On the existence of fractional brownian fields indexed by manifolds

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THÈSE

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Sur l'existence de champs browniens fractionnaires indexés par des variétés

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Pour réclamer un chat

Chez vous est né un chat à la tête de tigre,
Vous me l'avez promis, ne changez pas d'idée.
Ma demeure est trop froide pour les souris voleuses,
Je ne veux que le voir grimper en haut des arbres.

Xutang Zhiyu (1185-1269)

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Introduction

Subject of the thesis

In this thesis we study the existence of fractional Brownian fields indexed by metric spaces, focusing on the case of Riemannian manifolds. Our starting point is that an H-fractional Brownian field indexed by a metric space exists if and only if the distance to the power $2H$ is a kernel of negative type. We use elementary tools provided by the Riemannian framework to investigate the property of negative type for the power of the geodesic distance.

Motivation

The motivation to study fractional Brownian fields comes from the fact they generalise the fractional Brownian motion. The fractional Brownian motion is a collection of random variables indexed by the real line with features that make it a successful model in an important number of applied contexts such as Geography of coasts, Finance, Geology, Hydrology, Imagery, Signal Analysis, Telecommunications... A fractional Brownian *field* is a a collection of random variables indexed by a metric space, and a generalisation of the fractional Brownian motion. As such, fractional Brownian fields inherits important properties:

- The fractional Brownian motion is a **Gaussian process**, which means every linear combination of its marginals is a Gaussian random variable. The same is true for fractional Brownian field. Gaussian random variables play a central role in Probability and Statistics, because they appear as invariant in many situations (one can think of Gaussian variable as the solutions of the heat equation and as the limit law in the central limit theorem, for example), and also because they are in many ways easier to use than other distributions with a density.
- **Stationarity of increments** of the fractional Brownian motion $(B_t^H)_{t\in\mathbb{R}}$ means that the statistical properties of the increment $B_{t_2}^H - B_{t_1}^H$ depends only

on the elapsed time $t_2 - t_1$. Formally, for every real number s the random process

$$\left(B^H_{t_2+s} - B^H_{t_1+s}\right)_{t_2\in\mathbb{R}}$$

have the same finite-dimensional distributions that the random process

$$\left(B^H_{t_2} - B^H_{t_1}\right)_{t_2\in\mathbb{R}}.$$

This property persists for the fractional Brownian motion, for which the statistical properties of $B^H_{g(t_2)} - B^H_{g(t_1)}$ are the same for every isometry g of the index metric space.

- **Auto-similarity of the trajectories** for the fractional Brownian motion means that for every positive $\lambda$the statistical properties of the process $\left(B^H_{\lambda t}\right)_{t\in\mathbb{R}}$ are similar to those of B_t. In this case a scaling occurs and

$$\left(B^H_{\lambda t}\right)_{t\in\mathbb{R}}$$

have the same finite-dimensional distribution that

$$(\lambda^H B^H_t)_{t\in\mathbb{R}}.$$

In a general metric space there is no counterpart for the homothety map $t \mapsto \lambda t$ of the real line. However in a Riemannian manifold it is possible to define analogues of this map in a small enough neighbourhood of every point, and the property stays locally true for fractional Brownian fields indexed by Riemannian manifolds (we refer to the article [12] of Istas for an explicit statement).

Those three properties play a significant part in the success of the fractional Brownian motion. In particular auto-similarity appears in numerous natural phenomena which exhibit fractal regularity. Fractional Brownian motion indexed by Euclidean spaces have been studied in applications for geology and image analysis, but fractional Brownian fields indexed by general metric spaces allow to consider problems in which data is naturally indexed by spaces as different from each other as the sphere and a graph, for instance.

Let us mention that taking $H = 1/2$ gives the special case of Lévy Brownian fields, which generalise the Brownian motion.

Context: The existence problem

For a positive H and a metric space (E, d), H-fractional Brownian fields indexed by (E, d) do not always exist. Moreover there is no general method to check if they do. Since they are Gaussian random fields, they exist if and only if their covariance function is a kernel of positive type. In the case of fractional Brownian fields, this is equivalent to the fact that d^{2H} is a kernel of negative type.

In the case $H = 1/2$ this question goes back to Lévy which defines and show existence of Brownian fields indexed by the Euclidean space in [16]. In [3] Chentsov uses Gaussian white noise indexed by the sets of affine hyperplanes to give an explicit construction of Euclidean Lévy Brownian fields. Lévy adapts the method to construct the Lévy Brownian field indexed by the sphere $\mathbb{S}^2$ in [15]. In [23] Morozova and Chentsov discuss the generalisation of this construction to simply connected surfaces of nonpositive curvature (Cartan-Hadamard surfaces), which we detail in Chapter 2. When the space is very regular harmonic analysis is available and gives a characterisation of negative type kernels through Lévy-Kinchine formula (see [7]): Gangolli [8] and Molchan [21, 22, 20] investigate the existence of the Lévy Brownian fields indexed by symmetric spaces and in particular show existence in the cases of the spheres $\mathbb{S}^d$ and the hyperbolic spaces $\mathbb{H}^d$ and nonexistence in the case of the real and complex projective spaces. In the case of normed vector spaces let us mention the work of Lifshits [18] who generalises Chentsov representation, and Bretagnolle, Dacunha Castelle, Krivine [1] who relate the negative type property to isometric embedding in Hilbert spaces. See also Faraut and Harzallah [7] for the same point of view on geodesic distances on Riemannian manifolds.

Existence of H-fractional Brownian fields indexed by Euclidean spaces has been well known for some time (see for example Mandelbrot's work in [19]). In 2011 Istas defines the H-fractional Brownian fields indexed by metric spaces, together with the fractional index of a metric space (see [12]). He surveys and completes in his article the existing results, and gives new ones.

Results of the thesis

Recall that an H-fractional Brownian field indexed by a metric space (E, d) exists if and only if the kernel d^{2H} is of *negative type*, that is for every points $P_1, \cdots, P_n \in M$ and real coefficients $c_1, \cdots, c_n \in \mathbb{R}$ such that $\sum_{i=1}^n c_i = 0$,

$$\sum_{i,j=1}^{n} c_i c_j d^{2H}(P_i, P_j) \leq 0. \tag{1}$$

Every result we give is based on this criterion, and is about existence of fractional Brownian fields indexed by a Riemannian manifold M endowed with its geodesic distance d_M.

Perturbation of critical configurations In Chapter 3 we consider H-critical configurations, that is to say points $P_1, \cdots, P_n$ and coefficients $c_1, \cdots, c_n$

such that $\sum_{i=1}^{n} c_i = 0$ and

$$\sum_{i,j=1}^{n} c_i c_j d^{2H}(P_i, P_j) = 0. \tag{2}$$

The idea is to find a criterion to allow for a small perturbation of $(P_1, \cdots, P_n)$ and $(c_1, \cdots, c_n)$ such that (2) becomes positive, in which case an H-fractional Brownian field cannot exist (see (1)). We find that the existence of a curve orthogonal at some P_{i_0} to every minimal geodesic from P_{i_0} to the others points P_i is sufficient to have such a perturbation. We deduce that in order to have existence of a fractional Brownian field the minimal geodesics between points $P_1, \cdots, P_n$ of every critical configurations must span the whole tangent space with their speeds at every P_i. This necessary condition is given in Theorem 3.1.

- **Lévy Brownian field indexed by manifolds with closed minimal geodesics** On a circle every pair of antipodal points give an 1/2-critical configuration of four points. This stays true on a minimal closed geodesic of a Riemannian manifold. For the Lévy Brownian field (corresponding to $H = 1/2$) to exist Theorem 3.1 gives a necessary condition on the minimal geodesics between every such four points. By taking pairs of antipodal points as close to each other as needed we see that the same condition must hold for two antipodal points, which is a stronger result given in Theorem 3.2. We apply this criterion to show there exists no Lévy Brownian field indexed by a manifold with a loop of minimal length among loops which are not in the constant homotopy class (Theorem 3.3). In particular we show that there exists no Lévy Brownian field indexed by a nonsimply connected compact Riemannian manifold (Theorem 3.4). Let us notice that this result does not depend on the choice of Riemannian metric we make, but only on the topology of the underlying differentiable manifold.
- **Nondegeneracy of fractional Brownian fields indexed by the hyperbolic space** We use the existence of an H-fractional Brownian field X^H indexed by the d-dimensional real hyperbolic space $\mathbb{H}^d$ for every $0 < H \leq 1/2$ and every $d \in \mathbb{N}^*$ to show that there exists no critical configurations for those fields (otherwise Theorem 3.1 would apply and give nonexistence). This is equivalent to say there exists no points $P_1, \cdots, P_n$ and coefficients $c_1, \cdots, c_n$ such that $\sum_{i=1}^{n} c_i = 0$ and that the linear combination

$$\sum_{i=1}^{n} c_i X_{P_i}^H = 0 \text{ a.s.} \tag{3}$$

In the case of a fractional Brownian field with an origin $O \in \mathbb{H}^d$ such that $X_O^H = 0$ almost surely we are able to remove the condition $\sum_{i=1}^{n} c_i = 0$, which shows that the field is nondegenerate, that is every finite dimensional Gaussian vector $(X^H_{P_1}, \cdots, X^H_{P_n})$ admits a density with respect to the n-dimensional Lebesgue measure. The same argument applies to H-fractional Brownian fields indexed by $\mathbb{R}^d$ for every $0 < H \leq 1$. In this case the result was already known but the previous proof we are aware of is significantly longer (see for example [4]).

Nonexistence of fractional Brownian fields indexed by cylinders We show in Chapter 4 that for every positive H there exists no H-fractional Brownian motion indexed by the cylinder. We do so by exhibiting points $(P^H_{i,n})$ and coefficients (c_i) for every $1 \leq i \leq n$ such that $\sum_{i=1}^n c_i$ is always zero and

$$\lim_{n\to\infty} \sum_{i,j=1}^{n} c_i c_j d^{2H}(P^H_{i,n}, P^H_{j,n}) = +\infty, \tag{4}$$

which prevents (1) for n large enough. We start by investigating a collection of points on the circle, which we afterwards duplicate on two horizontal circles of the cylinder. Finally we consider the same collection of points on a number of circles depending on n. The behaviour of $\lim\limits_{n\to\infty} \sum_{i,j=1}^n c_i c_j d^{2H}(P^H_{i,n}, P^H_{j,n})$ when n goes to infinity is governed by the asymptotic regime of the distance z_n between two consecutive circles, which should be chosen carefully in order to obtain the desired divergence towards infinity. This adequate regime depends on H. Furthermore z_n tends towards zero when n goes to infinity, which means that all the points we consider converge towards points on the circle at height $z = 0$ when n goes to infinity. This allows the result to stay true for any cylinder $\mathbb{S}^1\times]0, \varepsilon[$. Let us notice that this example exhibits an obstruction to the existence of fractional Brownian motion which depends only on global features of the index space: indeed the cylinder and the Euclidean plane share the same local flat metric, but as one enjoys H-fractional Brownian fields for every $0 < H \leq 1$, the other admits none. As far as we know the only other examples of spaces with fractional exponent 0 are $\mathbb{R}^d$ endowed with the norms $\left(\sum_{i=1}^d |x_i|^q\right)^{1/q}$ for $d \geq 3, q > 2$ (see Koldobskiĭ [14]) and the quaternionic hyperbolic space endowed with its geodesic distance (see Faraut [7]).

- **Generalisation to Riemannian products** We show that the result holds for any Riemannian product $M \times N$ as long as there exists a minimal closed geodesic in the Riemannian manifold M (see Theorem 4.2).

- **A weak version of the result for surfaces close to the cylinder** On the set $\mathbb{S}^1\times]0, \varepsilon[$ we investigate the case of distances d' which converge to the classical product distance on the cylinder when $z \in]0,\varepsilon[$ tends towards zero. The idea is to consider the same points $(P_{i,n}^H)$ and to obtain again (4) for the other distance. The range of H for which the technique works depends on some rate δ of uniform convergence of d' towards the product distance: Theorem 4.3 gives a bound of the fractional index which depends on δ. In Theorem 4.4 we look at the case of a Riemannian manifold of dimension 2 in some chart $\mathbb{S}^1 \times]0,\varepsilon[$. We give examples among surfaces of revolution with increasing generating function, in particular in a case where the generating function which is flat at zero we recover nonexistence of every H-fractional Brownian motion for H positive.

- **Gromov-Hausdorff discontinuity of the fractional index** We notice that the fractional index of cylinders $\mathbb{S}^1 \times [0,\varepsilon]$ is zero, while the fractional index of the circle $\mathbb{S}^1 \times \{0\}$ is 1. We check that the metric space $\mathbb{S}^1 \times [0,\varepsilon]$ Gromov-Hausdorff converge towards $\mathbb{S}^1 \times \{0\}$ when ε tends towards zero, hence the fractional index is not continuous regarding Gromov-Hausdorff topology on the set of isometry classes of compact metric spaces (see Theorem 4.5).

Organisation of the thesis

Let us briefly describe the contents of the different chapters.

In Chapter 1 we provide the general definitions on random fields and Riemannian geometry we need, in an attempt to make this document as self-contained as possible. We also attempt to clarify the possible definition for complex-valued fractional Brownian motions.

In Chapter 2 we give some details on a argument by Morozova and Chentsov which proves the existence of the Lévy Brownian field indexed by simply connected surfaces with nonpositive curvature (Cartan-Hadamard surfaces).

In Chapter 3 we give a necessary condition for the existence of fractional Brownian fields indexed by Riemannian manifolds (Theorem 3.1). We investigate this necessary condition for Riemannian manifolds with closed geodesics (Theorem 3.2), from which we derive nonexistence of Lévy Brownian field indexed by manifolds with a loop of minimal length among loops which are not in the constant homotopy class (Theorem 3.3). In particular we show that there exists no Lévy Brownian field indexed by nonsimply connected compact manifolds (Theorem 3.4). Furthermore we derive from Theorem 3.1 the nondegeneracy of fractional Brownian fields indexed by the hyperbolic spaces (Theorem 3.5).

In Chapter 4 we show that for every H there exists no H-fractional Brownian motion indexed by the cylinder (Theorem 4.1). We generalise this result to the Riemannian product of the circle with any manifold (Theorem 4.2). We investigate the case of metric spaces for which the distance is asymptotically close to the cylinder distance (Theorem 4.3 and 4.4). From Theorem 4.1 we derive the discontinuity of the fractional index of a metric space with respect to the Gromov-Hausdorff convergence (Theorem 4.5).

Contents

Notations

$\mathbb{N}$	the set of nonnegative integers
$\mathbb{N}^*$	the set of positive integers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^*$	the set of nonzero real numbers
$\mathbb{C}$	the set of complex numbers
$\mathrm{Re}(z)$	the real part of the complex number z
$\mathrm{Im}(z)$	the imaginary part of the complex number z
a.s.	almost surely
$\mathbb{E}(X)$	the expectation of the random variable X
$\mathcal{N}(v, M)$	a Gaussian random vector with mean vector v and covariance matrix M
u^T	the transpose of the column vector $u \in \mathbb{R}^n$
$\mathbb{1}_A$	the indicator function of the set A

Chapter 1

Generalities

We give in this chapter some general definitions and results we need. All the material we cover here is standard, with the exception of Section 1.2.3 and Section 1.3.2 where we discuss the definition of complex-valued fractional fields.

1.1 Kernels of positive and negative type

Let us start with some definitions about kernels.

In all that follows, S is a set and K is a *kernel* on S, that is to say a map

$$K : S \times S \to \mathbb{R}$$

which is *symmetric*: for every $x, y \in S$,

$$K(x, y) = K(y, x).$$

Definition 1.1 (Kernel of positive type)**.** A kernel K on S is *of positive type* if for every $x_1, \cdots, x_n \in S$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$,

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j K(x_i, x_j) \geq 0.$$

Definition 1.2 (Kernel of negative type)**.** A kernel K on S is *of negative type* if

- for every $x \in S$, $K(x, x) = 0$ and
- for every $x_1, \cdots, x_n \in S$ and $c_1, \cdots, c_n \in \mathbb{R}$ such that $\displaystyle\sum_{i=1}^{n} c_i = 0$,

$$\sum_{i,j=1}^{n} c_i c_j K(x_i, x_j) \leq 0.$$

Remark 1.1. Some authors prefer to call kernels of positive type *positive definite kernels*, or *nonnegative definite kernels*. In a same way kernels of negative type are also called *negative definite kernels*, or *kernels conditionally of negative type*. Several definitions can be found, in particular a continuity assumption is sometimes added, and most of the time those definitions are given for complex-valued kernels.

Theorem 1.1. *Let K and R be kernels on S and $O \in S$ such that for every $x \in S$ $K(x,x) = 0$, and for every $x, y \in S$,*

$$R(x,y) = K(O,x) + K(O,y) - K(x,y).$$

Then R is of positive type if and only if K is of negative type.

Proof.

- It is clear that R of positive type implies K of negative type. Indeed for every $x_1, \cdots, x_n \in S$ and $c_1, \cdots, c_n \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} c_i = 0,$$

we have

$$\sum_{i,j=1}^{n} c_i c_j K(O, x_i) = \left(\sum_{j=1}^{n} c_j\right) \sum_{i=1}^{n} c_i K(O, x_i) = 0.$$

Hence

$$\sum_{i,j=1}^{n} c_i c_j K(x_i, x_j) = -\sum_{i,j=1}^{n} c_i c_j R(x_i, x_j).$$

- Conversely, for every $x_1, \cdots, x_n \in S$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, set $x_{n+1} = O$, and $\lambda_{n+1} = -\sum_{i=1}^{n} \lambda_i$. It is clear that $\sum_{i=1}^{n+1} \lambda_i = 0$. Furthermore

$$\begin{aligned}
\sum_{i,j=1}^{n+1} \lambda_i \lambda_j K(x_i, x_j) &= \sum_{i,j=1}^{n} \lambda_i \lambda_j K(x_i, x_j) + \sum_{i=1}^{n} \lambda_i \left(-\sum_{j=1}^{n} \lambda_j\right) K(x_i, O) \\
&\quad + \sum_{j=1}^{n} \left(-\sum_{i=1}^{n} \lambda_i\right) \lambda_j \ K(O, x_j) + \lambda_{n+1}^2 \underbrace{K(O,O)}_{=0} \\
&= -\sum_{i,j=1}^{n} \lambda_i \lambda_j \left(K(O, x_i) + K(O, x_j) - K(x_i, x_j)\right) \\
&= -\sum_{i,j=1}^{n} \lambda_i \lambda_j R(x_i, x_j). \quad \square
\end{aligned}$$

1.2 Random fields

1.2.1 General random fields

We give in this section some definitions about random fields and Kolmogorov existence theorem.

All the random variables we consider are measurable functions from a single probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a measurable space. We consider only random fields with values in $\mathbb{R}$ or $\mathbb{C}$, which we will denote by $\mathbb{K}$. When we do not precise it a random variable is implicitly with values in $\mathbb{K}$ endowed with its Borelian σ-algebra $\mathcal{B}$.

Definition 1.3 (Random field)**.** A *random field* indexed by a set S is a collection of random variables $(X_x)_{x\in S}$.

Definition 1.4 (Product σ-algebra)**.** The *product σ-algebra* $\mathcal{P}$ on $\mathbb{K}^S$ is the σ-algebra generated by all the *cylinder sets*, which are the

$$C_{A_1,\cdots,A_n,x_1\cdots x_n} = \left\{ f \in \mathbb{K}^S \text{ such that } \forall 1 \leq i \leq n,\ f(x_i) \in A_i \right\}, \tag{1.1}$$

for every $n \in \mathbb{N}^*$, $A_1, \cdots, A_n \in \mathcal{B}$ and $x_1 \cdots, x_n \in S$.

We now give an alternative definition of a random field, easily checked to be equivalent to the first one.

Definition 1.5 (Random field, alternative definition)**.** A *random field* indexed by S is a random variable with values in $\left(\mathbb{K}^S, \mathcal{P}\right)$.

Definition 1.6 (Image probability)**.** Given a random variable

$$f : (\Omega, \mathcal{A}, \mathbb{P}) \to (\Omega', \mathcal{A}')$$

we denote by $f_*(\mathbb{P})$ the distribution of f, that is to say the image probability of $\mathbb{P}$ by f on $(\Omega', \mathcal{A}')$, defined by:

$$\forall B \in \mathcal{A}',\ f_*(B) := \mathbb{P}(f^{-1}(B)).$$

Definition 1.7 (Distribution of a random field)**.** The distribution μ_X of a random field $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \left(\mathbb{K}^S, \mathcal{P}\right)$ is (as usual for a random variable) the image probability $X_*(\mathbb{P})$ on $\left(\mathbb{K}^S, \mathcal{P}\right)$.

Definition 1.8 (Equality in distribution)**.** We say that two random fields X and Y indexed by S are equal in distribution if $\mu_X = \mu_Y$. In this case we write

$$(X_P)_{P\in S} \overset{(d)}{=} (Y_P)_{P\in S}.$$

Definition 1.9 (Finite dimensional distributions)**.** Given a finite subset $K = \{x_1, \cdots, x_n\}$ of S we call a finite dimensional distribution of a random field X the distribution of the random vector $(X_{x_1}, \cdots, X_{x_n})$.

The following result follows from the fact that the set of all cylinders of the form (1.1) is a π-system:

Proposition 1.1. *The distribution μ_X of X is entirely characterised by all its finite dimensional distributions.*

For all $J \subset K \subset S$ let us denote by π_J^K the natural projection from $\mathbb{K}^K$ to $\mathbb{K}^J$.

Definition 1.10 (Consistency)**.** Let us denote by $\mathrm{Fin}(S)$ the set of all finite subsets of S. A collection $(\mu_K)_{K\in\mathrm{Fin}(S)}$ of distributions on $\mathbb{K}^K$ is said to be *consistent* if

$$\forall\ J \subset K \in \mathrm{Fin}(S),\ \mu_J = {\pi_J^K}_* (\mu_K)\,. \tag{1.2}$$

We now give Kolmogorov extension theorem for the random fields we consider. A more general version of this result is proved in [27].

Theorem 1.2 (Kolmogorov existence theorem)**.** *For a given consistent collection of distributions $(\mu_K)_{K\in\mathrm{Fin}(S)}$ on $\mathbb{K}^K$, there exists a random field X indexed by S which finite dimensional distributions are the μ_K.*

Remark 1.2. The product σ-algebra $\mathcal{P}$ is rather small: in particular when S is endowed with a topology, in general the set of continuous functions $C^0(S, \mathbb{K})$ from S to $\mathbb{K}$ is not in $\mathcal{P}$. This means we cannot check if a random field is almost surely with continuous sample paths by checking that $\mu_X(C^0(S, \mathbb{K})) = 1$. One overcomes this problem by checking that there exists a *modification* of the random field with continuous sample paths (see [13] for a discussion when $S = \mathbb{R}$). Let us stress out that we do not deal with such considerations here, which means we never consider the question of the existence of a continuous modification when we talk about existence of random fields.

1.2.2 Gaussian real-valued random fields

Definition 1.11 (Gaussian random variable)**.** A real-valued random variable X with mean μ and variance σ^2 is a *Gaussian random variable* if the distribution of X admits the density

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1.3}$$

with respect to the Lebesgue measure.

Definition 1.12 (Gaussian random field)**.** A *Gaussian random field* indexed by a set S is a random field $(X_x)_{x\in S}$ such that for all $n \in \mathbb{N}$, $x_1, \cdots, x_n \in S$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$, the random variable $\sum_{i=1}^{n} \lambda_i X_{x_i}$ is a Gaussian random variable.

Remark 1.3. Taking $S = \{1, \cdots, n\}$ we get the definition of *Gaussian vector.*

Proposition 1.2. *For a Gaussian random field $(X_x)_{x\in S}$ the mean*

$$m_X : x \mapsto \mathbb{E}(X_x) \tag{1.4}$$

and covariance functions

$$R_X : (x, y) \mapsto \mathbb{E}[X_x - m_X(x)][X_y - m_X(y)] \tag{1.5}$$

characterise the distribution of X.

Proof. From Proposition 1.1 we know that the distribution of X is characterised by its finite dimensional distributions. Now the joint distribution of the vector $(X_{x_1}, \cdots X_{x_n})$ is entirely determined by its characteristic function

$$\varphi_{x_1,\cdots,x_n}(\lambda_1, \cdots, \lambda_n) = \mathbb{E}\,\mathrm{Exp}\left(i \sum_{i=1}^{n} \lambda_j X_{x_i}\right). \tag{1.6}$$

It is well known that the characteristic function of a Gaussian vector $(X_{x_1}, \cdots X_{x_n})$ depends only on its mean $(m_X(x_1), \cdots, m_X(x_n))$ and covariance $(R_X(x_i, x_j))_{1\leq i,j\leq n})$, specifically:

$$\varphi_{x_1,\cdots,x_n}(\lambda_1, \cdots, \lambda_n) = \mathrm{Exp}\left(i \sum_{i=1}^{n} \lambda_i\ m_X(x_i) - \frac{1}{2} \sum_{i,j=1}^{n} \lambda_i \lambda_j\ R_X(x_i, x_j)\right), \tag{1.7}$$

which proves the Proposition. $\square$

Lemma 1.1. *Given a real-valued matrix $R = (R_{i,j})_{1\leq i,j\leq n}$ and $(m_i)_{1\leq i\leq n}$, there exists a Gaussian vector $(X_1, \cdots, X_n)$ such that*

$$\forall i \in \{1, \cdots, n\},\ \mathbb{E}(X_i) = m_i$$

and

$$\forall i, j \in \{1, \cdots, n\},\ \mathbb{E}(X_i - m_i)(X_j - m_j) = R_{i,j}$$

if and only if the matrix R is symmetric and positive semi-definite, *namely:*

$$\forall u \in \mathbb{R}^n,\ u^{\mathsf{T}} M u \geq 0.$$

Remark 1.4. Let us notice that the matrix R is symmetric and positive semi-definite if and only if $(u,v) \mapsto u^{\mathsf{T}} M v$ is a kernel of positive type on $\mathbb{R}^n$ (see Definition 1.1).

Proof. • If such a Gaussian vector $(X_1, \cdots, X_n)$ exists, for every $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{i=1}^n \lambda_i (X_i - m_i)\right)^2\right] &= \sum_{i,j=1}^n \lambda_i \lambda_j \, \mathbb{E}(X_i - m_i)(X_j - m_j) \\ &= \sum_{i,j=1}^n \lambda_i \lambda_j R_{i,j} \geq 0,\end{aligned}$$

which proves R is positive semi-definite.

• Reciprocally, if R is positive semi-definite, it is well known that there exists a square root of R, symmetric matrix with real coefficients

$$R^{1/2} = (r'_{i,j})_{1 \leq i,j \leq n}$$

such that

$$R^{1/2} R^{1/2} = R. \tag{1.8}$$

We now consider $(G_1, \cdots, G_n)$ independent standard Gaussian random variables and set

$$\forall i \in \{1, \cdots, n\}, \; X_i := m_i + \sum_{k=1}^n r'_{i,k} G_k.$$

It remains to check that

$$\begin{aligned}\mathbb{E}(X_i - m_i)(X_j - m_j) &= \mathbb{E}\left(\sum_{k=1}^n r'_{i,k} G_k\right)\left(\sum_{k=1}^n r'_{j,k} G_k\right) \\ &= \sum_{k=1}^n r'_{i,k} r'_{j,k} = R_{i,j},\end{aligned}$$

using $R^{1/2}$ symmetric and (1.8). □

Proposition 1.3. *Given $m : S \to \mathbb{R}$ and $R : S \times S \to \mathbb{R}$, there exists a real-valued Gaussian random field indexed by S with mean m and covariance R if and only if R is a kernel of positive type on S.*

Proof. Let us check that Theorem 1.2 applies. Since R is a kernel of positive type on S, for any $x_1, \cdots, x_n \in S$ the matrix $(R(x_i, x_j)_{1 \le i,j \le n})$ is symmetric and positive semi-definite. Using Lemma 1.1 we get a Gaussian vector $(X_{x_1}, \cdots, X_{x_n})$ which covariance and expectation agree with R and m. It is easy to check that this Gaussian vector induce a probability distribution on $\mathbb{R}^{\{x_1,\cdots,x_n\}}$.

Proceeding like this we get a collection of probability distributions

$$(\mu_K)_{K \in \mathrm{Fin}(S)}.$$

Because every finite dimensional distributions arises from the same covariance R, the consistency assumption of Theorem 1.2 is verified. There exists a random field X with finite dimensional distributions $(\mu_K)_{K \in \mathrm{Fin}(S)}$. It is clear that X have mean m , covariance R, and is a Gaussian field. Indeed those three properties involve only a finite number of random variables X_{x_i} at a time, thus they depend only on the finite dimensional distributions of X. □

1.2.3 Gaussian complex-valued random fields

We give here some material about Gaussian complex-valued random fields, in order to prepare for Section 1.3.2 where we discuss the definition of complex-valued fractional Brownian fields. All the definitions and properties we give are elementary, however let us notice that the definition of *isotropic* Gaussian random field is new as far as we know.

Definition 1.13 (Complex Gaussian random variable)**.** A random variable with complex values G is *Gaussian* if $(\mathrm{Re}(G), \mathrm{Im}(G))$ is a real-valued Gaussian vector.

Definition 1.14 (Isotropic variable)**.** A complex Gausian random variable G is *isotropic* if $\mathrm{Re}(G)$ and $\mathrm{Im}(G)$ are independent with same variance. Equivalently:

$$\forall \theta \in \mathbb{R},\ e^{i\theta}(G - \mathbb{E}\, G) \overset{(d)}{=} G - \mathbb{E}\, G. \tag{1.9}$$

Remark 1.5. From (1.9) it is clear that the isotropy of a Gaussian random variable does not depend on its expectation. Without loss of generality we will often check isotropy for centred random variables only.

Definition 1.15 (Gaussian (isotropic) complex-valued random fields)**.** A Gaussian (isotropic) complex-valued random field indexed by a set S is a complex-valued random field $(X_x)_{x \in S}$ such that for every $n \in \mathbb{N}$, $x_1, \cdots, x_n \in S$ and $\mu_1, \cdots, \mu_n \in \mathbb{C}$, $\sum_{i=1}^{n} \mu_i X_{x_i}$ is a Gaussian (isotropic) complex-valued random variable.

Remark 1.6. It is easy to check that those definitions are equivalents if we replace the complex μ_i by real numbers.

Remark 1.7. If we consider the disjoint union $S_1 \dot\cup S_2$ of two copies of S, $(X_x)_{x\in S}$ and $(Y_x)_{x\in S_1\dot\cup S_2}$ such that $Y_x := \mathrm{Re}(X_x)$ if $x \in S_1$ and $Y_x := \mathrm{Im}(X_x)$ if $x \in S_2$, $(X_x)_{x\in S}$ is a Gaussian complex-valued field if and only if $(Y_x)_{x\in S_1\dot\cup S_2}$ is a Gaussian real-valued field.

Remark 1.8. Unlike in the real-valued case, the mean and the covariance function $(x, y) \mapsto \mathbb{E}(X_x\overline{X_y})$ of a Gaussian complex-valued field are not sufficient to characterise its distribution. Consider a Gaussian real-valued vector

$$(A, B, C) \overset{(d)}{=} \mathcal{N}(0, I_3).$$

Let us set

$$\begin{aligned} X &= A + iB, \\ X' &= \frac{A+C}{\sqrt{2}} + iB, \\ X'' &= A + i\frac{B+C}{\sqrt{2}}. \end{aligned}$$

Since

$$\mathbb{E}[\mathrm{Im}(X)\,\mathrm{Im}(X')] \neq \mathbb{E}[\mathrm{Im}(X)\,\mathrm{Im}(X'')],$$

it is clear that the random vectors (X, X') and (X, X'') have distinct distributions, despite the fact that

$$\mathbb{E}(X\overline{X'}) = \mathbb{E}(X\overline{X''}) = 1 + \frac{1}{\sqrt{2}}.$$

Proposition 1.4. *For a complex-valued Gaussian field $(X_x)_{x\in S}$ the three functions*

$$m_X : x \mapsto \mathbb{E}(X_x), \tag{1.10}$$

$$R_X : (x, y) \mapsto \mathbb{E}[X_x - m(x)][X_y - m(y)], \tag{1.11}$$

$$S_X : (x, y) \mapsto \mathbb{E}[X_x - m(x)]\overline{[X_y - m(y)]} \tag{1.12}$$

characterise the distribution of X.

Proof. It is clear that the distribution of X determines m_X, R_X, S_X. Determining the distribution of X is equivalent to determining the distribution of the real-valued Gaussian field Y that we define in Remark 1.7. Let us write

$$X_x = m_X(x) + A_x + iB_x.$$

From Proposition 1.2 determining $m_X(x)$, $\mathbb{E}(A_xA_y)$, $\mathbb{E}(B_xA_y)$, and $\mathbb{E}(B_xB_y)$ is sufficient to characterise the distribution of Y. The following relations finish the proof:

$$\begin{aligned}\mathbb{E}(A_xA_y) &= 1/2\,\mathrm{Re}\left[R_X(x,y)+S_X(x,y)\right],\\ \mathbb{E}(B_xA_y) &= 1/2\,\mathrm{Im}\left[R_X(x,y)+S_X(x,y)\right],\\ \mathbb{E}(B_xB_y) &= 1/2\,\mathrm{Re}\left[R_X(x,y)-S_X(x,y)\right].\end{aligned}$$ □

Furthermore we have

Proposition 1.5. *Let $(X_x)_{x\in S}$ be a Gaussian complex-valued random field. The real-valued fields $(\mathrm{Re}(X_x))_{x\in S}$ and $(\mathrm{Im}(X_x))_{x\in S}$ are mutually independent if and only if S_X and R_X are real-valued.*

Proof. Since mutual independence does not depend on mean values we check the result for a centred field only. As before we write

$$X_x = A_x + iB_x.$$

We have

$$\begin{aligned}\mathrm{Im}\,S_X(x,y) &= \mathbb{E}(B_xA_y) - \mathbb{E}(A_xB_y),\\ \mathrm{Im}\,R_X(x,y) &= \mathbb{E}(B_xA_y) + \mathbb{E}(A_xB_y).\end{aligned}$$

If A_x and B_x are mutually independent it is clear that S_X and R_X are real-valued. Furthermore

$$\mathbb{E}\left(A_xB_y\right) = 1/2\left(\mathrm{Im}\,S_X(x,y) - \mathrm{Im}\,R_X(x,y)\right)$$

Since A and B are Gaussian fields they are mutually independent if and only if $\mathbb{E}\left(A_xB_y\right)=0$ for every x and y, hence the converse claim is true. □

Proposition 1.6. *A Gaussian complex-valued field $(X_x)_{x\in S}$ is isotropic if and only if the function R_X is identically zero. Equivalently,*

$$\forall\theta\in\mathbb{R},\ \left(e^{i\theta}[X_x-m_X(x)]\right)_{x\in S} \overset{(d)}{=} \left(X_x-m_X(x)\right)_{x\in S}. \tag{1.13}$$

Proof. Following Remark 1.5 we check the result for a centred field only.

- Let us write $X_x = A_x + iB_x$ and assume

  $$R_X(x,y) = \mathbb{E}(X_xX_y) = 0,$$

  that is to say

  $$\mathbb{E}(A_xA_y) - \mathbb{E}(B_xB_y) + i\left[\mathbb{E}(B_xA_y)+\mathbb{E}(A_xB_y)\right] = 0.$$

This is equivalent to

$$\mathbb{E}(A_x A_y) = \mathbb{E}(B_x B_y) \tag{1.14}$$

and

$$\mathbb{E}(B_x A_y) = -\,\mathbb{E}(A_x B_y). \tag{1.15}$$

We now check that for all $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ and $x_1, \cdots x_n \in S$ the Gaussian random variable

$$\sum_{i=1}^n \lambda_i X_{x_i} = \sum_{i=1}^n \lambda_i A_{x_i} + i \sum_{i=1}^n \lambda_i B_{x_i}$$

is isotropic. Following Definition 1.14 We start by checking independence:

$$\mathbb{E}\left[\left(\sum_{i=1}^n \lambda_i A_{x_i}\right)\left(\sum_{i=1}^n \lambda_i B_{x_i}\right)\right] = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j\, \mathbb{E}(A_{x_i} B_{x_j})$$
$$= \sum_{i=1}^n \lambda_i^2\, \mathbb{E}(A_{x_i} B_{x_i}) + \sum_{i<j} \lambda_i \lambda_j\, \mathbb{E}(A_{x_i} B_{x_j}) + \sum_{i>j} \lambda_i \lambda_j\, \mathbb{E}(A_{x_i} B_{x_j}) = 0.$$

Indeed using (1.15) we have

$$\mathbb{E}(A_{x_i} B_{x_i}) = -\,\mathbb{E}(A_{x_i} B_{x_i}) = 0$$

and

$$\mathbb{E}(A_{x_i} B_{x_j}) = -\,\mathbb{E}(A_{x_j} B_{x_i}),$$

thus the two last terms cancel each other. On the other side using (1.14) we obtain the desired equality for the variances:

$$\mathbb{E}\left(\sum_{i=1}^n \lambda_i A_{x_i}\right)^2 = \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j\, \mathbb{E}(A_{x_i} A_{x_j}) = \mathbb{E}\left(\sum_{i=1}^n \lambda_i B_{x_i}\right)^2,$$

which finishes to prove that the field is isotropic.

- Reciprocally if X is isotropic, in particular for every $x, y \in S$,

$$X_x + X_y = A_x + A_y + i(B_x + B_y)$$

is an isotropic random variable, hence

$$\mathbb{E}(A_x + A_y)^2 = \mathbb{E}(B_x + B_y)^2$$

and

$$\mathbb{E}(A_x + A_y)(B_x + B_y) = 0.$$

Expanding the two equations and using the isotropy of the random variables X_x et X_y we obtain

$$\mathbb{E}(A_x A_y) = \mathbb{E}(B_x B_y)$$

and

$$\mathbb{E}(B_x A_y) = -\,\mathbb{E}(A_x B_y),$$

therefore $\mathbb{E}(X_x X_y) = 0$.

- Let us now consider a field X such that

$$\forall \theta \in \mathbb{R},\ (e^{i\theta} X_x)_{x\in S} \overset{(d)}{=} (X_x)_{x\in S}\ .$$

In particular for every $\theta \in \mathbb{R}$, and $x, y \in S$,

$$\mathbb{E}(e^{i\theta} X_x e^{i\theta} X_y) = \mathbb{E}(X_x X_y).$$

This is equivalent to

$$e^{2i\theta}\,\mathbb{E}(X_x X_y) = \mathbb{E}(X_x X_y),$$

hence $\mathbb{E}(X_x X_y) = 0$, which proves that X is isotropic.

- Reciprocally if we consider X an isotropic field, for every $\theta \in \mathbb{R}$ we have

$$\mathbb{E}(e^{i\theta} X_x\ e^{i\theta}\overline{X_y}) = |e^{i\theta}|\,\mathbb{E}(X_x \overline{X_y}) = \mathbb{E}(X_x \overline{X_y})$$

and

$$\mathbb{E}(e^{i\theta} X_x\ e^{i\theta} X_y) = e^{2i\theta}\,\mathbb{E}(X_x X_y) = 0 = \mathbb{E}(X_x X_y).$$

Using Proposition 1.4 it is clear that the field $(e^{i\theta} X_x)_{x\in S}$ has same distribution as X. □

Remark 1.9. The real and imaginary part of a centred isotropic field have the same distribution. Indeed if we write $X_x = A + iB$ we have

$$\operatorname{Re}\mathbb{E}(X_x X_y) = \mathbb{E}(A_x A_y) - \mathbb{E}(B_x B_y)$$

which is identically zero from Proposition 1.6. This implies that the two real-valued Gaussian fields A and B have the same covariance and the same mean value, hence they have the same distribution (see Proposition 1.2).

Remark 1.10. Let us remark that the isotropy of a field is a stronger property than the isotropy of all the random variables X_x, $x \in E$, as (1.13) is stronger than

$$\forall x \in S,\ \forall \theta \in \mathbb{R},\ e^{i\theta} X_x \overset{(d)}{=} X_x. \tag{1.16}$$

Proposition 1.7. *If a Gaussian random field $X_x = A_x + iB_x$ is such that $A \overset{(d)}{=} B$ and A and B are mutually independent, then X is isotropic.*

Proof. We use Proposition 1.6 and

$$\mathbb{E}(X_x X_y) = \mathbb{E}(A_x A_y) - \mathbb{E}(B_x B_y) + i\Big(\mathbb{E}(B_x A_y) + \mathbb{E}(A_x B_y)\Big). \quad \square$$

Remark 1.11. The converse of Proposition 1.7 is not true. Indeed given a centred real-valued random vector $(A_1, \cdots, A_n, B_1, \cdots, B_n)$ with covariance matrix

$$\begin{pmatrix} \Gamma & \Sigma \\ \Sigma & \Gamma \end{pmatrix},$$

where Γ and Σ are $n \times n$ real matrices such that Γ is symmetric, it is clear that the complex Gaussian vector $X = (A_j + iB_j)_{1 \le j \le n}$ have real and imaginary part equal in distribution. Furthermore $(A_j)_{1 \le j \le n}$ and $(B_j)_{1 \le j \le n}$ are independent if and only if $\Sigma = 0$, while one can check that X is isotropic if and only if Σ is an antisymmetric matrix.

1.3 Fractional Brownian fields indexed by metric spaces

1.3.1 Definition

Definition 1.16 (Fractional Brownian field)**.** Let H be a positive real number. We call an *H-fractional Brownian field* indexed by a metric space (E, d) any Gaussian random field $\left(X_x^H\right)_{x \in E}$ such that:

$$\forall x \in E,\ \mathbb{E}\left(X_x^H\right) = 0, \tag{1.17}$$

and

$$\forall x, y \in E,\ \mathbb{E}\left[(X_x^H - X_y^H)^2\right] = \left[d(x, y)\right]^{2H}. \tag{1.18}$$

Remark 1.12. If there exists an H-fractional Brownian field X^H indexed by (E, d), for every centred Gaussian random variable G such that $X^H + G$ is a Gaussian field, $X^H + G$ is an H-fractional Brownian field. In particular if $G \sim \mathcal{N}(0,1)$ is independent from X^H,

$$\mathbb{E}(X_x^H + G)(X_y^H + G) = \mathbb{E}(X_x^H X_y^H) + \mathrm{Var}(G).$$

In this case the two $H-$fractional Brownian fields X^H and $X^H + G$ are not equal in distribution. The following definition overcomes this problem.

Definition 1.17. Let (E, d) be a metric space and $O \in E$. We call an H-fractional Brownian field indexed by (E, d) *with origin in* O any H-fractional Brownian field $\left(X^H\right)_{x \in E}$ such that $X_O^H = 0$ almost surely.

Proposition 1.8. *If X^H is an H-fractional Brownian fields with origin in $O \in E$, for every $x, y \in E$ we have*

$$\mathbb{E}(X_x^H X_y^H) = \frac{1}{2}\Big(d^{2H}(O,x) + d^{2H}(O,y) - d^{2H}(x,y)\Big). \tag{1.19}$$

Remark 1.13. In particular two H-fractional Brownian fields with origin in O are equal in distribution (see Proposition 1.2).

Proof. We simply expand

$$\mathbb{E}(X_x^H - X_y^H)^2 = \mathbb{E}(X_x^H)^2 + \mathbb{E}(X_y^H)^2 - 2\,\mathbb{E}(X_x^H X_y^H),$$

From $X_O = 0$ a.s. and (1.18) it is clear that $\mathbb{E}(X_x^H)^2 = [d(O,x)]^{2H}$ and $\mathbb{E}(X_y^H)^2 = [d(O,y)]^{2H}$. Using again (1.18) on the left-hand side of our equality we obtain the result. □

Proposition 1.9. *Given a metric space (E, d) and a positive H, the following assertions are equivalent:*

1. *There exists an H-fractional Brownian field indexed by (E, d).*
2. *For every $O \in E$, there exists an H-fractional Brownian field with origin in O.*
3. *For every $O \in E$, the kernel*

$$R_H(x,y) := \frac{1}{2}\Big(d^{2H}(O,x) + d^{2H}(O,y) - d^{2H}(x,y)\Big)$$

   on E is of positive type.

4. *The kernel d^{2H} on E is of negative type.*

Proof.
- 1 $\Leftrightarrow$ 2: from the existence of X^H H-fractional Brownian field we construct for every $O \in E$ the field $\tilde{X}_x^H := X_x^H - X_O^H$ which is an H-fractional Brownian field with origin in O.
- 2 $\Leftrightarrow$ 3 is a direct application of Proposition 1.3.
- 3 $\Leftrightarrow$ 4 is a direct application of Schoenberg's Theorem (see Theorem 1.1). $\square$

Definition 1.18. An isometry of (E, d) is a map $g : E \to E$ such that for every $x, y \in E$,

$$d(g(x), g(y)) = d(x, y).$$

Proposition 1.10 (Stationary increments)**.** *For every Fractional Brownian field X^H, every isometry g of (E, d), and every $y \in E$,*

$$\left(X_{g(x)}^H - X_{g(y)}^H\right)_{x\in E} \stackrel{(d)}{=} \left(X_x^H - X_y^H\right)_{x\in E}.$$

Proof. From Proposition 1.2 we only need to check that the two fields have same covariance. For every $x, x' \in E$ we have

$$\begin{aligned}
&\mathbb{E}\left(X_{g(x)}^H - X_{g(y)}^H\right)\left(X_{g(x')}^H - X_{g(y)}^H\right) \\
=\ &\mathbb{E}\left(X_{g(x)}^H X_{g(x')}^H\right) - \mathbb{E}\left(X_{g(x)}^H X_{g(y)}^H\right) - \mathbb{E}\left(X_{g(y)}^H X_{g(x')}^H\right) + \mathbb{E}\left(X_{g(y)}^H X_{g(y)}^H\right) \\
=\ &\frac{1}{2}\left(d^{2H}(O, g(x)) + d^{2H}(O, g(x')) - d^{2H}(g(x), g(x'))\right) \\
+\ &\frac{1}{2}\left(-d^{2H}(O, g(x)) - d^{2H}(O, g(y)) + d^{2H}(g(x), g(y))\right) \\
+\ &\frac{1}{2}\left(-d^{2H}(O, g(y)) - d^{2H}(O, g(x')) + d^{2H}(g(y), g(x'))\right) \\
+\ &\frac{1}{2}\left(d^{2H}(O, g(y)) + d^{2H}(O, g(y)) - d^{2H}(g(y), g(y))\right) \\
=\ &-d^{2H}(g(x), g(x')) + d^{2H}(g(x), g(y)) + d^{2H}(g(y), g(x')) - d^{2H}(g(y), g(y).
\end{aligned}$$

Using g isometry we have

$$-d^{2H}(x, x') + d^{2H}(x, y) + d^{2H}(y, x') - d^{2H}(y, y),$$

which is exactly what we obtain if we compute directly

$$\mathbb{E}\left(X_x^H - X_y^H\right)\left(X_{x'}^H - X_y^H\right). \quad \square$$

Remark 1.14. If we take $E = \mathbb{R}$ and g a translation we recover the classical property of stationary increments of the (real-indexed) fractional Brownian motion.

1.3.2 Complex-valued fractional Brownian field

Imitating Definition 1.16 we give a definition for a fractional Brownian fields with complex values:

Definition 1.19 (Complex-valued fractional Brownian fields)**.** Given a positive H and a metric space (E,d), a random field X^H indexed by E is a *complex-valued H-fractional Brownian field* if

$$\forall x \in E,\ \mathbb{E}\left(X_x^H\right) = 0, \tag{1.20}$$

and

$$\forall x,y \in E,\ \mathbb{E}\left[\left|X_x^H - X_y^H\right|^2\right] = [d(x,y)]^{2H}. \tag{1.21}$$

Definition 1.20. Let (E,d) be a metric space and $O \in E$. We call an H-fractional Brownian field indexed by (E,d) *with origin in O* any complex-valued H-fractional Brownian field $\left(X^H\right)_{x\in E}$ such that $X_O^H = 0$ almost surely.

In [11] Istas gives Definition 1.20 for a complex-valued fractional Brownian field. However in contrast with the real-valued case we will see that this definition is not enough to have uniqueness of the distribution of the random field.

Proposition 1.11. *Let A and B be two real-valued H-fractional Brownian fields indexed by the same metric space (E,d). For every $\theta \in [0,2\pi[$,*

$$X_x^H = \cos(\theta)A_x + i\sin(\theta)B_x \tag{1.22}$$

is a complex-valued H-fractional Brownian field.

Proof. It is clear that X^H is a centred field. For two complex-valued random values $X_1 = A_1 + iB_1$ and $X_2 = A_2 + iB_2$ we have

$$\mathbb{E}\left|X_1 - X_2\right|^2 = \mathbb{E}(A_1 - A_2)^2 + \mathbb{E}(B_1 - B_2)^2.$$

Using this and

$$\mathbb{E}(A_x - A_y)^2 = \mathbb{E}(B_x - B_y)^2 = d^{2H}(x,y),$$

we check (1.21):

$$\begin{aligned}\mathbb{E}\left|X_x^H - X_y^H\right|^2 &= \cos^2(\theta)\,\mathbb{E}(A_x - A_y)^2 + \sin^2(\theta)\,\mathbb{E}(B_x - B_y)^2\\ &= d^{2H}(x,y). \quad \square\end{aligned}$$

From Proposition 1.11 it seems clear that we could end up with complex-valued fractional Brownian fields with different distributions if we consider different coupling of two real-valued fractional Brownian fields A and B and set $X = A+iB$. In the next proposition we investigate the case where X is a linear coupling of two independent real-valued fractional Brownian fields.

Proposition 1.12. *1. Let A and B be two mutually independent real-valued H-fractional Brownian fields indexed by (E,d).*
For every $\theta,\varphi,\psi \in [0,2\pi[$,

$$X_x^H = \cos(\theta)\left(\cos(\varphi)A_x + \sin(\varphi)B_x\right) + i\sin(\theta)\left(\cos(\psi)A_x + \sin(\psi)B_x\right) \quad (1.23)$$

is a complex-valued H-fractional Brownian field.

2. Furthermore if A and B are equal in distribution, let us write

$$R(x,y) := \mathbb{E}(A_xA_y) = \mathbb{E}(B_xB_y).$$

We have

$$\mathbb{E}\left(X_x^H\overline{X_y^H}\right) = R(x,y), \quad (1.24)$$

and

$$\mathbb{E}\left(X_x^HX_y^H\right) = \big(\cos(2\theta) + i\sin(2\theta)\cos(\varphi-\psi)\big)R(x,y). \quad (1.25)$$

In particular each pair $(\theta,|\varphi-\psi|) \in [0,\pi[\times[0,\pi[$ yields a different distribution of random field.

Proof. 1. Let us check that the real part $Z_x := \cos(\varphi)A_x + \sin(\varphi)B_x$ of X_x^H is a fractional Brownian field. It is clear that Z_x is centred. Let us compute

$$\mathbb{E}(Z_x - Z_y)^2 = \mathbb{E}(\cos(\varphi)A_x + \sin(\varphi)B_x - \cos(\varphi)A_y - \sin(\varphi)B_y)^2.$$

Since A and B are mutually independent a lot of terms are null and we obtain

$$\begin{aligned}&\cos^2(\varphi)\left(\mathbb{E}\,A_x^2 + \mathbb{E}\,A_y^2 - 2\,\mathbb{E}\,A_xA_y\right) + \sin^2(\varphi)\left(\mathbb{E}\,B_x^2 + \mathbb{E}\,B_y^2 - 2\,\mathbb{E}\,B_xB_y\right)\\ &= \cos^2(\varphi)\,\mathbb{E}(A_x - A_y)^2 + \sin^2(\varphi)\,\mathbb{E}(B_x - B_y)^2.\end{aligned}$$

Because A and B are H-fractional Brownian fields we obtain $d^{2H}(x,y)$ and Z is an H-fractional Brownian field. The same argument carry on for the imaginary part of X_x^H. From Proposition 1.11 we deduce that X_x^H is a complex-valued H-fractional Brownian field.

2. Using

$$\mathbb{E}(A_xA_y) = \mathbb{E}(B_xB_y) = R(x,y)$$

we obtain

$$\begin{aligned}
&\mathbb{E}(X_x^H \overline{X_y^H}) \\
&= \cos^2(\theta)\big[\cos^2(\varphi)\,\mathbb{E}\,A_xA_y + \sin^2(\varphi)\,\mathbb{E}\,A_xA_y\big] \\
&+ \sin^2(\theta)\big[\cos^2(\psi)\,\mathbb{E}\,A_xA_y + \sin^2(\psi)\,\mathbb{E}\,A_xA_y\big] \\
&+ i\Big(\cos(\theta)\sin(\theta)\big[\cos(\varphi)\cos(\psi)\,\mathbb{E}\,A_xA_y + \sin(\varphi)\sin(\psi)\,\mathbb{E}\,B_xB_y\big] \\
&- \cos(\theta)\sin(\theta)\big[\cos(\varphi)\cos(\psi)\,\mathbb{E}\,A_xA_y + \sin(\varphi)\sin(\psi)\,\mathbb{E}\,B_xB_y\big]\Big) \\
&= R(x,y)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}(X_x^H X_y^H) \\
&= \cos^2(\theta)\big[\cos^2(\varphi)\,\mathbb{E}\,A_xA_y + \sin^2(\varphi)\,\mathbb{E}\,A_xA_y\big] \\
&- \sin^2(\theta)\big[\cos^2(\psi)\,\mathbb{E}\,A_xA_y + \sin^2(\psi)\,\mathbb{E}\,A_xA_y\big] \\
&+ 2i\Big(\cos(\theta)\sin(\theta)\big[\cos(\varphi)\cos(\psi)\,\mathbb{E}\,A_xA_y + \sin(\varphi)\sin(\psi)\,\mathbb{E}\,B_xB_y\big]\Big) \\
&= \big(\cos(2\theta) + i\sin(2\theta)\cos(\varphi-\psi)\big)R(x,y).
\end{aligned}$$

It is now easy to check that for every different pair

$$(\theta, |\varphi-\psi|) \in [0,\pi[\times[0,\pi[$$

we obtain a different function $\mathbb{E}(X_x^H X_y^H)$, hence a different distribution for the field X^H (see Proposition 1.4). □

Proposition 1.13. *1. The field X^H defined by (1.23) is isotropic if and only if its real and imaginary parts are mutually independent and equal in distribution.*

2. Furthermore if A and B are equal in distribution, X_x^H is isotropic if and only if $\theta \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ and $|\varphi-\psi| \in \{\pi/2, 3\pi/2\}$. In this case we have

$$X_x^H \overset{(d)}{=} \frac{A_x + iB_x}{\sqrt{2}}. \tag{1.26}$$

Proof. 1. Without further assumptions a direct computation shows that $\mathbb{E}\left(X_x^H\overline{X_y^H}\right)$ is real-valued. Let us now assume that X^H is isotropic. From Proposition 1.6 we know that $\mathbb{E}(X_x^H X_y^H)$ is identically zero. We apply Proposition 1.5 to deduce that the real and imaginary parts of X^H are mutually independent. Since X^H is isotropic and centred they are equal in distribution (see Remark 1.9).

The converse claim is just Proposition 1.7.

2. We apply Proposition 1.6 to check for which values of θ, φ, and ψ the field X^H is isotropic. From Proposition 1.12 we know that

$$\mathbb{E}\left(X_x^H X_y^H\right) = \big(\cos(2\theta) + i\sin(2\theta)\cos(\varphi - \psi)\big)R(x, y).$$

The function $\mathbb{E}(X_x^H X_y^H)$ is identically zero if and only if

$$\theta \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$$

and

$$|\varphi - \psi| \in \{\pi/2, 3\pi/2\}.$$

Proposition 1.4 implies that every such isotropic field have same distribution. Taking $\theta = \pi/4, \varphi = 0$, and $\psi = \pi/2$ we obtain

$$X_x^H \overset{(d)}{=} \frac{A_x + iB_x}{\sqrt{2}}. \quad \square$$

Remark 1.15. If we consider general linear combinations $X_x^H = aA_x + bB_x$ with $a, b \in \mathbb{R}$ instead of (1.22) in Proposition 1.12, or

$$X_x^H = a_1 A_x + b_1 B_x + i\,(a_1 A_x + b_2 B_x)$$

with $a_1, b_1, a_2, b_2 \in \mathbb{R}$ instead of (1.23) in Proposition 1.11 we end-up with "non-standard" complex-valued fractional Brownian field, that is to say

$$\mathbb{E}\left|X_x^H - X_y^H\right|^2 = C\ d^{2H}(x, y)$$

for some positive constant C, in which case $X^H/\sqrt{C}$ is a standard H-fractional Brownian field.

Remark 1.16. 1. An example of isotropic complex-valued H-fractional Brownian field with origin in O is given by

$$X_x^H = \frac{A_x + iB_x}{\sqrt{2}}, \tag{1.27}$$

where A and B are two mutually independent real-valued H-fractional Brownian fields with origin in $O \in E$. This is equivalent to say that X^H is centred and for every $x, y \in E$,

$$E(X_x \overline{X_y}) = \frac{1}{2}\Big(d^{2H}(O, x) + d^{2H}(O, y) - d^{2H}(x, y)\Big), \tag{1.28}$$

and

$$E(X_x X_y) = 0. \tag{1.29}$$

2. In [11] Istas considers the circle $\mathbb{S}^1$ parametrised by angle $x \in [0, 2\pi[$, the Fourier decomposition
$$d_{\mathbb{S}^1}(x,0) = \sum_{n\in\mathbb{Z}} f_n e^{inx}, \tag{1.30}$$
and notice that
$$d^{2H}(x,y) = \sum_{n\in\mathbb{Z}^*} f_n(e^{in(x-y)} - 1). \tag{1.31}$$
He shows that the Fourier coefficients f_n are negative for every $H \leq 1/2$ and defines
$$B_H(x) = \sum_{n\in\mathbb{Z}^*} d_n \varepsilon_n (e^{inx} - 1), \tag{1.32}$$
where ε_n are mutually independent *standard* complex-valued Gaussian random variables (that is to say $\mathbb{E}\,|\varepsilon_n|^2 = 1$ and $\mathbb{E}\,\varepsilon_n^2 = 0$) and
$$d_n = \frac{\sqrt{-f_n}}{2}. \tag{1.33}$$
Using $\mathbb{E}\,|\varepsilon_n|^2 = 1$ and (1.31) he checks that
$$\begin{aligned}
&\mathbb{E}(B_H(x)\overline{B_H(y)})\\
&= \sum_{n\in\mathbb{Z}^*} d_n^2 (e^{inx} - 1)(e^{-iny} - 1)\\
&= \sum_{n\in\mathbb{Z}^*} d_n^2 \left[(e^{in(x-y)} - 1) - (e^{inx} - 1) - (e^{-iny} - 1)\right]\\
&= \frac{1}{2}\left(-\sum_{n\in\mathbb{Z}^*} f_n(e^{in(x-y)} - 1) + \sum_{n\in\mathbb{Z}^*} f_n(e^{inx} - 1) + \sum_{n\in\mathbb{Z}^*} f_n(e^{-iny} - 1)\right)\\
&= \frac{1}{2}\left(d_{\mathbb{S}^1}^{2H}(0,x) + d_{\mathbb{S}^1}^{2H}(0,y) - d_{\mathbb{S}^1}^{2H}(x,y)\right).
\end{aligned}$$
Furthermore since $\mathbb{E}\,\varepsilon_n^2 = 0$ we have
$$\mathbb{E}(B_H(x)B_H(y)) = 0.$$
From Proposition 1.4 it is clear that the distribution of B_H is the same as the distribution of the field we consider in (1.27).

3. Let us also give as example the field
$$X_x^H = \frac{A_x + iA_x}{\sqrt{2}},$$

where A is a real-valued H-fractional Brownian field with origin in $O \in E$. Equivalently, X^H is centred and for every $x, y \in E$,

$$E(X_x\overline{X_y}) = \frac{1}{2}\Big(d^{2H}(O,x) + d^{2H}(O,y) - d^{2H}(x,y)\Big), \tag{1.34}$$

and

$$E(X_x X_y) = \frac{i}{2}\Big(d^{2H}(O,x) + d^{2H}(O,y) - d^{2H}(x,y)\Big). \tag{1.35}$$

1.3.3 Fractional index of a metric space

Theorem 1.3. *For every metric space (E,d), there exists $\beta_E \in [0,+\infty]$ such that the kernel d^{2H} is of negative type if and only if*

$$0 < 2H \leq \beta_E. \tag{1.36}$$

Definition 1.21 (Fractional index)**.** We call β_E the fractional index of the metric space (E,d).

From Proposition 1.9 and Proposition 1.12 it is clear that:

Proposition 1.14. *There exists a real-valued H-fractional Brownian field indexed by a metric space (E,d) if and only if $0 < 2H \leq \beta_E$. In this case there exists a complex-valued H-fractional Brownian field indexed by (E,d).*

Remark 1.17. From the definition it is clear that two isometric metric spaces have equal fractional indexes. Furthermore if (E,d) and (E',d') are two homothetic metric spaces, that is to say there exists a bijective map

$$\mathcal{H} : E \to E'$$

and $\lambda > 0$ such that for every $s, t \in E$,

$$d'(\mathcal{H}(s), \mathcal{H}(t)) = \lambda d(s,t),$$

then $\beta_E = \beta_{E'}$. Indeed

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j d^{2H}(t_j, t_j) = \frac{1}{\lambda^{2Hn^2}} \sum_{i,j=1}^{n} \lambda_i \lambda_j \left(d'(\mathcal{H}(t_i), \mathcal{H}(t_j)\right)^{2H}$$

hence it is clear that d^{2H} and d'^{2H} are kernels of negative type for the same values of H.

1.4 Riemannian geometry

This section aims at producing a consensus on the Riemannian geometry that we will use. In Section 1.4.1 we briefly evoke some definitions and give the general hypotheses on every Riemannian manifold we consider in this document. In 1.4.2 we give precise definitions about geodesics to avoid confusion and state all the classical results we need, along with the first variation formula in Section 1.4.3. Section 1.4.5 states elementary facts on Riemannian products, which will be useful in Chapter 4. We do not pretend to write anything new here, and everything we recall briefly is directly taken from [9] which we refer to for missing definitions, additional details, and proofs. We invite the geometer reader to skip this section.

1.4.1 Riemannian manifolds

We call *Riemannian manifold* a differentiable manifold which tangent space T_pM at any point p is endowed with a scalar product $\langle\ ,\ \rangle_p$ depending smoothly on p (See [9] for a definition). The family of scalar products $\langle\ ,\ \rangle_p$ is called a *Riemannian metric* on M. We will often drop the p to prefer the notation $\langle\ ,\ \rangle_M$, and denote by $||\ ||_M$ the family of norms defined by

$$||v||_M := \langle v, v\rangle_M^{1/2}. \tag{1.37}$$

Following [9] we consider only C^∞, connected, and countable at infinity manifolds in this whole document.

We recall the definition of the length $L(c)$ of any piecewise continuously differentiable curve $c : [a, b] \to M$:

$$L(c) := \int_a^b ||c'(t)||_M \, dt. \tag{1.38}$$

Given two points p, q in M, consider the set $\mathcal{C}(p, q)$ of all piecewise continuously differentiable maps c from some segment $[a, b]$ to M such that $c(a) = p$ and $c(b) = q$. The *geodesic distance* from p to q is given by

$$d_M(p, q) := \inf_{c\in\mathcal{C}(p,q)} L(c). \tag{1.39}$$

The Riemannian manifold M endowed with the geodesic distance d_M is a metric space.

Remark 1.18. Let us mention that the equality of the Riemannian metrics on subsets of two manifolds does not imply that the geodesic distances coincides on those subsets. Formally, if M and N are two Riemannian manifolds with subsets

$U_M \subset M$ and $U_N \subset N$, and a C^∞-diffeomorphism $\Phi : U_M \to U_N$ which preserves the Riemannian metric (namely, the pushforward of $\langle\ ,\rangle_M$ by Φ coincides with $\langle\ ,\ \rangle_N$ on U_N), $(U_M, d_{M|U_M})$ and $(U_N, d_{N|U_N})$ are not necessarily isometric metric spaces. This comes from the simple fact that a curve from two points of U_N can take values in $N \setminus U_N$ where the two Riemannian metrics do not coincide. In the case where $U_M = M$ and $U_N = N$ this problem is avoided: if Φ is a C^∞-diffeomorphism that preserves the Riemannian metric, it is an isometry of metric spaces between (M, d_M) and (N, d_N) (see [9]).

With the notable exception of cylinders of the form $\mathbb{S}^1 \times [0, \varepsilon]$ in Section 4.5, all the manifolds we consider are *without boundary*. Riemannian manifolds without boundary enjoy the nice properties of local existence, uniqueness and regularity of geodesics we describe in the next subsection, and which we will use extensively.

1.4.2 Geodesics

Definition 1.22 (Curve)**.** We call *(parametrised) curve* with values in M a continuous map from some interval $I \subset \mathbb{R}$ to M, and *reparametrisation* of c a curve $\tilde{c}$ defined on some interval J such that there exists $\varphi : J \mapsto I$ a continuous diffeomorphism,

$$\forall t \in J,\ \tilde{c}(t) = c(\varphi(t)).$$

Remark 1.19*.* We often make an abuse of notation and write c for the set of all points $\{c(t), t \in I\} \subset M$. This allows us to consider curves without specifying a parametrisation.

It is convenient to consider $TM := \{(p, v) \mid v \in T_pM\}$ the *tangent bundle* associated to the manifold M.

Definition 1.23 (Vector field along a curve)**.** Given a curve $c : I \to \mathbb{R}$ we call *vector field along the curve* $t \mapsto c(t)$ a continuous map

$$\begin{aligned} X : I &\to TM \\ t &\mapsto (c(t), v(t)). \end{aligned}$$

Given a differentiable vector field X along a curve c, it is possible to define $\frac{D}{dt}X$ a vector field along c that is the derivative of X along c, in some canonical sense associated to the Riemannian metric of M (See [9] for a definition).

Proposition 1.15 (Parallel transport)**.** *Given a C^1 curve $c : [a, b] \to M$ and $v \in T_{c(a)}M$, there exists a unique vector field X along c such that $\frac{D}{dt}X = 0$ and $X(a) = v$. The vector $X(t)$ is called the parallel transport of v from $c(a)$ to $c(t)$*

along c. Moreover for every $t \in [a,b]$

$$\begin{aligned} T_{c(a)}M &\to T_{c(t)}M \\ t &\mapsto X(t) \end{aligned}$$

is a linear isometry.

Definition 1.24 (Geodesic)**.** A differentiable curve $c : I \to M$ is a *geodesic* if for every $t \in I$ we have

$$\frac{D}{dt}c'(t) = 0. \tag{1.40}$$

Remark 1.20. In some sense (1.40) states the fact that $c'(t)$ "does not vary" along the geodesic. In the case of a surface embedded in $\mathbb{R}^3$, (1.40) implies that the acceleration vector $\frac{d^2c(t)}{dt^2}$ is normal to the surface at $c(t)$, which means there is no variation of the tangential component of $c'(t)$: the curve $t \mapsto c(t)$ travels "in a straight way" in the surface.

Remark 1.21. If we write (1.40) in a chart we obtain an ordinary differential equation on which Cauchy-Lipschitz theory applies. This has many consequences. In particular a geodesic $t \mapsto c(t)$ is

- entirely determined by its value and speed at $t = 0$,
- C^∞ with respect to $(t, c(0), c'(0))$.

Remark 1.22. Properties of the derivative of a vector field along a curve allow us to write

$$0 = \left\langle \frac{D}{dt}c'(t), c'(t) \right\rangle_M = \frac{1}{2}\frac{d}{dt}\left\langle c'(t), c'(t)\right\rangle_M = \frac{1}{2}\frac{d}{dt}||c'(t)||_M^2,$$

which proves that a geodesic curve has constant speed, that is to say it is parametrised proportionally to arc-length.

Definition 1.25 (Minimal geodesic)**.** Given $p, q \in M$, let us denote by $\mathcal{C}(p,q)$ the set of piecewise continuously differentiable curves c defined on some segment $[a,b]$ and with values in M such that $c(a) = p$, $c(b) = q$.

A curve $g \in \mathcal{C}(p,q)$ is a *minimal geodesic between p and q* if g is parametrised proportionally to arc-length and

$$L(g) = \min\{L(c),\ c \in \mathcal{C}(p,q)\}.$$

Given any real interval I, we say that a curve $g : I \to M$ is a *minimal geodesic* if it is a minimal geodesic between $g(t)$ and $g(t')$ for every $t, t' \in I$.

Remark 1.23. We will often use the notation PQ when there is no ambiguity on which minimal geodesic between P and Q we refer to.

Remark 1.24. Let us insist on the abuse of notation we already mentioned in Remark 1.19. We will sometime say that a curve $c \subset M$ is a geodesic (resp. minimal geodesic) without specifying the parametrisation: we mean that there exists a parametrisation of c which is a geodesic (resp. minimal geodesic). Notice that in this case every proportional to arc-length parametrisation of γ is a geodesic (resp. minimal geodesic).

Proposition 1.16. *1. Every minimal geodesic is a geodesic.*

2. For every geodesic $g : I \to M$ and every $t \in I$ there exists $\varepsilon > 0$ such that $c_{|[t-\varepsilon,t+\varepsilon]}$ is a minimal geodesic.

Definition 1.26 (Energy)**.** For every piecewise continuously differentiable curve $c : [a, b] \to M$ we define the energy of c

$$E_M(c) := \frac{1}{2}\int_a^b ||c'(t)||_M^2 dt.$$

Proposition 1.17. *Given $p, q \in M$ and $T > 0$ let us denote by $\mathcal{C}^T(p, q)$ the set of piecewise continuously differentiable curves $c : [a, b] \to M$ such that $b - a = T$, $c(a) = p$, and $c(b) = q$.*

A curve $g \in \mathcal{C}^T(p, q)$ is a minimal geodesic if and only if

$$E_M(g) = \min\left\{E_M(c),\ c \in \mathcal{C}^T(p, q)\right\}.$$

Remark 1.25. It is necessary to consider a fixed T and $\mathcal{C}^T(p, q)$ here. Indeed given a curve $c : [0, T] \to M$ and a positive λ, the reparametrisation

$$c_\lambda : [0, \lambda T] \to M$$

defined by $c_\lambda(t) = c(t/\lambda)$ is such that $E_M(c_\lambda) = E_M(c)/\lambda$.

Remark 1.26. If $c : [a, b] \to M$ is a piecewise continuously differentiable curve, Cauchy-Schwarz inequality in $L^2([a, b])$ with $f = ||c'(t)||_M$ and $g = 1$ gives

$$L(c)^2 = \left(\int_a^b ||c'(t)||_M dt\right)^2 \leq (b - a)\int_a^b ||c'(t)||_M^2 dt = 2(b - a)E_M(c).$$

We get

$$E_M(c) \geq \frac{L(c)^2}{2(b - a)} \tag{1.41}$$

and

$$E_M(c) = \frac{L(c)^2}{2(b-a)} \tag{1.42}$$

if and only if c is parametrised proportionally to arc-length.

In Chapter 3 we use the fact that small enough balls are geodesically convex:

Proposition 1.18. *For any $m \in M$, there exists $\varepsilon > 0$ such that any ball B (for the geodesic distance) with centre m and radius $R < \varepsilon$ is geodesically convex, that is to say for any two points $p, q \in B$ there exists a unique minimal geodesic*

$$g_{(p,q)} : [0,1] \to M$$

with $g(0) = p$ and $g(1) = q$, and this geodesic is contained in B. Furthermore $g_{(p,q)}(t)$ is C^∞ with respect to p, q, t.

Remark 1.27. An important consequence is that two curves c_0 and c_1 which take values in a sufficiently small ball are homotopic, that is to say we can find a *continuous* map

$$\begin{aligned}[0,1] \times [0,1] &\to M\\ (s,t) &\mapsto f_t(s)\end{aligned}$$

such that $f_0 = \tilde{c}_0$ and $f_1 = \tilde{c}_1$, with $\tilde{c}_0$ and $\tilde{c}_1$ reparametrisations of c_0 and c_1. In this case setting $f_t(s) = g_{(c_0(s), c_1(s))}(t)$ is sufficient.

Given a curve $\gamma : [0,T] \to M$ such that $\gamma(0) = \gamma(T)$ and $t_1 \le t_2 \in [0,T]$, we again make an abuse of notation and denote by $\gamma \setminus \gamma_{|[t_1,t_2]}$ the curve

$$\begin{aligned}\gamma \setminus \gamma_{|[t_1,t_2]} : [t_2, T+t_1] &\to M\\ t &\mapsto \begin{cases} \gamma(t) \text{ if } t \le T,\\ \gamma(t-T) \text{ elsewise.}\end{cases}\end{aligned} \tag{1.43}$$

Definition 1.27 (Minimal closed geodesic)**.** A curve $\gamma : [0,T] \to M$ is a *minimal closed geodesic* if $\gamma(0) = \gamma(T)$ and for every $t_1 \le t_2 \in [0,T]$, $\gamma_{|[t_1,t_2]}$ is a minimal geodesic or $\gamma \setminus \gamma_{|[t_1,t_2]}$ is a minimal geodesic.

Let us recall the expression of the geodesic distance on the circle $\mathbb{S}_1$ identified to $[0, 2\pi[$:

$$d_{\mathbb{S}^1}(\theta_1, \theta_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|). \tag{1.44}$$

The following proposition is elementary and explains why we are interested in minimal closed geodesics in Chapter 3.

Proposition 1.19. *Let $\gamma : [0,T] \to M$ be a minimal closed geodesic. For every θ_1 and θ_2 in $[0, 2\pi[$ we have*

$$d_M\left(\gamma\left(\frac{T\theta_1}{2\pi}\right), \gamma\left(\frac{T\theta_2}{2\pi}\right)\right) = \frac{L(\gamma)}{2\pi} \cdot d_{\mathbb{S}^1}(\theta_1, \theta_2).$$

Definition 1.28 (Exponential map)**.** Given $p \in M$ and $v \in T_pM$ we know there exists at most one geodesic $g : [0,1] \to M$ such that $g(0) = p$ and $g'(0) = v$ (see Remark 1.21). When g exists we set $\mathrm{Exp}_p(v) := g(1)$. Exp is called the *exponential map.*

Remark 1.28. In general the set of all (p, v) such that $\mathrm{Exp}_p(v)$ is defined is only a subset of the tangent bundle $TM := \{(p, v) \mid v \in T_pM\}$.

Remark 1.29. It is possible to show that $(p, v) \mapsto \mathrm{Exp}_p(v)$ is a C^∞ map (see again Remark 1.21).

In Chapter 3 we assume most of the time that Riemannian manifolds are complete. We recall the definition of completeness and state the classical results we need.

Definition 1.29 (Completeness)**.** A Riemannian manifold is said to be *complete* if every geodesic can be extended to a geodesic defined on all $\mathbb{R}$.

Proposition 1.20. *The following assertions are equivalent:*

1. *The Riemannian manifold M is complete.*
2. *The metric space (M, d_M) is complete.*
3. *The exponential map is defined on the whole tangent bundle.*

Proposition 1.21. *If M is complete, for every p and q in M there exists a minimal geodesic between p and q.*

1.4.3 First Variation of arc-length

We recall here the first variation formula, which we use in Chapter 3.

Definition 1.30 (Variation)**.** A *variation* of a C^∞ curve $c : [a, b] \to M$ is a C^∞ map $V : [a, b]\times] - \varepsilon, \varepsilon[\to M$ such that $V(s, 0) = c(s)$.

Let us denote by

$$T_{(s,t)}V : \mathbb{R}^2 \to T_{V(s,t)}M$$

the differential map of V at $(s, t) \in [a, b]\times] - \varepsilon, \varepsilon[$ (see [9] for a definition). Let Y be the vector field along c given for every $s \in [a, b]$ by

$$Y(s) := \left(T_{(s,0)}V\right)(0, 1).$$

Proposition 1.22 (First variation formula)**.** *For any variation*

$$(s,t) \mapsto V(s,t) = c_t(s)$$

of a C^∞ curve c parametrised by arc-length, the function $t \mapsto L(c_t)$ is differentiable at $t = 0$ and

$$\frac{d}{dt}L(c_t)\Big|_{t=0} = \Big[\Big\langle Y(s), c'(s)\Big\rangle_M\Big]_{s=a}^{s=b} - \int_a^b \Big\langle Y(s), \frac{D}{ds}c'(s)\Big\rangle_M ds.$$

1.4.4 Curvature

Curvature of Riemannian manifolds is a rich notion we will barely use. In general one can consider the curvature tensor of a Riemannian manifold from which it is possible to derive several notions of curvature. Let us precise what we mean when we refer to curvature in Chapter 2. *Sectional curvature* of a Riemannian manifold is a quantity $K(P, \mathcal{S})$ which depends on a point $P \in M$ and on $\mathcal{S}$ a linear subspace of dimension 2 of the tangent space at P (we refer to [9] for a definition). We say a manifold have nonpositive curvature if $K(P, \mathcal{S})$ is nonpositive for every P and $\mathcal{S}$.

In the case of surfaces let us mention that the sectional curvature depends only on the point P and coincides with the notion of *Gaussian curvature* $K(P)$. We refer to [6] for details on Gaussian curvature and a proof of the following result:

Theorem 1.4 (Gauss-Bonnet)**.** *Let S be a surface and ABC a triangle which edges are geodesics, and such that the interior I of ABC is simply connected. We have*

$$\hat{A} + \hat{B} + \hat{C} = \pi + \int_I K(P)dP,$$

where $\hat{A}, \hat{B}, \hat{C}$ denote the interior angles at the vertices A, B, C and dP is the area element on S.

1.4.5 Riemannian products

We recall in this section a few facts about Riemannian products.

Given two differential manifolds M and N, the Cartesian product $M \times N$ has a natural structure of differential manifold. Furthermore for every (p, q) in $M \times N$,

$$T_{(p,q)}(M \times N) = T_pM \times T_qN.$$

For every $u \in T_{(p,q)}(M \times N)$ we will write $u = (u_M, u_N)$.

Definition 1.31 (Riemannian product)**.** Let M, N be two Riemannian manifolds. For every $u, v \in T_{(p,q)}(M \times N)$ we define the *product metric*

$$\langle u, v\rangle_{M\times N} := \langle u_M, v_M\rangle_M + \langle u_N, v_N\rangle_N. \tag{1.45}$$

The Riemannian manifold $(M \times N, \langle\ ,\ \rangle_{M\times N})$ is called the *Riemannian product* of M and N.

Remark 1.30. From now on when we write $M \times N$ for two Riemannian manifolds we always refer to the Riemannian product of M by N, unless otherwise specified.

Proposition 1.23. *1. Given $T > 0$, a curve*

$$\begin{aligned} g : [0, T] &\to M \times N \\ t &\mapsto (m(t), n(t)) \end{aligned}$$

is a geodesic in $M \times N$ if and only if $m : [0, T] \to M$ is a minimal geodesic in M and $n : [0, T] \to N$ is a minimal geodesic in N.

2. $\forall (p_1, q_1), (p_2, q_2) \in M \times N$,

$$d_{M\times N}((p_1, q_1), (p_2, q_2)) = \left(d_M(p_1, p_2)^2 + d_N(q_1, q_2)^2\right)^{1/2}.$$

Proof. 1. Let us fix $T > 0$ and consider any piecewise differentiable curve

$$\begin{aligned} c : [0, T] &\to M \times N \\ t &\mapsto (c_M(t), c_N(t)). \end{aligned}$$

We have the following relations between energies

$$E_{M\times N}(c) = E_M(c_M) + E_N(c_N). \tag{1.46}$$

Indeed (1.45) and Definition 1.26 of the energy yields

$$\begin{aligned} E_{M\times N}(c) &= \frac{1}{2}\int_0^T \langle c'(t), c'(t)\rangle_{M\times N} dt \\ &= \frac{1}{2}\int_0^T \langle c_M'(t), c_M'(t)\rangle_M dt + \frac{1}{2}\int_0^T \langle c_N'(t), c_N'(t)\rangle_N dt. \end{aligned}$$

Among curves from $[0, T]$ to $M \times N$, M, or N, the minimal geodesics between two points are the curves that minimise the energy (see Proposition 1.17). From (1.46) we deduce directly point 1. of the proposition.

2. Let us consider a minimal geodesic

$$m \times n : t \mapsto (m(t), n(t)),$$

with $m : [0,T] \to M$ a minimal geodesic between p_1 and p_2 in M and $n : [0,T] \to N$ a minimal geodesic between q_1 and q_2 in N. Since m, n and $m \times n$ are geodesics they have constant speed and Cauchy-Schwarz equality case (see Remark 1.26) allows use to write

$$E_M(m) = \frac{L(m)^2}{2},$$

$$E_N(n) = \frac{L(n)^2}{2},$$

$$E_{M\times N}(m \times n) = \frac{L(m \times n)^2}{2}.$$

We deduce

$$L(m \times n) = \left(L(m)^2 + L(n)^2\right)^{1/2},$$

and we use the fact that $m, n, m \times n$ are minimal geodesics to conclude. □

Chapter 2

Existence of Lévy Brownian field indexed by Cartan-Hadamard surfaces

2.1 Introduction

While in Chapter 3 and Chapter 4 we give nonexistence results for fractional Brownian fields, in [23] Morozova and Chentsov give an argument to prove the existence of the Lévy Brownian field (corresponding to $H = 1/2$) indexed by any simply connected surface of nonpositive curvature (Cartan-Hadamard surfaces). In this chapter we give some details about this proof. Let us stress out that we do not claim any new result or method here: we simply give a more detailed version of Morozova and Chentsov's discussion.

2.2 Lévy-Chentsov construction for Gaussian fields

In this section we recall a general construction of Gaussian random fields from integration of Gaussian white noise. We refer to [17] for details and classical examples of this method.

Definition 2.1 (Gaussian white noise)**.** Let $(\mathcal{E}, \mathscr{P}, \mu)$ be a measure space and $\mathscr{P}_0 := \{A \in \mathscr{P},\ \mu(A) < \infty\}$. A *Gaussian white noise* with control measure μ is a real-valued centred random field $(\mathcal{W}(A))_{A\in\mathscr{P}_0}$ with covariance

$$\mathbb{E}(\mathcal{W}(A)\mathcal{W}(B)) = \mu(A \cap B). \tag{2.1}$$

Since the construction of the Gaussian fields we give here relies on a Gaussian white noise, we detail the following result.

Proposition 2.1 (Existence of Gaussian white noises)**.** *For every measure space $(\mathcal{E}, \mathscr{P}, \mu)$ there exists a Gaussian white noise with control measure μ.*

Proof. Let us prove that the covariance (2.1) is a kernel of negative type on $\mathscr{P}_0$. For every $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$ and $A_1, \cdots, A_n \in \mathscr{P}_0$ we have

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \mu(A_i \cap A_j) = \sum_{i,j=1}^{n} \lambda_i \lambda_j \int \mathbb{1}_{A_i} \mathbb{1}_{A_j} d\mu = \int \left(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i} \right)^2 d\mu \geq 0.$$

The existence of the Gaussian white noise is insured (see Proposition 1.3). □

The following Proposition is a simple consequence from the definition of a Gaussian white noise.

Proposition 2.2. *Let E be a set, $(\mathcal{E}, \mathscr{P}, \mu)$ a measure space and $\mathcal{W}$ a Gaussian white noise with control measure μ. For every map*

$$\Phi : E \to \mathscr{P},$$

the formula

$$X_P := \mathcal{W}\left(\Phi(P)\right)$$

defines a centred Gaussian random field $(X_P)_{P \in E}$ such that

$$\mathbb{E}(X_P X_Q) = \mu\left(\Phi(P) \cap \Phi(Q)\right).$$

Remark 2.1. Let us notice that

$$\begin{aligned} \mathbb{E}(X_P X_Q) &= \mu\left(\Phi(P) \cap \Phi(Q)\right) \\ &= \frac{1}{2}\Big(\mu(\Phi(P)) + \mu(\Phi(Q)) - \mu(\Phi(P)\Delta\Phi(Q))\Big), \end{aligned}$$

where $\Phi(P)\Delta\Phi(Q) := (\Phi(P) \setminus \Phi(Q)) \cup (\Phi(P) \setminus \Phi(Q))$ is the symmetric difference of the two sets $\Phi(P)$ and $\Phi(Q)$.

Remark 2.2. It is possible to define the integral $\int f d\mathcal{W}$ of any function $f \in L^2(\mathcal{E}, \mathscr{P}, \mu)$, starting with

$$\int \sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i} d\mathcal{W} := \sum_{i=1}^{n} \lambda_i \mathcal{W}(A_i)$$

and using the density of step functions $\sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i}$ in $L^2(\mathcal{E}, \mathscr{P}, \mu)$. Doing this one ends up with an Hilbert space of Gaussian random variables $\{\mathcal{W}(f)\}$ which is isometric to $L^2(\mathcal{E}, \mathscr{P}, \mu)$. In most situations $L^2(\mathcal{E}, \mathscr{P}, \mu)$ is separable and one can consider an orthogonal basis $(f_n)_{n\in\mathbb{N}}$ of $L^2(\Omega, \mathcal{A}, \mathbb{P})$, a sequence $(\varepsilon_n)_{\mathbb{N}}$ of i.i.d. standard Gaussian random variables, and set

$$\int f d\mathcal{W} := \sum_{i=0}^{\infty} \langle f, f_n \rangle_{L^2} \varepsilon_n.$$

Integration of L^2 functions with respect to a Gaussian white noise allows for construction of random fields indexed by a set S by considering $(f_P)_S$ in some L^2 space and setting $X_P := \int f_P d\mathcal{W}$. We mention this construction which is used to construct fractional fields (see for example [26]), however Proposition 2.2 is a simpler case which is sufficient in the case of Cartan-Hadamard surfaces. We again refer to [17] for details, proofs and examples of such constructions.

In [3] Chentsov uses a Gaussian white noise to construct a Lévy Brownian field indexed by the Euclidean spaces $\mathbb{R}^d$. In this case and with the notations of Proposition 2.2, $\mathcal{E}$ is the set of all hyperplanes (affine spaces of dimension $d-1$) and $\Phi(P)$ the set of hyperplanes intersecting the segment OP, where O is an arbitrary origin. The measure μ is the kinematic measure on the set of hyperplanes (see [24]). The idea of Chentsov and Morozova is to generalise the case $d = 2$ to a Cartan-Hadamard surface. In this case $\mathcal{E}$ is the manifold of oriented geodesics of S, and μ is the Liouville measure, which we briefly introduce in the next section.

Remark 2.3. As far as we know Lévy is the first author to use this kind of construction to show that there exists a Lévy Brownian field indexed by the sphere $\mathbb{S}^2$ in [15]. Let us mention that Takenaka generalised the method Chentsov uses in the Euclidean case to construct in a unified way Lévy Brownian fields indexed by the spheres $\mathbb{S}^d$, the Euclidean spaces $\mathbb{R}^d$, and the hyperbolic spaces $\mathbb{H}^d$ in [25], as well as stable fractional fields indexed by Euclidean spaces in [26].

2.3 Liouville measure on the manifold of geodesics

In all that follows S is a complete Riemannian manifold of dimension 2.

Duality between the unitary cotangent and tangent bundles Let us denote by SS the *unitary tangent bundle* of S: SS contains the elements (x, v) of the tangent bundle TS such that $\|v\|_S = 1$. Denote by T^*S the cotangent bundle of S: $(x, p) \in T^*S$ if and only if $x \in S$ and $p \in (T_xS)^*$ is a linear form on T_xS. We endow each $(T_xS)^*$ with the dual norm

$$\|p\|_S^* := \max_{\|u\|_S=1} \|p(u)\|_S, \tag{2.2}$$

and denote by S^*S the set of $(x, p) \in T^*S$ such that $\|p\|_S^* = 1$. S^*S is the *unitary cotangent bundle* of S.

Let us chose some local coordinates (x_1, x_2) in S. At each point $x \in S$ we denote by (v_1, v_2) the coordinates of $v \in T_xS$ in the canonical basis $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$, and (p_1, p_2) the coordinates of $p \in (T_xS)^*$ in the canonical basis (dx_1, dx_2). Define the map

$$\begin{aligned} \mathcal{L} : SS &\to S^*S \\ (x, v) &\mapsto \sum_{i=1}^{2} \frac{\partial \|v\|_S}{\partial v_i}\Big|_v dx_i, \end{aligned}$$

The definition of $\mathcal{L}$ does not depend on the choice of coordinates (x_1, x_2): indeed $\mathcal{L}(x, v)$ acts on T_xS as the differential of the norm $\| \, \|_S$ at point v.

The manifold of oriented geodesics Let us now associate to any curve

$$\begin{aligned} c : I &\to S \\ t &\mapsto c(t) \end{aligned}$$

such that for every $t \in I$, $\|c'(t)\|_S = 1$ the curve

$$\begin{aligned} \mathcal{L}(c) : I &\to S^*S \\ t &\mapsto \mathcal{L}(c(t), c'(t)) \ . \end{aligned}$$

Since S is complete every geodesic can be extended to geodesics defined on $\mathbb{R}$. In what follows we consider only geodesics defined on $\mathbb{R}$ and parametrised with speed equal to 1. Furthermore we look at those geodesics up to re-parametrisation by translation. We will call an *oriented geodesic* the resulting object (notice that there are two oriented geodesics for every geodesic curve $c \subset S$ we consider in Remark 1.24).

Since a geodesic is entirely determined by its position and speed at some point, it is clear that for every point (x, p) of S^*S there exists only one oriented geodesic g such that $\mathcal{L}(g)$ passes through (x, p). Given (x, p) and (x', p') we write $(x, p) \sim (x', p')$ if and only if there exists an oriented geodesic g such that $\mathcal{L}(g)$ goes through (x, p) and (x', p'). It is easy to check that $\sim$ is an equivalence relation. We denote by $\mathcal{G}$ the quotient $S^*S \,/\!\sim$. We admit that $\mathcal{G}$ has a canonical structure of differentiable manifold and that $\dim(\mathcal{G}) = 2$ (this follows from the fact that the $\mathcal{L}(g)$ are the leaves of a codimension 1 foliation of S^*S, see [24]).

The Liouville measure On S^*S let us consider the 2-form dG given by

$$dG = dp_1 \wedge dx_1 + dp_2 \wedge dx_2. \tag{2.3}$$

The following Lemma is proven in [24].

Lemma 2.1. *1. dG does not depend on the choice of coordinates (x_1, x_2).*

2. If we consider R a smooth domain in $\mathbb{R}^2$ and a C^∞ map

$$\mathcal{O} : R \to \mathcal{G},$$

choose for every $x \in R$ a parametrisation with unitary speed $t \mapsto g_x(t)$ of the geodesic $\mathcal{O}(x)$ such that the map $(x,t) \mapsto g_x(t)$ is C^∞ and set for every $t \in R$

$$\begin{aligned} \mathcal{O}_t : R &\to S^*S \\ x &\mapsto \mathcal{L}(g_x(t)), \end{aligned}$$

the quantity

$$\int_{\mathcal{O}_t(R)} dG$$

does not depend on $t \in \mathbb{R}$

From this Lemma we see that the integration of dG on S^*S is invariant by the geodesic flow, which allows to define without ambiguity the integration of dG on $\mathcal{G}$. Since $\dim(\mathcal{G}) = 2$ and dG is a 2-form, integration of $|dG|$ provides a measure μ_G on the σ-algebra $\mathcal{B}(\mathcal{G})$ of Borelian sets of $\mathcal{G}$, which is the Liouville measure μ_G (see for example [5] for details on integration of differential forms).

2.4 Construction on Cartan-Hadamard surfaces

Let us now consider S a Cartan-Hadamard surface: that is to say a complete and simply connected Riemannian manifold of dimension 2 with nonpositive curvature. Given any curve with values in S, let us denote by $\overline{c}$ the set of all oriented geodesics intersecting c.

The following properties will be useful:

Lemma 2.2. *1. For every $P, Q \in S$ there exists a unique (up to re-parametrisation) minimal geodesic PQ from P to Q.*

2. (Pasch's theorem) Every geodesic line intersecting a geodesic triangle ABC without passing through its vertices intersects exactly two of its edges. In particular for every distinct point A, B, C we have

$$\overline{AB} = \overline{AC} \; \Delta \; \overline{BC}.$$

3. For every geodesic $g : [a,b] :\to S$ we have

$$\mu_G(\overline{g}) = 4L(g).$$

Proof. 1. We admit this classical property of Cartan-Hadamard manifolds. See [9] for a proof.

2. Let us consider a geodesic $g(t)$ taking values outside the triangle ABC for any $t < 0$ and such that $g(0)$ belongs to the edge AB of the triangle. Let us remark that it is impossible that $g(t)$ take values inside the triangle for every $t \geq 0$: the interior of ABC is a compact set, hence we would obtain a converging sequence $g(t_n)$, which is impossible since for every t, $\|g'(t)\|_S = 1$. By the same argument we see that the geodesic g necessary intersects the triangle an even or infinite number of times. Let us finish the proof by showing that the geodesic g cannot intersect an edge more that once. If this happens, we have $t_1 < t_2$ such that $g(t_1)$ and $g(t_2)$ belongs to a geodesic segment. Taking any time $t_3 \in]t_1, t_2[$ we obtain a geodesic triangle $g(t_1)g(t_3)g(t_2)$ with an angle π at $g(t_3)$: hence the sum of the angles of the triangle is strictly more than π, in contradiction of Gauss-Bonnet theorem in nonpositive curvature (see Theorem 1.4).

3. Let us consider a geodesic g. Without loss of generality we set

$$g : [0, L] \to S$$

and assume g is parametrised by arc-length. Let us denote by $g^{s,\varphi} \in \mathcal{G}$ the oriented geodesic passing through $g(s)$ with an angle φ. Let us remark that a geodesic $g^{s,\varphi}$ intersects g at only one point (otherwise we get a geodesic triangle with the sum of its angles strictly greater that π which is impossible, see point 2 of the proof). This implies that we can identify $\overline{g}$ with $[0, L] \times [0, 2\pi[$.

We choose normal coordinates (x_1, x_2) in the neighbourhood of g such that x_1 coincides with the arc-length parameter s on g, and parametrise $g^{s,\varphi}$ such that $g^{s,\varphi}(0) = g(s)$). In the coordinates (x_1, x_2, p_1, p_2) on S^*S we have

$$\mathcal{L}((g^{s,\varphi})'(0)) = (s, 0, \cos(\varphi), \sin(\varphi)).$$

From Lemma 2.1 we know that $\mu_G(\overline{g})$ can be computed by integrating $|dG|$ at any time t of the geodesics $g^{s,\varphi}(t)$. In particular at $t = 0$ if we set

$$\overline{g}(0) := \{\mathcal{L}((g^{s,\varphi})'(0)), (s, \varphi) \in [0, L] \times [0, 2\pi]\}$$

we obtain

$$
\begin{aligned}
\mu_G(\overline{g}) &= \int_{\overline{g}(0)} |dG| \\
&= \int_{\overline{g}(0)} |dp_1 \wedge dx_1 + dp_2 \wedge dx_2| \\
&= \int_0^L \int_0^{2\pi} \left[\left| \frac{\partial(p_1, x_1)}{\partial(s, \varphi)} \right| + \left| \frac{\partial(p_2, x_2)}{\partial(s, \varphi)} \right| \right] ds d\varphi
\end{aligned}
$$

We compute the Jacobian determinants

$$\frac{\partial(p_1, x_1)}{\partial(s, \varphi)} = \sin(\varphi)$$

and

$$\frac{\partial(p_2, x_2)}{\partial(s, \varphi)} = 0$$

to obtain

$$\mu_G(\overline{g}) = \int_0^L \int_0^{2\pi} |\sin(\varphi)| ds d\varphi = 4L. \quad \square$$

Theorem 2.1. *There exists a Lévy Brownian field indexed by S.*

Proof. Let us choose an arbitrary origin $O \in S$. Define

$$
\begin{aligned}
\Phi : S &\to \mathcal{G} \\
P &\mapsto \overline{OP}.
\end{aligned}
$$

$\Phi(P)$ is the set of all the oriented geodesics intersecting the minimal geodesic OP. Consider $\mathcal{W}$ a Gaussian white noise with control measure μ_G the Liouville measure on $\mathcal{G}$. From Proposition 2.2 and Remark 2.1 we know that

$$X_P := \mathcal{W}\left(\Phi(P)\right)$$

is a centred Gaussian random field $(X_P)_{P \in E}$ with covariance function

$$
\begin{aligned}
\mathbb{E}(X_P X_Q) &= \frac{1}{2}\Big(\mu_G(\Phi(P)) + \mu_G(\Phi(Q)) - \mu_G(\Phi(P)\Delta\Phi(Q))\Big) \\
&= \frac{1}{2}\Big(\mu_G(\overline{OP}) + \mu_G(\overline{OQ}) - \mu_G(\overline{OP}\Delta\overline{OQ})\Big).
\end{aligned}
$$

From Lemma 2.2 we obtain

$$\begin{aligned}\mathbb{E}(X_P X_Q) &= \frac{1}{2}\Big(\mu_G(\overline{OP}) + \mu_G(\overline{OQ}) - \mu_G(\overline{PQ})\Big)\\ &= \frac{1}{2}\Big(4L(OP) + 4L(OQ) - 4L(PQ)\Big)\\ &= \Big(2d_S(O,P) + 2d_S(O,Q) - 2d_S(P,Q)\Big).\end{aligned}$$

This shows that $\frac{1}{2}(X_P)_{P\in S}$ is a Lévy Brownian field indexed by S with origin in O. $\square$

Remark 2.4. Let us remark that we do not need to consider 2-dimensional manifolds in order to define the manifold of oriented geodesics and the Liouville measure. Furthermore, Cartan-Hadamard manifolds of any dimension enjoy property 1 from Lemma 2.2. In a general Cartan-Hadamard manifolds it is not clear how to define the analogous of affine hyperplanes. In the case of constant curvature, let us recall that this construction is possible and was carried on by Takenaka in [25]. However Faraut and Harzallah showed in [7] that there exists no Lévy Brownian field indexed by the quaternionic hyperbolic space, which is a Cartan-Hadamard manifold. Hence it is clear that the construction cannot be generalised to any Cartan-Hadamard manifold.

Chapter 3

Perturbation of critical configurations

3.1 Introduction

In this chapter we give a necessary condition for the existence of fractional Brownian fields indexed by Riemannian manifolds (Theorem 3.1). We investigate this necessary condition for Riemannian manifolds with closed geodesics (Theorem 3.2), from which we derive nonexistence of Lévy Brownian field indexed by manifolds with a loop of minimal length among loops which are not in the constant homotopy class (Theorem 3.3). In particular we show that there exists no Lévy Brownian field indexed by nonsimply connected compact manifolds (Theorem 3.4). Furthermore we derive from Theorem 3.1 the nondegeneracy of fractional Brownian fields indexed by the hyperbolic spaces (Theorem 3.5).

3.2 Main result

We start by giving some definitions that will prove itselves handy. The first two are directly connected to the property that d^{2H} is of negative type.

Definition 3.1 (Configurations)**.** Given a metric space (E, d), we call a *configuration* and write $((P_1, \cdots, P_n), (c_1, \cdots, P_n))$ any finite collection of distinct points $(P_1, \cdots, P_n) \in E^n$ with $(c_1, \cdots, c_n) \in (\mathbb{R}^*)^n$ such that

$$\sum_{i=1}^{n} c_i = 0.$$

Definition 3.2 (Critical configurations)**.** Given $H > 0$, we say that a configuration is H-*critical* if

$$\sum_{i,j=1}^{n} c_i c_j d^{2H}(P_i, P_j) = 0.$$

Remark 3.1. Let us observe that if there exists an H-fractional Brownian field X^H indexed by (E, d) we have

$$\sum_{i,j=1}^{n} c_i c_j d^{2H}(P_i, P_j) = \mathrm{Var}\left(\sum_{i=1}^{n} c_i X^H_{P_i}\right).$$

In this case the configuration $((P_1, \cdots, P_n), (c_1, \cdots, c_n))$ is H-critical if and only if

$$\sum_{i=1}^{n} c_i X^H_{P_i} = 0 \textit{ almost surely.}$$

Definition 3.3 (Space of shortest directions)**.** Given a Riemannian manifold M, $P \in M$ and $S \subset M$, we define the *space of shortest directions from P to S*

$$T_{P\to S} = \mathrm{span}\left\{ \begin{array}{c} g'(0) \mid \exists Q \in S,\ g : [0,1] \to M \\ \text{minimal geodesic from } P \text{ to } Q \end{array} \right\},$$

where $\mathrm{span}(V)$ denotes the linear span of a set of vectors V: the space of shortest directions $T_{P\to S}$ is a vector subspace of the tangent space $T_P(M)$.

Let us state the main result of the chapter:

Theorem 3.1. *Let (M, d_M) be a complete Riemannian manifold and H in $]0, 1[$. If there exists an H-fractional Brownian field indexed by M, then for every H-critical configuration $((P_1, \cdots, P_n), (c_1, \cdots, c_n))$,*

$$\forall\ i \in \{1, \cdots, n\},\ \dim T_{P_i \to \{P_j, j\neq i\}} = \dim M. \tag{G}$$

Let us prove Theorem 3.1, starting with two lemmas.

Lemma 3.1. *Let M be a complete Riemannian manifold, $T > 0$, $A, B \in M$, and $g_m : [0, T] \to M$ a sequence of minimal geodesics in M such that $(g_m(0))_m$ and $(g_m(T))_m$ converge in M.*

There exists a minimal geodesic

$$g : [0, T] \to M$$

with $g(0) = \lim_{m\to+\infty} g_m(0)$ and $g(T) = \lim_{m\to+\infty} g_m(T)$, and a subsequence $g_{\varphi(m)}$ of g_m such that $g_{\varphi(m)}$ converges uniformly towards g. Furthermore $g'_{\varphi(m)}$ also converges towards g' uniformly.

Proof. If we take m is large enough the distance between $g_m(0)$ and $A := \lim\limits_{m\to+\infty} g_m(0)$ is short enough so there is a unique geodesic (up to reparametrisation) between those points (see Proposition 1.18). By parallel transport along it we identify the tangent space $T_{g_m(0)}$ to T_A. Because $g_m : [0,T] \to M$ is a minimal geodesic, $t \mapsto ||g'_m(t)||_M$ is constant. We deduce that

$$||g'_m(0)||_M = \frac{d_M(g_m(0, g_m(T))}{T}$$

is bounded in m. Recall that parallel transport along a curve is a linear isometry (see Proposition 1.15). The sequence $(g'_m(0))_{m\in\mathbb{N}}$, viewed as taking values in T_A is bounded and we extract $g'_{\varphi(m)}(0)$ converging to $v \in T_A$.

Since M is complete the exponential map is defined on the whole tangent bundle. For every t in $[0,T]$ we set

$$g(t) := \operatorname*{Exp}_A(tv).$$

As a linear isometry the parallel transport is C^∞, and so is Exp (see Remark 1.29), hence

$$g_{\varphi(m)}(t) = \operatorname*{Exp}_{g_{\varphi(m)}(0)}(tg'_{\varphi(m)}(0)) \underset{m\to\infty}{\longrightarrow} g(t),$$

as well as

$$g'_{\varphi(m)}(t) \underset{m\to\infty}{\longrightarrow} g'(t).$$

Because all arguments in the exponential belong to a compact set by Heine-Cantor theorem those convergences are uniform in t. As a consequence

$$L(g_{\varphi(m)}) \underset{m\to\infty}{\longrightarrow} L(g).$$

However

$$L(g_m) = d_M(g_m(0), g_m(T)) \underset{m\to\infty}{\longrightarrow} d_M(A,B)$$

hence g is a minimal geodesic. □

Lemma 3.2. *Let $P_1, \cdots, P_n$ be distinct points in a complete Riemannian manifold M and c be a C^∞ curve such that $c(0) = P_n$. There exists a sequence of positive ε_m converging towards zero such that for all $i \in \{1, \cdots, n-1\}$, there exists a minimal geodesic*

$$g_i : [0, d_M(P_i, P_n)] \to M$$

with $g_i(0) = P_i$ and $g_i(d_M(P_i, P_n)) = P_n$,

$$d_M(P_i, c(\varepsilon_m)) = d_M(P_i, P_n) + \langle c'(0), g'_i(d_M(P_i, P_n))\rangle_M \, \varepsilon_m + O\left(\varepsilon_m^2\right).$$

Proof. Let us set $T = d_M(P_1, P_n)$ and choose a decreasing sequence $\varepsilon_m > 0$ converging towards zero. Because M is complete there exists a sequence of minimal geodesics $g_{1,m} : [0,T] \to M$ between P_1 and $c(\varepsilon_m)$ (see Proposition 1.21). Using Lemma 3.1 we can assume that $g_{1,m}$ converges uniformly towards g_1 a minimal geodesic between P_1 and $c(0) = P_n$. Lemma 3.1 also gives the convergence of $\left(g'_{1,m}(0)\right)_m$ hence it is possible to find a C^∞ map

$$v :]-\varepsilon_0, \varepsilon_0[\to T_{P_1}M$$

such that $v(\varepsilon_m) = g'_{1,m}(0)$ for every $m \geq 0$. We now set

$$\begin{aligned} V : [0,T]\times]-\varepsilon_0,\varepsilon_0[&\to M \\ (s,\varepsilon) &\longmapsto \operatorname*{Exp}_{P_1}(v(\varepsilon)s), \end{aligned}$$

If we denote by $\mathcal{L}(\varepsilon)$ the length of the curve $s \mapsto V(s,\varepsilon)$, Proposition 1.22 gives

$$\frac{d}{d\varepsilon}\mathcal{L}(\varepsilon)\Big|_{\varepsilon=0} = \left[\left\langle \frac{\partial}{\partial\varepsilon}V(s,\varepsilon)\Big|_{(s,0)}, g'_1(s)\right\rangle_M\right]_{s=0}^{s=T} - \int_0^T \left\langle \frac{\partial}{\partial\varepsilon}V(s,\varepsilon)\Big|_{(s,0)}, \frac{D}{ds}g'_1(s)\right\rangle_M ds.$$

Because g_1 is a geodesic

$$\frac{D}{ds}g'_1(s) = 0,$$

hence the integral term is zero. Furthermore

- $\frac{\partial}{\partial\varepsilon}V(s,\varepsilon)\Big|_{(T,0)} = c'(0)$ since V is C^∞ and
$$V(T,\varepsilon_m) = c(\varepsilon_m) \text{ with } \lim_{m\to\infty}\varepsilon_m = 0.$$
- $\frac{\partial}{\partial\varepsilon}V(s,\varepsilon)\Big|_{(0,0)} = 0$ because $V(0,\varepsilon) = P_1$.

We obtain

$$\frac{d}{d\varepsilon}\mathcal{L}(\varepsilon)\Big|_{\varepsilon=0} = \left[\left\langle \frac{\partial}{\partial\varepsilon}V(s,\varepsilon)\Big|_{(s,0)}, g'_1(s)\right\rangle_M\right]_{s=0}^{s=T} = \langle c'(0), g'_1(T)\rangle_M,$$

hence we have

$$\mathcal{L}(\varepsilon) = \mathcal{L}(0) + \langle c'(0), g'_1(T)\rangle_M\,\varepsilon + O\left(\varepsilon^2\right).$$

Because $s \mapsto V(s,\varepsilon_m) = g_{1,m}(s)$ and $s \mapsto V(s,0) = g_1(s)$ are minimal geodesics we get

$$d_M(P_1, c(\varepsilon_m)) = d_M(P_1,P_n) + \langle c'(0), g'_1(T)\rangle_M\,\varepsilon_m + O\left(\varepsilon_m^2\right).$$

From the sequence ε_m we can extract $\varepsilon_{\varphi(m)}$ using Lemma 3.1 again and iterate the argument to get the result for every $d_M(P_i, c(\varepsilon_m))$. □

To prove Theorem 3.1 we will assume the existence of an H-critical configuration such that (G) does not hold and show that there exists no H-fractional Brownian field indexed by M.

Proof of Theorem 3.1. Let us consider H in $]0,1[$ and assume the existence of an H-critical configuration, that is to say distinct points $P_1, \cdots, P_n \in M$ and $c_1, \cdots, c_n \in \mathbb{R}^*$ such that

$$\sum_{i=1}^{n} c_i = 0$$

and

$$\sum_{i,j=1}^{n} c_i c_j d_M^{2H}(P_i, P_j) = 0.$$

Furthermore we suppose that the points $P_1, \cdots, P_n$ do not verify the geometrical condition (G), therefore

$$\exists i \in \{1, \cdots, n\},\ \dim T_{P_i \to \{P_j, j \neq i\}} < \dim M.$$

Without loss of generality we assume $i = n$ and consider a geodesic $g_\perp$ parametrised by arc-length with $g_\perp(0) = P_n$ and $g'_\perp(0) \in \left(T_{P_n \to \{P_j, j \neq n\}}\right)^\perp$. Using Lemma 3.2 we obtain a sequence of positive ε_m converging towards zero and geodesics $(g_i)_{1 \leq i \leq n-1}$ such that for every i in $\{1, \cdots, n-1\}$,

$$d_M(P_i, g_\perp(\varepsilon_m)) = d_M(P_i, P_n) + \langle g'_\perp(0), g'_i(d_M(P_i, P_n)) \rangle_M \, \varepsilon_m + O\left(\varepsilon_m^2\right).$$

Since $g'_\perp(0) \in \left(T_{P_n \to \{P_j, j \neq n\}}\right)^\perp$ we get for every $1 \leq i \leq n-1$

$$\langle g'_\perp(0), g'_i(d_M(P_i, P_n)) \rangle_M = 0,$$

Hence

$$d_M(P_i, g_\perp(\varepsilon_m)) = d_M(P_i, P_n) + O\left(\varepsilon_m^2\right), \tag{3.1}$$

from which

$$\begin{aligned} d_M^{2H}(P_i, g_\perp(\varepsilon_m)) &= \left(d_M(P_i, P_n) + O\left(\varepsilon_m^2\right)\right)^{2H} \\ &= d_M^{2H}(P_i, P_n) \left(1 + \frac{O\left(\varepsilon_m^2\right)}{d_M(P_i, P_n)}\right)^{2H} \\ &= d_M^{2H}(P_i, P_n) \left(1 + 2H \frac{O\left(\varepsilon_m^2\right)}{d_M(P_i, P_n)} + O\left(\varepsilon_m^4\right)\right). \end{aligned}$$

Using $H < 1$ we obtain

$$d_M^{2H}(P_i, g_\perp(\varepsilon_m)) = d_M^{2H}(P_i, P_n) + o\left(\varepsilon_m^{2H}\right). \tag{3.2}$$

Let us also notice that for m large enough, ε_m is small enough so that $g_\perp$ is a minimal geodesic between $g_\perp(\varepsilon_m)$ and P_n (see Proposition 1.16), hence

$$d_M^{2H}(P_n, g_\perp(\varepsilon_m)) = \varepsilon_m^{2H}. \tag{3.3}$$

We now set

$$P_{n+1} := g_\perp(\varepsilon_m) \tag{3.4}$$

and consider the configuration $((P_1, \cdots, P_n, P_{n+1}), (c'_1, \cdots, c'_{n+1}))$, with $(c'_1, \cdots, c'_{n-1}, c'_n, c'_{n+1}) = (c_1, \cdots, c_{n-1}, c_n/2, c_n/2)$ so that in particular

$$\sum_{i=1}^{n+1} c'_i = \sum_{i=1}^{n} c_i = 0.$$

Let us compute

$$\begin{aligned}
&\sum_{i,j=1}^{n+1} c'_i c'_j d_M^{2H}(P_i, P_j) \\
&= \sum_{i,j=1}^{n-1} c_i c_j d_M^{2H}(P_i, P_j) + 2\sum_{i=1}^{n-1} c_i \frac{c_n}{2} d_M^{2H}(P_i, P_n) + 2\sum_{i=1}^{n-1} c_i \frac{c_n}{2} d_M^{2H}(P_i, P_{n+1}) \\
&\qquad + 2\left(\frac{c_n}{2}\right)^2 d_M^{2H}(P_n, P_{n+1}).
\end{aligned}$$

Recalling (3.4) we use (3.2) and (3.3) to obtain

$$\begin{aligned}
&\sum_{i,j=1}^{n-1} c_i c_j d_M^{2H}(P_i, P_j) + 2\sum_{i=1}^{n-1} c_i \frac{c_n}{2} d_M^{2H}(P_i, P_n) + 2\sum_{i=1}^{n-1} c_i \frac{c_n}{2} \left(d_M^{2H}(P_i, P_n) + o\left(\varepsilon_m^{2H}\right)\right) \\
&+ 2\left(\frac{c_n}{2}\right)^2 \varepsilon_m^{2H} \\
&= \sum_{i,j=1}^{n} c_i c_j d_M^{2H}(P_i, P_j) + c_n \sum_{i=1}^{n-1} c_i \; o\left(\varepsilon_m^{2H}\right) + \frac{c_n^2}{2} \varepsilon_m^{2H}.
\end{aligned}$$

By hypothesis

$$\sum_{i,j=1}^{n} c_i c_j d_M^{2H}(P_i, P_j) = 0,$$

hence it is clear that

$$\sum_{i,j=1}^{n+1} c'_i c'_j d_M^{2H}(P_i, P_j) = \frac{c_n^2}{2}\,\varepsilon_m^{2H} + o\left(\varepsilon_m^{2H}\right)$$

is positive for m large enough. We conclude that d^{2H} is not of negative type and therefore there exists no H-fractional Brownian field indexed by (M, d_M). $\square$

Remark 3.2. To prove Theorem 3.1 we have exhibited a configuration $((P_1, \cdots, P_{n+1}), (c'_1, \cdots, c'_{n+1}))$ such that

$$\sum_{i=1}^{n+1} c'_i c'_j d^{2H}(P_i, P_j) > 0.$$

To obtain it from the critical configuration $((P_1, \cdots, P_n), (c_1, \cdots, c_n))$ we only added P_{n+1} as close as wanted to P_n and the coefficients c_i remained the same, except for P_n which "loses half of its coefficient to P_{n+1}", that is to say

$$c'_n = c'_{n+1} = c_n/2.$$

We can look at this new configuration as an infinitesimal perturbation of the critical configuration. Condition (G) from Theorem 3.1 is necessary to avoid that those perturbations of a critical configuration prevent the fractional Brownian motion to exist. Let us observe that while the perturbation happens in a neighbourhood of $(P_1, \cdots, P_n)$ it is impossible to decide whether $P_1, \cdots, P_n \in M$ verify (G) without considering the whole manifold (M, d), because condition (G) deals with minimal geodesics (see Remark 1.18).

3.3 Manifolds with a minimal closed geodesic

Let (M, d_M) be a Riemannian manifold with a minimal closed geodesic γ.

Definition 3.4 (Antipodal point)**.** For any P on γ, we call the antipodal point of P on γ, denoted by P^*, the unique point of γ such that

$$d_M(P, P^*) = L(\gamma)/2.$$

Many $1/2$-critical configurations on γ Consider now distinct points $P_1, \cdots, P_4 \in \gamma$ such that

$$P_3 = P_1^* \tag{3.5}$$

and

$$P_4 = P_2^*. \tag{3.6}$$

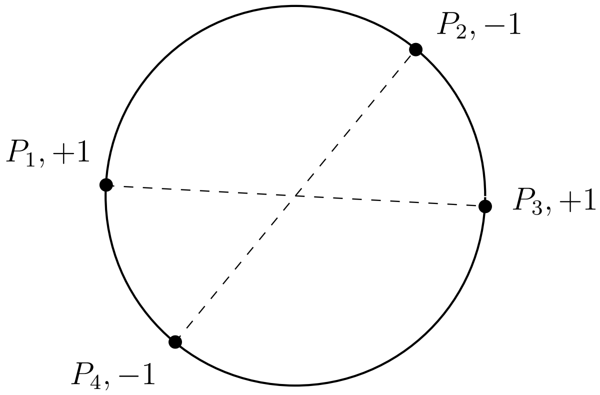

Figure 3.1: An 1/2-critical configuration on the circle

Because γ is a minimal closed geodesic we know that d_M restricted to γ is the distance on $\mathbb{S}^1$ up to a multiplication by $L(\gamma)/2\pi$ (see Proposition 1.19). Setting

$$c_1 = c_3 = 1, \text{ and } c_2 = c_4 = -1, \tag{3.7}$$

it is easy to check that

$$\sum_{i,j=1}^{4} c_i c_j d_M(P_i, P_j) = 0. \tag{3.8}$$

Observe that $c_1 + c_2 + c_3 + c_4 = 0$, so that $((P_1, P_2, P_3, P_4), (c_1, c_2, c_3, c_3))$ is a 1/2-critical configuration.

We already know that if there exists a Lévy Brownian field (*i.e.* a 1/2-fractional Brownian field) indexed by M, every distinct points $P_1, \cdots, P_4 \in \gamma$ verifying (3.5) and (3.6) must verify condition (G) from Theorem 3.1.

Furthermore it is possible to consider P_1, P_2, P_3, P_4 as required with $d_M(P_1, P_2)$ as small as wanted. Because P_1, P_3 and P_2, P_4 are respectively antipodal, $d_M(P_2, P_4) = d_M(P_1, P_3)$ is also as small as wanted. This allows us to give the following result.

Theorem 3.2. *Let (M, d_M) be a complete Riemannian manifold such that there exists a Lévy Brownian field indexed by (M, d_M). Then for every minimal closed geodesic γ and every $P \in \gamma$,*

$$\dim T_{P \to \{P^*\}} = \dim M.$$

Proof. Let us assume there exists $P \in \gamma$ such that $\dim T_{P \to \{P^*\}} < \dim M$. We take $P_4 = P$, $P_2 = P_4^*$, and choose for every $\eta \in]0, \pi[$ a point $P_3(\eta) \in \gamma$ such that $d_M(P_4, P_3(\eta)) = \eta > 0$. Finally we define $P_1(\eta) = (P_3(\eta))^*$ (see Figure 3.2 above).

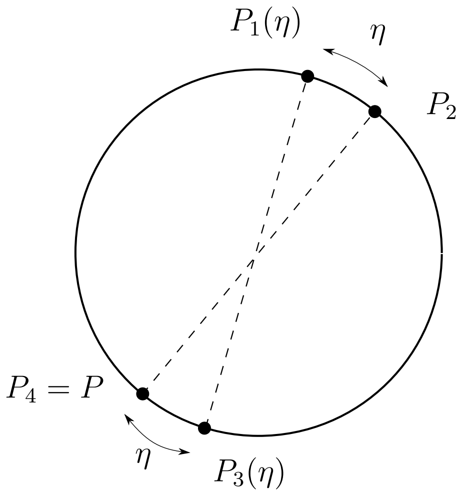

Figure 3.2: Disposition of the four points on γ

Please notice that we will sometimes write P_i with i taking values in $\{1, 2, 3, 4\}$: what we mean is $P_i(\eta)$ if $i \in \{1, 3\}$, and P_i if $i \in \{2, 4\}$.

Let $g_\perp$ be a geodesic parametrised by arc-length with $g_\perp(0) = P_4$ and $g'_\perp(0) \in \left(T_{P\to\{P^*\}}\right)^\perp$.

Using Lemma 3.2 we get a sequence of positive ε_m with $\lim \varepsilon_m = 0$ such that for each $i \in \{1, 2, 3\}$ there exists

$$g_i^\eta : [0, d_M(P_i, P_4)] \to M$$

a minimal geodesic with $g_i^\eta(0) = P_i$, $g_i^\eta(d_M(P_i, P_4)) = g_\perp(0) = P_4$ and such that

$$d_M(P_i, g_\perp(\varepsilon_m)) = d_M(P_i, P_4) + \left\langle g'_\perp(0), (g_i^\eta)'\left(d_M(P_i, P_4)\right)\right\rangle_M \varepsilon_m + O_\eta\left(\varepsilon_m^2\right). \quad (3.9)$$

Let us remark that the above expression is not uniform in m. Indeed the sequence (ε_m) depends on η, as much as the sequence $m \mapsto O_\eta(\varepsilon_m)$. This is not a problem as we fix η before we pass to the limit in m at the end of the proof.

We now distinguish three cases:

- i=1: Let us consider the following reparametrisations of the g_1^η:

$$\begin{aligned} \tilde{g}_1^\eta : [0, 1] &\to M \\ t &\mapsto g_1^\eta\big(t d_M\big(P_1(\eta), P_4\big)\big). \end{aligned}$$

Applying Lemma 3.1 we get a sequence of $\eta_n > 0$ converging towards zero such that $\tilde{g}_1^{\eta_n}$ converges to $\tilde{g}_1$ minimal geodesic between $\lim\limits_{n\to+\infty} P_1(\eta_n) = P_2$

and P_4 when n goes to infinity. We also have $\lim\limits_{n\to\infty} (\tilde{g}_1^{\eta_n})'(1) = \tilde{g}_1'(1)$ hence

$$(g_1^{\eta_n})'\big(d_M\big(P_1(\eta_n), P_4\big)\big) = \frac{(\tilde{g}_1^{\eta_n})'(1)}{d_M\big(P_1(\eta_n), P_4\big)} \underset{n\to+\infty}{\to} \frac{\tilde{g}_1'(1)}{d_M(P_2, P_4)}.$$

We obtain

$$\lim_{n\to\infty} \Big\langle g_\perp'(0), (g_1^{\eta_n})'\big(d_M\big(P_1(\eta_n), P_4\big)\big)\Big\rangle_M = \frac{\langle g_\perp'(0), \tilde{g}_1'(1)\rangle_M}{d_M(P_2, P_4)} = 0,$$

because $\tilde{g}_1$ is a minimal geodesic from P_2 to P_4 hence $\tilde{g}_1'(1) \in T_{P\to\{P^*\}}$.

- i=2: $\big\langle g_\perp'(0), (g_2^{\eta})'(d_M(P_2, P_4))\big\rangle_M = 0$ because

$$(g_2^{\eta})'(d_M(P_2, P_4)) \in T_{P\to\{P^*\}}.$$

- i=3: We again consider reparametrisations $\tilde{g}_3^{\eta_n} : [0,1] \to M$ of the $g_3^{\eta_n}$ and apply Lemma 3.1 to obtain the convergence of $\tilde{g}_3^{\eta_n}$ towards a minimising geodesic $\tilde{g}_3$ between $P_3(\eta_n)$ and P_4. For n large enough, $d_M(P_3(\eta_n), P_4) = \eta_n$ is small enough so that there exists a unique minimal geodesic between $P_3(\eta_n)$ and P_4 (up to reparametrisations). This proves that $\tilde{g}_3$ is included in γ when n is large enough hence $\tilde{g}_3'(1) \in T_{P\to\{P^*\}}$. Proceeding with the same computations we did for $i = 1$ we obtain

$$\lim_{n\to\infty} \big\langle g_\perp'(0), (g_3^{\eta_n})'\big(d_M\big(P_3(\eta_n), P_4\big)\big)\big\rangle_M = 0.$$

In the end for every i in $\{1, 2, 3\}$ we have

$$\lim_{n\to\infty} \big\langle g_\perp'(0), (g_i^{\eta_n})'(d_M(P_i, P_4))\big\rangle_M = 0. \tag{3.10}$$

We now follow exactly the proof of Theorem 3.1. Setting $P_5 = g_\perp(\varepsilon_m)$ and $(c_1', \cdots, c_5') = (c_1, c_2, c_3, c_4/2, c_4/2)$. Recall from (3.7) and (3.8) that $(c_1, \cdots, c_4) =$

$(1,-1,1,-1)$ so that $\sum_{i,j=1}^{4} c_i c_j d(P_i,P_j)=0$. We obtain

$$\begin{aligned}
&\sum_{i,j=1}^{5} c'_i c'_j d_M(P_i,P_j)\\
=&\sum_{i,j=1}^{3} c_i c_j d_M(P_i,P_j) + 2\sum_{i=1}^{3} c_i \frac{c_4}{2} d_M(P_i,P_4)\\
&+2\sum_{i=1}^{3} c_i \frac{c_4}{2}\Big[d_M(P_i,P_4) + \left\langle g'_\perp(0), (g_i^{\eta_n})'\left(d_M(P_i,P_4)\right)\right\rangle_M \varepsilon_m + O_{\eta_n}\left(\varepsilon_m^2\right)\Big]\\
&+2\left(\frac{c_4}{2}\right)^2 \varepsilon_m\\
=&\underbrace{\sum_{i,j=1}^{4} c_i c_j d_M(P_i,P_j)}_{=0} + c_4 \sum_{i=1}^{3} c_i \left(\left\langle g'_\perp(0), (g_i^{\eta_n})'\left(d_M(P_i,P_4)\right)\right\rangle_M \varepsilon_m + O_{\eta_n}\left(\varepsilon_m^2\right)\right)\\
&+2\left(\frac{c_4}{2}\right)^2 \varepsilon_m\\
=&\varepsilon_m\left(\frac{1}{2} - \sum_{i=1}^{3} c_i \left\langle g'_\perp(0), (g_i^{\eta_n})'\left(d_M(P_i,P_4)\right)\right\rangle_M\right) + O_{\eta_n}\left(\varepsilon_m^2\right).
\end{aligned}$$

Using (3.10) if we fix n large enough we have

$$\frac{1}{2} - \sum_{i=1}^{3} c_i \left\langle g'_\perp(0), (g_i^{\eta_n})'\left(d_M(P_i,P_4)\right)\right\rangle_M > 0.$$

In this case for m large enough

$$\sum_{i,j=1}^{5} c'_i c'_j d_M(P_i,P_j) > 0.$$

We conclude that d_M is not of negative type and therefore there exists no Lévy Brownian field indexed by (M, d_M). □

Example 3.1. Let us apply Theorem 3.2 to the ellipsoid of revolution $\mathcal{E}$ which a parametrisation is given by

$$\begin{aligned}
X : [0,2\pi]\times[-\pi/2,\pi/2] &\to \mathbb{R}^3\\
(\theta,\varphi) &\mapsto \begin{pmatrix} \cos(\varphi)\cos(\theta)\\ \cos(\varphi)\sin(\theta)\\ a\sin(\varphi)\end{pmatrix},
\end{aligned}$$

with $a > 1$.

We compute

$$\frac{\partial X}{\partial \theta} = \begin{pmatrix} -\cos(\varphi)\sin\theta \\ \cos(\varphi)\cos(\theta) \\ 0 \end{pmatrix},$$

$$\frac{\partial X}{\partial \varphi} = \begin{pmatrix} -\sin(\varphi)\cos(\theta) \\ -\sin(\varphi)\sin(\theta) \\ a\cos(\varphi) \end{pmatrix},$$

and deduce the coefficients of the first fundamental form of $\mathcal{E}$:

$$\begin{aligned} E_{\mathcal{E}} &= \left\langle \frac{\partial X}{\partial \theta}, \frac{\partial X}{\partial \theta} \right\rangle_{\mathbb{R}^3} = \cos^2(\varphi), \\ F_{\mathcal{E}} &= \left\langle \frac{\partial X}{\partial \theta}, \frac{\partial X}{\partial \varphi} \right\rangle_{\mathbb{R}^3} = 0, \\ G_{\mathcal{E}} &= \left\langle \frac{\partial X}{\partial \varphi}, \frac{\partial X}{\partial \varphi} \right\rangle_{\mathbb{R}^3} = \sin^2(\varphi) + a^2\cos^2(\varphi). \end{aligned}$$

We get the expression of the Riemannian metric

$$\langle\ ,\ \rangle_{\mathcal{E}} = \cos^2(\varphi)d\theta^2 + (\sin^2(\varphi)a + \cos^2(\varphi))d\varphi^2.$$

Let now consider the loop $t \mapsto \gamma(t) = (t, 0)$ in the coordinates (θ, φ) of $\mathcal{E}$. Consider $P = (\theta_P, 0)$ and $Q = (\theta_Q, 0)$ in the image of γ and

$$g : t \mapsto g(t) = (\theta(t), \varphi(t))$$

a minimal geodesic from P to Q in $\mathcal{E}$. We have

$$\begin{aligned} d_{\mathcal{E}}(P,Q) &= \int_0^1 \left[\cos^2(\varphi(t))\theta'(t)^2 + (\sin^2(\varphi(t)) + a^2\cos^2(\varphi(t)))\varphi'(t)^2\right]^{1/2} dt \\ &\geq \int_0^1 \left[\cos^2(\varphi(t))\theta'(t)^2 + \varphi'(t)^2\right]^{1/2} dt \\ &= L_{\mathbb{S}^2}(g) \geq d_{\mathbb{S}^2}(P,Q). \end{aligned}$$

Notice that if there exists t such that $\varphi'(t) \neq 0$, the first inequality is strict. For $Q = P^*$ antipodal to P on γ this would yield

$$d_{\mathcal{E}}(P, P^*) = \pi > d_{\mathbb{S}^2}(P, P^*) = \pi.$$

Since this is clearly impossible, for $Q = P^*$, φ' is identically zero and the geodesic g stays at every time included in γ. This shows that

$$\dim T_{P \to \{P^*\}} = 1 < \dim \mathcal{E} = 2.$$

From Theorem 3.2 we deduce that there exists no Lévy Brownian field indexed by $\mathcal{E}$.

Remark 3.3. This example has been previously dealt with by Chentsov and Morozova in [23], as a consequence of the following necessary condition for the Lévy Brownian field to exist when the index space is a manifold with minimal closed geodesics.

Theorem (Chentsov-Morozova). *Let M be a Riemannian manifold such that there exists a Lévy Brownian field indexed by (M, d_M). Then for all minimal closed geodesic γ and $Q \in M$ the quantity*

$$d_M(P, Q) + d_M(P^*, Q)$$

does not depend on $P \in \gamma$.

The proof by Chentsov and Morozova is based on the remark that for any Lévy Brownian field X_P indexed by $\mathbb{S}^1$,

$$\forall P \in \mathbb{S}^1, \; X_P + X_{P^*} = 2 \int_{\mathbb{S}^1} X_Q dQ,$$

where dQ denotes the uniform measure on $\mathbb{S}^1$. In particular this random variable does not depend on P. This directly implies that

$$\forall P, P' \in \mathbb{S}^1, \; X_P + X_{P^*} - X_{P'} - X_{P'_*} = 0 \text{ almost surely.}$$

Equivalently $((P, P^*, Q, Q^*), (1, -1, 1, -1))$ is a 1/2-critical configuration (see Remark 3.1), which is our starting point to prove Theorem 3.2.

This seems to suggest that there is a connexion between Theorem 3.2 and the condition by Chentsov and Morozova, though rigorously we are unable to say anything more about it. However using Theorem 3.2 we show there exists no Lévy Brownian field in cases where it seems uneasy to check whether Morozova and Chentsov's condition is verified or not (See Example 3.3, Theorem 3.3, and Theorem 3.4 below).

Example 3.2. Let us denote by $B(0,1) \subset \mathbb{R}^3$ the closed ball of center 0 and radius 1, and C a great circle of the unit sphere $\mathbb{S}^2$. Let $S \subset \mathbb{R}^3$ be a surface such that $S \cap B(0,1) = C$. Let us show that there exists no Lévy Brownian field indexed by S.

We consider

$$\begin{aligned} \Pi : \mathbb{R}^3 &\to \mathbb{R}^3 \\ X &\mapsto \frac{X}{\|X\|}. \end{aligned}$$

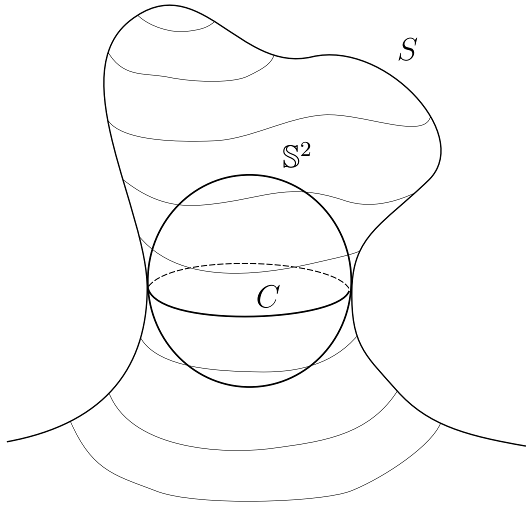

Figure 3.3: A surface S verifying our hypothesis

Lemma 3.3. For every curve c with values in $S \subset \mathbb{R}^3$,

$$L(c) \geq L\left(\Pi \circ c\right),$$

with equality if and only if c takes values in C.

Proof. For every $X = (x, y, z) \in \mathbb{R}^3$ we have

$$\Pi(X) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{1/2}}.$$

The map Π is differentiable on $\mathbb{R}^3 \setminus \{0\}$. For every $X \in \mathbb{R}^3 \setminus \{0\}$ we compute the Jacobian matrix

$$D\Pi_X = \frac{1}{\|X\|} \begin{pmatrix} 1 - \frac{x^2}{\|X\|^2} & \frac{xy}{\|X\|^2} & \frac{xz}{\|X\|^2} \\ \frac{yx}{\|X\|^2} & 1 - \frac{y^2}{\|X\|^2} & \frac{yz}{\|X\|^2} \\ \frac{zx}{\|X\|^2} & \frac{zy}{\|X\|^2} & 1 - \frac{z^2}{\|X\|^2} \end{pmatrix}.$$

We have

$$D\Pi_X = \frac{1}{\|X\|} \left(I_3 - n^{\mathsf{T}} n\right),$$

with

$$n = \overrightarrow{\nabla} \|X\| = \frac{1}{\|X\|} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let us notice that $I_3 - n^\mathsf{T} n$ is the matrix of the orthogonal projection on the plane of normal vector n. In particular for every $U \in \mathbb{R}^3$,

$$\|(I_3 - n^\mathsf{T} n)U\| \leq \|U\|,$$

hence for every X such that $\|X\| \geq 1$ and $U \in \mathbb{R}^3$,

$$\|D\Pi_X U\| \leq \|U\|,$$

with equality if and only if $\|X\| = 1$ and U is tangent to the unit sphere $\mathbb{S}^2$.

As a consequence for a curve $c : [a, b] \to S$, for every $t \in [a, b]$, $\|c(t)\| \geq 1$ and we obtain

$$L(\Pi \circ c) = \int_a^b \|(\Pi \circ c)'(t)\| dt = \int_a^b \|D\Pi_{c(t)} c'(t)\| dt \leq \int_a^b \|c'(t)\| dt = L(c). \quad \square$$

Let us now consider P and Q two points on the great circle C. Let c be a minimal geodesic from P to Q in S. From Lemma 3.3 we have

$$L(c) \geq L(\Pi \circ c).$$

Since $\Pi \circ c$ is a curve with values in $\mathbb{S}^2$ we know that $L(\Pi \circ c) \geq d_{\mathbb{S}^2}(P, Q)$. On the other the shorter arc of the great circle C joining P to P^* is a curve with values in S and length $d_{\mathbb{S}^2}(P, Q)$, which gives $d_{\mathbb{S}^2}(P, Q) \geq L(c)$ since c is a minimal geodesic. We obtain

$$d_{\mathbb{S}^2}(P, Q) \geq L(c) \geq L(\Pi \circ c) \geq d_{\mathbb{S}^2}(P, Q),$$

which means $L(c) = L(\Pi \circ c)$. The equality case of Lemma 3.3 states that c is included in the great circle C. We have shown that every minimal geodesic from P to Q is included in C, which shows that C is a closed minimal geodesic of S, together with

$$\dim T_{P \to \{P^*\}} = 1 < \dim S = 2$$

for every $P \in C$.

From Theorem 3.2 we know that there exists no Lévy Brownian field indexed by S.

Example 3.3 (rotation-invariant manifolds with a shortest parallel). Let us consider a complete Riemannian manifold N and consider $M = \mathbb{S}^1 \times N$ endowed with the Riemannian metric

$$\langle\ ,\ \rangle_M = f(z)\langle\ ,\ \rangle_{\mathbb{S}^1} + \langle\ ,\ \rangle_N,$$

where $f : N \to \mathbb{R}_+^*$ is a C^∞ function with a global minimum at $z_0 \in N$. We take two points P, Q on the parallel $\gamma = \mathbb{S}^1 \times \{z_0\}$ and a curve $g : [0, T] \to M$ with $g(0) = P,\ g(T) = Q$. We write $g(t) = (\theta(t), z(t))$ and compute the energy of g :

$$\begin{aligned} E_M(g) &= \frac{1}{2}\int_0^T \langle g'(t), g'(t)\rangle_M dt \\ &= \frac{1}{2}\int_0^T f(z)\,\langle \theta'(t), \theta'(t)\rangle_{\mathbb{S}^1} + \langle z'(t), z'(t)\rangle_N\, dt \\ &\geq \frac{1}{2}\int_0^T f(z_0)\,\langle \theta'(t), \theta'(t)\rangle_{\mathbb{S}^1}\, dt. \end{aligned}$$

From here it is clear that minimal geodesics between P and Q are included in γ (see Proposition 1.17). This shows that any proportional to arc-length parametrisation of γ is a minimal closed geodesic together with

$$\dim T_{P \to \{P*\}} = 1 < \dim M$$

for every $P \in \gamma$. We apply Theorem 3.2 to get the nonexistence of Lévy Brownian fields indexed by M.

Let us now recall some classical facts about free homotopy of loops.

Definition 3.5 (loops)**.** We call a loop any curve

$$\gamma : [0, T] \to M$$

with $\gamma(0) = \gamma(T)$.

Definition 3.6 (Free homotopy of loops)**.** We say that two loops γ_1, γ_2 are *freely homotopic* if there exists reparametrisations $\tilde{\gamma}_1, \tilde{\gamma}_2 : [0, 1] \to M$ and a homotopy of loops from $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$, that is to say a continuous map

$$\begin{aligned} f : [0,1] \times [0,1] &\to M \\ (s, t) &\longmapsto f_t(s) \end{aligned}$$

such that $f_0 = \tilde{\gamma}_1$ and $f_1 = \tilde{\gamma}_2$ and for every $t \in [0, 1],\ f_t : [0, 1] \to M$ is a loop. In this case we write $\gamma_1 \sim \gamma_2$.

Remark 3.4. The homotopy is said to be *free* in contrast to homotopy of loops with a fixed base point, which we do not discuss here.

We admit the following classical result about $\sim$, which proof is found in [10].

Proposition 3.1. $\sim$ *is an equivalence relation.*

In the following we denote by $\mathscr{L}(M)$ the set of all piecewise continuously differentiable loops with values in M. Notice that $L(\gamma)$ is properly defined for $\gamma \in \mathscr{L}(M)$. We denote by $\mathscr{C}_1(M) = \mathscr{L}(M)/\sim$ the set of all free homotopy classes of piecewise continuously differentiable loops in M. Furthermore we denote by $C \in \mathscr{C}_1(M)$ the class of piecewise continuously differentiable loops freely homotopic to any constant loop (recall that all the manifolds we consider are connected hence all the constant loops are freely homotopic).

Lemma 3.4. *Let M be a Riemannian manifold, and γ a loop of minimal length in $\mathscr{L}(M) \setminus C$. Then for all $P, Q \in \gamma$, all the minimal geodesics from P to Q are included in γ.*

Proof.

1. Let us assume that γ is of minimal length in $\mathscr{L}(M) \setminus C$ and show that every proportional to arc-length parametrisation of γ is a geodesic. Suppose that it is not the case. Clearly, there exists a proportional to arc-length parametrisation of γ and $t_1 < t_2$ with $t_2 - t_1$ as small as wanted such that $\gamma_{|[t_1,t_2]}$ is not a geodesic. Now we can take $t_2 - t_1$ small enough so that there exists a unique minimal geodesic $\gamma(t_1)\gamma(t_2) : [0, 1] \to M$ between $\gamma(t_1)$ and $\gamma(t_2)$ (see Proposition 1.18). It is clear that $\gamma(t_1)\gamma(t_2)$ is shorter than $\gamma_{|[t_1,t_2]}$ (otherwise $\gamma_{|[t_1,t_2]}$ is a minimal geodesic, which is impossible since it is not a geodesic). Hence the concatenation of $\gamma(t_1)\gamma(t_2)$ with $\gamma \setminus \gamma_{|[t_1,t_2]}$ (see (1.43) for a parametrisation) is a loop $\tilde{\gamma}$ with shorter length than γ. Now we can take $t_2 - t_1$ small enough to have $\gamma_{|[t_1,t_2]}$ and $\gamma(t_1)\gamma(t_2)$ taking values in a common geodesically convex ball. Therefore $\gamma(t_1)\gamma(t_2)$ is homotopic to $\gamma_{|[t_1,t_2]}$ (see Remark 1.27), and finally γ and $\tilde{\gamma}$ are homotopic. In the end $\tilde{\gamma} \in \mathscr{L}(M) \setminus C$ is shorter than γ. We have reached a contradiction.

2. Now let us assume there exist $P, Q \in \gamma$ and a minimal geodesic g between P and Q, not included in γ. Without loss of generality let us assume

$$\gamma : [0, 1] \to M$$

with

$$\gamma(0) = \gamma(1) = Q, \gamma(t_P) = P,$$

and

$$g : [0, 1] \to M$$

with

$$g(0) = P,\ g(1) = Q.$$

We define

$$\begin{aligned} \gamma_1 : [0, t_P] &\to M \\ t &\mapsto \gamma(t) \end{aligned} \qquad\qquad \begin{aligned} \gamma_2 : [t_P, 1] &\to M \\ t &\mapsto \gamma(t) \end{aligned}$$

the two halves of γ connecting P to Q. Let us consider $l_1 = \gamma_1 \cdot g$ (the concatenation of γ_1 and g), and $l_2 = \overleftarrow{g} \cdot \gamma_2$, where $\overleftarrow{g} : t \mapsto g(1-t)$. Because g is a minimal geodesic between P and Q we have

$$L(g) \leq L(\gamma_1) \text{ and } L(g) \leq L(\gamma_2),$$

hence

$$L(l_1) \leq L(\gamma) \text{ and } L(l_2) \leq L(\gamma). \tag{3.11}$$

Now l_1 is not C^∞ at P: indeed $g'(0)$ cannot be equal to $\gamma'(t_P)$, otherwise we would have

$$g(t) = \operatorname*{Exp}_p\left(\gamma'(t_P)t\right) = \begin{cases} \gamma(t_P + t) \text{ if } t_P + t, \leq 1 \\ \gamma(t_P + t - 1) \text{ elsewise.} \end{cases}$$

(Recall a geodesic is completely determined by some initial conditions on position and speed, see Remark 1.21). This is impossible since we assumed g is not included in γ. Since l_1 is not C^∞ it cannot be a geodesic, hence l_1 is not of minimal length in $\mathscr{L}(M) \setminus C$ (see part 1 of the proof). Because of (3.11), it is then clear that l_1 belongs to C. The same is true for l_2. We will now show that $\gamma \in C$ to get a contradiction.

Let us consider $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{g}, \overleftarrow{\tilde{g}}$ reparametrisations of $\gamma_1, \gamma_2, g, \overleftarrow{g}$ over $[0, 1/2]$. It is clear that $\tilde{l}_1 := \tilde{\gamma}_1 \cdot \tilde{g}$ and $\tilde{l}_2 := \overleftarrow{\tilde{g}} \cdot \tilde{\gamma}_2$ are reparametrisations over $[0, 1]$ of l_1 and l_2. In particular $\tilde{l}_1, \tilde{l}_2 \in C$ are homotopic to the constant loop $\overline{Q}$.

Let us consider $(s, t) \mapsto l^1_t(s)$ and $(s, t) \mapsto l^2_t(s)$ two free homotopies of loops such that

$$l^1_0 = \tilde{l}_1, \ l^1_1 = \overline{Q}$$

and

$$l^2_0 = \tilde{l}_2, \ l^2_1 = \overline{Q},$$

Let us now give an explicit free homotopy from γ to $\overline{Q}$. For every $0 \leq t \leq 1/2$ let us consider

$$\begin{aligned} \tilde{g}_t : [0, 1/2] &\to M \\ s &\mapsto \tilde{g}(st) \end{aligned} \qquad \begin{aligned} \overleftarrow{\tilde{g}_t} : [0, 1/2] &\to M \\ s &\mapsto \tilde{g}(t - st). \end{aligned}$$

and define

$$\gamma_t = \begin{cases} \tilde{\gamma}_1 \cdot \tilde{g}_t \cdot \overleftarrow{\tilde{g}_t} \cdot \tilde{\gamma}_2, & 0 \leq t \leq 1/2, \\ l^1_{2t-1} \cdot l^2_{2t-1}, & 1/2 \leq t \leq 1. \end{cases}$$

It is clear that for every $t \in [0, 1]$,

$$\begin{aligned} \gamma_t : [0, 2] &\to M \\ s &\mapsto \gamma_t(s) \end{aligned}$$

is a loop. We reparametrise γ_t over $[0,1]$ by setting $\tilde{\gamma}_t(s) = \gamma_t(s/2)$ to obtain a free homotopy of loops such that $\gamma_1 = \overline{Q}$ and γ_0 is a reparametrisation of γ. In the end γ is freely homotopic to $\overline{Q}$, which proves $\gamma \in C$: we have reached a contradiction. □

Theorem 3.3. *Let M be a Riemannian manifold of dimension at least 2 such that there exists γ of minimal length in $\mathscr{L}(M)\setminus C$. There exists no Lévy Brownian field indexed by M.*

Proof. From Lemma 3.4 we know that for every $P, Q \in \gamma$ all the minimal geodesics from P to Q are included in γ. In particular every proportional to arc-length parametrisation of γ is a closed minimal geodesic such that

$$\forall P \in \gamma,\ \dim T_{P \to \{P^*\}} = 1.$$

Since $\dim(M) \geq 2$, Theorem 3.2 shows that there exists no Lévy Brownian field indexed by M. □

For a compact manifold Cartan's theorem gives the existence of a closed minimal geodesic in every free homotopy class. We adapt its proof to obtain the following lemma.

Lemma 3.5. *Let M be a compact, nonsimply connected Riemannian manifold. There exists a loop γ of minimal length in $\mathscr{L}(M) \setminus C$.*

Proof. Let us consider

$$d := \inf\{L(l), l \in \mathscr{L}(M) \setminus C\}.$$

Since M is not simply connected $\mathscr{L}(M) \setminus C$ is not empty, hence $d > 0$ and there exists a sequence $(l_n)_{n\geq 0}$ of $\mathscr{L}(M) \setminus C$ such that

$$L(l_n) \underset{n\to\infty}{\longrightarrow} d. \tag{3.12}$$

Let us reparametrise l_n over $[0,1]$ and proportionally to arc-length in order to have $\forall t_1 \leq t_2 \in [0,1]$,

$$d_M(l_n(t_1), l_n(t_2)) \leq \int_{t_1}^{t_2} ||l_n'(t)||_M dt \leq \sup(L(l_n))(t_2 - t_1),$$

which shows that the set $\{l_n\}$ is equicontinuous. Because M is compact, for all $t \in [0,1]$, $\{l_n(t), n \in \mathbb{N}\}$ is relatively compact. By Arzelà-Ascoli theorem we conclude that $\{l_n\}$ is relatively compact in the uniform topology, hence we can extract a subsequence of l_n which converges uniformly to l_∞ a continuous loop in M. To obtain a piecewise continuously differentiable curve, we consider a partition of $[0,1]$

$$0 = t_0 < t_1 < \cdots < t_m = 1$$

with successive times t_k, t_{k+1} close enough to each other so that every $l_{\infty|[t_k,t_{k+1}]}$ takes value in a geodesically convex ball of M (see Proposition 1.18). We now consider the closed curve γ given by the concatenation of all the minimal geodesics $l_\infty(t_k)l_\infty(t_{k+1})$.

Let us notice that for n large enough $l_{n|[t_k,t_{k+1}]}$ takes values in the same geodesically convex ball as $\gamma_{|[t_k,t_{k+1}]} = l_\infty(t_k)l_\infty(t_{k+1})$, hence $l_{n|[t_k,t_{k+1}]}$ and $\gamma_{|[t_k,t_{k+1}]}$ are homotopic (see again Remark 1.27). In the end l_n is homotopic to γ, which shows that $\gamma \in \mathscr{L}(M) \setminus C$, hence $L(\gamma) \geq d$. We will now show that $L(\gamma) = d$ to finish the proof.

Let us assume that $L(\gamma) > d$. We write

$$L(\gamma) = L(\gamma) - d + d - L(l_n) + L(l_n).$$

We now use (3.12) to write $d - L(l_n) > -\varepsilon$ for n large enough. We obtain:

$$\begin{aligned} & L(\gamma) > L(\gamma) - d - \varepsilon + L(l_n) \\ \iff & \sum_{k=0}^{m-1} L(\gamma_{|[t_k,t_{k+1}]}) > \sum_{k=0}^{m-1} \left(\frac{L(\gamma) - d - \varepsilon}{m} + L(l_{n|[t_k,t_{k+1}]}) \right), \end{aligned}$$

which ensures there exists k such that

$$L(\gamma_{|[t_k,t_{k+1}]}) > \frac{L(\gamma) - d - \varepsilon}{m} + L(l_{n|[t_k,t_{k+1}]}). \tag{3.13}$$

Now let us recall that

$$\begin{aligned} l_n(t_k) &\underset{n\to\infty}{\longrightarrow} \gamma(t_k), \\ l_n(t_{k+1}) &\underset{n\to\infty}{\longrightarrow} \gamma(t_{k+1}), \end{aligned}$$

hence for n large enough

$$\frac{L(\gamma) - d - \varepsilon}{m} > d_M(l_n(t_k), \gamma(t_k)) + d_M(l_n(t_{k+1}), \gamma(t_{k+1})).$$

Together with (3.13) we obtain

$$L(\gamma_{|[t_k,t_{k+1}]}) > d_M(l_n(t_k), \gamma(t_k)) + d_M(l_n(t_{k+1}), \gamma(t_{k+1})) + L(l_{n|[t_k,t_{k+1}]}),$$

which contradicts the fact that $\gamma_{|[t_k,t_{k+1}]} = \gamma(t_k)\gamma(t_{k+1})$ is a minimal geodesic.

□

Theorem 3.4. *Let M be a compact, nonsimply connected Riemannian manifold of dimension at least 2. There exists no Lévy Brownian field indexed by M.*

Proof. Lemma 3.5 gives the existence of γ of minimal length in $\mathscr{L}(M) \setminus C$. Since M is compact it is complete as a metric space, hence a complete Riemannian manifold (see Proposition 1.20), and Theorem 3.3 applies. □

Example 3.4 (closed surfaces). In dimension 2 due to the classification of closed (*i.e.* compact without boundary) surfaces a closed surface admitting a Lévy Brownian field is homeomorphic to the sphere $\mathbb{S}^2$.

Remark 3.5. Let us notice that the ellipsoid from Example 3.1 is homeomorphic to $\mathbb{S}^2$ but does not admit a Lévy Brownian field. The spheres $\mathbb{S}^n$ are the only compact manifolds on which we know a Lévy Brownian field exists.

3.4 Nondegeneracy of fractional Brownian fields indexed by hyperbolic spaces

Let us recall some elementary facts on the hyperbolic spaces. We again refer to [9] for proofs and details.

Poincaré ball model There are many ways to present the hyperbolic space $\mathbb{H}^d$. We briefly introduce the Poincaré disk model.

Let us consider B_d the open ball of radius 1 in $\mathbb{R}^d$. We endow B_d with the Riemannian metric

$$\langle\ ,\ \rangle_d = \frac{4\displaystyle\sum_{i=1}^{d} dx_i^2}{\left(1-\displaystyle\sum_{i=1}^{d} x_i^2\right)^2}.$$

One can check that we obtain a complete Riemannian manifold of curvature -1, which we call the hyperbolic space of dimension d and denote by $\mathbb{H}^d$. It is well known that the geodesics of $\mathbb{H}^d$ are given by the arcs of the circles which intersect orthogonally the sphere $\mathbb{S}^{d-1}$ of radius 1 (By circles and sphere we mean: the usual circles and sphere from Euclidean geometry in $\mathbb{R}^d$).

Furthermore if we consider the unique circle of $\mathbb{R}^{d+1}$ passing through $P, Q \in B_d \times \{0\}$ which is orthogonal to $\mathbb{S}^d \times \{0\}$, we notice it is included in $\mathbb{R}^d \times \{0\}$ at all time and orthogonal to $\mathbb{S}^{d-1}$. This shows that the minimal geodesics between points of B_d are the same in $\mathbb{H}^d$ and $\mathbb{H}^{d+1}$, thus the inclusion $B_d \times \{0\} \subset B_{d+1}$

extends to an isometric immersion

$$\mathcal{I} : \mathbb{H}^d \hookrightarrow \mathbb{H}^{d+1},$$

that is to say

$$\forall P, Q \in \mathbb{H}^d,\ d_{\mathbb{H}^d}(P,Q) = d_{\mathbb{H}^{d+1}}(\mathcal{I}(P), \mathcal{I}(Q)). \tag{3.14}$$

Finally let us recall that for all $d \geq 1$ there exists an H-fractional Brownian field indexed by $\mathbb{H}^d$ if and only if $0 < H \leq 1/2$ (see [11]).

The following result is not surprising but should be quite tedious to prove with computations, if possible. Here we give a geometric proof of this fact.

Theorem 3.5. *1. For every $0 < H \leq 1/2$ there are no H-critical configurations in $\mathbb{H}^d$.*

2. Let $0 < H \leq 1/2$ and X^H be an H-fractional Brownian field indexed by the d-dimensional hyperbolic space $\mathbb{H}^d$, such that there exists $O \in \mathbb{H}^d$ and $X^H_O = 0$ a.s.. For all distinct $P_1, \cdots, P_n \in \mathbb{H}^d$ the Gaussian vector $(P_1, \cdots, P_n)$ is nondegenerate.

Proof. 1. Let us assume there exists an H-critical configuration $((P_1, \cdots, P_n), (c_1, \cdots, c_n))$ of $\mathbb{H}^d$. Using (3.14) it is clear that

$$\sum_{i,j=1}^{n} c_i c_j d^{2H}_{\mathbb{H}^d}(P_i, P_j) = 0$$
$$\Rightarrow \sum_{i,j=1}^{n} c_i c_j \left[d_{\mathbb{H}^{d+1}}(\mathcal{I}(P_i), \mathcal{I}(P_j))\right]^{2H} = 0,$$

which means $(\mathcal{I}(P_1), \cdots, \mathcal{I}(P_n), (c_1, \cdots, c_n))$ is an H-critical configuration of $\mathbb{H}^{d+1}$.

However we have seen that all the geodesics in $\mathbb{H}^{d+1}$ between the points of $\mathcal{I}(\mathbb{H}^d)$ are included in $\mathcal{I}(\mathbb{H}^d)$, therefore it is clear that

$$\forall\ i \in \{1, \cdots, n\},\ \dim T_{\mathcal{I}(P_i) \to \{\mathcal{I}(P_j), j \neq i\}} \leq \dim \mathcal{I}(\mathbb{H}^d) < \dim \mathbb{H}^{d+1}.$$

Condition (G) of Theorem 3.1 is not verified although there exists an H-fractional Lévy Brownian field indexed by $\mathbb{H}^{d+1}$. We have reached a contradiction.

2. Let us consider distinct $P_1, \cdots, P_n \in \mathbb{H}^d$ and $c_1, \cdots, c_n \in \mathbb{R}^*$ without further assumptions. Define $c_{n+1} = -\sum_{i=1}^n c_i$. Using $X^H_O = 0$ almost surely we can write

$$\sum_{i=1}^{n} c_i X^H_{P_i} = \sum_{i=1}^{n} c_i X^H_{P_i} + c_{n+1} X^H_O$$

which is not equal to zero almost surely using point 1. of the theorem. We have shown that the Gaussian vector $(P_1, \cdots, P_n)$ is nondegenerate.

$\square$

Remark 3.6. It is possible to follow the same argument to provide a short proof for the same facts about $\mathbb{R}^d$, which were already known (see for example [4]).

Chapter 4

Nonexistence of fractional fields indexed by cylinders

4.1 Introduction

In this chapter we show that for every H there exists no H-fractional Brownian motion indexed by the cylinder (Theorem 4.1). We generalise this result to the Riemannian product of the circle with any manifold (Theorem 4.2). We investigate the case of metric spaces for which the distance is asymptotically close to the cylinder distance (Theorem 4.3 and 4.4). From Theorem 4.1 we derive the discontinuity of the fractional index of a metric space with respect to the Gromov-Hausdorff convergence (Theorem 4.5).

Throughout this chapter we often use the term *configuration*, referring to Definition 3.1.

4.2 Main statement

In this section we consider the cylinder $\mathbb{S}^1 \times \mathbb{R}$ endowed with its Riemannian product metric

$$\langle\ ,\ \rangle_{\mathbb{S}^1\times\mathbb{R}} = d\theta^2 + dz^2. \tag{4.1}$$

From point 2. of Proposition 1.23 we know the expression of the geodesic distance

$$d_{\mathbb{S}^1\times\mathbb{R}}((\theta_1, z_1), (\theta_2, z_2)) = \left(d_{\mathbb{S}^1}(\theta_1, \theta_2)^2 + |z_1 - z_2|^2\right)^{1/2}, \tag{4.2}$$

where $d_{\mathbb{S}^1}$ is the geodesic distance on $\mathbb{S}^1$, given by

$$d_{\mathbb{S}^1}(\theta_1, \theta_2) = \min(|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|). \tag{4.3}$$

Remark 4.1. From point 1. of Proposition 1.23 we know that the geodesics in the cylinder are given by arcs of helices. In particular all the geodesics of the cylinder between points of $\mathbb{S}^1\times]0,\varepsilon[$ stay at all time in $\mathbb{S}^1\times]0,\varepsilon[$. As a consequence, the restriction of $d_{\mathbb{S}^1\times\mathbb{R}}$ to $\mathbb{S}^1\times]0,\varepsilon[$ and the geodesic distance associated to the metric (4.1) on $\mathbb{S}^1\times]0,\varepsilon[$ coincide.

Theorem 4.1. *For every $\varepsilon > 0$ and $H > 0$, there exists no H-fractional Brownian field indexed by the cylinder $\mathbb{S}^1\times]0,\varepsilon[$. In other terms,*

$$\beta_{\mathbb{S}^1\times]0,\varepsilon[} = 0.$$

Let us give an outline of the proof of the theorem. To prove the result we exhibit for every $0 < H < 1/2$ a sequence of configurations

$$((P^H_{1,n},\cdots,P^H_{n,n}),(c_1,\cdots,c_n))_{n\in\mathbb{N}}$$

such that

$$\lim_{n\to\infty}\sum_{i,j=1}^{n} c_i c_j d^{2H}_{\mathbb{S}^1\times]0,\varepsilon[}(P^H_{i,n},P^H_{j,n}) = +\infty,$$

which shows that $d^{2H}_{\mathbb{S}^1\times]0,\varepsilon[}$ is not a kernel of negative type. Hence there exists no H-fractional Brownian field indexed by $\mathbb{S}^1\times]0,\varepsilon[$ for every $H \in]0,1/2[$. To conclude for every $H > 0$ we recall that if d^{2H} is not a kernel of negative type then $d^{2H'}$ is not a kernel of negative type for every $H' \geq H$ (see Theorem 1.3).

We carry the proof with a cylinder of radius $\frac{1}{2\pi}$ in order to get parallel circles of perimeter 1 and lighten the computations. Such a cylinder is homothetic to a cylinder of radius 1 and therefore has the same fractional index (see Remark 1.17).

In Section 4.2.1 we work on a sequence of configurations with points in one circle. Section 4.2.2 deals with the same sequence duplicated on two parallel circles of the cylinder. We finish the proof in Section 4.2.3 by considering the same sequence of configurations on a diverging number of parallel circles of the cylinder.

4.2.1 A configuration on the circle

Let us consider a circle S of perimeter 1, parametrised by arc length $s \in [0,1[$. In this chart we have an explicit formula for the geodesic distance,

$$d_S(s,s') = \min(|s-s'|, 1-|s-s'|).$$

For every $N \in \mathbb{N}$ and $1 \leq i \leq 4N$ we define

$$P_{i,N} := \frac{i}{4N} \in S,$$

and the coefficients

$$c_i = (-1)^i.$$

Notice that for every N we have

$$\sum_{i=1}^{4N} c_i = 0,$$

so that $((P_{1,N}, \cdots P_{4N,N}), (c_1, \cdots, c_{4N}))$ is a configuration of $4N$ points in S.

We now deal with the asymptotic behaviour of

$$A_N := \sum_{i,j=1}^{4N} c_i c_j d_S^{2H}(P_{i,N}, P_{j,N}). \tag{4.4}$$

Lemma 4.1. *For every $H \in]0, 1/2[$,*

$$A_N \underset{N\to\infty}{\sim} \frac{N^{1-2H}}{4^{2H-1}} \sum_{p=0}^{\infty} \left[(2p)^{2H} - 2(2p+1)^{2H} + (2p+2)^{2H}\right].$$

Proof. We write P_i instead of $P_{i,N}$ when there is no ambiguity. The terms $d_S(P_i, P_j)$ appearing in the sum A_N are of the form $\frac{k}{4N}$ for $k \in \{1, \cdots, 2N\}$. Each one appears $8N$ times except the term for $k = 2N$. This last terms only appears $4N$ times corresponding to pairs of antipodal points.

Moreover $c_i c_j$ depends only on $d_S(P_i, P_j)$, therefore

$$\begin{aligned}
A_N &= 8N \sum_{k=1}^{2N-1} (-1)^k \left(\frac{k}{4N}\right)^{2H} + 4N \left(\frac{1}{2}\right)^{2H} \\
&= 8N \left(\sum_{p=1}^{N-1} \left(\frac{2p}{4N}\right)^{2H} - \sum_{p=0}^{N-1} \left(\frac{2p+1}{4N}\right)^{2H} \right) + 4N \left(\frac{1}{2}\right)^{2H} \\
&= 4N \left(\sum_{p=1}^{N-1} \left(\frac{2p}{4N}\right)^{2H} - 2 \sum_{p=0}^{N-1} \left(\frac{2p+1}{4N}\right)^{2H} + \sum_{p=0}^{N-2} \left(\frac{2p+2}{4N}\right)^{2H} \right) + 4N \left(\frac{1}{2}\right)^{2H} \\
&= 4N \sum_{p=0}^{N-1} \left[\left(\frac{2p}{4N}\right)^{2H} - 2 \left(\frac{2p+1}{4N}\right)^{2H} + \left(\frac{2p+2}{4N}\right)^{2H} \right] \\
&= \frac{4N^{1-2H}}{4^{2H}} \sum_{p=0}^{N-1} \left[(2p)^{2H} - 2(2p+1)^{2H} + (2p+2)^{2H}\right].
\end{aligned}$$

Because

$$(2p)^{2H} - 2(2p+1)^{2H} + (2p+2)^{2H} = O\left(\frac{1}{p^{2-2H}}\right)$$

and $H < 1/2$, the series above converge and we get the result. □

Remark 4.2. For $H < 1/2$ the sum of the series appearing in (4.1) is nonpositive by concavity of $x \mapsto x^{2H}$, hence $\lim_{N\to\infty} A_N = -\infty$. Because $\beta_S = \beta_{\mathbb{S}^1} = 1$, it is clear that no choice of configuration on the circle will give a positive result. It is then necessary to consider points at different heights on the cylinder in order to obtain our result. We start by duplicating our configuration on two circles.

4.2.2 Duplicating the circle configuration

We now turn to the cylinder $S \times \mathbb{R}$, considering again a circle S of perimeter 1 parametrised by arc length. In the entire proof of Theorem 4.1 we denote by d the geodesic distance $d_{S\times\mathbb{R}}$. Given two points $(s_1, z_1), (s_2, z_2) \in S \times \mathbb{R}$ we have

$$d((s_1, z_1), (s_2, z_2)) = \left(d_S(s_1, s_2)^2 + |z_1 - z_2|^2\right)^{1/2}.$$

Let us now consider a sequence of positive numbers $(z_N)_{N\in\mathbb{N}}$, and for every $N \in \mathbb{N}$,

$$P_{i,N} := \begin{cases} \left(\frac{i}{4N}, 0\right) & \text{if } 1 \le i \le 4N, \\ \left(\frac{i}{4N}, z_N\right) & \text{if } 4N+1 \le i \le 8N. \end{cases}$$

We set for every $1 \le i \le 8N$

$$c_i = (-1)^i,$$

and notice again that

$$\forall N \in \mathbb{N}, \ \sum_{i=1}^{8N} c_i = 0,$$

so that $((P_{1,N}, \cdots P_{8N,N}), (c_1, \cdots, c_{8N}))$ is a configuration of $8N$ points in $S \times \mathbb{R}$.

This time we deal with the asymptotic behaviour of

$$C_N := \sum_{i,j=1}^{8N} c_i c_j d^{2H}(P_{i,N}, P_{j,N}). \tag{4.5}$$

We write again P_i instead of $P_{i,N}$ when there is no ambiguity. Let us split

$$C_N = \sum_{i,j=1}^{4N} (-1)^{i+j}[d(P_i, P_j)]^{2H} + \sum_{i,j=4N+1}^{8N} (-1)^{i+j}[d(P_i, P_j)]^{2H}$$
$$+ \sum_{i=1}^{4N} \sum_{j=4N+1}^{8N} (-1)^{i+j}[d(P_i, P_j)]^{2H} + \sum_{i=4N+1}^{8N} \sum_{j=1}^{4N} (-1)^{i+j}[d(P_i, P_j)]^{2H}.$$

We now write

$$C_N = 2A_N + 2B_N(z_N),$$

with A_N as in (4.4) and

$$B_N(z_N) := \sum_{i=1}^{4N} \sum_{j=4N+1}^{8N} (-1)^{i+j}[d(P_i, P_j)]^{2H}. \tag{4.6}$$

Since we know from Lemma 4.1 how A_N behaves it remains to work on B_N under proper assumptions on the regime z_N. Because A_N is non positive, we aim to get a positive contribution from B_N. Asymptotic order of B_N is also crucial in order to outweigh A_N, which we have proven to have asymptotic order N^{1-2H}. From our investigations it seems that

- if z_N converges too quickly to zero B_N tends to behave like A_N. In particular setting
$$z_N = \frac{z_0}{N}$$
yields
$$B_N \underset{N\to\infty}{\sim} C(z)N^{1-2H},$$
with $C(z_0)$ continuous in z_0. Since setting $z_0 = 0$ gives $B_N = A_N$, it is clear that $C(z_0)$ is non positive for small values of z_0, which is problematic because we aim at considering cylinders of the form $S\times]0, \varepsilon[$ with ε as small as desired.

- Choosing z_N with slower regimes yields positive contribution from B_N at the expense of a less important asymptotic order. In particular setting
$$z_N = z_0 > 0$$
yields
$$B_N \underset{N\to\infty}{\longrightarrow} \frac{H}{2}\left(\frac{1}{4} + z_0^2\right)^{H-1}$$
which is negligeable in front of $|A_N|$.

We now give a class of regimes for z_N under which $B_N(z_N)$ converges to a positive constant independent of z_N, with uniform speed in z_N. We will later take advantage of this fact to consider an infinite number of circles and recover a dominant asymptotic order for $B_N(z_N)$.

Lemma 4.2. *Let us denote by $\mathcal{Z}_{\underline{\alpha},\overline{\alpha}}$ the set of all sequences of positive numbers $(z_N)_{N\geq 0}$ such that*

$$z_N N^{\underline{\alpha}} \underset{N\to\infty}{\longrightarrow} 0 \tag{H1}$$

and

$$z_N N^{\overline{\alpha}} \underset{N\longrightarrow\infty}{\longrightarrow} \infty. \tag{H2}$$

For every $0 < H < 1/2$ and $\underline{\alpha}, \overline{\alpha}$ such that $0 < \underline{\alpha} < \overline{\alpha} < 1$ we have

$$\lim_{N\to\infty} \sup_{(z_N)_{N\geq 0}\in\mathcal{Z}_{\underline{\alpha},\overline{\alpha}}} \left| B_N(z_N) - \frac{H}{2\cdot 4^{H-1}} \right| = 0.$$

Notations We introduce some notations we use in the whole proof of Lemma 4.2. Let us write

$$\left.\begin{aligned} \alpha_N &= \tfrac{-\ln(z_N)}{\ln(N)}, \\ \varphi &: x \mapsto (x^2+1)^H, \\ x_p &= \tfrac{2p+1}{4N^{1-\alpha_N}}, \\ h &= \tfrac{1}{4N^{1-\alpha_N}}, \\ \theta_l &= \alpha_N(l-1-2H) - l + 2. \end{aligned}\right\} \tag{4.7}$$

Because we aim for a result with uniformity in z_N, from now on we denote by

- $a(N, z_N) = O_u(b(N, z_N))$ the existence of $C > O$ and N_0 such that for every $z_N \in \mathcal{Z}_{\underline{\alpha},\overline{\alpha}}$ and $N \geq N_0$, $|a(N, z_N)| \leq C|b(N, z_N)|$.
- In a similar way, $a(N, z_N) = o_u(b(N, z_N))$ means that $\forall\varepsilon > 0$, $\exists N_0$, $\forall z_N \in \mathcal{Z}_{\underline{\alpha},\overline{\alpha}}$, $|a(N, z_N)| \leq \varepsilon|b(N, z_N)|$.

To prove Lemma 4.2 we proceed through Lemma 4.4, Lemma 4.5, and Lemma 4.6 to a Taylor-like expansion of $B_N(z_N)$ on the powers N^{θ_l}. Observe that for every l we have $N^{\theta_{l+1}} = o\left(N^{\theta_l}\right)$. Indeed

$$\frac{N^{\theta_{l+1}}}{N^{\theta_l}} = N^{\alpha_N - 1} = \frac{1}{z_N N} = \frac{1}{z_N N^{\overline{\alpha}}} \times \frac{N^{\overline{\alpha}}}{N}$$

converges towards zero when N goes to infinity (use (H2) and $\overline{\alpha} < 1$).

Let us now give Lemma 4.3 which we will use to estimate the asymptotic order of some remainders in the expansion.

Lemma 4.3. *With the notations from* (4.7), *for every* $H < 1/2$, *every integer* $q \geq 2$ *and*

$$y_p = x_p + h\delta_{p,N},$$

where $\delta_{p,N}$ *is any double-indexed sequence with values in* $[-1, 1]$,

$$\sum_{p=0}^{N-1} |\varphi^{(q)}(y_p)| = O_u\left(N^{1-\alpha_N}\right).$$

Proof. Along the proof we use the positive constants $C_1, \cdots, C_6$. We claim that they exist and are independent of N and the choice of $z_N \in \mathcal{Z}_{\underline{\alpha},\overline{\alpha}}$, though some may depend in q and H without altering the result. Let us notice that

$$\varphi^{(q)}(t) \underset{t\to\infty}{\sim} C_1\ t^{2H-q},$$

which yields

$$\varphi^{(q)}(t) \leq C_2\ t^{2H-q}.$$

We obtain

$$\sum_{p=0}^{N-1} |\varphi^{(q)}(y_p)| \leq \sum_{p=0}^{\lfloor N^{1-\alpha_N}\rfloor} ||\varphi^{(q)}||_\infty + \ C_2 \sum_{p=\lfloor N^{1-\alpha_N}\rfloor+1}^{N-1} (y_p)^{2H-q},$$

and

$$(y_p)^{2H-q} = x_p^{2H-q}\left(1+\frac{h\delta_{p,N}}{x_p}\right)^{2H-q} \leq C_3\ x_p^{2H-q}$$

because

$$\left(1+\frac{h\delta_{p,N}}{x_p}\right) = \left(1+\frac{\delta_{p,N}}{2p+1}\right)$$

is bounded and away from 0 as long as $p > 0$.

Finally

$$\begin{aligned}
\sum_{p=0}^{N-1} |\varphi^{(q)}(\delta_p)| &\leq (\lfloor N^{1-\alpha_N}\rfloor + 1)||\varphi^{(q)}||_\infty + \ C_2C_3 \sum_{p=\lfloor N^{1-\alpha_N}\rfloor+1}^{N-1} \left(\frac{2p+1}{4N^{1-\alpha_N}}\right)^{2H-q} \\
&\leq C_4N^{1-\alpha_N} + C_5\ \frac{1}{\left(N^{1-\alpha_N}\right)^{2H-q}} \sum_{p=\lfloor N^{1-\alpha_N}\rfloor+1}^{N-1} p^{2H-q} \\
&\leq C_4N^{1-\alpha_N} + C_6\ \frac{1}{\left(N^{1-\alpha_N}\right)^{2H-q}} \left(N^{1-\alpha_N}\right)^{2H-q+1} \\
&= O_u\left(N^{1-\alpha_N}\right). \quad \square
\end{aligned}$$

Lemma 4.4. *Under the assumptions* (H1), (H2) *and with the notations* (4.7) *we have for every* $M \geq 2$

$$B_N(z_N) = \sum_{n=2}^{M} b_n B_N^n + O_u\left(N^{\theta_{M+1}}\right),$$

with

$$B_N^n := N^{\theta_n} \sum_{p=0}^{N-1} \frac{1}{2N^{1-\alpha_N}} \varphi^{(n)}(x_p) \tag{4.8}$$

and

$$b_n := \frac{8}{n! 4^n}(1 + (-1)^n). \tag{4.9}$$

Proof. We start by reordering the terms in $B_N(z_N)$ in a similar way as we did for A_N in the proof of Lemma 4.1:

$$\begin{aligned}
B_N(z_N) &= 4N z_N^{2H} + 8N \sum_{k=1}^{2N-1} (-1)^k \left[\left(\frac{k}{4N}\right)^2 + z_N^2\right]^H + 4N \left[\frac{1}{2^2} + z_N^2\right]^H \\
&= 4N \sum_{p=0}^{N-1} \left(\left[\left(\frac{2p}{4N}\right)^2 + z_N^2\right]^H - 2\left[\left(\frac{2p+1}{4N}\right)^2 + z_N^2\right]^H + \left[\left(\frac{2p+2}{4N}\right)^2 + z_N^2\right]^H \right) \\
&= 4N \sum_{p=0}^{N-1} \left(\left[\left(\frac{2p}{4N}\right)^2 + \frac{1}{N^{2\alpha_N}}\right]^H - 2\left[\left(\frac{2p+1}{4N}\right)^2 + \frac{1}{N^{2\alpha_N}}\right]^H + \left[\left(\frac{2p+2}{4N}\right)^2 + \frac{1}{N^{2\alpha_N}}\right]^H \right) \\
&= \frac{4N}{N^{2\alpha_N H}} \sum_{p=0}^{N-1} \left(\left[\left(\frac{2p+1-1}{4N^{1-\alpha_N}}\right)^2 + 1\right]^H - 2\left[\left(\frac{2p+1}{4N^{1-\alpha_N}}\right)^2 + 1\right]^H + \left[\left(\frac{2p+1+1}{4N^{1-\alpha_N}}\right)^2 + 1\right]^H \right) \\
&= 4N^{1-2\alpha_N H} \sum_{p=0}^{N-1} \left[\varphi\left(x_p - h\right) - 2\varphi\left(x_p\right) + \varphi\left(x_p + h\right)\right],
\end{aligned}$$

Taylor expansions of φ up to an arbitrary order M give the following approximation of $B_N(z_N)$:

$$\begin{aligned}
&4N^{1-2\alpha_N H} \sum_{p=0}^{N-1} \sum_{n=2}^{M} \left[(-h)^n \frac{\varphi^{(n)}(x_p)}{n!} + h^n \frac{\varphi^{(n)}(x_p)}{n!}\right] \\
&= N^{1-2\alpha_N H} \sum_{p=0}^{N-1} \sum_{n=2}^{M} \frac{b_n}{2\left(N^{1-\alpha_N}\right)^n} \varphi^{(n)}(x_p) \\
&= \sum_{n=2}^{M} b_n \; N^{\theta_n} \sum_{p=0}^{N-1} \frac{1}{2N^{1-\alpha_N}} \varphi^{(n)}(x_p), \text{ with the remainder}
\end{aligned}$$

$$R_{M+1} := N^{1-2\alpha_N H} \sum_{p=0}^{N-1} \frac{C_M}{N^{(1-\alpha_N)(M+1)}} \left[\varphi^{(M+1)}(y_{p,1}) + (-1)^{(M+1)}\varphi^{(M+1)}(y_{p,2})\right],$$

where

$$y_{p,1} \in]x_p - h, x_p[$$

and

$$y_{p,2} \in]x_p, x_p + h[\ .$$

Using Lemma 4.3 with $y_p = y_{p,1}$ and again with $y_p = y_{p,2}$ shows that

$$R_{M+1} = O_u\big(N^{\theta_{M+1}}\big)\ . \quad \square$$

Lemma 4.5. *Under the assumptions* (H1), (H2) *and with the notations* (4.7), *for every $n \geq 3$ and $M \geq n$:*

$$B_N^n = \sum_{k=0}^{M-n} d_k N^{\theta_{n+k}} \varphi^{(n+k-1)}(0) + \sum_{k=1}^{M-n} a_k\ B_N^{n+k} + O_u\left(N^{\theta_{M+1}}\right) + o_u(1), \qquad (4.10)$$

while for every $M \geq 2$:

$$B_N^2 = \frac{H}{4^{H-1}} + \sum_{k=0}^{M-2} d_k N^{\theta_{2+k}} \varphi^{(2+k-1)}(0) + \sum_{k=1}^{M-2} a_k\ B_N^{2+k} + O_u\left(N^{\theta_{M+1}}\right) + o_u(1), \quad (4.11)$$

$$\textit{with} \quad d_k := -\frac{1}{4^k k!}\ , \qquad (4.12)$$

$$a_k := -\frac{1}{2^k(k+1)!}\ . \qquad (4.13)$$

Proof. Let us write

$$B_N^n = N^{\theta_n} \sum_{p=0}^{N-1} \frac{1}{2N^{1-\alpha_N}} \varphi^{(n)}(x_p) = N^{\theta_n} \sum_{p=0}^{N-1} \int_{x_p}^{x_{p+1}} \varphi^{(n)}(x_p) dt.$$

Proceeding to a Taylor expansion up to the order $M - n$ of $\varphi^{(n)}(t)$ for any $t \in [x_p, x_{p+1}]$, we write, calling R_{M+1}^n the remainder from the Taylor expansion:

$$\begin{aligned} B_N^n &= N^{\theta_n} \left(\sum_{p=0}^{N-1} \int_{x_p}^{x_{p+1}} \left[\varphi^{(n)}(t) - \sum_{k=1}^{M-n} \frac{(t-x_p)^k}{k!} \varphi^{(n+k)}(x_p) \right] dt \right) + R_{M+1}^n \\ &= N^{\theta_n} \left(\left[\varphi^{(n-1)}(t)\right]_{x_0}^{x_N} - \sum_{k=1}^{M-n} \frac{1}{2^k(k+1)!} \cdot \frac{1}{N^{(1-\alpha_N)k}} \sum_{p=0}^{N-1} \frac{1}{2N^{1-\alpha_N}} \varphi^{(n+k)}(x_p) \right) \\ &\qquad + R_{M+1}^n\ . \quad (4.14) \end{aligned}$$

For every p and $t \in [x_p, x_{p+1}]$, there exists $y_p(t)$ in $]x_p, x_{p+1}[$ and continuous in t such that

$$R^n_{M+1} = N^{\theta_n}\left(-\sum_{p=0}^{N-1}\int_{x_p}^{x_{p+1}}\frac{(t-x_p)^{M-n+1}}{(M-n+1)!}\varphi^{(M+1)}(y_p(t))dt\right) .$$

We have

$$\begin{aligned}\left|R^n_{M+1}\right| &\le N^{\theta_n}\sum_{p=0}^{N-1}\max_{t\in[x_p,x_{p+1}]}\left|\varphi^{(M+1)}(y_p(t))\right|\int_{x_p}^{x_{p+1}}\frac{(t-x_p)^{M-n+1}}{(M-n+1)!}dt\\ &= N^{\theta_n}\sum_{p=0}^{N-1}\left|\varphi^{(M+1)}(y'_p)\right|\ O_u\left(N^{(\alpha_N-1)(M-n+2)}\right)\\ &\qquad\text{for some } y'_p\in\operatorname*{argmax}_{t\in[x_p,x_{p+1}]}\left|\varphi^{(M+1)}(y_p(t))\right|\end{aligned}$$

Using Lemma 4.3 again we obtain

$$\left|R^n_{M+1}\right| = O_u\left(N^{\theta_{M+1}}\right). \tag{4.15}$$

Coming back to (4.14),

- for $n = 2$ it is easy to see that

  $$N^{\theta_n}\varphi^{(n-1)}(x_N) = N^{\theta_2}\varphi'(x_N) = \frac{H}{4^{H-1}} + O_u\left(z_N^2\right) + o_u(1).$$

  Using (H1) we obtain

  $$N^{\theta_n}\varphi^{(n-1)}(x_N) = \frac{H}{4^{H-1}} + o_u(1) \tag{4.16}$$

- while for $n \ge 3$

  $$N^{\theta_n}\varphi^{(n-1)}(x_N) = o_u(1). \tag{4.17}$$

In both cases, expanding $\varphi^{(n-1)}$ up to the order $M-n$ and using Lemma 4.3 again to deal with the remainder we get

$$-N^{\theta_n}\varphi^{(n-1)}(x_0) = \sum_{k=0}^{M-n} -\frac{N^{\theta_{n+k}}}{4^k k!}\varphi^{(n+k-1)}(0) + O_u\left(N^{\theta_{M+1}}\right). \tag{4.18}$$

It remains in (4.14) the term

$$\begin{aligned}&N^{\theta_N}\left(-\sum_{k=1}^{M-n}\frac{1}{2^k(k+1)!}\cdot\frac{1}{N^{(1-\alpha_N)k}}\sum_{p=0}^{N-1}\frac{1}{2N^{1-\alpha_N}}\varphi^{(n+k)}(x_p)\right)\\ &=\sum_{k=1}^{M-n} -\frac{1}{2^k(k+1)!}B_N^{n+k}.\end{aligned} \tag{4.19}$$

Putting together all the pieces of (4.14) from (4.15), (4.18), (4.19), and (4.16) or (4.17) whether $n = 2$ or $n \geq 3$, we get the result. $\square$

Lemma 4.6. *Under the assumptions* (H1), (H2) *and with the notations* (4.7), *for every* $M' \geq 2$ *we have*

$$B_N(z_N) = \frac{H}{2 \cdot 4^{H-1}} + \sum_{l'=1}^{M'-1} C_{2l'+1} N^{\theta_{2l'+1}} \varphi^{2l'}(0) + O_u\left(N^{\theta_{2M'+1}}\right) + o_u(1),$$

with

$$C_l := \sum_{n'=1}^{\lfloor l/2 \rfloor} \sum_{k=0}^{l-2n'} b_{2n'}\ A_{l-2n'-k}\ d_k, \tag{4.20}$$

where $A_0 := 1$ *and for every* $p \geq 1$,

$$A_p := \sum_{q=1}^{p} \sum_{\substack{m_1,\cdots,m_q>0 \\ m_1+\cdots+m_q=p}} a_{m_1} \cdots a_{m_q}. \tag{4.21}$$

Proof. Using (4.10) and (4.11) from Lemma 4.5 we get

$$\begin{aligned} B_N(z_N) = &\\ & b_2\left(\frac{H}{4^{H-1}} + \sum_{k=0}^{M-2} d_k N^{\theta_{2+k}} \varphi^{(2+k-1)}(0) + \sum_{k=1}^{M-2} a_k\ B_N^{2+k}\right) \\ & + \sum_{n=3}^{M} b_n \left(\sum_{k=0}^{M-n} d_k N^{\theta_{n+k}} \varphi^{(n+k-1)}(0) + \sum_{k=1}^{M-n} a_k\ B_N^{n+k}\right) \\ & \qquad + O_u\left(N^{\theta_{M+1}}\right) + o_u(1), \end{aligned}$$

gathering terms and using $b_2 = \frac{1}{2}$ we obtain

$$\begin{aligned} B_N(z_N) = \frac{H}{2 \cdot 4^{H-1}} + \sum_{n=2}^{M} b_n \left(\sum_{k=0}^{M-n} d_k N^{\theta_{n+k}} \varphi^{(n+k-1)}(0) + \sum_{m=1}^{M-n} a_m\ B_N^{n+m}\right) \\ + O_u\left(N^{\theta_{M+1}}\right) + o_u(1). \end{aligned}$$

We now recursively apply (4.10) to obtain an explicit expansion of $B_N(z_N)$: all the asymptotic terms of the form $O_u\left(N^{\theta_{M+1}}\right)$ and $o_u(1)$ gather because we only use (4.10) a finite number of times. Apart from $\frac{H}{2 \cdot 4^{H-1}}$, we only obtain terms of the form $CN^{\theta_l}\varphi^{l-1}(0)$. Furthermore :

1. if the term was obtained without using (4.10), $C = b_n d_k$ for some n and k such that $n + k = l$,
2. if the term was obtained after using (4.10) q times, $C = b_n a_{m_1} \cdots a_{m_q} d_k$ with $n + m_1 + \cdots + m_q + k = l$.

Hence the total constant before $N^{\theta_l} \varphi^{l-1}(0)$ equals

$$C_l = \sum_{n=2}^{l} \sum_{k=0}^{l-n} b_n \; A_{l-n-k} \; d_k.$$

Let us notice that $b_n = 0$ for odd n and $\varphi^{(l-1)}(0) = 0$ for even l. We therefore write $n = 2n'$, $l = 2l' + 1$ and $M' = \lceil M/2 \rceil$ and obtain the result. □

Proof of Lemma 4.2. We will now show that all the coefficients $C_{2l'+1}$ in Lemma 4.6 are vanishing. Let us write

$$C_l = \sum_{n'=1}^{\lfloor l/2 \rfloor} b_{2n'} Z_{l-2n'}$$

with

$$Z_r = \sum_{k=0}^{r} A_{r-k} d_k$$

for every $r \geq 1$. We are going to prove that $Z_r = 0$ when r is odd, which implies that $C_l = 0$ when l is odd. We do so by finding a formal power series associated to $(Z_r)_{r \geq 1}$ and showing that it converges to an even function.

$$Z_r = \sum_{k=0}^{r-1} A_{r-k} d_k + A_0 d_r = \sum_{k=0}^{r-1} \left(\sum_{q=1}^{r-k} \sum_{\substack{m_1, \cdots, m_q > 0 \\ m_1 + \cdots + m_q = r-k}} a_{m_1} \cdots a_{m_q} \right) \cdot d_k + d_r,$$

then we can write the formal expansion

$$\sum_{r=1}^{\infty} Z_r z^r = \left(\sum_{q=1}^{\infty} \left(\sum_{n=1}^{\infty} a_n z^n \right)^q \right) \cdot \left(\sum_{k=0}^{\infty} d_k z^k \right) + \sum_{r=1}^{\infty} d_r z^r.$$

It is easy to see that all series on the right side of the equality converges for z small enough and to compute explicitly

$$\sum_{r=1}^{\infty} Z_r z^r = \frac{z}{2 \; (e^{-z/4} - e^{z/4})} + 1$$

which is an even function of z.

Since all the coefficients in the expansion given in Lemma 4.6 are equal to 0 we get that

$$\forall M \in \mathbb{N}, \quad B_N(z_N) = \frac{H}{2 \cdot 4^{H-1}} + O_u\left(N^{\theta_M}\right) + o_u(1).$$

Let us now write

$$O_u\left(N^{\theta_M}\right) = O_u\left(N^{\alpha_N(M-1-2H)-M+2}\right) = O_u\left(z_N^{2H+1-M} N^{-M+2}\right).$$

From (H2) we have

$$z_N^{-1} = o_u(N^{\overline{\alpha}}),$$

therefore if we choose M large enough we have

$$O_u\left(N^{\theta_M}\right) = o_u\left(N^{\overline{\alpha}(M-2H-1)-M+2}\right) = o_u(1).$$

Finally

$$B_N(z_N) = \frac{H}{2 \cdot 4^{H-1}} + o_u(1)$$

and Lemma 4.2 is proven. □

4.2.3 Proof of the result

Proof of Theorem 4.1. In the cylinder $S \times \mathbb{R}$ we now consider a number of parallel circles depending on N. Each circle bear again the same configuration of points. Precisely, we choose

$$0 < \beta < \gamma < 1$$

and take $\lfloor N^\beta \rfloor$ circles at the heights

$$\frac{k}{N^\gamma}, \ k \in \{1, \cdots, \lfloor N^\beta \rfloor\}.$$

We put on the k-th of these circles $4N$ points $(P_i^k)_{i=1}^{4N}$ of coordinates

$$\left(\frac{i}{4N}, \frac{k}{N^\gamma}\right)_{i=1}^{4N}.$$

We associate to those points the usual coefficients

$$c_i^k = (-1)^i$$

and consider

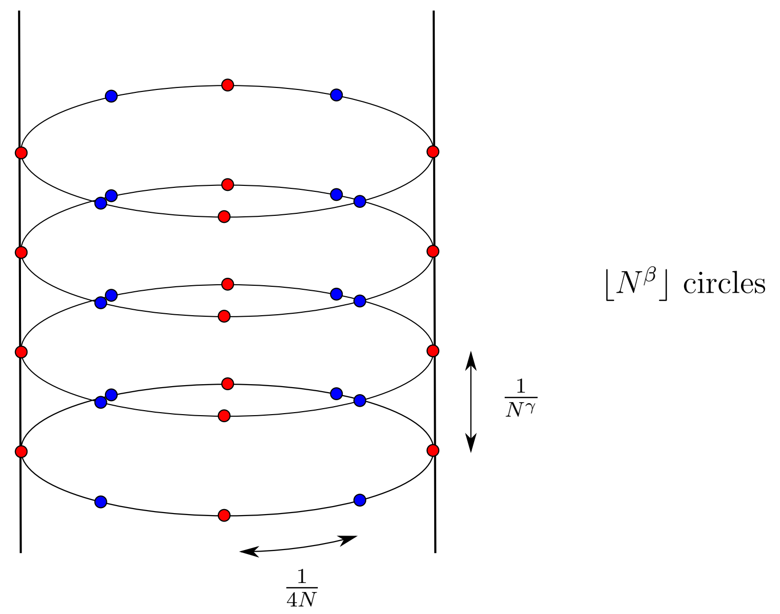

Figure 4.1: The configuration on the cylinder

$$\begin{aligned}
Q_N &= \sum_{k,l=1}^{\lfloor N^\beta \rfloor} \sum_{i,j=1}^{4N} c_i c_j \ d^{2H}(P_i^k, P_j^l) \\
&= \sum_{k=1}^{\lfloor N^\beta \rfloor} \sum_{i,j=1}^{4N} c_i c_j d^{2H}(P_i^k, P_j^k) + \sum_{k,l=1,k\neq l}^{\lfloor N^\beta \rfloor} \sum_{i,j=1}^{4N} c_i c_j d^{2H}(P_i^k, P_j^l) \\
&= \lfloor N^\beta \rfloor A_N + \sum_{k,l=1,k\neq l}^{\lfloor N^\beta \rfloor} B_N \left(z_N^{k,l} \right),
\end{aligned}$$

with

$$z_N^{k,l} = \frac{|k-l|}{N^\gamma}.$$

Let us observe that all the $z_N^{k,l}$ verify

$$\frac{1}{N^\gamma} \leq z_N^{k,l} \leq \frac{\lfloor N^\beta \rfloor}{N^\gamma}$$

and recall that $0 < \beta < \gamma < 1$, hence we can apply Lemma 4.2, since all $z_N^{k,l}$ verify (H1) together with (H2) as long as we choose $\underline{\alpha}, \overline{\alpha}$ such that

$$0 < \underline{\alpha} < \gamma < 1$$

and

$$0 < \gamma - \beta < \overline{\alpha} < 1 \ ,$$

which is always possible.

In the end we get that

$$Q_N = \lfloor N^\beta \rfloor A_N + \frac{\lfloor N^\beta \rfloor \left(\lfloor N^\beta \rfloor - 1 \right)}{2} \left(\frac{H}{2 \cdot 4^{H-1}} + o(1) \right) .$$

Recall from Lemma 4.1 that

$$A_N \underset{N \to \infty}{\sim} \frac{N^{1-2H}}{4^{2H-1}} \sum_{p=0}^{\infty} \left[(2p)^{2H} - 2(2p+1)^{2H} + (2p+2)^{2H} \right] ,$$

therefore if we choose $\beta > 1 - 2H$ we obtain

$$Q_N \underset{N \to \infty}{\sim} \frac{H}{4^H} \cdot N^{2\beta} \underset{N \to \infty}{\longrightarrow} +\infty \text{ as we wanted.} \tag{4.22}$$

Let us remark that for every positive ε the points $P_{i,N}$ belongs to $S \times]0, \varepsilon[$ for N large enough: Theorem 4.1 is proven. □

4.3 Extension of the result to Riemannian products

Theorem 4.2. *For every Riemannian manifolds M and N such that M contains a minimal closed geodesic, the Riemannian product $M \times N$ has fractional index*

$$\beta_{M \times N} = 0.$$

Proof. Let us consider

$$\gamma : [0, 2\pi] \to M$$

a closed minimal geodesic and

$$g : [0, T] \to N$$

any minimal geodesic in N, which we choose to parametrise by arc-length. Since γ is a minimal closed geodesic $\gamma([0, 2\pi])$ is isometric to the circle of radius $\frac{L(\gamma)}{2\pi}$. In the same way, g minimal geodesic implies that $g(]0, T[)$ is isometric to $]0, T[$. From Proposition 1.23 we deduce that

$$\gamma([0, 2\pi]) \times g(]0, T[) \subset M \times N$$

is isometric to the cylinder of radius $\frac{L(\gamma)}{2\pi}$ and height T. Since this cylinder is homothetic to

$$\mathbb{S}^1 \times \left]0, \frac{2\pi T}{L(\gamma)}\right[,$$

Theorem 4.1 shows that its fractional index is null (see Remark 1.17) and the proof is complete. □

Example 4.1. The d-dimensional flat torus $\mathbb{T}^d := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{\text{d times}}$ has fractional index 0.

Example 4.2. $\mathbb{S}^n \times \mathbb{R}$ has fractional index 0.

4.4 Perturbation of the product distance

In the following section we look at $\mathbb{S}^1\times]0,\varepsilon[$ endowed with a distance which converges to $d_{\mathbb{S}^1\times\mathbb{R}}$ as z is close to 0. We give in Theorem 4.3 a bound on the fractional index of such spaces, which depends on some rate of convergence towards the cylinder distance. In Section 4.4.1 we consider some surfaces of revolution as examples.

Theorem 4.3. *Let us consider a distance d' on $\mathbb{S}^1\times]0,\varepsilon[$ and denote by E' the resulting metric space. We define for very $h \in]0,\varepsilon[$*

$$\Delta(h) := \sup_{z_1,z_2\leq h} \sup_{\theta_1,\theta_2\in\mathbb{S}^1} \left|d'[(\theta_1,z_1),(\theta_2,z_2)] - d[(\theta_1,z_1),(\theta_2,z_2)]\right|.$$

where d denotes the classical distance on the cylinder. We call

$$\delta_{E'} := \sup\left\{\delta > 0,\ \Delta(h) \underset{h\to 0^+}{=} O\left(h^\delta\right)\right\}.$$

If $\delta_{E'}$ is finite we obtain that the fractional index of E' $\beta_{E'}$ verifies

$$\beta_{E'} \leq \frac{3}{\delta_{E'}+1},$$

and if $\delta_{E'} = +\infty$,

$$\beta_{E'} = 0.$$

Remark 4.3. If there is no $\delta > 0$ such that $\Delta(h) \underset{h\to 0^+}{=} O(h^\delta)$, the theorem states nothing.

Remark 4.4. The result is uninteresting if $0 < \delta_{E'} \leq 2$ since it is then clear that

$$\beta_{E'} \leq 1 \leq \frac{3}{\delta_{E'} + 1}.$$

Indeed $\mathbb{S}^1 \times \{0\} \subset E'$ is isometric to $\mathbb{S}^1$, which has fractional index 1 (see Istas [12]).

Proof. Let us assume there exists $\delta > 0$ such that

$$\Delta(h) \underset{h \to 0^+}{=} O\left(h^{\delta}\right)$$

which is true whether $\delta_{E'}$ is finite or $+\infty$. With the above remark the theorem is obvious for $0 < \delta_{E'} \leq 2$. From now on we assume that $\delta_{E'} > 2$. We consider $\delta < \delta_{E'}$ and

$$\frac{1}{2} > H > \frac{3}{2(\delta + 1)}. \tag{4.23}$$

We now apply the exact scheme of the proof of Theorem 4.1, with the new distance d'. Let us recall that the proof of Theorem 4.1 lies on the existence of β and γ such that

$$1 - 2H < \beta < \gamma < 1. \tag{4.24}$$

Our assumption (4.23) allows us to choose β and γ such that besides (4.24) we have

$$\delta(\beta - \gamma) < 2H - 3, \tag{4.25}$$

which will be useful later. With the notations of Section 4.2.3 we consider

$$\begin{aligned} Q'_N := & \sum_{k,l=1}^{\lfloor N^{\beta} \rfloor} \sum_{i,j=1}^{N} c_j c_j \left[d'(P_i^k, P_j^l) \right]^{2H} && (4.26) \\ = & \sum_{k,l=1}^{\lfloor N^{\beta} \rfloor} \sum_{i,j=1}^{N} c_j c_j \left[d(P_i^k, P_j^l) + d'(P_i^k, P_j^l) - d(P_i^k, P_j^l) \right]^{2H} \\ = & \sum_{k,l=1}^{\lfloor N^{\beta} \rfloor} \sum_{i,j=1}^{N} c_j c_j \left[d\left(P_i^k, P_j^l\right) \right]^{2H} \left[1 + \frac{d'(P_i^k, P_j^l) - d(P_i^k, P_j^l)}{d(P_i^k, P_j^l)} \right]^{2H}. && (4.27) \end{aligned}$$

As the maximum altitude of all points considered is $\frac{\lfloor N^{\beta} \rfloor}{N^{\gamma}}$, using (4.25) we obtain for every i, j, k, l

$$\left| d'(P_i^k, P_j^l) - d(P_i^k, P_j^l) \right| \leq \Delta\left(N^{\beta - \gamma}\right) = O\left(N^{\delta(\beta - \gamma)}\right) = o\left(N^{2H-3}\right), \tag{4.28}$$

moreover

$$d(P_i^k, P_j^l) \geq \frac{1}{4N} \tag{4.29}$$

so that $\dfrac{d'(P_i^k, P_j^l) - d(P_i^k, P_j^l)}{d(P_i^k, P_j^l)}$ tends towards 0 as N goes to infinity for every i, j, k, l.

Taylor expansions yields

$$Q'_N = Q_N + O\left(\sum_{k,l=1}^{\lfloor N^\beta \rfloor} \sum_{i,j=1}^{N} c_j c_j 2H d^{2H-1}(P_i^k, P_j^l) \ \left(d'\left(P_i^k, P_j^l\right) - d\left(P_i^k, P_j^l\right)\right) \right).$$

We compute

$$\begin{aligned}
|Q'_N - Q_N| &= O\left(\sum_{k,l=1}^{\lfloor N^\beta \rfloor} \sum_{i,j=1}^{N} H d^{2H-1}(P_i^k, P_j^l) \ \Delta\left(N^{\beta-\gamma}\right) \right) \\
&= O\left(\lfloor N^\beta \rfloor^2 (4N)^2 \left(\frac{1}{N}\right)^{2H-1} \right) \Delta\left(N^{\beta-\gamma}\right),
\end{aligned}$$

using (4.29) again and (4.28) we obtain

$$|Q'_N - Q_N| = O\left(N^{2\beta+2-2H+1}\right) o\left(N^{2H-3}\right) = o\left(N^{2\beta}\right). \tag{4.30}$$

Now given (4.24) and because $H < 1/2$ we still have (see (4.22))

$$Q_N \underset{N\to\infty}{\sim} \frac{H}{4^H} \cdot N^{2\beta},$$

hence

$$Q'_N \underset{N\to\infty}{=} \frac{H}{4^H} \cdot N^{2\beta} + o\left(N^{2\beta}\right)$$

is positive for N large enough, which implies that $(d')^{2H}$ is not of negative type and therefore $\beta_{E'} < 2H$. Since this is true for every $\delta < \delta_C$ and every $H > \frac{3}{2(\delta+1)}$, the theorem is proven. □

We now turn to the case of some Riemannian surfaces in a given chart.

Theorem 4.4. *Let I be an open real interval such that there exists $\varepsilon > 0$, $]0, \varepsilon[\subset I$ and consider the case where E' is $\mathbb{S}^1 \times I$ endowed with the Riemannian metric*

$$\langle\ ,\ \rangle' = (1 + f_1(\theta, z))d\theta^2 + (1 + f_2(\theta, z))dz^2,$$

with f_1 and f_2 C^∞ functions with values in $]-1, +\infty[$.

Let us assume that the Riemannian manifold E' is complete, and that

$$\sup_{P,Q\in\mathbb{S}^1\times]0,\varepsilon[}\ \sup\left\{\max\left(\int_{\gamma_{d'}}|d\theta|,\int_{\gamma_{d'}}|dz|\right),\ \begin{array}{r}\gamma_{d'}\ \textit{minimal geodesic in}\\ E'\ \textit{between}\ P\ \textit{and}\ Q\end{array}\right\}<\infty \tag{4.31}$$

For every $h\in I$ we define

$$z^+(h):=\sup_{P,Q\in\mathbb{S}^1\times]0,h]}\ \inf\left\{\begin{array}{c}\max_t(z(t))\ \textit{such that}\ t\mapsto(\theta(t),z(t))\ \textit{is a}\\ \textit{minimal geodesic in}\ E'\ \textit{between}\ P\ \textit{and}\ Q\end{array}\right\},$$

$$z^-(h):=\sup_{P,Q\in\mathbb{S}^1\times]0,h]}\ \sup\left\{\begin{array}{c}\min_t(z(t))\ \textit{such that}\ t\mapsto(\theta(t),z(t))\ \textit{is a}\\ \textit{minimal geodesic in}\ E'\ \textit{between}\ P\ \textit{and}\ Q\end{array}\right\},$$

$$F_1(h):=\sup_{z\in]z^-(h),z^+(h)[}\ \max_{\theta\in\mathbb{S}^1}\sqrt{|f_1(\theta,z)|},\ \ \delta_1:=\sup\left\{\delta>0,F_1(h)\underset{h\to0^+}{=}O\left(h^\delta\right)\right\},$$

$$F_2(h):=\sup_{z\in]z^-(h),z^+(h)[}\ \max_{\theta\in\mathbb{S}^1}\sqrt{|f_2(\theta,z)|},\ \ \delta_2:=\sup\left\{\delta>0,F_2(h)\underset{h\to0^+}{=}O\left(h^\delta\right)\right\}.$$

If $\min(\delta_1,\delta_2)$ *is finite we have*

$$\beta_{E'}\leq\frac{3}{\min(\delta_1,\delta_2)+1},$$

and if $\delta_1=\delta_2=+\infty$,

$$\beta_{E'}=0.$$

Proof. Let us consider $P_1=(\theta_1,z_1)$ and $P_2=(\theta_2,z_2)$ in $\mathbb{S}^1\times]0,\varepsilon[$. Let us denote by $\gamma_{d'}$ a minimal geodesic between P_1 and P_2 in E', and by γ_d a minimal geodesic between the same points in the cylinder endowed with its classical distance d. We also set

$$C=\sup_{P_1,P_2\in\mathbb{S}^1\times]0,\varepsilon[}\ \sup_{\gamma_d,\gamma_{d'}}\ \max\left\{\int_{\gamma_d}|d\theta|,\int_{\gamma_d}|dz|,\int_{\gamma_{d'}}|d\theta|,\int_{\gamma_{d'}}|dz|\right\}.$$

Let us notice that C is finite from hypothesis (4.31). Indeed the curves γ_d are minimal geodesics between points in $\mathbb{S}^1\times]0,\varepsilon[$, hence $\int_{\gamma_d}|dz|<\varepsilon$ and $\int_{\gamma_d}|dz|<\pi$.

We now assume that $z_1, z_2 \leq h$ and compute

$$\begin{aligned}
d'(P_1,P_2) &= \int_{\gamma_{d'}} \left(\langle \gamma'_{d'}, \gamma'_{d'} \rangle'\right)^{1/2} \leq \int_{\gamma_d} \left(\langle \gamma'_d, \gamma'_d \rangle'\right)^{1/2} \\
&= \int_{\gamma_d} \left((1+f_1(\theta,z))d\theta^2 + (1+f_2(\theta,z))dz^2\right)^{1/2} \\
&\leq \int_{\gamma_d} \left((1+\max|f_1\circ\gamma_d|)d\theta^2 + (1+\max|f_2\circ\gamma_d|)dz^2\right)^{1/2}, \\
&\text{using twice } (a+b)^{1/2} \leq a^{1/2}+b^{1/2} \text{ for } a,b>0: \\
&\leq \int_{\gamma_d} \left(d\theta^2+dz^2\right)^{1/2} + \max|f_1\circ\gamma_d|^{1/2}\int_{\gamma_d}|d\theta| + \max|f_2\circ\gamma_d|^{1/2}\int_{\gamma_d}|dz| \\
&\leq d(P_1,P_2) + C\left(\max|f_1\circ\gamma_d|^{1/2} + \max|f_2\circ\gamma_d|^{1/2}\right),
\end{aligned}$$

from which we deduce

$$d'(P_1,P_2) \leq d(P_1,P_2) + C(F_1(h)+F_2(h)). \tag{4.32}$$

In a similar way and with

$$f_i^-(\theta,z) := -\min(f_i(\theta,z),\, 0):$$

$$\begin{aligned}
d'(P_1,P_2) &= \int_{\gamma_{d'}} \left((1+f_1(\theta,z))d\theta^2 + (1+f_2(\theta,z))dz^2\right)^{1/2} \\
&\geq \int_{\gamma_{d'}} \left((1-f_1^-(\theta,z))d\theta^2 + (1-f_2^-(\theta,z))dz^2\right)^{1/2} \\
&\text{using } (a-b)^{1/2} \geq a^{1/2}-b^{1/2} \text{ for } a>b>0: \\
&\geq \int_{\gamma_{d'}} \left(d\theta^2+dz^2\right)^{1/2} - \max|f_1\circ\gamma_{d'}|^{1/2}\int_{\gamma_{d'}}|d\theta| - \max|f_2\circ\gamma_{d'}|^{1/2}\int_{\gamma_{d'}}|dz| \\
&\geq \int_{\gamma_d} \left(d\theta^2+dz^2\right)^{1/2} - C\left(\max|f_1\circ\gamma'_d|^{1/2} + \max|f_2\circ\gamma'_d|^{1/2}\right),
\end{aligned}$$

hence

$$d'(P_1,P_2) \geq d(P_1,P_2) + C(F_1(h)+F_2(h)). \tag{4.33}$$

Finally for every $P_1 = (\theta_1, z_1)$ and $P_2 = (\theta_2, z_2)$ with $z_1, z_2 \leq h$ we have

$$|d(P_1,P_2) - d'(P_1,P_2)| \leq C\left(F_1(h)+F_2(h)\right),$$

hence

$$\Delta(h) \leq C\left(F_1(h)+F_2(h)\right).$$

This implies that $\delta_{E'}$ (defined in Theorem 4.3) is such that

$$\delta_{E'} \geq \min(\delta_1, \delta_2),$$

and we apply Theorem 4.3 to conclude. $\square$

Remark 4.5. Assumption (4.31) is for example verified if E' a metric space of finite diameter and f_1 and f_2 are bounded below by $m > -1$. Indeed for every P, Q in $\mathbb{S}^1 \times]0, \varepsilon[$ and $\gamma_{d'}$ a minimal geodesic from P to Q in E' we have

$$\int_{\gamma_{d'}} \left((1 + f_1(\theta, z))d\theta^2 + (1 + f_2(\theta, z))dz^2\right)^{1/2} = d(P, Q),$$

hence

$$\int_{\gamma_{d'}} \left((1 + f_1(\theta, z))d\theta^2\right)^{1/2} \leq d(P, Q),$$

from which we deduce

$$\int_{\gamma_{d'}} |d\theta| \leq \frac{d(P, Q)}{\inf(1 + f_1(\theta, z))^{1/2}}.$$

The same argument gives

$$\int_{\gamma_{d'}} |dz| \leq \frac{d(P, Q)}{\inf(1 + f_2(\theta, z))^{1/2}}.$$

Remark 4.6. Let S be a complete, orientable Riemannian manifold of dimension 2, with

$$\gamma : [0, 2\pi] \to S$$

a minimal closed geodesic. Without loss of generality (see Remark 1.17) we assume that the minimal geodesic has length $L(\gamma) = 2\pi$ and is parametrised by arc-length. If we choose a C^∞ vector field v along γ such that for every θ, $\|v(\theta)\|_S = 1$ and $\langle v(\theta), \gamma'(\theta)\rangle_S = 0$, and define

$$\begin{aligned} \Phi : \mathbb{S}^1 \times \mathbb{R} &\to S \\ (\theta, z) &\mapsto \operatorname*{Exp}_{\gamma(\theta)}(zv(\theta)), \end{aligned}$$

it is possible to check that there exists $\varepsilon > 0$ such that the restriction of Φ to $\mathbb{S}^1 \times] - \varepsilon, \varepsilon[$ is a C^∞ diffeomorphism onto its image

$$\mathcal{V}_\varepsilon = \Phi(\mathbb{S}^1 \times] - \varepsilon, \varepsilon[).$$

Furthermore $\mathcal{V}_\varepsilon$ is a neighbourhood of γ. For every $p \in \mathcal{V}_\varepsilon$ we get the coordinates $(\theta, z) = \Phi^{-1}(p)$, and one can check that the inner product of S is given by

$$\langle \ , \ \rangle_S = (1 + f_1(\theta, z))\, d\theta^2 + dz^2,$$

where f_1 is a C^∞ function with values in $]-1, +\infty[$ such that $f_1(\theta, 0) = 0$ for every θ.

However it is not possible to apply Theorem 4.4 without global assumptions on S (which is not surprising, see Remark 1.18). In this case we need that all the minimal geodesics between points close enough to γ take values in $\mathcal{V}_\varepsilon$, in order to have $z^+(h)$ and $z^-(h)$ properly defined. Furthermore if we don't have $\lim_{h\to 0} z^+(h) = \lim_{h\to 0} z^-(h) = 0$, we don't have

$$\lim_{h\to 0} F_1(h) = \lim_{h\to 0} F_2(h) = 0,$$

hence $\delta_1 = \delta_2 = -\infty$ and Theorem 4.4 claims nothing. In the next section we consider revolution surfaces with increasing generating function and apply Theorem 4.4.

4.4.1 Some surfaces of revolution as examples

In all that follow, we consider a differentiable function $r : \mathbb{R}^+ \to \mathbb{R}^+_*$ and we call *the surface of revolution with generating function* r the surface Γ of $\mathbb{R}^3$ admitting the parametrisation

$$X : (\theta, z) \mapsto \begin{pmatrix} r(z)\cos(\theta) \\ r(z)\sin(\theta) \\ z \end{pmatrix}. \tag{4.34}$$

Lemma 4.7. *Let Γ be a surface of revolution with generating function r. If r is increasing, for every geodesic*

$$\begin{aligned} g : [0, T] &\to \Gamma \\ t &\mapsto (\theta(t), z(t)) \end{aligned}$$

and for every $t \in [0, T]$,

$$z(t) \leq \max(z(0), z(T)).$$

In particular for every $h \geq 0$,

$$z^+(h) = h.$$

Proof. We will use Clairaut's relation (see [6]) which states that along a given geodesic of a surface of revolution

$$r(z(t))\cos(\varphi(t)) = const.\ , \tag{4.35}$$

where $\varphi(t) \in [0, \pi/2]$ is the acute, nonoriented angle that makes the geodesic with the parallel that intersects it at $t = 0$.

Since any geodesic is differentiable, so is $t \mapsto z(t)$. Let us assume that $z(t)$ has a global maximum in $t_0 \in]0, T[$ and that there exists $t_1 \in]0, T[$ such that $z'(t_1) \neq 0$. Since $z(t_0)$ is a maximum we have $z'(t_0) = 0$, which is equivalent to $\varphi(t_0) = 0$. Because $z'(t_1) \neq 0$, $\varphi(t_1) \in]0, \pi/2]$. We have

$$\cos(\varphi(t_1)) < \cos(\varphi(t_0)) = 1.$$

Using r increasing and $z(t_1) \leq z(t_0)$ maximum, we obtain

$$r(z(t_1))\cos(\varphi(t_1)) < r(z(t_0))\cos(\varphi(t_0)),$$

which contradicts Clairaut's relation (4.35).

In the end, either $z'(t) = 0$ for every $t \in]0, T[$, which means $z(t) = const.$ and the result is clear, either the global maximum of z over $[0, T]$ (which exists since z is continuous) is reached in $t = 0$ or $t = T$. We have proven for every geodesic $t \mapsto (\theta(t), z(t))$ that

$$\forall t \in [0, T],\ z(t) \leq \max(z(0), z(T)).$$

Given the definition of z^+ (see Theorem 4.4) it is clear that $z^+(h) = h$ for every h. The lemma is proven. $\square$

Let us consider a surface of revolution Γ, admitting the parametrisation (4.34), with r an increasing, differentiable function such that $r(0) = 1$ Notice this last assumption is without loss of generality since fractional index is the same up to homothety (see Remark 1.17).

We compute

$$\frac{\partial X}{\partial \theta} = \begin{pmatrix} -r(z)\sin(\theta) \\ r(z)\cos(\theta) \\ 0 \end{pmatrix},$$

$$\frac{\partial X}{\partial z} = \begin{pmatrix} r'(z)\cos(\theta) \\ r'(z)\sin(\theta) \\ 1 \end{pmatrix},$$

and deduce the coefficients of the first fundamental form of Γ :

$$E_\Gamma = \left\langle \frac{\partial X}{\partial \theta}, \frac{\partial X}{\partial \theta} \right\rangle_{\mathbb{R}^3} = r^2(z),$$
$$F_\Gamma = \left\langle \frac{\partial X}{\partial \theta}, \frac{\partial X}{\partial z} \right\rangle_{\mathbb{R}^3} = 0,$$
$$G_\Gamma = \left\langle \frac{\partial X}{\partial z}, \frac{\partial X}{\partial z} \right\rangle_{\mathbb{R}^3} = r'(z)^2 + 1.$$

We get the expression of the Riemannian metric

$$\langle\ ,\ \rangle_\Gamma = r^2(z)d\theta^2 + (1 + r'(z)^2)dz^2.$$

Let us now fix a positive ε and apply Theorem 4.4 to $E' = \mathbb{S}^1 \times]0, \varepsilon[$ endowed with the inner product $\langle\ ,\ \rangle_\Gamma$. It is clear that E' is isometric to the Riemannian manifold $\Gamma_\varepsilon := X\,(\mathbb{S}^1 \times]0, \varepsilon[)$ endowed with $\langle\ ,\ \rangle_\Gamma$.

Remark 4.7. Since r is increasing, from Lemma 4.7 we have $z^+(h) = h$. As a consequence, every geodesic in Γ between points of $X(\mathbb{S}^1 \times [0, \varepsilon])$ stays in $X(\mathbb{S}^1 \times [0, \varepsilon])$, hence the geodesic distances d_Γ and d_{Γ_ε} coincide on Γ_ε. This allows to extend the conclusions of Theorem 4.4 from $\beta_{\Gamma_\varepsilon}$ to β_Γ.

Let us now check assumption (4.31), using Remark 4.5. It is clear that the Riemannian manifold $\Gamma_\varepsilon = X\,(\mathbb{S}^1 \times]0, \varepsilon[)$ endowed with $\langle\ ,\ \rangle_\Gamma$ is a metric space of finite diameter. We have

$$f_1(\theta, z) = r^2(z) - 1,$$
$$f_2(\theta, z) = r'(z)^2.$$

Since $r(0) = 1$ and r increasing we have $f_1(\theta, z) \geq 0 > -1$ and clearly $f_2(\theta, z) \geq 0 > -1$. From Remark 4.5, assumption (4.31) is verified.

Recall that $z^+(h) = h$ from Lemma 4.7, and clearly $z^-(h) = 0$. Since f_1 and f_2 do not depend on θ we get

$$F_1(h) = (r^2(h) - 1)^{1/2},$$
$$F_2(h) = \max_{z \in [0,h]} |r'(z)|.$$

Example 4.3. Taking

$$r(z) = 1 + z^a$$

with $a > 1$ we end up with

$$\begin{aligned} F_1(h) &= \left((1 + h^a)^2 - 1\right)^{1/2} \\ &= (1 + 2h^a + o(h^a) - 1)^{1/2} \\ &= \sqrt{2}h^{a/2} + o(h^{a/2}), \\ F_2(h) &= ah^{a-1}. \end{aligned}$$

Applying Theorem 4.4 we get

$$\beta_\Gamma \leq \frac{3}{\min(a-1, a/2)+1}.$$

If $a \leq 4$ the result is too weak to be interesting (see remark 4.4). For $a > 4$,

$$\beta_\Gamma \leq \frac{3}{a/2+1}.$$

As expected we observe that β_Γ decreases towards 0 as a increases.

Example 4.4. Consider now

$$r(z) = 1 + e^{-\frac{1}{z}}.$$

This time we get

$$F_1(h) = \sqrt{2}e^{-\frac{1}{2z}},$$

$$F_2(h) = \frac{e^{-\frac{1}{z}}}{z^2}.$$

We obtain $\delta_1 = \delta_2 = +\infty$ so that Theorem 4.4 gives

$$\beta_\Gamma = 0.$$

We exhibited a surface which is close enough to a cylinder (without being a Riemannian product) to show $\beta_\Gamma = 0$ with our technique.

4.5 Gromov-Hausdorff discontinuity of $E \mapsto \beta_E$

We recall that it is possible to endow the set[1] $\mathcal{M}$ of all isometry classes of compact metric spaces with the Gromow-Hausdorff distance d_{GH}.

Given two closed sets A, B in a metric space (E, d_E), the *Hausdorff distance* between A and B is

$$d_{\mathcal{H}}(A,B) := \max\{\sup_{x\in A} d_E(x,B), \sup_{y\in B} d_E(y,A)\}. \tag{4.36}$$

We now give the definition of the *Gromov-Hausdorff distance* between two isometry classes of compact metric spaces $\bar{E}$ and $\bar{F}$,

$$d_{\mathcal{GH}}(\bar{E},\bar{F}) := \inf_{i,j} d_{\mathcal{H}}(i(E), j(F)), \tag{4.37}$$

where E and F are any two representatives of $\bar{E}$ and $\bar{F}$, i and j run through all isometrics embeddings of E and F into any ambient metric space (X, d), and $d_{\mathcal{H}}$ denotes the Hausdorff distance on closed sets of (X, d).

It is known that $(\mathcal{M}, d_{\mathcal{GH}})$ is a metric space (see [2]).

[1]One can show that $\mathcal{M}$ is a set by checking its cardinal is inferior to card($\mathbb{R}$) (see [2]).

Theorem 4.5. *The map*

$$\begin{aligned}(\mathcal{M}, d_{\mathcal{GH}}) &\to [0, +\infty] \\ E &\mapsto \beta_E\end{aligned}$$

is not continuous at $E = \mathbb{S}^1$.

Proof. For the reasons we mentioned in Section 1.4.1 we do not deal with Riemannian manifolds with boundary in this document. Let us make an exception and consider $\mathbb{S}^1 \times [0, \varepsilon]$ endowed with the Riemannian product metric (4.1), which is nothing more than $\mathbb{S}^1 \times [0, \varepsilon]$ endowed with the restriction of $d_{\mathbb{S}^1 \times \mathbb{R}}$. (All that we say about $\mathbb{S}^1 \times]0, \varepsilon[$ in Remark 4.1 is true for $\mathbb{S}^1 \times [0, \varepsilon]$.)

It is clear that the isometry class of $\mathbb{S}^1 \times [0, \varepsilon]$ converges towards the isometry class of $\mathbb{S}^1$ regarding the Gromov-Hausdorff distance. Indeed if we denote by $C_\varepsilon = i(\mathbb{S}^1 \times [0, \varepsilon])$ the canonical embedding of $\mathbb{S}^1 \times [0, \varepsilon]$ in $(\mathbb{S}^1 \times \mathbb{R}, d)$,

$$\begin{aligned}d_{\mathcal{GH}}\left(\mathbb{S}^1, \mathbb{S}^1 \times [0, \varepsilon]\right) &\leq d_{\mathcal{H}}\left(C_0, C_\varepsilon\right) \\ &= \max\left(\sup_{y \in C_\varepsilon} d(x, C_0), \sup_{y \in C_0} d(y, C_\varepsilon)\right) \\ &= \max(\varepsilon, 0) = \varepsilon.\end{aligned}$$

Recall that $\beta_{\mathbb{S}^1} = 1$. From Theorem 4.1 we know that for every $\varepsilon > 0$, $\beta_{\mathbb{S}^1 \times [0,\varepsilon]} = 0$. The discontinuity at $E = \mathbb{S}^1$ is proven. $\square$

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Sur l'existence de champs browniens fractionnaires indexés par des variétés

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Résumé : cette thèse porte sur l'existence de champs browniens fractionnaires indexés par des variétés riemanniennes. Ces objets héritent des propriétés qui font le succès du mouvement brownien fractionnaire classique (H-autosimilarité des trajectoires ajustable, accroissements stationnaires), mais autorisent à considérer des applications où les données sont portées par un espace qui peut par exemple être courbé ou troué. L'existence de ces champs n'est assurée que lorsque la quantité $2H$ est inférieure à l'indice fractionnaire de la variété, qui n'est connu que dans un petit nombre d'exemples. Dans un premier temps nous donnons une condition nécessaire pour l'existence de champ brownien fractionnaire. Dans le cas du champ brownien (correspondant à $H = 1/2$) indexé par des variétés qui ont des géodésiques fermées minimales, cette condition s'avère très contraignante : nous donnons des résultats de non-existence dans ce cadre, et montrons notamment qu'il n'existe pas de champ brownien indexé par une variété compacte non simplement connexe. La condition nécessaire donne également une preuve courte d'un fait attendu qui est la non-dégénérescence du champ brownien indexé par les espaces hyperboliques réels. Dans un second temps nous montrons que l'indice fractionnaire du cylindre est nul, ce qui constitue un exemple totalement dégénéré. Nous en déduisons que l'indice fractionnaire d'un espace métrique n'est pas continu par rapport à la convergence de Gromov-Hausdorff. Nous généralisons ce résultat sur le cylindre à un produit cartésien qui possède une géodésique fermée minimale, et donnons une majoration de l'indice fractionnaire de surfaces asymptotiquement proches du cylindre au voisinage d'une géodésique fermée minimale.

Mots-clés : champ aléatoire, mouvement brownien, fractionnaire, exposant de Hurst, autosimilarité, variété riemannienne.

Abstract: the aim of the thesis is the study of the existence of fractional Brownian fields indexed by Riemannian manifolds. Those fields inherit key properties of the classical fractional Brownian motion (sample paths with self-similarity of adjustable parameter H, stationary increments), while allowing to consider applications with data indexed by a space which can be for example curved or with a hole. The existence of those fields is only insured when the quantity $2H$ is inferior or equal to the fractional index of the manifold, which is known only in a few cases. In a first part we give a necessary condition for the fractional Brownian field to exist. In the case of the Brownian field (corresponding to $H = 1/2$) indexed by a manifold with minimal closed geodesics this condition happens to be very restrictive. We give several nonexistence results in this situation. In particular we show that there exists no Brownian field indexed by a nonsimply connected compact manifold. Our necessary condition also gives a short proof of an expected result: we prove the nondegeneracy of fractional Brownian fields indexed by the real hyperbolic spaces. In a second part we show that the fractional index of the cylinder is null, which gives a totally degenerate case. We deduce from this result that the fractional index of a metric space is noncontinuous with respect to the Gromov-Hausdorff convergence. We generalise this result about the cylinder to a Cartesian product with a closed minimal geodesic. Furthermore we give a bound of the fractional index of surfaces asymptotically close to the cylinder in the neighbourhood of a closed minimal geodesic.

Keywords: random field, Brownian motion, fractional, Hurst exponent, autosimilarity, Riemannian manifold.