Metastability of the Blume-Capel model

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THESE

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Métastabilité du modèle de Blume-Capel

Présentée et soutenue par
Paul LEMIRE

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MIΣ

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Résumé

Les travaux de cette thèse portent sur l'étude de la métastabilité du modèle de Blume-Capel. Il s'agit d'un modèle introduit en 1966 dans lequel évoluent au cours du temps des spins à trois états +1, -1, 0, représentant respectivement une particule chargée positivement, négativement, et l'absence de particule, sur un réseau. La thèse est structurée en deux parties.

La première partie contient un travail en collaboration avec C. Landim qui est paru dans la revue *Journal of Statistical Physics*. L'article traite du comportement métastable du modèle de Blume-Capel lorsque la température tend vers 0, dans le cas où la taille du domaine dans lequel vit le processus est fixée durant l'évolution.

La seconde partie est consacrée à l'extension des résultats du premier papier au cas où la taille de la boite croît exponentiellement vite vers $+\infty$ lorsque la température décroît vers 0.

Pour ce modèle, sur une très grande échelle de temps, trois états métastables subsistent, à savoir les états où le tore est respectivement remplis par des spins négatifs, positifs, ou "nuls". Il est démontré qu'avec probabilité 1, partant de la configuration n'ayant que des spins négatifs, le processus visite la conguration n'ayant que des spins "nuls" avant de visiter la configuration n'ayant que des spins positifs. Les résultats de la thèse consistent notamment à caractériser les configurations critiques et à fournir des estimations précises des temps d'atteinte des états stables.

Mots-clés : Métastabilité, Blume-Capel, Configurations Critiques, Temps de Transition, Théorie du potentiel

Abstract

This thesis is about the study of the metastability of the Blume-Capel model. This model, introduced in 1966, is a nearest-neighbor spin system where the single spin variable takes three possible values +1, -1, 0. One can interpret it as a system of particles with spins. The value 0 of the spin corresponds to the absence of particle, whereas the values ± correspond to the presence of a particle with the respective spin. The thesis is divided in two parts.

The first part is an article published in *Journal of Statistical Physics* with C. Landim. We prove the metastable behavior of the Blume-Capel model when the temperature decreases to 0 on a fixed size torus.

The second part is dedicated to the generalization of these results to the case of a torus which size increases to $+\infty$ as the temperature decreases to 0.

For this model, three metastable states $\mathbf{-1}, \mathbf{0}, \mathbf{+1}$ remain on a very large time scale, where $\mathbf{-1}, \mathbf{0}, \mathbf{+1}$ stand for the configuration where the torus is respectively filled with -1's, 0's and +1's. We prove that starting from $\mathbf{-1}$, the process visits $\mathbf{0}$ before reaching $\mathbf{+1}$ with very high probability. We also caracterize the critical configurations and provide sharp estimates of the transition times.

Keywords : Metastability, Blume-Capel, critical configurations, transition times, potential theory

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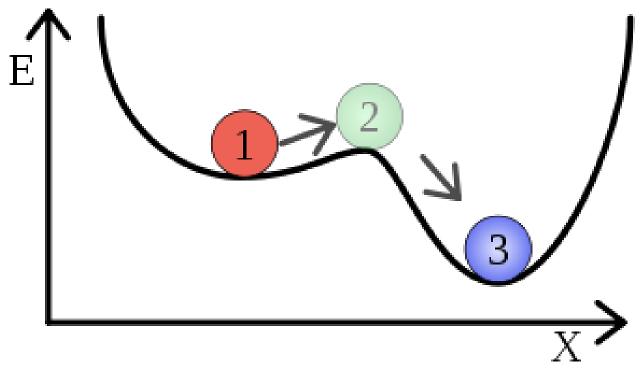
Introduction

La métastabilité est un phénomène observé dans la nature dans de nombreux modèles physiques, chimiques ou biologiques, en particulier dans des systèmes de particules dans le domaine de la mécanique statistique. Elle peut se définir comme la propriété, pour un système de particules, d'être stable d'un point de vue cinétique, mais instable d'un point de vue thermodynamique. En d'autres termes, elle met en étude l'existence d'états stables pour un système, dans lesquels ce dernier demeure pour une longue durée avant de rejoindre son état d'équilibre sous l'action d'une quantité suffisante d'énergie, appelée *énergie d'activation*. A titre d'illustration simple, voire simpliste, observons l'évolution d'une bille sur le schéma suivant :



Il est évident qu'en position (1), la bille se trouve dans un état stable ; sans aucune perturbation extérieure, elle restera immobile. Cependant, si l'on fournit suffisamment d'énergie pour qu'elle quitte l'état (1) pour rejoindre l'état (2) - appelé *critique* -, elle rejoindra l'état d'équilibre (3).

Une autre caractéristique de la métastabilité concerne le temps extrêmement long de transition entre un état stable et l'état d'équilibre. Nous pouvons en observer un exemple frappant avec le diamant. Ce dernier est composé d'atomes de carbone qui, sous cette forme, se trouvent dans un état stable. Une fois l'énergie d'activation fournie à ce système, dans le cas présent provenant naturellement des conditions normales de température et de pression, les atomes de carbone quittent l'état *diamant* pour rejoindre l'état d'équilibre *graphite*, mais ce après plusieurs milliards d'années.

La métastabilité est étudiée par bon nombre de scientifiques, qu'il s'agisse de physiciens ou encore de mathématiciens. Bien que les premières études mathématiques de la métastabilité datent du début du XX^{eme} siècle, avec les travaux de Eyring et Kramer, il a fallu attendre les approches de Freidlin et Wentzell [16] pour observer les premiers résultats obtenus dans les mathématiques modernes. La théorie des grandes déviations permet d'étudier le comportement asymptotique des systèmes métastables. L'approche utilisée est dite *trajectorielle*, ou *pathwise approach* dans la littérature.

Au début du XXI^{eme} siècle, Bovier, Eckhoff, Gayrard et Klein (BEGK) étudient la métastabilité sous un angle différent avec le développement de la théorie du potentiel. Le raffinement des résultats déjà obtenus est conséquent, puisque cette méthode permet pour la première fois d'estimer le temps d'atteinte d'états pertinents, en particulier les états *critiques*, qui constituent l'un des enjeux majeurs dans l'étude de la métastabilité, ainsi que la probabilité d'atteindre le voisinage de certains états métastables. A ce stade, il apparaît comme naturel que la théorie du potentiel présente de nombreux avantages dans l'étude de la métastabilité. En particulier, la notion de *capacité* se révèle être centrale, et il faut très souvent estimer cette quantité à l'aide des principes de Dirichlet et de Thomson (cf. (2.9) et (2.10)).

Au début des années 2010, Beltran et Landim [2–5] proposent une approche de type *martingale*, développant les travaux de BEGK et parviennent à décrire les comportements métastables de processus sous des hypothèses faibles : on suppose ici que le processus est irréductible, que son état stationnaire est réversible, et que les produits finis de taux de saut sont comparables (cf. [3] pour plus de détails). Pourtant, leur principal résultat assure de l'existence de différentes échelles de temps sur lesquelles nous pouvons observer un comportement métastable. Bien que ce soit encore une fois l'estimation de capacités, aidée par les deux principes précédemment cités, qui se trouve au centre des démonstrations, la méthode s'appuie également sur une réduction drastique du champ d'étude grâce à l'utilisation du processus-trace, permettant de restreindre l'observation du comportement du processus aux états pertinents, et donc de s'affranchir de données provenant d'évènements négligeables.

Contrairement à l'approche *pathwise*, celle-ci ne met pas en évidence le chemin précis parcouru par le processus d'un état stable x à un autre état stable y. En revanche, la caractérisation des états critiques permet un calcul exact de la profondeur des *puits* des états stables, le puits d'un état x étant un voisinage qui, une fois atteint par le processus, le conduit en x très rapidement. Ceci permet de donner une description exacte de la dynamique asymptotique entre les puits, en particulier de montrer l'existence de suites (θ_N) telles que T_N/θ_N converge vers une variable exponentielle de moyenne 1, où T_N est l'instant où le processus quitte un état métastable.

Cette thèse utilise les techniques développées par Beltrán et Landim pour le cas particulier du modèle de Blume-Capel.

Le modèle de Blume-Capel a été introduit en 1966 pour modéliser la transition He^3-He^4. Il s'agit d'un modèle de spins à trois états +1, -1, 0, représentant respectivement une particule chargée positivement, négativement, et l'absence de particule, sur un réseau. Bien que considérer $\mathbb{Z}^d$ comme réseau pour ce modèle ait du sens, cette thèse traite du cas du tore avec conditions périodiques au bord : $\Lambda_L = (\mathbb{Z}/L\mathbb{Z})^2$. L'espace d'états est donné par $\Omega_L = \{-1, 0, +1\}^{\Lambda_L}$, dont les éléments sont appelés des *configurations*. Chaque configuration $\sigma \in \Omega_L$ possède ainsi un certain niveau d'énergie représenté par l'hamiltonien $\mathbb{H}$ défini par

$$\mathbb{H}(\sigma) = \sum_{\substack{x,y\in\Lambda_L \\ ||x-y||=1}} (\sigma(x) - \sigma(y))^2 - \lambda \sum_{x\in\Lambda_L} (\sigma(x))^2 - h \sum_{x\in\Lambda_L} \sigma(x),$$

où la première somme ne compte les paires qu'une seule fois, $\lambda \in \mathbb{R}$ est le potentiel chimique, et $h \in \mathbb{R}$ est le champ magnétique extérieur. Nous observons alors qu'un tel système est paramétré par λ et h. Plus particulièrement, chaque somme traite d'une quantité pertinente pour le modèle. A l'évidence, la première indique le nombre total d'interfaces d'une configuration σ. Notons que les interfaces de spin +1 et de spin -1 ont une contribution élevée. Plus particulièrement, la seconde somme indique la magnétisation de la configuration σ, c'est-à-dire la valeur moyenne d'un spin de σ, tandis que la troisième somme indique sa concentration, autrement dit le nombre total de spins chargés. Nous considérons dans cette thèse le cas où $\lambda = 0$ et $0 < h < 2$ afin que les configurations présentant des rectangles de spins 0 soient des minima d'énergie locaux, ce qui s'avèrera être central. Pour β qui désignera l'inverse de la température (que l'on fera tendre vers l'infini), nous considérons une dynamique de Glauber, c'est-à-dire que les spins ne pourront être flippés qu'un par un, dont les taux de sauts sont donnés par :

$$R_\beta(\sigma, \sigma') = e^{-\beta[\mathbb{H}(\sigma') - \mathbb{H}(\sigma)]_+}, \; \sigma, \sigma' \in \Omega_L,$$

où $[\cdot]_+$ désigne la partie positive. La mesure stationnaire est la mesure de Gibbs associée à l'hamiltonien $\mathbb{H}$:

$$\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-\beta \mathbb{H}(\sigma)},$$

où Z_β est une constante de normalisation permettant à μ_β d'être une mesure de probabilité. En particulier, celle-ci satisfait la condition d'équilibre détaillée

$$\mu_\beta(\sigma) R_\beta(\sigma, \sigma') = R_\beta(\sigma', \sigma) \mu_\beta(\sigma').$$

Nous remarquons donc que notre modèle favorise les configurations dont l'énergie est faible ; en particulier, plus le nombre total d'interfaces d'une configuration σ est petit, plus le temps de séjour du processus en cette configuration sera élevé. On observe alors immédiatement que les trois états qui vont dessiner le paysage métastable du modèle sont les états **-1**, **0** et **+1**, correspondant aux configurations où tous les sites contiennent respectivement un spin -1, un spin 0 et un spin +1, qui n'ont toutes les trois aucune interface. Ces configurations ne sont départagées que par leur magnétisation ; on observe que

$$\mathbb{H}(\textbf{-1}) > \mathbb{H}(\mathbf{0}) > \mathbb{H}(\mathbf{+1}).$$

L'étude de la compétition entre ces trois états s'est alors naturellement imposée comme un sujet majeur pour la métastabilité de ce modèle.

Dans [13], Cirillo et Olivieri sont les premiers à travailler sur ce sujet, en s'intéressant notamment au cas où le tore est de taille fixe, et où le champ magnétique et le potentiel chimique sont faibles mais non-nuls. Comprenant que le système a une tendance à se diriger vers les états d'énergie faible lorsque la température tend vers 0, ils discutent des minima d'énergie locaux. On voit alors clairement que les configurations **0** et **-1** sont des états métastables dans lequel le modèle reste longtemps avant de tenter de rejoindre **+1**. C'est pourquoi ils se sont intéressés à la trajectoire typique du modèle partant de **-1** pour aller vers **+1** ; au milieu des spins -1 - on parlera d'une *mer* de -1 -, un amas de spins 0, aussi appelé *goutte*, est créé. Cette goutte grandit progressivement, au prix d'une augmentation significative de l'énergie, jusqu'à atteindre une taille dite *critique*, à partir de laquelle elle continue de grandir, cette fois en diminuant l'énergie jusqu'à atteindre la configuration **0**. La trajectoire de **0** à **+1** est alors analogue avec une goutte de +1 qui grandit de façon similaire.

L'un des enjeux majeurs lorsque l'on étudie la métastabilité est l'estimation du temps de passage d'un état métastable à un autre. C'est donc une question qui a suscité, et suscite encore, beaucoup d'intérêt. Concernant le modèle de Blume-Capel, on peut notamment trouver dans les travaux de Manzo et Olivieri [22], réalisés dans le cas où le tore est de volume croissant et où potentiel chimique et champ magnétique sont strictement positifs, des premières majorations et minorations du temps d'atteinte de **+1** partant de **-1**.

En 2013, Cirillo et Nardi [12] ont décrit de façon précise le paysage énergétique du modèle sur un tore de volume fixe, et ont surtout donné une meilleure estimation du temps d'atteinte de l'état **+1** lorsque le modèle part de **-1**, présentant une erreur exponentielle :

$$\lim_{\beta\to\infty} \frac{1}{\beta} \ln \mathbb{E}_{\mathbf{-1}}[H_{\mathbf{+1}}] = \Gamma_c,$$

où $H_A = \inf\{t > 0 \,:\, \sigma_t \in A\}$ représente le temps d'atteinte d'une configuration σ, $\Gamma_c = 4(n_0+1) - h[(n_0+1)n_0+1]$ est l'énergie nécessaire à la formation d'une goutte critique et $n_0 = [2/h]$, où $[\cdot]$ désigne la partie entière.

Dans cette thèse, nous montrons d'abord que, partant de **-1**, le processus atteint **0** avant d'atteindre **+1**, et nous caractérisons les configurations critiques. Comme mentionné ci-dessus, ces résultats ont été donné par Cirillo et Nardi. Néanmoins, nous présentons une nouvelle démonstration basée sur une inégalité fondamentale de la théorie du potentiel, faisant notamment le lien entre la probabilité d'atteindre **0** avant **+1**, et les capacités. De plus, les résultats précédents sur l'estimation des valeurs asymptotiques des temps d'atteinte s'appuient pour la plupart sur l'approche *pathwise*. Surtout, elles présentent des erreurs exponentielles ; nous étendons donc les résultats de Cirillo et Nardi en donnant des valeurs exactes. Nos travaux basés sur l'approche *martingale* de Beltrán et Landim présente donc deux avantages principaux : les estimations apparaissent comme plus fines, et la description du comportement métastable du modèle plus complète.

Cette thèse est constituée de deux parties. La première est constituée d'un article co-écrit avec Claudio Landim, et publié dans la *Journal of Statistical Physics*. La seconde est un article généralisant le premier au cas où le tore grandit au fur et à mesure que la température décroît.

Le chapitre 1 concerne le premier article. Dans celui-ci, nous étudions la métastabilité du modèle dans le cas où le tore est à taille fixée, principalement le comportement asymptotique du modèle. Le premier résultat principal concerne le passage de la configuration **-1** à la configuration **+1**. Nous démontrons que la probabilité de ne pas passer par la configuration **0** tend vers 0 quand l'inverse de la température β tend vers l'infini :

$$\lim_{\beta\to\infty} \mathbb{P}^{\beta}_{\mathbf{-1}}[H_{\mathbf{+1}} < H_{\mathbf{0}}] = 0.$$

La démonstration de ce résultat illustre parfaitement l'efficacité de l'approche martingale de Beltrán et Landim puisque, bien que la preuve d'un tel résultat soit évidemment divisée en étapes intermédiaires, la plupart d'entre elles concernent l'estimation de capacités. Or, dans le cas où le tore est de taille fixée, l'espace d'états est fixe. L'estimation de la capacité entre des ensembles A et B se rapporte alors à la recherche du chemin allant de A vers B avec la plus faible différence d'énergie, parfois appelée *hauteur* dans la littérature.

Le second résultat principal donne une estimation du temps d'atteinte. On démontre que, partant de $\mathbf{-1}$, le temps d'atteinte de l'état $\mathbf{+1}$ est d'ordre exponentiel :

$$\lim_{\beta\to\infty} \frac{1}{e^{\beta\Gamma_c}} \mathbb{E}_{\mathbf{-1}}[H_{\mathbf{+1}}] = 1,$$

où $e^{\beta\Gamma_c}$ représente l'échelle de temps sur laquelle le modèle évolue lors du passage de $\mathbf{-1}$ à $\mathbf{+1}$, avec Γ_c défini précédemment comme l'énergie permettant l'apparition d'une goutte critique. Là encore, la démonstration s'appuie essentiellement sur des résultats intermédiaires sur les capacités. Nous utilisons une identité dans [2], exprimant cette espérance en fonction d'une capacité et d'une somme de probabilités, puis l'on démontre que la contribution de la majorité des termes de la somme est négligeable devant les termes correspondant aux trois configurations $\mathbf{-1}, \mathbf{0}$ et $\mathbf{+1}$, dont les puits sont les plus profonds du modèle. La réversibilité ainsi que quelques manipulations sur les capacités permettent alors d'obtenir cette limite.

Dans de très nombreux systèmes observables dans la nature, l'évolution de la température influe sur la taille du domaine où évoluent les particules. Il apparaît donc comme pertinent de généraliser ces premiers résultats au cas où la taille du tore augmente quand la température décroît vers 0.

Dans le chapitre 2, nous nous intéressons cette fois-ci au cas où le tore est à taille croissante avec l'inverse de la température β. Si le cas où cette taille est fixe se ramène à l'estimation de capacités, celle-ci est rendue bien plus complexe par le fait que le nombre de chemins entre deux ensembles A et B soit désormais très grand, d'ordre exponentiel. Il s'agit alors de trouver des bornes fines pour ces capacités à l'aide des principes de Dirichlet et de Thomson.

Il découle de ces observations que la vitesse à laquelle la taille du tore augmente joue un rôle prépondérant. En effet, il est évident que le nombre total de configurations dans le modèle dépend de cette taille. Alors que, dans le cas où la taille du tore est fixe, il est suffisant de savoir que ce total est constant, il apparaît comme clair que cela ne l'est plus si la taille du tore dépend de β. Il est donc indispensable d'avoir une ou plusieurs conditions sur la vitesse de croissance. Plus précisément, nous supposons que la taille du tore croît exponentiellement rapidement :

$$\lim_{\beta\to\infty} |\Lambda_L|^{1/2} \left\{ e^{-[(n_0+1)h-2]\beta} + e^{-h\beta} \right\} \quad \text{et} \quad \lim_{\beta\to\infty} |\Lambda_L|^2 \, e^{-(2-h)\beta} \; = \; 0 \; ,$$

où $|\Lambda_L|$ désigne le nombre de sites sur le tore. De plus amples explications sur ces conditions sont disponibles à la suite du Théorème 2.2.2.

Une des principales difficultés réside encore dans l'estimations des capacités. En particulier, il faut être en mesure de compter le nombre de configurations d'un sous-ensemble donné. Lorsque la taille du tore est fixée, bien que le nombre total de configurations en dépende, celui-ci reste constant. Désormais, il est indispensable d'estimer le degré de dépendance en $|\Lambda_L|$ du nombre de configurations des sous-ensembles concernés par les capacités étudiées.

A cette difficulté s'ajoute celle de l'estimation de l'énergie de ces mêmes configurations. En effet, la finesse de nos conditions exponentielles sur la vitesse de croissance de la taille du tore repose sur notre capacité à donner l'énergie minimale d'un sous-ensemble $A \subset \Omega_L$ de configurations en fonction de la dépendance en $|\Lambda_L|$ du nombre total de configurations dans A.

Une première perspective possible pour ce travail est de tenter d'améliorer ces conditions. Nous donnons une première piste dans la Remarque 1.3.2 : il s'agirait de tenir compte des petites perturbations aléatoires pouvant se produire lorsqu'une *goutte* de spins 0 grandit. Ceci rendrait bien évidemment les estimations plus complexes.

Il serait également intéressant d'étudier la métastabilité dans d'autres cas concernant les valeurs de h et λ. Dans notre cas, nous supposions que $\lambda = 0$, ce qui avait pour effet de rendre symétriques les interactions de spins -1/0 et 0/+1. En particulier, la taille *critique* des *gouttes* de 0 est la même que celle des *gouttes* de +1. Si $\lambda > 0$, alors ces gouttes exhibent des tailles critiques différentes ; l'évolution d'une goutte de 0 est donc bien différente de celle d'une goutte de +1. Le comportement métastable s'en trouve alors bien plus difficile à étudier.

Chapitre 1

Metastability of the two-dimensional Blume-Capel model with zero chemical potential and small magnetic field

1.1 Introduction

Since its first rigorous mathematical treatment [7,10,16,24,25], metastability has been the subject of intensive investigation from different perspectives [1,6,14,15,23].

In [8,9] Bovier, Eckhoff, Gayrard and Klein, BEGK from now on, have shown that the potential theory of Markov chains can outperform large deviations arguments and provides sharp estimates for several quantities appearing in metastability, such as the expectation of the exit time from a well or the probability to hit a configuration before returning to the starting configuration.

Developing further BEGK's potential-theoretic approach, and with an intensive use of data reduction through trace processes, Beltrán and one of the authors of this paper, BL from now on, devised a scheme to describe the evolution of a Markov chain among the wells, particularly effective when the dynamics presents several valleys of the same depth [2,4,5]. The outcome of the method can be understood as a model reduction through coarse-graining, or as the derivation of the evolution of the slow variables of the chain.

In the case of finite state Markov chains [3,21], under minimal assumptions, BL's method permits the identification of all slow variables, the derivation of the time-scales at which they evolve and the characterization of their asymptotic dynamics.

In contrast with the pathwise approach [10,24] and the transition path theory [14,23], BEGK's and BL's approach do not attempt to describe the tube of typical trajectories in a transition between two valleys, nor does it identify the critical configurations which are visited with high probability in such transitions.

Nevertheless, under weak hypotheses, introduced in Section 1.3 below, potential-theoretic arguments together with data reduction through trace processes provide elementary identities and estimates which permit, without much effort, to characterize the critical configurations, and to compute the sub-exponential pre-factors of the expectation of hitting times. The purpose of this paper is to illustrate these assertions by examining the metastable behavior of the Blume-Capel model.

The Blume-Capel model is a two dimensional, nearest-neighbor spin system where the single spin variable takes three possible values : -1, 0 and $+1$. One can interpret it as a system of particles with spins. The value 0 of the spin at a lattice site corresponds to the absence of particles, whereas the values ± 1 correspond to the presence of a particle with the respective spin.

The metastability of the Blume-Capel model has been investigated by Cirillo and Olivieri [13], Manzo and Olivieri [22], and more recently by Cirillo and Nardi [12], and Cirillo, Nardi and Spitoni [11]. We refer to [13, 22] interest of the model and its role in the study of metastability.

We consider here a Blume-Capel model with zero chemical potential and a small positive magnetic field. We examine its metastable behavior in the zero-temperature limit in a large, but fixed, two-dimensional square with periodic boundary conditions. In this case, there are two metastable states, the configurations where all spins are equal to -1 or all spins equal to 0, and one ground state, the configuration where all spins are equal to $+1$.

The main results state that starting from $-\mathbf{1}$, the configuration where all spins are equal to -1, the chain visits $\mathbf{0}$ before hitting $+\mathbf{1}$. We also characterize the set of critical configurations. These results are not new and appeared in [12, 13], but we present a proof which relies on a simple inequality from the potential theory of Markov chains. We compute the exact asymptotic values of the transition times, which corresponds to the life-time of the metastable states. The previous results on the transition time, based on the pathwise approach which relies on large deviations arguments, presented estimates with exponential errors. To complete the picture, we show that the expectation of the hitting time of the configuration $\mathbf{0}$ starting from $-\mathbf{1}$ is much larger than the transition time. This phenomenon, which may seem contradicting the fact that the chain visits $\mathbf{0}$ before hitting $+\mathbf{1}$, occurs because the main contribution to the expectation comes from the event that the chain first hits $+\mathbf{1}$ and then visits $\mathbf{0}$. The very small probability of this event is compensated by the very long time the chain remains at $+\mathbf{1}$.

Finally, we prove the metastable behavior of the Blume-Capel model in the sense of BL. Let Σ be the set of configurations and let $\mathcal{V}_{-1}$, $\mathcal{V}_0$, $\mathcal{V}_{+1}$ be neighborhoods of the configurations $-\mathbf{1}$, $\mathbf{0}$, $+\mathbf{1}$, respectively. For instance, $\mathcal{V}_{-1} = \{-\mathbf{1}\}$, $\mathcal{V}_0 = \{\mathbf{0}\}$, $\mathcal{V}_{+1} = \{+\mathbf{1}\}$. Denote the inverse of the temperature by β, and let $\phi : \Sigma \to \{-1, 0, 1, \beta\}$ be the projection defined by

$$\phi(\sigma) \;=\; \sum_{a=-1,0,+1} a\, \mathbf{1}\{\sigma \in \mathcal{V}_a\} \;+\; \beta\, \mathbf{1}\Big\{\sigma \notin \bigcup_{a=-1,0,+1} \mathcal{V}_a\Big\}\,.$$

We prove that there exists a time scale θ_β for which $\phi(\sigma(t\theta_\beta))$ converges, as $\beta \to \infty$, to a Markov chain in $\{-1, 0, +1\}$. The point $+1$ is an absorbing point for this Markov chain, and the other jump rates are given by

$$r(-1,0) \;=\; r(0,1) \;=\; 1\;, \quad r(-1,1) \;=\; r(0,-1) \;=\; 0\;.$$

As we said above this result can be interpreted as a model reduction by coarse-graining, or as the identification of a slow variable, ϕ, whose evolution is asymptotically Markovian.

The article is divided as follows. In Section 1.2, we state the main results. In Section 1.3, we introduce the main tools used throughout the article and we present general results on finite-state reversible Markov chains. In section 1.4, we examine the transition from $-\mathbf{1}$ to $\mathbf{0}$, and in Section 1.5 the one from $\mathbf{0}$ to $+\mathbf{1}$. In Section 1.6, we analyze the hitting time of $\mathbf{0}$ starting from $-\mathbf{1}$. In the last section, we prove the metastable behavior of the Blume-Capel model with zero chemical potential as the temperature vanishes.

1.2 Notation and main results

Fix $L > 1$ and let $\Lambda_L = \mathbb{T}_L \times \mathbb{T}_L$, where $\mathbb{T}_L = \{1, \cdots, L\}$ is the discrete, one-dimensional torus of length L. Denote the configuration space by $\Omega = \{-1, 0, 1\}^{\Lambda_L}$, and by the Greek letters σ, η, ξ the configurations of Ω. Hence, $\sigma(x)$, $x \in \Lambda_L$, represents the spin at x of the configuration σ.

Fix an external field $0 < h < 1$, and denote by $\mathbb{H} : \Omega \to \mathbb{R}$ the Hamiltonian given by

$$\mathbb{H}(\sigma) \;=\; \sum \big(\sigma(y) - \sigma(x)\big)^2 \;-\; h \sum_{x \in \Lambda_L} \sigma(x) \;, \tag{1.1}$$

where the first sum is carried over all unordered pairs of nearest-neighbor sites of Λ_L. Let $n_0 = [2/h]$, where $[a]$ represents the integer part of $a \in \mathbb{R}_+$. We assume that $L > n_0 + 3$.

Denote by $\beta > 0$ the inverse of the temperature and by μ_β the Gibbs measure associated to the Hamiltonian $\mathbb{H}$ at inverse temperature β,

$$\mu_\beta(\sigma) \;=\; \frac{1}{Z_\beta}\, e^{-\beta \mathbb{H}(\sigma)} \;, \tag{1.2}$$

where Z_β is the partition function, the normalization constant which turns μ_β a probability measure.

Denote by $-\mathbf{1}$, $\mathbf{0}$, $+\mathbf{1}$ the configurations of Ω with all spins equal to -1, 0, $+1$, respectively. These three configurations are local minima of the energy $\mathbb{H}$, $\mathbb{H}(+\mathbf{1}) < \mathbb{H}(\mathbf{0}) < \mathbb{H}(-\mathbf{1})$, and $+\mathbf{1}$ is the unique ground state.

The Blume-Capel dynamics is the continuous-time Markov chain on Ω, denoted by $\{\sigma_t \;:\; t \ge 0\}$, whose infinitesimal generator L_β acts on functions $f : \Omega \to \mathbb{R}$ as

$$\begin{aligned} (L_\beta f)(\sigma) \;=\; & \sum_{x \in \Lambda_L} R_\beta(\sigma, \sigma^{x,+}) \big[f(\sigma^{x,+}) - f(\sigma) \big] \\ & + \sum_{x \in \Lambda_L} R_\beta(\sigma, \sigma^{x,-}) \big[f(\sigma^{x,-}) - f(\sigma) \big] \,. \end{aligned} \tag{1.3}$$

In this formula, $\sigma^{x,\pm}$ represents the configuration obtained from σ by modifying the spin at x as follows,

$$\sigma^{x,\pm}(z) := \begin{cases} \sigma(x) \pm 1 \bmod 3 & \text{if } \; z = x \\ \sigma(z) & \text{if } \; z \neq x \end{cases}$$

and the rates R_β are given by

$$R_\beta(\sigma, \sigma^{x,\pm}) \;=\; \exp\Big\{ -\beta \big[\mathbb{H}(\sigma^{x,\pm}) - \mathbb{H}(\sigma) \big]_+ \Big\} \,, \quad x \in \Lambda_L \,,$$

where a_+, $a \in \mathbb{R}$, stands for the positive part of $a : a_+ = \max\{a, 0\}$.

Clearly, the Gibbs measure μ_β satisfies the detailed balance condition

$$\mu_\beta(\sigma)R_\beta(\sigma, \sigma^{x,\pm}) \;=\; \min\big\{\mu_\beta(\sigma)\,,\, \mu_\beta(\sigma^{x,\pm})\big\} \;=\; \mu_\beta(\sigma^{x,\pm})R_\beta(\sigma^{x,\pm}, \sigma)\,,$$

and is therefore reversible for the dynamics.

Denote by $D(\mathbb{R}_+, \Omega)$ the space of right-continuous functions $\mathbf{x} : \mathbb{R}_+ \to \Omega$ with left-limits and by $\mathbb{P}_\sigma = \mathbb{P}^\beta_\sigma$, $\sigma \in \Omega$, the probability measure on the path space $D(\mathbb{R}_+, \Omega)$ induced by the Markov chain σ_t starting from σ. Expectation with respect to $\mathbb{P}_\sigma$ is represented by $\mathbb{E}_\sigma$.

Denote by $H_{\mathcal{A}}$, $H^+_{\mathcal{A}}$, $\mathcal{A} \subset \Omega$, the hitting time and the time of the first return to $\mathcal{A}$, respectively :

$$H_{\mathcal{A}} \;=\; \inf\big\{t > 0 : \sigma_t \in \mathcal{A}\big\}\;, \quad H^+_{\mathcal{A}} \;=\; \inf\big\{t > \tau_1 : \sigma_t \in \mathcal{A}\big\}\;, \tag{1.4}$$

where τ_1 represents the time of the first jump of the chain σ_t. We sometimes write $H(\mathcal{A})$, $H^+(\mathcal{A})$ instead of $H_{\mathcal{A}}$, $H^+_{\mathcal{A}}$.

Proposition 1.2.1. *Starting from $-\mathbf{1}$ the chain visits the state $\mathbf{0}$ in its way to the ground state $+\mathbf{1}$.*

$$\lim_{\beta\to\infty} \mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; 0\,.$$

Recall the definition of n_0 introduced just below (1.1). Denote by $\mathfrak{R}^l$ the set of configurations in $\{-1, 0, +1\}^{\Lambda_L}$ for which

(a) There is a $n_0 \times (n_0 + 1)$-rectangle or a $(n_0 + 1) \times n_0$-rectangle of 0-spins ;

(b) There is an extra 0 spin attached to the longest side of the rectangle ;

(c) All the remaining $|\Lambda_L| - [n_0(n_0 + 1) + 1]$ spins are equal to -1.

The configuration σ' in Figure 1.1 is an example of a configuration in $\mathfrak{R}^l$ with $n_0 = 5$.

The set of configurations $\mathfrak{R}^l_0$ is defined similarly, replacing 0-spins by +1-spins and −1-spins by 0-spins. In particular, configurations in $\mathfrak{R}^l_0$ do not have −1 spins and have $n_0(n_0 + 1) + 1$ spins equal to +1.

The next result states that, starting from $-\mathbf{1}$, the chain reaches the set $\mathfrak{R}^l$ before hitting $\mathbf{0}$.

Proposition 1.2.2. *We have that*

$$\lim_{\beta\to\infty} \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;=\; 1\;, \quad \lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{\mathfrak{R}^l_0} < H_{+\mathbf{1}}] \;=\; 1\,.$$

Denote by $\lambda_\beta(\sigma)$, $\sigma \in \Omega$, the holding rates of the Markov chain σ_t, and by $p_\beta(\eta, \xi)$, η, $\xi \in \Omega$, the jump probabilities, so that $R_\beta(\eta, \xi) = \lambda_\beta(\eta)p_\beta(\eta, \xi)$. Let $M_\beta(\eta) = \mu_\beta(\eta)\lambda_\beta(\eta)$ be the stationary measure for the embedded discrete-time Markov chain. The index β of M_β will be frequently omitted below.

Denote by $\text{cap}(\mathcal{A}, \mathcal{B})$ the capacity between two disjoint subsets $\mathcal{A}$, $\mathcal{B}$ of Ω :

$$\text{cap}(\mathcal{A}, \mathcal{B}) \;=\; \text{cap}_\beta(\mathcal{A}, \mathcal{B}) \;=\; \sum_{\sigma\in\mathcal{A}} M_\beta(\sigma)\, \mathbb{P}_\sigma[H_{\mathcal{B}} < H^+_{\mathcal{A}}]\;, \tag{1.5}$$

and let

$$\theta_\beta \;=\; \frac{\mu_\beta(-\mathbf{1})}{\text{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})}\,. \tag{1.6}$$

Lemma 4.2 in [3] states that $\text{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\}) / \exp\{-\beta H\}$ converges to a constant as $\beta \uparrow \infty$, where H is the energy of the saddle configuration separating $-\mathbf{1}$ from the set $\{\mathbf{0}, +\mathbf{1}\}$. In particular, $H - \mathbb{H}(-\mathbf{1})$ represents the energy barrier between $-\mathbf{1}$ and $\{\mathbf{0}, +\mathbf{1}\}$ and θ_β the time-scale at which the Blume-Capel model reaches the set $\{\mathbf{0}, +\mathbf{1}\}$ starting from the local minima $-\mathbf{1}$. In this sense, Proposition 1.2.3 below asserts that $\mathfrak{R}^l$ corresponds to the set of saddle or critical configurations between $-\mathbf{1}$ and $\{\mathbf{0}, +\mathbf{1}\}$, while $\mathfrak{R}^l_0$ the ones between $\mathbf{0}$ and $\{$**-1**$, +\mathbf{1}\}$.

Since we expect, by symmetric considerations, to reach $+\mathbf{1}$ before $-\mathbf{1}$ when starting from $\mathbf{0}$ (cf. Assertion 1.5.3), θ_β represents the time-scale in which the Blume-Capel model reaches the ground state $+\mathbf{1}$ starting from the local minima $-\mathbf{1}$ or $\mathbf{0}$.

Proposition 1.2.3. *For any configuration $\eta \in \mathfrak{R}^l$ and any configuration $\xi \in \mathfrak{R}^l_0$,*

$$\lim_{\beta\to\infty} \frac{\text{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})}{\mu_\beta(\eta)} \;=\; \frac{4(2n_0+1)}{3}\,|\Lambda_L| \;=\; \lim_{\beta\to\infty} \frac{\text{cap}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})}{\mu_\beta(\xi)}\ .$$

The first identity of this proposition is proved in Section 1.4 and the second one in Section 1.5.

Proposition 1.2.4. *The expected time to visit the ground state starting from $-\mathbf{1}$ and from $\mathbf{0}$ are given by*

$$\lim_{\beta\to\infty} \frac{1}{\theta_\beta}\,\mathbb{E}_{-\mathbf{1}}[H_{+\mathbf{1}}] \;=\; 2\ , \quad \lim_{\beta\to\infty} \frac{1}{\theta_\beta}\,\mathbb{E}_{\mathbf{0}}[H_{+\mathbf{1}}] \;=\; 1\ .$$

We have seen in Proposition 1.2.1 that starting from $-\mathbf{1}$ the process reaches $\mathbf{0}$ before visiting $+\mathbf{1}$. In contrast, the next proposition shows that the main contribution to the expectation $\mathbb{E}_{-\mathbf{1}}[H_{\mathbf{0}}]$ comes from the event in which the process, starting from $-\mathbf{1}$, first visits $+\mathbf{1}$, remains there for a very long time and then reaches $\mathbf{0}$.

Proposition 1.2.5. *We have that*

$$\frac{1}{\theta_\beta}\,\mathbb{E}_{-\mathbf{1}}[H_{\mathbf{0}}] \;=\; \big(1+o(1)\big)\,\frac{\mu_\beta(+\mathbf{1})}{\mu_\beta(\mathbf{0})}\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}]\ ,$$

where $o(1)$ is an expression which vanishes as $\beta \uparrow \infty$, and

$$\lim_{\beta\to\infty} \frac{\mu_\beta(+\mathbf{1})}{\mu_\beta(\mathbf{0})}\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; \infty\ .$$

A self-avoiding path γ from $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{A}$, $\mathcal{B} \subset \Omega$, $\mathcal{A} \cap \mathcal{B} = \varnothing$, is a sequence of configurations $(\xi_0, \xi_1, \dots, \xi_n)$ such that $\xi_0 \in \mathcal{A}$, $\xi_n \in \mathcal{B}$, $\xi_j \notin \mathcal{A} \cup \mathcal{B}$, $0 < j < n$, $\xi_i \neq \xi_j$, $i \neq j$, $R_\beta(\xi_i, \xi_{i+1}) > 0$, $0 \le i < n$. Denote by $\Gamma_{\mathcal{A},\mathcal{B}}$ the set of self-avoiding paths from $\mathcal{A}$ to $\mathcal{B}$ and let

$$\mathbb{H}(\mathcal{A}, \mathcal{B}) \;:=\; \min_{\gamma\in\Gamma_{\mathcal{A},\mathcal{B}}} \mathbb{H}(\gamma)\ , \quad \mathbb{H}(\gamma) \;:=\; \max_{0\le i\le n} \mathbb{H}(\xi_i)\ . \tag{1.7}$$

Thus, $\mathbb{H}(\gamma)$ represents the highest energy level attained by the path γ, while $\mathbb{H}(\mathcal{A}, \mathcal{B})$ denotes the energy of the saddle point between the sets $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{M} = \{-\mathbf{1}, \mathbf{0}, +\mathbf{1}\}$ be the set of ground configurations of the main wells, and let $\mathcal{V}_\eta$, $\eta \in \mathcal{M}$, be a neighborhood of the configuration η. We assume that all configurations $\sigma \in \mathcal{V}_\eta$, $\sigma \neq \eta$, fulfill the conditions

$$\mathbb{H}(\sigma) \ > \ \mathbb{H}(\eta) \ , \quad \mathbb{H}(\eta, \sigma) \ - \ \mathbb{H}(\eta) \ < \ \mathbb{H}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\}) \ - \ \mathbb{H}(-\mathbf{1}) \ . \tag{1.8}$$

The right hand side in the second condition represents the energetic barrier the chain needs to surmount to reach the set $\{\mathbf{0}, +\mathbf{1}\}$ starting from $-\mathbf{1}$, while the left hand side represents the energetic barrier to go from η to σ.

It follows from Proposition 1.2.3 that

$$\mathbb{H}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\}) - \mathbb{H}(-\mathbf{1}) \ = \ \mathbb{H}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\}) - \mathbb{H}(\mathbf{0}) \ . \tag{1.9}$$

Indeed, as we pointed out below (1.6), the set $\mathfrak{R}^l$ corresponds to the set of saddle points between $-\mathbf{1}$ and $\{\mathbf{0}, +\mathbf{1}\}$ so that $\mu_\beta(\eta) = \exp\{-\beta\, \mathbb{H}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})\}$ for $\eta \in \mathfrak{R}^l$. The same observation is in force for the set $\mathfrak{R}^l_0$ so that

$$\mu_\beta(\xi) = \exp\{-\beta\, \mathbb{H}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})\}$$

for $\xi \in \mathfrak{R}^l_0$. These two equations permit to compute the energies $\mathbb{H}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})$, $\mathbb{H}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})$ and to check the identy (1.9).

We may therefore replace the expression on the right hand side of (1.8) by the one on the right hand side of the previous formula.

Clearly, $\mathcal{V}_\eta = \{\eta\}$, $\eta \in \mathcal{M}$, is an example of neighborhoods satisfying (1.8). Let $\mathcal{V}$ be the union of the three neighborhoods, $\mathcal{V} = \cup_{\eta \in \mathcal{M}} \mathcal{V}_\eta$, and let $\pi : \mathcal{M} \to \{-1, 0, 1\}$ be the application which provides the magnetization of the states $-\mathbf{1}$, $\mathbf{0}$, $+\mathbf{1}$: $\pi(-\mathbf{1}) = -1$, $\pi(\mathbf{0}) = 0$, $\pi(+\mathbf{1}) = 1$. Denote by $\Psi = \Psi_\mathcal{V} : \Omega \to \{-1, 0, 1, [\beta]\}$ the projection defined by $\Psi(\sigma) = \pi(\eta)$ if $\sigma \in \mathcal{V}_\eta$, $\Psi(\sigma) = [\beta]$, otherwise :

$$\Psi(\sigma) \ = \ \sum_{\eta \in \mathcal{M}} \pi(\eta)\, \mathbf{1}\{\sigma \in \mathcal{V}_\eta\} \ + \ [\beta]\, \mathbf{1}\Big\{\sigma \notin \bigcup_{\eta \in \mathcal{M}} \mathcal{V}_\eta\Big\} \ .$$

Recall from [18] the definition of the soft topology.

Theorem 1.2.6. *The speeded-up, hidden Markov chain $X_\beta(t) = \Psi\big(\sigma(\theta_\beta t)\big)$ converges in the soft topology to the continuous-time Markov chain $X(t)$ on $\{-1, 0, 1\}$ in which 1 is an absorbing state, and whose jump rates are given by*

$$r(-1, 0) \ = \ r(0, 1) \ = \ 1 \ , \quad r(-1, 1) \ = \ r(0, -1) \ = \ 0 \ .$$

Remark 1.2.7. *Denote by $\mathcal{B}_\eta$, $\eta \in \mathcal{M}$, the basin of attraction of η :*

$$\mathcal{B}_\eta \ = \ \{\sigma : \lim_{\beta \to \infty} \mathbb{P}_\sigma[H_{\mathcal{M} \setminus \{\eta\}} < H_\eta] = 0\} \ .$$

We prove in (1.48) that $\mathcal{V}_\eta \subset \mathcal{B}_\eta$.

1.3 Metastability of reversible Markov chains

We present in this section some results on reversible Markov chains. Consider two nonnegative sequences $(a_N : N \geq 1)$, $(b_N : N \geq 1)$. The notation $a_N \prec b_N$ (resp.

$a_N \preceq b_N$) indicates that $\limsup_{N\to\infty} a_N/b_N = 0$ (resp. $\limsup_{N\to\infty} a_N/b_N < \infty$), while $a_N \approx b_N$ means that $a_N \preceq b_N$ and $b_N \preceq a_N$.

A set of nonnegative sequences $(a^r_N : N \ge 1)$, $r \in \mathfrak{R}$, is said to be *ordered* if for all $r \neq s \in \mathfrak{R}$ $\arctan(a^r_N/a^s_N)$ converges.

Fix a finite set E. Consider a sequence of continuous-time, E-valued Markov chains $\{\eta^N_t : t \ge 0\}$, $N \ge 1$. We assume, throughout this section, that the chain η^N_t is *irreducible*, that the unique stationary state, denoted by μ_N, is *reversible*, and that the jump rates of the chain η^N_t, denoted by $R_N(x,y)$, $x \neq y \in E$, satisfy the following hypothesis. Let $\mathbb{Z}_+ = \{0,1,2,\dots\}$, let $\mathbb{B}$ be the bonds of $E : \mathbb{B} = \{(x,y) \in E \times E : x \neq y \in E\}$, and let $\mathfrak{A}_m$, $m \ge 1$, be the set of functions $k : \mathbb{B} \to \mathbb{Z}_+$ such that $\sum_{(x,y)\in\mathbb{B}} k(x,y) = m$.

Assumption 1.3.1. *We assume that for every $m \ge 1$ the set of sequences*

$$\Big\{ \prod_{(x,y)\in\mathbb{B}} R_N(x,y)^{k(x,y)} : N \ge 1 \Big\} , \quad k \in \mathfrak{A}_m$$

is ordered.

This assumption is slightly weaker than the hypotheses (2.1), (2.2) in [3], but strong enough to derive all results presented in that article [21]. It is also not difficult to check that the Blume-Capel model introduced in the previous section fulfills Assumption 1.3.1.

We adopt here similar notation to the one introduced in the previous section. For example, $\lambda_N(x)$ represents the holding rates of the Markov chain η^N_t, $p_N(x,y)$, $x, y \in E$, the jump probabilities, and $M_N(x) = \mu_N(x)\lambda_N(x)$ a stationary measure of the embedded discrete-time Markov chain. Analogously, $\mathbb{P}_x = \mathbb{P}^N_x$, $x \in E$, represents the distribution of the Markov chain η^N_t starting from x and $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$.

Denote by H_A (resp. H^+_A), $A \subset E$, the hitting time of (resp. the return time to) the set A, introduced in (1.4), and by $\mathrm{cap}(A,B)$ the capacity between two disjoint subsets A, B of E, as defined in (1.5).

Identities (1.10)–(1.12) below are well known and will be used often (cf. [7, Lemma 8.4]). Let A, B be two disjoint subsets of E and let x be a point which does not belong to $A \cup B$. We claim that

$$\mathbb{P}_x[H_A < H_B] \;=\; \frac{\mathbb{P}_x[H_A < H^+_{B\cup\{x\}}]}{\mathbb{P}_x[H_{A\cup B} < H^+_x]} \,. \tag{1.10}$$

To prove this identity, intersect the event $\{H_A < H_B\}$ with the set $\{H^+_x < H_{A\cup B}\}$ and its complement, and then apply the strong Markov property to get that

$$\mathbb{P}_x[H_A < H_B] \;=\; \mathbb{P}_x[H^+_x < H_{A\cup B}]\, \mathbb{P}_x[H_A < H_B] \;+\; \mathbb{P}_x[H_A < H^+_{B\cup\{x\}}] \,.$$

To obtain (1.10) it remains to subtract the first term on the right hand side from the left hand side.

Multiply and divide the right hand side of (1.10) by $M(x)$ and recall the definition of the capacity to obtain that

$$\mathbb{P}_x[H_A < H_B] \;=\; \frac{M(x)\mathbb{P}_x[H_A < H^+_{B\cup\{x\}}]}{\mathrm{cap}(x, A\cup B)} \;\le\; \frac{M(x)\mathbb{P}_x[H_A < H^+_x]}{\mathrm{cap}(x, A\cup B)} \,. \tag{1.11}$$

Hence, by definition of capacity and since, by [17, Lemma 2.2], the capacity is monotone.

$$\mathbb{P}_x[H_A < H_B] \;\le\; \frac{\text{cap}(x,A)}{\text{cap}(x, A\cup B)} \;\le\; \frac{\text{cap}(x,A)}{\text{cap}(x,B)}\;. \tag{1.12}$$

For any disjoint subsets A and B of E,

$$\text{cap}(A,B) \;\approx\; \max_{x\in A}\max_{y\in B}\text{cap}(x,y)\;. \tag{1.13}$$

Indeed, on the one hand, by monotonicity of the capacity,

$$\text{cap}(A,B) \;\ge\; \max_{x\in A}\text{cap}(x,B) \;\ge\; \max_{x\in A}\max_{y\in B}\text{cap}(x,y)\;.$$

On the other hand, since by [17, Lemma 2.3] $\text{cap}(A,B) = \text{cap}(B,A)$, by definition of the capacity,

$$\text{cap}(A,B) \;=\; \text{cap}(B,A) \;=\; \sum_{y\in B} M(y)\mathbb{P}_y[H_A < H_B^+] \;\le\; \sum_{x\in A}\sum_{y\in B} M(y)\mathbb{P}_y[H_x < H_B^+]\;.$$

Therefore, since $\text{cap}(B,x) = \text{cap}(x,B)$,

$$\text{cap}(A,B) \;\le\; \sum_{x\in A}\text{cap}(x,B) \;\le\; |A|\,\max_{x\in A}\text{cap}(x,B)\;, \tag{1.14}$$

where $|A|$ stands for the cardinality of A. Repeating this argument for B in place of A, we conclude the proof of (1.13).

Let $G_N : E\times E \to \mathbb{R}_+$ be given by $G_N(x,y) = \mu_N(x)R_N(x,y)$ and note that G_N is symmetric. In the electrical network interpretation of reversible Markov chains, $G_N(x,y)$ represents the conductance of the bond (x,y). Recall that a self-avoiding path γ from A to B, A, $B\subset E$, $A\cap B = \varnothing$, is a sequence of sites $(x_0, x_1, \dots, x_n)$ such that $x_0\in A$, $x_n\in B$, $x_j\notin A\cup B$, $0<j<n$, $x_i\ne x_j$, $i\ne j$, $R_N(x_i,x_{i+1})>0$, $0\le i<n$. Denote by $\Gamma_{A,B}$ the set of self-avoiding paths from A to B and let

$$G_N(A,B) \;:=\; \max_{\gamma\in\Gamma_{A,B}} G_N(\gamma)\;,\quad G_N(\gamma) \;:=\; \min_{0\le i<n} G_N(x_i,x_{i+1})\;.$$

By [3, Lemma 4.2], for every disjoint subsets A, B of E, the limit

$$\lim_{N\to\infty}\frac{\text{cap}(A,B)}{G_N(A,B)} \;\text{ exists and belongs to } (0,\infty)\;. \tag{1.15}$$

Remark 1.3.2. *Suppose that the stationary state μ_N is a Gibbs measure associated to an energy $\mathbb{H}$, and that we are interested in the Metropolis dynamics : $\mu_N(x) = Z_N^{-1}\exp\{-N\mathbb{H}(x)\}$, where Z_N is the partition function, $R_N(x,y) = \exp\{-N[\mathbb{H}(y) - \mathbb{H}(x)]_+\}$. In this context, $G_N(x,y) = Z_N^{-1}\exp\{-N\max[\mathbb{H}(x),\mathbb{H}(y)]\}$. In particular, for a path $\gamma = (x_0,x_1,\dots,x_n)$, $G_N(\gamma) = Z_N^{-1}\exp\{-N\max_i H(x_i)\}$, and for two disjoint subsets A, B of E,*

$$\begin{aligned} G_N(A,B) \;&=\; \frac{1}{Z_N}\exp\Big\{-N\min_{\gamma\in\Gamma_{A,B}}\max_{x\in\gamma}\mathbb{H}(x)\Big\} \\ &=\; \frac{1}{Z_N}\exp\big\{-N\mathbb{H}(x_{A,B})\big\} \;=\; \mu_N(x_{A,B})\;, \end{aligned}$$

where $x_{A,B}$ represents the configuration with highest energy in the optimal path joining A to B.

Lemma 1.3.3. *Let E_1 be a subset of E. Assume that for every $y \notin E_1$, $z \in E_1$ such that $\mu(z) \preceq \mu(y)$,*

$$\frac{\operatorname{cap}(y,z)}{\mu(z)} \;\prec\; \frac{\operatorname{cap}(y,E_1)}{\mu(y)}\;. \tag{1.16}$$

Then, for any $B \subset E_1$, $x \in E_1 \setminus B$,

$$\mathbb{E}_x[H_B] \;=\; \big(1+o(1)\big)\,\frac{1}{\operatorname{cap}(x,B)}\sum_{y\in E_1}\mu(y)\,\mathbb{P}_y[H_x < H_B]\;.$$

Proof. By [2, Proposition 6.10],

$$\mathbb{E}_x[H_B] \;=\; \frac{1}{\operatorname{cap}(x,B)}\sum_{y\in E}\mu(y)\,\mathbb{P}_y[H_x < H_B]\;.$$

Denote by $\mathbb{P}^1_z$, $z \in E_1$, the distribution of the trace of $\sigma(t)$ on the set E_1 starting from z (cf. [2, Section 6] for the definition of trace), and let $q(y,z) = \mathbb{P}_y[H_{E_1} = H_z]$, $y \notin E_1$, $z \in E_1$. Decomposing the previous sum according to $y \in E_1$, $y \notin E_1$, since B and x are contained in E_1, we can write it as

$$\begin{aligned}
&\sum_{y\in E_1}\mu(y)\,\mathbb{P}^1_y[H_x<H_B] \;+\; \sum_{y\notin E_1}\sum_{z\in E_1}\mu(y)\,q(y,z)\,\mathbb{P}^1_z[H_x<H_B] \\
&\quad=\; \sum_{y\in E_1}\mu(y)\,\mathbb{P}^1_y[H_x<H_B] \;+\; \sum_{z\in E_1}\mathbb{P}^1_z[H_x<H_B]\sum_{y\notin E_1}\mu(y)\,q(y,z)\;.
\end{aligned} \tag{1.17}$$

We claim that for $y \notin E_1$, $z \in E_1$,

$$\mu(y)\,q(y,z) \;=\; \mu(y)\,\mathbb{P}_y[H_z < H_{E_1\setminus\{z\}}] \;\prec\; \mu(z)\;. \tag{1.18}$$

If $\mu(y) \prec \mu(z)$, there is nothing to prove. Assume that $\mu(z) \preceq \mu(y)$. In this case, by (1.12) and by (1.16), the second term in the previous expression is bounded by

$$\frac{\mu(y)\operatorname{cap}(y,z)}{\operatorname{cap}(y,E_1)} \;\prec\; \mu(z)\;.$$

This proves claim (1.18) and that the second term in the last equation of (1.17) is of smaller order than the first, as asserted. □

Remark 1.3.4. *In Lemma 1.3.3, the set E_1 has to be interpreted as the union of wells. In the set-up of the Metropolis dynamics introduced in Remark 1.3.2, by* (1.15) *and Remark 1.3.2, for two disjoint subsets A, B of E,* $\operatorname{cap}(A,B)/\mu_N(x_{A,B})$ *converges, as $N \uparrow \infty$, to a real number in* $(0,\infty)$. *Hence, assumption* (1.16) *requires that for all $z \in E_1$, $y \notin E_1$ such that $\mathbb{H}(y) \le \mathbb{H}(z)$,*

$$\mathbb{H}(x_{y,E_1}) \;-\; \mathbb{H}(y) \;<\; \mathbb{H}(x_{y,z}) \;-\; \mathbb{H}(z)\;. \tag{1.19}$$

In other words, it requires the energy barrier from y to E_1 to be smaller than the one from z to y.

The condition (1.19) *may seem unnatural, as one would expect on the right hand side $\mathbb{H}(x_{z,\breve{E}_z}) - \mathbb{H}(z)$ instead of $\mathbb{H}(x_{y,z}) - \mathbb{H}(z)$, where $\breve{E}_z$ represents the union of the wells which do not contain z. However, since in the applications the set E_1 represents the union of wells, and since $\mathbb{H}(y) \le \mathbb{H}(z)$, to reach y from z the chain has to jump from one well to another and therefore one should have $\mathbb{H}(x_{z,\breve{E}_z}) - \mathbb{H}(z) \le \mathbb{H}(x_{y,z}) - \mathbb{H}(z)$.*

Lemma 1.3.5. *Fix two points $a \neq b \in E$. The set of sequences $\mu_N(x)\mathbb{P}_x[H_a < H_b]$, $x \in E \setminus \{b\}$, is ordered.*

Proof. Fix two points $x \neq y \in E\setminus\{b\}$. We need to show that the ratio $\mu_N(x)\mathbb{P}_x[H_a < H_b]/\mu_N(y)\mathbb{P}_y[H_a < H_b]$ either converges to some value in $[0, \infty)$, or increases to ∞.

Assume that $x \neq a$, $y \neq a$, and consider the trace of the process $\eta^N(t)$ on $A = \{a, b, x, y\}$. By [2, Section 6], the stationary measure of the trace is the measure μ_N conditioned to A. Denote by $\mathbb{P}^A_z$ the distribution of the trace starting from z. It is clear that $\mathbb{P}_z[H_a < H_b] = \mathbb{P}^A_z[H_a < H_b]$. Therefore,

$$\frac{\mu_N(x)\mathbb{P}_x[H_a < H_b]}{\mu_N(y)\mathbb{P}_y[H_a < H_b]} \;=\; \frac{\mu^A_N(x)\mathbb{P}^A_x[H_a < H_b]}{\mu^A_N(y)\mathbb{P}^A_y[H_a < H_b]}\;,$$

where μ^A_N represents the measure μ_N conditioned to A. Since A has only four elements, it is not difficult to show that

$$\mathbb{P}^A_x[H_a < H_b] \;=\; \frac{p^A(x,a) + p^A(x,y)p^A(y,a)}{1 - p^A(x,y)p^A(y,x)}\;,$$

where $p^A(z, z')$ represents the jump probabilities of the trace process. In particular, multiplying the numerator and the denominator of the penultimate ratio by $\lambda^A(x)\lambda^A(y)$, where λ^A stands for the holding rates of the trace process, yields that the penultimate ratio is equal to

$$\frac{\mu^A_N(x)\{\lambda^A(y)R^A(x,a) + R^A(x,y)R^A(y,a)\}}{\mu^A_N(y)\{\lambda^A(x)R^A(y,a) + R^A(y,x)R^A(x,a)\}}\;,$$

where R^A is the jump rates of the trace process. By reversibility of the trace process, this expression is equal to

$$\frac{\lambda^A(y)R^A(a,x) + R^A(y,x)R^A(a,y)}{\lambda^A(x)R^A(a,y) + R^A(x,y)R^A(a,x)}\;.$$

By [3, Lemma 4.3], the set of jump rates $R^A(x, y)$ satisfies Assumption 1.3.1. Since $\lambda^A(z) = \sum_{z' \in A, z' \neq z} R^A(z, z')$, by Assumption 1.3.1, the previous expression either converges to some $a \in [0, \infty)$, or increases to $+\infty$. This completes the proof of the assertion in the case where x, $y \notin \{a, b\}$.

The case where $x = a$ or $y = a$ is simpler and left to the reader. □

1.4 Proofs of Propositions 1.2.1, 1.2.2 and 1.2.3

We examine in this section the metastable behavior of the Blume-Capel model starting from $-\mathbf{1}$. We first consider isovolumetric inequalities. Denote by $\|\cdot\|$ the Euclidean norm of $\mathbb{R}^2$. A subset A of $\mathbb{Z}^2$ is said to be connected if for every x, $y \in A$, there exists a path $\gamma = (x = x_0, x_1, \dots, x_n = y)$ such that $x_i \in A$, $\|x_{i+1} - x_i\| = 1$, $0 \leq i < n$. Denote by $\mathcal{C}_n$, $n \geq 1$, the class of connected subset of $\mathbb{Z}^2$ with n points and by $P(A)$ the perimeter of a set $A \in \mathcal{C}_n$:

$$P(A) \;=\; \#\{(x,y) \in \mathbb{Z}^2 : x \in A\,,\, y \notin A\,,\, \|x - y\| = 1\}\;.$$

where $\#B$ stands for the cardinality of B.

Assertion 1.4.1. *For every $A \in \mathcal{C}_n$, $n \geq 1$, $P(A) \geq 4\sqrt{n}$.*

Proof. For $A \in \mathcal{C}_n$, denote by R the smallest rectangle which contains A, and by $a \leq b$ the length of the sides of the rectangle R. Since A is connected, and since R is the smallest rectangle which contains A, $P(A) \geq 2(a+b) \geq 2\min\{\mathfrak{a}+\mathfrak{b} : \mathfrak{a}, \mathfrak{b} \in \mathbb{R}\,,\ \mathfrak{a}\mathfrak{b} \geq n\} = 4\sqrt{n}$. □

Assertion 1.4.2. *A set $A \in \mathcal{C}_m$, $m = n_0(n_0+1)$, is either a $n_0 \times (n_0+1)$ rectangle or has perimeter $P(A) \geq 4(n_0+1)$.*

Proof. Fix $A \in \mathcal{C}_m$, and recall the notation introduced in the proof of the previous assertion. We may restrict our study to the case where the length of the shortest side of R, denoted by a, is less than or equal to n_0, otherwise the perimeter is larger than or equal to $4(n_0+1)$. If $a = n_0$, either $b = n_0+1$, in which case, to match the volume, A must be a $n_0 \times (n_0+1)$ rectangle, or $b \geq n_0+2$, in which case the the perimeter is larger than or equal to $4(n_0+1)$. If $a = n_0 - j$ for some $j \geq 1$, then $b = n_0 + k$ for some $k \geq 1$ because the volume has to be at least n_0^2. Actually, we need $(n_0+k)(n_0-j) \geq n_0(n_0+1)$, i.e., $(k-j)n_0 \geq n_0 + kj$. This forces $k - j \geq 2$ and, in consequence, the perimeter $P \geq 4(n_0+1)$. □

We may extend the definition of the energy $\mathbb{H}$ introduced in (1.1) to configuration in $\{-1,0,1\}^{\mathbb{Z}^2}$. For such configurations, while $\mathbb{H}(\sigma)$ is not well defined, $\mathbb{H}(\sigma) - \mathbb{H}(-\mathbf{1})$ is well defined if $\sigma_x = -1$ for all but a finite number of sites.

Denote by $\partial_+ A$ the outer boundary of a connected finite subset A of $\mathbb{Z}^2$: $\partial_+ A = \{x \notin A : \exists\, y \in A \text{ s.t. } \|y - x\| = 1\}$.

Assertion 1.4.3. *Let $A \in \mathcal{C}_n$, $1 \leq n \leq (n_0+1)^2$, and let σ be a configuration of $\{-1,0,1\}^{\mathbb{Z}^2}$ whose spins in A are equal to $+1$ and whose spins in $\partial_+ A$ are either 0 or -1. Let $\sigma^\star$ b ?e the configuration obtained from σ by switching all spins in A to 0. Then, $\mathbb{H}(\sigma) \geq \mathbb{H}(\sigma^\star) + 2$.*

Proof. By definition of the energy and since A has n points, $\mathbb{H}(\sigma) - \mathbb{H}(\sigma^\star) = -hn + P_0 + 3P_{-1} \geq -hn + P$, where P_0 (resp. P_{-1}) represents the number of unordered pairs $\{x, y\}$ such that $x \in A$, $y \in \partial_+ A$, $\sigma_y = 0$ (resp. $\sigma_y = -1$), and where $P = P_0 + P_{-1}$ is the perimeter of the set A.

It remains to show that $P - hn \geq 2$. For $1 \leq n \leq 3$, this follows by inspecting all cases, keeping in mind that $h < 1 \leq 2$. Next, assume that $n \geq 4$. By hypothesis, and since $n_0 \geq 2$, $(2/3)\sqrt{n} \leq (2/3)(n_0+1) \leq n_0 < 2/h$ so that $hn < 3\sqrt{n}$. Hence, by Assertion 1.4.1, $hn < 4\sqrt{n} - \sqrt{n} \leq P - 2$. □

Let $A(\sigma) = \{x \in \mathbb{Z}^2 : \sigma_x \neq -1\}$, $\sigma \in \{-1,0,1\}^{\mathbb{Z}^2}$. Denote by $\mathfrak{B}$ the boundary of the valley of $-\mathbf{1}$ formed by the set of configurations with $n_0(n_0+1)$ sites with spins different from -1 :

$$\mathfrak{B} \;=\; \big\{\sigma \in \{-1,0,1\}^{\mathbb{Z}^2} : |A(\sigma)| = n_0(n_0+1)\big\}\ .$$

Sometimes, we consider $\mathfrak{B}$ as a subset of Ω. Denote by $\mathfrak{R}$ the subset of $\mathfrak{B}$ given by

$$\mathfrak{R} \;=\; \big\{\sigma \in \{-1,0\}^{\mathbb{Z}^2} : A(\sigma) \text{ is a } n_0 \times (n_0+1) \text{ rectangle } \big\}\ .$$

Note that the spins of a configuration $\sigma \in \mathfrak{R}$ are either -1 or 0 and that all configurations in $\mathfrak{R}$ have the same energy.

Assertion 1.4.4. *We have that* $\mathbb{H}(\sigma) \geq \mathbb{H}(\zeta) + 2$ *for all* $\sigma \in \mathfrak{B} \setminus \mathfrak{R}$, $\zeta \in \mathfrak{R}$.

Proof. Fix a configuration $\sigma \in \mathfrak{B}$. Let σ^* be the configuration of $\{-1,0\}^{\mathbb{Z}^2}$ obtained from σ by switching all $+1$ spins to 0. Applying Assertion 1.4.3 k times, where k is the number of connected components formed by $+1$ spins, we obtain that $\mathbb{H}(\sigma) \geq \mathbb{H}(\sigma^*) + 2k$. It is therefore enough to prove the lemma for configurations $\sigma \in \{-1,0\}^{\mathbb{Z}^2}$.

Let σ be a configuration in $\mathfrak{B} \cap \{-1,0\}^{\mathbb{Z}^2}$. If $A(\sigma)$ is not a connected set, by gluing the connected components of $A(\sigma)$, we reach a new configuration $\sigma^* \in \mathfrak{B} \cap \{-1,0\}^{\mathbb{Z}^2}$ such that $A(\sigma^*) \in \mathcal{C}_m$, $m = n_0(n_0+1)$. Since by gluing two components, the volume remains unchanged, but the perimeter decreases at least by 2, $\mathbb{H}(\sigma) \geq \mathbb{H}(\sigma^*) + 2$. It is therefore enough to prove the lemma for those configuration in $\mathfrak{B} \cap \{-1,0\}^{\mathbb{Z}^2}$ for which $A(\sigma)$ is a connected set.

Finally, fix a configuration in $\mathfrak{B} \cap \{-1,0\}^{\mathbb{Z}^2}$ for which $A(\sigma)$ is a connected set different from a $n_0 \times (n_0+1)$ rectangle. Since all spins of σ are either -1 or 0. By definition of the energy, $\mathbb{H}(\sigma) - \mathbb{H}(\zeta) = P(A(\sigma)) - P(A(\zeta))$, and the result follows from Assertion 1.4.2. □

Denote by $\mathfrak{R}^+$ the set of configurations in $\{-1,0,+1\}^{\mathbb{Z}^2}$ in which there are $n_0(n_0+1)+1$ spins which are not equal to -1. Of these spins, $n_0(n_0+1)$ form a $n_0 \times (n_0+1)$-rectangle of 0 spins. The remaining spin not equal to -1 is either 0 or $+1$.

It is clear that starting from $\mathbf{-1}$ the set $(\mathfrak{B} \setminus \mathfrak{R}) \cup \mathfrak{R}^+$ is hit before the chain attains the set $\{\mathbf{0}, \mathbf{+1}\}$:

$$H_{\mathfrak{B}^+} \;<\; H_{\{\mathbf{0},\mathbf{+1}\}} \quad \mathbb{P}_{\mathbf{-1}} \text{ a.s.}\;, \quad \text{where } \mathfrak{B}^+ = (\mathfrak{B} \setminus \mathfrak{R}) \cup \mathfrak{R}^+ \;. \tag{1.20}$$

Let $\mathfrak{R}^a \subset \mathfrak{R}^+$ be the set of configurations for which the remaining spin is a 0 spin attached to one of the sides of the rectangle. Note that all configurations of $\mathfrak{R}^a$ have the same energy and that $\mathbb{H}(\xi) = \mathbb{H}(\zeta) + 2 - h$ if $\xi \in \mathfrak{R}^a$, $\zeta \in \mathfrak{R}$. In particular,

$$\mathbb{H}(\sigma) \;\geq\; \mathbb{H}(\xi) \;+\; h\;, \quad \sigma \in \mathfrak{B} \setminus \mathfrak{R} \;,\; \xi \in \mathfrak{R}^a \;. \tag{1.21}$$

On the other hand, for a configuration $\eta \in \mathfrak{R}^+ \setminus \mathfrak{R}^a$, $\mathbb{H}(\eta) \geq \mathbb{H}(\zeta) + 4 - h$ if $\zeta \in \mathfrak{R}$, so that

$$\mathbb{H}(\eta) \;\geq\; \mathbb{H}(\xi) \;+\; 2\;, \quad \eta \in \mathfrak{R}^+ \setminus \mathfrak{R}^a \;,\; \xi \in \mathfrak{R}^a \;. \tag{1.22}$$

Recall the notation introduced just above Remark 1.3.2. For two disjoint subsets $\mathcal{A}$ to $\mathcal{B}$ of Ω, denote by $\xi_{\mathcal{A},\mathcal{B}}$, the configuration with highest energy in the optimal path joining $\mathcal{A}$ to $\mathcal{B}$. By Remark 1.3.2 and (1.15), the limit

$$\lim_{\beta\to\infty} \frac{\text{cap}(\mathcal{A},\mathcal{B})}{\mu_\beta(\xi_{\mathcal{A},\mathcal{B}})} \;\text{ exists and takes value in }\; (0,\infty)\;. \tag{1.23}$$

In particular, for every subset $\mathcal{B}$ of Ω and every configuration $\sigma \notin \mathcal{B}$,

$$\text{cap}(\sigma,\mathcal{B}) \;\preceq\; \frac{1}{Z_\beta}\, e^{-\beta\mathbb{H}(\sigma)}\;. \tag{1.24}$$

Assertion 1.4.5. *For all configurations* $\xi \in \mathfrak{R}^a$, $\text{cap}(\xi,\mathbf{-1}) \;\approx\; Z_\beta^{-1}\exp\{-\beta\mathbb{H}(\xi)\}$.

Proof. Fix $\xi \in \mathfrak{R}^a$. By (1.23), (1.24), it is enough to exhibit a path $\gamma = (\xi = \xi_0, \xi_1, \dots, \xi_n = -\mathbf{1})$ from ξ to $-\mathbf{1}$ such that $\max_i \mathbb{H}(\xi_i) = \mathbb{H}(\xi)$.

Consider the path $\gamma = (\xi = \xi_0, \xi_1, \dots, \xi_n = -\mathbf{1})$, $n = n_0(n_0+1)+1$, constructed as follows. ξ_1 is the configuration obtained from ξ by switching the attached spin from 0 to -1. Clearly, $\mathbb{H}(\xi_1) = \mathbb{H}(\xi) - 2 + h$, and ξ_1 consists of a $n_0 \times (n_0+1)$ rectangle of 0 spins.

The portion $(\xi_1, \dots, \xi_{n_0+1})$ of the path γ is constructed by flipping, successively, from 0 to -1, all spins of one of the shortest sides of the rectangle, keeping until the last step the perimeter of the set $A(\xi_i)$ equal to $4n_0(n_0+1)$. In particular, ξ_{n_0+1} consists of a $n_0 \times n_0$ square of 0 spins, $\mathbb{H}(\xi_{i+1}) = \mathbb{H}(\xi_i) + h$ for $1 \le i < n_0$, and $\mathbb{H}(\xi_{n_0+1}) = \mathbb{H}(\xi_{n_0}) - 2 + h$. The energy of this piece of the path attains its maximum at ξ_{n_0} and $\mathbb{H}(\xi_{n_0}) = \mathbb{H}(\xi_1) + (n_0-1)h = \mathbb{H}(\xi) - 2 + n_0 h < \mathbb{H}(\xi)$.

The path proceed in this way by always flipping from 0 to -1 all spins of one of the shortest sides. It is easy to check that $\mathbb{H}(\xi_i) < \mathbb{H}(\xi)$ for all $1 \le i \le n$, proving the assertion. □

Assertion 1.4.6. *For every $\sigma \in \mathfrak{B}^+ \setminus \mathfrak{R}^a$, $\lim_{\beta\to\infty} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] = 0$.*

Proof. Fix $\sigma \in \mathfrak{B}^+ \setminus \mathfrak{R}^a$ and $\xi \in \mathfrak{R}^a$. By (1.12), by the monotonicity of the capacity, stated in [17, Lemma 2.2], by (1.24), and by Assertion 1.4.5,

$$\mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \;\le\; \frac{\mathrm{cap}(\sigma, -\mathbf{1})}{\mathrm{cap}(\mathfrak{B}^+, -\mathbf{1})} \;\le\; \frac{C_0}{Z_\beta} \frac{e^{-\beta\mathbb{H}(\sigma)}}{\mathrm{cap}(\xi, -\mathbf{1})} \;\le\; C_0\, e^{-\beta\{\mathbb{H}(\sigma)-\mathbb{H}(\xi)\}}$$

for some finite constant C_0 independent of β. By (1.21), (1.22), this expression vanishes as $\beta \uparrow \infty$, proving the assertion. □

Denote by $\mathcal{S}$ the set of stable configurations :

$$\mathcal{S} \;=\; \Big\{\sigma \in \Omega : \lim_{\beta\to\infty} \lambda_\beta(\sigma) = 0\Big\}\ . \tag{1.25}$$

The stable configurations are formed as follows. On a background of negative spins, place vacant rectangles of length and width larger than or equal to 2, and vacant rings of length or width larger than 2. Note that inserting a vacant ring of size $L \times L$ is not excluded. We just require the graph distance between vacant rectangles and vacant rings to be greater than or equal to 3, and we do not allow the coexistence of horizontal and vertical vacant rings.

Inside the vacant rectangles and the vacant rings, embed positive rectangles along the same rules. This means that the length and the width of the positive rectangles should be larger than or equal to 2, that the length or the width of the positive rings should be larger than or equal to 2, that the graph distance between positive rectangles and positive rings should be greater than or equal to 3. Note that the previous rules do not allow the coexistence of horizontal and vertical positive rings. Furthermore, we do not allow positive spins to have negative spins as neighbors, a layer of vacant sites should separate positive from negative spins.

The next assertion is the only one in which capacities are not used to derive the needed bounds, because we estimate the probability of reaching a state which can be attained through paths in which the energy never increase. The argument, though, is fairly simple.

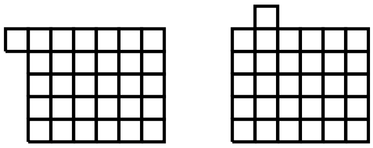


FIGURE 1.1 – Examples of configurations $\sigma \in \mathfrak{R}^c$ and $\sigma' \in \mathfrak{R}^i$ in the case where $n_0 = 5$. A 1×1 square centered at x has been placed at each site x occupied by a 0-spin. All the other spins are equal to -1.

Denote by $\mathfrak{R}^c$, $\mathfrak{R}^i$ the configurations of $\mathfrak{R}^a$ in which the extra particle is attached to the corner, interior of the rectangle, respectively (cf. Figure 1.1).

Assertion 1.4.7. *For $\sigma \in \mathfrak{R}^c$ and $\sigma' \in \mathfrak{R}^i$,*

$$\lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{\sigma_+} = H_{\mathcal{S}}] \;=\; 1/2 \quad and \quad \lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{\sigma_-} = H_{\mathcal{S}}] \;=\; 1/2\;,$$
$$\lim_{\beta\to\infty} \mathbb{P}_{\sigma'}[H_{\sigma_+} = H_{\mathcal{S}}] \;=\; 2/3 \quad and \quad \lim_{\beta\to\infty} \mathbb{P}_{\sigma'}[H_{\sigma_-} = H_{\mathcal{S}}] \;=\; 1/3\;,$$

where σ_- is the configuration obtained from σ or σ' by flipping to -1 the attached 0 spin, and σ_+ is the configuration whose set $A(\sigma_+)$, formed only by 0 spins, is the smallest rectangle which contains $A(\sigma)$.

Proof. Suppose that the extra 0 spin is not attached to the corner of the rectangle. Denote by σ_1, σ_2 the configurations obtained from σ' by flipping from -1 to 0 one of the two -1 spins which has two neighbor spins equal to 0, and let $\sigma_0 = \sigma_-$. By definition, $R_\beta(\sigma', \sigma_j) = 1$, $0 \le j \le 2$, and $R_\beta(\sigma', \sigma') = o(1)$ for all the other configurations, where $o(1)$ represents an expression which vanishes as $\beta \uparrow \infty$. This shows that $p_\beta(\sigma', \sigma_j)$ converges to $1/3$, $0 \le j \le 2$. We may repeat this argument to show that from σ_j, $j = 1, 2$, one reaches $\mathcal{S}$ at σ_+ with a probability asymptotically equal to 1. The argument is similar if the extra spin is attached to the corner. □

Assertion 1.4.8. *Fix a configuration $\sigma \in \mathcal{S}$ for which $A(\sigma)$ is a $m \times n$ rectangle of 0 spins in a sea of -1 spins. Assume that $m \le n$. Then,*

$$\lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{\mathcal{B}} = H_{\mathcal{S}\setminus\{\sigma\}}] \;=\; 1\;.$$

In this equation, if $n_0 < m$, $n \le L-3$, $\mathcal{B}$ is the set of four configurations in which a row or a column of 0 spins is added to the rectangle $A(\sigma)$. If $n_0 < m < n = L-2$, the set $\mathcal{B}$ is a triple which includes a band of 0 spins of width m and two configurations in which a row or a column of 0 spins of length n is added to the rectangle $A(\sigma)$. If $n_0 < m \le L-3$, $n = L$, the set $\mathcal{B}$ is a pair formed by two bands of 0 spins of width $m+1$. If $n_0 < m = n = L-2$, $\mathcal{B}$ is a pair of two bands of width $L-2$. If $n_0 < m = L-2$, $n = L$, $\mathcal{B} = \{\mathbf{0}\}$. Finally, if $2 \le m \le n_0$, $n \ge 3$, the set $\mathcal{B}$ is the pair (quaternion if $m = n$) of configurations in which a row or a column of 0 spins of length m is removed from the rectangle $A(\sigma)$, and if $m = n = 2$, $\mathcal{B} = \{-\mathbf{1}\}$. The case $m = 1$ is not considered because such a configuration is not stable.

Proof. The assertion follows from inequality (1.12), and from estimates of $\mathrm{cap}(\sigma, \mathcal{S} \setminus \{\sigma\})$, $\mathrm{cap}(\sigma, \mathcal{S}\setminus[\mathcal{B}\cup\{\sigma\}])$. In the case $m > n_0$, $\mathrm{cap}(\sigma, \mathcal{S}\setminus\{\sigma\}) \approx \mu_\beta(\sigma)\exp\{-\beta(2-h)\}$ and $\mathrm{cap}(\sigma, \mathcal{S}\setminus[\mathcal{B}\cup\{\sigma\}]) \prec \mu_\beta(\sigma)\exp\{-\beta(2-h)\}$, while in the case $m \le n_0$, $\mathrm{cap}(\sigma, \mathcal{S} \setminus \{\sigma\}) \approx \mu_\beta(\sigma)\exp\{-\beta(m-1)h\}$ and $\mathrm{cap}(\sigma, \mathcal{S} \setminus [\mathcal{B} \cup \{\sigma\}]) \prec \mu_\beta(\sigma)\exp\{-\beta(m-1)h\}$.
□

Lemma 1.4.9. *For every $\sigma \in \mathfrak{B}^+$,*

$$\lim_{\beta\to\infty} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \;=\; \frac{1}{|\mathfrak{R}^a|}\, \mathbf{1}\{\sigma \in \mathfrak{R}^a\}\;,$$

where $|\mathfrak{R}^a|$ represents the number of configurations in $\mathfrak{R}^a$.

Proof. In view of Assertion 1.4.6, we may restrict our attention to $\sigma \in \mathfrak{R}^a$. Fix a reference configuration σ^* in $\mathfrak{R}^a$. By (1.10) and by definition of the capacity,

$$\mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \;=\; \frac{M(-\mathbf{1})\, \mathbb{P}_{-\mathbf{1}}[H_\sigma = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}]}{\mathrm{cap}(-\mathbf{1}, \mathfrak{B}^+)}\;.$$

By reversibility, the numerator of this expression is equal to

$$M(\sigma)\, \mathbb{P}_\sigma[H_{-\mathbf{1}} = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}] \;=\; \mu_\beta(\sigma)\, \lambda(\sigma)\, \mathbb{P}_\sigma[H_{-\mathbf{1}} = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}]\;.$$

By Assertions 1.4.7 and 1.4.8, $\mathbb{P}_\sigma[H_{-\mathbf{1}} = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}] = \mathfrak{n}(\sigma) + o(1)$, where

$$\mathfrak{n}(\sigma) \;=\; \begin{cases} 1/2 & \text{if } \sigma \in \mathfrak{R}^c, \\ 1/3 & \text{if } \sigma \in \mathfrak{R}^i. \end{cases}$$

Since

$$\lambda(\sigma) \;=\; \begin{cases} 2 + o(1) & \text{if } \sigma \in \mathfrak{R}^c, \\ 3 + o(1) & \text{if } \sigma \in \mathfrak{R}^i, \end{cases}$$

and since $\mu_\beta(\sigma) = \mu_\beta(\sigma^*)$, we conclude that

$$\mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \;=\; \frac{\mu_\beta(\sigma^*)}{\mathrm{cap}(-\mathbf{1}, \mathfrak{B}^+)} \Big(\mathbf{1}\{\sigma \in \mathfrak{R}^a\} + o(1)\Big)\;.$$

Summing over $\sigma \in \mathfrak{B}^+$, we conclude that $\mu_\beta(\sigma^*)/\mathrm{cap}(-\mathbf{1}, \mathfrak{B}^+) = |\mathfrak{R}^a|^{-1}(1 + o(1))$, which completes the proof of the assertion. □

It follows from the proof of the previous lemma that for any configuration $\sigma^* \in \mathfrak{R}^a$,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1}, \mathfrak{B}^+)}{\mu_\beta(\sigma^*)} \;=\; |\mathfrak{R}^a|\;. \tag{1.26}$$

Denote by $\mathfrak{R}^l$, $\mathfrak{R}^s$ the configurations of $\mathfrak{R}^a$ in which the extra particle is attached to one of the longest, shortest sides, respectively, and let $\mathfrak{R}^{lc} = \mathfrak{R}^l \cap \mathcal{R}^c$, $\mathfrak{R}^{li} = \mathfrak{R}^l \cap \mathfrak{R}^i$. The next lemma is an immediate consequence of Assertions 1.4.7 and 1.4.8.

Lemma 1.4.10. *For $\sigma \in \mathfrak{R}^{lc}$, $\sigma' \in \mathfrak{R}^{li}$, and $\sigma'' \in \mathfrak{R}^s$,*

$$\lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{-\mathbf{1}} = H_{\mathbb{M}}] \;=\; 1/2 \quad \textit{and} \quad \lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{\mathbf{0}} = H_{\mathbb{M}}] \;=\; 1/2\;,$$
$$\lim_{\beta\to\infty} \mathbb{P}_{\sigma'}[H_{-\mathbf{1}} = H_{\mathbb{M}}] \;=\; 1/3 \quad \textit{and} \quad \lim_{\beta\to\infty} \mathbb{P}_{\sigma'}[H_{\mathbf{0}} = H_{\mathbb{M}}] \;=\; 2/3\;,$$
$$\lim_{\beta\to\infty} \mathbb{P}_{\sigma''}[H_{-\mathbf{1}} = H_{\mathbb{M}}] \;=\; 1\;.$$

It follows from this assertion that for every $\sigma \in \mathfrak{R}^a$,

$$\lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{\{-\mathbf{1},\mathbf{0}\}} < H_{+\mathbf{1}}] \;=\; 1\;, \tag{1.27}$$

Proof.[Proof of Proposition 1.2.3] We prove below the first idendity of the proposition. We first claim that

$$\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\}) \;=\; \mathrm{cap}(-\mathbf{1},\mathfrak{B}^+) \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\, \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]\;. \tag{1.28}$$

Indeed, since starting from $-\mathbf{1}$ the process hits $\mathfrak{B}^+$ before $\{\mathbf{0},+\mathbf{1}\}$, by the strong Markov property we have that

$$\mathbb{P}_{-\mathbf{1}}[H_{\{\mathbf{0},+\mathbf{1}\}} < H^+_{-\mathbf{1}}] \;=\; \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}]\, \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]\;.$$

By (1.10), we may rewrite the previous expression as

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{B}^+} < H^+_{-\mathbf{1}}] \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\, \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]\;.$$

This proves (1.28) in view of the definition (1.5) of the capacity.

By (1.28) and (1.26), for any configuration $\sigma^* \in \mathfrak{R}^a$,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})}{\mu_\beta(\sigma^*)} \;=\; |\mathfrak{R}^a| \lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\, \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]\;.$$

By Lemma 1.4.9, the right hand side is equal to

$$\lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]\;. \tag{1.29}$$

By Lemma 1.4.10, this expression is equal to $(1/2)|\mathfrak{R}^{lc}| + (2/3)|\mathfrak{R}^{li}| = 2|\Lambda_L|\{2 + (4/3)(n_0-1)\}$, which completes the proof of the first claim of the proposition. □

Assertion 1.4.11. *We have that*

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1},\mathbf{0})}{\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})} \;=\; 1\;.$$

Proof. Let σ^* be a configuration in $\mathfrak{R}^a$. By the proof of Proposition 1.2.3 up to (1.29),

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1},\mathbf{0})}{\mu_\beta(\sigma^*)} \;=\; \lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{\mathbf{0}} < H_{-\mathbf{1}}]\;.$$

By (1.27), this expression is equal to (1.29). This completes the proof of the assertion.
□

Proof.[Proof of Proposition 1.2.1] Let $q(\sigma) = \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]$, $\sigma \in \mathfrak{B}^+$. By Assertion 1.4.6, $q(\sigma) \to 0$ if $\sigma \notin \mathfrak{R}^a$. Hence, by (1.20),

$$\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; \sum_{\sigma\in\mathfrak{B}^+} q(\sigma)\,\mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; \sum_{\sigma\in\mathfrak{R}^a} q(\sigma)\,\mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;+\; o(1)\;.$$

By (1.27), for all $\sigma \in \mathfrak{R}^a$,

$$\mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; \mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}\,,\, H_{\{-\mathbf{1},\mathbf{0}\}} < H_{+\mathbf{1}}] \;=\; \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{+\mathbf{1}} < H_{\mathbf{0}}]\;.$$

Therefore, by the strong Markov property,

$$\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; \mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \sum_{\sigma\in\mathfrak{R}^a} q(\sigma)\,\mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\{\mathbf{0},+\mathbf{1}\}}] \;+\; o(1)\;.$$

By Lemma 1.4.10, for $\sigma \in \mathfrak{R}^l$, $\limsup_{\beta\to\infty} \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\{\mathbf{0},+\mathbf{1}\}}] \le 1/2$, which completes the proof of the proposition. □

Proof.[Proof of Proposition 1.2.2] We prove the proposition when the chain starts from $-\mathbf{1}$, the argument being analogous when it starts from $\mathbf{0}$. Since the chains hits $\mathfrak{B}^+$ before reaching $\mathbf{0}$ and $\mathfrak{R}^l$, by the strong Markov property,

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;=\; \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\,\mathbb{P}_\sigma[H_{\mathfrak{R}^l} < H_{\mathbf{0}}]\;.$$

By Lemma 1.4.9, this expression is equal to

$$\big(1 + o(1)\big)\frac{1}{|\mathfrak{R}^a|}\Big\{|\mathfrak{R}^l| \;+\; \sum_{\sigma\in\mathfrak{R}^s} \mathbb{P}_\sigma[H_{\mathfrak{R}^l} < H_{\mathbf{0}}]\Big\}\;.$$

By Assertions 1.4.7 and 1.4.8, for all $\sigma \in \mathfrak{R}^s$, $\sigma' \in \mathfrak{R}$

$$\lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{\mathfrak{R}} < H_{\mathfrak{R}^l\cup\{-\mathbf{1},\mathbf{0}\}}] \;=\; 1\;, \quad \lim_{\beta\to\infty} \mathbb{P}_{\sigma'}[H_{-\mathbf{1}} < H_{\mathfrak{R}^l\cup\{\mathbf{0}\}}] \;=\; 1\;.$$

Therefore, for all $\sigma \in \mathfrak{R}^s$,

$$\lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\mathfrak{R}^l\cup\{\mathbf{0}\}}] \;=\; 1\;.$$

Hence, by the strong Markov property and by the first two identities of this proof,

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;=\; \big(1 + o(1)\big)\frac{1}{|\mathfrak{R}^a|}\Big\{|\mathfrak{R}^l| \;+\; \sum_{\sigma\in\mathfrak{R}^s} \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}]\Big\}\;,$$

which completes the proof of the proposition. □

1.5 Proofs of Propositions 1.2.3 and 1.2.4

We examine in this section the metastable behavior of the Blume-Capel model starting from $\mathbf{0}$. The main observation is that the energy barrier from $\mathbf{0}$ to $-\mathbf{1}$ is larger that the one from $\mathbf{0}$ to $+\mathbf{1}$. We may therefore ignore $-\mathbf{1}$ and argue by symmetry that the passage from $\mathbf{0}$ to $+\mathbf{1}$ is identical to the one from $-\mathbf{1}$ to $\mathbf{0}$.

In analogy to the notation introduced right before Assertion 1.4.4, let $\mathfrak{B}_0$ be the set of configurations with $n_0(n_0+1)$ sites with spins different from 0, and let $\mathfrak{R}_0$ be the subset of $\mathfrak{B}_0$ given by

$$\mathfrak{R}_0 \;=\; \big\{\sigma \in \{0,+1\}^{\mathbb{Z}^2} : \{x : \sigma(x) \neq 0\} \text{ forms a } n_0 \times (n_0+1) \text{ rectangle } \big\}\;.$$

Denote by $\mathfrak{R}_0^+$ the set of configurations in $\{-1,0,+1\}^{\mathbb{Z}^2}$ in which there are $n_0(n_0+1)+1$ spins which are not equal to 0, and in which $n_0(n_0+1)$ spins of magnetization $+1$ form a $n_0 \times (n_0+1)$-rectangle. Finally, let $\mathfrak{R}_0^a \subset \mathfrak{R}_0^+$ be the set of configurations for which the remaining spin is a $+1$ spin attached to one of the sides of the rectangle, and let

$$\mathfrak{B}_0^+ \;=\; (\mathfrak{B}_0 \setminus \mathfrak{R}_0) \cup \mathfrak{R}_0^+\;.$$

Assertion 1.5.1. *For every $\sigma \in \mathfrak{B}_0^+$, $\lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_\sigma = H_{\mathfrak{B}_0^+}] = |\mathfrak{R}_0^a|^{-1}\mathbf{1}\{\sigma \in \mathfrak{R}_0^a\}$. Moreover, if σ^* represents a configuration in $\mathfrak{R}_0^a$,*

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(\mathbf{0}, \mathfrak{B}_0^+)}{\mu_\beta(\sigma^*)} \;=\; |\mathfrak{R}_0^a|\;.$$

Proof. As in Assertion 1.4.6, we may exclude all configurations $\sigma \in \{0,+1\}^{\Lambda_L}$ which do not belong to $\mathfrak{R}_0^a$. We may also exclude all configurations in $\mathfrak{B}_0^+$ which have a negative spin since by turning all negative spins into positive spins we obtain a new configurations whose energy is strictly smaller than the one of the original configuration. For the configurations in $\mathfrak{R}_0^a$ we may apply the arguments presented in the proof of Lemma 1.4.9. □

Denote by $\mathfrak{R}_0^c$, $\mathfrak{R}_0^i$ the configurations of $\mathfrak{R}_0^a$ in which the extra particle is attached to the corner, interior of the rectangle, respectively. Denote by $\mathfrak{R}_0^l$, $\mathfrak{R}_0^s$ the configurations of $\mathfrak{R}_0^a$ in which the extra particle is attached to one of the longest, shortest sides, respectively, and let $\mathfrak{R}_0^{lc} = \mathfrak{R}_0^l \cap \mathcal{R}_0^c$, $\mathfrak{R}_0^{li} = \mathfrak{R}_0^l \cap \mathfrak{R}_0^i$. The proof of the next assertion is analogous to the one of Lemma 1.4.10 since it concerns configurations with only 0 and $+1$ spins.

Assertion 1.5.2. *For $\sigma \in \mathfrak{R}_0^{lc}$, $\sigma' \in \mathfrak{R}_0^{li}$, and $\sigma'' \in \mathfrak{R}_0^s$,*

$$\lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{\mathbf{0}} = H_{\mathbb{M}}] \;=\; 1/2 \quad \textit{and} \quad \lim_{\beta\to\infty} \mathbb{P}_\sigma[H_{+\mathbf{1}} = H_{\mathbb{M}}] \;=\; 1/2\;,$$
$$\lim_{\beta\to\infty} \mathbb{P}_{\sigma'}[H_{\mathbf{0}} = H_{\mathbb{M}}] \;=\; 1/3 \quad \textit{and} \quad \lim_{\beta\to\infty} \mathbb{P}_{\sigma'}[H_{+\mathbf{1}} = H_{\mathbb{M}}] \;=\; 2/3\;,$$
$$\lim_{\beta\to\infty} \mathbb{P}_{\sigma''}[H_{\mathbf{0}} = H_{\mathbb{M}}] \;=\; 1\;.$$

The next claim follows from the previous two assertions.

Assertion 1.5.3. *We have that*

$$\lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}] \;=\; 0\;.$$

Proof. Since, starting from $\mathbf{0}$, the set $\mathfrak{B}_0^+$ is reached before the process hits $\{-\mathbf{1}, +\mathbf{1}\}$, by the strong Markov property,

$$\lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}] \;=\; \lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{B}_0^+} \mathbb{P}_{\mathbf{0}}[H_\sigma = H_{\mathfrak{B}_0^+}]\, \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{+\mathbf{1}}]\;.$$

By Assertion 1.5.1 and by the strong Markov property at time $H_{\mathcal{M}}$, this expression is equal to

$$\lim_{\beta\to\infty} \frac{1}{|\mathfrak{R}_0^a|} \sum_{\sigma\in\mathfrak{R}_0^a} \mathbb{E}_\sigma\Big[\mathbb{P}_{\sigma(H_{\mathcal{M}})}[H_{-\mathbf{1}} < H_{+\mathbf{1}}]\Big] \;=\; c_0 \lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}]\;.$$

where we applied Assertion 1.5.2 to derive the last identity. In this equation, $c_0 = \{|\mathfrak{R}_0^{lc}|/2|\mathfrak{R}_0^a|\} + \{|\mathfrak{R}_0^{li}|/3|\mathfrak{R}_0^a|\} < 1$. This completes the proof of the assertion. ☐

Proof.[Proof of Proposition 1.2.3, Part B] The proof is similar to the one of Part A, presented in the previous section. As in (1.28), we have that

$$\mathrm{cap}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\}) \;=\; \mathrm{cap}(\mathbf{0}, \mathfrak{B}_0^+) \sum_{\sigma\in\mathfrak{B}_0^+} \mathbb{P}_{\mathbf{0}}[H_\sigma = H_{\mathfrak{B}_0^+}]\, \mathbb{P}_\sigma[H_{\{-\mathbf{1},+\mathbf{1}\}} < H_{\mathbf{0}}]\;.$$

Hence, by Assertion 1.5.1, for any configuration $\sigma^* \in \mathfrak{R}_0^a$,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})}{\mu_\beta(\sigma^*)} \;=\; |\mathfrak{R}_0^a| \lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{B}_0^+} \mathbb{P}_{\mathbf{0}}[H_\sigma = H_{\mathfrak{B}_0^+}]\, \mathbb{P}_\sigma[H_{\{-\mathbf{1},+\mathbf{1}\}} < H_{\mathbf{0}}]\;.$$

By Assertion 1.5.1, the right hand side is equal to

$$\lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{R}_0^a} \mathbb{P}_\sigma[H_{\{-\mathbf{1},+\mathbf{1}\}} < H_{\mathbf{0}}]\;.$$

By Assertion 1.5.2, this expression is equal to $(1/2)|\mathfrak{R}_0^{lc}| + (2/3)|\mathfrak{R}_0^{li}| = 2|\Lambda_L|\{2 + (4/3)(n_0 - 1)\}$, which completes the proof of the second claim of the proposition. ☐

As in Assertion 1.4.11 we have that

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(\mathbf{0}, +\mathbf{1})}{\mathrm{cap}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})} \;=\; 1\;. \tag{1.30}$$

Assertion 1.5.4. *We have that*

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1}, +\mathbf{1})}{\mathrm{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})} \;=\; 1\;.$$

Proof. We repeat the proof of the part A of Proposition 1.2.3 up (1.29) to obtain that

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1}, +\mathbf{1})}{\mu_\beta(\sigma^*)} \;=\; \lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{-\mathbf{1}}]\;,$$

if σ^* represents a configuration in $\mathfrak{R}^a$. By (1.27), this expression is equal to

$$\begin{aligned} &\lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{\mathbf{0}} < H_{+\mathbf{1}} < H_{-\mathbf{1}}] \\ &\quad = \lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{+\mathbf{1}} < H_{-\mathbf{1}}] \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{\mathbf{0}} < H_{\{-\mathbf{1},+\mathbf{1}\}}]\;. \end{aligned}$$

where we used the strong Markov property in the last step. By Lemma 1.4.10 and Assertion 1.5.3, this limit is equal to $(1/2)|\mathfrak{R}^{lc}| + (2/3)|\mathfrak{R}^{li}|$, which completes the proof of the assertion. ☐

Assertion 1.5.5. *We have that*

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(+\mathbf{1}, \{-\mathbf{1}, \mathbf{0}\})}{\mathrm{cap}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})} \;=\; 1\;.$$

Proof. Indeed, by monotonicity of the capacity and by (1.14),

$$\mathrm{cap}(+\mathbf{1}, \mathbf{0}) \;\le\; \mathrm{cap}(+\mathbf{1}, \{-\mathbf{1}, \mathbf{0}\}) \;\le\; \mathrm{cap}(+\mathbf{1}, \mathbf{0}) \;+\; \mathrm{cap}(+\mathbf{1}, -\mathbf{1})\;.$$

By Assertion 1.5.4, by (1.30), and by Proposition 1.2.3, $\mathrm{cap}(+\mathbf{1}, -\mathbf{1})/\mathrm{cap}(\mathbf{0}, +\mathbf{1}) \to 0$ as $\beta \uparrow \infty$. Hence,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(+\mathbf{1}, \{-\mathbf{1}, \mathbf{0}\})}{\mathrm{cap}(\mathbf{0}, +\mathbf{1})} \;=\; 1\;.$$

To complete the proof, it remains to recall (1.30). □

We turn to the proof of Proposition 1.2.4. We first show that the assumptions of Lemma 1.3.3 are in force for $E_1 = \mathcal{M}$. Recall Remark 1.3.4.

Assertion 1.5.6. *Consider two configurations $\sigma \notin \mathcal{M}$ and $\eta \in \mathcal{M}$. If $\mathbb{H}(\sigma) \le \mathbb{H}(\eta)$, then $\mathbb{H}(\xi_{\sigma,\mathcal{M}}) - \mathbb{H}(\sigma) < \mathbb{H}(\xi_{\sigma,\eta}) - \mathbb{H}(\eta)$.*

Proof. We claim that for any configuration $\sigma \notin \mathcal{M}$, $\mathbb{H}(\xi_{\sigma,\mathcal{M}}) - \mathbb{H}(\sigma) \le 2 - h$. To prove this claim it is enough to exhibit a self-avoiding path from σ to $\mathcal{M}$ whose energy is kept below $\mathbb{H}(\sigma) + 2 - h$. This is easy. Starting from σ we may first reach the set $\mathcal{S}$ of stable configurations through a path whose energy does not increase. Denote by $\sigma^\star$ the configuration in $\mathcal{S}$ attained through this path. From $\sigma^\star$ we may reach the set $\mathcal{M}$ by removing all small droplets (the ones whose smaller side has length n_0 or less) and by increasing the large droplets (the ones whose both sides have length at least $n_0 + 1$) in such a way that the energy remains less than or equal to $\mathbb{H}(\sigma^\star) + 2 - h$. This proves the claim.

On the other hand, since $\mathbb{H}(\zeta) \ge \mathbb{H}(\eta) + 4 - h$ for any configuration ζ which differs from η at one site, $\mathbb{H}(\xi_{\sigma,\eta}) - \mathbb{H}(\eta) \ge 4 - h$, which proves the assertion. □

Assertion 1.5.7. *We have that*

$$\lim_{\beta\to\infty} \frac{M(-\mathbf{1})\,\mathbb{P}_{-\mathbf{1}}[H_{\mathbf{0}} < H^+_{\{-\mathbf{1},+\mathbf{1}\}}]}{\mathrm{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})} \;=\; 1\;.$$

Proof. Fix σ^* in $\mathfrak{R}^a$. In view of Proposition 1.2.3, it is enough to show that

$$\lim_{\beta\to\infty} \frac{M(-\mathbf{1})\,\mathbb{P}_{-\mathbf{1}}[H_{\mathbf{0}} < H^+_{\{-\mathbf{1},+\mathbf{1}\}}]}{\mu_\beta(\sigma^*)} \;=\; \frac{4(2n_0+1)}{3}\,|\Lambda_L|\;.$$

In the proof of Proposition 1.2.3.A, replace $\mathrm{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})$ by the numerator appearing in the statement of this assertion. The proof is identical up to formula (1.29). It remains to estimate

$$\lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{\mathbf{0}} < H_{\{-\mathbf{1},+\mathbf{1}\}}] \;\text{ which is equal to }\; \lim_{\beta\to\infty} \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]$$

in view of (1.27). This expression has been computed at the end of the proof of Proposition 1.2.3.A, which completes the proof of the assertion. □

Proof.[Proof of Proposition 1.2.4] We first assume that the chain starts from $\mathbf{0}$. By Lemma 1.3.3 and Assertion 1.5.6,

$$\mathbb{E}_{\mathbf{0}}[H_{+\mathbf{1}}] \;=\; \big(1+o(1)\big)\,\frac{1}{\mathrm{cap}(\mathbf{0},+\mathbf{1})}\Big\{\mu_\beta(\mathbf{0}) + \mu_\beta(-\mathbf{1})\mathbb{P}_{-\mathbf{1}}[H_{\mathbf{0}} < H_{+\mathbf{1}}]\Big\}\;.$$

Since the second term in the expression inside braces is bounded by $\mu_\beta(-\mathbf{1}) \prec \mu_\beta(\mathbf{0})$, the expectation is equal to $(1+o(1))\mu_\beta(\mathbf{0})/\mathrm{cap}(\mathbf{0},+\mathbf{1})$. By (1.30), we may replace $\mathrm{cap}(\mathbf{0},+\mathbf{1})$ by $\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})$. On the other hand, by Proposition 1.2.3,

$$\lim_{\beta\to\infty} \frac{\mu_\beta(\mathbf{0})}{\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})}\,\frac{\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})}{\mu_\beta(-\mathbf{1})} \;=\; 1\;.$$

This completes the proof of the proposition in the case in which the chain starts from $\mathbf{0}$.

We turn to the case in which the chain starts from $-\mathbf{1}$. By Lemma 1.3.3,

$$\mathbb{E}_{-\mathbf{1}}[H_{+\mathbf{1}}] \;=\; \big(1+o(1)\big)\,\frac{1}{\mathrm{cap}(-\mathbf{1},+\mathbf{1})}\Big\{\mu_\beta(-\mathbf{1}) + \mu_\beta(\mathbf{0})\mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}]\Big\}\;.$$

By (1.10), by reversibility and by definition of the capacity,

$$\begin{aligned}\mu_\beta(\mathbf{0})\,\mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}] \;&=\; \frac{M(\mathbf{0})\,\mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H^+_{\{\mathbf{0},+\mathbf{1}\}}]}{\lambda_\beta(\mathbf{0})\,\mathbb{P}_{\mathbf{0}}[H_{\{-\mathbf{1},+\mathbf{1}\}} < H^+_{\mathbf{0}}]}\\ &=\; \frac{M(-\mathbf{1})\,\mathbb{P}_{-\mathbf{1}}[H_{\mathbf{0}} < H^+_{\{-\mathbf{1},+\mathbf{1}\}}]}{\lambda_\beta(\mathbf{0})\,\mathbb{P}_{\mathbf{0}}[H_{\{-\mathbf{1},+\mathbf{1}\}} < H^+_{\mathbf{0}}]} \;=\; \frac{\mu_\beta(\mathbf{0})\,M(-\mathbf{1})\,\mathbb{P}_{-\mathbf{1}}[H_{\mathbf{0}} < H^+_{\{-\mathbf{1},+\mathbf{1}\}}]}{\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})}\;.\end{aligned}$$

Hence, by Assertion 1.5.7,

$$\mathbb{E}_{-\mathbf{1}}[H_{+\mathbf{1}}] \;=\; \big(1+o(1)\big)\,\frac{\mu_\beta(-\mathbf{1})}{\mathrm{cap}(-\mathbf{1},+\mathbf{1})}\Big\{1 + \frac{\mu_\beta(\mathbf{0})\,\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})}{\mu_\beta(-\mathbf{1})\,\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})}\Big\}\;.$$

To complete the proof it remains to recall the statements of Proposition 1.2.3 and Assertion 1.5.4. □

1.6 The hitting time of 0 starting from −1

We prove in this section Proposition 1.2.5. By Lemma 1.3.3 and Assertion 1.5.6,

$$\mathbb{E}_{-\mathbf{1}}[H_0] \;=\; \big(1+o(1)\big)\,\frac{1}{\mathrm{cap}(-\mathbf{1},\mathbf{0})}\,\Big\{\mu_\beta(-\mathbf{1}) + \mu_\beta(+\mathbf{1})\mathbb{P}_{+\mathbf{1}}[H_{-\mathbf{1}} < H_{\mathbf{0}}]\Big\}\;.$$

By (1.10) and the first identity in (1.11), and by reversibility, the second term inside braces is equal to

$$\mu_\beta(+\mathbf{1})\,M(+\mathbf{1})\frac{\mathbb{P}_{+\mathbf{1}}[H_{-\mathbf{1}} < H^+_{\{\mathbf{0},+\mathbf{1}\}}]}{\mathrm{cap}(+\mathbf{1},\{-\mathbf{1},\mathbf{0}\})} \;=\; \mu_\beta(+\mathbf{1})\,M(-\mathbf{1})\frac{\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H^+_{\{-\mathbf{1},\mathbf{0}\}}]}{\mathrm{cap}(+\mathbf{1},\{-\mathbf{1},\mathbf{0}\})}\;.$$

By the first identity in (1.11), this expression is equal to

$$\mu_\beta(+\mathbf{1})\,\frac{\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})}{\mathrm{cap}(+\mathbf{1},\{\mathbf{0},-\mathbf{1}\})}\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}]\;.$$

By Assertion 1.5.5 and Proposition 1.2.3, we may replace the ratio of the capacities by $\mu_\beta(-\mathbf{1})/\mu_\beta(\mathbf{0})$. Hence,

$$\mathbb{E}_{-\mathbf{1}}[H_{\mathbf{0}}] \;=\; \big(1+o(1)\big)\,\frac{\mu_\beta(-\mathbf{1})}{\mathrm{cap}(-\mathbf{1},\mathbf{0})}\Big\{1+\frac{\mu_\beta(+\mathbf{1})}{\mu_\beta(\mathbf{0})}\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}}<H_{\mathbf{0}}]\Big\}\;.$$

By Lemma 1.6.1 below, this expression is equal to

$$\mathbb{E}_{-\mathbf{1}}[H_{\mathbf{0}}] \;=\; \big(1+o(1)\big)\,\frac{\mu_\beta(-\mathbf{1})}{\mathrm{cap}(-\mathbf{1},\mathbf{0})}\,\frac{\mu_\beta(+\mathbf{1})}{\mu_\beta(\mathbf{0})}\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}}<H_{\mathbf{0}}]\;.$$

By Assertion 1.4.11, we may replace in the right hand side of the previous formula $\mathrm{cap}(-\mathbf{1},\mathbf{0})$ by $\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})$. This proves the first assertion of the proposition in view of the definition of θ_β. The second assertion of the proposition is the content of Lemma 1.6.1.

Lemma 1.6.1. *We have that*

$$\lim_{\beta\to\infty}\frac{\mu_\beta(+\mathbf{1})}{\mu_\beta(\mathbf{0})}\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}}<H_{\mathbf{0}}] \;=\; \infty\;.$$

The proof of this lemma is divided in several assertions. By (1.11), and by the definition of the capacity,

$$\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}}<H_{\mathbf{0}}] \;=\; \frac{\mu_\beta(-\mathbf{1})\,\lambda_\beta(-\mathbf{1})\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}}<H^+_{\{-\mathbf{1},\mathbf{0}\}}]}{\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})}\;. \tag{1.31}$$

We estimate the probability appearing in the numerator. This is done by proposing a path from $-\mathbf{1}$ to $+\mathbf{1}$ which does not visit $\mathbf{0}$. The obvious path is the optimal one from $-\mathbf{1}$ to $\mathbf{0}$ juxtaposed with the optimal one from $\mathbf{0}$ $+\mathbf{1}$, modified not to visit $\mathbf{0}$.

We describe the path in $\mathcal{S}$, the set of stable configurations introduced in (1.25). Let $\xi_0\in\mathcal{S}$ be the configuration formed by a $L\times(L-2)$ band of 0-spins and a $L\times 2$ band of -1 spins. The first piece of the path, denoted by γ_0, connects $-\mathbf{1}$ to ξ_0. It is formed by creating and increasing a droplet of 0-spins in a sea of -1-spins.

Let $\gamma_0=(-\mathbf{1}=\eta_0,\dots,\eta_N=\xi_0)$, where

- $N=2(L-3)$,
- η_1 is a 2×2 square of 0-spins in a background of negative spins,
- For $k<N-1$, η_{k+1} is obtained from η_k adding a line of 0-spins to transform a $j\times j$-square of 0-spins into a $(j+1)\times j$-square of 0-spins, or to transform $(j+1)\times j$-square of 0-spins into a $j\times j$-square of 0-spins.

Note that η_N is obtained from η_{N-1} transforming a $(L-2)\times(L-2)$ rectangle into a $L\times(L-2)$ band.

Let $\xi_1\in\mathcal{S}$ be the configuration formed by a 2×2 square of $+1$-spins in a background of 0-spins. The last piece of the path, denoted by γ_1, connects ξ_0 to $+\mathbf{1}$ and is constructed in a similar way as γ_0 so that $\gamma_1=(\xi_1=\zeta_0,\dots,\zeta_N=+\mathbf{1})$. Note that the length of γ_1 is the same as the one of γ_0.

Denote by $q(\eta,\xi)$ the jump probabilities of the trace of $\sigma(t)$ on $\mathcal{S}$: $q(\eta,\xi)=\mathbb{P}_\eta[H_\xi=H_{\mathcal{S}\setminus\{\eta\}}]$. Let

$$q(\gamma_0)\;=\;\prod_{k=0}^{N-1}q(\eta_k,\eta_{k+1})\;,\quad q(\gamma_1)\;=\;\prod_{k=0}^{N-1}q(\zeta_k,\zeta_{k+1})\;,$$

so that

$$\mathbb{P}_{-\mathbf{1}}\big[H_{+\mathbf{1}} < H^+_{\{-\mathbf{1},\mathbf{0}\}}\big] \;\ge\; q(\gamma_0)\, q(\xi_0,\xi_1)\, q(\gamma_1)\;. \tag{1.32}$$

We estimate the three terms on the right hand side.

Assertion 1.6.2. *There exists a positive constant c_0, independent of β, such that*

$$q(\gamma_0) \;\ge\; c_0\, e^{-\beta\{4(n_0-1)-[n_0(n_0+1)-2]h\}}\;.$$

Proof. By the arguments presented in the proof of Assertion 1.4.8, there exists a positive constant c_0, independent of β, such that $q(\eta_k,\eta_{k+1}) \ge c_0$ if $k \ge 2n_0 - 1$. Thus,

$$q(\gamma_0) \;\ge\; c_0 \prod_{k=0}^{2(n_0-1)} q(\eta_k,\eta_{k+1})\;,$$

and η_{2n_0-1} is a $(n_0+1)\times(n_0+1)$ square of 0 spins in a sea of -1-spins.

Denote by $\lambda_{\mathcal S}$ the holding rates of the trace of $\sigma(t)$ on $\mathcal S$, by $\mu_{\mathcal S}$ the invariant probability measure, and let $M_{\mathcal S}(\eta) = \lambda_{\mathcal S}(\eta)\mu_{\mathcal S}(\eta)$. The measure $M_{\mathcal S}$ is reversible for the dicrete-time chain which jumps from η to ξ with probability $q(\eta,\xi)$.

By the proof of Assertion 1.4.8, there exists a positive constant c_0, independent of β, such that $q(\eta_{k+1},\eta_k) \ge c_0$ if $k < 2(n_0-1)$. Thus, multiplying and dividing by $M_{\mathcal S}(-\mathbf{1})$, by reversibility

$$\begin{aligned}\prod_{k=0}^{2(n_0-1)} q(\eta_k,\eta_{k+1}) \;&=\; \frac{M_{\mathcal S}(\eta_{2(n_0-1)})}{M_{\mathcal S}(-\mathbf{1})} \prod_{k=0}^{2n_0-3} q(\eta_{k+1},\eta_k)\; q(\eta_{2(n_0-1)},\eta_{2n_0-1}) \\ &\ge\; c_0\, \frac{M_{\mathcal S}(\eta_{2(n_0-1)})}{M_{\mathcal S}(-\mathbf{1})}\, q(\eta_{2(n_0-1)},\eta_{2n_0-1})\;,\end{aligned}$$

where the configuration $\eta_{2(n_0-1)}$ is a $(n_0+1)\times n_0$ rectangle of 0-spins.

Recall that $M_{\mathcal S}(\eta) = \mu_{\mathcal S}(\eta)\,\lambda_{\mathcal S}(\eta)$. Since $\mu_{\mathcal S}(\eta) = \mu_\beta(\eta)/\mu_\beta(\mathcal S)$, by [2, Proposition 6.1], for any $\eta\in\mathcal S$,

$$M_{\mathcal S}(\eta) \;=\; \frac{\mu_\beta(\eta)}{\mu_\beta(\mathcal S)}\,\lambda_\beta(\eta)\,\mathbb{P}_\eta\big[H_{\mathcal S\setminus\{\eta\}} < H^+_\eta\big] \;=\; \frac{\mathrm{cap}(\eta,\mathcal S\setminus\{\eta\})}{\mu_\beta(\mathcal S)}\;. \tag{1.33}$$

We claim that

$$M_{\mathcal S}(\eta_{2(n_0-1)})\, q(\eta_{2(n_0-1)},\eta_{2n_0-1}) \;\ge\; c_0\, \mu_{\mathcal S}(\eta_{2(n_0-1)})\, e^{-\beta(2-h)}\;. \tag{1.34}$$

To keep notation simple, let $\eta = \eta_{2(n_0-1)}$, $\xi = \eta_{2n_0-1}$. By definition of q and by the first equality in (1.11), the jump probability appearing on the left hand side is equal to

$$\mathbb{P}_\eta\big[H_\xi = H_{\mathcal S\setminus\{\eta\}}\big] \;=\; \frac{\mu_\beta(\eta)\,\lambda_\beta(\eta)\,\mathbb{P}_\eta\big[H_\xi = H^+_{\mathcal S}\big]}{\mathrm{cap}(\eta,\mathcal S\setminus\{\eta\})}\;.$$

The denominator cancels the numerator in (1.33). On the other hand, to reach ξ from η without returning to η, the simplest way consists in creating a 0-spin attached to the longer side of the rectangle and to build a line of 0-spins from this first one. Only the first creation has a cost which vanishes as $\beta\uparrow\infty$. Hence, $\lambda_\beta(\eta)\,\mathbb{P}_\eta\big[H_\xi = H^+_{\mathcal S}\big] \ge c_0 R_\beta(\eta,\eta')$ where η' is a critical configuration in $\mathfrak{R}^l$. This completes the proof of (1.34) since $R_\beta(\eta,\eta') = \exp\{-\beta(2-h)\}$.

It remains to estimate $M_{\mathcal{S}}(-\mathbf{1})$. Recall (1.23). Since $\xi_{-\mathbf{1},\mathcal{S}\setminus\{-\mathbf{1}\}}$ is the configuration with three 0-spins included in a 2×2 square, $\text{cap}(-\mathbf{1},\mathcal{S}\setminus\{-\mathbf{1}\}) \le C_0 \exp\{-\beta[8-3h]\}\mu_\beta(-\mathbf{1})$. Hence, by (1.33),

$$M_{\mathcal{S}}(-\mathbf{1}) \ \le\ C_0\, e^{-\beta(8-3h)}\, \mu_{\mathcal{S}}(-\mathbf{1})\ . \tag{1.35}$$

Putting together all previous estimates, we obtain that

$$q(\gamma_0) \ \ge\ c_0\, \frac{\mu_\beta(\eta_{2(n_0-1)})}{\mu_\beta(-\mathbf{1})}\, e^{-\beta(2-h)}\, e^{\beta(8-3h)}\ ,$$

which completes the proof of the assertion in view of the definition of $\eta_{2(n_0-1)}$. ☐

Next result is proved similarly.

Assertion 1.6.3. *There exists a positive constant c_0, independent of β, such that*

$$q(\gamma_1) \ \ge\ c_0\, e^{-\beta\{4(n_0-1)-[n_0(n_0+1)-2]h\}}\ .$$

We turn to the probability $q(\xi_0,\xi_1)$. Recall that ξ_0 is the configuration formed by a $L\times(L-2)$ band of 0-spins and a $L\times 2$ band of -1 spins, and that ξ_1 is the configuration formed by a 2×2 square of $+1$-spins in a background of 0-spins.

Assertion 1.6.4. *There exists a positive constant c_0, independent of β, such that*

$$q(\xi_0,\xi_1) \ \ge\ c_0\, e^{-2\beta[2-h]}\ .$$

Proof. By definition of q and by (1.10),

$$q(\xi_0,\xi_1) \ =\ \mathbb{P}_{\xi_0}\big[H_{\xi_1} = H_{\mathcal{S}\setminus\{\xi_0\}}\big] \ =\ \frac{\mu_\beta(\xi_0)\,\lambda_\beta(\xi_0)\,\mathbb{P}_{\xi_0}\big[H_{\xi_1} = H^+_{\mathcal{S}}\big]}{\text{cap}(\xi_0,\mathcal{S}\setminus\{\xi_0\})}\ .$$

We claim that

$$\mathbb{P}_{\xi_0}\big[H_{\xi_1} = H^+_{\mathcal{S}}\big] \ \ge\ c_0\, e^{-2\beta[2-h]}\ . \tag{1.36}$$

To estimate this probability, we propose a path γ_3 from ξ_0 to ξ_1 which avoids $\mathcal{S}$. The path consists in filling the -1-spins with 0-spins, until one -1-spin is left. At this point, to avoid the configuration $\mathbf{0}$, we switch this -1-spin to $+1$. To complete the path we create a 2×2 square of $+1$-spins from the first $+1$-spin, as in the optimal path from $\mathbf{0}$ to ξ_1.

Hence, γ_3 as length $2L+3$. Denote this path by $\gamma_3 = (\xi_0 = \eta'_0, \eta'_1, \dots, \eta'_{2L+3} = \xi_1)$. From η'_0 to η'_{2L-2} the next configurations is obtained by flipping a -1 spin to a 0-spin as in an optimal path from ξ_0 to $\mathbf{0}$. In this piece of the path, all jumps have a probability bounded below by a positive constant. Therefore, there exists a positive constant c_0, independent of β, such that

$$\mathbb{P}_{\xi_0}\big[H_{\xi_1} = H^+_{\mathcal{S}}\big] \ \ge\ \prod_{j=0}^{2L+2} p_\beta(\eta'_j,\eta'_{j+1}) \ \ge\ c_0 \prod_{j=2L-1}^{2L+2} p_\beta(\eta'_j,\eta'_{j+1})\ .$$

The first and the last probabilities in this product, $p_\beta(\eta'_{2L-1},\eta'_{2L})$ and $p_\beta(\eta'_{2L+2},\eta'_{2L+3})$, are also bounded below by a positive constant. The other ones can be estimated easily, proving (1.36).

By (1.23), $\text{cap}(\xi_0,\mathcal{S}\setminus\{\xi_0\})/\mu_\beta(\xi_0)$ is bounded above by $C_0\exp\{-\beta[2-h]\}$, while an elementary computation shows that $\lambda_\beta(\xi_0)$ is bounded below by $c_0\exp\{-\beta[2-h]\}$. This completes the proof of the assertion. ☐

By (1.32) and Assertions 1.6.2, 1.6.3 and 1.6.4,

$$\mathbb{P}_{-\mathbf{1}}\big[H_{+\mathbf{1}} < H^{+}_{\{-\mathbf{1},\mathbf{0}\}}\big] \;\ge\; c_0\, e^{-2\beta\{2(2n_0-1)-[n_0(n_0+1)-1]h\}}\;. \tag{1.37}$$

Proof.[Proof of Lemma 1.6.1] Since $\lambda(-\mathbf{1}) \ge c_0 e^{-\beta[4-h]}$, by (1.31), (1.37),

$$\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;\ge\; c_0\, \frac{\mu_\beta(-\mathbf{1})}{\mathrm{cap}(-\mathbf{1},\{-\mathbf{1},\mathbf{0}\})}\, e^{-\beta\{8n_0-[2n_0(n_0+1)-1]h\}}\;.$$

Therefore, by Proposition 1.2.3,

$$\frac{\mu_\beta(+\mathbf{1})}{\mu_\beta(\mathbf{0})}\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;\ge\; c_0\, e^{\beta L^2}\, e^{-\beta\{4(n_0-1)-[n_0(n_0+1)-2]h\}}\;.$$

It remains to show that $L^2 > 4(n_0-1)-[n_0(n_0+1)-2]h$. By definition of n_0, $n_0 h > 2-h$, so that $4(n_0-1)-[n_0(n_0+1)-2]h \le 2n_0 - 6 + hn_0 + 3h$. As $hn_0 < 2$ and $h < 1$, this expression is less than or equal to $2n_0$. This expression is smaller than L^2 because $L \ge 2$ and $L > n_0$. □

1.7 Proof of Theorem 1.2.6

The statement of Theorem 1.2.6 follows from Proposition 1.7.3 and Lemma 1.7.6 below and from Theorem 5.1 in [18]. We start deriving some consequences of the assumption (1.8). Clearly, it follows from (1.8) and from (1.23) that for all $\eta \in \mathcal{M}$ and $\sigma \in \mathcal{V}_\eta$, $\sigma \neq \eta$,

$$\frac{\mu_\beta(\eta)}{\mathrm{cap}(\sigma,\eta)} \;\prec\; \theta_\beta\;. \tag{1.38}$$

Assertion 1.7.1. *For all $\eta \in \mathcal{M}$ and $\sigma \in \mathcal{V}_\eta$, $\sigma \neq \eta$,*

$$\mu_\beta(\sigma) \;\prec\; \mu_\beta(\eta)\;, \quad \mathrm{cap}(\eta,\mathcal{M}\setminus\{\eta\}) \;\prec\; \mathrm{cap}(\sigma,\eta)\;.$$

Proof. The first bound is a straightforward consequence of the hypothesis $\mathbb{H}(\sigma) > \mathbb{H}(\eta)$. In view of (1.23), to prove the second bound we have to show that $\mathbb{H}(\sigma,\eta) < \mathbb{H}(\eta,\mathcal{M}\setminus\{\eta\})$. The case $\eta = -\mathbf{1}$ is a consequence of the second hypothesis in (1.8), as the case $\eta = \mathbf{0}$ if one recalls (1.9). It remains to consider the case $\eta = +\mathbf{1}$. By condition (1.8) and (1.9),

$$\mathbb{H}(\sigma,+\mathbf{1}) \;<\; \mathbb{H}(+\mathbf{1}) \;+\; \mathbb{H}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\}) - \mathbb{H}(\mathbf{0}) \;<\; \mathbb{H}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})\;.$$

By Assertion 1.5.5 and by (1.23), $\mathbb{H}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\}) = \mathbb{H}(+\mathbf{1},\{-\mathbf{1},\mathbf{0}\})$. Therefore,

$$\mathbb{H}(\sigma,+\mathbf{1}) \;<\; \mathbb{H}(+\mathbf{1},\{-\mathbf{1},\mathbf{0}\})\;,$$

which proves the second claim of the assertion. □

Assertion 1.7.2. *For all $\eta \neq \xi \in \mathcal{M}$ and for all $\sigma \in \mathcal{V}_\eta$, $\sigma' \in \mathcal{V}_\xi$,*

$$\mathrm{cap}(\sigma, \sigma') \;\approx\; \mathrm{cap}(\eta, \xi)\;.$$

Proof. Fix $\eta \neq \xi \in \mathcal{M}$ and $\sigma \in \mathcal{V}_\eta$, $\sigma' \in \mathcal{V}_\xi$. We need to prove that $\mathbb{H}(\eta, \xi) = \mathbb{H}(\sigma, \sigma')$. On the one hand, by definition, $\mathbb{H}(\sigma, \sigma') \leq \max\{\mathbb{H}(\sigma, \eta), \mathbb{H}(\eta, \eta'), \mathbb{H}(\eta', \sigma')\}$. By the proof of Assertion 1.7.1, $\mathbb{H}(\sigma, \eta) < \mathbb{H}(\eta, \mathcal{M} \setminus \{\eta\})$, with a similar inequality replacing σ, η by σ', η', respectively. Since $\mathbb{H}(\mathcal{A}, \mathcal{B})$ is decreasing in each variable, $\mathbb{H}(\eta, \mathcal{M}\setminus\{\eta\})$ and $\mathbb{H}(\eta', \mathcal{M} \setminus \{\eta'\})$ are less than or equal to $\mathbb{H}(\eta, \eta')$, which shows that $\mathbb{H}(\sigma, \sigma') \leq \mathbb{H}(\eta, \eta')$.

Conversely, $\mathbb{H}(\eta, \eta') \leq \max\{\mathbb{H}(\eta, \sigma), \mathbb{H}(\sigma, \sigma'), \mathbb{H}(\sigma', \eta')\}$. By the previous paragraph, $\mathbb{H}(\eta, \sigma) < \mathbb{H}(\eta, \eta')$ and $\mathbb{H}(\sigma', \eta') < \mathbb{H}(\eta, \eta')$ so that $\mathbb{H}(\eta, \eta') \leq \mathbb{H}(\sigma, \sigma')$. This completes the proof of the assertion. □

We conclude this preamble with two simple remarks. Fix $\eta \in \mathcal{M}$ and $\sigma \in \mathcal{V}_\eta$. By (1.13) and Assertion 1.7.2,

$$\mathrm{cap}(\sigma, \cup_{\xi\neq\eta}\mathcal{V}_\xi) \;\approx\; \max_{\sigma' \in \cup_{\xi\neq\eta}\mathcal{V}_\xi} \mathrm{cap}(\sigma, \sigma') \;\approx\; \max_{\xi \in \mathcal{M}\setminus\{\eta\}} \mathrm{cap}(\eta, \xi)\;.$$

Applying (1.13) once more, we conclude that

$$\mathrm{cap}(\sigma, \cup_{\xi\neq\eta}\mathcal{V}_\xi) \;\approx\; \mathrm{cap}(\eta, \mathcal{M} \setminus \{\eta\})\;. \tag{1.39}$$

In particular, by Assertion 1.7.1,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(\sigma, \cup_{\xi\neq\eta}\mathcal{V}_\xi)}{\mathrm{cap}(\sigma, \eta)} \;=\; 0\;. \tag{1.40}$$

Denote by $\sigma^{\mathcal{A}}(t)$, $\mathcal{A} \subset \Omega$, the trace of $\sigma(t)$ on $\mathcal{A}$. By [2, Proposition 6.1], $\sigma^{\mathcal{A}}(t)$ is a continuous-time Markov chain. Moreover, for $\mathcal{B} \subset \mathcal{A}$, $\sigma^{\mathcal{B}}(t)$ is the trace of $\sigma^{\mathcal{A}}(t)$ on $\mathcal{B}$. When $\mathcal{A} = \mathcal{M}$, we represent $\sigma^{\mathcal{A}}(t)$ by $\eta(t)$. Denote $R^{\mathcal{A}}_\beta(\sigma, \sigma')$, $\sigma \neq \sigma' \in \mathcal{A}$, the jump rates of the Markov chain $\sigma^{\mathcal{A}}(t)$.

Recall the definition of the map $\pi : \mathcal{M} \to \{-1, 0, 1\}$, introduced just before the statement of Theorem 1.2.6. Denote by $\psi = \psi_{\mathcal{V}} : \mathcal{V} \to \{-1, 0, 1\}$ the projections defined by $\psi(\sigma) = \pi(\eta)$ if $\sigma \in \mathcal{V}_\eta$:

$$\psi(\sigma) \;=\; \sum_{\eta\in\mathcal{M}} \pi(\eta)\, \mathbf{1}\{\sigma \in \mathcal{V}_\eta\}\;.$$

Recall also the definition of the time-scale θ_β introduced in (1.6).

Proposition 1.7.3. *As $\beta \uparrow \infty$, the speeded-up, hidden Markov chain $\psi(\sigma^{\mathcal{V}}(\theta_\beta t))$ converges to the continuous-time Markov chain $X(t)$ introduced in Theorem 1.2.6.*

We first prove Proposition 1.7.3 in the case where the wells $\mathcal{V}_\eta$ are singletons : $\mathcal{V}_\eta = \{\eta\}$. In this case, ψ is a bijection, and $\psi(\eta(t))$ is a Markov chain on $\{-1, 0, 1\}$.

Lemma 1.7.4. *As $\beta \uparrow \infty$, the speeded-up chain $\eta(\theta_\beta t)$ converges to the continuous-time Markov chain on $\mathcal{M}$ in which $+\mathbf{1}$ is an absorbing state, and whose jump rates $\mathbf{r}(\eta, \xi)$, are given by*

$$\mathbf{r}(-\mathbf{1}, \mathbf{0}) \;=\; \mathbf{r}(\mathbf{0}, +\mathbf{1}) \;=\; 1\;, \quad \mathbf{r}(-\mathbf{1}, +\mathbf{1}) \;=\; \mathbf{r}(\mathbf{0}, -\mathbf{1}) \;=\; 0\;.$$

Proof. Denote by $r_\beta(\eta,\xi)$ the jump rates of the chain $\eta(\theta_\beta t)$. It is enough to prove that

$$\lim_{\beta\to\infty} r_\beta(\eta,\xi) \;=\; \mathbf{r}(\eta,\xi) \tag{1.41}$$

for all $\eta\neq\xi\in\mathcal{M}$.

By [2, Proposition 6.1], the jump rates $r_\beta(\eta,\xi)$, $\eta\neq\xi\in\mathcal{M}$, of the Markov chain $\eta_\beta(t)$ are given by

$$r_\beta(\eta,\xi) \;=\; \theta_\beta\,\lambda(\eta)\,\mathbb{P}_\eta[H_\xi = H^+_{\mathcal{M}}]\;.$$

Dividing and multiplying the previous expression by $\mathbb{P}_\eta[H_{\mathcal{M}\setminus\{\eta\}} < H^+_\eta]$, in view of [2, Lemma 6.6] and of (1.10), we obtain that

$$r_\beta(\eta,\xi) \;=\; \frac{\theta_\beta}{\mu_\beta(\eta)}\,\mathrm{cap}(\eta,\mathcal{M}\setminus\{\eta\})\,\mathbb{P}_\eta[H_\xi < H_{\mathcal{M}\setminus\{\eta,\xi\}}]\;.$$

For $\eta=+\mathbf{1}$ and $\xi=-\mathbf{1}$, $\mathbf{0}$, by Assertion 1.5.5 and by Proposition 1.2.3,

$$\lim_{\beta\to\infty} r_\beta(+\mathbf{1},\xi) \;\le\; \lim_{\beta\to\infty}\frac{\theta_\beta}{\mu_\beta(+\mathbf{1})}\,\mathrm{cap}(+\mathbf{1},\mathcal{M}\setminus\{+\mathbf{1}\}) \;=\; \lim_{\beta\to\infty}\frac{\mu_\beta(\mathbf{0})}{\mu_\beta(+\mathbf{1})} \;=\; 0\;.$$

On the other hand, by Proposition 1.2.3,

$$\lim_{\beta\to\infty}\frac{\theta_\beta}{\mu_\beta(\mathbf{0})}\,\mathrm{cap}(\mathbf{0},\mathcal{M}\setminus\{\mathbf{0}\}) \;=\; 1\;,$$

while $\theta_\beta\mathrm{cap}(-\mathbf{1},\mathcal{M}\setminus\{-\mathbf{1}\})/\mu_\beta(-\mathbf{1}) \;=\; 1$. Furthermore, by Proposition 1.2.1 and Assertion 1.5.3,

$$\lim_{\beta\to\infty}\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; \lim_{\beta\to\infty}\mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}] \;=\; 0\;.$$

This yields (1.41) and completes the proof of the lemma. □

Denote by $\mathbb{P}^{\mathcal{V}}_\sigma$, $\sigma\in\mathcal{V}$, the probability measure on the path space $D(\mathbb{R}_+,\mathcal{V})$ induced by the Markov chain $\sigma^{\mathcal{V}}(t)$ starting from σ. Expectation with respect to $\mathbb{P}^{\mathcal{V}}_\sigma$ is represented by $\mathbb{E}^{\mathcal{V}}_\sigma$. Clearly, for any disjoint subsets $\mathcal{A}$, $\mathcal{B}$ of $\mathcal{V}$,

$$\mathbb{P}^{\mathcal{V}}_\sigma[H_{\mathcal{A}} < H_{\mathcal{B}}] \;=\; \mathbb{P}_\sigma[H_{\mathcal{A}} < H_{\mathcal{B}}]\;. \tag{1.42}$$

The hitting time of a subset $\mathcal{A}$ of $\mathcal{V}$ by the trace chain $\sigma^{\mathcal{V}}$ can be represented in terms of the original chain $\sigma(t)$. Under $\mathbb{P}_\sigma$,

$$H^{\mathcal{V}}_{\mathcal{A}} \;=\; \inf\{t>0 : \sigma^{\mathcal{V}}(t)\in\mathcal{A}\} \;=\; \int_0^{H_{\mathcal{A}}}\mathbf{1}\{\sigma(t)\in\mathcal{V}\}\,dt\;. \tag{1.43}$$

Let

$$\breve{\mathcal{V}}(\eta) \;=\; \breve{\mathcal{V}}_\eta \;=\; \bigcup_{\zeta\neq\eta}\mathcal{V}_\zeta\;.$$

Denote by $\{T_j : j\ge 0\}$ the jump times of the hidden chain $\psi(\sigma^{\mathcal{V}}(t))$:

$$T_0 \;=\; 0\;,\quad T_{j+1} \;=\; \inf\{t\ge T_j : \sigma^{\mathcal{V}}(t)\in\breve{\mathcal{V}}(\sigma^{\mathcal{V}}(T_j))\}\;,\quad j\ge 0\;,$$

Similarly, denote by $\{\tau_j : j\ge 0\}$ the successive jump times of the chain $\eta(t)$.

Lemma 1.7.5. *Fix $\sigma \in \mathcal{V}_{-\mathbf{1}}$. There exists a sequence $\varepsilon_\beta \to 0$ such that for $j = 1, 2$*

$$\lim_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_\sigma\big[\,|\tau_j - T_j| \ge \theta_\beta \varepsilon_\beta\big] \;=\; 0\;, \quad \lim_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_\sigma\big[\,T_3 - T_2 \le \theta_\beta \varepsilon_\beta^{-1}\big] \;=\; 0\;. \tag{1.44}$$

Moreover,

$$\lim_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_\sigma\big[\sigma(T_1) \notin \mathcal{V}_{\mathbf{0}}\big] \;=\; 0\;, \quad \lim_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_\sigma\big[\sigma(T_2) \notin \mathcal{V}_{+\mathbf{1}}\big] \;=\; 0\;. \tag{1.45}$$

Proof. Fix a configuration $\sigma \in \mathcal{V}_{-\mathbf{1}}$. By (1.42), (1.12) and (1.40),

$$\limsup_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_\sigma\big[H_{\breve{\mathcal{V}}(-\mathbf{1})} < H_{-\mathbf{1}}\big] \;\le\; \lim_{\beta\to\infty} \frac{\mathrm{cap}(\sigma, \breve{\mathcal{V}}(-\mathbf{1}))}{\mathrm{cap}(\sigma, -\mathbf{1})} \;=\; 0\;. \tag{1.46}$$

On the other hand, under $\mathbb{P}^{\mathcal{V}}_\sigma$,

$$\tau_1 \;=\; \int_{H_{-\mathbf{1}}}^{H_{\mathbf{0},+\mathbf{1}}} \mathbf{1}\{\sigma(s) = -\mathbf{1}\}\, ds\;.$$

Hence, under $\mathbb{P}^{\mathcal{V}}_\sigma$ and on the event $\{H_{-\mathbf{1}} < H_{\breve{\mathcal{V}}(-\mathbf{1})}\}$ we have that

$$\begin{aligned} T_1 \;&=\; H_{-\mathbf{1}} \;+\; \int_{H_{-\mathbf{1}}}^{H(\breve{\mathcal{V}}_{-\mathbf{1}})} \big\{\mathbf{1}\{\sigma(s) = -\mathbf{1}\} + \mathbf{1}\{\sigma(s) \ne -\mathbf{1}\}\big\}\, ds \\ &=\; \tau_1 \;+\; H_{-\mathbf{1}} \;+\; \int_{H_{-\mathbf{1}}}^{H(\breve{\mathcal{V}}_{-\mathbf{1}})} \mathbf{1}\{\sigma(s) \ne -\mathbf{1}\}\, ds \;-\; \int_{H(\breve{\mathcal{V}}_{-\mathbf{1}})}^{H_{\mathbf{0},+\mathbf{1}}} \mathbf{1}\{\sigma(s) = -\mathbf{1}\}\, ds\;. \end{aligned}$$

It remains to estimate the last three terms.

To bound the first term, by (1.42), (1.12), and (1.40),

$$\limsup_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_\sigma[H_{\breve{\mathcal{V}}_{-\mathbf{1}}} < H_{-\mathbf{1}}] \;=\; \limsup_{\beta\to\infty} \mathbb{P}_\sigma[H_{\breve{\mathcal{V}}_{-\mathbf{1}}} < H_{-\mathbf{1}}] \;\le\; \lim_{\beta\to\infty} \frac{\mathrm{cap}(\sigma, \breve{\mathcal{V}}_{-\mathbf{1}})}{\mathrm{cap}(\sigma, -\mathbf{1})} \;=\; 0\;.$$

Hence, to prove that $\mathbb{P}^{\mathcal{V}}_\sigma[H_{-\mathbf{1}} > \theta_\beta\varepsilon_\beta] \to 0$, it is enough to show that

$$\lim_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_\sigma[H_{\breve{\mathcal{V}}_{-\mathbf{1}}\cup\{-\mathbf{1}\}} > \theta_\beta\varepsilon_\beta] \;=\; 0\;.$$

By Tchebycheff inequality, by Lemma 6.9 and Proposition 6.10 in [2], and by (1.42), the previous probability is less than or equal to

$$\frac{1}{\theta_\beta\varepsilon_\beta}\, \frac{1}{\mathrm{cap}(\sigma, \breve{\mathcal{V}}_{-\mathbf{1}} \cup \{-\mathbf{1}\})} \sum_{\eta\in\mathcal{V}} \mu_\beta(\eta)\, \mathbb{P}_\eta[H_\sigma < H_{\breve{\mathcal{V}}_{-\mathbf{1}}\cup\{-\mathbf{1}\}}]\;.$$

By definition of θ_β, since the capacity is monotone, and since we may restrict the sum to $\mathcal{V}_{-\mathbf{1}}$, the previous expression is less than or equal to

$$\frac{1}{\varepsilon_\beta}\, \frac{\mathrm{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})}{\mathrm{cap}(\sigma, -\mathbf{1})}\, \frac{\mu_\beta(\mathcal{V}_{-1})}{\mu_\beta(-\mathbf{1})}\;.$$

By Assertion 1.7.1, $\mu_\beta(\mathcal{V}_{-1})/\mu_\beta(-\mathbf{1})$ is bounded and $\mathrm{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})/\mathrm{cap}(\sigma, -\mathbf{1})$ vanishes as $\beta \uparrow \infty$. Hence, the previous expression converges to 0 for every sequence ε_β which does not decrease too fast.

We turn to the second term of the decomposition of T_1. By the strong Markov property and by (1.43), we need to estimate,

$$\mathbb{P}^{\mathcal{V}}_{-\mathbf{1}}\Big[\int_0^{H(\breve{\mathcal{V}}_{-\mathbf{1}})} \mathbf{1}\{\sigma(s)\neq -\mathbf{1}\}\, ds > \theta_\beta\varepsilon_\beta\Big] \\ =\; \mathbb{P}_{-\mathbf{1}}\Big[\int_0^{H(\breve{\mathcal{V}}_{-\mathbf{1}})} \mathbf{1}\{\sigma(s)\in \mathcal{V}\setminus\{-\mathbf{1}\}\}\, ds > \theta_\beta\varepsilon_\beta\Big]\;.$$

By Tchebycheff inequality and by [2, Proposition 6.10], the previous probability is less than or equal to

$$\frac{1}{\theta_\beta\varepsilon_\beta}\,\frac{1}{\mathrm{cap}(-\mathbf{1},\breve{\mathcal{V}}_{-\mathbf{1}})}\sum_{\eta\in\mathcal{V}\setminus\{-\mathbf{1}\}}\mu_\beta(\eta)\,\mathbb{P}_\eta[H_{-\mathbf{1}}<H_{\breve{\mathcal{V}}_{-\mathbf{1}}}]\;.$$

By (1.39), $\mathrm{cap}(-\mathbf{1},\breve{\mathcal{V}}_{-\mathbf{1}})\approx\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})$. Hence, by definition of θ_β, and since the sum can be restricted to the set $\mathcal{V}_{-\mathbf{1}}\setminus\{-\mathbf{1}\}$, the previous expression is less than or equal to

$$\frac{C_0}{\varepsilon_\beta}\,\frac{1}{\mu_\beta(-\mathbf{1})}\,\mu_\beta(\mathcal{V}_{-\mathbf{1}}\setminus\{-\mathbf{1}\})$$

for some finite constant C_0. By Assertion 1.7.1, the ratio of the measures vanishes as $\beta\uparrow\infty$. In particular, the previous expression converges to 0 as $\beta\uparrow\infty$ if ε_β does not decrease too fast.

The third term in the decomposition of T_1 is absolutely bounded by $H_{\mathbf{0},+\mathbf{1}} - H(\breve{\mathcal{V}}_\eta)$ and can be handled as the first one. This proves the first assertion of (1.44) for $j=1$.

In a similar way one proves that T_2-T_1 is close to $\tau_2-\tau_1$. The first assertion of (1.44) for $j=2$ follows from this result and from the bound for $T_1-\tau_1$. The details are left to the reader.

We turn to the proof of the first assertion in (1.45). Since $\breve{\mathcal{V}}_{-\mathbf{1}}=\mathcal{V}_{\mathbf{0}}\cup\mathcal{V}_{+\mathbf{1}}$,

$$\mathbb{P}^{\mathcal{V}}_\sigma\big[\sigma(T_1)\notin\mathcal{V}_{\mathbf{0}}\big]\;=\;\mathbb{P}_\sigma\big[\sigma(H)\in\mathcal{V}_{+\mathbf{1}}\big]\;,$$

where $H=H(\breve{\mathcal{V}}_{-\mathbf{1}})$. We may rewrite the previous probability as

$$\mathbb{P}_\sigma\big[\sigma(H)\in\mathcal{V}_{+\mathbf{1}}\,,\,H_{\mathbf{0}}<H_{+\mathbf{1}}\big]\;+\;\mathbb{P}_\sigma\big[\sigma(H)\in\mathcal{V}_{+\mathbf{1}}\,,\,H_{\mathbf{0}}>H_{+\mathbf{1}}\big]\;.$$

Both expression vanishes as $\beta\uparrow\infty$. The second one is bounded by $\mathbb{P}_\sigma[H_{+\mathbf{1}}<H_{\mathbf{0}}]$, which vanishes by Proposition 1.2.1. Since $H<\min\{H_{\mathbf{0}},H_{+\mathbf{1}}\}$, by the strong Markov property, the first term is less than or equal to

$$\max_{\sigma'\in\mathcal{V}_{+1}}\mathbb{P}_{\sigma'}\big[H_{\mathbf{0}}<H_{+\mathbf{1}}\big]\;.$$

This expression converges to 0 as $\beta\uparrow\infty$ because $\mathcal{V}_{+\mathbf{1}}$ is contained in the basin of attraction of $+\mathbf{1}$. The proof of the second assertion in (1.45) is similar and left to the reader.

We finally examine the second assertion of (1.44). By the second assertion of (1.45), it is enough to prove that

$$\lim_{\beta\to\infty}\mathbb{P}^{\mathcal{V}}_\sigma\big[\,T_3-T_2\le\theta_\beta\varepsilon_\beta^{-1}\,,\,\sigma(T_2)\in\mathcal{V}_{+\mathbf{1}}\big]\;=\;0\;.$$

By the strong Markov property, this limit holds if

$$\lim_{\beta\to\infty} \max_{\sigma'\in\mathcal{V}_{+1}} \mathbb{P}^{\mathcal{V}}_{\sigma'}\big[\, T_1 \le \theta_\beta \varepsilon_\beta^{-1} \big] \;=\; 0 \;.$$

Since $\mathcal{V}_{+\mathbf{1}}$ is contained in the basin of attraction of $+\mathbf{1}$, it is enough to show that

$$\lim_{\beta\to\infty} \max_{\sigma'\in\mathcal{V}_{+1}} \mathbb{P}^{\mathcal{V}}_{\sigma'}\big[\, T_1 \le \theta_\beta \varepsilon_\beta^{-1} \;,\; H_{+\mathbf{1}} < T_1 \big] \;=\; 0 \;.$$

On the event $\{H_{+\mathbf{1}} < T_1\}$, $\{T_1 \le \theta_\beta \varepsilon_\beta^{-1}\} \subset \{T_1 \circ \theta_{H_{+\mathbf{1}}} \le \theta_\beta \varepsilon_\beta^{-1}\}$, where $\{\theta_t : t \ge 0\}$ represents the semigroup of time translations. In particular, by the strong Markov property, we just need to show that

$$\lim_{\beta\to\infty} \mathbb{P}^{\mathcal{V}}_{+\mathbf{1}}\big[\, T_1 \le \theta_\beta \varepsilon_\beta^{-1} \big] \;=\; 0 \;.$$

Let $\{\mathfrak{e}_j : j \ge 1\}$ be the length of the sojourn times at $+\mathbf{1}$. Hence, $\{\mathfrak{e}_j : j \ge 1\}$ is a sequence of i.i.d. exponential random variables with parameter $\lambda(+\mathbf{1})$. Denote by $\mathcal{A}$ the set of configurations with at least $n_0(n_0+1)$ sites with spins not equal to $+1$. Each time the process leaves the state $+\mathbf{1}$ it attempts to reach $\mathcal{A}$ before it returns to $+\mathbf{1}$. Let δ be the probability of success :

$$\delta \;=\; \mathbb{P}^{\mathcal{V}}_{+\mathbf{1}}\big[\, H_{\mathcal{A}} < H^+_{+\mathbf{1}} \big] \;.$$

Let $N \ge 1$ be the number of attempts up to the first success so that $\sum_{1\le j\le N} \mathfrak{e}_j$ represents the total time the process $\sigma(t)$ remained at $+\mathbf{1}$ before it reached $\mathcal{A}$. It is clear that under $\mathbb{P}^{\mathcal{V}}_{+\mathbf{1}}$,

$$\sum_{j=1}^{N} \mathfrak{e}_j \;\le\; T_1 \;,$$

and that N is a geometric random variable of parameter δ independent of the sequence $\{\mathfrak{e}_j : j \ge 1\}$. In view of the previous inequality, it is enough to prove that

$$\lim_{\beta\to\infty} \mathbb{P}_{+\mathbf{1}}\Big[\sum_{j=1}^{N} \mathfrak{e}_j \le \theta_\beta \varepsilon_\beta^{-1} \Big] \;=\; 0 \;.$$

The previous probability is less than or equal to

$$\lambda(+\mathbf{1})\, \theta_\beta \varepsilon_\beta^{-1}\, \mathbb{P}^{\mathcal{V}}_{+\mathbf{1}}\big[\, H_{\mathcal{A}} < H^+_{+\mathbf{1}} \big] \;=\; \frac{1}{\varepsilon_\beta}\, \frac{\theta_\beta}{\mu_\beta(+\mathbf{1})}\, \mathrm{cap}(\mathcal{A}, +\mathbf{1}) \;.$$

Since $\theta_\beta \mathrm{cap}(\mathcal{A}, +\mathbf{1})/\mu_\beta(+\mathbf{1}) \prec 1$, the previous expression vanishes if ε_β does not decrease too fast to 0. This completes the proof of the lemma. □

Proof.[Proof of Proposition 1.7.3] The assertion of the proposition is a straightforward consequence of Lemmas 1.7.4 and 1.7.5.

Fix $\sigma \in \mathcal{V}_{-\mathbf{1}}$ and recall the notation introduced in Lemma 1.7.5. Let $\mathcal{A} = \{\sigma(T_1) \in \mathcal{V}_{\mathbf{0}}\} \cap \{\sigma(T_2) \in \mathcal{V}_{+\mathbf{1}}\}$. By (1.45), $\mathbb{P}^{\mathcal{V}}_{\sigma}[\mathcal{A}^c] \to 0$. On the set $\mathcal{A}$,

$$\psi(\sigma^{\mathcal{V}}(\theta_\beta t)) \;=\; -\mathbf{1}\{t < T_1/\theta_\beta\} \;+\; \mathbf{1}\{T_2/\theta_\beta \le t < T_3/\theta_\beta\}$$

By Lemma 1.7.4, $(\tau_1/\theta_\beta, (\tau_2 - \tau_1)/\theta_\beta)$ converges to a pair of independent, mean 1, exponential random variables. Hence, by (1.44), $(T_1/\theta_\beta, (T_2 - T_1)/\theta_\beta, (T_3 - T_2)/\theta_\beta)$ converges in distribution to $(\mathfrak{e}_1, \mathfrak{e}_2, \infty)$, where $(\mathfrak{e}_1, \mathfrak{e}_2)$ is a pair of independent, mean 1, exponential random variables. This completes the proof. □

Lemma 1.7.6. *Let $\Delta = \Omega \setminus \mathcal{V}$. For all $\xi \in \mathcal{V}$, $t > 0$,*

$$\lim_{\beta\to\infty} \mathbb{E}_\xi \Big[\int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta\}\, ds \Big] \;=\; 0\,.$$

Proof. Fix $\xi \in \mathcal{V}_{+\mathbf{1}}$. On the one hand, by [2, Proposition 6.10],

$$\frac{1}{\theta_\beta} \mathbb{E}_\xi \Big[\int_0^{H_{+\mathbf{1}}} \mathbf{1}\{\sigma(s) \in \Delta\}\, ds \Big] \;\le\; \frac{1}{\theta_\beta} \frac{\mu_\beta(+\mathbf{1})}{\mathrm{cap}(\xi, +\mathbf{1})} \frac{\mu_\beta(\Delta)}{\mu_\beta(+\mathbf{1})}\,.$$

This expression vanishes as $\beta \uparrow \infty$ because, by (1.38), $\mu_\beta(+\mathbf{1})/\mathrm{cap}(\xi, +\mathbf{1}) \preceq \theta_\beta$, and because $\mu_\beta(\Delta) \prec \mu_\beta(+\mathbf{1})$, as $+\mathbf{1}$ is the unique ground state.

On the other hand, by the strong Markov property,

$$\frac{1}{\theta_\beta} \mathbb{E}_\xi \Big[\int_{H_{+\mathbf{1}}}^{t\theta_\beta} \mathbf{1}\{\sigma(s) \in \Delta\}\, ds \Big] \;\le\; \frac{1}{\theta_\beta} \mathbb{E}_{+\mathbf{1}} \Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s) \in \Delta\}\, ds \Big]\,.$$

Therefore, to prove the lemma for $\xi \in \mathcal{V}_{+\mathbf{1}}$ it is enough to show that

$$\lim_{\beta\to\infty} \mathbb{E}_{+\mathbf{1}} \Big[\int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta\}\, ds \Big] \;=\; 0\,.$$

This last assertion follows from Lemma 1.7.7 below.

Similar arguments permit to reduce the statement of the lemma for $\xi \in \mathcal{V}_{\mathbf{0}}$ (resp. $\xi \in \mathcal{V}_{-\mathbf{1}}$) to the verification that

$$\lim_{\beta\to\infty} \mathbb{E}_\zeta \Big[\int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta\}\, ds \Big] \;=\; 0\,,$$

for $\zeta = \mathbf{0}$ (resp. $\zeta = -\mathbf{1}$), which follows from Lemma 1.7.7 below.

□

Lemma 1.7.7. *Let $\Delta^* = \Omega \setminus \mathcal{M}$. For all $\xi \in \mathcal{M}$, $t > 0$,*

$$\lim_{\beta\to\infty} \mathbb{E}_\xi \Big[\int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta^*\}\, ds \Big] \;=\; 0\,.$$

Proof. Consider first the case $\xi = +\mathbf{1}$. Clearly,

$$\mathbb{E}_{+\mathbf{1}} \Big[\int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta^*\}\, ds \Big] \;\le\; \frac{1}{\mu_\beta(+\mathbf{1})} \sum_{\sigma\in\Omega} \mu_\beta(\sigma)\, \mathbb{E}_\sigma \Big[\int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta^*\}\, ds \Big]\,.$$

Since μ_β is the stationary state, the previous expression is equal to

$$\frac{t\mu_\beta(\Delta^*)}{\mu_\beta(+\mathbf{1})}\,,$$

which vanishes as $\beta \to \infty$ because $+\mathbf{1}$ is the unique ground state.

We turn to the case $\xi = \mathbf{0}$. We first claim that

$$\lim_{\beta\to\infty} \mathbb{E}_{\mathbf{0}} \Big[\frac{1}{\theta_\beta} \int_0^{H_{\{-\mathbf{1},+\mathbf{1}\}}} \mathbf{1}\{\sigma(s) \in \Delta^*\}\, ds \Big] \;=\; 0\,. \tag{1.47}$$

Indeed, by [2, Proposition 6.10], the previous expectation is equal to

$$\frac{1}{\theta_\beta}\,\frac{\langle V\mathbf{1}\{\Delta^*\}\rangle_{\mu_\beta}}{\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})} \;=\; \frac{\mu_\beta(\mathbf{0})}{\theta_\beta\,\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})}\,\frac{\langle V\mathbf{1}\{\Delta^*\}\rangle_{\mu_\beta}}{\mu_\beta(\mathbf{0})}\;,$$

where V is the harmonic function $V(\sigma) = \mathbb{P}_\sigma[H_{\mathbf{0}} < H_{\{-\mathbf{1},+\mathbf{1}\}}]$. By definition of θ_β, the first fraction on the right hand side is bounded.

It remains to estimate the second fraction, which is equal to

$$\frac{1}{\mu_\beta(\mathbf{0})}\sum_{\sigma\in\Delta^*_0}\mu_\beta(\sigma)\mathbb{P}_\sigma[H_{\mathbf{0}} < H_{\{-\mathbf{1},+\mathbf{1}\}}]\;,$$

where $\Delta^*_0 = \{\sigma\in\Delta^* : \mu_\beta(\sigma)\succeq\mu_\beta(\mathbf{0})\}$. By (1.12), this sum is less than or equal to

$$\sum_{\sigma\in\Delta^*_0}\frac{\mathrm{cap}(\sigma,\mathbf{0})}{\mu_\beta(\mathbf{0})}\,\frac{\mu_\beta(\sigma)}{\mathrm{cap}(\sigma,\mathcal{M})}\;.$$

Each term of this sum vanishes as $\beta\uparrow\infty$. Indeed, as σ belongs to Δ^*_0, to reach σ from $\mathbf{0}$ the chain has to escape from the well of $\mathbf{0}$ so that $\mathrm{cap}(\sigma,\mathbf{0})/\mu_\beta(\mathbf{0})\approx\theta_\beta^{-1}$. On the other hand, $\mu_\beta(\sigma)/\mathrm{cap}(\sigma,\mathcal{M})$ is the time scale in which the process reaches one of the configurations in $\mathcal{M}$ starting from σ, a time scale of smaller order than the one in which it jumps between configurations in $\mathcal{M}$.

By (1.47), to prove the lemma for $\xi = \mathbf{0}$, we just have to show that

$$\lim_{\beta\to\infty}\mathbb{E}_{\mathbf{0}}\Big[\frac{1}{\theta_\beta}\int_0^{t\theta_\beta}\mathbf{1}\{\sigma(s)\in\Delta^*\}\,ds\,\mathbf{1}\{H_{\{-\mathbf{1},+\mathbf{1}\}}\le t\theta_\beta\}\Big] \;=\; 0\;.$$

Since $\mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}]\to 0$, we may add in the previous expectation the indicator of the set $\{H_{+\mathbf{1}} < H_{-\mathbf{1}}\}$. Rewrite the integral over the time interval $[0,t\theta_\beta]$ as the sum of an integral over $[0,H_{\{-\mathbf{1},+\mathbf{1}\}}]$ with one over the time interval $[H_{\{-\mathbf{1},+\mathbf{1}\}},t\theta_\beta]$. The expectation of the first one is handled by (1.47). The expectation of the second one, by the strong Markov property, on the set $\{H_{\{-\mathbf{1},+\mathbf{1}\}}\le t\theta_\beta\}\cap\{H_{+\mathbf{1}} < H_{-\mathbf{1}}\}$, is less than or equal to

$$\mathbb{E}_{+\mathbf{1}}\Big[\frac{1}{\theta_\beta}\int_0^{t\theta_\beta}\mathbf{1}\{\sigma(s)\in\Delta^*\}\,ds\Big]\;.$$

By the first part of the proof this expectation vanishes as $\beta\uparrow\infty$.

It remains to consider the case $\xi = -\mathbf{1}$. As in the case $\xi = \mathbf{0}$, we first estimate the expectation in (1.47), with $H_{\{\mathbf{0},+\mathbf{1}\}}$ instead of $H_{\{-\mathbf{1},+\mathbf{1}\}}$. Then, we repeat the arguments presented for $\xi = \mathbf{0}$, with obvious modifications, to reduce the case $\xi = -\mathbf{1}$ to the case $\xi = \mathbf{0}$, which has already been examined. □

We conclude this section proving the assertion of Remark 1.2.7. Fix $\eta\in\mathcal{M}$, $\sigma\in\mathcal{V}_\eta$, $\sigma\neq\eta$. By (1.12), (1.13) and Assertion 1.7.2,

$$\mathbb{P}_\sigma[H_{\mathcal{M}\setminus\{\eta\}} < H_\eta] \;\le\; \frac{\mathrm{cap}(\sigma,\mathcal{M}\setminus\{\eta\})}{\mathrm{cap}(\sigma,\eta)} \;\approx\; \sum_{\xi\in\mathcal{M}\setminus\{\eta\}}\frac{\mathrm{cap}(\sigma,\xi)}{\mathrm{cap}(\sigma,\eta)} \;\approx\; \sum_{\xi\in\mathcal{M}\setminus\{\eta\}}\frac{\mathrm{cap}(\eta,\xi)}{\mathrm{cap}(\sigma,\eta)}\;. \tag{1.48}$$

By monotonicity of the capacity, the previous expression is bounded by $2\mathrm{cap}(\eta,\mathcal{M}\setminus\{\eta\})/\mathrm{cap}(\sigma,\eta)$, which vanishes in view of Assertion 1.7.1.

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Chapitre 2

Metastability of the two-dimensional Blume-Capel model with zero chemical potential and small magnetic field on a large torus

2.1 Introduction

The Blume–Capel model is a two dimensional, nearest-neighbor spin system where the single spin variable takes three possible values : 1, 0 and +1. One can interpret it as a system of particles with spins. The value 0 of the spin at a lattice site corresponds to the absence of particles, whereas the values ±1 correspond to the presence of a particle with the respective spin.

Denote by $\mathbb{T}_L = \{1, \dots, L\}$ the discrete, one-dimensional torus of length L, and let $\Lambda_L = \mathbb{T}_L \times \mathbb{T}_L$, $\Omega_L = \{-1, 0, 1\}^{\Lambda_L}$. Elements of Ω_L are called configurations and are represented by the Greek letter σ. For $x \in \Lambda_L$, $\sigma(x) \in \{-1, 0, 1\}$ stands for the value at x of the configuration σ and is called the spin at x of σ.

We consider in this article a Blume–Capel model with zero chemical potential and a small positive magnetic field. Fix an external field $0 < h < 2$, and denote by $\mathbb{H} = \mathbb{H}_{L,h} : \Omega_L \to \mathbb{R}$ the Hamiltonian given by

$$\mathbb{H}(\sigma) = \sum (\sigma(y) - \sigma(x))^2 - h \sum_{x \in \Lambda_L} \sigma(x), \tag{2.1}$$

where the first sum is carried over all unordered pairs of nearest-neighbor sites of Λ_L. We assumed that $h < 2$ for the configuration whose 0-spins form a rectangle in a background of -1 spins to be a local minima of the Hamiltonian.

The continuous-time Metropolis dynamics at inverse temperature β is the Markov chain on Ω_L, denoted by $\{\sigma_t : t \geq 0\}$, whose infinitesimal generator L_β acts on functions $f : \Omega_L \to \mathbb{R}$ as

$$(L_\beta f)(\sigma) = \sum_{x\in\Lambda_L} R_\beta(\sigma,\sigma^{x,+})\,[f(\sigma^{x,+}) - f(\sigma)] + \sum_{x\in\Lambda_L} R_\beta(\sigma,\sigma^{x,-})\,[f(\sigma^{x,-}) - f(\sigma)]\ .$$

In this formula, $\sigma^{x,\pm}$ represents the configuration obtained from σ by modifying the spin at x as follows,

$$\sigma^{x,\pm}(z) := \begin{cases} \sigma(x) \pm 1 \bmod 3 & \text{if } \ z = x\ , \\ \sigma(z) & \text{if } \ z \neq x\ , \end{cases}$$

where the sum is taken modulo 3, and the jump rates R_β are given by

$$R_\beta(\sigma,\sigma^{x,\pm}) \;=\; \exp\Big\{ -\beta\,\big[\mathbb{H}(\sigma^{x,\pm}) - \mathbb{H}(\sigma)\big]_+ \Big\}\,, \quad x\in\Lambda_L\,,$$

where a_+, $a \in \mathbb{R}$, stands for the positive part of a : $a_+ = \max\{a, 0\}$. We often write R instead of R_β.

Denote by μ_β the Gibbs measure associated to the Hamiltonian $\mathbb{H}$ at inverse temperature β,

$$\mu_\beta(\sigma) = \frac{1}{Z_\beta} e^{-\beta\mathbb{H}(\sigma)}, \tag{2.2}$$

where Z_β is the partition function, the normalization constant which turns μ_β into a probability measure. We often write μ instead of μ_β.

Clearly, the Gibbs measure μ_β satisfies the detailed balance conditions

$$\mu_\beta(\sigma)\,R_\beta(\sigma,\sigma^{x,\pm}) \;=\; \min\big\{\mu_\beta(\sigma)\,,\,\mu_\beta(\sigma^{x,\pm})\big\} \;=\; \mu_\beta(\sigma^{x,\pm})\,R_\beta(\sigma^{x,\pm},\sigma)\,,$$

$\sigma \in \Omega_L$, $x \in \Lambda_L$, and is therefore reversible for the dynamics.

Denote by $\mathbf{\text{-}1}, \mathbf{0}, \mathbf{+1}$ the configurations of Ω_L with all spins equal to $-1, 0, +1$, respectively. The configurations $\mathbf{\text{-}1}$, $\mathbf{0}$ are local minima of the Hamiltonian, while the configuration $\mathbf{+1}$ is a global minimum. Moreover, $\mathbb{H}(\mathbf{0}) < \mathbb{H}(\mathbf{\text{-}1})$.

The existence of several local minima of the energy turns the Blume-Capel model a perfect dynamics to be examined by the theory developed by Beltrán and Landim in [2, 4] for metastable Markov chains.

Let $\mathcal{M} = \{-\mathbf{1}, \mathbf{0}, +\mathbf{1}\}$, and denote by $\Psi : \Omega_L \to \{-1, 0, 1, \mathfrak{d}\}$ the projection defined by

$$\Psi(\sigma) \;=\; \sum_{\eta\in\mathcal{M}} \pi(\eta)\,\mathbf{1}\{\sigma = \eta\} \;+\; \mathfrak{d}\,\mathbf{1}\{\sigma \not\in \mathcal{M}\}\ ,$$

where $\mathfrak{d}$ is a point added to the set $\{-1, 0, 1\}$ and $\pi : \mathcal{M} \to \{-1, 0, 1\}$ is the application which provides the magnetization of the states $-\mathbf{1}$, $\mathbf{0}$, $+\mathbf{1}$: $\pi(-\mathbf{1}) = -1$, $\pi(\mathbf{0}) = 0$, $\pi(+\mathbf{1}) = 1$.

A scheme has been developed in [2, 4, 20] to derive the existence of a time-scale θ_β for which the the finite-dimensional distributions of the hidden Markov chain $\Psi(\sigma(t\theta_\beta))$ converge to the ones of a $\{-1, 0, 1\}$-valued, continuous-time Markov chain. Note that the limiting process does not take the value $\mathfrak{d}$.

The approach consists in proving first that in the time-scale θ_β the trace of the process σ_t on $\mathcal{M}$ converges in the Skorohod topology to a continuous-time Markov chain. Then, to prove that in this time scale the time spent on $\Omega_L \setminus \mathcal{M}$ is negligible. Finally, to show that, at any fixed, the probability to be in $\Omega_L \setminus \mathcal{M}$ is negligible.

This is the content of the main result of this article. We are also able to describe the path which drives the process from the highest local minima, **-1** to the ground state **+1**. We not only characterize the critical droplet but we also describe precisely how it grows until it invades the all space. In this process, we show that starting from **-1**, the model visits **0** on its way to **+1**.

We consider in this article the situation in which the length of the torus increases with the inverse of the temperature. The case in which L is fixed has been considered by Cirillo and Nardi [12], by us [19] and by Cirillo, Nardi and Spitoni [11].

The method imposes a limitation on the rate at which the space grows, as we need the energy to prevail over the entropy created by the multitude of configurations. In particular, the conditions on the growth impose that the stationary state restricted to the valleys of **-1**, **0** or **+1**, defined at the beginning of the next section, is concentrated on these configurations (cf. (2.5)).

The study of the metastability of the Blume–Capel model has been initiated by Cirillo and Olivieri [13] and Manzo and Olivieri [22]. We refer to these papers for the interest of the model and its role in the understanding of metastability.

2.2 Notation and Results

Denote by $D(\mathbb{R}_+, \Omega_L)$ the space of right-continuous functions $\omega : \mathbb{R}_+ \to \Omega_L$ with left-limits and by $\mathbb{P}_\sigma = \mathbb{P}^{\beta,L}_\sigma$, $\sigma \in \Omega_L$, the probability measure on the path space $D(\mathbb{R}_+, \Omega_L)$ induced by the Markov chain $(\sigma_t : t \geq 0)$ starting from σ. Sometimes, we write $\sigma(t)$ instead of σ_t.

Denote by H_A, H_A^+, $A \subset \Omega_L$, the hitting time of A and the time of the first return to A respectively :

$$H_A \;=\; \inf\{t > 0 : \sigma_t \in A\} \;, \quad H_A^+ \;=\; \inf\{t > T_1 : \sigma_t \in A\} \;, \tag{2.3}$$

where T_1 represents the time of the first jump of the chain σ_t.

Critical droplet. We have already observed that **+1** is the ground state of the dynamics and that **-1** and **0** are local minima of the Hamiltonian. The first main result of this article characterizes the critical droplet in the course from **-1** and **0** to **+1**. Let $n_0 = \lfloor 2/h \rfloor$, where $\lfloor a \rfloor$ stands for the integer part of $a \in \mathbb{R}_+$.

Denote by $\mathcal{V}_{\mathbf{-1}}$ the valley of **-1**. This is the set constituted of all configurations which can be attained from **-1** by flipping $n_0(n_0 + 1)$ or less spins from **-1**. If after $n_0(n_0 + 1)$ flips we reached a configuration where $n_0(n_0 + 1)$ 0-spins form a $[n_0 \times (n_0+1)]$-rectangle, we may flip one more spin. Hence, all configurations of $\mathcal{V}_{\mathbf{-1}}$ differ from **-1** in at most $n_0(n_0 + 1) + 1$ sites.

The valley $\mathcal{V}_{\mathbf{0}}$ of **0** is defined in a similar way, a $n_0 \times (n_0+1)$-rectangle of +1-spins replace the one of 0-spins. Here and below, when we refer to a $[n_0 \times (n_0+1)]$-rectangle, n_0 may be its length or its height.

Denote by $\mathfrak{R}^l = \mathfrak{R}^l_L$ the set of configurations with $n_0(n_0+1)+1$ 0-spins forming, in a background of -1-spins, a $n_0 \times (n_0+1)$ rectangle with an extra 0-spin attached to the longest side of this rectangle. This means that the extra 0-spin is surrounded by three -1-spins and one 0-spins which belongs to the longest side of the rectangle. The set $\mathfrak{R}^l_0 = \mathfrak{R}^l_{0,L}$ is defined analogously, the -1-spins, 0-spins being replaced by 0-spins, $+1$-spins, respectively.

We show in the next theorem that, starting from **-1**, resp. **0**, the process visits $\mathfrak{R}^l$, resp. $\mathfrak{R}^l_0$, before hitting $\{\mathbf{0}, \mathbf{+1}\}$, resp. $\{\mathbf{\text{-}1}, \mathbf{+1}\}$. An assumption on the growth of the torus is necessary to avoid the entropy of configurations with high energy to prevail over the local minima of the Hamiltonian. We assume that

$$\lim_{\beta\to\infty} |\Lambda_L|\, e^{-2\beta} \;=\; 0\ . \tag{2.4}$$

We prove in Lemma 2.8.2 that under this condition,

$$\lim_{\beta\to\infty} \frac{\mu_\beta(\mathcal{V}_{\mathbf{\text{-}1}} \setminus \{\mathbf{\text{-}1}\})}{\mu_\beta(\mathbf{\text{-}1})} \;=\; 0\ . \tag{2.5}$$

Theorem 2.2.1. *Assume that* $0 < h < 1$*, that* $2/h$ *is not an integer and that* (2.4) *is in force. Then,*

$$\lim_{\beta\to\infty} \mathbb{P}_{\mathbf{\text{-}1}}[H_{\mathfrak{R}^l} < H_{\{\mathbf{0},\mathbf{+1}\}}] \;=\; 1\ , \quad \lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{\mathfrak{R}^l_0} < H_{\{\mathbf{\text{-}1},\mathbf{+1}\}}] \;=\; 1\ .$$

On the other hand, under the condition that

$$\lim_{\beta\to\infty} |\Lambda_L|^{1/2} \left\{ e^{-[(n_0+1)h-2]\beta} \,+\, e^{-h\beta} \right\} \quad\text{and}\quad \lim_{\beta\to\infty} |\Lambda_L|^2\, e^{-(2-h)\beta} \;=\; 0\ , \tag{2.6}$$

it follows from Proposition 2.5.1 that

$$\lim_{\beta\to\infty} \inf_{\eta\in\mathfrak{R}^l} \mathbb{P}_\eta[H_{\{\mathbf{0},\mathbf{+1}\}} < H_{\mathbf{\text{-}1}}] \;>\; 0\ , \quad \lim_{\beta\to\infty} \inf_{\xi\in\mathfrak{R}^l_0} \mathbb{P}_\xi[H_{\{\mathbf{\text{-}1},\mathbf{+1}\}} < H_{\mathbf{0}}] \;>\; 0\ .$$

The route from -1 to +1. The second main result of the article asserts that starting from **-1**, the processes visits **0** in its way to **+1**. Actually, in Section 2.5, we describe in detail how the critical droplet growths until it invades the all space.

Theorem 2.2.2. *Assume that* $0 < h < 1$*, that* $2/h$ *is not an integer and that condition* (2.6) *is in force. Then,*

$$\lim_{\beta\to\infty} \mathbb{P}_{\mathbf{\text{-}1}}[H_{\mathbf{+1}} < H_{\mathbf{0}}] \;=\; 0 \quad\textit{and}\quad \lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{\mathbf{\text{-}1}} < H_{\mathbf{+1}}] \;=\; 0\ .$$

The strategy of the proof relies on the assumption that while the critical droplet increases, invading the entire space, nothing else relevant happens in other parts of the torus. A new row or column is added to a supercritical droplet at rate $e^{-(2-h)\beta}/|\Lambda_L|$, because $e^{-(2-h)\beta}$ is the rate at which a negative spin is flipped to 0 when it is surrounded by three negative spins and one 0-spin, and L is the time needed for a rate one asymmetric random walk to reach L starting from the origin. We need L^2 because we need an extra 0-spin to complete a row, and then to repeat this procedure L times for the droplet to fill the torus. This rate has to be confronted to the rate at which a 0-spin appears somewhere in the space. The rate at which a negative spin is flipped to 0 when it is surrounded by four negative spins is $e^{-(4-h)\beta}$. Since this may happen at $|\Lambda_L|$ different positions, the method of the proof requires at least $|\Lambda_L| e^{-(4-h)\beta}$ to be much smaller than $e^{-(2-h)\beta}/|\Lambda_L|$, that is, $|\Lambda_L|^2 e^{-2\beta} \to 0$. This almost explains the main hypothesis of the theorem. The extra conditions appear because we need to take care of other details to lengthy to explain here.

Metastability. For two disjoint subsets $\mathcal{A}$, $\mathcal{B}$ of Ω_L, denote by $\text{cap}(\mathcal{A}, \mathcal{B})$ the capacity between $\mathcal{A}$ and $\mathcal{B}$:

$$\text{cap}(\mathcal{A}, \mathcal{B}) \;=\; \sum_{\sigma\in\mathcal{A}} \mu_\beta(\sigma)\lambda_\beta(\sigma)\, \mathbb{P}_\sigma[H_{\mathcal{B}} < H^+_{\mathcal{A}}]\;,$$

where $\lambda_\beta(\sigma) = \sum_{\sigma'\in\Omega_L} R_\beta(\sigma, \sigma')$ represents the holding times of the Blume-Capel model. Let

$$\theta_\beta \;=\; \frac{\mu_\beta(-\mathbf{1})}{\text{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})}\;. \tag{2.7}$$

We prove in Proposition 2.7.3 that under the hypotheses of Theorem 2.2.2 for any configuration $\eta \in \mathfrak{R}^l$,

$$\lim_{\beta\to\infty} \frac{\text{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})}{\mu_\beta(\eta)\, |\Lambda_L|} \;=\; \frac{4(2n_0+1)}{3}\;.$$

Theorem 2.2.3. *Under the hypotheses of Theorem 2.2.2, the finite-dimensional distributions of the speeded-up, hidden Markov chain $X_\beta(t) = \Psi\big(\sigma(\theta_\beta t)\big)$ converge to the ones of the $\{-1, 0, 1\}$-valued, continuous-time Markov chain $X(t)$ in which* 1 *is an absorbing state, and whose jump rates are given by*

$$r(-1, 0) \;=\; r(0, 1) \;=\; 1\;, \quad r(-1, 1) \;=\; r(0, -1) \;=\; 0\;.$$

Note that the limit chain does not take the value $\mathfrak{d}$, in contrast with $X_\beta(t)$ since $\Psi(\sigma) = \mathfrak{d}$ for all $\sigma \notin \mathcal{M}$.

The paper is organized as follows. In the next section, we recall some general results on potential theory of reversible Markov chains and we prove a lemma on asymmetric birth and death chains which is used later in the article. In Section 2.4, we examine the formation of a critical droplet and, in Section 2.5, the growth of a supercritical droplet. Theorems 2.2.1 and 2.2.2 are proved in Section 2.6. In the following two sections, we prove that the trace of σ_t on $\mathcal{M}$ converges to a three-state Markov chain and that the time spent outside $\mathcal{M}$ is negligible. In the final section we prove Theorem 2.2.3.

2.3 Metastability of reversible Markov chains

In this section, we present general results on reversible Markov chains used in the next sections. Fix a finite set E. Consider a continuous-time, E-valued, Markov chain $\{X_t : t \geq 0\}$. Assume that the chain X_t is irreducible and that the unique stationary state π is reversible.

Elements of E are represented by the letters x, y. Let $\mathbb{P}_x$, $x \in E$, be the distribution of the Markov chain X_t starting from x. Recall from (2.3) the definition of the hitting time and the return time to a set.

Denote by $R(x, y)$, $x \neq y \in E$, the jump rates of the Markov chain X_t, and let $\lambda(x) = \sum_{y\in E} R(x, y)$ be the holding rates. Denote by $p(x, y)$ the jump probabilities, so that $R(x,y) = \lambda(x)\, p(x,y)$. The stationary state of the embedded discrete-time Markov chain is given by $M(x) = \pi(x)\,\lambda(x)$.

Potential theory. Fix two subsets A, B of E such that $A \cap B = \varnothing$. Recall that the capacity between A and B, denoted by $\mathrm{cap}(A,B)$, is given by

$$\mathrm{cap}(A,B) \;=\; \sum_{x\in A} M(x)\, \mathbb{P}_x[H_B < H_A^+]\;. \tag{2.8}$$

Denote by L the generator of the Markov chain X_t and by $D(f)$ the Dirichlet form of a function $f : E \to \mathbb{R}$:

$$D(f) \;=\; -\sum_{x\in E} f(x)\,(Lf)(x)\,\pi(x) \;=\; \frac{1}{2}\sum_{x,y} \pi(x)\, R(x,y)\, [f(y) - f(x)]^2\;.$$

In this later sum, each unordered pair $\{a,b\} \subset E$, $a \neq b$, appears twice. The Dirichlet principle provides a variational formula for the capacity :

$$\mathrm{cap}(A,B) \;=\; \inf_f D(f)\;, \tag{2.9}$$

where the infimum is carried over all functions $f : E \to [0,1]$ such that $f = 1$ on A and $f = 0$ on B.

Denote by $\mathcal{P}$ the set of oriented edges of E : $\mathcal{P} = \{(x,y) \in E\times E : R(x,y) > 0\}$. An anti-symmetric function $\phi : \mathcal{P} \to \mathbb{R}$ is called a flow. The divergence of a flow ϕ at $x \in E$ is defined as

$$(\mathrm{div}\,\phi)(x) \;=\; \sum_{y:(x,y)\in\mathcal{P}} \phi(x,y)\;.$$

Let $\mathcal{F}_{A,B}$ be the set of flows such that

$$\sum_{x\in A} (\mathrm{div}\ \phi)(x) \;=\; 1\;, \quad \sum_{y\in B} (\mathrm{div}\ \phi)(y) \;=\; -1\;, \quad (\mathrm{div}\ \phi)(z) \;=\; 0\;, \quad z \notin A\cup B\;.$$

The Thomson principle provides an alternative variational formula for the capacity :

$$\frac{1}{\mathrm{cap}(A,B)} \;=\; \inf_{\phi\in\mathcal{F}_{A,B}} \frac{1}{2} \sum_{(x,y)\in\mathcal{P}} \frac{1}{\pi(x)\, R(x,y)}\, \phi(x,y)^2\;. \tag{2.10}$$

We refer to [7] for a proof of the Dirichlet and the Thomson principles.

In the Blume-Capel model, by definition of the rate function $R_\beta(\sigma, \sigma^{x,\pm})$,

$$\mu_\beta(\sigma)\, R_\beta(\sigma, \sigma^{x,\pm}) \;=\; \mu_\beta(\sigma) \wedge \mu_\beta(\sigma^{x,\pm})\;.$$

This identity will be used throughout the paper, without further notice, to replace the left-hand side, which appears in the Dirichlet and in the Thomson principle, by the right-hand side.

We turn to an estimate of hitting times in terms of capacities. Fix $x \in E\setminus(A\cup B)$. Then,

$$\mathbb{P}_x[H_A < H_B] \;=\; \frac{\mathbb{P}_x[\, H_A < H^+_{B\cup\{x\}}\,]}{\mathbb{P}_x[\, H_{A\cup B} < H^+_x\,]}\;. \tag{2.11}$$

Indeed, intersecting the event $\{H_A < H_B\}$ with $\{H_x^+ < H_{A\cup B}\}$ and its complement, by the Strong Markov property,

$$\mathbb{P}_x[H_A < H_B] \;=\; \mathbb{P}_x[\,H_x^+ < H_{A\cup B}\,]\,\mathbb{P}_x[H_A < H_B] \;+\; \mathbb{P}_x[\,H_A < H^+_{B\cup\{x\}}\,]\;,$$

which proves (2.11) by substracting the first term on the right hand side from the left hand side.

Recall the definition of the capacity introduced in (2.8). Multiplying and dividing the right hand side of (2.11) by $M(x)$ yields that

$$\mathbb{P}_x[H_A < H_B] \;=\; \frac{M(x)\,\mathbb{P}_x[\,H_A < H^+_{B\cup\{x\}}\,]}{\mathrm{cap}(x, A\cup B)} \;\le\; \frac{M(x)\,\mathbb{P}_x[\,H_A < H^+_x\,]}{\mathrm{cap}(x, A\cup B)}\;.$$

Therefore, by definition of the capacity and since, by (2.9), the capacity is monotone,

$$\mathbb{P}_x[H_A < H_B] \;\le\; \frac{\mathrm{cap}(x,A)}{\mathrm{cap}(x, A\cup B)} \;\le\; \frac{\mathrm{cap}(x,A)}{\mathrm{cap}(x, B)}\;. \tag{2.12}$$

Trace process. We recall in this subsection the definition of the trace of a Markov process on a proper subset of the state space. Fix $F \subsetneq E$ and denote by $T_F(t)$ the time the process X_t spent on the set F in the time-interval $[0,t]$:

$$T_F(t) \;=\; \int_0^t \chi_F(X_s)\,ds\;,$$

where χ_F represents the indicator function of the set F. Denote by $S_F(t)$ the generalized inverse of the additive functional $T_F(t)$:

$$S_F(t) \;=\; \sup\{s\ge 0 : T_F(s)\le t\}\,.$$

The recurrence guarantees that for all $t>0$, $S_F(t)$ is finite almost surely.

Denote by $X^F(t)$ the *trace* of the chain X_t on the set F, defined by $X^F(t) := X(S_F(t))$. It can be proven [2] that $X^F(t)$ is an irreducible, recurrent, continuous-time, F-valued Markov chain. The jump rates of the chain $X^F(t)$, denoted by $r_F(x,y)$, are given by

$$r_F(x,y) \;=\; \lambda(x)\,\mathbb{P}_x\big[\,H_F^+ = H_y\,\big]\;, \quad x\,,\,y\,\in\,F\;, \quad x\neq y\;.$$

The unique stationary probability measure for the trace chain, denoted by π_F, is given by $\pi_F(x) = \pi(x)/\pi(F)$. Moreover, π_F is reversible if so is π.

Estimates of an eigenfunction. We derive in this subsection an estimate needed in the next sections. Consider the continuous-time Markov chain X_t on $E = \{0,\dots,n\}$ which jumps from k to $k+1$ at rate ε and from $k+1$ to k at rate 1, $0\le k<n$.

Denote by $\mathbf{P}_k$ the distribution of the Markov chain X_t starting from $k\in E$. Expectation with respect to $\mathbf{P}_k$ is represented by $\mathbf{E}_k$.

Denote by H_n the hitting time of n. Fix $\theta>0$, and let $f : E\to\mathbb{R}_+$ be given by

$$f(k) \;=\; \mathbf{E}_k\big[e^{-\theta\,H_n}\big]\;.$$

An elementary computation based on the strong Markov property shows that f is the solution of the boundary-valued elliptic problem

$$\begin{cases} (Lf)(k) \;=\; \theta\, f(k)\;, \quad 0 \le k < n\;, \\ f(n) \;=\; 1\;, \end{cases}$$

where L stands for the generator of the Markov chain X_t.

Lemma 2.3.1. *We have that $f(0) \le \varepsilon^n/\theta$.*

Proof. Multiplying the identity $(Lf)(k) = \theta\, f(k)$ by ε^k and summing over $0 \le k < n$ yields that

$$\sum_{k=0}^{n-1} \varepsilon^{k+1}\, [f(k+1) - f(k)] \;+\; \sum_{k=1}^{n-1} \varepsilon^{k}\, [f(k-1) - f(k)] \;=\; \theta \sum_{k=0}^{n-1} f(k)\, \varepsilon^k\;.$$

On the left-hand side, all terms but one cancel so that

$$\varepsilon^n\, [f(n) - f(n-1)] \;=\; \theta \sum_{k=0}^{n-1} f(k)\, \varepsilon^k\;.$$

Since $f(k) \ge 0$ and $f(n) = 1$, we have that

$$\theta\, f(0) \;\le\; \theta \sum_{k=0}^{n-1} f(k)\, \varepsilon^k \;=\; \varepsilon^n\, [1 - f(n-1)] \;\le\; \varepsilon^n\;,$$

as claimed. □

This result has a content only in the case $\varepsilon < 1$, but we did not use this condition in the proof.

2.4 The emergence of a critical droplet

In this section, we prove that starting from **-1**, the process creates a droplet of 0-spins on its way to $\{\mathbf{0}, \mathbf{+1}\}$, that is, a configuration σ with a $n_0 \times (n_0+1)$ rectangle of 0-spins (or *0-rectangle*) and an extra 0-spin attached to one of the sides of the rectangle, in a background of negative spins.

In the next section, we prove that if this extra 0-spin is attached to one of the longest sides of the rectangle, with a positive probability the process hits **0** before $\{\mathbf{-1}, \mathbf{+1}\}$, while if it is attached to one of the shortest sides, with probability close to 1, the process returns to **-1** before hitting $\{\mathbf{0}, \mathbf{+1}\}$. An important feature of this model is that the size of a critical droplet is independent of β and L.

Throughout this section, C_0 is a large constant, which does not depend on β or L but only on h, and whose value may change from line to line.

Recall the definition of the valley $\mathcal{V}_{\mathbf{-1}}$ introduced just above equation (2.4). Note that there are few configurations in $\mathcal{V}_{\mathbf{-1}}$ which differ from **-1** at $n_0(n_0+1)+1$ sites. Moreover, such configurations

$$\text{may not have two spins equal to } +1. \tag{2.13}$$

To define the boundary of the valley of $\mathcal{V}_{\mathbf{-1}}$, fix L large, and denote by $\mathfrak{B}$ the set of configurations with $n_0(n_0+1)$ spins different from -1 :

$$\mathfrak{B} \;=\; \big\{\sigma \in \Omega_L : |A(\sigma)| = n_0(n_0+1)\big\} \;,$$

where

$$A(\sigma) \;=\; \{x \in \Lambda_L : \sigma(x) \neq -1\} \;.$$

Denote by $\mathfrak{R}$ the subset of $\mathfrak{B}$ given by

$$\mathfrak{R} \;=\; \big\{\sigma \in \{-1,0\}^{\Lambda_L} : A(\sigma) \text{ is a } n_0 \times (n_0+1)\text{rectangle } \big\} \;.$$

Note that the spins of a configuration $\sigma \in \mathfrak{R}$ are either -1 or 0 and that all configurations in $\mathfrak{R}$ have the same energy.

Denote by $\mathfrak{R}^+$ the set of configurations in Ω_L in which there are $n_0(n_0+1)+1$ spins which are not equal to -1. Of these spins, $n_0(n_0+1)$ form a $n_0 \times (n_0+1)$-rectangle of 0 spins. The remaining spin not equal to -1 is either 0 or $+1$. Figure 2.1 present some examples of configurations in $\mathfrak{R}^+$.

FIGURE 2.1 – Examples of two configurations in $\mathfrak{R}^+$ in the case where $n_0 = 5$. An empty (resp. filled) 1×1 square centered at x has been placed at each site x occupied by a 0-spin (resp. positive spin). All the other spins are equal to -1.

Let $\mathfrak{B}^+$ be the boundary of $\mathcal{V}_{\mathbf{-1}}$. This set consists of all configurations σ in $\mathcal{V}_{\mathbf{-1}}$ which have a neighbor [that is, a configuration σ' which differs from σ at one site] which does not belong to $\mathcal{V}_{\mathbf{-1}}$. By definition of $\mathcal{V}_{\mathbf{-1}}$,

$$\mathfrak{B}^+ \;=\; (\mathfrak{B} \setminus \mathfrak{R}) \;\cup\; \mathfrak{R}^+ \;.$$

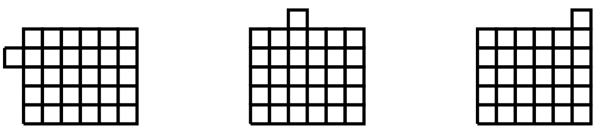

FIGURE 2.2 – Example of three configurations in $\mathfrak{R}^a$ in the case where $n_0 = 5$. An 1×1 empty square centered at x has been placed at each site x occupied by a 0-spin. All the other spins are equal to -1. The one on the left belongs to $\mathfrak{R}^s$. According to the notation introduced at the beginning of Section 2.5, the one on the center belongs to $\mathfrak{R}^{li}$ and the one on the right to $\mathfrak{R}^{lc}$.

Let $\mathfrak{R}^a \subset \mathfrak{R}^+$ be the set of configurations for which the remaining spin is a 0 spin attached to one of the sides of the rectangle. Figure 2.2 present some examples of configurations in $\mathfrak{R}^a$. We write the boundary $\mathfrak{B}^+$ as

$$\mathfrak{B}^+ \;=\; (\mathfrak{B} \setminus \mathfrak{R}) \;\cup\; (\mathfrak{R}^+ \setminus \mathfrak{R}^a) \;\cup\; \mathfrak{R}^a \;. \tag{2.14}$$

Since $\mathfrak{B}^+$ is the boundary of the valley $\mathcal{V}_{\mathbf{-1}}$, starting from $-\mathbf{1}$, it is reached before the chain attains the set $\{\mathbf{0}, +\mathbf{1}\}$:

$$H_{\mathfrak{B}^+} \;<\; H_{\{\mathbf{0},+\mathbf{1}\}} \quad \mathbb{P}_{-\mathbf{1}} \text{ a.s.} \tag{2.15}$$

Note that all configurations of $\mathfrak{R}^a$ have the same energy and that $\mathbb{H}(\xi) = \mathbb{H}(\zeta) + 2 - h$ if $\xi \in \mathfrak{R}^a$, $\zeta \in \mathfrak{R}$. In particular, by Assertion 4.D in [19],

$$\mathbb{H}(\sigma) \;\geq\; \mathbb{H}(\xi) \;+\; h\,, \quad \sigma \in \mathfrak{B} \setminus \mathfrak{R} \;\;,\;\; \xi \in \mathfrak{R}^a\;. \tag{2.16}$$

On the other hand, for a configuration $\eta \in \mathfrak{R}^+ \setminus \mathfrak{R}^a$, $\mathbb{H}(\eta) \geq \mathbb{H}(\zeta) + 4 - h$ if $\zeta \in \mathfrak{R}$, so that

$$\mathbb{H}(\eta) \;\geq\; \mathbb{H}(\xi) \;+\; 2\,, \quad \eta \in \mathfrak{R}^+ \setminus \mathfrak{R}^a \;\;,\;\; \xi \in \mathfrak{R}^a\;. \tag{2.17}$$

In particular, at the boundary $\mathfrak{B}^+$ the energy is minimized by configurations in $\mathfrak{R}^a$. This means that σ_t should attained $\mathfrak{B}^+$ at $\mathfrak{R}^a$. This is the content of the main result of this section. Let

$$\varepsilon(\beta) \;=\; |\Lambda_L|\, e^{-2\beta} \;+\; e^{-h\beta}\;. \tag{2.18}$$

Proposition 2.4.1. *There exists a finite constant C_0 such that*

$$\mathbb{P}_{\mathbf{-1}}[H_{\mathfrak{B}^+} < H_{\mathfrak{R}^a}] \;\leq\; C_0\, \varepsilon(\beta)$$

for all $\beta \geq C_0$.

Proof. The proof of this lemma is divided in several steps. Denote by $\{\eta_t : t \geq 0\}$ the process obtained from the Blume-Capel model by forbiding any jump from the valley $\mathcal{V}_{\mathbf{-1}}$ to its complement. This process is sometimes called the reflected process.

It is clear that η_t is irreducible and that its stationary state, denoted by $\mu_{\mathcal{V}}$ is given by $\mu_{\mathcal{V}}(\sigma) = (1/Z_{\mathcal{V}})\, \exp\{-\beta\mathbb{H}(\sigma)\}$, where $Z_{\mathcal{V}}$ is a normalizing constant.

Moreover, starting from **-1**, we may couple σ_t with η_t in such a way that $\sigma_t = \eta_t$ until they hit the boundary. Hence, if we denote by $\mathbb{P}^{\mathcal{V}}_{\mathbf{-1}}$ the distribution of η_t,

$$\mathbb{P}_{\mathbf{-1}}[H_{\mathfrak{B}^+} < H_{\mathfrak{R}^a}] \;=\; \mathbb{P}^{\mathcal{V}}_{\mathbf{-1}}[H_{\mathfrak{B}^+} < H_{\mathfrak{R}^a}]\;.$$

By (2.12),

$$\mathbb{P}^{\mathcal{V}}_{\mathbf{-1}}[H_{\mathfrak{B}^+} < H_{\mathfrak{R}^a}] \;=\; \mathbb{P}^{\mathcal{V}}_{\mathbf{-1}}[H_{\mathfrak{B}^+\setminus\mathfrak{R}^a} < H_{\mathfrak{R}^a}] \;\leq\; \frac{\mathrm{cap}_{\mathcal{V}}(\mathfrak{B}^+ \setminus \mathfrak{R}^a, \mathbf{-1})}{\mathrm{cap}_{\mathcal{V}}(\mathfrak{R}^a, \mathbf{-1})}\;,$$

where $\mathrm{cap}_{\mathcal{V}}$ represents the capacity with respect to the process η_t. The lemma now follows from Lemmata 2.4.2 and 2.4.3 below. □

Denote by Γ_c the energy of a configuration $\sigma \in \mathfrak{R}^a$:

$$\Gamma_c \;=\; 4\,(n_0+1) \;-\; h\left[\, n_0(n_0+1) \;+\; 1 \;-\; |\Lambda_L|\,\right]. \tag{2.19}$$

Lemma 2.4.2. *There exists a finite constant C_0 such that*

$$\frac{1}{\mathrm{cap}_{\mathcal{V}}(\mathfrak{R}^a, \mathbf{-1})} \;\leq\; C_0\, \frac{1}{|\Lambda_L|}\, Z_{\mathcal{V}}\, e^{\beta\Gamma_c}\;.$$

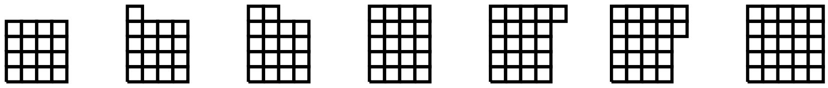

FIGURE 2.3 – We present in this figure some configurations $\zeta_{x,k}$ introduced in the proof of Lemma 2.4.2. Let $m = 16$. The figure represent the configurations $\zeta_{x,m}$, $\zeta_{x,m+1}$, $\zeta_{x,m+2}$. Then, $\zeta_{x,m+4}$, $\zeta_{x,m+5}$, $\zeta_{x,m+6}$, and $\zeta_{x,m+9}$. An 1×1 empty square centered at x has been placed at each site x occupied by a 0-spin. All the other spins are equal to -1.

Proof. We use the Thomson principle to bound this capacity by constructing a flow from **-1** to $\mathfrak{R}_a$. The flow is constructed in two stages.

To explain the procedure we interpret a flow as a mass transport, $\phi(\eta, \xi)$ representing the total mas transported from η to ξ. The goal is to define the transport of a mass equal to 1 from **-1** to $\mathfrak{R}^a$. The first step consists in transferring the mass from **-1** to $\mathfrak{R}$.

This is done as follows. Consider the sequence of points in $\mathbb{Z}^2$ which forms a succesion of squares of length 1, 2 up to n_0. It is given by $u_1 = (1,1)$, $u_2 = (1,2)$, $u_3 = (2,2)$, $u_4 = (2,1)$, $u_5 = (1,3)$ and so on until $u_{n_0^2} = (n_0, 1)$. Hence, we first add a new line on the upper side of the square from left to right, and then a new column on the right side from top to bottom. Once we arrived at the $(n_0 \times n_0)$-square, we add a final row at the upper side of the square : Let $u_{n_0^2+k} = (k, n_0+1)$, $1 \le k \le n_0$, to obtain a $n_0 \times (n_0+1)$-rectangle.

Note that we reach through this procedure only rectangles whose height is larger than the length. We could have defined flows which reach both types of rectangles, but the bound would not improve significantly.

Let $A_k = \{u_1, \dots, u_k\}$ and denote by $A_{x,k}$ the set A_k translated by $x \in \mathbb{Z}^2$: $A_{x,k} = x + A_k$. Denote by $\zeta_{x,k}$ the configuration with 0-spins at $A_{x,k}$ and -1-spins elsewhere. Figure 2.3 presents some of these configurations. Let $\varepsilon = 1/|\Lambda_L|$. The first stage of the flow consists in transferring a mass ε from **-1** to each $\zeta_{x,1}$ and then transfer this mass from $\zeta_{x,k}$ to $\zeta_{x,k+1}$ for $1 \le k < n_0(n_0+1)$.

Let $u_{n_0^2+n_0+1} = (1, n_0+2)$, and consider the configurations $\zeta_{x,n_0^2+n_0+1}$ obtained through the correspondance adopted above. The final stage consists in transferring the mass ε from $\zeta_{x,n_0^2+n_0}$ to $\zeta_{x,n_0^2+n_0+1}$.

Since each configuration $\zeta_{x,n_0^2+n_0+1}$ belongs to $\mathfrak{R}^a$, the total effect of this procedure is to transport a mass equal to 1 from the configuration **-1** to the set $\mathfrak{R}^a$.

Denote this flow by ϕ, so that $\phi(\mathbf{-1}, \zeta_{x,1}) = \phi(\zeta_{x,k}, \zeta_{x,k+1}) = \varepsilon$. We extend this flow by imposing it to be anti-symmetric and to vanish on the other bonds. It is clear that this flow belongs to $\mathcal{F}_{\mathbf{-1},\mathfrak{R}^a}$, the set of flows defined above 2.10. Therefore, by the Thomson principle,

$$\frac{1}{\mathrm{cap}_{\mathcal{V}}(\mathfrak{R}^a, \mathbf{-1})} \;\le\; \varepsilon^2\, |\Lambda_L| \sum_{k=0}^{n_0(n_0+1)} \frac{1}{\mu_{\mathcal{V}}(\zeta_{1,k}) \,\wedge \mu_{\mathcal{V}}(\zeta_{1,k+1})}\;.$$

Since $\varepsilon = 1/|\Lambda_L|$ and $\mu_{\mathcal{V}}(\zeta_{1,k}) \ge \mu_{\mathcal{V}}(\zeta_{1,n_0(n_0+1)+1})$, the previous expression is bounded by

$$\frac{1}{\mathrm{cap}_{\mathcal{V}}(\mathfrak{R}^a, \mathbf{-1})} \;\le\; C_0\, \frac{1}{|\Lambda_L|}\, \frac{1}{\mu_{\mathcal{V}}(\zeta_{1,n_0(n_0+1)+1})}\;,$$

which completes the proof of the lemma because the energy of the configuration $\zeta_{1,n_0(n_0+1)+1}$ is Γ_c. $\square$

We turn to the upper bound for $\mathrm{cap}(\mathfrak{B}^+ \setminus \mathfrak{R}^a, \mathbf{-1})$.

Lemma 2.4.3. *There exists a finite constant C_0 such that*

$$\mathrm{cap}_{\mathcal{V}}(\mathbf{-1}, \mathfrak{B}^+ \setminus \mathfrak{R}^a) \;\le\; C_0\, \frac{1}{Z_{\mathcal{V}}}\, |\Lambda_L|\, e^{-\beta\Gamma_c} \Big\{ |\Lambda_L|\, e^{-2\beta} \,+\, e^{-h\beta} \Big\}\,.$$

for all $\beta \ge C_0$.

The proof of this lemma is divided in several steps. Let $B := \mathfrak{B}^+ \setminus \mathfrak{R}^a$ and let $\chi_B : \Omega_L \to \mathbb{R}$ be the indicator of the set B. Since $\chi_B(-\mathbf{1}) = 0$ and $\chi_B(\sigma) = 1$ for $\sigma \in B$, by the Dirichlet principle,

$$\mathrm{cap}_{\mathcal{V}}(\mathfrak{B}^+ \setminus \mathfrak{R}^a, -\mathbf{1}) \;\le\; D_{\mathcal{V}}(\chi_B)\,, \tag{2.20}$$

where $D_{\mathcal{V}}(f)$ stands for the Dirichlet form of f for the reflected process η_t. An elementary computation yields that

$$D_{\mathcal{V}}(\chi_B) \;=\; \sum_{\sigma\in B}\; \sum_{\sigma'\in \mathcal{V}_{\mathbf{-1}}\setminus B} \mu_{\mathcal{V}}(\sigma) \wedge \mu_{\mathcal{V}}(\sigma')\,, \tag{2.21}$$

where the second sum is carried over all configurations σ' which belong to $\mathcal{V}_{\mathbf{-1}} \setminus B$ and which differ from σ at exactly one spin. We denote this relation by $\sigma' \sim \sigma$.

Let

$$B_1 \;:=\; \mathfrak{B} \setminus \mathfrak{R}\,, \quad B_2 \;:=\; \mathfrak{R}^+ \setminus \mathfrak{R}^a\,, \tag{2.22}$$

so that $B = B_1 \cup B_2$, and consider separately the sums over B_1 and B_2. We start with B_2.

Assertion 2.4.4. *We have that*

$$\sum_{\sigma\in B_2} \sum_{\sigma'\sim\sigma} \mu_{\mathcal{V}}(\sigma) \wedge \mu_{\mathcal{V}}(\sigma') \;\le\; C_0\, \frac{1}{Z_{\mathcal{V}}}\, |\Lambda_L|\, e^{-\beta\Gamma_c} \Big\{ |\Lambda_L|\, e^{-2\beta} \,+\, e^{-[10-h]\beta} \Big\}\,.$$

Proof. A configuration $\eta \in B_2$ has a $n_0 \times (n_0+1)$ rectangle of 0-spins, and an extra-spin. If this extra-spin is attached to the rectangle it is equal to $+1$, while it may be 0 or $+1$ if it is not. We study the two cases separately.

Fix a configuration $\eta \in B_2$ where the extra-spin is attached to the rectangle, so that $\mathbb{H}(\eta) = \Gamma_c + (10-h)$. Consider a configuration $\sigma' \in \mathcal{V}_{\mathbf{-1}} \setminus B$ such that $\sigma' \sim \eta$. As σ' may have at most $n_0(n_0+1)+1$ spins different from -1, this excludes the possibility that σ' is obtained from η by flipping a -1. By (2.13), configurations in $\mathfrak{B}$ with $n_0(n_0+1)+1$ spins different from -1 may not have two spins equal to $+1$. This excludes flipping a 0 to $+1$. Finally, we may not flip the $+1$ to 0 because by doing so we obtain a configuration in $\mathfrak{R}^a$, and thus not in $B = \mathfrak{B}^+ \setminus \mathfrak{R}^a$.

Hence, either σ' is obtained from η by flipping the $+1$ to -1, or it is obtained by flipping a 0 to -1. In the first case $\mathbb{H}(\sigma') < \mathbb{H}(\eta)$, while in the second case if the 0-spin belongs to the corner, $\mathbb{H}(\sigma') > \mathbb{H}(\eta)$. Since the number of configurations obtained by these flips is bounded by a finite constant, the contribution to the sum appearing in the statement of the assertion is bounded by

$$C_0 \sum_{\eta} \mu_{\mathcal{V}}(\eta) \;\le\; C_0 \, \frac{1}{Z_{\mathcal{V}}} \, e^{-\beta\,\Gamma_c} \, |\Lambda_L| \, e^{-[10-h]\beta} \;,$$

where the factor $|\Lambda_L|$ comes from the number of possible positions of the rectangle, while the constant C_0 absorbs the number of positions of the positive spin.

Fix now a configuration $\eta \in B_2$ where the extra-spin is not attached to the rectangle. Then $\mathbb{H}(\eta) \ge \Gamma_c + 2$. Consider a configuration $\sigma' \in \mathcal{V}_{\mathbf{-1}} \setminus B$ such that $\sigma' \sim \eta$. As before, since σ' may have at most $n_0(n_0+1)+1$ spins different from -1, this excludes the possibility that σ' is obtained from η by flipping a -1. By excluding this possibility, we are left with a finite number [depending on n_0] of possible jumps. Hence, the contribution of configurations of this type to the sum appearing in the statement of the assertion of is bounded by

$$C_0 \sum_{\eta} \mu_{\mathcal{V}}(\eta) \;\le\; C_0 \, \frac{1}{Z_{\mathcal{V}}} \, e^{-\beta\Gamma_c} \, |\Lambda_L|^2 \, e^{-2\beta} \;,$$

where the factor $|\Lambda_L|^2$ appeared to take into account the possible positions of the rectangle and of the extra particle. This proves the assertion. □

It remains to examine the sum over B_1. Denote by $N(\sigma)$ the number of spins of the configuration σ which are different from -1 :

$$N(\sigma) \;=\; \#A(\sigma) \;=\; \#\{x \in \Lambda_L : \sigma_x \;\ne\; -1\} \;. \tag{2.23}$$

Next assertion states that we can restrict our attention to configurations σ which have no spin equal to $+1$. For a configuration σ such that $N(\sigma) \le n_0(n_0+1)+1$, let σ^o be the one obtained from σ by replacing all spins equal to $+1$ by 0-spins : $\sigma^o_x = \sigma_x \wedge 0$.

Assertion 2.4.5. *For all $\sigma \in \Omega_L$ such that $N(\sigma) \le n_0(n_0+1)+1$,*

$$\mathbb{H}(\sigma^o) \;\le\; \mathbb{H}(\sigma) \;.$$

Proof. This result is clearly not true in general because $+\mathbf{1}$ is the ground state. It holds because we are limiting the number of spins different from -1.

For a configuration $\sigma \in \Omega_L$, denote by $I_{a,b}(\sigma)$, $-1 \le a < b \le 1$, the number of unordered pairs $\{x, y\}$ of Ω_L such that $\|x-y\| = 1$, $\{\sigma_x, \sigma_y\} = \{a, b\}$, where $\|z\|$ stands for the Euclidean norm of $z \in \mathbb{R}^2$.

An elementary computation yields that

$$\mathbb{H}(\sigma) \;-\; \mathbb{H}(\sigma^o) \;=\; I_{0,1}(\sigma) \;+\; 3\,I_{-1,1}(\sigma) \;-\; hN_1(\sigma) \;,$$

where $N_1(\sigma)$ stands for the total number of spins equal to $+1$ in the configuration σ. To prove the assertion, it is therefore enough to show that $hN_1(\sigma) \le I_{0,1}(\sigma) + I_{-1,1}(\sigma)$.

By [19, Assertion 4.A], $I_{0,1}(\sigma) + I_{-1,1}(\sigma) \ge 4\sqrt{N_1(\sigma)}$. It remains to obtain that $hN_1(\sigma) \le 4\sqrt{N_1(\sigma)}$, i.e., that $h\sqrt{N_1(\sigma)} \le 4$. Indeed, since $N(\sigma) \le n_0(n_0+1)+1$, $N_1(\sigma) \le n_0(n_0+1)+1$ so that, by definition of n_0, $h\sqrt{N_1(\sigma)} \le h(n_0+1) \le 2+h \le 3$. □

Recall that $A(\sigma) = \{x \in \Lambda_L : \sigma_x \neq -1\}$. A set $A \subset A(\sigma)$ is said to be a connected component of $A(\sigma)$ if (a) for any x, $y \in A$, there exists a path $(x_0 = x, x_1, \dots, x_m = y)$ such that $x_i \in A$, $\|x_{i+1} - x_i\| = 1$, $0 \leq i < m$ and (b) for any $x \in A$, $y \notin A$, such a path does not exists.

Next assertion gives an estimation of the energy of a configuration $\sigma \in \Omega_L$ such that $N(\sigma) \leq n_0(n_0+1)+1$ in terms of the number of connected components.

Assertion 2.4.6. *Let $\sigma \in \Omega_L$ be a configuration such that $N(\sigma) = n_0(n_0+1)$, and denote by k, $1 \leq k \leq n_0(n_0+1)$ the number of connected components of σ. Then,*

$$\mathbb{H}(\sigma) \;\geq\; \Gamma_c \;+\; 2\,(k-1) \;+\; h\;.$$

Proof. By Assertion 2.4.5, we can assume that σ has no spin equal to $+1$. For such a configuration and by definition of Γ_c,

$$\mathbb{H}(\sigma) \;=\; \big[\, I_{-1,0}(\sigma) \;-\; (4n_0+4)\,\big] \;+\; \Gamma_c \;+\; h\;.$$

To complete the proof of the assertion, we have to show that $I_{-1,0}(\sigma) \geq (4n_0+4) + 2(k-1)$.

By moving 2 of the connected components of σ, and gluing them together, we reach a new configuration σ^1 such that $N(\sigma^1) = N(\sigma)$, while the size of the interface has decreased at least by 2 :

$$I_{-1,0}(\sigma) \;\geq\; I_{-1,0}(\sigma^1) \;+\; 2\;.$$

Iterating this argument $k-1$ times, we finally reach a configuration σ^* with only one connected component and such that

$$I_{-1,0}(\sigma) \;\geq\; I_{-1,0}(\sigma^*) \;+\; 2(k-1)\;. \tag{2.24}$$

The last connected component is glued to the set formed by the previous ones in such a way that the set $A(\sigma^*)$ is not a $n_0 \times (n_0+1)$ rectangle. This is always possible.

Since the connected set $A(\sigma^*)$ is not a $n_0 \times (n_0+1)$ rectangle, by [2, Assertion 4.B], $I_{-1,0}(\sigma^*) \geq 4n_0+4$, so that

$$I_{-1,0}(\sigma) \geq (4n_0+4) + 2(k-1),$$

which proves the assertion. □

We estimate the sum over $\sigma \in B_1$ on the right-hand side of (2.21) in the next assertion.

Assertion 2.4.7. *There exists a finite constant C_0 such that*

$$\sum_{\sigma \in B_1} \sum_{\sigma' \sim \sigma} \mu_{\mathcal{V}}(\sigma) \wedge \mu_{\mathcal{V}}(\sigma') \;\leq\; C_0\, \frac{1}{Z_{\mathcal{V}}}\, |\Lambda_L|\, e^{-\beta\Gamma_c}\, e^{-h\beta}$$

for all $\beta \geq C_0$.

Proof. The proof of this assertion is divided in three steps. The first one consists in applying Assertion 2.4.5 to restrict the first sum to configurations with no spin equal to +1. Indeed, as configuration in B_1 have at most $n_0(n_0+1)$ spins different from -1, by this assertion,

$$\sum_{\sigma\in B_1}\sum_{\sigma'\sim\sigma}\mu_{\mathcal{V}}(\sigma)\wedge\mu_{\mathcal{V}}(\sigma')\ \le\ 2^{n_0(n_0+1)}\sum_{\sigma\in B_{1,0}}\sum_{\sigma'\sim\sigma}\mu_{\mathcal{V}}(\sigma)\wedge\mu_{\mathcal{V}}(\sigma')\ ,$$

where $B_{1,0}$ represents the set of configurations in $\{-1,0\}^{\Lambda}$ which belong to B_1.

The second step consists in characterizing all configurations σ' which may appear in the second sum. Recall that it is performed over configurations $\sigma'\in\mathcal{V}_{\mathbf{-1}}\setminus B$ which can be obtained from σ by one flip. In particular, $N(\sigma')$, introduced in (2.23), can differ from $N(\sigma)=n_0(n_0+1)$ by at most by 1.

If $N(\sigma')=N(\sigma)+1$, as $\sigma'\not\ni B\supset\mathfrak{R}^+$, we have that $\sigma'\in\mathfrak{R}^a$. Since σ does not belong to $\mathfrak{R}$, the 0-spins of the configuration σ form a $n_0\times(n_0+1)$-rectangle in which one site has been removed and one site at the boundary of the rectangle has been added. In this case $\mathbb{H}(\sigma)>\mathbb{H}(\sigma')$ and $\mathbb{H}(\sigma)\ge\Gamma_c+h$. The last bound is attained if the site removed from the rectangle to form σ is a corner. Hence, restricting the second sum to configurations σ' such that $N(\sigma')=N(\sigma)+1$ yields that

$$\sum_{\sigma\in B_{1,0}}\sum_{\sigma'\sim\sigma}\mu_{\mathcal{V}}(\sigma)\wedge\mu_{\mathcal{V}}(\sigma')\ \le\ C_0\,\frac{1}{Z_{\mathcal{V}}}\,|\Lambda_L|\,e^{-\beta\Gamma_c}\,e^{-h\beta}\ ,$$

where the factor $|\Lambda_L|$ takes into account the possible positions of the rectangle and the constant C_0 the positions of the erased and added sites.

If $N(\sigma')=N(\sigma)$, resp. $N(\sigma')=N(\sigma)-1$, as σ has no spin equal to +1, this means that one 0-spin has been flipped to +1, resp. to -1. In both cases, there are $n_0(n_0+1)$ such configurations σ'. Hence, the sum restricted to such configurations σ' is less than or equal to

$$C_0\sum_{\sigma\in B_{1,0}}\mu_{\mathcal{V}}(\sigma)\ .$$

Let $N:=n_0(n_0+1)$, and denote by $\mathcal{C}_k$, $1\le k\le N$, the set of configurations in $B_{1,0}$ which have k connected components. Rewrite the previous sum according to the number of components and apply Assertion 2.4.6 to obtain that it is bounded by

$$\sum_{k=1}^{N}\sum_{\sigma\in\mathcal{C}_k}\mu_{\mathcal{V}}(\sigma)\ \le\ C_0\frac{1}{Z_{\mathcal{V}}}\,|\Lambda_L|\,e^{-\beta\Gamma_c}\,e^{-h\beta}\sum_{k=1}^{N}|\Lambda_L|^{k-1}\,e^{-\beta[2(k-1)]}\ ,$$

where $|\Lambda_L|^k$ takes into account the number of positions of the k components, and C_0 the form of each component. By the assumption of the theorems, $|\Lambda_L|\,e^{-2\beta}$ is bounded by 1/2 for β large enough, so that the sum is bounded by 2. To complete the proof of the assertion it remains to recollect the previous estimates. □

Proof.[Proof of Lemma 2.4.3] This Lemma is a consequence of Assertions 2.4.4, 2.4.7, and from the fact that $h<5$. □

2.5 The growth of a supercritical droplet

In the previous section we have seen that starting from **-1** we hit the boundary of the valley $\mathcal{V}_{\mathbf{-1}}$ at $\mathfrak{R}^a$. In this section we show that starting from $\mathfrak{R}^a$ the process either returns to **-1**, if the extra 0-spin is attached to one of the shortest sides of the rectangle, or it invades the all space with positive probability, if the extra 0-spin is attached to one of the longest sides of the rectangle.

Denote by $\mathfrak{R}^l$, $\mathfrak{R}^s$ the configurations of $\mathfrak{R}^a$ in which the extra particle is attached to one of the longest, shortest sides of the rectangle, respectively, and by $\mathfrak{R}^c$ the configurations of $\mathfrak{R}^a$ in which the extra particle is attached to one corner of the rectangle. Let $\mathfrak{R}^i = \mathfrak{R}^a \setminus \mathfrak{R}^c$, $\mathfrak{R}^{lc} = \mathfrak{R}^l \cap \mathcal{R}^c$, $\mathfrak{R}^{li} = \mathfrak{R}^l \cap \mathfrak{R}^i$.

Recall that $\mathcal{M} = \{\mathbf{-1}, \mathbf{0}, \mathbf{+1}\}$, and let

$$\delta(\beta) \;=\; |\Lambda_L|^{1/2}\, e^{-[(n_0+1)h-2]\beta} \;+\; |\Lambda_L|^{1/2}\, e^{-h\beta} \;+\; |\Lambda_L|^2\, e^{-(2-h)\beta} \;. \tag{2.25}$$

Proposition 2.5.1. *There exists a finite constant C_0 such that for all $\sigma \in \mathfrak{R}^{lc}$, $\sigma' \in \mathfrak{R}^{li}$, and $\sigma'' \in \mathfrak{R}^s$,*

$$\begin{gathered}\big|\, \mathbb{P}_\sigma[H_{\mathbf{-1}} = H_{\mathcal{M}}] \,-\, 1/2 \,\big| \;\le\; C_0\, \delta(\beta) \quad \text{and} \quad \big|\, \mathbb{P}_\sigma[H_{\mathbf{0}} = H_{\mathcal{M}}] \,-\, 1/2 \,\big| \;\le\; C_0\, \delta(\beta)\;, \\ \big|\, \mathbb{P}_{\sigma'}[H_{\mathbf{-1}} = H_{\mathcal{M}}] \,-\, 1/3 \,\big| \;\le\; C_0\, \delta(\beta) \quad \text{and} \quad \big|\, \mathbb{P}_{\sigma'}[H_{\mathbf{0}} = H_{\mathcal{M}}] \,-\, 2/3 \,\big| \;\le\; C_0\, \delta(\beta)\;, \\ \mathbb{P}_{\sigma''}[H_{\mathbf{-1}} = H_{\mathcal{M}}] \;\ge\; 1 \,-\, C_0\, \delta(\beta)\end{gathered}$$

for all $\beta \ge C_0$.

The proof of this proposition is divided in several lemmata. The first result describes what happens when there is a 0-spin attached to the side of a rectangle of 0-spins in a sea of -1-spins. From such a configuration, either the attached 0-spin is flipped to -1 or an extra 0-spin is created at the neighborood of the attached 0-spin.

For $n \ge 1$, let

$$\kappa_n(\beta) \;:=\; e^{-h\beta} \;+\; n\, e^{-(2-h)\beta} \;+\; |\Lambda_L|\, e^{-(4-h)\beta} \;. \tag{2.26}$$

Since $n^2 \le |\Lambda_L|$, $\kappa_n(\beta) \le \delta_1(\beta)$, where

$$\delta_1(\beta) \;:=\; e^{-h\beta} \;+\; |\Lambda_L|^{1/2}\, e^{-(2-h)\beta} \;+\; |\Lambda_L|\, e^{-(4-h)\beta} \;.$$

Note that for β large enough, $|\Lambda_L|\, e^{-(4-h)\beta} \le |\Lambda_L|^{1/2}\, e^{-(2-h)\beta}$.

Assertion 2.5.2. *Fix $n_0 \le m \le n \le L-3$. Consider a configuration σ with $nm+1$ 0-spins, all the other ones being -1. The 0-spins form a $(n \times m)$-rectangle and the extra 0-spin has one neighbor 0-spin which sits at one corner of the rectangle [there is only one -1-spin with two 0-spins as neighbors]. Let σ_-, resp. σ_+, be the configuration obtained from σ by flipping to -1 the attached 0-spin, resp. by flipping to 0 the unique -1 spin with two 0-spins as neighbors. Then, there exists a constant C_0 such that*

$$\Big|\, p_\beta(\sigma, \sigma_-) \,-\, \frac{1}{2} \,\Big| \;\le\; C_0\, \delta_1(\beta)\;, \qquad \Big|\, p_\beta(\sigma, \sigma_+) \,-\, \frac{1}{2} \,\Big| \;\le\; C_0\, \delta_1(\beta)\;.$$

Proof. We prove the lemma for σ_-, the proof for σ_+ being identical. Clearly, $R_\beta(\sigma,\sigma_-) = R_\beta(\sigma,\sigma_+) = 1$, so that

$$p_\beta(\sigma,\sigma_-) \;=\; \frac{R_\beta(\sigma,\sigma_-)}{\lambda_\beta(\sigma)} \;=\; \frac{1}{2+\sum_{\sigma'\neq\sigma_-,\sigma_+} R_\beta(\sigma,\sigma')}\;.$$

Consider the last sum. There are three terms, corresponding to the corners of the rectangle, for which $R_\beta(\sigma,\sigma') \le e^{-\beta h}$. There are $4(n+m)-2$ terms, corresponding to the inner and outer boundaries of the rectangle, such that $R_\beta(\sigma,\sigma') \le e^{-\beta(2-h)}$. All the remaining rates are bounded by $e^{-\beta(4-h)}$. Hence,

$$\sum_{\sigma'\neq\sigma_-,\sigma_+} R_\beta(\sigma,\sigma') \;\le\; C_0\, \kappa_n(\beta)\;,$$

where $\kappa_n(\beta)$ has been introduced in (2.26). This proves the assertion. □

In the next assertion we consider the case in which the extra 0-spin does not sit at the corner of the rectangle, but in its interior. The proof of this result, as well as the one of the next assertion, is similar to the previous proof.

Assertion 2.5.3. *Under the same hypotheses of the previous assertion, assume now that the extra* 0*-spin has one neighbor* 0*-spin which does not sit at one corner of the rectangle [there are exactly two* -1*-spins with two* 0*-spins as neighbors]. Let* σ_-*, resp.* σ_+^+*,* σ_+^-*, be the configuration obtained from* σ *by flipping to* -1 *the attached* 0*-spin, resp. by flipping to* 0 *one of the two* -1*-spins with two* 0*-spins as neighbors. Then, there exists a constant* C_0 *such that*

$$\Big|\, p_\beta(\sigma,\sigma_-) \,-\, \frac{1}{3}\,\Big| \;\le\; C_0\,\delta_1(\beta)\;, \quad \Big|\, p_\beta(\sigma,\sigma_+^\pm) \,-\, \frac{1}{3}\,\Big| \;\le\; C_0\,\delta_1(\beta)\;.$$

The next lemma states that once there are two adjacent 0-spins attached to one of the sides of the rectangle, this additional rectangle increases with very high probability. This result will permit to enlarge a $(p\times 1)$-rectangle to a $(2n_0\times 1)$-rectangle. To enlarge it further we will apply Lemma 2.5.8 below.

This result will be used in three different situations :

(A1) To increase in any direction a rectangle with 2 adjacent 0-spins whose distance from the corners is larger than $2n_0$ to a rectangle with $2n_0$ adjacent 0-spins ;

(A2) To increase in the direction of the corner a rectangle with $k\ge 2$ adjacent 0-spins which is at distance n_0 or less than from one of the corners to a rectangle with adjacent 0-spins which goes up to the corner ;

(A3) To increase a rectangle with $k<2n_0$ adjacent 0-spins which includes one of the corners to a rectangle with $2n_0$ adjacent 0-spins.

Fix $n_0\le m\le n\le L-3$. Denote by σ a configuration in which nm 0-spins form a $(m\times n)$-rectangle in a sea of -1-spins. Recall that we denote this rectangle by $A(\sigma)$, and assume, without loss of generality, that m is the length and n the height of $A(\sigma)$. Let (x,y) be the position of the upper-left corner of $A(\sigma)$.

We attach to one of the sides of $A(\sigma)$ and extra $(p\times 1)$-rectangle of 0-spins, where $p\ge 2$. To fix ideas, suppose that the extra 0-spins are attached to the upper side of length m of the rectangle.

More precisely, denote by $\eta^{(c,d)}$, $0 \le c < d \le m$, $d - c \ge 2$, the configuration obtained from σ by flipping from -1 to 0 the $([d-c] \times 1)$-rectangle, denoted by $R_{c,d}$, given by $\{(x+c, y+1), \dots, (x+d, y+1)\}$. The next lemma asserts that before anything else happens the rectangle $R_{c,d}$ increases at least by n_0 units at each side.

For a pair (c,d) as above, denote by $\mathcal{S}_{c,d}$ the set of configurations given by

$$\mathcal{S}_{c,d} \;=\; \{\eta^{(a,b)} : 0 \le a \le c \text{ and } d \le b \le m\}\;,$$

and by $E_{c,d}$ the exit time from $\mathcal{S}_{c,d}$,

$$E_{c,d} \;=\; \inf\big\{t > 0 : \sigma_t \notin \mathcal{S}_{c,d}\big\}\;.$$

Let $c^* = \max\{0, c - n_0\}$, $d^* = \min\{m, d + n_0\}$. Denote by $H_{c,d}$ the hitting time of the set $\mathcal{S}_{c^*,d^*}$:

$$H_{c,d} \;=\; \inf\big\{t > 0 : \sigma_t \in \mathcal{S}_{c^*,d^*}\big\}\;.$$

Lemma 2.5.4. *There exists a constant C_0 such that*

$$\mathbb{P}_{\eta^{(c,d)}}\big[E_{c,d} < H_{c,d}\big] \;\le\; C_0\, \delta_1(\beta)\;.$$

Proof. Consider a configuration $\eta^{(a,b)}$ in $\mathcal{S}_{c,d}$. To fix ideas assume that $a > 0$, $b < m$. At rate 1 the -1-spins at $(x+a-1, y+1)$, $(x+b+1, y+1)$ flip to 0. Consider all other possible spin flips. There are less than $2\,|\Lambda_L|$ flips whose rates are bounded by $e^{-[4-h]\beta}$, $4(n+m)$ flips whose rates are bounded by $e^{-[2-h]\beta}$ and 4 flips whose rates are bounded by $e^{-h\beta}$. Since all these jumps are independent, the probability that the -1-spin at $(x+a-1, y+1)$ flips to 0 before anything else happens is bounded below by $1 - C_0\,[\,|\Lambda_L|\, e^{-[4-h]\beta} + n e^{-[2-h]\beta} + e^{-h\beta}\,]$. Iterating this argument n_0-times yields the lemma. □

Applying Assertion 2.5.2 or 2.5.3 and then Lemma 2.5.4 to a configuration $\sigma \in \mathfrak{R}^a$ yields that either the process returns to $\mathfrak{R}$ or an extra row or line of 0-spins is added to the rectangle of 0-spins. The next two lemmata describe how the process evolves after reaching such a configuration.

Denote by $m \le n$ the length of the rectangle of 0-spins. If the shortest side has length n_0 or less, the configuration evolves to a $(m \times [n-1])$ rectangle of of 0-spins. If both sides are supercritical, that is if $m > n_0$, a -1-spin next to the rectangle is flipped to 0.

Denote by $\mathcal{S}_L$ the set of stable configurations of Ω_L, i.e., the ones which are local minima of the energy :

$$\mathcal{S}_L \;=\; \big\{\sigma \in \Omega_L : \mathbb{H}(\sigma) < \mathbb{H}(\sigma^{x,\pm}) \text{ for all } x \in \Lambda_L\big\}\;.$$

Let

$$\delta_2(\beta) \;=\; |\Lambda_L|\, e^{-[4-n_0 h]\beta} \;+\; e^{-h\beta}\;.$$

Fix $2 \le m \le n_0$, $2 \le n \le n_0 + 1$, $m \le n$. Consider a configuration σ with nm 0-spins forming a $(n \times m)$-rectangle, all the other ones being -1. If $m = n = 2$, let $\mathcal{S}(\sigma) = \{\mathbf{-1}\}$. If this is not the case, let $\mathcal{S}(\sigma)$ be the pair (quaternion if $m = n$) of configurations in which a row or a column of 0-spins of length m is removed from the rectangle $A(\sigma)$.

We define the valley of σ, denoted by $\mathcal{V}_\sigma$, as follows. Let $\mathcal{G}_k$, $0 \le k \le m$, be the configurations which can be obtained from σ by flipping to -1 a total of k 0-spins surrounded, at the moment they are switched, by two -1-spins. In particular, the elements of $\mathcal{G}_1$ are the four configurations obtained by flipping to -1 a corner of $A(\sigma)$.

Let $\mathcal{G} = \cup_{0\le k<m}\mathcal{G}_k$. Note that we do not include $\mathcal{G}_m$ in this union. Let $\mathcal{B}$ be the configurations which do not belong to $\mathcal{G}$, but which can be obtained from a configuration in $\mathcal{G}$ by flipping one spin. Clearly, $\mathcal{S}(\sigma)$ and $\mathcal{G}_m$ are contained in $\mathcal{B}$. Finally, let $\mathcal{V}(\sigma) = \mathcal{G} \cup \mathcal{B}$ be the neighborhood of σ.

Lemma 2.5.5. *Fix $2 \le m \le n_0$, $2 \le n \le n_0 + 1$, $m \le n$. Consider a configuration σ with nm 0-spins forming a $(n \times m)$-rectangle, all the other ones being -1. Then, there exists a constant C_0 such that*

$$\mathbb{P}_\sigma\big[\, H_{\mathcal{B}\setminus\mathcal{S}(\sigma)} < H_{\mathcal{S}(\sigma)} \,\big] \;\le\; C_0\, \delta_2(\beta)\ .$$

Proof. Assume that $2 < m < n$. The other cases are treated in a similar way. As in the proof of Proposition 2.4.1, denote by η_t the process σ_t reflected at $\mathcal{V}_\sigma$, and by $\mathbb{P}^{\mathcal{V}}_\sigma$ its distribution starting from σ. By (2.12),

$$\mathbb{P}_\sigma\big[\, H_{\mathcal{B}\setminus\mathcal{S}(\sigma)} < H_{\mathcal{S}(\sigma)} \,\big] \;=\; \mathbb{P}^{\mathcal{V}}_\sigma\big[\, H_{\mathcal{B}\setminus\mathcal{S}(\sigma)} < H_{\mathcal{S}(\sigma)} \,\big] \;\le\; \frac{\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{B}\setminus\mathcal{S}(\sigma))}{\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{S}(\sigma))}\ ,$$

where $\mathrm{cap}_{\mathcal{V}}$ represents the capacity with respect to the process η_t.

We estimate separately these two capacities. Let $\eta^{(k)}$, $0 \le k \le m$, be a sequence of configurations such that $\eta^{(0)} = \sigma$, $\eta^{(m)} \in \mathcal{S}(\sigma)$, and $\eta^{(k+1)}$ is obtained from $\eta^{(k)}$ by flipping to -1 a 0-spin surrounded by two -1-spins.

Consider the flow φ from σ to $\mathcal{S}(\sigma)$ given by $\varphi(\eta^{(k)}, \eta^{(k+1)}) = 1$ and $\varphi = 0$ for all the other bonds. By Thomson's principle,

$$\frac{1}{\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{S}(\sigma))} \;\le\; m\, Z_{\mathcal{V}}\, e^{[\mathbb{H}(\sigma)+(m-1)h]\beta}\ .$$

To estimate the capacity on the numerator, denote by $\chi = \chi_{\mathcal{B}\setminus\mathcal{S}(\sigma)}$ the indicator function of the set $\mathcal{B}\setminus\mathcal{S}(\sigma)$. By the Dirichlet principle,

$$\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{B}\setminus\mathcal{S}(\sigma)) \;\le\; D_{\mathcal{V}}(\chi) \;\le\; \sum_{\sigma'\in\mathcal{B}\setminus\mathcal{S}(\sigma)} \sum_{\sigma''} \mu_{\mathcal{V}}(\sigma') \,\wedge\, \mu_{\mathcal{V}}(\sigma'')\ ,$$

where the last sum is performed over all configurations $\sigma'' \in \mathcal{V}_\sigma \setminus [\mathcal{B}\setminus\mathcal{S}(\sigma)]$ which can be obtained from σ' by one flip.

To estimate the last sum we examine all elements of $\mathcal{B}\setminus\mathcal{S}(\sigma)$. There are at most $C_0\,|\Lambda_L|$ configurations σ' obtained from a configuration in $\mathcal{G}$ by flipping a spin at distance 2 or more from the [inner or outer] boundary of $A(\sigma)$. These configurations have only one neighbor σ'' in $\mathcal{V}_\sigma$ and their energy is bounded below by $\mathbb{H}(\sigma)+4-h$.

There are at most C_0 configurations σ' not in $\mathcal{G}_m$ and obtained from a configuration in $\mathcal{G}$ by flipping a spin [surrounded by three spins of the same type] at the boundary of $A(\sigma)$. These configurations have only one neighbor σ'' in $\mathcal{V}_\sigma$ and their energy is bounded below by $\mathbb{H}(\sigma) + 2 - h$.

Finally there are at most C_0 configurations σ' in $\mathcal{G}_m \setminus \mathcal{S}(\sigma)$ or obtained from a configuration in $\mathcal{G}$ by flipping a spin [surrounded by two spins of the same type] at the boundary of $A(\sigma)$. These configurations have at most C_0 neighbors σ'' in $\mathcal{V}_\sigma$ and their energy is bounded below by $\mathbb{H}(\sigma) + mh$. It follows from the previous estimates that

$$D_{\mathcal{V}}(\chi) \;\le\; C_0 \,\frac{1}{Z_{\mathcal{V}}} e^{-\mathbb{H}(\sigma)\beta} \big\{ \, |\Lambda_L| \, e^{-[4-h]\beta} \;+\; e^{-mh\beta} \, \big\} \;.$$

Putting together the previous estimates on the capacity, we conclude that

$$\mathbb{P}_\sigma\big[\, H_{\mathcal{B}\setminus\mathcal{S}(\sigma)} < H_{\mathcal{S}(\sigma)} \,\big] \;\le\; C_0 \,\big\{ \, |\Lambda_L| \, e^{-[4-n_0 h]\beta} \;+\; e^{-h\beta} \big\} \;.$$

This completes the proof of the lemma. □

Applying the previous result repeatedly yields that starting from a configuration σ with nm 0-spins forming a $(n \times m)$-rectangle in a sea of -1-spins the process converges to **-1** if the shortest side has length $m \le n_0$.

Corollary 2.5.6. *Let σ be a configuration with $n_0(n_0+1)$ 0-spins which form a $n_0 \times (n_0+1)$-rectangle in a background of -1. Then,*

$$\mathbb{P}_\sigma[H_{-\mathbf{1}} = H_{\mathcal{M}}] \;\ge\; 1 \;-\; C_0 \, \delta_2(\beta) \;.$$

The next results shows that, in constrast, if $m > n_0$, then the rectangle augments. We first characterize how the process leaves the neighborhood of such a configuration σ.

Fix $n_0 < m \le n \le L-3$. Consider a configuration σ with nm 0-spins forming a $(n \times m)$-rectangle in a sea of -1's. Recall that we denote by $A(\sigma)$ the rectangle of 0-spins. Let $\mathcal{V}_\sigma$ be the valley of σ whose elements can be constructed from σ as follows.

Fix $0 \le k \le n_0$. We first flip sequentially k spins of $A(\sigma)$ from 0 to -1. At each step we only flip a 0-spin if it is surrounded by two -1-spins. The set of all configurations obtained by such a sequence of k flips is represented by $\mathcal{G}_k$. In particular, since at the beginning we may only flip the corners of $A(\sigma)$, $\mathcal{G}_1$ is composed of the four configurations obtained by flipping to -1 one corner of $A(\sigma)$. On the other hand, since $m > n_0$, all configurations of $\mathcal{G}_k$ have an energy equal to $\mathbb{H}(\sigma) + kh$. Denote by $\mathcal{G}_{-1}$ the configuration obtained from σ by flipping to 0 a -1-spin which is surrounded by one 0-spin. Let $\mathcal{G} = \cup_{-1 \le k < n_0} \mathcal{G}_k$, and note that $\mathcal{G}_{n_0}$ has not been included in the union.

The second and final stage in the construction of the valley $\mathcal{V}_\sigma$ consists in flipping a spin of a configuration in $\mathcal{G}$. More precisely, denote by $\mathcal{B}$ all configurations which are not in $\mathcal{G}$, but which can be obtained from a configuration in $\mathcal{G}$ by flipping one spin. The set $\mathcal{B}$ is interpreted as the boundary of the valley $\mathcal{V}_\sigma := \mathcal{G} \cup \mathcal{B}$.

Note that all configurations in $\mathcal{V}_\sigma$ can be obtained from σ by at most n_0 flips. Conversely, if $(\eta^{(0)} = \sigma, \eta^{(1)}, \dots, \eta^{(n_0)})$ is a sequence of configurations starting from σ in which each element is obtained from the previous one by flipping a different spin, one of the configurations $\eta^{(k)}$ belong to the boundary of $\mathcal{V}_\sigma$.

Denote by $\mathcal{R}_2$ the set of $2(m+n-2)$ configurations obtained from σ by flipping to 0 two adjacent -1-spins, each of which is surrounded by a 0-spin. Clearly, $\mathcal{R}_2$ is contained in $\mathcal{B}$, and the energy of a configuration in $\mathcal{R}_2$ is equal to $\mathbb{H}_{-1} = \mathbb{H}_0 - h$, where $\mathbb{H}_0 := \mathbb{H}(\sigma) + (2-h)$ is the configuration in which only one -1-spin has flipped to 0. As $n_0 h > 2 - h$, an inspection shows that all the elements of $\mathcal{A} := \mathcal{B} \setminus \mathcal{R}_2$ have an energy strictly larger that $\mathbb{H}_0$. In particular, starting from σ, the process reaches the boundary $\mathcal{B}$ at $\mathcal{R}_2$. This is the content of the next lemma.

Let

$$\delta_3(\beta) \;=\; e^{-[(n_0+1)h-2]\beta} \;+\; |\Lambda_L|^{1/2}\, e^{-(2-h)\beta} \;+\; |\Lambda_L|\, e^{-2\beta} \;. \tag{2.27}$$

Lemma 2.5.7. *Fix $n_0 < m \le n \le L-3$. Consider a configuration σ with nm 0-spins forming a $(n \times m)$-rectangle in a sea of -1's. Recall that $\mathcal{A} := \mathcal{B} \setminus \mathcal{R}_2$. Then, there exists a constant C_0 such that*

$$\mathbb{P}_\sigma[H_{\mathcal{A}} < H_{\mathcal{R}_2}] \;\le\; C_0\, \delta_3(\beta) \;.$$

Proof. Since we may not leave the set $\mathcal{V}_\sigma$ without crossing its boundary $\mathcal{B}$, the probability appearing in the statement of the lemma is equal to the one for the reflected process at $\mathcal{V}_\sigma$, that is, the one in which we forbid jumps from $\mathcal{V}_\sigma$ to its complement. We estimate the probability for this later dynamics which is restricted to $\mathcal{V}_\sigma$.

By (2.12), the probability appearing in the statement of the lemma is bounded above by $\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{A})/\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{R}_2)$, where $\mathrm{cap}_{\mathcal{V}}$ stands for the capacity with respect to the reflected process. We estimate the numerator by the Dirichlet principle and the denominator by the Thomson principle.

We start with the denominator. Denote by $\eta^{(1)}, \dots, \eta^{(2(n+m-2))}$, the configurations of $\mathcal{R}_2$, and by $\boldsymbol{x}_j$, $\boldsymbol{y}_j \in \mathbb{Z}^2$ the positions of the two extra 0-spins of $\eta^{(j)}$. Assume that $\boldsymbol{x}_j \neq \boldsymbol{x}_k$ for $j \neq k$. Consider the flow φ from σ to $\mathcal{R}_2$ such that $\varphi(\sigma, \sigma^{\boldsymbol{x}_j}) = 1/[2(n+m-2)]$, $\varphi(\sigma^{\boldsymbol{x}_j}, \eta^{(j)}) = 1/[2(n+m-2)]$, and $\varphi = 0$ at all the other bonds. By the Thomson principle, since $\mu_{\mathcal{V}}(\sigma^{\boldsymbol{x}_j})$ is less than or equal to $\mu_{\mathcal{V}}(\sigma)$ and $\mu_{\mathcal{V}}(\eta^{(j)})$,

$$\frac{1}{\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{R}_2)} \;\le\; \frac{1}{n+m-2}\, Z_{\mathcal{V}}\, e^{\beta \mathbb{H}_0} \;. \tag{2.28}$$

We turn to the numerator. Denote by f the indicator function of the set $\mathcal{A}$. Since f vanishes at σ and is equal to 1 at $\mathcal{A}$, by the Dirichlet principle, $\mathrm{cap}_{\mathcal{V}}(\sigma, \mathcal{A}) \le D_{\mathcal{V}}(f)$. On the other hand,

$$D_{\mathcal{V}}(f) \;=\; \sum_{\eta \in \mathcal{A}} \sum_{\xi \sim \eta} \mu_{\mathcal{V}}(\eta) \wedge \mu_{\mathcal{V}}(\xi) \;, \tag{2.29}$$

where the second sum is performed over all configurations in $\mathcal{V}_\sigma \setminus A$ which can be obtained from η by one spin flip. This relation is represented by $\xi \sim \eta$.

We first consider the configuration η in $\mathcal{A}$ which have a neighbor in $\mathcal{G}_{-1}$. Fix $\xi \in \mathcal{G}_{-1}$. Consider the configurations obtained from ξ by flipping a spin which is not at the boundary of $A(\sigma)$. There are at most $|\Lambda_L|$ of such spins, and the energy of the configurations obtained by this spin flip is bounded below by $\mathbb{H}_0 + 4 - h$. There is one special spin, though, the one which is next to the extra spin and not at the boundary of $A(\sigma)$. The energy of the configuration obtained by flipping this spin to 0 or to +1 is bounded below by $\mathbb{H}_0 + 2 - h$. The contribution of these terms to (2.29) is thus bounded above by

$$2\,(n+m)\, \frac{1}{Z_{\mathcal{V}}}\, e^{-\beta \mathbb{H}_0}\, \big\{\, |\Lambda_L|\, e^{-(4-h)\beta} \;+\; e^{-(2-h)\beta} \,\big\} \;,$$

where the factor $2\,(n+m)$ comes from the total number of configurations in $\mathcal{G}_{-1}$.

We turn to the configurations obtained from ξ by flipping a spin at the boundary of $A(\sigma)$. Since the configuration resulting from this flip can not be in $\mathcal{R}_2$, their energy is bounded below by $\mathbb{H}_0 + 2 - h$. The contribution of these terms to the sum (2.29) is thus bounded by

$$4\,(n+m)^2\,\frac{1}{Z_{\mathcal{V}}}\,e^{-\beta\mathbb{H}_0}\,e^{-(2-h)\beta}\ ,$$

the extra factor $2(n+m)$ coming from the possible positions of the extra spin flip at the boundary.

Consider now configurations η in $\mathcal{A}$ which have a neighbor in a set $\mathcal{G}_k$, $0 \le k < n_0$. Fix $0 \le k < n_0$ and $\xi \in \mathcal{G}_k$. The configuration ξ is formed by a connected set $A(\xi) \subset A(\sigma)$ of 0-spins in a sea of -1-spins.

There is one special case which is examined separately. Suppose that ξ belongs to $\mathcal{G}_{n_0-1}$ and η to $\mathcal{G}_{n_0}$. There are $C(n_0)$ of such pairs, and the energy of η is equal to $\mathbb{H}(\sigma) + n_0 h = \mathbb{H}_0 + (n_0+1)h - 2$. We exclude from now in the analysis these pairs.

Apart from this case, there are two types of configurations $\eta \in \mathcal{A}$ which can be obtained from ξ by a spin flip. The first ones are the ones in which η and ξ differ by a spin which belongs to the inner or outer boundary of $A(\xi)$. There are at most $4(n+m) \le 8n$ of such configurations. The energy of these configurations is bounded below by $\mathbb{H}(\xi) + 2 - h = \mathbb{H}(\sigma) + kh + 2 - h = \mathbb{H}_0 + kh$. The minimal case occurs when a -1-spin which has a 0-spin as neighbor is switched to 0.

The previous estimate is not good enough in the case $k = 0$ because in the argument we did not exclude the configurations in $\mathcal{G}_{-1}$. For $k = 0$ if η belongs to $\mathcal{B} \setminus \mathcal{G}_{-1}$, we obtain that $\mathbb{H}(\eta) \ge \mathbb{H}(\sigma) + 2 + h = \mathbb{H}_0 + 2h$. The right-hand side of this inequality corresponds to the case in which a 0-spin surrounded by three 0-spins has been changed to -1. In conclusion, if the flip occurs at the boundary of $A(\xi)$, there are at most $8n$ configurations and the energy of such a configuration is bounded below by $\mathbb{H}_0 + h$.

If the flip did not occur at the boundary of $A(\xi)$, there are at most $|\Lambda_L|$ possible configurations, and the energy of these configurations is bounded below by $\mathbb{H}(\xi) + 4 - h = \mathbb{H}(\sigma) + kh + 4 - h = \mathbb{H}_0 + 2 + kh$.

The previous estimates yield that the Dirichlet form (2.29) is bounded by

$$\frac{C_0}{Z_{\mathcal{V}}}e^{-\beta\mathbb{H}_0}\Big\{n\,|\Lambda_L|\,e^{-(4-h)\beta} \;+\; n^2\,e^{-(2-h)\beta} \;+\; e^{-[(n_0+1)h-2]\beta} + ne^{-\beta h} + |\Lambda_L|e^{-2\beta}\Big\}\ .$$

Multiplying this expression by (2.28) yields that the probability appearing in the statement of the lemma is bounded above by

$$C_0\,\Big\{\,n\,e^{-(2-h)\beta} \;+\; e^{-[(n_0+1)h-2]\beta} \;+\; |\Lambda_L|\,e^{-2\beta}\Big\}$$

because $(n_0+1)h - 2 < h$ and $4 - h > 2$. We bounded $1/n$ by 1 when n appeared in the denominator because n can be as small as $n_0 + 1$. This completes the proof of the lemma since $n^2 \le |\Lambda_L|$. □

The previous lemma asserts that the process leaves the neighborhood of a large rectangle of 0-spins in a sea of -1 spins by switching from -1 to 0 two adjacent spins at the outer boundary of the rectangle. At this point, applying Lemma 2.5.4 yields that with a probability close to 1 these two adjacent 0-spins will increase to $2n_0$ adjacent 0-spins. To increase it further, we apply the next lemma.

This result will be used in two different situations :

(B1) To increase in any direction a rectangle with $2n_0$ adjacent 0-spins whose distance from the corners is larger than $2n_0$ to a rectangle of adjacent 0-spins which is at distance less than $2n_0$ from one of the corners;

(B2) To increase a rectangle with $k \geq 2n_0$ adjacent 0-spins which contains one corner and is at a distance larger than $2n_0$ from the other corner to a rectangle of adjacent 0-spins which is at distance less than $2n_0$ from this later corner.

To avoid a too strong assumption on the rate at which the cube Λ_L increases, we do not impose [as in the Assertions 2.5.2–2.5.3 and Lemma 2.5.4] the extra rectangle of 0-spins to grow without never shrinking or to grow while the spins at the corners stay put.

As in the proof of Lemma 2.5.7, we construct a set of configurations in two stages. We consider below the case in which the extra rectangle is far from the corners. The case in which it contains one of the corners can be handled similarly.

Fix $n_0 \leq m \leq n \leq L-3$. Denote by σ a configuration in which nm 0-spins form a $(n \times m)$-rectangle in a sea of -1-spins. Denote this rectangle by $A(\sigma)$, and assume, without loss of generality, that m is the length and n the height of $A(\sigma)$. Let (x, y) be the position of the upper-left corner of $A(\sigma)$.

We attach to one of the sides of $A(\sigma)$ an extra $(p \times 1)$-rectangle of 0-spins, where $p > n_0$. In particular, the length of the side to which this extra rectangle is attached has to be larger than n_0. To fix ideas, suppose that the extra 0-spins are attached to the upper side of length m of the rectangle and assume that $m > 5n_0$. As explained previously, the case $m \leq 5n_0$ is handled by Lemma 2.5.4.

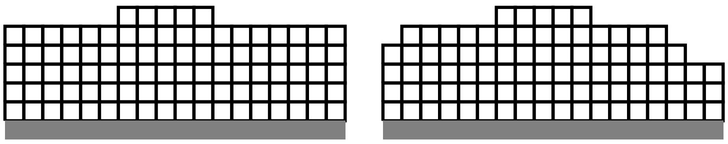

Figure 2.4 – Assume that $n_0 = 3$. The first picture provides an example of a configuration $\eta^{(c,d)}$. Here, $m = 18 \leq n$, $c = 6$, $d = 10$ and $p = 5$. The gray portion indicates that the rectangle continues below as its height is larger than 18. The second picture presents a configuration in $\mathcal{G}_{c,d,6}$. We chose $k = 6 > n_0$ to make the definition clear.

Denote by $\eta^{(c,d)}$, $2n_0 \leq c < d \leq m - 2n_0$, $d - c > n_0$, the configuration obtained from σ by flipping from -1 to 0 the $([d-c] \times 1)$-rectangle, denoted by $R_{c,d}$, given by $\{(x+c, y+1), \ldots, (x+d, y+1)\}$. Denote by $\mathcal{G}_{c,d,k}$, $0 \leq k \leq n_0$, the configurations obtained from $\eta^{(c,d)}$ by sequentially flipping to -1, close to the corners of $A(\sigma)$, a total of k 0-spins surrounded, at the moment they are switched, by two -1 spins. We do not flip spins in $R_{c,d}$.

In the case [not considered below] where the rectangle $R_{c,d}$ includes one corner, say $c = 0$, we treat the spins at $(x, y+1), \ldots, (x+n_0, y+1)$ as belonging to the corner and we allow them to be flipped.

Let $\mathcal{G} = \cup_{c,d} \cup_{0 \leq k < n_0} \mathcal{G}_{c,d,k}$, where the first union is performed over all indices such that $2n_0 \leq c < d \leq m - 2n_0$, $d - c > n_0$. Note that we excluded $k = n_0$ in this union. Denote by $\mathcal{B}$ the configurations which do not belong to $\mathcal{G}$ and which can be obtained from a configuration in $\mathcal{G}$ by flipping one spin. The set $\mathcal{B}$ is treated as the boundary of $\mathcal{G}$.

Note that $\mathcal{B}$ contains configurations in $\mathcal{G}_{c,d,n_0}$ and also configurations in $\mathcal{G}_{c,d,k}$ in which $d-c=n_0$. Let $\mathcal{A}_1$, $\mathcal{A}_2$ be such configurations :

$$\mathcal{A}_1 \;:=\; \bigcup_{c,d} \mathcal{G}_{c,d,n_0}\;, \quad \mathcal{A}_2 \;:=\; \bigcup_{c',d'} \bigcup_{0\le k<n_0} \mathcal{G}_{c',d',k}\;, \quad \mathcal{A} \;:=\; \mathcal{A}_1 \cup \mathcal{A}_2\;,$$

where the first union is performed over all indices such that $2n_0 \le c < d \le m-2n_0$, $d-c > n_0$, and the second one is performed over all indices such that $2n_0 < c' < d' \le m-2n_0$, $d'-c' = n_0$. The set $\mathcal{B}$ also contains configurations in which a 0-spin in a rectangle $R_{c,d}$ surrounded by 3 0-spins is flipped to ± 1.

All configurations in $\mathcal{G}$ are similar to the ones represented in Figure 2.4. They are obtained by adding a $(p\times 1)$-rectangle of 0-spins to the upper side of $A(\sigma)$ and by switching to -1 some of the spins of $A(\sigma)$ close to the corners.

For $t < H_{\mathcal{B}}$, denote by c_t, resp. d_t, the position at time t of the leftmost, resp. rightmost, 0-spin of the upper rectangle. Let τ^* be the first time $c_t \le 2n_0$ or $d_t \ge m-2n_0$:

$$\tau^* \;:=\; \inf\{t\ge 0 : c_t \le 2n_0 \text{ or } d_t \ge m-2n_0\}\;.$$

and let $\delta'_4(\beta) = |\Lambda_L|\, e^{-[4-h]\beta} \;+\; |\Lambda_L|^{1/2}\, e^{-[2-h]\beta}$,

$$\delta_4(\beta) \;:=\; |\Lambda_L|^{3/2}\, e^{-[4-h]\beta} \;+\; |\Lambda_L|\, e^{-[2-h]\beta}\;. \tag{2.30}$$

Lemma 2.5.8. *Let $\sigma' = \eta^{(c_0,d_0)}$, for some $2n_0 < c_0 < d_0 < m-2n_0$, $d_0 - c_0 \ge 2n_0$. Then, there exists a finite constant C_0 such that*

$$\mathbb{P}_{\sigma'}[H_{\mathcal{B}} < \tau^*] \;\le\; C_0\, \delta_4(\beta)$$

for all $\beta \ge C_0$.

Proof. Let C_t be the set of spins in $A(\sigma)$ close to the corners which takes the value -1 at time t. In the right picture of Figure 2.4 the set C_t consists of the 6 squares at the corners which have been removed from the left picture. Set C_t to be Λ_L for $t \ge H_{\mathcal{B}}$. Before hitting $\mathcal{B}$, the total number of sites of C_t, represented by $|C_t|$, is strictly bounded by n_0. Moreover, Before hitting $\mathcal{B}$, $|C_t|$, which starts from 0, is bounded by a Markov process m_t which jumps from $k\ge 0$ to $k+1$ at rate $n_0 e^{-\beta h}$ and from $k+1$ to k at rate 1.

Let $b_t = \mathbf{1}\{\sigma_t \in \mathcal{G}^c \setminus \mathcal{A}\}$. This process starts from 0 and jumps to 1 when σ_t reaches $\mathcal{B}$ through a configuration which is not in $\mathcal{A}$. Inspecting all possible jumps yields that the process b_t is bounded by a process z_t which starts from 0 and jumps to 1 at rate $|\Lambda_L| e^{-[4-h]\beta} + 2(n+m) e^{-[2-h]\beta} \le \delta'_4(\beta)$, where $\delta'_4(\beta)$ has been introduced just above (2.30)

The key observation in the proof of this lemma is that the processes (c_t, d_t), C_t and b_t are independent until the set $\mathcal{B}$ is attained because they involve different spin jumps.

Let $x_t = d_t - c_t$. Before hitting $\mathcal{B}$, x_t evolves as a random walk in $\mathbb{Z}$ which starts from $d-c\ge 2n_0$ and jumps from k to $k+1$ at rate 2 and from $k+1$ to k at rate $2e^{-\beta h}$. Let τ_0 be the first time $x_t \le n_0$.

Let H^b_1 be the hitting time of 1 by the process b_t, and let $H^C_{n_0}$ be the first time $|C_t|$ attains n_0. The event $\{H_{\mathcal{B}} < \tau^*\}$ is contained in the event $\{\tau_0 < \tau^*\} \cup \{H^b_1 < \tau^*\} \cup \{H^C_{n_0} < \tau^*\}$.

Consider three independent Markov chains, X_t, Y_t, Z_t. The first one takes value in $\{0,\dots,m\}$, it starts from $2n_0$, and jumps from k to $k+1$ at rate 2 and from $k+1$ to k at rate $2e^{-\beta h}$. The process Y_t takes value in $\{0,\dots,n_0\}$, it starts from 0, and jumps from k to $k+1$ at rate $2e^{-\beta h}$ and from $k+1$ to k at rate 1. The last one takes value in $\{0,1\}$, it starts from 0, and jumps from 0 to 1 at rate $\delta_4'(\beta)$.

Before time $H_{\mathcal{B}}$ we may couple $(x_t,|C_t|,b_t)$ with (X_t,Y_t,Z_t) in such a way that $X_t = x_t$, $|C_t|\le Y_t$ and $b_t \le Z_t$. In particular, $\{\tau_0<\tau^*\}\subset\{H_0^X<H_m^X\}$, $\{H_{n_0}^C<\tau^*\}\subset\{H_{n_0}^Y<H_m^X\}$, $\{H_1^b<\tau^*\}\subset\{H_1^Z<H_m^X\}$. In these formulas, H^W stands for the hitting time of the process W. Hence, the probability appearing in the statement of the lemma is bounded above by

$$P\big[H_{n_0}^X<H_m^X\big] \;+\; P\big[H_{n_0}^Y<H_m^X<H_{n_0}^X\big] \;+\; P\big[H_1^Z<H_m^X<H_{n_0}^X\big]\;. \tag{2.31}$$

We estimate each term separately.

The first one is easy. Denote by P_k^X the distribution of X_t starting from k. Let $f(k)=P_k^X[H_{n_0}^X<H_m^X]$, which is harmonic. It can be computed explicitly and one gets that

$$P\big[H_{n_0}^X<H_m^X\big] \;=\; P_{2n_0}^X\big[H_{n_0}^X<H_m^X\big] \;\le\; 2e^{-n_0h\beta}$$

provided $\beta\ge C_0$.

We turn to the second term of (2.31). On the set $\{H_m^X<H_{n_0}^X\}$ we may replace X by a random walk on $\mathbb{Z}$ and estimate $P\big[H_{n_0}^Y<H_m^X\big]$. As X and Y are independent, we condition on Y and treat $H_{n_0}^Y$ as a positive real number. The set $\{H_{n_0}^Y<H_m^X\}$ is contained in $\{X_J\le m\}$ where $J=H_{n_0}^Y$. Fix $\theta>0$. By the exponential Chebyshev inequality and since $m\le n$ [the sizes of the rectangle $A(\sigma)$],

$$P_{2n_0}^X\big[X_J\le m\big] \;\le\; P_0^X\big[X_J\le n\big] \;\le\; e^{\theta n}E_0^X\big[e^{-\theta X_J}\big]\;.$$

Choose $\theta=1/n$ and compute the expectation to obtain that the previous expression is bounded by $3\,\exp\{-(2/n)H_{n_0}^Y\}$ provided $\beta\ge C_0$. By Lemma 2.3.1,

$$E\big[e^{-(2/n)H_{n_0}^Y}\big] \;\le\; C_0\,n\,e^{-n_0h\beta}\;,$$

where E represents the expectation with respect to P.

The third expression in (2.31) is estimated similarly. The argument yields that it is bounded by

$$3\,E\big[e^{-(2/n)H_1^Z}\big] \;\le\; 3\,\frac{\delta_4'(\beta)}{(2/n)+\delta_4'(\beta)} \;\le\; 2\,n\,\delta_4'(\beta)\;,$$

where $\delta_4'(\beta)$ has been introduced just above (2.30). This completes the proof of the lemma because $n_0h>2-h$. □

Remark 2.5.9. *One could improve the previous argument and obtain a better estimate by allowing the spins at the boundary of $A(\sigma)$ to flip while the rectangle $R_{c,d}$ fills the upper side.*

The next result describes how the supercritical droplet of 0-spins grows. Let

$$\delta_5(\beta) \;:=\; e^{-[(n_0+1)h-2]\beta} \;+\; e^{-h\beta} \;+\; |\Lambda_L|^{3/2}\,e^{-(2-h)\beta}\;.$$

A simple computation based on the bound $|\Lambda_L| e^{-2\beta} \le 1$, which holds for β large enough, shows that there exists C_0 such that

$$\delta_1(\beta) \;+\; \delta_3(\beta) \;+\; |\Lambda_L|^{1/2}\, \delta_4(\beta) \;\le\; C_0\, \delta_5(\beta) \tag{2.32}$$

for all $\beta \ge C_0$.

Proposition 2.5.10. *Fix $n_0 < m \le n \le L$. Let σ be a configuration with mn 0-spins which form a $m \times n$-rectangle in a background of -1-spins. Then, there exists a constant C_0 such that*

$$\mathbb{P}_\sigma[H_{\mathcal{S}_L \setminus \{\sigma\}} = H_{\mathcal{S}(\sigma)}] \;\ge\; 1 \;-\; C_0\, \delta_5(\beta)\;,$$

for all $\beta \ge C_0$. In this equation, if $n \le L-3$, $\mathcal{S}(\sigma)$ is the set of four configurations in which a row or a column of 0 spins is added to the rectangle $A(\sigma)$. If $m < n = L-2$, the set $\mathcal{S}(\sigma)$ is a triple which includes a band of 0 spins of width m and two configurations in which a row or a column of 0 spins of length n is added to the rectangle $A(\sigma)$. If $m \le L-3$, $n = L$, the set $\mathcal{S}(\sigma)$ is a pair formed by two bands of 0 spins of width $m+1$. If $m = n = L-2$, $\mathcal{S}(\sigma)$ is a pair of two bands of width $L-2$. If $n_0 < m = L-2$, $n = L$, $\mathcal{S}(\sigma) = \{\mathbf{0}\}$.

Proof. Consider the first case, the proof of the other ones being similar. By Lemma 2.5.7, with a probability close to 1, the process σ_t escapes from the valley $\mathcal{V}_\sigma$ of σ by flipping to 0 two adjacent spins at the outer boundary of $A(\sigma)$. By Lemma 2.5.4, with a probability close to 1, these two adjacent spins will become $2n_0$ adjacent 0-spins. Of course, if the length of the side is smaller than $2n_0$, this simply means that the 0-spins fill the side.

Denote by R_e the $(2n_0 \times 1)$-rectangle of adjacent 0-spins. At this point, if R_e is at distance less than $2n_0$ of one of the corners of $A(\sigma)$, we apply Lemma 2.5.4 again to extend it up to the corner. After this step, or if R_e is at distance greater than $2n_0$ of one of the corners of $A(\sigma)$, we apply Lemma 2.5.8 to increase R_e up to the point that one of its extremities is at a distance less than $2n_0$ of one of the corners of $A(\sigma)$. We fill the $2n_0$ sites with 0-spins by applying again Lemma 2.5.4. We repeat the procedure applying Lemma 2.5.8 to reach a position close to the corner and then Lemma 2.5.4 to fill the gap.

The probability that something goes wrong in the way is bounded by the sum of the probabilities that each step goes wrong. This is given by $C_0\{\delta_3(\beta) + 6n_0\delta_1(\beta) + 2|\Lambda_L|^{1/2}\delta_4(\beta)\}$, which completes the proof of the proposition in view of (2.32). □

Corollary 2.5.11. *Let σ be a configuration with $n_0(n_0+1)+2$ 0-spins which form a $n_0 \times (n_0+1)$-rectangle in a background of -1, with two additional adjacent 0-spins attached to the longest side of the rectangle. Then, there exists a constant C_0 such that*

$$\mathbb{P}_\sigma[H_{\mathbf{0}} = H_{\mathcal{M}}] \;\ge\; 1 \;-\; C_0\, |\Lambda_L|^{1/2}\, \delta_5(\beta)$$

for all $\beta \ge C_0$.

Proof. Denote by $\{T_j : j \ge 1\}$ the jump times of the process σ_t. Let σ_+ be the configuration obtained from σ by flipping to 0 all -1-spins in the same row or column of the two adjacent 0-spins. Hence, σ_+ has $(n_0+1)^2$ 0-spins in a background of -1 spins. According to Lemma 2.5.4, $\mathbb{P}_\sigma[\sigma(T_{n_0-1}) = \sigma_+] \ge 1 - C_0\,\delta_1(\beta)$. Since $T_{n_0-1} \le H_{\mathcal{M}}$, we may apply the strong Markov property to conclude that

$$\mathbb{P}_\sigma[H_{\mathbf{+1}} = H_{\mathcal{M}}] \;\ge\; \mathbb{P}_{\sigma_+}[H_{\mathbf{+1}} = H_{\mathcal{M}}] \;-\; C_0\,\delta_1(\beta)\ .$$

At this point, apply Proposition 2.5.10 $(2|\Lambda_L|^{1/2})$ times to complete the proof. □

Proof.[Proof of Proposition 2.5.1] We prove the first statement, the argument for the other ones being analogous. Fix $\sigma \in \mathfrak{R}^{lc}$. Recall that we denote by T_1 the time of the first jump. By the strong Markov property at time T_1 and by Assertion 2.5.2,

$$\begin{aligned}\mathbb{P}_\sigma[H_{\mathbf{-1}} = H_{\mathcal{M}}] \;&=\; \mathbb{E}_\sigma\big[\,\mathbb{P}_{\sigma_{T_1}}[H_{\mathbf{-1}} = H_{\mathcal{M}}]\,\big]\\ &=\; \frac{1}{2}\,\big\{\mathbb{P}_{\sigma_+}[H_{\mathbf{-1}} = H_{\mathcal{M}}] \;+\; \mathbb{P}_{\sigma_-}[H_{\mathbf{-1}} = H_{\mathcal{M}}]\,\big\} \;+\; R_\beta\ ,\end{aligned}$$

where R_β is a remainder whose absolute values is bounded by $C_0\,\delta_1(\beta)$, and σ_-, resp. σ_+, is the configuration obtained from σ by flipping to -1 the attached 0-spin, resp. by flipping to 0 the unique -1 spin with two 0-spins as neighbors.

The configuration σ_- has $n_0(n_0+1)$ 0-spins which form a $n_0 \times (n_0+1)$-rectangle in a background of -1. Hence, by Corollary 2.5.6, $\mathbb{P}_{\sigma_-}[H_{\mathbf{-1}} = H_{\mathcal{M}}] \ge 1 - C_0\,\delta_2(\beta)$. On the other hand, by Corollary 2.5.11, $\mathbb{P}_{\sigma_+}[H_{\mathbf{+1}} = H_{\mathcal{M}}] \ge 1 - C_0\,|\Lambda_L|^{1/2}\,\delta_5(\beta)$. The first statement of the proposition follows from these estimates and from the fact that $\delta_2(\beta) < \delta_5(\beta)$, because $4 - n_0 h > 2$, and $\delta_3(\beta) < C_0\,\delta_5(\beta)$ for β large enough by (2.32). □

2.6 Proof of Theorems 2.2.1 and 2.2.2

The proofs of Theorems 2.2.2 and 2.2.1 are based on Propositions 2.4.1 and 2.5.1. By Proposition 2.5.1, there exists a finite constant C_0 such that

$$\begin{aligned}\max_{\sigma\in\mathfrak{R}^a} \mathbb{P}_\sigma[H_{\mathbf{+1}} < H_{\{\mathbf{-1},\mathbf{0}\}}] \;&\le\; C_0\,\delta(\beta)\ ,\\ \max_{\sigma\in\mathfrak{R}^l} \mathbb{P}_\sigma[H_{\mathbf{-1}} < H_{\{\mathbf{0},\mathbf{+1}\}}] \;&\le\; (1/2) \;+\; C_0\,\delta(\beta)\end{aligned} \tag{2.33}$$

for all $\beta \ge C_0$, where $\delta(\beta)$ has been introduced in (2.25).

Proof.[Proof of Theorem 2.2.2] We prove the first statement of the theorem, the argument for the second one being identical. Recall the definition of the boundary $\mathfrak{B}^+$ of the valley of **-1** introduced in (2.14). By (2.15) and by the strong Markov property at time $H_{\mathfrak{B}^+}$,

$$\mathbb{P}_{\mathbf{-1}}\big[H_{\mathbf{+1}} < H_{\mathbf{0}}\big] \;=\; \mathbb{E}_{\mathbf{-1}}\big[\,\mathbb{P}_{\sigma(H_{\mathfrak{B}^+})}\big[H_{\mathbf{+1}} < H_{\mathbf{0}}\big]\,\big]\ .$$

Let $q(\sigma) = \mathbb{P}_{\mathbf{-1}}[H_\sigma = H_{\mathfrak{B}^+}]$, $\sigma \in \mathfrak{B}^+$. By Proposition 2.4.1, the previous expectation is equal to

$$\sum_{\sigma\in\mathfrak{B}^+} q(\sigma)\,\mathbb{P}_\sigma[H_{\mathbf{+1}} < H_{\mathbf{0}}] \;\le\; \sum_{\sigma\in\mathfrak{R}^a} q(\sigma)\,\mathbb{P}_\sigma[H_{\mathbf{+1}} < H_{\mathbf{0}}] \;+\; C_0\,\varepsilon(\beta)\ ,$$

where $\varepsilon(\beta)$ has been introduced in (2.18).

By the first estimate in (2.33), uniformly in $\sigma \in \mathfrak{R}^a$,

$$\begin{aligned} \mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;&\le\; \mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{\mathbf{0}} \,,\, H_{\{-\mathbf{1},\mathbf{0}\}} < H_{+\mathbf{1}}] \;+\; C_0\,\delta(\beta) \\ &=\; \mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{+\mathbf{1}} < H_{\mathbf{0}}] \;+\; C_0\,\delta(\beta)\;. \end{aligned}$$

Therefore, by the strong Markov property at time $H_{-\mathbf{1}}$,

$$\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;\le\; \mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \sum_{\sigma\in\mathfrak{R}^a} q(\sigma)\,\mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\{\mathbf{0},+\mathbf{1}\}}] \;+\; C_0\,\delta(\beta)\;.$$

because $\varepsilon(\beta) \le \delta(\beta)$.

By the second bound in (2.33), as $\delta(\beta)\to 0$, for $\sigma\in\mathfrak{R}^l$, $\mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\{\mathbf{0},+\mathbf{1}\}}] \le 2/3$ provided $\beta \ge C_0$. Hence, for β large enough,

$$\begin{aligned} \mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;&\le\; (2/3)\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \sum_{\sigma\in\mathfrak{R}^a} q(\sigma) \;+\; C_0\,\delta(\beta) \\ &\le\; (2/3)\,\mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;+\; C_0\,\delta(\beta)\;. \end{aligned}$$

This completes the proof of the theorem since $\delta(\beta)\to 0$. □

Proof.[Proof of Theorem 2.2.1] Since the chains hits $\mathfrak{B}^+$ before reaching $\mathbf{0}$ and $\mathfrak{R}^l$, by the strong Markov property,

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;=\; \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\,\mathbb{P}_\sigma[H_{\mathfrak{R}^l} < H_{\mathbf{0}}]\;.$$

Recall the definition of $q(\sigma)$ introduced in the previous proof. By Proposition 2.4.1, this expression is equal to

$$\sum_{\sigma\in\mathfrak{R}^l} q(\sigma) \;+\; \sum_{\sigma\in\mathfrak{R}^s} q(\sigma)\,\mathbb{P}_\sigma[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;+\; R(\beta)\;. \tag{2.34}$$

where the absolute value of the remainder $R(\beta)$ is bounded by $C_0\,\varepsilon(\beta)$. By Assertion 2.5.2, Lemma 2.5.4 and by the proof Lemma 2.5.5, uniformly in $\sigma\in\mathfrak{R}^s$, $\sigma'\in\mathfrak{R}$

$$\begin{aligned} \mathbb{P}_\sigma[H_{\mathfrak{R}} < H_{\mathfrak{R}^l\cup\{-\mathbf{1},\mathbf{0}\}}] \;&\ge\; 1 \,-\, C_0\,[\delta_1(\beta) \,+\, \delta_2(\beta)]\;, \\ \mathbb{P}_{\sigma'}[H_{-\mathbf{1}} < H_{\mathfrak{R}^l\cup\{\mathbf{0}\}}] \;&=\; 1 \,-\, C_0\,\delta_2(\beta)\;. \end{aligned}$$

Hence, uniformly in $\sigma\in\mathfrak{R}^s$,

$$\mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\mathfrak{R}^l\cup\{\mathbf{0}\}}] \;\ge\; 1 \,-\, C_0\,[\delta_1(\beta) \,+\, \delta_2(\beta)]\;, \tag{2.35}$$

and we may introduce the set $\{H_{-\mathbf{1}} < H_{\mathfrak{R}^l\cup\{\mathbf{0}\}}\}$ inside the probability appearing in (2.34) by paying a cost bounded by $C_0[\delta_1(\beta) \,+\, \delta_2(\beta)]$.

Up to this point, we proved that

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;=\; \sum_{\sigma\in\mathfrak{R}^l} q(\sigma) \;+\; \sum_{\sigma\in\mathfrak{R}^s} q(\sigma)\,\mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;+\; R(\beta)\;,$$

where the absolute value of the remainder $R(\beta)$ is bounded by $C_0\,[\varepsilon(\beta) \,+\, \delta_1(\beta) \,+\, \delta_2(\beta)]$. By the strong Markov property this expression is equal to

$$\sum_{\sigma\in\mathfrak{R}^l} q(\sigma) \;+\; \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \sum_{\sigma\in\mathfrak{R}^s} q(\sigma)\,\mathbb{P}_\sigma[H_{-\mathbf{1}} < H_{\mathfrak{R}^l\cup\{\mathbf{0}\}}] \;+\; R(\beta)\;.$$

By (2.35), this expression is equal to

$$\sum_{\sigma\in\mathfrak{R}^l} q(\sigma) \;+\; \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \sum_{\sigma\in\mathfrak{R}^s} q(\sigma) \;+\; R(\beta)\;,$$

where the value of $R(\beta)$ has changed but not its bound. Therefore,

$$\Big(1 \,-\, \sum_{\sigma\in\mathfrak{R}^s} q(\sigma)\Big)\, \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;=\; \sum_{\sigma\in\mathfrak{R}^l} q(\sigma) \;+\; R(\beta)\;.$$

Since, by Proposition 2.4.1,

$$\sum_{\sigma\in\mathfrak{R}^l\cup\mathfrak{R}^s} q(\sigma) \;=\; \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^a} = H_{\mathfrak{B}^+}] \;\geq\; 1 \,-\, \varepsilon(\beta)\;,$$

replacing on the right-hand side $\sum_{\sigma\in\mathfrak{R}^l} q(\sigma)$ by $1 - \sum_{\sigma\in\mathfrak{R}^s} q(\sigma) - R'(\beta)$, where the absolute value of $R'(\beta)$ is bounded by $\varepsilon(\beta)$, we conclude that

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^l} < H_{\mathbf{0}}] \;=\; 1 \;+\; R(\beta)\;,$$

as claimed. □

2.7 The convergence of the trace process

In this section, we examine the evolution of the trace of σ_t on $\mathcal{M} = \{-\mathbf{1}, \mathbf{0}, +\mathbf{1}\}$ under the hypotheses of Theorem 2.2.2. Denote by η_t the trace of σ_t on $\mathcal{M}$. We refer to Section 2.3 for a precise definition. By [2, Proposition 6.1], η_t is an $\mathcal{M}$-valued, continuous-time Markov chain. Recall the definition of θ_β given in (2.7).

Proposition 2.7.1. *As $\beta \uparrow \infty$, the speeded-up Markov chain $\eta(\theta_\beta t)$ converges to the continuous-time Markov chain on $\mathcal{M}$ in which $+\mathbf{1}$ is an absorbing state, and whose jump rates $\mathbf{r}(\eta, \xi)$, are given by*

$$\mathbf{r}(-\mathbf{1}, \mathbf{0}) \;=\; \mathbf{r}(\mathbf{0}, +\mathbf{1}) \;=\; 1\;, \quad \mathbf{r}(-\mathbf{1}, +\mathbf{1}) \;=\; \mathbf{r}(\mathbf{0}, -\mathbf{1}) \;=\; 0\;.$$

The proof of this proposition, presented at the end of this section, relies on estimation of capacities. We start characterizing the distribution of $\sigma(H_{\mathfrak{B}^+})$ when the process starts from **-1**. Recall the definition of $\delta_2(\beta)$ introduced just before Lemma 2.5.5.

Lemma 2.7.2. *There exists a finite constant C_0 such that for every $\sigma \in \mathfrak{R}^a$,*

$$\Big|\, |\mathfrak{R}^a|\, \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \,-\, 1 \Big| \;\leq\; C_0\, [\, \varepsilon(\beta) + \delta_1(\beta)\,]$$

for all $\beta \geq C_0$.

Proof. Fix a reference configuration σ^* in $\mathfrak{R}^a$. By (2.11) and by definition of the capacity,

$$\mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \;=\; \frac{M(-\mathbf{1})\, \mathbb{P}_{-\mathbf{1}}[H_\sigma = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}]}{\mathrm{cap}(-\mathbf{1}, \mathfrak{B}^+)}\;.$$

By reversibility, the numerator of this expression is equal to

$$M(\sigma)\,\mathbb{P}_\sigma[H_{-\mathbf{1}} = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}] \;=\; \mu_\beta(\sigma)\,\lambda_\beta(\sigma)\,\mathbb{P}_\sigma[H_{-\mathbf{1}} = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}]\;.$$

By Assertions 2.5.2 and 2.5.3, with a probability close to 1, either a negative spin next to the attached 0-spin flips to 0 or the attached 0-spin flips to -1. In the first case, as the process left the valley $\mathcal{V}_{\mathbf{-1}}$ introduced at the beginning of Section 2.4, it will hit $\mathfrak{B}^+$ before reaching **-1**. In the second case, applying Lemma 2.5.5 repeatedly yields that the process reaches **-1** before hitting $\mathfrak{B}^+$. Hence, by these three results,

$$\big|\,\mathbb{P}_\sigma[H_{-\mathbf{1}} = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}] \;-\; \mathfrak{n}(\sigma)\,\big| \;\le\; C_0\,[\,\delta_1(\beta) \;+\; \delta_2(\beta)\,]\;,$$

where

$$\mathfrak{n}(\sigma) \;=\; \begin{cases} 1/2 & \text{if } \sigma\in\mathfrak{R}^c,\\ 1/3 & \text{if } \sigma\in\mathfrak{R}^i. \end{cases}$$

Since

$$\lambda(\sigma) \;=\; \begin{cases} 2+\delta_1(\beta) & \text{if } \sigma\in\mathfrak{R}^c,\\ 3+\delta_1(\beta) & \text{if } \sigma\in\mathfrak{R}^i, \end{cases}$$

and since $\mu_\beta(\sigma) = \mu_\beta(\sigma^*)$, we conclude that

$$\mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}] \;=\; \frac{\mu_\beta(\sigma^*)}{\mathrm{cap}(-\mathbf{1},\mathfrak{B}^+)}\,\big(1+R_\beta\big)\;,$$

where the absolute value of R_β is bounded by $C_0[\delta_1(\beta)+\delta_2(\beta)]$. Summing over $\sigma\in\mathfrak{R}^a$, it follows from Proposition 2.4.1 that for any configuration $\sigma^*\in\mathfrak{R}^a$,

$$\Big|\,\frac{\mu_\beta(\sigma^*)\,|\mathfrak{R}^a|}{\mathrm{cap}(-\mathbf{1},\mathfrak{B}^+)} \;-\; 1\,\Big| \;\le\; C_0\,[\varepsilon(\beta)+\delta_1(\beta)]$$

because $\delta_2(\beta)\le\varepsilon(\beta)$. To complete the proof of the lemma, it remains to multiply both sides of the penultimate identity by $|\mathfrak{R}^a|$. □

It follows from the proof of the previous lemma and the identity $|\mathfrak{R}^a| = 4(2n_0+1)|\Lambda_L|$ that there exists a finite constant C_0 such that for all $\sigma\in\mathfrak{R}^a$,

$$\Big|\,\frac{\mathrm{cap}(-\mathbf{1},\mathfrak{B}^+)}{\mu_\beta(\sigma)\,|\Lambda_L|} \;-\; 4(2n_0+1)\,\Big| \;\le\; C_0\,[\varepsilon(\beta)+\delta_1(\beta)] \tag{2.36}$$

for all $\beta\ge C_0$.

Proposition 2.7.3. *For any configuration $\eta\in\mathfrak{R}^l$ and any configuration $\xi\in\mathfrak{R}^l_0$,*

$$\lim_{\beta\to\infty}\frac{\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})}{\mu_\beta(\eta)\,|\Lambda_L|} \;=\; \frac{4(2n_0+1)}{3} \;=\; \lim_{\beta\to\infty}\frac{\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})}{\mu_\beta(\xi)\,|\Lambda_L|}\;.$$

Proof. We prove below the first idendity of the proposition, the one of the second being analogous. We first claim that

$$\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\}) \;=\; \mathrm{cap}(-\mathbf{1},\mathfrak{B}^+)\sum_{\sigma\in\mathfrak{B}^+}\mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\,\mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]\;. \tag{2.37}$$

Indeed, since starting from $-\mathbf{1}$ the process hits $\mathfrak{B}^+$ before $\{\mathbf{0}, +\mathbf{1}\}$, by the strong Markov property we have that

$$\mathbb{P}_{-\mathbf{1}}[H_{\{\mathbf{0},+\mathbf{1}\}} < H^+_{-\mathbf{1}}] \;=\; \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H^+_{\mathfrak{B}^+\cup\{-\mathbf{1}\}}]\, \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}] \;.$$

By (2.11), we may rewrite the previous expression as

$$\mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{B}^+} < H^+_{-\mathbf{1}}] \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\, \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}] \;.$$

This proves (2.37) in view of the definition (2.8) of the capacity.

By (2.36) and (2.37), for any configuration $\sigma^* \in \mathfrak{R}^a$,

$$\frac{\mathrm{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})}{|\Lambda_L|\, \mu_\beta(\sigma^*)} \;=\; [4(2n_0+1) + R^{(1)}_\beta] \sum_{\sigma\in\mathfrak{B}^+} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\, \mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}] \;.$$

where $|R^{(1)}_\beta| \le C_0\, [\varepsilon(\beta) + \delta_1(\beta)]$ for β large.

Consider the sum. By Proposition 2.4.1, we may ignore the terms $\sigma \not\in \mathfrak{R}^a$. On the other hand, Proposition 2.5.1 provides the asymptotic value of $\mathbb{P}_\sigma[H_{\{\mathbf{0},+\mathbf{1}\}} < H_{-\mathbf{1}}]$ for $\sigma \in \mathfrak{R}^a$. Putting together these two result yields that the sum is equal to

$$\frac{1}{2}\, \mathbb{P}_{-\mathbf{1}}[H_{\mathfrak{R}^{lc}} = H_{\mathfrak{B}^+}] \;+\; \frac{2}{3}\, \mathbb{P}_\sigma[H_{\mathfrak{R}^{li}} = H_{\mathfrak{B}^+}] \;+\; R^{(2)}_\beta \;,$$

where the absolute value of the remainder $R^{(2)}_\beta$ is bounded by $C_0[\varepsilon(\beta) + \delta(\beta)]$. By Lemma 2.7.2, this expression is equal to

$$\frac{1}{2}\, \frac{2}{2n_0+1} \;+\; \frac{2}{3}\, \frac{n_0-1}{2n_0+1} \;+\; R^{(3)}_\beta \;,$$

where $|R^{(3)}_\beta| \;\le\; C_0\, [\varepsilon(\beta) + \delta(\beta)]$ because $\delta_1(\beta) \;\le\; \delta(\beta)$. The first assertion of the proposition follows from the previous estimates. □

The same proof yields that for any configuration $\eta \in \mathfrak{R}^l$, $\xi \in \mathfrak{R}^l_0$,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1}, \mathbf{0})}{\mu_\beta(\eta)\, |\Lambda_L|} \;=\; \frac{4(2n_0+1)}{3} \;=\; \lim_{\beta\to\infty} \frac{\mathrm{cap}(\mathbf{0}, +\mathbf{1})}{\mu_\beta(\xi)\, |\Lambda_L|} \;.$$

In particular,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1}, \mathbf{0})}{\mathrm{cap}(-\mathbf{1}, \{\mathbf{0}, +\mathbf{1}\})} \;=\; 1 \;, \quad \lim_{\beta\to\infty} \frac{\mathrm{cap}(\mathbf{0}, +\mathbf{1})}{\mathrm{cap}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})} \;=\; 1 \;. \tag{2.38}$$

Corollary 2.7.4. *We have that*

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(\mathbf{0}, \mathcal{M}\setminus\{\mathbf{0}\})}{\mu_\beta(\mathbf{0})}\, \theta_\beta \;=\; 1 \;,$$

where θ_β has been introduced in (2.7).

Proof. Fix $\eta \in \mathfrak{R}^a$ and $\xi \in \mathfrak{R}^a_0$. By definition of θ_β, the expression appearing in the statement of the corollary can be written as

$$\frac{\mu_\beta(\xi)}{\mu_\beta(\mathbf{0})}\, \frac{\mathrm{cap}(\mathbf{0}, \mathcal{M}\setminus\{\mathbf{0}\})}{\mu_\beta(\xi)\, |\Lambda_L|}\, \frac{\mu_\beta(\eta)\, |\Lambda_L|}{\mathrm{cap}(-\mathbf{1}, \mathcal{M}\setminus\{-\mathbf{1}\})}\, \frac{\mu_\beta(-\mathbf{1})}{\mu_\beta(\eta)} \;.$$

By the previous lemma, the product of the second and third expression converges to 1, while the first and fourth term cancel. □

Lemma 2.7.5. *We have that*

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1},+\mathbf{1})}{\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})} \;=\; 1\;.$$

Proof. Fix a configuration σ^* in $\mathfrak{R}^a$. By the proof of Proposition 2.7.3,

$$\frac{\mathrm{cap}(-\mathbf{1},+\mathbf{1})}{\mu_\beta(\sigma^*)\,|\Lambda_L|} \;=\; 4(2n_0+1)\sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\,\mathbb{P}_\sigma[H_{+\mathbf{1}} < H_{-\mathbf{1}}] \;+\; R^{(1)}_\beta\;,$$

where $|R^{(1)}_\beta| \le C_0[\varepsilon(\beta)+\delta(\beta)]$ for some finite constant C_0.

By Proposition 2.5.1, starting from $\sigma\in\mathfrak{R}^a$ we reach $\{\textbf{-1},\mathbf{0}\}$ before $+\mathbf{1}$ with a probability close to 1. Hence, up to a small error, we may include the event $H_{\{-\mathbf{1},\mathbf{0}\}} < H_{+\mathbf{1}}$ inside the second probability which becomes $\{H_{\mathbf{0}} < H_{+\mathbf{1}} < H_{-\mathbf{1}}\}$. Applying the strong Markov property, the right-hand side of the previous expression becomes

$$4(2n_0+1)\,\mathbb{P}_{\mathbf{0}}[H_{+\mathbf{1}} < H_{-\mathbf{1}}]\sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_{-\mathbf{1}}[H_\sigma = H_{\mathfrak{B}^+}]\,\mathbb{P}_\sigma[H_{\mathbf{0}} < H_{\{\textbf{-1},+\mathbf{1}\}}] \;+\; R^{(1)}_\beta$$

for a new remainder $R^{(1)}_\beta$ whose absolute value is bounded by $C_0[\varepsilon(\beta)+\delta(\beta)]$.

By Theorem 2.2.2, $\mathbb{P}_{\mathbf{0}}[H_{+\mathbf{1}} < H_{-\mathbf{1}}]$ converges to 1. On the other hand, the sum can be handled as in the proof of Proposition 2.7.3 to yield that

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(-\mathbf{1},+\mathbf{1})}{\mu_\beta(\sigma^*)\,|\Lambda_L|} \;=\; \frac{4(2n_0+1)}{3}\;.$$

This completes the proof of the lemma in view of Proposition 2.7.3. □

Lemma 2.7.6. *We have that*

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(+\mathbf{1},\{-\mathbf{1},\mathbf{0}\})}{\mathrm{cap}(\mathbf{0},\{-\mathbf{1},+\mathbf{1}\})} \;=\; 1\;.$$

Proof. By monotonicity of the capacity and by equation (3.5) in [19],

$$\mathrm{cap}(+\mathbf{1},\mathbf{0}) \;\le\; \mathrm{cap}(+\mathbf{1},\{-\mathbf{1},\mathbf{0}\}) \;\le\; \mathrm{cap}(+\mathbf{1},\mathbf{0}) \;+\; \mathrm{cap}(+\mathbf{1},-\mathbf{1})\;.$$

We claim that $\mathrm{cap}(-\mathbf{1},+\mathbf{1})/\mathrm{cap}(\mathbf{0},+\mathbf{1}) \to 0$. By Lemma 2.7.5, we may replace the numerator by $\mathrm{cap}(-\mathbf{1},\{\mathbf{0},+\mathbf{1}\})$, and by the second identity of (2.38), we may replace the denominator by $\mathrm{cap}(\mathbf{0},\{\textbf{-1},+\mathbf{1}\})$. At this point, the claim follows from Proposition 2.7.3.

Therefore,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(+\mathbf{1},\{-\mathbf{1},\mathbf{0}\})}{\mathrm{cap}(\mathbf{0},+\mathbf{1})} \;=\; 1\;.$$

To complete the proof, it remains to recall again the second identity in (2.38). □

It follows from the previous result that

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}(+\mathbf{1}, \mathcal{M}\setminus\{+\mathbf{1}\})}{\mu_\beta(+\mathbf{1})}\, \theta_\beta \;=\; 0\;. \tag{2.39}$$

Indeed, by the previous lemma, this limit is equal to

$$\lim_{\beta\to\infty} \theta_\beta\, \frac{\mathrm{cap}(\mathbf{0}, \{-\mathbf{1}, +\mathbf{1}\})}{\mu_\beta(\mathbf{0})}\, \frac{\mu_\beta(\mathbf{0})}{\mu_\beta(+\mathbf{1})}\;.$$

This expression vanishes in view of Corollary 2.7.4 and because $\mu_\beta(\mathbf{0})/\mu_\beta(+\mathbf{1}) \to 0$.
Proof.[Proof of Proposition 2.7.1] Denote by $r_\beta(\eta,\xi)$ the jump rates of the chain $\eta_{\theta_\beta t}$. It is enough to prove that

$$\lim_{\beta\to\infty} r_\beta(\eta,\xi) \;=\; \mathbf{r}(\eta,\xi) \tag{2.40}$$

for all $\eta \neq \xi \in \mathcal{M}$.

By [2, Proposition 6.1], the jump rates $r_\beta(\eta,\xi)$, $\eta\neq\xi\in\mathcal{M}$, of the Markov chain $\eta_{\theta_\beta t}$ are given by

$$r_\beta(\eta,\xi) \;=\; \theta_\beta\, \lambda(\eta)\, \mathbb{P}_\eta[H_\xi = H^+_{\mathcal{M}}]\;.$$

Dividing and multiplying the previous expression by $\mathbb{P}_\eta[H_{\mathcal{M}\setminus\{\eta\}} < H^+_\eta]$, by definition of the capacity and by (2.11),

$$r_\beta(\eta,\xi) \;=\; \frac{\theta_\beta}{\mu_\beta(\eta)}\, \mathrm{cap}(\eta, \mathcal{M}\setminus\{\eta\})\, \mathbb{P}_\eta[H_\xi < H_{\mathcal{M}\setminus\{\eta,\xi\}}]\;.$$

It follows from this identity and from (2.39) that for $\xi = -\mathbf{1}$, $\mathbf{0}$,

$$\lim_{\beta\to\infty} r_\beta(+\mathbf{1},\xi) \;\le\; \lim_{\beta\to\infty} \frac{\theta_\beta}{\mu_\beta(+\mathbf{1})}\, \mathrm{cap}(+\mathbf{1}, \mathcal{M}\setminus\{+\mathbf{1}\}) \;=\; 0\;.$$

On the other hand, by Corollary 2.7.4,

$$\lim_{\beta\to\infty} \frac{\theta_\beta}{\mu_\beta(\mathbf{0})}\, \mathrm{cap}(\mathbf{0}, \mathcal{M}\setminus\{\mathbf{0}\}) \;=\; 1\;,$$

while, by definition, $\theta_\beta\, \mathrm{cap}(-\mathbf{1}, \mathcal{M}\setminus\{-\mathbf{1}\})/\mu_\beta(-\mathbf{1}) = 1$. Furthermore, by Theorem 2.2.2,

$$\lim_{\beta\to\infty} \mathbb{P}_{-\mathbf{1}}[H_{+\mathbf{1}} < H_{\mathbf{0}}] \;=\; \lim_{\beta\to\infty} \mathbb{P}_{\mathbf{0}}[H_{-\mathbf{1}} < H_{+\mathbf{1}}] \;=\; 0\;.$$

This yields (2.40) and completes the proof of the lemma. □

2.8 The time spent out of $\mathcal{M}$

We prove in this section that the total time spent out of $\mathcal{M}$ by the process $\sigma(t\theta_\beta)$ is negligible. Unless otherwise stated, we assume that the hypotheses of Theorem 2.2.2 are in force.

Proposition 2.8.1. *Let $\Delta = \Omega_L \setminus \mathcal{M}$. For every $\xi \in \mathcal{M}$, $t > 0$,*

$$\lim_{\beta\to\infty} \mathbb{E}_\xi \Big[\int_0^t \mathbf{1}\{\sigma(s\theta_\beta) \in \Delta\}\, ds \Big] \;=\; 0\;.$$

The proof of this proposition is divided in several steps. Suppose that the process starts from **-1**. In this case, we divide the time interval $[0, t]$ in 5 pieces, and we prove that the time spent in Δ in each time-interval $[0, H_{\mathfrak{B}^+}]$, $[H_{\mathfrak{B}^+}, H_{\mathbf{0}}]$, $[H_{\mathbf{0}}, H_{\mathfrak{B}_0^+}]$, $[H_{\mathfrak{B}_0^+}, H_{\mathbf{+1}}]$ and $[H_{\mathbf{+1}}, \infty)$ is negligible.

The proof of the last step requires the introduction of the valley of $\mathbf{+1}$ which is slightly different from $\mathcal{V}_{\mathbf{-1}}$ and $\mathcal{V}_{\mathbf{0}}$. Denote by $\mathcal{V}_{\mathbf{+1}}$ the valley of $\mathbf{+1}$. This is the set constituted of all configurations which can be attained from $\mathbf{+1}$ by flipping $n_0(n_0+1)$ or less spins of $\mathbf{+1}$. The boundary of this set, denoted by $\mathfrak{B}^+_{\mathbf{+1}}$, is formed by all configurations which differ from $\mathbf{+1}$ at exactly $n_0(n_0+1)$ sites. The configuration with minimal energy in $\mathfrak{B}^+_{\mathbf{+1}}$ is the one where $n_0(n_0+1)$ 0-spins form a $n_0 \times (n_0+1)$-rectangle. Denote the set of these configurations by $\mathfrak{R}_{\mathbf{+1}}$. Fix $\eta \in \mathfrak{R}_{\mathbf{+1}}$ and $\xi \in \mathfrak{R}^a$ and note that

$$\mathbb{H}(\eta) \;-\; \mathbb{H}(\mathbf{+1}) \;>\; \mathbb{H}(\xi) \;-\; \mathbb{H}(\mathbf{-1})\;. \tag{2.41}$$

Thus $\mathcal{V}_{\mathbf{+1}}$ is a deeper valley than $\mathcal{V}_{\mathbf{-1}}$ or $\mathcal{V}_{\mathbf{0}}$.

Indeed, according to [2, Theorem 2.6], the depth of the valley $\mathcal{V}_{\mathbf{+1}}$ is given by $\mu_\beta(\mathbf{+1})/\mathrm{cap}(\mathbf{+1}, \mathfrak{B}^+_{\mathbf{+1}})$. As in the proof of (2.36), or by applying the Dirichlet and the Thomson principles, we have that $\mathrm{cap}(\mathbf{+1}, \mathfrak{B}^+_{\mathbf{+1}})$ is of the order of $|\Lambda_L|\, \mu_\beta(\eta)$ for $\eta \in \mathfrak{R}_{\mathbf{+1}}$. Hence the depth of the valley $\mathcal{V}_{\mathbf{+1}}$ is of the order of $e^{[\mathbb{H}(\eta) - \mathbb{H}(\mathbf{+1})]\beta}/|\Lambda_L|$. By (2.41), the definition (2.7) of θ_β, and Proposition 2.7.3, this expression is much larger than θ_β, which is of the same order of the depth of the valley $\mathcal{V}_{\mathbf{-1}}$.

In particular, for all $t > 0$

$$\lim_{\beta\to\infty} \mathbb{P}_{\mathbf{+1}} \big[H_{\mathfrak{B}^+_{\mathbf{+1}}} < t\, \theta_\beta \big] \;=\; 0\;. \tag{2.42}$$

We turn to the proof of Proposition 2.8.1. We first show that conditioned to the valley $\mathcal{V}_{\mathbf{-1}}$, the measure μ_β is concentrated on the configuration **-1**. The same argument yields that this result holds for the pairs $(\mathbf{0}, \mathcal{V}_{\mathbf{0}})$, $(\mathbf{+1}, \mathcal{V}_{\mathbf{+1}})$.

Lemma 2.8.2. *Suppose that* (2.4) *holds. Then, there exists a constant C_0 such that*

$$\frac{\mu_\beta(\mathcal{V}_{\mathbf{-1}} \setminus \{\mathbf{-1}\})}{\mu_\beta(\mathbf{-1})} \;\le\; C_0\, |\Lambda_L|\, e^{-2\beta}$$

for all $\beta \ge C_0$.

Proof. Fix $1 \le k \le N = n_0(n_0+1) + 1$, and denote by $\mathcal{A}_k$ the configurations in $\mathcal{V}_{\mathbf{-1}}$ with k spins different from -1. The ratio appearing in the statement of the assertion is equal to

$$\sum_{k=1}^{N} \sum_{\sigma\in\mathcal{A}_k} \frac{\mu_\beta(\sigma)}{\mu_\beta(\mathbf{-1})} \;\le\; 2^N \sum_{k=1}^{N} \sum_{\sigma\in\mathcal{A}^0_k} \frac{\mu_\beta(\sigma)}{\mu_\beta(\mathbf{-1})}\;\cdot \tag{2.43}$$

In this equation, $\mathcal{A}^0_k$ represents the configurations in $\mathcal{V}_{\mathbf{-1}}$ with k spins equal to 0, and we applied Assertion 2.4.5.

Fix $k < N$ and consider the set $\mathcal{A}^0_{k,j}$ of configurations in $\mathcal{A}^0_k$ for which the 0-spins have j connected components. There are at most $C_0|\Lambda_L|^j$ of such configurations, and the energy of one of them, denoted by σ, is equal to $\mathbb{H}(\mathbf{-1}) - kh + I_{-1,0}(\sigma)$, where $I_{-1,0}(\sigma)$ represents the size of the interface. By (2.24), $I_{-1,0}(\sigma) \ge I_{-1,0}(\sigma^*) + 2(j-1)$, where σ^* is configuration obtained from σ by gluing the connected components. By [2, Assertion 4.A], $I_{-1,0}(\sigma^*) \ge 4\sqrt{k}$. Therefore, the previous sum for $k < N$ is bounded above by

$$C_0 \sum_{k=1}^{N-1} e^{-[4\sqrt{k}-kh]\beta} \sum_{j=1}^{k} |\Lambda_L|^j \, e^{-2(j-1)\beta} \,.$$

Since $|\Lambda_L| \, e^{-2\beta} \le 1/2$ for β sufficiently large and since $4\sqrt{k} - kh \ge \min\{4 - h, 4\sqrt{N-1} - (N-1)h\}$, the previous sum is less than or equal to

$$C_0 \, |\Lambda_L| \, \Big(e^{-[4-h]\beta} + e^{-2(n_0-1)\beta} \Big) \;\le\; C_0 \, |\Lambda_L| \, \Big(e^{-[4-h]\beta} + e^{-2\beta} \Big) \;\le\; C_0 \, |\Lambda_L| \, e^{-2\beta}$$

because $4\sqrt{N-1} - (N-1)h = 4\sqrt{n_0(n_0+1)} - n_0(n_0+1)h \ge 4n_0 - 2(n_0+1) = 2(n_0-1) \ge 2$.

It remains to consider the contribution of the set $\mathcal{A}_N$. There are at most $C_0 \, |\Lambda_L|$ configurations in this set, and each configuration has the same energy. The contribution of these terms to the sum (2.43) is bounded by $C_0 \, |\Lambda_L| e^{-(2n_0+1)\beta} \le C_0 \, |\Lambda_L| e^{-2\beta}$. This completes the proof of the lemma. □

Assertion 2.8.3. *We have that*

$$\lim_{\beta\to\infty} \frac{1}{\theta_\beta} \, \mathbb{E}_{\mathbf{-1}} \Big[\int_0^{H_{\mathfrak{B}^+}} \mathbf{1}\{\sigma(s) \in \Delta\} \, ds \Big] \;=\; 0 \,.$$

Proof. As the process σ_t is stopped at time $H_{\mathfrak{B}^+}$, we may replace Δ by $\Delta \cap \mathcal{V}_{\mathbf{-1}}$. By [2, Proposition 6.10], and by definition of θ_β, the expression appearing in the statement of the lemma is bounded above by

$$\frac{1}{\theta_\beta} \, \frac{\mu_\beta(\Delta \cap \mathcal{V}_{\mathbf{-1}})}{\mathrm{cap}(\mathbf{-1}, \mathfrak{B}^+)} \;=\; \frac{\mathrm{cap}(\mathbf{-1}, \mathcal{M} \setminus \{\mathbf{-1}\})}{\mathrm{cap}(\mathbf{-1}, \mathfrak{B}^+)} \, \frac{\mu_\beta(\Delta \cap \mathcal{V}_{\mathbf{-1}})}{\mu_\beta(\mathbf{-1})} \,.$$

By Proposition 2.7.3 and (2.36), the first ratio converges to 1/3, while by Lemma 2.8.2 the second one converges to 0. □

A similar result holds for the pairs $(\mathbf{0}, \mathcal{V}_{\mathbf{0}})$, $(\mathbf{+1}, \mathcal{V}_{\mathbf{+1}})$, where the valleys are defined analogously as $\mathcal{V}_{\mathbf{-1}}$.

Denote by $\mathfrak{R}^l_2$, resp, $\mathfrak{R}^s_2$, the set of configurations with $n_0(n_0+1)+2$ 0-spins in a background of -1-spins. The 0-spins form a $[n_0 \times (n_0+1)]$-rectangle with two extra contiguous 0-spins attached to one of the longest, resp. shortest, sides of the rectangle.

Lemma 2.8.4. *For every $t > 0$,*

$$\lim_{\beta\to\infty} \frac{1}{\theta_\beta} \max_{\xi \in \mathfrak{R}^s_2 \cup \mathfrak{R}} \mathbb{E}_\xi \big[H_{\mathbf{-1}} \wedge t\theta_\beta \big] \;=\; 0 \,.$$

Proof. Consider a configuration $\sigma \in \mathfrak{R}$. Applying Lemma 2.5.5 repeatedly yields that $\mathbb{P}_\sigma[H_{\mathfrak{B}^+} < H_{\mathbf{-1}}] \to 0$. We may therefore include the indicator of the set $\{H_{\mathbf{-1}} < H_{\mathfrak{B}^+}\}$ inside the expectation appearing in the statement of the lemma. After this inclusion, we may replace $H_{\mathbf{-1}}$ by $H_{\mathfrak{B}^+\cup\{\mathbf{-1}\}}$. At this point, it remains to estimate $(1/\theta_\beta)\,\mathbb{E}_\sigma[H_{\mathfrak{B}^+\cup\{\mathbf{-1}\}}]$. Since the process is stopped as it reaches $\mathfrak{B}^+$, we may replace σ_t by the reflected process at $\mathcal{V}_{\mathbf{-1}}$. After this replacement, bound $H_{\mathfrak{B}^+\cup\{\mathbf{-1}\}}$ by $H_{\mathbf{-1}}$.

We need therefore to estimate $(1/\theta_\beta)\,\mathbb{E}^{\mathcal{V}}_\sigma[H_{\mathbf{-1}}]$. By [2, Proposition 6.10] and by definition of θ_β, this expression is bounded by

$$\frac{\mathrm{cap}(\sigma^*, \mathbf{-1})}{\mu_\beta(\mathbf{-1})}\,\frac{1}{\mathrm{cap}_{\mathcal{V}}(\sigma, \mathbf{-1})} \;=\; \frac{\mathrm{cap}_{\mathcal{V}}(\sigma^*, \mathbf{-1})}{\mu_{\mathcal{V}}(\mathbf{-1})}\,\frac{1}{\mathrm{cap}_{\mathcal{V}}(\sigma, \mathbf{-1})}\;,$$

where σ^* is a configuration in $\mathfrak{R}^a$. By Lemma 2.8.2, $\mu_{\mathcal{V}}(\mathbf{-1}) \to 1$, while by the proofs of Lemma 2.4.2 and 2.4.3,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}_{\mathcal{V}}(\sigma^*, \mathbf{-1})}{\mathrm{cap}_{\mathcal{V}}(\sigma, \mathbf{-1})} \;\le\; C_0 \lim_{\beta\to\infty} \frac{\mu_{\mathcal{V}}(\sigma^*)}{\mu_{\mathcal{V}}(\sigma)} \;=\; 0\;.$$

Consider now a configuration $\sigma \in \mathfrak{R}^s_2$. Denote by A the $[n_0 \times (n_0+2)]$-rectangle obtained from the set of 0-spins of σ by completing the line or the row where the two extra spins sit. Denote by ξ the configuration where each site in A has a 0-spin, all the other ones being -1.

As in the proof of Lemma 2.5.5, we define the valley $\mathcal{V}_\xi$ of ξ in two stages. Fix $0 \le k \le n_0$. We first flip sequentially k spins of A from 0 to -1. At each step we only flip a 0-spin if it is surrounded by two -1-spins. The set of all configurations obtained by such a sequence of k flips is represented by $\mathcal{G}_k$. In particular, since at the beginning we may only flip the corners of A, $\mathcal{G}_1$ is composed of the four configurations obtained by flipping to -1 one corner of A. Let $\mathcal{G} = \cup_{0\le k<n_0}\mathcal{G}_k$, and note that $\mathcal{G}_{n_0}$ has not been included in the union.

The second stage in the construction of the valley $\mathcal{V}_\xi$ consists in flipping a spin of a configuration in $\mathcal{G}$. More precisely, denote by $\mathcal{B}$ all configurations which are not in $\mathcal{G}$, but which can be obtained from a configuration in $\mathcal{G}$ by flipping one spin. The set $\mathcal{B}$ is interpreted as the boundary of the valley $\mathcal{V}_\xi := \mathcal{G} \cup \mathcal{B}$ and it contains $\mathcal{G}_{n_0}$.

Note that σ belongs to $\mathcal{G}$ and that starting from σ the process hits $\mathcal{B}$ before $\mathbf{-1}$, so that $H_{\mathbf{-1}} = H_{\mathcal{B}} + H_{\mathbf{-1}} \circ H_{\mathcal{B}}$. Since $(a+b)\wedge t \le a + (b\wedge t)$, a, $b > 0$,

$$\mathbb{E}_\sigma\big[H_{\mathbf{-1}} \wedge t\theta_\beta\big] \;\le\; \mathbb{E}_\sigma\big[H_{\mathcal{B}}\big] \;+\; \mathbb{E}_\sigma\big[\,(H_{\mathbf{-1}}\circ H_{\mathcal{B}}) \wedge t\theta_\beta\big]\;.$$

Replacing σ_t by the process reflected at $\mathcal{V}_\xi$, applying [2, Proposition 6.10], and estimating the capacities yield that the first term divided by θ_β converges to 0 as $\beta \to \infty$.

We turn to the second term. We may insert the indicator function of the set $\{H_{\mathcal{B}} = H_{\{\eta^{(1)},\eta^{(2)}\}}\}$, where $\eta^{(1)}$, $\eta^{(2)}$ are the configurations obtained from ξ by flipping to -1 a line or a row of length n_0 of the rectangle A. After this insertion, the strong Markov property yields that the second term of the previous displayed equation is bounded by

$$\mathbb{E}_\sigma\Big[\,\mathbf{1}\{H_{\mathcal{B}} = H_{\{\eta^{(1)},\eta^{(2)}\}}\}\,\mathbb{E}_{\sigma(H_{\mathcal{B}})}\big[H_{\mathbf{-1}} \wedge t\theta_\beta\,\big]\,\Big]\;.$$

Since the configurations $\eta^{(1)}$, $\eta^{(2)}$ belong to $\mathfrak{R}$, the result follows from the first part of the proof. □

Lemma 2.8.5. *For every $t > 0$,*

$$\limsup_{\beta\to\infty} \frac{1}{\theta_\beta}\, \mathbb{E}_{\mathbf{-1}}\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big] \;\le\; 3\, \limsup_{\beta\to\infty} \max_{\xi\in\mathfrak{R}^l_2} \frac{1}{\theta_\beta}\, \mathbb{E}_\xi\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big]\, .$$

Proof. By Assertion 2.8.3,

$$\lim_{\beta\to\infty} \frac{1}{\theta_\beta}\, \mathbb{E}_{\mathbf{-1}}\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\, \mathbf{1}\big\{H_{\mathfrak{B}^+} \ge t\theta_\beta\big\}\Big] \;=\; 0\, .$$

We may therefore insert the indicator of the set $\{H_{\mathfrak{B}^+} < t\theta_\beta\}$ inside the expectation appearing in the statement of the assertion at a negligible cost. By Proposition 2.4.1, we may also insert the indicator of the set $\{H_{\mathfrak{B}^+} = H_{\mathfrak{R}^a}\}$. After inserting these indicator functions, by the strong Markov property, we get that the expectation appearing in the statement of the lemma is bounded by

$$\frac{1}{\theta_\beta}\, \mathbb{E}_{\mathbf{-1}}\Big[\mathbf{1}\big\{H_{\mathfrak{R}^a} = H_{\mathfrak{B}^+} \le t\theta_\beta\big\}\, \mathbb{E}_{\sigma(H_{\mathfrak{B}^+})}\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big]\Big] \;+\; R_\beta\, ,$$

where $R_\beta \to 0$.

The previous expectation is bounded by

$$\frac{1}{\theta_\beta} \sum_{\sigma\in\mathfrak{R}^a} \mathbb{P}_{\mathbf{-1}}\big[H_\sigma = H_{\mathfrak{B}^+}\big]\, \mathbb{E}_\sigma\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big]\, .$$

Since $\lambda_\beta(\sigma) \ge 1$ for $\sigma\in\mathfrak{R}^a$ and since $\mathfrak{R}^a \subset \Delta$, by removing from the integral the interval $[0,\tau_1]$, where τ_1 represents the time of the first jump, the previous expectation is less than or equal to

$$\frac{1}{\theta_\beta} \sum_{\sigma\in\mathfrak{R}^a} \sum_{\xi} \mathbb{P}_{\mathbf{-1}}\big[H_\sigma = H_{\mathfrak{B}^+}\big]\, p_\beta(\sigma,\xi)\, \mathbb{E}_\xi\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big] \;+\; \frac{1}{\theta_\beta}\, \cdot$$

By Assertions 2.5.2 and 2.5.3, we may disregard all configurations ξ which do not have two contiguous attached 0-spins and which do not belong to $\mathfrak{R}$. The previous expression is thus bounded above by

$$\max_{\xi\in\mathfrak{R}^l_2} \frac{1}{\theta_\beta}\, \mathbb{E}_\xi\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big] + \frac{2}{3}\, \frac{1}{\theta_\beta} \max_{\xi\in\mathfrak{R}^s_2\cup\mathfrak{R}} \mathbb{E}_\xi\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big] + R_\beta\, ,$$

where $\mathfrak{R}^s_2$ represents the set of configurations with two contiguous 0-spins attached to the shortest side of the rectangle, and $R_\beta \to 0$.

The second expectations is bounded by

$$\frac{1}{\theta_\beta} \max_{\xi\in\mathfrak{R}^s_2\cup\mathfrak{R}} \mathbb{E}_\xi\big[H_{\mathbf{-1}} \wedge t\theta_\beta\big] \;+\; \frac{2}{3}\, \frac{1}{\theta_\beta}\, \mathbb{E}_{\mathbf{-1}}\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big]\, .$$

By Lemma 2.8.4, the first term vanishes as $\beta\to\infty$. The second one can be absorbed in the left-hand side of the expression appearing in the statement of the lemma, which completes the proof of the lemma. □

Lemma 2.8.6. *For every $t > 0$,*

$$\limsup_{\beta\to\infty} \max_{\xi\in\mathfrak{R}_2^l} \frac{1}{\theta_\beta} \mathbb{E}_\xi\big[H_{\mathbf{0}} \wedge t\theta_\beta\big] \;=\; 0\;.$$

Proof. The proof of this lemma follows the steps of Section 2.5 where we described the growth of the supercritical droplet. Fix a configuration ξ in $\mathfrak{R}_2^l$ and recall Lemma 2.5.4. By this result,

$$\frac{1}{\theta_\beta} \mathbb{E}_\xi\big[\, H_{\mathbf{0}} \wedge t\theta_\beta\,\big] \;\le\; \frac{1}{\theta_\beta} \mathbb{E}_\xi\big[\, \mathbf{1}\{E_{c,d} = H_{c,d}\}\,(H_{\mathbf{0}} \wedge t\theta_\beta)\,\big] \;+\; t\,\delta_1(\beta)\;.$$

Since $E_{c,d} < H_{\mathbf{0}}$, we may write $H_{\mathbf{0}}$ as $E_{c,d} + H_{\mathbf{0}} \circ E_{c,d}$ and bound $H_{\mathbf{0}} \wedge t\theta_\beta$ by $E_{c,d} + [(H_{\mathbf{0}} \circ E_{c,d}) \wedge t\theta_\beta]$. Hence, by the strong Markov property, the previous expression is less than or equal to

$$t\,\delta_1(\beta) \;+\; \frac{1}{\theta_\beta} \mathbb{E}_\xi\big[E_{c,d}\big] \;+\; \frac{1}{\theta_\beta} \mathbb{E}_\xi\Big[\mathbf{1}\{E_{c,d} = H_{c,d}\}\, \mathbb{E}_{\sigma(E_{c,d})}\big[\, H_{\mathbf{0}} \wedge t\theta_\beta\big]\,\Big]\;.$$

On the set $\{E_{c,d} = H_{c,d}\}$, $\sigma(E_{c,d})$ is a configuration with $(n_0+1)^2$ 0-spins forming a square in a sea of -1-spins. The previous expression is thus bounded by

$$t\,\delta_1(\beta) \;+\; \frac{1}{\theta_\beta} \mathbb{E}_\xi\big[E_{c,d}\big] \;+\; \frac{1}{\theta_\beta}\, \max_{\eta} \mathbb{E}_\eta\big[\, H_{\mathbf{0}} \wedge t\theta_\beta\,\big]\;.$$

Compare the previous expression with the one on the statement of the lemma. We replaced the set $\mathfrak{R}_2^l$ by the set of (n_0+1) 0-squares at the cost of the error term $t\,\delta_1(\beta)$, and the expectation of the hitting time of the boundary of the neighborhood of ξ. Proceeding in this way until hitting $\mathbf{0}$ will bring the sum of all errors and the sum of the expectations of all hitting times. The assumptions of Theorem 2.2.2 where inserted to guarantee that the sum of the error terms converge to 0 as $\beta\to\infty$. The hitting times of the boundaries involve either the creation of two contiguous 0-spin at the boundary of a rectangle, whose order is $e^{(2-h)\beta}$, or the filling of a side of a rectangle, which corresponds to the hitting time of an asymmetric one-dimensional random walk, whose order is proportional to the length of the rectangle. Both orders are much smaller than θ_β, which completes the proof of the lemma. □

Lemma 2.8.7. *For every $t > 0$,*

$$\limsup_{\beta\to\infty} \frac{1}{\theta_\beta} \mathbb{E}_{+\mathbf{1}}\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\sigma(s)\in\Delta\}\, ds\Big] \;=\; 0\;.$$

Proof. By (2.42), we may insert the indicator of the set $\{H_{\mathfrak{B}^+_{+\mathbf{1}}} \ge t\,\theta_\beta\}$ inside the expectation. After this insertion, we may replace the process σ_t by the reflected process at $\mathcal{V}_{+\mathbf{1}}$, denoted by η_t, and then bounded the expression by

$$\frac{1}{\theta_\beta} \mathbb{E}^{\mathcal{V}}_{+\mathbf{1}}\Big[\int_0^{t\theta_\beta} \mathbf{1}\{\eta(s)\in\Delta\}\, ds\Big]\;.$$

This term is equal to

$$\frac{1}{\theta_\beta} \int_0^{t\theta_\beta} \mathbb{P}^{\mathcal{V}}_{+\mathbf{1}}\big[\eta(s)\in\Delta\big]\, ds \;\le\; \frac{1}{\theta_\beta} \int_0^{t\theta_\beta} \frac{1}{\mu_{\mathcal{V}}(+\mathbf{1})} \sum_{\xi\in\mathcal{V}_{+\mathbf{1}}} \mu_{\mathcal{V}}(\xi)\, \mathbb{P}^{\mathcal{V}}_\xi\big[\eta(s)\in\Delta\big]\, ds\;.$$

As $\mu_{\mathcal{V}}$ is the stationary state for the reflected process, this expression is equal to

$$t\,\frac{\mu_{\mathcal{V}}(\Delta)}{\mu_{\mathcal{V}}(+\mathbf{1})} \;=\; t\,\frac{\mu_\beta(\mathcal{V}_{+\mathbf{1}}\setminus\{+\mathbf{1}\})}{\mu_\beta(+\mathbf{1})}\;.$$

By Lemma 2.8.2, for $\mathcal{V}_{+\mathbf{1}}$ instead of $\mathcal{V}_{\text{-}\mathbf{1}}$, this expression vanishes as $\beta\to\infty$. □

Proof.[Proof of Proposition 2.8.1] Lemmata 2.8.5 and 2.8.6 show that the time spent on Δ until the process reaches $\mathbf{0}$ is negligible. We may repeat the same argument to extend the result up to the time where the process reaches $+\mathbf{1}$. Once the process reaches $+\mathbf{1}$, we apply Lemma 2.8.7. □

2.9 Proof of Theorem 2.2.3

Unless otherwise stated, we assume throughout this section that the hypotheses of Theorem 2.2.3 are in force. According to [20, Proposition 2.1], we have to show that for all $\eta\in\mathcal{M}$,

$$\lim_{\delta\to 0}\limsup_{\beta\to\infty}\sup_{2\delta\le s\le 3\delta}\mathbb{P}_\eta[\sigma(s\theta_\beta)\in\Delta] \;=\; 0\;. \tag{2.44}$$

We present the proof for $\eta=\text{-}\mathbf{1}$. The proof for $\eta=\mathbf{0}$ is identical. The one for $\eta=+\mathbf{1}$ is even simpler because the valley is deeper.

Lemma 2.9.1. *Under $\mathbb{P}_{\text{-}\mathbf{1}}$, the random variable $3\,H_{\mathfrak{B}^+}/\theta_\beta$ converges to a mean one exponential random variable.*

Proof. As we are interested in $H_{\mathfrak{B}^+}$, we may replace the process σ_t by the reflected process at $\mathcal{V}_{\text{-}\mathbf{1}}$, denoted by η_t.

The proof is based on [2, Theorem 2.6] applied to the triple $(\{\text{-}\mathbf{1}\},\mathcal{V}_{\text{-}\mathbf{1}}\setminus\mathfrak{B}^+,\text{-}\mathbf{1})$. We claim that the process η_t fulfills all the hypotheses of this theorem. Condition (2.14) is satisfied because the set $\{\text{-}\mathbf{1}\}$ is a singleton, and condition (2.15) is in force in view of Lemma 2.8.2. Therefore, by this theorem, the triple $(\{\text{-}\mathbf{1}\},\mathcal{V}_{\text{-}\mathbf{1}}\setminus\mathfrak{B}^+,\text{-}\mathbf{1})$ is a valley of depth

$$\frac{\mu_{\mathcal{V}}(\text{-}\mathbf{1})}{\mathrm{cap}_{\mathcal{V}}(\text{-}\mathbf{1},\mathfrak{B}^+)} \;=\; \frac{\mu_\beta(\text{-}\mathbf{1})}{\mathrm{cap}(\text{-}\mathbf{1},\mathfrak{B}^+)} \;=\; \frac{\mathrm{cap}(\text{-}\mathbf{1},\{\mathbf{0},+\mathbf{1}\})}{\mathrm{cap}(\text{-}\mathbf{1},\mathfrak{B}^+)}\,\theta_\beta\;,$$

where the last identity follows from the definition of θ_β given in (2.7). By (2.36) and Proposition 2.7.3, the last ratio converges to $1/3$. Hence, by property (V2) of [2, Definition 2.1], $3\,H_{\mathfrak{B}^+}/\theta_\beta$ converges to a mean-one exponential random variable, as claimed. □

By the previous lemma, for $2\delta\le s\le 3\delta$,

$$\mathbb{P}_{\text{-}\mathbf{1}}\big[\,\sigma(s\theta_\beta)\in\Delta\big] \;=\; \mathbb{P}_{\text{-}\mathbf{1}}\big[\,\sigma(s\theta_\beta)\in\Delta\,,\,H_{\mathfrak{B}^+}\ge 4\delta\theta_\beta\big] \;+\; R_{\beta,\delta}\;,$$

for some remainder $R_{\beta,\delta}$ which vanishes as $\beta\to\infty$ and then $\delta\to 0$. On the set $\{H_{\mathfrak{B}^+}\ge 4\delta\}$, we may replace the process σ_t by the reflected process η_t and bound the first term by

$$\mathbb{P}^{\mathcal{V}}_{\text{-}\mathbf{1}}\big[\,\eta(s\theta_\beta)\in\Delta\big] \;\le\; \frac{1}{\mu_{\mathcal{V}}(\text{-}\mathbf{1})}\sum_{\sigma\in\mathcal{V}_{\text{-}\mathbf{1}}}\mu_{\mathcal{V}}(\sigma)\,\mathbb{P}^{\mathcal{V}}_\sigma\big[\,\eta(s\theta_\beta)\in\Delta\big] \;=\; \frac{\mu_{\mathcal{V}}(\Delta)}{\mu_{\mathcal{V}}(\text{-}\mathbf{1})}$$

because $\mu_{\mathcal{V}}$ is the stationary state. By Lemma 2.8.2 this expression vanishes as $\beta\to\infty$, which proves (2.44) for $\eta=\text{-}\mathbf{1}$.

Bibliographie

[1] J. Barrera, O. Bertoncini, and R. Fernández. Abrupt convergence and escape behavior for birth and death chains. *J. Stat. Phys.*, 137(4) :595–623, 2009.

[2] J. Beltrán and C. Landim. Tunneling and metastability of continuous time Markov chains. *J. Stat. Phys.*, 140(6) :1065–1114, 2010.

[3] J. Beltrán and C. Landim. Metastability of reversible finite state Markov processes. *Stochastic Process. Appl.*, 121(8) :1633–1677, 2011.

[4] J. Beltrán and C. Landim. Tunneling and metastability of continuous time Markov chains II, the nonreversible case. *J. Stat. Phys.*, 149(4) :598–618, 2012.

[5] J. Beltrán and C. Landim. A martingale approach to metastability. *Probab. Theory Related Fields*, 161(1-2) :267–307, 2015.

[6] Alessandra Bianchi and Alexandre Gaudillière. Metastable states, quasi-stationary distributions and soft measures. *Stochastic Process. Appl.*, 126(6) :1622–1680, 2016.

[7] Anton Bovier and Frank den Hollander. *Metastability*, volume 351 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Cham, 2015. A potential-theoretic approach.

[8] Anton Bovier, Michael Eckhoff, Véronique Gayrard, and Markus Klein. Metastability in stochastic dynamics of disordered mean-field models. *Probab. Theory Related Fields*, 119(1) :99–161, 2001.

[9] Anton Bovier, Michael Eckhoff, Véronique Gayrard, and Markus Klein. Metastability and low lying spectra in reversible Markov chains. *Comm. Math. Phys.*, 228(2) :219–255, 2002.

[10] Marzio Cassandro, Antonio Galves, Enzo Olivieri, and Maria Eulália Vares. Metastable behavior of stochastic dynamics : a pathwise approach. *J. Statist. Phys.*, 35(5-6) :603–634, 1984.

[11] E. N. M. Cirillo, F. R. Nardi, and C. Spitoni. Sum of exit times in series of metastable states in probabilistic cellular automata. In *Cellular automata and discrete complex systems*, volume 9664 of *Lecture Notes in Comput. Sci.*, pages 105–119. Springer, [Cham], 2016.

[12] Emilio N. M. Cirillo and Francesca R. Nardi. Relaxation height in energy landscapes : an application to multiple metastable states. *J. Stat. Phys.*, 150(6) :1080–1114, 2013.

[13] Emilio N. M. Cirillo and Enzo Olivieri. Metastability and nucleation for the Blume-Capel model. Different mechanisms of transition. *J. Statist. Phys.*, 83(3-4) :473–554, 1996.

[14] Weinan E and Eric Vanden-Eijnden. Towards a theory of transition paths. *J. Stat. Phys.*, 123(3) :503–523, 2006.

[15] R. Fernandez, F. Manzo, F. R. Nardi, E. Scoppola, and J. Sohier. Conditioned, quasi-stationary, restricted measures and escape from metastable states. *Ann. Appl. Probab.*, 26(2) :760–793, 2016.

[16] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1998. Translated from the 1979 Russian original by Joseph Szücs.

[17] A. Gaudillière and C. Landim. A Dirichlet principle for non reversible Markov chains and some recurrence theorems. *Probab. Theory Related Fields*, 158(1-2) :55–89, 2014.

[18] C. Landim. A topology for limits of Markov chains. *Stochastic Process. Appl.*, 125(3) :1058–1088, 2015.

[19] C. Landim and P. Lemire. Metastability of the two-dimensional Blume-Capel model with zero chemical potential and small magnetic field. *J. Stat. Phys.*, 164(2) :346–376, 2016.

[20] C. Landim, M. Loulakis, and M. Mourragui. Metastable Markov chains : from the convergence of the trace to the convergence of the finite-dimensional distributions. *ArXiv e-prints*, March 2017.

[21] Claudio Landim and Tiecheng Xu. Metastability of finite state Markov chains : a recursive procedure to identify slow variables for model reduction. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(2) :725–751, 2016.

[22] F. Manzo and E. Olivieri. Dynamical Blume-Capel model : competing metastable states at infinite volume. *J. Statist. Phys.*, 104(5-6) :1029–1090, 2001.

[23] Philipp Metzner, Christof Schütte, and Eric Vanden-Eijnden. Transition path theory for Markov jump processes. *Multiscale Model. Simul.*, 7(3) :1192–1219, 2008.

[24] Enzo Olivieri and Maria Eulália Vares. *Large deviations and metastability*, volume 100 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[25] O. Penrose and J. L. Lebowitz. Rigorous treatment of metastable states in the van der Waals-Maxwell theory. *J. Statist. Phys.*, 3 :211–236, 1971.