Bifurcations locales et instabilités dans des modèles issus de l'optique et de la mécanique des fluides

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Bifurcations locales et instabilités dans des modèles issus de l'optique et de la mécanique des fluides

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Introduction

Cette thèse présente quelques contributions à l'étude qualitative de solutions d'équations aux dérivées partielles non linéaires, dans des modèles issus de l'optique et de la mécanique des fluides. Nous nous intéressons plus particulièrement à deux types de questions, l'existence de solutions et leurs propriétés structurelles d'une part, et la stabilité temporelle de ces solutions d'autre part. Nous nous demandons notamment si ces deux problèmes peuvent être traités simultanément, en utilisant une même formulation des équations.

Le Chapitre 1 est consacré à l'équation de Lugiato–Lefever, qui est une variante de l'équation de Schrödinger non linéaire et qui a été dérivée dans plusieurs contextes en optique non linéaire. En utilisant des outils issus de la théorie des bifurcations et des formes normales, nous procédons à une étude systématique des solutions stationnaires de cette équation, et prouvons notamment l'existence de solutions périodiques et localisées. Dans le Chapitre 2, nous présentons un critère simple d'instabilité linéaire pour des ondes non linéaires. Nous appliquons ensuite ce résultat aux équations de Lugiato-Lefever, de Kadomtsev-Petviashvili-I et de Davey-Stewartson. Ces deux dernières équations sont des équations modèles dérivées en mécanique des fluides. Enfin, dans le Chapitre 3, nous établissons un critère d'instabilité pour des solutions périodiques de petite amplitude, par rapport à certaines perturbations. Ce résultat est ensuite appliqué à l'équation de Lugiato-Lefever. Nous concluons en évoquant quelques perspectives et problèmes ouverts.

1 Bifurcations locales pour l'équation de Lugiato–Lefever

1.1 Le problème physique

Le Chapitre 1 est consacré à l'étude de l'équation de Lugiato–Lefever

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi\, |\psi|^2 + F, \tag{1}$$

où la fonction inconnue ψ est à valeurs complexes, et dépend du temps t et d'une variable d'espace x et où α, β et F sont des paramètres réels dont la signification physique sera donnée plus loin. Notons d'abord que l'équation (1) est une variante de l'équation Schrödinger non linéaire avec non linéarité cubique, donnée par

$$\frac{\partial \psi}{\partial t} = i\frac{\partial^2 \psi}{\partial x^2} \pm i\psi\, |\psi|^2 ,$$

où les symboles $+$ et $-$ correspondent respectivement aux cas focalisant et défocalisant.

L'équation de Lugiato-Lefever (1) a été dérivée en optique non linéaire dans plusieurs contextes [11, 46] et a fait l'objet de nombreuses études d'un point de vue physique et numérique (voir en particulier les références de [19]). Notre étude de l'équation de Lugiato–Lefever est plus particulièrement motivée par le problème physique évoqué dans [11], qui concerne la formation de peignes de fréquences optiques par effet Kerr dans des résonateurs optiques à modes de galerie. Un résonateur optique est, d'une manière générale, un dispositif expérimental dans lequel des photons sont susceptibles de rester confinés. Dans le contexte de [11], ces photons proviennent d'un laser, et sont envoyés dans le résonateur via une fibre optique. Une partie de ces photons est alors déviée dans le résonateur, et y créent un champ électrique ψ, qui est solution de l'équation (1) (voir la Figure 1 ci-après pour une représentation schématique de ce dispositif expérimental).

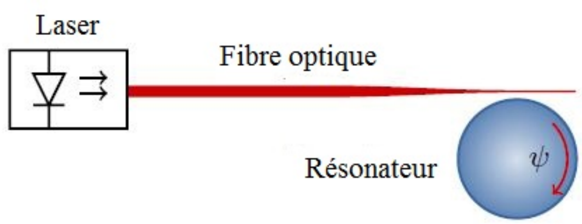


Figure 1 – Représentation schématique d'un résonateur optique à modes de galerie

En revenant à l'équation (1), le paramètre $\alpha \in \mathbb{R}$ représente le décalage du laser, et le paramètre $F > 0$ sa puissance. Le coefficient $\beta \in \mathbb{R}^*$ est quant à lui un paramètre de dispersion. Quitte à effectuer un changement d'échelle pour x, nous supposerons $\beta = \pm 1$. Par la suite, dans le cas $\beta = 1$ la dispersion sera qualifiée de normale, et d'anormale dans le cas $\beta = -1$. Notre but est d'étudier l'existence de solutions stationnaires pour l'équation (1) en fonction des paramètres α et F, qui sont des paramètres physiquement modifiables et que nous considérerons par la suite comme paramètres de bifurcation.

Signalons enfin que les applications technologiques de ce type de dispositif sont potentiellement prometteuses et variées, notamment dans le cadre de la métrologie du temps et de fréquences, ainsi que de la fabrication d'horloges ultra-précises.

1.2 État de l'art

D'un point de vue mathématique, les solutions de l'équation (1) ont été peu étudiées, et de nombreuses questions concernant leur dynamique sont ouvertes. Ces questions concernent notamment l'existence de solutions stationnaires et leur stabilité.

Une première étude d'existence de solutions stationnaires pour l'équation (1) a été faite dans le cas $\beta < 0$ dans [48]. Les auteurs étudient dans un premier temps certaines bifurcations de solutions périodiques, qui émergent à partir de solutions constantes. La seconde partie de cette étude consiste à montrer l'existence d'autres types de solutions stationnaires, notamment localisées. Elle repose sur une formulation de l'équation stationnaire comme système dynamique du premier ordre, comme nous le faisons dans le Chapitre 1. La différence majeure réside dans le choix des paramètres de bifurcation. Ces paramètres ne sont pas les paramètres physiques α et F, mais α et ρ, où $\rho = |\psi|^2$. En utilisant des outils issus de la théorie des formes normales, les auteurs obtiennent l'existence de plusieurs types de solutions stationnaires (en particulier des solutions périodiques et localisées), pour ρ au voisinage de 1. L'étude de la stabilité d'une famille de solutions périodiques trouvée dans [48] est également faite dans [49] (voir la Section 1.6).

Dans [47], les auteurs trouvent des bornes *a priori* pour les solutions stationnaires 2π-périodiques de (1), dans le cas $\beta = 1$. Ces bornes sont explicites, et dépendent des paramètres α et F. Par ailleurs, il est également prouvé que les solutions stationnaires de (1) sont constantes, pour des valeurs du paramètre α suffisament grandes. L'existence de solutions stationnaires 2π-périodiques, satisfaisant des conditions de bords de type Neumann, y est également prouvée, sous réserve que les coefficients α et F appartiennent à des intervalles appropriés. Cette étude utilise des méthodes de bifurcations globales, reposant entre autres sur le théorème de Crandall-Rabinowitz [41, Chapitre 1, Théorème I.5.1].

Par ailleurs, une analyse rigoureuse du problème dépendant du temps a été menée dans [36]. Plus précisément, les auteurs montrent que dans le cas $\beta = 1$, le problème de Cauchy associé à l'équation (1.1) possède une unique solution, pour des données initiales suffisament régulières. De plus, de la même manière que dans [47], les auteurs trouvent des bornes pour les solutions dépendantes du temps.

Dans [19], nous avons étudié, dans les deux cas $\beta = 1$ et $\beta = -1$, l'existence de bifurcations locales pour l'équation stationnaire, en fonction des paramètres α et F. Nous avons utilisé une formulation de l'équation comme système dynamique du premier ordre dans lequel x est la variable d'évolution. Cette analyse a été complétée par des simulations numériques et expériences, qui ont conduit à l'observation de plusieurs types de solutions stationnaires. En particulier, l'existence de solutions périodiques et de plusieurs types d'ondes solitaires a été conjecturée (voir la Figure 2 ci-dessous) pour différentes valeurs de α et F.

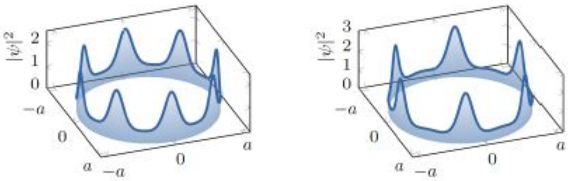
Une étude systématique de ces bifurcations locales a été menée dans [18] et est reprise dans le Chapitre 1 de cette thèse. Nous en donnons ci-après les principales étapes et les résultats obtenus.

1.3 Dynamique spatiale et bifurcations locales

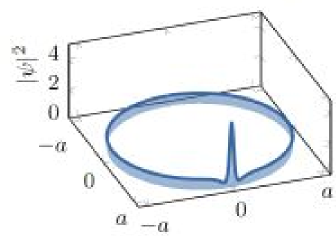
Notre but est de généraliser le travail initié dans [48] et d'étudier l'existence de solutions stationnaires de l'équation (1), c'est-à-dire de solutions de l'équation

$$\beta \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = (i - \alpha)\psi + \psi|\psi|^2 - iF, \tag{2}$$

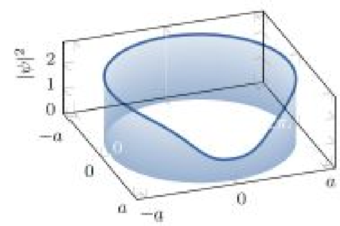
afin de confirmer l'existence des solutions observées dans [19]. Plus précisément, nous procédons à une analyse systématique des bifurcations locales trouvées dans [19] et prouvons l'existence de solutions bornées de l'équation (2), qui bifurquent à partir de solutions constantes. Nous employons pour cela une formulation de l'équation (2) comme système dynamique du



(a) Solutions périodiques de l'équation (2), observées pour $\alpha = 1$, $\beta = -0.04$ et $F^2 = 1.248$ (à gauche) et $\alpha = 1.5$, $\beta = -0.04$ et $F^2 = 1.388$ (à droite).



(b) Onde solitaire pour l'équation (2), observée pour $\alpha = 2$, $\beta = -0.004$ et $F^2 = 1.883$.



(c) Onde solitaire pour l'équation (2), observée pour $\alpha = 2$, $\beta = -0.004$ et $F^2 = 1.883$.

FIGURE 2 – Différents types de solutions stationnaires de l'équation (1) observées numériquement.

premier ordre, ainsi que des outils issus de la théorie des formes normales et des variétés centrales.

Cette approche, appelée dynamique spatiale, a été introduite par K. Kirchgässner [42, 43]. Elle est basée sur une formulation d'une équation aux dérivées partielles stationnaire sous forme d'un système dynamique dans lequel la variable d'évolution est une variable d'espace non bornée. Le problème de Cauchy pour ce système d'évolution est typiquement mal posé, mais des méthodes de la théorie des systèmes dynamiques et des bifurcations permettent d'étudier l'existence de solutions bornées. Cette méthode a été largement utilisée pour prouver l'existence d'ondes progressives dans des contextes variés (voir les références de [27, Chapitre 5]), et en particulier celui des ondes hydrodynamiques de gravité ou de gravité-capillarité de surface (le problème des vagues).

Les outils utilisés ici, à savoir les formes normales et la réduction à une variété centrale, sont décrits plus précisément dans les Sections 1.4 et 1.5. Ils ont eux aussi été utilisés dans des contextes très variés comme les ondes hydrodynamiques [7, 9, 21], ou encore les équations de réaction-diffusion [29, 30, 31, 32].

Solutions constantes

Commençons par quelques rappels sur les solutions constantes de l'équation (1), qui sont bien connues et caractérisées dans la littérature physique [19]. D'un point de vue mathématique, elles sont solutions de l'équation algébrique

$$i\psi|\psi|^2 - (1 + i\alpha)\psi + F = 0. \tag{3}$$

On peut montrer que si $\alpha \leqslant \sqrt{3}$, l'équation (3) possède une unique solution, pour tout $F > 0$. L'équation (1) a donc une unique solution constante. Par ailleurs, lorsque $\alpha > \sqrt{3}$, il existe un intervalle ouvert $(F_-^2(\alpha), F_+^2(\alpha))$, tel que si $F^2 \in (F_-^2(\alpha), F_+^2(\alpha))$, alors l'équation (1) possède trois solutions constantes ψ_1, ψ_2 et ψ_3. Les bornes de cet intervalle peuvent être calculées de manière explicite, et sont données par

$$F_\pm^2(\alpha) = \frac{2\alpha \mp \sqrt{\alpha^2 - 3}}{3} \left(1 + \left(\frac{\sqrt{\alpha^2 - 3} \pm \alpha}{3}\right)^2\right).$$

Par ailleurs, si $F^2 = F_\pm^2(\alpha)$, l'équation a deux solutions constantes, et si $F^2 \notin [F_-^2(\alpha), F_+^2(\alpha)]$, elle en possède une. Ces résultats sont résumés dans le diagramme ci-dessous.

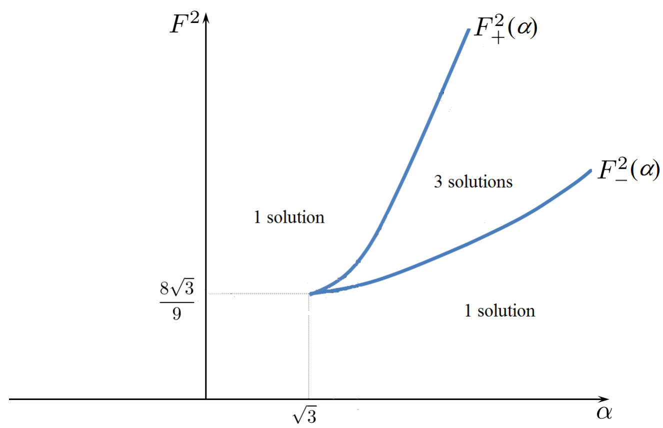


FIGURE 3 – Nombre de solutions constantes de l'équation (1) en fonction des paramètres α et F. Le long des courbes d'équation $F^2 = F_\pm^2(\alpha)$, l'équation a deux solutions.

Système dynamique

La première étape de notre analyse consiste à écrire l'équation différentielle du second ordre (2) dans $\mathbb{C}$ comme système dynamique du premier ordre dans $\mathbb{R}^4$, de la forme

$$\frac{\mathrm{d}U}{\mathrm{d}x} = L_\pm U + G_\pm(U, \alpha, F), \tag{4}$$

où $L_\pm$ est une application linéaire de $\mathbb{R}^4$, $G_\pm$ représente la partie non linéaire du système, et où les symboles $+$ et $-$ correspondent respectivement aux cas $\beta = 1$ et $\beta = -1$. Une propriété qui sera capitale dans l'étude des bifurcations locales menée ci-après est la réversibilité de ce système, plus précisément, il existe une symétrie linéaire $S \neq \mathrm{id}$, telle que $S^2 = \mathrm{id}$, qui anticommute avec le champ de vecteurs dans (4), c'est-à-dire

$$L_\pm S = -SL_\pm, \quad G_\pm(SU, \alpha, F) = -SG_\pm(U, \alpha, F), \quad \forall\ U \in \mathbb{R}^4,\ \forall\ (\alpha, F) \in \mathbb{R}^2.$$

Bifurcations locales

Considérons un système dynamique de la forme

$$\frac{\mathrm{d}U}{\mathrm{d}x} = F(U, \mu), \tag{5}$$

où $\mu \in \mathbb{R}$ est un paramètre, et $F(0,0) = 0$. Ainsi, 0 est un équilibre du système pour $\mu = 0$. D'une manière générale, une bifurcation correspond à un changement dans la structure de l'ensemble des solutions de ce système, lorsque le paramètre μ varie : apparition ou disparition de certaines solutions ou familles de solutions, ou changements dans la stabilité de ces solutions. Nous nous intéressons plus précisément aux bifurcations locales, c'est-à-dire aux bifurcations qui apparaissent dans un voisinage de l'équilibre 0, lorsque μ varie.

Le but du travail initié dans [19] est l'étude systématique des bifurcations locales dans le cas de l'équation (4), afin de déterminer l'ensemble des solutions bornées qui bifurquent depuis les solutions constantes, en fonction des paramètres α et F. Rappelons que d'un point de vue physique, ces paramètres sont modifiables et peuvent être considérés comme des paramètres de bifurcation.

La première étape dans l'étude des bifurcations locales consiste à déterminer les valeurs de α^* et de F^* pour lesquelles une bifurcation locale peut avoir lieu. Le théorème de Hartmann-Grobman [12, Chapitre 1, Théorème 1.27] affirme que si le spectre imaginaire pur de la matrice $L_\pm$ est vide, alors les portraits de phase de l'équation (5) sont qualitativement les mêmes lorsque μ varie au voisinage de 0. En particulier, aucune bifurcation n'a lieu dans ce cas. Si ce spectre est non vide, une bifurcation peut apparaître.

Le calcul du spectre imaginaire pur des matrices $L_\pm$ effectué dans [19] (voir la Figure 4), montre que, suivant les valeurs de α^* et F^*, il existe quatre types de bifurcations locales pour l'équation de Lugiato-Lefever stationnaire (2).

Plus précisément, on trouve :

- une bifurcation de type $(i\omega)^2$, lorsque les matrices $L_\pm$ possèdent une paire de valeurs propres imaginaires pures $\pm i\omega$ algébriquement doubles et géométriquement simples ;
- une bifurcation de type $0^2(i\omega)$, lorsque les matrices $L_\pm$ possèdent une paire de valeurs propres algébriquement simples $\pm i\omega$, et où 0 est une valeur propre algébriquement double et géométriquement simple ;
- une bifurcation de type 0^2 lorsque 0 est une valeur propre algébriquement double et géométriquement simple de $L_\pm$;
- une bifurcation de codimension 2, de type 0^4, lorsque 0 est une valeur propre de $L_\pm$ de multiplicité algébrique 4 et géométriquement simple.

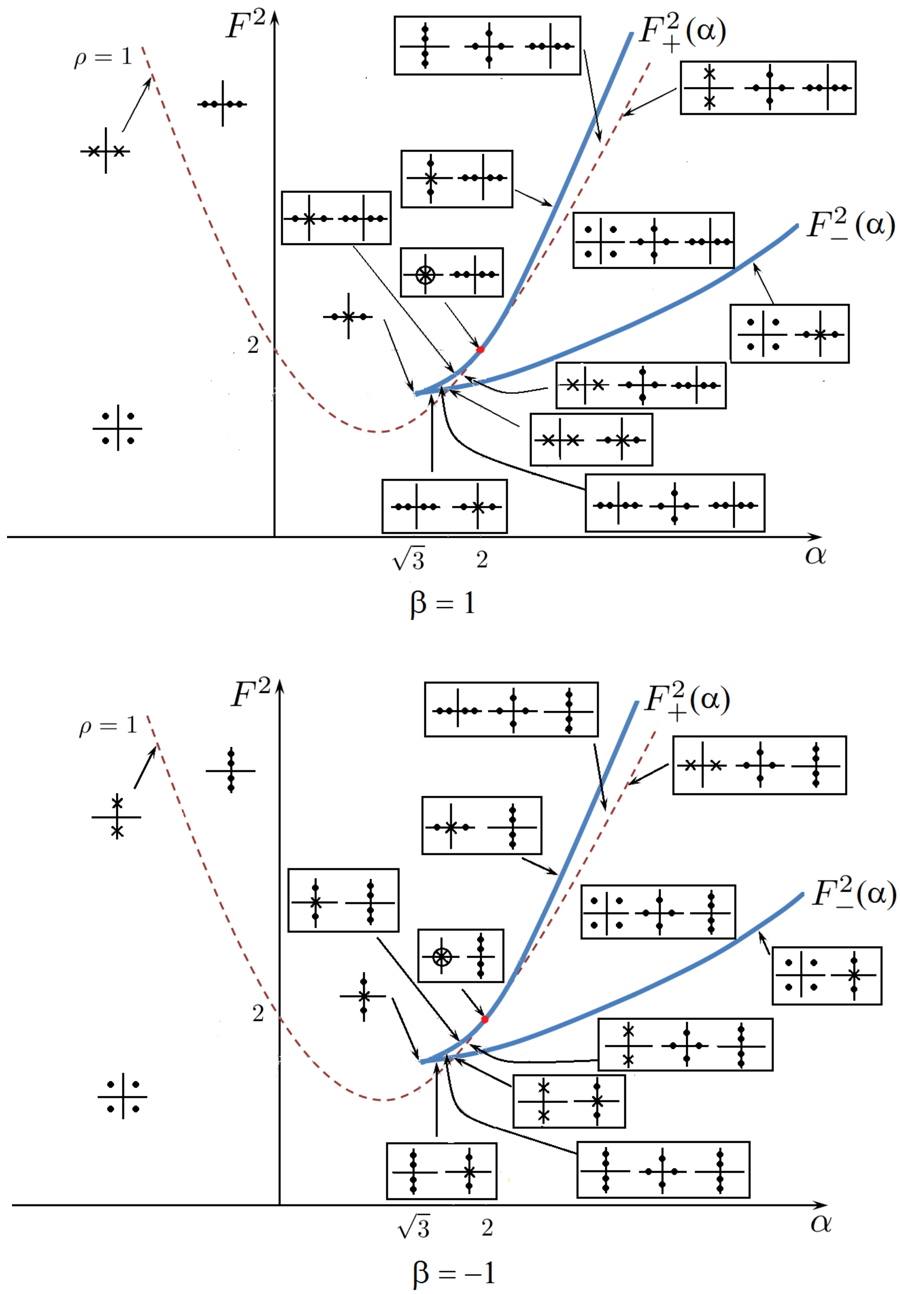


FIGURE 4 – Diagrammes montrant la localisation du spectre des matrices L_+ (en haut) et L_- (en bas). Lorsque l'équation (1) possède plusieurs solutions constantes, les pictogrammes des spectres correspondants sont rangés de gauche à droite par modules croissants.

Ces bifurcations apparaissent le long des courbes représentées dans la Figure 5. Notons que les bifurcations $(i\omega)^2$, $0^2(i\omega)$ et 0^2 apparaissent le long de courbes dans le plan des paramètres (α, F^2), et mettent en jeu un seul paramètre de bifurcation. Ces bifurcations sont dites de codimension 1. En revanche, la bifurcation 0^4 apparait en un point, et met en jeu deux

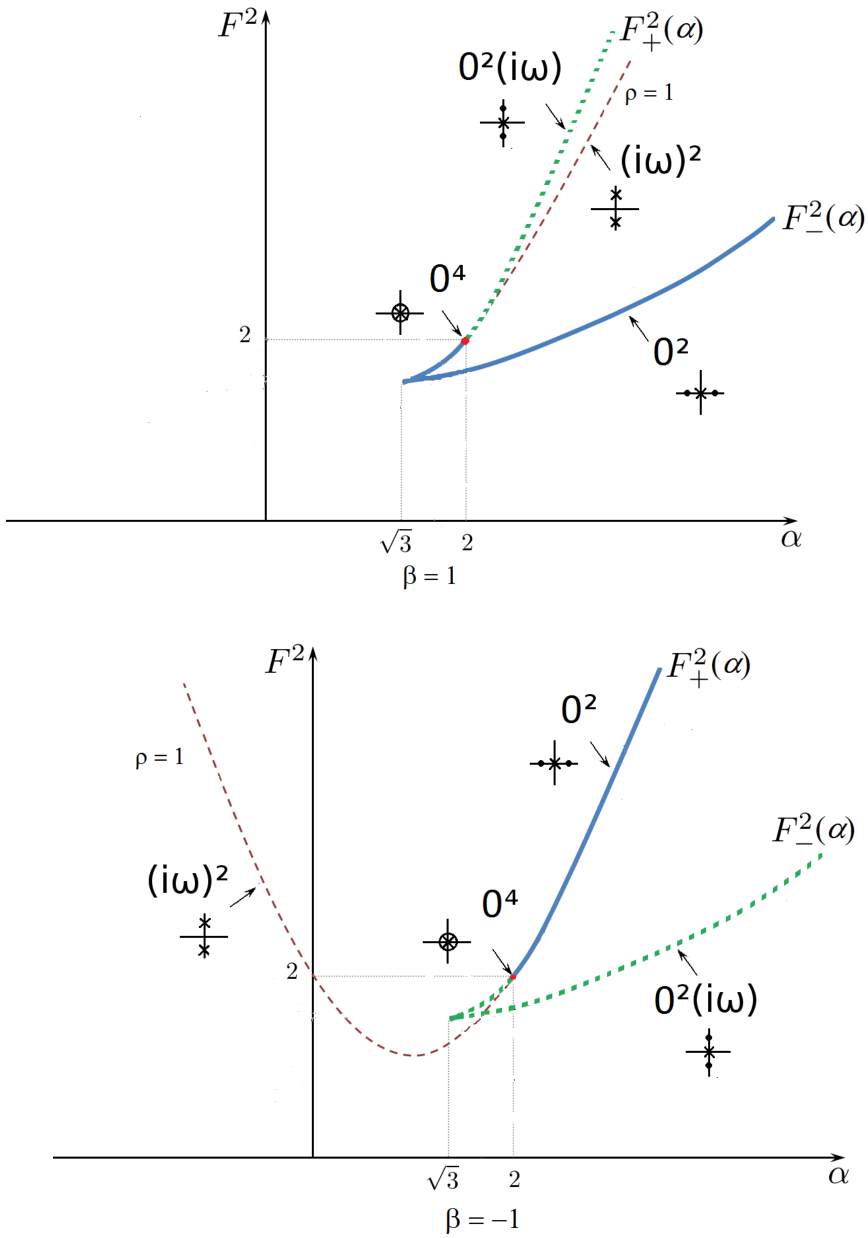


FIGURE 5 – Diagrammes de bifurcation dans les cas $\beta = 1$ (en haut) et $\beta = -1$ (en bas). Les valeurs propres simples sont indiquées par un point, les valeurs propres doubles par une croix et les valeurs propres de multiplicité 4 par une croix entourée.

paramètres de bifurcation. Il s'agit alors d'une bifurcation de codimension 2.

Dans notre étude, nous choisissons de fixer le paramètre F^*, et de prendre α comme paramètre de bifurcation. Plus, précisément, nous posons $\alpha = \alpha^* + \mu$, $F = F^*$, où le paramètre de bifurcation $\mu \in \mathbb{R}$ est supposé petit. Le système (4) s'écrit alors sous la forme (5)

$$U_x = L_\pm U + R_\pm(U, \mu). \tag{6}$$

La dynamique du système (6) dépend du type de bifurcation. Dans la suite, nous nous restreignons à l'étude des trois bifurcations de codimension 1, qui sont bien connues et caractérisées dans la littérature (voir en particulier [27, Chapitre 4]).

1.4 Formes normales

L'étude des bifurcations $(i\omega)^2$ et $0^2(i\omega)$ repose sur le calcul d'une forme normale du système (6). Nous rappelons ici la méthode générale, ainsi que les résultats obtenus pour l'équation de Lugiato-Lefever.

La méthode

Considérons un système de la forme

$$\frac{\mathrm{d}U}{\mathrm{d}x} = LU + R(U, \mu), \tag{7}$$

où L est une application linéaire de $\mathbb{R}^n$ et $R : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ une application non linéaire suffisament régulière, et où le paramètre de bifurcation μ est supposé petit.

On suppose que 0 est un équilibre du système (7) pour $\mu = 0$, c'est-à-dire $R(0, 0) = 0$, et que $\mathrm{d}_U R(0, 0) = 0$. Le but de la théorie des formes normales est de trouver une forme simplifiée pour la partie non linéaire R de (7), permettant ensuite de déterminer l'ensemble des solutions bornées du système (7) qui sont proches de 0, pour μ assez petit. L'idée générale consiste à trouver un changement de variable polynomial qui simplifie le développement de Taylor de la partie non linéaire du système (7) (voir par exemple [15]). Plus précisément, d'après [27, Chapitre 2, Théorème 2.2], pour tout entier $p \geqslant 2$, il existe un polynôme $\Phi_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ de degré p tel que le changement de variable

$$U = V + \Phi_\mu(V)$$

transforme le système (7) au voisinage de 0 en le système

$$\frac{\mathrm{d}V}{\mathrm{d}x} = LV + N_\mu(V) + \rho(V, \mu), \tag{8}$$

où $N_\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ est un polynôme de degré p, et le reste ρ est petit, $\rho(V, \mu) = o\left(\|V\|^p\right)$. On a de plus l'égalité suivante, pour tout V au voisinage de 0,

$$N_\mu(e^{xL^*}V) = e^{xL^*}N_\mu(V), \tag{9}$$

où L^* désigne l'adjoint de L. Cette dernière propriété s'utilise notamment pour calculer de manière explicite le système sous forme normale (8).

Par ailleurs, dans le cas où le système (7) est réversible, c'est-à dire lorsqu'il existe une symétrie qui anticommute avec le champ de vecteurs dans (7), alors le système (8) est également réversible avec la même symétrie (voir [27, Chapitre 3, Théorème 3.4]).

Les formes normales facilitent l'étude de la dynamique du système (7) au voisinage de 0. L'idée consiste à étudier d'abord le système tronqué

$$\frac{\mathrm{d}V}{\mathrm{d}x} = LV + N_\mu(V). \tag{10}$$

Dans les applications, ce système est très souvent tronqué à l'ordre $p = 2$ ou $p = 3$. Par la suite, des techniques perturbatives permettent, selon les cas, de montrer la persistance de cette dynamique tronquée, et en particulier de solutions bornées pour le système complet (7). Des propriétés de symétrie, comme la réversibilité, s'avèrent très utiles dans ces arguments (voir notamment [27, Chapitre 4]).

La bifurcation $(i\omega)^2$

Dans le cas de la bifurcation $(i\omega)^2$, on peut montrer, en utilisant notamment la caractérisation (9), que le système tronqué à l'ordre 3 s'écrit

$$\frac{\mathrm{d}A}{\mathrm{d}x} = i\omega^* A + B \tag{11}$$

$$\frac{\mathrm{d}B}{\mathrm{d}x} = i\omega^* B + a_2 \mu A + b_2 |A|^2 A \tag{12}$$

et que sa dynamique dépend du signe des coefficients a_2 et b_2. Ces coefficients peuvent être calculés en suivant, par exemple, la méthode donnée dans [27, Chapitre 4]. En appliquant ensuite les résultats de [27, Chapitre 4, Théorème 3.21], nous obtenons le théorème d'existence suivant.

Théorème 1.1. *Supposons que $\beta = 1$ et que le point (α^*, F^{*2}) appartienne à la courbe correspondant à la bifurcation $(i\omega)^2$ dans la Figure 5. Pour tout $\mu \in \mathbb{R}$ assez petit, le système dynamique (6) possède un équilibre symétrique, une famille d'orbites périodiques à un paramètre, une famille d'orbites quasipériodiques à deux paramètres. De plus, lorsque $\mu > 0$, le système dynamique (6) possède une paire d'orbites homoclines réversibles à l'équilibre symétrique.*

Nous obtenons un résultat analogue dans le cas $\beta = -1$. Dans ce cas, il se trouve que le coefficient b_2 change de signe pour $\alpha^* = 41/30$, et la dynamique du système (6) est différente suivant que $\alpha^* > 41/30$ ou que $\alpha^* < 41/30$.

Théorème 1.2. *Supposons que $\beta = -1$ et que (α^*, F^{*2}) appartienne à la courbe correspondant à la bifurcation $(i\omega)^2$ dans la Figure 5.*

(i) Si $\alpha^* > 41/30$, alors pour tout $\mu \in \mathbb{R}$ assez petit, le système dynamique (6) possède les mêmes types de solutions que dans le Théorème 1.1.

(ii) Si $\alpha^* < 41/30$, alors pour tout μ assez petit tel que $a_2\mu < 0$, le système dynamique (6) possède un équilibre symétrique, une famille à un paramètre d'orbites périodiques, une famille à deux paramètres d'orbites quasipériodiques, et une famille à un paramètre d'orbites homoclines aux orbites périodiques. Si $a_2\mu > 0$, alors le système dynamique (6) possède un équilibre symétrique et pas d'autre solution bornée.

La bifurcation $0^2(i\omega)$

Dans le cas de la bifurcation $0^2(i\omega)$, le système tronqué à l'ordre 3 s'écrit

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}x} &= B\\ \frac{\mathrm{d}B}{\mathrm{d}x} &= a_1\mu + b_1A^2 + c_1|C|^2\\ \frac{\mathrm{d}C}{\mathrm{d}x} &= i\omega^*C,\end{aligned}$$

et sa dynamique dépend du signe des coefficients a_1, b_1 et c_1. D'après [27, Chapitre 4, Théorème 3.10], nous obtenons pour l'équation de Lugiato-Lefever le théorème d'existence suivant (les énoncés précis seront donnés dans le Chapitre 1, Théorèmes 1.3.2 et 1.3.3).

Théorème 1.3. *Supposons que $\beta = 1$ et que (α^*, F^{*2}) appartienne à la courbe correspondant à la bifurcation $0^2(i\omega)$ dans la Figure 5. Pour tout $\mu > 0$ assez petit, le système dynamique (6) possède deux équilibres, un centre et un point-selle, ainsi que deux familles d'orbites périodiques qui tendent vers les équilibres lorsque leur taille tend vers 0. Pour toute orbite périodique dans la famille qui tend vers le point-selle, de taille $r > r_*(\mu)$ pas trop petite, il existe une paire d'orbites homoclines réversibles connectant cette orbite avec elle-même. Par ailleurs, il existe une troisième famille d'orbites périodiques, ainsi que des orbites quasipériodiques. Si $\mu < 0$, alors le système dynamique (6) n'a pas de solutions bornées.*

Dans le cas $\beta = -1$, deux régimes de paramètres pour α^* et F^* doivent être envisagés. Lorsque le point (α^*, F^{*2}) appartient à la courbe correspondant à la bifurcation $0^2(i\omega)$ dans la Figure 5, avec $F^{*2} = F_+^*(\alpha^*)$, le système (6) possède les mêmes types de solutions que dans le Théorème 1.3. Lorsque $F^{*2} = F_-^2(\alpha^*)$, nous avons le résultat suivant.

Théorème 1.4. *Supposons que $\beta = -1$ et que (α^*, F^{*2}) appartienne à la courbe correspondant à la bifurcation $0^2(i\omega)$ dans la Figure 5, avec $F^{*2} = F_-^*(\alpha^*)$.*

(i) Pour tout $\mu > 0$ assez petit, le système dynamique (6) possède deux familles d'orbites périodiques. Pour toute orbite périodique dans l'une de ces deux familles, il existe une paire d'orbites homoclines réversibles connectant cette orbite avec elle-même. Par ailleurs, il existe des orbites quasipériodiques.

(ii) Pour tout $\mu < 0$ assez petit, le système dynamique (6) possède les mêmes types de solutions que dans le Théorème 1.3, pour $\mu > 0$.

1.5 Réduction à une variété centrale

Dans le cas de la bifurcation 0^2, le spectre des matrices $L_\pm$ n'est pas entièrement situé sur l'axe imaginaire pur. Plus précisément, il est constitué de 0, qui est une valeur propre algébriquement double et géométriquement simple, ainsi que d'une paire de valeurs propres réelles opposées. L'étude de cette bifurcation repose alors sur une réduction à une variété centrale. L'idée de cette méthode est de trouver une variété localement invariante, qui contient l'ensemble des solutions bornées. Ici, cette variété sera de dimension 2, ce qui correspond à la multiplicité algébrique de la valeur propre 0.

La méthode

Reprenons le système (7). On suppose comme précedemment que 0 est un équilibre de (7) pour $\mu = 0$, et que $\mathrm{d}_U R(0,0) = 0$. On fait également l'hypothèse que le spectre imaginaire pur de L est non vide. Notons P_0 le projecteur spectral associé. On a alors la décomposition spectrale

$$\mathbb{R}^n = X_0 \oplus X_1,$$

avec $X_0 = P_0(\mathbb{R}^n)$ et $X_1 = (\mathrm{id} - P_0)(\mathbb{R}^n)$. Sous certaines hypothèses de régularité du champ de vecteurs de (7), d'après, par exemple, [27, Chapitre 2, Théorème 3.3]), il existe une variété localement invariante de dimension égale à celle de X_0, qui contient toutes les solutions bornées et petites du système (7), pour tout μ assez petit. De plus, ces solutions sont de la forme

$$U = U_0 + \Psi(U_0, \mu),$$

où $\Psi : X_0 \times \mathbb{R} \to X_1$ est une application assez régulière, et U_0 est solution du système réduit

$$\frac{\mathrm{d}U_0}{\mathrm{d}x} = L_0 U + P_0 R(U_0 + \Psi(U_0, \mu), \mu), \tag{13}$$

L_0 étant la restriction de L à X_0. Pour étudier l'existence de solutions bornées pour le système (7), il suffit donc d'étudier le système (13).

Signalons que cette méthode de réduction à une variété centrale peut également s'appliquer dans des espaces de dimension infinie [27, Chapitre 2, Théorème 2.9]. Des hypothèses spectrales supplémentaires sont alors requises. Le spectre imaginaire pur de l'opérateur L doit en particulier être constitué d'un nombre fini de valeurs propres isolées. Sous des hypothèses appropriées, ce résultat permet de réduire localement un système en dimension infinie à un système d'équations différentielles de dimension finie. La dimension de ce système est égale à la somme des multiplicités algébriques des valeurs propres imaginaires pures de L.

La bifurcation 0^2

Dans le cas de la bifurcation 0^2, le système réduit (7) peut être calculé en utilisant les arguments de [27, Chapitre 2]. Il se trouve que le système obtenu est déjà sous forme normale, et le système tronqué à l'ordre 2 correspondant est de la forme

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} &= B \\ \frac{\mathrm{d}B}{\mathrm{d}x} &= a\mu + bA^2. \end{aligned}$$

La dynamique de ce système dépend du signe des coefficients a et b. D'après [27, Chapitre 4, Théorème 1.8], nous obtenons pour l'équation de Lugiato-Lefever le résultat d'existence suivant, valable dans les deux cas $\beta = 1$ et $\beta = -1$.

Théorème 1.5. *Supposons que le point (α^*, F^{*2}) appartienne à la courbe correspondant à la bifurcation 0^2 dans la Figure 5. Pour tout μ assez petit tel que $ab\mu < 0$, le système* *dynamique (6) possède deux équilibres symétriques, un point-selle et un centre. Le centre est entouré par une famille à un paramètre d'orbites périodiques, qui tendent vers une orbite homocline connectant le point-selle avec lui-même, quand la période tend vers l'infini. Si* $ab\mu > 0$*, alors le système dynamique (6) n'a pas de solutions bornées.*

1.6 Comparaison avec les résultats expérimentaux et les simulations numériques

L'analyse des bifurcations locales menée ici permet de trouver les valeurs des paramètres α et F pour lesquelles des solutions stationnaires bornées de l'équation (1) bifurquent depuis des solutions constantes. Nous obtenons en particulier l'existence de solutions périodiques et localisées. Il se trouve que certaines de ces solutions ont été observées expérimentalement et numériquement [19]. Plus précisément, dans les cas $\beta = 1$ et $\beta = -1$, pour des valeurs de α et F proches de celles conduisant aux bifurcations $(i\omega)^2$, des solutions périodiques sont observées, ce qui confirme expérimentalement une partie des résultats des Théorèmes 1.1 et 1.2 (voir les Figures 2 (a) et 6). Dans les cas $\beta = 1$ et $\beta = -1$, des ondes solitaires sont également observées, avec une forme correspondant aux ondes solitaires trouvées dans la bifurcation de type 0^2 (voir le Théorème 1.5, ainsi que les Figures 2 (c) et 6). Un autre type d'ondes solitaires a été observé dans le cas $\beta = -1$, correspondant aux orbites homoclines trouvées dans les bifurcations de type $0^2(i\omega)$ (voir le Théorème 1.3, ainsi que la Figure 2 (b)). En particulier, la différence d'allure entre les deux types d'ondes solitaires observées s'explique par le fait qu'elles proviennent de deux bifurcations distinctes. Il y a donc une bonne corrélation entre nos résultats théoriques d'existence et les simulations numériques et expériences de [19].

Cependant, nous mettons également en évidence des solutions stationnaires qui n'ont pas été observées numériquement et/ou expérimentalement, en particulier les solutions périodiques trouvées dans les bifurcations 0^2 et $0^2(i\omega)$, ainsi que les ondes solitaires trouvées dans la bifurcation $(i\omega)^2$. Il pourrait être judicieux d'étudier la stabilité de ces solutions, le fait qu'elles soient difficiles à observer pouvant indiquer qu'elles sont instables.

D'autre part, des solutions périodiques en temps ont été observées dans les expériences et simulations de [19] (voir la Figure 6 ci-après). Parmi ces solutions, une catégorie d'ondes localisées en espace, appelées breathers, sont particulièrement intéressantes d'un point de vue physique [19]. Il se trouve que l'on peut étendre l'analyse de bifurcations que nous avons menée pour inclure des solutions périodiques en temps. Nous montrons cependant dans la Section 1.5 du Chapitre 1 que cette analyse n'apporte pas de résultats nouveaux, car les solutions obtenues sont constantes en temps, et elles coïncident avec celles que nous avons trouvées précedemment. L'existence de ces solutions pourrait être établie de manière rigoureuse via une étude de bifurcations secondaires.

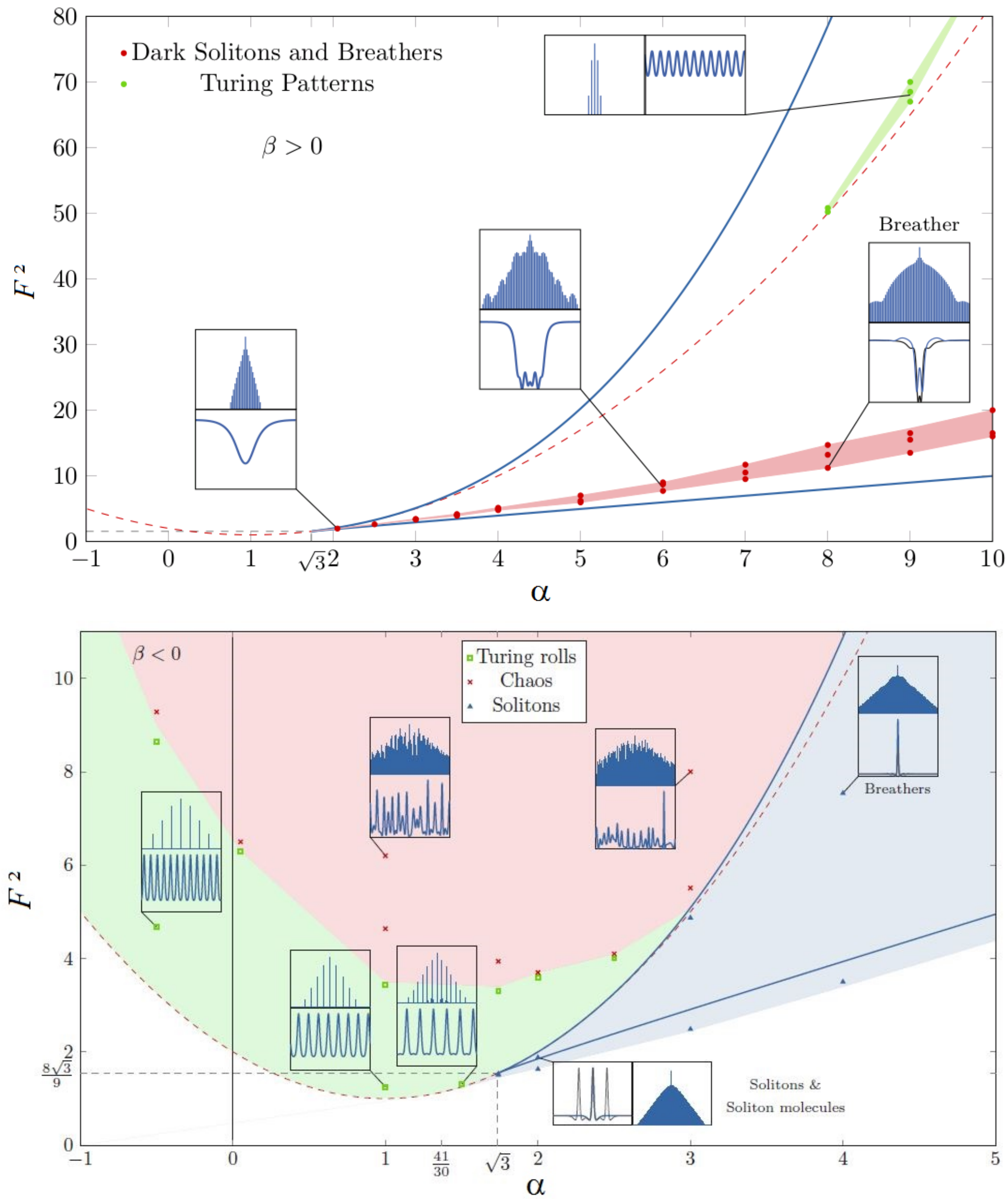


FIGURE 6 – Diagrammes de bifurcations obtenus numériquement dans les cas $\beta > 0$ (en haut) et $\beta < 0$ (en bas).

1.7 Résultats de stabilité

La question de la stabilité des solutions stationnaires est un problème important, tant d'un point de vue mathématique que physique. Les résultats obtenus pourraient notamment expliquer le fait que certaines solutions dont nous avons montré l'existence n'ont pas été observées expérimentalement. Nous discutons ici quelques résultats existants de stabilité pour l'équation (1), ces questions étant néanmoins largement ouvertes.

Tout d'abord, la stabilité des solutions constantes de l'équation (1) vis-à-vis de perturbations constantes est bien connue d'un point de vue physique [19]. Plus précisément, lorsque l'équation (1) possède une ou deux solutions constantes, ces solutions sont stables par rapport à des perturbations constantes, et lorsqu'elle possède trois solutions constantes ψ_1, ψ_2 et ψ_3, avec $|\psi_1| < |\psi_2| < |\psi_3|$, alors ψ_1, ψ_3 sont stables, et ψ_2 est stable. Nous étudions également la stabilité spectrale de ces solutions vis-à-vis de perturbations localisées ou bornées (voir la Section 2.1 ci-après pour quelques rappels sur la notion de stabilité spectrale). Les résultats obtenus dans ce cas contrastent avec ceux obtenus dans le cadre des perturbations constantes, et diffèrent notamment dans les cas $\beta = 1$ et $\beta = -1$. Plus précisément, dans le cas $\beta = 1$, lorsque l'équation (1) possède une solution constante, celle-ci est stable. Lorsque qu'elle possède deux solutions constantes ψ_1 et ψ_2, avec $|\psi_1| < |\psi_2|$, ψ_2 est toujours stable, et ψ_1 devient instable si $|\psi_1| > 1$ et $\alpha > 2$. Enfin, dans le cas où (1) a trois solutions constantes ψ_1, ψ_2 et ψ_3, avec $|\psi_1| < |\psi_2| < |\psi_3|$, ψ_2 est instable, ψ_3 est stable et ψ_1 devient instable si $|\psi_1| > 1$ et $\alpha > 2$. Dans le cas $\beta = -1$, le résultat est plus simple à décrire, puisque toutes les solutions constantes de module strictement supérieur à 1 sont instables, sauf si $\alpha > 2$ et $1 < |\psi| < \rho_+(\alpha)$. Notons que l'instabilité vis-à-vis de perturbations non constantes de solutions qui sont stables par rapport à des perturbations constantes est une instabilité connue sous le nom d'instabilité de Turing.

La stabilité des solutions non constantes, en particulier celle des solutions stationnaires trouvées dans le Chapitre 1, est un problème largement ouvert. Les seuls résultats existants concernent une famille de solutions périodiques, obtenue avec la formulation de [48] dans un voisinage de $(\rho, \alpha) = (1, 41/30)$ dans le cas $\beta = -1$. Les perturbations considérées sont des perturbations périodiques ayant la même période que celle de la solution périodique, et les résultats de [48, Théorème 3.1] et de [49, Théorème 1.1] montrent que ces solutions sont stables si $\alpha < 41/30$ et instables si $\alpha > 41/30$.

2 Un critère d'instabilité linéaire

Nous exposons dans cette section les résultats du Chapitre 2. Le but de ce chapitre est d'obtenir des conditions suffisantes simples d'instabilité linéaire pour des solutions stationnaires d'équations aux dérivées partielles non linéaires. Nous partons d'une formulation dynamique du système, dans laquelle la variable d'évolution est une variable spatiale, contrairement aux études usuelles où cette variable est une variable temporelle. Le résultat principal obtenu dans ce chapitre nous permet alors d'étudier l'instabilité linéaire de plusieurs types de solutions dans différentes équations modèles.

2.1 Instabilité spectrale et linéaire

Les études de stabilité de solutions stationnaires des équations aux dérivées partielles reposent le plus souvent sur une formulation de l'équation sous la forme

$$U_t = G(U), \tag{14}$$

dans laquelle l'inconnue U appartient à un certain espace de fonctions dépendant des variables spatiales. Dans cette formulation, les solutions stationnaires sont des équilibres U_* du système (14), vérifiant $G(U_*) = 0$.

Une question naturelle et importante concerne la stabilité de ces équilibres. Intuitivement, une solution U_* est stable si les solutions du système (14) restent proches de U_*, pour des données initiales proches de U_*, c'est-à-dire de la forme $U_0 = U_* + V_0$, avec V_0 petit. Dans le cas contraire, U_* est instable. Pour les équations aux dérivées partielles, il existe plusieurs notions de stabilité, dont les notions de stabilité spectrale et linéaire, qui concernent les propriétés du système linéarisé, et différentes notions de stabilité non linéaire (orbitale, asymptotique) qui concernent les propriétés du système complet non linéaire.

Nous nous restreignons ici aux questions de stabilité spectrale et linéaire, qui concernent donc le système linéarisé

$$U_t = \mathcal{L}U, \tag{15}$$

où l'opérateur $\mathcal{L}$ désigne la différentielle $\mathrm{d}G(U_*)$ de G en U_*. Suivant, par exemple, [39, Chapitre 4, Définition 4.1.7], l'équilibre U_* est dit **spectralement stable** si le spectre $\sigma(\mathcal{L})$ est tel que

$$\sigma(\mathcal{L}) \subset \{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) \leqslant 0\},$$

et **spectralement instable** sinon. Cette notion donne des conditions nécessaires de stabilité, mais pas de conditions suffisantes, cela même dans le cas simple où l'équation (14) est une équation différentielle.

Supposons maintenant que U_* soit spectralement instable, au sens de la définition précédente. Il existe donc $\lambda \in \sigma(\mathcal{L})$ tel que $\mathrm{Re}(\lambda) > 0$. Si λ est une valeur propre de $\mathcal{L}$, de vecteur propre associé U_0, la fonction U définie par

$$U(t) = e^{\lambda t} U_0$$

est solution du système linéarisé (15), et cette solution est exponentiellement croissante en temps. L'équilibre U_* est alors également **linéairement instable**. Nous utilisons cette propriété pour caractériser l'instabilité linéaire dans notre approche, qui repose sur une formulation différente de (14).

2.2 Dynamique spatiale

Dans notre approche des questions d'instabilité, nous considérons des systèmes écrits sous la forme

$$U_x = DU_t + F(U), \tag{16}$$

où l'inconnue U, qui dépend du temps t et d'une variable d'espace non bornée x, prend ses valeurs dans un espace de Banach, D est un opérateur linéaire et F est une application non linéaire suffisament régulière. Notons d'abord que cette formulation est inhabituelle dans l'étude d'un problème de stabilité, qui est classiquement écrit comme problème d'évolution en temps, de la forme (14). Elle est inspirée de l'approche dynamique spatiale utilisée dans les études d'existence de solutions, comme celle faite pour l'équation de Lugiato-Lefever dans le Chapitre 1. En particulier, la formulation (16) permet d'étudier l'existence de solutions stationnaires pour le système (16), c'est-à-dire de solutions du système

$$U_x = F(U),$$

en utilisant notamment les outils issus de la théorie des bifurcations et des formes normales présentés dans la Section 1. On peut dès lors se demander si une telle formulation peut permettre d'obtenir des informations sur la stabilité des solutions obtenues. L'avantage serait alors d'étudier simultanément l'existence de solutions et leur stabilité en utilisant une seule formulation du problème.

Nous montrons qu'il est effectivement possible de trouver des conditions suffisantes d'instabilité linéaire pour les solutions indépendantes de x du système (16), par rapport à des perturbations dépendantes de x. Plus précisément, le système linéarisé autour d'une solution U_* indépendante de x, vérifiant donc $F(U_*) = 0$, s'écrit sous la forme

$$U_x = DU_t + LU, \tag{17}$$

où $L = \mathrm{d}F(U_*)$ désigne la différentielle de F en U_*. Cette solution U_* est **linéairement instable** par rapport à des perturbations dépendantes de x, si le système (17) possède des solutions de la forme

$$U(x,t) = e^{\lambda t}V(x), \tag{18}$$

où λ est un nombre complexe de partie réelle strictement positive et V une fonction bornée. Nous donnons ci-après des conditions suffisantes pour obtenir ce type d'instabilité, en construisant des perturbations $U(x,t)$ de la forme (18), avec V une fonction périodique.

Notons que dans le cas où U_* est une fonction dépendant d'autres variables spatiales $y_1, \dots y_n$, cette instabilité est également appelée **instabilité transverse**.

2.3 Le résultat principal

Outre des hypothèses techniques simples sur les opérateurs D et L dans (17), notre première hypothèse principale concerne l'existence d'une paire de valeurs propres imaginaires pures simples et isolées $\pm ik_*$, $k_* > 0$, pour l'opérateur L. Cette hypothèse est complétée par une hypothèse de symétrie du système (17), plus précisément nous supposons que ce système est réversible, c'est-à-dire qu'il existe une symétrie linéaire S, $S \neq \mathrm{id}$, $S^2 = \mathrm{id}$, qui anticommute avec D et L ($SD = -DS$ et $SL = -LS$).

Le théorème suivant, qui a fait l'objet de la publication [16], affirme que ces hypothèses impliquent l'instabilité linéaire de U_*.

Théorème 2.1. *Sous les hypothèses 2.1.1 données dans le Chapitre 2, pour tout $\lambda > 0$ suffisamment petit, le système (17) admet une solution de la forme*

$$U(x,t) = e^{\lambda t}V(x),$$

où la fonction V est continue et périodique. Par conséquent, U_* est linéairement instable.

La preuve de ce résultat consiste dans un premier temps à remarquer qu'il suffit de montrer que l'opérateur $\lambda D + L$ possède une paire de valeurs propres imaginaires pures simples, si $\lambda \in \mathbb{R}$. Un argument de perturbation montre ensuite que pour λ réel suffisament petit, il existe deux voisinages des valeurs propres ik_* et $-ik_*$ de l'opérateur L, contenant chacun une valeur propre simple de l'opérateur $\lambda D + L$, notées $ik_\pm(\lambda)$. D'autre part, la réversibilité du système (17), ainsi que le fait que l'opérateur $\lambda D + L$ soit réel, impliquent que le spectre de $\lambda D + L$ est symétrique par rapport à l'axe imaginaire et à l'axe réel. On en déduit alors que les valeurs propres $ik_\pm(\lambda)$ sont complexes conjuguées et imaginaires pures.

Notons que ce résultat peut être généralisé au cas l'opérateur L possède une paire de valeurs propres imaginaires pures et isolées, de multiplicité algébrique impaire. Il peut également s'appliquer pour des systèmes de la forme

$$U_x = PU + F(U), \quad P = \sum_{k=1}^{n} D_k \partial_t^{(k)},$$

où $D_1, \ldots, D_n$ sont des opérateurs linéaires.

2.4 Applications

Nous appliquons le résultat obtenu dans le Théorème 2.1 pour étudier l'instabilité linéaire de solutions stationnaires dans trois exemples différents : les solutions constantes de l'équation de Lugiato-Lefever, les ondes progressives solitaires et périodiques de l'équation de Kadomtsev-Petviashvili-I, et une famille de solutions périodiques des équations de Davey-Stewartson. Notons que ce résultat a été récemment utilisé dans [22] pour montrer l'instabilité linéaire d'une onde solitaire des équations d'Euler par rapport à des perturbations périodiques de période grande, et dans [23] pour montrer l'instabilité linéaire d'une onde solitaire des équations de Davey-Stewartson.

Solutions constantes de l'équation de Lugiato-Lefever

Le critère d'instabilité du Théorème 2.1 permet de retrouver les résultats d'instabilité des solutions constantes de l'équation de Lugiato-Lefever (1) par rapport à des perturbations localisées ou bornées décrits dans la Section 1.7. L'équation (1) s'écrit en effet sous la forme (16), et dans ce cas l'opérateur L dans (16) correspond aux matrices $L_\pm$ dans (4), dont le spectre est connu. De plus, le système obtenu est réversible, tout comme le système stationnaire (4). Le Théorème 2.1 permet alors d'affirmer qu'une solution constante ψ^* est linéairement instable dès que les matrices $L_\pm$ possèdent au moins une paire de valeurs propres imaginaires pures simples. Les valeurs des paramètres α et F qui mènent à une telle instabilité peuvent être lues dans les diagrammes de bifurcations de la Figure 4, déjà utilisés pour déterminer les bifurcations locales dans la Section 1.3.

Ondes progressives de l'équation de Kadomtsev-Petviashvili I

Considérons l'équation de Kadomtsev-Petviashvili I (KP-I),

$$(u_t + u_{xxx} + uu_x)_x - u_{yy} = 0, \tag{19}$$

où l'inconnue u dépend ici du temps t, et des variables d'espace x et y. Notons que l'équation (19) est une version 2D de l'équation de Korteweg-de Vries

$$u_t + u_{xxx} + uu_x = 0. \tag{20}$$

L'équation (19) a été dérivée pour étudier l'instabilité transverse des ondes solitaires et périodiques de l'équation de Korteweg-de Vries [38]. Il existe également une autre version de l'équation de KP-I, appelée équation de KP-II, qui est donnée par

$$(u_t + u_{xxx} + uu_x)_x + u_{yy} = 0. \tag{21}$$

Les deux équations (19) et (21) apparaissent toutes deux comme des modèles pour les ondes longues de surface dans le problème des vagues, respectivement dans le cas d'une tension superficielle grande et petite (voir [44, Chapitre 7]). Cependant, les propriétés de stabilité transverse des ondes solitaires et périodiques diffèrent selon les deux versions. Pour l'équation de KP-I, ces résultats de stabilité sont bien connus. Plus précisément, les ondes solitaires et périodiques sont transversalement instables (voir [55, 57] pour les ondes solitaires et [25, 37, 59] pour les ondes périodiques). Pour l'équation de KP-II, la stabilité transverse non linéaire des ondes solitaires par rapport à des perturbations périodiques a été récemment prouvée dans [51], et leur stabilité non linéaire par rapport à des perturbations localisées dans [50]. La stabilité des ondes périodiques est, quant à elle, une question largement ouverte. Signalons enfin que certains de ces résultats concernant les équations de KP ont été étendus aux équations d'Euler, qui modélisent le problème hydrodynamique complet [8, 26, 56].

Notre but est de retrouver les résultats d'instabilité des ondes progressives unidimensionnelles de l'équation (19), c'est-à-dire les solutions de (19) de la forme

$$u(x, y, t) = v(x - ct).$$

où $c > 0$ est la vitesse de propagation de l'onde. Notre critère d'instabilité linéaire donné par le Théorème 2.1 permet de retrouver ces résultats, à la fois pour les ondes progressives périodiques et solitaires. Plus précisément, on peut écrire l'équation (19) sous la forme

$$U_y = DU_t + F(U). \tag{22}$$

Remarquons qu'avec cette formulation, les ondes progressives de (19), qui ne dépendent pas de y, sont des équilibres du système (22), vérifiant $F(U) = 0$. En appliquant le Théorème 2.1, nous montrons ici que les ondes unidimensionnelles solitaires et périodiques de l'équation (19) sont linéairement transversalement instables.

Notons que dans [55] et dans [59], l'instabilité transverse linéaire des ondes solitaires et périodiques de l'équation (19) est obtenue grâce au critère d'instabilité de [55]. La différence majeure avec le critère du Théorème 2.1 réside dans la formulation des équations, qui sont écrites comme systèmes d'évolution par rapport au temps, et non en utilisant la dynamique spatiale. Outre la différence de formulation, des similitudes existent dans les hypothèses du critère de [55] et du Théorème 2.1. Dans les deux cas, une hypothèse spectrale est requise, ainsi qu'une hypothèse de symétrie sur le système. Plus précisément, le critère de [55] utilise une formulation hamiltonienne du système, et notre critère s'applique dans le cas des systèmes réversibles. En particulier, le critère de [55] ne s'applique pas pour l'équation de Lugiato-Lefever (1), qui ne possède pas de structure hamiltonienne. Ces critères d'instabilité sont toutefois utilisés dans des cadres différents. La formulation de notre critère d'instabilité est notamment bien adaptée pour l'étude des bifurcations qui peuvent apparaître suite à l'instabilité. Le résultat d'instabilité linéaire de [55] peut être utilisé pour obtenir des résultats d'instabilité non linéaire pour des systèmes hamiltoniens [53, 54, 56].

Solutions périodiques des équations de Davey-Stewartson

Notre troisième exemple est le système de Davey-Stewartson,

$$\begin{aligned} iA_t + A_{xx} + \alpha A_{yy} + \lambda A + \delta B_x A + \gamma |A|^2 A &= 0 && (23) \\ B_{xx} + \nu B_{yy} + \mu \left(|A|^2\right)_x &= 0, && (24) \end{aligned}$$

qui modélise la propagation des ondes tridimensionnelles de capillarité-gravité dans le problème des vagues, tout comme les équations de Kadomtsev-Petviashvili mais dans un autre régime de paramètres [1]. Les fonctions A et B dépendent du temps t, et de deux variables spatiales x et y, et sont respectivement à valeurs complexes et réelles, et les coefficients α, λ, δ, γ, μ et ν sont réels. La dynamique des solutions du système (23)-(24) dépend fortement du signe des coefficients α, δ, μ et ν. Nous nous restreignons aux trois cas suivants, qui sont les cas pertinents dans le problème hydrodynamique sous-jacent :

- **Cas 1 :** $\alpha > 0$, $\delta < 0$, $\mu > 0$ et $\nu > 0$;
- **Cas 2 :** $\alpha > 0$, $\delta < 0$, $\mu < 0$ et $\nu < 0$;
- **Cas 3 :** $\alpha < 0$, $\delta > 0$, $\mu > 0$ et $\nu > 0$.

Les équations (23)-(24) possèdent une famille de solutions périodiques unidimensionnelles données par

$$A^*(x) = a_0 e^{ikx}, \qquad B^*_x = \chi - \mu |a_0|^2, \tag{25}$$

avec $k^2 = (\gamma - \delta\mu) |a_0|^2 + \lambda + \delta\chi$, χ étant une constante arbitraire, que l'on peut supposer nulle sans perte de généralité. Notre but est d'étudier l'instabilité linéaire de ces solutions périodiques par rapport à des perturbations dépendantes de y, de longueur d'onde K. Comme dans le cas de l'équation de KP-I, cette instabilité est appelé instabilité transverse linéaire.

Les équations (23)-(24) s'écrivent facilement sous la forme (16), et en appliquant le Théorème 2.1, on montre le résultat suivant, énoncé dans [16].

Théorème 2.2. *La solution périodique (A^*, B^*) est transversalement linéairement instable dans le Cas 1 si $K^2 - 2(3k^2 - \lambda) < 0$, ou si $K^2 - 2(3k^2 - \lambda) > 0$ et $\gamma > 0$, ainsi que dans les Cas 2 et 3.*

2.5 Bifurcations induites

Le résultat principal du Chapitre 2 est un critère d'instabilité linéaire. Une question liée est celle des bifurcations induites par cette instabilité. Dans le cas où l'instabilité est une instabilité transverse, comme pour les équations de KP-I et de Davey-Stewartson, ces bifurcations sont appelées bifurcations de type **rupture de dimension**. Ce problème consiste à étudier l'existence de solutions du système stationnaire

$$U_x = F(U) \tag{26}$$

qui sont proches d'un équilibre U_* linéairement instable au sens du Théorème 2.1. Dans cette situation, on peut s'attendre à des bifurcations de solutions périodiques pour le système (26), à cause de la présence d'une paire de valeurs propres imaginaires pures dans le spectre de $\mathrm{d}F(U_*)$. Cette seule hypothèse ne suffit cependant pas à étudier de telles bifurcations.

Une réduction à une variété centrale (voir la Section 1.5), avec des hypothèses supplémentaires, peut mettre en évidence de telles bifurcations. Cette méthode a été employée par exemple dans [14], dans le cas des équations de Davey-Stewartson. En particulier, une réduction à une variété centrale montre l'existence de plusieurs types de solutions bidimensionnelles, émergeant à partir des solutions périodiques unidimensionnelles (25).

Une autre méthode classique employée pour mener l'étude de ces bifurcations repose sur le théorème du centre de Lyapunov (voir par exemple [2, 5, 41]). Ce résultat affirme que si le spectre de l'opérateur $\mathrm{d}F(U_*)$ contient une paire de valeurs propres imaginaires pures simples $\pm ik_*$, $k_* > 0$, alors, sous des hypothèses additionnelles, le système stationnaire (26) possède une famille de solutions périodiques de petite amplitude proches de U_*, et de longueurs d'onde proches de k_*. Ce théorème a été largement utilisé dans le cadre du problème des vagues, modélisé par les équations d'Euler avec frontière libre [20, 22, 26]. En particulier, il est utilisé dans [26] pour prouver l'émergence d'une famille de solutions périodiques tridimensionnelles, à partir de solutions périodiques bidimensionnelles, qui se trouvent être transversalement linéairement instables.

Nous utilisons le théorème du centre de Lyapunov dans le cas de l'équation de KP-I, pour établir l'existence d'une famille de solutions périodiques par rapport aux variables x et y, émergeant à partir d'une onde progressive périodique. Dans le cas d'une onde progressive solitaire, 0 appartient au spectre essentiel de l'opérateur L, et le théorème du centre de Lyapunov ne peut pas s'appliquer. L'étude des bifurcations de type rupture de dimension pourrait alors s'appuyer sur le théorème de Lyapunov-Iooss (voir [5, Chapitre 2, Théorème 2.5.2] et [34, Section 4, Théorème 4] pour l'énoncé de ce résultat, ainsi que [22] pour une application au problème des vagues).

Par la suite, se pose la question de la stabilité des solutions périodiques issues de ces bifurcations. Le Chapitre 3 est consacré à ce problème, dans le cas particulier où le spectre imaginaire pur de l'opérateur $\mathrm{d}F(U_*)$ est constitué de deux paires de valeurs propres simples.

3 Instabilité linéaire de solutions périodiques

3.1 Le problème d'instabilité

Dans le Chapitre 3, qui reprend les travaux de l'article [17] en préparation, nous employons la même formulation des équations que dans le Chapitre 2, et considérons des systèmes de la forme

$$U_x = DU_t + LU + R(U). \tag{27}$$

La fonction U dépend du temps t et d'une variable d'espace x, et prend ses valeurs dans un espace de Hilbert, D et L sont des opérateurs linéaires réels, et R une application non linéaire supposée assez régulière. On suppose de plus que 0 est un équilibre de (27), donc $R(0) = 0$, et que $\mathrm{d}R(0) = 0$.

Les hypothèses formulées dans ce chapitre sont proches de celles du théorème du centre de Lyapunov. En particulier, nous supposons que le système (27) est réversible. D'autre part, on suppose que le spectre imaginaire pur de L est constitué de deux paires de valeurs propres simples $\pm ik_1$ et $\pm ik_2$, $k_1, k_2 \neq 0$, qui sont non résonantes, dans le sens où, pour tous $m, n \in \mathbb{Z}^*$, on a $mk_1 \neq nk_2$.

Le critère d'instabilité du Théorème 2.1 donné dans le Chapitre 2 s'applique dans ce cas et permet d'établir que l'équilibre 0 est linéairement instable. Sous ces hypothèses, le théorème du centre de Lyapunov s'applique également. Le système stationnaire

$$U_x = LU + R(U) \tag{28}$$

possède alors deux familles de solutions périodiques de petite amplitude et de longueurs d'onde proches de k_1 et k_2, notées

$$u_{\varepsilon,j}(x) = \varepsilon U_{\varepsilon,j}(k_{\varepsilon,j}x),\ k_{\varepsilon,j} = k_j + \varepsilon l_{\varepsilon,j}, \quad \varepsilon \in \mathbb{R} \text{ petit}, \quad j = 1, 2. \tag{29}$$

Le but du Chapitre 3 est d'étudier l'instabilité linéaire de ces solutions vis-à-vis de certaines perturbations. Pour cela, considérons le linéarisé du système (27) en la solution périodique $u_{\varepsilon,j}$, et effectuons le changement d'échelle $x' = k_{\varepsilon,j}x$ afin de se ramener des fonctions 2π-périodiques. Ce système linéarisé s'écrit

$$k_{\varepsilon,j}U_x = DU_t + LU + \varepsilon L_{\varepsilon,j}U, \tag{30}$$

où $L_{\varepsilon,j} = \mathrm{d}R(U_{\varepsilon,j})$, et où on a écrit x à la place de x' par souci de simplification des notations. Alors l'onde périodique $U_{\varepsilon,j}$ est **linéairement instable par rapport à des perturbations de longueur d'onde** K si le système linéarisé (30) possède une solution de la forme

$$U(x,t) = e^{\lambda t}e^{iKx}V(x), \tag{31}$$

où λ est un nombre complexe de partie réelle strictement positive, $K \in [0, 1)$ et V est une fonction 2π-périodique. Notons que cette longueur d'onde correspond à $k_{\varepsilon,j}K$ avant le changement de variable $x' = k_{\varepsilon,j}x$.

3.2 Le résultat principal

Le résultat suivant affirme que les Hypothèses 3.1.1 données au Chapitre 3 sont suffisantes pour obtenir l'instabilité linéaire de $U_{\varepsilon,j}$ par rapport à des perturbations d'une certaine longueur d'onde. Nous démontrons alors le résultat suivant.

Théorème 3.1. *Sous les Hypothèses 3.1.1 données dans le Chapitre 3, pour $\varepsilon > 0$, $\lambda > 0$ suffisamment petits, il existe $K = K(\varepsilon, \lambda) \in [0, 1)$ tel que le système (30) possède une solution de la forme (31). L'onde périodique $U_{\varepsilon,j}$ est donc linéairement instable par rapport à des perturbations de longueur d'onde K.*

On donne ci-après une ébauche de preuve du Théorème 3.1. Supposons $j = 1$, sans perte de généralité. En substituant l'ansatz (31) dans le système linéarisé (30), on montre qu'il suffit, pour prouver ce théorème, de vérifier que l'opérateur

$$\mathcal{L}_{\varepsilon,\lambda} = k_1\partial_x - L + \varepsilon(l_{\varepsilon,1}\partial_x - L_{\varepsilon,1}) - \lambda D$$

possède une paire de valeurs propres imaginaires pures. Pour cela, on montre que l'opérateur $\mathcal{L}_{0,0} = k_1\partial_x - L$ possède une paire de valeurs propres imaginaires pures simples. Un argument de perturbation permet alors d'affirmer que pour ε, λ réels assez petits, l'opérateur $\mathcal{L}_{\varepsilon,\lambda}$ possède une paire de valeurs propres complexes conjuguées, qui sont imaginaires pures par réversibilité du système. L'étude du spectre de l'opérateur $\mathcal{L}_{0,0}$ s'avère délicate en dimension infinie, et repose sur les résultats de [4]. On montre alors, en utilisant des séries de Fourier, que $\mathcal{L}_{0,0}$ possède deux familles de valeurs propres imaginaires pures, données par

$$i\mu_{1,n}^{\pm} = ink_1 \pm ik_1, \quad i\mu_{2,n}^{\pm} = ink_1 \pm ik_2, \quad n \in \mathbb{Z}.$$

Les valeurs propres $i\mu_{1,n}^{\pm}$ sont doubles, tandis que les valeurs propres $i\mu_{2,n}^{\pm}$ sont simples par non résonance de $\pm ik_1$ et $\pm ik_2$, ce qui prouve le résultat.

Signalons que la preuve du Théorème 3.1, notamment l'hypothèse de non résonance des valeurs propres $\pm ik_1$ et $\pm ik_2$, montre que l'onde périodique $U_{\varepsilon,j}$ est linéairement instable par rapport à une famille de perturbations avec des longueurs d'onde proches de k_2/k_1. La solution $u_{\varepsilon,j}$ est donc linéairement instable par rapport à une famille de perturbations avec des longueurs d'onde proches de k_2. Ceci contraste avec l'équilibre 0, qui lui est linéairement instable par rapport à deux familles de perturbations périodiques, de longueurs d'onde proches de k_1 et k_2.

3.3 Application à l'équation de Lugiato-Lefever

Nous appliquons ce résultat à l'équation de Lugiato-Lefever (1). En effet, lorsque les paramètres α et F appartiennent aux zones identifiées dans les diagrammes de la Figure 7, le Théorème 2.3.3 prouvé dans le Chapitre 2 montre l'existence de deux familles de solutions périodiques de la forme (29). De plus, l'équation (1) peut s'écrire sous la forme (27), et le système obtenu est réversible. On en déduit alors que les solutions périodiques de la forme (29) sont linéairement instables au sens du Théorème 3.1.

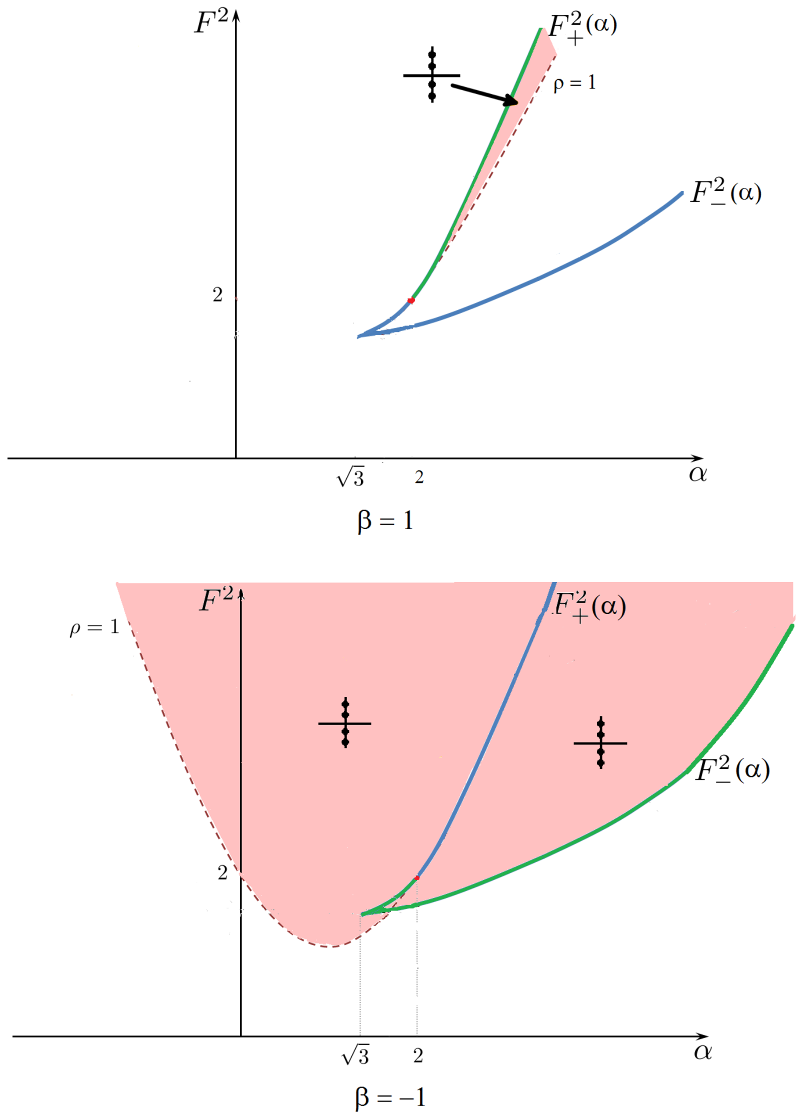


FIGURE 7 – Diagrammes montrant les zones du plan (α, F^2) dans lesquelles l'hypothèse spectrale du Théorème 3.1 est vérifiée pour l'équation de Lugiato-Lefever, dans les cas $\beta = 1$ (en haut) et $\beta = -1$ (en bas). Dans la région $F_-^2(\alpha) < F^2 < F_+^2(\alpha)$, où l'équation (1) a trois solutions constantes, la solution constante concernée est celle de plus petit module lorsque $\beta = 1$, et celle de plus grand module lorsque $\beta = -1$.

4 Perspectives et problèmes ouverts

Nous terminons cette introduction en évoquant quelques questions ouvertes.

Dans le Chapitre 1 nous avons procédé à une analyse systématique des bifurcations locales pour l'équation de Lugiato-Lefever (1). Nous en avons déduit l'existence de différents types de solutions stationnaires pour l'équation (1). Il se trouve que les expériences et simulations numériques de [19] mettent en évidence des solutions périodiques en temps, qui ne rentrent pas dans le cadre de notre analyse de bifurcations, et requièrent d'autres outils mathématiques. Parmi ces solutions, une catégorie de solutions localisées en espace, appelées breathers, font l'objet d'un intérêt particulier d'un point de vue physique [19]. Un prolongement possible de notre travail consisterait à prouver rigoureusement l'existence de ces solutions. Pour cela, une étude de bifurcations secondaires pourrait être envisagée. Le travail de ce chapitre pourrait aussi se prolonger par l'étude de la stabilité des solutions stationnaires dont nous avons prouvé l'existence. Les résultats obtenus expliqueraient alors probablement pourquoi certaines solutions ne sont pas observées dans [19].

Le Chapitre 2 est consacré à l'obtention d'un critère simple d'instabilité linéaire de solutions d'équations aux dérivées partielles non linéaires. Nous avons appliqué ce critère dans différents contextes, notamment à des questions d'instabilité transverse. Nous pourrions nous demander si l'hypothèse de réversibilité requise par le Théorème 2.1 peut être remplacée par d'autres propriétés de symétrie du système (16), par exemple par une structure hamiltonienne. Une possibilité serait une structure hamiltonienne "globale", incluant la dépendance en t, c'est-à-dire l'existence d'un opérateur antiadjoint J et d'une application H, tels que le second membre du système (16) s'écrive

$$DU_t + F(U) = J\nabla_U H(U).$$

Cependant, une telle structure est difficile à mettre en application. On peut dès lors se demander si elle peut être assouplie. Les mêmes questions se posent également pour le Théorème 3.1, énoncé dans le Chapitre 3.

Une motivation pour étendre les critères d'instabilité des Chapitres 2 et 3 à un cadre hamiltonien est l'étude du problème des vagues, pour lequel nous avons obtenu un certain nombre de résultats, mais pour des équations modèles uniquement (équations de KP-I et de Davey-Stewartson). Le problème hydrodynamique est régi par les équations d'Euler, décrivant les ondes non linéaires à la surface d'une couche bidimensionnelle de fluide parfait incompressible, de densité constante, soumise à l'action de la pesanteur. En coordonnées cartésiennes $(x, \widetilde{y})$, le fluide occupe le domaine

$$D_\eta = \{(x, \widetilde{y}),\ x \in \mathbb{R}, \widetilde{y} \in (0, \eta(x,t))\}.$$

La surface libre η et le potentiel de vitesse ϕ du fluide vérifient alors l'équation de Laplace

$$\phi_{xx} + \phi_{\widetilde{y}\widetilde{y}} = 0,\ \text{dans } D_\eta, \tag{32}$$

à laquelle s'ajoute des conditions aux bords non linéaires

$$\phi_{\widetilde{y}} = 0,\ \text{sur } \widetilde{y} = 0, \tag{33}$$

$$\phi_{\widetilde{y}} = \eta_t - \eta_x + \eta_x \phi_x,\ \text{sur } \widetilde{y} = \eta, \tag{34}$$

$$\phi_t - \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_{\widetilde{y}}^2\right) + \alpha(\eta - 1) - \beta \left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x = 0,\ \text{sur } \widetilde{y} = \eta. \tag{35}$$

En utilisant les méthodes développées dans [21, 24], les équations (32)-(35) peuvent s'écrire sous la forme (27). De plus, en utilisant les arguments de [7, 9], on peut montrer que le système (27) possède, pour des valeurs des coefficients α et β bien choisies, deux familles de solutions périodiques de la forme (29). Cependant, le système (27) obtenu n'est pas réversible, et le Théorème 3.1 ne peut pas être utilisé pour étudier l'instabilité de ces solutions. La question de la stabilité de ces solutions périodiques de petite amplitude est un problème ouvert, et la structure hamiltonienne du système (27) dans ce cas pourrait en fournir une approche possible.

Chapter 1

A local bifurcation analysis for the Lugiato–Lefever equation

In this chapter, we consider the Lugiato-Lefever equation,

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi\, |\psi|^2 + F, \tag{1.1}$$

where the unknown function ψ is complex valued and depends upon the time t and the space variable x. The coefficient $\alpha \in \mathbb{R}$ is a detuning parameter, $F > 0$ is a driving term and $\beta \in \mathbb{R}^*$ denotes a dispersion parameter, which can be supposed to be equal to ± 1, up to rescaling x. In the sequel of the chapter we refer to the case $\beta = 1$ (respectively $\beta = -1$) as the normal (respectively anomalous) dispersion case.

Our goal is to prove the existence of several types of stationary solutions of the equation (1.1), *i. e.*, solutions of the differential equation

$$\beta \frac{\mathrm{d}^2 \psi}{\mathrm{d}x^2} = (i - \alpha)\psi + \psi|\psi|^2 - iF. \tag{1.2}$$

In Section 1.1 we describe the set of constant solutions of the equation (1.1) and recall some results of the bifurcation analysis in [19]. In particular, we formulate (1.2) as a first order dynamical system. In Sections 1.2, 1.3 and 1.4, using tools from normal form and center manifold theory, we perform a systematic bifurcation analysis for the equation (1.2), and prove the existence of several types of stationary solutions. We also compute the leading order term in the Taylor expansions of the periodic and homoclinic solutions. We conclude with a discussion in Section 1.5 of some further issues about stability and time periodic solutions.

1.1 Spatial dynamics and bifurcation diagrams

In this section we recall the main results of the bifurcation analysis performed in [19]. We first describe the set of constant solutions of the equation (1.1). Then we formulate the equation (1.2) as a dynamical system. We use this formulation to prove the existence of four local bifurcations, depending on the values of the coefficients α and F.

1.1.1 Constant solutions

Let $\psi = \psi_r + i\psi_i$ be a constant solution of (1.1), where ψ_r and ψ_i denote the real and imaginary part of ψ, respectively. Then ψ satisfies the algebraic equation

$$i\psi|\psi|^2 - (1+i\alpha)\psi + F = 0.$$

Notice that ψ does not depend upon β. A straightforward computation shows that ψ_r and ψ_i are given implicitly by

$$\psi_r = \frac{F}{1+(\rho-\alpha)^2}, \qquad \psi_i = \frac{F(\rho-\alpha)}{1+(\rho-\alpha)^2},$$

where $\rho = |\psi|^2$. In particular, we have

$$F^2 = \rho\left(1+(\rho-\alpha)^2\right). \tag{1.3}$$

This relation relates the parameters α, F^2 and the square modulus ρ of a constant solution, and allows to determine the number of constant solutions of (1.1) in terms of the parameters α and F^2.

More precisely, for $\rho > 0$, the derivative of the cubic polynomial

$$P_1 : \rho \longmapsto \rho\left(1+(\rho-\alpha)^2\right) \tag{1.4}$$

is given by

$$P_1' : \rho \longmapsto 3\rho^2 - 4\alpha\rho + \alpha^2 + 1, \tag{1.5}$$

and the discriminant of P_1' is $\Delta = 4(\alpha^2 - 3)$. For $\alpha \leqslant \sqrt{3}$, the cubic polynomial P_1 has no positive critical point and is monotonically increasing for $\rho > 0$, so that the equation (1.3) has precisely one solution, and therefore the equation (1.1) has one constant solution, for any $F > 0$. For $\alpha > \sqrt{3}$, the polynomial P has two positive critical points, which are the roots of P',

$$\rho_\pm(\alpha) = \frac{2\alpha \mp \sqrt{\alpha^2-3}}{3};$$

(see Figure 1.1). The corresponding values $F_\pm^2(\alpha)$ of F^2 are then given by

$$\begin{aligned} F_\pm^2(\alpha) &= \rho_\pm(\alpha)(1+(\rho_\pm(\alpha)-\alpha)^2) \\ &= \frac{2\alpha \mp \sqrt{\alpha^2-3}}{3}\left(1+\left(\frac{\sqrt{\alpha^2-3}\pm\alpha}{3}\right)^2\right). \end{aligned}$$

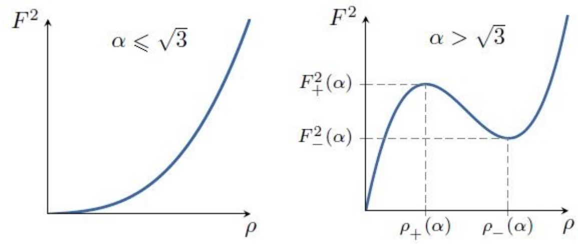


FIGURE 1.1 – Monotonicity of the polynomial $\rho \longmapsto \rho\,(1+(\rho-\alpha)^2)$ for $\alpha \leqslant \sqrt{3}$ (to the left) and $\alpha > \sqrt{3}$ (to the right).

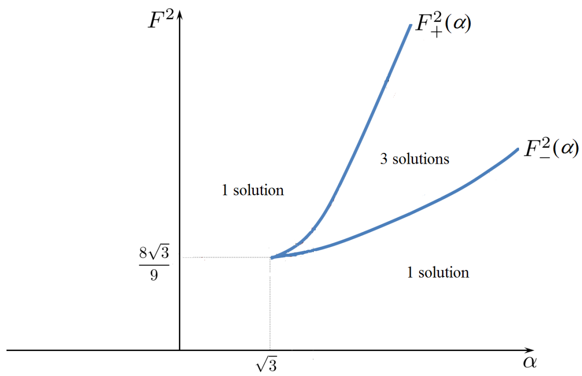


FIGURE 1.2 – Number of constant solutions of the equation (1.1) in terms of α and F^2. Along the curves $F^2 = F_\pm^2(\alpha)$, the equation has precisely two constant solutions.

Consequently, for any $\alpha > \sqrt{3}$ and any $F^2 \in (F_-^2(\alpha), F_+^2(\alpha))$, the equation (1.3) has three solutions, so that the equation (1.1) has three constant solutions ψ_1, ψ_2 and ψ_3. Their respective square modulus ρ_1, ρ_2 and ρ_3 satisfy

$$\rho_1 < \rho_+(\alpha) < \rho_2 < \rho_-(\alpha) < \rho_3.$$

For $F^2 = F_\pm^2(\alpha)$, the equation (1.3) has two solutions. One of this solution is double, and its square modulus ρ satisfies $\rho = \rho_\pm(\alpha)$. Finally, for $F^2 \notin [F_-^2(\alpha), F_+^2(\alpha)]$, the equation (1.1) has precisely one constant solution. We sum up these results in Figure 1.2.

1.1.2 Spatial dynamics

Let $\psi^* = \psi_r^* + i\psi_i^*$ be a constant solution of (1.1), and let α^*, F^* be the corresponding values of the parameters. We set

$$\psi = \psi^* + \widetilde{\psi}, \qquad \widetilde{\psi} = \widetilde{\psi_r} + i\widetilde{\psi}_i, \qquad \frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}x} = \widetilde{\varphi_r} + i\widetilde{\varphi}_i,$$

and we rewrite the stationary equation (1.2) as a dynamical system in $\mathbb{R}^4$,

$$\frac{\mathrm{d}U}{\mathrm{d}x} = L_\pm U + G_\pm(U, \alpha, F), \tag{1.6}$$

in which

$$U = \begin{pmatrix} \widetilde{\psi_r} \\ \widetilde{\varphi_r} \\ \widetilde{\psi_i} \\ \widetilde{\varphi_i} \end{pmatrix}, \qquad L_\pm = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \pm\left(3\psi_r^{*2} + \psi_i^{*2} - \alpha^*\right) & 0 & \pm\left(2\psi_r^*\psi_i^* - 1\right) & 0 \\ 0 & 0 & 0 & 1 \\ \pm\left(2\psi_r^*\psi_i^* + 1\right) & 0 & \pm\left(\psi_r^{*2} + 3\psi_i^{*2} - \alpha^*\right) & 0 \end{pmatrix},$$

and

$$G_\pm(U,\alpha,F) = \pm \begin{pmatrix} 0 \\ \widetilde{\psi_r}^3 + \widetilde{\psi_r}\widetilde{\psi_i}^2 + 3\psi_r^*\widetilde{\psi_r}^2 + 2\psi_i^*\widetilde{\psi_r}\widetilde{\psi_i} + \psi_r^*\widetilde{\psi_i}^2 - (\alpha-\alpha^*)(\widetilde{\psi_r} + \psi_r^*) \\ 0 \\ \widetilde{\psi_i}^3 + \widetilde{\psi_r}^2\widetilde{\psi_i} + \psi_i^*\widetilde{\psi_r}^2 + 2\psi_r^*\widetilde{\psi_r}\widetilde{\psi_i} + 3\psi_i^*\widetilde{\psi_i}^2 - (\alpha-\alpha^*)(\widetilde{\psi_i} + \psi_i^*) - (F - F^*) \end{pmatrix}.$$

Here the symbols $+$ and $-$ in $L_\pm$ and $G_\pm$ stand for the case of normal ($\beta = 1$) and anomalous ($\beta = -1$) dispersion, respectively. Notice that the matrices $L_\pm$ are obtained by linearizing the system (1.6) at ψ^*.

An important property of the system (1.6) is its reversibility, *i.e.*, there exists a symmetry

$$S = \mathrm{diag}(1, -1, 1, -1)$$

which anticommutes with the vector field in (1.6) :

$$L_\pm SU = -SL_\pm U, \quad G_\pm(SU,\alpha,F) = -SG_\pm(U,\alpha,F),$$

for all $U \in \mathbb{R}^4$ and $(\alpha, F) \in \mathbb{R} \times (0, +\infty)$. This property will play a key role in our analysis.

1.1.3 Local bifurcations

We are interested in finding the values of the parameters α^* and F^{*2} for which a local bifurcation occurs. Local bifurcations are determined by the purely imaginary eigenvalues of the matrix $L_\pm$. A direct calculation shows that the eigenvalues of $L_\pm$ are the complex roots X of the characteristic polynomial

$$P_\pm(X) = X^4 \mp 2(2\rho^* - \alpha^{*2})X^2 + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1, \quad \rho^* = \psi_r^{*2} + \psi_i^{*2}. \tag{1.7}$$

Notice that the spectrum of the real matrix $L_\pm$ is on the one hand symmetric with respect to the real axis, and on the other hand symmetric with respect to the origin, because of the reversibility of the system (1.6).

We discuss the location of the eigenvalues of $L_\pm$ in the complex plane. For notational simplicity, we suppose that $\beta = 1$ (the arguments in the case $\beta = -1$ are similar). First, the polynomial (1.7) is quadratic in X^2, so we can consider the polynomial

$$\widetilde{P_+}(Y) = Y^2 - 2(2\rho^* - \alpha^*)Y + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1, \quad \rho^* = \psi_r^{*2} + \psi_i^{*2}, \tag{1.8}$$

with discriminant $\widetilde{\Delta} = 4(\rho^{*2} - 1)$. Notice that $\widetilde{P_+}(0) = 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1$. This implies that the product of the roots of $\widetilde{P_+}(Y)$ is precisely the derivative P_1' of the polynomial P_1 given by (1.5), so that its sign is the same as P_1'.

(i) Suppose that $\rho^* > 1$. In this case, the polynomial $\widetilde{P_+}$ has two real roots given by $2\rho^* - \alpha^* \pm \sqrt{\rho^{*2} - 1}$.

(a) If $3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 > 0$, *i.e.*, if $\rho^* < \rho_+(\alpha^*)$ or $\rho^* > \rho_-(\alpha^*)$, the product of these roots is positive, so they have the same sign. If $2\rho^* - \alpha^* > 0$ (respectively $2\rho^* - \alpha^* < 0$), they are positive (respectively negative), and the polynomial P_+ possesses four real simple roots (respectively four simple purely imaginary roots).

(b) If $3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 = 0$, *i.e.*, if $\rho^* = \rho_\pm(\alpha^*)$, 0 is root of $\widetilde{P_+}$ and a double root of P_+. Additionally, if $2\rho^* - \alpha^* > 0$ (respectively $2\rho^* - \alpha^* < 0$), the polynomial $\widetilde{P_+}$ has a positive root (respectively a negative root) and P_+ has a pair of simple real roots (respectively a pair of purely imaginary roots).

(c) If $3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 < 0$, *i.e.*, if $\rho^* \in (\rho_+(\alpha^*), \rho_-(\alpha^*))$, the polynomial $\widetilde{P_+}$ has a positive and a negative root. Consequently, P_+ has a pair of simple real roots and a pair of simple purely imaginary roots.

(ii) Suppose that $\rho^* = 1$. In this case, the polynomial $\widetilde{P_+}$ has a double root given by $Y = 2 - \alpha^*$. Consequently, the polynomial P_+ possesses a pair of double purely imaginary roots if $\alpha^* > 2$, and a pair of double real roots if $\alpha^* < 2$. In the case $\alpha^* = 2$, 0 is a quadruple root.

(iii) Suppose that $\rho^* < 1$. In this case, the polynomial $\widetilde{P_+}$ has two complex conjugate roots, given by $2\rho^* - \alpha^* \pm i\sqrt{1 - \rho^{*2}}$, and the polynomial P_+ has a quadruplet of complex roots of the form $a \pm ib$, $-a \pm ib$.

These results are summarized in Figure 1.3.

The diagrams in the (α, ρ)-plane of Figure 1.3 allows us to obtain the diagrams in the (α, F^2)-plane given in Figure 1.4. First, notice that $3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 = 0$ if and only if $F^{*2} = F_\pm^2(\alpha^*)$, so that the curve of equation $3\rho^2 - 4\alpha\rho + \alpha^2 + 1 = 0$ in the (α, ρ)-plane corresponds to the curves of equations $F^2 = F_\pm^2(\alpha)$ in the (α, F^2)-plane. Moreover, in the area of the (α, F^2)-plane in which the equation (1.1) has three constant solutions ψ_1^*, ψ_2^* and ψ_3^*, we use the fact that their respective square modulus ρ_1^*, ρ_2^* and ρ_3^* satisfy

$$\rho_1^* < \rho_+(\alpha^*) < \rho_2^* < \rho_-(\alpha^*) < \rho_3^*.$$

We can now deduce the values of the parameters α^* and F^{*2} for which a bifurcation occurs. We have four types of bifurcations, depending upon the structure of the purely imaginary spectrum $\sigma_{\mathrm{im}}(L_\pm)$ of the matrices $L_\pm$:

– an $\boldsymbol{(i\omega)^2}$ **bifurcation** when $\sigma_{\mathrm{im}}(L_\pm) = \{\pm i\omega\}$, where $\pm i\omega$ are algebraically double and geometrically simple eigenvalues. In the case of normal dispersion ($\beta = 1$), this bifurcation occurs for $\alpha^* > 2$ and $F^{*2} = 1 + (1 - \alpha^*)^2$, when $\rho^* = 1$. In the case of anomalous dispersion ($\beta = -1$), it occurs for $\alpha^* < 2$, $F^{*2} = 1 + (1 - \alpha^*)^2$ and $\rho^* = 1$.

– a $\boldsymbol{0^2(i\omega)}$ **bifurcation** when $\sigma_{\mathrm{im}}(L_\pm) = \{0, \pm i\omega\}$, where 0 is an algebraically double and geometrically simple eigenvalue. In the case of normal dispersion, this bifurcation occurs for $\alpha^* > 2$, and $F^{*2} = F_+^2(\alpha^*)$, when $\rho^* = \rho_+(\alpha^*)$. In the case of anomalous dispersion, it occurs for $\alpha^* \in (\sqrt{3}, 2)$ $F^{*2} = F_+^2(\alpha^*)$, and $\rho^* = \rho_+(\alpha^*)$, and for $\alpha^* \geqslant \sqrt{3}$, $F^{*2} = F_-^2(\alpha^*)$ and $\rho^* = \rho_-(\alpha^*)$.

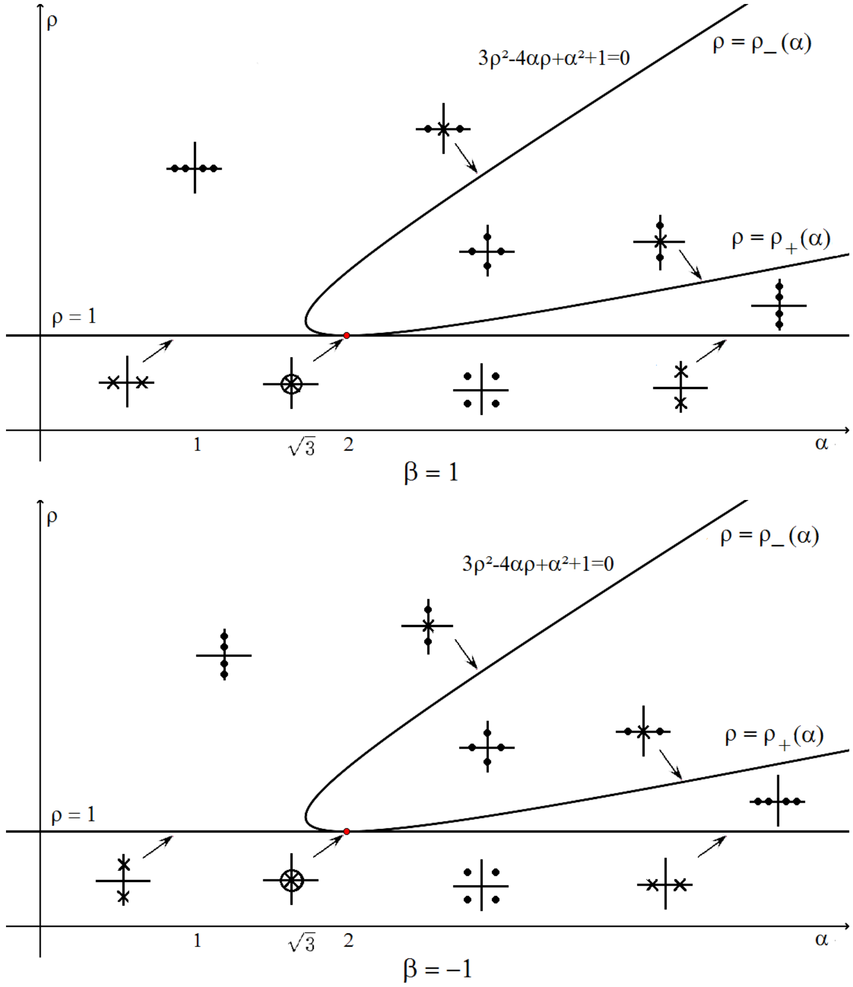
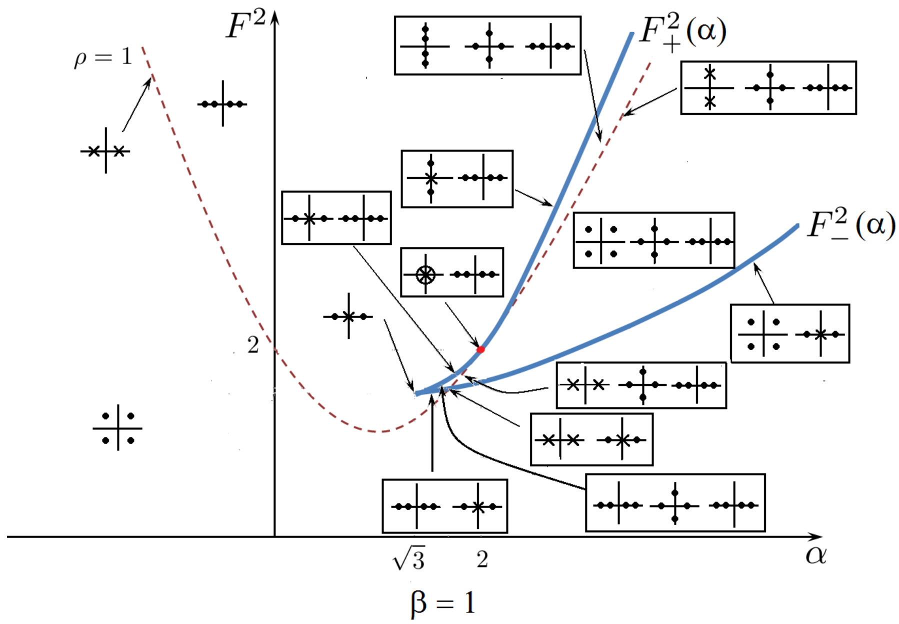


FIGURE 1.3 – Location of the eigenvalues of the matrices L_+ (up) and L_- (down) in the (α, ρ)-plane. Simple, double and quadruple eigenvalues are indicated with a dot, a cross and a circled cross, respectively.

- a $\mathbf{0^2}$ **bifurcation** when $\sigma_{\rm im}(L_\pm) = \{0\}$, where 0 is an algebraically double and geometrically simple eigenvalue. In the case of normal dispersion, it occurs for $\alpha^* \in (\sqrt{3}, 2)$ and $F^{*2} = F_+^2(\alpha^*)$, when $\rho^* = \rho_+(\alpha^*)$, and for $\alpha^* \geqslant \sqrt{3}$, $F^{*2} = F_-^2(\alpha^*)$, when $\rho^* = \rho_-(\alpha^*)$. In the case of anomalous dispersion, this bifurcation occurs for $\alpha^* > 2$, $F^{*2} = F_+^2(\alpha^*)$ and $\rho^* = \rho_+(\alpha^*)$.
- a $\mathbf{0^4}$ **bifurcation** when $\sigma_{\rm im}(L_\pm) = \{0\}$, where 0 is an algebraically quadruple and geometrically simple eigenvalue. In both cases, this bifurcation occurs for $\alpha^* = F^{*2} = 2$, when $\rho^* = 1$.

These results are summarized in Figure 1.5.

Remark 1.1.1. *The* $(i\omega)^2$, $0^2(i\omega)$ *and* 0^2 *bifurcations are codimension 1 bifurcations, involving only one bifurcation parameter. In order to describe the values of the parameters* α *and* F^2 *in a neighborhood of the point* (α^*, F^{*2}), *which is set on the curves in Figure 1.5, we can*



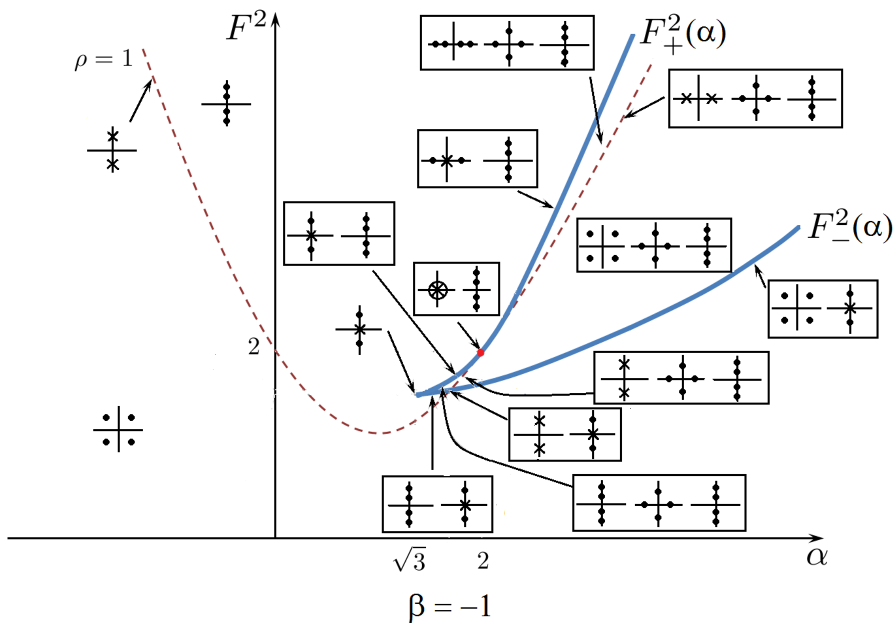


FIGURE 1.4 – Location of the eigenvalues of the matrices L_+ (up) and L_- (down) in the (α, F^2)-plane. Along the curves $F^2 = F_\pm^2(\alpha)$, where the equation (1.1) has two constant solutions, and in the area in which (1.1) has three constant solutions, the constant solutions are ordered from left to right by increasing modulus.

fix one parameter and take the other one as a bifurcation parameter. More precisely, we can set $\alpha = \alpha^* + \mu$ and $F^2 = F^{*2}$, or $\alpha = \alpha^*$ and $F^2 = F^{*2} + \mu$, where μ is a real and small parameter.

In contrast, the 0^4 bifurcation, which occurs at the point $(\alpha^*, F^{*2}) = (2, 2)$, is a codimension 2 bifurcation. Iin order to describe the values of α and F^2 around this point, we need two bifurcation parameters. We may set for example $\alpha = \alpha^* + \mu$, $F^2 = F^{*2} + \nu$, where μ and ν are real and small.

Remark 1.1.2. *The normal form analysis in the sequel will show that in the case of the $0^2(i\omega)$ and 0^2 bifurcations found at the cusp point $(\alpha^*, F^{*2}) = (\sqrt{3}, 8\sqrt{3}/9)$, one of the leading order coefficient of the normal form system vanishes (see Sections 1.3.3 and 1.4.2). Consequently, the analysis of these bifurcations at this point requires a higher order expansion of the normal form than for the other values of α^*, and we will exclude this case from our analysis.*

The dynamics of the system (1.6) depends on the type of bifurcation. In the sequel we restrict to the codimension 1 bifurcations : $(i\omega)^2$, $0^2(i\omega)$ and 0^2. We fix the parameter $F = F^*$, take α as a bifurcation parameter by setting $\alpha = \alpha^* + \mu$, with small μ. We rewrite the system (1.6) as

$$\frac{\mathrm{d}U}{\mathrm{d}x} = L_\pm U + R_\pm(U, \mu), \tag{1.9}$$

where U, $L_\pm$ are the same as in (1.6) and

$$R_\pm(U,\mu) = \pm \begin{pmatrix} 0 \\ \widetilde{\psi_r}^3 + \widetilde{\psi_r}\widetilde{\psi_i}^2 + 3\psi_r^*\widetilde{\psi_r}^2 + 2\psi_i^*\widetilde{\psi_r}\widetilde{\psi_i} + \psi_r^*\widetilde{\psi_i}^2 - \mu(\widetilde{\psi_r} + \psi_r^*) \\ 0 \\ \widetilde{\psi_i}^3 + \widetilde{\psi_r}^2\widetilde{\psi_i} + \psi_i^*\widetilde{\psi_r}^2 + 2\psi_r^*\widetilde{\psi_r}\widetilde{\psi_i} + 3\psi_i^*\widetilde{\psi_i}^2 - \mu(\widetilde{\psi_i} + \psi_i^*) \end{pmatrix}.$$

The maps $R_\pm$ are smooth, $R_\pm \in \mathcal{C}^\infty(\mathbb{R}^4 \times \mathbb{R}, \mathbb{R}^4)$, and

$$R_\pm(0,0) = 0, \qquad \mathrm{d}_U R_\pm(0,0) = 0,$$

where d_U denotes their differential with respect to U. Notice that the Taylor expansion of $R_\pm$ is given by

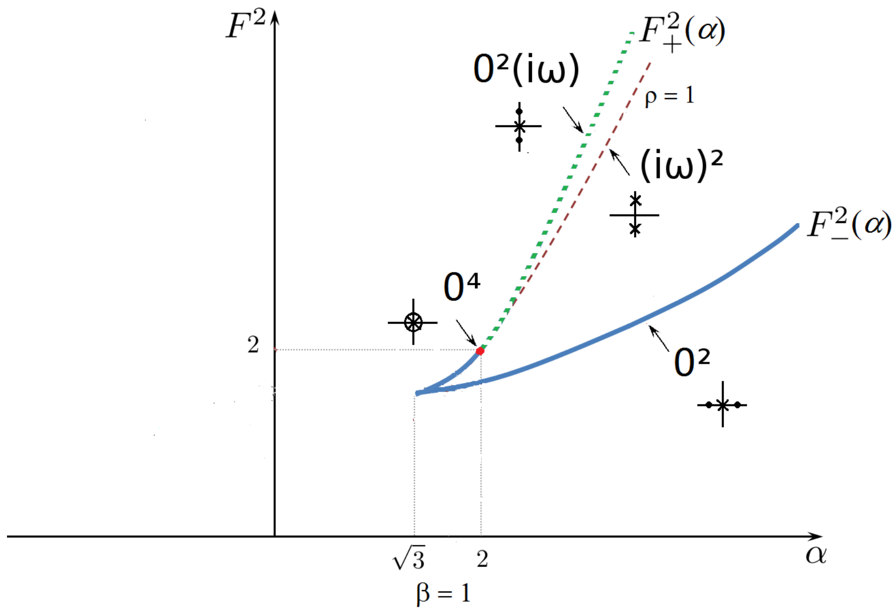
$$R_\pm(U,\mu) = \mu R_\pm^{0,1} + \mu R_\pm^{1,1}(U) + R_\pm^{2,0}(U,U) + R_\pm^{3,0}(U,U,U), \tag{1.10}$$

where

$$R_\pm^{0,1} = \pm \begin{pmatrix} 0 \\ -\psi_r^* \\ 0 \\ -\psi_i^* \end{pmatrix},$$

and where the maps $R_\pm^{1,1}$, $R_\pm^{2,0}$, $R_\pm^{3,0}$ are linear, bilinear and trilinear, respectively, and are given by

$$R_\pm^{1,1}(U) = \pm \begin{pmatrix} 0 \\ -\widetilde{\psi_r} \\ 0 \\ -\widetilde{\psi_i} \end{pmatrix}, \quad R_\pm^{2,0}(U_1,U_2) = \pm \begin{pmatrix} 0 \\ 3\psi_r^*\psi_{r,1}\psi_{r,2} + \psi_i^*(\psi_{r,1}\psi_{i,2} + \psi_{r,2}\psi_{i,1}) + \psi_r^*\psi_{i,1}\psi_{i,2} \\ 0 \\ \psi_i^*\psi_{r,1}\psi_{r,2} + \psi_r^*(\psi_{r,1}\psi_{i,2} + \psi_{r,2}\psi_{i,1}) + 3\psi_i^*\psi_{i,1}\psi_{i,2} \end{pmatrix},$$



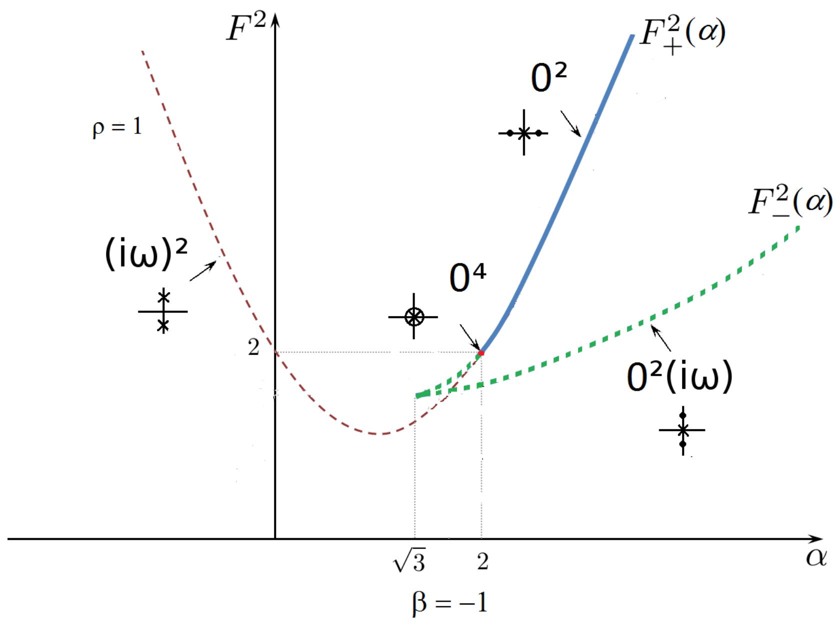


FIGURE 1.5 – Bifurcation diagrams for the case of normal (up) and anomalous (down) dispersion. $(i\omega)^2$ bifurcations occur along the dashed lines (of equation $\rho = 1$) ; $0^2(i\omega)$ and 0^2 bifurcations occur along the dotted lines, and the continuous lines, respectively (of equations $F^2 = F_\pm^2(\alpha)$). In both cases a 0^4 bifurcation occurs at the point $(\alpha^*, F^{*2}) = (2, 2)$.

$$R_\pm^{3,0}(U_1, U_2, U_3) = \pm \begin{pmatrix} 0 \\ \psi_{r,1}\psi_{r,2}\psi_{r,3} + \frac{1}{3}\left(\psi_{r,1}\psi_{i,2}\psi_{i,3} + \psi_{i,1}\psi_{r,2}\psi_{i,3} + \psi_{i,1}\psi_{i,2}\psi_{r,3}\right) \\ 0 \\ \psi_{i,1}\psi_{i,2}\psi_{i,3} + \frac{1}{3}\left(\psi_{r,1}\psi_{r,2}\psi_{i,3} + \psi_{r,1}\psi_{i,2}\psi_{r,3} + \psi_{i,1}\psi_{r,2}\psi_{r,3}\right) \end{pmatrix},$$

in which

$$U_j = \begin{pmatrix} \psi_{r,j} \\ \varphi_{r,j} \\ \psi_{i,j} \\ \varphi_{i,j} \end{pmatrix}, \quad j = 1, 2, 3.$$

Furthermore, the system (1.9) is reversible, just as the system (1.6).

1.2 Reversible $(i\omega)^2$ bifurcation

In this section we treat the case of the reversible $(i\omega)^2$ bifurcation and compute the normal form in the cases of normal ($\beta = 1$) and anomalous ($\beta = -1$) dispersion. We then discuss the existence of bounded solutions of (1.9), focusing on periodic and localized solutions.

This bifurcation occurs, for $\beta = 1$, along the half curve

$$\alpha^* > 2, \quad F^{*2} = 1 + (1 - \alpha^*)^2$$

in the (α, F^2)-parameter space, and for $\beta = -1$ along the half curve

$$\alpha^* < 2, \quad F^{*2} = 1 + (1 - \alpha^*)^2.$$

Recall that $\alpha = \alpha^* + \mu$ is the bifurcation parameter. In both cases, for such values of the parameters, we have $\rho^* = 1$, and the matrices $L_\pm$ possess one pair of algebraically double and geometrically simple purely imaginary eigenvalues denoted by $\pm i\omega^*$, with $\omega^* = \sqrt{|\alpha^* - 2|}$.

1.2.1 Normal form

Following [27, Chapter 4, Lemma 3.15], we consider a basis $\{\text{Re}\ \zeta_0, \text{Im}\ \zeta_0, \text{Re}\ \zeta_1, \text{Im}\ \zeta_1\}$ of $\mathbb{R}^4$ such that

$$(L_\pm - i\omega^*)\zeta_0 = 0,\ (L_\pm - i\omega^*)\zeta_1 = \zeta_0,\ (L_\pm + i\omega^*)\overline{\zeta_0} = 0,\ (L_\pm + i\omega^*)\overline{\zeta_1} = \overline{\zeta_0},$$

$$S\zeta_0 = \overline{\zeta_0},\ S\zeta_1 = -\overline{\zeta_1}.$$

In this basis we represent a vector $U \in \mathbb{R}^4$ by

$$U = A\zeta_0 + B\zeta_1 + \overline{A\zeta_0} + \overline{B\zeta_1},$$

with $A, B \in \mathbb{C}$, by identifying $\mathbb{R}^4$ with the complex vector space $\widetilde{\mathbb{R}^4} = \{(A, B, \overline{A}, \overline{B}), A, B \in \mathbb{C}\}$. In the sequel we use the dual vector ζ_1^* satisfying

$$\langle \zeta_0, \zeta_1^* \rangle = 0, \quad \langle \overline{\zeta_0}, \zeta_1^* \rangle = 0, \quad \langle \zeta_1, \zeta_1^* \rangle = 1, \quad \langle \overline{\zeta_1}, \zeta_1^* \rangle = 0, \quad S\zeta_1^* = -\overline{\zeta_1^*}, \quad L_\pm^*\zeta_1^* = -i\omega^*\zeta_1^*,$$

where $L_\pm^*$ is the adjoint operator of $L_\pm$ and $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product on $\widetilde{\mathbb{R}^4}$.

The vector field in the reversible system (1.9) is of class $\mathcal{C}^\infty$ on $\mathbb{R}^4 \times \mathbb{R}$ and satisfies

$$R_\pm(0, 0) = 0, \quad \mathrm{d}_U R_\pm(0, 0) = 0.$$

We can then apply to the system (1.9) the normal form theorem for reversible systems in the case of an $(i\omega)^2$ bifurcation (see [27, Chapter 3, Theorem 3.4] and [27, Chapter 4, Lemma 3.17]). We obtain the following result.

Theorem 1.2.1. *For any positive integer $N \geqslant 2$, there exist two neighborhoods V_1 and V_2 of 0 in $\widetilde{\mathbb{R}^4}$ and $\mathbb{R}$, respectively, such that for any $\mu \in V_2$ there exists a polynomial $\Phi(\cdot,\mu) : \widetilde{\mathbb{R}^4} \longrightarrow \widetilde{\mathbb{R}^4}$ of degree N with the following properties :*

(i) the coefficients of the monomials of degree q in $\Phi(\cdot,\mu)$ are functions of μ of class $\mathcal{C}^\infty$,

$$\Phi(0,0,0,0,0) = 0, \qquad \partial_{(A,B,\overline{A},\overline{B})}\Phi(0,0,0,0,0) = 0,$$

and

$$\Phi(\overline{A},-\overline{B},A,-B,\mu) = S\Phi(A,B,\overline{A},\overline{B},\mu);$$

(ii) for $(A,B,\overline{A},\overline{B}) \in V_1$ the change of variable

$$U = A\zeta_0 + B\zeta_1 + \overline{A\zeta_0} + \overline{B\zeta_1} + \Phi(A,B,\overline{A},\overline{B},\mu) \tag{1.11}$$

transforms the system (1.9) into the normal form

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}x} &= i\omega^* A + B + iAP\left(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu\right) + \rho_A(A,B,\overline{A},\overline{B},\mu)\\ \frac{\mathrm{d}B}{\mathrm{d}x} &= i\omega^* B + iBP\left(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu\right)\\ &\quad + AQ\left(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu\right) + \rho_B(A,B,\overline{A},\overline{B},\mu),\end{aligned} \tag{1.12}$$

where P and Q are real-valued polynomials of degree $N-1$ in $(A,B,\overline{A},\overline{B})$. Moreover the remainders ρ_A and ρ_B satisfy

$$\begin{aligned}\rho_A(\overline{A},-\overline{B},A,-B,\mu) &= -\overline{\rho_A}(A,B,\overline{A},\overline{B},\mu),\\ \rho_B(\overline{A},-\overline{B},A,-B,\mu) &= \overline{\rho_B}(A,B,\overline{A},\overline{B},\mu)\end{aligned}$$

and

$$|\rho_A(A,B,\overline{A},\overline{B},\mu)| + |\rho_B(A,B,\overline{A},\overline{B},\mu)| = o\left(|A|^N + |B|^N\right).$$

A key role in our analysis is played by the leading order terms in the Taylor expansions of P and Q. We set

$$\begin{aligned}P\left(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu\right) &= a_1\mu + b_1|A|^2 + \frac{ic_1}{2}(A\overline{B} - \overline{A}B) + O(|\mu|^2 + |A|^4 + |B|^4),\\ Q\left(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu\right) &= a_2\mu + b_2|A|^2 + \frac{ic_2}{2}(A\overline{B} - \overline{A}B) + O(|\mu|^2 + |A|^4 + |B|^4).\end{aligned} \tag{1.13}$$

According to [27, Chapter 4], the dynamics of the system (1.12) is determined by the signs of the coefficients a_2 and b_2. In the case of reversible bifurcations, these coefficients have been explicitly computed in [27, Appendix D]. We recall the following result.

Lemma 1.2.1. *The coefficients a_2 and b_2 in the expansion (1.13) of the polynomial Q are given by*

$$a_2 = \langle R_{\pm}^{1,1}\zeta_0 + 2R_{\pm}^{2,0}(\zeta_0, \Phi_{00001}), \zeta_1^* \rangle, \tag{1.14}$$

$$b_2 = \left\langle 2R_{\pm}^{2,0}(\zeta_0, \Phi_{10100}) + 2R_{\pm}^{2,0}(\overline{\zeta_0}, \Phi_{20000}) + 3R_{\pm}^{3,0}(\zeta_0, \zeta_0, \overline{\zeta_0}), \zeta_1^* \right\rangle , \tag{1.15}$$

where the vectors Φ_{00001}, Φ_{20000} and Φ_{10100} solve the equations

$$L_{\pm}\Phi_{00001} = -R_{\pm}^{0,1}, \quad (L_{\pm} - 2i\omega^*)\Phi_{20000} + R_{\pm}^{2,0}(\zeta_0, \zeta_0) = 0$$

and

$$L_{\pm}\Phi_{10100} + 2R_{\pm}^{2,0}(\zeta_0, \overline{\zeta_0}) = 0.$$

Proof. We start with the Taylor expansion of the map Φ in (1.11)

$$\Phi(A, B, \overline{A}, \overline{B}, \mu) = \sum_{1 \leqslant n+p+q+r+s \leqslant N} A^n B^p \overline{A}^q \overline{B}^r \mu^s \Phi_{npqrs}.$$

We use the change of variable (1.11) with the systems (1.6) and (1.12), which leads to the equality

$$\begin{aligned} &(i\omega^* A + B)\partial_A \Phi + i\omega^* B \partial_B \Phi + (-i\omega^* \overline{A} + \overline{B})\partial_{\overline{A}} \Phi - i\omega^* \overline{B} \partial_{\overline{B}} \Phi \\ &+ \left(iA(\zeta_0 + \partial_A \Phi) - i\overline{A}(\overline{\zeta_0} + \partial_{\overline{A}}\Phi)\right) P + (\zeta_1 + \partial_B \Phi)(iBP + AQ) + (\overline{\zeta_1} + \partial_{\overline{B}}\Phi(-i\overline{B}P + \overline{A}Q) \\ &\qquad = L_{\pm}\Phi + R_{\pm}(A\zeta_0 + B\zeta_1 + \overline{A\zeta_0} + \overline{B\zeta_1} + \Phi, \mu). \end{aligned} \tag{1.16}$$

Computation of a_2. Using the expansions of $R_{\pm}$, P and Q and identifying the terms of order $O(A\mu)$ in (1.16), we obtain the equality

$$a_1\zeta_0 + a_2\zeta_1 = (L_{\pm} - i\omega^*)\Phi_{10001} + R_{\pm}^{1,1}\zeta_0 + 2R_{\pm}^{2,0}(\zeta_0, \Phi_{00001}).$$

The formula (1.14) directly follows, using the fact that the vector ζ_1^* is orthogonal to the range of $L_{\pm} - i\omega^*$. Finally, by identifying the terms of order $O(\mu)$ in (1.16), we find that Φ_{00001} solve the linear system

$$L_{\pm}\Phi_{00001} = -R_{\pm}^{0,1}.$$

Computation of b_2. Using the expansions of $R_{\pm}$, P and Q and identifying the terms of order $O(A^2\overline{A})$ in (1.16), we obtain the equality

$$ib_1\zeta_0 + b_2\zeta_1 = (L_{\pm} - i\omega^*)\Phi_{20100} + 2R_{\pm}^{2,0}(\zeta_0, \Phi_{10100}) + 2R_{\pm}^{2,0}(\overline{\zeta_0}, \Phi_{20000}) + 3R_{\pm}^{3,0}(\zeta_0, \zeta_0, \overline{\zeta_0}). \tag{1.17}$$

Again, using the fact that the vector ζ_1^* is orthogonal to the range of $L_{\pm} - i\omega^*$, we obtain the formula (1.15). Finally, by identifying in (1.16) the terms of order $O(A^2)$ and $O(A\overline{A})$, respectively, we find that Φ_{20000} and Φ_{10100} solve the equations

$$(L_{\pm} - 2i\omega^*)\Phi_{20000} + R_{\pm}^{2,0}(\zeta_0, \zeta_0) = 0,$$

$$L_{\pm}\Phi_{10100} + 2R_{\pm}^{2,0}(\zeta_0, \overline{\zeta_0}) = 0.$$

This completes the proof of the lemma. □

1.2.2 Case of normal dispersion ($\beta = 1$)

We suppose that $\beta = 1$, $\alpha^* > 2$ and $F^{*2} = 1 + (1-\alpha^*)^2$. In this case, we have $\omega^* = \sqrt{\alpha^* - 2}$. A direct computation shows that the vectors ζ_0, ζ_1 and the dual vector ζ_1^* are given by

$$\zeta_0 = \begin{pmatrix} M \\ i\omega^* M \\ 1 \\ i\omega^* \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} \frac{2i\omega^*}{1+2\psi_r^*\psi_i^*} \\ M - \frac{2\omega^{*2}}{1+2\psi_r^*\psi_i^*} \\ 0 \\ 1 \end{pmatrix}, \quad \zeta_1^* = \frac{2-\alpha^*}{4F^{*2}} \begin{pmatrix} -i\omega^* \\ 1 \\ i\omega^* M \\ -M \end{pmatrix},$$

where, using the formulas for ψ_r^* and ψ_i^* and the fact that $\psi_r^{*2} + \psi_i^{*2} = 1$, we obtain

$$M = \frac{1 - 2\psi_r^*\psi_i^*}{3\psi_r^{*2} + \psi_i^{*2} - 2} = \frac{\psi_r^* - \psi_i^*}{\psi_r^* + \psi_i^*} = \frac{\alpha^*}{2-\alpha^*}.$$

The following lemma gives the expression for the coefficients a_2 and b_2 above, in terms of the parameters α^* and F^*.

Lemma 1.2.2. *In the case $\beta = 1$, the coefficients a_2 and b_2 in (1.14) and (1.15) are given by*

$$a_2 = \frac{\alpha^* - 1}{(\alpha^* - 2)^3} > 0, \quad b_2 = \frac{2F^{*2}(41 - 30\alpha^*)}{9(\alpha^* - 2)^5} < 0.$$

Proof. **Computation of a_2.** According to formula (1.14), we have

$$a_2 = \langle R_+^{1,1}\zeta_0 + 2R_+^{2,0}(\zeta_0, \Phi_{00001}), \zeta_1^* \rangle, \quad L_+\Phi_{00001} = -R_+^{0,1}. \tag{1.18}$$

Using the expression of $R_+^{0,1}$ given in Section 1.1.3, we find

$$\Phi_{00001} = \frac{1}{(\alpha^* - 2)^2} \begin{pmatrix} (1-\alpha^*)\psi_r^* + \psi_i^* \\ 0 \\ -\psi_r^* + (1-\alpha^*)\psi_i^* \\ 0 \end{pmatrix}.$$

Next, we compute

$$R_+^{1,1}\zeta_0 = \begin{pmatrix} 0 \\ -M \\ 0 \\ -1 \end{pmatrix}$$

and

$$R_+^{2,0}(\zeta_0, \Phi_{00001}) = \begin{pmatrix} 0 \\ \frac{(3M(1-\alpha^*)-1)\psi_r^{*2} + 2(M+1-\alpha^*)\psi_r^*\psi_i^* + (M(1-\alpha^*)+1)\psi_i^{*2}}{(\alpha^*-2)^2} \\ 0 \\ \frac{(1-\alpha^*-M)\psi_r^{*2} + 2(M(1-\alpha^*)-1)\psi_r^*\psi_i^* + (M+3(1-\alpha^*))\psi_i^{*2}}{(\alpha^*-2)^2} \end{pmatrix}.$$

We obtain from (1.18)

$$a_2 = \frac{(2M(1-\alpha^*)+M^2-1)(\psi_r^{*2}-\psi_i^{*2})+2(2M+(1-\alpha^*)(1-M^2))\psi_r^*\psi_i^*}{2F^{*2}(2-\alpha^*)}.$$

Finally, using the symbolic package Maple, we find

$$a_2 = \frac{\alpha^*-1}{(\alpha^*-2)^3} > 0.$$

Computation of b_2. According to formula (1.15), we have

$$b_2 = \left\langle 2R_+^{2,0}(\zeta_0, \Phi_{10100}) + 2R_+^{2,0}(\overline{\zeta_0}, \Phi_{20000}) + 3R_+^{3,0}(\zeta_0,\zeta_0,\overline{\zeta_0}), \zeta_1^* \right\rangle. \qquad (1.19)$$

Using the formulas for $R_+^{2,0}$ in Section 1.1.3, we find

$$R_+^{2,0}(\zeta_0,\zeta_0) = R_+^{2,0}(\zeta_0,\overline{\zeta_0}) = \begin{pmatrix} 0 \\ (3M^2+1)\psi_r^*+2M\psi_i^* \\ 0 \\ 2M\psi_r^*+(M^2+3)\psi_i^* \end{pmatrix},$$

and then

$$\Phi_{20000} = \frac{1}{9(\alpha^*-2)^2}\begin{pmatrix} D \\ 2i\omega^* D \\ E \\ 2i\omega^* E \end{pmatrix}, \quad \Phi_{10100} = \frac{1}{(\alpha^*-2)^2}\begin{pmatrix} G \\ 0 \\ H \\ 0 \end{pmatrix},$$

where the quantities D, E, G, H are given by

$$D = -(2\psi_i^{*2}+3\alpha^*-7)((3M^2+1)\psi_r^*+2M\psi_i^*)+(2\psi_r^*\psi_i^*-1)(2M\psi_r^*+(M^2+3)\psi_i^*),$$

$$E = -(2\psi_r^{*2}+3\alpha^*-7)(2M\psi_r^*+(M^2+3)\psi_i^*)+(2\psi_r^*\psi_i^*+1)((3M^2+1)\psi_r^*+2M\psi_i^*)$$

$$G = -2(2\psi_i^{*2}+1-\alpha^*)((3M^2+1)\psi_r^*+2M\psi_i^*)-2((3M^2+1)\psi_i^*+2M\psi_r^*)(\psi_r^*-\psi_i^*)^2,$$

and

$$H = -2(2\psi_r^{*2}+1-\alpha^*)(2M\psi_r^*+(M^2+3)\psi_i^*)+2((3M^2+1)\psi_r^*+2M\psi_i^*)(2\psi_r^*\psi_i^*+1).$$

Next, we compute

$$R_+^{2,0}(\zeta_0, \Phi_{10100}) = \begin{pmatrix} 0 \\ (3MG+H)\psi_r^*+(MH+G)\psi_i^* \\ 0 \\ (MH+G)\psi_r^*+(MG+3H)\psi_i^* \end{pmatrix},$$

$$R_+^{2,0}(\overline{\zeta_0}, \Phi_{20000}) = \begin{pmatrix} 0 \\ (3MD+E)\psi_r^* + (ME+D)\psi_i^* \\ 0 \\ (ME+D)\psi_r^* + (MD+3E)\psi_i^* \end{pmatrix}, \quad R_+^{3,0}(\zeta_0, \zeta_0, \overline{\zeta_0}) = \begin{pmatrix} 0 \\ M^3+M \\ 0 \\ M^2+1 \end{pmatrix}.$$

Finally, using the symbolic package Maple, we obtain from (1.19)

$$\begin{aligned} b_2 &= \frac{2-\alpha^*}{2F^{*2}}\left((D+G)\left(2M\psi_r^* + (1-M^2)\psi_i^*\right) + (E+M)\left((1-M^2)\psi_r^* - 2M\psi_i^*\right)\right) \\ &= \frac{2F^{*2}(41-30\alpha^*)}{9(\alpha^*-2)^5} < 0. \end{aligned}$$

This completes the proof of the lemma. □

Bounded solutions of (1.9). The bounded solutions of the system are found via perturbation arguments from solutions of the truncated normal norm (see the system (1.20)-(1.21) below). While the persistence of equilibria and of periodic solutions is proved using the implicit function theorem, the reversibility of the system plays a key role in the proofs of persistence of other solutions, in particular of homoclinic solutions. For these solutions, persistence is shown for the symmetric solutions of the truncated normal form, *i.e.*, solutions satisfying $A(x) = \overline{A}(-x)$ and $B(x) = -\overline{B}(-x)$ (see [35, Section III] for more details). According to [27, Chapter 4, Theorem 3.21], we obtain the following result.

Theorem 1.2.2. *Suppose that $\beta = 1$, $\alpha^* > 2$ and $F^{*2} = 1 + (1-\alpha^*)^2$.*

(i) For any $\mu > 0$ sufficiently small, the dynamical system (1.9) possesses one symmetric equilibrium, a one-parameter family of periodic orbits, a two-parameter family of quasiperiodic orbits and a pair of reversible homoclinic orbits to the symmetric equilibrium.

(ii) For any $\mu < 0$ sufficiently small, the dynamical system (1.9) possesses one symmetric equilibrium, a one-parameter family of periodic orbits and a two-parameter family of quasiperiodic orbits.

Taylor expansions of periodic and localized solutions of (1.1). Going back to the equation (1.1), we can compute the leading order terms in the Taylor expansions of the bounded solutions corresponding to the ones given by the previous theorem. We focus on the periodic and localized solutions, which correspond to periodic and homoclinic orbits, respectively, of the system (1.9).

First, the origin is an equilibrium of the normal form (1.12), which corresponds to a symmetric equilibrium of the system (1.9), and to a **constant solution** of (1.1), given by

$$\psi = \psi^* + O(\mu).$$

This solution is precisely the constant solution of the equation (1.1) given in Section 1.1.1, with corresponding parameters $\alpha = \alpha^* + \mu$ and $F = F^*$.

The leading order terms of the non-constant solutions are computed from the truncated normal form

$$\frac{\mathrm{d}A}{\mathrm{d}x} = i\omega^* A + B \tag{1.20}$$

$$\frac{\mathrm{d}B}{\mathrm{d}x} = i\omega^* B + a_2\mu A + b_2|A|^2 A, \tag{1.21}$$

and the relationship

$$\psi(x) = \psi^* + 2(M+i)\mathrm{Re}\ A(x) - \frac{4\omega^*}{1+2\psi_r^*\psi_i^*}\mathrm{Im}\ B(x) + O(|\mu| + (|A(x)|+|B(x)|)^2),$$

obtained by going back to the normal form calculations in Section 1.2.1.

In order to find the family of **periodic solutions**, we consider the ansatz

$$A(x) = e^{i(\omega^*+K)x}\widehat{A}(x), \quad B(x) = e^{i(\omega^*+K)x}\widehat{B}(x), \quad K \in \mathbb{R},$$

which transforms the system (1.20)-(1.21) into the system

$$\frac{\mathrm{d}\widehat{A}}{\mathrm{d}x} = -iK\widehat{A} + \widehat{B} \tag{1.22}$$

$$\frac{\mathrm{d}\widehat{B}}{\mathrm{d}x} = -iK\widehat{B} + a_2\mu\widehat{A} + b_2|\widehat{A}|^2\widehat{A}. \tag{1.23}$$

The one-parameter families of periodic solutions are found, in both cases, $\mu >$ and $\mu < 0$, from the symmetric nontrivial equilibria of this system. When K is such that $a_2\mu + K^2 > 0$, the system (1.22)-(1.23) has two nontrivial equilibria, given by

$$\widehat{A} = \pm\sqrt{\frac{-a_2\mu - K^2}{b_2}}, \quad \widehat{B} = iK\widehat{A}.$$

These equilibria correspond to a family of periodic solutions of the truncated normal form (1.20)-(1.21) of the form

$$A(x) = \pm\sqrt{\frac{-a_2\mu - K^2}{b_2}}e^{i(\omega^*+K)x}, \quad B(x) = iKA(x).$$

For sufficiently small K with $a_2\mu + K^2 > 0$, these solutions persist for the full normal form. The corresponding solutions of the Lugiato-Lefever equation (1.1) have the form

$$\psi_{\mu,K}(x) = P(kx), \tag{1.24}$$

where the $2\pi-$periodic function P is given by

$$P(y) = \psi^* + 2\sqrt{\frac{-a_2\mu - K^2}{b_2}}\left(M - \frac{4\omega^*}{1+2\psi_r^*\psi_i^*} + i\right)\cos(y) + O(|\mu|),$$

and

$$k = \omega^* + K + O(|\mu|^{\frac{1}{2}}).$$

In the case $\mu > 0$, the pair of **reversible homoclinic solutions** of the system (1.20)-(1.21) have the form

$$A(x) = \pm\sqrt{\frac{-2a_2\mu}{b_2}}\,\mathrm{sech}(\sqrt{a_2\mu}x)e^{i\omega^* x}, \quad B(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x) - i\omega^* A(x).$$

The corresponding solutions of (1.1) are solitary waves with the expansion

$$\psi_\mu(x) = \psi^* \pm 2(M+i)\sqrt{\frac{-2a_2}{b_2}}\,\mathrm{sech}(\sqrt{a_2\mu}x)\cos(\omega^* x)\mu^{\frac{1}{2}} + O(\mu).$$

1.2.3 Case of anomalous dispersion ($\beta = -1$)

We consider the case $\beta = -1$ and keep the same notations as in the previous section. We suppose that $\alpha^* < 2$, $F^{*2} = 1 + (1-\alpha^*)^2$ and $\rho^* = 1$. In this case we have $\omega^* = \sqrt{2-\alpha^*}$. A direct computation shows that the vectors ζ_0, ζ_1 and the dual vector ζ_1^* are given by

$$\zeta_0 = \begin{pmatrix} M \\ i\omega^* M \\ 1 \\ i\omega^* \end{pmatrix}, \qquad \zeta_1 = \begin{pmatrix} -\frac{2i\omega^*}{1+2\psi_r^*\psi_i^*} \\ M + \frac{2\omega^{*2}}{1+2\psi_r^*\psi_i^*} \\ 0 \\ 1 \end{pmatrix}, \qquad \omega^* = \sqrt{2-\alpha^*},$$

and

$$\zeta_1^* = \frac{2-\alpha^*}{4F^{*2}}\begin{pmatrix} -i\omega^* \\ 1 \\ i\omega^* M \\ -M \end{pmatrix}, \qquad M = \frac{\alpha^*}{2-\alpha^*}.$$

Notice that the formula for the eigenvector ζ_0 is the same as in the case $\beta = 1$.

Lemma 1.2.3. *In the case $\beta = -1$, the coefficients a_2 and b_2 in (1.14) and (1.15) are given by*

$$a_2 = \frac{\alpha^* - 1}{(2-\alpha^*)^3}, \qquad b_2 = \frac{2F^{*2}(41 - 30\alpha^*)}{9(2-\alpha^*)^5}.$$

Notice that the signs of these coefficients depend upon α^*. They change at $\alpha^* = 1$ and $\alpha^* = 41/30$.

Bounded solutions of (1.9). Using [27, Chapter 4, Theorem 3.21] we obtain the following result.

Theorem 1.2.3. *Assume that $\beta = -1$.*

(i) Suppose that $41/30 < \alpha^* < 2$ and $F^{*2} = 1+(1-\alpha^*)^2$. Then $a_2 > 0, b_2 < 0$, and for any μ sufficiently small, the dynamical system (1.9) possesses the same types of solutions as in Theorem 1.2.2.

(ii) Suppose that $\alpha^* < 41/30$, $\alpha^* \neq 1$ and $F^{*2} = 1 + (1-\alpha^*)^2$. Then $b_2 > 0$.

(a) For any μ sufficiently small such that $a_2\mu > 0$, there is one symmetric equilibrium and no other bounded solution.

(b) For any μ sufficiently small such that $a_2\mu < 0$, the dynamical system (1.9) has one symmetric equilibrium, a one-parameter family of periodic orbits and a two-parameter family of quasiperiodic orbits. Moreover, there exists a one-parameter family of pairs of reversible homoclinic orbits to periodic orbits.

Remark 1.2.1. *The case $\alpha^* = 1$ does not enter in this analysis, since the coefficient a_2 vanishes. This issue can be avoided by fixing $\alpha = \alpha^*$ and taking $F^2 = F^{*2} + \mu$ as a bifurcation parameter.*

Taylor expansions of periodic and localized solutions of (1.1). As before we can compute the leading order terms in the expansions of the corresponding solutions of the equation (1.1). The truncated normal form is the same system (1.20)-(1.21) obtained in the previous case. We give hereafter the Taylor expansions of the corresponding bounded solutions of the Lugiato-Lefever equation (1.1).

In the case $41/30 < \alpha^* < 2$, the bounded orbits are the same as the ones found in the previous section for $\beta = 1$, and their expansions are also the same. In the case $\alpha^* < 41/30$ and $a_2\mu < 0$, the family of periodic solutions of the system (1.6) has also the same form as in Section 1.2.2, and the corresponding periodic solutions are given by formula (1.24). In this case, the truncated system (1.20)-(1.21) possesses also a one-parameter family of pairs of homoclinic orbits to periodic solutions. We restrict here to the simplest pair of solutions in this family, given by

$$A(x) = \pm\sqrt{\frac{-a_2\mu}{b_2}}\tanh\left(\sqrt{\frac{-a_2\mu}{2}}x\right)e^{i\omega^* x}, \quad B(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x) + i\omega^* A(x).$$

The corresponding solution of (1.1) has the form

$$\psi_\mu(x) = \psi^* \pm 2(M+i)\sqrt{\left|\frac{a_2}{b_2}\right|}\tanh\left(\sqrt{\frac{-a_2\mu}{2}}x\right)\cos(\omega^* x)|\mu|^{\frac{1}{2}} + O(|\mu|).$$

1.3 Reversible $0^2(i\omega)$ bifurcation

In this section we consider the $0^2(i\omega)$ bifurcation. We compute the normal form of the system (1.9) in the cases of normal ($\beta = 1$) and anomalous ($\beta = -1$) dispersion, and then discuss the existence of bounded stationary solutions of the equation (1.1). As for the previous bifurcation, we focus on periodic and localized solutions, which are found as periodic orbits and homoclinic to periodic orbits of the normal form. We point out that this bifurcation occurs in a neighborhood of the constant solution of (1.1) which is a critical point of the polynomial P_1 given by (1.4).

In the case of normal dispersion ($\beta = 1$), this bifurcation occurs for

$$\alpha^* > 2, \quad F^{*2} = F_+^2(\alpha^*) = \frac{2\alpha^* - \gamma^*}{3}\left(1 + \left(\frac{\gamma^* + \alpha^*}{3}\right)^2\right),$$

where $\gamma^* = \sqrt{\alpha^{*2} - 3}$. In this case, we have $\rho^* = \rho_+(\alpha^*) = (2\alpha^* - \gamma^*)/3$.

In the case of anomalous dispersion ($\beta = -1$), we consider two cases for the parameters α^* and F^* :

(i) $\alpha^* \in \left(\sqrt{3}, 2\right)$ and $F^{*2} = F_+^2(\alpha^*)$. In this case we have $\rho^* = \rho_+(\alpha^*)$.

(ii) $\alpha^* > \sqrt{3}$ and

$$F^{*2} = F_-^2(\alpha^*) = \frac{2\alpha^* + \gamma^*}{3}\left(1 + \left(\frac{\gamma^* - \alpha^*}{3}\right)^2\right),$$

when $\rho^* = \rho_-(\alpha^*) = (2\alpha^* + \gamma^*)/3$.

Recall that we exclude the cusp point $(\alpha^*, F^{*2}) = (\sqrt{3}, 8\sqrt{3}/9)$, because at this point one of the coefficients of the normal form computed in Section 1.3.3 below vanishes.

In both cases of normal and anomalous dispersion, we have

$$3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 = 0,$$

so that 0 is an algebraically double eigenvalue of $L_\pm$, and the other two eigenvalues $\pm i\omega^*$ of $L_\pm$ satisfy

$$\omega^{*2} = \mp(4\rho^* - 2\alpha^*).$$

1.3.1 Normal form

As in Section 1.2.1, we start with a convenient basis of $\mathbb{R}^4$. According to [27, Chapter 4, Lemma 3.3], there exists a basis $\{\zeta_0, \zeta_1, \operatorname{Re}\zeta, \operatorname{Im}\zeta\}$ of $\mathbb{R}^4$, such that

$$L_\pm\zeta_0 = 0, \quad L_\pm\zeta_1 = \zeta_0, \quad L_\pm\zeta = i\omega^*\zeta, \quad S\zeta_0 = \zeta_0, \quad S\zeta_1 = -\zeta_1, \quad S\zeta = \overline{\zeta}.$$

In this basis we represent a vector $U \in \mathbb{R}^4$ by

$$U = A\zeta_0 + B\zeta_1 + C\zeta + \overline{C\zeta},$$

with $A, B \in \mathbb{R}$ and $C \in \mathbb{C}$, by identifying $\mathbb{R}^4$ with the space $\mathbb{R}^2 \times \widetilde{\mathbb{R}^2}$, where $\widetilde{\mathbb{R}^2} = \left\{(C, \overline{C}), C \in \mathbb{C}\right\}$. We also consider a vector ζ_1^* satisfying

$$\langle\zeta_0, \zeta_1^*\rangle = 0 \qquad \langle\zeta_1, \zeta_1^*\rangle = 1, \qquad L_\pm^*\zeta_1^* = 0,$$

where $L_\pm^*$ is the adjoint operator of $L_\pm$ and $\langle\cdot,\cdot\rangle$ denotes the Hermitian inner product on $\mathbb{R}^2 \times \widetilde{\mathbb{R}^2}$.

The vector field in the reversible system (1.9) is of class $\mathcal{C}^\infty$ on $\mathbb{R}^4 \times \mathbb{R}$ and satisfies

$$R_\pm(0,0) = 0, \quad \mathrm{d}_U R_\pm(0,0) = 0.$$

We apply to the system (1.9) the normal form theorem for reversible systems in the case of a $0^2(i\omega)$ bifurcation (see [27, Chapter 3, Theorem 3.4] and [27, Chapter 4, Lemma 3.5]), and obtain the following theorem.

Theorem 1.3.1. *For any positive integer $N \geqslant 2$, there exist two neighborhoods V_1 and V_2 of 0 in $\mathbb{R}^2 \times \widetilde{\mathbb{R}^2}$ and $\mathbb{R}$, respectively, and a polynomial $\Phi : V_1 \times V_2 \longrightarrow \mathbb{R}^4$ of degree N with the properties :*

(i) Φ satisfies

$$\Phi(0,0,0,0,0) = 0, \quad \partial_{(A,B,C,\overline{C})}\Phi(0,0,0,0,0) = 0,$$

and

$$\Phi(A, -B, \overline{C}, C, \mu) = S\Phi(A, B, C, \overline{C}, \mu).$$

(ii) For $(A, B, C, \overline{C}) \in V_1$, the change of variable

$$U = A\zeta_0 + B\zeta_1 + C\zeta + \overline{C\zeta} + \Phi(A, B, C, \overline{C}, \mu) \tag{1.25}$$

transforms the equation (1.9) into the normal form

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} &= B \\ \frac{\mathrm{d}B}{\mathrm{d}x} &= P(A, |C|^2, \mu) + \rho_B(A, B, C, \overline{C}, \mu) \\ \frac{\mathrm{d}C}{\mathrm{d}x} &= i\omega^* C + iCQ(A, |C|^2, \mu) + \rho_C(A, B, C, \overline{C}, \mu), \end{aligned} \tag{1.26}$$

where P and Q are real-valued polynomials of degree N and $N-1$ in $(A, B, C, \overline{C})$, respectively. Moreover the remainders ρ_B and ρ_C satisfy

$$\begin{aligned} \rho_B(A, -B, \overline{C}, C, \mu) &= \rho_B(A, B, C, \overline{C}, \mu), \\ \rho_C(A, -B, \overline{C}, C, \mu) &= -\overline{\rho_C}(A, B, C, \overline{C}, \mu), \end{aligned}$$

with the estimate

$$\left|\rho_B(A, B, C, \overline{C}, \mu)\right| + \left|\rho_C(A, B, C, \overline{C}, \mu)\right| = o\left((|A| + |B| + |C|)^N\right).$$

As before, we consider the Taylor expansions of P and Q at order 2 :

$$\begin{aligned} P(A, |C|^2, \mu) &= a_1\mu + b_1A^2 + c_1|C|^2 + O(|\mu|^2 + (|A| + |C|)^3), \\ Q(A, |C|^2, \mu) &= a_2\mu + b_2A + c_2|C|^2 + O((|\mu| + |A| + |C|^2)^2). \end{aligned} \tag{1.27}$$

The dynamics of the system (1.26) depends on the signs of the coefficients a_1, b_1 and c_1, provided that they do not vanish [27, Chapter 4, Theorem 3.10].

Lemma 1.3.1. *The coefficients a_1, b_1 and c_1 in the expansion of P in (1.27) are given by*

$$\begin{aligned} a_1 &= \left\langle R_{\pm}^{0,1}, \zeta_1^* \right\rangle, & (1.28)\\ b_1 &= \left\langle R_{\pm}^{2,0}(\zeta_0,\zeta_0), \zeta_1^* \right\rangle, & (1.29)\\ c_1 &= \left\langle 2R_{\pm}^{2,0}(\zeta,\overline{\zeta}), \zeta_1^* \right\rangle. & (1.30)\end{aligned}$$

Proof. We start with the Taylor expansion of the map Φ in (1.25) :

$$\Phi(A,B,C,\overline{C},\mu) = \sum_{1\leqslant n+p+q+r+s\leqslant N} A^n B^p C^q \overline{C}^r \mu^s \Phi_{npqrs}.$$

The change of variable (1.25) in the systems (1.6) and (1.26) gives

$$\begin{aligned} B\partial_A\Phi + i\omega^* C\partial_C\Phi - i\omega^*\overline{C}\partial_{\overline{C}}\Phi + (\zeta_1 + \partial_B\Phi)P(A,|C|^2,\mu) + \\ \left(iC(\zeta+\partial_C\Phi) - i\overline{C}(\overline{\zeta}+\partial_{\overline{C}}\Phi)\right) Q(A,|C|^2,\mu) \\ = L_{\pm}\Phi + R_{\pm}(A\zeta_0 + B\zeta_1 + C\zeta + \overline{C\zeta} + \Phi,\mu). \quad (1.31)\end{aligned}$$

Then using the expansions of $R_{\pm}$, Φ, P and Q and identifying the terms of orders $O(\mu)$, $O(A^2)$ and $O(C\overline{C})$ in (1.31), respectively, we obtain the following equalities :

$$\begin{aligned} a_1\zeta_1 &= L_{\pm}\Phi_{00001} + R_{\pm}^{0,1}, & (1.32)\\ b_1\zeta_1 &= L_{\pm}\Phi_{20000} + R_{\pm}^{2,0}(\zeta_0,\zeta_0), & (1.33)\\ c_1\zeta_1 &= L_{\pm}\Phi_{00110} + 2R_{\pm}^{2,0}(\zeta,\overline{\zeta}). & (1.34)\end{aligned}$$

Using the fact that the dual vector ζ_1^* belongs to the kernel of the adjoint operator $L_{\pm}^*$, we obtain from (1.32), (1.33) and (1.34),

$$a_1 = \left\langle R_{\pm}^{0,1}, \zeta_1^* \right\rangle, \quad b_1 = \left\langle R_{\pm}^{2,0}(\zeta_0,\zeta_0), \zeta_1^* \right\rangle, \quad c_1 = \left\langle 2R_{\pm}^{2,0}(\zeta,\overline{\zeta}), \zeta_1^* \right\rangle.$$

□

1.3.2 Case of normal dispersion ($\beta = 1$)

We suppose that $\alpha^* > 2$ and $F^{*2} = F_+^2(\alpha^*)$. In this case, we have

$$\omega^{*2} = 2\alpha^* - 4\rho^* = \frac{2}{3}\left(2\sqrt{\alpha^{*2}-3} - \alpha^*\right).$$

A direct computation shows that the vectors ζ_0, ζ_1, ζ and the dual vector ζ_1^* are given by

$$\zeta_0 = \begin{pmatrix}1\\0\\D\\0\end{pmatrix}, \quad \zeta_1 = \begin{pmatrix}0\\1\\0\\D\end{pmatrix}, \quad \zeta = \begin{pmatrix}1\\i\omega^*\\0\\0\end{pmatrix}, \quad \zeta_1^* = \frac{1}{D}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad (1.35)$$

in which

$$D = \frac{3\psi_r^{*2} + \psi_i^{*2} - \alpha^*}{1 - 2\psi_r^*\psi_i^*} = -\frac{\omega^{*2}}{2}.$$

The explicit expressions of the coefficients a_1, b_1 and c_1 in terms of the parameters α^* and F^* can be computed using Lemma 1.3.1.

Lemma 1.3.2. *In the case $\beta = 1$, the coefficients a_1, b_1 and c_1 in (1.28), (1.29) and (1.30) are given by*

$$a_1 = -\frac{9F^*(\alpha^* + \gamma^*)}{2(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} < 0,$$

$$b_1 = \frac{3F^*(2\alpha^{*5} - 18\alpha^* + (2\alpha^{*4} + 3\alpha^{*2} + 9)\gamma^*)}{2(\alpha^{*2} + 3 + \alpha^*\gamma^*)(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} > 0,$$

$$c_1 = \frac{9F^*(\alpha^* + \gamma^*)}{\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*} > 0.$$

Proof. According to Lemma 1.3.1, the coefficients a_1, b_1 and c_1 are given by

$$a_1 = \left\langle R_+^{0,1}, \zeta_1^* \right\rangle, \quad b_1 = \left\langle R_+^{2,0}(\zeta_0, \zeta_0), \zeta_1^* \right\rangle, \quad c_1 = \left\langle 2R_+^{2,0}(\zeta, \overline{\zeta}), \zeta_1^* \right\rangle.$$

Using the Taylor expansion of the map R_+ in Section 1.1.3, we have

$$R_+^{0,1} = \begin{pmatrix} 0 \\ -\psi_r^* \\ 0 \\ -\psi_i^* \end{pmatrix}, \quad R_+^{2,0}(\zeta_0, \zeta_0) = \begin{pmatrix} 0 \\ (D^2 + 3)\psi_r^* + 2D\psi_i^* \\ 0 \\ 2D\psi_r^* + (3D^2 + 1)\psi_i^* \end{pmatrix}, \quad R_+^{2,0}(\zeta, \overline{\zeta}) = \begin{pmatrix} 0 \\ 3\psi_r^* \\ 0 \\ \psi_i^* \end{pmatrix},$$

and a direct computation gives

$$a_1 = -\frac{\psi_i^*}{D}, \qquad b_1 = \frac{2D\psi_r^* + (3D^2 + 1)\psi_i^*}{D}, \qquad c_1 = \frac{2\psi_i^*}{D}.$$

The explicit expressions of a_1, b_1 and c_1 in terms of α^* and F^* and their signs are obtained with the symbolic package Maple. □

Bounded solutions of (1.9). Here again, the reversibility of the system (1.9) is a crucial argument in the proofs of the existence of solutions. Following [27, Chapter 4, Theorem 3.10], and taking into account the signs of the coefficients a_1, b_1 and c_1, we obtain the following result :

Theorem 1.3.2. *Suppose that $\beta = 1$, $\alpha^* > 2$ and $F^{*2} = F_+^2(\alpha^*)$.*

(i) For any $\mu > 0$ sufficiently small, the dynamical system (1.9) possesses two equilibria, a center and a saddle-center, together with two one-parameter families of periodic orbits, called periodic orbits of the first kind, parametrized by their size r, for $r < r^*(\mu) = O(\mu^{\frac{1}{2}})$, which tend to the two equilibria when r tends to 0. For any periodic orbit in the family which tends to the saddle-equilibrium, with size r not too small, $r > r_*(\mu)$, there is a pair of reversible homoclinic orbits connecting the periodic orbit to itself. In addition, the system possesses periodic orbits called periodic orbits of the second kind and quasiperiodic orbits.

(ii) For any $\mu < 0$ sufficiently small, the dynamical system (1.9) has no bounded solution.

Taylor expansions of periodic and localized solutions of (1.1). As for the $(i\omega)^2$ bifurcation before, the leading order terms of bounded solutions of the normal form (1.26) are computed from the truncated system

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} &= B \\ \frac{\mathrm{d}B}{\mathrm{d}x} &= a_1\mu + b_1A^2 + c_1|C|^2 \\ \frac{\mathrm{d}C}{\mathrm{d}x} &= i\omega^* C. \end{aligned} \tag{1.36}$$

We restrict to $\mu > 0$, since the system has no bounded solutions for $\mu < 0$. Going back to the normal form computations in Section 1.3.1, the corresponding solutions of the Lugiato-Lefever equation (1.1) have the form

$$\psi(x) = \psi^* + (1 + iD)A(x) + 2\mathrm{Re}\ C(x) + O(|\mu| + (|A(x)| + |B(x)| + |C(x)|)^2).$$

Notice that $K = |C|^2 > 0$ is a first integral of the system (1.36).

First, the truncated system (1.36) has two **equilibria** $(\pm A_0, 0, 0)$, with $A_0 = \sqrt{-a_1\mu/b_1}$. The equilibrium $(A_0, 0, 0)$ is a saddle-center and $(-A_0, 0, 0)$ is a center. These equilibria correspond to the constant solutions of the equation (1.1) discussed in Section 1.1.1, with parameters $\alpha = \alpha^* + \mu$ and $F = F^*$, and read

$$\psi(x) = \psi^* \pm (1 + iD)\sqrt{\frac{-a_1}{b_1}}\mu^{\frac{1}{2}} + O(\mu).$$

Next, the truncated system (1.36) possesses two families of **periodic orbits of the first kind** $(\pm A_K, 0, C(x))$, where

$$A_K = \sqrt{\frac{-a_1\mu - c_1K}{b_1}}, \quad C(x) = \sqrt{K}e^{i\omega^* x}, \tag{1.37}$$

in which K is such that $c_1K < |a_1\mu|$, in particular their size is of order $O(|\mu|^{\frac{1}{2}})$. These periodic orbits are called periodic orbits of the first kind, because their (A, B) components are constant functions. The corresponding solutions of the equation (1.1) are periodic solutions of the form

$$\psi_{\mu,\pm,K}(x) = P_1(kx),$$

where the $2\pi-$periodic function P_1 has the expansion

$$P_1(y) = \psi^* \pm (1+Di)\sqrt{\frac{-a_1\mu - c_1K}{b_1}} + 2\sqrt{K}\cos(y) + O(\mu),$$

and $k = \omega^* + O(\mu^{\frac{1}{2}})$.

In addition, the system (1.36) admits a third family of periodic orbits, which are obtained for $C = 0$ and are called **periodic orbits of the second kind**. These solutions have the form $(A(x), B(x), 0)$, and (A, B) satisfy a system of two differential equations, which are the two first equations in (1.36) with $C = 0$. In this system, the equilibrium $(-A_0, 0)$ is a center, and is surrounded by a one-parameter family of periodic orbits. The solutions which are close to the center have the expansion

$$A(x) = -\sqrt{\frac{-a_1\mu}{b_1}} + \varepsilon\sqrt{\mu}\cos(\sqrt{2}(-a_1b_1\mu)^{\frac{1}{4}}x) + O(\varepsilon^2), \quad B(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x),$$

where ε is a real and small parameter. The corresponding solutions of the equation (1.1) have the form

$$\psi_{\mu,\varepsilon}(x) = P_2(kx),$$

where the $2\pi-$periodic function P_2 is given by

$$P_2(y) = \psi^* + (1+iD)\left(-\sqrt{\frac{-a_1}{b_1}} + \varepsilon\cos(y)\right)\mu^{\frac{1}{2}} + O(\mu+\varepsilon^2)$$

and $k = \sqrt{2}(-a_1b_1\mu)^{\frac{1}{4}} + O(\mu^{\frac{1}{2}})$.

Finally, to each of the periodic orbits of the first kind of the truncated normal form surrounding the saddle-center $(A_0, 0, 0)$, there exists a one-parameter family of **homoclinic orbit** connecting this periodic orbit to itself. This family of homoclinic orbits is explicitly given by

$$A(x) = A_K\left(1 - 3\,\mathrm{sech}^2\left(\delta_K x\right)\right), \qquad B(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x), \qquad C(x) = \sqrt{K}e^{i(\omega^* x + \lambda\tanh(\delta_K x) + \varphi)},$$

where $K > 0$, $\delta_K = \sqrt{b_1A_K/2}$, $\lambda = O(1)$ and $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$.

The question of the persistence of these solutions as solutions of the full system (1.26) is a delicate issue, which has been studied in [45, Chapter 7, Section 7.3]. It turns out that the reversible orbits (*i.e.*, the orbits obtained with $\varphi = 0, \pi$) persist, provided that K is not too small. In particular, the homoclinic solution $(A(x), B(x), 0, 0)$ of (1.36), with

$$A(x) = A_0\left(1 - 3\,\mathrm{sech}^2\left(\delta_0 x\right)\right), \qquad B(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x),$$

does not persist as a homoclinic solution of the system (1.26) (see [45, Chapter 7, Section 7.4]).

The solutions of the equation (1.1) corresponding to the homoclinic solutions which persist, hence for K not too small, have the form

$$\psi_{\mu,K,\pm}(x) = \psi^* + (1+iD)\sqrt{\frac{-a_1\mu - c_1 K}{b_1}}\left(1 - 3\,\mathrm{sech}^2\,(\delta_K x)\right) \\ \pm 2\sqrt{K}\cos(kx + \lambda \tanh(\delta_K x)) + O(\mu),$$

where $k = \omega^* + O(\mu^{\frac{1}{2}})$.

1.3.3 Case of anomalous dispersion ($\beta = -1$)

We keep the notations of Section 1.3.2, and compute the normal form coefficients in the cases (i), $\alpha^* \in (\sqrt{3}, 2)$, $F^{*2} = F_+^2(\alpha^*)$ and (ii), $\alpha^* > \sqrt{3}$, $F^{*2} = F_-^2(\alpha^*)$. In particular, notice that the formulas for the vectors ζ_0, ζ_1, ζ and ζ_1^* are the same as in (1.35).

Case $\alpha^* \in (\sqrt{3}, 2)$, $F^{*2} = F_+^2(\alpha^*)$.

In this case $\rho^* = \rho_+(\alpha^*) = (2\alpha^* - \gamma^*)/3$ and

$$\omega^{*2} = 4\rho^* - 2\alpha^* = \frac{2}{3}(\alpha^* + 2\sqrt{\alpha^{*2} - 3}).$$

The system (1.9) has the normal form (1.26), and the coefficients a_1, b_1 and c_1 can be computed as in the proof of Lemma 1.3.2. We obtain the following result :

Lemma 1.3.3. *Suppose that $\beta = -1$ and $\alpha^* \in (\sqrt{3}, 2)$, $F^{*2} = F_+^2(\alpha^*)$. Then the coefficients a_1, b_1 and c_1 in (1.28), (1.29) and (1.30) are given by*

$$a_1 = \frac{9F^*(\alpha^* + \gamma^*)}{2(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} < 0,$$

$$b_1 = -\frac{3F^*(2\alpha^{*5} - 18\alpha^* + (2\alpha^{*4} + 3\alpha^{*2} + 9)\gamma^*)}{2(\alpha^{*2} + 3 + \alpha^*\gamma^*)(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} > 0,$$

$$c_1 = -\frac{9F^*(\alpha^* + \gamma^*)}{\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*} > 0.$$

Notice that the signs of the coefficients a_1, b_1 and c_1 are the same as in the case $\beta = 1$, so that the system (1.9) possesses the same types of solutions as in Theorem 1.3.2. The leading order terms of the corresponding solutions of the equation (1.1) are also the same as the one found in Section 1.3.2.

Case $\alpha^* > \sqrt{3}$, $F^{*2} = F_-^2(\alpha^*)$.

In this case $\rho^* = \rho_-(\alpha^*) = (2\alpha^* + \gamma^*)/3$ and

$$\omega^{*2} = \frac{2}{3}(\alpha^* + 2\sqrt{\alpha^{*2} - 3}).$$

By arguing as before we obtain the following lemma :

Lemma 1.3.4. *Suppose that $\beta = -1$ and $\alpha^* > \sqrt{3}$, $F^{*2} = F_-^2(\alpha^*)$. Then the coefficients a_1, b_1 and c_1 in (1.28), (1.29) and (1.30) are given by*

$$a_1 = \frac{9F^*(\alpha^* - \gamma^*)}{2(\alpha^{*3} - 9\alpha^* - (\alpha^{*2} + 6)\gamma^*)} < 0,$$

$$b_1 = -\frac{3F^*(2\alpha^{*5} - 18\alpha^* - (2\alpha^{*4} + 3\alpha^{*2} + 9)\gamma^*)}{2(\alpha^{*2} + 3 - \alpha^*\gamma^*)(\alpha^{*3} - 9\alpha^* - (\alpha^{*2} + 6)\gamma^*)} < 0,$$

$$c_1 = -\frac{9F^*(\alpha^* - \gamma^*)}{\alpha^{*3} - 9\alpha^* - (\alpha^{*2} + 6)\gamma^*} > 0.$$

Notice that the sign of b_1 changes in this case, which will lead to a different situation.

Bounded solutions of (1.9). According to [27, Chapter 4, Theorem 3.10], we obtain the following theorem :

Theorem 1.3.3. *Suppose that $\beta = -1$, $\alpha^* > \sqrt{3}$ and $F^{*2} = F_-^2(\alpha^*)$. In a neighborhood of 0 in $\mathbb{R}^4$ the following properties hold :*

(i) For any $\mu > 0$ sufficiently small, the system (1.9) possesses two families of periodic orbits of the first kind, parametrized by their size r, with $r > r^*(\mu) = O(\mu^{\frac{1}{2}})$. To any periodic orbit in one of these families, there is a pair of reversible homoclinic orbits connecting the periodic orbit to itself. In addition, there are quasiperiodic orbits.

(ii) For any $\mu < 0$ sufficiently small, the system (1.9) possesses the same types of solutions as in Theorem 1.3.2 (i).

Taylor expansions of periodic and localized solutions of (1.1). We start with the truncated system (1.36) and keep the notations of Section 1.3.2.

When $\mu < 0$, the system (1.36) has two **equilibria** $(\pm A_0, 0, 0)$, where A_0 is the same as in the case $\beta = 1$. However, in this case, the stability of these equilibria is changed. More precisely, $(A_0, 0, 0)$ is a center, and $(-A_0, 0, 0)$ is a saddle-center. The formulas for the corresponding constant solutions of the equation (1.1) are the same as in Section 1.3.2.

Next, the formulas (1.37) for the **periodic orbits of the first kind** are still true in this case. Notice that there is no restriction for the size of K, since $(-a_1\mu - c_1 K)/b_1 > 0$, for any $K > 0$. The formulas for the solutions of (1.1) corresponding to the periodic orbits of the first kind of the system (1.9) are the same as in Section 1.3.2.

The system (1.36) also have periodic orbits of the second kind. The solutions of the equation (1.1) corresponding to these orbits have the expansion

$$\psi_{\mu,\varepsilon}(x) = P_2(kx),$$

where

$$P_2(y) = \psi^* + (1 + Di)\left(-\sqrt{\frac{a_1\mu}{b_1}} + \varepsilon\cos(y)\right)|\mu|^{\frac{1}{2}} + O(\mu + \varepsilon^2)$$

with $k = \sqrt{2}(-a_1 b_1 \mu)^{\frac{1}{4}} + O(|\mu|^{\frac{1}{2}})$ and ε is a real and small parameter.

Finally, the system (1.36) possesses homoclinic connections to the periodic orbits surrounding the saddle-center $(-A_0, 0, 0)$. The solutions of (1.1) corresponding to the homoclinic connections which persist, hence for K not too small, have the expansion

$$\psi_{\mu,K,\pm}(x) = \psi^* - (1 + iD)\sqrt{\frac{-a_1\mu - c_1 K}{b_1}}\left(1 - 3\,\mathrm{sech}^2(\delta_K x)\right) \\ \pm 2\sqrt{K}\cos(kx + \lambda\tanh(\delta_K x)) + O(|\mu|),$$

where $k = \omega^* + O(|\mu|^{\frac{1}{2}})$, $\lambda = O(1)$ and $\delta_K = \sqrt{-b_1 A_K/2}$.

Now suppose that $\mu > 0$. In this case, the system (1.36) has no equilibria, since $a_1\mu < 0$, $b_1 < 0$, and no periodic orbits of the second kind. However, there are still periodic orbits of the first kind, when $|a_1\mu| < c_1 K$. The size of these orbits is then at least $O(|\mu|^{\frac{1}{2}})$, and they are explicitely given by formula (1.37), and the corresponding solutions of the equation (1.1) have the same expansion as in Section 1.3.2.

Finally, any periodic orbits of the first kind in this case has a pair of homoclinic connections. The solutions of (1.1) corresponding to these homoclinic connections have the expansion

$$\psi_{\mu,K,\pm}(x) = \psi^* \pm (1 + iD))\sqrt{\frac{-a_1\mu - c_1 K}{b_1}}\left(1 - 3\,\mathrm{sech}^2(\delta_K x)\right) \\ \pm 2\sqrt{K}\cos(kx + \lambda\tanh(\delta_K x)) + O(\mu),$$

where $k = \omega^* + O(\mu^{\frac{1}{2}})$, $\lambda = O(1)$ and $\delta_K = \sqrt{|b_1 A_K|/2}$.

1.4 Reversible 0^2 bifurcation

In this section we analyse the 0^2 bifurcation in the cases of normal and anomalous dispersion. In contrast to the previous two bifurcations, this analysis relies upon a center manifold reduction. We prove the existence of periodic and localized solutions for the Lugiato-Lefever equation (1.1).

Recall that in the case of normal dispersion ($\beta = 1$), the 0^2 bifurcation occurs in the following cases :

(i) $\alpha^* \in \left(\sqrt{3}, 2\right)$ and $F^{*2} = F_+^2(\alpha^*)$, when $\rho^* = \rho_+(\alpha^*)$;

(ii) $\alpha^* > \sqrt{3}$ and $F^{*2} = F_-^2(\alpha^*)$, when $\rho^* = \rho_-(\alpha^*)$.

The cusp point $(\alpha^*, F^{*2}) = (\sqrt{3}, 8\sqrt{3}/9)$ will be excluded here, because at this point one of the coefficients of the normal form computed in Section 1.4.2 below vanishes. In the case of anomalous dispersion ($\beta = -1$), this bifurcation occurs for $\alpha^* > 2$ and $F^{*2} = F_+^2(\alpha^*)$, when $\rho^* = \rho_+(\alpha^*)$.

In both cases, 0 is an non-semisimple eigenvalue of $L_\pm$ with algebraic multiplicity 2 and $L_\pm$ has no other purely imaginary eigenvalue. Indeed, when $\beta = 1$, the two nonzero eigenvalues $\pm\omega^*$ of L_+ satisfy, in Case (i),

$$\omega^{*2} = \frac{2}{3}(\alpha^{*2} - 2\sqrt{\alpha^{*2} - 3}) > 0,$$

and in Case (ii),

$$\omega^{*2} = \frac{2}{3}(\alpha^{*2} + 2\sqrt{\alpha^{*2} - 3}) > 0.$$

When $\beta = -1$, the eigenvalues $\pm\omega^*$ of L_- satisfy

$$\omega^{*2} = \frac{2}{3}(\alpha^{*2} - 2\sqrt{\alpha^{*2} - 3}) > 0.$$

1.4.1 Center manifold reduction

Let ζ_0 and ζ_1 be two generalized eigenvectors of $L_\pm$, satisfying

$$L_\pm \zeta_0 = 0, \quad L_\pm \zeta_1 = \zeta_0, \quad S\zeta_0 = \zeta_0, \quad S\zeta_1 = -\zeta_1.$$

Consider the spectral decomposition $\mathbb{R}^4 = X_0 \oplus X_1$, with

$$X_0 = \mathrm{span}(\zeta_0, \zeta_1), \quad X_1 = (\mathrm{id} - P_0)(\mathbb{R}^4),$$

where P_0 is the unique projection onto X_0 which commutes with $L_\pm$. The projection P_0 can be computed with the formula

$$P_0 V = \langle V, \zeta_0^* \rangle \zeta_0 + \langle V, \zeta_1^* \rangle \zeta_1, \tag{1.38}$$

for any $V \in \mathbb{R}^4$. Here the vectors $\zeta_0^*,\ \zeta_1^* \in \mathbb{R}^4$ satisfy

$$\langle \zeta_1^*, \zeta_0 \rangle = 0, \quad \langle \zeta_1^*, \zeta_1 \rangle = 1, \quad L_\pm^* \zeta_1^* = 0, \quad L_\pm^* \zeta_0^* = \zeta_1^*,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^4$ and $L_\pm^*$ is the adjoint operator of $L_\pm$.

Recall that the maps $R_\pm$ are of class $\mathcal{C}^\infty$, and $R_\pm(0,0) = 0$, $\mathrm{d}_U R_\pm(0,0) = 0$. We can then apply the center manifold theorem (see for example [27, Chapter 2, Theorem 3.3]), and we obtain the following reduction result for the system (1.9).

Theorem 1.4.1. *Assume that the parameters α^* and F^* satisfy, in the case $\beta = 1$, $\alpha^* \in (\sqrt{3}, 2)$ and $F^{*2} = F_+^2(\alpha^*)$, or $\alpha^* > \sqrt{3}$ and $F^{*2} = F_-^2(\alpha^*)$, and in the case $\beta = -1$, $\alpha^* > 2$ and $F^{*2} = F_+^2(\alpha^*)$. For any integer $k \geqslant 2$, there exist a map $\Psi \in \mathcal{C}^k(X_0 \times \mathbb{R}, X_1)$, satisfying*

$$\Psi(0,0) = 0, \quad \mathrm{d}_U\Psi(0,0) = 0,$$

and a neighborhood $O_1 \times O_2$ of $(0,0)$ in $\mathbb{R}^4 \times \mathbb{R}$ such that for $\mu \in O_2$, the manifold

$$\mathcal{M}(\mu) = \{U_0 + \Psi(U_0, \mu),\ U_0 \in X_0\}$$

is locally invariant, and contains the set of bounded solutions of the system (1.9) staying in O_1 for all $x \in \mathbb{R}$. Consequently, any small bounded solution U of (1.9) has the form

$$U = U_0 + \Psi(U_0, \mu), \quad U_0 = P_0 U, \quad \Psi(U_0, \mu) = (\mathrm{id} - P_0)U,$$

with

$$U_0(x) = A(x)\zeta_0 + B(x)\zeta_1, \quad \Psi(U_0, \mu) = O(|\mu| + \|U_0\| \,(|\mu| + \|U_0\|)).$$

where A, B are real-valued functions.

As an immediate consequence of this result, we have that small bounded solutions of the system (1.9) can be found by solving the reduced system

$$\frac{\mathrm{d}U_0}{\mathrm{d}x} = L_{\pm,0}U_0 + P_0 R_\pm(U_0 + \Psi(U_0, \mu), \mu), \tag{1.39}$$

where $L_{\pm,0}$ is the restriction of $L_\pm$ to X_0, $L_{\pm,0} = P_0 L_\pm$ (see for example [27, Chapter 2, Corollary 2.12]). In the basis $(\zeta_0,\ \zeta_1)$ of X_0 considered above, this is a system of two ordinary differential equations for A and B. Our goal is now to compute the leading order part of the system (1.39) in the cases $\beta = 1$ and $\beta = -1$.

1.4.2 Case of normal dispersion ($\beta = 1$)

We suppose that $\beta = 1$. The following lemma provides an explicit expression of the reduced system (1.39).

Lemma 1.4.1. *Assume that $\alpha^* \in (\sqrt{3}, 2)$ and $F^{*2} = F_+^2(\alpha^*)$, or $\alpha^* > \sqrt{3}$ and $F^{*2} = F_-^2(\alpha^*)$. For any μ sufficiently small, the reduced system (1.39) is given by*

$$\frac{\mathrm{d}A}{\mathrm{d}x} = B \tag{1.40}$$

$$\frac{\mathrm{d}B}{\mathrm{d}x} = a\mu + bA^2 + O\left((|A| + |B|)\left(|\mu| + \left(|A|^2 + |B|^2\right)\right)\right), \tag{1.41}$$

where

$$a = -\frac{\psi_i^*}{D}, \qquad b = \frac{2D\psi_r^* + (3D^2+1)\psi_i^*}{D},$$

and $D = -\omega^{*2}/2$.

Proof. First, straightforward computations give

$$\zeta_0 = \begin{pmatrix} 1 \\ 0 \\ D \\ 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ D \end{pmatrix}, \quad D = \frac{3\psi_r^{*2} + \psi_i^{*2} - \alpha^*}{1 - 2\psi_r^* \psi_i^*} = -\frac{\omega^{*2}}{2}$$

and

$$\zeta_0^* = \frac{1}{D} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \zeta_1^* = \frac{1}{D} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, according to (1.38), we obtain

$$P_0 R_+(U_0 + \Psi(U_0, \mu), \mu) = \langle R_+(U_0 + \Psi(U_0, \mu), \mu), \zeta_0^* \rangle \, \zeta_0 + \langle R_+(U_0 + \Psi(U_0, \mu), \mu), \zeta_1^* \rangle \, \zeta_1.$$

Since the second and the fourth components of ζ_0^* and the first and the third components of R_+ vanish, we have

$$\langle R_+(U_0 + \Psi(U_0, \mu), \mu), \zeta_0^* \rangle = 0,$$

so that

$$P_0 R_+(U_0 + \Psi(U_0, \mu), \mu) = \langle R_+(U_0 + \Psi(U_0, \mu), \mu), \zeta_1^* \rangle \, \zeta_1. \tag{1.42}$$

Furthermore, since $\Psi(U_0, \mu) = O(|\mu| + \|U_0\| \, (|\mu| + \|U_0\|))$ and $R_+(U, \mu) = O(|\mu| + \|U\|^2)$, we obtain

$$P_0 R_+(U_0 + \Psi(U_0, \mu), \mu) = \langle R_+(U_0, \mu), \zeta_1^* \rangle \, \zeta_1 + O(\|U_0\| \, (|\mu| + \|U_0\|^2)).$$

Using the expression of R_+ given in (1.10), we have

$$\begin{aligned} R_+(U_0, \mu) &= R_+(A\zeta_0 + B\zeta_1, \mu) && (1.43) \\ &= \begin{pmatrix} 0 \\ -\mu\psi_r^* + ((D^2+3)\psi_r^* + 2D\psi_i^*)A^2 \\ 0 \\ -\mu\psi_i^* + (2D\psi_r^* + (3D^2+1)\psi_i^*)A^2 \end{pmatrix} + O(\|U_0\| \, (|\mu| + \|U_0\|^2)). && (1.44) \end{aligned}$$

Then (1.40)-(1.41) follows from (1.39), (1.42), (1.43) and the explicit formula for ζ_1^*. $\square$

We point out that the reduced system (1.40)-(1.41) is already in normal form at order 2. Again, its dynamics depends on the signs of the coefficients a and b, provided that they do not vanish [27, Chapter 4, Section 4.1.1, Theorem 1.8]. It turns out that their signs and expressions in terms of parameters depend on the two cases (i) and (ii).

Case $\alpha^* \in (\sqrt{3}, 2)$**,** $F^{*2} = F_+^2(\alpha^*)$**.**

In this case $\rho^* = \rho_+(\alpha^*)$. A direct computation, using the symbolic package Maple, gives

$$a = -\frac{9F^*(\alpha^* + \gamma^*)}{2(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} > 0,$$

$$b = \frac{3F^*(2\alpha^{*5} - 18\alpha^* + (2\alpha^{*4} + 3\alpha^{*2} + 9)\gamma^*)}{2(\alpha^{*2} + 3 + \alpha^*\gamma^*)(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} < 0.$$

Bounded solutions of (1.9). According to [27, Chapter 4, Theorem 1.8], we have the following result, which is valid in a neighborhood of the origin in $\mathbb{R}^4$, for small μ :

Theorem 1.4.2. *Assume that* $\alpha^* \in (\sqrt{3}, 2)$ *and* $F^{*2} = F_+^2(\alpha^*)$*.*

(i) For any $\mu > 0$ *sufficiently small, the system (1.9) possesses two equilibria of order* $O(\mu^{\frac{1}{2}})$*, a saddle and a center. The center is surrounded by a one-parameter family of periodic orbits, which tend to a homoclinic orbit connecting the saddle equilibrium to itself, as the period tends to* ∞*.*

(ii) For any $\mu < 0$ *sufficiently small, the system (1.9) has no bounded solution.*

Taylor expansions of periodic and localized solutions of (1.1). The leading order terms of periodic and homoclinic solutions of the system (1.40)-(1.41) are computed from the truncated normal form

$$\frac{\mathrm{d}A}{\mathrm{d}x} = B \tag{1.45}$$

$$\frac{\mathrm{d}B}{\mathrm{d}x} = a\mu + bA^2. \tag{1.46}$$

The corresponding solutions of the equation (1.1) have the form

$$\psi(x) = \psi^* + (1 + iD)A(x) + O\left(|\mu| + |A|^2 + |B|^2\right).$$

As stated in Theorem 1.4.2, the system (1.9) possesses bounded solutions only when $\mu > 0$, and the same is true for the truncated system (1.45)-(1.46).

The two **equilibria** of (1.9) are to leading order given by $(\pm A_0, 0)$, where $A_0 = \sqrt{-a\mu/b}$. The equilibrium $(A_0, 0)$ is a center and $(-A_0, 0)$ is a saddle. These equilibria correspond to constant solutions of the equation (1.1) with corresponding parameters $\alpha^* + \mu$ and F^*, and read

$$\psi_\mu = \psi^* \pm \sqrt{\frac{-a}{b}}(1 + iD)\mu^{\frac{1}{2}} + O(\mu).$$

Next, the small **periodic solutions** (A, B) of (1.45)-(1.46) which are close to the center have the form

$$A(x) = A_0 + \varepsilon\sqrt{\mu}\cos\left(\sqrt{2}(-ab\mu)^{\frac{1}{4}}x\right) + O(\varepsilon^2), \quad B(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x),$$

where ε is a real and small parameter. The corresponding solutions of (1.1) have the form

$$\psi_{\mu,\varepsilon}(x) = P(kx),$$

where P is the 2π-periodic function defined by

$$P(y) = \psi^* + (1 + Di)\left(\sqrt{\frac{-a}{b}} + \varepsilon\cos(y)\right)\mu^{\frac{1}{2}} + O(\mu + \varepsilon^2),$$

and $k = \sqrt{2}(-ab\mu)^{\frac{1}{4}} + O(\mu^{\frac{1}{2}})$.

Finally, the **homoclinic solution** (A, B) of the system (1.45)-(1.46) is given by

$$A(x) = -A_0\left(1 - 3\,\mathrm{sech}^2(\delta x)\right), \quad B(x) = \frac{\mathrm{d}A}{\mathrm{d}x}(x),$$

where

$$\delta = \sqrt{\frac{-bA_0}{2}}.$$

The corresponding solution of (1.1) reads

$$\psi_\mu(x) = \psi^* - \sqrt{\frac{-a}{b}}(1 + Di)\left(1 - 3\,\mathrm{sech}^2(\delta x)\right)\mu^{\frac{1}{2}} + O(\mu).$$

Case $\alpha^* > \sqrt{3}$, $F^{*2} = F_-^2(\alpha^*)$.

In this case $\rho^* = \rho_-(\alpha^*)$, and a direct computation gives the coefficients a and b,

$$a = -\frac{9F^*(\alpha^* - \gamma^*)}{2(\alpha^{*3} - 9\alpha^* - (\alpha^{*2} + 6)\gamma^*)} > 0$$

and

$$b = \frac{3F^*(2\alpha^{*5} - 18\alpha^* - (2\alpha^{*4} + 3\alpha^{*2} + 9)\gamma^*)}{2(\alpha^{*2} + 3 - \alpha^*\gamma^*)(\alpha^{*3} - 9\alpha^* - (\alpha^{*2} + 6)\gamma^*)} > 0.$$

Bounded solutions of (1.9). The set of bounded solutions of the system (1.9) is very similar to the previous case. The only difference is in the sign of μ, *i. e.*, the side of the bifurcation curve where nontrivial bounded solutions exist. More precisely, for $\mu > 0$, the system has no bounded solutions, as in Theorem 1.4.2 (ii) and for $\mu < 0$ we have exactly the same solutions as in Theorem 1.4.2 (i).

Taylor expansions of periodic and localized solutions of (1.1). We consider the truncated system (1.45)-(1.46) in the case $\mu < 0$, when this system possesses bounded solutions. As before, we have two **equilibria** $(\pm A_0, 0)$, where $A_0 = \sqrt{-a\mu/b}$, but in this case $(A_0, 0)$ is a saddle and $(-A_0, 0)$ is a center. These equilibria correspond to constant solutions of the equation (1.1) with corresponding parameters $\alpha^* + \mu$ and F^*, and read

$$\psi = \psi^* \pm \sqrt{\frac{a}{b}}(1 + Di)|\mu|^{\frac{1}{2}} + O(\mu).$$

The **periodic solutions** of the truncated system are as above, and we give the periodic solutions of (1.1),

$$\psi_{\mu,\varepsilon}(x) = P(kx),$$

where $k = \sqrt{2}(-ab\mu)^{\frac{1}{4}} + O(|\mu|^{\frac{1}{2}})$, and the 2π−periodic function P is given by

$$P(y) = \psi^* + (1 + Di)\left(-\sqrt{\frac{a}{b}} + \varepsilon \cos(y)\right)|\mu|^{\frac{1}{2}} + O(\mu + \varepsilon^2),$$

in which ε is a real and small parameter, again. Finally, the truncated system possesses a **homoclinic solution**, which corresponds to a localized wave of (1.1) with expansion

$$\psi_\mu(x) = \psi^* + (1 + Di)\sqrt{\frac{a}{b}}\left(1 - 3\,\mathrm{sech}^2(\delta x)\right)|\mu|^{\frac{1}{2}} + O(\mu),$$

where $\delta = \sqrt{bA_0/2}$.

1.4.3 Case of anomalous dispersion ($\beta = -1$)

We keep the notations of Section 1.4.2. We suppose that $\alpha^* > 2$ and $F^{*2} = F_+^2(\alpha^*)$, so that $\rho^* = \rho_+(\alpha^*)$. The reduced system has the same form as the one found in 1.4.2, and the coefficients at leading order are now given by

$$a = \frac{9F^*(\alpha^* + \gamma^*)}{2(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} > 0$$

and

$$b = -\frac{3F^*(2\alpha^{*5} - 18\alpha^* + (2\alpha^{*4} + 3\alpha^{*2} + 9)\gamma^*)}{2(\alpha^{*2} + 3 + \alpha^*\gamma^*)(\alpha^{*3} - 9\alpha^* + (\alpha^{*2} + 6)\gamma^*)} < 0.$$

In this case the dynamical system (1.9) has the same types of solutions as in Theorem 1.4.2, and the Taylor expansions for the corresponding solutions of the equation (1.1) are the same as the ones found for $\alpha^* \in (\sqrt{3}, 2)$ and $F^{*2} = F_+^2(\alpha^*)$ (see Section 1.4.2).

1.5 Discussion

We conclude this chapter with a discussion about stability results for the constant solutions of the Lugiato-Lefever (1.1), and about time-periodic solutions.

1.5.1 Stability of constant solutions

We start by a discussion of the stability properties of constant solutions of the equation (1.1). Recall that (1.1) possesses one, two or three constant solutions depending upon the values of the parameters α anf F (see Figure 1.2). We write the equation (1.1) as an evolution system with respect to the time t,

$$\Psi_t = A\Psi_{xx} + G(\Psi), \tag{1.47}$$

where

$$\Psi = \begin{pmatrix} \psi_r \\ \psi_i \end{pmatrix}, \quad A = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad G(\Psi) = \begin{pmatrix} -\psi_r + \alpha\psi_i - (\psi_r^2 + \psi_i^2)\psi_i + F \\ -\alpha\psi_r - \psi_i + (\psi_r^2 + \psi_i^2)\psi_r \end{pmatrix},$$

in which the symbols $+$ and $-$ stand for the case of normal ($\beta = 1$) and anomalous dispersion ($\beta = -1$), respectively. Let $\psi^* = \psi_r^* + i\psi_i^*$ be a constant solution of (1.1), with corresponding parameters α^* and F^*. We consider hereafter the stability of ψ^* with respect to constant, localized and bounded perturbations.

Stability with respect to constant perturbations. The stability of the constant solution ψ^* of the equation (1.1) with respect to constant perturbations, as obtained from the spectral stability of the constant solution $\Psi^* = (\psi_r^*, \psi_i^*)$ of the ordinary differential equation

$$\Psi_t = G(\Psi), \tag{1.48}$$

where $\Psi(t) \in \mathbb{R}^2$, is well known in the literature. For completeness we recall these results here. Notice that the stability of Ψ^* does not depend upon the sign of β in this case.

Consider the linearized system at Ψ^*,

$$\Psi_t = \mathrm{d}G(\Psi^*)\Psi,$$

with

$$\mathrm{d}G(\Psi^*) = \begin{pmatrix} -1 - 2\psi_r^*\psi_i^* & \alpha - \psi_r^{*2} - 3\psi_i^{*2} \\ -\alpha + 3\psi_r^{*2} + \psi_i^{*2} & -1 + 2\psi_r^*\psi_i^* \end{pmatrix}.$$

The stability of Ψ^* is determined by the location of the spectrum of the matrix $\mathrm{d}G(\Psi^*)$ in the complex plane. More precisely, Ψ^* is asymptotically stable if the two eigenvalues of $\mathrm{d}G(\Psi^*)$ have negative real parts, stable if they have nonpositive real parts and at least one eigenvalue lying on the imaginary axis, and unstable otherwise. Notice that these spectral criteria also imply nonlinear (in)stability in the first and third case, since (1.48) is an ordinary differential equation.

We have the following result.

Theorem 1.5.1. *(i) Suppose that $\alpha^* \leqslant \sqrt{3}$, or $\alpha^* > \sqrt{3}$ and $F^{*2} \notin (F_-^2(\alpha^*), F_+^2(\alpha^*))$. In this case the equation (1.1) possesses precisely one constant solution ψ^*, which is asymptotically stable with respect to constant perturbations, except for $\alpha^* = \sqrt{3}$, $F^{*2} = 8\sqrt{3}/9$, where ψ^* is stable.*

(ii) Suppose that $\alpha^* > \sqrt{3}$ and $F^{*2} = F^2_\pm(\alpha^*)$. In this case the equation (1.1) possesses two constant solutions ψ_1^* and ψ_2^*, one asymptotically stable and one stable.

(iii) Suppose that $\alpha^* > \sqrt{3}$ and $F^{*2} \in (F^2_-(\alpha^*), F^2_+(\alpha^*))$. Then the equation (1.1) possesses three constant solutions ψ_1^*, ψ_2^* and ψ_3^*, with $|\psi_1^*| < \psi_2^*| < |\psi_3^*|$. The solutions ψ_1^*, ψ_3^* are asymptotically stable, and ψ_2^* is unstable.

Proof. Let ψ^* be a constant solution of the equation (1.1). The stability of ψ^* is given by the eigenvalues of $\mathrm{d}G(\Psi^*)$, which are the roots of the polynomial

$$Q(X) = X^2 + 2X + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1, \qquad \rho^* = |\psi^*|^2.$$

A direct calculation shows that the discriminant of Q is given by $\Delta = 4(3\rho^* - \alpha^*)(\alpha^* - \rho^*)$.

If $\Delta < 0$, *i. e.* , if $\alpha^* < 0$, or if $\alpha^* \geqslant 0$ and $\rho^* \notin (\alpha^*/3, \alpha^*)$, the roots of P are complex and have real part -1. If $\Delta = 0$, *i. e.* , if $\rho^* = \alpha^*/3$ or $\rho^* = \alpha^*$, -1 is a double root of Q. Finally, if $\Delta > 0$, then Q has two real roots, and their sum is equal to -2, so at least one of these roots is negative. Their signs depend upon the sign of their product $3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1$, which turns out to be the derivative of the polynomial P in Section 1.1.1. If $3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 \geqslant 0$, *i.e.* if $\rho^* \leqslant \rho_+(\alpha^*)$ or $\rho^* \geqslant \rho_-(\alpha^*)$, the real roots of P are nonpositive, and if $3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 < 0$, *i.e.* if $\rho^* \in (\rho_+(\alpha^*), \rho_-(\alpha^*))$, the polynomial P possesses one positive root. These results are summarized in Figure 1.6.

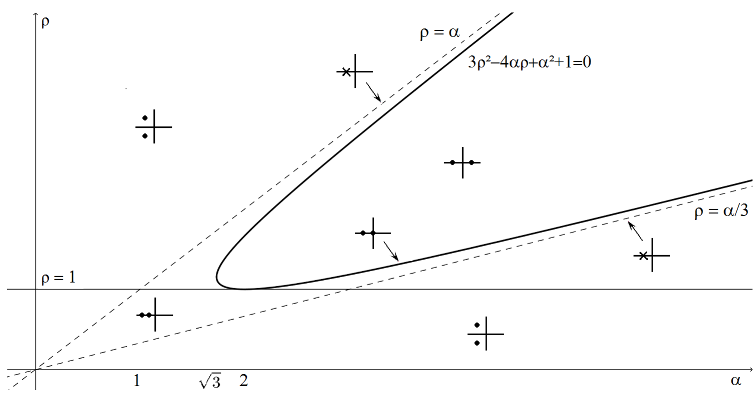


FIGURE 1.6 – Location of the eigenvalues of the matrix $\mathrm{d}G(\Psi^*)$ in terms of α and ρ.

We now discuss the stability of ψ^* in term of the parameters α^* and F^*. If $\alpha^* < \sqrt{3}$, then the equation (1.1) has one constant solution ψ^*. According to the diagram in Figure 1.6, the matrix $\mathrm{d}G(\Psi^*)$ has negative eigenvalues only and ψ^* is asymptotically stable. If $\alpha^* > \sqrt{3}$ and $F^{*2} \notin [F^2_-(\alpha^*), F^2_+(\alpha^*)]$, then the equation (1.1) possesses one constant solution, as well. In this case, the square modulus ρ^* of ψ^* satisfies $\rho^* \notin (\rho_+(\alpha^*), \rho_-(\alpha^*))$, and according to Figure 1.6, $\mathrm{d}G(\Psi^*)$ has two negative eigenvalues so that ψ^* is asymptotically stable in this case too.

If $\alpha^* > \sqrt{3}$ and $F^{*2} = F^2_\pm(\alpha^*)$, the equation (1.1) possesses two constant solutions ψ_1^* and ψ_2^*, with respective square modulus ρ_1^* and ρ_2^*. Without loss of generality, we suppose $\rho_1^* < \rho_2^*$. According to Figure 1.1, either $\rho_1^* = \rho_+(\alpha^*)$ and $\rho_2^* > \rho_-(\alpha^*)$, or $\rho_1^* < \rho_+(\alpha^*)$ and

$\rho_2^* = \rho_-(\alpha^*)$. It turns out that 0 is an eigenvalue of $\mathrm{d}G(\Psi^*)$ when $\rho^* = \rho_\pm(\alpha^*)$, the second eigenvalue being negative, so that ψ_1^* in the first case and ψ_2^* in the second case are stable. For the other solution, the two eigenvalues are negative, so that they are asymptotically unstable.

Finally, we suppose that $\alpha > \sqrt{3}$ and $F^{*2} \in (F_-^2(\alpha^*), F_+^2(\alpha^*))$. In this case the equation (1.1) possesses three constant solutions ψ_1^*, ψ_2^* and ψ_3^*, with respective square modulus ρ_1^*, ρ_2^* and ρ_3^*, satisfying $\rho_1^* < \rho_+(\alpha^*) < \rho_2^* < \rho_-(\alpha^*) < \rho_3^*$. According to Figure 1.6, the matrices $\mathrm{d}G(\Psi_1^*)$ and $\mathrm{d}G(\Psi_3^*)$ have negative eiganvalues and the constant solutions ψ_1^*, ψ_3^* are asymptotically stable, whereas the matrix $\mathrm{d}G(\Psi_2^*)$ has a positive eigenvalue, so that ψ_2^* is unstable. This completes the proof of the theorem. □

Spectral stability with respect to localized or bounded perturbations. We now study the spectral stability of the constant solution ψ^* with respect to localized or bounded perturbations. We regard the system (1.47) in the space $Z = (L^2(\mathbb{R}))^2$ or $(\mathcal{C}_b(\mathbb{R}))^2$. As in the previous case, we consider the linearized system at Ψ^*,

$$\Psi_t = \mathcal{L}_\pm \Psi,$$

where $\mathcal{L}_\pm$ is the differential operator

$$\mathcal{L}_\pm = \begin{pmatrix} -1 - 2\psi_r^*\psi_i^* & \pm\partial_{xx} + \alpha^* - \psi_r^{*2} - 3\psi_i^{*2} \\ \mp\partial_{xx} - \alpha^* + 3\psi_r^{*2} + \psi_i^{*2} & -1 + 2\psi_r^*\psi_i^* \end{pmatrix}.$$

Our goal is now to find the spectrum of the operator $\mathcal{L}_\pm$. In this case we say that Ψ^* is stable if the spectrum of $\mathcal{L}_\pm$ lies in the closed left half complex plane, and unstable otherwise. We prove the following result.

Theorem 1.5.2. *Let ψ^* be a constant solution of (1.1), with corresponding parameters α^* and F^*. Set $\rho^* = |\psi^*|^2$.*

(i) In the case $\beta = 1$, ψ^* is stable with respect to localized perturbations if and only if one of the following assumptions holds :

(a) $\alpha^* \in \mathbb{R}$ and $F^{*2} \leqslant 1 + (1-\alpha^*)^2$, when $\rho^* \leqslant 1$.

(b) $\alpha^* \in (\sqrt{3}, 2)$ and $F^{*2} \in (1 + (1-\alpha)^2, F_+^2(\alpha^*))$, when $1 < \rho^* < \rho_-(\alpha^*)$.

(c) $\alpha^* \leqslant \sqrt{3}$ and $F^{*2} > 1 + (1-\alpha^*)^2$, when $\rho^* > 1$.

(d) $\alpha^* > \sqrt{3}$ and $F^{*2} \geqslant F_-^2(\alpha^*)$, when $\rho^* \geqslant \rho_-(\alpha^*)$.

(ii) In the case $\beta = -1$, ψ^* is stable with respect to localized perturbations if and only if one of the following assumptions holds :

(a) $\alpha^* \in \mathbb{R}$ and $F^{*2} \leqslant 1 + (1-\alpha^*)^2$, when $\rho^* \leqslant 1$.

(b) $\alpha > 2$ and $F^{*2} \in (1 + (1-\alpha^*)^2, F_+^2(\alpha^*)]$, when $1 < \rho^* \leqslant \rho_+(\alpha^*)$.

Proof. Let $\lambda \in \mathbb{C}$. Then λ belongs to the spectrum $\sigma(\mathcal{L}_\pm)$ of $\mathcal{L}_\pm$ if and only if the operator $\mathcal{L}_\pm - \lambda$ is not invertible. Equivalently, the complex number λ belongs to the resolvent set of

$\mathcal{L}_\pm$ if and only if, for any $V \in Z$, there exists a unique solution U in the domain of $\mathcal{L}_\pm$ such that

$$(\mathcal{L}_\pm - \lambda)U = V, \tag{1.49}$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Using Fourier transform in (1.49), we find that λ belongs to the resolvent set of $\mathcal{L}_\pm$ if and only if the linear system

$$\begin{cases} (-1 - 2\psi_r^* \psi_i^* - \lambda)u_1 + (\mp k^2 + \alpha^* - \psi_r^{*2} - 3\psi_i^{*2})u_2 = v_1 \\ (\pm k^2 - \alpha^* + 3\psi_r^{*2} + \psi_i^{*2})u_1 + (-1 + 2\psi_r^* \psi_i^* - \lambda)u_2 = v_2 \end{cases}$$

has a unique solution, for any $k \in \mathbb{R}$, which is the case if and only if

$$\lambda^2 + 2\lambda + k^4 \pm k^2(4\rho^* - 2\alpha^*) + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 \neq 0.$$

Consequently, the spectrum of the operator $\mathcal{L}_\pm$ is given by

$$\begin{aligned} \sigma(\mathcal{L}_\pm) &= \{\lambda \in \mathbb{C},\ \lambda^2 + 2\lambda + k^4 \pm (4\rho^* - 2\alpha^*)k^2 + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + 1 = 0,\ k \in \mathbb{R}\} \\ &= \{\lambda \in \mathbb{C},\ \lambda^2 + 2\lambda + P_\mp(k) = 0,\ k \in \mathbb{R}\}, \end{aligned}$$

in which $P_\mp(k)$ are the characteristic polynomials of the matrices $L_\mp$ defined in (1.6). Notice that $\sigma(\mathcal{L}_\pm)$ is symmetric with respect to the axis $\mathrm{Re}(\lambda) = -1$. Indeed, a direct computation shows that if $\lambda \in \sigma(\mathcal{L}_\pm)$, then $-2 - \lambda \in \sigma(\mathcal{L}_\pm)$.

We now discuss the location of $\sigma(\mathcal{L}_\pm)$ in the complex plane. For notational simplicity, we focus on the case $\beta = 1$. The complex roots λ_1, λ_2 of the quadratic polynomial

$$\lambda^2 + 2\lambda + P_-(k) \tag{1.50}$$

satisfy $\lambda_1 + \lambda_2 = -2$, so that at least one of these roots has a negative real part, and their product is given by $P_-(k)$.

If $P_-(k) < 0$, we have three possible cases, depending upon the sign of the discriminant Δ_k of the polynomial (1.50) :

- if $\Delta_k > 0$, λ_1, λ_2 are negative real numbers ;
- if $\Delta_k = 0$, -1 is a double root of the polynomial (1.50) ;
- if $\Delta_k < 0$, λ_1, λ_2 are conjugate complex numbers, with real part -1.

If $P_-(k) = 0$, the roots of the polynomial (1.50) are -2 and 0, and finally if $P_-(k) > 0$, λ_1 and λ_2 are real, because the discriminant of the polynomial (1.50) is positive, and they have opposite signs.

Consequently, the location of $\sigma(\mathcal{L}_+)$ can be deduced from the sign of the polynomial P_-, which depends upon the values of α^* and ρ^*. The diagram in Figure 1.7 shows the sign of $P_-(k)$ in the (α, ρ)-plane. In order to build this diagram, we used the result of the bifurcation

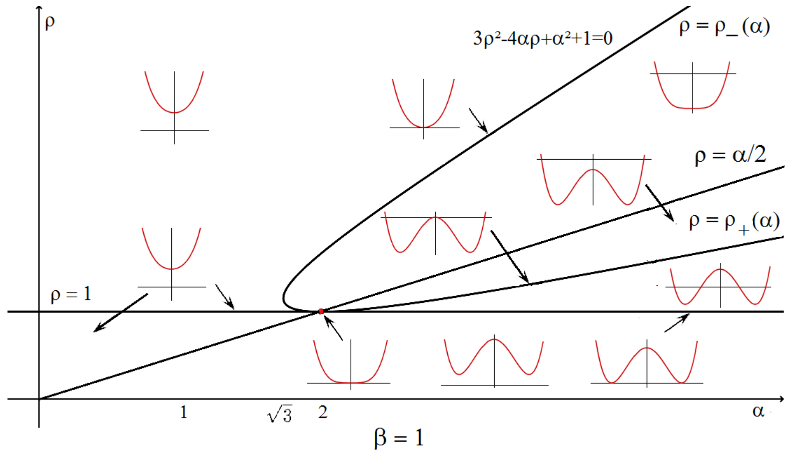


FIGURE 1.7 – Sign of the polynomial P_- in terms of α and ρ.

analysis performed in Section 1.1.3, and the fact that P_- has nonzero critical points if and only if $\rho^* \leqslant \alpha^*/2$. Indeed, the derivative of P_- is given by

$$P'_-(k) = 4k^3 + 4(2\rho^* - \alpha^*)k,$$

which possesses nonzero real roots if and only if $\rho^* \leqslant \alpha^*/2$.

We obtain the diagram in Figure 1.8, showing the location of $\sigma(\mathcal{L}_+)$ in the (α, ρ)-plane. Consequently, the operator $\mathcal{L}_\pm$ possesses stable spectrum only in the following cases :

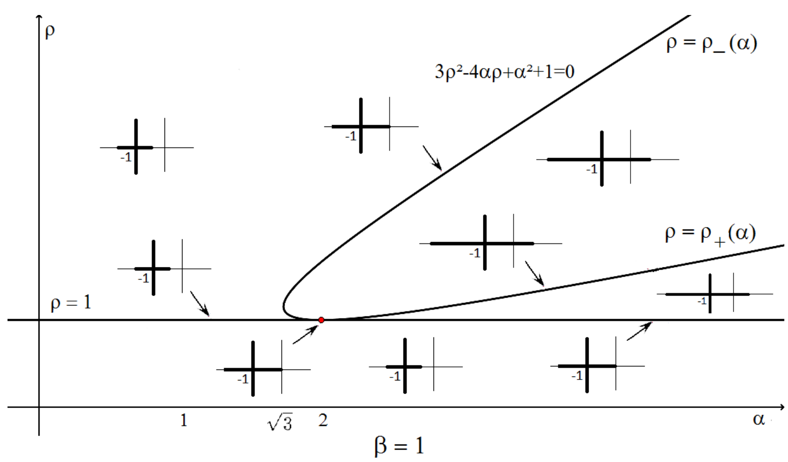


FIGURE 1.8 – Location of the spectrum of $\mathcal{L}_+$ in terms of α and ρ.

(i) $\alpha^* \in \mathbb{R}$ and $\rho^* \leqslant 1$. In this case, $F^{*2} \leqslant 1 + (1-\alpha^*)^2$.

(ii) $\alpha^* \in (\sqrt{3}, 2)$ and $1 < \rho^* < \rho_-(\alpha^*)$. In this case, $F^{*2} \in (1 + (1-\alpha)^2, F_+^2(\alpha^*))$.

(iii) $\alpha^* \leqslant \sqrt{3}$ and $\rho^* > 1$. In this case, $F^{*2} > 1 + (1-\alpha^*)^2$.

(iv) $\alpha^* > \sqrt{3}$ and $\rho^* \geqslant \rho_-(\alpha^*)$. In this case, $F^{*2} \geqslant F_-^2(\alpha^*)$.

This proves the theorem for $\beta = 1$. Similarly, for $\beta = -1$, we obtain the diagram given in Figure 1.9, which implies the result in the second part of the theorem. Arguing as in the case

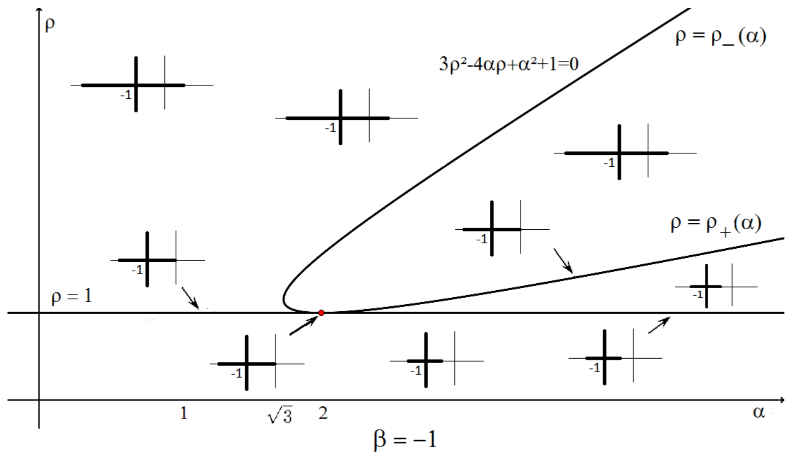


FIGURE 1.9 – Location of the spectrum of $\mathcal{L}_-$ in terms of α and ρ.

$\beta = 1$, we find that the operator $\mathcal{L}_-$ has unstable spectrum for the values of the parameters α^* and F^* given in Theorem 1.5.2. $\square$

Remark 1.5.1. *Going back to the diagrams in Figure 1.3, we see that there is a connection between the spectra of the operators $\mathcal{L}_\pm$ and the spectra of the matrices $L_\pm$ defined in Section 1.1.3. More precisely, $\sigma(\mathcal{L}_\pm)$ contains unstable spectrum if and only if the matrix $L_\pm$ possesses at least one pair of purely imaginary eigenvalues. This fact will be partially recovered in Chapter 2, Section 2.2.1, using the instability criterion given in Theorem 2.1.1.*

1.5.2 Time periodic solutions

In this section, we show that there are no time periodic solutions of the Lugiato-Lefever equation (1.1) bifurcating from constant solutions. The spatial dynamics approach used in Section 1.1.3 for stationary solutions also works for time periodic solutions, the main difference being the phase space, which will be now infinite-dimensional, instead of $\mathbb{R}^4$. We conclude that the only small time-periodic solutions are time-independent.

Let ψ^* be a constant solution of (1.1). We look for time periodic solutions of (1.1), which are close to ψ^*. We set $\psi = \psi^* + \widetilde{\psi}$, where $\widetilde{\psi}$ is a time periodic solution of (1.1) with period $2\pi/\omega$, and

$$\alpha = \alpha^* + \mu, \qquad \widetilde{\psi} = \widetilde{\psi_r} + i\widetilde{\psi_i}, \qquad \frac{\partial \widetilde{\psi}}{\partial x} = \widetilde{\varphi_r} + i\widetilde{\varphi_i}.$$

In order to consider $2\pi-$periodic functions, we set $t' = \omega t$. Dropping the primes for notational simplicity, we rewrite the equation (1.1) as a first order infinite-dimensional dynamical system

$$\frac{\mathrm{d}U}{\mathrm{d}x} = \widehat{L_{\pm,\omega}} U + R_\pm(U, \mu),$$

where

$$U = \begin{pmatrix} \widetilde{\psi_r} \\ \widetilde{\varphi_r} \\ \widetilde{\psi_i} \\ \widetilde{\varphi_i} \end{pmatrix},$$

$\widehat{L_{\pm,\omega}}$ is the differential operator,

$$\widehat{L_{\pm,\omega}} = L_\pm \pm \omega \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\partial_t & 0 \\ 0 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 0 \end{pmatrix}.$$

in which $L_\pm$ are the matrices defined in Section 1.1.2, ∂_t is the differentiation operator with respect to t, and $R_\pm(U,\mu)$ is the same as in Section 1.1.2.

We regard this system as a dynamical system in the phase space $\left(L^2_{\text{per}}(0,2\pi)\right)^4$, where

$$L^2_{\text{per}}(0,2\pi) = \left\{v \in L^2_{\text{loc}}(\mathbb{R}), v(t+2\pi) = v(t), \text{ for a.e. } t \in \mathbb{R}\right\}.$$

The operator $\widehat{L_{\pm,\omega}}$ is closed in $\left(L^2_{\text{per}}(0,2\pi)\right)^4$, with domain $\left(H^1_{\text{per}}(0,2\pi) \times L^2_{\text{per}}(0,2\pi)\right)^2$, where $H^1_{\text{per}}(0,2\pi)$ is the function space

$$H^1_{\text{per}}(0,2\pi) = \left\{v \in H^1_{\text{loc}}(\mathbb{R}), v(t+2\pi) = v(t), \text{ for all } t \in \mathbb{R}\right\}.$$

Using a Fourier transform, we find that the spectrum of the operator $\widehat{L_{\pm,\omega}}$ is the union of the spectra of the matrices

$$\widehat{L_{\pm,\omega,k}} = L_\pm \pm \omega \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -ik & 0 \\ 0 & 0 & 0 & 0 \\ ik & 0 & 0 & 0 \end{pmatrix}.$$

A direct computation gives

$$\sigma\left(\widehat{L_{\pm,\omega}}\right) = \left\{\lambda \in \mathbb{C},\ \lambda^4 \mp (4\rho^* - 2\alpha^*)\lambda^2 + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + (1+i\omega k)^2, k \in \mathbb{R}\right\}.$$

Notice that finding purely imaginary eigenvalues of the polynomial

$$\lambda^4 \mp (4\rho^* - 2\alpha^*)\lambda^2 + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + (1+i\omega k)^2$$

is equivalent to finding negative eigenvalues of the quadratic polynomial

$$\widetilde{\lambda}^2 \mp (4\rho^* - 2\alpha^*)\widetilde{\lambda} + 3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + (1+i\omega k)^2.$$

Remark that the product of the roots of this polynomial is equal to

$$3\rho^{*2} - 4\alpha^*\rho^* + \alpha^{*2} + (1+i\omega k)^2,$$

which is real number if and only if $k = 0$. We deduce that $\sigma\left(\widehat{L_{\pm,\omega}}\right)$ contains purely imaginary complex numbers for $k = 0$ only. Consequently, a bifurcation can only occur for $k = 0$, and the resulting solutions are the ones discussed in Sections 1.2, 1.3 and 1.4.

Chapter 2

Linear instability of spatially homogeneous waves

In this chapter, we consider partial differential equations of the form

$$U_x = DU_t + F(U), \tag{2.1}$$

where the unknown function U depends upon the time variable $t \in \mathbb{R}$ and a space variable $x \in \mathbb{R}$, with values in a real Banach space Z, D is a linear operator acting in Z and F a nonlinear map. In Section 2.1 we prove a simple criterion for linear instability of x-independent steady solutions of (2.1) with respect to x-dependent perturbations. The main hypotheses of this result are spectral assumptions and a symmetry hypothesis. We use this criterion in Section 2.2 and prove linear instability of constant solutions of the Lugiato-Lefever equation, traveling waves of the Kadomtsev-Petviashvili-I equation and periodic waves of the Davey-Stewartson equations. In Section 2.3, we study the bifurcations which are induced by this instability. With the help of the Lyapunov center theorem, we study the bifurcations for the Lugiato-Lefever equation and for the KP-I equation.

2.1 The main result

2.1.1 Instability problem and hypotheses

Consider the partial differential equation (2.1), in which F is a smooth map defined on a subspace $X \subset Z$, and D is a linear operator with domain $\mathrm{dom}(D)$ such that $X \subset \mathrm{dom}(D)$. Suppose that $U_* \in X$ is an x-independent steady solution of the equation (2.1), *i.e.*, $F(U_*) = 0$. Consider the linearized equation (2.2) at U_*,

$$U_x = DU_t + LU, \tag{2.2}$$

where $L = \mathrm{d}F(U_*)$ is the differential of F at U_*. Our goal is to study the linear instability of U_*, with respect to x-dependent perturbations. This problem concerns the existence of exponentially growing in time solutions for the linear system (2.2). More precisely, we say that

U_* is linearly unstable with respect to x-dependent perturbations if there exists a complex number λ, with $\mathrm{Re}(\lambda) > 0$, such that the linear equation (2.2) possesses a solution of the form

$$U(x,t) = e^{\lambda t} V(x),$$

with $V : \mathbb{R} \to X$ a differentiable and bounded function.

We make the following assumptions.

Hypothesis 2.1.1. *(i) L and D are closed real operators in Z, with domains* $\mathrm{dom}(L) = X \subset \mathrm{dom}(D)$. *Moreover, we suppose that the embedding $X \subset Z$ is continuous.*

(ii) The spectrum $\sigma(L)$ of the linear operator L contains a pair of isolated simple purely imaginary eigenvalues $\pm ik_*, k_* > 0$.

(iii) The equation (2.2) is reversible, i.e., there exists a linear symmetry $S \in \mathcal{L}(Z) \cap \mathcal{L}(X)$, $S \neq \mathrm{id}$, such that $S^2 = \mathrm{id}$ and $SDU = -DSU$, $SLU = -LSU$, for all $U \in X$.

2.1.2 The instability criterion

Our main result is the following instability criterion.

Theorem 2.1.1 (Linear instability criterion)**.** *Under the assumptions in Hypothesis 2.1.1, there exists $\lambda_* > 0$, such that for any $\lambda \in (0, \lambda^*)$, the system (2.2) has a solution of the form $U : (x,t) \mapsto e^{\lambda t} V(x)$, where $V(x) \in X$, for any $x \in \mathbb{R}$ and the map $x \mapsto V(x)$ is smooth and periodic. Consequently, the equilibrium U_* is linearly unstable.*

Proof. We claim that it is enough to prove that for any sufficiently small $\lambda > 0$ the spectrum $\sigma(\lambda D + L)$ of the real operator $\lambda D + L$ contains a pair of purely imaginary eigenvalues $\pm ik, k > 0$. Indeed, if W denotes an eigenvector associated to ik, its complex conjugate $\overline{W}$ is an eigenvector associated to $-ik$, and the function V defined through $V(x) = e^{ikx}W + e^{-ikx}\overline{W}$, for all $x \in \mathbb{R}$, is a real, smooth, periodic solution of the equation

$$V_x = (\lambda D + L)V.$$

Consequently, the function U defined through $U(x,t) = e^{\lambda t} V(x)$ satisfies (2.2), which proves the claim.

We show now that the operator $\lambda D + L$ possesses a pair of isolated purely imaginary eigenvalues, for any sufficiently small $\lambda > 0$. More precisely, we prove that there exist $\lambda_* > 0$ and $\varepsilon > 0$ such that, for any $\lambda \in (0, \lambda_*)$,

$$\sigma(\lambda D + L) \cap B(\pm ik_*, \varepsilon) = \{\pm ik(\lambda)\},$$

where $B(\pm ik_*, \varepsilon)$ denotes the open ball of radius ε centered in $\pm ik_*$ and $\pm ik(\lambda)$ are simple complex-conjugate eigenvalues of $\lambda D + L$, with $k(\lambda) > 0$.

By Hypothesis 2.1.1 (ii), the eigenvalues $\pm ik_*$ are isolated points in the spectrum of L so that there exists $\varepsilon > 0$ such that $\sigma(L) \cap B(\pm ik_*, \varepsilon) = \{\pm ik_*\}$, and $B(\pm ik_*, \varepsilon) \subset \{\nu \in$

$\mathbb{C}, \pm\mathrm{Im}(\nu) > 0\}$. We denote by $\Gamma_\pm$ the boundaries of these open balls, oriented counterclockwise, and by $P_\pm^*$ the spectral projections defined through

$$P_\pm^* = \frac{1}{2\pi i}\int_{\Gamma_\pm} (\nu - L)^{-1}\, \mathrm{d}\nu.$$

First, we prove that $\Gamma_\pm \subset \rho(\lambda D + L)$, where $\rho(\lambda D + L)$ is the resolvent set of the operator $\lambda D + L$, if λ is small enough. Indeed, let $\nu \in \Gamma_\pm$, so that $\nu - L$ is invertible. We write

$$\nu - (\lambda D + L) = (\mathrm{id} - \lambda D(\nu - L)^{-1})(\nu - L), \tag{2.3}$$

where id denotes the identity operator. Assume that λ satisfies the inequality

$$|\lambda| \leqslant \inf_{\nu\in\Gamma_\pm} \frac{1}{2\left\|D(\nu - L)^{-1}\right\|}, \tag{2.4}$$

where $\|\cdot\|$ denotes the operator norm in $\mathcal{L}(Z)$. Remark that the infimum in the right hand side of (2.4) is a positive number. Indeed, we first claim that the quantity $\|D(\nu - L)^{-1}\|$ is finite. Since the domain of D contains the domain of L, and since these operators are closed in Z, we obtain that D is a relatively bounded perturbation of L (see [40, Chapter IV, Remark 1.5]). In particular, $D \in \mathcal{L}(X, Z)$, and the operator norm $\|D\|_{X\to Z}$ is finite. On the other hand, since the complex number ν belongs to the resolvent set of L, we have that $(\nu - L)^{-1} \in \mathcal{L}(Z, X)$, and $\left\|(\nu - L)^{-1}\right\|_{Z\to X}$ is finite as well. The claim follows from the inequality

$$\left\|D(\nu - L)^{-1}\right\|_{Z\to Z} \leqslant \|D\|_{X\to Z} \left\|(\nu - L)^{-1}\right\|_{Z\to X}.$$

Moreover $\|D(\nu - \mathcal{L})^{-1}\|$ is uniformly bounded for $\nu \in \Gamma_\pm$ because the function

$$\nu \mapsto \left\|(\nu - \mathcal{L})^{-1}\right\|_{Z\to X}$$

is continuous on the compact set Γ_+. This proves that the infimum in (2.4) is finite.

As a consequence of the inequality (2.4), we have that

$$\left\|\lambda D(\nu - L)^{-1}\right\| \leqslant \frac{1}{2},$$

which implies that the operator $\mathrm{id} - \lambda D(\nu - L)^{-1}$ in the right hand side of (2.3) is invertible. Since $\nu \in \rho(L)$, the operator $\nu - L$ is invertible as well, and (2.3) implies that the operator $\nu - (\lambda D + L)$ is invertible. Consequently $\nu \in \rho(\lambda D + L)$, if λ is sufficiently small, showing that $\Gamma_\pm \subset \rho(\lambda D + \mathcal{L})$.

Next, consider the spectral projections $P_\pm(\lambda)$ defined through

$$P_\pm(\lambda) = \frac{1}{2i\pi}\int_{\Gamma_\pm} (\nu - (\lambda D + L))^{-1} \mathrm{d}\nu,$$

which are well-defined because $\Gamma_\pm \subset \rho(\lambda D + L)$. If we prove that

$$\left\|P_\pm^* - P_\pm(\lambda)\right\| < \min\left(\frac{1}{\|P_\pm^*\|}, \frac{1}{\|P_\pm(\lambda)\|}\right), \tag{2.5}$$

then Lemma 2.1.1 below will show that the projections $P_\pm^*$ and $P_\pm(\lambda)$ have the same finite rank, so that

$$\sigma\left(\lambda D+L\right)\cap B\left(\pm ik_*,\varepsilon\right)=\left\{ik_\pm(\lambda),k_\pm(\lambda)\in\mathbb{C}\right\}, \tag{2.6}$$

with $ik_\pm(\lambda)$ simple eigenvalues of $\lambda D+L$.

In order to prove (2.5), we write

$$\begin{aligned}
(\nu-(\lambda D+L))^{-1} &= (\nu-L)^{-1}(\mathrm{id}-\lambda D(\nu-L)^{-1})^{-1} \\
&= (\nu-L)^{-1}\sum_{n=0}^{+\infty}(\lambda D(\nu-L)^{-1})^n \\
&= (\nu-L)^{-1}+(\nu-L)^{-1}\sum_{n=1}^{+\infty}(\lambda D(\nu-L)^{-1})^n.
\end{aligned}$$

Then

$$\begin{aligned}
\left\|P_\pm(\lambda)-P_\pm^*\right\| &= \left\|\frac{1}{2i\pi}\int_{\Gamma_\pm}(\nu-L)^{-1}\sum_{n=1}^{+\infty}\lambda^n(D(\nu-L)^{-1})^n\mathrm{d}\nu\right\| \\
&\leqslant \frac{1}{2\pi}\int_{\Gamma_\pm}\left\|(\nu-L)^{-1}\right\|\sum_{n=1}^{+\infty}|\lambda^n|\left\|D(\nu-L)^{-1}\right\|^n|\mathrm{d}\nu| \\
&= \frac{1}{2\pi}\int_{\Gamma_\pm}\left\|(\nu-L)^{-1}\right\|\frac{|\lambda|\,\|D(\nu-L)^{-1}\|}{1-|\lambda|\,\|D(\nu-L)^{-1}\|}|\mathrm{d}\nu| \\
&\leqslant \frac{|\lambda|}{\pi}\int_{\Gamma_\pm}\left\|(\nu-L)^{-1}\right\|\left\|D(\nu-L)^{-1}\right\||\mathrm{d}\nu| \\
&\leqslant C\,|\lambda|,
\end{aligned}$$

for some constant $C>0$, where we have used (2.4) and the boundedness of the maps $\nu\mapsto\left\|D(\nu-L)^{-1}\right\|$ and $\nu\mapsto\|(\nu-L)^{-1}\|$, for $\nu\in\Gamma_\pm$.

Furthermore, we have

$$\|P_\pm(\lambda)\|\leqslant\left\|P_\pm^*\right\|+\left\|P_\pm(\lambda)-P_\pm^*\right\|\leqslant\left\|P_\pm^*\right\|+C\,|\lambda|.$$

Assuming that λ is small enough, we obtain on the one hand

$$\frac{1}{\|P_\pm^*\|}\geqslant C|\lambda|,$$

and on the other hand,

$$\frac{1}{\|P_\pm(\lambda)\|}>\frac{1}{\|P_\pm^*\|+C\,|\lambda^*|}>\frac{1}{|\lambda|},$$

which implies

$$\left\|P_\pm^*-P_\pm(\lambda)\right\|<\min\left(\frac{1}{\|P_\pm^*\|},\frac{1}{\|P_\pm(\lambda)\|}\right).$$

This proves the inequality (2.5) and implies the property (2.6).

Finally, the fact the eigenvalues $ik_\pm(\lambda)$ are purely imaginary and complex conjugate results from Hypothesis 2.1.1 (iii) and the fact that $\lambda D + L$ is a real operator. Indeed, the reversibility of the system (2.2) implies that the spectrum of the operator $\lambda D + L$ is symmetric with respect to the imaginary axis (see Lemma 2.1.2 below). Since the operator has precisely one simple eigenvalue $ik_\pm(\lambda)$ in $B(\pm ik_*, \varepsilon)$, this implies that this eigenvalue is purely imaginary, so that $k_\pm(\lambda) \in \mathbb{R}$. In addition, since $\lambda D + L$ is a real operator, its spectrum is symmetric with respect to the real axis, so that $k_\pm(\lambda) = \pm k(\lambda)$, for some $k(\lambda) > 0$. This proves that the operator $\lambda D + L$ has a pair of simple purely imaginary eigenvalues and completes the proof of the theorem. □

We give without proof the lemma from [28], used in the proof of Theorem 2.1.1.

Lemma 2.1.1. *Let Z be a Banach space and P_1, P_2 be two projections in $\mathcal{L}(Z)$. Denote by E_1 and E_2 the ranges of P_1 and P_2, respectively. Assume that P_1 has finite rank, and that*

$$\|P_1 - P_2\|_{Z\to Z} < \min\left(\frac{1}{\|P_1\|_{Z\to Z}}, \frac{1}{\|P_2\|_{Z\to Z}}\right).$$

Then $P_1|_{E_2} : E_2 \longrightarrow E_1$ and $P_2|_{E1} : E_1 \longrightarrow E_2$ are isomorphisms. Consequently P_1 and P_2 have the same rank.

The following lemma was also used in the proof of Theorem 2.1.1, and shows the symmetry of the spectrum of the operator $\lambda D + L$ with respect to the imaginary axis, for any $\lambda \in \mathbb{R}$.

Lemma 2.1.2. *Suppose that the linearized system (2.2) is reversible, i.e., there exists a linear symmetry $S \in \mathcal{L}(Z) \cap \mathcal{L}(X)$, $S \neq \mathrm{id}$, $S^2 = \mathrm{id}$, satisfying*

$$SDU = -DSU, \;\; SLU = -LSU, \;\; \forall\, U \in X.$$

Then the spectrum of the real operator $\lambda D + L$ is symmetric with respect to the imaginary axis, for any $\lambda \in \mathbb{R}$.

Proof. First, when $\lambda \in \mathbb{R}$, the operator $\lambda D + L$ is real, so that its spectrum is symmetric with respect to the real axis.

Next, we show that the spectrum of $\lambda D + L$ is symmetric with respect to the origin. This is equivalent to prove that the resolvent set $\rho(\lambda D + L)$ of the operator $\lambda D + L$ is symmetric with respect to the imaginary axis. Let $\gamma \in \rho(\lambda D + L)$. We have to prove that $-\gamma \in \rho(\lambda D + L)$. Since $\gamma \in \rho(\lambda D + L)$, the operator $\gamma - (\lambda D + L)$ is invertible, and in particular, we have

$$(\gamma - (\lambda D + L))^{-1}(\gamma - (\lambda D + L)) = \mathrm{id}.$$

We multiply this equality by S from the right side. Using the fact that S anticommutes with L and D, we obtain

$$(\gamma - (\lambda D + L))^{-1} S(\gamma + (\lambda D + L)) = S.$$

Since the operator S is invertible, with inverse $S^{-1} = S$, we have

$$(\gamma - (\lambda D + L))^{-1} S = \left(S(\gamma - (\lambda D + L))\right)^{-1} = S(\gamma + (\lambda D + L))^{-1},$$

where we have used again the fact that S anticommutes with L and D. We deduce

$$(\gamma + (\lambda D + L))^{-1}(\gamma + (\lambda D + L)) = \mathrm{id}.$$

Similarly, we find that

$$(\gamma + (\lambda D + L))(\gamma + (\lambda D + L))^{-1} = \mathrm{id},$$

which shows that the operator $\gamma + (\lambda D + L)$ is invertible, and that $-\gamma \in \rho(\lambda D + L)$. This proves the symmetry of $\rho(\lambda D + L)$ and completes the proof of the lemma. □

We conclude this section with some remarks on the hypotheses and some possible extensions.

Remark 2.1.1. *We assumed that the purely imaginary eigenvalues of L are nonzero. When 0 is a simple eigenvalue of the operator L, then similar arguments show that 0 persists as a simple eigenvalue of $\lambda D+L$, for sufficiently small real numbers λ. Indeed, the fact that $\lambda D+L$ is a real operator implies that its spectrum is symmetric with respect to the real axis, so that 0 persists as a real eigenvalue. This real eigenvalue is necessarily 0, due to the symmetry of the spectrum of $\lambda D + L$ with respect to the imaginary axis. As a consequence, for $\lambda > 0$ sufficiently small, the linearized system (2.2) has a solution of the form $U(x,t) = e^{\lambda t} V$, for some $V \in X$. This implies that the equilibrium U_* is linearly unstable with respect to constant perturbations in this case.*

Remark 2.1.2. *The perturbation argument in the proof of Theorem 2.1.1 also holds when the purely imaginary eigenvalues $\pm ik_*$ in Hypothesis 2.1.1 (i) have odd algebraic multiplicities, and Theorem 2.1.1 is true in this case as well. Indeed, using the same arguments as in the proof of Theorem 2.1.1, we obtain that for sufficiently small $\lambda > 0$, the spectral projections $P_\pm(\lambda)$ have the same odd rank, so that there exist two neighborhoods of $\pm ik_*$, each containing an odd number of eigenvalues of $\lambda D + L$, counted with their algebraic multiplicity. At least one pair of these eigenvalues are purely imaginary, since the spectrum of $\lambda D+L$ is symmetric with respect to the imaginary axis.*

Remark 2.1.3. *Theorem 2.1.1 can be easily extended to systems of the form*

$$U_x = PU + F(U), \quad P = \sum_{k=1}^{n} D_k \partial_t^{(k)}, \tag{2.7}$$

in which $D_1, \ldots, D_n$ are closed linear operators whose domains contain X and $\partial_t^{(k)}$ represents the differential operator of order k with respect to t (see [22] for an example). Indeed, for the system (2.7), it is enough to prove that for sufficiently small $\lambda > 0$, the operator $\sum_{k=1}^{n} \lambda^k D_k + L$ possesses a pair of purely imaginary eigenvalues. This can be done using the same perturbation arguments as in the proof of Theorem 2.1.1.

Remark 2.1.4. *A key argument in the proof of Theorem 2.1.1 is the fact that the spectrum of the operator $\lambda D + L$ is symmetric with respect to the imaginary axis, for $\lambda \in \mathbb{R}$. In particular, if Hypothesis 2.1.1 (iii) is not satisfied, the conclusion of Theorem 2.1.1 may be wrong. As a counterexample consider the Korteweg-de Vries equation*

$$u_t = u_{xxx} + u_x + uu_x, \tag{2.8}$$

and the constant solution $u_* = 0$. Equation (2.8) can be written in the form

$$U_x = DU_t + F(U),$$

where

$$U = \begin{pmatrix} u \\ u_1 \\ u_2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(U) = \begin{pmatrix} u_1 \\ u_2 \\ -u_1 - uu_1 \end{pmatrix}.$$

The linearized system around 0 reads

$$U_x = DU_t + LU, \tag{2.9}$$

in which L is the 3×3 matrix given by

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Hypothesis 2.1.1 (i) is true, by setting $Z = X = \mathbb{R}^3$. Moreover, a direct computation shows that the spectrum of L is given by $\sigma(L) = \{0, \pm i\}$, where $\pm i$ are simple eigenvalues, so that Hypothesis 2.1.1 (ii) is satisfied.

However, Hypothesis 2.1.1 (iii) does not hold, because the only 3×3 matrix which anti-commutes with L and D is the null matrix. Furthermore, the spectrum of $\lambda D + L$, for $\lambda \in \mathbb{R}$, is not symmetric with respect to the imaginary axis. Indeed, the eigenvalues of the operator $\lambda D + L$ are the complex roots of the characteristic polynomial

$$X^3 + X - \lambda.$$

If $\mu \in \mathbb{C}$ is a root of this polynomial, i. e., if $\mu^3 + \mu - \lambda = 0$, then for any nonzero real number λ, we have $-\overline{\mu}^3 - \overline{\mu} - \lambda \neq 0$, and $-\overline{\mu}$ is not an eigenvalue of $\lambda D + L$. Consequently, Theorem 2.1.1 cannot be applied to state the linear instability of u_*.

In fact, the solution u_* is spectrally stable. Indeed, the linear operator obtained by linearizing (2.8) around u_* is

$$\mathcal{L} = \partial_{xxx} + \partial_x,$$

which is closed in $L^2(\mathbb{R})$, with domain $H^3(\mathbb{R})$. Using a Fourier transform, we obtain that the spectrum of $\mathcal{L}$ is given by

$$\sigma(\mathcal{L}) = \{(ik)^3 + ik, k \in \mathbb{R}\} = i\mathbb{R},$$

which shows that u_* is spectrally stable.

2.2 Applications

In this section, we apply the instability criterion in Theorem 2.1.1 to several examples.

2.2.1 Constant solutions of the Lugiato-Lefever equation

We consider the Lugiato-Lefever equation,

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi |\psi|^2 + F, \tag{2.10}$$

discussed in Chapter 1. We set

$$\psi = \psi_r + i\psi_i, \qquad \frac{\mathrm{d}\psi}{\mathrm{d}x} = \varphi_r + i\varphi_i,$$

and write equation (2.10) in the form (2.1),

$$U_x = D_\pm U_t + F_\pm(U), \tag{2.11}$$

where

$$U = \begin{pmatrix} \psi_r \\ \varphi_r \\ \psi_i \\ \varphi_i \end{pmatrix}, \qquad D_\pm U = \pm \begin{pmatrix} 0 \\ -\psi_i \\ 0 \\ \psi_r \end{pmatrix}, \qquad F_\pm(U) = \begin{pmatrix} \varphi_r \\ \pm(\psi_r^3 + \psi_i^2\psi_r - \alpha\psi_r - \psi_i) \\ \varphi_i \\ \pm(\psi_i^3 + \psi_r^2\psi_i + \psi_r - \alpha\psi_i - F) \end{pmatrix},$$

in which the symbols $+$ and $-$ stand for the case of normal ($\beta = 1$) and anomalous dispersion ($\beta = -1$), respectively. Our goal is to apply the instability criterion in Theorem 2.1.1 to the constant solutions of (2.10), and recover the instability result in Theorem 1.5.2 of Chapter 1.

Let $\psi^* = \psi_r^* + i\psi_i^*$ be a constant solution of (2.10), with corresponding parameters α^* and F^*. The set of these solutions has been described in Section 1.1.1 of Chapter 1. Linearizing (2.11) around $U^* = (\psi_r^*, 0, \psi_i^*, 0)^T$, we find the system

$$U_x = D_\pm U_t + L_\pm U, \tag{2.12}$$

where the matrices

$$L_\pm = \mathrm{d}F_\pm(U^*) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \pm(3\psi_r^{*2} + \psi_i^{*2} - \alpha^*) & 0 & \pm(2\psi_r^*\psi_i^* - 1) & 0 \\ 0 & 0 & 0 & 1 \\ \pm(2\psi_r^*\psi_i^* + 1) & 0 & \pm(\psi_r^{*2} + 3\psi_i^{*2} - \alpha^*) & 0 \end{pmatrix}$$

are precisely the ones found in Chapter 1, Section 1.1.2. We prove the following result.

Theorem 2.2.1. *Let ψ^* be a constant solution of (2.10), with corresponding parameters α^* and F^*. Set $\rho^* = |\psi^*|^2$.*

(i) In the case $\beta = 1$, the constant solution ψ^* is linearly unstable with respect to localized perturbations in the following cases :

(a) $\alpha^* > \sqrt{3}$ and $F^{*2} \in (F_-(\alpha^*), F_+(\alpha^*))$, when $\rho^* \in (\rho_+(\alpha^*), \rho_-(\alpha^*))$.

(b) $\alpha^* > 2$ and $F^{*2} \in (1 + (1-\alpha^*)^2, F_+^2(\alpha^*)]$, when $\rho^* \in (1, \rho_+(\alpha^*)]$.

(ii) In the case $\beta = -1$, the constant solution ψ^* is linearly unstable with respect to localized perturbations in the following cases :

(a) $\alpha^* \leqslant \sqrt{3}$ and $F^{*2} > 1 + (1-\alpha^*)^2$, when $\rho > 1$.

(b) $\alpha^* \in (\sqrt{3}, 2)$ and $F^{*2} \in (1 + (1-\alpha^*)^2, F_+^2(\alpha^*)]$, when $\rho^* \in (1, \rho_+(\alpha^*)]$.

(c) $\alpha^* > \sqrt{3}$ and $F^* \in (F_-^2(\alpha^*), F_+(\alpha^*))$, when $\rho^* \in (\rho_+(\alpha^*), \rho_-(\alpha^*))$.

(d) $\alpha^* > \sqrt{3}$ and $F^{*2} \geqslant F_-(\alpha^*)$, when $\rho^* \geqslant \rho_-(\alpha^*)$.

Proof. We check the hypotheses of Theorem 2.1.1. First, the operators $L_\pm$ and $D_\pm$ above are bounded in $\mathbb{R}^4$, so that Hypothesis 2.1.1 (i) is satisfied. Moreover, the system (2.12) is reversible, since the linear symmetry $S = \mathrm{diag}(1, -1, 1, -1)$ anticommutes with $L_\pm$ and $D_\pm$. Finally, the location of the eigenvalues of the matrices $L_\pm$ in the complex plane has been discussed in Chapter 1, Section 1.1.3. Going back to the diagrams in Figures 1.3 and 1.4, for the values of the parameters α^* and F^* as given in Theorem 2.2.1, the matrices $L_\pm$ possess at least one pair of simple purely imaginary eigenvalues. This proves that Hypothesis 2.1.1 (ii) holds, and completes the proof of the theorem. □

Remark 2.2.1. *The sufficient conditions given in Theorem 2.2.1 are in fact necessary to obtain linear instability of constant solutions of the equation (2.10), as shown in Theorem 1.5.2 proved in Chapter 1.*

2.2.2 Traveling waves of the KP-I equation

In this section, we consider the Kadomtsev-Petviashvili I (KP-I) equation,

$$(u_t + u_{xxx} + uu_x)_x - u_{yy} = 0, \tag{2.13}$$

where the function u depends on the two space variables $x, y \in \mathbb{R}$ and on the time $t \in \mathbb{R}$. This equation is a two-dimensional generalization of the KdV equation (see 2.8), and was derived in 1970 by the Soviet physicists Boris B. Kadomtsev and Vladimir I. Petviashvili to model the propagation of water waves when the surface tension is strong [38]. Our goal is to use the criterion in Theorem 2.1.1 and show the transverse instability of line solitary and periodic waves. We recover known transverse instability results [25, 37, 55, 59].

We start by studying the existence of one-dimensional, y-independent, solitary and periodic traveling waves of (2.13). Then we write (2.13) in the form

$$U_y = DU_t + F(U), \tag{2.14}$$

and prove the linear instability of solitary and periodic traveling waves of (2.13) with respect to y-dependent perturbations, using Theorem 2.1.1. We refer to such a linear instability as transverse linear instability.

One-dimensional traveling waves of the KP-I equation

One-dimensional traveling waves are solutions of (2.13) of the form

$$u(x, y, t) = v(x - ct),$$

where c denotes the speed of the wave. We can set $c = 1$, because the equation (2.13) has the scaling invariance

$$u(x, y, t) \longmapsto cu(\sqrt{c}x, cy, c\sqrt{c}t).$$

Writing x instead of $x - t$, we obtain that v satisfies the differential equation

$$(v_{xxx} - v_x + vv_x)_x = 0.$$

Integrating this equation twice leads to the equation

$$v_{xx} - v + \frac{1}{2}v^2 = \alpha x + \beta. \tag{2.15}$$

Since we focus on periodic and solitary waves, we can set $\alpha = 0$, and we can set $\beta = 0$ due to Galilean invariance. Setting $w = v_x$, we write (2.15) as a first order system

$$V_x = F(V),$$

where

$$V = \begin{pmatrix} v \\ w \end{pmatrix}, \ F(V) = \begin{pmatrix} w \\ v - \frac{1}{2}v^2 \end{pmatrix}.$$

This system has two equilibria, $(0, 0)$, which is a saddle, and $(2, 0)$, which is a center. As it is well known, this system has a first integral

$$(x, y) \mapsto \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

Using this first integral, we obtain the phase portrait given in Figure 2.1. We conclude that

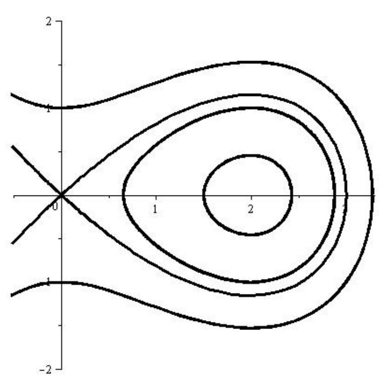


FIGURE 2.1 – Phase portrait of (2.15) for $\alpha = \beta = 0$.

the equation (2.15) possesses a family of periodic orbits and one homoclinic orbit. These correspond to a family of periodic traveling waves and a solitary traveling wave of (2.13). We point out that explicit formulas are available for these waves, since the equation is integrable. Without loss of generality, we assume that these waves are even.

Transverse linear instability of solitary waves

We start by writing the equation (2.13) (in which we write x instead of $x - ct$) in the form (2.14). We set $v = u_y$, and

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 \\ \partial_x & 0 \end{pmatrix}, \; F(U) = \begin{pmatrix} v \\ u_{xxxx} - u_{xx} + \left(\frac{1}{2}u^2\right)_{xx} \end{pmatrix},$$

so that (2.13) becomes

$$U_y = DU_t + F(U). \tag{2.16}$$

For the function space Z we choose $Z = H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

Consider an even solitary wave P of (2.13), and the corresponding linearized system

$$U_y = DU_t + LU, \tag{2.17}$$

where the operator $L = \mathrm{d}F(P)$ is given by

$$L = \begin{pmatrix} 0 & 1 \\ \partial_{xxxx} - \partial_{xx} + \partial_{xx}(P\cdot) & 0 \end{pmatrix}$$

with domain $X = H^4(\mathbb{R}) \times H^1(\mathbb{R})$. Notice that in this formulation, the traveling wave P, which does not depend upon y, is an equilibrium of the system (2.1).

Applying the general criterion in Theorem 2.1.1, we obtain the following result.

Theorem 2.2.2. *The even solitary traveling wave P of the KP-I equation (2.13) is transversely linearly unstable.*

Proof. We check the hypotheses of Theorem 2.1.1. First, the choice of spaces for Z and X above ensures that L is a closed operator in Z, with domain X. Moreover, the operator D is bounded in Z. Consequently, Hypothesis 2.1.1 (i) is satisfied.

Next, notice that the system (2.17) is reversible. Indeed, the linear symmetry $S = \mathrm{diag}(1, -1)$ anticommutes with D and L, because P is an even function. This proves that Hypothesis 2.1.1 (iii) is satisfied.

It remains to prove the spectral assumption in Hypothesis 2.1.1 (ii), *i.e.*, that the operator L has a pair of simple purely imaginary eigenvalues. Consider the operator

$$L_0 = \partial_{xxxx} - \partial_{xx} + \partial_{xx}(P\cdot),$$

which is closed in $L^2(\mathbb{R})$, with domain $H^4(\mathbb{R})$. We claim that it is enough to prove that L_0 has a simple negative eigenvalue. Indeed, for $\mu \in \mathbb{C}$, the eigenvalue problem

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} u \\ v \end{pmatrix}$$

is equivalent to

$$v = \mu u, \quad u_{xxxx} - u_{xx} + (Pu)_{xx} = \mu^2 u,$$

so that μ is an eigenvalue of L if and only if μ^2 is an eigenvalue of L_0.

The arguments in the proof of Theorem 1 in [33] show that the spectrum of the operator L_0 is given by

$$\sigma(L_0) = \{-\omega_0^2\} \cup [0, +\infty),$$

where $-\omega_0^2 < 0$ is a simple eigenvalue. Consequently, $\pm i\omega_0$ is a pair of simple purely imaginary eigenvalues of the operator L_0. We deduce that Hypothesis 2.1.1 (ii) is satisfied.

These arguments show that the hypotheses of Theorem 2.1.1 hold, so that the even solitary traveling wave P of the KP-I equation (2.13) is transversely linearly unstable. $\square$

Transverse linear instability of periodic waves

We now consider the case where P is an even periodic traveling wave of the equation (2.13). We denote by T its minimal period. We set $Z = H^1_{\text{per}} \times L^2_{\text{per}}$, where

$$L^2_{\text{per}} = \left\{v \in L^2_{\text{loc}}(\mathbb{R}), v(x+T) = v(x), \text{for a.e. } x \in \mathbb{R}\right\}, \tag{2.18}$$

and the Sobolev space H^n_{per} is defined for any $n \in \mathbb{N}^*$ by

$$H^n_{\text{per}} = \{v \in H^n_{\text{loc}}(\mathbb{R}), v(x+T) = v(x), \text{ for all } x \in \mathbb{R}\}. \tag{2.19}$$

Equation (2.13) can be written in the form (2.16), where U, D and F are the same as in the previous case. Linearizing around the periodic wave P, we obtain the operator

$$L = \mathrm{d}F(P) = \begin{pmatrix} 0 & 1 \\ \partial_{xxxx} - \partial_{xx} + \partial_{xx}(P\cdot) & 0 \end{pmatrix}$$

with domain $X = H^4_{\text{per}} \times H^1_{\text{per}}$.

Theorem 2.2.3. *The even periodic travelling wave P of the KP-I equation (2.13) is transversely linearly unstable.*

Proof. We check the assumptions of Theorem 2.1.1. With the choice of the spaces above, the operator L is closed in Z, with domain X. Moreover, the operator D is bounded in Z. This shows that Hypothesis 2.1.1 (i) holds. Next, Hypothesis 2.1.1 (iii) can be checked as in the proof of Theorem 2.2.2.

It remains to prove that the operator L has a pair of simple purely imaginary eigenvalues. Notice that in this case, the operator L has compact resolvent, because the embedding $X \subset Z$ is compact. According to Theorem [40, Chapter III, Theorem 6.29], the spectrum of L consists in isolated eigenvalues with finite algebraic multiplicities. Then, as in the proof of Theorem 2.2.2, it is enough to prove that the operator $L_0 = \partial_{xxxx} - \partial_{xx} + \partial_{xx}(P\cdot)$, which is closed in L^2_{per}, with domain H^4_{per}, has a simple negative eigenvalue. According to the proof of Theorem 1 of [59], the operator L_0 has precisely one negative eigenvalue, which is simple. Consequently, Hypothesis 2.1.1 (ii) is satisfied.

We conclude that the the hypotheses of Theorem 2.1.1 hold, so that the even periodic traveling wave P of the KP-I equation (2.13) is transversely linearly unstable. $\square$

2.2.3 Periodic solutions of the Davey-Stewartson equations

Consider the Davey-Stewartson equations

$$
\begin{aligned}
iA_t + A_{xx} + \alpha A_{yy} + \lambda A + \delta B_x A + \gamma |A|^2 A &= 0 \qquad (2.20)\\
B_{xx} + \nu B_{yy} + \mu \left(|A|^2\right)_x &= 0, \qquad (2.21)
\end{aligned}
$$

where the unknowns A and B are complex- and real-valued functions, respectively, depending on the space variables x, y and the time t, and where the coefficients α, λ, δ, γ, μ and ν are real. The system (2.20)-(2.21) arises as a model in the water-wave problem, approximately describing three-dimensional gravity-capillary waves (see [1]). The dynamics of the solutions of (2.20)-(2.21) strongly depends on the signs of the coefficients α, δ, μ and ν. Following [1], we focus on the three following cases, which are relevant for the water-wave problem :

- **Case 1 :** $\alpha > 0$, $\delta < 0$, $\mu > 0$ and $\nu > 0$.
- **Case 2 :** $\alpha > 0$, $\delta < 0$, $\mu < 0$ and $\nu < 0$.
- **Case 3 :** $\alpha < 0$, $\delta > 0$, $\mu > 0$ and $\nu > 0$.

Our goal is to study the linear instability of a family of y-independent periodic steady waves of (2.20)-(2.21), with respect to y-dependant perturbations. As in Section 2.2.2, the corresponding linear instability will be called transverse linear instability.

The system (2.20)-(2.21) has a family of one-dimensional periodic solutions (A^*, B^*), given by

$$
A^*(x) = a_0 e^{ikx}, \qquad B_x^* = \chi - \mu |a_0|^2,
$$

with $k^2 = (\gamma - \delta\mu)|a_0|^2 + \lambda + \delta\chi$, where χ is an arbitrary constant (see [14]). Since the system (2.20)-(2.21) is invariant under the change $B_x \longmapsto B_x + \chi$, $\lambda \longmapsto \lambda + \delta\chi$, without loss of generality, we may assume that $\chi = 0$. Notice that the system (2.20)-(2.21) is invariant under multiplication of the complex variable A by $e^{i\theta}$, $\theta \in \mathbb{R}$, so that we can restrict to the case $a_0 \in \mathbb{R}$.

In order to study the stability of the periodic solution (A^*, B^*), we set

$$
A(x,y,t) = A^*(x)\left(1 + a(x,y,t)\right), \quad B_x(x,y,t) = B_x^* + b(x,y,t), \tag{2.22}
$$

where the functions a and b are assumed to be periodic with respect to x, with wavenumber K, and b has zero mean. Notice that the perturbations A^*a and b belong to a rather general class. In particular, these perturbations are periodic in x with the same period as (A^*, B^*) if $K = nk$, for some $n \in \mathbb{Z}$. A direct calculation shows that a and b satisfy the nonlinear system

$$
\begin{aligned}
a_{yy} &= -\frac{1}{\alpha}\left(ia_t + a_{xx} + 2ika_x + \gamma a_0^2 (a + \overline{a}) + \delta b + \delta ab + \gamma a_0^2 \left(a^2 + 2a\overline{a} + a^2\overline{a}\right)\right) \qquad (2.23)\\
b_{yy} &= -\frac{1}{\nu}\left(b_{xx} + \mu a_0^2 (a + \overline{a})_{xx} + \mu a_0^2 (a\overline{a})_{xx}\right). \qquad (2.24)
\end{aligned}
$$

With this formulation, the periodic solution (A^*, B^*) of the system (2.20)-(2.21) corresponds to the trivial equilibrium $(0, 0)$ of (2.23)-(2.24).

Next, we set $a = a_r + ia_i$, $a_y = \widetilde{a}_r + i\widetilde{a}_i$, $b_y = \widetilde{b}$ and $U = \left(a_r,\ a_i,\ b,\ \widetilde{a}_r,\ \widetilde{a}_i,\ \widetilde{b}\right)^T$, so that the system (2.23)-(2.24) is written in the form

$$U_y = DU_t + F(U),$$

where

$$D = -\frac{1}{\alpha}\begin{pmatrix} 0_3 & 0_3 \\ E & 0_3 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

in which 0_3 is the zero matrix of order 3, and

$$F(U) = \begin{pmatrix} \widetilde{a}_r \\ \widetilde{a}_i \\ \widetilde{b} \\ -\frac{1}{\alpha}\left((a_r)_{xx} + 2\gamma a_0^2 a_r - 2k(a_i)_x + \delta b + \delta a_r b + \gamma a_0^2\left(3a_r^2 + a_i^2 + a_r(a_r^2 + a_i^2)\right)\right) \\ -\frac{1}{\alpha}\left(2k(a_r)_x + (a_i)_{xx} + \delta a_i b + \gamma a_0^2 a_i\left(2a_r + (a_r^2 + a_i^2)\right)\right) \\ -\frac{1}{\nu}\left(2\mu a_0^2(a_r)_{xx} + b_{xx} + \mu(a_r^2 + a_i^2)\right) \end{pmatrix}.$$

Linearizing this system at the origin we find a system of the form

$$U_y = DU_t + LU, \tag{2.25}$$

with

$$L = \mathrm{d}F(0) = \begin{pmatrix} 0_3 & I_3 \\ C & 0_3 \end{pmatrix}, \quad C = \begin{pmatrix} -\frac{1}{\alpha}(\partial_{xx} + 2\gamma a_0^2) & \frac{2}{\alpha}k\partial_x & -\frac{\delta}{\alpha} \\ -\frac{2}{\alpha}k\partial_x & -\frac{1}{\alpha}\partial_{xx} & 0 \\ -\frac{2}{\nu}\mu a_0^2\partial_{xx} & 0 & -\frac{1}{\nu}\partial_{xx} \end{pmatrix},$$

where I_3 is the identity matrix of order 3. For the function space Z we choose the set

$$Z = \left(H^1_{\text{per}}\right)^2 \times \widetilde{H^1_{\text{per}}} \times \left(L^2_{\text{per}}\right)^2 \times \widetilde{L^2_{\text{per}}},$$

where

$$L^2_{\text{per}} = \left\{v \in L^2_{\text{loc}}(\mathbb{R}), v\left(x + \frac{2\pi}{K}\right) = v(x),\ \text{for a.e. } x \in \mathbb{R}\right\},$$

$$H^j_{\text{per}} = \left\{v \in H^j_{\text{loc}}(\mathbb{R}), v\left(x + \frac{2\pi}{K}\right) = v(x), \forall\ x \in \mathbb{R}\right\},\ \text{for } j \in \mathbb{N}^*,$$

and $\widetilde{L^2_{\text{per}}} \subset L^2_{\text{per}}$, $\widetilde{H^j_{\text{per}}} \subset H^j_{\text{per}}$ are the subspaces consisting of functions with zero mean.

Theorem 2.2.4. *The periodic wave (A^*, B^*) is transversely linearly unstable with respect to perturbations of the form (2.22) in Case 1 if $K^2 - 2(3k^2 - \lambda) < 0$, or if $K^2 - 2(3k^2 - \lambda) > 0$ and $\gamma > 0$, and in Cases 2 and 3.*

Proof. We apply Theorem 2.1.1 and check that the assumptions in Hypothesis 2.1.1 are satisfied by L and D. With the choice of Z above, the operator L is closed, with domain

$$X = \left(H^2_{\mathrm{per}}\right)^2 \times \widetilde{H^2_{\mathrm{per}}} \times \left(H^1_{\mathrm{per}}\right)^2 \times \widetilde{H^1_{\mathrm{per}}}.$$

Furthermore D is a bounded operator in Z, so that Hypothesis 2.1.1 (i) is satisfied.

Next, remark that the linearized system (2.25) is reversible, since the linear symmetry $S = \mathrm{diag}\,(1,1,1,-1,-1,-1)$ anticommutes with D and L. Consequently, Hypothesis 2.1.1 (iii) holds.

Now we prove that the spectrum of the operator L contains at least one pair of simple purely imaginary eigenvalues. The compact embedding $X \subset Z$ ensures that the spectrum $\sigma(L)$ of L is a discrete set of isolated eigenvalues with finite multiplicities. Using Fourier series, we have that

$$\sigma(L) = \bigcup_{n\in\mathbb{Z}} \Sigma_n,$$

where Σ_n is the spectrum of the matrix

$$L_n = \begin{pmatrix} 0_3 & I_3 \\ C_n & 0_3 \end{pmatrix}, \quad C_n = \begin{pmatrix} -\frac{1}{\alpha}(-n^2K^2 + 2\gamma a_0^2) & \frac{2}{i} n\alpha kK & -\frac{\delta}{\alpha} \\ -\frac{2}{\alpha} inkK & \frac{1}{\alpha} n^2K^2 & 0 \\ \frac{2}{\nu}\mu a_0^2 n^2 K^2 & 0 & \frac{1}{\nu} n^2K^2 \end{pmatrix}. \tag{2.26}$$

A direct computation shows that Σ_n is the set of the roots of the polynomial

$$P_n = s^6 - p_4 s^4 + p_2 s^2 - \frac{n^4K^4(n^2K^2 - 2(3k^2 - \lambda))}{\nu\alpha^2},$$

where

$$p_4 = \frac{n^2K^2(\alpha + 2\nu) - 2\nu\gamma a_0^2}{\nu\alpha},$$

and

$$p_2 = \frac{n^2K^2\left[2\alpha(n^2K^2 - (\gamma - \delta\mu)a_0^2) + \nu(n^2K^2 - 2(2k^2 + \gamma a_0^2))\right]}{\nu\alpha^2}.$$

Notice that $P_n = P_{-n}$, so that the eigenvalues of L_n and L_{-n} are the same, for $n \neq 0$. In particular, the eigenvalues of L have even algebraic multiplicity. In order to check Hypothesis (ii), which requires simple purely imaginary eigenvalues, we restrict the analysis to the invariant subspace

$$Z_r = \left\{ (a_r,\ a_i,\ b,\ ,\widetilde{a_r},\ \widetilde{a_i},\ \widetilde{b}) \in Z \,\middle|\, a_r,\ \widetilde{a_r},\ b,\ \widetilde{b} \text{ even functions},\ a_i,\ \widetilde{a_i} \text{ odd functions} \right\}.$$

With this restriction, for $n \neq 0$, we can count the multiplicity of the eigenvalues from those of L_n with $n > 0$. When $n = 0$ we are left with only the two eigenvalues $\pm\sqrt{-2\gamma a_0^2/\alpha}$, which are purely imaginary if $\alpha\gamma > 0$.

Case 1. Suppose that $\alpha > 0$, $\delta < 0$, $\mu > 0$ and $\nu > 0$. This case has been analyzed in [14]. We regard P_n as a cubic polynomial in s^2. A direct computation shows that this polynomial has no negative critical point. If $K^2 - 2(3k^2 - \lambda) > 0$, then $n^2K^2 - 2(3k^2 - \lambda) > 0$ for any $n \geqslant 1$, and the product of the roots of the cubic polynomial P_n is positive. Then the polynomial in s^2 has no negative roots, so that the matrices L_n have no purely imaginary eigenvalues for $n \geqslant 1$. When $\gamma > 0$, the matrix L_0 possesses a pair of simple purely imaginary eigenvalues, given by $\pm\sqrt{2\gamma a_0^2/\alpha}\, i$. Finally, the spectrum of L contains a pair of simple purely imaginary eigenvalues if and only if $\gamma > 0$.

If $K^2-2(3k^2-\lambda) > 0$, then the product of the roots of the cubic polynomial P_1 is negative, so that it possesses a unique negative root. Consequently, the spectrum of L possesses at least a pair of purely imaginary eigenvalues. Following the arguments of [14], these eigenvalues are simple.

Case 2. Suppose that $\alpha > 0$, $\delta < 0$, $\mu < 0$ and $\nu < 0$. We set $s^2 = n^2K^2t$. Then s is a root of P_n if and only if t is a root of the cubic polynomial

$$P_\varepsilon = t^3 - \varepsilon p_4 t^2 + \varepsilon^2 p_2 t - \varepsilon^3 p_0,$$

where we set $\varepsilon = 1/n^2K^2$. We regard the polynomial P_ε as a perturbation of the asymptotic polynomial

$$Q = t^3 - \left(\frac{1}{\nu} + \frac{2}{\alpha}\right) t^2 + \left(\frac{2}{\nu\alpha} + \frac{1}{\alpha^2}\right) - \frac{1}{\nu\alpha^2}.$$

In this case, notice that $1/\nu$ is a negative simple root and $1/\alpha$ is a positive double root of Q. Using an implicit functions argument, we obtain that for ε sufficiently small, the polynomial P_ε possesses a simple negative root, given by

$$t_{0,\varepsilon} = \frac{1}{\nu} + O(\varepsilon).$$

Consequently, if n is large enough, the cubic polynomial P_n possesses a simple negative root $s_{0,n}^2$ and two positive roots $s_{1,n}^2$ and $s_{2,n}^2$. The negative root $s_{0,n}^2$ has the asymptotic expansion

$$s_{0,n}^2 = \frac{n^2K^2}{\nu} + O\left(1\right),$$

as n tends to $+\infty$. In particular, the sequence $\left(s_{0,n}^2\right)_{n\geqslant 1}$ is strictly decreasing when n is large enough, so that the negative roots s_{0,n_1}^2 and s_{0,n_2}^2 are distinct for $n_1 \neq n_2$ sufficiently large. Consequently, for n large enough, the matrices L_n have a pair of simple purely imaginary eigenvalues, which are distinct for $n_1 \neq n_2$. This implies that the spectrum of L contains at least one pair of simple purely imaginary eigenvalues.

Case 3. Suppose that $\alpha < 0, \delta > 0, \mu > 0$ and $\nu > 0$. We keep the notations of Case 2. In this case, $1/\alpha$ is a double negative root and $1/\nu$ is a simple positive root of Q. Using the Newton polygon method (see for example [13, Chapter 2]), we find that P_ε has two simple negative roots, given by

$$t_{1,\varepsilon} = \frac{1}{\alpha} - \frac{2k}{\alpha}|\varepsilon|^{\frac{1}{2}} + O(\varepsilon), \quad t_{2,\varepsilon} = \frac{1}{\alpha} + \frac{2k}{\alpha}|\varepsilon|^{\frac{1}{2}} + O(\varepsilon).$$

Consequently, for n sufficiently large, the cubic polynomial P_n possesses two distinct negative roots $s_{1,n}^2$, $s_{2,n}^2$ and one positive root $s_{0,n}^2$. The negative roots have the asymptotic expansion

$$s_{1,n}^2 = \frac{n^2K^2}{\alpha} - \frac{2k}{\alpha}|nK| + O(1), \quad s_{2,n}^2 = \frac{n^2K^2}{\alpha} + \frac{2k}{\alpha}|nK| + O(1),$$

as n tends to $+\infty$. Using the fact that the sequences $\left(s_{1,n}^2\right)_{n\geqslant 1}$ and $\left(s_{2,n}^2\right)_{n\geqslant 1}$ are strictly decreasing when n is large enough, we conclude that these roots are simple and distinct for $n_1 \neq n_2$. As in the previous case, this implies that L possesses at least one pair of simple purely imaginary eigenvalues.

This shows that Hypothesis 2.1.1 (ii) holds in Z_r for the values of the parameters in the theorem, and completes the proof of the theorem. □

2.3 Induced bifurcations

In this section, we consider the stationary system

$$U_x = F(U), \tag{2.27}$$

and we suppose that U_* is an equilibrium of (2.27). We are interested in bifurcations of x-dependent solutions, which are close to the unstable x-independent solution U_*. In the case where U_* is unstable with respect to transverse perturbations, these bifurcations are called dimension breaking bifurcations. The result in Theorem 2.1.1 shows, under the assumptions in Hypothesis 2.1.1, that the equilibrium U_* is linearly unstable, and in particular we expect bifurcations of periodic solutions for the system (2.27), due to the presence of a pair of purely imaginary eigenvalues in the spectrum $L = \mathrm{d}F(U_*)$. However, the spectral assumption in Hypothesis 2.1.1 (ii) is too weak to ensure the existence of such bifurcations. In Section 2.3.1, we recall the Lyapunov center theorem, which, under stronger assumptions, shows the existence of small periodic solutions bifurcating from U_*. Then we apply this theorem to the Lugiato-Lefever equation and to the KP-I equation.

2.3.1 Lyapunov center theorem

We recall the following theorem (see for example [2, 5, 41]), which gives sufficient conditions for the existence of periodic solutions for the stationary system (2.27).

Theorem 2.3.1 (Lyapunov)**.** *Let X and Z be two real Banach spaces such that X is continuously and densely embedded in Z. Consider the system (2.27), where $F \in \mathcal{C}^3(\Omega, Z)$, in which Ω is an open subset of X. Let $U_* \in \Omega$ be an equilibrium of (2.27), and set $L = \mathrm{d}F(U_*)$. Assume that the following properties hold :*

(i) $\pm i\omega_0 \neq 0$ are simple eigenvalues of L ;

(ii) $ik\omega_0$ belongs to the resolvent set of L, for all $k \in \mathbb{Z} \setminus \{-1, 1\}$;

(iii) there exists $a_* > 0$, such that the resolvent estimates

$$\left\|(ia - L)^{-1}\right\|_{Z\to Z} \leqslant \frac{C}{|a|}, \quad \left\|(ia - L)^{-1}\right\|_{Z\to X} \leqslant C$$

hold for any $a \in \mathbb{R}$, $|a| > a_*$;

(iv) the system (2.27) is reversible, i.e. there exists a linear symmetry $S \neq \mathrm{id}$ such that $S^2 = \mathrm{id}$ and $SF(U) = -F(SU)$, for any $U \in X$.

Then there exists an open interval $I \subset \mathbb{R}$, $0 \in \mathbb{I}$, a continuously differentiable curve $\{U_s\}_{s\in I}$ of periodic solutions of the system (2.27), where $U_0 = U_*$, and a continuously differentiable curve of frequencies $\{\omega_s\}_{s\in I}$, such that the period of the solution U_s is $2\pi/\omega_s$.

2.3.2 Application to the Lugiato-Lefever equation

We consider the stationary Lugiato-Lefever equation,

$$i\beta\frac{\partial^2\psi}{\partial x^2} = -(1 + i\alpha)\psi + i\psi\,|\psi|^2 + F, \tag{2.28}$$

for which we proved in Section 2.2.1, that certain constant solutions are linearly unstable (see Theorem 2.2.1). Using the Lyapunov center theorem stated before, we prove that x-periodic solutions bifurcate from some of these unstable constant solutions. This completes the bifurcation results in Chapter 1, where we restricted to parameters close to the curves $F^2 = 1 + (1 - \alpha)^2$ and $F^2 = F_\pm^2(\alpha)$. However, the analysis here is restricted to periodic solutions.

Let $\psi^* = \psi_r^* + i\psi_i^*$ be a constant solution of (2.28), with corresponding parameters α^* and F^*, and denote by ρ^* the square modulus of ψ^*. As in Section 1.1.2, we set

$$\psi = \psi^* + \widetilde{\psi}, \qquad \widetilde{\psi} = \widetilde{\psi_r} + i\widetilde{\psi}_i, \qquad \frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}x} = \widetilde{\varphi_r} + i\widetilde{\varphi}_i,$$

and rewrite the equation (2.28) in the form

$$U_x = L_\pm U + F_\pm(U), \tag{2.29}$$

where $L_\pm$ are the matrices in (1.6) and $F_\pm(U) = G_\pm(U, \alpha^*, F^*)$.

Our goal is now to prove the existence of periodic solutions of the system (2.29) with small amplitude. For this purpose, we use Theorem 2.3.1. We state the following result.

Theorem 2.3.2. *In both cases $\beta = 1$ and $\beta = -1$, assume that $\alpha^* > \sqrt{3}$ and $F^{*2} \in (F_-^2(\alpha^*), F_+^2(\alpha^*))$, and take $\rho^* \in (\rho_+(\alpha^*), \rho_-(\alpha^*))$. Then the matrices $L_\pm$ have one pair of simple purely imaginary eigenvalues $\pm ik$, $k > 0$, and the system (2.29) possesses a family of periodic solutions u_ε of the form*

$$u_{\varepsilon,j}(x) = \varepsilon U_\varepsilon(k_\varepsilon x),\ k_\varepsilon = k + \varepsilon l_\varepsilon$$

for ε small enough.

Proof. We check the hypotheses of Theorem 2.3.1. First, we set $Z = X = \mathbb{R}^4$. Next, for values of the coefficients α^* and F^* given in Theorem 2.3.2, the bifurcation analysis performed in Chapter 1, Section 1.1.3 ensures that the matrix $L_\pm$ possesses one pair of simple purely imaginary eigenvalues $\pm ik$, $k \neq 0$, and no other purely imaginary eigenvalues (see Figure 1.4 in Chapter 1). Hypotheses (i) and (ii) are then satisfied.

We now prove that the resolvent estimates in Hypothesis (iii). The estimate

$$\|(ia - L_\pm)^{-1}\|_{\mathbb{R}^4 \to \mathbb{R}^4} \leqslant C,$$

for $|a| > a_*$ sufficiently large and some suitable constant $C > 0$, is automatically satisfied, since the operators $L_\pm$ are bounded in the finite-dimensional space $\mathbb{R}^4$. This resolvent estimate implies the second one in Hypothesis (iii) in Theorem 2.3.1. Indeed, if

$$(L_\pm - ia)U = V,$$

then

$$|a|\|U\| \leqslant \|L_\pm U\| + \|V\| \leqslant \|L_\pm\|\|U\| + \|V\|.$$

For $|a| > a_*$ sufficiently large and for some constant C satisfying $C \geqslant |a|/(|a| - \|L_\pm\|)$, we have

$$\|U\| \leqslant \frac{C}{|a|}\|V\|.$$

Consequently, Hypothesis (iii) holds.

Finally, remark that the linear symmetry $S = \mathrm{diag}(1, -1, 1, -1)$ anticommutes with the vector field in (2.29), so that the system (2.29) is reversible, and Hypothesis (iv) is satisfied.

Theorem 2.3.1 then proves the existence of a family of periodic solutions of the system (2.29), with small amplitude and wavelengths close to k. □

Remark 2.3.1. *Theorem 2.3.1 cannot be applied to the system (2.29) when the purely imaginary spectrum of the matrices $L_\pm$ consists in a pair of simple nonzero eigenvalues and when 0 is a double eigenvalue. In this case, an $0^2(i\omega)$ bifurcation occurs, which has been studied in Chapter 1, Section 1.3.*

We also state the following result.

Theorem 2.3.3. *We make the following assumptions on α^*, F^* and ρ^* :*

(i) $\alpha^* > 2$, $F^{*2} \in (1 + (1-\alpha^*)^2, F_+^2(\alpha^*))$ and $\rho^* \in (1, \rho_+(\alpha^*))$, if $\beta = 1$;

(ii) $\alpha^* < \sqrt{3}$, $F^{*2} > 1 + (1-\alpha^*)^2$ and $\rho^* > 1$, $\alpha^* > \sqrt{3}$, $F^{*2} > F_-^2(\alpha^*)$ and $\rho^* > \rho_-(\alpha^*)$ or $\alpha^* \in (\sqrt{3}, 2)$, $F^{*2} \in (1 + (1-\alpha^*)^2, F_+^2(\alpha^*))$, and $1 < \rho^* < \rho_+(\alpha^*)$, if $\beta = -1$.

For such values of the parameters α^* and F^{*2}, the matrices $L_\pm$ have two pairs of simple purely imaginary eigenvalues $\pm ik_1$, $\pm ik_2$. On a dense subset of the areas of the (α, F^2)-plane in which the conditions (i) and (ii) above are satisfied, the system (2.29) possesses two families of periodic solutions $u_{\varepsilon,j}$ of the form

$$u_{\varepsilon,j}(x) = \varepsilon U_{\varepsilon,j}(k_{\varepsilon,j}x),\ k_{\varepsilon,j} = k_j + \varepsilon l_{\varepsilon,j}, \quad j = 1, 2,$$

for ε small enough.

Proof. Here again, we check the hypotheses of Theorem 2.3.1. For values of the coefficients α^* and F^* given in Theorem 2.3.3, the bifurcation analysis performed in Chapter 1, Section 1.1.3 ensures that the matrix $L_\pm$ possesses two pairs of simple purely imaginary eigenvalues $\pm ik_1$, $\pm ik_2$, $k_1, k_2 \neq 0$. Hypothesis (i) is then satisfied. Moreover, up to a restriction to a dense subset of the areas in Figure 1.4 in Chapter 1 in which Hypothesis (i) holds, we can suppose that $\pm ik_1$ and $\pm ik_2$ are non-resonant eigenvalues, in the sense that $nk_1 \neq mk_2$, for any $n, m \in \mathbb{Z}^*$. Consequently, ink_1 and ink_2 belong to the resolvent set of L for any $n \in \mathbb{Z} \setminus \{-1, 1\}$, and Hypothesis (ii) holds. Finally, Hypotheses (iii) and (iv) can be checked as in the proof of Theorem 2.3.2. □

Remark 2.3.2. *In Chapter 3, we will prove the linear instability of the periodic solutions in Theorem 2.3.3, which occurs when $L_\pm$ has a pair of nonresonant purely imaginary eigenvalues (see Section 3.3).*

2.3.3 Application to the KP-I equation

We now apply Theorem 2.3.1 to the periodic solutions KP-I equation. We consider an even periodic traveling wave P of (2.13), as described in Section 2.2.2, and write the stationary KP-I equation in the form

$$U_y = F(U), \tag{2.30}$$

where F is the same as in (2.16).

Theorem 2.3.4. *Assume that P is an even periodic traveling wave of the KP-I equation (2.13). Then a family of doubly periodic solutions $U_s(x, y)$ of the KP-I equation, with s small, emerges from the periodic wave P in a dimension breaking bifurcation :*
- *$U_s(x, y) = P(x) + O(|s|)$,*
- *U_s is periodic with respect to y, with period $2\pi/\omega_s$, where $\lim_{s\to 0} \omega_s = \omega_0$.*

Proof. As in the previous example, it is enough to check the hypotheses of the Lyapunov center theorem.

For convenience, we set $Z = H^2_{\rm per} \times L^2_{\rm per}$ and $X = H^4_{\rm per} \times H^2_{\rm per}$, where the spaces $L^2_{\rm per}$ and $H^n_{\rm per}$, $n \in \mathbb{N}^*$, are defined by (2.18) and (2.19), respectively. Then the embedding $X \subset Z$ is dense and compact. With this choice of spaces, the operator $L = \mathrm{d}F(P)$ given by

$$L = \begin{pmatrix} 0 & 1 \\ \partial_{xxxx} - \partial_{xx} + \partial_{xx}(P\cdot) & 0 \end{pmatrix}$$

is closed in Z with domain X. Moreover it has compact resolvent, thus the spectrum of L consists of isolated eigenvalues. We have previously seen, in the proof of Theorem 2.2.3, that the operator

$$L_0 = \partial_{xxxx} - \partial_{xx} + \partial_{xx}(P\cdot)$$

is closed in $L^2_{\rm per}$, with domain $H^4_{\rm per}$. Moreover, the two first eigenvalues of L_0 are $-\omega_0^2 < 0$, which is simple, and 0. Consequently, $\pm i\omega_0$ are the only nonzero purely imaginary eigenvalues of L, and they are simple too. In addition, $0 \in \sigma(L)$, and in order to apply Theorem 2.3.1 we have to restrict the space Z and the domain of L, because Hypothesis (ii) of Theorem 2.3.1 does not hold.

We consider the invariant subspace $\widetilde{Z} = \widetilde{H^2_{\text{per}}} \times \widetilde{L^2_{\text{per}}}$, in which $\widetilde{H^n_{\text{per}}}$ (respectively $\widetilde{L^2_{\text{per}}}$) denotes the subspace of H^n_{per} (respectively the subspace of L^2_{per}) of even functions with zero mean. We claim that the kernel of L restricted to the subspace $\widetilde{X} = X \cap \widetilde{Z} = \widetilde{H^4_{\text{per}}} \times \widetilde{H^2_{\text{per}}}$ is trivial. Indeed, we observe that

$$\ker L = \ker L_0 \times \{0\} .$$

We write $L_0 = \partial_{xx} C$, where $A = \partial_{xx} - 1 + P$. We have the equality $\ker(\partial_{xx} C) = \ker(\partial_x C)$ because of the equivalences

$$\partial_{xx} C u = 0 \Longleftrightarrow Cu = \text{const.} \Longleftrightarrow \partial_x C u = 0,$$

since we consider only periodic functions. According to [59], the kernel of $\partial_x C$ is two-dimensional, and is spanned by P_x, which is an odd function, and an even function with non-zero mean. Thus the kernel of $L_0 = \partial_{xx} C$ restricted to the subspace $\widetilde{H^4_{\text{per}}}$ is trivial. Moreover, following the arguments of [59], the eigenfunction of L_0 associated with its negative eigenvalue is an even function. We conclude that

$$\sigma(L) \cap i\mathbb{R} = \{\pm i\omega_0\} ,$$

so that the two first hypotheses of Theorem 2.3.1 are satisfied.

We now check the resolvent estimates of Theorem 2.3.1. We write $L = L_1 + L_2$, where

$$L_1 = \begin{pmatrix} 0 & 1 \\ \partial_{xxxx} - \partial_{xx} & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 \\ (P\cdot)_{xx} & 0 \end{pmatrix}.$$

The operator L_2 is a relatively bounded perturbation of the operator with constant coefficients L_1. First, we prove the resolvent estimates in Theorem 2.3.1 for the operator L_1. Consider $a \in \mathbb{R}$, and

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in X, \qquad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in Z,$$

such that

$$(ia - L_1)U = V. \tag{2.31}$$

Let $c_n(u_1)$ (respectively $c_n(u_2), c_n(v_1), c_n(v_2)$) be the n^{th} Fourier coefficient of u (respectively u_2, v_1, v_2). Then, expanding (2.31) into Fourier series, we obtain that $c_n(u_1)$ and $c_n(u_2)$ satisfy the linear system

$$\begin{aligned} iac_n(u_1) - c_n(u_2) &= c_n(v_1) \\ -(n^4 + n^2)c_n(u_1) + iac_n(u_2) &= c_n(v_2). \end{aligned}$$

Solving this system we find

$$c_n(u_1) = \frac{iac_n(v_1) + c_n(v_2)}{n^4 + n^2 + a^2}$$

and

$$c_n(u_2) = \frac{(n^4+n^2)c_n(v_1) + iac_n(v_2)}{n^4+n^2+a^2}.$$

Since the system is linear we can estimate for $v_1 = 0$ and $v_2 = 0$ separately. Taking $v_2 = 0$, we find :

$$\begin{aligned} \|u_1\|^2_{H^2_{\mathrm{per}}} &= \sum_{n\in\mathbb{Z}} (1+n^2)^2 \left|c_n(u_1)\right|^2 = \sum_{n\in\mathbb{Z}} \frac{(1+n^2)^2 \left|iac_n(v_1)\right|^2}{(n^4+n^2+a^2)^2} \\ &\leqslant \sum_{n\in\mathbb{Z}} \frac{(1+n^2)^2 a^2 \left|c_n(v_1)\right|^2}{a^4} = \frac{1}{a^2} \|v_1\|^2_{H^2_{\mathrm{per}}}. \end{aligned}$$

Similarly, taking $v_1 = 0$, we have

$$\|u_1\|^2_{H^2_{\mathrm{per}}} = \sum_{n\in\mathbb{Z}} \frac{(1+n^2)^2 \left|c_n(v_2)\right|^2}{n^4+n^2+a^2}.$$

The rational fraction

$$n \longmapsto \frac{a^2(1+n^2)^2}{(n^4+n^2+a^2)^2}$$

is uniformly bounded in a, so that

$$\|u_1\|^2_{H^2_{\mathrm{per}}} \leqslant \frac{C_1}{a^2} \|v_2\|^2_{L^2_{\mathrm{per}}},$$

for some $C_1 > 0$ and a large enough. Hence we have, for a constant $C_2 > 0$,

$$\|u_1\|_{H^2_{\mathrm{per}}} \leqslant \frac{C_2}{|a|} \left(\|v_1\|_{H^2_{\mathrm{per}}} + \|v_2\|_{L^2_{\mathrm{per}}} \right).$$

Using similar arguments, we also find

$$\|u_2\|^2_{L^2_{\mathrm{per}}} \leqslant \frac{D_1}{a^2} \|v_1\|^2_{H^2_{\mathrm{per}}},$$

for some $D_1 > 0$ and

$$\|u_2\|^2_{L^2_{\mathrm{per}}} \leqslant \frac{1}{a^2} \|v_2\|^2_{L^2_{\mathrm{per}}},$$

which leads to

$$\|u_2\|_{L^2_{\mathrm{per}}} \leqslant \frac{D_2}{|a|} \left(\|v_1\|_{H^2_{\mathrm{per}}} + \|v_2\|_{L^2_{\mathrm{per}}} \right)$$

for some $D_2 > 0$. As a result we have the estimate for the norm of the resolvent of L_1,

$$\left\|(ia - L_1)^{-1}\right\|_{Z\to Z} \leqslant \frac{C}{|a|}, \tag{2.32}$$

for some $C > 0$. The inequality (2.32) implies the second estimate for $\left\|(ia - L_1)^{-1}\right\|_{Z\to X}$. Indeed, for $U \in X$ and $V \in Z$, satisfying $(ia - L_1)U = V$, estimate (2.32) implies

$$\|L_1 U\|_Z \leqslant \|V\|_Z + |a| \, \|U\|_Z \leqslant (1 + C) \, \|V\|_Z \, .$$

Since the embedding $X \subset Z$ is continuous and the operator L_1 is closed in Z, with domain X, the norm on X is equivalent to the graph norm of L_1 (see for example [52, Theorem 2.1]). Consequently, there exists a constant C' such that

$$\|U\|_X \leqslant C' \left(\|L_1 U\|_Z + \|U\|_Z\right),$$

and finally there exists a constant C'' such that

$$\|U\|_X \leqslant C'' \, \|V\|_Z \, .$$

This implies the second resolvent estimate for L_1

$$\left\|(ia - L_1)^{-1}\right\|_{Z\to X} \leqslant C''. \tag{2.33}$$

In order to prove these estimates for L, we write

$$ia - L = \left(\mathrm{id} - L_2(ia - L_1)^{-1}\right)(ia - L_1)\, ,$$

and suppose that a is large enough to ensure that $\left\|L_2\,(ia - L_1)^{-1}\right\|_{Z\to Z} \leqslant 1/2$, which is possible due to the estimate (2.32). Then $\mathrm{id} - L_2(ia - L_1)^{-1}$ is invertible, and we have

$$\begin{aligned}
\left\|(ia - L)^{-1}\right\|_{Z\to Z} &\leqslant \left\|(ia - L_1)^{-1}\right\|_{Z\to Z} \left\|\left(\mathrm{id} - L_2(ia - L_1)^{-1}\right)^{-1}\right\|_{Z\to Z} \\
&\leqslant 2\left\|(ia - L_1)^{-1}\right\|_{Z\to Z} .
\end{aligned}$$

Using the inequality (2.32), we have

$$\left\|(ia - L)^{-1}\right\|_{Z\to Z} \leqslant \frac{2C}{|a|}.$$

Similarly, using (2.33), we obtain the estimate

$$\left\|(ia - L)^{-1}\right\|_{Z\to X} \leqslant \left\|(ia - L_1)^{-1}\right\|_{Z\to X} \left\|\left(\mathrm{id} - L_2(ia - L_1)^{-1}\right)^{-1}\right\|_{Z\to Z} \leqslant 2C.$$

Finally, the stationary system (2.30) is reversible, since the linear symmetry $S = \mathrm{diag}(1, -1)$ anticommutes with F. This proves that the hypotheses of the Lyapunov center theorem hold, and the result follows from Theorem 2.3.1. $\square$

Remark 2.3.3. *In the case of a solitary traveling wave, Theorem 2.3.1 cannot be used to study dimension breaking bifurcations, since* 0 *belongs to the essential spectrum of* L *(see the proof of Theorem 2.2.2). In this case, the dimension breaking bifurcations induced by the linear instability of* P *may be investigated with the help of a Lyapunov-Iooss theorem (see for example [5, Chapter 2, Theorem 2.5.2] and [34, Section 4, Theorem 4]).*

Chapter 3

Finite-wavelength instability of periodic waves

In this chapter, we consider systems of the form

$$U_x = DU_t + LU + R(U), \tag{3.1}$$

where the unknown function U depends upon a space variable x and the time t, D and L are linear operators, and R is a smooth nonlinear map. We suppose that $U_* = 0$ is an equilibrium of (3.1). Using the same formulation as in Chapter 2, we give sufficient conditions in Section 3.1 for the linear instability of 0. Under these hypotheses, periodic solutions with small amplitude emerge from 0, and the goal of Section 3.2 below is to investigate the linear instability of these periodic solutions, with respect to certain perturbations. In Section 3.3, we apply the instability criterion proved in Section 3.2 to the Lugiato-Lefever equation. We conclude with a discussion about some open questions in the water-wave problem.

3.1 The instability problem

We consider the system (3.1), in which the unknown function U takes values in a Hilbert space Z. We denote by $\langle\cdot,\cdot\rangle$ the inner product on Z, and by $\|\cdot\|_Z$ the associated norm. We suppose that R is a smooth map defined on a Hilbert space $(X, \langle\cdot,\cdot\rangle_X)$ such that $X \subset Z$, satisfying $R(0) = 0, \mathrm{d}R(0) = 0$ and D, L are linear operators acting in Z, with respective domains $\mathrm{dom}(D)$ and $\mathrm{dom}(L) = X$. More precisely, we make the following assumptions, which are very similar to the ones of the Lyapunov center theorem (see Theorem 2.3.1).

Hypothesis 3.1.1. *(i) L and D are closed real operators in Z, with domains* $\mathrm{dom}(L) = X \subset \mathrm{dom}(D)$*, and $R : X \to Z$ is a function of class $\mathcal{C}^3$, satisfying $R(0) = 0$,* $\mathrm{d}R(0) = 0$*. Moreover, we suppose that the embedding $X \subset Z$ is dense and compact.*

(ii) The purely imaginary spectrum of L consists of two pairs of isolated, simple eigenvalues $\pm ik_1$, $\pm ik_2$, $k_1, k_2 \neq 0$. Additionaly, we suppose that these eigenvalues are non-resonant, i. e., for any $n, m \in \mathbb{Z}^*$, $nk_1 \neq mk_2$.

(iii) There exists $a_* > 0$ such that the resolvent estimate

$$\|(ia - L)^{-1}\|_{Z \to Z} \leqslant \frac{C}{|a|}. \tag{3.2}$$

holds for any $a \in \mathbb{R}$, $|a| > a_*$.

(iv) The system (3.1) is reversible, i.e., there exists a linear symmetry $S \in \mathcal{L}(X) \cap \mathcal{L}(Z)$, $S \neq \mathrm{id}$, $S^2 = \mathrm{id}$, satisfying

$$SDU = -DSU, \ SLU = -LSU, \ SR(U) = -R(SU), \ \forall\, U \in X.$$

Remark 3.1.1. *The compactness of the embedding $X \subset Z$ implies that the operator L has compact resolvent, so that its spectrum consists of isolated eigenvalues with finite multiplicities (see [40, Chapter III, Theorem 6.29]).*

The following theorem, showing the linear instability of 0 and the existence of periodic solutions bifurcating from 0, is a straightforward consequence of Theorems 2.1.1 and 2.3.1.

Theorem 3.1.1. *Under the assumptions in Hypothesis 3.1.1, the following properties hold.*

(i) The constant solution 0 of the system (3.1) is linearly unstable.

(ii) The stationary system

$$U_x = LU + R(U) \tag{3.3}$$

possesses two families of periodic solutions with small amplitude and wavelengths close to k_1 and k_2 respectively,

$$u_{\varepsilon,j}(x) = \varepsilon U_{\varepsilon,j}(k_{\varepsilon,j}x), \ k_{\varepsilon,j} = k_j + \varepsilon l_{\varepsilon,j}, \quad j = 1, 2, \tag{3.4}$$

for ε small enough, where the function $U_{\varepsilon,j}$ is 2π-periodic and of class $\mathcal{C}^1$.

Proof. (i) The assumptions (i), (ii) and (iv) in Hypothesis 3.1.1 imply that Theorem 2.1.1 in Chapter 2 can be applied to the constant solution 0 of the system (3.1). We conclude that 0 is linearly unstable.

(ii) We check the hypotheses of Lyapunov center theorem 2.3.1, given in Chapter 2, Section 2.3, which then implies the result in the second part of the theorem. First, according to Hypothesis (ii), the spectrum of L contains two pairs of simple nonzero purely imaginary eigenvalues $\pm ik_1$, $\pm ik_2$. These eigenvalues are non-resonant, so that ink_1 and ink_2 belong to the resolvent set of L, for any $n \in \mathbb{Z} \setminus \{-1, 1\}$.

Next, the resolvent estimate (3.2) implies the second estimate in Theorem 2.3.1,

$$\|(ia - L)^{-1}\|_{Z \to X} \leqslant C.$$

Indeed, for $U \in X$ and $V \in Z$, satisfying $(ia - L)U = V$, estimate (3.2) implies

$$\|LU\|_Z \leqslant \|V\|_Z + |a| \, \|U\|_Z \leqslant (1 + C) \, \|V\|_Z \, .$$

Since the embedding $X \subset Z$ is continuous and the operator L is closed in Z, with domain X, the norm on X is equivalent to the graph norm of L (see for example [52, Theorem 2.1]). Consequently, there exists a constant C' such that

$$\|U\|_X \leqslant C' \left(\|LU\|_Z + \|U\|_Z\right),$$

and a constant C'' such that

$$\|U\|_X \leqslant C'' \|V\|_Z .$$

Finally, the stationary system (3.3) is reversible, since the system (3.1) is reversible. This shows that the hypotheses of the Lyapunov center theorem are satisfied for the two pairs of eigenvalues $\pm ik_1$, $\pm ik_2$ and completes the proof.

□

Our goal is to study the stability of the periodic solutions (3.4) in Theorem 3.1.1. More precisely, we will show that each family of periodic waves is unstable with respect to certain perturbations.

Consider the linearized system about $u_{\varepsilon,j}$,

$$U_x = DU_t + LU + \mathrm{d}R(u_{\varepsilon,j})U.$$

For convenience, we work with 2π-periodic functions and therefore set $x' = k_{\varepsilon,j}x$. This leads to the following system,

$$k_{\varepsilon,j}U_x = DU_t + LU + \varepsilon L_{\varepsilon,j}U, \tag{3.5}$$

where $L_{\varepsilon,j} = \mathrm{d}R(U_{\varepsilon,j})$, and where we have dropped the primes for notational simplicity.

Definition 3.1.1. *We say that the periodic wave $U_{\varepsilon,j}$ is* ***linearly unstable with respect to perturbations of wavelength*** *K if the linearized system (3.5) possesses a solution of the form*

$$U(x,t) = e^{\lambda t}e^{iKx}V(x), \tag{3.6}$$

where $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$, $K \in [0,1)$, and $V : \mathbb{R} \to X$ is a 2π-periodic function.

Notice that the 2π-periodic function V in (3.6) satisfies the eigenvalue problem

$$\mathcal{L}_{\varepsilon,\lambda}V = -ik_{\varepsilon,1}KV, \tag{3.7}$$

where $\mathcal{L}_{\varepsilon,\lambda}$ is the linear operator defined through

$$\mathcal{L}_{\varepsilon,\lambda} = k_1\partial_x - L + \varepsilon\left(l_{\varepsilon,1}\partial_x - L_{\varepsilon,1}\right) - \lambda D. \tag{3.8}$$

This property will play a key role in our proof of linear instability in the next section.

3.2 Linear instability of small periodic waves

In this section, we prove that the assumptions given in Hypothesis 3.1.1 are sufficient conditions to obtain linear instability of the periodic waves $U_{\varepsilon,j}$, in the sense of Definition 3.1.1. We first recall some preliminary results.

3.2.1 Preliminary results

We give here some results that will be needed in the proof of Theorem 3.2.1 below. These results are well-known, so we recall them without proof. First, we define the function spaces

$$\widetilde{Z} = L^2_{\text{per}}(0, 2\pi; Z), \quad \widetilde{X} = H^1_{\text{per}}(0, 2\pi; Z) \cap L^2_{\text{per}}(0, 2\pi; X),$$

where, for a Hilbert space Y, we define

$$L^2_{\text{per}}(0, 2\pi; Y) = \left\{ f : \mathbb{R} \to Y \text{ measurable, } f(x + 2\pi) = f(x) \text{ a.e. and } \int_0^{2\pi} \|f(x)\|_Y^2 \, \mathrm{d}x < \infty \right\}$$

and

$$H^1_{\text{per}}(0, 2\pi; Y) = \left\{ f \in L^2_{\text{per}}(0, 2\pi; Y), f' \in L^2_{\text{per}}(0, 2\pi; Y) \right\},$$

in which f' denotes the derivative of f in the sense of distributions with values in Y.

We set, for any $U, V \in \widetilde{Z}$,

$$\langle U, V \rangle_{\widetilde{Z}} = \int_0^{2\pi} \langle U(x), V(x) \rangle_Z \, \mathrm{d}x \tag{3.9}$$

and for any $U, V \in \widetilde{X}$,

$$\langle U, V \rangle_{\widetilde{X}} = \int_0^{2\pi} \langle U_x(x), V_x(x) \rangle_Z + \langle U(x), V(x) \rangle_X \, \mathrm{d}x. \tag{3.10}$$

In Lemmas 3.2.1 and 3.2.2 hereafter, we recall some properties of the spaces $\widetilde{Z}$ and $\widetilde{X}$.

Lemma 3.2.1. *(i) Formula (3.9) defines an inner product on $\widetilde{Z}$. Moreover, the space $\widetilde{Z}$, endowed with this inner product, is a Hilbert space.*

(ii) Formula (3.10) defines an inner product on $\widetilde{X}$. Moreover, the space $\widetilde{X}$, endowed with this inner product, is a Hilbert space.

Notice that the proofs of this result can be easily adapted from the proofs of [6, Chapter 4, Theorem 4.8]) and [6, Chapter 8, Proposition 8.1].

In the sequel of this chapter, we endow $\widetilde{Z}$ with the norm defined by

$$\|U\|_{\widetilde{Z}} = \left(\int_0^{2\pi} \|U(x)\|_Z^2 \mathrm{d}x \right)^{\frac{1}{2}}, \ \forall \, U \in \widetilde{Z},$$

which is the Hilbertian norm associated with the inner product (3.9) on $\widetilde{Z}$. We also endow $\widetilde{X}$ with the Hilbertian norm

$$\|U\|_{\widetilde{X}} = \left(\|U_x\|_{\widetilde{Z}}^2 + \|U\|^2_{L^2_{\text{per}}([0,2\pi],X)} \right)^{\frac{1}{2}}, \ \forall \, U \in \widetilde{X}.$$

We also recall the following result, which can be deduced from [58, Theorem 1].

Lemma 3.2.2. *The injection $\widetilde{X} \subset \widetilde{Z}$ is compact.*

In the proof of Theorem 3.2.1, which is the main result of this chapter, we use expansions in Fourier series of functions of the space $\widetilde{Z}$. Theorem 1.6 of [3], which is recalled hereafter, ensures the convergence of such series.

Proposition 3.2.1. *For any $f \in \widetilde{Z}$, the Fourier series $\sum_{n\in\mathbb{Z}} c_n(f)e^{inx}$ converges to f in $\widetilde{Z}$, in which the Fourier coefficients $c_n(f)$ are given by*

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\mathrm{d}x.$$

Remark 3.2.1. *The result in Proposition 3.2.1 stays true when Z is a Banach space with UMD (Uniform Martingale Difference) property. More precisely, let $p \in (1, +\infty)$. A Banach space Y has the UMD$_p$ property if there exists a constant $C > 0$ such that, for any martingale $(f_n)_{n\in\mathbb{N}}$ converging in $L^p(Y)$ and for any choice of sign $\varepsilon_n = \pm 1$,*

$$\sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} \varepsilon_k (f_k - f_{k-1}) \right\|_{L^p(Y)} \leqslant C \sup_n \|f_n\|_{L^p(Y)},$$

with the convention $f_{-1} = 0$. We say that Y has the UMD property if there exists $p \in (1, +\infty)$ such that Y has the UMD$_p$ property. In particular, any Hilbert space has the UMD$_2$ property (see for example [10])

We finally recall the following results of [4, Lemma 11] and [4, Corollary 1], respectively.

Proposition 3.2.2. *Consider $k \in \mathbb{R}^*$, L a closed linear operator in Z with domain X, and $\mathcal{L} = k\partial_x - L$. Assume that L satisfies Hypothesis 3.1.1 (i) and (iii). Then the following properties hold.*

(i) There exists a constant $M > 0$ such that

$$\|U\|_{\widetilde{X}} \leqslant M \left(\|U\|_{\widetilde{Z}} + \|\mathcal{L}U\|_{\widetilde{Z}}\right), \ \forall\, U \in \widetilde{X}. \tag{3.11}$$

(ii) The operator $\mathcal{L}$ is closed in $\widetilde{Z}$ with domain $\widetilde{X}$.

3.2.2 The main result

Our main result is the Theorem 3.2.1 below, which shows that Hypothesis 3.1.1 is sufficient to obtain linear instability of the small periodic waves of Theorem 3.1.1.

Theorem 3.2.1. *Assume that Hypothesis 3.1.1 holds, and consider the families of periodic waves $U_{\varepsilon,j}$ found in Theorem 3.1.1. Then there exist $\varepsilon_* > 0$, $\lambda_* > 0$, such that for any sufficiently small positive real numbers $\varepsilon \in (0, \varepsilon_*)$ and $\lambda \in (0, \lambda_*)$, there exists $K = K(\varepsilon, \lambda) \in [0, 1)$ such that the system (3.5) possesses a solution of the form $U(x, t) = e^{\lambda t}e^{iKx}V(x)$, where $V : \mathbb{R} \to X$ is a 2π-periodic function. Consequently, the periodic wave $U_{\varepsilon,j}$, for $j = 1, 2$, is linearly unstable with respect to perturbations of wavelength K.*

Proof. Without loss of generality, we set $j=1$. We make the ansatz (3.6) in the linearized equation (3.5). Then the 2π-periodic function V satisfies the eigenvalue problem (3.7).

Our goal is to prove that for sufficiently small positive numbers ε and λ, the operator $\mathcal{L}_{\varepsilon,\lambda}$ possesses purely imaginary eigenvalues. Indeed, if $i\nu$ is a purely imaginary eigenvalue of $\mathcal{L}_{\varepsilon,\lambda}$, with eigenvector $V \in \widetilde{X}$, then the function defined through

$$U(x,t) = e^{\lambda t} e^{-i\frac{\nu}{k_{\varepsilon,1}}x} V(x)$$

is a solution of the system (3.5). Consequently, the periodic wave $U_{\varepsilon,1}$ is linearly unstable with respect to perturbations of wavelength $\nu/k_{\varepsilon,1}$.

First, we prove that the operator $\mathcal{L}_{\varepsilon,\lambda}$ is closed in $\widetilde{Z}$ with domain $\widetilde{X}$. For this purpose, we prove that $\mathcal{L}_{\varepsilon,\lambda}$ is a relatively bounded perturbation of the operator $\mathcal{L}_{0,0} = k_1\partial_x - L$. Then we use a perturbation argument to show that it is enough to prove that $\mathcal{L}_{0,0}$ possesses simple purely imaginary eigenvalues. We finally study the spectrum of $\mathcal{L}_{0,0}$, using Fourier series.

Closeness of the operator $\mathcal{L}_{\varepsilon,\lambda}$. First, notice that the operator $\mathcal{L}_{0,0}$ is closed in $\widetilde{Z}$ with domain $\widetilde{X}$, by Proposition 3.2.2. We prove that the operator

$$\mathcal{L}_{\varepsilon,\lambda} - \mathcal{L}_{0,0} = \varepsilon\left(l_{\varepsilon,1}\partial_x - L_{\varepsilon,1}\right) - \lambda D$$

is relatively bounded with respect to $\mathcal{L}_{0,0}$, with relative bound smaller than 1 when ε and λ are real and small enough. This will imply, using [40, Chapter IV, Theorem 1.1], that $\mathcal{L}_{\varepsilon,\lambda}$ is closed in $\widetilde{Z}$ with domain $\widetilde{X}$.

We have to check that there exists a constant $C = C(\varepsilon,\lambda)$, which is smaller than 1, for sufficiently small ε, λ, such that for any $U \in \widetilde{X}$,

$$\|\varepsilon l_{\varepsilon,1}U_x - \varepsilon L_{\varepsilon,1}U - \lambda DU\|_{\widetilde{Z}} \leqslant C\left(\|U\|_{\widetilde{Z}} + \|\mathcal{L}_{0,0}U\|_{\widetilde{Z}}\right).$$

According to the inequality (3.11) given in Proposition 3.2.2, it is enough to prove that for some constant C', which can be made sufficiently small, for small enough ε and λ, the following inequality holds

$$\|\varepsilon l_{\varepsilon,1}U_x - \varepsilon L_{\varepsilon,1}U - \lambda DU\|_{\widetilde{Z}} \leqslant C'\|U\|_{\widetilde{X}}, \ \forall\, U \in \widetilde{X}.$$

Let $U \in \widetilde{X}$. We have the estimate

$$\|\varepsilon l_{\varepsilon,1}U_x - \varepsilon L_{\varepsilon,1}U - \lambda DU\|_{\widetilde{Z}} \leqslant |\varepsilon l_{\varepsilon,1}|\|U_x\|_{\widetilde{Z}} + |\varepsilon|\|L_{\varepsilon,1}U\|_{\widetilde{Z}} + |\lambda|\|DU\|_{\widetilde{Z}}.$$

First, remark that

$$\|U_x\|_{\widetilde{Z}} \leqslant \|U\|_{\widetilde{X}}. \tag{3.12}$$

Next, since $R : X \to Z$ is a smooth function, the operator $L_{\varepsilon,1} : X \to Z$ is bounded. Thus, there exists a constant $C_1 > 0$ such that for any $U \in \widetilde{X}$,

$$\|L_{\varepsilon,1}U\|_{\widetilde{Z}} = \left(\int_0^{2\pi} \|L_{\varepsilon,1}U(x)\|_Z^2\,\mathrm{d}x\right)^{\frac{1}{2}} \leqslant C_1\left(\int_0^{2\pi} \|U(x)\|_X^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Consequently,

$$\|L_{\varepsilon,1}U\|_{\widetilde{Z}} \leqslant C_1\|U\|_{L^2_{\mathrm{per}}([0,2\pi],X)} \leqslant C_1\|U\|_{\widetilde{X}}. \tag{3.13}$$

We now treat the term $\|DU\|_{\widetilde{Z}}$. The operators D and L are closed in Z and the domain of D contains the domain of L. Consequently, the operator D is $L-$bounded, and there exists a constant $C_2 > 0$, such that for any $U(x) \in \mathrm{dom}(D)$,

$$\|DU(x)\|_Z \leqslant C_2 \left(\|U(x)\|_Z + \|LU(x)\|_Z\right).$$

Next, the continuity of the embedding $X \subset Z$ and the fact that L is closed imply that the norm on X is equivalent to the graph norm of L, so that there exists a constant $C_3 > 0$, such that for any $U(x) \in \mathrm{dom}(D)$,

$$\|DU(x)\|_Z \leqslant C_3\|U(x)\|_X.$$

We deduce that

$$\|DU\|_{\widetilde{Z}} \leqslant C_3\|U\|_{L^2([0,2\pi],X)} \leqslant C_3\|U\|_{\widetilde{X}}. \tag{3.14}$$

Combining the inequalities (3.12), (3.13) and (3.14), we conclude that there exists a constant $C_4 > 0$ such that, for any $U \in \widetilde{Z}$,

$$\|\varepsilon l_{\varepsilon,1}U_x - \varepsilon L_{\varepsilon,1}U - \lambda DU\|_{\widetilde{Z}} \leqslant (2|\varepsilon| + |\lambda|)C_4\|U\|_{\widetilde{X}}.$$

As a result, there exist $\varepsilon_*, \lambda_* > 0$ sufficiently small, such that when $\varepsilon \in (0, \varepsilon_*)$ and $\lambda \in (0, \lambda_*)$, the operator $\varepsilon\left(l_{\varepsilon,1}\partial_x - L_{\varepsilon,1}\right) - \lambda D$ is a relatively bounded perturbation of $\mathcal{L}_{0,0}$, with relatively bound smaller than 1. Then $\mathcal{L}_{\varepsilon,\lambda}$ is closed in $\widetilde{Z}$, with domain $\widetilde{X}$.

Perturbation argument. We claim that it is enough to prove that the operator $\mathcal{L}_{0,0} = k_1\partial_x - L$ possesses a pair of simple complex conjugated purely imaginary eigenvalues, which we denote by $\pm i\mu$. Indeed, when ε and λ are real and small, the perturbation argument used in the proof of Theorem 2.1.1 in Chapter 2 implies that there exist two neighborhoods of $i\mu$ and $-i\mu$, each containing a simple eigenvalue of the operator $\mathcal{L}_{\varepsilon,\lambda}$. Lemma 3.2.3 below shows that the spectrum of the operator $\mathcal{L}_{\varepsilon,\lambda}$ is symmetric with respect to both the real and imaginary axis, so that these two eigenvalues are purely imaginary and complex conjugated.

Spectrum of $\mathcal{L}_{0,0}$. Finally, we prove that the operator $\mathcal{L}_{0,0}$ possesses simple purely imaginary eigenvalues. According to Lemma 3.2.2, the embedding $\widetilde{X} \subset \widetilde{Z}$ is compact, so that the spectrum of $\mathcal{L}_{0,0}$ consists of isolated eigenvalues with finite multiplicities.

Let μ be a complex number. Then μ belongs to the resolvent set of the operator $\mathcal{L}_{0,0}$ if and only if the operator $\mathcal{L}_{0,0} - \mu$ is invertible in $\widetilde{Z}$, *i.e.*, if and only if for any $V \in \widetilde{Z}$, there exists a unique $U \in \widetilde{X}$, satisfying

$$k_1\partial_x U - LU - \mu U = V. \tag{3.15}$$

According to Proposition 3.2.1, we can expand (3.15) into Fourier series, which leads to the equality

$$\sum_{n=-\infty}^{+\infty} (ink_1 - L - \mu)c_n(U)e^{inx} = \sum_{n=-\infty}^{+\infty} c_n(V)e^{inx}.$$

We deduce that the operator $\mathcal{L}_{0,0}-\mu$ is invertible in Z if and only if the operator $ink_1-L-\mu$ is invertible, for any $n \in \mathbb{Z}$. The spectrum $\sigma(\mathcal{L}_{0,0})$ of $\mathcal{L}_{0,0}$ is therefore given by

$$\sigma(\mathcal{L}_{0,0}) = \bigcup_{n\in\mathbb{Z}} \sigma(ink_1 - L).$$

Then the purely imaginary eigenvalues of $\mathcal{L}_{0,0}$ are found from the formulas

$$i\mu_{1,n}^{\pm} = ink_1 \pm ik_1, \quad i\mu_{2,n}^{\pm} = ink_1 \pm ik_2, \quad n \in \mathbb{Z}.$$

We remark that the eigenvalues $i\mu_{2,n}^{\pm}$ are simple, indeed the equality

$$imk_1 + ik_2 = ink_1 - ik_2$$

implies

$$(n-m)k_1 = 2k_2,$$

which is not the case since the eigenvalues $\pm ik_1$ and $\pm ik_2$ are supposed to be nonresonnant. For example, $\pm ik_2$ is a pair of simple purely imaginary eigenvalues of $\mathcal{L}_{0,0}$. This completes the proof of the theorem. □

Remark 3.2.2. *The arguments of the proof of Theorem 3.2.1 show that the periodic wave $u_{\varepsilon,1}$ (respectively $u_{\varepsilon,2}$) is linearly unstable with respect to a family of quasiperiodic perturbations, which are superposition of perturbations of wavelength close to* 1 *and k_2 (respectively* 1 *and k_1). In contrast, the equilibrium* 0 *is linearly unstable with respect to two families of periodic perturbations, with wavelengths close to k_1 and k_2.*

We give hereafter the lemma used in the proof of Theorem 3.2.1, which ensures the symmetry of the spectrum of the operator $\mathcal{L}_{\varepsilon,\lambda}$ with respect to the imaginary axis.

Lemma 3.2.3. *Suppose that the system (3.1) is reversible, i.e., there exists a linear symmetry $S \in \mathcal{L}(X) \cap \mathcal{L}(Z)$, $S \neq \mathrm{id}$, $S^2 = \mathrm{id}$, such that*

$$SDU = -DSU, \ SLU = -LSU, \ SR(U) = -R(SU), \ \forall\ U \in X.$$

Then the spectrum of the operator $\mathcal{L}_{\varepsilon,\lambda}$ defined by (3.8) is symmetric with respect to the imaginary axis.

Proof. Recall that the injection $\widetilde{X} \subset \widetilde{Z}$ is compact, so that the spectrum of $\mathcal{L}_{\varepsilon,\lambda}$ consists in isolated eigenvalues with finite multiplicities. We prove that if $\nu \in \mathbb{C}$ is an eigenvalue of $\mathcal{L}_{\varepsilon,\lambda}$, then $-\overline{\nu}$ is also an eigenvalue of $\mathcal{L}_{\varepsilon,\lambda}$.

Let $\nu \in \mathbb{C}$ be an eigenvalue of $\mathcal{L}_{\varepsilon,\lambda}$, with associate eigenvector V, hence satisfying

$$\mathcal{L}_{\varepsilon,\lambda} V = \nu V.$$

The function W defined through $W(x) = SV(-x)$ satisfies

$$\mathcal{L}_{\varepsilon,\lambda} W = -k_{\varepsilon,j} SV_x - LSV - \varepsilon L_{\varepsilon,j} SV - \lambda DSV.$$

Using the fact that S anticommtes with R and D, we have

$$\begin{aligned}\mathcal{L}_{\varepsilon,\lambda}W &= S(k_{\varepsilon,j}V_x + LV + \varepsilon L_{\varepsilon,j}V + \lambda DV) \\ &= S(-\mathcal{L}_{\varepsilon,\lambda}V) \\ &= S(-\nu W) \\ &= -\nu W.\end{aligned}$$

Consequently, $-\nu$ is an eigenvalue of $\mathcal{L}_{\varepsilon,\lambda}$, which implies that the spectrum of $\mathcal{L}_{\varepsilon,\lambda}$ is symmetric with respect to the origin. Since the operator is real, its spectrum is also symmetric with respect to the real axis, so that $-\overline{\nu}$ is an eigenvalue of $\mathcal{L}_{\varepsilon,\lambda}$, and we conclude that the spectrum of $\mathcal{L}_{\varepsilon,\lambda}$ is symmetric with respect to the imaginary axis. $\square$

3.3 Application to the Lugiato–Lefever equation

In this section, we consider the Lugiato–Lefever equation,

$$\frac{\partial\psi}{\partial t} = -i\beta\frac{\partial^2\psi}{\partial x^2} - (1+i\alpha)\psi + i\psi\,|\psi|^2 + F, \tag{3.16}$$

for which we have shown, in Section 2.3.2 of Chapter 2, the existence of periodic solutions, induced by the instability of some constant solutions. We show here that the result in Theorem 3.2.1 can be applied to the periodic waves found in Theorem 2.3.3.

We start with a constant solution $\psi^* = \psi_r^* + i\psi_i^*$ of (3.16), with corresponding parameters α^* and F^{*2}, and square modulus ρ^*. Upon setting

$$\psi = \psi^* + \widetilde{\psi}, \qquad \widetilde{\psi} = \widetilde{\psi_r} + i\widetilde{\psi}_i, \qquad \frac{\mathrm{d}\widetilde{\psi}}{\mathrm{d}x} = \widetilde{\varphi_r} + i\widetilde{\varphi}_i,$$

we rewrite the equation (3.16) in the form

$$U_x = D_\pm U_t + L_\pm U + R_\pm(U), \tag{3.17}$$

in which U, $L_\pm$ and $R_\pm$ are the same as in (1.9), and $D_\pm$ is the same as in (2.11).

The next theorem shows that the small periodic waves found in Theorem 2.3.3 are unstable in the sense of Definition 3.1.1.

Theorem 3.3.1. *Consider the periodic waves $U_{\varepsilon,j}$ in Theorem 2.3.3. There exist $\varepsilon_* > 0$, $\lambda_* > 0$, such that for any sufficiently small positive real numbers $\varepsilon \in (0,\varepsilon_*)$ and $\lambda \in (0,\lambda_*)$, there exists $K = K(\varepsilon,\lambda) \in [0,1)$ such that the periodic wave $U_{\varepsilon,j}$, $j = 1,2$, is linearly unstable with respect to periodic perturbations of wavelength K.*

Proof. We check that the assumptions in Hypothesis 3.1.1 hold for the system (3.17).

First, Hypothesis (i) is easily checked by setting $Z = X = \mathbb{R}^4$. Then, for the values of the parameters α^* and F^* given in Theorem 2.3.3, the matrices $L_\pm$ have two pairs of simple purely imaginary eigenvalues $\pm ik_1$, $\pm ik_2$. Up to restricting to a dense subset of the areas of the (α, F^2)-plane in which these eigenvalues are nonresonant, Hypothesis (ii) holds. The resolvent estimate (3.2) can be checked as in the proof of Theorem 2.3.3. Finally, the system (3.3) is reversible, since the linear symmetry $S = \mathrm{diag}(1, -1, 1, -1)$ anticommutes with the vector field in (3.3), so that Assumption (iv) of Hypothesis 3.1.1 is true. The result follows from Theorem 3.2.1.

$\square$

3.4 Discussion : the water-wave problem

In this discussion, we consider the two-dimensional Euler's equations, modeling the propagation of a two-dimensional perfect fluid. These equations can be written in the form (3.1), and it turns out that the stationary system (3.3) has periodic solutions with small amplitude. However, Theorem 3.2.1 cannot be used to study the linear instability of these periodic solutions.

Euler's equations. Consider a two-dimensional flow of a perfect fluid of unit density subject to gravity and surface tension, with mean depth h. In Cartesian coordinates $(x, \widetilde{y})$, the fluid occupies the domain

$$D_\eta = \{(x, \widetilde{y}),\ x \in \mathbb{R}, \widetilde{y} \in (0, \eta(x,t))\},$$

where $\eta > 0$ is a function depending upon the spatial coordinate x and the time t. The flow is supposed to be irrotational and is then described by an Eulerian velocity potential $\phi(x, y, t)$. The mathematical problem consists in solving Laplace's equation

$$\phi_{xx} + \phi_{\widetilde{y}\widetilde{y}} = 0 \text{ in } D_\eta, \tag{3.18}$$

with boundary conditions

$$\phi_{\widetilde{y}} = 0 \text{ on } \widetilde{y} = 0, \tag{3.19}$$

$$\eta_t = \phi_{\widetilde{y}} - \eta_x \phi_x \text{ on } \widetilde{y} = \eta, \tag{3.20}$$

$$\phi_t = -\frac{1}{2}\left(\phi_x^2 + \phi_{\widetilde{y}}^2\right) - g\eta + \sigma\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x + B \text{ on } \widetilde{y} = \eta, \tag{3.21}$$

where $\sigma \geqslant 0$ is the coefficient of surface tension and B is the Bernoulli constant (see for example [21, 24]).

Since we are interested in waves that are uniformly translating in the $x-$direction with speed c, we look for solutions of the form $\eta(x,t) = \eta'(x - ct, t)$, $\phi(x,y,t) = \phi'(x - ct, y, t)$. We introduce the dimensionless variables

$$(x', \widetilde{y}', t') = \frac{1}{h}(x, \widetilde{y}, t), \quad \eta'(x', t') = \frac{1}{h}\eta(x,t), \quad \phi'(x', \widetilde{y}', t') = \frac{1}{ch}\phi(x, \widetilde{y}, t).$$

Dropping the primes and writing x instead of $x - ct$, we obtain the equations

$$\phi_{xx} + \phi_{\widetilde{y}\widetilde{y}} = 0 \text{ in } D_\eta, \tag{3.22}$$

with boundary conditions

$$\phi_{\widetilde{y}} = 0 \text{ on } \widetilde{y} = 0, \tag{3.23}$$

$$\phi_{\widetilde{y}} = \eta_t - \eta_x + \eta_x \phi_x \text{ on } \widetilde{y} = \eta, \tag{3.24}$$

$$\phi_t - \phi_x + \frac{1}{2}\left(\phi_x^2 + \phi_{\widetilde{y}}^2\right) + \alpha(\eta - 1) - \beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x = 0 \text{ on } \widetilde{y} = \eta, \tag{3.25}$$

where $\alpha = gh/c^2$, $\beta = \sigma/hc$ and B has been set to gh.

Dynamical system. In order to write equations (3.22)–(3.25) in the form (3.1), we first transform D_η into the fixed domain $\mathbb{R} \times (0,1)$, using the change of variable $\widetilde{y} = \eta(x,t)\ y$. Consider the new variable Φ defined by

$$\Phi(x,y,t) = \phi(x,\widetilde{y},t),$$

and set

$$\eta = \rho + 1,$$

$$\omega = -\int_0^1 \left((\rho+1)(\Phi_x - 1) - y\rho_x\Phi_y + 1\right)\frac{y\Phi_y}{\rho+1}\mathrm{d}y + \beta\frac{\rho_x}{\sqrt{1+\rho_x^2}},$$

$$\Psi = (\rho+1)(\Phi_x - 1) - y\rho_x\Phi_y + 1.$$

Using the Hamiltonian formulation of [21, 24], we write the equations (3.22)–(3.25) as a dynamical system

$$U_x = DU_t + G(U), \tag{3.26}$$

where

$$U = \begin{pmatrix} \rho \\ \omega \\ \Phi \\ \Psi \end{pmatrix}, \quad DU = \begin{pmatrix} 0 \\ \Phi|_{y=1} \\ 0 \\ 0 \end{pmatrix},$$

$$G(U) = \begin{pmatrix} \frac{W}{\sqrt{\beta^2 - W^2}} \\ \int_0^1 \frac{(\Psi-1)^2 - \Phi_y^2}{2(\rho+1)}\mathrm{d}y + \alpha\rho - \frac{1}{2} + \frac{W}{(\rho+1)^2\sqrt{\beta^2-W^2}}\int_0^1 (\Psi-1)y\Psi_y\mathrm{d}y \\ \frac{\Psi-1}{\rho+1} + 1 + \frac{Wy\Phi_y}{(\rho+1)\sqrt{\beta^2-W^2}} \\ -\frac{\Phi_{yy}}{\rho+1} + \frac{W}{(\rho+1)\sqrt{\beta^2-W^2}}\left(y(\Psi-1)\right)_y \end{pmatrix},$$

$$W = \omega + \int_0^1 \frac{(\Psi-1)y\Phi_y}{\rho+1}\mathrm{d}y = \frac{\beta\rho_x}{\sqrt{1+\rho_x^2}}.$$

with boundary conditions

$$\Phi_y = b(U)_t + g(U) \text{ on } y = 0, 1, \tag{3.27}$$

where

$$b(U) = \frac{1}{2}y(\rho+1)^2, \quad g(U) = \frac{Wy(\Psi-1)}{\sqrt{\beta^2-W^2}}.$$

Remark that $G(0) = 0$, and that G is smooth. We then write (3.26) in the form

$$U_x = DU_t + LU + F(U), \tag{3.28}$$

where the operator L is the linearization of G at 0 and is given by

$$LU = \begin{pmatrix} \frac{1}{\beta}\left(\omega - \int_0^1 y\Phi_y \mathrm{d}y\right) \\ (\alpha-1)\rho \\ \rho + \Psi \\ -\Phi_{yy} - \frac{1}{\beta}\left(\omega - \int_0^1 y\Phi_y \mathrm{d}y\right) \end{pmatrix},$$

with boundary conditions

$$\Phi_y = 0 \text{ on } y = 0,$$

$$\Phi_y + \frac{1}{\beta}\left(\omega - \int_0^1 y\Phi_y \mathrm{d}y\right) = 0 \text{ on } y = 1,$$

and $F(U) = G(U) - LU$.

Existence of periodic solutions. Lyapunov center theorem can be used to show that the reversible stationary system

$$U_x = LU + F(U), \tag{3.29}$$

with nonlinear boundary conditions

$$\Phi_y = g(U) \text{ on } y = 0, 1, \tag{3.30}$$

possesses two families of small periodic waves. The first step, using the method of [21], is to find a change of variable, which transforms the problem (3.29)-(3.30) into an equivalent problem in a linear space.

Next, according to the arguments given in [7, 9], it turns out that for suitable values of the parameters α and β, the operator L has two pairs of simple purely imaginary eigenvalues $\pm ik_1$, $\pm ik_2$, which can be supposed to be nonresonant. The resolvent estimate in assumption (iii) of Hypothesis 3.1.1 also follows from [7, 9]. Consequently, the system (3.29) possesses two families of periodic solutions with small amplitude and wavelength close to k_1 and k_2.

Linear instability of small periodic waves. However, Theorem 3.2.1 cannot be used to state the linear instability of these small solutions with respect to periodic perturbations, since the system (3.28) is not reversible. Indeed, notice that the linear symmetry $S = \mathrm{diag}(1, -1, -1, 1)$ anticommutes with L and F in (3.28), but commutes with D.

As already discussed in the introduction, the stability of these solutions is an open problem, which might be investigated with the help of the Hamiltonian structure of the system (3.28).

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