gamma\sigma)} + \widehat{\Omega}^{\mathfrak{z}(\gamma\sigma)}\rangle)\bigg]^{\max} \\ \det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^\perp(\gamma\sigma)} + \mathcal{O}(d^{-1}). \end{aligned} \tag{9.2.16}$$

Equation (9.2.16) extends to $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$.

By (9.2.8) - (9.2.14), we get

$$\begin{aligned} d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big] \\ = \frac{\exp(-2\pi^2 t|\lambda|^2)}{(2t)^{p/2}} \sum_{\kappa_\ell \in C\mathcal{J}(\gamma\sigma)} \rho^{E_\lambda}(h^{j_\ell})^d \\ \int_{\mathfrak{k}(\gamma\sigma)} \bigg[\int^{\widehat{B}} L\exp(\langle Y_0^{\mathfrak{k}}, \Omega^{\mathfrak{z}(\gamma\sigma)} + \widehat{\Omega}^{\mathfrak{z}(\gamma\sigma)}\rangle)\bigg]^{\max} R^{\kappa_\ell}_{\gamma\sigma,\lambda}(-iY_0^{\mathfrak{k}}) \\ \exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}. \end{aligned} \tag{9.2.17}$$

Using the same arguments as in [BMZ17, eq.(8.143) - eq.(8.154)], we get

$$\frac{\exp(-2\pi^2 t|\lambda|^2)}{(2t)^{p/2}}\int_{\mathfrak{k}(\gamma\sigma)}\Big[\int^{\widehat{B}} L\exp(\langle Y_0^{\mathfrak{k}},\Omega^{\mathfrak{z}(\gamma\sigma)}+\widehat{\Omega}^{\mathfrak{z}(\gamma\sigma)}\rangle)\Big]^{\max} R^{\kappa_\ell}_{\gamma\sigma,\lambda}(-iY_0^{\mathfrak{k}}) \exp(-|Y_0^{\mathfrak{k}}|^2/2t)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}} = 2[e^{\kappa_\ell}_{t/2}]^{\max}. \tag{9.2.18}$$

By (9.2.17), (9.2.18), we get the first identity in (9.2.7). By (8.3.27), we get the second identity in (9.2.7). The proof of our theorem is completed. □

Proposition 9.2.2. *There exists $C>0$ such that for $d\in\mathbb{N}_{>0}$, $0<t\leq 1$,*

$$\begin{aligned}&\Big|d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\big]\Big|\leq C/\sqrt{t}\\ &\Big|d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\big[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)(1-t\mathbf{D}^{X,F_d,2}/d^2)\\ &\qquad\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\big]\Big|\leq C\sqrt{t}.\end{aligned} \tag{9.2.19}$$

Proof. The integral in the right-hand side of (9.2.8) can be rewritten as

$$\begin{aligned}&\int_{\mathfrak{k}(\gamma\sigma)} J_{\gamma\sigma}\big(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d}\big)\mathrm{Tr}_s^{\Lambda^\cdot(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)}-\frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)e^{-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(\sqrt{t}Y_0^{\mathfrak{k}}/d)}\big]\\ &\mathrm{Tr}^{E_d}\big[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(\sqrt{t}Y_0^{\mathfrak{k}}/d+\frac{t}{2d^2}C^{\mathfrak{g},E_d}))\big]\exp(-\frac{|Y_0^{\mathfrak{k}}|^2}{2})\frac{dY_0^{\mathfrak{k}}}{(2\pi)^{q/2}}.\end{aligned} \tag{9.2.20}$$

By (9.2.13), (9.2.14), if $Y_0^{\mathfrak{k}}\in\mathfrak{t}(\gamma\sigma)$, when d is large and t is small, we get,

$$\begin{aligned}&\frac{d^{p-1}}{t^{(p-1)/2}}\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}\big[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)}-\frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(\sqrt{t}Y_0^{\mathfrak{k}}/d))\big]\\ &=-\det(i\mathrm{ad}(Y_0^{\mathfrak{k}}))|_{\mathfrak{b}^\perp(\gamma\sigma)}\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^\perp(\gamma\sigma)}+\mathcal{O}(\sqrt{t}d^{-1}).\end{aligned} \tag{9.2.21}$$

Then we use the estimates in the proof of Theorem 9.2.1, we get the first estimate of (9.2.19).

If $Y_0^{\mathfrak{k}}\in\mathfrak{k}(\gamma\sigma)$, set

$$\begin{aligned}f(Y_0^{\mathfrak{k}})=J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\det(1-\mathrm{Ad}(k^{-1}\sigma)\exp(-i\mathrm{ad}(Y_0^{\mathfrak{k}})))|_{\mathfrak{p}^\perp(\gamma\sigma)}\\ d^{-n(\gamma\sigma)}\mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(Y_0^{\mathfrak{k}}))].\end{aligned} \tag{9.2.22}$$

Then $f(Y_0^{\mathfrak{k}})$ is analytic function on $\mathfrak{k}(\gamma\sigma)$. Let $\nabla f(Y_0^{\mathfrak{k}})$ be the gradient of f on $\mathfrak{k}(\gamma\sigma)$ with respect to the Euclidean scalar product of $\mathfrak{k}(\gamma\sigma)$. Then if $t>0$,

$$\frac{\partial}{\partial t}f\big(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d}\big)=\frac{1}{t}\langle\nabla f\big(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d}\big),\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{2d}\rangle. \tag{9.2.23}$$

If $Y_0^{\mathfrak{k}} \in \mathfrak{t}(\gamma\sigma)$, by (9.2.13), (9.2.22), we have

$$\begin{aligned}&\frac{d^{p-n(\gamma\sigma)-1}}{t^{p/2}} J_{\gamma\sigma}(\sqrt{t}Y_0^{\mathfrak{k}}/d)\\ &\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}[\big(N^{\Lambda^\cdot(\mathfrak{p}^*)} - \frac{m}{2}\big)\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(k^{-1}\sigma)\exp(-i\rho^{\Lambda^\cdot(\mathfrak{p}^*)}(\sqrt{t}Y_0^{\mathfrak{k}}/d))]\\ &\mathrm{Tr}^{E_d}[\rho^{E_d}(k^{-1}\sigma)\exp(-i\rho^{E_d}(\sqrt{t}Y_0^{\mathfrak{k}}/d) + \frac{t}{2d^2}C^{\mathfrak{g},E_d})]\\ &= \frac{1}{\sqrt{t}} f(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d})\frac{d^{p-1}}{t^{(p-1)/2}} \det(1-\exp(-i\mathrm{ad}(\sqrt{t}Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{b}^\perp(\gamma\sigma)}.\end{aligned} \tag{9.2.24}$$

Put

$$\begin{aligned}\tilde{f}(t,Y_0^{\mathfrak{k}},d) &= \frac{\partial}{\partial t}\left(f(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d})\frac{d^{p-1}}{t^{(p-1)/2}} \det(1-\exp(-i\mathrm{ad}(\sqrt{t}Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{b}^\perp(\gamma\sigma)}\right)\\ &= \frac{1}{t}\langle \nabla f(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d}), \frac{\sqrt{t}Y_0^{\mathfrak{k}}}{2d}\rangle\\ &\quad \frac{d^{p-1}}{t^{(p-1)/2}} \det(1-\exp(-i\mathrm{ad}(\sqrt{t}Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{b}^\perp(\gamma\sigma)}\\ &\quad + f(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d})\frac{\partial}{\partial t}\left(\frac{d^{p-1}}{t^{(p-1)/2}} \det(1-\exp(-i\mathrm{ad}(\sqrt{t}Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{b}^\perp(\gamma\sigma)}\right)\end{aligned} \tag{9.2.25}$$

A simple computation shows that there exists $c' > 0$, $C' > 0$, for $d \in \mathbb{N}_{>0}$, $0 < t \leq 1$, and $Y_0^{\mathfrak{k}} \in \mathfrak{t}(\gamma\sigma)$,

$$\frac{\partial}{\partial t}\left(\frac{d^{p-1}}{t^{(p-1)/2}} \det(1-\exp(-i\mathrm{ad}(\sqrt{t}Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{b}^\perp(\gamma\sigma)}\right) \leq C'\exp(c'|Y_0^{\mathfrak{k}}|). \tag{9.2.26}$$

Each part in the right-hand side of (9.2.25) lifts to a central function of $\mathfrak{k}(\gamma\sigma)$. Then the estimate as in (9.2.26) still holds for $Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma\sigma)$.

Also since $\dim \mathfrak{b}^\perp(\gamma\sigma)$ is even, when taking the Taylor expansion of the function as follows

$$\frac{1}{t}\langle \nabla f(\frac{\sqrt{t}Y_0^{\mathfrak{k}}}{d}), \frac{\sqrt{t}Y_0^{\mathfrak{k}}}{2d}\rangle \frac{d^{p-1}}{t^{(p-1)/2}} \det(1-\exp(-i\mathrm{ad}(\sqrt{t}Y_0^{\mathfrak{k}}/d)))|_{\mathfrak{b}^\perp(\gamma\sigma)}, \tag{9.2.27}$$

the terms of even power of $Y_0^{\mathfrak{k}}$ have no negative powers of the parameter t in their coefficient. Then by (9.2.25), (9.2.26), we get that there exist $C > 0$ such that for $d \in \mathbb{N}_{>0}$, $0 < t \leq 1$,

$$\left| \int_{\mathfrak{k}(\gamma\sigma)} \tilde{f}(t,Y_0^{\mathfrak{k}},d)\exp(-|Y_0^{\mathfrak{k}}|^2/2)dY^{\mathfrak{k}} \right| \leq C. \tag{9.2.28}$$

Using the fact that the two quantities in (9.2.19) are related by the operator $1 + 2t\frac{\partial}{\partial t}$, and by (9.2.24), (9.2.28), we get the second estimate in (9.2.19). This completes the proof of our proposition. □

Remark 9.2.3. The estimates in (9.2.19) for the twisted orbital integrals can be viewed as an analogue of the estimates in [BMZ17, Theorem 6.5]

9.3. **Asymptotics of the equivariant Ray-Singer analytic torsions.** In this subsection, we assume that Γ is torsion free. Recall that $Z=\Gamma\backslash X$ is now a compact manifold equipped with a group action of Σ^σ. We will use the notation in subsections 4.5, 7.8. Then the flat vector bundle F_d descends to a flat vector bundle on Z, which we still denote by F_d. Also the operator $\mathbf{D}^{X,F_d}$ descends to the corresponding operator $\mathbf{D}^{Z,F_d}$. Moreover, the action of Σ^σ lifts to F_d so that $\mathbf{D}^{Z,F_d}$ commutes with Σ^σ. For $d\in\mathbb{N}_{>0}$, let

$$\vartheta_\sigma(g^{TZ},\nabla^{F_d,f},g^{F_d})(s) \tag{9.3.1}$$

be the function defined in Definition 7.8.1 for flat vector bundle F_d, which is holomorphic near $s=0$.

Recall that the equivariant Ray-Singer analytic torsion of the de Rham complex $(\Omega^\cdot(Z,F_d),d^{Z,F_d})$ is given by

$$\mathcal{T}_\sigma(g^{TZ},\nabla^{F_d,f},g^{F_d})=\frac{1}{2}\frac{\partial\vartheta_\sigma(g^{TZ},\nabla^{F_d},g^{F_d})}{\partial s}(0). \tag{9.3.2}$$

If μ is nondegenerate with respect to $\omega^{\mathfrak{p}}$, then by (9.1.10), we have

$$\mathbf{D}^{Z,F_d,2}\geq cd^2-C. \tag{9.3.3}$$

Then if d is large enough, we have

$$H^\cdot(Z,F_d)=0. \tag{9.3.4}$$

By (7.8.6), (7.8.8), if d is large enough, we have

$$\chi_\sigma(F_d)=0,\quad \chi'_\sigma(F_d)=0. \tag{9.3.5}$$

Let $b_t(F_d,g^{F_d})$, $t>0$ be the function defined in (7.8.7) for the flat vector bundle F_d. Then by (7.8.11), (9.3.5), we have

$$b_\infty(F_d,g^{F_d})=0. \tag{9.3.6}$$

By (7.8.9), as $t\to 0$,

$$b_t(F_d,g^{F_d})=\mathcal{O}(\sqrt{t}). \tag{9.3.7}$$

By (7.8.10), as $t\to+\infty$,

$$b_t(F_d,g^{F_d})=\mathcal{O}(1/\sqrt{t}). \tag{9.3.8}$$

Then by (7.8.12), we have

$$\mathcal{T}_\sigma(g^{TZ},\nabla^{F_d,f},g^{F_d})=-\int_0^{+\infty}b_t(F_d,g^{F_d})\frac{dt}{t}. \tag{9.3.9}$$

Proposition 9.3.1. *If μ is nondegenerate with respect to $\omega^{\mathfrak{p}}$, then there exists $c > 0$ such that for d large enough,*

$$\begin{aligned}&\mathcal{T}_\sigma(g^{TZ}, \nabla^{F_d,f}, g^{F_d})\\&= -\frac{1}{2}\sum_{\underline{[\gamma]_\sigma}\in\underline{E}} \mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma))\\&\int_0^d \mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\big)(1 - t\mathbf{D}^{X,F_d,2}/2d^2)\exp(-t\mathbf{D}^{X,F_d,2}/4d^2)\Big]\frac{dt}{t}\\&\qquad + \mathcal{O}(e^{-cd}).\end{aligned} \tag{9.3.10}$$

Proof. By [BMZ17, eq.(7.3)], (9.3.9), we can write

$$\begin{aligned}\mathcal{T}_\sigma(g^{TZ}, \nabla^{F_d,f}, g^{F_d}) = &-\int_{1/d}^{+\infty} b_t(F_d, g^{F_d})\frac{dt}{t}\\&-\int_0^d b_{t/d^2}(F_d, g^{F_d})\frac{dt}{t}.\end{aligned} \tag{9.3.11}$$

By (9.3.3) and using the same arguments as in [BMZ17, Subsection 7.2], we can get that there exists $c > 0$ such that

$$\int_{1/d}^{+\infty} b_t(F_d, g^{F_d})\frac{dt}{t} = \mathcal{O}(e^{-cd}). \tag{9.3.12}$$

By (4.5.5), (7.8.7), (9.1.15), we get

$$b_t(F_d, g^{F_d}) = (1 + 2t\frac{\partial}{\partial t})\int_Z \sum_{\gamma\in\Gamma} v_t(E_d, \gamma\sigma, z)dz. \tag{9.3.13}$$

We split the sum in (9.3.13) into two parts:

$$\sum_{\gamma\in\Gamma, \gamma\sigma \text{ elliptic}} + \sum_{\gamma\in\Gamma,\gamma\sigma \text{ non-elliptic}} \tag{9.3.14}$$

By (4.5.8), (4.5.10), (4.5.14), we get the integral of the first part of sum in (9.3.14) is just the sum on elliptic classes $\underline{E}$ in the left-hand side of (9.3.10).

If $x \in X$, put

$$h_t(F_d, g^{F_d}, x) = \sum_{\gamma\in\Gamma,\gamma\sigma \text{ non-elliptic}} v_t(E_d, \gamma\sigma, x). \tag{9.3.15}$$

Then it is enough to prove that

$$\int_0^d (1 + 2t\frac{\partial}{\partial t})\int_Z h_{t/d^2}(E_d, \gamma\sigma, z)dz\frac{dt}{t} = \mathcal{O}(e^{-cd}). \tag{9.3.16}$$

Indeed, using Lemma 9.1.2 and by (9.1.4), there exists $C > 0$, $c' > 0$, $c'' > 0$ such that if d is large enough, $0 < t \leq d$, then

$$|h_{t/d^2}(F_d, g^{F_d}, x)| \leq C\dim(E_d)e^{-c't}\exp(-c''d^2/t). \tag{9.3.17}$$

By (8.1.19), there exists $C_0 > 0$, $l \in \mathbb{N}$ such that

$$\dim(E_d) \leq C_0 d^l. \tag{9.3.18}$$

It is clear that

$$\int_Z h_{1/d}(F_d, g^{F_d}, z)dz = \mathcal{O}(e^{-cd}). \tag{9.3.19}$$

Also

$$\begin{aligned} &\int_0^d h_{t/d^2}(F_d, g^{F_d}, x)\frac{dt}{t} \\ &= \int_0^1 h_{t/d^2}(F_d, g^{F_d}, x)\frac{dt}{t} + \int_1^d h_{t/d^2}(F_d, g^{F_d}, x)\frac{dt}{t} \end{aligned} \tag{9.3.20}$$

By (9.3.17), (9.3.18), then

$$\begin{aligned} &|\int_0^1 h_{t/d^2}(F_d, g^{F_d}, x)\frac{dt}{t}| \leq Ce^{-c''d^2/2}\dim(E_d)\int_0^1 e^{-c''d^2/2t}\frac{dt}{t} = \mathcal{O}(e^{-cd}), \\ &|\int_1^d h_{t/d^2}(F_d, g^{F_d}, x)\frac{dt}{t}| \leq Ce^{-c''d}\dim(E_d)\int_1^d e^{-c't}\frac{dt}{t} = \mathcal{O}(e^{-cd}). \end{aligned} \tag{9.3.21}$$

Combining (9.3.19), (9.3.21), we get (9.3.16). This completes the proof of our proposition. □

Remark 9.3.2. As in [BMZ17, Remark 8.15], by (5.1.13), (7.7.21), if $\gamma\sigma$ is not elliptic, i.e., $a \neq 0$, then there exists $C' > 0$, $c' > 0$ such that, for $t > 0$,

$$\begin{aligned} &\left|\mathrm{Tr_s}^{[\gamma\sigma]}\left[\left(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\right)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\right]\right| \\ &\qquad\qquad \leq C'_{\gamma\sigma}\exp(-\frac{|a|^2}{4t}d^2)\exp(-c't). \end{aligned} \tag{9.3.22}$$

In particular, the constant c' does not depend on $\gamma\sigma$. Also by (9.1.14), we have

$$|a| \geq c_{\Gamma,\sigma}. \tag{9.3.23}$$

We can see that the estimate (9.3.16) is compatible with (9.3.22).

For each class $\underline{e}_i = \underline{[\gamma_i]}_\sigma$ in $\underline{E}$, we fix a $k_i^{-1} \in K$ such that the element $\gamma_i \in \Gamma$ is C^σ-conjugate to k_i^{-1} by an element in G. Recall $\epsilon(\underline{e}_i)$ is given by (7.8.15) such that $\epsilon(\underline{e}_i) = \dim \mathfrak{b}(k_i^{-1}\sigma)$.

Put

$$\mathfrak{C}(\sigma) = \{\underline{e}_i \in \underline{E} \ : \ \epsilon(\underline{e}_i) = 1\}. \tag{9.3.24}$$

Recall that $n(k_i^{-1}\sigma)$ is defined as in (8.2.58) for $k_i^{-1}\sigma$. Set

$$m(\sigma) = \max\{n(k_i^{-1}\sigma) \ : \ \underline{e}_i \in \mathfrak{C}(\sigma)\}, \tag{9.3.25}$$

and set

$$\mathfrak{C}'(\sigma) = \{\underline{e}_i \in \underline{E} \ : \ \epsilon(\underline{e}_i) = 1, \ n(k_i^{-1}\sigma) = m(\sigma)\}. \tag{9.3.26}$$

If $\mathfrak{C}(\sigma)$ is an empty set, we may set $m(\sigma) = -1$.

Note that for each k_i^{-1}, we have a finite set of differential forms $W^{\kappa_\ell}_{k_i^{-1}\sigma}$ defined as in subsection 9.2. We use the corresponding notation as in subsection 8.2. Note that given $f_1(d)$, $f_2(d)$ two functions in $d\in\mathbb{N}$, we say that $f_1(d)=f_2(d)+o(1)$ as $d\to+\infty$ if $f_1(d)-f_2(d)\to 0$ as $d\to+\infty$.

Theorem 9.3.3. *If μ is nondegenerate with respect to $\omega^{\mathfrak{p}}$, as $d\to+\infty$,*

$$\begin{aligned}&d^{-m(\sigma)-1}\mathcal{T}_\sigma(g^{TZ},\nabla^{F_d,f},g^{F_d})\\&\quad=\sum_{\underline{e}_i\in\mathfrak{C}'(\sigma)}\mathrm{Vol}(\Gamma\cap Z(\gamma_i\sigma)\backslash X(\gamma_i\sigma))\Big[\sum_{\kappa_\ell\in C\mathcal{J}(k_i^{-1}\sigma)}\rho^{E_\lambda}(h^{j_\ell})^d[W^{\kappa_\ell}_{k_i^{-1}\sigma}]^{\max}\Big]+o(1).\end{aligned}\tag{9.3.27}$$

In particular, if $\mathcal{C}(\sigma)=\emptyset$, as $d\to+\infty$,

$$\mathcal{T}_\sigma(g^{TZ},\nabla^{F_d,f},g^{F_d})=\mathcal{O}(e^{-cd}).\tag{9.3.28}$$

Proof. If $\underline{[\gamma]}_\sigma$ is an elliptic class and $\underline{[\gamma]}_\sigma\notin\mathcal{C}(\sigma)$, then by Theorem 7.7.3, Corollary 7.7.4, we get

$$\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)\exp(-t\mathbf{D}^{X,F_d,2}/4d^2)\Big]=0.\tag{9.3.29}$$

If $\underline{e}_i\in\mathfrak{C}'(\sigma)$, using Theorem 9.2.1 and by (9.2.19), (9.3.29), as $d\to+\infty$, we have

$$\begin{aligned}&\int_0^1\mathrm{Tr_s}^{[\gamma_i\sigma]}\Big[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)(1-t\mathbf{D}^{X,F_d,2}/2d^2)\exp(-t\mathbf{D}^{X,F_d,2}/4d^2)\Big]\frac{dt}{t}\\&\quad=2d^{n(k_i^{-1}\sigma)+1}\sum_{\kappa_\ell\in C\mathcal{J}(k_i^{-1}\sigma)}\rho^{E_\lambda}(h^{j_\ell})^d\int_0^1[d^{\kappa_\ell}_{t/4}]^{\max}\frac{dt}{t}+\mathcal{O}(d^{n(k_i^{-1}\sigma)}).\end{aligned}\tag{9.3.30}$$

For $t>0$, if $x\in X$, we have

$$q^{X,E_d}_{t/2}(x,\gamma_i\sigma(x))\gamma_i\sigma=\int_X q^{X,E_d}_{t/4}(x,x')q^{X,E_d}_{t/4}(x',\gamma_i\sigma(x))\gamma_i\sigma dx'.\tag{9.3.31}$$

The identity in (9.3.31) is equivalent to

$$q^{X,E_d}_{t/2}(x,\gamma_i\sigma(x))\gamma_i\sigma=\big[\exp(-t\mathbf{D}^{X,F_d,2}/8)q^{X,E_d}_{t/4}(\cdot,\gamma_i\sigma(x))\gamma_i\sigma\big](x).\tag{9.3.32}$$

By (9.1.10), (9.3.32), there exists $C>0$, $c>0$ such that if d is large enough,

$$\begin{aligned}&||\mathbf{D}^{X,F_d,2}_x q^{X,E_d}_{t/2}(x,\gamma_i\sigma(x))\gamma_i\sigma||\\&\qquad\le C\exp(-cd^2t)||\mathbf{D}^{X,F_d,2}_x q^{X,E_d}_{t/4}(x,\gamma_i\sigma(x))\gamma_i\sigma||.\end{aligned}\tag{9.3.33}$$

By (4.2.6), (9.1.4), (9.1.8), (9.3.33), and using the same arguments in the proof of Lemma (9.1.2) and the estimates of derivatives of heat kernels as in (9.1.18), there exists $c>0$, $C>0$ such that , for d large enough, and $1\le t\le d$,

$$\begin{aligned}\Big|&d^{-n(k_i^{-1}\sigma)-1}\mathrm{Tr_s}^{[\gamma_i\sigma]}\big[\big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\big)(1-t\mathbf{D}^{X,F_d,2}/2d^2)\\&\qquad\exp(-t\mathbf{D}^{X,F_d,2}/4d^2)\big]\Big|\le C\exp(-ct).\end{aligned}\tag{9.3.34}$$

By (8.4.5), (9.3.34), and using the dominated convergence theorem, as $d \to +\infty$,

$$\begin{aligned} d^{-n(k_i^{-1}\sigma)-1} \int_1^d \mathrm{Tr_s}^{[\gamma_i\sigma]} \Big[\big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2} \big) (1 - t\mathbf{D}^{X,F_d,2}/2d^2) \\ \exp(-t\mathbf{D}^{X,F_d,2}/4d^2) \Big] \frac{dt}{t} \\ = 2 \sum_{\kappa_\ell \in C\mathcal{J}(k_i^{-1}\sigma)} \rho^{E_\lambda}(h^{j_\ell})^d \int_1^{+\infty} [d^{\kappa_\ell}_{t/4}]^{\max} \frac{dt}{t} + o(1). \end{aligned} \tag{9.3.35}$$

By (8.4.5), (9.3.10), (9.3.24), (9.3.25), (9.3.26), (9.3.30), (9.3.35), we get (9.3.27).

Equation (9.3.28) follows from Proposition 9.3.1. This completes the proof of our theorem. □

Remark 9.3.4. Using the estimates in [BMZ17, Section 7.3], we can refine the result of Theorem 9.3.3 to

$$\begin{aligned} & d^{-m(\sigma)-1} \mathcal{T}_\sigma(g^{TZ}, \nabla^{F_d,f}, g^{F_d}) \\ & = \sum_{\underline{e}_i \in \mathfrak{C}'(\sigma)} \mathrm{Vol}(\Gamma \cap Z(\gamma_i\sigma) \backslash X(\gamma_i\sigma)) \big[\sum_{\kappa_\ell \in C\mathcal{J}(k_i^{-1}\sigma)} \rho^{E_\lambda}(h^{j_\ell})^d [W^{\kappa_\ell}_{k_i^{-1}\sigma}]^{\max} \big] + \mathcal{O}(d^{-1}). \end{aligned} \tag{9.3.36}$$

Remark 9.3.5. The proofs of results in subsections 9.2 - 9.3 hold even if Γ has elliptic elements, then the above results can be extended easily to the case where Γ is just cocompact discrete and not torsion free, so that Z is a compact orbifold.

As explained in subsection 0.8, the results in Proposition 9.3.1, Theorem 9.3.3 are compatible with the results of Ksenia Fedosova [Fed15], where she considered the asymptotics of Ray-Singer analytic torsions for compact hyperbolic orbifolds, i.e., $G = \mathrm{Spin}(1, 2n + 1)$ and Γ may have elliptic elements.

References

[AB67] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. I*, Ann. of Math. (2) **86** (1967), 374–407. MR 0212836

[AB68] ———, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*, Ann. of Math. (2) **88** (1968), 451–491. MR 0232406

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), no. suppl. 1, 3–38. MR 0167985

[ArC89] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989. MR 1007299

[ALR07] A. Adem, J. Leida, and Y. Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007. MR 2359514

[BG04] J.-M. Bismut and S. Goette, *Equivariant de Rham torsions.*, Ann. Math. (2) **159** (2004), no. 1, 53–216 (English).

[BGV04] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original. MR 2273508

[B05] J.-M. Bismut, *The hypoelliptic Laplacian on the cotangent bundle*, Journal of the American Mathematical Society **18** (2005), no. 2, 379–476.

[B11] ———, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080

[B17] ———, *Eta invariants and the hypoelliptic Laplacian*, J.E.M.S. (2017), Accepted and to appear.

[BL95] J.-M. Bismut and J. Lott, *Flat vector bundles, direct images and higher real analytic torsion*, J. Amer. Math. Soc. **8** (1995), no. 2, 291–363. MR 1303026

[BL08] J.-M. Bismut and G. Lebeau, *The hypoelliptic Laplacian and Ray-Singer metrics*, Annals of Mathematics Studies, vol. 167, Princeton University Press, Princeton, NJ, 2008. MR 2441523

[BeLi17] N. Bergeron and M. Lipnowski, *Twisted limit formula for torsion and cyclic base change*, J. Éc. polytech. Math. **4** (2017), 435–471. MR 3646025

[BMZ17] J.-M. Bismut, X. Ma, and W. Zhang, *Asymptotic torsion and Toeplitz operators*, J. Inst. Math. Jussieu **16** (2017), no. 2, 223–349. MR 3615411

[Bou87] A. Bouaziz, *Sur les caractères des groupes de Lie réductifs non connexes*, Journal of Functional Analysis **70** (1987), no. 1, 1 – 79.

[Bo04] N. Bourbaki, *Lie groups and Lie algebras: Chapters 7-9*, Elements of mathematics, Springer Berlin Heidelberg, 2004.

[BtD85] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985. MR 781344

[BV85] N. Berline and M. Vergne, *The equivariant index and Kirillov's character formula*, Amer. J. Math. **107** (1985), no. 5, 1159–1190. MR 805808

[BeV13] N. Bergeron and A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu **12** (2013), no. 2, 391–447. MR 3028790

[C84] L. Clozel, *Théorème d'Atiyah-Bott pour les variétés $\mathfrak{p}$-adiques et caractères des groupes réductifs*, Mém. Soc. Math. France (N.S.) (1984), no. 15, 39–64, Harmonic analysis on Lie groups and symmetric spaces (Kleebach, 1983). MR 789080

[DH82] J. J. Duistermaat and G. J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math. **69** (1982), no. 2, 259–268. MR 674406

[DH83] ______, *Addendum to: "On the variation in the cohomology of the symplectic form of the reduced phase space"*, Invent. Math. **72** (1983), no. 1, 153–158. MR 696693

[DHV84] M. Duflo, G. Heckman, and M. Vergne, *Projection d'orbites, formule de Kirillov et formule de Blattner*, Mém. Soc. Math. France (N.S.) (1984), no. 15, 65–128, Harmonic analysis on Lie groups and symmetric spaces (Kleebach, 1983). MR 789081

[DK00] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000. MR 1738431

[Don79] H. Donnelly, *Asymptotic expansions for the compact quotients of properly discontinuous group actions*, Illinois J. Math. **23** (1979), no. 3, 485–496. MR 537804

[Duf72] M. Duflo, *Généralités sur les représentations induites, représentations des groupes de Lie résolubles*, Monographies de la Societe Mathematique de France. 4. Paris: Dunod. X, 272 p. F 88.00 (1972)., 1972, pp. 93–119.

[E96] P. B. Eberlein, *Geometry of nonpositively curved manifolds*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996. MR 1441541

[Fed15] K. Fedosova, *On the asymptotics of the analytic torsion for compact hyperbolic orbifolds*, ArXiv e-prints (2015).

[Fli82] Y. Z. Flicker, *The trace formula and base change for* GL(3), Lecture Notes in Mathematics, vol. 927, Springer-Verlag, Berlin-New York, 1982. MR 663002

[H79] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, vol. 80, Academic press, 1979.

[Hoc52] G. Hochschild, *The automorphism group of Lie group*, Trans. Amer. Math. Soc. **72** (1952), 209–216. MR 0045735

[Hör67] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Mathematica **119** (1967), no. 1, 147–171.

[K86] A. W. Knapp, *Representation theory of semisimple groups*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986, An overview based on examples. MR 855239

[K02] ______, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1920389

[Kos97] B. Kostant, *Clifford algebra analogue of the Hopf-Koszul-Samelson Theorem, the ρ-Decomposition* $C(\mathfrak{g}) = \mathrm{End}V_\rho \otimes C(P)$*, and the* $\mathfrak{g}$*-Module Structure of* $\Lambda\mathfrak{g}$, Advances in Mathematics **125** (1997), no. 2, 275 – 350.

[Kos99] ______, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), no. 3, 447–501.

[L80] R. P. Langlands, *Base change for* GL(2), Annals of Mathematics Studies, vol. 96, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980. MR 574808

[Lic62] A. Lichnerowicz, *Laplacien sur une variété riemannienne et spineurs*, Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali **33** (1962), no. 0155268, 187–191.

[Lip15] M. Lipnowski, *Equivariant torsion and base change*, Algebra Number Theory **9** (2015), no. 10, 2197–2240. MR 3437760

[LM89] B. H. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR 1031992

[Lot94] J. Lott, *Equivariant analytic torsion for compact Lie group actions*, Journal of Functional Analysis **125** (1994), no. 2, 438 – 451.

[M17] X. Ma, *Geometric hypoelliptic Laplacian and orbital integrals (after Bismut, Lebeau and Shen)*, Séminaire Bourbaki (69ème année, 2016-2017, n° 1130) (2017).

[MM15] X. Ma and G. Marinescu, *Exponential estimate for the asymptotics of Bergman kernels*, Mathematische Annalen **362** (2015), no. 3, 1327–1347.

[MüP13] W. Müller and J. Pfaff, *On the asymptotics of the Ray-Singer analytic torsion for compact hyperbolic manifolds*, Int. Math. Res. Not. IMRN (2013), no. 13, 2945–2983. MR 3072997

[McS67] H. P. McKean, Jr. and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967), no. 1, 43–69. MR 0217739

[MS91] H. Moscovici and Robert J. Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Invent. Math. **105** (1991), no. 1, 185–216. MR 1109626

[Mül12] W. Müller, *The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3-manifolds*, Metric and differential geometry, Progr. Math., vol. 297, Birkhäuser/Springer, Basel, 2012, pp. 317–352. MR 3220447

[P09] M. Peter, *The orbit method for compact connected Lie groups*, arXiv:0906.4915v1 (2009).

[RS71] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210. MR 0295381

[RS73] ———, *Analytic torsion*, Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), Amer. Math. Soc., Providence, R.I., 1973, pp. 167–181. MR 0339293

[See67] R. T. Seeley, *Complex powers of an elliptic operator*, Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), Amer. Math. Soc., Providence, R.I., 1967, pp. 288–307. MR 0237943

[Seg68] G. Segal, *The representation ring of a compact Lie group*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 113–128. MR 0248277

[Sel56] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) **20** (1956), 47–87. MR 0088511

[Sel60] ———, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164. MR 0130324

[S18] S. Shen, *Analytic torsion, dynamical zeta functions, and the Fried conjecture*, Anal. PDE **11** (2018), no. 1, 1–74. MR 3707290

[V96] M. Vergne, *Equivariant index formulas for orbifolds*, Duke Math. J. **82** (1996), no. 3, 637–652. MR 1387687

[W73] N. R. Wallach, *Harmonic analysis on homogeneous spaces*, Marcel Dekker, Inc., New York, 1973, Pure and Applied Mathematics, No. 19. MR 0498996

[Wit82] E. Witten, *Supersymmetry and Morse theory*, Journal of Differential Geometry **17** (1982), no. 4, 661–692.

Index

université PARIS-SACLAY | École doctorale de mathématiques Hadamard (EDMH)

Titre : Laplacien hypoelliptique et formule des traces tordue

Mots Clefs : laplacian hypoelliptique, intégrale orbitale tordue, formule des traces tordue, torsion analytique équivariante.

Résumé :
Dans cette thèse, on donne une formule géométrique explicite pour les intégrales orbitales semisimples tordues du noyau de la chaleur sur un espace symétrique, en utilisant la méthode du laplacien hypoelliptique développée par Bismut. On montre que nos résultats sont compatible avec les résultats classiques de la théorie de l'indice équivariant local sur les espaces localement symétriques compacts.
On utilise notre formule explicite pour évaluer le terme dominant dans l'asymptotique quand $d \to +\infty$ de la torsion analytique équivariante de Ray-Singer associée à une famille de fibrés vectoriels plats F_d sur un espace localement symétrique compact. On montre que le terme dominant peut être calculé à l'aide de W-invariants au sens de Bismut-Ma-Zhang.

Title : Hypoelliptic Laplacian and twisted trace formula

Keywords : Hypoelliptic Laplacian, twisted orbital integral, twisted trace formula, equivariant analytic torsion.

Abstract :
In this thesis, we give an explicit geometric formula for the twisted semisimple orbital integrals associated with the heat kernel on symmetric spaces. For that purpose, we use the method of the hypoelliptic Laplacian developed by Bismut. We show that our results are compatible with classical results in local equivariant index theory.
We also use this formula to evaluate the leading term of the asymptotics as $d \to +\infty$ of the equivariant Ray-Singer analytic torsion associated with a family of flat vector bundles F_d on a compact locally symmetric space. We show that the leading term can be evaluated in terms of the W-invariants constructed by Bismut-Ma-Zhang.