Dimension géométrique propre et espaces classifiants des groupes arithmétiques

Cyril Lacoste

ANNÉE 2018

THÈSE / UNIVERSITÉ DE RENNES 1
sous le sceau de l'Université Bretagne Loire

pour le grade de

DOCTEUR DE L'UNIVERSITÉ DE RENNES 1

Mention : Mathématiques et leurs interactions

Ecole doctorale MathSTIC

présentée par

Cyril Lacoste

Préparée à l'unité de recherche UMR CNRS 6625 (IRMAR)
Institut de Recherche Mathématique de Rennes
U.F.R de Mathématiques

Dimension géométrique propre et espaces classifiants des groupes arithmétiques

Thèse soutenue à Rennes le 15 juin 2018

devant le jury composé de :

Christophe BAVARD
Professeur à l'Université de Bordeaux / *rapporteur*

Françoise DAL'BO
Professeure à l'Université de Rennes 1 / *examinatrice*

Vincent GUIRARDEL
Professeur à l'Université de Rennes 1 / *examinateur*

Ian LEARY
Professeur à l'Université de Southampton / *rapporteur*

Gilbert LEVITT
Professeur à l'Université de Caen / *examinateur*

Conchita MARTÍNEZ-PÉREZ
Professeure à l'Université de Zaragoza / *examinatrice*

Juan SOUTO
DR CNRS à l'Université de Rennes 1 / *directeur de thèse*

Remerciements

Mes premiers remerciements sont bien sûr adressés à mon directeur de thèse Juan Souto. Merci pour ta patience, ta disponibilité, ton investissement, merci d'avoir relu de nombreuses fois mes articles et d'être resté à l'écoute durant ces trois années. Merci à Christophe Bavard et Ian Leary d'avoir accepté d'être rapporteurs de ma thèse, et merci également à Françoise Dal'Bo, Vincent Guirardel, Gilbert Levitt et Conchita Martínez-Pérez pour leur présence au sein du jury.

Je vais maintenant passer aux différentes figures mathématiques qui m'ont accompagné depuis plus de dix ans. Merci tout d'abord à mon professeur de terminale Tarkan Gezer de m'avoir véritablement donné le goût des maths, de m'avoir incité à faire une CPGE et de m'avoir inscrit au concours général de mathématiques. Merci ensuite aux professeurs que j'ai eus au cours de mes 3 années de CPGE, Messieurs Fayard, Chillès et Choimet, pour leur riche enseignement de grande qualité, tant sur le plan mathématique qu'humain. J'ai bien sûr côtoyé d'excellents enseignants à l'ENS Rennes et à l'Université de Rennes 1 au cours des 7 dernières années, je ne vais pas tous les citer mais je remercie plus particulièrement Arnaud Debussche, Benoît Cadre, Michel Pierre, Rozenn Texier-Picard, Jérémy Le Borgne, Thibaut Deheuvels et François Schwarzentruber. Merci en particulier pour l'excellente préparation à l'agrégation que j'ai reçue, ainsi que pour ma bourse de thèse que j'ai eue par un contrat doctoral spécifique normalien. Je remercie aussi Rozenn et Jérémy de m'avoir fait participer à MathC2+.

Durant cette thèse, je faisais partie des équipes de Théorie ergodique et Géométrie analytique du laboratoire IRMAR. Même si je me demandais parfois ce que je faisais dans ces équipes, tant les domaines de recherche de leurs membres sont vastes, j'ai apprécié durant ces 3 années les séminaires, groupes de travail et cours doctoraux où j'ai appris plein de choses. Merci donc à leurs membres et tout particulièrement Rémi, Vincent, Ludovic, Barbara, Ania, François, Sébastien, Juliette, Françoise, Dominique. Merci aussi à Serge et Jürgen de m'avoir proposé de faire un exposé aux Journées Louis Antoine, une expérience encore très enrichissante. Merci aussi à Roger, Stéphane, Matthieu et Christophe avec qui j'ai eu l'opportunité de travailler au cours de ma mission d'enseignement. Merci enfin à toute l'équipe administrative et technique de l'IRMAR, en particulier Marie-Aude (nos discussions dans le bus vont me manquer !), Chantal, Nicole, Aude, Xhensila, Hélène, Marie-Annick, Olivier. Sans oublier Elodie Cottrel que je remercie chaleureusement de m'avoir aidé au cours du long périple administratif d'inscription puis de soutenance de thèse...

Je ne peux évidemment pas oublier les anciens élèves de prépa, de l'ENS Rennes et les doctorants dont j'ai croisé la route. Ce serait bien trop long de tous les citer, je m'excuse pour les personnes que je vais oublier. Tout d'abord merci à ma petite bande de taupins du lycée du Parc pour les bons moments passés malgré le stress des concours : Jérémy, Joséphine, Pauline, Carole, Yoann, Matthias, Guillaume, Alexandre. Merci ensuite à la promo 2011 du département Mathématiques de l'ENS Rennes, en particulier mon éternel binôme Théo (merci d'avoir fait tous les TP à ma place !) et toute la dreamteam de l'agreg Olivier, Yon, Salim, Antonin, sans oublier Marine, Coralie, Isaline, Lauriane, Lucas, Philippe, Valentin, Grégory, Jean-Jérôme, David et les deux Tristan. Merci aussi aux membres du BDE pour leurs soirées, les WEI, toute la bonne ambiance de l'école, merci aussi à ma bizute Maud, Fred, Hugo, Emilie, Fabrice, Octave, Antoine. Merci à tous les doctorants que j'ai pu croiser dans les couloirs ou à midi, en premier lieu mes cobureaux Rémi, Trang et Simon (merci en particulier à toi d'avoir relu mon introduction de thèse !). Merci aussi à ceux qui m'ont supporté pendant les pauses jeux du midi et qui ont fait de délicieux gâteaux pour le fat day : Arnaud, Ophélie, Blandine, Julie, les deux Hélène, Benoît, Axel, Alexandre, Camille, les deux Florian, Thi, Jesus David, Mercedes, Marine, Tristan, Olivier, Yon (certains ont déjà été cités mais ils ne m'en voudront pas). Merci aussi à mon frère de thèse Andrés ainsi qu'à Basile, Adrien, Vincent, Yvan, Nestor, Maria, Youenn, Türkü, Richard, Christian, Federico, Kévin, Adrien, Ninon, Pierre, Zoïs, Arnaud, Joackim, Arame, Paul, Shengyan, Phuong. Merci aussi à James et Matthieu avec qui j'ai eu l'occasion de discuter en conférence.

J'en viens maintenant aux remerciements plus personnels, et je ne peux pas passer à côté de tous les membres de ce qu'on appelle maintenant la Commuz de l'ENS Rennes, un club de comédie musicale que j'ai vu éclore et évoluer en 7 ans. Encore une fois pardon aux personnes que je vais oublier. Une pensée d'abord à la toute première troupe à laquelle j'ai appartenu, celle de Grease. Merci aussi à Alice d'avoir fait de ce club une association et pour tout ce que tu as apporté, ainsi qu'à Edouard, Isaline, Lauriane, Christy, Julie, Alexandre, les deux Sylvain, Ambroise, Pierre, Dan, Olivier, Claire, Anne-Sophie, Estelle, Agathe (j'en profite aussi pour te remercier pour le club d'impro !), Delphine, Simon, Kellian, Manon, Léa, Sarah, Alexandre, Baptiste, tant de personnes formidables que j'ai appris à connaître et avec qui j'ai partagé des moments inoubliables, sur scène ou en dehors. J'ai une pensée émue et particulière à la troupe de Friends que j'ai eu l'immense joie de mettre en scène pendant ma première année de thèse, un énorme merci à Marie et Noémie pour cette merveilleuse année et en particulier les weeks-ends papier peint à l'ENS, merci aux autres respos Hortense, Pauline, Xavier, Charlie, Grégoire, et bien sûr à nos talents, Lisa, Alizée, Manon, Marion, Louis, Amel, Maud, Valère, Lucile. Merci et bravo aussi à ceux qui ont repris le flambeau, Emilie, Pierre, bon courage à Clémentine et aux nouveaux respos pour l'an prochain, et longue vie à la Commuz !

Je terminerai en remerciant mon entourage proche de m'avoir soutenu et épaulé pendant toutes ces années, même s'il ne comprenait pas trop ce que je faisais. Merci à mes amies de longue date Violette, Aurélia et Déborah. Merci à mes parents Charles et Lydie, mon frère Gaël, ma soeur Léa, ainsi qu'à Nicolas. Merci à mon adorable boule de poils tachetée Marius. Et bien sûr merci à toi Maximilien, une rencontre inespérée et dont je ne pourrai plus me passer.

Table des matières

Avant-propos

L'objectif de cette thèse est d'étudier la dimension géométrique propre et les espaces classifiants pour les actions propres des réseaux des groupes de Lie semisimples. De tels espaces peuvent par exemple servir à calculer la cohomologie des groupes correspondants, en effet il est toujours préférable d'avoir un espace de la plus petite dimension possible. La thèse s'articule autour de deux grandes questions. La première est de calculer la dimension géométrique propre, la deuxième de construire concrètement un espace classifiant pour les actions propres (appelé aussi modèle de type $\underline{E}\Gamma$) de dimension minimale.

La première question fait l'objet de la première partie de la thèse. En poursuivant le travail d'Aramayona, Degrijse, Martinez-Perez et Souto dans [1], nous montrons que si Γ est commensurable à un réseau dans le groupe d'isométries d'un espace symétrique S de type non-compact sans facteur euclidien (en particulier si Γ est un réseau d'un groupe de Lie semisimple), alors sa dimension géométrique propre $\underline{\mathrm{gd}}(\Gamma)$ est égale à sa dimension cohomologique virtuelle $\mathrm{vcd}(\Gamma)$. L'intérêt de cette dernière est que l'on peut la calculer avec la formule de Borel-Serre dans le cas où Γ est arithmétique (ce qui est le cas notamment si Γ est un réseau irréductible d'un groupe de Lie semisimple de rang réel supérieur ou égal à 2) :

$$\mathrm{vcd}(\Gamma) = \dim S - \mathrm{rg}_{\mathbb{Q}}\Gamma,$$

où $\mathrm{rg}_{\mathbb{Q}}\Gamma$ désigne le rang rationnel de Γ. La preuve de l'égalité entre les deux dimensions utilise une troisième notion de dimension, la dimension cohomologique de Bredon $\underline{\mathrm{cd}}(\Gamma)$, qui peut être vue comme un analogue algébrique de la dimension géométrique propre $\underline{\mathrm{gd}}(\Gamma)$. Par un résultat de Lück et Meintrup dans [35], on a presque toujours l'égalité $\underline{\mathrm{cd}}(\Gamma) = \underline{\mathrm{gd}}(\Gamma)$, il nous reste alors à montrer l'égalité entre les dimensions cohomologiques $\underline{\mathrm{cd}}(\Gamma)$ et $\mathrm{vcd}(\Gamma)$. Nous pouvons montrer que cela revient à calculer la dimension des points fixes S^α de l'espace symétrique S par les éléments $\alpha \in \Gamma$ d'ordre fini non centraux. Nous utilisons alors la classification des automorphismes extérieurs des algèbres de Lie simples, ainsi que celle des espaces symétriques. La preuve repose donc essentiellement sur des arguments algébriques, et n'est en aucun cas constructive, elle ne donne pas de moyen de construire concrètement un modèle de type $\underline{E}\Gamma$ de dimension $\mathrm{vcd}(\Gamma)$.

La deuxième partie de la thèse concerne donc la construction de ces espaces classifiants pour les actions propres. Notons tout d'abord que l'espace symétrique S est lui-même un modèle de type $\underline{E}\Gamma$. Nous allons donc chercher notre modèle de dimension minimale comme rétract par déformation de S, ce que nous appellerons une épine. Nous ne savons pas s'il est possible de réaliser une telle rétraction pour tout réseau Γ. Nous ne connaissons en réalité que très peu d'exemples, essentiellement pour les groupes de rang rationnel 1 (voir [49]) et le groupe $\mathrm{SL}(n,\mathbb{Z})$. Pour ce dernier cas, le rétract en question a été construit par Ash dans [3], et s'appelle le "rétract bien équilibré". Pour le construire, commençons par identifier l'espace symétrique associé $S_n = \mathrm{SO}(n)\backslash \mathrm{SL}(n,\mathbb{R})$ et les réseaux de $\mathbb{R}^n$ de covolume 1, modulo isométries et munis d'une $\mathbb{Z}$-base. Un réseau Λ de $\mathbb{R}^n$ est dit bien équilibré si ses systoles (ou vecteurs minimaux) engendrent $\mathbb{R}^n$. Ash a montré que l'ensemble des réseaux bien équilibrés est une épine pour $\mathrm{SL}(n,\mathbb{Z})$. Nous montrerons qu'il est impossible d'utiliser les mêmes méthodes pour le groupe symplectique $\mathrm{Sp}(2n,\mathbb{Z})$ ainsi que pour le groupe d'automorphismes $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$, ce qui peut paraître surprenant car $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$ et $\mathrm{SL}(n,\mathbb{Z})$ ne diffèrent que d'un groupe fini. Ces résultats reflètent ainsi la difficulté de trouver des épines.

La thèse est organisée comme suit :

- Le chapitre 1 donne une vue d'ensemble de la thèse. Nous rappelons d'abord les différentes notions qui nous servirons : espaces classifiants, dimensions géométrique, cohomologique, géométrique propre et cohomologique virtuelle. Puis nous donnons l'exemple de l'épine du groupe $\mathrm{SL}(n,\mathbb{Z})$ et énonçons les résultats qui seront prouvés dans les chapitres suivants.

- Le chapitre 2 est consacré à la preuve de l'égalité $\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$ pour les réseaux $\Gamma \subset \mathrm{Isom}(S)$ où S est un espace symétrique de type non-compact sans facteur euclidien.

- Dans le chapitre 3, nous essayons de construire concrètement des modèles de type $\underline{E}\Gamma$ de dimension minimale. Nous montrons que les techniques utilisées pour l'épine de $\mathrm{SL}(n,\mathbb{Z})$ ne s'adaptent pas aux cas des groupes $\mathrm{Sp}(2n,\mathbb{Z})$ (pour $n \geqslant 2$) et $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$ (pour $n \geqslant 3$).

Cette thèse a en outre donné lieu à deux articles :

- *Dimension rigidity of lattices in semisimple Lie groups* [30], accepté pour publication dans la revue *Groups, Geometry and Dynamics*, et reprenant les résultats du Chapitre 2.
- *On the difficulty of finding spines* [31], publié dans la revue en ligne *Comptes rendus Mathématique*, basé sur le Chapitre 3.

Tous les groupes de Lie seront supposés réels (ou complexes) linéaires, et les algèbres de Lie de dimension finie sur $\mathbb{R}$ (ou $\mathbb{C}$).

Chapitre 1

Introduction générale

1.1 Espaces classifiants, dimensions géométrique et cohomologique

La notion principale de notre étude est celle d'espace classifiant. Nous renvoyons à [12] pour plus d'informations sur les notions que nous allons introduire et les démonstrations des propositions énoncées. Si Γ est un groupe discret, un espace classifiant pour Γ, ou modèle de type $E\Gamma$, est un Γ-complexe cellulaire X contractile sur lequel Γ agit librement. Le quotient X/Γ est alors un modèle de type $K(\Gamma, 1)$, c'est-à-dire que son groupe fondamental est isomorphe à Γ et son revêtement universel (qui est X) est contractile. De tels espaces existent toujours, et la dimension géométrique de Γ, notée $\mathrm{gd}(\Gamma)$, est la dimension minimale d'un modèle de type $E\Gamma$ (ou de type $K(\Gamma, 1)$), sachant que celle-ci peut être infinie. Il est toujours utile de chercher un espace de dimension minimale sur lequel faire agir Γ, car cela peut simplifier l'étude des propriétés du groupe, par exemple le calcul de ses groupes de cohomologie.

La première propriété importante de ces espaces est la suivante :

Proposition. Deux espaces de type $K(\Gamma, 1)$ sont homotopiquement équivalents. De même, deux espaces de type $E\Gamma$ sont Γ-homotopiquement équivalents.

Exemple 1. Considérons le groupe libre à n générateurs F_n. Un espace de type $K(F_n, 1)$ est la fleur à n pétales. Son revêtement universel, qui est le graphe de Cayley de F_n, est un espace classifiant pour F_n. Comme ces espaces sont de dimension 1, on a $\mathrm{gd}(F_n) = 1$ (cela caractérise d'ailleurs les groupes libres).

Exemple 2. Considérons le groupe $\mathbb{Z}^n$. Il agit librement par translation sur l'espace $\mathbb{R}^n$, qui est contractile, donc c'est un espace de type $E\mathbb{Z}^n$, et le tore $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ est un espace de type $K(\mathbb{Z}^n, 1)$. Sont-ils de dimension minimale ? Si Y est un autre espace de type $K(\mathbb{Z}^n, 1)$, il est homotopiquement équivalent à $\mathbb{T}^n$, en particulier il a la même cohomologie, donc :

$$H^n(Y, \mathbb{R}) \cong H^n(\mathbb{T}^n, \mathbb{R}) \cong \mathbb{R},$$

et on en déduit que la dimension de Y est au moins n. Par conséquent, $\mathrm{gd}(\mathbb{Z}^n) = n$.

On voit avec ce deuxième exemple apparaître une deuxième notion de dimension plus algébrique et qui repose sur l'étude des groupes de cohomologie de Γ :

Définition. La dimension cohomologique de Γ est définie par :

$$\mathrm{cd}(\Gamma) = \sup\{n \mid H^n(\Gamma, A) \neq 0 \text{ pour un certain } \mathbb{Z}\Gamma\text{-module } A\}.$$

La définition précise de la cohomologie des groupes se trouve dans [12], où une autre définition de la dimension cohomologique à l'aide des résolutions projectives est donnée, nous n'en aurons pas besoin ici. Ce que l'on peut retenir, c'est que si Y est un espace de type $K(\Gamma, 1)$ on a $H^*(\Gamma, \mathbb{R}) \cong H^*(Y, \mathbb{R})$.

Dans le cas où il existe un $K(\Gamma, 1)$ qui soit un complexe cellulaire fini, le calcul de la dimension cohomologique est simplifié, il n'est en effet pas nécessaire de regarder les groupes de cohomologie pour tous les $\mathbb{Z}\Gamma$-modules:

Proposition. S'il existe un $K(\Gamma, 1)$ qui soit un complexe cellulaire fini, on a :

$$\mathrm{cd}(\Gamma) = \sup\{n \mid H^n(\Gamma, \mathbb{Z}\Gamma) \neq 0\}.$$

On a immédiatement la relation suivante entre les dimensions géométrique et cohomologique :

$$\mathrm{cd}(\Gamma) \leqslant \mathrm{gd}(\Gamma),$$

car un espace de dimension n a tous ses groupes de cohomologie strictement supérieurs à n triviaux. Il se trouve que l'inégalité inverse est presque toujours vraie (voir [18]) :

Théorème (Eilenberg-Ganea). Si $\mathrm{cd}(\Gamma) \geqslant 3$, on a $\mathrm{cd}(\Gamma) = \mathrm{gd}(\Gamma)$.

Le cas de la dimension 1 est traitée par Stallings et Swan (voir [47]) :

Théorème (Stallings-Swan). Les propositions suivantes sont équivalentes :

1. $\mathrm{cd}(\Gamma) = 1$.
2. $\mathrm{gd}(\Gamma) = 1$.
3. Γ est un groupe libre non trivial.

Pour le cas $\mathrm{cd}(\Gamma) = 2$, la seule chose que l'on puisse dire est que $\mathrm{gd}(\Gamma) \in \{2, 3\}$, la conjecture d'Eilenberg-Ganea affirme que $\mathrm{gd}(\Gamma) = 3$ mais n'a pas encore été prouvée.

Signalons que les deux dimensions introduites sont croissantes pour l'inclusion:

Proposition. Si H est un sous-groupe de G, alors $\mathrm{gd}(H) \leqslant \mathrm{gd}(G)$ et $\mathrm{cd}(H) \leqslant \mathrm{cd}(G)$.

Ces notions de dimensions ne vont cependant pas nous satisfaire pour les groupes qui nous intéressent, en effet, si Γ a des éléments de torsion, on a $\mathrm{cd}(\Gamma) = \mathrm{gd}(\Gamma) = \infty$ (par la proposition qui précède, il suffit de le voir pour les groupes finis). Par exemple, pour $\Gamma = \mathbb{Z}/2\mathbb{Z}$, on sait que Γ agit librement sur les sphères $\mathbb{S}^n$ mais elles ne sont pas contractiles, il faut donc considérer la sphère $\mathbb{S}^\infty$ qui est bien un espace classifiant, mais de dimension infinie, et il n'en existe pas de dimension finie. Tous les groupes que nous allons étudier seront cependant virtuellement sans torsion, c'est-à-dire admettent un sous-groupe d'indice fini sans torsion, dans ce cas nous allons pouvoir introduire de nouvelles notions de dimension.

1.2 Dimension géométrique propre et dimension cohomologique virtuelle

Soit Γ un groupe discret virtuellement sans torsion. Un Γ-complexe cellulaire X est un modèle de type $\underline{E}\Gamma$, ou espace classifiant pour les actions propres, si pour tout sous-groupe H de Γ, l'ensemble des points fixes X^H est contractile si H est fini et vide sinon. Remarquons que, en considérant le sous-groupe trivial, cela implique que X est lui-même contractile. Comme dans le cas précédent, un tel modèle existe toujours, et deux modèles de type $\underline{E}\Gamma$ sont Γ-homotopiquement équivalents. La dimension géométrique propre, notée $\underline{\mathrm{gd}}(\Gamma)$, est la dimension minimale d'un modèle de type $\underline{E}\Gamma$. Notons que si Γ est sans torsion, les notions d'espace classifiant et d'espace classifiant pour les actions propres coïncident. Nous renvoyons à [34] et [11] pour plus de détails sur ces notions.

Exemples.

1. Le plan hyperbolique $\mathbb{H}^2$ est un espace classifiant pour les actions propres du groupe $\mathrm{SL}(2,\mathbb{Z})$.
2. L'espace de Teichmüller $\mathcal{T}_{g,n}$ d'une surface $S_{g,n}$ de genre g avec n points marqués est un espace classifiant pour les actions propres du groupe modulaire $\mathrm{Mod}(S_{g,n})$ (voir [2]).
3. L'outre-espace X_n est un espace classifiant pour les actions propres du groupe des automorphismes extérieurs du groupe libre $\mathrm{Out}(F_n)$ (voir [48]).

Plus généralement, si S est un espace symétrique (riemannien) de type non-compact sans facteur euclidien, c'est-à-dire de la forme G/K où G est un groupe de Lie semisimple et K un sous-groupe compact maximal, et si Γ est un réseau du groupe d'isométries $\mathrm{Isom}(S)$, alors S est un espace de type $\underline{E}\Gamma$.

L'autre dimension que nous introduisons pour remédier à la torsion dans le cas où Γ est virtuellement sans torsion est la dimension cohomologique virtuelle de Γ, notée $\mathrm{vcd}(\Gamma)$. C'est la dimension cohomologique virtuelle d'un sous-groupe d'indice fini sans torsion Γ' de Γ, qui ne dépend pas du sous-groupe Γ' (voir [12]) :

$$\mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma') = \sup\{n \mid H^n(\Gamma', A) \neq 0 \text{ pour un certain } \mathbb{Z}\Gamma'\text{-module } A\}.$$

Cette dernière peut s'exprimer plus simplement si l'on dispose d'un modèle cocompact X de type $\underline{E}\Gamma$ (voir [12, Cor. VIII.7.6]) :

$$\mathrm{vcd}(\Gamma) = \max\{m \in \mathbb{N} \mid \mathrm{H}_c^m(X) \neq 0\}, \tag{1.2.1}$$

où $\mathrm{H}_c^m(X)$ désigne la cohomologie à support compact de X. Notons enfin que dans le cas des groupes arithmétiques comme $\mathrm{SL}(n,\mathbb{Z})$, celle-ci se calcule très facilement (voir [10]) :

Théorème (Borel-Serre). Soit G un groupe de Lie semisimple, $K \subset G$ un sous-groupe compact maximal et $\Gamma \subset G$ un réseau arithmétique. Alors :

$$\mathrm{vcd}(\Gamma) = \dim(G/K) - \mathrm{rg}_{\mathbb{Q}}\Gamma.$$

Nous renvoyons au chapitre 1 et à [8] pour une définition précise d'un groupe arithmétique et du rang rationnel, on peut retenir ici que si $\mathbb{G}$ est un groupe algébrique semisimple défini sur $\mathbb{Q}$, $\mathbb{G}_{\mathbb{Z}}$ est l'archétype d'un groupe arithmétique (par exemple $\mathrm{SL}(n, \mathbb{Z})$ ou $\mathrm{Sp}(2n, \mathbb{Z})$).

Précisons que dans la preuve de leur théorème, Borel et Serre ont introduit une bordification X de l'espace symétrique $S = G/K$, qui est un modèle cocompact de type $\underline{E}\Gamma$, et l'on peut donc utiliser l'espace X dans la formule (1.2.1). Enfin, la plupart des réseaux qui nous intéressent sont arithmétiques, en effet par le théorème d'arithméticité de Margulis (voir [37, Ch. IX]), si Γ est irréductible et G est de rang réel supérieur ou égal à 2, alors Γ est arithmétique.

Les deux nouvelles notions de dimension que nous avons introduites sont reliées, on a en effet toujours l'inégalité :

$$\mathrm{vcd}(\Gamma) \leqslant \underline{\mathrm{gd}}(\Gamma),$$

en effet si X est un espace de type $\underline{E}\Gamma$ et $\Gamma' \subset \Gamma$ est un sous-groupe d'indice fini sans torsion, X est aussi un espace de type $E\Gamma'$.

Remarque importante : L'inégalité $\mathrm{vcd}(\Gamma) \leqslant \underline{\mathrm{gd}}(\Gamma)$ peut être stricte, voir des exemples dans [32],[11], [33], [39], [14], [15]. Il y a également des cas où c'est une égalité, voir [13], [1], [2], [34] [48].

1.3 Le cas du groupe $\mathrm{SL}(n, \mathbb{Z})$

Calculons la dimension géométrique propre du groupe $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. Dans un premier temps, nous allons obtenir une borne inférieure sans utiliser la formule de Borel-Serre. Soit N le sous-groupe de $\mathrm{SL}(n, \mathbb{R})$ des matrices triangulaires supérieures unipotentes. Il n'est pas difficile de voir que $N \cap \mathrm{SL}(n, \mathbb{Z})$ est un sous-groupe cocompact de N. Dès lors, si $\Gamma' \subset \mathrm{SL}(n, \mathbb{Z})$ est un sous-groupe d'indice fini sans torsion, $N/(N \cap \Gamma')$ est une sous-variété fermée, qui est un $K(N \cap \Gamma', 1)$ car N est contractile (homéomorphe à $\mathbb{R}^{\frac{n(n-1)}{2}}$). Donc on a la minoration :

$$\underline{\mathrm{gd}}(\Gamma) \geqslant \mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma') \geqslant \mathrm{cd}(N \cap \Gamma') = \dim(N/(N \cap \Gamma')) = \frac{n(n-1)}{2},$$

la dernière égalité venant du fait que le n-ième groupe de cohomologie d'une variété fermée orientable de dimension n est non trivial.

Pour prouver l'inégalité inverse, nous allons construire explicitement un modèle de type $\underline{E}\Gamma$ de dimension $\frac{n(n-1)}{2}$, qui sera un rétract par déformation Γ-équivariant de l'espace symétrique associé $S_n = \mathrm{SO}(n) \setminus \mathrm{SL}(n, \mathbb{R})$ (dans cette partie nous quotientons à gauche par $\mathrm{SO}(n)$ pour faciliter l'identification avec les réseaux à venir), ce que nous appelons une épine. Plus précisément, une épine pour Γ est un rétract par déformation Γ-équivariant de l'espace symétrique associé (en particulier, c'est un modèle de type $\underline{E}\Gamma$), de dimension $\underline{\mathrm{gd}}(\Gamma)$ et sur lequel Γ agit de manière cocompacte (cette dernière condition facilite notamment les calculs de la cohomologie de Γ).

L'épine pour $\mathrm{SL}(n, \mathbb{Z})$ a été construite par Ash (voir [3]) et se généralise aux espaces symétriques linéaires.

Nous commençons par identifier l'espace symétrique $S_n = \mathrm{SO}(n)\backslash\mathrm{SL}(n, \mathbb{R})$ avec l'ensemble des réseaux de $\mathbb{R}^n$ de covolume 1 modulo isométries munis d'une $\mathbb{Z}$-base. La classe d'une matrice $A \in \mathrm{SL}(n, \mathbb{R})$ est associée au réseau $A\mathbb{Z}^n$. La multiplication à droite par une matrice de $\mathrm{SL}(n, \mathbb{Z})$ ne change pas le réseau mais change la $\mathbb{Z}$-base.

Définition. Soit $A \in \mathrm{SL}(n, \mathbb{R})$ et $\Lambda = A\mathbb{Z}^n$ le réseau associé. On définit la systole de A (ou Λ) par :

$$\mathrm{syst}(A) = \min_{x \in \mathbb{Z}^n, x \neq 0} |Ax|,$$

et nous appelons également systoles (ou vecteurs minimaux) les vecteurs $x \in \mathbb{Z}^n$ qui réalisent ce minimum, et notons $\mathcal{S}(A)$ leur ensemble.

Ash a prouvé le théorème suivant :

Théorème (Ash). L'ensemble des réseaux dont les vecteurs minimaux engendrent $\mathbb{R}^n$ est une épine pour $\mathrm{SL}(n, \mathbb{Z})$.

Plus précisément, si $\mathcal{X}_k$ est l'ensemble des réseaux dont les vecteurs minimaux engendrent un sous-espace de dimension au moins k (pour $k = 1, \ldots, n$), Ash a montré que l'ensemble $\mathcal{X}_{k+1}$ est un rétract par déformation $\mathrm{SL}(n, \mathbb{Z})$-équivariant de $\mathcal{X}_k$ pour tout $i = 1, \ldots, n-1$. L'idée pour cette construction est d'augmenter la taille de l'image des systoles et de réduire celle de son orthogonal :

Définition. Pour $A \in \mathcal{X}_k$ et $\lambda \in \mathbb{R}$ on définit $T_A^\lambda \in \mathrm{SL}(n, \mathbb{R})$ par :

$$T_A^\lambda v = \begin{cases} e^{(n-k)\lambda} v \text{ si } v \in A\langle S(A)\rangle_{\mathbb{R}} \\ e^{-k\lambda} v \text{ si } v \in (A\langle S(A)\rangle_{\mathbb{R}})^\perp \end{cases} ,$$

où $\langle S(A)\rangle_{\mathbb{R}}$ est l'espace vectoriel réel engendré par les vecteurs minimaux de A.

On a pour $U \in \mathrm{SO}(n)$, $T_{UA}^\lambda UA = UT_A^\lambda A$, donc $T_A^\lambda A \in S_n$ ne dépend pas du représentant de A. De plus $T_A^0 A = A$, et l'on voit avec un argument géométrique qu'il existe $\lambda \geqslant 0$ tel que $T_A^\lambda A \in \mathcal{X}_{k+1}$.

On pose alors pour $A \in \mathcal{X}_k$: $\tau(A) = \inf\{\lambda, T_A^\lambda A \in \mathcal{X}_{k+1}\}$. On a $\tau(A) = 0$ si et seulement si $A \in \mathcal{X}_{k+1}$. Le rétract de $\mathcal{X}_k$ sur $\mathcal{X}_{k+1}$ est donné par : $(t, A) \mapsto T_A^{t\tau(A)} A$. Pour montrer que c'est bien un rétract par déformation, il reste à établir la continuité de τ.

Notons que dans le cas $n = 2$, en identifiant l'espace symétrique S_2 au plan hyperbolique $\mathbb{H}^2$, le réseau associé à $\tau \in \mathbb{H}^2$ est celui engendré par 1 et τ normalisé pour avoir volume 1, et le rétract bien équilibré correspond à l'arbre de Bass-Serre de $\mathrm{SL}(2, \mathbb{Z})$. Dans le cas $n = 3$, Soulé a donné dans [45] une description concrète du rétract bien équilibré de $\mathrm{SL}(3, \mathbb{Z})$.

Nous pouvons de plus montrer que le rétract de Ash est minimal, au sens où il n'existe pas d'épine pour $\mathrm{SL}(n, \mathbb{Z})$ incluse strictement dedans, voir [44] et [43].

1.4 Le cas du groupe symplectique

Il y a eu par la suite des tentatives de constructions similaires à celle de Ash pour d'autres groupes arithmétiques tels que le groupe symplectique $\mathrm{Sp}(2g, \mathbb{Z})$. Dans le cas $g = 2$, MacPherson et McConnell ont construit dans [36] ce que l'on appelle une épine au sens faible, c'est-à-dire pour tout sous-groupe $\Gamma \subset \mathrm{Sp}(4, \mathbb{Z})$ d'indice fini sans torsion, un rétract par déformation Γ-équivariant cocompact de l'espace symétrique $\mathrm{Sp}(4, \mathbb{R})/\mathrm{U}(2)$, en utilisant la décomposition de Voronoi de l'espace $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$ (voir le Chap.VII de [38] pour plus d'informations sur cette décomposition). Cependant leur construction ne s'étend pas à tout le groupe $\mathrm{Sp}(4, \mathbb{Z})$, mais c'est le premier exemple de construction d'une épine (au sens faible) pour un espace symétrique non linéaire de rang strictement supérieur à 1.

Pour g quelconque, Bavard a montré dans [6] un résultat similaire à celui de Ash, en identifiant l'espace symétrique $\mathfrak{h}_g = \mathrm{Sp}(2g, \mathbb{R})/\mathrm{U}(g)$ (appelé aussi espace de Siegel) avec l'ensemble des réseaux symplectiques de $\mathbb{R}^{2g}$, c'est-à-dire les réseaux munis d'une $\mathbb{Z}$-base symplectique.

Théorème (Bavard). L'ensemble des réseaux symplectiques dont les systoles engendrent un sous-espace non-isotrope pour la forme symplectique est un rétract $\mathrm{Sp}(2g, \mathbb{Z})$-équivariant de l'espace de Siegel.

Malheureusement ce rétract est de codimension 1, car il existe des réseaux symplectiques avec seulement deux systoles non-isotropes. On pourrait alors essayer de rétracter sur l'ensemble des réseaux symplectiques avec au moins trois systoles linéairement indépendantes. Nous montrerons dans le chapitre 2 que ce l'on ne peut pas faire cela :

Théorème. L'ensemble $(\mathcal{X}_3 \cap \mathfrak{h}_g)$ des réseaux symplectiques dans $\mathfrak{h}_g$ dont les systoles engendrent un sous-espace de dimension au moins 3 dans $\mathbb{R}^{2g}$ ne contient aucun modèle de type $\underline{E}\mathrm{Sp}(2g, \mathbb{Z})$. En particulier, il ne contient aucun rétract par déformation $\mathrm{Sp}(2g, \mathbb{Z})$-équivariant de $\mathfrak{h}_g$.

Nous obtenons par une preuve similaire le même résultat en remplaçant $\mathrm{Sp}(2g, \mathbb{Z})$ par $\mathrm{Aut}(\mathrm{SL}(n, \mathbb{Z}))$:

Théorème. L'ensemble $\mathcal{X}_3$ des réseaux de $\mathbb{R}^n$ dont les systoles engendrent un sous-espace de dimension au moins 3 ne contient aucun modèle de type $\underline{E}\mathrm{Aut}(\mathrm{SL}(n, \mathbb{Z}))$.

Ce résultat est remarquable car $\mathrm{SL}(n, \mathbb{Z})$ et $\mathrm{Aut}(\mathrm{SL}(n, \mathbb{Z}))$ ne diffèrent que d'un groupe fini, et agissent tous les deux par isométries sur le même espace symétrique.

Ces résultats ne prouvent cependant pas qu'il n'existe aucune épine pour les groupes $\mathrm{Sp}(2n, \mathbb{Z})$ et $\mathrm{Aut}(\mathrm{SL}(n, \mathbb{Z}))$. Une idée pour en trouver serait de considérer les systoles d'ordre supérieur, qui comme la fonction systole sont des exponentielles de fonctions de Busemann (voir [5]).

1.5 Calcul de la dimension géométrique propre pour les réseaux des groupes de Lie semisimples

Nous avons vu que dans le cas de $\mathrm{SL}(n, \mathbb{Z})$, la dimension géométrique propre est égale à la dimension cohomologique virtuelle, et en particulier est calculable avec la formule de Borel-Serre. Cependant la preuve utilisait la construction d'une épine, ce qui n'est pas toujours facile comme on l'a vu avec le cas $\mathrm{Sp}(2n, \mathbb{Z})$. Nous démontrerons dans le chapitre 1 le résultat principal de cette thèse, à savoir que l'inégalité $\text{vcd}(\Gamma) \leqslant \underline{\text{gd}}(\Gamma)$ est une égalité dans le cas qui nous intéresse :

Théorème. Soit S un espace symétrique de type non-compact sans facteur euclidien. Alors :

$$\text{vcd}(\Gamma) = \underline{\text{gd}}(\Gamma),$$

pour tout réseau $\Gamma \subset \text{Isom}(S)$.

Le cas où Γ est un réseau dans un groupe de Lie simple classique a déjà été traité dans [1] et nous utiliserons les mêmes méthodes pour étendre le résultat.

Ce théorème a plusieurs conséquences importantes. La première est que la conclusion reste vraie pour des réseaux commensurables aux réseaux ci-dessus (on parle de rigidité dimensionnelle) :

Corollaire. Si Γ est commensurable à un réseau du groupe d'isométries d'un espace symétrique de type non-compact sans facteur euclidien, alors :

$$\text{vcd}(\Gamma) = \underline{\text{gd}}(\Gamma).$$

Rappelons que deux groupes Γ_1 et Γ_2 sont dits commensurables s'ils admettent des sous-groupes d'indice fini isomorphes. Notons que le résultat du corollaire est faux dans le cas général : la dimension géométrique propre se comporte mal vis-à-vis de la commensurabilité.

Plus généralement, en utilisant [34, Thm. 5.16] on peut montrer :

Corollaire. Si Γ est un réseau du groupe d'isométries d'un espace symétrique de type non-compact sans facteur euclidien, et si

$$1 \to \Gamma \to G \to Q \to 1$$

est une suite exacte courte, alors $\underline{\text{gd}}(G) \leqslant \underline{\text{gd}}(\Gamma) + \underline{\text{gd}}(Q)$.

Le dernier corollaire est plus géométrique et est une conséquence directe du fait que l'espace symétrique est un modèle de type $\underline{E}\Gamma$ et que deux modèles de ce type sont Γ-homotopiquement équivalents :

Corollaire. Si S est un espace symétrique de type non-compact sans facteur euclidien, et $\Gamma \subset \text{Isom}(S)$ un réseau, alors S est Γ-homotopiquement équivalent à un Γ-complexe cellulaire propre cocompact de dimension $\text{vcd}(\Gamma)$.

Ce corollaire est uniquement théorique et ne donne aucun moyen constructif de réaliser un tel espace.

Donnons les principaux arguments de la preuve du théorème. Tout d'abord, remarquons que si $S = G/K$ est un espace symétrique de type non-compact sans facteur euclidien, le groupe G a un nombre fini de composantes connexes et un centre fini, on peut donc le supposer connexe de centre trivial. Dans ce cas le groupe d'isométries $\mathrm{Isom}(S)$ est égal au groupe $\mathrm{Aut}(G)$ d'automorphismes de G, lui-même égal au groupe d'automorphismes de son algèbre de Lie $\mathrm{Aut}(\mathfrak{g})$. C'est un groupe de Lie linéaire semisimple mais pas nécessairement connexe. La composante connexe de l'identité est le groupe d'automorphismes intérieurs $\mathrm{Int}(\mathfrak{g})$ isomorphe à G et le quotient $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ est le groupe d'automorphismes extérieurs $\mathrm{Out}(\mathfrak{g})$. Le théorème se reformule donc de la manière suivante :

Théorème-bis. Soit $\mathfrak{g}$ une algèbre de Lie semisimple. Alors:

$$\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma),$$

pour tout réseau $\Gamma \subset \mathrm{Aut}(\mathfrak{g})$.

Pour montrer l'égalité entre les deux dimensions, nous en introduisons une troisième, la dimension cohomologique de Bredon $\underline{\mathrm{cd}}(\Gamma)$. Nous renvoyons au chapitre 2 ainsi qu'à [35], [41] et [13] pour une définition précise. Retenons qu'elle peut être vue comme l'analogue algébrique de la dimension géométrique propre, et dans la plupart des cas, elles sont égales (voir [35]) :

Théorème (Lück-Meintrup). Si Γ est un groupe discret tel que $\underline{\mathrm{cd}}(\Gamma) \geqslant 3$, alors $\underline{\mathrm{gd}}(\Gamma) = \underline{\mathrm{cd}}(\Gamma)$.

Nous montrons donc que $\mathrm{vcd}(\Gamma) = \underline{\mathrm{cd}}(\Gamma)$, et pour cela nous allons utiliser une caractérisation cohomologique de $\underline{\mathrm{cd}}(\Gamma)$ similaire à l'expression (1.2.1). Rappelons que la bordification de Borel-Serre X est un modèle cocompact de type $\underline{E}\Gamma$. Si on note $\mathcal{F}_0$ la famille des sous-groupes finis de Γ, alors on a (voir [13, Th 1.1]) :

$$\underline{\mathrm{cd}}(\Gamma) = \max\{n \in \mathbb{N} \mid \exists K \in \mathcal{F}_0 \ t.q. \ \mathrm{H}^n_c(X^K, X^K_{sing}) \neq 0\}, \tag{1.5.1}$$

où X^K_{sing} désigne le sous-complexe cellulaire de X^K composé des cellules dont le stabilisateur contient strictement K.

Cas des algèbres de Lie simples

Les formules (1.2.1) et (1.5.1) nous amènent au lemme-clé de la démonstration du théorème dans le cas des algèbres de Lie simples (voir [1, Cor. 3.4]):

Lemme-clé. Soit G le groupe des points réels d'un groupe algébrique semisimple $\mathbb{G}$ de rang réel au moins 2, $\Gamma \subset G$ un réseau non cocompact de G, $K \subset G$ un sous-groupe compact maximal et $S = G/K$ l'espace symétrique riemannien associé. Si $\dim S^\alpha < \mathrm{vcd}(\Gamma)$ pour tout $\alpha \in \Gamma$ d'ordre fini non central, alors $\underline{\mathrm{cd}}(\Gamma) = \mathrm{vcd}(\Gamma)$.

Pour résumer, la démonstration du théorème repose en grande partie sur le calcul de dimensions des points fixes S^α. Nous utiliserons notamment la classification des automorphismes extérieurs des algèbres de Lie simples. La plupart d'entre eux étant d'ordre 2, le quotient S/S^α est dans ce cas un espace symétrique (pas nécessairement riemannien), nous utiliserons donc aussi la classification des espaces symétriques.

Regardons en exemple le cas de l'algèbre de Lie $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Soit $\Gamma \subset \mathrm{Aut}(\mathfrak{g})$ un réseau non cocompact, et considérons l'espace symétrique $S = \mathrm{PSL}(n, \mathbb{C})/\mathrm{PSU}(n)$. En utilisant le lemme-clé, la formule de Borel-Serre et le fait que le rang rationnel est inférieur au rang réel, pour montrer que $\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$ il suffit d'établir la relation :

$$\dim S^\alpha < \dim S - \mathrm{rg}_{\mathbb{R}}(\mathrm{PSL}(n, \mathbb{C})) = (n^2 - 1) - (n - 1) \qquad (1.5.2)$$

pour tout $\alpha \in \Gamma$ d'ordre fini non central. Notons d'abord que α est la composée d'un automorphisme extérieur et d'un automorphisme intérieur. Par la classification des automorphismes extérieurs (voir par exemple [22]), on voit que chaque élément de $\mathrm{Out}(\mathfrak{g})$ est d'ordre 2, et donc α^2 est un automorphisme intérieur. S'il est non trivial, en remarquant que les points fixes de α sont aussi des points fixes de α^2, nous sommes ramenés à des calculs de commutants, qui ont été faits dans la Section 6.1 de [1]. Plus précisément, si $\alpha = \mathrm{Ad}(A)$ (avec $A \in G = \mathrm{PSL}(n, \mathbb{C})$) est un automorphisme intérieur non trivial de $\mathfrak{sl}(n, \mathbb{C})$, l'ensemble des points fixes S^α est l'espace symétrique riemannien associé au centralisateur $C_G(A)$.

Il nous reste à traiter le cas où α^2 est trivial, c'est-à-dire α est d'ordre 2. Alors $\alpha \in \mathrm{Aut}(\mathfrak{g})$ est induit par un automorphisme de $G = \mathrm{PSL}(n, \mathbb{C})$ que l'on note toujours α et qui est aussi une involution. L'ensemble des points fixes S^α est l'espace symétrique riemannien associé à G^α. Comme α est une involution, le quotient G/G^α est un espace symétrique. Les espaces symétriques associés aux groupes de Lie simples ont été classifiés par

Berger dans [7]. Dans le cas du groupe $G = \mathrm{PSL}(n, \mathbb{C})$, on obtient par cette classification que l'algèbre de Lie du groupe G^α est soit compacte soit isomorphe à l'une des algèbres de Lie suivantes: $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{s}(\mathfrak{gl}(k, \mathbb{C}) \oplus \mathfrak{gl}(n-k, \mathbb{C}))$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{su}(p, n-p)$, $\mathfrak{sl}(\frac{n}{2}, \mathbb{H})$, où $\mathfrak{sp}(n, \mathbb{C})$ et $\mathfrak{sl}(\frac{n}{2}, \mathbb{H})$ n'apparaissent que si n est pair. Puis, nous vérifions (1.5.2) pour chacune de ces algèbres de Lie ce qui mène à la preuve du théorème dans le cas $\mathfrak{g} = \mathfrak{sl}(\mathfrak{n}, \mathbb{C})$. Cet argument sera appliqué pour toutes les algèbres de Lie simples (complexes comme réels, classiques comme exceptionnelles). Seuls quelques cas poseront problème (notamment $\mathfrak{sl}(n, \mathbb{R})$ ou $\mathfrak{so}(p, q)$), ils feront l'objet de parties à part.

Cas des algèbres de Lie semisimples

Rappelons qu'une algèbre de Lie semisimple est somme directe d'algèbres de Lie simples. Un automorphisme α de $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \mathfrak{g}_\mathfrak{n}$ est la composée d'un automorphisme diagonal $\rho = \rho_1 \oplus \cdots \oplus \rho_n$ et d'une permutation σ de facteurs isomorphes.

La stratégie utilisée dans le cas des algèbres de Lie simples ne s'applique plus ici. En effet, l'inégalité

$$\dim S^\alpha \leqslant \dim S - \mathrm{rg}_\mathbb{R} G$$

pour $\alpha \in \mathrm{Aut}(\mathfrak{g})$ nécessaire pour appliquer le lemme-clé n'est plus valable même dans les cas les plus simples.

Par exemple, si $G = \mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$ et $A = \left(I_3, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$, on a $\dim S^A = 5 + 3 = 8 > 6 = \dim S - \mathrm{rg}_\mathbb{R} G$.

Nous contournons ce problème en revenant à la formule de Borel-Serre :

$$\mathrm{vcd}(\Gamma) = \dim S - \mathrm{rg}_\mathbb{Q} \Gamma,$$

on veut donc majorer $\mathrm{rg}_\mathbb{Q} \Gamma$. Nous nous restreignons dans un premier temps aux réseaux irréductibles et nous prouvons le résultat suivant :

Proposition. Soit $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ une algèbre de Lie semisimple et G_i le groupe adjoint de $\mathfrak{g}_i$ pour $i = 1 \ldots n$. Alors

$$\mathrm{rg}_\mathbb{Q} \Gamma \leqslant \min_{i=1 \ldots n} \mathrm{rg}_\mathbb{R} G_i$$

pour tout réseau arithmétique irréductible $\Gamma \subset G = \mathrm{Aut}(\mathfrak{g})$.

L'extension aux réseaux non-irréductibles se fait ensuite aisément.

Chapter 2

Dimension rigidity of lattices in semisimple Lie groups

2.1 Introduction

Let Γ be a discrete virtually torsion-free group. There exist several notions of "dimension" for Γ. One of them is the *virtual cohomological dimension* $\mathrm{vcd}(\Gamma)$, which is the cohomological dimension of any torsion-free finite index subgroup of Γ. Due to a result by Serre, it does not depend on the choice of such a subgroup (see [12]). Another one is the *proper geometric dimension.* A Γ-CW-complex X is said to be a model for $\underline{E}\Gamma$ if the stabilizers of the action of Γ on X are finite and for every finite subgroup H of Γ, the fixed point space X^H is contractible. Note that two models for $\underline{E}\Gamma$ are Γ-equivariantly homotopy equivalent to each other. The *proper geometric dimension* $\underline{\mathrm{gd}}(\Gamma)$ of Γ is the smallest possible dimension of a model for $\underline{E}\Gamma$.

These two notions are related. In fact, we always have the inequality

$$\mathrm{vcd}(\Gamma) \leqslant \underline{\mathrm{gd}}(\Gamma)$$

but this inequality may be strict, see for instance the construction of Leary and Nucinkis in [32], or other examples in [11], [33], [39], [14], [15].

However there are also many examples of virtually torsion-free groups Γ with $\mathrm{vcd}(\Gamma) = \underline{\mathrm{gd}}(\Gamma)$. For instance in [13] Degrijse and Martinez-Perez prove that this is the case for a large class of groups. Other examples for equality can be found in [1], [2], [34] and [48].

In this paper we will prove that equality holds for groups acting by isometries, discretely and with finite covolume on symmetric spaces of non-compact type without euclidean factors:

Theorem 2.1.1. *Let S be a symmetric space of non-compact type without euclidean factors. Then*

$$\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$$

for every lattice $\Gamma \subset \mathrm{Isom}(S)$.

Recall that a symmetric space of non-compact type without euclidean factors is of the form G/K where G is a semisimple Lie group, which can be assumed to be connected and centerfree, and $K \subset G$ is a maximal compact subgroup. Then $\mathrm{Isom}(S) = \mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(G)$ where $\mathfrak{g}$ is the Lie algebra of G, and note that this group is semisimple, linear and algebraic but may be not connected. In [1] the authors prove Theorem 3.1.1 for lattices in classical simple Lie groups G. We will heavily rely on their results and techniques.

We discuss now some applications of Theorem 3.1.1. First note that the symmetric space S is a model for $\underline{E}\Gamma$. Theorem 3.1.1 yields then that:

Corollary 2.1.1. *If S is a symmetric space of non-compact type and without euclidean factors, and if $\Gamma \subset \mathrm{Isom}(S)$ is a lattice, then S is Γ-equivariantly homotopy equivalent to a proper cocompact Γ-CW complex of dimension* $\mathrm{vcd}(\Gamma)$*.*

We stress again that in the setting of Theorem 3.1.1 we are considering the full group of isometries of S. This has the consequence that we are able to deduce that there is equality between the virtual cohomological dimension and the proper geometric dimension not only for lattices in $\mathrm{Isom}(S)$, but also for groups abstractly commensurable to them. Here, two groups Γ_1 and Γ_2 are said *abstractly commensurable* if for $i = 1, 2$, there exists a subgroup $\widetilde{\Gamma_i}$ of finite index in Γ_i, such that $\widetilde{\Gamma_1}$ is isomorphic to $\widetilde{\Gamma_2}$. Then we obtain from Theorem 3.1.1 that:

Corollary 2.1.2. *If a group Γ is abstractly commensurable to a lattice in the group of isometries of a symmetric space of non-compact type without euclidean factors, then $\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$.*

Remark 2.1.1 Note that in general the equality between the proper geometric dimension and the virtual cohomological dimension behaves badly under commensuration. For instance, the fact that there exist virtually torsion-free groups Γ with $\mathrm{vcd}(\Gamma) \geqslant 3$ and such that $\mathrm{vcd}(\Gamma) < \underline{\mathrm{gd}}(\Gamma)$ proves that if Γ' is a torsion-free subgroup of Γ of finite index, then $\mathrm{vcd}(\Gamma') = \mathrm{cd}(\Gamma') = \mathrm{gd}(\Gamma') = \underline{\mathrm{gd}}(\Gamma')$, whereas Γ is commensurable to Γ' and $\mathrm{vcd}(\Gamma) < \underline{\mathrm{gd}}(\Gamma)$. In fact, we have concrete examples of groups for which Corollary 2.1.2 fails among familiar classes of groups. For instance, in [13] the authors prove that if Γ is a finitely generated Coxeter group then $\mathrm{vcd}(\Gamma) = \underline{\mathrm{gd}}(\Gamma)$ and in [33] the authors construct finite extensions of certain right-angled Coxeter groups such that $\mathrm{vcd}(\Gamma) < \underline{\mathrm{gd}}(\Gamma)$.

Returning to the applications of Theorem 3.1.1, we obtain from Corollary 2.1.2 that lattices in $\mathrm{Isom}(S)$ are dimension rigid in the sense of [15]: we say that a virtually torsion-free group Γ is *dimension rigid* if one has $\underline{\mathrm{gd}}(\widetilde{\Gamma}) = \mathrm{vcd}(\widetilde{\Gamma})$ for every group $\widetilde{\Gamma}$ which contains Γ as a finite index normal subgroup.

Dimension rigidity has a strong impact on the behaviour of the proper geometric dimension under group extensions, and we obtain from Corollary 2.1.2 and [34, Thm. 5.16] that:

Corollary 2.1.3. *If Γ is a lattice in the group of isometries of a symmetric space of non-compact type without euclidean factors and*

$$1 \to \Gamma \to G \to Q \to 1$$

is a short exact sequence, then $\underline{\mathrm{gd}}(G) \leqslant \underline{\mathrm{gd}}(\Gamma) + \underline{\mathrm{gd}}(Q)$. ■

We sketch now the strategy of the proof of Theorem 3.1.1. To begin with, note that while symmetric spaces, both Riemannian and non-Riemannian, will play a key role in our considerations, most of the time we will be working in the ambient Lie group. In fact it will be convenient to reformulate Theorem 3.1.1 as follows:

Main Theorem *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then*

$$\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$$

for every lattice $\Gamma \subset \mathrm{Aut}(\mathfrak{g})$.

The key ingredient in the proof of the Main Theorem, and hence of Theorem 3.1.1, is a result of Lück and Meintrup [35], which basically asserts that the proper geometric dimension $\underline{\mathrm{gd}}(\Gamma)$ equals the Bredon cohomological dimension $\underline{\mathrm{cd}}(\Gamma)$ - see Theorem 2.2.3 for a precise statement. In the light of this theorem it suffices to prove that the two cohomological notions of dimension $\mathrm{vcd}(\Gamma)$ and $\underline{\mathrm{cd}}(\Gamma)$ coincide. In [1] the authors noted that to prove the equality $\mathrm{vcd}(\Gamma) = \underline{\mathrm{cd}}(\Gamma)$ it suffices to ensure that the fixed point sets S^α of finite order elements $\alpha \in \Gamma$ are of small dimension - see Section 2.8 for details. Still in [1] the authors checked that this was the case for lattices contained in the classical simple Lie groups. We will use a similar strategy to prove the Main Theorem for lattices in groups of automorphisms of all simple Lie algebras. Recall that any non-compact finite dimensional simple Lie algebra over $\mathbb{R}$ is either isomorphic to one of the classical types or to one of the exceptional ones. The classical Lie algebras are the complex ones

$$\mathfrak{sl}(n,\mathbb{C}), \mathfrak{so}(n,\mathbb{C}), \mathfrak{sp}(2n,\mathbb{C})$$

and their real forms

$$\mathfrak{sl}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{H}), \mathfrak{so}(p,q), \mathfrak{su}(p,q), \mathfrak{sp}(p,q), \mathfrak{sp}(2n,\mathbb{R}), \mathfrak{so}^*(2n).$$

Similarly, the exceptional Lie algebras are the five complex ones

$$\mathfrak{g}_2^{\mathbb{C}}, \mathfrak{f}_4^{\mathbb{C}}, \mathfrak{e}_6^{\mathbb{C}}, \mathfrak{e}_7^{\mathbb{C}}, \mathfrak{e}_8^{\mathbb{C}}$$

and their twelve real forms

$$\mathfrak{g}_{2(2)}, \mathfrak{f}_{4(4)}, \mathfrak{f}_{4(-20)}, \mathfrak{e}_{6(6)}, \mathfrak{e}_{6(2)}, \mathfrak{e}_{6(-14)}, \mathfrak{e}_{6(-26)},$$

$$\mathfrak{e}_{7(7)}, \mathfrak{e}_{7(-5)}, \mathfrak{e}_{7(-25)}, \mathfrak{e}_{8(8)}, \mathfrak{e}_{8(-24)}.$$

Here the number in brackets is the difference between the dimension of the adjoint group and twice the dimension of a maximal compact subgroup (which equals 0 for a complex Lie group).

We illustrate now the basic steps of the proof of the Main Theorem in the example of $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$. Suppose that $\Gamma \subset \mathrm{Aut}(\mathfrak{g})$ is a lattice, and consider the symmetric space $S = \mathrm{PSL}(n,\mathbb{C})/\mathrm{PSU}_n$. To prove that $\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$, it will suffice to establish that

$$\dim S^\alpha < \dim S - \mathrm{rk}_{\mathbb{R}}(\mathrm{PSL}(n,\mathbb{C})) = (n^2-1) - (n-1) \tag{2.1.1}$$

for every $\alpha \in \Gamma$ of finite order and non central (see Lemma 2.2.2). First note that α is the composition of an inner automorphism and an outer automorphism. Since every non-trivial element in $\mathrm{Out}(\mathfrak{g})$ has order 2, it follows that α^2 is an inner automorphism. If it is non trivial then we use the results of Section 6.1 in [1]. In general, if $\alpha = \mathrm{Ad}(A)$ is a non-trivial inner automorphism of $\mathfrak{sl}(n,\mathbb{C})$, we get from [1] that (2.1.1) holds.

We are reduced to the case where α^2 is trivial, meaning that α is of order 2. Then the automorphism $\alpha \in \mathrm{Aut}(\mathfrak{g})$ is induced by an automorphism of the adjoint group $G_{ad} = \mathrm{PSL}(n,\mathbb{C})$ which is still denoted α and is also an involution. The fixed point set S^α is the Riemannian symmetric space associated to G_{ad}^α, where G_{ad}^α is the set of fixed points of α. Now, notice that the quotient G_{ad}/G_{ad}^α is a (non-Riemannian) symmetric space. The symmetric spaces associated to simple groups have been classified by Berger in [7]. In the case of $G_{ad} = \mathrm{PSL}(n,\mathbb{C})$, we obtain from this classification that the Lie algebra of G_{ad}^α is either compact or isomorphic to $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{s}(\mathfrak{gl}(k,\mathbb{C}) \oplus \mathfrak{gl}(n-k,\mathbb{C}))$, $\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}(p,n-p)$ or $\mathfrak{sl}(\frac{n}{2},\mathbb{H})$, where $\mathfrak{sp}(n,\mathbb{C})$ and $\mathfrak{sl}(\frac{n}{2},\mathbb{H})$ only appear if n is even. Armed with this information, we check (2.1.1) for every involution α, which leads to the Main Theorem for $\mathfrak{g} = \mathfrak{sl}(\mathfrak{n},\mathbb{C})$. The argument we just sketched will be applied in Section 3 to all complex simple Lie algebras and in Section 4 to the real ones. Since the arguments are similar, and since the complex case is somewhat easier, we advise the reader to skip Section 4 in a first reading.

Having dealt with the simple Lie algebras, we treat in Section 5 the semisimple case. The method for the simple algebras will not work at first sight, but the proof will eventually be simpler. The idea is to restrict to irreducible lattices, i.e. those who cannot be decomposed into a product. Then we will show that the rational rank of an irreducible lattice is lower than the real rank of any factor of the adjoint group, meaning that we get a much improved bound than in (2.1.1). This fact will lead rapidly to the Main Theorem.

Finally note that in the proof of the Main Theorem, we do not construct a concrete model for $\underline{E}\Gamma$ of dimension $\mathrm{vcd}(\Gamma)$, we just prove its existence. It is however worth mentioning that in a few cases such models are known. For instance if $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, the symmetric space $S = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}_n$ admits a Γ-equivariant deformation retract of dimension $\mathrm{vcd}(\Gamma)$ called the "well-rounded retract" (see [3], [44] and [43]). It will be interesting to do the same for groups such as $\mathrm{Sp}(2n, \mathbb{Z})$.

2.2 Preliminaries

In this section we recall some basic facts and definitions about algebraic groups, Lie groups and Lie algebras, symmetric spaces, lattices and arithmetic groups, virtual cohomological dimension and Bredon cohomology.

Algebraic groups and Lie groups

An *algebraic group* is a subgroup $\mathbb{G}$ of $\mathrm{SL}(N, \mathbb{C})$ determined by a collection of polynomials. It is *defined over* a subfield k of $\mathbb{C}$ if those polynomials can be chosen to have coefficients in k. The Galois criterion (see [9, Prop. 14.2 p.30]) says that $\mathbb{G}$ is defined over k if and only if $\mathbb{G}$ is stable under the Galois group $\mathrm{Gal}(\mathbb{C}/k)$. If $\mathbb{G}$ is an algebraic group and $R \subset \mathbb{C}$ is a ring we denote $\mathbb{G}_R$ the set of elements of $\mathbb{G}$ with entries in R. If $\mathbb{G}$ is an algebraic group defined over $\mathbb{R}$, it is well-known that the groups $\mathbb{G}_\mathbb{C}$ and $\mathbb{G}_\mathbb{R}$ are Lie groups with finitely many connected components. In fact, $\mathbb{G}$ is Zariski connected if and only if $\mathbb{G}_\mathbb{C}$ is a connected Lie group, whereas $\mathbb{G}_\mathbb{R}$ may not be connected in this case. A non-abelian algebraic group (or Lie group) is said to be *simple* if every connected normal subgroup is trivial, and *semisimple* if every connected normal abelian subgroup is trivial. Note that if $\mathbb{G}$ is a semisimple algebraic group defined over $k = \mathbb{R}$ or $\mathbb{C}$ then $\mathbb{G}_k$ is a semisimple Lie group. Any connected semisimple complex linear Lie group is algebraic and any connected semisimple real linear Lie group is the identity component of the group of real points of an algebraic group defined over $\mathbb{R}$. Recall that two Lie groups G_1 and G_2 are *isogenous* if they are locally isomorphic, meaning that there exist finite normal subgroups $N_1 \subset G_1^0$ and $N_2 \subset G_2^0$ of the identity components of G_1 and G_2 such that G_1^0/N_1 is isomorphic to G_2^0/N_2. A semisimple linear Lie group is isogenous to a product of simple Lie groups.

The center $Z(\mathbb{G})$ of a semisimple algebraic group $\mathbb{G}$ is finite (it is also the case for semisimple linear Lie groups but not for semisimple Lie groups in general) and the quotient $\mathbb{G}/Z(\mathbb{G})$ is again a semisimple algebraic group (see [9, Thm. 6.8 p.98]). Moreover, if $\mathbb{G}$ is defined over k then so is $\mathbb{G}/Z(\mathbb{G})$.

A connected algebraic group $\mathbb{T} \subset \mathrm{SL}(N, \mathbb{C})$ is a *torus* if it is diagonalizable, meaning there exists $A \in \mathrm{SL}(N, \mathbb{C})$ such that for every $B \in \mathbb{T}$, ABA^{-1} is diagonal. A torus is in particular abelian and isomorphic, as an algebraic group, to a product $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$. If $\mathbb{T}$ is defined over k, it is said to be *k-split* if the conjugating element A can be chosen in $\mathrm{SL}(N, k)$. A torus in an algebraic group $\mathbb{G}$ is a subgroup that is a torus. It is said to be *maximal* if it is not strictly contained in any other torus. An important fact is that any two maximal tori in $\mathbb{G}$ are conjugate in $\mathbb{G}$, and that if $\mathbb{G}$ is defined over k, then any two maximal k-split tori are conjugate by an element in $\mathbb{G}_k$. The *k-rank* of $\mathbb{G}$ (or of $\mathbb{G}_k$), denoted by $\mathrm{rk}_k\mathbb{G}$ (or $\mathrm{rk}_k\mathbb{G}_k$), is the dimension of any maximal k-split torus in $\mathbb{G}$, and the *rank* of $\mathbb{G}$ is just the $\mathbb{C}$-rank.

We refer to [9], [29] and [42] for basic facts about algebraic groups and Lie groups.

Lie algebras and their automorphisms

Recall that the Lie algebra $\mathfrak{g}$ of a Lie group G is the set of left-invariant vector fields. A *subalgebra* of $\mathfrak{g}$ is a subspace closed under Lie bracket. An *ideal* is a subalgebra I such that $[\mathfrak{g}, I] \subset I$. The Lie algebra $\mathfrak{g}$ is *simple* if it is not abelian and has no non-trivial ideals, and *semisimple* if it has no non-zero abelian ideals. A Lie group is simple (resp. semisimple) if and only if its Lie algebra is simple (resp. semisimple). A semisimple Lie algebra is isomorphic to a finite direct sum of simple ones.

By Lie's third theorem, if $\mathfrak{g}$ is a finite dimensional real Lie algebra (which will be always the case here), there exists a connected Lie group, unique up to covering, whose Lie algebra is $\mathfrak{g}$. This means that there exists a unique simply connected Lie group G associated to $\mathfrak{g}$, and every other connected Lie group whose Lie algebra is $\mathfrak{g}$ is a quotient of G by a subgroup contained in the center. In particular, $G_{ad} = G/Z(G)$ is the unique connected centerfree

Lie group associated to $\mathfrak{g}$. The group G_{ad} is called the *adjoint group* of $\mathfrak{g}$. The adjoint group is a linear algebraic group, whereas its universal cover may be not linear (see for instance the universal cover of $\mathrm{PSL}(2,\mathbb{R})$). It follows that the classification of simple Lie algebras is in correspondence with that of simple Lie groups. A Lie algebra is said to be *compact* if the adjoint group is.

An automorphism of a Lie algebra $\mathfrak{g}$ is a bijective linear endomorphism which preserves the Lie bracket. The group of automorphisms of $\mathfrak{g}$ is denoted $\mathrm{Aut}(\mathfrak{g})$, it is linear and algebraic but not connected in general. If G is a Lie group associated to $\mathfrak{g}$, then the differential of a Lie group automorphism of G is an automorphism of $\mathfrak{g}$. Conversely, if G is either simply connected or connected and centerfree, any automorphism of $\mathfrak{g}$ comes from an automorphism of G. In this case, we will often identify these two automorphisms and denote them by the same letter. An *inner automorphism* is the derivative of the conjugation in G by an element $A \in G$ - we denote it $\mathrm{Ad}(A)$. The group $\mathrm{Inn}(\mathfrak{g})$ of inner automorphisms is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$. It is also the identity component of $\mathrm{Aut}(\mathfrak{g})$ and is isomorphic to the adjoint group G_{ad}. If $\mathfrak{g}$ is semisimple, the subgroup $\mathrm{Inn}(\mathfrak{g})$ is of finite index in $\mathrm{Aut}(\mathfrak{g})$ and the quotient $\mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is the (finite) group of *outer automorphisms* $\mathrm{Out}(\mathfrak{g})$. Moreover if $\mathfrak{g}$ is simple, $\mathrm{Out}(\mathfrak{g})$ can be seen as a subgroup of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g})$ is the semidirect product of $\mathrm{Out}(\mathfrak{g})$ and $\mathrm{Inn}(\mathfrak{g})$, that is $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}) \rtimes \mathrm{Out}(\mathfrak{g})$ (see [21]).

Note that even if $\mathfrak{g}$ is complex, we let $\mathrm{Aut}(\mathfrak{g})$ be the group of real automorphisms. If $\mathfrak{g}$ is complex and simple then $\mathrm{Aut}(\mathfrak{g})$ contains the complex automorphism group $\mathrm{Aut}_{\mathbb{C}}(\mathfrak{g})$ as a subgroup of index 2, the quotient being generated by complex conjugation (see [16, Prop. 4.1]).

Recall that if $\mathfrak{g}$ is a complex Lie algebra, a *real form* of $\mathfrak{g}$ is a real Lie algebra whose complexification is $\mathfrak{g}$. Any real form is the group of fixed points by a *conjugation* of $\mathfrak{g}$, meaning an involutive real automorphism which is antilinear over $\mathbb{C}$.

We refer to [42], [29], [22] and [16] for other facts about Lie algebras and their automorphisms.

Simple Lie groups, simple Lie algebras and their outer automorphisms

As mentioned in the previous section, the classifications of simple Lie groups (up to isogeny) and of simple Lie algebras are in correspondence. Both are due to Cartan. We will now see that of simple Lie groups. Every linear simple Lie group is isogenous to either a classical group or to one of the finitely many exceptional groups. We denote the transpose of a matrix A by A^t and its conjugate transpose by A^* and we consider the particular matrices

$$J_n = \begin{pmatrix} & \mathrm{Id}_n \\ -\mathrm{Id}_n & \end{pmatrix}, Q_{p,q} = \begin{pmatrix} -\mathrm{Id}_p & \\ & \mathrm{Id}_q \end{pmatrix}.$$

The non-compact classical simple Lie groups are the groups in the following list

$\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid \det A = 1\}$	$n \geqslant 2$
$\mathrm{SO}(n,\mathbb{C}) = \{A \in \mathrm{SL}(n,\mathbb{C}) \mid A^tA = \mathrm{Id}\}$	$n \geqslant 3, n \neq 4$
$\mathrm{Sp}(2n,\mathbb{C}) = \{A \in \mathrm{SL}(2n,\mathbb{C}) \mid A^tJ_nA = J_n\}$	$n \geqslant 1$
$\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A = 1\}$	$n \geqslant 2$
$\mathrm{SL}(n,\mathbb{H}) = \{A \in \mathrm{GL}(n,\mathbb{H}) \mid \det A = 1\}$	$n \geqslant 2$
$\mathrm{SO}(p,q) = \{A \in \mathrm{SL}(p+q,\mathbb{R}) \mid A^tQ_{p,q}A = Q_{p,q}\}$	$1 \leqslant p \leqslant q, p+q \geqslant 3$
$\mathrm{SU}(p,q) = \{A \in \mathrm{SL}(p+q,\mathbb{C}) \mid A^*Q_{p,q}A = Q_{p,q}\}$	$1 \leqslant p \leqslant q, p+q \geqslant 3$
$\mathrm{Sp}(p,q) = \{A \in \mathrm{GL}(p+q,\mathbb{H}) \mid A^*Q_{p,q}A = Q_{p,q}\}$	$1 \leqslant p \leqslant q, p+q \geqslant 3$
$\mathrm{Sp}(2n,\mathbb{R}) = \{A \in \mathrm{SL}(2n,\mathbb{R}) \mid A^tJ_nA = J_n\}$	$n \geqslant 1$
$\mathrm{SO}^*(2n) = \{A \in \mathrm{SU}(n,n) \mid A^tQ_{n,n}J_nA = Q_{n,n}J_n\}$	$n \geqslant 2$

Similarly we give the list of the compact ones

$\mathrm{SO}_n = \{A \in \mathrm{SL}(n,\mathbb{R}) \mid A^tA = \mathrm{Id}\}$	$\mathrm{O}_n = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^tA = \mathrm{Id}\}$
$\mathrm{SU}_n = \{A \in \mathrm{SL}(n,\mathbb{C}) \mid A^*A = \mathrm{Id}\}$	$\mathrm{U}_n = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid A^*A = \mathrm{Id}\}$
$\mathrm{Sp}_n = \{A \in \mathrm{GL}(n,\mathbb{H}) \mid A^*A = \mathrm{Id}\}$	

The compact exceptional Lie groups are

$$G_2, F_4, E_6, E_7, E_8$$

and the non-compact ones are the complex ones (which are the complexifications of the previous compact groups)

$$G_2^{\mathbb{C}}, F_4^{\mathbb{C}}, E_6^{\mathbb{C}}, E_7^{\mathbb{C}}, E_8^{\mathbb{C}}$$

and their real forms

$$G_{2(2)}, F_{4(4)}, F_{4(-20)}, F_{4(4)}, E_{6(6)}, E_{6(2)}, E_{6(-14)}, E_{6(-26)},$$

$$E_{7(7)}, E_{7(-5)}, E_{7(-25)}, E_{8(8)}, E_{8(-24)}.$$

We refer to [50] for definitions and complete descriptions of the simply connected versions of the exceptional Lie groups. Note that in this paper we will always consider the centerless versions with the same notations.

As usual, the simple Lie algebra associated to a simple Lie group will be denoted by gothic caracters, for instance $\mathfrak{sl}(n,\mathbb{R})$ is the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$. Note that the adjoint group of $\mathfrak{sl}(n,\mathbb{R})$ is $\mathrm{PSL}(n,\mathbb{R})$. The classification of simple Lie algebras runs in parallel to that of simple Lie groups. The following table summarizes the structure of the outer automorphisms groups of simple Lie algebras (see [22] section 3.2). We denote by S_n the symmetric group and D_{2n} the dihedral group.

$\mathfrak{g}$	$\mathrm{Out}(\mathfrak{g})$
$\mathfrak{sl}(n,\mathbb{C}),\ n \geqslant 3$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathfrak{so}(8,\mathbb{C})$	$\mathcal{S}_3 \times \mathbb{Z}_2$
$\mathfrak{so}(2n,\mathbb{C}),\ n \geqslant 5$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathfrak{e}_6^{\mathbb{C}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
all others complex Lie algebras	$\mathbb{Z}_2$
$\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{Z}_2$
$\mathfrak{sl}(n,\mathbb{R}),\ n \geqslant 3$ odd	$\mathbb{Z}_2$
$\mathfrak{sl}(n,\mathbb{R}),\ n \geqslant 4$ even	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathfrak{su}(p,q),\ p \neq q$	$\mathbb{Z}_2$
$\mathfrak{su}(p,p),\ p \geqslant 2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathbb{Z}_2$
$\mathfrak{so}(p,q),\ p+q$ odd	$\mathbb{Z}_2$
$\mathfrak{so}(p,q)$, p and q odd, $p \neq q$	$\mathbb{Z}_2$
$\mathfrak{so}(p,q)$ p and q even, $p \neq q$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathfrak{so}(p,p)$ $p \geqslant 5$ odd	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathfrak{so}(p,p)$ $p \geqslant 6$ even	$\mathcal{D}_4$
$\mathfrak{so}(4,4)$	$\mathcal{S}_4$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathbb{Z}_2$
$\mathfrak{sp}(p,p)$	$\mathbb{Z}_2$
$\mathfrak{so}^*(2n)$	$\mathbb{Z}_2$
$\mathfrak{e}_{6(j)},\ j = 6, 2, -14, -26$	$\mathbb{Z}_2$
$\mathfrak{e}_{7(j)},\ j = 7, -25$	$\mathbb{Z}_2$
all others real Lie algebras	1

Table 1: Outer automorphisms groups of simple Lie algebras

Note that we have the isomorphisms: $\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sp}(2,\mathbb{C})$, $\mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C})$, $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$ and the corresponding ones between their real forms, and that $\mathfrak{so}(4,\mathbb{C})$ is not simple but isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$.

Symmetric spaces

Let G be a Lie group. A *symmetric space* is a space of the form G/G^ρ where ρ is an involutive automorphism of G and G^ρ its fixed points set. It is said to be *irreducible* if it can not be decomposed as a product. From an algebraic point of view, the irreducibility of G/H implies that the Lie algebra $\mathfrak{h}$ of H is a maximal subalgebra of the Lie algebra $\mathfrak{g}$ of G. Equivalently, the irreducibility of G/H implies that the identity component of H is a maximal connected Lie subgroup of the identity component of G.

Another point of view on symmetric spaces is based on Lie algebras. If G/G^ρ is a symmetric space and $\mathfrak{g}$ is the Lie algebra of G, the involutive automorphism ρ induces an involutive automorphism of $\mathfrak{g}$ whose fixed point set is the Lie algebra $\mathfrak{h}$ of $H = G^\rho$. We can thus always associate to a symmetric space G/H a linear space $\mathfrak{g}/\mathfrak{h}$, called a *local symmetric space*. The Lie subalgebra $\mathfrak{h}$ is called the *isotropy algebra* of $\mathfrak{g}/\mathfrak{h}$ (more generally we say that $\mathfrak{h}$ is an isotropy algebra if it is the fixed point set of an involutive automorphism). Conversely, if $\mathfrak{g}$ is a Lie algebra, G a simply connected or connected and centerless Lie group whose Lie algebra is $\mathfrak{g}$, and $\mathfrak{h} \subset \mathfrak{g}$ an isotropy algebra, the local symmetric space $\mathfrak{g}/\mathfrak{h}$ lifts to a symmetric space G/H, because $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}(G)$. So the classification of symmetric spaces are in correspondence with those of local symmetric spaces, and has been done by Berger in [7].

Note that if $\mathfrak{g}$ is simple and complex, and if $\rho \in \mathrm{Aut}(\mathfrak{g})$ is an involution, then ρ is either $\mathbb{C}$-linear and in this case $\mathfrak{h} = \mathfrak{g}^\rho$ is also complex, or ρ is anti-linear (that means it is a conjugation) and $\mathfrak{h} = \mathfrak{g}^\rho$ is a real form of $\mathfrak{g}$. Note also that if $\mathfrak{g}$ is real and ρ is an involution, then ρ can be extended to a $\mathbb{C}$-linear involution $\rho^\mathbb{C}$ of the complexification $\mathfrak{g}^\mathbb{C}$ of $\mathfrak{g}$, and the isotropy algebra $(\mathfrak{g}^\mathbb{C})^{\rho^\mathbb{C}}$ is the complexification of $\mathfrak{h} = \mathfrak{g}^\rho$, that is $(\mathfrak{g}^\mathbb{C})^{\rho^\mathbb{C}} = \mathfrak{h}^\mathbb{C}$.

We give now the list of the non-compact isotropy algebras of the local symmetric spaces associated to $\mathfrak{sl}(n,\mathbb{C})$ and its real forms.

$\mathfrak{g}^\mathbb{C} = \mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{h}^\mathbb{C} = \mathfrak{so}(n,\mathbb{C})$	$\mathfrak{h}^\mathbb{C} = \mathfrak{s}(\mathfrak{gl}(k,\mathbb{C}) \oplus \mathfrak{gl}(l,\mathbb{C}))$	$\mathfrak{h}^\mathbb{C} = \mathfrak{sp}(n,\mathbb{C})$
$\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{h} = \mathfrak{so}(k,l)$	$\mathfrak{h} = \mathfrak{s}(\mathfrak{gl}(k,\mathbb{R}) \oplus \mathfrak{gl}(l,\mathbb{R}))$ $\mathfrak{h} = \mathfrak{gl}\left(\frac{n}{2},\mathbb{C}\right)$	$\mathfrak{h} = \mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{g} = \mathfrak{su}(p,q)$	$\mathfrak{h} = \mathfrak{so}(p,q)$ $(p=q)$ $\mathfrak{h} = \mathfrak{so}^*(2p)$	$\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(k_p,k_q) \oplus \mathfrak{u}(l_p,l_q))$ $\mathfrak{h} = \mathfrak{gl}(p,\mathbb{C})$	$\mathfrak{h} = \mathfrak{sp}\left(\frac{p}{2},\frac{q}{2}\right)$ $\mathfrak{h} = \mathfrak{sp}(2p,\mathbb{R})$
$\mathfrak{g} = \mathfrak{sl}\left(\frac{n}{2},\mathbb{H}\right)$	$\mathfrak{h} = \mathfrak{so}^*\left(2\left(\frac{n}{2}\right)\right)$	$\mathfrak{h} = \mathfrak{s}(\mathfrak{gl}(\frac{k}{2},\mathbb{H}) \oplus \mathfrak{gl}(\frac{l}{2},\mathbb{H}))$ $\mathfrak{gl}\left(\frac{n}{2},\mathbb{C}\right)$	$\mathfrak{h} = \mathfrak{sp}\left(\frac{k}{2},\frac{l}{2}\right)$

Table 2: Non-compact isotropy algebras of $\mathfrak{sl}(n,\mathbb{C})$ and its real forms

Table 2 is organized as follows: in the first line we give the complex isotropy algebras $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{sl}(n, \mathbb{C})$ (fixed by a complex involution). Each column consists of real forms of the complex algebra in the first entry. The local symmetric spaces associated to $\mathfrak{sl}(n, \mathbb{C})$ are then those of the form $\mathfrak{sl}(n, \mathbb{C})/\mathfrak{h}^{\mathbb{C}}$, for instance $\mathfrak{sl}(n, \mathbb{C})/\mathfrak{so}(n, \mathbb{C})$, or of the form $\mathfrak{sl}(n, \mathbb{C})/\mathfrak{g}$, for instance $\mathfrak{sl}(n, \mathbb{C})/\mathfrak{sl}(n, \mathbb{R})$. The ones associated to a real form $\mathfrak{g}$ are of the form $\mathfrak{g}/\mathfrak{h}$, for instance $\mathfrak{sl}(n, \mathbb{R})/\mathfrak{so}(k, l)$ with $k + l = n$.

The following tables summarize the classification for other simple Lie algebras. They are organized in a similar way.

$\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{so}(k, \mathbb{C}) \oplus \mathfrak{so}(l, \mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{gl}\left(\frac{n}{2}, \mathbb{C}\right)$
$\mathfrak{g} = \mathfrak{so}(p, q)$	$\mathfrak{h} = \mathfrak{so}(k_p, k_q) \oplus \mathfrak{so}(l_p, l_q))$ $(p = q)$ $\mathfrak{h} = \mathfrak{so}(p, \mathbb{C})$	$\mathfrak{h} = \mathfrak{u}\left(\frac{p}{2}, \frac{q}{2}\right)$ $\mathfrak{h} = \mathfrak{gl}(p, \mathbb{R})$
$\mathfrak{g} = \mathfrak{so}^*\left(2\left(\frac{n}{2}\right)\right)$	$\mathfrak{h} = \mathfrak{so}^*\left(2\left(\frac{k}{2}\right)\right) \oplus \mathfrak{so}^*\left(2\left(\frac{l}{2}\right)\right)$ $\mathfrak{h} = \mathfrak{so}\left(\frac{n}{2}, \mathbb{C}\right)$	$\mathfrak{h} = \mathfrak{u}\left(\frac{k}{2}, \frac{l}{2}\right)$ $\mathfrak{h} = \mathfrak{gl}\left(\frac{n}{4}, \mathbb{H}\right)$

Table 3: Non-compact isotropy algebras of $\mathfrak{so}(n, \mathbb{C})$ and its real forms

$\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2n, \mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{sp}(2k, \mathbb{C}) \oplus \mathfrak{sp}(2l, \mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$
$\mathfrak{g} = \mathfrak{sp}(p, q)$	$\mathfrak{h} = \mathfrak{sp}(k_p, k_q) \oplus \mathfrak{sp}(l_p, l_q)$ $(p = q)$ $\mathfrak{h} = \mathfrak{sp}(p, \mathbb{C})$	$\mathfrak{h} = \mathfrak{u}(p, q)$ $\mathfrak{h} = \mathfrak{gl}(p, \mathbb{H})$
$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{h} = \mathfrak{sp}(2k, \mathbb{R}) \oplus \mathfrak{sp}(2l, \mathbb{R})$ $\mathfrak{h} = \mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{h} = \mathfrak{u}(k, l)$ $\mathfrak{h} = \mathfrak{gl}(n, \mathbb{R})$

Table 4: Non-compact isotropy algebras of $\mathfrak{sp}(2n, \mathbb{C})$ and its real forms

$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_2^{\mathbb{C}}$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
$\mathfrak{g} = \mathfrak{g}_{2(2)}$	$\mathfrak{h} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$

Table 5: Non-compact isotropy algebras of $\mathfrak{g}_2^{\mathbb{C}}$ and its real form

$\mathfrak{g}^{\mathbb{C}} = \mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{sp}(6, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{so}(9, \mathbb{C})$
$\mathfrak{g} = \mathfrak{f}_{4(4)}$	$\mathfrak{h} = \mathfrak{sp}(6, \mathbb{R}) \oplus \mathfrak{sp}(2, \mathbb{R})$ $\mathfrak{h} = \mathfrak{sp}(1, 2) \oplus \mathfrak{sp}(1)$	$\mathfrak{h} = \mathfrak{so}(4, 5)$
$\mathfrak{g} = \mathfrak{f}_{4(-20)}$	$\mathfrak{h} = \mathfrak{sp}(1, 2) \oplus \mathfrak{sp}(1)$	$\mathfrak{h} = \mathfrak{so}(1, 8)$

Table 6: Non-compact isotropy algebras of $\mathfrak{f}_4^{\mathbb{C}}$ and its real forms

$\mathfrak{g}^{\mathbb{C}} = \mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{sp}(8,\mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{so}(10,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{f}_4^{\mathbb{C}}$
$\mathfrak{g} = \mathfrak{e}_{6(6)}$	$\mathfrak{h} = \mathfrak{sp}(2,2)$ $\mathfrak{h} = \mathfrak{sp}(8,\mathbb{R})$	$\mathfrak{h} = \mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{h} = \mathfrak{sl}(3,\mathbb{H}) \oplus \mathfrak{su}(2)$	$\mathfrak{h} = \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)$	$\mathfrak{h} = \mathfrak{f}_{4(4)}$
$\mathfrak{g} = \mathfrak{e}_{6(2)}$	$\mathfrak{h} = \mathfrak{sp}(1,3)$ $\mathfrak{h} = \mathfrak{sp}(8,\mathbb{R})$	$\mathfrak{h} = \mathfrak{su}(2,4) \oplus \mathfrak{su}(2)$ $\mathfrak{h} = \mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{h} = \mathfrak{so}(4,6) \oplus \mathfrak{so}(2)$ $\mathfrak{h} = \mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$	$\mathfrak{h} = \mathfrak{f}_{4(4)}$
$\mathfrak{g} = \mathfrak{e}_{6(-14)}$	$\mathfrak{h} = \mathfrak{sp}(2,2)$	$\mathfrak{h} = \mathfrak{su}(2,4) \oplus \mathfrak{su}(2)$ $\mathfrak{h} = \mathfrak{su}(1,5) \oplus \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{h} = \mathfrak{so}(2,8) \oplus \mathfrak{so}(2)$ $\mathfrak{h} = \mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$	$\mathfrak{h} = \mathfrak{f}_{4(-20)}$
$\mathfrak{g} = \mathfrak{e}_{6(-26)}$	$\mathfrak{h} = \mathfrak{sp}(1,3)$	$\mathfrak{h} = \mathfrak{sl}(3,\mathbb{H}) \oplus \mathfrak{sp}(1)$	$\mathfrak{h} = \mathfrak{so}(1,9) \oplus \mathfrak{so}(1,1)$	$\mathfrak{h} = \mathfrak{f}_{4(-20)}$

Table 7: Non-compact isotropy algebras of $\mathfrak{e}_6^{\mathbb{C}}$ and its real forms

$\mathfrak{g}^{\mathbb{C}} = \mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{sl}(8,\mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{so}(12,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{e}_6^{\mathbb{C}} \oplus \mathfrak{so}(2,\mathbb{C})$
$\mathfrak{g} = \mathfrak{e}_{7(7)}$	$\mathfrak{h} = \mathfrak{su}(4,4)$ $\mathfrak{h} = \mathfrak{sl}(8,\mathbb{R})$ $\mathfrak{h} = \mathfrak{sl}(4,\mathbb{H})$	$\mathfrak{h} = \mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{h} = \mathfrak{so}^*(12) \oplus \mathfrak{sp}(1)$	$\mathfrak{h} = \mathfrak{e}_{6(6)} \oplus \mathfrak{so}(1,1)$ $\mathfrak{h} = \mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$
$\mathfrak{g} = \mathfrak{e}_{7(-5)}$	$\mathfrak{h} = \mathfrak{su}(4,4)$ $\mathfrak{h} = \mathfrak{su}(2,6)$	$\mathfrak{h} = \mathfrak{so}(4,8) \oplus \mathfrak{su}(2)$ $\mathfrak{h} = \mathfrak{so}^*(12) \oplus \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{h} = \mathfrak{e}_{6(2)} \times \mathfrak{so}(2)$ $\mathfrak{h} = \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$
$\mathfrak{g} = \mathfrak{e}_{7(-25)}$	$\mathfrak{h} = \mathfrak{sl}(4,\mathbb{H})$ $\mathfrak{h} = \mathfrak{su}(2,6)$	$\mathfrak{h} = \mathfrak{so}(2,10) \oplus \mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{h} = \mathfrak{so}^*(12) \oplus \mathfrak{sp}(1)$	$\mathfrak{h} = \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$ $\mathfrak{h} = \mathfrak{e}_{6(-26)} \oplus \mathfrak{so}(1,1)$

Table 8: Non-compact isotropy algebras of $\mathfrak{e}_7^{\mathbb{C}}$ and its real forms

$\mathfrak{g}^{\mathbb{C}} = \mathfrak{e}_8^{\mathbb{C}}$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{e}_7^{\mathbb{C}} \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{h}^{\mathbb{C}} = \mathfrak{so}(16,\mathbb{C})$
$\mathfrak{g} = \mathfrak{e}_{8(8)}$	$\mathfrak{h} = \mathfrak{e}_{7(7)} \oplus \mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{h} = \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$	$\mathfrak{h} = \mathfrak{so}(8,8)$ $\mathfrak{h} = \mathfrak{so}^*(16)$
$\mathfrak{g} = \mathfrak{e}_{8(-24)}$	$\mathfrak{h} = \mathfrak{e}_{7(-25)} \oplus \mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{h} = \mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$	$\mathfrak{h} = \mathfrak{so}(4,12)$ $\mathfrak{h} = \mathfrak{so}^*(16)$

Table 9: Non-compact isotropy algebras of $\mathfrak{e}_8^{\mathbb{C}}$ and its real forms

Note that not all the symmetric spaces given in these tables are irreducible. For instance $\mathfrak{sl}(n,\mathbb{C})/\mathfrak{s}(\mathfrak{gl}(k,\mathbb{C}) \oplus \mathfrak{gl}(l,\mathbb{C}))$ is not. The results of [7] are more precise and we refer to them for the list of the irreducible symmetric spaces and the non-irreducibles ones.

We refer to [23] and [7] for facts about symmetric spaces and local symmetric spaces.

Riemannian symmetric spaces

We stress that the symmetric spaces G/H associated to the isotropy subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ in Tables 2 to 9 are non-Riemannian. We discuss now a few features about Riemannian symmetric spaces, which are of the form G/G^ρ with G^ρ compact. The symmetric spaces which are Riemannian spaces of non-positive curvature are called symmetric spaces *of non-compact type*. They are all of the form $S = G/K$ where $G = \mathrm{Isom}(S)^0$ and K is a maximal compact subgroup. If it has no euclidean factors, then G is semisimple, linear and centerless.

Recall that if G is a Lie group, all maximal compact subgroups are conjugate. If G is semisimple, or more generally reductive, and if K is a maximal compact subgroup, then the symmetric space G/K is called the *Riemannian symmetric space associated to* G. It follows that we can identify the smooth manifold

$$S = G/K$$

with the set of all maximal compact subgroups of G. Remark that isogenous Lie groups have isometric associated Riemannian symmetric spaces. In particular, if G is a semisimple linear Lie group, the associated Riemannian symmetric space is the same as that associated to its identity component G^0 or to $G/Z(G)$. We can thus assume that G is connected and centerless. In this case, as the image of a maximal compact subgroup by an automorphism of G is again a maximal compact subgroup, we have an action of $\mathrm{Aut}(G) = \mathrm{Aut}(\mathfrak{g})$ by isometries on $S = G/K$. Finally we have that the group of isometries of a symmetric space $S = G/K$ of non-compact type without euclidean factors is $\mathrm{Aut}(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G.

An important part of our work will be to compute dimensions of fixed point sets

$$S^\alpha = \{x \in S \mid \alpha(x) = x\}$$

where $\alpha \in \mathrm{Isom}(S) = \mathrm{Aut}(\mathfrak{g})$. Assuming that G is connected and centerless, the fixed point set S^α is the Riemannian symmetric space associated to G^α (recall that we denote by the same letter the automorphism of $\mathfrak{g}$ and that of G). If $A \in G$ we will denote by S^A the fixed point set of the inner automorphism $\mathrm{Ad}(A)$. In the case where A is of finite order, it can be conjugate in the maximal compact subgroup K. Then the fixed point set of G by $\mathrm{Ad}(A)$ is the centralizer of A in G, that is $G^{\mathrm{Ad}(A)} = C_G(A) = \{B \in G \mid AB = BA\}$. A maximal compact subgroup of $C_G(A)$ is $C_K(A)$, the centralizer of A in K. So we can identify S^A with $C_G(A)/C_K(A)$, and we can write:

$$\dim S^A = \dim C_G(A) - \dim C_K(A).$$

We refer to [23] for other facts about Riemannian symmetric spaces.

Lattices and arithmetic groups

A discrete subgroup Γ of a Lie group G is said to be a *lattice* if the quotient $\Gamma\backslash G$ has finite Haar measure. It is said to be *uniform* (or cocompact) if this quotient is compact and *non-uniform* otherwise. The Borel density theorem (see [40, Cor. 4.5.6]) says that if G is the group of real points of a connected semisimple algebraic group defined over $\mathbb{R}$, and if a lattice $\Gamma \subset G$ projects densely into the maximal compact factor of G, then Γ is Zariski-dense in G. For instance, if $\mathbb{G}$ is a connected semisimple algebraic group defined over $\mathbb{Q}$, then the group $\mathbb{G}_{\mathbb{Z}}$ is a lattice in $\mathbb{G}_{\mathbb{R}}$ and thus Zariski-dense. The group $\mathbb{G}_{\mathbb{Z}}$ is the paradigm of an arithmetic group, which will be defined now.

Let G be a semisimple Lie group with identity component G^0 and $\Gamma \subset G$ a lattice. The lattice Γ is said to be *arithmetic* if there are a connected algebraic group $\mathbb{G}$ defined over $\mathbb{Q}$, compact normal subgroups $K \subset G^0$, $K' \subset \mathbb{G}^0_{\mathbb{R}}$ and a Lie group isomorphism

$$\varphi : G^0/K \to \mathbb{G}^0_{\mathbb{R}}/K',$$

such that $\varphi(\overline{\Gamma})$ is commensurable to $\overline{\mathbb{G}_{\mathbb{Z}}}$, where $\overline{\Gamma}$ and $\overline{\mathbb{G}_{\mathbb{Z}}}$ are the images of $\Gamma \cap G^0$ and $\mathbb{G}_{\mathbb{Z}} \cap \mathbb{G}^0_{\mathbb{R}}$ in G^0/K and $\mathbb{G}^0_{\mathbb{R}}/K'$ (recall that two subgroups H and H' of G are commensurable if their intersection is of finite index in both subgroups).

We say that the lattice $\Gamma \subset G$ is *irreducible* if ΓN is dense in G for every non-compact, closed, normal subgroup N of G^0. The Margulis arithmeticity theorem (see [37, Ch. IX] and [40, Thm. 5.2.1]) tells us that in a way, most irreducible lattices are arithmetic.

Theorem 2.2.1 (Margulis arithmeticity theorem)**.** *Let G be the group of real points of a semisimple algebraic group defined over $\mathbb{R}$ and $\Gamma \subset G$ an irreducible lattice. If G is not isogenous to $\mathrm{SO}(1,n) \times K$ or $\mathrm{SU}(1,n) \times K$ for any compact group K, then Γ is arithmetic.*

Observe that $\mathrm{SO}(1,n) \times K$ and $\mathrm{SU}(1,n) \times K$ have real rank 1, so the arithmeticity theorem applies to every irreducible lattice in a group of real rank at least 2.

The definition of arithmeticity can be simplified in some cases. If G is connected, centerfree and has no compact factors, the compact subgroup K in the definition must be trivial. Moreover, if Γ is non-uniform and irreducible, then the compact subgroup K' is not needed either (see [40, Cor. 5.3.2]). Under the same assumptions, we can also assume that the algebraic group $\mathbb{G}$ is centerfree, and in this case the commensurator of $\mathbb{G}_{\mathbb{Z}}$ in $\mathbb{G}$ is $\mathbb{G}_{\mathbb{Q}}$ and $\varphi(\Gamma) \subset \mathbb{G}_{\mathbb{Q}}$. Under the same hypotheses on G, if Γ is non irreducible, it is almost a product of irreducible lattices. In fact (see [40, Prop. 4.3.3]), there is a direct decomposition $G = G_1 \times \cdots \times G_r$ such that Γ is commensurable to $\Gamma_i \times \cdots \times \Gamma_r$ where $\Gamma_i = \Gamma \cap G_i$ is an irreducible lattice in G_i.

The rational rank of the arithmetic group Γ, denoted by $\mathrm{rk}_{\mathbb{Q}}\Gamma$, is by definition the $\mathbb{Q}$-rank of the algebraic group $\mathbb{G}$ in the definition of arithmeticity, and we have

$$\mathrm{rk}_{\mathbb{Q}}\Gamma \leqslant \mathrm{rk}_{\mathbb{R}}\mathbb{G}.$$

Note that $\mathrm{rk}_{\mathbb{Q}}\Gamma = 0$ if and only if Γ is cocompact (see [8, Thm. 8.4]).

We refer to [8] and [40] for other facts about lattices and arithmetic groups.

Virtual cohomological dimension and proper geometric dimension

Recall that the virtual cohomological dimension of a virtually torsion-free discrete subgroup Γ is the cohomological dimension of any torsion-free subgroup Γ' of finite index of Γ, that is

$$\mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma') = \max\{n \mid H^n(\Gamma', A) \neq 0 \text{ for a certain } \mathbb{Z}\Gamma'\text{-module } A\}.$$

If X is a cocompact model for $\underline{E}\Gamma$, we can compute the virtual cohomological dimension of Γ as

$$\mathrm{vcd}(\Gamma) = \max\{n \in \mathbb{N} \mid \mathrm{H}^n_c(X) \neq 0\} \tag{2.2.1}$$

where $\mathrm{H}^n_c(X)$ denotes the compactly supported cohomology of X (see [12, Cor. VIII.7.6]).

Recall that the proper geometric dimension $\underline{\mathrm{gd}}(\Gamma)$ is the smallest possible dimension of a model for $\underline{E}\Gamma$.

If G is the group of real points of a semisimple algebraic group, $K \subset G$ a maximal compact subgroup, $S = G/K$ the associated Riemannian symmetric space and $\Gamma \subset G$ a uniform lattice of G, S is a model for $\underline{E}\Gamma$ and has dimension $\mathrm{vcd}(\Gamma)$, so we have $\mathrm{vcd}(\Gamma) = \underline{\mathrm{gd}}(\Gamma)$, that is why we will be mostly interested in non-uniform lattices.

We will also rule out the case when the adjoint group G_{ad} of $\mathfrak{g}$ has real rank 1, in fact we have the following (see [1, Cor. 2.8])

Proposition 2.2.1. *Let $\mathbb{G}$ be an algebraic group defined over $\mathbb{R}$ and $\Gamma \subset \mathbb{G}_{\mathbb{R}}$ a lattice. If $\mathrm{rk}_{\mathbb{R}}\mathbb{G} = 1$, then $\mathrm{vcd}(\Gamma) = \underline{\mathrm{gd}}(\Gamma)$.*

For the case of higher real rank, recall that by Margulis arithmeticity theorem, Γ is arithmetic as long as it is irreducible.

If Γ is non-uniform, $\Gamma \setminus S$ is not compact. However Borel and Serre constructed in [10] a Γ-invariant bordification of S called the *Borel-Serre bordification* X which is a cocompact model for $\underline{E}\Gamma$ (see [26, Th. 3.2]).

Using their bordification, Borel and Serre proved in [10] the following theorem which links the virtual cohomological dimension and the rational rank of such an arithmetic lattice.

Theorem 2.2.2 (Borel-Serre)**.** *Let G be a semisimple Lie group, $K \subset G$ a maximal compact subgroup and $\Gamma \subset G$ an arithmetic lattice. Then:*

$$\mathrm{vcd}(\Gamma) = \dim(G/K) - \mathrm{rk}_{\mathbb{Q}}\Gamma.$$

In particular:

$$\mathrm{vcd}(\Gamma) \geqslant \dim(G/K) - \mathrm{rk}_{\mathbb{R}}G.$$

Before moving on, note that we will often in this article consider groups up to isogeny, and the philosophy behind it is that normal finite subgroups do not change the dimensions, indeed we have:

Lemma 2.2.1. *Let Γ be a virtually torsion-free discrete group, and $N \subset \Gamma$ a finite normal subgroup. Then:*

$$\underline{\mathrm{gd}}(\Gamma) = \underline{\mathrm{gd}}\left(\Gamma/N\right),$$

$$\mathrm{vcd}(\Gamma) = \mathrm{vcd}\left(\Gamma/N\right).$$

Proof. For the first equality: if X is a model for $\underline{E}\Gamma$ it follows easily that X^N is a model for $\underline{E}(\Gamma/N)$ of dimension lower than those of X, so:

$$\underline{\mathrm{gd}}(\Gamma) \geqslant \underline{\mathrm{gd}}\left(\Gamma/N\right).$$

Reciprocally, a model for $\underline{E}(\Gamma/N)$ is also a model for $\underline{E}\Gamma$ and we have the other inequality.

For the second equality, it suffices to recall that $\mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma')$ where $\Gamma' \subset \Gamma$ is a torsion-free subgroup of finite index, and in this case $\Gamma' N/N$ is a torsion-free subgroup of finite index of Γ/N isomorphic to Γ'. ■

We refer to [12] and [10] for facts about the (virtual) cohomological dimension and geometric dimension.

Bredon cohomology

The Bredon cohomological dimension $\underline{\mathrm{cd}}(\Gamma)$ is an algebraic counterpart to the proper geometric dimension $\underline{\mathrm{gd}}(\Gamma)$. We recall how $\underline{\mathrm{cd}}(\Gamma)$ is defined and a few of its properties.

Let Γ be a discrete group and $\mathcal{F}$ be the family of finite subgroups of Γ. The *orbit category* $\mathcal{O}_{\mathcal{F}}\Gamma$ is the category whose objects are left coset spaces Γ/H with $H \in \mathcal{F}$ and where the morphisms are all Γ-equivariant maps between them. An $\mathcal{O}_{\mathcal{F}}\Gamma$-*module* is a contravariant functor

$$M : \mathcal{O}_{\mathcal{F}}\Gamma \to \mathbb{Z}\text{-mod}$$

to the category of $\mathbb{Z}$-modules. The *category of* $\mathcal{O}_{\mathcal{F}}\Gamma$-*modules*, denoted by Mod-$\mathcal{O}_{\mathcal{F}}\Gamma$, has as objects all the $\mathcal{O}_{\mathcal{F}}\Gamma$-modules and all the natural transformations between them as morphisms. One can show that Mod-$\mathcal{O}_{\mathcal{F}}\Gamma$ is an abelian category and that we can construct projective resolutions on it. The *Bredon cohomology of* Γ with coefficients in $M \in$ Mod-$\mathcal{O}_{\mathcal{F}}\Gamma$, denoted by $\mathrm{H}^*_{\mathcal{F}}(\Gamma, M)$, is by definition the cohomology associated to the cochain of complexes $\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}\Gamma}(C_*, M)$ where $C_* \to \underline{\mathbb{Z}}$ is a projective resolution of the functor $\underline{\mathbb{Z}}$ which maps all objects to $\mathbb{Z}$ and all morphisms to the identity map. If X is a model for $\underline{E}\Gamma$, the augmented cellular chain complexes $C_*(X^H) \to \mathbb{Z}$ of the fixed points sets X^H for $H \in \mathcal{F}$ form such a projective (even free) resolution $C_*(X^-) \to \underline{\mathbb{Z}}$. Thus we have

$$\mathrm{H}^n_{\mathcal{F}}(\Gamma, M) = \mathrm{H}^n(\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}\Gamma}(C_*(X^-), M)).$$

The *Bredon cohomological dimension of* Γ *for proper actions*, denoted by $\underline{\mathrm{cd}}(\Gamma)$ is defined as

$$\underline{\mathrm{cd}}(\Gamma) = \sup\{n \in \mathbb{N} \mid \exists M \in \text{Mod-}\mathcal{O}_{\mathcal{F}}\Gamma : \mathrm{H}^n_{\mathcal{F}}(\Gamma, M) \neq 0\}.$$

As we said above, this invariant can be viewed as an algebraic counterpart to $\underline{\mathrm{gd}}(\Gamma)$. Indeed Lück and Meintrup proved in [35] the following theorem:

Theorem 2.2.3 (Lück-Meintrup)**.** *If* Γ *is a discrete group with* $\underline{\mathrm{cd}}(\Gamma) \geqslant 3$*, then* $\underline{\mathrm{gd}}(\Gamma) = \underline{\mathrm{cd}}(\Gamma)$*.*

We explain now the strategy to prove that $\mathrm{vcd}(\Gamma) = \underline{\mathrm{cd}}(\Gamma)$, beginning with some material and definitions. Recall that if G is the group of real points of a semisimple algebraic group and $\Gamma \subset G$ is a lattice, then the Borel-bordification X is a model for $\underline{E}\Gamma$. Note also that if H is a finite subgroup of Γ, $\dim(X^H) = \dim(S^H)$. If we denote $\mathcal{F}_0$ the family of finite subgroups of Γ containing properly the kernel of Γ (that is the kernel of the action of Γ on X), X_{sing} the subspace of the Borel-bordification X consisting of points whose stabilizer is stricly larger than the kernel of Γ, and

$$\mathcal{S} = \{X^H \mid H \in \mathcal{F}_0 \text{ and } \nexists H' \in \mathcal{F}_0 \text{ with } X^H \subsetneq X^{H'}\},$$

then we have

$$X_{sing} = \bigcup_{X^H \in \mathcal{S}} X^H.$$

Also every fixed point set $X^H \in \mathcal{S}$ is of the form X^α where $\alpha \in \Gamma$ is of finite order and non-central.

In general, computing $\underline{\mathrm{cd}}(\Gamma)$ is not an easy task. However, if Γ admits a cocompact model X for $\underline{E}\Gamma$, then there is a version of the formula (2.2.1) for the Bredon cohomological dimension. In fact, from [13, Th. 1.1] we get that

$$\underline{\mathrm{cd}}(\Gamma) = \max\{n \in \mathbb{N} \mid \exists K \in \mathcal{F}_0 \; s.t. \; \mathrm{H}^n_c(X^K, X^K_{sing}) \neq 0\}$$

where X^K is the fixed point set of X under K and X^K_{sing} is the subcomplex of X^K consisting of those cells that are fixed by a finite subgroup of Γ that strictly contains K.

Using the above characterisations of $\mathrm{vcd}(\Gamma)$ and $\underline{\mathrm{cd}}(\Gamma)$, one can show (see [1, Prop. 3.3])

Proposition 2.2.2. *Let G be the group of real points of a semisimple algebraic group $\mathbb{G}$ of real rank at least two, $\Gamma \subset G$ a non-uniform lattice of G, $K \subset G$ a maximal compact subgroup and $S = G/K$ the associated Riemannian symmetric space. If*

1. $\dim(X^\alpha) \leqslant \mathrm{vcd}(\Gamma)$ *for every* $X^\alpha \in \mathcal{S}$, *and*
2. *the homomorphism* $\mathrm{H}_c^{\mathrm{vcd}(\Gamma)}(X) \to \mathrm{H}_c^{\mathrm{vcd}(\Gamma)}(X_{sing})$ *is surjective*

then $\mathrm{vcd}(\Gamma) = \underline{\mathrm{cd}}(\Gamma)$.

Note that in [1] the authors assume that $\mathbb{G}$ is connected but this hypothesis is not needed as the Borel-Serre bordification is still a model for $\underline{E}\Gamma$ if $\mathbb{G}$ is not connected (see [26, Th. 3.2]). As $\dim(X^\alpha) = \dim(S^\alpha)$ we have immediately the following lemma as a corollary of the previous proposition (see [1, Cor. 3.4])

Lemma 2.2.2. *With the same notations as above, if* $\dim S^\alpha < \mathrm{vcd}(\Gamma)$ *for all* $\alpha \in \Gamma$ *of finite order and non central, then* $\underline{\mathrm{cd}}(\Gamma) = \mathrm{vcd}(\Gamma)$.

This lemma will be the key argument to prove the Main Theorem. However, as it is the case in [1], in some cases we will need the following result (see [1, Cor 3.7])

Lemma 2.2.3. *With the same notations as above, suppose that*

1. $\dim(S^\alpha) \leqslant \mathrm{vcd}(\Gamma)$ *for every non-central finite order element* $\alpha \in \Gamma$*,*
2. $\dim(S^\alpha \cap S^\beta) \leqslant \mathrm{vcd}(\Gamma) - 2$ *for any distinct* $S^\alpha, S^\beta \in \mathcal{S}$*, and*
3. *for any finite set of non-central finite order elements* $\alpha_1, \ldots, \alpha_r$ *with* $S^{\alpha_i} \neq S^{\alpha_j}$ *for* $i \neq j$*,* $\dim(S^{\alpha_i}) = \mathrm{vcd}(\Gamma)$*, and such that* $C_\Gamma(\alpha_i)$ *is a cocompact lattice in* $C_G(\alpha_i)$*, there exists a rational flat* F *in* $\mathcal{S}$ *that intersects* S^{α_i} *in exactly one point and is disjoint from* S^{α_i} *for* $i \in \{2, \ldots, n\}$*.*

Then $\mathrm{vcd}(\Gamma) = \underline{\mathrm{cd}}(\Gamma)$*.*

We refer to [35] and [13] for other facts about Bredon cohomology.

2.3 Complex simple Lie algebras

In this section we prove the Main Theorem for all complex simple Lie algebras:

Proposition 2.3.1. *Let* $\mathfrak{g}$ *be a complex simple Lie algebra,* $G = \mathrm{Aut}(\mathfrak{g})$ *its group of automorphisms and* S *the associated Riemannian symmetric space. We assume that* $\mathrm{rk}_{\mathbb{R}} G \geqslant 2$*. Then*

$$\dim S^\alpha < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every $\alpha \in G$ *of finite order and non central. In particular*

$$\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$$

for every lattice $\Gamma \subset G$*.*

Recall that the adjoint group G_{ad} is the identity component of $G = \mathrm{Aut}(\mathfrak{g})$. It agrees with the group of inner automorphisms, that is $G_{ad} = \mathrm{Inn}(\mathfrak{g})$. Note that G_{ad} is centerfree has the same dimension and real rank as G, and their associated Riemannian symmetric spaces agree. The quotient $\mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ is the group of outer automorphism $\mathrm{Out}(\mathfrak{g})$ and can be realized as a subgroup of $\mathrm{Aut}(\mathfrak{g})$. The group $\mathrm{Aut}(\mathfrak{g})$ is then the semi-direct product of $\mathrm{Inn}(\mathfrak{g})$ and $\mathrm{Out}(\mathfrak{g})$ (see [21]). Recall also that if $A \in G_{ad}$, S^A is the fixed point set of the inner automorphism $\mathrm{Ad}(A)$.

For further use, note that if $\rho \in \mathrm{Aut}(\mathfrak{g})$ is an involution, then it is induced by an involution on G_{ad} that will be still denoted ρ. The group of its fixed points G_{ad}^ρ has Lie algebra $\mathfrak{g}^\rho$ and the fixed point set S^ρ is the associated Riemannian symmetric space. In particular, $\dim S^\rho = 0$ if G_{ad}^ρ is compact.

The proof of Proposition 2.3.1 relies on the following lemmas:

Lemma 2.3.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra such that every element of* $\mathrm{Out}(\mathfrak{g})$ *has order at most 2. Let G be the group of automorphisms of $\mathfrak{g}$ and let S be the associated Riemannian symmetric space. If*

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G \text{ and } \dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G \tag{2.3.1}$$

for all $A \in G_{ad}$ non trivial of finite order, and for all involutions $\rho \in G$, then we also have

$$\dim S^\alpha < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every $\alpha \in G$ of finite order and non central.

Proof. Every element $\alpha \in \mathrm{Aut}(\mathfrak{g})$ is of the form $\mathrm{Ad}(A) \circ \rho$ where $A \in G_{ad}$ and $\rho \in \mathrm{Out}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$. We know that ρ is of order at most 2 by hypothesis. Then $\alpha^2 = \mathrm{Ad}(A\rho(A))$ is an inner automorphism and we have the inclusion $S^\alpha \subset S^{\alpha^2} = S^{A\rho(A)}$. So, if $A\rho(A)$ is not central in G_{ad}, then we have

$$\dim S^\alpha \leqslant \dim S^{A\rho(A)} < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

Note now that if $A\rho(A)$ is central, then it is actually the identity because G_{ad} is centerfree. This means that $\alpha^2 = Id$. In other words, α is an involution, and we again have

$$\dim S^\alpha < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

by assumption. We have proved the claim. ∎

To check the first part of (2.3.1) we will use the following:

Lemma 2.3.2. *Let G be the group of complex points of a semisimple connected algebraic group and $K \subset G$ a maximal compact subgroup. Suppose that there exists a Lie group H isogenous to a subgroup H' of K such that K/H' is an irreducible symmetric space,* $\mathrm{rk} K = \mathrm{rk} H$*,* $\dim H < \dim K - \mathrm{rk}_{\mathbb{R}} G$*, and satisfying*

$$\dim C_H(A) < \dim H - \mathrm{rk}_{\mathbb{R}} G$$

for all $A \in H$ of finite order and non central. Then we have

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every $A \in G$ of finite order and non central.

Proof. As all maximal compact subgroups are conjugate, we can conjugate such an $A \in G$ into K. Since K is connected, A is then contained in a maximal torus. Since all maximal tori are conjugate, we can conjugate A into any one of them. Since the subgroup H' has the same rank as K, a maximal torus in H' is also maximal in K. We can then assume, up to replacing A by a conjugate element, that $A \in H'$.

Taking now into account that G is the group of complex points of a reductive algebraic group, $C_G(A)$ is the complexification of its maximal compact subgroup $C_K(A)$ and then its dimension is twice that of $C_K(A)$. As a result we get

$$\dim S^A = \dim C_G(A) - \dim C_K(A) = \dim C_K(A)$$

because $S^A \simeq C_G(A)/C_K(A)$ as seen in section 2.5.

Similarly we have

$$\dim S = \dim(G/K) = \dim K.$$

In particular, the claim follows once we show that

$$\dim C_K(A) < \dim K - \mathrm{rk}_{\mathbb{R}} G.$$

Now, as H and H' are isogenous, assume for simplicity that $H = H'/F$ with F a finite normal subgroup of H'. We denote by $\overline{A}$ the class of A in H'/F. As F is finite, we have

$$\dim C_{H'}(A) = \dim C_H(\overline{A}).$$

In fact, the elements in $C_H(\overline{A})$ are the classes of elements $B \in H'$ such that the commutator $[A, B]$ belongs to the finite group F, so there are finitely many connected components of the same dimension as $C_{H'}(A)$ in $C_H(\overline{A})$.

Suppose for a moment that A is non central in H', then we write

$$\dim C_K(A) \leqslant \dim C_{H'}(A) + \dim K - \dim H = \dim C_H(\overline{A}) + \dim K - \dim H$$

and by assumption we have

$$\dim C_H(\overline{A}) < \dim H - \mathrm{rk}_{\mathbb{R}} G$$

and finally

$$\dim C_K(A) < \dim K - \mathrm{rk}_{\mathbb{R}} G.$$

It remains to treat the case that A is central in H' but not in K, that is $H' \subset C_K(A) \subsetneq K$. Since the symmetric space K/H' is irreducible, it follows that the identity component of H' is a maximal connected Lie group of K, so $\dim C_K(A) = \dim H' = \dim H$, and we have

$$\dim H < \dim K - \mathrm{rk}_{\mathbb{R}} G$$

by assumption. ∎

In the course of the proof of Proposition 2.3.1, the subgroups H will all be classical groups of the forms $\mathrm{SO}(n)$ or $\mathrm{SU}(n)$ and we will need the following bounds for the dimension of centralizers in those groups (see [1] section 5)

1. Let $A \in \mathrm{SO}(n)$ $(n \geqslant 3)$ of finite order and non central, then

$$\dim C_{\mathrm{SO}(n)}(A) \leqslant \frac{(n-1)(n-2)}{2}. \tag{2.3.2}$$

2. Let $A \in \mathrm{SU}(n)$ $(n \geqslant 2)$ of finite order and non central, then

$$C_{\mathrm{SU}(n)}(A) \leqslant (n-1)^2. \tag{2.3.3}$$

For more simplicity we will sometimes consider H as a subgroup of K and denote K/H the symmetric space K/H'. For the convenience of the reader, we summarize in the following table the information we need to prove Proposition 2.3.1 for exceptional Lie algebras. We refer to [50] for explicit descriptions.

G_{ad}	K	H	$\dim K$	$\dim H$	$\mathrm{rk}H = \mathrm{rk}K$	$\mathrm{rk}_{\mathbb{R}} G_{ad}$
$G_2^{\mathbb{C}}$	G_2	SO(4)	14	6	2	2
$F_4^{\mathbb{C}}$	F_4	SO(9)	52	36	4	4
$E_6^{\mathbb{C}}$	E_6	$\mathrm{U}(1) \times \mathrm{SO}(10)$	78	46	6	6
$E_7^{\mathbb{C}}$	E_7	SU(8)	133	63	7	7
$E_8^{\mathbb{C}}$	E_8	SO(16)	248	120	8	8

Table 10: Exceptional complex simple centerless Lie groups G_{ad}, maximal compact subgroups K, classical subgroups H, dimensions and ranks.

We are now ready to launch the proof of Proposition 2.3.1.

Proof of Proposition 2.3.1. The second claim follows from Lemma 2.2.2 because we have

$$\mathrm{vcd}(\Gamma) \geqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every lattice $\Gamma \subset G$ by Theorem 2.2.2. So it suffices to prove the first claim.

Recall that every complex simple Lie algebra is isomorphic to either one of the classical algebras $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sp}(2n, \mathbb{C})$ (with conditions on n to ensure simplicity), or to one of the 5 exceptional ones: $\mathfrak{g}_2^{\mathbb{C}}$, $\mathfrak{f}_4^{\mathbb{C}}$, $\mathfrak{e}_6^{\mathbb{C}}$, $\mathfrak{e}_7^{\mathbb{C}}$ and $\mathfrak{e}_8^{\mathbb{C}}$.

To prove Proposition 2.3.1 we will consider all these cases individually.

Classical complex simple Lie algebras:

Let $\mathfrak{g}$ be a classical complex simple Lie algebra and $\Gamma \subset G = \mathrm{Aut}(\mathfrak{g})$ a lattice. From a brief inspection of Table 1 in section 2.3 we obtain that, unless $\mathfrak{g} = \mathfrak{so}(8,\mathbb{C})$, every outer automorphism of $\mathfrak{g}$ has order 2. We will assume that $\mathfrak{g} \neq \mathfrak{so}(8,\mathbb{C})$ for a while, treating this case later.

To begin with note that we get from parts 6.1, 6.2 and 6.3 of [1] that

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for all $A \in G_{ad}$ non trivial and of finite order. In other words, the first part of condition (2.3.1) in Lemma 2.3.1 holds. To check the second part we make use of the classification of local symmetric spaces in [7]. For instance if $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ with $n \geqslant 3$ because of our assumption on the rank, and if $\rho \in G$ is an involution, then the Lie algebra $\mathfrak{g}^\rho$ is isomorphic to either a Lie algebra whose adjoint group is compact or to one of the following: $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{s}(\mathfrak{gl}(k,\mathbb{C}) \oplus \mathfrak{gl}(n-k,\mathbb{C}))$, $\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}(p,n-p)$ and $\mathfrak{sl}(\frac{n}{2},\mathbb{H})$, where $\mathfrak{sp}(n,\mathbb{C})$ and $\mathfrak{sl}(\frac{n}{2},\mathbb{H})$ only appear if n is even. The associated Riemannian symmetric space S^ρ is obtained by taking the quotient of the adjoint group by a maximal compact subgroup, for example in the case of $\mathfrak{so}(n,\mathbb{C})$ it is $\mathrm{PSO}(n,\mathbb{C})/\mathrm{PSO}_n$. The Lie algebra $\mathfrak{g}^\rho$ for which $\dim S^\rho$ is maximal is $\mathfrak{s}(\mathfrak{gl}(1,\mathbb{C}) \oplus \mathfrak{gl}(n-1,\mathbb{C}))$ for $n \neq 4$ (where $\dim S^\rho = (n-1)^2$), and $\mathfrak{sp}(4,\mathbb{C})$ for $n = 4$ (where $\dim S^\rho = 10$). In all these cases we have

$$\dim S^\rho < (n^2-1) - (n-1) = \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

We get then by Lemma 2.3.1 that the first claim of Proposition 2.3.1 holds. The cases of $\mathfrak{sp}(2n,\mathbb{C})$ and $\mathfrak{so}(n,\mathbb{C})$ for $n \neq 8$ are similar, we leave the details to the reader.

Now we treat the case of the Lie algebra $\mathfrak{so}(8,\mathbb{C})$. Its group of complex outer automorphisms is isomorphic to the symmetric group S_3 and contains an order 3 element τ called triality. (See section 1.14 in [50] for an interpretation of triality in terms of octonions.) The group $\mathrm{Out}(\mathfrak{so}(8,\mathbb{C}))$ of real outer automorphisms is then isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$ where the second factor corresponds to complex conjugation. Consequently the only order 3 outer automorphisms are τ and τ^{-1} and those of order 6 are their compositions with complex conjugation. If $\rho \in \mathrm{Aut}(\mathfrak{so}(8,\mathbb{C}))$ is of order 6 and $\alpha = \mathrm{Ad}(A) \circ \rho$, then α^3 is the composition of an inner automorphism and ρ^3. As ρ^3 is of order 2 and we have the inclusions $S^\alpha \subset S^{\alpha^3}$, we can consider α^3 instead of α and we just have to treat the cases when ρ is of order 2 or 3.

If ρ is of order 2 we apply the same method than for other classical simple Lie algebras, using the classification of local symmetric spaces. It remains to treat the case when $\rho = \tau$ the triality automorphism (or its inverse). In this case $\alpha = \mathrm{Ad}(A) \circ \tau$ is a complex automorphism. Proceeding like in the proof of Lemma 2.3.1, α^3 is an inner automorphism and the result follows if it is non trivial. If $\alpha^3 = 1$ then α belongs to the set $\mathrm{Aut}^3_{\mathbb{C}}(\mathfrak{so}(8,\mathbb{C}))$ of complex automorphisms of order 3. A result of Gray and Wolf (see [20, Thm. 5.5]) says that if $\sim_i$ is the equivalence relation of conjugation by an inner automorphism in $\mathrm{Aut}^3_{\mathbb{C}}(\mathfrak{so}(8,\mathbb{C}))$, then $\mathrm{Aut}^3_{\mathbb{C}}(\mathfrak{so}(8,\mathbb{C}))/\sim_i$ contains, besides the classes of inner automorphisms, four other classes: those of τ and τ^{-1} and two others represented by order 3 automorphisms τ' and τ'^{-1}. The Lie algebra of the fixed point set of triality is the exceptional Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$, and that of the fixed point set of τ' is isomorphic to the Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. In both cases we have

$$\dim S^{\alpha} < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

and Proposition 2.3.1 holds for $\mathfrak{g} = \mathfrak{so}(8,\mathbb{C})$.

Exceptional complex simple Lie algebras:

The proof for exceptional complex simple Lie algebras mainly relies on Lemmas 2.3.1 and 2.3.2, Tables 5 to 10 and inequalites (2.3.2) and (2.3.3).

Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{g}_2^{\mathbb{C}}$. Its outer automorphism group is of order 2 and its adjoint group is $G_{ad} = G_2^{\mathbb{C}}$, a connected algebraic group of real rank 2 and complex dimension 14. The compact group G_2 is a maximal compact subgroup of $G_2^{\mathbb{C}}$.

The group G_2 contains a subgroup H isomorphic to SO(4), fixed by an involution of G_2 which extends to a conjugation of $G_2^{\mathbb{C}}$, giving the split real form $G_{2(2)}$ (see section 1.10 of [50] for an explicit description). Then $G_2/\mathrm{SO}(4)$ is an irreducible symmetric space, $\mathrm{rk}(G_2) = \mathrm{rk}(\mathrm{SO}(4)) = 2$ and

$$\dim H + \mathrm{rk}_{\mathbb{R}} G_{ad} = 6 + 2 = 8 < 14 = \dim K.$$

Moreover if $A \in H$ is of finite order and non central in $H \simeq \mathrm{SO}(4)$ we have by inequality (2.3.2)

$$\dim C_H(A) + \mathrm{rk}_{\mathbb{R}} G_{ad} \leqslant 3 + 2 = 5 < 6 = \dim H.$$

So by Lemma 2.3.2, the first part of (2.3.1) in Lemma 2.3.1 holds.

To check the second part we have to list the local symmetric spaces associated to an involution $\rho \in \mathrm{Aut}(\mathfrak{g}_2^{\mathbb{C}})$. By the classification of Berger in [7], the only non compact cases are when $\mathfrak{g}^{\rho}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ or $\mathfrak{g}_{2(2)}$. The associated Riemannian symmetric spaces are $S^{\rho} = (\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}))/(\mathrm{PSU}_2 \times \mathrm{PSU}_2)$ and $G_{2(2)}/\mathrm{SO}(4)$ and we have in both cases

$$\dim S^{\rho} < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.3.1 holds for $G = \mathrm{Aut}(\mathfrak{g}_2^{\mathbb{C}})$.

Lie algebra $\mathfrak{f}_4^{\mathbb{C}}$:

We proceed like previously for the simple Lie algebra $\mathfrak{g} = \mathfrak{f}_4^{\mathbb{C}}$, with $|\mathrm{Out}(\mathfrak{f}_4^{\mathbb{C}})| = 2$, $G_{ad} = F_4^{\mathbb{C}}$ of maximal compact subgroup $K = F_4$, $\mathrm{rk}_{\mathbb{R}} G_{ad} = 4$ and $\dim K = 52$. We know that there exists a subgroup $H \subset K$ isogenous to $\mathrm{SO}(9)$, with $\mathrm{rk} H = \mathrm{rk} K = 4$, and such that K/H is an irreducible symmetric space. In addition to that

$$\dim H + \mathrm{rk}_{\mathbb{R}} G_{ad} = 36 + 4 = 40 < 52 = \dim K$$

and if $A \in H$ is of finite order and non central in H, we have

$$\dim C_H(A) + \mathrm{rk}_{\mathbb{R}} G_{ad} \leqslant 28 + 4 = 32 < 36 = \dim H$$

by inequality (2.3.2).

So by Lemma 2.3.2, the first part of (2.3.1) in Lemma 2.3.1 holds.

Then by the classification of local symmetric spaces, the ones we have to study is those when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{sp}(6, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C})$, $\mathfrak{so}(9, \mathbb{C})$, $\mathfrak{f}_{4(4)}$ or $\mathfrak{f}_{4(-20)}$, in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.3.1 holds for $G = \mathrm{Aut}(\mathfrak{f}_4^{\mathbb{C}})$.

Lie algebra $\mathfrak{e}_6^{\mathbb{C}}$:

For the algebra $\mathfrak{g} = \mathfrak{e}_6^{\mathbb{C}}$, the outer automorphism group is a product of two groups of order 2, $G_{ad} = E_6^{\mathbb{C}}$ of maximal compact subgroup $K = E_6$, $\mathrm{rk}_{\mathbb{R}} G_{ad} = 6$ and $\dim K = 78$. We know there exists $H \subset K$ isogenous to $\mathrm{U}(1) \times \mathrm{SO}(10)$, with $\mathrm{rk} H = \mathrm{rk} K = 6$, K/H is an irreducible symmetric space and we have the following

$$\dim H + \mathrm{rk}_{\mathbb{R}} G_{ad} = 46 + 6 = 52 < 78 = \dim K$$

and if $A \in H$ is of finite order and non central in H, by inequality (2.3.2) we have

$$\dim C_H(A) + \mathrm{rk}_{\mathbb{R}} G_{ad} \leqslant 1 + 36 + 6 = 43 < 46 = \dim H.$$

So by Lemma 2.3.2, the first part of (2.3.1) in Lemma 2.3.1 holds.

Then by the classification of local symmetric spaces, the ones we have to study is those when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{sp}(8, \mathbb{C})$, $\mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{so}(10, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$, $\mathfrak{f}_4^{\mathbb{C}}$, $\mathfrak{e}_{6(6)}$, $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{6(-14)}$ or $\mathfrak{e}_{6(-26)}$, in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.3.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_6^{\mathbb{C}})$.

Lie algebra $\mathfrak{e}_7^{\mathbb{C}}$:

Consider now the simple Lie algebra $\mathfrak{g} = \mathfrak{e}_7^{\mathbb{C}}$, of order 2 outer automorphism group, and of adjoint group $G_{ad} = E_7^{\mathbb{C}}$, whose compact maximal subgroup is $K = E_7$, $\mathrm{rk}_{\mathbb{R}} G_{ad} = 7$ and $\dim K = 133$. We know there exists $H \subset K$ isogenous to $\mathrm{SU}(8)$ with $\mathrm{rk} H = \mathrm{rk} K = 7$ and K/H is an irreducible symmetric space. We have the inequality

$$\dim H + \mathrm{rk}_{\mathbb{R}} G_{ad} = 63 + 7 = 70 < 133 = \dim K$$

and if $A \in H$ is of finite order and non central in H, by inequality (2.3.3) we have

$$\dim C_H(A) + \mathrm{rk}_{\mathbb{R}} G_{ad} \leqslant 49 + 7 < 63 = \dim H.$$

So by Lemma 2.3.2, the first part of (2.3.1) in Lemma 2.3.1 holds.

Then by the classification of local symmetric spaces, the ones we have to study is those when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{sl}(8, \mathbb{C})$, $\mathfrak{so}(12, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C})$, $\mathfrak{e}_6^{\mathbb{C}} \oplus \mathfrak{so}(2, \mathbb{C})$, $\mathfrak{e}_{7(7)}$, $\mathfrak{e}_{7(5)}$ or $\mathfrak{e}_{7(-25)}$, in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.3.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_7^{\mathbb{C}})$.

Lie algebra $\mathfrak{e}_8^{\mathbb{C}}$:

The last exceptional Lie algebra is $\mathfrak{g} = \mathfrak{e}_8^{\mathbb{C}}$, again its outer automorphism group is of order 2, its adjoint group is $G_{ad} = E_8^{\mathbb{C}}$, of maximal compact subgroup $K = E_8$, $\mathrm{rk}_{\mathbb{R}} G_{ad} = 8$ and $\dim K = 248$. We know there exists $H \subset K$ isogenous to $\mathrm{SO}(16)$, with $\mathrm{rk} H = \mathrm{rk} K = 8$ and K/H is an irreducible symmetric space. We have also the inequality

$$\dim H + \mathrm{rk}_{\mathbb{R}} G_{ad} = 120 + 8 = 128 < 248 = \dim K$$

and if $A \in H$ is of finite order and non central in H, by inequality (2.3.2) we have

$$\dim C_H(A) + \mathrm{rk}_{\mathbb{R}} G_{ad} \leqslant 105 + 8 < 120 = \dim H.$$

So by Lemma 2.3.2, the first part of (2.3.1) in Lemma 2.3.1 holds.

Then by the classification of local symmetric spaces, the ones we have to study is those when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{so}(16, \mathbb{C})$, $\mathfrak{e}_7^{\mathbb{C}} \oplus \mathfrak{sp}(2, \mathbb{C})$, $\mathfrak{e}_{8(8)}$ or $\mathfrak{e}_{8(-24)}$, in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.3.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_8^{\mathbb{C}})$ and it concluded its proof. ■

2.4 Real simple Lie algebras

We will in this section extend the previous proposition to the real simple Lie algebras. They are the real forms of the complex ones studied in the previous section. The ideas of the proof are similar to those of the complex case, although we face some additional difficulties. Maybe the reader can skip this section in a first reading.

Proposition 2.4.1. *Let $\mathfrak{g}$ be a real simple Lie algebra, $G = \mathrm{Aut}(\mathfrak{g})$ its group of automorphisms and S the associated Riemannian symmetric space. Then*

$$\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$$

for every lattice $\Gamma \subset G$. Moreover

$$\dim S^\alpha \leqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every $\alpha \in G$ of finite order and non central.

We will again use Lemma 2.3.1, but in the case of exceptional real simple Lie algebras, we cannot use Lemma 2.3.2 to establish inequalities of the form

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for A in the adjoint group G_{ad} of $\mathfrak{g}$. The difficulty is that the dimension of G_{ad} is not anymore twice that of a maximal compact subgroup. To some extent, we will bypass this problem using the following lemma:

Lemma 2.4.1. *Let G be a connected Lie group which is the group of real points of a semisimple algebraic group defined over $\mathbb{R}$, and $K \subset G$ a maximal compact subgroup. Suppose there exists a subgroup $\overline{G} \subset G$ such that $G/\overline{G}$ is an irreducible symmetric space and whose compact maximal subgroup $\overline{K} \subset K$ has the same rank as K. Let $S = G/K$ and $\overline{S} = \overline{G}/\overline{K}$ be the associated Riemannian symmetric spaces. If we have*

$$\dim \overline{S} < \dim S - \mathrm{rk}_{\mathbb{R}} G,$$

and

$$\dim \overline{S}^A < \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G$$

for every $A \in \overline{K}$ of finite order non central, then we also have

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every $A \in G$ of finite order and non central.

Proof. As in the proof of Lemma 2.3.2 we can conjugate such an A into a maximal torus in $\overline{K}$.

If A is central in $\overline{G}$: as $G/\overline{G}$ is irreducible, we have that the identity component of $\overline{G}$ is a maximal connected Lie subgroup of G. It follows thus from $\overline{G} \subset C_G(A) \subsetneq G$ that the Riemannian symmetric spaces of $C_G(A)$ and $\overline{G}$ are the same, that is $S^A = \overline{S}$. Then the result follows from the assumption

$$\dim \overline{S} < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

Suppose now that A is non central in $\overline{G}$, then we have

$$\dim S - \dim S^A \geqslant \dim \overline{S} - \dim \overline{S}^A$$

as $\overline{S}^A = \overline{S} \cap S^A$. Then the result follows because

$$\dim \overline{S} - \dim \overline{S}^A > \mathrm{rk}_{\mathbb{R}} G$$

by assumption. ∎

In all cases of interest, the group $\overline{G}$ will be a classical group. We will use the following inequalities to majorate $\dim \overline{S}^A$ (see [1] sections 6.5, 6.6 and 6.8):

1. Let $A \in \mathrm{SU}(p,q)$ ($p \leqslant q$, $p+q \geqslant 3$) of finite order non central, and $S = \mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ the associated symmetric space, then

$$\dim S^A \leqslant 2p(q-1). \tag{2.4.1}$$

2. Let $A \in \mathrm{Sp}(p,q)$ ($p \leqslant q$, $p+q \geqslant 3$) of finite order non central, and $S = \mathrm{Sp}(p,q)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q))$ the associated symmetric space, then

$$\dim S^A \leqslant 4p(q-1). \tag{2.4.2}$$

3. Let $A \in \mathrm{SO}^*(2n)$ ($n \geqslant 2$) of finite order non central, and $S = \mathrm{SO}^*(2n)/\mathrm{U}(n)$ the associated symmetric space, then

$$\dim S^A \leqslant n^2 - n - 2(n-1). \tag{2.4.3}$$

The tables below list exceptional real simple Lie groups, the subgroups $\overline{G}$ we will use and the informations we need to know for the proof of Proposition 2.4.1. Note that for more simplicity, the compact maximal subgroups K are given up to isogeny. We refer to [50] for explicit descriptions.

G_{ad}	K	$\overline{G}$	$\overline{K}$
$E_{6(6)}$	Sp(4)	Sp(2, 2)	Sp(2) × Sp(2)
$E_{6(2)}$	SU(6) × SU(2)	SO*(10) × SO(2)	U(5) × SO(2)
$E_{6(-14)}$	SO(10) × SO(2)	SO*(10) × SO(2)	U(5) × SO(2)
$E_{6(-26)}$	F_4	Sp(1, 3)	Sp(1) × Sp(3)
$E_{7(7)}$	SU(8)	$E_{6(2)}$ × SO(2)	SU(6) × SU(2) × SO(2)
$E_{7(-5)}$	SO(12) × SU(2)	SU(4, 4)	S(U(4) × U(4))
$E_{7(-25)}$	E_6 × SO(2)	SU(2, 6)	S(U(2) × U(6))
$E_{8(8)}$	SO(16)	SO*(16)	U(8)
$E_{8(-24)}$	E_7 × SU(2)	SO*(16)	U(8)
$G_{2(2)}$	SO(4)	SL(2, ℝ) × SL(2, ℝ)	SO(2) × SO(2)
$F_{4(4)}$	Sp(3) × Sp(1)	SO(4, 5)	S(O(4) × O(5))

Table 11: Real exceptional simple centerless Lie groups G_{ad}, certain classical subgroups $\overline{G} \subset G_{ad}$ and the respective maximal compact subgroups

G_{ad}	$\dim S$	$\dim \overline{S}$	$\mathrm{rk}\overline{K} = \mathrm{rk}K$	$\mathrm{rk}_{\mathbb{R}} G_{ad}$
$E_{6(6)}$	42	16	4	6
$E_{6(2)}$	40	20	6	4
$E_{6(-14)}$	32	20	6	2
$E_{6(-26)}$	26	12	4	2
$E_{7(7)}$	70	40	7	7
$E_{7(-5)}$	64	32	7	4
$E_{7(-25)}$	54	24	7	3
$E_{8(8)}$	128	56	8	8
$E_{8(-24)}$	112	56	8	4
$G_{2(2)}$	8	4	2	2
$F_{4(4)}$	28	20	4	4

Table 12: With the same notations as in Table 11, dimensions of the Riemannian symmetric spaces associated to G_{ad} and $\overline{G}$, together with the ranks of K, $\overline{K}$ and G_{ad}

We are now ready to prove Proposition 2.4.1.

Proof of Proposition 2.4.1. Recall that the first claim holds when the adjoint group has real rank 1 by Proposition 2.2.1, that is when $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{sp}(2,\mathbb{R})$, $\mathfrak{sl}(2,\mathbb{H})$, $\mathfrak{so}^*(4)$, $\mathfrak{so}^*(6)$, $\mathfrak{so}(1,n)$, $\mathfrak{su}(1,n)$, $\mathfrak{sp}(1,n)$ and $\mathfrak{f}_{4(-20)}$. The second claim is also true because if $\alpha \in \mathrm{Aut}(\mathfrak{g})$ is of finite order and non central, then S^α is a strict submanifold of S so we have

$$\dim S^\alpha \leqslant \dim S - 1.$$

We suppose from now on that $\mathrm{rk}_{\mathbb{R}} G \geqslant 2$. By inspection of Table 1 in section 2.3, we see that every outer automorphism of $\mathfrak{g}$ has order 2, except if $\mathfrak{g} = \mathfrak{so}(p,p)$ with $p \geqslant 4$ even.

As in the proof of Proposition 2.3.1, we will again do a case-by-case analysis.

Classical real simple Lie algebras other that $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{so}(p,q)$: We start dealing with the classical Lie algebras $\mathfrak{su}(p,q)$, $\mathfrak{sl}(n,\mathbb{H})$, $\mathfrak{sp}(2n,\mathbb{R})$ and $\mathfrak{so}^*(2n)$. Note that we rule out $\mathfrak{su}(2,2) \cong \mathfrak{so}(2,4)$ and $\mathfrak{sp}(4,\mathbb{R}) \cong \mathfrak{so}(2,3)$.

We use again Lemma 2.3.1. We want then to establish

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G \text{ and } \dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every A in the adjoint group G_{ad} of finite order and non central and for every involution $\rho \in G = \mathrm{Aut}(\mathfrak{g})$. The first condition holds by the computations in sections 6.4 to 6.7 of [1]. Using the classification of local symmetric spaces, we can check the second condition as we did in the complex case. For instance, if $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{R})$ with $n \geqslant 3$, then $\mathfrak{g}^\rho$ is either compact or isomorphic to one of the following: $\mathfrak{sp}(2k,\mathbb{R}) \oplus \mathfrak{sp}(2(n-k),\mathbb{R})$, $\mathfrak{u}(k,n-k)$, $\mathfrak{gl}(n,\mathbb{R})$ or $\mathfrak{sp}(n,\mathbb{C})$, the last case only appearing if n is even. The Lie algebra $\mathfrak{g}^\rho$ for which $\dim S^\rho$ is maximal is $\mathfrak{sp}(2(n-1),\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R})$, for which we have

$$\dim S^\rho = n^2 - n + 2 < n^2 = \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

Hence by Lemma 2.3.1 and Lemma 2.2.2, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{sp}(2n,\mathbb{R}))$. The cases of $\mathfrak{su}(p,q)$, $\mathfrak{sl}(n,\mathbb{H})$ and $\mathfrak{so}^*(2n)$ are similar.

Lie algebras $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{so}(p,q)$ The remaining classical cases are $\mathfrak{sl}(n,\mathbb{R})$ and $\mathfrak{so}(p,q)$. If $p \geqslant 6$ is even, then $\mathrm{Out}(\mathfrak{so}(p,p))$ is isomorphic to D_4 so every outer automorphism ρ has order 2 or 4. The only case where we have order 3 outer automorphisms is $\mathfrak{so}(4,4)$, as $\mathrm{Out}(\mathfrak{so}(4,4))$ is isomorphic to S_4.

As already noted in [1], where the argument for lattices in $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SO}(p,q)$ was more involved than in the other cases, we face the problem that there exists $\alpha \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\dim S^\alpha = \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

Our next goal is to characterize the automorphisms α for which this happens.

Lemma 2.4.2. *If $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ with $n \neq 4$ or $\mathfrak{so}(p,q)$ with $p+q = n$ and $(p,q) \neq (3,3)$, and $\alpha \in \mathrm{Aut}(\mathfrak{g})$, then*

$$\dim S^\alpha \leqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

with equality if and only if n is odd and $S^\alpha = S^A$ with A conjugate to

$$Q_{n-1,1} = \begin{pmatrix} -I_{n-1} & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{PSL}(n,\mathbb{R})$ *or* $\mathrm{PSO}(p,q)$*, or n is even and α is conjugate to the outer automorphism corresponding to the conjugation by $Q_{n-1,1}$.*

By abusing notations we will still denote S^A the fixed point set by the automorphism corresponding to the conjugation by A with A conjugate to $Q_{n-1,1}$ even if it is not an inner automorphism.

Proof. We begin with the case $\mathfrak{g} \neq \mathfrak{so}(4,4)$ and we use the same strategy that for the proof of Lemma 2.3.1. An automorphism α of $\mathfrak{g}$ is the composition of an inner automorphism and an outer automorphism ρ of order 2 or 4. If ρ has order 4 then α^2 is the composition of an inner automorphism and ρ^2 which is of order 2, and we have the inclusion $S^\alpha \subset S^{\alpha^2}$ so we can replace α by α^2 and it will suffice to consider the outer automorphisms of order 2. Then if ρ has order 2, α^2 is an inner automorphism, that is $\alpha^2 = \mathrm{Ad}(A)$ with $A \in G_{ad}$. If A is not trivial, by the computations in sections 7 and 8 in [1] we have

$$\dim S^{\alpha^2} \leqslant \dim S^A \leqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

with equality in the last inequality if and only if n is odd and A conjugate by $Q_{n-1,1}$. The first inequality is an equality if and only if $S^{\alpha^2} = S^A$. We have proved the claim if A is not trivial. If A is trivial, then α is an involution, so we use the classification of local symmetric spaces. For instance if $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ with $n \neq 4$, the non-compact associated isotropy algebra $\mathfrak{h} = \mathfrak{g}^\alpha$ are $\mathfrak{so}(k, n-k)$, $\mathfrak{s}(\mathfrak{gl}(k,\mathbb{R}) \oplus \mathfrak{gl}(n-k,\mathbb{R}))$, $\mathfrak{gl}\left(\frac{n}{2},\mathbb{C}\right)$ or $\mathfrak{sp}(n,\mathbb{R})$, the last two cases only appearing if n is even. In all these cases we have

$$\dim S^\alpha \leqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

with equality if and only if $\mathfrak{h} = \mathfrak{s}(\mathfrak{gl}(1,\mathbb{R}) \oplus \mathfrak{gl}(n-1,\mathbb{R}))$, which corresponds to an automorphism α conjugate to the inner automorphism $\mathrm{Ad}(Q_{n-1,1})$ if n is odd or to the outer automorphism of conjugation by $Q_{n-1,1}$ if n if even.

It remains to consider the case $\mathfrak{g} = \mathfrak{so}(4,4)$. In this case the group of outer automorphism is isomorphic to the symmetric group $\mathcal{S}_4$ so we can have elements of order 2, 3 or 4. If ρ is an outer automorphism of order 2 or 4, and $\alpha = \mathrm{Ad}(A) \circ \tau$, we apply the same method using the classification of local symmetric spaces and we see that

$$\dim S^\alpha \leqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

with equality if and only if α is an outer automorphism corresponding to the conjugation by a matrix conjugate to $Q_{7,1}$. If ρ is of order 3 and $\alpha = \mathrm{Ad}(A)\circ\tau$ then α^3 is inner and we have just to treat the case where it is trivial, that is α is of order 3. Then its complexification $\alpha^{\mathbb{C}}$ is an order 3 complex automorphism of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(8, \mathbb{C})$, a case already treated in the previous section. We know that the fixed point set $(\mathfrak{g}^{\mathbb{C}})^{\alpha^{\mathbb{C}}}$ is isomorphic to $\mathfrak{g}_2^{\mathbb{C}}$, $\mathfrak{sl}(3, \mathbb{C})$ or is compact. As $\mathfrak{g}^\alpha$ is a real form of $(\mathfrak{g}^{\mathbb{C}})^{\alpha^{\mathbb{C}}}$, it is isomorphic to $\mathfrak{g}_{2(2)}$, $\mathfrak{sl}(3, \mathbb{R})$, $\mathfrak{su}(2, 1)$, $\mathfrak{s}(\mathfrak{u}(d_p, d_q) \oplus \mathfrak{o}(s_p, s_q))$ with $2d_p + s_p = 4$ and $2d_q + s_q = 4$ or is compact. In all these cases we have

$$\dim S^\alpha < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

so we have proved the claim. ■

Let us assume for a while that $\mathfrak{g} \neq \mathfrak{sl}(3, \mathbb{R})$ and $\mathfrak{g} \neq \mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3)$. We will conclude that $\mathrm{vcd}(\Gamma) = \underline{\mathrm{gd}}(\Gamma)$ using Lemma 2.2.3. The first condition in the said lemma holds by Lemma 2.4.2. To check the second condition, take S^α and S^β maximal and distinct. We want to establish

$$\dim(S^\alpha \cap S^\beta) \leqslant \mathrm{vcd}(\Gamma) - 2.$$

First remark that by maximality S^α and S^β are not contained in each other. If one of them is not of the form S^A with A conjugate to $Q_{n-1,1}$, let us say S^α, then $\dim S^\alpha \leqslant \mathrm{vcd}(\Gamma) - 1$ and $S^\alpha \cap S^\beta$ is a strict submanifold of S^α so the result holds. If we have $S^\alpha = S^A$ and $S^\beta = S^B$ with A and B conjugate to $Q_{n-1,1}$, then we refer to the computations in the proofs of Lemma 7.2 and Lemma 8.4 in [1]. The proof of the third point is the same as for Lemma 7.5 and Lemma 8.8 in [1]. Note that in [1] the authors consider only inner automorphisms, so the case n odd, but their argument also works without modifications of any kind for n even.

It must be enlighted why the argument we just gave fails for $\mathfrak{sl}(3, \mathbb{R})$ and $\mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3)$. For $\mathfrak{sl}(3, \mathbb{R})$, the second condition of Lemma 2.2.3 does not hold anymore. In the case that $\mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3)$, the conclusion of Lemma 2.4.2 does not apply, because we have that

$$\dim S^\alpha = \dim S - \mathrm{rk}_{\mathbb{R}} G$$

when α is either the conjugation by $Q_{5,1}$ in $\mathrm{PSO}(3, 3)$ or the conjugation by $Q_{3,1}$ in $\mathrm{PSL}(4, \mathbb{R})$. These two are not conjugate, as the conjugation by $Q_{5,1}$ in $\mathrm{PSO}(3, 3)$ corresponds in $\mathrm{PSL}(4, \mathbb{R})$ to an outer automorphism whose fixed point set is isomorphic to $\mathrm{PSp}(4, \mathbb{R})$.

However the proof of Lemma 7.6 in [1] concerning lattices in $\mathrm{SL}(3, \mathbb{R})$ can be adapted to $\mathrm{Aut}(\mathfrak{sl}(3, \mathbb{R}))$ and $\mathrm{Aut}(\mathfrak{sl}(4, \mathbb{R}))$. In fact it can be adapted to

$\mathrm{Aut}(\mathfrak{sl}(n,\mathbb{R}))$ for all $n \geqslant 3$, because a lattice in $\mathrm{PSL}(n,\mathbb{R})$ of $\mathbb{Q}$-rank $n-1$ can be conjugate to a lattice commensurable to $\mathrm{PSL}(n,\mathbb{Z})$ (see the classification of arithmetic groups of classical groups in Section 18.5 in [40]).

As a result Proposition 2.4.1 holds for all real classical simple Lie algebras.

Exceptional real simple Lie algebras:

The proof for exceptional real simple Lie algebras relies on Lemmas 2.3.1 and 2.4.1, Tables 5 to 9 and 11-12 and inequalities (2.4.1), (2.4.2) and (2.4.3). As in the complex case, the proofs are quite similar, although the cases of $\mathfrak{g}_{2(2)}$ and $\mathfrak{f}_{4(4)}$ are specific and treated separately in the end of this section.

Lie algebra $\mathfrak{e}_{6(6)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{6(6)}$. Its outer automorphism group is of order 2 and its adjoint group is $G_{ad} = E_{6(6)}$, which is the group of real points of an algebraic group of real rank 6. This group contains a maximal compact subgroup K isogenous to $\mathrm{Sp}(4)$. We will use Lemma 2.4.1 to check the first condition of Lemma 2.3.1.

The group $G_{ad} = E_{6(6)}$ contains a subgroup $\overline{G}$ isogenous to $\mathrm{Sp}(2,2)$ whose maximal compact subgroup is $\overline{K} = \mathrm{Sp}(2) \times \mathrm{Sp}(2)$. We see in [7] that $E_{6(6)}/\mathrm{Sp}(2,2)$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 4$ and

$$\dim \overline{S} = 16 < 42 - 6 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.2)

$$\dim \overline{S}^A \leqslant 8 < 16 - 6 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

Lemma 2.4.1 applies and shows that the first condition of Lemma 2.3.1 holds.

To check the second condition we list the local symmetric spaces associated to an involution $\rho \in \mathrm{Aut}(\mathfrak{e}_{6(6)})$. By the classification of Berger in [7], the only non compact cases are when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{sp}(2,2), \mathfrak{sp}(8,\mathbb{R}), \mathfrak{sl}(6,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{sl}(3,\mathbb{H}) \oplus \mathfrak{su}(2), \mathfrak{so}(5,5) \oplus \mathfrak{so}(1,1)$ or $\mathfrak{f}_{4(4)}$. We have in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{6(6)})$.

Lie algebra $\mathfrak{e}_{6(2)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{6(2)}$. Its outer automorphism group is of order 2 and its adjoint group is $G_{ad} = E_{6(2)}$, which is the group of real points of an algebraic group of real rank 4. This group contains a maximal compact subgroup K isogenous to $\mathrm{SU}(6) \times \mathrm{SU}(2)$. We will use Lemma 2.4.1 to check the first condition of Lemma 2.3.1.

The group $G_{ad} = E_{6(2)}$ contains a subgroup $\overline{G}$ isogenous to $\mathrm{SO}^*(10) \times \mathrm{SO}(2)$ whose maximal compact subgroup is $\overline{K} = \mathrm{U}(5) \times \mathrm{SO}(2)$. We see in [7] that $E_{6(2)}/(\mathrm{SO}^*(10) \times \mathrm{SU}(2))$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 6$ and

$$\dim \overline{S} = 20 < 40 - 4 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.3)

$$\dim \overline{S}^A \leqslant 12 < 20 - 4 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

Lemma 2.4.1 applies and shows that the first condition of Lemma 2.3.1 holds.

To check the second condition we list the local symmetric spaces associated to an involution $\rho \in \mathrm{Aut}(\mathfrak{e}_{6(2)})$. By the classification of Berger in [7], the only non compact cases are when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{sp}(1,3)$, $\mathfrak{sp}(8,\mathbb{R})$, $\mathfrak{su}(2,4) \oplus \mathfrak{su}(2)$, $\mathfrak{su}(3,3) \oplus \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(4,6) \oplus \mathfrak{so}(2)$, $\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$ or $\mathfrak{f}_{4(4)}$. We have in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{6(2)})$.

Lie algebra $\mathfrak{e}_{6(-14)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{6(-14)}$. Its outer automorphism group is of order 2 and its adjoint group is $G_{ad} = E_{6(-14)}$, which is the group of real points of an algebraic group of real rank 2. This group contains a maximal compact subgroup K isogenous to $\mathrm{SO}(10) \times \mathrm{SO}(2)$. We will use Lemma 2.4.1 to check the first condition of Lemma 2.3.1.

The group $G_{ad} = E_{6(-14)}$ contains a subgroup $\overline{G}$ isogenous to $\mathrm{SO}^*(10) \times \mathrm{SO}(2)$ whose maximal compact subgroup is $\overline{K} = \mathrm{U}(5) \times \mathrm{SO}(2)$. We see in [7] that $E_{6(-14)}/(\mathrm{SO}^*(10) \times \mathrm{SU}(2))$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 6$ and

$$\dim \overline{S} = 20 < 32 - 2 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.3)

$$\dim \overline{S}^A \leqslant 12 < 20 - 2 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

Lemma 2.4.1 applies and shows that the first condition of Lemma 2.3.1 holds.

To check the second condition we list the local symmetric spaces associated to an involution $\rho \in \mathrm{Aut}(\mathfrak{e}_{6(-14)})$. By the classification of Berger in [7], the only non compact cases are when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{sp}(2,2), \mathfrak{su}(2,4) \oplus \mathfrak{su}(2), \mathfrak{su}(1,5) \oplus \mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(2,8) \oplus \mathfrak{so}(2)$, $\mathfrak{so}^*(10) \oplus \mathfrak{so}(2)$ or $\mathfrak{f}_{4(-20)}$. We have in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{6(-14)})$.

Lie algebra $\mathfrak{e}_{6(-26)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{6(-26)}$. Its outer automorphism group is of order 2 and its adjoint group is $G_{ad} = E_{6(-26)}$, which is the group of real points of an algebraic group of real rank 2. This group contains a maximal compact subgroup K isogenous to F_4. We will use Lemma 2.4.1 to check the first condition of Lemma 2.3.1.

The group $G_{ad} = E_{6(-26)}$ contains a subgroup $\overline{G}$ isogenous to $\mathrm{Sp}(1,3)$ whose maximal compact subgroup is $\overline{K} = \mathrm{Sp}(1) \times \mathrm{Sp}(3)$. We see in [7] that $E_{6(-26)}/\mathrm{Sp}(1,3)$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 4$ and

$$\dim \overline{S} = 12 < 26 - 2 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.2)

$$\dim \overline{S}^A \leqslant 8 < 12 - 2 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

Lemma 2.4.1 applies and shows that the first condition of Lemma 2.3.1 holds.

To check the second condition we list the local symmetric spaces associated to an involution $\rho \in \mathrm{Aut}(\mathfrak{e}_{6(-26)})$. By the classification of Berger in [7], the only non compact cases are when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{sp}(1,3), \mathfrak{sl}(3,\mathbb{H}) \oplus \mathfrak{sp}(1), \mathfrak{so}(1,9) \oplus \mathfrak{so}(1,1)$, or $\mathfrak{f}_{4(-20)}$. We have in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{6(-26)})$.

Lie algebra $\mathfrak{e}_{7(7)}$:
Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{7(7)}$. Its outer automorphism group is of order 2 and its adjoint group is $G_{ad} = E_{7(7)}$, which is the group of real points of an algebraic group of real rank 7. This group contains a maximal compact subgroup K isogenous to $\mathrm{SU}(8)$. We will use Lemma 2.4.1 to check the first condition of Lemma 2.3.1.

The group $G_{ad} = E_{7(7)}$ contains a subgroup $\overline{G}$ isogenous to $E_{6(2)} \times \mathrm{SO}(2)$ whose maximal compact subgroup is $\overline{K} = \mathrm{SU}(6) \times \mathrm{SU}(2) \times \mathrm{SO}(2)$. We see in [7] that $E_{7(7)}/(E_{6(2)} \times \mathrm{SO}(2))$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 7$ and

$$\dim \overline{S} = 40 < 70 - 7 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from the results about $\mathfrak{e}_{6(2)}$

$$\dim \overline{S} - \dim \overline{S}^A \geqslant 20 - 12 = 8 > \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

Lemma 2.4.1 applies and shows that the first condition of Lemma 2.3.1 holds.

To check the second condition we list the local symmetric spaces associated to an involution $\rho \in \mathrm{Aut}(\mathfrak{e}_{7(7)})$. By the classification of Berger in [7], the only non compact cases are when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{su}(4,4), \mathfrak{sl}(8,\mathbb{R}), \mathfrak{sl}(4,\mathbb{H}), \mathfrak{so}(6,6) \oplus \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}^*(12) \oplus \mathfrak{sp}(1), \mathfrak{e}_{6(6)} \oplus \mathfrak{so}(1,1)$ or $\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$. We have in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

So by Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{7(7)})$.

Lie algebra $\mathfrak{e}_{7(-5)}$:
Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{7(-5)}$. Its outer automorphism group is trivial so $G = \mathrm{Aut}(\mathfrak{g})$ is equal to the adjoint group $G_{ad} = E_{7(7)}$, which is the group of real points of an algebraic group of real

rank 4. Thus we only have to check the first condition of Lemma 2.3.1 and we will again use Lemma 2.4.1.

The group $G_{ad} = E_{7(-5)}$ contains a maximal compact subgroup K isogenous to $\mathrm{SO}(12) \times \mathrm{SU}(2)$. It also contains a subgroup $\overline{G}$ isogenous to $\mathrm{SU}(4,4)$ whose maximal compact subgroup is $\overline{K} = \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(4))$. We see in [7] that $E_{7(-5)}/\mathrm{SU}(4,4)$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 7$ and

$$\dim \overline{S} = 32 < 64 - 4 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.1)

$$\dim \overline{S}^A \leqslant 24 < 32 - 4 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

So by Lemma 2.4.1 and Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{7(-5)})$.

Lie algebra $\mathfrak{e}_{7(-25)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{7(-25)}$. Its outer automorphism group is of order 2 and its adjoint group is $G_{ad} = E_{7(-25)}$, which is the group of real points of an algebraic group of real rank 3. This group contains a maximal compact subgroup K isogenous to $E_6 \times \mathrm{SO}(2)$. We will use Lemma 2.4.1 to check the first condition of Lemma 2.3.1.

The group $G_{ad} = E_{7(-25)}$ contains a subgroup $\overline{G}$ isogenous to $\mathrm{SU}(2,6)$ whose maximal compact subgroup is $\overline{K} = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(6))$. We see in [7] that $E_{7(-25)}/\mathrm{SU}(2,6)$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 7$ and

$$\dim \overline{S} = 24 < 54 - 3 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.1)

$$\dim \overline{S}^A \leqslant 20 < 24 - 3 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

Lemma 2.4.1 applies and shows that the first condition of Lemma 2.3.1 holds.

To check the second condition we list the local symmetric spaces associated to an involution $\rho \in \mathrm{Aut}(\mathfrak{e}_{7(-25)})$. By the classification of Berger in [7], the only non compact cases are when $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{su}(2,6), \mathfrak{sl}(4,\mathbb{H}), \mathfrak{so}(2,10) \oplus \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}^*(12) \oplus \mathfrak{sp}(1), \mathfrak{e}_{6(-14)} \oplus \mathfrak{so}(2)$ or $\mathfrak{e}_{6(-26)} \oplus \mathfrak{so}(1,1)$. We have in all cases

$$\dim S^\rho < \dim S - \mathrm{rk}_\mathbb{R} G.$$

So by Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{7(-25)})$.

Lie algebra $\mathfrak{e}_{8(8)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{8(8)}$. Its outer automorphism group is trivial so $G = \mathrm{Aut}(\mathfrak{g})$ is equal to the adjoint group $G_{ad} = E_{8(8)}$, which is the group of real points of an algebraic group of real rank 8. Thus we only have to check the first condition of Lemma 2.3.1 and we will again use Lemma 2.4.1.

The group $G_{ad} = E_{8(8)}$ contains a maximal compact subgroup K isogenous to SO(16). It also contains a subgroup $\overline{G}$ isogenous to $\mathrm{SO}^*(16)$ whose maximal compact subgroup is $\overline{K} = \mathrm{U}(8)$. We see in [7] that $E_{8(8)}/\mathrm{SO}(16)$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 8$ and

$$\dim \overline{S} = 56 < 128 - 8 = \dim S - \mathrm{rk}_\mathbb{R} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.3)

$$\dim \overline{S}^A \leqslant 42 < 56 - 8 = \dim \overline{S} - \mathrm{rk}_\mathbb{R} G_{ad}.$$

So by Lemma 2.4.1 and Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{8(8)})$.

Lie algebra $\mathfrak{e}_{8(-24)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{e}_{8(-24)}$. Its outer automorphism group is trivial so $G = \mathrm{Aut}(\mathfrak{g})$ is equal to the adjoint group $G_{ad} = E_{8(-24)}$, which is the group of real points of an algebraic group of real rank 4. Thus we only have to check the first condition of Lemma 2.3.1 and we will again use Lemma 2.4.1.

The group $G_{ad} = E_{8(-24)}$ contains a maximal compact subgroup K isogenous to $E_7 \times \mathrm{SU}(2)$. It also contains a subgroup $\overline{G}$ isogenous to $\mathrm{SO}^*(16)$ whose maximal compact subgroup is $\overline{K} = \mathrm{U}(8)$. We see in [7] that $E_{8(-24)}/\mathrm{SO}(16)$

is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 8$ and

$$\dim \overline{S} = 56 < 112 - 4 = \dim S - \mathrm{rk}_{\mathbb{R}} G_{ad}$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively G_{ad} and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we get from inequality (2.4.3)

$$\dim \overline{S}^A \leqslant 42 < 56 - 4 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G_{ad}.$$

So by Lemma 2.4.1 and Lemma 2.3.1, Proposition 2.4.1 holds for $G = \mathrm{Aut}(\mathfrak{e}_{8(-24)})$.

Lie algebra $\mathfrak{g}_{2(2)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{g}_{2(2)}$. Its outer automorphism group is trivial, so the group $G = \mathrm{Aut}(\mathfrak{g})$ equals the adjoint group $G_{ad} = G_{2(2)}$, which is the group of real points of an algebraic group of real rank 2. Thus we only have to check the conditions of Lemma 2.3.2.

The group $G_{2(2)}$ contains a maximal compact subgroup K isomorphic to $\mathrm{SO}(4)$. It also contains a subgroup $\overline{G}$ isogenous to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ whose maximal compact subgroup is $\overline{K} = \mathrm{SO}(2) \times \mathrm{SO}(2)$. We see in [7] that $G_{2(2)}/(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 2$ and

$$\dim \overline{S} = 4 < 8 - 2 = \dim S - \mathrm{rk}_{\mathbb{R}} G$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively $G = G_{ad}$ and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we have

$$\dim \overline{S}^A \leqslant 2 = 4 - 2 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G.$$

The equality case in the last inequality happens when A is conjugate to a matrix of the form:

$$\begin{pmatrix} \pm I_2 & 0 \\ 0 & R_\theta \end{pmatrix}$$

with

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Assume that A is of this form and that the first block is I_2, we will prove directly that

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

First of all, $C_K(A) = C_{\mathrm{SO}(4)}(A) = \mathrm{SO}(2) \times \mathrm{SO}(2)$ is of dimension 2.

To study $C_G(A)$, we have to know to which element of $G = G_{2(2)}$ this matrix corresponds. Recall that $G_{2(2)}$ is the group of automorphisms of the (non associative) algebra $\mathbb{O}'$ of split octonions, which is of dimension 8 over $\mathbb{R}$, and equiped with a quadratic form of signature (4,4) (see section 1.13 of in [50]). We can decompose $\mathbb{O}'$ into the direct sum $\mathbb{H} \oplus \mathbb{H}e'_4$ where $\mathbb{H} = \mathrm{Vect}\{1, e_1, e_2, e_3\}$ is the quaternion algebra.

So G is a subgroup of the special orthogonal group $\mathrm{SO}(3,4)$ which preserves the standard form of signature (4,4) over $\mathbb{R}^8$ and fixes 1.

The maximal compact subgroup K corresponds to the stabilizer of $\mathbb{H}$, meaning the elements $\alpha \in G$ such that $\alpha(\mathbb{H}) = \mathbb{H}$. Automatically we have $\alpha(\mathbb{H}e'_4) = \mathbb{H}e'_4$ as this is the orthogonal of $\mathbb{H}$. K is isomorphic to $\mathrm{SO}(4)$ via the isomorphism who sends α to its restriction to $\mathbb{H}e'_4$.

Consequently, the matrix A we consider corresponds to the matrix of the restriction of an element $\alpha \in K$ to $\mathbb{H}e'_4$. This element α is entirely determined by the matrix A, indeed for example we have:

$$\alpha(e_1e'_4) = \alpha(e_1)\alpha(e'_4) = \alpha(e_1)e'_4 = e_1e'_4$$

so we deduce $\alpha(e_1) = e_1$. Similarly we find:

$$\alpha(e_2) = \cos(\theta)e_2 + \sin(\theta)e_3,$$

$$\alpha(e_3) = -\sin(\theta)e_2 + \cos(\theta)e_3.$$

Knowing $\alpha(1) = 1$, we have completely described α. The matrix of $\mathrm{SO}(3,4)$ which corresponds to is:

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_\theta & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & R_\theta \end{pmatrix}.$$

Then we remark that $C_G(A) \subset C_{\mathrm{SO}(3,4)}(\widetilde{A})$ so:

$$\dim C_G(A) \leqslant \dim C_{\mathrm{SO}(3,4)}(\widetilde{A}) = \dim \mathrm{S}(\mathrm{O}(1,2) \times \mathrm{U}(1,1)) = 6.$$

Finally:

$$\dim S^A \leqslant 6 - 2 = 4 < 6 = \dim S - \mathrm{rk}_{\mathbb{R}} G.$$

Thus we have that

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every $A \in G$ of finite order and non central. Then by Lemma 2.2.2, Proposition 2.4.1 holds for $\mathfrak{g} = \mathfrak{g}_{2(2)}$.

Lie algebra $\mathfrak{f}_{4(4)}$:

Here we consider the simple exceptional Lie algebra $\mathfrak{g} = \mathfrak{f}_{4(4)}$. Its outer automorphism group is trivial, so the group $G = \mathrm{Aut}(\mathfrak{g})$ equals the adjoint group $G_{ad} = F_{4(4)}$, which is the group of real points of an algebraic group of real rank 4. Thus we only have to check the conditions of Lemma 2.3.2.

The group $F_{4(4)}$ contains a maximal compact subgroup K isogenous to $\mathrm{Sp}(3) \times \mathrm{Sp}(1)$. It also contains a subgroup $\overline{G}$ isogenous to $\mathrm{SO}(4,5)$ whose maximal compact subgroup is $\overline{K} = \mathrm{S}(\mathrm{O}(4) \times \mathrm{O}(5))$. We see in [7] that $F_{4(4)}/\mathrm{SO}(4,5)$ is an irreducible symmetric space, furthermore we have $\mathrm{rk}(K) = \mathrm{rk}(\overline{K}) = 4$ and

$$\dim \overline{S} = 20 < 28 - 4 = \dim S - \mathrm{rk}_{\mathbb{R}} G$$

where S and $\overline{S}$ are the Riemannian symmetric spaces associated to respectively $G = G_{ad}$ and $\overline{G}$.

Moreover, if $A \in \overline{K}$ is of finite order and non central, we have by the computations in section 8 of [1]

$$\dim \overline{S}^A \leqslant 16 = 20 - 4 = \dim \overline{S} - \mathrm{rk}_{\mathbb{R}} G.$$

The equality case in the last inequality happens when A is conjugate to the matrix:

$$\begin{pmatrix} -I_8 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SO}(4) \times \mathrm{SO}(5).$$

Assuming that A is of this form, the conjugation by A is an involutive automorphism of $G = G_{ad} = F_{4(4)}$ so the quotient G/G^A is a symmetric space and we know by the classification in [7] that G^A is isogenous to either $\mathrm{Sp}(6,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$, $\mathrm{Sp}(2,1) \times \mathrm{Sp}(1)$ or $\mathrm{SO}(4,5)$. In all these cases the inequality

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

holds (in fact G^A is isogenous to $\mathrm{SO}(4,5)$).

Thus we have that

$$\dim S^A < \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for every $A \in G$ of finite order and non central. Then by Lemma 2.2.2, Proposition 2.4.1 holds for $\mathfrak{g} = \mathfrak{f}_{4(4)}$, and it concludes the proof. ■

2.5 Semisimple Lie algebras

We prove in this section the Main Theorem:

Main Theorem *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $G = \mathrm{Aut}(\mathfrak{g})$. *Then*

$$\underline{\mathrm{gd}}(\Gamma) = \mathrm{vcd}(\Gamma)$$

for every lattice $\Gamma \subset G$.

Recall that if $\mathfrak{g}$ is semisimple, it is isomorphic to a sum of simple Lie algebras $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$. The adjoint group G_{ad} of $\mathfrak{g}$ is then isomorphic to a product of simple Lie groups, that is $G_{ad} = G_1 \times G_2 \times \cdots \times G_n$ where the G_i are the adjoint groups of the $\mathfrak{g}_i$. We can also assume that G_{ad} has no compact factors, indeed the symmetric spaces S and S^α do not change if we replace G_{ad} by its quotient by the compact factors. An automorphism of $\mathfrak{g}$ is the composition of a permutation σ of the isomorphic factors of $\mathfrak{g}$ and a diagonal automorphism ρ of the form $\rho = \rho_1 \oplus \cdots \oplus \rho_r$ with $\rho_i \in \mathrm{Aut}(\mathfrak{g}_i)$.

We now explain why the strategy used in the previous sections does not work. The point is that the inequality

$$\dim S^\alpha \leqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

for $\alpha \in \mathrm{Aut}(\mathfrak{g})$ needed to apply Lemma 2.2.3 does not hold even in the simplest cases.

In fact, if $G_{ad} = \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ and $A = \left(I_3, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$, we have $\dim S^A = 5 + 3 = 8 > 6 = \dim S - \mathrm{rk}_{\mathbb{R}} G$.

We bypass this this problem by improving the lower bound

$$\mathrm{vcd}(\Gamma) \geqslant \dim S - \mathrm{rk}_{\mathbb{R}} G$$

used above. Remember that by Theorem 2.2.2

$$\mathrm{vcd}(\Gamma) = \dim S - \mathrm{rk}_{\mathbb{Q}} \Gamma$$

as long as Γ is arithmetic, so we want to majorate $\mathrm{rk}_{\mathbb{Q}}\Gamma$. To do that we will restrict our study to irreducible lattices. Recall that in this context, a lattice Γ in G is irreducible if ΓH is dense in G for every non-compact, closed, normal subgroup H of G_{ad}.

We prove the following result, which is probably known to experts:

Proposition 2.5.1. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be a semisimple Lie algebra and G_i the adjoint group of $\mathfrak{g}_i$ for $i = 1 \dots n$. Then*

$$\mathrm{rk}_{\mathbb{Q}}\Gamma \leqslant \min_{i=1\dots n} \mathrm{rk}_{\mathbb{R}} G_i$$

for every irreducible arithmetic lattice $\Gamma \subset G = \mathrm{Aut}(\mathfrak{g})$.

Proposition 2.5.1 will follow from the following theorem proved in [27]

Theorem 2.5.1. *Let $G = G_1 \times \cdots \times G_n$ ($n \geqslant 2$) be a product of non-compact connected simple Lie groups. The following statements are equivalent:*

1. *G contains an irreducible lattice.*
2. *G is isomorphic as a Lie group to $(\mathbb{G}_{\mathbb{R}})^0$, where $\mathbb{G}$ is an $\mathbb{Q}$-simple algebraic group.*
3. *G is isotypic, that is the complexifications of the Lie algebras of the G_i are isomorphic.*

In addition to that, in this case G contains both cocompact and non cocompact irreducible lattices.

Recall that an algebraic group $\mathbb{G}$ defined over $\mathbb{Q}$ is said to be $\mathbb{Q}$-*simple* if it does not contain non-trivial connected normal subgroups defined over $\mathbb{Q}$.

Then we can prove Proposition 2.5.1.

Proof of Proposition 2.5.1. Remark that $\mathrm{rk}_{\mathbb{Q}}\Gamma = \mathrm{rk}_{\mathbb{Q}}(\Gamma \cap G_{ad})$. Moreover if Γ is an irreducible lattice of G then $\Gamma \cap G_{ad}$ is an irreducible lattice of G_{ad}, so we can assume that $\Gamma \subset G_{ad}$. Remember that $G_{ad} = G_1 \times \cdots \times G_n$ and that we assumed that none of the G_i is compact. If $n = 1$ the result is trivial so assume that $n \geqslant 2$. Then $\mathrm{rk}_{\mathbb{R}} G_{ad} \geqslant 2$ so Γ is arithmetic by Theorem 2.2.1 and there exists a Lie group isomorphism $\varphi : G_{ad} \to (\mathbb{G}_{\mathbb{R}})^0$, where $\mathbb{G}$ is $\mathbb{Q}$-simple by Theorem 2.5.1. Then we have $\mathrm{rk}_{\mathbb{Q}}\Gamma = \mathrm{rk}_{\mathbb{Q}}\mathbb{G}$. The algebraic group $\mathbb{G}$ is isomorphic to a product $\mathbb{G}_1 \times \cdots \times \mathbb{G}_n$ where the $\mathbb{G}_i$ are $\mathbb{R}$-groups with $(\mathbb{G}_i)_{\mathbb{R}}$ isomorphic to G_i for $i = 1 \dots n$ (we can define $\mathbb{G}_i$ as the centralizer in $\mathbb{G}$ of the product $\prod_{k \neq i} G_k$). We denote π_i the canonical projection of $\mathbb{G}$ on $\mathbb{G}_i$. Let $\mathbb{T} \subset \mathbb{G}$ be maximal $\mathbb{Q}$-split torus. Our goal is to prove that the restriction $\pi_i|_{\mathbb{T}}$ is of finite kernel. On the one hand, $\mathrm{Ker}(\pi_i|_{\mathbb{G}_{\mathbb{Q}}}) \subset \mathbb{G}_{\mathbb{Q}}$ is a normal subgroup of $\mathbb{G}_{\mathbb{Q}}$ and $\mathbb{G}$ is defined over $\mathbb{Q}$, so the Zariski closure of $\mathrm{Ker}(\pi_i|_{\mathbb{G}_{\mathbb{Q}}})$ is defined over $\mathbb{Q}$ by the Galois rationality criterion. However it is a non trivial normal subgroup of $\mathbb{G}$ which is $\mathbb{Q}$-simple so $\mathrm{Ker}(\pi_i|_{\mathbb{G}_{\mathbb{Q}}})$ is finite (it may be not connected). So $\mathrm{Ker}(\pi_i|_{\mathbb{T}_{\mathbb{Q}}})$ is finite too. But $\mathrm{Ker}(\pi_i|_{\mathbb{T}})$ is a subgroup of the $\mathbb{Q}$-split torus $\mathbb{T}$, so its identity component is a $\mathbb{Q}$-split torus, and we have just seen its group of rational points is finite, so $\mathrm{Ker}(\pi_i|_{\mathbb{T}})$ is finite too.

Then the image of $\mathbb{T}$ by π_i is a torus of $\mathbb{G}_i$ of the same dimension as $\mathbb{T}$ (see [9, Cor. 8.4 p.114]). It may not be $\mathbb{Q}$-split because the projection is not defined over $\mathbb{Q}$, but it is $\mathbb{R}$-split as the projection is defined over $\mathbb{R}$, so:

$$\mathrm{rk}_{\mathbb{Q}}\mathbb{G} = \dim \mathbb{T} \leqslant \mathrm{rk}_{\mathbb{R}}\mathbb{G}_i = \mathrm{rk}_{\mathbb{R}}G_i.$$

■

We can now conclude the proof of our Main Theorem.

Proof of the Main Theorem. If $\mathfrak{g}$ is simple then the result follows from Propositions 2.2.1, 2.3.1 and 2.4.1. Then we assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with $n \geqslant 2$. We can also assume that the adjoint group G_{ad} of $\mathfrak{g}$ is of the form $G_{ad} = G_1 \times \cdots \times G_n$ where the G_i are simple, non-compact and G_i is the adjoint group of $\mathfrak{g}_i$.

We begin with the case where Γ is irreducible. Then we have

$$\mathrm{rk}_{\mathbb{Q}}\Gamma \leqslant \min_{i=1\ldots r} \mathrm{rk}_{\mathbb{R}}G_i.$$

As $\mathrm{rk}_{\mathbb{R}}G \geqslant 2$, Γ is arithmetic by Theorem 2.2.1. Remember that $\Gamma \cap G_{ad}$ is also an irreducible arithmetic lattice of G_{ad}. We can then assume that $G_{ad} = (\mathbb{G}_{\mathbb{R}})^0$, where $\mathbb{G} = \mathbb{G}_1 \times \cdots \times \mathbb{G}_n$ is a semisimple $\mathbb{Q}$-group, which is $\mathbb{Q}$-simple by Theorem 2.5.1. As G_{ad} has trivial center, we can assume that $\mathbb{G}$ is centerfree. In this case we have $\Gamma \cap G_{ad} \subset \mathbb{G}_{\mathbb{Q}}$.

We want to use Lemma 2.2.2. Let $\alpha \in \Gamma$ of finite order non central. Then α is of the form $\sigma \circ \rho$ where σ is a permutation of the isomorphic factors of $\mathfrak{g}$ and $\rho = \rho_1 \oplus \cdots \oplus \rho_r$ with $\rho_i \in \mathrm{Aut}(\mathfrak{g}_i)$. Assume for a while that σ is trivial. We identify $\alpha \in \mathrm{Aut}(\mathfrak{g})$ (resp. $\rho_i \in \mathrm{Aut}(\mathfrak{g}_i)$) with the corresponding automorphism of G_{ad} (resp. G_i). The key point is to remark that for all i between 1 and n, the automorphism ρ_i is not trivial. In fact if $A \in \Gamma \cap G_{ad} \subset \mathbb{G}_{\mathbb{Q}}$, we can identify it with the inner automorphism $\mathrm{Ad}(A)$ and we have

$$\alpha \circ \mathrm{Ad}(A) \circ \alpha^{-1} = \mathrm{Ad}(\alpha(A)) \in \Gamma \cap G_{ad} \subset \mathbb{G}_{\mathbb{Q}},$$

so $\alpha(A)$ lies also in $\mathbb{G}_{\mathbb{Q}}$. Recall that we have seen in the proof of Proposition 2.5.1 that the projections $\pi_i|_{\mathbb{G}_{\mathbb{Q}}} : \mathbb{G}_{\mathbb{Q}} \to G_i$ are injective. So if ρ_i is trivial, we have $\pi_i(A) = \pi_i(\alpha(A))$ for each $A \in \Gamma \cap G_{ad} \subset \mathbb{G}_{\mathbb{Q}}$, which leads to $\alpha(A) = A$. Then α is trivial on $\Gamma \cap G_{ad}$ which is Zariski-dense in G_{ad}, so α is trivial.

Finally, $\alpha = \rho = \rho_1 \oplus \cdots \oplus \rho_n$ where each ρ_i is a non trivial automorphism of G_i. By Proposition 2.5.1 we also have

$$\mathrm{rk}_{\mathbb{Q}}\Gamma \leqslant \min_{i=1\ldots n} \mathrm{rk}_{\mathbb{R}} G_i.$$

Then if we denote S the symmetric space associated to G and S_i those associated to G_i and we have by Propositions 2.3.1 and 2.4.1

$$\begin{aligned}\dim S^{\alpha} = \sum_{i=1}^{n} \dim S_i^{\rho_i} &\leqslant \sum_{i=1}^{n} (\dim S_i - \mathrm{rk}_{\mathbb{R}} G_i) \\ &\leqslant \dim S - \sum_{i=1}^{n} \mathrm{rk}_{\mathbb{R}} G_i \\ &< \dim S - \mathrm{rk}_{\mathbb{Q}}\Gamma.\end{aligned}$$

as we assumed $n \geqslant 2$.

By Theorem 2.2.2, $\dim S^A < \mathrm{vcd}(\Gamma)$ and Lemma 2.2.2 gives us the result.

If σ is not trivial, the fixed point set will be even smaller. Indeed, assume for simplicity that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, and $\alpha \in \mathrm{Aut}(\mathfrak{g})$ is of the form

$$\alpha(x_1, x_2) = (\rho_1(x_2), \rho_2(x_1))$$

with $\rho_1, \rho_2 \in \mathrm{Aut}(\mathfrak{g}_1) = \mathrm{Aut}(\mathfrak{g}_2)$. Then the fixed point set S^{α} is $S_1^{\rho_1\rho_2}$ where S_1 is the symmetric space associated to $\mathfrak{g}_1$. In fact the elements fixed by α are of the form $(x_1, \rho_2(x_1))$ where x_1 is a fixed point of $\rho_1\rho_2$. So we have

$$\dim S^{\alpha} \leqslant \dim S_1 < \dim S - \mathrm{rk}_{\mathbb{R}} G_1 \leqslant \dim S - \mathrm{rk}_{\mathbb{Q}}\Gamma = \mathrm{vcd}(\Gamma)$$

as $\dim S = 2 \dim S_1$ and $\dim S_1 > \mathrm{rk}_{\mathbb{R}} G_1$. The same argument works for a higher number of summands by decomposing the permutation σ into disjoint cycles.

Finally if Γ is reducible, there exists a decomposition of $G = H_1 \times H_2$ such that the projections $\pi_1(\Gamma)$ and $\pi_2(\Gamma)$ are lattices in H_1 and H_2, and then $\Gamma \subset \pi_1(\Gamma) \times \pi_2(\Gamma)$ (see the proof of [40, Prop. 4.3.3]). It follows by induction that Γ is contained in a product of irreducible lattices of factors of G. We will treat the case where $G = H_1 \times H_2$, $\Gamma \subset \Gamma_1 \times \Gamma_2$ and Γ_1 and Γ_2 are irreducible lattices of H_1 and H_2. As Γ is of finite index in G, it is also of finite index in $\Gamma_1 \times \Gamma_2$ so $\mathrm{vcd}(\Gamma) = \mathrm{vcd}(\Gamma_1 \times \Gamma_2)$. If we denote S, S_1, S_2 the symmetric spaces associated to G, H_1, H_2, by Theorem 2.2.2 we have:

$$\begin{aligned}\mathrm{vcd}(\Gamma) = \mathrm{vcd}(\Gamma_1 \times \Gamma_2) &= \dim S_1 + \dim S_2 - \mathrm{rk}_{\mathbb{Q}}\Gamma_1 - \mathrm{rk}_{\mathbb{Q}}\Gamma_2 \\ &= \mathrm{vcd}(\Gamma_1) + \mathrm{vcd}(\Gamma_2).\end{aligned}$$

Finally, we have:

$$\underline{\mathrm{gd}}(\Gamma) \leqslant \underline{\mathrm{gd}}(\Gamma_1 \times \Gamma_2) \leqslant \underline{\mathrm{gd}}(\Gamma_1) + \underline{\mathrm{gd}}(\Gamma_2)$$

because if X_1 and X_2 are models for $\underline{E}\Gamma_1$ and $\underline{E}\Gamma_2$, $X_1 \times X_2$ is a model for $\underline{E}(\Gamma_1 \times \Gamma_2)$. As Γ_1 and Γ_2 are irreducible, we have $\underline{\mathrm{gd}}(\Gamma_1) = \mathrm{vcd}(\Gamma_1)$ and $\underline{\mathrm{gd}}(\Gamma_2) = \mathrm{vcd}(\Gamma_2)$ so:

$$\underline{\mathrm{gd}}(\Gamma) \leqslant \mathrm{vcd}(\Gamma).$$

The other inequality is always true, so it concludes the proof of the Main Theorem. ■

We will end with the proof of Corollaries 2.1.1 and 2.1.2.

Proof of Corollary 2.1.1. The case of real rank 1 is treated in Proposition 2.6 in [1]. For higher real rank, we know by the Main Theorem that there exists a model for $\underline{E}\Gamma$ of dimension $\mathrm{vcd}(\Gamma)$. We also know that the Borel-Serre bordification is a cocompact model for $\underline{E}\Gamma$. Then using the same construction as in the proof of Corollary 1.1 in [1], one has a cocompact model for $\underline{E}\Gamma$ of dimension $\mathrm{vcd}(\Gamma)$. As all models of $\underline{E}\Gamma$ are Γ-equivariantly homotopy equivalent and the symmetric space S is also a model for $\underline{E}\Gamma$, we conclude that S is Γ-equivariantly homotopy equivalent to a cocompact model for $\underline{E}\Gamma$ of dimension $\mathrm{vcd}(\Gamma)$. ■

Proof of Corollary 2.1.2. We have to prove that if $\Gamma_1 \subset \mathrm{Aut}(\mathfrak{g})$ and Γ_2 have a common subgroup $\widetilde{\Gamma}$ of finite index, then $\underline{\mathrm{gd}}(\Gamma_2) = \mathrm{vcd}(\Gamma_2)$. To that end, we will prove that Γ_2 is essentially also a lattice in $\mathrm{Aut}(\mathfrak{g})$. First note that $\widetilde{\Gamma}$ is a lattice in $\mathrm{Aut}(\mathfrak{g})$, so we can assume that $\widetilde{\Gamma} = \Gamma_1 \subset \Gamma_2$. We can also assume that Γ_1 is a normal finite index subgroup of Γ_2. Then Γ_2 acts by conjugation on Γ_1. By Mostow rigidity Theorem (see for example [40, Thm. 15.1.2]), automorphisms of Γ_1 can be extended to automorphisms of G_{ad}, so we have a morphism $\Gamma_2 \to \mathrm{Aut}(G_{ad})$. The kernel N of this morphism does not intersect Γ_1 (since Γ_1 is centerfree, as it is a lattice and thus it is Zariski-dense in G_{ad}) and Γ_1 is of finite index in Γ_2, so N is finite. Then Γ_2/N is isomorphic to a lattice in $\mathrm{Aut}(G_{ad})$. The result follows now from the Main Theorem and Lemma 2.2.1.

Note that Mostow rigidity theorem does not apply to the group $\mathrm{PSL}(2,\mathbb{R})$, whose associated symmetric space is the hyperbolic plane. In this case the lattice Γ_1 is either a virtually free group or a virtually surface group. In the first case the group Γ_2 is also virtually free, so there exists a model for $\underline{E}\Gamma_2$ which is a tree (see [28]), and $\underline{\mathrm{gd}}(\Gamma_2) = \mathrm{vcd}(\Gamma_2) = 1$. In the second case, Γ_2 acts as a convergence group on $\mathbb{S}^1 = \partial_\infty\Gamma_1$, so it is also a Fuchsian group (see [19]), that is Γ_2 is isomorphic to a cocompact lattice of $\mathrm{PSL}(2,\mathbb{R})$. Finally we have $\underline{\mathrm{gd}}(\Gamma_2) = \mathrm{vcd}(\Gamma_2) = 2$. ■

Chapter 3

On the difficulty of finding spines

3.1 Introduction

Let Γ be an infinite discrete group. A *model for* $\underline{E}\Gamma$, or a classifying space for proper actions, is a Γ-CW-complex W such that for every subgroup $H \subset \Gamma$, the fixed point set W^H is contractible if H is finite and empty otherwise. Models for $\underline{E}\Gamma$ always exist, and the minimal possible dimension of such a model is the *proper geometric dimension* of Γ, denoted $\underline{\mathrm{gd}}(\Gamma)$. Completing earlier work in [1], we proved in [30] that if G is a semisimple linear Lie group and $\Gamma \subset G$ is a lattice, then $\underline{\mathrm{gd}}(\Gamma)$ equals the *virtual cohomological dimension* $\mathrm{vcd}(\Gamma)$ of Γ, that is the cohomological dimension of any torsionfree finite index subgroup of Γ. If Γ is arithmetic and $K \subset G$ is a maximal compact subgroup, then $\mathrm{vcd}(\Gamma)$ equals the dimension of G/K minus the rational rank of Γ (see [10]).

With the same notations, note that the symmetric space $X = G/K$ is itself a model for $\underline{E}\Gamma$, but not of minimal dimension unless Γ is cocompact. It is then a question to find concretely a cocompact model W for $\underline{E}\Gamma$ of dimension $\mathrm{vcd}(\Gamma)$. Besides the intrinsic interest of the problem, one can use such a model to compute the cohomology of Γ (see examples in [45], [4], [24], [17]).

Because of the lack of another starting point, it is natural to try to construct W as a subspace of the symmetric space X. In this case we call it a *spine*. More precisely a spine for Γ is a Γ-equivariant deformation retract of the symmetric space $X = G/K$, of dimension $\mathrm{vcd}(\Gamma)$ and on which Γ acts cocompactly.

It might be surprising to the reader that such spines are known only in very few cases: basically only for $\mathbb{Q}$-rank 1 groups (see [49]) and for $\mathrm{SL}(n, \mathbb{Z})$ (and somewhat more generally for linear symmetric spaces, see [46], [3], [43] and [44]) . The aim of this note is to maybe explain why it might not be easy to find spines.

First recall the construction of $\mathrm{SL}(n, \mathbb{Z})$'s spine. Indentify the symmetric space $S_n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ with the space of lattices in $\mathbb{R}^n$ of covolume 1 modulo isometries with a $\mathbb{Z}$-basis. The *systole* of a lattice $\Lambda = A\mathbb{Z}^n$, with $A \in \mathrm{SL}(n, \mathbb{R})$, is defined as

$$\mathrm{syst}(\Lambda) = \min_{v \in \mathbb{Z}^n \setminus \{0\}} |Av|.$$

We will also call systoles (or minimal vectors) the vectors v which realize the minimum. A lattice is *well-rounded* if its minimal vectors span $\mathbb{R}^n$. Generalizing a result of Soulé in [46], Ash proved in [3] that the well-rounded retract, that is the set of all well-rounded lattices, is a spine for $\mathrm{SL}(n, \mathbb{Z})$. The idea to realize the retraction is as follows: given a non well-rounded lattice in $\mathbb{R}^n$, expand the space spanned by the shortest vectors and contract its orthogonal complement until we find an additional systole. In this way, one can prove that if $\mathcal{X}_i$ is the set of lattices whose systoles span a subspace of dimension at least i (for $i = 1, \dots, n$), then $\mathcal{X}_{i+1}$ is a $\mathrm{SL}(n, \mathbb{Z})$-equivariant deformation retract of $\mathcal{X}_i$ for every $i = 1, \dots, n-1$. Remark that $\mathcal{X}_1 = S_n$ and $\mathcal{X}_n$ is the set of well-rounded lattices, that is our well-rounded retract.

Some effort has been devoted to mimic the construction of the well-rounded retract in other situations. For instance in [26] Ji constructed well-rounded retracts for mapping class groups acting on Teichmüller spaces, and Bavard proved in [6] that the symmetric space $\mathrm{Sp}(2g, \mathbb{R})/\mathrm{U}(g)$ (also known as the Siegel space $\mathfrak{h}_g$), identified with the set of symplectic lattices in $\mathbb{R}^{2g}$ endowed with a symplectic basis, admits a $\mathrm{Sp}(2g, \mathbb{Z})$-equivariant deformation retract, consisting of the set of symplectic lattices whose systoles span a non-isotropic subspace. In both cases, the retract has not minimal dimension. For example, in the case of $\mathrm{Sp}(2g, \mathbb{Z})$ the virtual cohomological dimension is g^2 and Bavard's retract has codimension one[1]: there are lattices there with only two systoles which are non-isotropic. To retract into a higher codimension set it would be reasonable to expect that one should be able to retract into the set of symplectic lattices with more linearly independent systoles. We prove that we cannot do this:

Theorem 3.1.1. *The set $(\mathcal{X}_3 \cap \mathfrak{h}_g)$ of symplectic lattices in $\mathfrak{h}_g$ whose systoles generate a vector space of dimension at least 3 in $\mathbb{R}^{2g}$ does not contain*

[1]Recall that in general, if $\Gamma \subset G$ is not cocompact, one can always construct a cocompact model for $\underline{E}\Gamma$ of codimension 1 in G/K, see Proposition 2.6 in [1].

any model for $\underline{E}\mathrm{Sp}(2g,\mathbb{Z})$*. In particular, it does not contain any* $\mathrm{Sp}(2g,\mathbb{Z})$*-equivariant deformation retract of* $\mathfrak{h}_g$*.*

We will also obtain that the same results holds if we replace $\mathrm{Sp}(2g,\mathbb{Z})$ by $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$:

Theorem 3.1.2. *The set* $\mathcal{X}_3 \subset S_n = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ *(*$n \geqslant 3$*) of lattices in* $\mathbb{R}^n$ *whose systoles generate a subspace of dimension greater than 3 does not contain any model for* $\underline{E}\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$*.*

This second result is noteworthy because $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$ only differ by a finite group and both act by isometries on S_n.

The remaining of this note is devoted to the proofs of Theorem 3.1.1 and Theorem 3.1.2.

3.2 Proofs of the results

We begin with the proof of Theorem 3.1.1:

Proof of Theorem 3.1.1. Recall that the group $\mathrm{Sp}(2g,\mathbb{R})$ is the set of matrices A of size $2g$ such that ${}^tAJA = J$ where J is the block-diagonal matrix $(J_2, J_2, \ldots, J_2)$ and $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In fact, J is the matrix of a symplectic form ω of $\mathbb{R}^{2g}$ in the canonical basis. A basis of $\mathbb{R}^{2g}$ is said to be symplectic if the associated matrix is symplectic, which is equivalent to the fact that the matrix of the symplectic form ω in this basis is J. To prove the theorem, we will show that there exists a finite subgroup $H \subset \mathrm{Sp}(2g,\mathbb{Z})$ such that the fixed point set W^H is not contractible, for every subset $W \subset (\mathcal{X}_3 \cap \mathfrak{h}_g)$.

The result holds trivially for $g = 1$, but let us recall some facts about the systole function in $\mathfrak{h}_1 = \mathbb{H}^2$. First recall that we identify a point $\tau \in \mathbb{H}^2$ with the lattice generated by 1 and τ rescaled to have covolume 1. The well-rounded retract in $\mathbb{H}^2$ is then the Bass-Serre tree of $\mathrm{SL}(2,\mathbb{Z})$. Then, note that the lattices with maximal systole in $\mathbb{R}^2$ are the hexagonal ones. They are the translates by $\mathrm{SL}(2,\mathbb{Z})$ of the standard hexagonal lattice Λ_0, associated in $\mathbb{H}^2$ with $\tau_0 = e^{i\frac{\pi}{3}}$. The point τ_0 is the only fixed point by the homography $z \mapsto 1 - \frac{1}{z}$ associated to the matrix $A_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$.

Let us continue with the case $g = 2$. A way to construct a symplectic lattice in $\mathbb{R}^4$, with its canonical basis (e_1, e_2, e_3, e_4), is to sum two lattices in $\mathrm{Vect}\{e_1, e_2\}$ and $\mathrm{Vect}\{e_3, e_4\}$, which are orthogonal for both the symplectic form ω and the usual euclidean scalar product. The corresponding element in $\mathfrak{h}_2 = \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$ belongs to a subspace homeomorphic to

$\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ which can be identified with the product of two copies of the hyperbolic plane $\mathbb{H}^2$. This subspace is also the fixed point set of the finite subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ generated by the diagonal matrix $(I_2, -I_2)$.

A lattice Λ in $\mathbb{R}^4$ corresponding to an element in $\mathbb{H}^2 \times \mathbb{H}^2$ is a product of two lattices $\Lambda_1 \times \Lambda_2$ which are orthogonal, so the systoles of Λ belong to the subset $(\Gamma_1 \times \{0\}) \cup (\{0\} \times \Gamma_2)$. Λ has three linearly independent systoles if and only if Λ_1 and Λ_2 have the same systole and one of them is well-rounded.

Claim: The fixed point set W^H, with H being the subgroup of $\mathrm{Sp}(4,\mathbb{Z})$ generated by (I_2, A_0), is either empty or not connected.

Proof of the claim. The fixed point set of $\mathbb{H}^2 \times \mathbb{H}^2$ by the pair $(I_2, A_0) \in \mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{Sp}(4,\mathbb{Z})$ is $\mathbb{H}^2 \times \{\tau_0\}$. If $\Lambda = \Lambda_1 \times \Lambda_0$ has 3 linearly independent systoles and lie in this set, then Λ_0 is hexagonal and has maximal systole, so Λ_1 has to be hexagonal too. It follows that W^H is either empty or homeomorphic to the set of translates of τ_0 by $\mathrm{SL}(2,\mathbb{Z})$ and hence is discrete. ■

So W^H is not contractible and W is not a model for $\underline{E}\mathrm{Sp}(4,\mathbb{Z})$. We have proved the theorem in the case $g = 2$.

For the general case, we will explain which finite subgroup H of $\mathrm{Sp}(2g,\mathbb{Z})$ we will take. It will be generated by some finite order matrices in $\mathrm{SL}(2,\mathbb{Z}) \times \cdots \times \mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{Sp}(2g,\mathbb{Z})$. First consider the g diagonal matrices of the form $(I_2, I_2, \ldots, -I_2, I_2, \ldots, I_2)$. The fixed point set of the finite subgroup generated by them is $\mathbb{H}^2 \times \cdots \times \mathbb{H}^2$. Add to this subgroup the matrix $(I_2, A_0, \ldots, A_0)$. The fixed point set $(\mathfrak{h}_g)^H$ is then homeomorphic to $\mathbb{H}^2 \times \{\tau_0\} \times \cdots \times \{\tau_0\}$ and we can apply the preceeding argument. ■

Remark 3.2.1 The symmetric space $\mathbb{H}^2 \times \mathbb{H}^2$ admits a $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$-equivariant deformation retract of dimension $\mathrm{vcd}(\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})) = 2$ which is just the product of two copies of the well-rounded retract of $\mathbb{H}^2$, but the associated lattices in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ are not well-rounded in general.

Remark 3.2.2 It is worth mentioning that in the case $g = 2$, MacPherson and McConnell have constructed in [36] a *weak spine* of $\mathfrak{h}_2$, that is, for every finite index torsionfree subgroup $\Gamma \subset \mathrm{Sp}(4,\mathbb{Z})$, a cocompact Γ-equivariant deformation retract W_Γ of $\mathfrak{h}_2$ of dimension $\mathrm{vcd}(\mathrm{Sp}(4,\mathbb{Z})) = \mathrm{cd}(\Gamma) = 4$. The methods they used are slightly different as the ones for the well-rounded retract, but do not yield a $\mathrm{Sp}(4,\mathbb{Z})$-equivariant deformation retract. They used the Voronoi decomposition of the symmetric space $\mathrm{SL}(4,\mathbb{R})/\mathrm{SO}(4)$ (identified with the set of positive definite quadratic forms in $\mathbb{R}^4$ modulo homotheties) and studied the intersection of the cells with $\mathfrak{h}_2 = \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$.

It is the first example of a (weak) spine of a nonlinear symmetric space of real rank greater than 1. Note that we can also use the Voronoi sets of $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)=\mathbb{H}^2$ to construct the well-rounded retract for $\mathrm{SL}(2,\mathbb{Z})$. See Chap.VII of [38] for more about the Voronoi sets.

We continue with the proof of Theorem 3.1.2:

Proof of Theorem 3.1.2. Recall that for $n \geqslant 3$, the group $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$ is the semidirect product of the group of inner automorphism, which is isomorphic to $\mathrm{PSL}(n,\mathbb{Z})$, and the outer automorphism group generated by the conjugation by the diagonal matrix with entries $(-1,1,\ldots,1)$ and the automorphism σ defined by $\sigma(X)=({}^tX)^{-1}$ for $X\in\mathrm{SL}(n,\mathbb{Z})$ (see [25]). The usual action of $\mathrm{PSL}(n,\mathbb{Z})$ on S_n extends to an isometric action of $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$.

We begin with the case where $n=2p$ is even. We see that the Siegel space $\mathfrak{h}_p=\mathrm{Sp}(2p,\mathbb{R})/\mathrm{U}(p)$ is the fixed point set of the automorphism α defined by:

$$\alpha(X)=J^{-1}\sigma(X)J,$$

that is the composition of σ and an inner automorphism. Then we can take the subgroup of $\mathrm{Aut}(\mathrm{SL}(2p,\mathbb{Z}))$ generated by α and the subgroup H in the proof of Theorem 3.1.1 and apply the same argument as before.

If $n=2p+1$ is odd, we see that the fixed point set by the subgroup $\widetilde{H}$ of $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$ generated by the inner automorphism of conjugation by the diagonal matrix $(1,-1,\ldots,-1)$ and the outer automorphism $\widetilde{\alpha}$ defined by:

$$\widetilde{\alpha}(X)=\begin{pmatrix}1 & 0\\ 0 & J^{-1}\end{pmatrix}({}^tX)^{-1}\begin{pmatrix}1 & 0\\ 0 & J\end{pmatrix},$$

consists of all matrices of the form $\begin{pmatrix}1 & 0\\ 0 & B\end{pmatrix}$ with $B\in\mathrm{Sp}(2p,\mathbb{R})$. Then if we consider the finite subgroup $\hat{H}$ generated by $\widetilde{H}$, the diagonal matrices $(1,I_2,I_2,\ldots,-I_2,I_2,\ldots,I_2)$ and $(1,A_0,\ldots,A_0)$, the fixed point set is reduced to the lattice $\{1\}\times\Lambda_0\times\cdots\times\Lambda_0$ which has only one systole, so its intersection with $\mathcal{X}_3$ is empty. ∎

Remark 3.2.3 The proof of Theorem 3.1.2 only involves the outer automorphism σ. In fact, the well-rounded retract is also a spine for $\mathrm{PGL}(n,\mathbb{Z})$, which is of index 2 in $\mathrm{Aut}(\mathrm{SL}(n,\mathbb{Z}))$. Note also that in the case $n=2$, σ is an inner automorphism and the Bass-Serre tree is the unique minimal spine for $\mathrm{PGL}(2,\mathbb{Z})$.

Remark 3.2.4 It follows from the proof of Theorem 3.1.2, that if W is a $\mathrm{Aut}(\mathrm{SL}(2g,\mathbb{Z}))$-equivariant deformation retract of S_{2g}, then its intersection with $\mathfrak{h}_g$ is a model for $\underline{E}\mathrm{Sp}(2g,\mathbb{Z})$. Then, to construct a spine for $\mathrm{Sp}(2g,\mathbb{Z})$, we could try to find one for $\mathrm{Aut}(\mathrm{SL}(2g,\mathbb{Z}))$. As we just saw, one cannot do this using just the systole function, but one can hope to succeed by using other classical functions on the space of lattices. For instance we can think to the k-systoles functions, which measure the volume of the k-dimensional sections of the lattice (see [5] for definitions and properties). Note that like the usual systole, the k-systoles are exponentials of Busemann functions.

Bibliography

[1] J. Aramayona, D. Degrijse, C. Martínez-Pérez, and J. Souto. *Geometric dimension of lattices in classical simple Lie groups.* Journal of Topology 10, 632-667, 2017.

[2] J. Aramayona and C. Martínez-Pérez. *The proper geometric dimension of the mapping class group.* Algebraic and Geometric Toplogy 14, 217-227, 2014.

[3] A. Ash. *Small-dimensional classifying spaces for arithmetic subgroups of general linear groups.* Duke Math. J. 51, no. 2, 459-468, 1984.

[4] A. Ash and M. McConnell. *Cohomology at infinity and the well-rounded retract for general linear groups.* Duke Math J. 90, no. 3, 549-576, 1997.

[5] C. Bavard. *Systole et invariant d'Hermite.* J. reine angew. Math. 482, 93-120, 1997.

[6] C. Bavard. *Classes minimales de réseaux et rétractions géométriques équivariantes dans les espaces symétriques.* J. London Math. Soc. (2) 64, 275-286, 2001.

[7] M. Berger. *Les espaces symétriques non compacts.* Annales scientifiques de l'ENS, 1957.

[8] A. Borel. *Introduction aux groupes arithmétiques.* Hermann, 1969.

[9] A. Borel. *Linear Algebraic Groups.* Springer, 1969.

[10] A. Borel and J-P. Serre. *Corners and arithmetic groups.* Comment.Math.Helv.48, 436-491, 1973.

[11] N. Brady, I. Leary, and B. Nucinkis. *On algebraic and geometric dimensions for group with torsion.* London. Math. Soc.(2), 64, 489–500, 2001.

[12] K.S. Brown. *Cohomology of groups.* Graduate texts in Mathematics 77, Springer, 1982.

[13] D. Degrijse and C. Martínez-Pérez. *Dimension invariants for groups admitting a cocompact model for proper actions.* Journal für reine und angewandte Mathematik: Crelle's Journal, 2014.

[14] D. Degrijse and N. Petrosyan. *Geometric dimension of groups for the family of virtually cyclic subgroups.* J. Topol. 7, 697–726, 2014.

[15] D. Degrijse and J. Souto. *Dimension invariants of outer automorphism groups.* arXiv:1602.04354, to appear in Groups, Geometry and Dynamics, 2016.

[16] D.Z. Djokovic. *On real form of complex semisimple Lie algebras.* Aequationes Math. 58, 73-84, 1999.

[17] M. Dutour-Sikiriĉ, G. Ellis, and A. Schürmann. *On the integral homology of* $\mathrm{PSL}(4,\mathbb{Z})$ *and other arithmetic groups.* J. Number Theory 131, no. 12, 2368-2375, 2011.

[18] S. Eilenberg and T.Ganea. *On the Lusternik–Schnirelmann category of abstract groups.* Annals of Mathematics. 2nd Ser. 65, 517-518, 1957.

[19] D. Gabai. *Convergence groups are Fuchsian groups.* Annals of Mathematics, 136, 447-510, 1992.

[20] A. Gray and J.A. Wolf. *Homogeneous spaces defined by Lie groups automorphisms.* I. J. Diff. Geom. 2, 77-114, 1968.

[21] H. Gündogan. *The Component Group of the Automorphism Group of a Simple Lie Algebra and the Splitting of the Corresponding Short Exact Sequence.* Journal of Lie Theory 20, 709-737, 2010.

[22] H. Gündogan. *Classification and structure theory of Lie algebras of smooth section.* Logos Verlag Berlin GmbH, 2011.

[23] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric spaces.* American Mathematical Society, 2001.

[24] H-W. Henn. *The cohomology of* $\mathrm{SL}(3,\mathbb{Z}[1/2])$. K-theory 16, no. 4, 299-359, 1999.

[25] L.K. Hua and I. Reiner. *Automorphisms of the unimodular group.* Trans. Amer. Math. Soc. 71, 331–348, 1951.

[26] L. Ji. *Integral Novikov conjectures and arithmetic groups containing torsion elements.* Communications in analysis and geometry, Volume 15, Number 3, 509–533, 2007.

[27] F.E.A. Johnson. *On the existence of irreducible discrete subgroups in isotypic Lie groups of classical type.* Proc.London Math.Soc., 1988.

[28] A. Karass, A. Pietrowski, and D. Solitar. *Finite and infinite cyclic extensions of free group.* J. Austral. Math. Soc. 16, 458–466, 1973.

[29] A.W. Knapp. *Lie groups beyond an introduction.* Progress in Mathematics, Birkäuser, 2002.

[30] C. Lacoste. *Dimension rigidity of lattices in semisimple Lie groups.* to appear in "Groups, Geometry and Dynamics", arXiv:1607.03652.

[31] C. Lacoste. *On the difficulty of finding spines.* Comptes-rendus Mathématique 356 pp. 141-145, 2018.

[32] I. Leary and B. Nucinkis. *Some groups of type VF.* Invent. Math. 151, 135–165, 2001.

[33] I. Leary and N. Petrosyan. *On dimensions of groups with cocompact classifying spaces for proper actions.* Advances in Mathematics, 311, 730-747, 2017.

[34] W. Lück. *Survey on classifying spaces for families of subgroups.* Infinite Groups: Geometric, Combinatorial and Dynamical Aspects, Springer, 269-322, 2005.

[35] W. Lück and D. Meintrup. *On the universal space for group actions with compact isotropy.* Proc. of the conference "Geometry and Topology" in Aarhus, 293-305, 1998.

[36] R. MacPherson and M. McConnell. *Explicit reduction theory for Siegel modular threefolds.* Invent. math. 111, 575-625, 1993.

[37] G.A. Margulis. *Discrete subgroups of semisimple Lie groups.* Springer-Verlag, 1989.

[38] J. Martinet. *Les réseaux parfaits des espaces euclidiens.* Masson, 1996.

[39] C. Martínez-Pérez. *Euler classes and Bredon cohomology for groups with restricted families of finite torsion.* Math. Z. 2013, 761–780, 2013.

[40] D. Witte Morris. *Introduction to arithmetic groups.* arXiv:math/0106063, 1990.

[41] B. Nucinkis. *On dimensions in Bredon homology.* Homology, Homotopy and Applications Vol.6 Nr.1, 33–47, 2004.

[42] A.L. Onishchik and E.B. Vinberg. *Lie groups and Algebraic groups.* Springer-Verlag, 1980.

[43] A. Pettet and J. Souto. *Minimality of the well-rounded retract.* Geom. Topol. 12, no. 3, 1543–1556, 2008.

[44] A. Pettet and J. Souto. *The spine which was no spine.* Enseign. Math. (2) 54, no. 3-4, 273–285, 2008.

[45] C. Soulé. *The cohomology of* $\mathrm{SL}(3,\mathbb{Z})$. Topology 17, no. 1, 1-22, 1978.

[46] C. Soulé. *Thèse de Doctorat d'État.* Université Paris VII, 1978.

[47] R.G. Swan. *Groups of cohomological dimension one.* Journal of Algebra 12, 585-601, 1969.

[48] K. Vogtmann. *Automorphisms of free groups and outer space.* Geometriae Dedicata, Vol.94 (1), 1-31, 2002.

[49] D. Yasaki. *On the existence of spines for* $\mathbb{Q}$*-rank 1 groups.* Selecta Math. (N.S.) 12, no. 3-4, 541-564, 2006.

[50] I. Yokota. *Exceptional Lie Groups.* arXiv:0902.0431, 2009.