Exclusion process with long jumps in contact with reservoirs

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Processus d'exclusion avec des sauts longs en contact avec des réservoirs

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Résumé: Cette thèse est consacrée à dériver la limite hydrodynamique et hydrostatique du processus d'exclusion avec des sauts longs dans la boîte $\Lambda_N = \{1, \cdots, N-1\}$, pour $N \geq 2$, en contact avec une infinité de réservoirs de densité α à gauche et β à droite de la boîte Λ_N. Le taux de saut est décrit par une probabilité de transition p qui est symétrique et a une queue lourde, proportionnelle à $|\cdot|^{-(1+\gamma)}$ pour $\gamma > 1$. Les réservoirs ajoutent ou enlèvent des particules avec un taux proportionnel à $\kappa N^{-\theta}$, où $\kappa > 0$ et $\theta \in \mathbb{R}$. Nous considérons les deux cas suivants:

i) Le cas $\gamma > 2$. La probabilité de transition p a une variance finie. Si $\theta < 0$ (resp. $\theta > 0$) les réservoirs ajoutent ou enlèvent rapidement (resp. lentement). D'après la valeur de θ, nous prouvons que l'évolution temporelle de la densité spatiale des particules est décrite par certaines équations aux dérivées partielles avec différentes conditions aux limites.

ii) Le cas $\gamma \in (1, 2)$. La probabilité de transition p a une variance infinie. Si $\theta = 0$ nous obtenons une collection d'équations de réaction-diffusion fractionnaires régionales indexées par le paramètre κ et les conditions aux limites de Dirichlet. Nous analysons également la convergence de l'unique solution faible de ces équations lorsque nous envoyons le paramètre κ à zéro et à l'infini. Lorsque nous considérons $\theta \neq 0$, nous conjecturons que le profil limite lorsque $\kappa \to 0$ est celui que nous devrions obtenir en prenant des réservoirs peu lents (petites valeurs positives de θ) et le profil limite quand $\kappa \to \infty$ est celui que nous devrions obtenir en prenant des réservoirs très rapides ($\theta < 0$). Si $\theta < 0$ nous prouvons que l'évolution temporelle de la densité spatiale des particules est décrite par une équation de réaction avec conditions aux limites de Dirichlet, que coïncide avec la limite $\kappa \to \infty$ précédente.

Mots-clés. La limite hydrodynamique, l'équation réaction-diffusion, conditions aux limites, processus d'exclusion avec des longs sauts.

Exclusion process with long jumps in contact with reservoirs

Abstract. This thesis is devoted to the derivation of the hydrodynamic and the hydrostatic limit of the exclusion process with long jumps in the box $\Lambda_N = \{1, \ldots, N-1\}$, for $N \geq 2$, in contact with infinitely many reservoirs with density α on the left and β at the right of the box Λ_N. The jump rate is described by a transition probability p which is symmetric and has a long tail, proportional to $|\cdot|^{-(1+\gamma)}$ for $\gamma > 1$. The reservoirs add or remove particles with rate proportional to $\kappa N^{-\theta}$, where $\kappa > 0$ and $\theta \in \mathbb{R}$. We consider the following two cases:

i) The case $\gamma > 2$. The transition probability rate p has finite variance. If $\theta < 0$ (resp. $\theta > 0$) the reservoirs fastly (resp. slowly) add or remove particles in the bulk. According to the value of θ we prove that the time evolution of the spatial density of particles is described by some partial differential equations with various boundary conditions.

ii) The case $\gamma \in (1, 2)$. The probability transition rate p has infinite variance. If $\kappa = 0$ we obtain a collection of regional fractional reaction-diffusion equations indexed by the parameter κ and with Dirichlet boundary conditions. We also analyze the convergence of the unique weak solution of these equations when we send the parameter κ to zero and to infinity. When considering $\theta \neq 0$, we conjecture that the limiting profile when $\kappa \to 0$ is the one that we should obtain when taking not very slow reservoirs (small positive values of θ) and the limiting profile when $\kappa \to \infty$ is the one that we should obtain when taking very fast reservoirs ($\theta < 0$). If $\theta < 0$ we prove that the time evolution of the spatial density of particles is described by a reaction equation with Dirichlet boundary conditions, which coincides with the previous limit as $\kappa \to \infty$.

Key words. Hydrodynamic limit, Reaction-diffusion equation, Boundary conditions, Exclusion process with long jumps.

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Contents

Frequently used notation

Λ_N	The finite set of points $\{1, \cdots, N-1\}$ for $N \geq 2$
Ω_N	The state space of the process: $\{0,1\}^{\Lambda_N}$
x, y, z	Microscopic variables
u, v	Macroscopic variables
t	Macroscopic time
$\Theta(N)$	Time scale
$\Theta(N)t$	Microscopic time
p	Transition probability function
π^N	Empirical measure
$\eta(t)$	The Markov process : exclusion process
$\sigma^{x,y}$	Exchange of occupation variables at x, y
σ^x	Creation/annihilation of a particle at site x
α, β, γ, κ, θ	Parameters
δ_a	Dirac mass at a
L_N	Generator of the process
Δ	Laplacian
$-(-\Delta)^{\gamma/2}$	Fractional Laplacian of exponent $\gamma/2$
$\mathbb{L}$	Regional Fractional Laplacian on $(0,1)$

Chapter 1

Introduction

Contents

1.1 From Microscopic to Macroscopic

One of the things that can be perceived with astonishment is the fact that our world looks differently depending on the scale we look at it, for example at the macroscopic and at the microscopic scales.

> For look closely, whenever rays are let in and pour the sun's light through the dark places in houses: for you will see many tiny bodies mingle in many ways all through the empty space right in the light of the rays, and as though in some everlasting strife wage war and battle, struggling troop against troop, nor ever crying a halt, harried with constant meetings and partings; so that you may guess from this what it means that the first-beginnings of things are fore ever tossing in the great void. So far as maybe, a little thing can give a picture of great things and afford traces of a concept.

Lucretius (c. 99 BC - c. 55 BC)[1]

A macroscopic system is constituted by a large number of particles, typically of order 10^{23} (the Avogadro's number). For example, a glass of water, the air in the room. We can characterize such system by a small number of macroscopic quantities called *thermodynamics variables*: density, temperature, pressure, volume or others. Thermodynamics variables do not depend on the behavior of few individual particles, but on the statistical properties of many particles.

In the microscopic world the particles are governed by Newton's equation of motions (or Schrödinger's equations if quantum effects are taken into account). Due to the large number of particles it does not seem viable to try to understand the system by studying the deterministic behavior of each particle according to Newton's (or Schrödinger's) equations. Such particles behave savagely and chaotically despite their deterministic nature. The most interesting is that this erratic microscopic behavior is reflected in a coherent behavior of the thermodynamic variables. The fundamental question is: How do these particles manage to organize themselves in order to do this latter? At this point is where *statistical mechanics* appears and takes a preponderant role. Its goal is to begin with the microscopic laws of physics that govern the behavior of the individual particles of the system and deduce the macroscopic properties of the system. So, statistical mechanics is a bridge between the microscopic and macroscopic worlds.

We say that a macroscopic system is in *equilibrium* if there is no net macroscopic flow of matter (or energy) within the system, otherwise the system is said to be out of the equilibrium (*non-equilibrium state*). Moreover, we say that a macroscopic system is in its *stationary* (or *steady*) state when its thermodynamics variables do not change with time.

In order to introduce some important concepts, let us analyze an example using the statistical mechanics approach introduced by Ludwig Boltzmann (one of the fathers of statistical mechanics) in the nineteenth century. Consider a gas confined to a finite volume V. We are interested in the study of the temporal evolution of the system. The first step is to examine the equilibrium states of the system and characterize them by a small number of thermodynamics variables. Suppose, for simplicity, that the unique thermodynamic variable of interest is the density ρ_0. Now, suppose that the system is starting out of equilibrium. Denote by V_u a neighborhood of $u \in V$, such that it is small with respect the whole volume, but large enough to have a huge number of particles. The latter assumption allows to believe that in each neighborhood V_u the system is very close to an equilibrium state characterized by a density $\rho_0(u)$ (in this case the density may depend on u): this is called *local equilibrium* property. Then, we can analyze the temporal evolution of the gas in the volume, assuming that the local equilibrium property is propagated in a smooth way in time: after an elapsed time t we look at V_u and note that the system is in a new local equilibrium characterized by a density $\rho_t(u)$, which does not only depend on the space variable u but on t. Then, the equation of the density is described by a partial differential equation, called *hydrodynamic equation*. The approach that allows to obtain (from the microscopic dynamics) this partial differential equation is called *hydrodynamic limit*. Now, the non-equilibrium state reaches a stationary state after an elapsed time large enough ($t \to \infty$) characterized by a stationary density profile. The derivation of the macroscopic stationary profiles (from the microscopic dynamics) is done through of the *hydrostatic limit* and

[1] Lucretius. De rerum natura

the stationary partial equation that governs this stationary profile is called *hydrostatic equation* [15, 13, 16, 40, 57].

Two assumptions can be considered in order to derive rigorously the propagation of the local equilibrium. The first simplification consists in considering a system governed by deterministic equations of motion but with a low density of particles, in such way that we just have a finite number of collisions in a finite lapse of time. In the second simplification, we do not assume a low density of particles but the particles are no longer governed by deterministic equations of motion but by stochastic rules (see [57] for details of these two assumptions). We are interested in the latter simplification, which is at the origin of the introduction of (*stochastic*) *interacting particle systems*. This term begins to emerge in the late 1960's, the pioneers in this field being F. Spitzer [56] in the United states and R. L Dobrushin in the Soviet Union [24, 25]. More specifically, an interacting particle system consists of many particles which evolve like a Markov process. Some examples of interacting particles systems are the stochastic Ising model, the voter model, the contact process and the exclusion process [47]. In this work we are interested in the exclusion process.

1.2 Symmetric exclusion process

The exclusion process is a continuous time interacting particle system introduced in the mathematical literature during the seventies by Frank Spitzer [56]. Despite the simplicity of its dynamics, it captures the main features of more realistic diffusive systems driven out of equilibrium [47, 48, 57]. It is a system of identical particles which perform jumps on the lattice $\mathbb{Z}$ like random walks, each one being independent of the others and following the exclusion rule: a jump is suppressed if the target site of the jump is already occupied. This latter rule allows just single occupancy per site and introduce interaction between particles. More precisely, the dynamics is the following: particles are distributed on $\mathbb{Z}$, each site being occupied by at most one particle. Fix a transition probability function $p : \mathbb{Z} \to [0,1]$. Associated to each pair of sites $\{x, y\} \subset \mathbb{Z}$ there is a Poisson clock of parameter 1, independent from the others. When it rings the state of the occupation of sites x and y are interchanged with probability $p(x-y)$. If both sites are occupied or both sites are vacant, nothing happens. If one of the sites is occupied and the other is vacant, the interchange is seen as a jump of the particle from the occupied site to the empty site (see Figure 1.1). We say that the process is finite (resp. long) range if $p(x) = 0$ for $|x|$ large enough (resp. otherwise). In the particular case where the exclusion process has range 1, i.e. $p(x) = 0$ for $|x| > 1$, it is called the simple exclusion process. We say that it is a *symmetric (resp. asymmetric) exclusion process* in the case where p is symmetric, i.e. $p(-x) = p(x)$ (resp. $\exists x, p(x) \neq p(-x)$).

1.2.1 Boundary driven symmetric exclusion process

Now, in order to take into account the interaction of the system with its environment, the symmetric exclusion process can also be considered in contact with reservoirs. Such reservoirs work macroscopically at different particle densities and microscopically create or annihilate particles in the system. For instance, in Figure 1.2, we consider the one dimensional symmetric

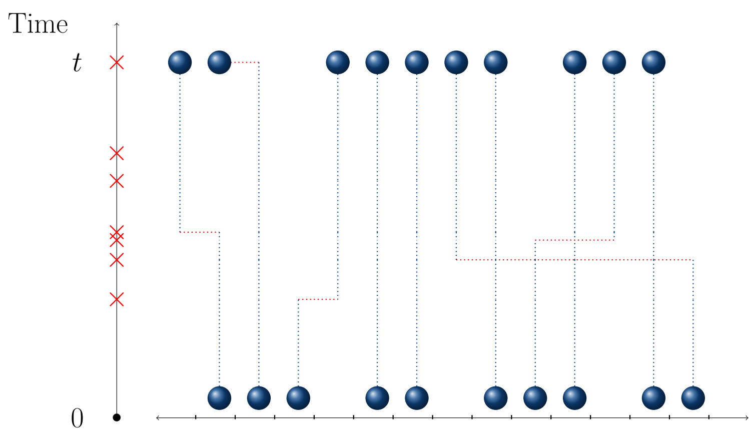

Figure 1.1: Time evolution on $\mathbb{Z}$.

simple exclusion process in an open lattice of length $N-1$, called the *bulk*, in contact with two reservoirs. Fix $\alpha, \beta \in (0,1)$. At the left boundary, particles are created with rate α and annihilated with rate $1-\alpha$. At the right boundary this is done with rates β and $1-\beta$. Consider first the system in its steady state. Observe that in the case where $\alpha = \beta$ the system is in equilibrium, i.e there is an absence of flux of particles (mathematically, since p is symmetric the invariant measure is reversible). On the other hand, if $\alpha < \beta$ there is a flow of matter from the right boundary to the left one (or in the other way around if $\alpha > \beta$). The presence of this exchange of matter between the system and the boundaries creates a non-equilibrium stationary state with a steady flux of particles through the system. The hydrodynamic limit of this process is studied in [30], and the hydrodynamic equations correspond to the heat equation with Dirichlet boundary conditions imposed by reservoir densities. The hydrostatic profile is given by a linear profile connecting these densities α, β at $0, 1$ respectively: $\bar{\rho}(u) = (\beta-\alpha)u+\alpha$. One can go further in the study of the system. For instance, after having obtained the hydrodynamics (law of large numbers), it is natural to ask about the Gaussian fluctuations around the solution of the hydrodynamic or hydrostatic equation [10, 12, 34, 35, 44]. We can also study large deviations, which describe the probability of large fluctuations around the solution of the hydrodynamic or hydrostatic equation [5, 6, 26, 31, 43, 51, 52].

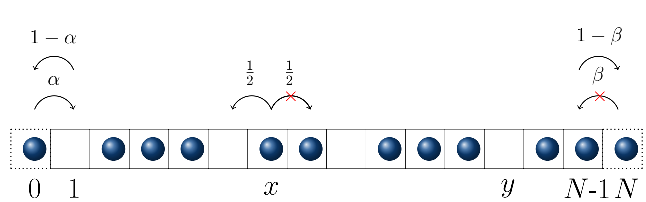

Figure 1.2: Symmetric simple exclusion process in contact with reservoirs.

A variant of the process introduced above is when the process is in the presence of a mechanism (placed in the bonds or in the reservoirs) that regulates (decreasing/increasing) the rate of the passage of particles through them. Recently a series of work have been devoted to the study of the simple exclusion process whose dynamics is perturbed by the presence of a slow bond [33], a slow site [35], slow boundary effects [1] and current boundary effects [14, 17, 18, 49]. The behavior of the system is then strongly affected and new conditions may be derived at the macroscopic level. In Subsection 1.2.2 below, we see how the boundary conditions of the hydrodynamic equation may be affected due to the presence of slow/fast boundaries effects in the context of the simple exclusion process.

1.2.2 A warming-up example

In order to illustrate the type of results that we are going to prove, we derive heuristically the hydrodynamic and hydrostatic equations of the symmetric simple exclusion process with slow/fast boundary (for details of slow boundaries see [1]). Fix $N \geq 2$, which represents the inverse of the distance between neighboring sites and that will increase to infinity. Denote by $\eta_x \in \{0,1\}$ the particle occupancy at the site $\frac{x}{N} \in (0,1)$. Therefore, the configuration $\{\eta_1, \cdots, \eta_{N-1}\}$ is an element of the space $\Omega_N := \{0,1\}^{\Lambda_N}$, where $\frac{1}{N}\Lambda_N$ is a discretization with mesh $\frac{1}{N}$ of the continuous space $(0,1)$ through the map $x \in \Lambda_N \to \frac{x}{N} \in (0,1)$ where $\Lambda_N = \{1, \cdots, N-1\}$. From the latter we see how the macroscopic space $(0,1)$ and the microscopic space Λ_N are naturally connected. The simple exclusion process with slow/fast boundaries can be defined as follows: on each site of Λ_N there exists at most one particle, which can jump to one of its nearest neighbors according to the exclusion rule. Fix $\kappa > 0$, $\theta \in \mathbb{R}$ and $0 < \alpha \leq \beta < 1$. A particle at the site 1 (resp. $N-1$) can get out from the system Λ_N with rate $\alpha\kappa N^{-\theta}$ (resp. $\beta\kappa N^{-\theta}$), whereas if the site 1 (resp. $N-1$) is empty a particle from the reservoir can get into the site 1 (resp. $N-1$) with rate $(1-\alpha)\kappa N^{-\theta}$ (resp. $(1-\beta)\kappa N^{-\theta}$). This dynamics corresponds to a Markov process $\{\eta(t)\}_{t\geq 0}$ defined on Ω_N whose infinitesimal generator is given by

$$L_N := L_N^B + \kappa N^{-\theta} L_N^b.$$

Here the generator L_N^B corresponds to the bulk dynamics and its action on functions $f : \Omega_N \to \mathbb{R}$ is

$$(L_N^B f)(\eta) = \sum_{x=1}^{N-2} [f(\sigma^{x,x+1}\eta) - f(\eta)],$$

where for $x, y \in \Lambda_N$, $\sigma^{x,y}\eta$ is the configuration in Ω_N which is obtained from η by exchanging the values of η_x and η_y:

$$(\sigma^{x,y}\eta)_z = \begin{cases} \eta_z, \ z \neq x, y, \\ \eta_y, \ z = x, \\ \eta_x, \ z = y. \end{cases} \tag{1.2.1}$$

The generator L_N^b, corresponding to non-conservative boundary dynamics, acts on functions $f : \Omega_N \to \mathbb{R}$ as

$$\begin{aligned}(L_N^b f)(\eta) =& [\alpha(1-\eta_1)+\eta_1(1-\alpha)][f(\sigma^1\eta)-f(\eta)]\\ &+[\beta(1-\eta_{N-1})+\eta_{N-1}(1-\beta)][f(\sigma^{N-1}\eta)-f(\eta)],\end{aligned}$$

where for any $x \in \Lambda_N$, the function σ^x corresponds to the creation/annihilation of a particle at site x:

$$(\sigma^x\eta)_z = \begin{cases}\eta_z, \; z \neq x,\\ 1-\eta_x, \; z = x.\end{cases} \tag{1.2.2}$$

We consider the Markov process speeded up in the time scale N^2, so that $\{\eta^N(t)\}_{t\geq 0} := \{\eta(tN^2)\}_{t\geq 0}$ has infinitesimal generator $N^2 L_N$. This time re-normalization is in order to observe non-trivial hydrodynamic phenomena.

First we analyze the hydrostatic behavior. Since the Markov process $\{\eta^N(t)\}_{t\geq 0}$ is irreducible, it is well know that there exists a unique measure $\bar{\mu}_N$ which is invariant under the evolution of $\eta^N(t)$. The expectation under $\bar{\mu}_N$ is denoted by $\mathbb{E}_{\bar{\mu}_N}$. In the case where $\alpha = \beta$ the measure $\bar{\mu}_N$ is reversible and can be computed easily (Bernoulli product measure with parameter α). In such case we say that the system is in equilibrium, since there is no flow of matter in the system. If $\alpha \neq \beta$ the invariant measure is irreversible ($\bar{\mu}_N$ is expressed in [19] in a semi-explicit matrix product form), i.e. the system is no longer in equilibrium. Recall that even if the system is not in equilibrium, due to the huge number of particles and the "strong" interaction between them, we expect that the system is "close" (in some sense) to a local equilibrium state. Thus, we can use the local equilibrium property in order to associate a macroscopic profile to this invariant measure, i.e. there should exist a profile $\bar{\rho} : [0,1] \to \mathbb{R}$ such that for all $u \in [0,1]$ we have that

$$\mathbb{E}_{\bar{\mu}_N}[\eta_{[uN]}] =: \bar{\rho}_N([uN]) \sim \bar{\rho}\left(\tfrac{[uN]}{N}\right). \tag{1.2.3}$$

Above $[\cdot]$ stands for the integer part. Note that applying the generator to the function $\eta \to \eta_x$ for $x \in \Lambda_N$, we have that

$$L_N\eta_x = \begin{cases}\eta_{x-1}-2\eta_x+\eta_{x+1}, & \text{if } x \in \{2,\cdots,N-2\},\\ \eta_2-\eta_1+\kappa N^{-\theta}(\alpha-\eta_1), & \text{if } x = 1,\\ \eta_{N-2}-\eta_{N-1}+\kappa N^{-\theta}(\beta-\eta_{N-1}), & \text{if } x = N-1.\end{cases} \tag{1.2.4}$$

Since the measure is invariant we know that $\mathbb{E}_{\bar{\mu}_N}[L_N\eta_x] = 0$, for all $x \in \Lambda_N$ (see [47] for details). Then, we have a linear system of equations whose solution is given by $\bar{\rho}_N(x) = a_N x + b_N$, where

$$a_N = \frac{\kappa(\beta-\alpha)}{2N^\theta-2\kappa+\kappa N} \text{ and } b_N = a_N\left(\frac{N^\theta}{\kappa}-1\right)+\alpha.$$

This shows that (1.2.3) is valid with

$$\bar{\rho}(u)=\begin{cases}(\beta-\alpha)u+\alpha, & \text{if } \theta<1,\\ \dfrac{\kappa(\beta-\alpha)}{2+\kappa}u+\alpha+\dfrac{(\beta-\alpha)}{2+\kappa}, & \text{if } \theta=1,\\ \dfrac{\beta+\alpha}{2}, & \text{if } \theta>1.\end{cases} \tag{1.2.5}$$

Now, we focus on the hydrodynamic behavior. For $t\geq 0$ we denote by $\mu_{N,t}$ the law of the process η^N at time t. The expectation under $\mu_{N,t}$ is denoted by $\mathbb{E}_{\mu_{N,t}}$. We assume that at time $t=0$ a local equilibrium holds and we expect that the local equilibrium property is conserved a time t, i.e. there exists a profile $\rho:[0,\infty)\times[0,1]\to\mathbb{R}$ such that for all $(t,u)\in[0,\infty)\times[0,1]$ we have that

$$\mathbb{E}_{\mu_{N,t}}[\eta_{[uN]}(tN^2)]\sim\rho_t\left(\tfrac{[uN]}{N}\right).$$

In order to derive the form of ρ, we proceed as follows. Fix a smooth function $G:[0,\infty)\times[0,1]\to\mathbb{R}$. We know by Dynkin's formula (see A.5 in [40]) that

$$M_t^N(G):=N^{-1}\sum_{x\in\Lambda_N}G_t(\tfrac{x}{N})\eta_x^N(t)-N^{-1}\sum_{x\in\Lambda_N}G_0(\tfrac{x}{N})\eta_x(0)-N^{-1}\int_0^t\sum_{x\in\Lambda_N}N^2G_s(\tfrac{x}{N})L_N\eta_x^N(s)ds, \tag{1.2.6}$$

is a martingale with respect to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$, where for each $t\geq 0$, $\mathscr{F}_t:=\sigma(\{\eta^N(s)\}_{s\leq t})$. After summation by parts and using (1.2.4) we have that

$$\begin{aligned}M_t^N(G)=&N^{-1}\sum_{x\in\Lambda_N}G_t(\tfrac{x}{N})\eta_x^N(t)-N^{-1}\sum_{x\in\Lambda_N}G_0(\tfrac{x}{N})\eta_x(0)\\ &-N^{-1}\int_0^t\sum_{x=2}^{N-2}\eta_x^N(s)\Delta_NG_s(\tfrac{x}{N})-\eta_{N-1}^N(s)\nabla_N^+G_s(\tfrac{N-2}{N})+\eta_1^N(s)\nabla_N^+G_s(\tfrac{1}{N})ds\\ &+\kappa N^{1-\theta}\int_0^t(\alpha-\eta_1^N(s))G_s(\tfrac{1}{N})+(\beta-\eta_{N-1}^N(s))G_s(\tfrac{N-1}{N})ds,\end{aligned} \tag{1.2.7}$$

where, for $u\in[0,1]$, $\Delta_NG(u)=N^2[G(u-\frac{1}{N})-2G(u)+G(u+\frac{1}{N})]$ and $\nabla_N^+G(u)=N[G(u+\frac{1}{N})-G(u)]$ stand, respectively, for the discrete Laplacian and the discrete derivative of a function G. Take the expectation with respect to $\mu_{N,t}$ in last expression. Since $M_t^N(G)$ is a martingale vanishing a time 0 and the expectation of martingales is constant, $\mathbb{E}_{\mu_{N,t}}[M_t^N(G)]=0$. By invoking the local equilibrium property we have that the macroscopic profile should satisfy

$$\begin{aligned}&N^{-1}\sum_{x\in\Lambda_N}G_t(\tfrac{x}{N})\rho_t(\tfrac{x}{N})-N^{-1}\sum_{x\in\Lambda_N}G_0(\tfrac{x}{N})\rho_0(\tfrac{x}{N})\\ &-N^{-1}\int_0^t\sum_{x=2}^{N-2}\rho_s(\tfrac{x}{N})\Delta_NG_s(\tfrac{x}{N})-\rho_s(\tfrac{N-1}{N})\nabla_N^+G_s(\tfrac{N-2}{N})+\rho_s(\tfrac{1}{N})\nabla_N^+G_s(\tfrac{1}{N})ds\\ &+\kappa N^{1-\theta}\int_0^t(\alpha-\rho_s(\tfrac{1}{N}))G_s(\tfrac{1}{N})+(\beta-\rho_s(\tfrac{N-1}{N}))G_s(\tfrac{N-1}{N})ds\approx 0.\end{aligned} \tag{1.2.8}$$

Case $\theta < 1$: In this regime, since $\theta < 1$ and we are going to take N large in (1.2.8), we see that it is necessary to assume that $\rho_s(\frac{1}{N}) \sim \alpha$ and $\rho_s(\frac{N-1}{N}) \sim \beta$ for all $s \in [0,t]$. Then, letting $N \to \infty$ we get that

$$\int_0^1 G_t(u)\rho_t(u)du - \int_0^1 G_0(u)\rho_0(u)du - \int_0^t \int_0^1 \Delta G_s(u)\rho_s(u)duds$$
$$+ \int_0^t \beta \partial_u G_s(1) - \alpha \partial_u G_s(0) ds = 0.$$

Therefore we obtain that ρ is the weak solution of the heat equation with Dirichlet boundary condition:

$$\begin{cases} \partial_t \rho_t(u) = \Delta \rho_t(u), & (t,u) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, & t \in (0,T], \end{cases}$$

and with initial condition ρ_0.

Case $\theta = 1$: Letting $N \to \infty$ in (1.2.8) we get that

$$\int_0^1 G_t(u)\rho_t(u)du - \int_0^1 G_0(u)\rho_0(u)du - \int_0^t \int_0^1 \Delta G_s(u)\rho_s(u)duds$$
$$+ \int_0^t \rho_s(1)\partial_u G_s(1) - \rho_s(0)\partial_u G_s(0) ds + \kappa \int_0^t (\alpha - \rho_s(0))G_s(0) + (\beta - \rho_s(0))G_s(1) ds = 0.$$

Then we obtain that ρ is the weak solution of the heat equation with Robin boundary conditions:

$$\begin{cases} \partial_t \rho_t(u) = \Delta \rho_t(u), & (t,u) \in [0,T] \times (0,1), \\ \partial_u \rho_t(0) = \kappa(\rho_t(0) - \alpha), \quad \partial_u \rho_t(1) = \kappa(\beta - \rho_t(1)), & t \in (0,T], \end{cases} \tag{1.2.9}$$

and with initial condition ρ_0.

Case $\theta > 1$: Letting $N \to \infty$ in (1.2.8) we have that

$$\int_0^1 G_t(u)\rho_t(u)du - \int_0^1 G_0(u)\rho_0(u)du - \int_0^t \int_0^1 \rho_s(u)\Delta G_s(u)duds$$
$$+ \int_0^t \rho_s(1)\partial_u G_s(1) - \rho_s(0)\partial_u G_s(0) ds = 0$$

Then we obtain that ρ is the weak solution of (1.2.9) with $\kappa = 0$ (the heat equation with Neumann boundary conditions).

Note that the profile $\bar{\rho}$ given in (1.2.5) is a stationary solution of the corresponding hydrodynamic equation [2], which are different if $\theta < 1$, $\theta = 1$ or $\theta > 1$. Moreover, note that the slow boundaries at the microscopic level have an effect on the system at the macroscopic level. In fact, in the case $\theta = 0$ we see that such interaction allows to have a fixed density

[2] The stationary solution is unique if $\theta \leq 1$.

at the boundaries (α on the left and β on the right). Now, increasing the value of θ, we see that the previous behavior is maintained up to values of θ lower than 1, i.e. the interaction with the boundary is not slowed enough to give a new behavior. In the case where $\theta > 1$, the reservoirs are sufficiently slowed to not permit any exchange of mass between the system and the reservoir, i.e. the boundary conditions at the macroscopic level describe a system isolated from the environment. Now, in the case $\theta = 1$ we have a transition phase between the two cases above, i.e. the boundary conditions are a combination between Dirichlet and Neumann boundary conditions.

We have already studied the role of θ, but in the definition of the dynamics of the process, we said that the reservoirs add or remove particles with rate proportional to $\kappa N^{-\theta}$. Why is the value of κ important? It is because it can be used as a tool in order to understand the transition from one phase ($\theta < 1$) to another ($\theta > 1$) at the macroscopic level. For instance, suppose that in the example above we only know the behavior for $\theta = 1$ and that we would like to know the hydrodynamic behavior of the system for values of θ around 1. We note that in the case $\theta = 1$ the hydrodynamic equation depends on κ. Intuitively we can see that if we take $\kappa \to \infty$ in (1.2.9) we obtain the heat equation with Dirichlet boundary conditions. On the other hand, if we take $\kappa \to 0$ in (1.2.9) we get the heat equation with Neumann boundary conditions. The same conclusion also applies to the stationary profile (see (1.2.5)). What we learn from the above simple example is that the behavior of the system for values of $\theta \in (1-a, 1)$ (resp. $\theta \in (1, 1+a)$) for some $a > 0$ may be obtained taking at the macroscopic level κ very large (resp. κ very small) in the hydrodynamic equation (1.2.9) obtained for $\theta = 1$. This approach does not give us the optimal value of a and for that reason we only deduce the behavior for values of θ close to 1. If we take this approach at the formal level it would be very useful to recognize new phases, which could be difficult to obtain directly at the microscopic level (see Theorem 3.2.10).

The purpose of this work is to extend formally the scenario informally explained above to a process with long range interactions. We call this process *the exclusion process with long jumps in contact with reservoirs*.

1.3 The model

In this work we are interested in the case where the probability transition function p has a heavy tail proportional to $|\cdot|^{-(1+\gamma)}$ for $\gamma > 1$. Curiously it is only very recently that the investigation of the exclusion process with long jumps has started [3, 38, 39, 36, 55, 58].

More precisely, for $N \geq 2$ let $\Lambda_N = \{1, \dots, N-1\}$ be a finite lattice of size $N-1$ called the *bulk*. We consider the exclusion process in contact with reservoirs, which is a Markov process $\{\eta(t)\}_{t\geq 0}$ with state space $\Omega_N = \{0,1\}^{\Lambda_N}$. The configurations of the state space Ω_N are denoted by η, so that for $x \in \Lambda_N$, $\eta_x = 0$ means that the site x is vacant while $\eta_x = 1$ means that the site x is occupied. The translation invariant transition probability on $\mathbb{Z}$ is defined by

$$p(z) = \begin{cases} \dfrac{c_\gamma}{|z|^{\gamma+1}}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0 \end{cases} \tag{1.3.1}$$

where $c_\gamma^{-1} = 2\zeta_{\gamma+1}$ (ζ_s is the Riemann zeta function defined for $s > 1$). In fact the results obtained in this thesis could probably be generalized to the case where p is such that $p(z) \sim L(z)|z|^{-(1+\gamma)}$ as $z \to \pm\infty$ for some slowly varying function L. Moreover, the model can be defined in higher dimensions and we should expect similar results. However the proofs could be much more technical.

1.3.1 Infinitely extended reservoirs

We consider the process in contact with infinitely many stochastic reservoirs at all the negative integer sites $z \le 0$ and at all the integer sites $z \ge N$. We fix four parameters $\alpha, \beta \in (0,1)$, $\kappa > 0$ and $\theta \in \mathbb{R}$. Particles can get into (resp. exit) the bulk of the system from any site at the left of 0 at rate $\alpha\kappa N^{-\theta} p(z)$ (resp. $(1-\alpha)\kappa N^{-\theta} p(z)$), where z is the jump size (see Figure 1.3). The stochastic reservoir at the right acts in the same way as the left reservoir but with the intensity α replaced by β.

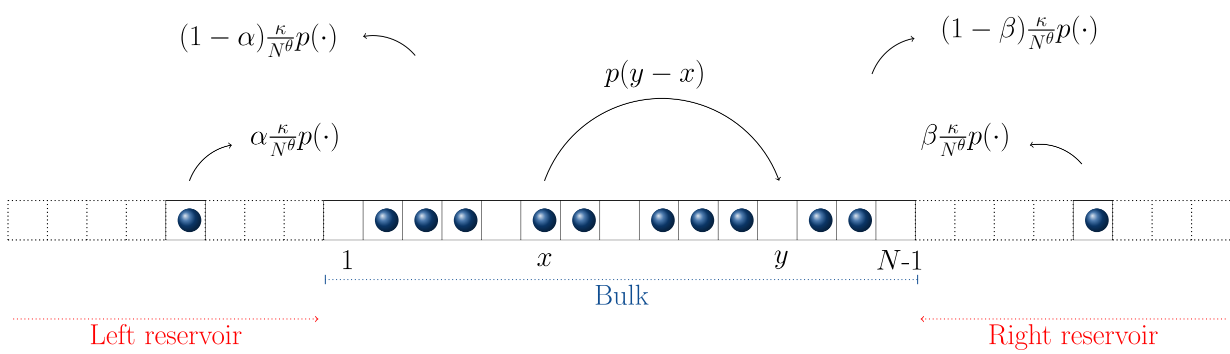

Figure 1.3: Exclusion process with long jumps and infinitely extended reservoirs.

Hence, we have the presence of two dynamics: bulk and boundary dynamics. The dynamics in the bulk is defined as follows. Each pair of sites of the bulk $\{x, y\} \subset \Lambda_N$ carries a Poisson process of intensity one. The Poisson processes associated to different bonds are independent. If for the configuration η, the clock associated to the bond $\{x, y\}$ rings, then we exchange the values η_x and η_y with rate $p(y-x)/2$. It is clear that this dynamics conserves the number of particles of the system. Now we explain the dynamics at the boundaries. Each pair of sites $\{x, y\}$ with $x \in \Lambda_N$ and $y \in \mathbb{Z} - \Lambda_N$ carries a Poisson process of intensity one, all of them being independent. Recall that the coupling with the reservoirs is regulated by a prefactor $\kappa N^{-\theta}$, $\kappa > 0$, $\theta \in \mathbb{R}$. If for the configuration η, the clock associated to the bond $\{x, y\}$ rings and $y \le 0$ then we change η_x into $1 - \eta_x$ with rate $\kappa N^\theta p(x-y)[(1-\alpha)\eta_x + \alpha(1-\eta_x)]$. At the right boundary the dynamics is similar but instead of α the density is given by β. Observe that the reservoirs add and remove particles on all the sites of the bulk Λ_N, and not only at the boundaries, but with rates which decrease as the distance from the corresponding reservoir increases. The process is characterized by its infinitesimal generator

$$L_N = L_N^0 + \kappa N^{-\theta} L_N^\ell + \kappa N^{-\theta} L_N^r. \tag{1.3.2}$$

Here the generator L_N^0 corresponds to the bulk dynamics and its action on functions $f : \Omega_N \to \mathbb{R}$ is given by

$$\begin{aligned}(L_N^0 f)(\eta) &= \sum_{x,y\in\Lambda_N} p(x-y)\eta_x(1-\eta_y)[f(\sigma^{x,y}\eta)-f(\eta)] \\ &= \frac{1}{2}\sum_{x,y\in\Lambda_N} p(x-y)[f(\sigma^{x,y}\eta)-f(\eta)],\end{aligned} \tag{1.3.3}$$

where for $x, y \in \Lambda_N$, $\sigma^{x,y}\eta$ is the configuration in Ω_N is given in (1.2.1). The generators L_N^ℓ and L_N^r corresponding to non-conservative boundary dynamics act on a function $f : \Omega_N \to \mathbb{R}$ as

$$\begin{aligned}(L_N^\ell f)(\eta) &= \sum_{\substack{x\in\Lambda_N \\ y\le 0}} p(x-y)c_x(\eta;\alpha)[f(\sigma^x\eta)-f(\eta)], \\ (L_N^r f)(\eta) &= \sum_{\substack{x\in\Lambda_N \\ y\ge N}} p(x-y)c_x(\eta;\beta)[f(\sigma^x\eta)-f(\eta)]\end{aligned} \tag{1.3.4}$$

where $\sigma^x\eta$ was introduced in (1.2.2) and $c_x(\eta;\alpha) = [\eta_x(1-\alpha)+(1-\eta_x)\alpha]$ and $c_x(\eta;\beta) = [\eta_x(1-\beta)+(1-\eta_x)\beta]$.

We would like to characterize the collective behavior of the microscopic process described above. In order to obtain this characterization we need to establish a connection between the microscopic and the macroscopic system, by using a limit procedure. Such a limit procedure give us the convergence of the spatial density of particles (called empirical measure associated to the process) to the solution of a macroscopic equation. As we said before, this is called hydrodynamic limit.

The next two chapters are devoted to analyze the repercussions at the macroscopic level by changing γ in the probability transition function and the strength of the reservoirs by changing θ and κ.

1.3.2 Other kinds of reservoirs

In this work we decided to consider in details only one kind of reservoirs. However, since a reservoir model is not universal, other natural models are of interest. We explain three possible cases where the boundary conditions are linear, but there exist very interesting and more complicated models for which the boundary conditions are non-linear [18, 49].

Case 1: The reservoir consists on the left (resp. on the right) of a single Glauber dynamics whose action of the generator on a function $f:\Omega_N\to\mathbb{R}$ is

$$(L_N^\ell f)(\eta)=\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}c_x(\eta;\alpha)p(x)[f(\sigma^x\eta)-f(\eta)],$$
$$\Big(\text{resp. } (L_N^r f)(\eta)=\frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N}c_x(\eta;\beta)p(N-x)[f(\sigma^x\eta)-f(\eta)]\Big).$$

Thus it creates a particle at the site $x\in\Lambda_N$ with rate $\frac{\kappa}{N^\theta}\alpha p(x)$ (resp. $\frac{\kappa}{N^\theta}\beta p(N-x)$) if the site x is empty and it removes a particle at the site x with rate $\frac{\kappa}{N^\theta}(1-\alpha)p(x)$ (resp. $\frac{\kappa}{N^\theta}(1-\beta)p(N-x)$) if the site x is occupied. The bulk dynamics is unmodified (see Figure 1.4).

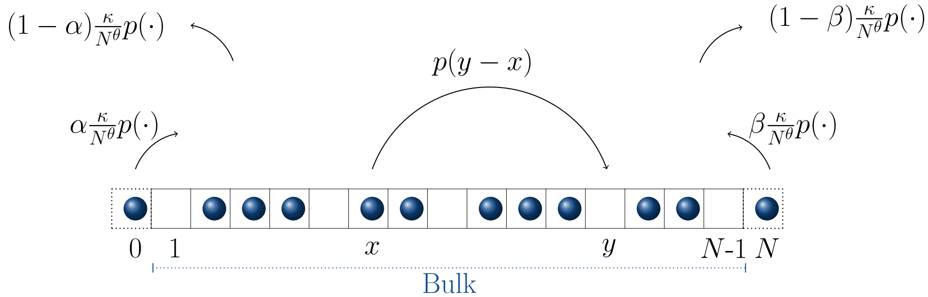

Figure 1.4: Case 1

Case 2: The reservoir consists on the left (resp. on the right) of a single Glauber dynamics whose action of the generator on a function $f:\Omega_N\to\mathbb{R}$ is

$$(L_N^\ell f)(\eta)=\frac{\kappa}{N^\theta}c_1(\eta;\alpha)[f(\sigma^1\eta)-f(\eta)],$$
$$\Big(\text{resp. } (L_N^r f)(\eta)=\frac{\kappa N}{N^\theta}c_{N-1}(\eta;\beta)[f(\sigma^{N-1}\eta)-f(\eta)]\Big).$$

Thus it creates a particle at the site 1 with rate $\frac{\kappa}{N^\theta}\alpha$ (resp. $\frac{\kappa}{N^\theta}\beta$) if the site 1 (resp. $N-1$) is empty and it removes a particle at the site 1 with rate $\frac{\kappa}{N^\theta}(1-\alpha)$ (resp. $\frac{\kappa}{N^\theta}(1-\beta)$) if the site 1 (resp. $N-1$) is occupied. The bulk dynamics is unmodified (see Figure 1.5).

Case 3: The reservoir consists on the left (resp. on the right) of an infinite number of Glauber dynamics whose action of the generator on a local function $f:\{0,1\}^{\mathbb{Z}}\to\mathbb{R}$ is

$$(L_N^\ell f)(\eta)=\frac{\kappa}{N^\theta}\sum_{x\le 0}c_x(\eta;\alpha)[f(\sigma^x\eta)-f(\eta)],$$
$$\Big(\text{resp. } (L_N^r f)(\eta)=\frac{\kappa}{N^\theta}\sum_{x\ge N}c_x(\eta;\beta)[f(\sigma^x\eta)-f(\eta)]\Big).$$

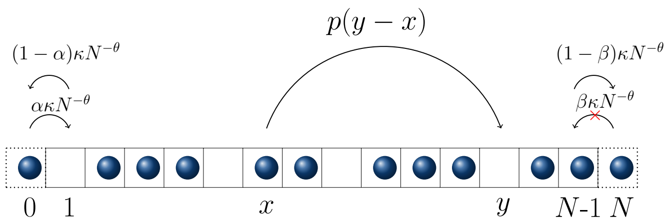

Figure 1.5: Case 2

Thus it creates a particle at the site $x \leq 0$ (resp. $x \geq N$) with rate $\frac{\kappa}{N^\theta}\alpha$ (resp. $\frac{\kappa}{N^\theta}\beta$) if the site x is empty and it removes a particle at the site $x \leq 0$ (resp. $x \geq N$) with rate $\frac{\kappa}{N^\theta}(1-\alpha)$ (resp. $\frac{\kappa}{N^\theta}(1-\beta)$) if the site x is occupied (see Figure 1.6). Moreover, in this case we assume that the long jumps are not restricted to sites $x, y \in \Lambda_N$ but may occur in all the lattice $\mathbb{Z}$, i.e. the action of the bulk dynamics generator on a local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is now described by

$$(L^0 f)(\eta) = \frac{1}{2}\sum_{x,y\in\mathbb{Z}} p(x-y)[f(\sigma^{x,y}\eta) - f(\eta)].$$

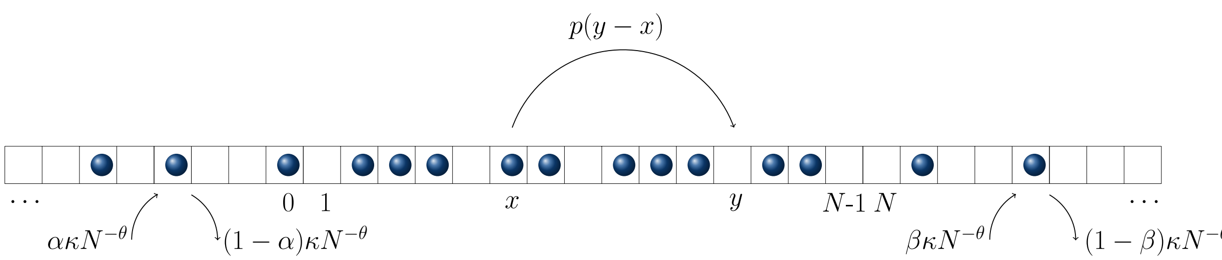

Figure 1.6: Case 3

1.4 Diffusive case

In Chapter 2 we consider the probability transition function p for $\gamma > 2$. In such case we can give a complete scenario for all $\theta \in \mathbb{R}$ of the behavior of the system. By a complete scenario we mean that we can describe the macroscopic behavior of the system for any value of $\theta \in \mathbb{R}$. Since θ ranges from $-\infty$ to ∞ the interaction of the particles with the boundaries can be slowed down or fasted up.

In this case the transition function p has mean zero and finite variance , i.e.

$$\sum_{z\in\mathbb{Z}} z p(z) = 0, \;\; \sigma^2 := \sum_{z\in\mathbb{Z}} z^2 p(z) < \infty.$$

Then, in spite our process has long jumps we will see that it is a diffusive system. However the presence of long jumps allows to obtain new phases that in the diffusive system of short range are not reached.

1.4.1 Hydrodynamic and hydrostatic limit

The problem we address is to characterize the hydrodynamic behavior of the process described above and to analyze the repercussions at the macroscopic level of slowing down or fasting up the interaction with the reservoirs, by changing the values of θ. Usually the characterization of the hydrodynamic limit is formulated in terms of a weak solution of some partial differential equation, namely, the hydrodynamic equation. Depending on the intensity of the coupling with the reservoirs we will observe a phase transition for profiles which are solutions of the hydrodynamic equation with different types of boundary conditions, depending on the range of the parameter θ.

We extend the results of the symmetric simple exclusion process with slow boundary that was studied in [1] and reviewed in Subsection 1.2.2 by considering long jumps, infinitely extended reservoirs and also fast reservoirs ($\theta < 0$). In the case $\theta \geq 0$ (slow reservoirs) we recover in our model a similar hydrodynamical behavior to the one obtained in [1], since we imposed that the probability transition rate to be symmetric and with finite variance. The presence of long jumps and the fast boundaries ($\theta < 0$) generate two new phases of transition which do not occur in the case of short range. In fact, for the short range case of [1] it is possible to extend the results of [1] to the case $\theta < 0$ (see Subsection 1.2.2). However, in such case we will not see a new behavior: in the regime $\theta < 0$ the heat equation with Dirichlet boundary conditions appears, as in the case $\theta \in [0,1)$.

More specifically, let us consider first the case $\theta < 1$ for which three phases can be identified. We will show that the empirical measure associated to the particle systems that we described above converges to the weak solution of the *reaction-diffusion equation with inhomogeneous Dirichlet boundary conditions*:

$$\begin{cases} \partial_t \rho_t(u) = \frac{\hat{\sigma}^2}{2} \Delta \rho_t(u) + \hat{\kappa} \Big\{ \frac{\alpha - \rho_t(u)}{u^\gamma} + \frac{\beta - \rho_t(u)}{(1-u)^\gamma} \Big\}, & (t,u) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in [0,T], \\ \rho_0(u) = \rho_0(u), \quad u \in (0,1), \end{cases} \tag{1.4.1}$$

with $\hat{\sigma}$ and $\hat{\kappa}$ being parameters specified below. When $\theta < 2-\gamma$ the boundaries are enough fasted to make the diffusion part disappear, i.e. the profile is solution of (1.4.1) with $\hat{\sigma} = 0$ and $\hat{\kappa} = \kappa c_\gamma \gamma^{-1}$. Physically it means that the particles are entering and leaving the system so fast that they do not have time to diffuse. For $\theta = 2-\gamma$ we have a transition phase. In such case we get that the profile satisfies (1.4.1) with $\hat{\sigma} = \sigma$ and $\hat{\kappa} = \kappa c_\gamma \gamma^{-1}$. If $\theta \in (2-\gamma, 1)$, the boundary effects are such that we get the classical heat equation with Dirichlet boundary conditions, that means that the profile is solution of (1.4.1) with $\hat{\sigma} = \sigma$ and $\hat{\kappa} = 0$.

Now we consider the case $\theta \geq 1$ for which two phases can be identified. In this case the empirical measure associated to the process converges to the weak solution of the *heat equation* *with Robin boundary conditions*:

$$\begin{cases} \partial_t \rho_t(u) = \frac{\hat{\sigma}^2}{2} \Delta \rho_t(u), \quad (t,u) \in [0,T] \times (0,1), \\ \partial_u \rho_t(0) = \frac{2\hat{m}}{\hat{\sigma}^2}(\rho_t(0) - \alpha), \quad \partial_u \rho_t(1) = \frac{2\hat{m}}{\hat{\sigma}^2}(\beta - \rho_t(1)), \quad t \in [0,T] \\ \rho_0(u) = g(u), \quad u \in (0,1), \end{cases} \tag{1.4.2}$$

where $\hat{m}$ is a parameter specified below. If $\theta = 1$, the reservoirs are slowed enough so that we obtain that the profile satisfies (1.4.2) with $\hat{\sigma} = \sigma$ and $\hat{m} = \kappa \sum_{z \geq 1} z p(z)$. For $\theta \in (1, \infty)$, the reservoirs are sufficiently slowed so that we get the heat equation with Neumann boundary conditions, that is, the profile solves equation (1.4.2) with $\hat{\sigma} = \sigma$ and $\hat{m} = 0$. Physically it means that the particles almost do not interact with the reservoirs (isolated system). We resume all the results in Figure 1.7.

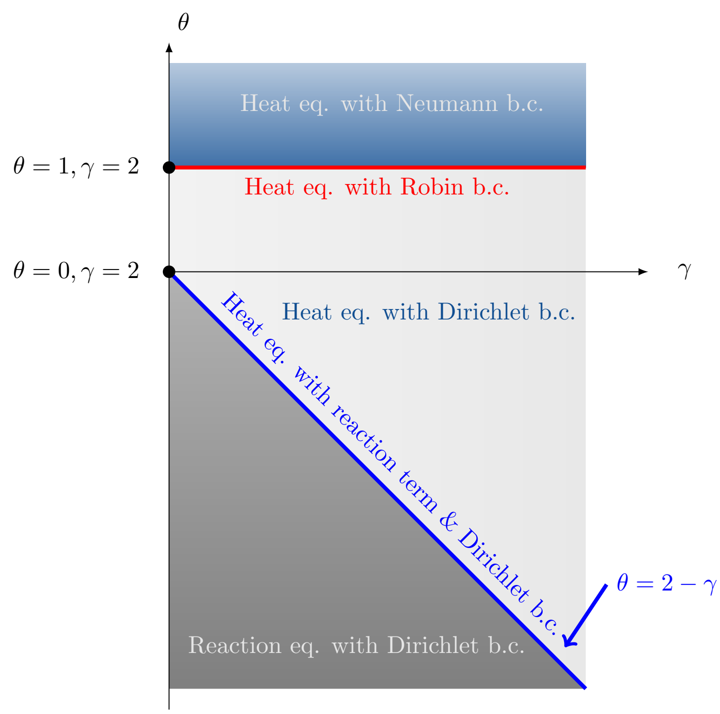

Figure 1.7: The five different hydrodynamic regimes in terms of γ and θ.

After having established the hydrodynamic behavior, we can study their stationary solutions which should describe the mean density profile in the non-equilibrium stationary state of the microscopic system in the thermodynamic limit $N \to \infty$ (see Figure 1.8). Since in the case $\theta > 2 - \gamma$ we recover the same hydrodynamic behavior as in [1] for $\theta \geq 0$, then the hydrostatic behavior coincides as well (see (1.2.5)). For $\theta \in (2-\gamma, 1)$ (heat equation with Dirichlet boundary conditions) the stationary solution is the linear profile connecting α at 0 to β at 1. For $\theta = 1$ (heat equation with Robin boundary conditions) the profile is still linear but the values at the boundaries are different and given by:

$$\bar{\rho}(0) = \frac{(\alpha+\beta)\sigma^2 + 2\alpha m\kappa}{2(m\kappa+\sigma^2)} \text{ and } \bar{\rho}(1) = \frac{(\alpha+\beta)\sigma^2 + 2\beta m\kappa}{2(m\kappa+\sigma^2)}. \tag{1.4.3}$$

For $\theta > 1$ (heat equation with Neumann boundary conditions) we expect that if we compute directly the stationary profile in the non-equilibrium stationary state of the microscopic system in the thermodynamic limit, the stationary profile will be flat with the value $\frac{\alpha+\beta}{2}$. Note that by taking $t \to \infty$ in (1.4.2) (with $\hat{\sigma} = \sigma$ and $\hat{m} = 0$) we obtain that the stationary solution of the heat equation with Neumann boundary conditions is given by $\int_0^1 \rho_0(v)dv$, which in general differs from $\frac{\alpha+\beta}{2}$. However, the latter constant is the appropriate one since it is the limit of the empirical density profile in the stationary state for $\theta > 1$. In fact, this constant can be recovered from the stationary solution with Robin boundary condition by sending $\kappa \to 0$ (see (1.4.3)).

On the other hand, the form of the stationary solution in the reaction equation, i.e. when $\theta < 2-\gamma$, is explicitly given by $\bar{\rho}^\infty(u) := \frac{V_0(u)}{V_1(u)}$ for $u \in [0,1]$ and where

$$V_0(u) = \alpha r^-(u) + \beta r^+(u), \quad V_1(u) = r^-(u) + r^+(u), \tag{1.4.4}$$

where the functions $r^\pm : (0,1) \to (0,\infty)$ are defined by

$$r^-(u) = c_\gamma \gamma^{-1} u^{-\gamma}, \quad r^+(u) = c_\gamma \gamma^{-1}(1-u)^{-\gamma}. \tag{1.4.5}$$

It is not difficult so see that this profile is increasing, non-linear, convex on $(0,1/2)$ and concave on $(1/2,1)$, with $\bar{\rho}^\infty(0) = \alpha$ and $\bar{\rho}^\infty(1) = \beta$. Finally, obtaining the properties of the stationary solution of the reaction-diffusion equation, i.e. $\theta = 2-\gamma$, is more tricky. We will see that the solution of the stationary reaction-diffusion equation satisfies the properties of $\bar{\rho}^\infty$, described above. These stationary profiles obtained from the hydrodynamic limit may also be obtained directly from the microscopic model (hydrostatic limit). We will do it in the particular case $\theta = 2-\gamma$ (see Chapter 2, Section 2.4).

Now, we explain, without proofs, how our results have to be modified considering the three different kind of reservoirs given in Subsection 1.3.2. In the two first cases, the density profile will be described by a function $\rho_t(u)$ where $u \in [0,1]$ while in the third case it will be described by a function $\rho_t(u)$ where $u \in \mathbb{R}$, since the system evolves on $\mathbb{Z}$.

Case 1: We still have five different regimes. The changes with respect to our results are:

a) the value of θ for which we obtain the reaction-diffusion equation (now is $\theta = \gamma - 1$ instead of $\theta = 2-\gamma$) and the reaction equation (now is for $\theta < 1-\gamma$ instead of $\theta < 2-\gamma$);

b) the term that depends on γ in (1.4.1) are the same as before but the exponent in this case is $1+\gamma$ instead of γ. We note that all the other regimes are not affected.

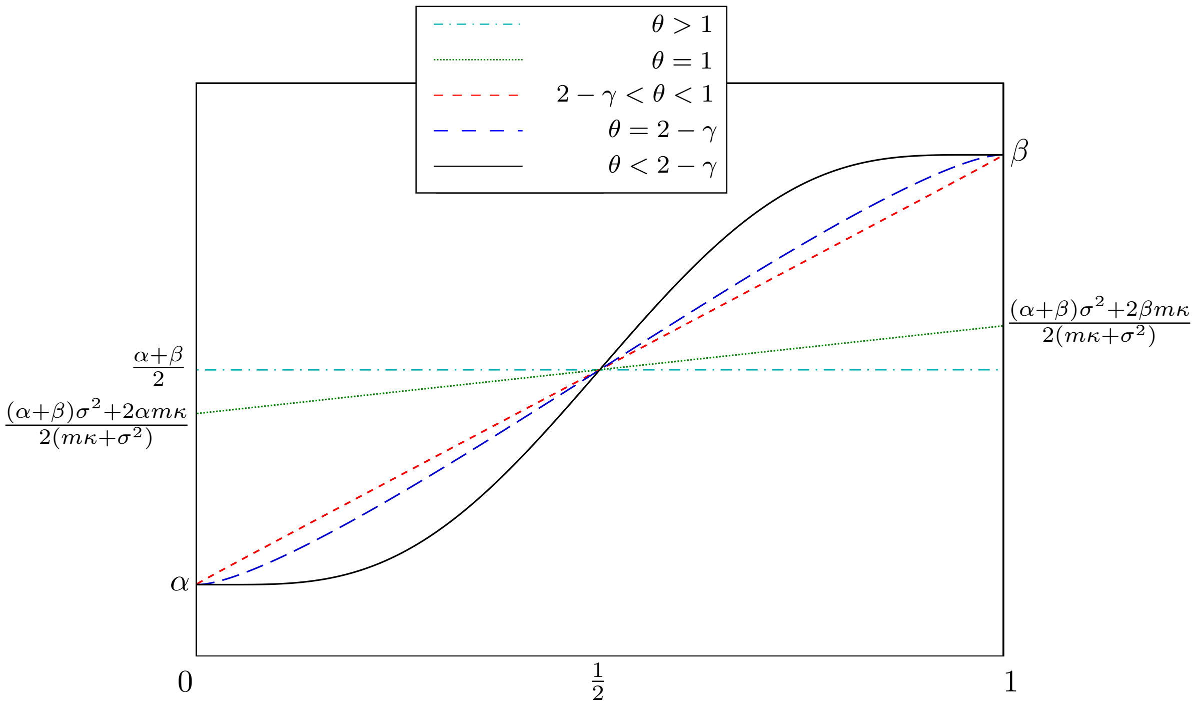

Figure 1.8: Stationary solution of the hydrodynamic equations according to the value of θ and $\gamma > 2$.

Case 2: We have now only three different regimes. If $\theta > 1$ the macroscopic behavior is described by the heat equation with Neumann boundary conditions; if $\theta = 1$, it is described by the heat equation with Robin boundary condition; if $\theta < 1$ (positive or negative) it is described by the heat equation with Dirichlet boundary conditions.

Case 3: We have now only three different regimes.

a) If $\theta > 2$ the reservoirs are too weak and the density profile evolves in the diffusive scaling according to the heat equation on $\mathbb{R}$

$$\partial \rho_t(u) = \tfrac{\sigma^2}{2}\Delta \rho_t(u),$$

without any boundary conditions.

b) If $\theta = 2$, the density profile evolves in the diffusive scaling according to the reaction-diffusion equation on $\mathbb{R}$ given by

$$\partial_t \rho_t(u) = \tfrac{\sigma^2}{2}\Delta \rho_t(u) - \kappa \mathbf{1}_{u\leq 0}(\rho_t(u)-\alpha) - \kappa \mathbf{1}_{u\geq 1}(\rho_t(u)-\beta).$$

c) If $\theta < 2$, the reservoirs are so fast that in the diffusive time scale they fix the density profile to be α at the left of 0 and β at the right of 1. In the bulk $(0,1)$, the density profile evolves according to the heat equation restricted to $(0,1)$ with these inhomogeneous Dirichlet boundary conditions.

1.4.2 Fick's law

In the steady state an other issue of physical interest that we can study is the derivation of the *Fick's law*. Fick's law is a phenomenological law stating that the flux of matter (current) due to the diffusion goes from regions of high concentration to regions of low concentration, with a magnitude that is proportional to the concentration gradient:

$$W = -D\partial_u \bar{\rho}(u), \tag{1.4.6}$$

where W denotes the current, and D is the diffusion coefficient. For instance, for our model in the case where $\theta \in (2-\gamma, 1)$, we know from the hydrodynamic equation that $D = \frac{\sigma^2}{2}$. Then, we expect that Fick's law holds in the steady state with

$$W = -\tfrac{\sigma^2}{2}(\beta - \alpha),$$

since we know that $\bar{\rho}(u) = (\beta - \alpha)u + \alpha$ for $u \in [0,1]$. Since for $\theta = 2-\gamma$ we have a reaction-diffusion equation, the classical Fick's law has to be replaced by the *generalized Fick's law*. In fact, we should have a first term coming from the diffusion part and the other one coming from the reaction term due to the reservoirs. Then we expect to have

$$W = -\tfrac{\sigma^2}{2}\partial_v \bar{\rho}(v) + \hat{\kappa}\int_v^1 \frac{\alpha - \bar{\rho}(u)}{u^\gamma}du - \hat{\kappa}\int_0^v \frac{\beta - \bar{\rho}(u)}{(1-u)^\gamma}du, \tag{1.4.7}$$

for $v \in [0,1]$ and where $\bar{\rho}$ is the stationary solution of (1.4.2). Above we have just considered the Fick's law at a macroscopic level. We can also try to obtain the Fick's law directly at the microscopic level. Since we are considering $\alpha \leq \beta$, we know that there exists a flux of particles. We consider the microscopic current W_x for $x \in \Lambda_N \cup \{N\}$, which is defined as the rate of particles crossing $x - \frac{1}{2}$ from the left to the right, minus the rate of particles crossing $x - \frac{1}{2}$ from the right to the left. If $\theta < 1$, we expect that the expectation of the current W_x under the stationary measure is of order N^{-1}. In the case $\theta \in (2-\gamma, 1)$ (resp. $\theta = 2-\gamma$) the first order correction (i.e. N^{-1}) to the expectation of the current $W_{[vN]}$ under the stationary measure should converge to (1.4.6) (resp. (1.4.7)). As we said before, in Chapter 2 we study some properties of the case $\theta = 2-\gamma$ and we complete this study deriving the generalized Fick's law from the microscopic system. Since similar arguments can be also done in the case $\theta \in (2-\gamma, 1)$ to derive the classical Fick's law from the microscopic system, we omitted its proof.

1.5 Super-diffusive case

Normal (diffusive) transport phenomena are described by standard random walk models. Anomalous transport, in particular transport phenomena giving rise to super-diffusion, are nowadays encapsulated in the Lévy flights or Lévy walks framework [27, 60] and appear in physics, finance, biology ... A Lévy flight is nothing but a random walk in which the step-lengths have a probability distribution that is heavy tailed. A (one-dimensional) Lévy walker moves with a constant velocity v for a heavy-tailed random time τ on a distance $x = v\tau$ in either direction with equal probability and then chooses a new direction and moves again. One then easily shows that for Lévy flights or Lévy walks, the space-time scaling limit $P(x,t)$ of the probability distribution of the particle position $x(t)$ is solution of the fractional diffusion equation

$$\partial_t P = -c(-\Delta)^{\gamma/2} P \tag{1.5.1}$$

where c is a constant and $\gamma \in (1,2)$. In physics, the description of anomalous transport phenomena by Lévy walks instead of Lévy flights is sometimes preferred despite the two models have the same scaling limit form provided by (1.5.1) because the first ones have a finite speed of propagation (see [60] for more details).

While Lévy walks and Lévy flights are today well known and are popular models to describe super-diffusion in infinite systems in various application fields, there has been recently several physical studies pointing out that it would be desirable to have a better understanding of Lévy walks in bounded domains. For bounded domains, boundary conditions and exchange with reservoirs or environment have to be taken into account. A particular interest for this problem is related to the description of anomalous diffusion of energy in low-dimensional lattices [20, 45] in contact with reservoirs [21, 22, 46]. It is well established that superdiffusive systems are much more sensitive to the reservoirs and boundaries than diffusive systems but quantitative informations, like the form of the singularities of the profiles at the boundaries, are still missing.

In Chapter 3, motivated by these studies, we consider the boundary driven exclusion process with long jumps whose distribution is in the form of (1.3.1) with $1 < \gamma < 2$, which may be considered as a substitute to Lévy flights in bounded domains with reservoirs when Lévy flights are moreover *interacting*. As we will see, the main operator emerging from the microscopic dynamic is a non-local operator, namely, the regional fractional Laplacian. For that reason we recall the definition and basic properties of the regional fractional Laplacian. Details can be found in [37, 9].

1.5.1 Regional fractional Laplacian

Consider an open subset I of $\mathbb{R}$. Let $L^1\left(I, \frac{du}{(1+|u|^{1+\gamma})}\right)$ be the space of all Borel functions G on I satisfying

$$\int_I \frac{|G(u)|}{(1+|u|)^{1+\gamma}} du < \infty. \tag{1.5.2}$$

The regional fractional Laplacian $-(-\Delta)_I^{\gamma/2}$ is defined on the set of functions $G \in L^1\left(I, \frac{du}{(1+|u|^{1+\gamma})}\right)$ by

$$-(-\Delta)_I^{\gamma/2} G(u) = c_\gamma \lim_{\varepsilon \to 0} \int_I \mathbf{1}_{|v-u|\geq\varepsilon} \frac{G(v)-G(u)}{|v-u|^{1+\gamma}} dv \tag{1.5.3}$$

provided the limit exists.

When $I=\mathbb{R}$, we get that the previous definition coincides with the fractional Laplacian denoted by $-(-\Delta)^{\gamma/2}$ (see [9, 23]). Up to a multiplicative constant, $-(-\Delta)^{\gamma/2}$ is the generator of a γ-Lévy stable process. The fractional Laplacian can also be defined in an equivalent way as a pseudo-differential operator of symbol $|\xi|^\gamma$ (up to a multiplicative constant). We have the following identification

$$\forall u\in I,\ -(-\Delta)_I^{\gamma/2}G(u)=-(-\Delta)^{\gamma/2}G(u)+V_I(u)G(u), \tag{1.5.4}$$

for all smooth function $G:[0,1]\to\mathbb{R}$ with compact support included in I and where $V_I(u)=c_\gamma\int_{I^c}\dfrac{dv}{|u-v|^{\gamma+1}}$ for all $u\in I$ (see [8]). In this work we are interested in the cases $I=(0,1)$. In order to simplify the notation we use

$$\mathbb{L}:=-(-\Delta)_{(0,1)}^{\gamma/2}$$

and $V_{(0,1)}(u)=r^-(u)+r^+(u)=V_1(u)$ for all $u\in(0,1)$ (see (1.4.4)).

1.5.2 Hydrodynamic and hydrostatic limit

We have the natural stimulus of trying to extend the results obtained in the previous section. Our main result is the derivation of the hydrodynamic and hydrostatic limit for the density of particles for this system. If $\theta=0$ the limiting PDE depends on the value of κ (we use the notation ρ^κ for indicating the dependence on κ of the solution) and takes the form of a fractional heat equation with a singular reaction term:

$$\begin{cases} \partial_t\rho_t^\kappa(u)=\mathbb{L}\rho_t^\kappa(u)+\kappa c_\gamma\gamma^{-1}\left\{\frac{\alpha-\rho_t^\kappa(u)}{u^\gamma}+\frac{\beta-\rho_t^\kappa(u)}{(1-u)^\gamma}\right\}, & (t,u)\in[0,T]\times(0,1),\\ \rho_t^\kappa(0)=\alpha,\quad \rho_t^\kappa(1)=\beta,\quad t\in[0,T]. \end{cases} \tag{1.5.5}$$

The singular reaction term fixes the density at 0 to be α and at 1 to be β (see remark 3.2.4). We obtain in this way a new family operators indexed by κ, taking the form

$$\mathbb{L}_\kappa=\mathbb{L}-\kappa V_1,$$

where V_1 is given in (1.4.4). These operators are symmetric non-positive when restricted to the set of smooth functions compactly supported in $(0,1)$. For $\kappa=1$, we recover the so-called restricted fractional Laplacian, where the fractional Laplacian $-(-\Delta)^{\gamma/2}$ is restricted to act only on functions that are zero outside $(0,1)$ (see [59]) while in the limit $\kappa\to 0$ we get the so-called regional fractional Laplacian. We recall that since the fractional Laplacian is a non-local operator, the definition of a fractional Laplacian with Dirichlet boundary conditions is not obvious from a modeling point of view. In the PDE's literature several candidates have been proposed, for instance, "restricted fractional Laplacian", "spectral fractional Laplacian", "Neumann Fractional Laplacian " (see [59, 2]), but often without a clear physical interpretation. A probabilistic interpretation of these operators is possible and may enlighten their meaning. The restricted fractional Laplacian ($\kappa=1$) corresponds to the generator of a γ-Lévy stable process killed outside of $(0,1)$, while the regional fractional Laplacian ($\kappa=0$) corresponds to the generator of a censored γ-Lévy stable process on $(0,1)$ (see [8, 37]). For $\kappa \neq 0,1$ we could rely on the Feynman-Kac formula but we do not pursue this issue here. As mentioned above our reservoirs are regulated by the parameters $\kappa N^{-\theta}$, $\kappa > 0$, and in this regime of γ we focus on the case $\theta \leq 0$. For $\theta < 0$ the hydrodynamic equation is given by

$$\begin{cases} \partial_t \rho_t^\kappa(u) = -\kappa \rho_t^\kappa(u) V_1(u) + \kappa V_0(u), \quad (t,u) \in [0,T] \times (0,1), \\ \rho_t^\kappa(0) = \alpha, \quad \rho_t^\kappa(1) = \beta, \quad t \in [0,T], \\ \rho_0^\kappa(u) = g(u), \quad u \in (0,1), \end{cases}$$

where $V_0(u)$ is given in (1.4.4). This equation can be explicitly solved (see Remark 3.2.7). Moreover, we can interpret this case saying that the reservoirs are fast enough in such a way that the particles do not have time to perform anomalous diffusion. We resume our panorama in Figure 1.9.

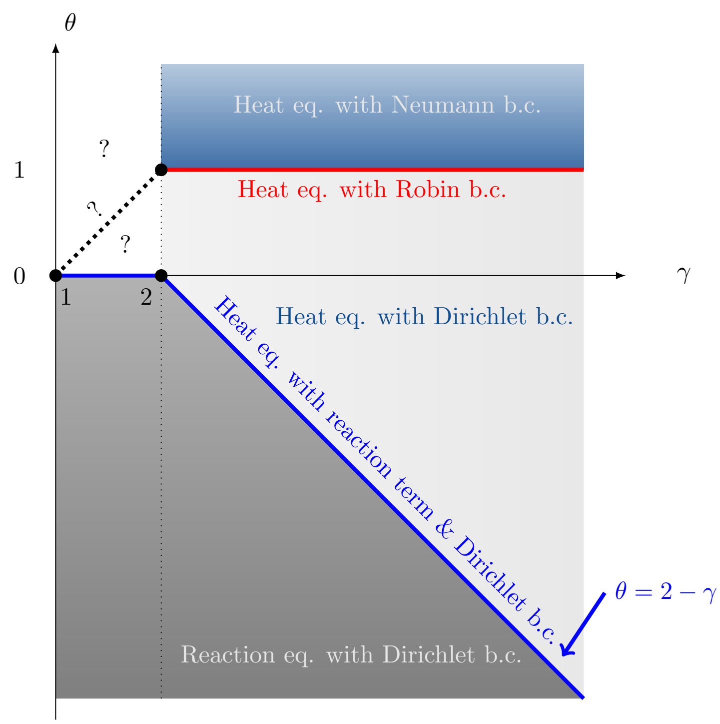

Figure 1.9: Regimes in terms of γ and θ.

Now, we conjecture that for small values of $\theta > 0$ the hydrodynamic equation is given by

$$\begin{cases} \partial_t \rho_t^0(u) = \mathbb{L} \rho_t^0(u), \quad (t,u) \in [0,T] \times (0,1), \\ \rho_t^0(0) = \alpha, \quad \rho_t^0(1) = \beta, \quad t \in [0,T], \\ \rho_0^0(u) = g(u), \quad u \in (0,1). \end{cases} \tag{1.5.6}$$

Remember that we explained in the end of Subsection 1.2.2 the importance of the parameter κ, which in this case will aid to support the conjecture above. Indeed, in Theorem 3.2.10 of

Chapter 3, we analyze the convergence of the profile (that we obtained for $\theta = 0$ and which is indexed in κ) when $\kappa \to 0$. Indeed, when $\kappa \to 0$ we obtain that the limiting profile is a weak solution of the equation above. This approach gives us a conjecture of the hydrodynamic behavior for small values of $\theta > 0$. However, we cannot discard other possible transitions. In fact, we believe that the fractional versions of the heat equation with Robin and Neumann boundary conditions could appear for θ sufficiently large. For the "Robin case" we do not have any conjecture for the form of the hydrodynamic equation but we expect it occurs for $\theta = \gamma - 1$. For the "Neumann case" ($\theta > \gamma - 1$) the hydrodynamic equation could be the following

$$\begin{cases} \partial_t \rho_t(u) = \mathbb{L}\rho_t(u), \quad (t,u) \in [0,T] \times (0,1), \\ \rho_0(u) = g(u), \quad u \in (0,1). \end{cases} \tag{1.5.7}$$

Observe the difference of (1.5.7) and (1.5.6), in (1.5.7) we do not impose any boundary condition for ρ_t. This case can be interpreted as the extension of the heat equation with Neumann boundary conditions to the $1 < \gamma < 2$ case. In fact the boundary conditions are encapsulated in $\mathbb{L}$. We leave these open problems for a future work.

After having obtained the hydrodynamic limit for $\theta = 0$, we study their stationary solutions $\bar{\rho}^\kappa$, which are not explicit apart from the case $\kappa = 1$ and the case $\kappa = \infty$, i.e. $\bar{\rho}^\infty = \lim_{\kappa \to \infty} \bar{\rho}^\kappa$. These profiles coincide with the profiles of the microscopic system in their non-equilibrium stationary states (see Chapter 3 for the $\kappa = 1$ case). The bounded continuous function $\bar{\rho}^\kappa$ has α and β as boundary conditions and is such that it solves in a distributional sense the equation

$$\mathbb{L}_\kappa \bar{\rho}^\kappa = -\kappa V_0. \tag{1.5.8}$$

We prove that as $\kappa \to 0$, $\bar{\rho}^\kappa \to \bar{\rho}^0$ in a suitable topology where $\bar{\rho}^0$ is a weakly harmonic function of the regional fractional Laplacian $\mathbb{L}$, i.e. we can take $\kappa = 0$ in (1.5.8).

In Chapter 3 we also show that (for $\kappa = 1$, $\theta = 0$) the stationary density profile is described by the stationary solution of a fractional diffusion equation with Dirichlet boundary conditions:

$$\begin{cases} \mathbb{L}_1 \bar{\rho}^1 + V_0 = 0, \quad u \in (0,1), \\ \bar{\rho}^1(0) = \alpha, \ \bar{\rho}^1(1) = \beta. \end{cases} \tag{1.5.9}$$

There are many recent studies focusing on the regularization properties of fractional operators in bounded domains [50, 54, 32]. Even in this one dimensional setup, the question is in general non trivial. For $\kappa = \infty$ we have an explicit expression given by $\bar{\rho}^\infty(u) = \dfrac{V_0(u)}{V_1(u)}$ for all $u \in [0,1]$, which has Hölder regularity equal to γ at the boundaries. For $\kappa = 1$, the profile $\bar{\rho}^1$ is given in terms of a Poisson kernel and it has Hölder regularity equal to $\frac{\gamma}{2}$ at the boundaries (see [4]). In the case $\kappa = 0$ we just know that the profile $\bar{\rho}^\kappa$ is at least $\frac{\gamma-1}{2}$-Hölder on $[0,1]$. For $\kappa \neq 1$, it should be possible to prove the interior regularity of $\bar{\rho}^\kappa$ by some existing methods [50] but the boundary regularity that numerical simulations seem to indicate to depend on κ is much more challenging.

1.5.3 Fractional Fick's law

In this case the "classical" generalized Fick's law is violated and shall be replaced by a *fractional Fick's law* of particle transport. In Chapter 3 we prove the validity of the fractional Fick's law for $\theta = 0$ and $\kappa = 1$ (see Theorem 3.2.16). Consider the current W_x introduced in Subsection 1.4.1. We first prove that the expectation of the current W_x under the stationary measure is of order $N^{1-\gamma}$. Then, we prove that the fractional order correction (i.e. $N^{1-\gamma}$) to the expectation of the current $W_{[vN]}$ under the stationary measure converges to a semi-explicit expression given by

$$\int_0^v \int_v^1 \frac{\bar{\rho}(w) - \bar{\rho}(u)}{|w-u|^{\gamma+1}} du dw + \kappa c_\gamma \gamma^{-1} \left[\int_v^1 \frac{\alpha - \bar{\rho}(u)}{u^\gamma} du - \int_0^v \frac{\beta - \bar{\rho}(u)}{(1-u)^\gamma} du \right],$$

for $v \in [0,1]$ where $\bar{\rho}$ is the unique stationary solution of (1.5.5). Recall that Fick's law represents the simplest relationship between the flux and the gradient of the density, which turns out to be local. However in this case, it is replaced by a non-local law, describing a different transport process.

Chapter 2

Diffusive case

Contents

2.1 Introduction

We consider an exclusion process with long jumps in the bulk $\Lambda_N = \{1,\ldots,N-1\}$, for $N \geq 2$, in contact with infinitely extended reservoirs on the left and on the right of the bulk. The jump rate is described by a transition probability p which is symmetric, with infinite support but with finite variance. The reservoirs add or remove particles with rate proportional to $\kappa N^{-\theta}$, where $\kappa > 0$ and $\theta \in \mathbb{R}$. If $\theta < 0$ (resp. $\theta > 0$) the reservoirs fastly (resp. slowly) add and remove particles in the bulk. According to the value of θ we prove that the time evolution of the spatial density of particles is described by some partial differential equations with various boundary conditions.

2.2 Statement of results

In this chapter we consider the process introduced in Section 1.3, whose generator L_N is given in (1.3.2). In this case the probability transition function p (see (1.3.1)) depends on a parameter $\gamma > 2$. Since p is symmetric it is mean zero, that is: $\sum_{z\in\mathbb{Z}} zp(z) = 0$. We denote $m = \sum_{z\geq 1} zp(z)$. Moreover, p has finite variance that is $\sigma^2 := \sum_{z\in\mathbb{Z}} z^2 p(z) < \infty$.

Remark 2.2.1. *Along this chapter we will note that many of our results are true, in the case where p has finite variance, in the more general setting where we only assume p to be translation invariant and mean zero.*

We consider the Markov process speeded up in the time scale $\Theta(N)$ and we use the notation $\eta^N(t) := \eta(t\Theta(N))$, so that $\{\eta^N(t)\}_{t\geq 0}$ has infinitesimal generator $\Theta(N)L_N$. Although $\eta^N(t)$ depends on α, β, κ and θ, we shall omit these indexes in order to simplify notation.

Let us denote by $\bar{\mu}_N$ the unique invariant measure of $\{\eta(t)\}_{t\geq 0}$. If $\alpha = \beta = \rho$ then $\bar{\mu}_N$ is equal to the Bernoulli product measure with density ρ. It is denoted by ν_ρ. The expectation of a function f with respect to $\bar{\mu}_N$ (resp. ν_ρ) is denoted by $\langle f \rangle_N$ (resp. $\langle f \rangle_\rho$) or $\mu_N(f)$ (resp. $\nu_\rho(f)$). For any $\rho \in (0,1)$ the density of $\bar{\mu}_N$ with respect to ν_ρ is denoted by $f_{N,\rho}$.

2.2.1 Notation

From now on up to the rest of this chapter we fix a finite time horizon $[0,T]$. To properly state the hydrodynamic and hydrostatic limit, we need to introduce some notations and definitions. The Hilbert space $L^2([0,1]^d, h(u)du)$ for $d = 1,2$ is abbreviated by $L^2_h([0,1]^d)$ and we denote its inner product by $\langle \cdot,\cdot \rangle_h$ and the corresponding norm by $\|\cdot\|_h$. When $h \equiv 1$ we simply write $L^2([0,1]^d)$, $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$. The set $C^\infty([0,1]^d)$ denotes the set of smooth functions on $[0,1]^d$. The supremum norm is denoted by $\|\cdot\|_\infty$. We denote by $C_c^\infty((0,1)^d)$ the set of all smooth real-valued functions defined in $(0,1)^d$ with compact support included in $(0,1)^d$. For an interval I in $\mathbb{R}$ and integers m and n, we denote by $C^{m,n}([0,T]\times I)$ the set of functions defined on $[0,T]\times I$ that are m times differentiable on the first variable and n times differentiable on the second variable. An index on a function will always denote a fixed variable, not a derivative. For example, $G_t(u)$ means $G(t,u)$. The derivative of $G \in C^{m,n}([0,T]\times I)$ will be denoted by $\partial_t G$ (first variable) and $\partial_u G$ (second variable). We also consider the set $C_c^{m,n}([0,T]\times[0,1])$ of functions $G \in C^{m,n}([0,T]\times[0,1])$ such that G_t has a compact support included in $(0,1)$ for any time t.

We denote by Δ the Laplacian operator: $\Delta = \sum_{i=1}^{d} \partial_{u_i}^2$. The semi inner-product $\langle \cdot, \cdot \rangle_1$ is defined on the set $C^\infty([0,1]^d)$ by

$$\langle F, G \rangle_1 = \int_{[0,1]^d} \sum_{i=1}^{d} (\partial_{u_i} F)(u)\, (\partial_{u_i} G)(u)\, du.$$

The corresponding semi-norm is denoted by $\| \cdot \|_1$.

Definition 2.2.2. *The Sobolev space $\mathscr{H}^1([0,1]^d)$ is the Hilbert space defined as the completion of $C^\infty([0,1]^d)$ for the norm*

$$\| \cdot \|^2_{\mathscr{H}^1([0,1]^d)} := \| \cdot \|^2 + \| \cdot \|_1^2.$$

Its elements elements coincide a.e. with continuous functions. The completion of $C_c^\infty((0,1)^d)$ for this norm is denoted by $\mathscr{H}_0^1([0,1]^d)$. This is a Hilbert space whose elements coincide a.e. with continuous functions vanishing at the boundary of $[0,1]^d$. On $\mathscr{H}_0^1([0,1]^d)$, the two norms $\| \cdot \|_{\mathscr{H}^1([0,1]^d)}$ and $\| \cdot \|_1$ are equivalent. We also define the spaces $\mathscr{H}_h^1([0,1]^d) := \mathscr{H}^1([0,1]^d) \cap L_h^2([0,1]^d)$ and $\mathscr{H}_{0,h}^1([0,1]^d) := \mathscr{H}_0^1([0,1]^d) \cap L_h^2([0,1]^d)$ and the space $L^2(0,T;\mathscr{H}^1([0,1]))$ is the set of measurable functions $f : [0,T] \to \mathscr{H}^1([0,1])$ such that

$$\int_0^T \|f_s\|^2_{\mathscr{H}^1([0,1])} ds < \infty.$$

The space $L^2(0,T;\mathscr{H}_0^1([0,1]))$ is defined similarly.

We write $f(u) \lesssim g(u)$ if there exists a constant C independent of u such that $f(u) \le C g(u)$ for every u. We will also write $f(u) = O(g(u))$ if the condition $|f(u)| \lesssim |g(u)|$ is satisfied. Sometimes, in order to stress the dependence of a constant C on some parameter a, we write $C(a)$.

2.2.2 Hydrodynamic equations

We can now give the definition of the weak solutions of the hydrodynamic equations that will be derived in this chapter.

Definition 2.2.3. *Let $\hat{\sigma} \ge 0$ and $\hat{\kappa} \ge 0$ be some parameters. Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the reaction-diffusion equation with inhomogeneous Dirichlet boundary conditions*

$$\begin{cases} \partial_t \rho_t(u) = \frac{\hat{\sigma}^2}{2} \Delta \rho_t(u) + \hat{\kappa} \left\{ \frac{\alpha - \rho_t(u)}{u^\gamma} + \frac{\beta - \rho_t(u)}{(1-u)^\gamma} \right\}, & (t,u) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, \quad t \in [0,T], \\ \rho_0(u) = g(u), \quad u \in (0,1), \end{cases} \tag{2.2.1}$$

if the following three conditions hold:

i) $\rho \in L^2(0,T;\mathscr{H}^1([0,1]))$ *if* $\hat{\sigma} > 0$ *and* $\int_0^T \int_0^1 \left\{ \frac{(\alpha - \rho_t(u))^2}{u^\gamma} + \frac{(\beta - \rho_t(u))^2}{(1-u)^\gamma} \right\} du\, dt < \infty$ *if* $\hat{\kappa} > 0$, *ii)* ρ *satisfies the weak formulation:*

$$
\begin{aligned}
F_{RD}(t,\rho,G,g) := &\int_0^1 \rho_t(u)G_t(u)\,du - \int_0^1 g(u)G_0(u)\,du \\
&- \int_0^t \int_0^1 \rho_s(u)\Big(\tfrac{\hat{\sigma}^2}{2}\Delta + \partial_s\Big)G_s(u)\,du\,ds \\
&- \hat{\kappa}\int_0^t \int_0^1 G_s(u)\Big(\frac{\alpha-\rho_s(u)}{u^\gamma} + \frac{\beta-\rho_s(u)}{(1-u)^\gamma}\Big)\,du\,ds = 0,
\end{aligned}
$$

for all $t \in [0,T]$ *and any function* $G \in C_c^{1,2}([0,T]\times[0,1])$*,*

iii) if $\hat{\sigma} > 0$ *and* $\hat{\kappa} = 0$*, then* $\rho_t(0) = \alpha$ *and* $\rho_t(1) = \beta$ *for* t *a.s in* $[0,T]$*.*

Remark 2.2.4. *Observe that in the case* $\hat{\sigma} > 0$ *and* $\hat{\kappa} = 0$ *we recover the heat equation with Dirichlet inhomogeneous boundary conditions. If* $\hat{\sigma} = 0$ *the equation does not have a diffusion part and the solution is fully explicit. Despite in the weak formulation we do not require any boundary condition (except the second part of item i) nor any regularity assumption, it turns out that the (unique) weak solution is smooth and satisfies the boundary conditions of item iii).*

Remark 2.2.5. *Observe that in the case* $\hat{\sigma} > 0$ *and* $\hat{\kappa} > 0$ *the item i) of the previous definition implies that* $\rho_t(0) = \alpha$ *and* $\rho_t(1) = \beta$*, for almost every* t *in* $[0,T]$*. Indeed, first note that by item i) we know that* ρ_t *is* $\frac{1}{2}$*-Hölder for almost every* t *in* $[0,T]$ *since a function in* $\mathscr{H}^1([0,1])$ *is* $\frac{1}{2}$*-Hölder. Now, taking* $\varepsilon \in (0,1)$ *we note that*

$$
\int_0^T \frac{(\rho_t(0)-\alpha)^2}{\gamma-1}dt = \int_0^T \lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1 \frac{(\rho_t(0)-\alpha)^2}{u^\gamma}du\,dt.
$$

By summing and subtracting $\rho_t(u)$ *inside the square on the right hand side in the previous equality and using the inequality* $(a+b)^2 \le 2a^2+2b^2$ *we get that the term on the right the hand side of last equality is bounded from above by*

$$
2\int_0^T \lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1 \frac{(\rho_t(0)-\rho_t(u))^2}{u^\gamma}du\,dt + 2\int_0^T \lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1 \frac{(\rho_t(u)-\alpha)^2}{u^\gamma}du\,dt.
$$

Since ρ_t *is* $\frac{1}{2}$*-Hölder for almost every* t *in* $[0,T]$ *the first term in the previous expression vanishes. Now, the second term in the previous expression is bounded from above by*

$$
2\lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_0^T\int_0^1 \frac{(\rho_t(u)-\alpha)^2}{u^\gamma}du\,dt,
$$

which vanishes since the second claim of item i) holds. Thus, we have that

$$
\int_0^T \frac{(\rho_t(0)-\alpha)^2}{\gamma-1}dt = 0,
$$

whence we get that $\rho_t(0) = \alpha$ *for almost every* t *in* $[0,T]$*. Showing that* $\rho_t(1) = \beta$ *for almost every* t *in* $[0,T]$ *is completely analogous.*

Definition 2.2.6. *Let $\hat{\sigma} > 0$ and $\hat{m} \geq 0$ be some parameters. Let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the heat equation with Robin boundary conditions*

$$\begin{cases} \partial_t \rho_t(u) = \frac{\hat{\sigma}^2}{2} \Delta \rho_t(u), \quad (t,u) \in [0,T] \times (0,1), \\ \partial_u \rho_t(0) = \frac{2\hat{m}}{\hat{\sigma}^2}(\rho_t(0) - \alpha), \quad \partial_u \rho_t(1) = \frac{2\hat{m}}{\hat{\sigma}^2}(\beta - \rho_t(1)), \quad t \in [0,T] \\ \rho_0(u) = g(u), \quad u \in (0,1), \end{cases} \tag{2.2.2}$$

if the following two conditions hold:

i) $\rho \in L^2(0,T;\mathscr{H}^1([0,1]))$*,*

ii) ρ *satisfies the weak formulation:*

$$\begin{aligned} F_{Rob}(t,\rho,G,g) := & \int_0^1 \rho_t(u) G_t(u)\, du - \int_0^1 g(u) G_0(u)\, du \\ & - \int_0^t \int_0^1 \rho_s(u) \Big(\frac{\hat{\sigma}^2}{2} \Delta + \partial_s \Big) G_s(u)\, du\, ds \\ & + \frac{\hat{\sigma}^2}{2} \int_0^t \{ \rho_s(1) \partial_u G_s(1) - \rho_s(0) \partial_u G_s(0) \}\, ds \\ & - \hat{m} \int_0^t \{ G_s(0)(\alpha - \rho_s(0)) + G_s(1)(\beta - \rho_s(1)) \}\, ds = 0, \end{aligned} \tag{2.2.3}$$

for all $t \in [0,T]$, any function $G \in C^{1,2}([0,T] \times [0,1])$.

Remark 2.2.7. *Observe that in the case $\hat{m} = 0$ the PDE above is the heat equation with Neumann boundary conditions.*

2.2.3 Hydrodynamic Limit

Let $\mathscr{M}^+$ be the space of positive measures on $[0,1]$ with total mass bounded by 1 equipped with the weak topology. For any configuration $\eta \in \Omega_N$ we define the empirical measure $\pi^N(\eta, du)$ on $[0,1]$ by

$$\pi^N(\eta, du) = \frac{1}{N-1} \sum_{x \in \Lambda_N} \eta_x \delta_{\frac{x}{N}}(du), \tag{2.2.4}$$

where δ_a is a Dirac mass on $a \in [0,1]$, and

$$\pi_t^N(\eta, du) := \pi^N(\eta^N(t), du).$$

Fix $T > 0$ and $\theta \in \mathbb{R}$. We denote by $\mathbb{P}_{\mu_N}$ the probability measure in the Skorohod space $\mathscr{D}^T_{\Omega_N} := \mathscr{D}([0,T], \Omega_N)$ induced by the Markov process $\{\eta^N(t)\}_{t \geq 0}$ and the initial probability measure μ_N and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$. Let $\{\mathbb{Q}_N\}_{N \geq 1}$ be the sequence of probability measures on $\mathscr{D}^T_{\mathscr{M}^+} := \mathscr{D}([0,T], \mathscr{M}^+)$ induced by the Markov process $\{\pi_t^N\}_{t \geq 0}$ and by $\mathbb{P}_{\mu_N}$.

Definition 2.2.8. *Let $g : [0,1] \to [0,1]$ be a measurable function. We say that a sequence of probability measures $\{\mu_N\}_{N\geq 1}$ in Ω_N is associated to the profile g if for any continuous function $G : [0,1] \to \mathbb{R}$ and every $\delta > 0$*

$$\lim_{N\to\infty} \mu_N \left(\eta \in \Omega_N : \left| \frac{1}{N-1} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) \eta_x - \int_0^1 G(u)g(u)du \right| > \delta \right) = 0. \tag{2.2.5}$$

The main result of this chapter is summarized in the following theorem (see Figure 1.7).

Theorem 2.2.9. *(Hydrodynamic limit) Let $g : [0,1] \to [0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures in Ω_N associated to g. Then, for any $0 \leq t \leq T$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\Big(\eta^N(\cdot) \in \mathscr{D}^T_{\Omega_N} : \left| \frac{1}{N-1} \sum_{x\in\Lambda_N} G\left(\tfrac{x}{N}\right) \eta^N_x(t) - \int_0^1 G(u)\rho_t(u)du \right| > \delta \Big) = 0,$$

where the time scale is given by

$$\Theta(N) = \begin{cases} N^2, & \text{if } \theta \geq 2-\gamma, \\ N^{\gamma+\theta}, & \text{if } \theta < 2-\gamma, \end{cases} \tag{2.2.6}$$

and ρ is the unique weak solution of :

- (2.2.1) *with $\hat{\sigma} = 0$ and $\hat{\kappa} = \kappa c_\gamma \gamma^{-1}$, if $\theta \in (-\infty, 2-\gamma)$;*
- (2.2.1) *with $\hat{\sigma} = \sigma$ and $\hat{\kappa} = \kappa c_\gamma \gamma^{-1}$, if $\theta = 2-\gamma$;*
- (2.2.1) *with $\hat{\sigma} = \sigma$ and $\hat{\kappa} = 0$, if $\theta \in (2-\gamma, 1)$;*
- *(2.2.2) with $\hat{\sigma} = \sigma$ and $\hat{m} = m\kappa$, if $\theta = 1$;*
- *(2.2.2) with $\hat{\sigma} = \sigma$ and $\hat{m} = 0$, if $\theta \in (1, \infty)$.*

It is not always possible to write fully explicit expressions for the solutions of these hydrodynamic equations. The form of the corresponding stationary solutions is of interest since the latter are expected to describe, in general, the mean density profile in the non-equilibrium stationary state of the microscopic system in the thermodynamic limit $N \to \infty$. Observe that this is not a trivial fact since it requires to exchange the limit $t \to \infty$ with $N \to \infty$ (and for $\theta > 1$ this is for example false, see below).

The stationary solutions of the hydrodynamic limits in the $\theta > 2-\gamma$ case are standard. On the other hand, the form and properties of the stationary solutions in the $\theta \leq 2-\gamma$ case are original and more tricky to obtain in the $\theta = 2-\gamma$ case. This problem is studied in more details in Section 2.2.5.

For $\theta \in (2-\gamma, 1)$ (heat equation with Dirichlet boundary conditions) the stationary solution is the linear profile connecting α at 0 to β at 1. For $\theta = 1$ (heat equation with Robin boundary conditions) the profile is still linear but the values at the boundaries are different. Observe that if $\kappa \to 0$ these values converge to $\frac{\alpha+\beta}{2}$ so that the profile becomes flat and equal to $\frac{\alpha+\beta}{2}$. For

$\theta > 1$ (heat equation with Neumann boundary conditions) the stationary solution is constant equal to $\int_0^1 g(u)du$ where g is the initial condition. In fact, for $\theta > 1$, we expect that if we compute directly the stationary profile in the non-equilibrium stationary state of the microscopic system in the thermodynamic limit, the stationary profile will be flat with the value $\frac{\alpha+\beta}{2}$. This value is therefore memorized in the form of the hydrodynamic limit for $\theta = 1$, despite the fact that it has been forgotten in the hydrodynamic limit for $\theta > 1$. In the case $\theta < 2-\gamma$ (reaction equation) the stationary profile is fully explicit and given by $\bar{\rho}^{\infty}(u) = \frac{V_0(u)}{V_1(u)}$ (recall (1.4.4)). Observe that this profile is increasing, non-linear, convex on $(0, 1/2)$ and concave on $(1/2, 1)$ and connects α at 0 to β at 1. At the boundaries the profile is very flat. In Subsection 2.2.5 we claim that these properties remain valid for the stationary solution of the hydrodynamic equation in the case $\theta = 2-\gamma$ and in Section 2.5 we give the respective proof.

2.2.4 Hydrostatic equation for $\theta = 2-\gamma$

For the case $\theta = 2-\gamma$ we use the notation $\bar{\rho}^{\kappa}$, for indicating the dependence on κ of the stationary density profile.

Definition 2.2.10. *Let $\kappa > 0$. We say that $\bar{\rho}^{\kappa} : [0,1] \to [0,1]$ is a weak solution of the stationary reaction-diffusion equation with Dirichlet conditions*

$$\begin{cases} -\frac{\sigma^2}{2}\Delta\bar{\rho}^{\kappa}(u) + \kappa V_1(u)\Big\{\bar{\rho}^{\kappa}(u) - \bar{\rho}^{\infty}(u)\Big\} = 0, \quad u \in (0,1), \\ \bar{\rho}^{\kappa}(0) = \alpha, \quad \bar{\rho}^{\kappa}(1) = \beta, \end{cases} \tag{2.2.7}$$

where $\bar{\rho}^{\infty}(u) = \dfrac{V_0(u)}{V_1(u)}$, if

i) $\bar{\rho}^{\kappa} \in \mathscr{H}^1([0,1])$.

ii) $\int_0^1 \left\{ \frac{(\alpha - \bar{\rho}^{\kappa}(u))^2}{u^{\gamma}} + \frac{(\beta - \bar{\rho}^{\kappa}(u))^2}{(1-u)^{\gamma}} \right\} du < \infty$.

iii) For any function $G \in C_c^{\infty}((0,1))$ we have that

$$-\langle \bar{\rho}^{\kappa}, \tfrac{\sigma^2}{2}\Delta G\rangle + \kappa \langle \bar{\rho}^{\kappa}, G\rangle_{V_1} - \kappa\langle V_0, G\rangle = 0. \tag{2.2.8}$$

Remark 2.2.11. *Observe that items i) and ii) of the previous definition imply that $\bar{\rho}^{\kappa}(0) = \alpha$ and $\bar{\rho}^{\kappa}(1) = \beta$ (see Remark 2.2.5).*

Remark 2.2.12. *Since $\bar{\rho}^{\infty}$ is a continuous function such that*

$$\int_0^1 \left\{ \frac{(\alpha - \bar{\rho}^{\infty}(u))^2}{u^{\gamma}} + \frac{(\beta - \bar{\rho}^{\infty}(u))^2}{(1-u)^{\gamma}} \right\} du < \infty$$

and $\bar{\rho}^{\infty}(0) = \alpha$ and $\bar{\rho}^{\infty}(1) = \beta$, it is easy to see that from item i) and item ii) in Definition 2.2.10 we have that $\bar{\rho}^{\kappa} - \bar{\rho}^{\infty} \in \mathscr{H}^1_{0,V_1}([0,1])$.

Proposition 2.2.13. *There exists a unique weak solution to (2.2.7).*

Proof. First note that we can rewrite (2.2.8) as

$$-\langle\varphi^\kappa, \tfrac{\sigma^2}{2}\Delta G\rangle + \kappa\langle\varphi^\kappa, G\rangle_{V_1} = \langle\bar\rho^\infty, \tfrac{\sigma^2}{2}\Delta G\rangle, \tag{2.2.9}$$

where $\varphi^\kappa(u) = \bar\rho^\kappa(u) - \bar\rho^\infty(u)$.

Let $a^\kappa : \mathscr{H}^1_{0,V_1}([0,1]) \times \mathscr{H}^1_{0,V_1}([0,1]) \to \mathbb{R}$ be the bilinear form defined as

$$a^\kappa(\varphi, \varrho) = \langle\varphi, \varrho\rangle_1 + \kappa\langle\varphi, \varrho\rangle_{V_1},$$

for any functions $\varphi, \varrho \in \mathscr{H}^1_{0,V_1}([0,1])$. We note that a^κ is coercive. Indeed

$$a^\kappa(\varphi, \varphi) = \|\varphi\|_1^2 + \kappa\|\varphi\|_{V_1}^2 \geq \min\{1, \kappa V_1(\tfrac{1}{2})\}\|\varphi\|^2_{\mathscr{H}^1([0,1])}$$

and trivially we have that $a^\kappa(\varphi,\varphi) \geq \kappa\|\varphi\|^2_{V_1}$. By using the Cauchy-Schwarz inequality we can get that

$$|a^\kappa(\varphi, \varrho)| \leq \|\varphi\|_1\|\varrho\|_1 + \kappa\|\varphi\|_{V_1}\|\varrho\|_{V_1}.$$

The latter allows to conclude that the bilinear form a^κ is also continuous. Now we consider the linear form $I_{\bar\rho^\infty} : \mathscr{H}^1_{0,V_1}([0,1]) \to \mathbb{R}$ defined by $I_{\bar\rho^\infty}(\varphi) = -\frac{\sigma^2}{2}\langle\bar\rho^\infty, \varphi\rangle_1$. This linear form is continuous. Indeed, first note that $\bar\rho^\infty \in C^2([0,1])$. Using the Cauchy-Schwarz inequality we get that

$$|I_{\bar\rho^\infty}(\varphi)| \leq \tfrac{\sigma^2}{2}\|\bar\rho^\infty\|_1\|\varphi\|_1.$$

On the other hand, using integration by parts and the Cauchy-Schwarz inequality we have that

$$|I_{\bar\rho^\infty}(\varphi)| = \tfrac{\sigma^2}{2}|\langle\Delta\bar\rho^\infty V_1^{-1/2}, V_1^{1/2}\varphi\rangle| \leq \tfrac{\sigma^2}{2}\|\bar\rho^\infty V_1^{-1/2}\|\|\varphi\|_{V_1}.$$

Now we can apply Lax-Milgram's Theorem to guarantee that there exists a unique function $\varphi^\kappa \in \mathscr{H}^1_{0,V_1}([0,1])$, which satisfies (2.2.9) for any function $G \in C_c^\infty((0,1))$. Then in order to conclude the proof it is enough to take $\bar\rho^\kappa(u) = \varphi^\kappa(u) + \bar\rho^\infty(u)$ which clearly satisfies Definition 2.2.10. □

As an immediate consequence of the uniqueness result we have that the graph of $\bar\rho^\kappa$ has a rotational symmetry with respect to the point $(\frac{1}{2}, \frac{\alpha+\beta}{2})$ (see Lemma 2.2.14). This result will be used in the proof of Theorem 2.2.17.

Lemma 2.2.14. *Let $\bar\rho^\kappa$ be the weak solution of (2.2.7). Then we have that $\bar\rho^\kappa(u) + \bar\rho^\kappa(1-u) = \alpha + \beta$.*

Proof. Note that $\alpha + \beta - \bar\rho^\kappa(1-u)$ is a weak solution of (2.2.7). Then, by uniqueness we have that $\bar\rho^\kappa(u) = \alpha + \beta - \bar\rho^\kappa(1-u)$. □

2.2.5 Hydrostatic Limit and generalized Fick's law for $\theta = 2-\gamma$

The first result is the following law of large numbers for the empirical density under the stationary measure $\bar{\mu}_N$.

Theorem 2.2.15. *(Hydrostatic limit) For any continuous function $G : [0,1] \to \mathbb{R}$ and for any $\delta > 0$*

$$\lim_{N\to\infty} \bar{\mu}_N \left[\left| \frac{1}{N-1} \sum_{x\in\Lambda_N} G(\tfrac{x}{N})\eta_x - \int_0^1 G(u)\bar{\rho}^\kappa(u)du \right| > \delta \right] = 0,$$

where $\bar{\rho}^\kappa$ is the unique weak solution of (2.2.7).

In order to state our second result, which is the "generalized Fick's law", we must introduce the concept of current. Then, given $x \in \Lambda_N \cup \{N\}$ and a configuration η, we denote by $W_x(\eta)$ the current over the value $x - \frac{1}{2}$ which is defined as the rate of particles crossing $x - \frac{1}{2}$ from the left to the right minus the rate of particles crossing $x - \frac{1}{2}$ from the right to the left. Then, the current can be written as

$$\begin{aligned} W_x(\eta) = & \sum_{\substack{1\le y\le x-1 \\ x-1<z\le N-1}} p(z-y)[\eta_y - \eta_z] \\ & + \frac{\kappa}{N^\theta} \sum_{\substack{x\le z\le N-1 \\ y\le 0}} p(z-y)(\alpha - \eta_z) - \frac{\kappa}{N^\theta} \sum_{\substack{1\le y\le x-1 \\ z\ge N}} p(z-y)(\beta - \eta_y) \\ & := W_x^0(\eta) + \frac{\kappa}{N^\theta} W_x^{\ell,r}(\eta). \end{aligned} \tag{2.2.10}$$

We will often omit the dependence of W_x on η. Note that for any $x \in \Lambda_N$ we have the following microscopic continuity equation

$$L_N \eta_x = -(W_{x+1} - W_x).$$

Recall (1.4.5).

Theorem 2.2.16. *(Generalized Fick's law.) For all $v \in (0,1)$ the following Fick's law holds*

$$\begin{aligned} \lim_{N\to\infty} N\langle W_{[vN]}\rangle_N = & -\tfrac{\sigma^2}{2}\partial_v \bar{\rho}^\kappa(v) + \kappa \int_v^1 (\alpha - \bar{\rho}^\kappa(u))r^-(u)du \\ & - \kappa \int_0^v (\beta - \bar{\rho}^\kappa(u))r^+(u)du, \end{aligned} \tag{2.2.11}$$

where $\bar{\rho}^\kappa$ is the unique weak solution of (2.2.7).

Observe that (2.2.11) does not depend on v. Indeed, it can be proved by taking the derivative with respect to v of the right hand side of (2.2.11) and showing that it vanishes thanks to $\bar{\rho}^\kappa$ being the unique solution of (2.2.7). Then, we have that

$$\lim_{N\to\infty} N\langle W_1\rangle_N = \kappa \int_0^1 (\alpha - \bar{\rho}^\kappa(u))r^-(u)du.$$

Our last result is about the behavior of the weak solution of (2.2.7). Namely, we prove that this solution is increasing, convex on $[0, \frac{1}{2}]$ and concave on $[\frac{1}{2}, 1]$. Those facts follow directly from a description of the dependence of the profile on the parameter κ that we prove thanks to an adaptation of the maximum principle. In a second step, we will see that those properties induce a precise description of the behavior of the profile near the boundary, it will allow us to improve the regularity given by the existence theory, based on Lax-Milgram theorem (see Lemma 2.2.13) and enlarge the space where we have uniqueness ($\mathscr{H}^1_{V_1}([0,1])$ to $C([0,1])$).

Recall $\bar{\rho}^\infty$ from Definition 2.2.10 and let $\bar{\rho}^0(u) = (\beta - \alpha)u + \alpha$.

Theorem 2.2.17. *(Stationary solution) Let $\bar{\rho}^\kappa$ be the unique stationary weak solution of (2.2.7). Then,*

i) $\bar{\rho}^\kappa$ *increases on* $[0,1]$*, it is convex on* $[0, \frac{1}{2}]$ *and concave on* $[\frac{1}{2}, 1]$*. In the midle,* $\bar{\rho}^\kappa(\frac{1}{2}) = \frac{\alpha+\beta}{2}$ *and* $(\beta - \alpha) \le (\bar{\rho}^\kappa)'(\frac{1}{2}) \le \gamma(\beta - \alpha)$.

ii) If $\kappa < \iota$ *and* $\bar{\rho}^\kappa$*,* $\bar{\rho}^\iota$ *are the respective solutions of* (2.2.7) *then we have*

- $\bar{\rho}^0(u) > \bar{\rho}^\kappa(u) > \bar{\rho}^\iota(u) > \bar{\rho}^\infty(u)$ *if* $u \in (0, \frac{1}{2})$*,*
- $\bar{\rho}^0(u) < \bar{\rho}^\kappa(u) < \bar{\rho}^\iota(u) < \bar{\rho}^\infty(u)$ *if* $u \in (\frac{1}{2}, 1)$*.*

iii) $\bar{\rho}^\kappa \in C^2([0,1]) \cap C^\infty((0,1))$*, its behavior at the boundary is precisely described:*

$$\bar{\rho}^\kappa(u) \underset{u\to 0}{\sim} \alpha + (\beta - \alpha)u^\gamma + o(u^\gamma) \quad \text{and} \quad \bar{\rho}^\kappa(u) \underset{u\to 1}{\sim} \beta - (\beta - \alpha)(1-u)^\gamma + o\left((1-u)^\gamma\right).$$

Note that in the general case $\gamma > 2$, the regularity of $\bar{\rho}^\kappa$ on $[0,1]$ is optimal: if $2 \le n < \gamma < n+1$, $\bar{\rho}^\kappa$ cannot be in $C^{n+1}([0,1])$ by item iii) of Theorem 2.2.17. The function $\bar{\rho}^\kappa$ can possibly be a smooth function only if γ is an integer. It is easy to see that $\bar{\rho}^\kappa$ depends linearly on the boundary conditions. Since κ and σ can be associated in a single parameter, Corollary 2.5.2 ends the description of the dependence of $\bar{\rho}^\kappa$ in all the parameters. Moreover, in Corollary 2.5.2, we also prove that

$$\bar{\rho}^\kappa \in C([0,1]) \cap C^\infty((0,1)),$$

independently of Theorem 2.2.17 and this result will be useful in the proof of such Theorem.

2.3 Proof of the Hydrodynamic limit

The proof of Theorem 2.2.9 follows the usual approach of convergence in distribution of stochastic processes. In Subsection 2.3.2, we show that the sequence $\{\mathbb{Q}_N\}_{N\ge1}$ is tight and in Subsection 2.3.8 we prove that all limiting points of the sequence $\{\mathbb{Q}_N\}_{N\ge1}$ are concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure, that is $\pi_t(du) = \rho_t(u)du$ for all $u \in [0,1]$. Now we argue that the density ρ is a weak solution of the corresponding hydrodynamic equation for each regime of θ. The precise proof of this result is given ahead in Proposition 2.3.18.

Before beginning the steps of the proof, in the following subsection we give the main ideas which are behind the identification of limit points as weak solutions of the partial differential equations given in Section 2.2.2.

2.3.1 Heuristics for the hydrodynamic equations

The identification of the density ρ as a weak solution of the hydrodynamic equation is obtained by using auxiliary martingales. For that purpose, and to make the exposition simpler, we fix a function $G:[0,1]\to\mathbb{R}$ which does not depend on time and is two times continuously differentiable. If $\theta<1$ we will assume further that it has a compact support included in $(0,1)$ and for $\theta\geq 1$ we assume that it has a compact support but not necessarily contained in $[0,1]$ so that G has a good decay at infinity. In the last case observe that G can take non-zero values at 0 and 1. We know by Dynkin's formula that

$$M_t^N(G)=\langle\pi_t^N,G\rangle-\langle\pi_0^N,G\rangle-\int_0^t\Theta(N)L_N\langle\pi_s^N,G\rangle\,ds,\tag{2.3.1}$$

is a martingale with respect to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ where $\mathscr{F}_t:=\sigma(\{\eta(s)\}_{s\leq t})$ for all $t\in[0,T]$. Above the notation $\langle\pi_s^N,G\rangle$ represents the integral of G with respect the measure π_s^N. This notation should not be mistaken with the notation used for the inner product in $L^2([0,1])$. A simple computation, based on (2.3.17) and the discussion after this equation, shows that $\mathbb{E}_{\mu_N}\Big[\big(M_t^N(G)\big)^2\Big]$ vanishes as $N\to\infty$. Now we look at the integral term in (2.3.1). A simple computation shows that

$$\begin{aligned}\int_0^t\Theta(N)L_N(\langle\pi_s^N,G\rangle)\,ds=&\frac{\Theta(N)}{N-1}\int_0^t\sum_{x\in\Lambda_N}\mathscr{L}_NG(\tfrac{x}{N})\eta_x^N(s)\,ds\\&+\frac{\kappa\Theta(N)}{(N-1)N^\theta}\int_0^t\sum_{x\in\Lambda_N}(Gr_N^-)(\tfrac{x}{N})(\alpha-\eta_x^N(s))\,ds\\&+\frac{\kappa\Theta(N)}{(N-1)N^\theta}\int_0^t\sum_{x\in\Lambda_N}(Gr_N^+)(\tfrac{x}{N})(\beta-\eta_x^N(s))\,ds,\end{aligned}\tag{2.3.2}$$

where for all $x\in\Lambda_N$

$$\begin{aligned}(\mathscr{L}_NG)(\tfrac{x}{N})&=\sum_{y\in\Lambda_N}p(y-x)\big[G(\tfrac{y}{N})-G(\tfrac{x}{N})\big],\\ r_N^-(\tfrac{x}{N})&=\sum_{y\geq x}p(y),\quad r_N^+(\tfrac{x}{N})=\sum_{y\leq x-N}p(y).\end{aligned}\tag{2.3.3}$$

Now, we want to extend the first sum in (2.3.2) to all the integers. For that purpose we extend the function G to $\mathbb{R}$ in such a way that it remains two times continuously differentiable.

By the definition of $\mathscr{L}_N$, we get that

$$\begin{aligned}\frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N} \mathscr{L}_N G(\tfrac{x}{N})\eta_x^N(s)\,ds &= \frac{\Theta(N)}{N-1}\int_0^t \sum_{x\in\Lambda_N} (K_N G)(\tfrac{x}{N})\eta_x^N(s)\,ds \\ &\quad -\frac{\Theta(N)}{N-1}\int_0^t \sum_{\substack{x\in\Lambda_N\\ y\leq 0}} \big[G(\tfrac{y}{N})-G(\tfrac{x}{N})\big]p(x-y)\eta_x^N(s)\,ds \\ &\quad -\frac{\Theta(N)}{N-1}\int_0^t \sum_{\substack{x\in\Lambda_N\\ y\geq N}} \big[G(\tfrac{y}{N})-G(\tfrac{x}{N})\big]p(x-y)\eta_x^N(s)\,ds,\end{aligned} \tag{2.3.4}$$

where

$$(K_N G)(\tfrac{x}{N}) = \sum_{y\in\mathbb{Z}} p(y-x)\big[G(\tfrac{y}{N})-G(\tfrac{x}{N})\big].$$

Now we state some required convergence.

Lemma 2.3.1. *Let $G:\mathbb{R}\to\mathbb{R}$ be a two times continuously differentiable function with compact support. We have*

$$\limsup_{N\to\infty}\sup_{x\in\Lambda_N}\left|N^2(K_N G)(\tfrac{x}{N})-\tfrac{\sigma^2}{2}\Delta G(\tfrac{x}{N})\right|=0.$$

Proof. See Appendix 4.2. □

Lemma 2.3.2. *Let $\gamma>0$ and $a\in(0,1)$. Then we have the following uniform convergence on $[a,1-a]$*

i) $\lim\limits_{N\to\infty} N^\gamma r_N^-(u)=r^-(u)$,

ii) $\lim\limits_{N\to\infty} N^\gamma r_N^+(u)=r^+(u)$.

Proof. See Appendix 4.3. □

Now, we are going to analyze all the terms in (2.3.4) and the boundary terms in (2.3.2) for the different regimes of θ. Thus, we will be able to see how the different boundary conditions appear on the hydrodynamic equations given in Subsection 2.2.2 from the underlying particle system.

2.3.1.1 The case $\theta<2-\gamma$

In this regime we take initially a function $G:(0,1)\to\mathbb{R}$ two times continuously differentiable and with compact support in $(0,1)$ (so that we can choose an extension by 0 outside of $(0,1)$).

Now we start by analyzing the first term on the right hand side of (2.3.4). Recall (2.2.6). Since $\Theta(N)=N^{\gamma+\theta}$, a simple computation, shows that the first term on the right hand side of

(2.3.4) vanishes for $\theta < 2-\gamma$. Indeed, by a Taylor expansion on G and the fact that p is mean zero, we have that

$$N^{\gamma+\theta}\sum_{y\in\mathbb{Z}}(G(\tfrac{y+x}{N})-G(\tfrac{x}{N}))p(y)$$

is of same order as

$$N^{\gamma+\theta-2}G''(\tfrac{x}{N})\sum_{y\in\mathbb{Z}}y^2p(y)$$

and since $\theta < 2-\gamma$ and p has finite variance last expression vanishes as $N\to\infty$.

Moreover, a simple computation shows that the second and third terms on the right hand side of (2.3.4) vanish as $N\to\infty$, since $\Theta(N)=N^{\gamma+\theta}$ and $\theta<2-\gamma$. Indeed we can bound from above, for example the second term in (2.3.4) by tN^{θ} times

$$\frac{1}{N-1}\sum_{x\in\Lambda_N}N^{\gamma}r_N^-(\tfrac{x}{N})\,|G(\tfrac{x}{N})|$$

because G vanishes outside $(0,1)$ and $|\eta_x^N(s)|\leq 1$ for all $s>0$. Since $\theta<0$ and that the previous sum converges to the (finite) integral of $|G|r^-$ on $(0,1)$, by Lemma 2.3.2, the previous display vanishes as $N\to\infty$. Now we look at the boundary terms in (2.3.2). The second term on the right hand side of (2.3.2) can be written, for the choice of $\Theta(N)=N^{\gamma+\theta}$, as:

$$\frac{\kappa N^{\gamma}}{N-1}\int_0^t\sum_{x\in\Lambda_N}G\big(\tfrac{x}{N}\big)r_N^-(\tfrac{x}{N})(\alpha-\eta_x^N(s))\,ds$$

which can be replaced, thanks to Lemma 2.3.2 and the fact that G has compact support, by

$$\kappa\int_0^t\langle\alpha-\pi_s^N,Gr^-\rangle\,ds\to\kappa\int_0^t\int_0^1 G(u)r^-(u)(\alpha-\rho_s(u))du\,ds$$

as $N\to\infty$. The last convergence holds because G has a compact support included in $(0,1)$ so that Gr^- is a continuous function. For the remaining term we can perform exactly the same analysis.

2.3.1.2 The case $\theta=2-\gamma$

In this case, and as above, we take initially a function $G:(0,1)\to\mathbb{R}$ two times continuously differentiable and with compact support in $(0,1)$ (so that we can choose a two times continuously differentiable extension which is 0 outside of $(0,1)$). In this case, since $\Theta(N)=N^2$, by Lemma 2.3.1, the first term on the right hand side of (2.3.4) can be replaced, for N sufficiently big, by

$$\frac{1}{N-1}\int_0^t\sum_{x\in\Lambda_N}\tfrac{\sigma^2}{2}\Delta G(\tfrac{x}{N})\,\eta_x^N(s)\,ds.$$

Moreover, a computation similar to the one in the previous case shows that the second and third terms on the right hand side of (2.3.4) vanish as $N \to \infty$ (recall that $\Theta(N) = N^2$ and $\gamma > 2$). Finally, the first term on the right hand side of (2.3.2) can be rewritten as

$$\frac{\kappa N^\gamma}{(N-1)} \int_0^t \sum_{x \in \Lambda_N} (Gr_N^-)(\tfrac{x}{N})(\alpha - \eta_x^N(s))\, ds$$

which can be replaced, thanks to Lemma 2.3.2 and the fact that G has compact support, by

$$\kappa \int_0^t \langle \alpha - \pi_s^N, Gr^- \rangle\, ds \to \kappa \int_0^t \int_0^1 G(u) r^-(u)(\alpha - \rho_s(u)) du\, ds$$

as $N \to \infty$ because Gr^- is a continuous function. The same computation can be done for the remaining term.

2.3.1.3 The case $\theta \in (2-\gamma, 1)$

In this case we take again a function $G : (0,1) \to \mathbb{R}$ two times continuously differentiable and with compact support in $(0,1)$ and extend it by 0 outside of $(0,1)$. As above, we can easily show that the last two terms on the right hand side of (2.3.2) vanish as $N \to \infty$, since we can transform each one of them into $N^{2+\gamma-\theta}$ times a converging integral, which vanishes since $\theta > 2-\gamma$. Analogously, the second and third terms on the right hand side of (2.3.4) also vanish because, for example, the second term on the right hand side of (2.3.4)

$$\frac{N^2}{N-1} \int_0^t \sum_{x \in \Lambda_N} G(\tfrac{x}{N}) r_N^-(\tfrac{x}{N}) \eta_x^N(s)\, ds$$

can be bounded from above by a constant times $tN^{2-\gamma}$ times a sum converging to the integral of $|G|r^-$ on $(0,1)$. The estimate of the third term is analogous. Therefore since $\gamma > 2$, both vanish as $N \to \infty$.

Remark 2.3.3. *Observe that in the three previous cases, we imposed to G to have a compact support included in $(0,1)$. This was used in order to extend smoothly the function G by 0 outside of $(0,1)$ (the condition $G(0) = G(1) = 0$ would not have been sufficient) and this was fundamental to ensure that the functions Gr^-, Gr^+ do not have singularities at the boundaries. On the other hand, in the two next cases, it will be fundamental to consider test functions $G : [0,1] \to \mathbb{R}$ which are not necessarily 0 at the boundaries in order to "see" the boundaries in the weak formulation.*

2.3.1.4 The case $\theta = 1$

In this case we consider an arbitrary function $G : [0,1] \to \mathbb{R}$ which is two times continuously differentiable and we extend it on $\mathbb{R}$ in a two times continuously differentiable function with compact support. Its support strictly (a priori) contains $[0,1]$ since G can take non-zero values at 0 and 1. We start by looking at the terms coming from the boundary, namely the two last terms on the right hand side of (2.3.2). Then, in the second term on the right hand side of (2.3.2) (resp. the third term) we perform at first a Taylor expansion on G and then we replace η_x by the average $\overrightarrow{\eta}_0^{\varepsilon N}$ (resp. η_x by $\overleftarrow{\eta}_N^{\varepsilon N}$) defined in (2.3.31), which can be done as a consequence of Lemma 2.3.12 as pointed out in Remark 2.3.13. Moreover, note that

$$\sum_{x\in\Lambda_N} r_N^-(\tfrac{x}{N}) \xrightarrow[N\to\infty]{} \sum_{y\geq 1} yp(y) = m, \qquad \sum_{x\in\Lambda_N} r_N^+(\tfrac{x}{N}) \xrightarrow[N\to\infty]{} \sum_{y\geq 1} yp(y) = m. \tag{2.3.5}$$

Therefore, we can write the last two terms in (2.3.2) as

$$m\kappa \int_0^t \{(\alpha - \overleftarrow{\eta}_0^{\varepsilon N}(sN^2))G(0) + (\beta - \overrightarrow{\eta}_N^{\varepsilon N}(sN^2))G(1)\}\, ds,$$

plus lower-orders terms (with respect to N). Since (in some sense that we will see in the proof of Proposition 2.3.18),

$$\overrightarrow{\eta}_0^{\varepsilon N}(sN^2) \xrightarrow[N\to\infty]{} \rho_s(0), \quad \overleftarrow{\eta}_N^{\varepsilon N}(sN^2) \xrightarrow[N\to\infty]{} \rho_s(1)$$

last term writes as

$$m\kappa \int_0^t \{(\alpha - \rho_s(0))G(0) + (\beta - \rho_s(1))G(1)\}\, ds. \tag{2.3.6}$$

Now we look at the remaining terms, namely, the two last terms on the right hand side of (2.3.4). Recall that the function G has been extended into a two times continuously differentiable function on $\mathbb{R}$. By a Taylor expansion on G we can write those terms as

$$\frac{N}{N-1}\sum_{x\in\Lambda_N} G'(\tfrac{x}{N})\Theta_x^-\eta_x^N(s) - \frac{N}{N-1}\sum_{x\in\Lambda_N} G'(\tfrac{x}{N})\Theta_x^+\eta_x^N(s) \tag{2.3.7}$$

plus lower-order terms (with respect to N). Above for $x \in \Lambda_N$,

$$\Theta_x^- = \sum_{y\leq 0}(x-y)p(x-y) \quad \text{and} \quad \Theta_x^+ = \sum_{y\geq N}(y-x)p(x-y).$$

Note that

$$\sum_{x\in\Lambda_N} \Theta_x^\pm \lesssim 1 \quad \text{and} \quad \frac{1}{N}\sum_{x\in\Lambda_N} x\Theta_x^\pm \xrightarrow[N\to\infty]{} 0. \tag{2.3.8}$$

Moreover, note that

$$\sum_{x\in\Lambda_N}\Theta_x^- = \sum_{x\in\Lambda_N}\sum_{y\geq x} yp(y) \xrightarrow[N\to\infty]{} \tfrac{\sigma^2}{2}, \quad \sum_{x\in\Lambda_N}\Theta_x^+ = \sum_{x\in\Lambda_N}\sum_{y\geq N-x} yp(y) \xrightarrow[N\to\infty]{} \tfrac{\sigma^2}{2}. \tag{2.3.9}$$

In order to prove the convergence of $\sum_{x\in\Lambda_N}\Theta_x^-$ (or of $\sum_{x\in\Lambda_N}\Theta_x^+$ in (2.3.9)) we use Fubini's theorem to get that

$$\sum_{x\in\Lambda_N}\Theta_x^- = \sum_{y\in\Lambda_N}\sum_{x=1}^{y} yp(y) + \sum_{y\geq N}\sum_{x\in\Lambda_N} yp(y) = \sum_{y\in\Lambda_N} y^2p(y) + (N-1)\sum_{y\geq N} yp(y),$$

and since $\gamma > 2$ the result follows. By another Taylor expansion on G we can write (2.3.7) as

$$\frac{N}{N-1}G'(0)\sum_{x\in\Lambda_N}\Theta_x^-\eta_x^N(s)-\frac{N}{N-1}G'(1)\sum_{x\in\Lambda_N}\Theta_x^+\eta_x^N(s) \tag{2.3.10}$$

plus lower-order terms (with respect to N). Thanks to Lemma 2.3.12 we can replace in the term on the left (resp. right) hand side of last expression $\eta_x(sN^2)$ by $\overrightarrow{\eta}_0^{\varepsilon N}(sN^2)$ (resp. $\overleftarrow{\eta}_N^{\varepsilon N}(sN^2)$). Therefore, (2.3.10) can be replaced, for N sufficiently big and then ε sufficiently small, by

$$G'(0)\tfrac{\sigma^2}{2}\overrightarrow{\eta}_0^{\varepsilon N}(sN^2)-G'(1)\tfrac{\sigma^2}{2}\overleftarrow{\eta}_N^{\varepsilon N}(sN^2).$$

Now, since (in some sense that we will see in the proof of Proposition 2.3.18) we have that $\overrightarrow{\eta}_0^{\varepsilon N}(sN^2)\xrightarrow[N\to\infty]{}\rho_s(0)$ and $\overleftarrow{\eta}_N^{\varepsilon N}(sN^2)\xrightarrow[N\to\infty]{}\rho_s(1)$, last term tends to

$$G'(0)\tfrac{\sigma^2}{2}\rho_s(0)-G'(1)\tfrac{\sigma^2}{2}\rho_s(1). \tag{2.3.11}$$

Putting together (2.3.6) and (2.3.11) we see the boundary terms that appear at the right hand side of (2.2.3).

2.3.1.5 The case $\theta \in (1,\infty)$

In this case we consider an arbitrary function $G:[0,1]\to\mathbb{R}$ which is two times continuously differentiable and we extend it on $\mathbb{R}$ in a two times continuously differentiable function with compact support. Its support may strictly contain $[0,1]$ since G can take non-zero values at 0 and 1. The last two terms on the right hand side of (2.3.2) vanish, as $N\to\infty$ since, we can bound, for example, the first term on the right hand side of (2.3.2) by a constant times

$$N^{1-\theta}\sum_{x\in\Lambda_N}r_N^-(\tfrac{x}{N}).$$

Since $\gamma > 2$ last expression vanishes if $\theta > 1$. Thus, we only need to look at (2.3.4). Therefore, in order to see the boundaries terms that appear in (2.2.3), we can use exactly the computations already done in the case $\theta = 1$ from which we obtain (2.3.11).

2.3.2 Tightness

In this section we prove that the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$, defined in Section 2.2.3, is tight.

Proposition 2.3.4. *The sequence of measures $\{\mathbb{Q}_N\}_{N\geq1}$ is tight with respect to the Skorohod topology of $\mathcal{D}^T_{\mathcal{M}^+}$.*

Proof. In order to prove the assertion see, for example, Proposition 1.6 of Chapter 4 in [40], it is enough to show that, for all $\varepsilon > 0$

$$\lim_{\delta\to0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\bar\tau\leq\delta}\mathbb{P}_{\mu_N}\Big(\eta^N(\cdot)\in\mathcal{D}^T_{\Omega_N}:\big|\langle\pi^N_{\tau+\bar\tau},G\rangle-\langle\pi^N_\tau,G\rangle\big|>\varepsilon\Big)=0, \tag{2.3.12}$$

holds for any function G belonging to $C([0,1])$. Here $\mathscr{T}_T$ is the set of stopping times bounded by T and we implicitly assume that all the stopping times are bounded by T, thus, $\tau+\bar{\tau}$ should be read as $(\tau+\bar{\tau})\wedge T$. In fact it is enough to prove the assertion for functions G in a dense subset of $C([0,1])$, with respect to the uniform topology.

We split the proof according to two different regimes of θ, namely $\theta \geq 1$ and $\theta < 1$. When $\theta \geq 1$ we prove (2.3.12) directly for functions $G \in C^2([0,1])$ and we conclude that the sequence is tight. When $\theta < 1$, we prove (2.3.12) first for functions $G \in C_c^2((0,1))$ and then we extend it, by a $L^1([0,1])$ approximation procedure which is explained below, to functions $G \in C^1([0,1])$, the latter space being dense in $C([0,1])$ for the uniform topology.

Recall from (2.3.1) that $M_t^N(G)$ is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. In order to prove (2.3.12) it is enough to show that

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar{\tau}\leq\delta}\mathbb{E}_{\mu_N}\Big[\Big|\int_\tau^{\tau+\bar{\tau}}\Theta(N)L_N\langle\pi_s^N,G\rangle ds\Big|\Big]=0 \tag{2.3.13}$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar{\tau}\leq\delta}\mathbb{E}_{\mu_N}\Big[\big(M_\tau^N(G)-M_{\tau+\bar{\tau}}^N(G)\big)^2\Big]=0. \tag{2.3.14}$$

<u>Proof of (2.3.13)</u>: Given a function G, we claim that

$$|\Theta(N)L_N(\langle\pi_s^N,G\rangle)|\lesssim 1 \tag{2.3.15}$$

for any $s \leq T$, which trivially implies (2.3.13). To prove it, we recall (2.3.2) and start to prove that the last two terms of (2.3.2) are bounded. For example, the absolute value of the second term at the right hand side of (2.3.2) is bounded from above by

$$\int_0^t\Big|\frac{\Theta(N)\kappa}{(N-1)N^\theta}\sum_{x\in\Lambda_N}(Gr_N^-)(\tfrac{x}{N})(\alpha-\eta_x^N(s))\Big|ds. \tag{2.3.16}$$

For $\theta < 1$, we use the fact that $G \in C_c^2((0,1))$ and that $|\eta_x^N(s)| \leq 1$ is bounded, and we bound from above this last term by a constant times $\Theta(N)N^{-\theta-\gamma}$. Using the definition of $\Theta(N)$ it is easy to see, for $\theta < 2-\gamma$ and for $2-\gamma \leq \theta < 1$, that (2.3.16) is bounded from above by a constant. This proves (2.3.15) in the case $\theta < 1$. In the case $\theta \geq 1$, we use the fact that the sum in (2.3.16) is uniformly bounded in N to conclude that (2.3.16) is bounded from above even if G does not have a compact support included in $(0,1)$. A similar argument can be done for the last term at the right hand side of (2.3.2).

Now we need to bound the first term at the right hand side of (2.3.2). For $\theta < 1$ we use the fact that $G \in C_c^2((0,1))$ so that $\Big|\frac{\Theta(N)}{N-1}\langle\pi_s^N,\mathscr{L}_N G\rangle\Big|$ is less or equal than

$$\frac{\Theta(N)}{N-1}\sum_{x\in\Lambda_N}|(K_NG)(\tfrac{x}{N})|+\frac{\Theta(N)}{N-1}\sum_{x\in\Lambda_N}|G(\tfrac{x}{N})|r_N^-\Big(\tfrac{x}{N}\Big)+\frac{\Theta(N)}{N-1}\sum_{x\in\Lambda_N}|G(\tfrac{x}{N})|r_N^+\Big(\tfrac{x}{N}\Big).$$

The two terms at the right hand side of the previous expression can be bounded from above by a constant times $\Theta(N)N^{-\gamma}$. It is clearly bounded in the case $\theta \geq 2-\gamma$ since then $\Theta(N) = N^2$ (recall $\gamma > 2$). In the case $\theta < 2-\gamma$, $\Theta(N) = N^{\theta+\gamma}$ and thus $\Theta(N)N^{-\gamma}$ is bounded. This together with Lemma 2.3.1 shows that

$$\left|\frac{\Theta(N)}{N-1}\langle \pi_s^N, \mathscr{L}_N G\rangle\right| \lesssim 1,$$

which proves the claim (2.3.15) in the case $\theta < 1$. Now, in the case $\theta \geq 1$, since $\Theta(N) = N^2$, we have that the first term at the right hand side of (2.3.2) is bounded from above by a constant times

$$\frac{N^2}{N-1}\sum_{x\in\Lambda_N} |K_N G(\tfrac{x}{N})| + \frac{N^2}{N-1}\sum_{x\in\Lambda_N}\sum_{y\leq 0}\left|G(\tfrac{y}{N}) - G(\tfrac{x}{N})\right| p(x-y)$$
$$+\frac{N^2}{N-1}\sum_{x\in\Lambda_N}\sum_{y\geq N}\left|G(\tfrac{y}{N}) - G(\tfrac{x}{N})\right| p(x-y).$$

By the Mean Value Theorem, the last two terms of the previous expression can be bounded from above by

$$\|G'\|_\infty \sum_{x\in\Lambda_N}\sum_{y\leq 0}|y-x|p(x-y) \lesssim \sum_{x\in\Lambda_N}\frac{1}{x^{\gamma-1}}$$

which is finite since $\gamma > 2$. This together with Lemma 2.3.1 proves (2.3.15) in the case $\theta \geq 1$.

Proof of (2.3.14): We know by Dynkin's formula that

$$\left(M_t^N(G)\right)^2 - \int_0^t \Theta(N)\left[L_N\langle \pi_s^N, G\rangle^2 - 2\langle \pi_s^N, G\rangle L_N\langle \pi_s^N, G\rangle\right] ds,$$

is a martingale with respect to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$. From the computations of Appendix 4.1 we get that the term inside the time integral in the previous display is equal to

$$\frac{\Theta(N)}{(N-1)^2}\sum_{x<y\in\Lambda_N}\left(G\left(\tfrac{x}{N}\right) - G\left(\tfrac{y}{N}\right)\right)^2 p(x-y)(\eta_y^N(s) - \eta_x^N(s))^2$$
$$+\quad \frac{\Theta(N)\kappa}{N^\theta(N-1)^2}\sum_{x\in\Lambda_N}(G(\tfrac{x}{N}))^2 r_N^-(\tfrac{x}{N})(\alpha - \eta_x^N(s))(1-2\eta_x^N(s))$$
$$+\quad \frac{\Theta(N)\kappa}{N^\theta(N-1)^2}\sum_{x\in\Lambda_N}(G(\tfrac{x}{N}))^2 r_N^+(\tfrac{x}{N})(\beta - \eta_x^N(s))(1-2\eta_x^N(s)).$$

Since $\Theta(N) \leq N^2$ and G' is bounded it is easy to see that the absolute value of the previous display is bounded from above by a constant times

$$\frac{1}{(N-1)^2}\sum_{x,y\in\Lambda_N}(x-y)^2 p(x-y) + \frac{\Theta(N)}{N^\theta(N-1)^2}\sum_{x\in\Lambda_N}(G(\tfrac{x}{N}))^2\Big(r_N^-(\tfrac{x}{N}) + r_N^+(\tfrac{x}{N})\Big). \tag{2.3.17}$$

Since $\sum_{x,y\in\Lambda_N}(x-y)^2p(x-y)=O(N)$ the first term in (2.3.17) is $O(N^{-1})$. For the second term at the right hand side of (2.3.17), we split the argument according to the cases $\theta\ge 1$ and $\theta<1$. First when $\theta\ge 1$, by using the fact that $\gamma>2$ and G is bounded so that the sum in that term is finite, and since $\Theta(N)=N^2$, we conclude that the term is $O(N^{-\theta})\le O(N^{-1})$. From this we obtain (2.3.14). Now if $\theta<1$, recall that G has compact support and Lemma 2.3.2. We then write

$$\frac{\Theta(N)}{N^\theta(N-1)^2}\sum_{x\in\Lambda_N}(G(\tfrac{x}{N}))^2\Big(r_N^-(\tfrac{x}{N})+r_N^+(\tfrac{x}{N})\Big)=\frac{\Theta(N)}{N^{\theta+\gamma}(N-1)}I_N(G)$$

where $I_N(G)$ is a Riemann sum converging to $\int_0^1(G(u))^2\big[r^-(u)+r^+(u)\big]du<\infty$. Therefore the second term in (2.3.17) is of order $\Theta(N)N^{-1-\theta-\gamma}=O(N^{-1})$ by (2.2.6). This ends the proof of tightness in the case $\theta\ge 1$, since $C^2([0,1])$ is a dense subset of $C([0,1])$ with respect to the uniform topology.

Nevertheless, for $\theta<1$, we have proved (2.3.13) and (2.3.14), and thus (2.3.12), only for functions $G\in C_c^2((0,1))$ and we need to extend this result to functions in $C^1([0,1])$. To accomplish that, we take a function $G\in C^1([0,1])\subset L^1([0,1])$, and we take a sequence of functions $\{G_k\}_{k\ge 0}\in C_c^2((0,1))$ converging to G with respect to the L^1-norm as $k\to\infty$. Now, since the probability in (2.3.12) is less or equal than

$$\begin{aligned}&\mathbb{P}_{\mu_N}\Big(\eta_\cdot^N\in\mathscr{D}^T_{\Omega_N}:\big|\langle\pi^N_{\tau+\bar\tau},G_k\rangle-\langle\pi^N_\tau,G_k\rangle\big|>\frac{\varepsilon}{2}\Big)\\ +\;&\mathbb{P}_{\mu_N}\Big(\eta_\cdot^N\in\mathscr{D}^T_{\Omega_N}:\big|\langle\pi^N_{\tau+\bar\tau},G-G_k\rangle-\langle\pi^N_\tau,G-G_k\rangle\big|>\frac{\varepsilon}{2}\Big)\end{aligned}$$

and since G_k has compact support, from the computation above, it remains only to check that the last probability vanishes as $N\to\infty$ and then $k\to\infty$. For that purpose, we use the fact that

$$\big|\langle\pi^N_{\tau+\bar\tau},G-G_k\rangle-\langle\pi^N_\tau,G-G_k\rangle\big|\le\frac{2}{N}\sum_{x\in\Lambda_N}\big|(G-G_k)(\tfrac{x}{N})\big|,$$

and we use the estimate

$$\begin{aligned}\frac{1}{N}\sum_{x\in\Lambda_N}\big|(G-G_k)(\tfrac{x}{N})\big|&\le\sum_{x\in\Lambda_N}\int_{x/N}^{(x+1)/N}\big|(G-G_k)(\tfrac{x}{N})-(G-G_k)(u)\big|\,du+\int_0^1|(G-G_k)(u)|du\\&\le\frac{1}{N}\|(G-G_k)'\|_\infty+\int_0^1|(G-G_k)(u)|du.\end{aligned}$$

We conclude the result by taking first the limsup in $N\to\infty$ and then in $k\to\infty$. □

2.3.3 Replacement lemmas and auxiliary results

In this section we establish some technical results needed in the proof of the hydrodynamic limit. In what follows, we will suppose without loss of generality that $\alpha\le\beta$. Let $h:[0,1]\to$

$[0,1]$ be a Lipschitz function such that $\alpha \le h(u) \le \beta$, for all $u \in [0,1]$. Let ν_h^N be the Bernoulli product measure on Ω_N with marginals given by

$$\nu_h^N\{\eta : \eta_x = 1\} = h\left(\tfrac{x}{N}\right). \tag{2.3.18}$$

Given two functions $f, g : \Omega_N \to \mathbb{R}$ and a probability measure μ on Ω_N, we denote here by $\langle f, g\rangle_\mu$ the scalar product between f and g in $L^2(\Omega_N, \mu)$, that is,

$$\langle f, g\rangle_\mu = \int f(\eta)g(\eta)\, d\mu.$$

The notation above should note be mistaken to the notation that we introduced in Subsection 2.2.1. We denote by $H_N(\mu|\nu_h^N)$ the relative entropy of a probability measure μ on Ω_N with respect to the probability measure ν_h^N on Ω_N. It is easy to prove the existence of a constant $C_0 := C_0(\alpha, \beta)$, such that

$$H_N(\mu|\nu_h^N) \le NC_0. \tag{2.3.19}$$

In fact, using the explicit formula for the entropy and the definition of the product measure ν_h^N, we get that

$$\begin{aligned} H(\mu|\nu_h^N) &= \sum_{\eta\in\Omega_N} \mu(\eta)\log\left(\frac{\mu(\eta)}{\nu_h^N(\eta)}\right) \le \sum_{\eta\in\Omega_N} \mu(\eta)\log\left(\frac{1}{\nu_h^N(\eta)}\right) \\ &\le \log\left(\left[\frac{1}{\alpha\wedge(1-\beta)}\right]^N\right)\sum_{\eta\in\Omega_N}\mu(\eta) \le N\log\left(\frac{1}{\alpha\wedge(1-\beta)}\right) \le NC_0. \end{aligned}$$

Remark 2.3.5. *We note that above we use the fact that $\alpha \neq 0$ and $\beta \neq 1$ since in last estimate the constant $C_0 = -\log(\alpha\wedge(1-\beta))$.*

2.3.4 Estimates on Dirichlet forms

For a probability measure μ on Ω_N, $x, y \in \Lambda_N$ and a density function $f : \Omega_N \to [0,\infty)$ with respect to μ we introduce

$$\begin{aligned} I_{x,y}(\sqrt{f}, \mu) &:= \int \left(\sqrt{f(\sigma^{x,y}\eta)} - \sqrt{f(\eta)}\right)^2 d\mu, \\ I_x^\alpha(\sqrt{f}, \mu) &:= \int c_x(\eta;\alpha)\left(\sqrt{f(\sigma^x\eta)} - \sqrt{f(\eta)}\right)^2 d\mu. \end{aligned}$$

Then we define

$$D_N(\sqrt{f}, \mu) := (D_N^0 + D_N^\ell + D_N^r)(\sqrt{f}, \mu)$$

where

$$D_N^0(\sqrt{f}, \mu) := \frac{1}{2}\sum_{x,y\in\Lambda_N} p(y-x)\, I_{x,y}(\sqrt{f}, \mu),$$

$$D_N^\ell(\sqrt{f},\mu) := \frac{\kappa}{N^\theta}\sum_{\substack{x\in\Lambda_N\\ y\leq 0}} p(y-x)\, I_x^\alpha(\sqrt{f},\mu) = \frac{\kappa}{N^\theta}\sum_{x\in\Lambda_N} r_N^-(\tfrac{x}{N}) I_x^\alpha(\sqrt{f},\mu)$$

and $D_N^r(\sqrt{f},\mu)$ is the same as $D_N^\ell(\sqrt{f},\mu)$ but in $I_x^\alpha(\sqrt{f},\mu)$ the parameter α is replaced by β and r_N^- is replaced by r_N^+.

Our first goal is to express, for the measure ν_h^N, a relation between the Dirichlet form defined by $\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}$ and $D_N(\sqrt{f},\nu_h^N)$. More precisely, we claim that for any positive constant B, there exists a constant $C>0$ such that

$$\begin{aligned}\frac{1}{BN}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N} \leq & -\frac{1}{4BN}D_N(\sqrt{f},\nu_h^N) + \frac{C}{BN}\sum_{x,y\in\Lambda_N} p(y-x)\Big(h(\tfrac{x}{N})-h(\tfrac{y}{N})\Big)^2\\ &+\frac{C\kappa}{BN^{1+\theta}}\sum_{x\in\Lambda_N}\Big\{\Big(h(\tfrac{x}{N})-\alpha)^2 r_N^-(\tfrac{x}{N}) + \Big(h(\tfrac{x}{N})-\beta\Big)^2 r_N^+(\tfrac{x}{N})\Big\}.\end{aligned} \tag{2.3.20}$$

Our aim is then to choose h in order to minimize the error term, i.e. the two last terms at the right hand side of the previous inequality.

If h is such that $h(0)=\alpha$ and $h(1)=\beta$, since it is assumed to be Lipschitz, we get the estimate

$$\frac{N}{B}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N} \leq -\frac{N}{4B}D_N(\sqrt{f},\nu_h^N) + \frac{C}{B}\sigma^2 + \frac{C\kappa}{BN^{1+\theta}}\sum_{x\in\Lambda_N}\Big\{x^2 r_N^-(\tfrac{x}{N}) + \big(x-N\big)^2 r_N^+(\tfrac{x}{N})\Big\}. \tag{2.3.21}$$

Moreover, if the function h is such that $h(0)=\alpha$ and $h(1)=\beta$, Hölder of parameter $\gamma/2$ at the boundaries and Lipschitz inside, then we have

$$\frac{N}{B}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N} \leq -\frac{N}{4B}D_N(\sqrt{f},\nu_h^N) + \frac{C}{B}\sigma^2 + \frac{C\kappa}{BN^{\gamma+\theta-2}}. \tag{2.3.22}$$

On the other hand if the function h is constant, equal to α (or to β), then we have

$$\frac{N}{B}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_\alpha} \leq -\frac{N}{4B}D_N(\sqrt{f},\nu_\alpha) + \frac{C\kappa}{B}N^{1-\theta}. \tag{2.3.23}$$

In order to prove (2.3.20) we need some intermediate results. In what follows C is a constant depending on α and β whose value can change from line to line.

Lemma 2.3.6. *Let $T:\eta\in\Omega_N\to T(\eta)\in\Omega_N$ be a transformation and $c:\eta\to c(\eta)$ be a positive local function. Let f be a density with respect to a probability measure μ on Ω_N. Then, we have that*

$$\begin{aligned}&\Big\langle c(\eta)[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}]\,,\,\sqrt{f(\eta)}\Big\rangle_\mu\\ &\leq -\frac{1}{4}\int c(\eta)\Big(\Big[\sqrt{f(T(\eta))}\Big]-\Big[\sqrt{f(\eta)}\Big]\Big)^2 d\mu\\ &+\frac{1}{16}\int\frac{1}{c(\eta)}\Big[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\Big]^2\Big(\Big[\sqrt{f(T(\eta))}\Big]+\Big[\sqrt{f(\eta)}\Big]\Big)^2 d\mu.\end{aligned} \tag{2.3.24}$$

Proof. By writing the term at the left hand side of (2.3.24) as its half plus its half and summing and subtracting the term needed to complete the square as written in the first term at the right hand side of (2.3.24), we have that

$$\begin{aligned}
&\int c(\eta)\Big[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}\Big]\sqrt{f(\eta)}\,d\mu\\
&=-\frac{1}{2}\int c(\eta)\Big[\sqrt{f(T(\eta))}-\sqrt{f(\eta)}\Big]^2\,d\mu\\
&+\frac{1}{2}\int\Big[\sqrt{f(T(\eta))}\Big]^2\Big[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\Big]\,d\mu.
\end{aligned}$$

Repeating again the same argument, the second term at the right hand side of last expression can be written as

$$\frac{1}{4}\int\Big(\Big[\sqrt{f(T(\eta))}\Big]^2-\Big[\sqrt{f(\eta)}\Big]^2\Big)\Big[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\Big]d\mu.$$

By Young's inequality and the elementary equality $a^2-b^2=(a-b)(a+b)$, last expression is bounded from above by

$$\begin{aligned}
&\frac{1}{4}\int c(\eta)\Big(\Big[\sqrt{f(T(\eta))}\Big]-\Big[\sqrt{f(\eta)}\Big]\Big)^2\,d\mu\\
+\;&\frac{1}{16}\int\frac{1}{c(\eta)}\Big[c(\eta)-c(T(\eta))\frac{\mu(T(\eta))}{\mu(\eta)}\Big]^2\Big(\Big[\sqrt{f(T(\eta))}\Big]+\Big[\sqrt{f(\eta)}\Big]\Big)^2\,d\mu,
\end{aligned}$$

which finishes the proof. □

Lemma 2.3.7. *There exists a constant $C:=C(h)$ such that for any $N\geq 1$ and any density f with respect to ν_h^N*

$$\sup_{x\neq y\in\Lambda_N}\int f(\sigma^{x,y}\eta)\,d\nu_h^N(\eta)\;\leq\;C,\qquad\sup_{x\in\Lambda_N}\int f(\sigma^{x}\eta)\,d\nu_h^N(\eta)\;\leq\;C.$$

Proof. Let us prove only the left hand side bound since the proof of the second one is similar. We perform in the left hand side integral above the change of variables $\omega=\sigma^{x,y}\eta$ and we use that, uniformly in $x,y\in\Lambda_N$ and ω, we have

$$\theta^{x,y}(\omega)=\frac{\nu_h^N(\sigma^{x,y}\omega)}{\nu_h^N(\omega)}=1+O(\tfrac{1}{N}).\tag{2.3.25}$$

By using the fact that f is a density it is easy to conclude. □

Now, let us look at some consequences of these lemmas. We start with the bulk generator L_N^0 given in (1.3.3).

Corollary 2.3.8. *There exists a constant $C>0$ (independent of f and N) such that*

$$\Big\langle L_N^0\sqrt{f},\sqrt{f}\Big\rangle_{\nu_h^N} \;\le\; -\frac{1}{4}D_N^0(\sqrt{f},\nu_h^N)+C\sum_{x,y\in\Lambda_N}p(y-x)\Big(h(\tfrac{x}{N})-h(\tfrac{y}{N})\Big)^2$$

for any density f with respect to ν_h^N.

Proof. To prove this we note that

$$\Big\langle L_N^0\sqrt{f},\sqrt{f}\Big\rangle_{\nu_h^N}=\frac{1}{2}\sum_{x,y\in\Lambda_N}p(y-x)\,\Big\langle\Big[\sqrt{f(\sigma^{x,y}\eta)}-\sqrt{f(\eta)}\Big],\,\sqrt{f(\eta)}\Big\rangle_{\nu_h^N}.$$

Now, by Lemma 2.3.6 with $c\equiv 1$, $T=\sigma^{x,y}$, and Lemma 2.3.7, last expression is bounded from above by

$$-\frac{1}{4}D_N^0(\sqrt{f},\nu_h^N)+C\sum_{x,y\in\Lambda_N}p(y-x)\Big(h\big(\tfrac{x}{N}\big)-h\big(\tfrac{y}{N}\big)\Big)^2,$$

because $|\theta^{x,y}(\eta)-1|^2\lesssim(h(\frac{x}{N})-h(\frac{y}{N}))^2$. □

Now we look at the generators of the reservoirs given in (1.3.4).

Corollary 2.3.9. *Let $\theta\in\mathbb{R}$ be fixed. There exists a constant $C>0$ (independent of f and N) such that*

$$\langle L_N^\ell\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}\le-\frac{1}{4}D_N^\ell(\sqrt{f},\nu_h^N)+\frac{C\kappa}{N^\theta}\sum_{x\in\Lambda_N}r_N^-(\tfrac{x}{N})\Big(h(\tfrac{x}{N})-\alpha\Big)^2,$$
$$\langle L_N^r\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}\le-\frac{1}{4}D_N^r(\sqrt{f},\nu_h^N)+\frac{C\kappa}{N^\theta}\sum_{x\in\Lambda_N}r_N^+(\tfrac{x}{N})\Big(h(\tfrac{x}{N})-\beta\Big)^2$$

for any density f with respect to ν_h^N.

Proof. We present the proof for the first inequality but we note that the proof of the second one is analogous. First observe that $\big\langle L_N^\ell\sqrt{f},\sqrt{f}\big\rangle_{\nu_h^N}$ is equal to

$$\frac{\kappa}{N^\theta}\sum_{\substack{x\in\Lambda_N\\ y\le 0}}p(y-x)\Big\langle c_x(\eta;\alpha)\Big[\sqrt{f(\sigma^x\eta)}-\sqrt{f(\eta)}\Big],\,\sqrt{f(\eta)}\Big\rangle_{\nu_h^N}.$$

Now, by using Lemma 2.3.6 with $c(\eta)=c_x(\eta;\alpha)$, $T=\sigma^x$ and Lemma 2.3.7, last expression is bounded from above by

$$-\frac{1}{4}D_N^\ell(\sqrt{f},\nu_h^N)+\frac{C\kappa}{N^\theta}\sum_{\substack{x\in\Lambda_N\\ y\le 0}}p(y-x)\Big(h(\tfrac{x}{N})-\alpha\Big)^2.$$

□

From the two previous corollaries the claim (2.3.20) follows.

2.3.5 Replacement Lemmas

Lemma 2.3.10. *For any density f with respect to ν_h^N, any $x \in \Lambda_N$ and any positive constant A_x, we have that*

$$\left|\left\langle t_x^\alpha, f\right\rangle_{\nu_h^N}\right| \lesssim \frac{1}{A_x} I_x^\alpha(\sqrt{f}, \nu_h^N) + A_x + |h(\tfrac{x}{N}) - \alpha|,$$

where $t_x^\alpha(\eta) = \eta_x - \alpha$. The same result holds if α is replaced by β.

Proof. By a simple computation we have that:

$$\left|\left\langle t_x^\alpha, f\right\rangle_{\nu_h^N}\right| \leq \frac{1}{2}\left|\int t_x^\alpha(\eta)(f(\eta) - f(\sigma^x\eta))\, d\nu_h^N\right| + \frac{1}{2}\left|\int [f(\sigma^x\eta) + f(\eta)] t_x^\alpha(\eta)\, d\nu_h^N\right|,$$

where σ^x is the flip given in (1.2.2). By Young's inequality, using the fact that $(a-b) = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$ for all $a, b \geq 0$ and Lemma 2.3.7, the first term at the right side of (2.3.26) is bounded from above, for any positive constant A_x, by

$$\frac{A_x}{4}\int \frac{(t_x^\alpha(\eta))^2}{c_x(\eta;\alpha)}\Big(\Big[\sqrt{f(\sigma^x\eta)}\Big] + \Big[\sqrt{f(\eta)}\Big]\Big)^2 d\nu_h^N + \frac{I_x^\alpha(\sqrt{f}, \nu_h^N)}{4A_x} \lesssim A_x + \frac{I_x^\alpha(\sqrt{f}, \nu_h^N)}{A_x}.$$

Now, we look at the second term on the right hand side of (2.3.26). By using the fact that ν_h^N is product and denoting by $\bar{\eta}$ the configuration η removing its value at x so that $(\eta_x, \bar{\eta}) = \eta$, we have that the second term at the right side of (2.3.26) is equal to

$$\begin{aligned}
&\frac{1}{2}\Big|\sum_{\bar{\eta}}\Big((1-\alpha)(f(1,\bar{\eta}) + f(0,\bar{\eta}))\nu_h^N(\eta_x = 1) - \alpha(f(0,\bar{\eta}) + f(1,\bar{\eta}))\nu_h^N(\eta_x = 0)\Big)\, \nu_h^N(\bar{\eta})\Big| \\
=&\frac{1}{2}\Big|\sum_{\bar{\eta}}\Big(h(\tfrac{x}{N}) - \alpha\Big)(f(0,\bar{\eta}) + f(1,\bar{\eta}))\nu_h^N(\bar{\eta})\Big| \\
\lesssim& |h(\tfrac{x}{N}) - \alpha| \sum_{\bar{\eta}} h(\tfrac{x}{N}) f(1,\bar{\eta})\nu_h^N(\bar{\eta}) + \Big(1 - h(\tfrac{x}{N})\Big) f(0,\bar{\eta})\nu_h^N(\bar{\eta}) \\
=&|h(\tfrac{x}{N}) - \alpha| \sum_{\eta \in \Omega_N} f(\eta)\nu_h^N(\eta) = \Big(h(\tfrac{x}{N}) - \alpha\Big)
\end{aligned}$$

because $\max_{x \in \Lambda_N}\left\{\frac{1}{2h(\frac{x}{N})}, \frac{1}{2\left(1 - h(\frac{x}{N})\right)}\right\}$ is bounded from above by a constant depending only on α and β. Above $f(1,\bar{\eta})$ (resp. $f(0,\bar{\eta})$) means that we are computing $f(\eta)$ with $\eta_x = 1$ (resp. $\eta_x = 0$). □

Lemma 2.3.11. *Let $\theta > 1$. For any $t > 0$, we have that*

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t N^{1-\theta} \sum_{x \in \Lambda_N} Gr_N^-(\tfrac{x}{N})(\eta_x^N(s) - \alpha)\, ds\right|\right] = 0, \\
&\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\left[\left|\int_0^t N^{1-\theta} \sum_{x \in \Lambda_N} Gr_N^+(\tfrac{x}{N})(\eta_x^N(s) - \beta)\, ds\right|\right] = 0,
\end{aligned} \tag{2.3.26}$$

for any bounded function $G : \mathbb{R} \to \mathbb{R}$.

Proof. We present the proof for the first term, but we note that the proof for the second term is completely analogous.

We start by fixing a Lipschitz profile h such that $h(0)=\alpha\leq h(u)\leq\beta=h(1)$, for all $u\in[0,1]$. By the entropy and Jensen's inequalities, for any $B>0$, the first expectation of (2.3.26) is bounded from above by

$$\frac{H(\mu_N|\nu_h^N)}{BN}+\frac{1}{BN}\log\mathbb{E}_{\nu_h^N}\left[e^{BN\left|\int_0^t N^{1-\theta}\sum_{x\in\Lambda_N}Gr_N^-\left(\frac{x}{N}\right)(\eta_x^N(s)-\alpha)ds\right|}\right]. \tag{2.3.27}$$

We can remove the absolute value inside the exponential since $e^{|u|}\leq e^u+e^{-u}$ and

$$\limsup_{N\to\infty}N^{-1}\log(a_N+b_N)=\max\left\{\limsup_{N\to\infty}N^{-1}\log(a_N),\limsup_{N\to\infty}N^{-1}\log(b_N)\right\}. \tag{2.3.28}$$

By (2.3.19) and Feynman-Kac's formula, (2.3.27) is bounded from above by

$$\frac{C_0}{B}+t\sup_f\left\{N^{1-\theta}\sum_{x\in\Lambda_N}\left|Gr_N^-(\tfrac{x}{N})\langle t_x^\alpha,f\rangle_{\nu_h^N}\right|+\frac{N}{B}\left\langle L_N\sqrt{f},\sqrt{f}\right\rangle_{\nu_h^N}\right\},$$

where the supremum is carried over all the densities f with respect to ν_h^N. We recall that $t_x^\alpha(\eta)=\eta_x-\alpha$. From Lemma 2.3.10 we have that there exists a constant $C:=C(\alpha,\beta,\gamma)>0$ such that

$$\begin{aligned}N^{1-\theta}\sum_{x\in\Lambda_N}\left|(Gr_N^-)(\tfrac{x}{N})\langle t_x^\alpha,f\rangle_{\nu_h^N}\right|&\leq CN^{1-\theta}\sum_{x\in\Lambda_N}|(Gr_N^-)(\tfrac{x}{N})|\left[A_x+\frac{I_x^\alpha(\sqrt{f},\nu_h^N)}{A_x}+\frac{x}{N}\right]\\&\leq 4C^2\kappa^{-1}BN^{1-\theta}\sum_{x\in\Lambda_N}G^2(\tfrac{x}{N})r_N^-(\tfrac{x}{N})+\frac{N}{4B}D_N^\ell(\sqrt{f},\nu_h^N)\\&\quad+CN^{-\theta}\sum_{x\in\Lambda_N}|G(\tfrac{x}{N})|r_N^-(\tfrac{x}{N})x.\end{aligned} \tag{2.3.29}$$

The last inequality is obtained by choosing $A_x=4\kappa^{-1}C|G(\frac{x}{N})|B$. Recall (2.3.21).

Since $\theta>1$ and the function G is bounded, we use (2.3.29) and (2.3.21) and we estimate from above (2.3.27) by a constant times

$$\frac{1}{B}+\frac{1}{BN^{1+\theta}}\sum_{x\in\Lambda_N}\left\{x^2r_N^-(\tfrac{x}{N})+\left(x-N\right)^2r_N^+(\tfrac{x}{N})\right\}+BN^{1-\theta}\sum_{x\in\Lambda_N}r_N^-(\tfrac{x}{N})+N^{-\theta}\sum_{x\in\Lambda_N}r_N^-(\tfrac{x}{N})x,$$

which, by

$$\sum_{x\in\Lambda_N}x^2r_N^-(\tfrac{x}{N})\lesssim\begin{cases}N^{3-\gamma}, & \gamma\in(2,3),\\ \log N, & \gamma=3,\\ 1, & \gamma>3,\end{cases} \tag{2.3.30}$$

and (2.3.5), goes to zero, taking first $N\to\infty$ and then $B\to\infty$. □

Let us define for $\ell\in\mathbb{N}$ the following empirical averages

$$\overrightarrow{\eta}_0^\ell:=\frac{1}{\ell}\sum_{y=1}^{\ell}\eta_y\qquad\text{and}\qquad\overleftarrow{\eta}_N^\ell:=\frac{1}{\ell}\sum_{y=N-1-\ell}^{N-1}\eta_y. \tag{2.3.31}$$

Lemma 2.3.12. *For any $t > 0$ and any $\theta \geq 1$ we have that*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t \sum_{x\in\Lambda_N}\Theta_x^-(\eta_x^N(s)-\overrightarrow{\eta}_0^{\varepsilon N}(sN^2))\,ds\right|\right]=0,$$
$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t \sum_{x\in\Lambda_N}\Theta_x^+(\eta_x^N(s)-\overleftarrow{\eta}_N^{\varepsilon N}(sN^2))\,ds\right|\right]=0.$$

Proof. We present the proof for the first term, but we note that the proof for the second one is analogous. Here we take as reference measure the Bernoulli product measure with constant parameter (for example α) and we recall (2.3.23). By the entropy and Jensen's inequalities the expectation in the statement of the lemma is bounded from above, for any $B > 0$, by

$$\frac{H(\mu_N|\nu_\alpha^N)}{BN}+\frac{1}{BN}\log\mathbb{E}_{\nu_\alpha^N}\left[e^{BN\left|\int_0^t\sum_{x\in\Lambda_N}\Theta_x^-(\eta_x^N(s)-\overrightarrow{\eta}_0^{\varepsilon N}(sN^2))\,ds\right|}\right].$$

As in the previous proof, we can remove the absolute value inside the exponential, so that by (2.3.19) and by Feynman-Kac's formula last expression can be estimated from above by

$$\frac{C_0}{B}+t\sup_f\Big\{\sum_{x\in\Lambda_N}\Theta_x^-\langle\tau_x^{\varepsilon N},f\rangle_{\nu_\alpha^N}+\frac{N}{B}\Big\langle L_N\sqrt{f},\sqrt{f}\Big\rangle_{\nu_\alpha^N}\Big\},$$

where the supremum is carried over all the densities f with respect to ν_α^N. Here $\tau_x^{\varepsilon N}(\eta)=\eta_x-\overrightarrow{\eta}_0^{\varepsilon N}$.

Now we have to split the sum in x, depending on whether $N-1\geq x\geq\varepsilon N$ or $x\leq\varepsilon N-1$. We start by the first case and we have

$$\langle\tau_x^{\varepsilon N},f\rangle_{\nu_\alpha^N}=\frac{1}{\varepsilon N}\sum_{y=1}^{\varepsilon N}\int(\eta_x-\eta_y)f(\eta)\,d\nu_\alpha^N=\frac{1}{\varepsilon N}\sum_{y=1}^{\varepsilon N}\sum_{z=y}^{x-1}\int(\eta_{z+1}-\eta_z)f(\eta)\,d\nu_\alpha^N.$$

By writing the previous term as its half plus its half and by performing in one of the terms the change of variables η into $\sigma^{z,z+1}\eta$, for which the measure ν_α^N is invariant, we write it as

$$\frac{1}{2\varepsilon N}\sum_{y=1}^{\varepsilon N}\sum_{z=y}^{x-1}\int(f(\eta)-f(\sigma^{z,z+1}\eta))(\eta_{z+1}-\eta_z)\,d\nu_\alpha^N.$$

By using the fact that $(a-b)=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ for any $a,b\geq 0$ and since $ab\leq\dfrac{Aa^2}{2}+\dfrac{b^2}{2A}$ for all $A>0$, we have that

$$\begin{aligned}\sum_{x=\varepsilon N}^{N-1}\Theta_x^-\langle\tau_x^{\varepsilon N},f\rangle_{\nu_\alpha^N}\leq{}&\frac{A}{4\varepsilon N}\sum_{x=\varepsilon N}^{N-1}\Theta_x^-\sum_{y=1}^{\varepsilon N}\sum_{z=y}^{x-1}\int\left(\sqrt{f(\eta)}-\sqrt{f(\sigma^{z,z+1}\eta)}\right)^2d\nu_\alpha^N\\&+\frac{1}{4A\varepsilon N}\sum_{x=\varepsilon N}^{N-1}\Theta_x^-\sum_{y=1}^{\varepsilon N}\sum_{z=y}^{x-1}\int\left(\sqrt{f(\eta)}+\sqrt{f(\sigma^{z,z+1}\eta)}\right)^2(\eta_{z+1}-\eta_z)^2d\nu_\alpha^N.\end{aligned}\tag{2.3.32}$$

By neglecting the jumps of size bigger than one, we see that

$$D^{NN}(\sqrt{f},\nu_\alpha^N)=\sum_{z\in\Lambda_N}\int\left(\sqrt{f(\eta)}-\sqrt{f(\sigma^{z,z+1}\eta)}\right)^2 d\nu_\alpha^N \lesssim D_N^0(\sqrt{f},\nu_\alpha^N).$$

Therefore, by using also (2.3.8), the first term at the right hand side of (2.3.32) can be bounded from above by

$$\frac{A}{4}\sum_{x=\varepsilon N}^{N-1}\Theta_x^- \, D^{NN}(\sqrt{f},\nu_\alpha^N) \lesssim AD^{NN}(\sqrt{f},\nu_\alpha^N) \lesssim AD_N^0(\sqrt{f},\nu_\alpha^N). \tag{2.3.33}$$

Recall (2.3.23) and observe that $D_N(\sqrt{f},\nu_\alpha^N)\geq D_N^0(\sqrt{f},\nu_\alpha^N)$. Then we choose the constant A in the form $A=CN/B$ for some suitable constant C in order that one half of the term $-\frac{N}{4B}D_N(\sqrt{f},\nu_\alpha^N)$ appearing in (2.3.23) counterbalances negatively the term at the right hand side of (2.3.33). Moreover we can bound from above the last term at the right hand side of (2.3.32) by (use Lemma 2.3.7)

$$\begin{aligned}&\frac{CB}{N}\sum_{x=\varepsilon N}^{N-1}\Theta_x^- \, \frac{1}{2\varepsilon N}\sum_{y=1}^{\varepsilon N}\sum_{z=y}^{x-1}\int(\sqrt{f(\eta)}+\sqrt{f(\sigma^{z,z+1}\eta)})^2(\eta_{z+1}-\eta_z)^2 d\nu_\alpha^N\\ &\lesssim \frac{B}{N}\sum_{x\in\Lambda_N}x\Theta_x^-\end{aligned}$$

which vanishes as $N\to\infty$ by (2.3.23). Therefore we proved that uniformly in ε

$$\limsup_{B\to\infty}\limsup_{N\to\infty}\sup_f\left\{\sum_{x=\varepsilon N}^{N-1}\Theta_x^-\langle\tau_x^{\varepsilon N},f\rangle_{\nu_\alpha^N}+\frac{N}{B}\left\langle L_N\sqrt{f},\sqrt{f}\right\rangle_{\nu_\alpha^N}\right\}=0.$$

It remains to prove that

$$\limsup_{B\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_f\left\{\sum_{x=1}^{\varepsilon N-1}\Theta_x^-\langle\tau_x^{\varepsilon N},f\rangle_{\nu_\alpha^N}+\frac{N}{B}\left\langle L_N\sqrt{f},\sqrt{f}\right\rangle_{\nu_\alpha^N}\right\}=0. \tag{2.3.34}$$

If $x\leq\varepsilon N-1$, we write

$$\begin{aligned}\langle\tau_x^{\varepsilon N},f\rangle_{\nu_\alpha^N}&=\frac{1}{\varepsilon N}\sum_{y=1}^{\varepsilon N}\int(\eta_x-\eta_y)f(\eta)\,d\nu_\alpha^N\\ &=\frac{1}{\varepsilon N}\sum_{y=1}^{x-1}\sum_{z=y}^{x-1}\int(\eta_{z+1}-\eta_z)f(\eta)\,d\nu_\alpha^N-\frac{1}{\varepsilon N}\sum_{y=x+1}^{\varepsilon N}\sum_{z=x}^{y-1}\int(\eta_{z+1}-\eta_z)f(\eta)\,d\nu_\alpha^N,\end{aligned}$$

and the same estimates as before give that there exists a constant $C>0$ such that for any $A>0$,

$$\sum_{x=1}^{\varepsilon N-1}\Theta_x^-\langle\tau_x^{\varepsilon N},f\rangle_{\nu_\alpha^N}\leq C\left[AD_N(\sqrt{f},\nu_\alpha^N)+\frac{\varepsilon N}{A}\sum_{x=1}^{\varepsilon N-1}\Theta_x^-\right].$$

Recall (2.3.23) and (2.3.8). Then, we choose $A = N/8CB$ and we get that (2.3.34). This finishes the proof.

$\square$

Remark 2.3.13. *We note that above, if we change in the statement of the lemma $\Theta_x^{\pm}$ by $r_N^{\pm}$, then the same result holds by performing exactly the same estimates as above, because what we need is that*

$$\sum_{x\in\Lambda_N}\Theta_x^{\pm}\lesssim 1 \quad \text{and} \quad \frac{1}{N}\sum_{x\in\Lambda_N}x\Theta_x^{\pm}\to 0$$

which also holds for $r_N^{\pm}$ instead of $\Theta_x^{\pm}$ since $\gamma>2$.

2.3.6 Fixing the profile at the boundary

Let $\mathbb{Q}$ be a limit point of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$, whose existence follows from Proposition 2.3.4 and assume, without lost of generality, that $\{\mathbb{Q}_N\}_{N\geq 1}$ converges to $\mathbb{Q}$. We note that since our model is an exclusion process, it is standard [40] to show that $\mathbb{Q}$ almost surely the trajectories of measures are absolutely continuous with respect to the Lebesgue measure, that is: $\pi_t(du)=\rho_t(u)du$ for any $t\in[0,T]$. In Subsection 2.3.7 we prove that the density ρ belongs to $L^2(0,T;\mathscr{H}^1([0,1]))$ if $\theta\geq 2-\gamma$. In particular, for almost every t, ρ_t can be identified with a continuous function on $[0,1]$.

In this section we prove iii) of Definition 2.2.3, that is, for $\theta\in[2-\gamma,1)$ we show that the profile satisfies $\rho_t(0)=\alpha$ and $\rho_t(1)=\beta$ for $t\in[0,T]$ a.s. Recall (2.3.31). Observe that

$$\mathbb{E}_{\mu_N}\left[\left|\int_0^t(\overrightarrow{\eta}_0^{\epsilon N}(sN^2)-\alpha)\,ds\right|\right]=\mathbb{E}_{\mathbb{Q}_N}\left[\left|\int_0^t(\langle\pi_s,\iota_\varepsilon^0\rangle-\alpha)\,ds\right|\right]$$

where $\iota_\varepsilon^0(u)=\varepsilon^{-1}\mathbf{1}_{(0,\varepsilon)}(u)$ for all $u\in(0,1)$. Therefore we have that for any $\delta>0$,

$$\mathbb{Q}_N\left[\left|\int_0^t(\langle\pi_s,\iota_\varepsilon^0\rangle-\alpha)\,ds\right|>\delta\right]\;\leq\;\delta^{-1}\,\mathbb{E}_{\mu_N}\left[\left|\int_0^t(\overrightarrow{\eta}_0^{\epsilon N}(sN^2)-\alpha)\,ds\right|\right].$$

By Portemanteau's Theorem [1] we conclude that

$$\mathbb{Q}\left[\left|\int_0^t(\langle\pi_s,\iota_\varepsilon^0\rangle-\alpha)\,ds\right|>\delta\right]\;\leq\;\delta^{-1}\liminf_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t(\overrightarrow{\eta}_0^{\epsilon N}(sN^2)-\alpha)\,ds\right|\right].$$

Now, if we are able to prove that the right hand side of the previous inequality is equal to zero, since we have that $\mathbb{Q}$ a.s. $\pi_s(du)=\rho_s(u)du$ with ρ_s a continuous function in 0 for almost every s, by taking the limit as $\varepsilon\to 0$, we can deduce that $\mathbb{Q}$ a.s. $\rho_s(0)=\alpha$ for a.e. $s\in[0,T]$. A similar argument applies for the right boundary. Therefore it is sufficient to prove the following lemma.

[1] In fact, since ι_ε^0 is not a continuous function it is not given for free that the set $\left\{\pi\,;\,\left|\int_0^t(\langle\pi_s,\iota_\varepsilon^0\rangle-\alpha)\,ds\right|>\delta\right\}$ is an open set in the Skorohod topology. A simple argument based on a L^1-approximation of ι_ε^0 by continuous functions permits to bypass this difficulty.

Lemma 2.3.14. *Let $\theta < 1$. For any $t \in [0, T]$ we have that*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t (\overrightarrow{\eta}_0^{\epsilon N}(sN^2)-\alpha)\,ds\right|\right]=0,$$
$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t (\overleftarrow{\eta}_N^{\epsilon N}(sN^2)-\beta)\,ds\right|\right]=0.$$

Last lemma is a consequence of the next two results.

Lemma 2.3.15. *Let $\theta < 1$. For any $t \in [0, T]$ we have that*

$$\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t (\eta_1^N(s)-\alpha)\,ds\right|\right]=0,$$
$$\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\left[\left|\int_0^t (\eta_{N-1}^N(s)-\beta)\,ds\right|\right]=0.$$

Proof. We give the proof for the first display, but we note that for the other one it is similar. Fix a Lipschitz profile h such that $h(0) = \alpha \le h(u) \le \beta = h(1)$, for all $u \in [0, 1]$ and h is $\gamma/2$-Hölder at the boundary. By the entropy and Jensen's inequalities, for any $B > 0$, the previous expectation is bounded from above by

$$\frac{H(\mu_N|\nu_h^N)}{BN}+\frac{1}{BN}\log\mathbb{E}_{\nu_h^N}\left[e^{BN\left|\int_0^t(\eta_1^N(s)-\alpha)\,ds\right|}\right].$$

By (2.3.19), Feynman-Kac's formula and noting, as we did in the proof of Lemma 2.3.11, that we can remove the absolute value inside the exponential, last display can be estimated from above by

$$\frac{C_0}{B}+t\sup_f\left\{\left\langle t_1^\alpha, f\right\rangle_{\nu_h^N}+\frac{N}{B}\left\langle L_N\sqrt{f},\sqrt{f}\right\rangle_{\nu_h^N}\right\},\tag{2.3.35}$$

where the supremum is carried over all the densities f with respect to ν_h^N. Here we recall that $t_1^\alpha(\eta) = \eta_1 - \alpha$. By Lemma 2.3.10, since h is Lipschitz, for any $A > 0$, the first term in the supremum in (2.3.35) is bounded from above by

$$C\left[\frac{1}{A}I_1^\alpha(\sqrt{f},\nu_h^N)+A+\frac{1}{N}\right]$$

for some constant $C > 0$ independent of f and A. Moreover from (2.3.22), since

$$D_N(\sqrt{f},\nu_h^N)\ge D_N^\ell(\sqrt{f},\nu_h^N)$$

and $\gamma + \theta - 2 \ge 0$, we know that there exists a constant $C' > 0$ such that

$$\frac{N}{B}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}\ \le\ -\frac{N^{1-\theta}}{4B}\sum_{x\in\Lambda_N}I_x^\alpha(\sqrt{f},\nu_h^N)r_N^-(\tfrac{x}{N})+\frac{C'}{B}.$$

To get an upper bound, at the right hand side of the previous inequality, we only keep the term coming from $x = 1$ in the sum. Now, by choosing $A = 4C(r_N^-(\frac{1}{N}))^{-1}BN^{\theta-1}$, we get then that the expression inside brackets in (2.3.35) is bounded by

$$4C^2\frac{BN^{\theta-1}}{r_N^-(\frac{1}{N})} + \frac{C}{N} + \frac{C'}{B}.$$

Now since $r_N^-(\frac{1}{N})$ is bounded from below by a constant independent of N and $\theta < 1$, the proof follows by sending first $N \to \infty$ and then $B \to \infty$. $\square$

Lemma 2.3.16. *Let $\theta \in \mathbb{R}$. For any $t > 0$ we have that*

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t \overrightarrow{\eta}_0^{\epsilon N}(sN^2) - \eta_1^N(s))\,ds\Big|\Big] = 0,$$
$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mu_N}\Big[\Big|\int_0^t \overleftarrow{\eta}_N^{\epsilon N}(sN^2) - \eta_{N-1}^N(s))\,ds\Big|\Big] = 0.$$

Proof. We present the proof of the first item, but we note that for the second it is exactly the same. Fix a Lipcshitz profile h such that $h(0) = \alpha \le h(u) \le \beta = h(1)$, for all $u \in [0,1]$ and h is $\gamma/2$-Hölder at the boundary. By the entropy and Jensen's inequalities, for any $B > 0$, the previous expectation is bounded from above by

$$\frac{H(\mu_N|\nu_h^N)}{BN} + \frac{1}{BN}\log\mathbb{E}_{\nu_h^N}\Big[e^{BN|\int_0^t \overrightarrow{\eta}_0^{\epsilon N}(sN^2) - \eta_1^N(s)\,ds|}\Big].$$

By (2.3.19), Feynman-Kac's formula, and using the same argument as in the proof of the previous lemma, the estimate of the previous expression can be reduced to bound

$$\frac{C_0}{B} + t\sup_f\Big\{\frac{1}{\ell}\sum_{y=2}^{\ell+1}|\langle v_y^1, f\rangle_{\nu_h^N}| + \frac{N}{B}\Big\langle L_N\sqrt{f}, \sqrt{f}\Big\rangle_{\nu_h^N}\Big\}, \tag{2.3.36}$$

where $\ell = \epsilon N$ and $v_y^1(\eta) = \eta_y - \eta_1$. Here the supremum is carried over all the densities f with respect to ν_h^N. Note that since $y \in \Lambda_N$ we know that $v_y^1(\eta) = \sum_{z=1}^{y-1}(\eta_{z+1} - \eta_z)$. Observe now that

$$\begin{aligned}\sum_{z=1}^{y-1}\int(\eta_{z+1}-\eta_z)f(\eta)d\nu_h^N &= \frac{1}{2}\sum_{z=1}^{y-1}\int(\eta_{z+1}-\eta_z)(f(\eta)-f(\sigma^{z,z+1}\eta))d\nu_h^N\\ &+\frac{1}{2}\sum_{z=1}^{y-1}\int(\eta_{z+1}-\eta_z)(f(\eta)+f(\sigma^{z,z+1}\eta))d\nu_h^N.\end{aligned}$$

By using the fact that for any $a,b\geq 0$, $(a-b)=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and Young's inequality, we have, for any positive constant A, that

$$\begin{aligned}\frac{1}{\ell}\sum_{y=2}^{\ell+1}|\langle v_y^1,f\rangle_{\nu_h^N}| \leq\ & \frac{1}{2A\ell}\sum_{y=2}^{\ell+1}\sum_{z=1}^{y-1}\int(\eta_{z+1}-\eta_z)^2\left(\sqrt{f(\eta)}+\sqrt{f(\sigma^{z,z+1}\eta)}\right)^2 d\nu_h^N\\ &+\frac{A}{2\ell}\sum_{y=2}^{\ell+1}\sum_{z=1}^{y-1}\int\left(\sqrt{f(\eta)}-\sqrt{f(\sigma^{z,z+1}\eta)}\right)^2 d\nu_h^N\\ &+\frac{1}{2\ell}\sum_{y=2}^{\ell+1}\left|\sum_{z=1}^{y-1}\int\left(\eta_{z+1}-\eta_z\right)\left(f(\eta)+f(\sigma^{z,z+1}\eta)\right)d\nu_h^N\right|.\end{aligned} \tag{2.3.37}$$

By neglecting the jumps of size bigger than one, we see that

$$D^{NN}(\sqrt{f},\nu_h^N)=\sum_{z\in\Lambda_N}\int\left(\sqrt{f(\eta)}-\sqrt{f(\sigma^{z,z+1}\eta)}\right)^2 d\nu_h^N \lesssim D_N^0(\sqrt{f},\nu_h^N).$$

Then, the second term on the right hand side of (2.3.37) is bounded from above by

$$\begin{aligned}&\frac{A}{2\ell}\sum_{y=2}^{\ell+1}D^{NN}(\sqrt{f},\nu_h^N)\leq A\,D^{NN}(\sqrt{f},\nu_h^N)\\ &\leq CA\,D_N^0(\sqrt{f},\nu_h^N)\leq CA\,D_N(\sqrt{f},\nu_h^N)\end{aligned}$$

where C is a positive constant independent of A,ℓ,f. Then, for the choice $A=N(4BC)^{-1}$ and from (2.3.22), since $\gamma+\theta-2\geq 0$, we can bound from above (2.3.36) by

$$\begin{aligned}&\frac{2BC}{N\ell}\sum_{y=2}^{\ell+1}\sum_{z=1}^{y-1}\int(\eta_{z+1}-\eta_z)^2\left(\sqrt{f(\eta)}+\sqrt{f(\sigma^{z,z+1}\eta)}\right)^2 d\nu_h^N\\ &+\frac{1}{2\ell}\sum_{y=2}^{\ell+1}\left|\sum_{z=1}^{y-1}\int\left(\eta_{z+1}-\eta_z\right)\left(f(\eta)+f(\sigma^{z,z+1}\eta)\right)d\nu_h^N\right|+\frac{C'}{B}\\ &\lesssim\frac{B\ell}{N}+\frac{1}{B}+\frac{1}{2\ell}\sum_{y=2}^{\ell+1}\left|\sum_{z=1}^{y-1}\int\left(\eta_{z+1}-\eta_z\right)\left(f(\eta)+f(\sigma^{z,z+1}\eta)\right)d\nu_h^N\right|\end{aligned} \tag{2.3.38}$$

for some constant $C'>0$. For the last inequality we used Lemma 2.3.7. Observe that $B\ell/N=B\varepsilon$ vanishes as $\epsilon\to 0$. It remains to estimate the third term on the right hand side of the last inequality. For that purpose we make a similar computation to the one of Lemma 2.3.10. Let

$$C_z=\max\left\{\frac{1}{h\left(\frac{z}{N}\right)\left(1-h\left(\frac{z+1}{N}\right)\right)},\frac{1}{h\left(\frac{z+1}{N}\right)\left(1-h\left(\frac{z}{N}\right)\right)}\right\}$$

which is bounded from above by a constant depending only on α and β. By using the fact that ν_h^N is product and denoting by $\tilde{\eta}$ the configuration η removing its value at z and $z+1$ so that

$(\eta_z, \eta_{z+1}, \tilde{\eta}) = \eta$, we have that

$$\begin{aligned}
&\sum_{z=1}^{y-1}\left|\int (\eta_{z+1}-\eta_z)(f(\eta)+f(\sigma^{z,z+1}\eta))d\,\nu_h^N\right| \\
=&\sum_{z=1}^{y-1}\Bigg|\sum_{\tilde{\eta}}(f(0,1,\tilde{\eta})+f(1,0,\tilde{\eta}))h(\tfrac{z+1}{N})(1-h(\tfrac{z}{N}))\ \nu_h^N(\tilde{\eta}) \\
&-\sum_{\tilde{\eta}}(f(1,0,\tilde{\eta})+f(0,1,\tilde{\eta}))h(\tfrac{z}{N})(1-h(\tfrac{z+1}{N}))\ \nu_h^N(\tilde{\eta})\Bigg|.
\end{aligned}$$

By regrouping terms, the last expression is equal to

$$\begin{aligned}
=&\sum_{z=1}^{y-1}\left|\sum_{\tilde{\eta}}\Big(h\Big(\tfrac{z+1}{N}\Big)-h\Big(\tfrac{z}{N}\Big)\Big)(f(0,1,\tilde{\eta})+f(1,0,\tilde{\eta}))\ \nu_h^N(\tilde{\eta})\right| \\
\leq&\frac{1}{2}\sum_{z=1}^{y-1}C_z\left|h\Big(\tfrac{z+1}{N}\Big)-h\Big(\tfrac{z}{N}\Big)\Big)\right|\ \sum_{\tilde{\eta}}\Big\{f(1,0,\tilde{\eta})\,h\Big(\tfrac{z}{N}\Big)\Big(1-h\Big(\tfrac{z+1}{N}\Big)\Big)\ \nu_h^N(\tilde{\eta}) \\
&+f(0,1,\tilde{\eta})\,\Big(1-h\Big(\tfrac{z}{N}\Big)\Big)h\Big(\tfrac{z+1}{N}\Big)\ \nu_h^N(\tilde{\eta})\Big\} \\
\lesssim&\sum_{z=1}^{y-1}\left|h\Big(\tfrac{z+1}{N}\Big)-h\Big(\tfrac{z}{N}\Big)\Big)\right|.
\end{aligned}$$

Above, for example, $f(1,0,\tilde{\eta})$ (resp. $f(0,1,\tilde{\eta})$) means that we are computing $f(\eta)$ with η such that $\eta_z = 1$ and $\eta_{z+1} = 0$ (resp. $\eta_z = 0$ and $\eta_{z+1} = 1$). Since h is Lipschitz, by (2.3.38), this estimate provides an upper bound for (2.3.36) which is in the form of a constant times

$$\frac{B\ell}{N}+\frac{1}{B}+\frac{1}{N\ell}\sum_{y=2}^{\ell+1}y\ \lesssim\ B\varepsilon+B^{-1}+\varepsilon$$

which vanishes, as $\varepsilon \to 0$ and then $B \to \infty$. This ends the proof. □

2.3.7 Energy Estimates

Let $\mathbb{Q}$ be a limit point of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$, whose existence follows from Proposition 2.3.4 and assume, without lost of generality, that $\{\mathbb{Q}_N\}_{N\geq 1}$ converges to $\mathbb{Q}$. We note that since our model is an exclusion process, it is standard (see [40]) to show that, $\mathbb{Q}$ almost surely, the trajectories of measures are absolutely continuous with respect to the Lebesgue measure, that is: $\pi_t(du) = \rho_t(u)du$ for any $t \in [0, T]$.

2.3.7.1 The case $\theta \geq 2-\gamma$

Recall that in this case the system is speeded up in the diffusive time scale so that $\Theta(N) = N^2$. In this section we prove that the density ρ belongs to $L^2(0, T; \mathscr{H}^1([0,1]))$, see Definition 2.2.2.

For that purpose, we define the linear functional ℓ_ρ on $C_c^{0,1}([0,T]\times(0,1))$ by

$$\ell_\rho(G)=\int_0^T\int_0^1 \partial_u G_s(u)\rho_s(u)\,duds=\int_0^T\int_0^1 \partial_u G_s(u)\,d\pi_s(u)ds.$$

By Proposition 2.3.17 below we have that ℓ_ρ is, $\mathbb{Q}$ almost surely, continuous, thus we can extend this linear functional to $L^2([0,T]\times(0,1))$. Moreover, by Riesz's Representation Theorem we find $\zeta\in L^2([0,T]\times(0,1))$ such that

$$\ell_\rho(G)=-\int_0^T\int_0^1 G_s(u)\zeta_s(u)duds,$$

for all $G\in C_c^{0,1}([0,T]\times(0,1))$, which implies that $\rho\in L^2(0,T;\mathscr{H}^1([0,1]))$.

Proposition 2.3.17. *For all $\theta\geq 2-\gamma$. There exist positive constants C and c such that*

$$\mathbb{E}\Big[\sup_G\big\{\ell_\rho(G)-c\|G\|_2^2\big\}\Big]\leq C<\infty,$$

where the supremum above is taken on the set $C_c^{0,1}([0,T]\times(0,1))$. Here we denote by $\|G\|_2$ the norm of a function $G\in L^2([0,T]\times(0,1))$.

Proof. By density it is enough to prove the result for a countable dense subset $\{G^m\}_{m\in\mathbb{N}}$ on $C_c^{0,2}([0,T]\times(0,1))$ and by the Monotone Convergence Theorem it is enough to prove that

$$\mathbb{E}\Big[\sup_{k\leq m}\big\{\ell_\rho(G^k)-c\|G^k\|_2^2\big\}\Big]\leq K_0,$$

for any m and for K_0 independent of m. Now, we define $\Phi:\mathscr{D}^T_{\mathscr{M}^+}\to\mathbb{R}$ by

$$\Phi(\pi_\cdot)=\max_{k\leq m}\left\{\int_0^T\int_0^1 \partial_u G_s^k(u)\,d\pi_s(u)ds-c\|G^k\|_2^2\right\},$$

which is a continuous and bounded function for the Skorohod topology of $\mathscr{D}^T_{\mathscr{M}^+}$. Thus we have that

$$\mathbb{E}[\Phi]=\lim_{N\to\infty}\mathbb{E}_{\mu_N}\left[\max_{k\leq m}\left\{\int_0^T\frac{1}{N-1}\sum_{x=1}^{N-1}\partial_u G_s^k(\tfrac{x}{N})\eta_x^N(s)ds-c\|G^k\|_2^2\right\}\right].$$

By the entropy inequality, Jensen's inequality and the fact that $e^{\max_{k\leq m}a_k}\leq\sum_{k=1}^m e^{a_k}$ the previous display is bounded from above by

$$C_0+\frac{1}{N}\log\mathbb{E}_{\nu_h^N}\left[\sum_{k=1}^m e^{\int_0^T\sum_{x\in\Lambda_N}\partial_u G_s^k\left(\frac{x}{N}\right)\eta_x^N(s)ds-cN\|G^k\|_2^2}\right],$$

where h is Lipschitz such that $h(0)=\alpha\le h(u)\le\beta=h(1)$, for all $u\in[0,1]$ and it is $\frac{\gamma}{2}$-Hölder at the boundary. In order to deal with the second term in the previous display we use (2.3.28) and it is enough to bound

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{E}_{\nu_h^N}\left[e^{\int_0^T\sum_{x\in\Lambda_N}\partial_u G_s\left(\frac{x}{N}\right)\eta_x^N(s)ds-cN\|G\|_2^2}\right],$$

for a fixed function $G\in C_c^{0,2}([0,T]\times(0,1))$, by a constant independent of G. By Feynman-Kac's formula, the last expression is bounded from above by

$$\limsup_{N\to\infty}\int_0^T\sup_f\Big\{\frac{1}{N}\int\sum_{x\in\Lambda_N}\partial_u G_s(\tfrac{x}{N})\eta_x f(\eta)d\nu_h^N-c\|G\|_2^2+\tfrac{\Theta(N)}{N}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}\Big\}ds$$

where the supremum is carried over all the densities f with respect to ν_h^N. Let us now focus on the first term inside braces in the previous expression. Observe first that the space derivative of G_s can be replaced by the discrete gradient $\nabla_N G_s(\frac{x-1}{N})=N\big[G_s(\frac{x}{N})-G_s(\frac{x-1}{N})\big]$ of G_s with an error $R_N(G)$ satisfying uniformly the bound $|R_N(G)|\lesssim 1/N$ since $G\in C_c^{0,2}([0,T]\times(0,1))$. By summing and subtracting the term $\nabla_N G_s(\frac{x-1}{N})$ inside the sum, and doing a summation by parts, we can write

$$\frac{1}{N}\int\sum_{x\in\Lambda_N}\partial_u G_s(\tfrac{x}{N})\eta_x f(\eta)d\nu_h^N=\int\sum_{x=1}^{N-2}G_s(\tfrac{x}{N})(\eta_x-\eta_{x+1})f(\eta)d\nu_h^N+R_N(G).$$

A simple computation shows that we can write the first term at the right hand side of the previous display as

$$\begin{aligned}&\frac{1}{2}\int\sum_{x=1}^{N-2}G_s(\tfrac{x}{N})(\eta_x-\eta_{x+1})(f(\eta)-f(\sigma^{x,x+1}\eta))d\nu_h^N\\&+\frac{1}{2}\int\sum_{x=1}^{N-2}G_s(\tfrac{x}{N})(\eta_x-\eta_{x+1})f(\sigma^{x,x+1}\eta)(1-\theta^{x,x+1}(\eta))d\nu_h^N.\end{aligned}\tag{2.3.39}$$

Recall that for $u,v\ge0$, $u-v=(\sqrt{u}-\sqrt{v})(\sqrt{u}+\sqrt{v})$ and the inequality $ab\le\dfrac{Ba^2}{2}+\dfrac{b^2}{2B}$ which is valid for any $B>0$. Taking $B=\dfrac{N}{\Theta(N)}$ and using Lemma 2.3.7 we bound the first term in (2.3.39) by

$$\begin{aligned}&\frac{N}{4\Theta(N)}\int\sum_{x=1}^{N-2}(G_s(\tfrac{x}{N}))^2(\sqrt{f(\eta)}+\sqrt{f(\sigma^{x,x+1}\eta)})^2d\nu_h^N\\+\;&\frac{\Theta(N)}{4N}\int\sum_{x=1}^{N-2}(\sqrt{f(\eta)}-\sqrt{f(\sigma^{x,x+1}\eta)})^2d\nu_h^N\\\le\;&\frac{\Theta(N)}{4N}D_N^0(\sqrt{f},\nu_h^N)+\frac{CN}{\Theta(N)}\sum_{x\in\Lambda_N}(G_s(\tfrac{x}{N}))^2\end{aligned}$$

for some $C > 0$. Similarly we can estimate the second term in (2.3.39) from above by

$$\begin{aligned}
&\frac{1}{4N}\int\sum_{x=1}^{N-2}(G_s(\tfrac{x}{N}))^2(\eta_x-\eta_{x+1})^2 f(\sigma^{x,x+1}\eta)d\nu_h^N\\
+&\ \frac{N}{4}\int\sum_{x=1}^{N-2} f(\sigma^{x,x+1}\eta)(\theta^{x,x+1}(\eta)-1)^2 d\nu_h^N\\
\lesssim&\ \frac{1}{N}\sum_{x\in\Lambda_N}(G_s(\tfrac{x}{N}))^2+1.
\end{aligned}$$

We use now (2.3.22) with $B=1$ there and observe that last two terms at the right hand side of (2.3.22) are bounded from above by a constant since $\gamma+\theta-2\geq 0$. Observe also that $D_N^0(\sqrt{f},\nu_h^N)\leq D_N(\sqrt{f},\nu_h^N)$. Recalling that $\Theta(N)=N^2$ we get then that (2.3.39) is bounded from above by

$$C\int_0^T\Big[1+\frac{1}{N}\sum_{x\in\Lambda_N}(G_s(\tfrac{x}{N}))^2\Big]ds\ -\ c\|G\|_2^2\ +\ R_N(G)$$

where C is a positive constant independent of G. We then choose $c > C$ in order to conclude that

$$\limsup_{N\to\infty}\ \Big\{C\int_0^T\Big[1+\frac{1}{N}\sum_{x\in\Lambda_N}(G_s(\tfrac{x}{N}))^2\Big]ds\ -\ c\|G\|_2^2\ +\ R_N(G)\Big\}\ \lesssim\ 1.$$

This finishes the proof. □

2.3.7.2 The case $\theta\leq 2-\gamma$

In this section we prove that the function $(t,u)\to\rho_t(u)-\alpha$ (resp. $(t,u)\to\rho_t(u)-\beta$) belongs to $L^2([0,T]\times(0,1),dt\otimes d\mu)$ (resp. $L^2([0,T]\times(0,1),dt\otimes d\mu')$), where μ (resp. μ') is the measure that has the density with respect to the Lebesgue measure given by

$$u\in[0,1]\to\frac{1}{u^\gamma}\Big(\text{resp. }\frac{1}{(1-u)^\gamma}\Big). \tag{2.3.40}$$

Let ν_h^N be as above, where $h:[0,1]\to[0,1]$ is a profile such that $h(0)=\alpha\leq h(u)\leq\beta=h(1)$, for all $u\in[0,1]$, Hölder of parameter $\gamma/2$ at the boundary and Lipschitz inside. Let $G\in C_c^{1,\infty}([0,T]\times[0,1])$. By the entropy and Jensen's inequalities and the Feynmann-Kac's formula, we have that

$$\begin{aligned}
&\mathbb{E}_{\mu_N}\Bigg(\int_0^T dt\,N^{\gamma-1}\sum_{x\in\Lambda_N}G_t(\tfrac{x}{N})r_N^-\big(\tfrac{x}{N}\big)(\eta_x^N(t)-\alpha)\Bigg)\\
\leq&C_0+\int_0^T\sup_f\Big\{N^{\gamma-1}\sum_{x\in\Lambda_N}G_t(\tfrac{x}{N})r_N^-\big(\tfrac{x}{N}\big)\langle t_x^\alpha,f\rangle_{\nu_h^N}+\frac{\Theta(N)}{N}\Big\langle L_N\sqrt{f},\sqrt{f}\Big\rangle_{\nu_h^N}\Big\}dt
\end{aligned}\tag{2.3.41}$$

where the supremum is taken over all the densities f on Ω_N with respect to ν_h^N. Below C is a constant that may change from line to line. Since the profile is Hölder of parameter $\gamma/2$ at the boundary and Lipschitz inside, and from (2.3.22) the term at the right hand side of last expression is bounded from above by

$$-\frac{\Theta(N)}{4N}D_N(\sqrt{f},\nu_h^N)+\frac{\Theta(N)}{N^2}C+\frac{\Theta(N)}{N^{\gamma+\theta}}C.$$

Repeating the proof of Lemma 2.3.14, we get that (2.3.41) is bounded from above by

$$CN^{\gamma-1}\sum_{x\in\Lambda_N} r_N^-\left(\tfrac{x}{N}\right)\left(G_t\left(\tfrac{x}{N}\right)\right)^2+C+\frac{\Theta(N)}{N^2}C+\frac{\Theta(N)}{N^{\gamma+\theta}}C.$$

We take the limit $N\to\infty$. We conclude that there exist constants $C>0$ independent of G such that

$$\mathbb{E}\left[\int_0^T\int_0^1\frac{(\rho_t(u)-\alpha)G_t(u)}{|u|^\gamma}\,du\,dt\;-\;C\int_0^T\int_0^1\frac{(G_t(u))^2}{|u|^\gamma}\,du\,dt\right]\lesssim 1.$$

By using a similar method as in the proof of the previous lemma we see that the supremum over G can be inserted in the expectation so that

$$\mathbb{E}\left[\sup_G\left\{\int_0^T\int_0^1\frac{(\rho_t(u)-\alpha)G_t(u)}{|u|^\gamma}\,du\,dt\;-\;C\int_0^T\int_0^1\frac{(G_t(u))^2}{|u|^\gamma}\,du\,dt\right\}\right]\lesssim 1.$$

The previous formula implies that

$$\mathbb{E}\left[\int_0^T\int_0^1\frac{(\rho_t(u)-\alpha)^2}{|u|^\gamma}\,du\,dt\right]\lesssim 1,$$

which proves the claim.

2.3.8 Characterization of limit points

We prove in this section that for each range of θ, all limit points $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ are concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure whose density ρ is a weak solution of the corresponding hydrodynamic equation. Let $\mathbb{Q}$ be a limit point of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$, whose existence follows from Proposition 2.3.4 and assume, without lost of generality, that $\{\mathbb{Q}_N\}_{N\geq 1}$ converges to $\mathbb{Q}$. As mentioned above, since there is at most one particle per site, it is easy to show that $\mathbb{Q}$ is concentrated on trajectories $\pi_t(du)$ which are absolutely continuous with respect to the Lebesgue measure, that is, $\pi_t(du)=\rho_t(u)du$ (for more details see [40]). Below, we prove, for each range of θ, that the density ρ is a weak solution of the corresponding hydrodynamic equation.

Proposition 2.3.18. *If $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ then*

1. *if $\theta<1$:*

$$\mathbb{Q}\Big(\pi_\cdot\in\mathcal{D}^T_{\mathcal{M}^+}:F_{RD}(t,\rho,G,g)=0,\forall t\in[0,T],\forall G\in C_c^{1,2}([0,T]\times[0,1])\Big)=1.$$

2. *if* $\theta \geq 1$*:*

$$\mathbb{Q}\Big(\pi_{\cdot} \in \mathscr{D}^T_{\mathscr{M}^+} : F_{Rob}(t,\rho,G,g) = 0, \forall t \in [0,T], \forall G \in C^{1,2}([0,T]\times[0,1])\Big) = 1.$$

Remark 2.3.19. *In this proposition, the constants $\hat{\kappa}, \hat{\sigma}, \hat{m}$ appearing in F_{RD} and F_{Rob} are fixed in Theorem 2.2.9.*

Proof. Note that in order to prove the proposition, it is enough to verify, for $\delta > 0$ and G in the corresponding space of test functions, that

$$\mathbb{Q}\left(\pi_{\cdot} \in \mathscr{D}^T_{\mathscr{M}^+} : \sup_{0\leq t\leq T} |F_{\bullet}(t,\rho,G,g)| > \delta\right) = 0,$$

for each θ, where $F_{\bullet}$ stands for F_{RD} if $\theta < 1$ and F_{Rob} if $\theta \geq 1$. From here on, in order to simplify notation, we will erase $\pi_{\cdot}$ from the sets that we have to look at.

• We start with the case $\theta \geq 1$. Recall $F_{Rob}(t,\rho,G,g)$ from Definition 2.2.3. Observe that, due to the boundary terms that involve $\rho_s(1)$ and $\rho_s(0)$, the set inside last probability is not an open set in the Skorohod topology, therefore we cannot use directly Portmanteau's Theorem as we would like to. In order to avoid this problem, we fix $\varepsilon > 0$ and we consider two approximations of the identity given by $\iota^0_\varepsilon(u) = \frac{1}{\varepsilon}1_{(0,\varepsilon)}(u)$ and $\iota^1_\varepsilon(u) = \frac{1}{\varepsilon}1_{(1-\varepsilon,1)}(u)$ and we sum and subtract to $\rho_s(0)$ (resp. $\rho_s(1)$) the mean $\langle \pi_s, \iota^0_\varepsilon\rangle = \frac{1}{\varepsilon}\int_0^\varepsilon \rho_s(u)du$ (resp. $\langle \pi_s, \iota^1_\varepsilon\rangle = \frac{1}{\varepsilon}\int_{1-\varepsilon}^\varepsilon \rho_s(u)du$). Thus, we bound last probability from above by the sum of the following four terms

$$\begin{aligned}\mathbb{Q}\Bigg(\sup_{0\leq t\leq T}\Big|\langle \rho_t, G_t\rangle - \langle \rho_0, G_0\rangle - \int_0^t \langle \rho_s, \Big(\tfrac{\hat{\sigma}^2}{2}\Delta + \partial_s\Big)G_s\rangle ds - \int_0^t \langle \pi_s, \iota^0_\varepsilon\rangle\Big(\tfrac{\hat{\sigma}^2}{2}\partial_u G_s(0) - \hat{m}G_s(0)\Big)ds \\ + \int_0^t \langle \pi_s, \iota^1_\varepsilon\rangle\Big(\tfrac{\hat{\sigma}^2}{2}\partial_u G_s(1) + \hat{m}G_s(1)\Big)ds - \hat{m}\int_0^t G_s(0)\alpha + G_s(1)\beta\, ds\Big| > \frac{\delta}{4}\Bigg),\end{aligned} \tag{2.3.42}$$

$$\mathbb{Q}\left(\Big|\langle(\rho_0 - g), G_0\rangle\Big| > \frac{\delta}{4}\right), \tag{2.3.43}$$

$$\mathbb{Q}\left(\sup_{0\leq t\leq T}\Big|\int_0^t \big[\rho_s(0) - \langle \pi_s, \iota^0_\varepsilon\rangle\big]\Big[\hat{m}G_s(0) - \tfrac{\hat{\sigma}^2}{2}\partial_u G_s(0)\Big]ds\Big| > \frac{\delta}{4}\right), \tag{2.3.44}$$

and

$$\mathbb{Q}\left(\sup_{0\leq t\leq T}\Big|\int_0^t \big[\rho_s(1) - \langle \pi_s, \iota^1_\varepsilon\rangle\big](\hat{m}G_s(1) + \tfrac{\hat{\sigma}^2}{2}\partial_u G_s(1))ds\Big| > \frac{\delta}{4}\right). \tag{2.3.45}$$

We note that the terms (2.3.44) and (2.3.45) converge to 0 as $\varepsilon \to 0$ since we are comparing $\rho_s(0)$ (resp. $\rho_s(1)$) with the corresponding average around the boundary points 0 (resp. 1) and (2.3.43) is equal to zero since $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ and $\mathbb{Q}_N$ is induced by μ_N which satisfies (2.2.5). Therefore it remains only to consider (2.3.42). We still cannot use Portmanteau's Theorem, since the functions ι^0_ε and ι^1_ε are not continuous. Nevertheless, we can approximate each one of these functions by continuous functions in such a way that the error vanishes as $\varepsilon \to 0$. Then, from Proposition A.3 of [33] we can use Portmanteau's Theorem and bound (2.3.42) from above by

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{Q}_N\Bigg(\sup_{0\le t\le T}\Big|\langle\rho_t,G_t\rangle-\langle\rho_0,G_0\rangle-\int_0^t\langle\rho_s,\big(\tfrac{\hat\sigma^2}{2}\Delta+\partial_s\big)G_s\rangle ds\\
&-\int_0^t\langle\pi_s,\iota_\varepsilon^0\rangle\big(\tfrac{\hat\sigma^2}{2}\partial_u G_s(0)-\hat m G_s(0)\big)\,ds+\int_0^t\langle\pi_s,\iota_\varepsilon^1\rangle\big(\tfrac{\hat\sigma^2}{2}\partial_u G_s(1)+\hat m G_s(1)\big)\,ds\\
&-\hat m\int_0^t G_s(0)\alpha+G_s(1)\beta\,ds\Big|>\frac{\delta}{2^4}\Bigg).
\end{aligned}\tag{2.3.46}$$

Summing and subtracting $\int_0^t N^2 L_N\langle\pi_s^N,G_s\rangle ds$ to the term inside the supremum in (2.3.46), recalling (2.3.1) and (2.3.31), the definition of $\mathbb{Q}_N$, we bound (2.3.46) from above by the sum of the next two terms

$$\liminf_{N\to\infty}\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\big|M_t^N(G)\big|>\frac{\delta}{2^5}\Bigg),\tag{2.3.47}$$

and

$$\begin{aligned}
&\liminf_{N\to\infty}\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Big|\int_0^t N^2 L_N\langle\pi_s^N,G_s\rangle\,ds-\frac{\hat\sigma^2}{2}\int_0^t\langle\rho_s,\Delta G_s\rangle ds\\
&-\int_0^t \overrightarrow{\eta}_0^{\varepsilon N}(s)\big(\tfrac{\hat\sigma^2}{2}\partial_u G_s(0)-\hat m G_s(0)\big)\,ds+\int_0^t \overleftarrow{\eta}_{N-1}^{\varepsilon N}(s)\big(\tfrac{\hat\sigma^2}{2}\partial_u G_s(1)+\hat m G_s(1)\big)\,ds\\
&-\hat m\int_0^t G_s(0)\alpha+G_s(1)\beta\,ds\Big|>\frac{\delta}{2^5}\Bigg).
\end{aligned}\tag{2.3.48}$$

From Doob's inequality together with (2.3.17), (2.3.47) goes to 0 as $N\to\infty$. Finally, (2.3.48) can be rewritten as

$$\begin{aligned}
&\liminf_{N\to\infty}\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Big|\int_0^t N^2 L_N\langle\pi_s^N,G_s\rangle\,ds-\frac{\hat\sigma^2}{2}\int_0^t\langle\pi_s^N,\Delta G_s\rangle\,ds\\
&-\int_0^t \overrightarrow{\eta}_0^{\varepsilon N}(s)\big(\tfrac{\hat\sigma^2}{2}\partial_u G_s(0)-\hat m G_s(0)\big)\,ds+\int_0^t \overleftarrow{\eta}_{N-1}^{\varepsilon N}(s)\big(\tfrac{\hat\sigma^2}{2}\partial_u G_s(1)+\hat m G_s(1)\big)\,ds\\
&-\hat m\int_0^t G_s(0)\alpha+G_s(1)\beta\,ds\Big|>\frac{\delta}{2^5}\Bigg).
\end{aligned}\tag{2.3.49}$$

Now, from (2.3.2) and (2.3.4) we can bound from above the probability in (2.3.49) by the sum of the five following terms

$$\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Big|\frac{N^2}{N-1}\int_0^t\sum_{x\in\Lambda_N}(K_N G_s)(\tfrac{x}{N})\eta_x^N(s)ds-\frac{\hat\sigma^2}{2}\int_0^t\big\langle\pi_s^N,\Delta G_s\big\rangle\,ds\Big|>\frac{\delta}{2^6}\Bigg),\tag{2.3.50}$$

$$\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Big|\frac{N^2}{N-1}\int_0^t\sum_{x\in\Lambda_N}\sum_{y\le 0}\big[G_s(\tfrac{y}{N})-G_s(\tfrac{x}{N})\big]p(x-y)\eta_x^N(s)ds \\ +\frac{\hat\sigma^2}{2}\int_0^t\overrightarrow{\eta}_0^{\varepsilon N}(sN^2)\partial_u G_s(0)\,ds\Big|>\frac{\delta}{2^6}\Bigg), \tag{2.3.51}$$

and

$$\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Big|\int_0^t\frac{N\kappa}{N-1}\sum_{x\in\Lambda_N}(G_s r_N^-)(\tfrac{x}{N})(\alpha-\eta_x^N(s))\,ds-m\kappa\int_0^t G_s(0)(\alpha-\overrightarrow{\eta}_0^{\varepsilon N}(sN^2))ds\Big|>\frac{\delta}{2^6}\Bigg) \tag{2.3.52}$$

and the sum of two terms which are very similar to the two previous ones but which are concerned with the right boundary. Thus, to conclude we have to show that these five terms go to 0 as $N\to\infty$. Applying Lemma 2.3.1 and noting that $|\eta_x^N(s)|\le 1$ for any x and any $s\ge 0$, we conclude that (2.3.50) goes to 0 as $N\to\infty$. Note also that by Taylor expansion, we can bound from above (2.3.51) by

$$\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Big|\int_0^t\partial_u G_s(0)\sum_{x\in\Lambda_N}\Theta_x^-\big[\eta_x^N(s)-\overrightarrow{\eta}_0^{\varepsilon N}(sN^2)\big]ds\Big|>\frac{\delta}{2^8}\Bigg). \tag{2.3.53}$$

Using Lemma 2.3.12 we see that (2.3.53) vanishes as $N\to\infty$. Now we look at (2.3.52) and we prove that it vanishes as $N\to\infty$. Performing a Taylor expansion on G_s at 0 and using (2.3.5) the probability in (2.3.52) is bounded from above by

$$\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Big|\int_0^t G_s(0)\sum_{x\in\Lambda_N}r_N^-(\tfrac{x}{N})\big[\overrightarrow{\eta}_0^{\varepsilon N}(sN^2)-\eta_x^N(s)\big]ds\Big|>\frac{\delta}{2^8}\Bigg),$$

plus lower-order terms (with respect to N). From Lemma 2.3.12 and Remark 2.3.13 last display vanishes as $N\to\infty$. Similarly the two terms which are similar to (2.3.51) and (2.3.52) but which are concerned with the right boundary vanish as $N\to\infty$. Thus the proof is finished.

• Now we treat the case $\theta<1$. We have to prove that

$$\mathbb{Q}\Big(\pi_\cdot\in\mathscr{D}^T_{\mathscr{M}^+}:\sup_{0\le t\le T}|F_{RD}(t,\rho,G,g)|>\delta\Big)=0$$

for any $G\in C_c^{1,2}([0,T]\times[0,1])$. We can bound from above the previous probability by

$$\mathbb{Q}\Bigg(\sup_{0\le t\le T}\Big|\langle\rho_t,G_t\rangle-\langle\rho_0,G_0\rangle-\int_0^t\langle\rho_s,\Big(\tfrac{\hat\sigma^2}{2}\Delta+\partial_s\Big)G_s\rangle ds \\ -\hat\kappa\int_0^t\langle G_s,V_0\rangle ds+\hat\kappa\int_0^t\langle G_s,\rho_s\rangle_{V_1}ds\Big|>\frac{\delta}{2}\Bigg), \tag{2.3.54}$$

and

$$\mathbb{Q}\Big(\Big|\langle(\rho_0-g),G_0\rangle\Big|>\frac{\delta}{2}\Big). \tag{2.3.55}$$

We note that (2.3.55) is equal to zero since $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\in\mathbb{N}}$ and $\mathbb{Q}_N$ is induced by μ_N which satisfies (2.2.5). We note that from Proposition A.3 of [33], the set inside the probability in (2.3.54) is an open set in the Skorohod space (the singularities of V_0 and V_1 are not present because G_s has compact support included in $(0,1)$). From Portmanteau's Theorem we bound (2.3.54) from above by

$$\begin{aligned}\liminf_{N\to\infty} \mathbb{Q}_N \Big(\sup_{0\le t\le T} \Big| \langle \rho_t, G_t\rangle - \langle \rho_0, G_0\rangle - \int_0^t \langle \rho_s, \Big(\tfrac{\hat\sigma^2}{2}\Delta + \partial_s\Big) G_s\rangle ds \\ - \hat\kappa \int_0^t \langle G_s, V_0\rangle\, ds + \hat\kappa \int_0^t \langle G_s, \rho_s\rangle_{V_1}\, ds \Big| > \frac{\delta}{2}\Big).\end{aligned} \tag{2.3.56}$$

Summing and subtracting $\int_0^t \Theta(N) L_N \langle \pi_s^N, G_s\rangle ds$ to the term inside the previous absolute value, recalling (2.3.1) and the definition of $\mathbb{Q}_N$, we can bound the previous probability from above by the sum of the next two terms

$$\mathbb{P}_{\mu_N}\Big(\sup_{0\le t\le T} \big|M_t^N(G)\big| > \frac{\delta}{4}\Big),$$

and

$$\begin{aligned}\mathbb{P}_{\mu_N}\Big(\sup_{0\le t\le T} \Big| \int_0^t \Theta(N) L_N \langle \pi_s^N, G_s\rangle ds - \int_0^t \Big\langle \pi_s^N, \tfrac{\sigma^2}{2}\Delta G_s\Big\rangle\, ds \\ - \hat\kappa \int_0^t \langle G_s, V_0\rangle\, ds + \hat\kappa \int_0^t \langle G_s, \rho_s\rangle_{V_1}\, ds \Big| > \frac{\delta}{4}\Big).\end{aligned} \tag{2.3.57}$$

The first term above can be estimated as in the case $\theta \ge 1$ and it vanishes as $N \to \infty$. It remains to prove that (2.3.57) vanishes as $N \to \infty$. For that purpose, we recall Lemma 2.3.2 and we use (2.3.2), (2.3.4) to bound it from above by the sum of the following terms

$$\mathbb{P}_{\mu_N}\Big(\sup_{0\le t\le T} \Big| \int_0^t \frac{\Theta(N)}{N-1} \sum_{x\in\Lambda_N} (K_N G_s)(\tfrac{x}{N}) \eta_x^N(s) ds - \frac{\hat\sigma^2}{2} \int_0^t \big\langle \pi_s^N, \Delta G_s\big\rangle\, ds \Big| > \frac{\delta}{2^4}\Big), \tag{2.3.58}$$

$$\begin{aligned}\mathbb{P}_{\mu_N}\Big(\sup_{0\le t\le T} \Big| \int_0^t \Big\{ \frac{\kappa\Theta(N)}{(N-1)N^\theta} \sum_{x\in\Lambda_N} (G_s r_N^-)(\tfrac{x}{N})(\alpha - \eta_x^N(s)) \\ - \hat\kappa \int_0^1 (G_s r^-)(u)(\alpha - \rho_s(u)) du \Big\}\, ds \Big| > \frac{\delta}{2^4}\Big),\end{aligned} \tag{2.3.59}$$

and

$$\begin{aligned}\mathbb{P}_{\mu_N}\Big(\sup_{0\le t\le T} \Big| \int_0^t \Big\{ \frac{\kappa\Theta(N)}{(N-1)N^\theta} \sum_{x\in\Lambda_N} (G_s r_N^+)(\tfrac{x}{N})(\beta - \eta_x^N(s)) \\ - \hat\kappa \int_0^1 (G_s r^+)(u)(\beta - \rho_s(u)) du \Big\}\, ds \Big| > \frac{\delta}{2^4}\Big),\end{aligned} \tag{2.3.60}$$

In the case $\theta \in [2-\gamma, 1)$, since $\Theta(N) = N^2$ and $\hat{\sigma} = \sigma$, from Lemma 2.3.1 we have that (2.3.58) goes to 0 as $N \to \infty$. In the case $\theta < 2-\gamma$, since $\Theta(N) = N^{\theta+\gamma}$ and $\hat{\sigma} = 0$, from Lemma 2.3.1 we also have that (2.3.58) goes to 0 as $N \to \infty$.

In order to see that the boundary terms (2.3.59) and (2.3.60) go to 0 as $N \to \infty$ it is enough to note that since G_s has compact support in $(0,1)$ we know by Lemma 2.3.2 that $N^\gamma G_s r_N^-(u)$ and $N^\gamma G_s r_N^+(u)$ converge uniformly to $(G_s r^-)(u)$ and $(G_s r^+)(u)$, respectively, as $N \to \infty$. This ends the proof. □

2.4 Proof of Hydrostatic Limit and generalized Fick's law

In this section we prove Theorems 2.2.15 and 2.2.16. Let $\mathcal{M}_2^+$, be the space of positive measures on $[0,1]^2$ with total mass bounded by 1 equipped with the weak topology. For any $\eta \in \Omega_N$ the empirical measure $\hat{\pi}^N(\eta) \in \mathcal{M}_2^+$ is defined by

$$\hat{\pi}^N(\eta) = \frac{1}{(N-1)^2} \sum_{x,y=1}^{N-1} \eta_x \eta_y \delta_{(x/N, y/N)}$$

where $\delta_{(u,v)}$ is the Dirac mass on $(u,v) \in [0,1]^2$. Recall $\mathcal{M}^+$ introduced in Subsection 2.2.3. Let $\mathbb{P}^N$ be the law on $\mathcal{M}^+ \times \mathcal{M}_2^+$ induced by $(\pi^N, \hat{\pi}^N) : \Omega_N \to \mathcal{M}^+ \times \mathcal{M}_2^+$ when Ω_N is equipped with the non-equilibrium stationary state $\bar{\mu}_N$. To simplify notations, we denote $\hat{\pi}^N(\eta)$ by $\hat{\pi}^N$ and the action of $\pi \in \mathcal{M}_2^+$ on a continuous function $G : [0,1]^2 \to \mathbb{R}$ by $\langle \pi, G \rangle = \int_{[0,1]^2} G(u)\pi(du)$.

Our goal is to prove that every limit point $\mathbb{P}^*$ of the sequence $\{\mathbb{P}^N\}_{N\geq 2}$ is concentrated on the set of measures $(\pi, \hat{\pi})$ of $\mathcal{M}^+ \times \mathcal{M}_2^+$ such that π (resp. $\hat{\pi}$) is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ (resp. $[0,1]^2$) and with a density $\bar{\rho}^\kappa(u)$ for all $u \in (0,1)$ (resp. $\bar{\rho}^\kappa(u)\bar{\rho}^\kappa(v)$ for all $(u,v) \in (0,1)^2$) where $\bar{\rho}^\kappa$ is a weak solution of (2.2.7).

Lemma 2.4.1. *The sequence $\{\mathbb{P}^N\}_{N\geq 2}$ is tight. Let $\mathbb{P}^*$ be a limit point of the sequence $\{\mathbb{P}^N\}_{N\geq 2}$. Then $\mathbb{P}^*$ is concentrated on absolutely continuous measures. The density π is a positive function in $\mathcal{H}^1([0,1])$ and satisfies that $\int_0^1 \left\{ \frac{(\alpha-\pi(u))^2}{u^\gamma} + \frac{(\beta-\pi(u))^2}{(1-u)^\gamma} \right\} du < \infty$.*

Proof. Since $\mathcal{M}^+ \times \mathcal{M}_2^+$ is compact in the weak topology we have that the sequence $\{\mathbb{P}^N\}_{N\geq 2}$ is tight on $\mathcal{M}^+ \times \mathcal{M}_2^+$ (see e.g [7]). $\mathbb{P}^*$ is concentrated on absolutely continuous measures because the process allows at most one particle per site. By construction we get that the densities of $\hat{\pi}$ are product.

The proof that the density π belongs to $\mathcal{H}^1([0,1])$ and satisfies $\int_0^1 \left\{ \frac{(\alpha-\pi(u))^2}{u^\gamma} + \frac{(\beta-\pi(u))^2}{(1-u)^\gamma} \right\} du < \infty$ is analogous to the one done in Section 2.3.7, and for this reason it is reported to Appendix 4.7. □

Let $\mathbb{P}^*$ be a limit point of the sequence $\{\mathbb{P}^N\}_{N\geq 2}$ whose existence follows from the previous lemma. Hereinafter, we assume without lost of generality that $\{\mathbb{P}^N\}_{N\geq 2}$ converges weakly to $\mathbb{P}^*$.

Lemma 2.4.2. *Let $\bar{\rho}^\kappa$ be the unique weak solution of (2.2.7). For any F,G in $C_c^\infty([0,1])$ we have*

$$\int_{[0,1]^2} \Big\{F(u)\Big[-\tfrac{\sigma^2}{2}\Delta G(v)+\kappa G(v)V_1(v)\Big] \\ + G(v)\Big[-\tfrac{\sigma^2}{2}\Delta F(u)+\kappa F(u)V_1(u)\Big]\Big\}\; I^\kappa(u,v)dudv = 0 \tag{2.4.1}$$

where

$$I^\kappa(u,v) = \mathbb{E}^*\left[(\pi(u)-\bar{\rho}^\kappa(u))(\pi(v)-\bar{\rho}^\kappa(v))\right].$$

Proof. We have that

$$\begin{aligned} N^2L_N(\langle\pi^N,G\rangle) =& \frac{1}{N-1}\sum_{x\in\Lambda_N}\left(N^2\sum_{y\in\mathbb{Z}}\big(G(\tfrac{y+x}{N})-G(\tfrac{x}{N})\big)p(y)\right)\eta_x \\ &+\frac{N^2}{N-1}\sum_{x\in\Lambda_N}\big(G(\tfrac{x}{N})r_N^-(\tfrac{x}{N})+G(\tfrac{x}{N})r_N^+(\tfrac{x}{N})\big)\,\eta_x \\ &+\frac{N^2}{N-1}\sum_{x\in\Lambda_N}\big(G(\tfrac{x}{N})r_N^-(\tfrac{x}{N})(\alpha-\eta_x)+G(\tfrac{x}{N})r_N^+(\tfrac{x}{N})(\beta-\eta_x)\big). \end{aligned} \tag{2.4.2}$$

Taking the expectation with respect to $\bar{\mu}_N$ on both sides of (2.4.2), by stationarity, the left hand side vanishes. By using Lemma 2.3.1, Lemma 2.3.2 and weak convergence we have that

$$\mathbb{E}^*\left[\int_0^1 -\frac{\sigma^2}{2}\Delta G(u)+G(u)V_1(u)\pi(u)du\right]-\kappa\int_0^1 V_0(u)G(u)du = 0. \tag{2.4.3}$$

Now we compute $L_N(\langle\hat{\pi}^N,J\rangle)$ where $J:[0,1]^2\to\mathbb{R}$ is a smooth test function with compact support strictly included in $[0,1]^2$ and which is identically equal to 0 on the diagonal. Consider a small $\delta>0$ and take a smooth even function $H_\delta:\mathbb{R}\to[0,1]$ which is equal to 0 on $[-\delta,\delta]$ and equal to 1 outside of $[-2\delta,2\delta]$. Let then $J_\delta(u,v)=F(u)G(v)H_\delta(v-u)$, $(u,v)\in[0,1]^2$.

For $u\in[0,1]$ let

$$F_{\delta,u}(v)=F(v)H_\delta(v-u), \quad G_{\delta,u}(v)=G(v)H_\delta(v-u). \tag{2.4.4}$$

By using Lemma 4.1.1 (see Appendix 4.1) we get that

$$\begin{aligned} N^2L_N(\langle\hat{\pi}^N,J_\delta\rangle) =& \frac{1}{N-1}\sum_{x\in\Lambda_N}F(\tfrac{x}{N})N^2L_N(\langle\pi^N,G_{\delta,\frac{x}{N}}\rangle)\eta_x \\ &+\frac{1}{N-1}\sum_{y\in\Lambda_N}G(\tfrac{y}{N})N^2L_N(\langle\pi^N,F_{\delta,\frac{y}{N}}\rangle)\eta_y \\ &-\frac{N^2}{(N-1)^2}\sum_{x,y\in\Lambda_N}p(y-x)(\eta_y-\eta_x)^2J_\delta(\tfrac{x}{N},\tfrac{y}{N}). \end{aligned} \tag{2.4.5}$$

Since $J_\delta(u,v)$ is equal to 0 for $|u-v|\le\delta$, we have that

$$\bar\mu_N\left(\frac{-N^2}{(N-1)^2}\sum_{x,y\in\Lambda_N}p(y-x)(\eta_y-\eta_x)^2J_\delta(\tfrac{x}{N},\tfrac{y}{N})\right)=O(N^{1-\gamma}).$$

We multiply (2.4.5) by N^2 and take the expectation with respect to $\bar\mu_N$ on both sides, the left hand side being then equal to 0 by stationarity. By using Lemmas 2.3.1 and 2.3.2, (2.4.3) and weak convergence we conclude that

$$\begin{aligned}&\mathbb{E}^*\left[\int_{[0,1]^2}\left\{F(u)(-\tfrac{\sigma^2}{2}\Delta G_{\delta,u}(v)+\kappa V_1(v)G_{\delta,u}(v))\right\}\pi(u)\pi(v)dudv\right]\\ &+\mathbb{E}^*\left[\int_{[0,1]^2}\left\{G(v)(-\tfrac{\sigma^2}{2}\Delta F_{\delta,v}(u)+\kappa V_1(u)F_{\delta,v}(u))\right\}\pi(u)\pi(v)dudv\right]\\ &-\mathbb{E}^*\left[\int_{[0,1]^2}\kappa\left\{F(u)G_{\delta,u}(v)V_0(v)\,\pi(u)+F_{\delta,v}(u)G(v)V_0(u)\pi(v)\right\}dudv\right]=0.\end{aligned}$$

We can take the limit $\delta\to0$ and since H_δ converges to the function identically equal to 1, we get

$$\begin{aligned}&\mathbb{E}^*\left[\int_{[0,1]^2}\left\{F(u)(-\tfrac{\sigma^2}{2}\Delta G(v)+\kappa V_1(v)G(v))\right\}\pi(u)\pi(v)dudv\right]\\ &+\mathbb{E}^*\left[\int_{[0,1]^2}\left\{G(v)(-\tfrac{\sigma^2}{2}\Delta F(u)+\kappa V_1(u)F(u))\right\}\pi(u)\pi(v)dudv\right]\\ &-\mathbb{E}^*\left[\int_{[0,1]^2}\kappa\left\{F(u)G(v)V_0(v)\,\pi(u)+F(u)G(v)V_0(u)\pi(v)\right\}dudv\right]=0.\end{aligned}\tag{2.4.6}$$

Let $\bar\rho$ be the unique weak solution of (2.2.7). Then we have

$$\int_0^1-\tfrac{\sigma^2}{2}\Delta G(u)\bar\rho^\kappa(u)+\kappa V_1(u)\bar\rho^\kappa(u)G(u)du-\kappa\int_0^1G(u)V_0(u)\,du=0,\tag{2.4.7}$$

for all $G\in C_c^\infty((0,1))$. By using (2.4.3) and (2.4.7) in equation (2.4.6), then (2.4.1) follows. □

Now consider the following definition

Definition 2.4.3. *We say that $\bar I^\kappa:[0,1]^2\to[0,1]$ is a weak solution of*

$$\begin{cases}-\frac{\sigma^2}{2}\Delta\bar I^\kappa(u,v)+\kappa\bar I^\kappa(u,v)\hat V(u,v)=0, & (u,v)\in(0,1)^2,\\ \bar I^\kappa(u,v)=0, & (u,v)\in\partial[0,1]^2\end{cases}\tag{2.4.8}$$

where $\hat V(u,v)=V_1(u)+V_1(v)$ and $\partial[0,1]^2$ denotes the boundary of the set $[0,1]^2$, if *i)* $\bar{I}^\kappa \in \mathscr{H}^1_{0,\hat{V}}([0,1]^2)$,

ii) *For any function* $G \in C_c^\infty((0,1)^2)$ *we have that*

$$-\langle \bar{I}^\kappa, \tfrac{\sigma^2}{2}\Delta G\rangle + \kappa\langle \bar{I}^\kappa, G\rangle_{\hat{V}} = 0. \tag{2.4.9}$$

Lemma 2.4.4. *The unique weak solution of (2.4.8) is the constant function equal to zero.*

Proof. It is clear that the function equal to zero is a weak solution of (2.4.8). Now, we use the Lax-Milgram's Theorem in order to prove uniqueness.

Let $a^\kappa : \mathscr{H}^1_{0,\hat{V}}([0,1]^2) \times \mathscr{H}^1_{0,\hat{V}}([0,1]^2) \to \mathbb{R}$ be the bilinear form defined as

$$a^\kappa(\varphi, \varrho) = \langle \varphi, \varrho\rangle_1 + \kappa\langle \varphi, \varrho\rangle_{\hat{V}},$$

for functions $\varphi, \varrho \in \mathscr{H}^1_{0,\hat{V}}([0,1]^2)$. We note that a^κ is coercive, indeed

$$a^\kappa(\varphi, \varphi) = \|\varphi\|_1^2 + \kappa\|\varphi\|_{\hat{V}}^2 \geq \min\{1, \kappa\hat{V}(\tfrac{1}{2})\}\|\varphi\|^2_{\mathscr{H}^1([0,1]^2)}$$

and trivially we have that $a^\kappa(\varphi,\varphi) \geq \kappa\|\varphi\|^2_{\hat{V}}$. By using the Cauchy-Schwarz's inequality we get that

$$|a^\kappa(\varphi, \varrho)| \leq \|\varphi\|_1\|\varphi\|_1 + \kappa\|\varrho\|_{\hat{V}}\|\varrho\|_{\hat{V}}.$$

The latter allows to conclude that the bilinear form a^κ is also continuous. Then the Lax-Milgram's Theorem guarantees that there exists a unique function $\bar{I}^\kappa$, which satisfies (2.4.9) for any function $G \in C_c^\infty((0,1)^2)$. □

2.4.1 Proof of Theorem 2.2.15

Let $\bar{\rho}^\kappa$ the weak solution of (2.2.7) and recall the definition of the function $I^\kappa : [0,1]^2 \to \mathbb{R}$ introduced in Lemma 2.4.2. We want to prove that I^κ is a weak solution of (2.4.8). First, we claim that $I^\kappa \in \mathscr{H}^1_{0,\hat{V}}([0,1]^2) = \mathscr{H}^1_0([0,1]^2) \cap L^2_{\hat{V}}([0,1]^2)$. Indeed, since $\bar{\rho}^\kappa, \pi \in \mathscr{H}^1([0,1])$ (see Definition 2.2.10 and Lemma 2.4.1) then we have $I^\kappa \in \mathscr{H}^1_0([0,1]^2)$. In order to show that $I^\kappa \in L^2_{\hat{V}}([0,1]^2)$, note that

$$\begin{aligned}\int_{[0,1]^2} (I^\kappa(u,v))^2\hat{V}(u,v)dudv &\leq \mathbb{E}^*\left[\int_{[0,1]^2} P^2(u,v)\hat{V}(u,v)dudv\right]\\ &\leq 2\mathbb{E}^*\left[\int_{[0,1]^2} P^2(u,v)V_1(v)dudv\right],\end{aligned} \tag{2.4.10}$$

where $P(u,v) = (\pi(u) - \bar{\rho}^\kappa(u))(\pi(v) - \bar{\rho}^\kappa(v))$. In the first inequality above we used Jensen's inequality and in the last one we performed a change of variables. Note that the last term on the right hand side of (2.4.10) is bounded from above by a constant times

$$\mathbb{E}^*\left[\int_0^1 (\pi(u) - \bar{\rho}^\kappa(u))du \int_0^1 \left((\pi(v) - \bar{\rho}^\infty(v))^2 + (\bar{\rho}^\infty(v) - \bar{\rho}^\kappa(v))^2\right)V_1(v)dv\right]. \tag{2.4.11}$$

Since we know that $\pi, \bar{\rho}^\kappa$ satisfy items i) and ii) then by Remarks 2.2.11 and 2.2.12 we have that (2.4.11) is finite. Therefore we get that $I^\kappa \in L^2_{\hat{V}}([0,1]^2)$. Now, by Lemma 2.4.2 we have that the function I^κ is a weak solution of (2.4.8). By Lemma 2.4.4 we have that $I^\kappa \equiv 0$. Whence we conclude that $I(u,u) = 0$ for all $u \in (0,1)$ or equivalently $\mathbb{P}^*$ almost surely $\pi = \bar{\rho}^\kappa$. This concludes the proof of Theorem 2.2.15.

□

An important step in the proof of Theorem 2.2.16 is the stationarity of $\bar{\mu}_N$ in order to derive an upper bound of the average current. Recall that the expectation with respect to $\bar{\mu}_N$ is denoted by $\langle \cdot \rangle_N$.

Lemma 2.4.5. *Fix $N \geq 2$. There exists a constant $C > 0$ such that $\langle W_1 \rangle_N \leq CN^{-1}$.*

Proof. By stationarity of $\bar{\mu}_N$ we have that

$$\langle W_1 \rangle_N = \frac{1}{N-1}\sum_{x=1}^{N-1}\langle W_x \rangle_N = \frac{1}{N-1}\sum_{x=1}^{N-1}\langle W_x^0 \rangle_N + \frac{1}{N-1}\sum_{k=1}^{N-1}\langle W_x^{\ell,r} \rangle_N = (I) + (II).$$

For (I) we observe that

$$(I) = \frac{1}{N-1}\sum_{\substack{y<z \\ y,z\in\Lambda_N}} p(z-y)(z-y)[\langle \eta_y \rangle_N - \langle \eta_z \rangle_N] = -\frac{1}{N-1}\sum_{y=1}^{N-2}\sum_{x=1}^{N-1-y} xp(x)[\langle \eta_{y+x} \rangle_N - \langle \eta_y \rangle_N].$$

Now, using Fubini's Theorem we get

$$(I) = -\frac{1}{N-1}\sum_{x=1}^{N-2} xp(x) \sum_{y=1}^{N-1-x}[\langle \eta_{y+x} \rangle_N - \langle \eta_y \rangle_N].$$

Observe that for any sequence $(f(x))_{x\in\mathbb{Z}}$ and any $n, k \geq 1$ we have

$$\sum_{x=1}^{n}[f(x+k) - f(x)] = \sum_{x=1}^{k}[f(n+1+k-x) - f(x)]. \tag{2.4.12}$$

It follows that

$$(I) = -\frac{1}{N-1}\sum_{x=1}^{N-2} xp(x)\sum_{y=1}^{x}[\langle \eta_{N-y} \rangle_N - \langle \eta_y \rangle_N]$$

so that

$$|I| \quad \leq \quad \frac{2}{N-1}\sum_{x=1}^{N-2} x^2 p(x) \leq \sigma^2 (N-1)^{-1}.$$

The last inequality is obtained using the fact that p has finite variance.

For (II) we first use Fubini's theorem which permits to rewrite (II) as

$$\frac{\kappa}{N^{\theta+1}}\sum_{x=1}^{N-1} x r_N^-(\tfrac{x}{N})(\alpha-\langle\eta_x\rangle_N)+\frac{\kappa}{N^{\theta+1}}\sum_{x=1}^{N-1}(N-1-x)r_N^+(\tfrac{x}{N})(\langle\eta_x\rangle_N-\beta).$$

We will just analyze the first term on the right hand side of the latter expression, because analogous arguments can be done for the other one. Fix $a\in(0,\frac{1}{2})$. Note that

$$\left|\frac{\kappa}{N^{\theta+1}}\sum_{x=1}^{N-1} x r_N^-(\tfrac{x}{N})(\alpha-\langle\eta_x\rangle_N)\right|\leq\frac{\kappa}{N^{\theta+1}}\left(\sum_{x=1}^{[aN]-1} x r_N^-(\tfrac{x}{N})|\alpha-\langle\eta_x\rangle_N|+2\sum_{x=[aN]}^{N-1} x r_N^-(\tfrac{x}{N})\right).$$

Using Lemma 2.3.2 we get that

$$\frac{2\kappa}{N^{\theta+1}}\sum_{x=[aN]}^{N-1} x r_N^-(\tfrac{x}{N})\lesssim N^{-1}.$$

Since the measure $\bar{\mu}_N$ is invariant, by writing $\langle L_N\eta_x\rangle_N=0$, it is easy to see that

$$\langle\eta_x\rangle_N=\frac{\sum_{y\in\Lambda_N}p(x,y)\langle\eta_y\rangle_N+\frac{\kappa}{N^\theta}\beta r_N^+\left(\frac{x}{N}\right)+\alpha\frac{\kappa}{N^\theta}r_N^-\left(\frac{x}{N}\right)}{\sum_{y\in\Lambda_N}p(x,y)+\frac{\kappa}{N^\theta}r_N^+\left(\frac{x}{N}\right)+\frac{\kappa}{N^\theta}r_N^-\left(\frac{x}{N}\right)},$$

for any $x\in\Lambda_N$. Then we have that

$$\langle\eta_x\rangle_N-\alpha=\frac{\sum_{y\in\Lambda_N}p(x,y)(\langle\eta_y\rangle_N-\alpha)+\frac{\kappa}{N^\theta}(\beta-\alpha)r_N^+\left(\frac{x}{N}\right)}{\sum_{y\in\Lambda_N}p(x,y)+\frac{\kappa}{N^\theta}r_N^+\left(\frac{x}{N}\right)+\frac{\kappa}{N^\theta}r_N^-\left(\frac{x}{N}\right)}.$$

By neglecting terms in the denominator and bounding from above $|\langle\eta_y\rangle_N-\alpha|$ by 2 , then for any $x\in\{1,\cdots,[aN]-1\}$ we have that

$$\begin{aligned}|\langle\eta_x\rangle_N-\alpha| &\leq \frac{2\sum_{y\in\Lambda_N}p(x,y)+\frac{\kappa}{N^\theta}(\beta-\alpha)r_N^+\left(\frac{x}{N}\right)}{\frac{\kappa}{N^\theta}p(1)}\leq\frac{2N^\theta}{\kappa p(1)}+\frac{(\beta-\alpha)r_N^+\left(\frac{x}{N}\right)}{p(1)}\\ &= \frac{N^\theta}{c_\gamma\kappa}+c_\gamma^{-1}(\beta-\alpha)r_N^-\left(\frac{N-x}{N}\right)\leq\frac{N^\theta}{c_\gamma\kappa}+\gamma^{-1}(\beta-\alpha)(N-[aN])^{-\gamma}.\end{aligned}$$

Then, using the previous result and the fact that $\gamma>2$, we have

$$\begin{aligned}\frac{\kappa}{N^{\theta+1}}\sum_{x=1}^{[aN]-1} x r_N^-(\tfrac{x}{N})|\alpha-\langle\eta_x\rangle_N| &\leq \left(\frac{1}{N}+\frac{\kappa(\beta-\alpha)}{\gamma N}\frac{N^\gamma}{[N-aN]^\gamma}\right)\frac{c_\gamma\zeta_{\gamma-1}}{\gamma}\\ &\leq \left(\frac{c_\gamma\zeta_{\gamma-1}}{\gamma}+\frac{2^\gamma\kappa(\beta-\alpha)c_\gamma\zeta_{\gamma-1}}{\gamma^2}\right)N^{-1},\end{aligned}$$

where ζ_s is the Riemann zeta function defined for $s>1$. It is clear by the previous inequality that

$$\left|\frac{\kappa}{N^{\theta+1}}\sum_{x=1}^{N-1} x r_N^-(\tfrac{x}{N})(\alpha-\langle\eta_x\rangle_N)\right| \lesssim N^{-1},$$

and we are done. □

2.4.2 Proof of Theorem 2.2.16

For any $\delta>0$ we define the function $G_\delta\in C_c^\infty((0,1))$ such that $0\le G_\delta(u)\le 1$ and $G_\delta(u)=1$ for $u\in[\delta,1-\delta]$. By stationarity of $\bar\mu_N$ we have that

$$\begin{aligned} N\langle W_{[vN]}\rangle_N=\sum_{x=1}^{N}\langle W_x\rangle_N &= \sum_{x=1}^{N}G_\delta(\tfrac{x}{N})\langle W_x\rangle_N+\sum_{x=1}^{N}(1-G_\delta(\tfrac{x}{N}))\langle W_x\rangle_N\\ &= \sum_{x=1}^{N}G_\delta(\tfrac{x}{N})\langle W_x\rangle_N+\langle W_1\rangle_N\sum_{x=1}^{N}(1-G_\delta(\tfrac{x}{N}))\\ &= \sum_{x=1}^{N}G_\delta(\tfrac{x}{N})\Big(\langle W_x^0\rangle_N+\frac{\kappa}{N^\theta}\langle W_x^{\ell,r}\rangle_N\Big)+O(\delta), \end{aligned} \tag{2.4.13}$$

where in the last equality we used the definition of G_δ and the fact that $\langle W_1\rangle_N=O(N^{-1})$ (see Lemma 2.4.5 above). We first consider the term in (2.4.13) with $\frac{\kappa}{N^\theta}\langle W_x^{\ell,r}\rangle_N$. Since G_δ has compact support included in $(0,1)$ we use Lemma 2.3.2 and Riemann sum to get easily that

$$\begin{aligned} \lim_{N\to\infty}\frac{\kappa}{N^2}\sum_{x\in\Lambda_N}G_\delta(\tfrac{x}{N})\langle N^\gamma W_x^{\ell,r}\rangle_N &= \kappa\int_0^1 G_\delta(v)\Bigg[\int_v^1(\alpha-\bar\rho_\kappa(u))r_N^-(u)du\\ &\quad -\int_0^v(\beta-\bar\rho_\kappa(u))r_N^+(u)du\Bigg]dv. \end{aligned}$$

On the other hand, by stationarity of $\bar\mu_N$ it is easy to see that $\sum_{x=1}^N G_\delta(\frac{x}{N})\langle W_x^0\rangle_N$ is equal to

$$\frac{\kappa}{N^\theta}\sum_{x=1}^{N}G_\delta(\tfrac{x}{N})\sum_{y=1}^{x-1}\Big((\alpha-\langle\eta_y\rangle_N)r_N^-(\tfrac{y}{N})+(\beta-\langle\eta_y\rangle_N)r_N^+(\tfrac{y}{N})\Big).$$

Taking N to infinity we have that last expression converges to

$$\kappa\int_0^1 G_\delta(v)\Bigg[\int_0^v(\alpha-\bar\rho_\kappa(u))r_N^-(u)du+\int_0^v(\beta-\bar\rho_\kappa(u))r_N^+(u)du\Bigg]dv.$$

By summing the two previous displays we get that

$$\lim_{N\to 0}N\langle W_{[vN]}\rangle_N=\kappa\left[\int_0^1(\alpha-\bar\rho_\kappa(u))r_N^-(u)du\right]\int_0^1 G_\delta(v)dv.$$

We conclude the proof by taking $\delta \to 0$.

2.5 Proof of Theorem 2.2.17

In this section we present some properties of the solution of (2.2.7) given in Theorem 2.2.17 which will give us an idea of its behavior. The variation of $\bar{\rho}^\kappa$ and its derivative can be summarized in Figure 2.1.

u	0		$\frac{1}{2}$		1
$(\bar{\rho}^\kappa)''(u)$	0	$+$	0	$-$	0
$(\bar{\rho}^\kappa)'(u)$	0	$\nearrow$	$(\bar{\rho}^\kappa)'(\frac{1}{2})$	$\searrow$	0
$\bar{\rho}^\kappa(u)$	α	$\nearrow$	$\frac{\alpha+\beta}{2}$	$\nearrow$	β

Figure 2.1: The variations of $\bar{\rho}^\kappa$.

As we will see, the properties of $\bar{\rho}^\kappa$ are very related to the ones satisfied by $\bar{\rho}^\infty$. We now set up those properties in the following lemma.

Lemma 2.5.1. *Setting $\bar{\rho}^\infty(u) = \alpha \frac{(1-u)^\gamma}{u^\gamma+(1-u)^\gamma} + \beta \frac{u^\gamma}{u^\gamma+(1-u)^\gamma}$, $\bar{\rho}^\infty$ has the following properties*

i) $\bar{\rho}^\infty(u)$ is a solution of (2.2.7) for $\sigma = 0$.

ii) $\bar{\rho}^\infty(u) + \bar{\rho}^\infty(1-u) = \alpha + \beta$.

iii) $(\bar{\rho}^\infty)'(u) = \gamma(\beta-\alpha)\frac{(1-u)^{\gamma-1}u^{\gamma-1}}{(u^\gamma+(1-u)^\gamma)^2}$, in particular $\bar{\rho}^\infty$ is increasing.

iv) $\bar{\rho}^\infty$ is convex on $[0,1/2]$ and concave on $[1/2,1]$.

Proof. The computations which lead to prove $i)-iii)$ are not difficult to verify, since $\bar{\rho}^\infty(u)$ has an explicit form. From $iii)$, we get that $\frac{(\bar{\rho}^\infty)''(u)}{\beta-\alpha}$ is equal to

$$\gamma(\gamma-1)\frac{u^{\gamma-2}(1-u)^{\gamma-2}}{(u^\gamma+(1-u)^\gamma)^2}(1-2u) + 2\gamma^2\frac{u^{\gamma-1}(1-u)^{\gamma-1}}{(u^\gamma+(1-u)^\gamma)^3}((1-u)^{\gamma-1}-u^{\gamma-1}).$$

We check that those two terms are both non negative for $u \in [0,1/2]$ and both non positive for $u \in [1/2,1]$. $\square$

In items $iii)$ and $iv)$ of Lemma 2.5.1, we recognize the properties that we will prove for $\bar{\rho}^\kappa$. We now start the proof of the properties listed in Theorem 2.2.17. The methods used in the proof of the item iii) are quite different from the ones used for the two first ones. Thus, we have decided to split the proof in two parts by seeking of lightness.

Proof of item $i)$ and $ii)$ of Theorem 2.2.17. We split the proof in four steps.
***First step**: Position of $\bar{\rho}^\kappa$ related to $\bar{\rho}^\infty$.*
We first prove the inequalities in item $ii)$ of Theorem 2.2.17 between $\bar{\rho}^\kappa$ and $\bar{\rho}^\infty$ by contradiction. Suppose that we can find $\hat{u} \in (\frac{1}{2}, 1)$ such that $\bar{\rho}^\kappa(\hat{u}) > \bar{\rho}^\infty(\hat{u})$. We set

$$\begin{aligned} \overline{u} &:= \max\{v \in [\hat{u}, 1] | \forall u \in [\hat{u}, v), \bar{\rho}^\kappa(u) > \bar{\rho}^\infty(u)\}, \\ \underline{u} &:= \min\{v \in [0, \hat{u}] | \forall u \in (v, \hat{u}], \bar{\rho}^\kappa(u) > \bar{\rho}^\infty(u)\}. \end{aligned} \tag{2.5.1}$$

According to Lemmas 2.2.14 and 2.5.1, $\bar{\rho}^\kappa(\frac{1}{2}) = \bar{\rho}^\infty(\frac{1}{2}) = \frac{\alpha+\beta}{2}$, therefore, $\underline{u} \geq \frac{1}{2}$. The continuity of $\bar{\rho}^\kappa - \bar{\rho}^\infty$ allow us to deduce that $\bar{\rho}^\kappa(\underline{u}) = \bar{\rho}^\infty(\underline{u})$. For all $h \in [0, \hat{u} - \underline{u}]$, we get

$$\frac{\bar{\rho}^\kappa(\underline{u}+h) - \bar{\rho}^\kappa(\underline{u})}{h} \geq \frac{\bar{\rho}^\infty(\underline{u}+h) - \bar{\rho}^\infty(\underline{u})}{h}.$$

Letting h go to 0, we deduce that $(\bar{\rho}^\kappa)'(\underline{u}) \geq (\bar{\rho}^\infty)'(\underline{u})$, which is well defined since $\underline{u} \leq \hat{u} < 1$. Take $u \in [\underline{u}, \overline{u})$, since $1 > u \geq \frac{1}{2}$ we have $(\bar{\rho}^\infty)''(u) \leq 0$ by item $iv)$ in Lemma 2.5.1. On the other hand, since $\bar{\rho}^\kappa(u) \geq \bar{\rho}^\infty(u)$, we have $(\bar{\rho}^\kappa)''(u) \geq 0$ by (2.2.7), then $(\bar{\rho}^\kappa - \bar{\rho}^\infty)''$ is non negative on $[\underline{u}, \overline{u})$. Integrating this property, we get for all $u \in [\underline{u}, \overline{u})$

$$(\bar{\rho}^\kappa)'(u) - (\bar{\rho}^\infty)'(u) \geq (\bar{\rho}^\kappa)'(\underline{u}) - (\bar{\rho}^\infty)'(\underline{u}) \geq 0.$$

Since $\overline{u} > \hat{u}$, by a second integration over $[\hat{u}, \bar{u}]$ we deduce that

$$\bar{\rho}^\kappa(\overline{u}) - \bar{\rho}^\infty(\overline{u}) \geq \bar{\rho}^\kappa(\hat{u}) - \bar{\rho}^\infty(\hat{u}) > 0.$$

By continuity of $\bar{\rho}^\kappa - \bar{\rho}^\infty$, last expression is only possible when $\overline{u} = 1$. We deduce that $\bar{\rho}^\kappa(1) > \bar{\rho}^\infty(1)$ which is wrong since both terms are equal to β. On $[\frac{1}{2}, 1]$, we have proved that $\bar{\rho}^\kappa \leq \bar{\rho}^\infty$.
If we can find $u \in (\frac{1}{2}, 1)$ such that $\bar{\rho}^\kappa(u) = \bar{\rho}^\infty(u)$, then u is a local maximum of $\bar{\rho}^\kappa - \bar{\rho}^\infty$. According to Lemma 2.5.1 and (2.2.7), $(\bar{\rho}^\kappa)''(u) - (\bar{\rho}^\infty)''(u) = -(\bar{\rho}^\infty)''(u) > 0$, which is a contradiction. On $(\frac{1}{2}, 1)$, we have proved that $\bar{\rho}^\kappa < \bar{\rho}^\infty$. The opposite inequality on $(0, 1/2)$ can be easily deduced from Lemma 2.2.14 and item $ii)$ of Lemma 2.5.1. Finally we have

$$\bar{\rho}^\kappa(u) > \bar{\rho}^\infty(u) \text{ for all } u \in (0, \tfrac{1}{2}) \text{ and } \bar{\rho}^\kappa(u) < \bar{\rho}^\infty(u) \text{ for all } u \in (\tfrac{1}{2}, 1). \tag{2.5.2}$$

***Second step**: Position of $\bar{\rho}^\kappa$ related to $\bar{\rho}^\iota$.*
The proof of that point is very similar to the previous one, we just point out the differences. As previously we argue by contradiction and suppose that for $\hat{u} \in (\frac{1}{2}, 1)$, we have that $\bar{\rho}^\kappa(\hat{u}) > \bar{\rho}^\iota(\hat{u})$. Changing ∞ for ι in the last step, we define $\frac{1}{2} \leq \underline{u} < \hat{u} < \overline{u} \leq 1$ as we did in (2.5.1). As before, we have $\bar{\rho}^\kappa \geq \bar{\rho}^\iota$ on $[\underline{u}, \overline{u}]$, $\bar{\rho}^\kappa(\underline{u}) = \bar{\rho}^\iota(\underline{u})$ and $(\bar{\rho}^\kappa)'(\underline{u}) \geq (\bar{\rho}^\iota)'(\underline{u})$. For all $u \in [\underline{u}, \overline{u}]$, we have by (2.2.7)

$$(\bar{\rho}^\iota)''(u) = \frac{2\iota}{\sigma^2} V_1(u)(\bar{\rho}^\iota(u) - \bar{\rho}^\infty(u)) \leq \frac{2\iota}{\sigma^2} V_1(u)(\bar{\rho}^\kappa(u) - \bar{\rho}^\infty(u)) \leq (\bar{\rho}^\kappa)''(u)$$

(note that we needed to know that $\bar{\rho}^\kappa(u) - \bar{\rho}^\infty(u) \leq 0$). As previously, we get $\bar{\rho}^\kappa(1) > \bar{\rho}^\iota(1)$ which is wrong.
If we can find $u \in (\frac{1}{2}, 1)$ such that $\bar{\rho}^\kappa(u) = \bar{\rho}^\iota(u)$, then u is a local maximum of $\bar{\rho}^\kappa - \bar{\rho}^\iota$.

According to (2.2.7), we get

$$(\bar\rho^\kappa - \bar\rho^\iota)''(u) = \frac{2(\kappa-\iota)}{\sigma^2} V_1(u)(\bar\rho^\kappa(u) - \bar\rho^\infty(u)).$$

Since $\bar\rho^\kappa(u) - \bar\rho^\infty(u) < 0$, we deduce $(\bar\rho^\kappa - \bar\rho^\iota)''(u) > 0$ which is a contradiction.
Using Lemma 2.2.14 again, we finally get

$$\bar\rho^\kappa(u) > \bar\rho^\iota(u) \text{ for all } u \in (0, \tfrac{1}{2}) \text{ and } \bar\rho^\kappa(u) < \bar\rho^\iota(u) \text{ for all } u \in (\tfrac{1}{2}, 1). \tag{2.5.3}$$

Third step: *Proof of* ii).
On $[\frac{1}{2}, 1]$ we have proved that $\bar\rho^\kappa$ is strictly concave, $\bar\rho^0$ is a linear function given by

$$\bar\rho^0(u) = (\beta - \alpha)u + \alpha. \tag{2.5.4}$$

Since $\bar\rho^\kappa(\frac{1}{2}) = \bar\rho^0(\frac{1}{2})$ and $\bar\rho^\kappa(1) = \bar\rho^0(1)$, we deduce by concavity that $\bar\rho^0 < \bar\rho^\kappa$ on $(\frac{1}{2}, 1)$. Using Lemma 2.2.14 again, we get

$$\bar\rho^0(u) > \bar\rho^\kappa(u) \text{ for all } u \in (0, \tfrac{1}{2}) \text{ and } \bar\rho^0(u) < \bar\rho^\kappa(u) \text{ for all } u \in (\tfrac{1}{2}, 1). \tag{2.5.5}$$

Putting (2.5.2), (2.5.3) and (2.5.5) together, we have proved item ii) of Theorem 2.2.17.
Fourth step: *Proof of* i).
According to (2.5.2) and (2.2.7), it is clear that $(\bar\rho^\kappa)'$ increases on $[0, \frac{1}{2}]$ and decreases on $[\frac{1}{2}, 1]$. The convexity and the concavity of $\bar\rho^\kappa$ on these sets is established. Since $(\bar\rho^\kappa)'' \le 0$ on $[\frac{1}{2}, 1)$, $(\bar\rho^\kappa)'(u)$ goes to a limit $\ell \in \mathbb{R} \cup \{-\infty\}$ when u goes to 1. By (2.5.2), for all u in $[\frac{1}{2}, 1]$, we also have $\bar\rho^\kappa(u) \le \bar\rho^\infty(u) \le \beta = \bar\rho^\kappa(1)$, then ℓ cannot be negative. Using Lemma 2.2.14 to deduce what happens at 0, we have

$$\lim_{u\to 0} (\bar\rho^\kappa)'(u) = \lim_{u\to 1} (\bar\rho^\kappa)'(u) = \ell \in \mathbb{R}_+. \tag{2.5.6}$$

From the variations of $(\bar\rho^\kappa)'$, we deduce that $(\bar\rho^\kappa)'(u) \ge \ell \ge 0$ on $[0, 1]$.
According to Lemmas 2.2.14 and 2.5.1 and (2.5.4), it is clear that we have $\bar\rho^\infty\left(\frac{1}{2}\right) = \bar\rho^\kappa\left(\frac{1}{2}\right) = \bar\rho^0\left(\frac{1}{2}\right) = \frac{\alpha+\beta}{2}$. For all u in $[\frac{1}{2}, 1]$ we have established $\bar\rho^0 \le \bar\rho^\kappa \le \bar\rho^\infty$, by item iii) in Lemma 2.5.1 and (2.5.4), we deduce that

$$(\beta - \alpha) = (\bar\rho^0)'\left(\tfrac{1}{2}\right) \le (\bar\rho^\kappa)'\left(\tfrac{1}{2}\right) \le (\bar\rho^\infty)'\left(\tfrac{1}{2}\right) = \gamma(\beta - \alpha).$$

It ends the proof of item i) of Theorem 2.2.17. □

We end by investigating the behavior of $\bar\rho^\kappa$ at the boundary.

Proof of item iii) of Theorem 2.2.17. According to (2.5.6), it is clear that $\bar\rho^\kappa \in C^1([0, 1])$ with $(\bar\rho^\kappa)'(0) = (\bar\rho^\kappa)'(1) = \ell$. Using the first order Taylor approximation of $\bar\rho^\kappa$ around 0, we get from (2.2.7) that

$$(\bar\rho^\kappa)''(u) \underset{u\to 0}{=} \frac{2c_\gamma\kappa}{\gamma\sigma^2}\left(\frac{\ell u + o(u)}{u^\gamma} + \frac{\alpha - \beta + \ell u + o(u)}{(1-u)^\gamma}\right) \underset{u\to 0}{=} \frac{2c_\gamma\kappa\ell u^{1-\gamma}}{\gamma\sigma^2} + o\left(u^{1-\gamma}\right).$$

Since $\gamma > 2$, we deduce that $(\bar\rho^\kappa)''$ is integrable at 0 if and only if $\ell = 0$. If not, we have $\lim_{u\to 0} (\bar\rho^\kappa)'(u) = +\infty$ which is wrong by (2.5.6). We have proved

$$\lim_{u\to 0} (\bar\rho^\kappa)'(u) = \lim_{u\to 1} (\bar\rho^\kappa)'(u) = 0. \tag{2.5.7}$$

The next part of the proof is devoted to show that $(\bar\rho^\kappa)''$ satisfies the same property. Since we

do not have any clear information about $(\bar{\rho}^\kappa)^{(3)}$, the proof is more complex and we will have to split it in several steps.
First step: *proof of* $\liminf_{u\to 0}(\bar{\rho}^\kappa)''(u) = \limsup_{u\to 1}(\bar{\rho}^\kappa)''(u) = 0$.
We now suppose that $\liminf_{u\to 0}(\bar{\rho}^\kappa)''(u) \neq 0$. According to item i) of Theorem 2.2.17, $(\bar{\rho}^\kappa)''$ is positive on $(0, \frac{1}{2})$. Then, we can find $M > 0$ such that $(\bar{\rho}^\kappa)''(u) \geq M$ in a neighborhood of 0. In other words for $n = 0$, there exists $\varepsilon, M > 0$ such that for all $u \in (0, \varepsilon)$ we have that

$$(\bar{\rho}^\kappa)''(u) \geq \frac{M}{u^{n(\gamma-2)}}. \tag{2.5.8}$$

If (2.5.8) is satisfied for $0 \leq n < \frac{1}{\gamma-2}$, we integrate it two times and since $\bar{\rho}^\kappa(0) = \alpha$ and $(\bar{\rho}^\kappa)'(0) = 0$, we get for any $u \in (0, \varepsilon)$ that

$$\bar{\rho}^\kappa(u) \geq \alpha + Cu^{2-n(\gamma-2)}$$

where $C := M\left[(1-n(\gamma-2))(2-n(\gamma-2))\right]^{-1}$. Using (2.2.7), we have that for all $u \in (0, \varepsilon)$,

$$(\bar{\rho}^\kappa)''(u) \geq \frac{2c_\gamma\kappa}{\gamma\sigma^2}\left(\frac{Cu^{2-n(\gamma-2)}}{u^\gamma} + \frac{\alpha + Cu^{2-n(\gamma-2)} - \beta}{(1-u)^\gamma}\right) \underset{u\to 0}{\sim} \frac{2c_\gamma\kappa C}{\gamma\sigma^2}\frac{1}{u^{(n+1)(\gamma-2)}}.$$

Then, changing M for $\dfrac{c_\gamma\kappa C}{\gamma\sigma^2}$ and taking potentially ϵ a bit smaller, it is clear that (2.5.8) is also satisfied for $n+1$. Finally, we take $m < \frac{1}{\gamma-2} \leq m+1$ such that (2.5.8) is satisfied for $n = m+1$ by induction. Since $(m+1)\gamma > 1$ we deduce that $(\bar{\rho}^\kappa)''$ is not integrable in $[0, \varepsilon]$ and contradicts (2.5.7). We have proved that $\liminf_{u\to 0}(\bar{\rho}^\kappa)''(u) = 0$ and using Theorem 2.2.14 we deduce that $\limsup_{u\to 1}(\bar{\rho}^\kappa)''(u) = 0$.
Second step: *Choosing the good neighborhood of* 1.
For simplicity we first suppose that $\alpha < \beta = 0$ and point out that in this situation $\bar{\rho}^\kappa$ and $\bar{\rho}^\infty$ are non positive because of item i) of Theorem 2.2.17. According to Lemma 2.5.1 and (2.2.7), it is clear that we have

$$\bar{\rho}^\infty(u) \underset{u\to 1}{\sim} \alpha(1-u)^\gamma, \quad (\bar{\rho}^\infty)'(u) \underset{u\to 1}{\sim} -\gamma\alpha(1-u)^{\gamma-1} \tag{2.5.9}$$

and

$$(\bar{\rho}^\kappa)''(u) \underset{u\to 1}{\sim} \frac{2\alpha c_\gamma\kappa}{\gamma\sigma^2}\left(\frac{\bar{\rho}^\kappa(u) - \rho_\infty(u)}{\rho_\infty(u)}\right). \tag{2.5.10}$$

We now fix $\varepsilon > 0$ and we set $A = \frac{2\gamma^2(1+\varepsilon)\sigma^2}{\kappa c_\gamma \varepsilon}$. From the equivalences given in (2.5.9) and (2.5.10), we can find $\lambda > 0$ such that for all $u \in (\lambda, 1)$, we have the following inequalities

$$\begin{array}{ll} i)\ (\bar{\rho}^\kappa)''(u) \leq \left(\frac{\alpha c_\gamma\kappa}{\gamma\sigma^2}\right)\frac{\bar{\rho}^\kappa(u) - \bar{\rho}^\infty(u)}{\bar{\rho}^\infty(u)} \leq 0 & ii)\ (\bar{\rho}^\infty)'(u) \leq -2\gamma\alpha(1-u)^{\gamma-1} \\ iii)\ \dfrac{\bar{\rho}^\infty(u)}{(1+\varepsilon)} \geq \alpha(1-u)^{\gamma-1} \geq (1+\varepsilon)\bar{\rho}^\infty(u) & iv)\ \left(\frac{1}{1-A(1-u)^{\gamma-2}}\right)^\gamma \leq (1+\varepsilon)\ . \end{array} \tag{2.5.11}$$

Since $\limsup_{u\to 1}(\bar{\rho}^\kappa)''(u) = 0$, according with i) in (2.5.11) , we can find $u_1 \in (\lambda, 1)$ such that $\frac{\bar{\rho}^\kappa(u_1)}{\bar{\rho}^\infty(u_1)} \leq (1+\varepsilon)$. We now prove that in $(u_1, 1)$, we have $\frac{\bar{\rho}^\kappa}{\bar{\rho}^\infty} \leq (1+\varepsilon)^4$. If in $(u_1, 1)$ we have $\bar{\rho}^\kappa \geq (1+\varepsilon)\bar{\rho}^\infty$, it is obvious. If not, then we take $\hat{u} \in (u_1, 1)$ such that $\bar{\rho}^\kappa(\hat{u}) < (1+\varepsilon)\bar{\rho}^\infty(\hat{u})$. We set $\underline{u} = \min\{v \in (u_1, \hat{u}) | \forall u \in (v, \hat{u}]\bar{\rho}^\kappa(u) < (1+\varepsilon)\bar{\rho}^\infty(u)\}$.
Third step: *estimate on* $\hat{u} - \underline{u}$.

By continuity, it is clear that $\bar{\rho}^\kappa(\underline{u}) = (1+\varepsilon)\bar{\rho}^\infty(\underline{u})$. Thus we have for all $h \in [0, \hat{u}-\underline{u}]$,

$$\frac{\bar{\rho}^\kappa(\underline{u}+h)-\bar{\rho}^\kappa(\underline{u})}{h} \leq \frac{(1+\varepsilon)(\bar{\rho}^\infty(\underline{u}+h)-\bar{\rho}^\infty(\underline{u}))}{h}$$

and taking $h \to 0$ we get that

$$(\bar{\rho}^\kappa)'(\underline{u}) \leq (1+\varepsilon)(\bar{\rho}^\infty)'(\underline{u}).$$

By definition of $\underline{u}$ and by (2.5.11), for all $v \in [\underline{u}, \hat{u}]$ we have

$$(\bar{\rho}^\kappa)''(v) \leq \frac{\alpha c_\gamma \kappa}{\gamma\sigma^2}\left(\frac{\bar{\rho}^\kappa(v)-\bar{\rho}^\infty(v)}{\bar{\rho}^\infty(v)}\right) \leq \frac{\varepsilon\alpha c_\gamma\kappa}{\gamma\sigma^2} \leq 0. \tag{2.5.12}$$

We now integrate (2.5.12) on $[\underline{u}, \hat{u}]$ to get

$$\begin{aligned}(\bar{\rho}^\kappa)'(\hat{u}) &= (\bar{\rho}^\kappa)'(\underline{u}) + \int_{\underline{u}}^{\hat{u}} (\bar{\rho}^\kappa)''(s)ds \leq (1+\varepsilon)(\bar{\rho}^\infty)'(\underline{u}) + \frac{\varepsilon\alpha c_\gamma\kappa}{\gamma\sigma^2}(\hat{u}-\underline{u}) \\ &\leq -2(1+\varepsilon)\gamma\alpha(1-\underline{u})^{\gamma-1} + \frac{\varepsilon\alpha c_\gamma\kappa}{\gamma\sigma^2}(\hat{u}-\underline{u}).\end{aligned}$$

Since $(\bar{\rho}^\kappa)'$ is positive, we get

$$\hat{u}-\underline{u} \leq \underbrace{\frac{2\gamma^2(1+\varepsilon)\sigma^2}{\kappa c_\gamma \varepsilon}}_{=A} (1-\underline{u})^{\gamma-1}. \tag{2.5.13}$$

***Fourth step**: estimate on $\frac{\bar{\rho}^\kappa(\hat{u})}{\bar{\rho}^\infty(\hat{u})}$ and conclusion.*
Thanks to (2.5.11), we deduce that this distance is small enough to get a good estimate of $\frac{\bar{\rho}^\kappa(\hat{u})}{\bar{\rho}^\infty(\hat{u})}$. Since $\bar{\rho}^\kappa$ increase and $\bar{\rho}^\infty$ is negative, we have:

$$\begin{aligned}\frac{\bar{\rho}^\kappa(\hat{u})}{\bar{\rho}^\infty(\hat{u})} &\leq \frac{\bar{\rho}^\kappa(\underline{u})}{\bar{\rho}^\infty(\hat{u})} \leq \frac{\bar{\rho}^\kappa(\underline{u})}{\bar{\rho}^\infty(\underline{u})}\frac{\bar{\rho}^\infty(\underline{u})}{\bar{\rho}^\infty(\hat{u})} \leq (1+\varepsilon)^3\frac{\alpha(1-\underline{u})^\gamma}{\alpha(1-\hat{u})^\gamma} \\ &\leq (1+\varepsilon)^3\left(\frac{1-\underline{u}}{1-\underline{u}-A(1-\underline{u})^{\gamma-1}}\right)^\gamma \leq (1+\varepsilon)^4.\end{aligned}$$

We have proved that we could find $u_1 < 1$ such that for all $u \in [u_1, 1)$, we have $\frac{\bar{\rho}^\kappa(u)}{\bar{\rho}^\infty(\hat{u})} \leq (1+\varepsilon)^4$, in other words $\bar{\rho}^\kappa(u) \underset{u\to 1}{\sim} \bar{\rho}^\infty(u)$. Using (2.5.9) and (2.5.10), we get

$$\lim_{u\to 1}(\bar{\rho}^\kappa)''(u) = 0 \text{ and } \bar{\rho}^\kappa(u) \underset{u\to 1}{\sim} \alpha(1-u)^\gamma.$$

If $\beta \neq 0$, we can check that $\bar{\rho}^\kappa - \beta$ is solution of (2.2.7) for the boundary conditions $(\alpha-\beta, 0)$, we get $\bar{\rho}^\kappa(u) \underset{u\to 1}{=} \beta + (\alpha-\beta)(1-u)^\gamma + o((1-u)^\gamma)$. We deduce the similar property when u goes to 0 by Lemma 2.2.14. □

Corollary 2.5.2. *The solution $\bar{\rho}^\kappa$ is unique in $C([0,1])$ and the mapping $\kappa \mapsto \bar{\rho}^\kappa$ is continuous from $[0,+\infty]$ to $C([0,1])$.*

Proof. ***Step 1** : uniqueness.* Previously, we have proved in Proposition 2.2.13 that there was a unique solution $\bar{\rho}^{\kappa}$ of (2.2.7) such that

$$\bar{\rho}^{\kappa}-\bar{\rho}^{\infty}\in\mathcal{H}^1_{0,V_1}([0,1]).$$

It is well known that $\mathcal{H}^1_0([0,1])\hookrightarrow C^{1/2}([0,1])$ (see [29]), since we also have $\bar{\rho}^{\infty}\in C^2([0,1])$, it is clear that $\bar{\rho}^{\kappa}\in C([0,1])$.

From now until the end of this step, we just consider $\bar{\rho}^{\kappa}$ as a weak solution of (2.2.7) such that $\bar{\rho}^{\kappa}\in C([0,1])$. From (2.2.7), the second weak derivative $\Delta\bar{\rho}^{\kappa}$ is continuous on $(0,1)$. Therefore, it is enough to deduce that $\Delta\bar{\rho}^{\kappa}$ is actually a classical second derivative, the argument is standard, we briefly explain how we proceed. We fix $\varepsilon>0$. For $\tau<\varepsilon$ we define $\bar{\rho}^{\kappa,\tau}=\bar{\rho}^{\kappa}*(\frac{1}{\tau}\theta(\frac{\cdot}{\tau}))$ where θ is an even non negative smooth function supported in $(0,1)$ such that $\int\theta(u)du=1$. The function $\bar{\rho}^{\kappa,\tau}$ is smooth and well defined on $[2\varepsilon,1-2\varepsilon]$ and its second derivative is $(\Delta\bar{\rho}^{\kappa})*(\frac{1}{\tau}\theta(\frac{\cdot}{\tau}))$. Using the uniform continuity of $\bar{\rho}^{\kappa}$ and $\Delta\bar{\rho}^{\kappa}$ on $[\varepsilon,1-\varepsilon]$, we prove that $\bar{\rho}^{\kappa,\tau}$ and $(\bar{\rho}^{\kappa,\tau})''$ converge, respectively, to $\bar{\rho}^{\kappa}$ and $\Delta\bar{\rho}^{\kappa}$ in $L^{\infty}([2\varepsilon,1-2\varepsilon])$ as τ goes to 0. Since $C^2([2\varepsilon,1-2\varepsilon])$ is a Banach space for that convergence, we deduce that $\bar{\rho}^{\kappa}\in C^2([2\varepsilon,1-2\varepsilon])$ and its weak and classical second derivative are both $\Delta\bar{\rho}^{\kappa}$.

Letting ε go to 0 we get that $\bar{\rho}^{\kappa}\in C^2((0,1))$ and its classical second derivative is given by (2.2.7). By induction, we get immediately that

$$\bar{\rho}^{\kappa}\in C([0,1])\cap C^{\infty}((0,1)). \tag{2.5.14}$$

One can check that in the proof of Theorem 2.2.17 we have only used (2.5.14). It is easy to check that the regularity and behavior of $\bar{\rho}^{\kappa}$ near the boundary given in $iii)$ of Theorem 2.2.17 are enough to ensure $\bar{\rho}^{\kappa}-\bar{\rho}^{\infty}\in\mathcal{H}^1_{0,V_1}([0,1])$. By Proposition 2.2.13 we deduce that the solutions are unique in $C([0,1])$.

***Step 2**: continuity.* Take a sequence of real numbers $\{\kappa_n\}_{n\in\mathbb{N}}$ monotone such that $\kappa_n\xrightarrow[n\to\infty]{}\kappa\in[0,+\infty]$. According to item $ii)$ of Theorem 2.2.17, for all u in $[0,1]$, the mapping $\iota\mapsto\bar{\rho}^{\iota}(u)$ is monotone and bounded, then $\{\bar{\rho}^{\kappa_n}(u)\}_{n\in\mathbb{N}}$ is also monotonous and bounded for all u, thus it converges. We set $\hat{\rho}(u):=\lim_{n\to\infty}\bar{\rho}^{\kappa_n}(u)\quad\forall u\in[0,1]$. According to item $i)$ of Theorem 2.2.17, for all $n\in\mathbb{N}$ we have

$$||(\bar{\rho}^{\kappa_n})'||_{L^{\infty}([0,1])}=(\bar{\rho}^{\kappa_n})'(\tfrac{1}{2})\leq\gamma(\beta-\alpha).$$

By the Arzela-Ascoli's Theorem, we can find a subsequence $\{n(k)\}_{k\in\mathbb{N}}$ such that $||\bar{\rho}_{\kappa_{n(k)}}-\hat{\rho}||_{L^{\infty}([0,1])}$ goes to 0 as $k\to\infty$. For all u, since $\{\bar{\rho}^{\kappa_n}(u)\}_{n\in\mathbb{N}}$ is monotonous and convergent, if $m>n$ we have $|\bar{\rho}_{\kappa_m}(u)-\hat{\rho}(u)|\leq|\bar{\rho}_{\kappa_n}(u)-\hat{\rho}(u)|$. Taking the supremum on all $u\in[0,1]$, we deduce that $\left\{||\bar{\rho}^{\kappa_n}-\hat{\rho}||_{L^{\infty}([0,1])}\right\}_{n\in\mathbb{N}}$ decreases. We get

$$\lim_{n\to\infty}||\bar{\rho}^{\kappa_n}-\hat{\rho}||_{L^{\infty}([0,1])}=0. \tag{2.5.15}$$

In order to conclude, we just have to identify $\hat{\rho}$. When $\kappa=+\infty$, we come back to item $i)$ of Theorem 2.2.17. It allows us to deduce that for all n, we have the uniform estimate $||(\bar{\rho}^{\kappa_n})''||_{L^1([0,1])}=2(\bar{\rho}^{\kappa_n})'(\frac{1}{2})\leq 2\gamma(\beta-\alpha)$. Dividing (2.2.7) by κ_nV_1, we get that

$$||\bar{\rho}^{\kappa_n}-\bar{\rho}^{\infty}||_{L^1(0,1)}\leq\frac{\sigma^2}{2\kappa_n}\left|\left|\frac{1}{V_1}\right|\right|_{L^{\infty}(0,1)}||(\bar{\rho}^{\kappa_n})''||_{L^1(0,1)}\leq\frac{\sigma^2}{2\kappa_n}2^{\gamma+1}\gamma(\beta-\alpha)\xrightarrow[n\to\infty]{}0.$$

By uniqueness of the limit in the distribution space, we deduce from (2.5.15) that $\hat{\rho} = \bar{\rho}^{\infty}$.

When κ belongs to $[0, +\infty)$, we end proving that $\hat{\rho}$ is the unique solution of (2.2.7). Take $\varepsilon > 0$ and $K_\varepsilon = [\varepsilon, 1-\varepsilon]$. From (2.2.7) and (2.5.15) it is clear that $(\bar{\rho}^{\kappa_n})''$ goes to the mapping $u \mapsto \frac{2\kappa}{\sigma^2} V_1(u)(\hat{\rho}(u) - \bar{\rho}^{\infty}(u))$ in $L^{\infty}(K_\varepsilon)$. Since $C^2(K_\varepsilon)$ is a Banach space for the norm $f \mapsto ||f||_{L^\infty(K_\varepsilon)} + ||f''||_{L^\infty(K_\varepsilon)}$, we deduce that $\hat{\rho} \in C^2(K_\varepsilon)$ and its second derivative is $\hat{\rho}''(u) = \frac{2\kappa}{\sigma^2} V_1(u)(\hat{\rho}(u) - \bar{\rho}^{\infty}(u))$. Letting ε go to 0 and getting the boundary conditions from (2.5.15), we deduce that $\hat{\rho}$ is the unique solution of (2.2.7) for the limit parameter κ. $\square$

Chapter 3

Super-diffusive case

Contents

3.1 Introduction

In this chapter we prove the hydrodynamic limit for the symmetric exclusion process with long jumps given by a mean zero probability transition rate with infinite variance and in contact with infinitely many reservoirs with density α at the left of the system and β at the right of the system. The strength of the reservoirs is ruled by $\kappa N^{-\theta} > 0$. Here N is the size of the system, $\kappa > 0$ and $\theta \in \mathbb{R}$. Our results are valid for $\theta \leq 0$. For $\theta = 0$, we obtain a collection of fractional reaction-diffusion equations indexed by the parameter κ and with Dirichlet boundary conditions. Their solutions also depend on κ. For $\theta < 0$, the hydrodynamic equation corresponds to a reaction equation with Dirichlet boundary conditions. The case $\theta > 0$ is still open. For that reason we also analyze the convergence of the unique weak solution of the equation in the case $\theta = 0$ when we send the parameter κ to zero. Indeed, we conjecture that the limiting profile when $\kappa \to 0$ is the one that we should obtain when taking small values of $\theta > 0$. Comparing with the case $\gamma > 2$, we do not rule out the possible presence of other transition phases.

3.2 Statement of results

In this chapter we consider the process introduced in Section 1.3, whose generator L_N is given by (1.3.2). We assume that $\gamma \in (1,2)$. Thus, we have that p has infinite variance but finite mean (see (1.3.1)).

To study the hydrodynamic limit we will consider the Markov process speeded up in the time scale $\Theta(N)$, so that $\{\eta^N(t)\}_{t\geq 0} := \{\eta(t\Theta(N))\}_{t\geq 0}$ has infinitesimal generator $\Theta(N)L_N$. Recall from Chapter 2 that $\bar{\mu}_N$ is the unique invariant measure of $\{\eta(t)\}_{t\geq 0}$ and that if $\alpha = \beta = \rho$ then $\bar{\mu}_N = \nu_\rho$. The expectation of a function f with respect to $\bar{\mu}_N$ (resp. ν_ρ) is denoted by $\langle f \rangle_N$ (resp. $\langle f \rangle_\rho$) or $\mu_N(f)$ (resp. $\nu_\rho(f)$). For any $\rho \in (0,1)$ the density of $\bar{\mu}_N$ with respect to ν_ρ is denoted by $f_{N,\rho}$

3.2.1 Notation

From now on up to the rest of this chapter we fix a finite time horizon $[0,T]$. To properly state the hydrodynamic and hydrostatic limits, we need to introduce some notations and definitions.

We recall that the fractional Laplacian $-(-\Delta)^{\gamma/2} := -(-\Delta)^{\gamma/2}_{\mathbb{R}}$ of exponent $\gamma/2$ is defined on the set of functions $G : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} \frac{|G(u)|}{(1+|u|)^{1+\gamma}} du < \infty \tag{3.2.1}$$

by

$$-(-\Delta)^{\gamma/2} G(u) = c_\gamma \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathbf{1}_{|u-v|\geq\varepsilon} \frac{G(v)-G(u)}{|u-v|^{1+\gamma}} dv$$

provided the limit exists (which is the case, for example, if G is in the Schwartz space). Above c_γ is set in (1.3.1). Up to a multiplicative constant, $-(-\Delta)^{\gamma/2}$ is the generator of a γ-Lévy stable process (see Subsection 1.5.1 in Chapter 1).

We define the operator $\mathbb{L}$ by its action on functions $G \in C^\infty((0,1))$, by

$$\forall u \in (0,1), \quad (\mathbb{L}G)(u) = c_\gamma \lim_{\varepsilon\to 0} \int_0^1 \mathbf{1}_{|u-v|\geq\varepsilon} \frac{G(v)-G(u)}{|u-v|^{1+\gamma}} dv.$$

To see that the right hand side above is well defined we perform a second order Taylor expansion of G at u, we observe by a symmetry argument that for ε sufficiently small

$$\int_0^1 \mathbf{1}_{|v-u|\geq\varepsilon} \frac{v-u}{|v-u|^{1+\gamma}} dv = \int_u^{1-u} \frac{v}{|v|^{1+\gamma}} dv$$

and we conclude by using that the remainder term is integrable. The operator $\mathbb{L}$ is called the *regional fractional Laplacian* on $(0,1)$. The semi inner-product $\langle \cdot,\cdot \rangle_{\gamma/2}$ is defined on the set $C^\infty((0,1))$ by

$$\langle G,H\rangle_{\gamma/2} = \frac{c_\gamma}{2} \iint_{[0,1]^2} \frac{(H(u)-H(v))(G(u)-G(v))}{|u-v|^{1+\gamma}} du dv.$$

The corresponding semi-norm is denoted by $\|\cdot\|_{\gamma/2}$. Observe that for any $G,H \in C^\infty((0,1))$ we have that

$$\langle G, -\mathbb{L}H\rangle = \langle -\mathbb{L}G, H\rangle = \langle G,H\rangle_{\gamma/2}.$$

Recall (1.4.4). We also introduce a family of operators indexed by κ and taking the form

$$\mathbb{L}_\kappa = \mathbb{L} - \kappa V_1. \tag{3.2.2}$$

Acting on $C_c^\infty((0,1))$ these operators are symmetric and non-positive. For $\kappa = 1$, we recover the so-called restricted fractional Laplacian (see [59]):

$$\forall u \in (0,1), \quad -(-\Delta)^{\gamma/2} G(u) = (\mathbb{L}G)(u) - V_1(u)G(u) := (\mathbb{L}_1 G)(u), \tag{3.2.3}$$

while in the limit $\kappa \to 0$ we get the regional fractional Laplacian.

Definition 3.2.1. *The Sobolev space $\mathcal{H}^{\gamma/2} := \mathcal{H}^{\gamma/2}([0,1])$ consists of all square integrable functions $g : (0,1) \to \mathbb{R}$ such that $\|g\|_{\gamma/2} < \infty$. This is a Hilbert space for the norm $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ defined by*

$$\|g\|^2_{\mathcal{H}^{\gamma/2}} := \|g\|^2 + \|g\|^2_{\gamma/2}.$$

Its elements coincide a.e. with continuous functions. The completion of $C_c^\infty((0,1))$ for this norm is denoted by $\mathcal{H}_0^{\gamma/2} := \mathcal{H}_0^{\gamma/2}([0,1])$. This is a Hilbert space whose elements coincide a.e. with continuous functions vanishing at 0 and 1. On $\mathcal{H}_0^{\gamma/2}$, the two norms $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ and $\|\cdot\|_{\gamma/2}$ are equivalent.

The space $L^2(0,T;\mathcal{H}^{\gamma/2})$ is the set of measurable functions $f : [0,T] \to \mathcal{H}^{\gamma/2}$ such that

$$\int_0^T \|f_t\|^2_{\mathcal{H}^{\gamma/2}} dt < \infty.$$

The spaces $L^2(0,T;\mathcal{H}_0^{\gamma/2})$ and $L^2(0,T;L^2_h)$ are defined similarly.

We now extend the definition of the regional fractional Laplacian on $(0,1)$, which has been defined on $C^\infty((0,1))$, to the space $\mathcal{H}^{\gamma/2}$.

Definition 3.2.2. *For $\rho \in \mathscr{H}^{\gamma/2}$ we define the distribution $\mathbb{L}\rho$ by*

$$\langle \mathbb{L}\rho, G\rangle = \langle \rho, \mathbb{L}G\rangle, \quad G \in C_c^\infty((0,1)).$$

Let us check that $\mathbb{L}\rho$ is indeed a well defined distribution. Consider a sequence $\{G_n\}_{n\geq 1} \in C_c^\infty((0,1))$ converging to 0 in the usual topology of the test functions. By the integration by parts formula for the regional fractional Laplacian (see Theorem 3.3 in [37]) we have for any $\rho \in \mathscr{H}^{\gamma/2}$ that $\langle \mathbb{L}\rho, G_n\rangle = \langle \rho, G_n\rangle_{\gamma/2}$. Now using the Cauchy-Schwarz inequality and the mean value theorem, we get that $\langle \mathbb{L}\rho, G_n\rangle$ is bounded from above by a constant times

$$\|\rho\|_{\gamma/2}\|G_n\|_{\gamma/2} \lesssim \|\rho\|_{\gamma/2}\|G_n'\|_\infty^2 \iint_{[0,1]^2} |u-v|^{1-\gamma} du dv$$

which goes to 0 as $n \to \infty$ since $\gamma \in (1,2)$. Therefore $\mathbb{L}\rho$ is a well defined distribution.

3.2.2 Hydrodynamic equations

We can now give the definition of the weak solutions of the hydrodynamic equations that will be derived in this chapter. Recall V_0 from (1.4.4).

Definition 3.2.3. *Let $\hat{\kappa} \geq 0$ be some parameter and let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho^{\hat{\kappa}} : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the non-homogeneous regional fractional reaction-diffusion equation with Dirichlet boundary conditions given by*

$$\begin{cases} \partial_t \rho_t^{\hat{\kappa}}(u) = \mathbb{L}_{\hat{\kappa}} \rho_t^{\hat{\kappa}}(u) + \hat{\kappa} V_0(u), \quad (t,u) \in [0,T] \times (0,1), \\ \rho_t^{\hat{\kappa}}(0) = \alpha, \quad \rho_t^{\hat{\kappa}}(1) = \beta, \quad t \in [0,T], \\ \rho_0^{\hat{\kappa}}(u) = g(u), \quad u \in (0,1), \end{cases} \tag{3.2.4}$$

if :

i) $\rho^{\hat{\kappa}} \in L^2(0,T;\mathscr{H}^{\gamma/2})$.

ii) $\int_0^T \int_0^1 \left\{ \frac{(\alpha - \rho_t^{\hat{\kappa}}(u))^2}{u^\gamma} + \frac{(\beta - \rho_t^{\hat{\kappa}}(u))^2}{(1-u)^\gamma} \right\} du\, dt < \infty$ *for* $\hat{\kappa} > 0$; $\rho_t^{\hat{\kappa}}(0) = \alpha$, $\rho_t^{\hat{\kappa}}(1) = \beta$ *for almost every* $t \in [0,T]$, *for* $\hat{\kappa} = 0$.

iii) For all $t \in [0,T]$ *and all functions* $G \in C_c^{1,\infty}([0,T] \times (0,1))$ *we have that*

$$F_{Dir}(t, \rho^{\hat{\kappa}}, G, g) := \left\langle \rho_t^{\hat{\kappa}}, G_t \right\rangle - \langle g, G_0\rangle - \int_0^t \left\langle \rho_s^{\hat{\kappa}}, \left(\partial_s + \mathbb{L}_{\hat{\kappa}}\right) G_s \right\rangle ds - \hat{\kappa} \int_0^t \langle G_s, V_0\rangle \, ds = 0. \tag{3.2.5}$$

Remark 3.2.4. *Note that item ii) is different for $\hat{\kappa} > 0$ and $\hat{\kappa} = 0$. We can see that the condition for $\hat{\kappa} = 0$ is weaker than the condition for $\hat{\kappa} > 0$. In fact, item i) and item ii) for $\hat{\kappa} > 0$ of the previous definition imply that $\rho_t^{\hat{\kappa}}(0) = \alpha$ and $\rho_t^{\hat{\kappa}}(1) = \beta$, for almost every t in $[0,T]$. Indeed,* *first note that by item i) we know that ρ_t is $\frac{\gamma-1}{2}$-Hölder for almost every t in $[0,T]$ (see Theorem 8.2 of [23]). Then, we note that*

$$\int_0^T \frac{(\rho_t^{\hat{\kappa}}(0)-\alpha)^2}{\gamma-1}dt = \int_0^T \lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1 \frac{(\rho_t^{\hat{\kappa}}(0)-\alpha)^2}{u^\gamma}dudt.$$

By summing and subtracting $\rho_t^{\hat{\kappa}}(u)$ inside the square in the expression on the right hand side in the previous equality and using the inequality $(a+b)^2 \le 2a^2+2b^2$ we get that the right hand side of the previous equality is bounded from above by

$$2\int_0^T \lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1 \frac{(\rho_t^{\hat{\kappa}}(0)-\rho_t^{\hat{\kappa}}(u))^2}{u^\gamma}dudt + 2\int_0^T \lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1 \frac{(\rho_t^{\hat{\kappa}}(u)-\alpha)^2}{u^\gamma}dudt.$$

Since ρ_t is $\frac{\gamma-1}{2}$-Hölder for almost every t in $[0,T]$ the term on the left hand side in the previous expression vanishes. Now, the term on the right hand side in the previous expression is bounded from above by

$$2\lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_0^T\int_0^1 \frac{(\rho_t^{\hat{\kappa}}(u)-\alpha)^2}{u^\gamma}dudt,$$

which vanishes as a consequence of item ii). Thus, we have that

$$\int_0^T \frac{(\rho_t^{\hat{\kappa}}(0)-\alpha)^2}{\gamma-1}dt = 0,$$

whence we get that $\rho_t^{\hat{\kappa}}(0)=\alpha$ for almost every t in $[0,T]$. Showing that $\rho_t^{\hat{\kappa}}(1)=\beta$ for almost every t in $[0,T]$ is completely analogous.

Moreover, the existence and uniqueness of a weak solution to the equation above, for $\hat{\kappa}>0$ does not require the strong form of ii). Nevertheless, in order to prove Theorem 3.2.10 we need to impose that condition.

Remark 3.2.5. *Observe that in the case $\hat{\kappa}=1$, since $\mathbb{L}_1 = -(-\Delta)^{\gamma/2}$ we obtain in Definition 3.2.3 the fractional heat equation with reaction and Dirichlet boundary conditions, i.e.*

$$\begin{cases} \partial_t \rho_t^1(u) = \mathbb{L}_1\rho_t^1(u) + V_0(u), \quad (t,u)\in[0,T]\times(0,1), \\ \rho_t^1(0)=\alpha, \quad \rho_t^1(1)=\beta, \quad t\in[0,T], \\ \rho_0^1(u)=g(u), \quad u\in(0,1), \end{cases}$$

by (3.2.3) and (3.2.2) the notion of item iii) is reduced to

$$F_{Dir}(t,\rho^1,G,g) := \left\langle \rho_t^1, G_t\right\rangle - \langle g, G_0\rangle - \int_0^t \left\langle \rho_s^1, \left(\partial_s - (-\Delta)^{\gamma/2}\right)G_s\right\rangle ds - \int_0^t \langle G_s, V_0\rangle\, ds = 0,$$

for all $t\in[0,T]$ and all functions $G\in C_c^{1,\infty}([0,T]\times(0,1))$.

Definition 3.2.6. *Let $\hat{\kappa} > 0$ be some parameter and let $g : [0,1] \to [0,1]$ be a measurable function. We say that $\rho^{\hat{\kappa}} : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the non-homogeneous reaction equation with Dirichlet boundary conditions given by*

$$\begin{cases} \partial_t \rho_t^{\hat{\kappa}}(u) = -\hat{\kappa}\rho_t^{\hat{\kappa}}(u)V_1(u) + \hat{\kappa}V_0(u), & (t,u) \in [0,T] \times (0,1), \\ \rho_t^{\hat{\kappa}}(0) = \alpha, \quad \rho_t^{\hat{\kappa}}(1) = \beta, & t \in [0,T], \\ \rho_0^{\hat{\kappa}}(u) = g(u), \quad u \in (0,1), \end{cases} \tag{3.2.6}$$

if:

i) $\int_0^T \int_0^1 \left\{ \frac{(\alpha - \rho_t^{\hat{\kappa}}(u))^2}{u^\gamma} + \frac{(\beta - \rho_t^{\hat{\kappa}}(u))^2}{(1-u)^\gamma} \right\} du\, dt < \infty.$

ii) For all $t \in [0,T]$ and all functions $G \in C_c^{1,\infty}([0,T] \times (0,1))$ we have

$$\begin{aligned} F_{Reac}(t, \rho^{\hat{\kappa}}, G, g) := &\left\langle \rho_t^{\hat{\kappa}}, G_t \right\rangle - \langle g, G_0 \rangle - \int_0^t \left\langle \rho_s^{\hat{\kappa}}, \partial_s G_s \right\rangle ds \\ &+ \hat{\kappa} \int_0^t \left\langle \rho_s^{\hat{\kappa}}, G_s \right\rangle_{V_1} ds - \hat{\kappa} \int_0^t \langle G_s, V_0 \rangle\, ds = 0. \end{aligned}$$

Remark 3.2.7. *Note that the explicit solution $\rho^{\infty,\hat{\kappa}}$ of (3.2.6) is given by*

$$\rho_t^{\infty,\hat{\kappa}}(u) = \bar{\rho}^{\infty}(u) + (g(u) - \bar{\rho}^{\infty}(u))e^{-t\hat{\kappa}V_1(u)},$$

where $\bar{\rho}^{\infty}(u) = \dfrac{V_0(u)}{V_1(u)}$. As we will see, the function $\bar{\rho}^{\infty}$ plays an important role in the proof of some of our main results, namely, Theorems 3.2.10 and 3.2.17.

Lemma 3.2.8. *The weak solutions of (3.2.4) and (3.2.6) are unique.*

Aiming to concentrate in the main facts, the proof of this lemma is reported to Appendix 4.6.

3.2.3 Hydrodynamic limit

First we want to state the hydrodynamic limit of the process $\{\eta^N(t)\}_{t \geq 0}$ speeded up in time scale $\Theta(N)$, with state space Ω_N and with infinitesimal generator $\Theta(N)L_N$ defined in (1.3.2). Recall (2.2.4). We denote by $\mathbb{P}_{\mu_N}$ the probability measure in the Skorohod space $\mathcal{D}_{\Omega_N}^T := \mathcal{D}([0,T], \Omega_N)$ induced by the Markov process $\{\eta^N(t)\}_{t \geq 0}$ and the initial measure μ_N in Ω_N and we denote by $\mathbb{E}_{\mu_N}$ the expectation with respect to $\mathbb{P}_{\mu_N}$. Let $\{\mathbb{Q}_N\}_{N \geq 1}$ be the sequence of probability measures on the Skorohod space $\mathcal{D}_{\mathcal{M}^+}^T := \mathcal{D}([0,T], \mathcal{M}^+)$ induced by the Markov process $\{\pi_t^N\}_{t \geq 0}$ and by $\mathbb{P}_{\mu_N}$.

Recall Definition 2.2.8. At this point we are ready to state the hydrodynamic limit of the process $\{\eta^N(t)\}_{t \geq 0}$.

Theorem 3.2.9. *(Hydrodynamic limit) Let $g:[0,1]\to[0,1]$ be a measurable function and let $\{\mu_N\}_{N\geq 1}$ be a sequence of probability measures on Ω_N associated to g. Then, for any $0\leq t\leq T$,*

$$\lim_{N\to\infty}\mathbb{P}_{\mu_N}\left(\eta^N(\cdot)\in\mathscr{D}^T_{\Omega_N}:\left|\frac{1}{N-1}\sum_{x\in\Lambda_N}G\left(\tfrac{x}{N}\right)\eta^N_x(t)-\int_0^1 G(u)\rho^\kappa_t(u)du\right|>\delta\right)=0,$$

where the time scale is given by $\Theta(N)=N^{\gamma+\theta}$ and ρ^κ_t is the unique weak solution of:

- (3.2.6) *with $\hat{\kappa}=\kappa$, if $\theta<0$;*
- *(3.2.4) with $\hat{\kappa}=\kappa$, if $\theta=0$.*

At this point it is very natural to ask about the case $\theta>0$. In the case discussed in Chapter 2 we obtained a complete panorama ($\theta\in\mathbb{R}$). In the super-diffusive case the study is more interesting and more difficult. Remember Theorem 2.2.9 in Chapter 2 and Theorem 3.2.9 above. We see that the hydrodynamic behavior obtained in the diffusive case ($\gamma>2$) with $\theta<2-\gamma$ is also valid in the super-diffusive case with $\theta<0$. In fact, these two cases are governed by a reaction equation with Dirichlet boundary conditions. The behavior in the diffusive case with $\theta=2-\gamma$ is given by the reaction-diffusion equation with Dirichlet boundary conditions. Here, it is replaced by a fractional reaction-diffusion equation with Dirichlet boundary conditions in the super-diffusive case ($\gamma\in(1,2)$), the condition $\theta=2-\gamma$ for $\gamma>2$ being replaced by the condition $\theta=0$ for $\gamma\in(1,2)$. So far, for $\theta>0$ we do not know what is the hydrodynamic behavior of the system. Now, recall that in the end of the Subsection 1.2.2, we discussed about the importance of κ in the macroscopic equations. We showed intuitively that by letting κ go to ∞ (or 0) it is possible to get the transition from one phase to another at the macroscopic level. For instance, in the diffusive case we have two phases of transition: when $\theta=2-\gamma$ and $\theta=1$. The hydrodynamic equation for the former is the reaction-diffusion equation with Dirichlet boundary conditions and depends on κ. Taking $\kappa\to 0$ (resp. $\kappa\to\infty$) it is no difficult to get the heat (resp. a reaction) equation with Dirichlet boundary conditions. Then we get a similar behavior at the macroscopic level when $\theta\in(2-\gamma,1)$ (resp. $\theta\in(-\infty,2-\gamma)$) and κ is large (resp. small) enough. Now, the hydrodynamic equation for $\theta=1$ is the heat equation with Robin boundary condition depending on κ. Taking $\kappa\to 0$ (resp. $\kappa\to\infty$) it is not difficult to get the heat equation with Neumann (resp. Dirichlet) boundary conditions. Then we get a similar behavior at the macroscopic level when $\theta\in(1,\infty)$ (resp. $(2-\gamma,1)$ and κ is enough large (resp. small).

For the super-diffusive case we have that item ii) of Theorem 3.2.10 stated below confirms that taking κ large enough in the regional fractional reaction-diffusion equation with Dirichlet boundary conditions we get a reaction equation with Dirichlet boundary conditions. Then, with the idea above in mind from item i) of Theorem 3.2.10 we conjecture that when taking κ small enough in the regional fractional reaction-diffusion equation with Dirichlet boundary conditions we pass to the fractional heat equation with Dirichlet boundary conditions. It is clear that we are not discarding other phases. In fact, we believe that all the phases obtained in the diffusive case can be extended to their fractional versions. However, the fact that the operator that governs the macroscopic state is a non-local operator makes it difficult to understand properly the corresponding boundary conditions (see Figure 1.9).

Theorem 3.2.10. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. Further, let ρ^κ be the unique weak solution of (3.2.4), with initial condition ρ_0 which is independent of κ and let $\hat{\rho}^\kappa_t := \rho^\kappa_{t/\kappa}$, for all $t \in [0,T]$. Then*

i) ρ^κ converges strongly to ρ^0 in $L^2(0,T;\mathscr{H}^{\gamma/2})$ as κ goes to 0, where ρ^0 is the weak solution of (3.2.4) with $\hat{\kappa}=0$ and initial condition ρ_0.

ii) If $\rho_0 - \bar{\rho}^\infty \in \mathscr{H}^{\gamma/2}$ then $\hat{\rho}^\kappa$ converges strongly to ρ^∞ in $L^2(0,T;L^2_{V_1})$ as κ goes to ∞, where ρ^∞ is the weak solution of (3.2.6).

Remark 3.2.11. *The convergence in Theorem 3.2.10 is also true in $L^2(0,T;L^2)$. In fact, we will see that a crucial step in the proof of the theorem is to show that ρ^κ converges strongly in $L^2(0,T;L^2)$. The convergence in i) is also true in $L^2(0,T;L^2_{V_1})$ and it is a consequence of the fractional Hardy's inequality (see (3.4.2)).*

3.2.4 Hydrostatic equation

In order to state the hydrostatic limit and fractional Fick's law we first define the Hydrostatic equation.

Definition 3.2.12. *Let $\hat{\kappa} \geq 0$ be some parameter. We say that $\bar{\rho}^{\hat{\kappa}} : [0,1] \to [0,1]$ is a weak solution of the stationary regional fractional reaction-diffusion equation with non-homogeneous Dirichlet boundary conditions given by*

$$\begin{cases} \mathbb{L}_{\hat{\kappa}} \bar{\rho}^{\hat{\kappa}}(u) + \kappa V_0(u) = 0, \quad u \in (0,1), \\ \bar{\rho}^{\hat{\kappa}}(0) = \alpha, \quad \bar{\rho}^{\hat{\kappa}}(1) = \beta, \end{cases} \tag{3.2.7}$$

if:

i) $\bar{\rho}^{\hat{\kappa}} \in \mathscr{H}^{\gamma/2}$.

ii) $\int_0^1 \left\{ \frac{(\alpha - \bar{\rho}^{\hat{\kappa}}(u))^2}{u^\gamma} + \frac{(\beta - \bar{\rho}^{\hat{\kappa}}(u))^2}{u^\gamma} \right\} du < \infty$ if $\hat{\kappa} > 0$ and $\bar{\rho}^{\hat{\kappa}}(0) = \alpha$, $\bar{\rho}^{\hat{\kappa}}(1) = \beta$ if $\hat{\kappa} = 0$.

iii) For any function $G \in C_c^\infty((0,1))$ we have

$$\bar{F}_{Dir}(\bar{\rho}^{\hat{\kappa}}, G) := \left\langle \bar{\rho}^{\hat{\kappa}}, \mathbb{L}_{\hat{\kappa}} G \right\rangle + \hat{\kappa} \left\langle G, V_0 \right\rangle = 0.$$

Remark 3.2.13. *We observe that $\bar{\rho}^0$ is a weak harmonic function for $\mathbb{L}$.*

Lemma 3.2.14. *There exists a unique weak solution of (3.2.7).*

Proof. See Appendix 4.5. $\square$

3.2.5 Hydrostatic limit and Fractional Fick's law

We study in this subsection the asymptotic behavior of the empirical measure under the stationary state $\bar{\mu}_N$ (hydrostatic limit) for the case where $\kappa = 1$ and $\theta = 0$. However this result could work for values of $\kappa > 0$. As a result of hydrostatic limit we obtain a fractional version of the Fick's law. Moreover, in order to understand the hydrostatic behavior for small values of θ, we study in Theorem 3.2.17 the limit of $\bar{\rho}^{\kappa}$ as $\kappa \to \infty$.

Theorem 3.2.15. *(Hydrostatic limit) Let $\gamma \in (1,2)$. For any continuous function $G : [0,1] \to \mathbb{R}$ we have that*

$$\lim_{N\to\infty} \frac{1}{N-1}\sum_{z=1}^{N-1} G(\tfrac{z}{N})\eta_z = \int_0^1 G(u)\bar{\rho}^1(u)du$$

in probability under $\bar{\mu}_N$ defined in Chapter 2.

The classic Fick's law describes diffusion phenomena. In the standard case, the diffusion turns out to be described locally. However, in this chapter we are considering a model which presents a non-standard diffusion and will not be described locally. Our second result is the following "fractional Fick's law". Recall the definition of the current W_x (see (2.2.10)) introduced in Chapter 2.

Theorem 3.2.16. *(Fractional Fick's law) The following fractional Fick's law holds*

$$\lim_{N\to\infty} N^{\gamma-1}\langle W_1\rangle_N = c_\gamma \int_{-\infty}^{u}\int_{u}^{\infty} \frac{\bar{\rho}^1(v)-\bar{\rho}^1(w)}{(w-v)^{1+\gamma}}dwdv \;+\; \frac{c_\gamma}{\gamma(\gamma-1)}(\beta-\alpha) \tag{3.2.8}$$

where $\bar{\rho}^1 : \mathbb{R} \to [0,1]$ is the unique solution of (3.2.7) for $\kappa = 1$ and u is arbitrary in $(0,1)$.

Observe that the current is a non-local function of the density. The right hand side of (3.2.8) does not depend on u. This can be proved by taking the derivative with respect to u on the right hand side of (3.2.8) and showing that it vanishes thanks to (3.2.12).

Our last result is about the behavior of $\bar{\rho}^{\kappa}$ as κ goes to 0 or ∞. In the case $\kappa \to \infty$, the profile converges to the explicit function given in Remark 3.2.7, verifying that for values of κ large enough we obtain a behavior similar to the case $\theta < 0$ (microscopically, it means that the interaction between the system and the reservoirs is very intense, not allowing anomalous diffusion). Similarly, we conjecture that the profile obtained taking $\kappa \to 0$ describes the hydrostatic behavior of our model for small values of $\theta > 0$.

Theorem 3.2.17. *Let $\bar{\rho}^{\kappa}$ be the unique weak solution of (3.2.7). Then,*

i) $\bar{\rho}^{\kappa}$ converges strongly to $\bar{\rho}^0$ in $\mathcal{H}^{\gamma/2}$ as κ goes to 0, where $\bar{\rho}^0$ is the weak solution of (3.2.7) with $\hat{\kappa} = 0$.

ii) $\bar{\rho}^{\kappa}$ converges strongly to $\bar{\rho}^{\infty}$ in $L^2_{V_1}$ as κ goes to ∞.

3.3 Proof of Theorem 3.2.9: Hydrodynamic limit

The proof of this theorem follows the usual approach of convergence in distribution of stochastic processes: recall the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ defined similarly as in Subsection 2.2.3 in Chapter 2. We prove tightness of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ and then we prove uniqueness of the limiting point, which we denote by $\mathbb{Q}$. These two results combined give the convergence of $\{\mathbb{Q}_N\}_{N\geq 1}$ to $\mathbb{Q}$, as $N\to\infty$. In order to characterize the limiting point $\mathbb{Q}$, we prove that all limiting points of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ are concentrated on trajectories of measures that are absolutely continuous with respect to the Lebesgue measure and whose density ρ_t^κ is a weak solution of the hydrodynamic equation as given in Definition 3.2.3. From the uniqueness of the weak solutions of this equation, namely Lemma 3.2.14, we conclude that $\{\mathbb{Q}_N\}_{N\geq 1}$ has a unique limit point $\mathbb{Q}$.

First, in following subsection we explain how item iii) in Definition 3.2.3 appears. In Subsection 3.3.2 we prove that $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight, then in Subsection 3.3.3 we obtain energy estimates which provides some regularity of the limiting trajectories, allowing to identify and fix the boundary conditions. The latter is crucial to ensure the uniqueness of the limiting point. We conclude with the characterization of the limiting point in Subsection 3.3.4.

3.3.1 Heuristics for the hydrodynamic equations

In order to make the presentation simple, let us fix a function $G:[0,1]\to\mathbb{R}$ which does not depend on time.

By Dynkin's formula (see Lemma A.5.1 in [40]) we have that

$$M_t^N(G)=\langle\pi_t^N,G\rangle-\langle\pi_0^N,G\rangle-\int_0^t\Theta(N)L_N\langle\pi_s^N,G\rangle ds,\tag{3.3.1}$$

is a martingale with respect to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ where $\mathscr{F}_t:=\sigma(\{\eta^N(s)\}_{s\leq t})$ for all $t\in[0,T]$.

Above, for an integrable function $G:[0,1]\to\mathbb{R}$, recall we used the notation $\langle\pi_t^N,G\rangle$ to represent the integral of G with respect the measure π_t^N:

$$\left\langle\pi_t^N,G\right\rangle=\frac{1}{N-1}\sum_{x\in\Lambda_N}G\left(\tfrac{x}{N}\right)\eta_x(t\Theta(N)).$$

Recall that $L_N\eta_x$ is equal to

$$\sum_{y\in\Lambda_N}p(x-y)[\eta_y-\eta_x]+\frac{\kappa}{N^\theta}\sum_{y\leq 0}p(x-y)[\alpha-\eta_x]+\frac{\kappa}{N^\theta}\sum_{y\geq N}p(x-y)[\beta-\eta_x].$$

Therefore, a simple computation shows that

$$\begin{aligned}\Theta(N)L_N(\langle\pi^N,G\rangle)=&\frac{\Theta(N)}{N-1}\sum_{x\in\Lambda_N}(\mathscr{L}_NG)(\tfrac{x}{N})\eta_x\\&+\frac{\kappa\Theta(N)}{(N-1)N^\theta}\sum_{x\in\Lambda_N}G(\tfrac{x}{N})\left(r_N^-(\tfrac{x}{N})(\alpha-\eta_x)+r_N^+(\tfrac{x}{N})(\beta-\eta_x)\right),\end{aligned}\tag{3.3.2}$$

where, we denote by $\mathscr{L}_N G$ the continuous function on $[0,1]$ which is defined as the linear interpolation of the functions

$$(\mathscr{L}_N G)(\tfrac{x}{N}) = \sum_{y\in\Lambda_N} p(y-x)\big[G(\tfrac{y}{N}) - G(\tfrac{x}{N})\big],$$

for all $x \in \Lambda_N$ with $(\mathscr{L}_N G)(0) = (\mathscr{L}_N G)(1) = 0$. We also define the functions $r_N^{\pm} : [0,1] \to \mathbb{R}$ as the linear interpolation of the function

$$r_N^-(\tfrac{x}{N}) = \sum_{y\geq x} p(y), \quad r_N^+(\tfrac{x}{N}) = \sum_{y\leq x-N} p(y),$$

for all $x \in \Lambda_N$ with $r_N^{\pm}(0) = r_N^{\pm}(\tfrac{1}{N})$ and $r_N^{\pm}(1) = r_N^{\pm}(\tfrac{N-1}{N})$. Finally, let $\mathscr{K}_N$ be the operator defined by

$$\mathscr{K}_N = \mathscr{L}_N - r_N^- - r_N^+ \tag{3.3.3}$$

which, for functions G with compact support in $[0,1]$, satisfies

$$(\mathscr{K}_N G)(\tfrac{x}{N}) = \sum_{y\in\mathbb{Z}} p(y-x)\big[G(\tfrac{y}{N}) - G(\tfrac{x}{N})\big].$$

Lemma 3.3.1. *Let G be a smooth function with compact support included in $[a,1-a]$ where $a \in (0,1)$. Then we have the following uniform convergence on $[a,1-a]$*

i) $\lim_{N\to\infty} N^{\gamma} r_N^-(u) = r^-(u)$,

ii) $\lim_{N\to\infty} N^{\gamma} r_N^+(u) = r^+(u)$,

iii) $\lim_{N\to\infty} N^{\gamma}(\mathscr{K}_N G)(u) = -\left[(-\Delta)^{\gamma/2} G\right](u)$.

Proof. This Lemma establishes uniform convergence of Riemann sums to the corresponding integrals. But since the uniformity statement requires a bit of technical work it is postponed to Appendix. The two first items of the previous lemma are in fact valid for $\gamma \in (0,\infty)$. See the proof in Appendix 4.3. For the proof of item iii) see Appendix 4.4. □

We also can deduce from the previous lemma that

$$\lim_{N\to\infty} N^{\gamma}(\mathscr{L}_N G)(u) = (\mathbb{L}G)(u) \tag{3.3.4}$$

uniformly in $[a,1-a]$, for all functions G with compact support included in $[a,1-a]$. Now, we are going to analyze all the terms in (3.3.2) for $\theta \leq 0$. Thus, we will be able to see how the different boundary conditions appear on the hydrodynamic equations given in Subsection 3.2.2 from the underlying particle system.

3.3.1.1 The case $\theta < 0$

In this regime we take $\Theta(N) = N^{\gamma+\theta}$ and a function $G \in C_c^\infty(0,1)$. By using (3.3.4) we have that the first term on the right hand side of (3.3.2) vanishes since $\theta < 0$. Now, the second term on the right hand side in (3.3.2) is equal to $\kappa\langle \alpha - \pi_t^N, Gr_N^-\rangle + \kappa\langle \beta - \pi_t^N, Gr_N^+\rangle$. By Lemma 3.3.1 the previous expression converges, as N goes to ∞, to

$$
\begin{aligned}
&\kappa \int_0^1 (\alpha - \rho_t^\kappa(u))G(u)r^-(u)du + \kappa \int_0^1 (\beta - \rho_t^\kappa(u))G(u)r^+(u)du \\
&= -\kappa \int_0^1 \rho_t^\kappa(u)G(u)V_1(u)du + \kappa \int_0^1 G(u)V_0(u)du.
\end{aligned}
$$

3.3.1.2 The case $\theta = 0$

In this regime we take $\Theta(N) = N^\gamma$ and a function $G \in C_c^\infty(0,1)$. The first term on the right hand side in (3.3.2) can be replaced, thanks to (3.3.4) by

$$
\langle \pi_t^N, \mathbb{L}G\rangle \to \int_0^1 (\mathbb{L}G)(u)\rho_t^\kappa(u)du,
$$

as N goes to ∞. Similarly, the second term on the right hand side of (3.3.2) is equal to $\kappa\langle \alpha - \pi_t^N, Gr^-\rangle + \kappa\langle \beta - \pi_t^N, Gr^+\rangle$ which converges, as N goes to ∞, to

$$
\begin{aligned}
&\kappa \int_0^1 (\alpha - \rho_t^\kappa(u))G(u)r^-(u)du + \kappa \int_0^1 (\beta - \rho_t^\kappa(u))G(u)r^+(u)du \\
&= -\kappa \int_0^1 \rho_t^\kappa(u)G(u)V_1(u)du + \kappa \int_0^1 G(u)V_0(u)du.
\end{aligned}
$$

This intuitive argument is rigorously proved in Subsection 3.3.4.

3.3.2 Tightness

In this subsection we prove that the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight. We use the usual approach, which says that is enough to show (2.3.12) for any function G belonging to $C([0,1])$. In fact, it is enough to prove it for a dense set of $C([0,1])$, e.g. $C^2([0,1])$. Above $\mathscr{T}_T$ is the set of stopping times bounded by T and we implicitly assume that all the stopping times are bounded by T, thus, $\tau + \bar{\tau}$ should be read as $(\tau+\bar{\tau})\wedge T$. We prove (2.3.12) directly for functions $G \in C^2([0,1])$ and we conclude that the sequence is tight.

Proposition 3.3.2. *The sequence of measures $\{\mathbb{Q}_N\}_{N\geq 1}$ is tight with respect to the Skorohod topology of $\mathscr{D}^T_{\mathscr{M}^+}$.*

Proof. Recall from (3.3.1) that $M_t^N(G)$ is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. In order to prove (2.3.12) it is enough to show that

$$
\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T, \bar{\tau}\leq\delta} \mathbb{E}_{\mu_N}\left[\left|\int_\tau^{\tau+\bar{\tau}} \Theta(N)L_N\langle \pi_s^N, G\rangle ds\right|\right] = 0 \tag{3.3.5}
$$

and

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathscr{T}_T,\bar\tau\le\delta}\mathbb{E}_{\mu_N}\left[\left(M^N_\tau(G)-M^N_{\tau+\bar\tau}(G)\right)^2\right]=0. \tag{3.3.6}$$

By using (3.3.4) for any function $G\in C^2([0,1])$ we can bound the first term at the right hand side in (3.3.2) by a constant. By using the fact that $|\eta^N_x(s)|\le 1$ and

$$\sum_{x\ge 1}\left(r^-_N(\tfrac{x}{N})+r^+_N(\tfrac{x}{N})\right)<\infty \tag{3.3.7}$$

(since $\gamma>1$), we can bound from above the second term at the right hand side in (3.3.2) by a constant times $\Theta(N)N^{-1-\theta}$. Considering the different values of θ we see that such term is bounded from above by a constant. Then we have that

$$|\Theta(N)L_N(\langle\pi^N_s,G\rangle)|\lesssim 1 \tag{3.3.8}$$

for any $s\le T$, which trivially implies (3.3.5).

In order to prove (3.3.6), by Dynkin's formula (see Appendix 1 in [40]) we know that

$$\left(M^N_t(G)\right)^2-\int_0^t\Theta(N)\left[L_N\langle\pi^N_s,G\rangle^2-2\langle\pi^N_s,G\rangle L_N\langle\pi^N_s,G\rangle\right]ds,$$

is a martingale with respect to the natural filtration $\{\mathscr{F}_t\}_{t\ge 0}$. By Lemma 4.1.1 in Appendix 4.1 we get that the term inside the time integral in the previous expression is equal to

$$\begin{aligned}&\frac{\Theta(N)}{(N-1)^2}\sum_{x<y\in\Lambda_N}\left(G\left(\tfrac{x}{N}\right)-G\left(\tfrac{y}{N}\right)\right)^2p(x-y)(\eta^N_y(s)-\eta^N_x(s))^2\\&+\frac{\kappa\Theta(N)}{N^\theta(N-1)^2}\sum_{x\in\Lambda_N}\left(G\left(\tfrac{x}{N}\right)\right)^2(1-2\eta^N_x(s))\left(r^-_N(\tfrac{x}{N})(\alpha-\eta^N_x(s))\right)\\&+\frac{\kappa\Theta(N)}{N^\theta(N-1)^2}\sum_{x\in\Lambda_N}\left(G\left(\tfrac{x}{N}\right)\right)^2(1-2\eta^N_x(s))\left(r^+_N(\tfrac{x}{N})(\beta-\eta^N_x(s))\right).\end{aligned} \tag{3.3.9}$$

Since the first derivative of G is bounded it is easy to see that the absolute value of (3.3.9) is bounded from above by a constant times

$$\frac{\Theta(N)}{(N-1)^4}\sum_{x,y\in\Lambda_N}(x-y)^2p(x-y)+\frac{\kappa\Theta(N)}{N^\theta(N-1)^2}\sum_{x\in\Lambda_N}\left(G\left(\tfrac{x}{N}\right)\right)^2\left(r^-_N(\tfrac{x}{N})+r^+_N(\tfrac{x}{N})\right). \tag{3.3.10}$$

Note that $(x-y)^2p(x-y)\lesssim 1$ because $\gamma>1$, so that

$$\frac{\Theta(N)}{(N-1)^4}\sum_{x,y\in\Lambda_N}(x-y)^2p(x-y)\lesssim\Theta(N)N^{-2}=O(N^{\gamma-2}).$$

By (3.3.7), the remaining terms in (3.3.10) are $O(\Theta(N)N^{-\theta-2})$ so that (3.3.10) is $O(N^{\gamma-2})$.

Thus, since τ is a stopping time and $\gamma < 2$ we have that

$$
\begin{aligned}
&\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\bar\tau\le\delta}\mathbb{E}_{\mu^N}\Big[\big(M_\tau^{N,G}-M_{\tau+\bar\tau}^{N,G}\big)^2\Big]\\
=\;&\lim_{\delta\to 0}\limsup_{N\to\infty}\sup_{\tau\in\mathcal{T}_T,\bar\tau\le\delta}\mathbb{E}_{\mu^N}\left[\int_\tau^{\tau+\bar\tau}\Theta(N)\big[L_N\langle\pi_s^N,G\rangle^2-2\langle\pi_s^N,G\rangle L_N\langle\pi_s^N,G\rangle\big]ds\right]\\
=\;&0.
\end{aligned}
$$

□

3.3.3 Energy Estimates

We prove in this subsection that any limit point $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\ge 1}$ is concentrated on trajectories $\pi_t^\kappa(u)du$ with finite energy, i.e. π^κ belongs to $L^2(0,T;\mathcal{H}^{\gamma/2})$. Moreover, we prove that π_t^κ satisfies item ii) in Definition 3.2.3. The latter is the content of Theorem 3.3.3 stated below. Fix a limit point $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\ge1}$ and assume, without of loss of generality, that the sequence $\mathbb{Q}_N$ converges to $\mathbb{Q}$ as N goes to ∞.

Theorem 3.3.3. *The probability measure $\mathbb{Q}$ is concentrated on trajectories of measures of the form $\pi_t^\kappa(u)du$, such that for any interval $I\subset[0,T]$ the density π^κ satisfies*

i) $\int_I \|\pi_t^\kappa\|^2_{\gamma/2}dt \lesssim |I|(\kappa+1)$, *if* $\theta=0$.

ii) $\displaystyle\int_I\int_0^1\left\{\frac{(\alpha-\pi_t^\kappa(u))^2}{u^\gamma}+\frac{(\beta-\pi_t^\kappa(u))^2}{(1-u)^\gamma}\right\}du\,dt\lesssim |I|\frac{\kappa+1}{\kappa}$, *if* $\theta\le 0$.

Remark 3.3.4. *It follows from item i) of the previous theorem and from Theorem 8.2 of [23] that π_t^κ is, $\mathbb{P}$ almost surely, $\frac{\gamma-1}{2}$-Hölder for all $t\in I$.*

By taking $I=[0,T]$ in item i) of Theorem 3.3.3 we see that $\pi^\kappa\in L^2(0,T;\mathcal{H}^{\gamma/2})$. Moreover, from item ii) of Theorem 3.3.3, we claim that

$$\int_I\|\pi_t^\kappa-\bar\rho^\infty\|^2_{V_1}dt\lesssim|I|\frac{\kappa+1}{\kappa}$$

where $\bar\rho^\infty$ is given in Remark 3.2.7. Note that

$$\int_I\|\pi_t^\kappa-\bar\rho^\infty\|^2_{V_1}dt=c_\gamma\int_I\int_0^1\left\{\frac{(\pi_t^\kappa(u)-\bar\rho^\infty(u))^2}{\gamma u^\gamma}+\frac{(\pi_t^\kappa(u)-\bar\rho^\infty(u))^2}{\gamma(1-u)^\gamma}\right\}dudt. \tag{3.3.11}$$

By summing and subtracting α inside the first square in the expression on the right hand side in (3.3.11), β in the second one and using the fact that $(a+b)^2\le 2(a^2+b^2)$ we get that (3.3.11) is bounded from above by

$$
\begin{aligned}
&2c_\gamma\gamma^{-1}\int_I\int_0^1\left\{\frac{(\pi_t^\kappa(u)-\alpha)^2}{u^\gamma}+\frac{(\pi_t^\kappa(u)-\beta)^2}{(1-u)^\gamma}\right\}dudt\\
&+2c_\gamma\gamma^{-1}\int_I\int_0^1\left\{\frac{(\alpha-\bar\rho^\infty(u))^2}{u^\gamma}+\frac{(\beta-\bar\rho^\infty(u))^2}{(1-u)^\gamma}\right\}dudt.
\end{aligned}
\tag{3.3.12}
$$

Now, by using item ii) of Theorem 3.3.3 we have that the first term in the previous expression is bounded by constant times $|I|\dfrac{\kappa+1}{\kappa}$. Finally, using the definition of $\bar{\rho}^{\infty}$ (see Remark 3.2.7) the second term in (3.3.12) *is equal to*

$$2c_{\gamma}\gamma^{-1}(\beta-\alpha)^2|I|\int_0^1 (u^{\gamma}+(1-u)^{\gamma})^{-1}du \lesssim 1.$$

Before we prove Theorem 3.3.3, we recall some estimates on the Dirichlet form (introduced in Subsections 2.3.3, 2.3.4) which are needed in due course.

3.3.3.1 Estimates on the Dirichlet form

In this subsection and in the proof of Theorem 3.3.3 we use h, ν_h^N, H_N and D_N introduced in Subsection 2.3.4. Our goal is to express, for the measure ν_h^N, a relation between the Dirichlet form defined by $\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}$ and the quantity D_N. More precisely, we have the following result.

Lemma 3.3.5. *For any positive constant B and any density function f with respect to ν_h^N, there exists a constant $C>0$ (independent of f and N) such that*

$$\begin{aligned}\frac{\Theta(N)}{NB}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N} \leq& -\frac{\Theta(N)}{4NB}D_N(\sqrt{f},\nu_h^N)+\frac{C\Theta(N)}{NB}\sum_{x,y\in\Lambda_N}p(y-x)\Big(h(\tfrac{x}{N})-h(\tfrac{y}{N})\Big)^2\\ &+\frac{C\kappa\Theta(N)}{N^{\theta+1}B}\sum_{x\in\Lambda_N}\left\{\Big(h(\tfrac{x}{N})-\alpha\Big)^2 r_N^-(\tfrac{x}{N})+\Big(h(\tfrac{x}{N})-\beta\Big)^2 r_N^+(\tfrac{x}{N})\right\}.\end{aligned} \tag{3.3.13}$$

The proof of this statement is similar to the one in Section 2.3.3 and thus it is omitted. Moreover, note that as a consequence of the previous lemma, for a Lipschitz function h such that $\alpha\leq h(u)\leq\beta$ we have that

$$\frac{\Theta(N)}{NB}\langle L_N\sqrt{f},\sqrt{f}\rangle_{\nu_h^N}\leq -\frac{\Theta(N)}{4NB}D_N(\sqrt{f},\nu_h^N)+\Theta(N)N^{-\gamma}\frac{C(\kappa N^{-\theta}+1)}{B}. \tag{3.3.14}$$

Note that taking $h(0)=\alpha\leq h(u)\leq\beta=h(1)$ for all $u\in[0,1]$ and h Lipschitz in Lemma 2.3.10 we also get for any positive constant A_x that

$$\Big|\langle\eta_x-\alpha,f\rangle_{\nu_h^N}\Big| \lesssim \frac{1}{A_x}I_x^{\alpha}(\sqrt{f},\nu_h^N)+A_x+\frac{x}{N}. \tag{3.3.15}$$

3.3.3.2 Proof of of Theorem 3.3.3

Firs step: $\pi^{\kappa}\in L^2(0,T;\mathcal{H}^{\gamma/2})$ $\mathbb{Q}$ almost surely. Recall that in this case ($\theta=0$) the system is speeded up in the sub-diffusive time scale $\Theta(N)=N^{\gamma}$. Let $\varepsilon>0$ be a small real number.

Let $F \in C_c^{0,\infty}(I \times [0,1]^2)$, where the I is a subinterval of $[0,T]$. By the entropy and Jensen's inequality and Feynman-Kac's formula (see Lemma A.7.2 in [40]), we have that

$$\begin{aligned}
&\mathbb{E}_{\mu_N}\Big[\int_I N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)(\eta_y^N(t)-\eta_x^N(t))\Big]dt\\
&\leq C_0 + \int_I \sup_f \Big\{N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int(\eta_y-\eta_x)f(\eta)d\nu_h^N + N^{\gamma-1}\Big\langle L_N\sqrt{f},\sqrt{f}\Big\rangle_{\nu_h^N}\Big\}dt
\end{aligned} \tag{3.3.16}$$

where the supremum is taken over all densities f on Ω_N with respect to ν_h^N. Note that, by a change of variables, we have that

$$\begin{aligned}
&N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int(\eta_y-\eta_x)f(\eta)d\nu_h^N\\
=&N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int(\eta_y-\eta_x)f(\eta)d\nu_h^N\\
=&N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int \eta_y\left(f(\eta)-f(\sigma^{x,y}\eta)\right)d\nu_h^N\\
+&N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F_t^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\int \eta_x f(\eta)\left(\theta^{x,y}(\eta)-1\right)d\nu_h^N
\end{aligned} \tag{3.3.17}$$

where $\theta^{x,y}(\eta) = \frac{d\nu_h^N(\sigma^{x,y}\eta)}{d\nu_h^N(\eta)}$ and F^a is the antisymmetric part of F, i.e. for all $t \in I$ and $(u,v) \in [0,1]^2$

$$F_t^a(u,v) = \frac{1}{2}\Big[F_t(u,v) - F_t(v,u)\Big].$$

Observe that $F_t^a(u,u) = 0$. By Young's inequality, the fact that f is a density and $|\eta_y| \leq 1$, we have that, for any $A > 0$, the third term in (3.3.17) is bounded from above by a constant times

$$\begin{aligned}
&N^{\gamma-1}A \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} \left(F_t^a\left(\tfrac{x}{N},\tfrac{y}{N}\right)\right)^2 p(y-x) + \frac{N^{\gamma-1}}{A} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} p(y-x)I_{x,y}(\sqrt{f},\nu_h^N)\\
\leq&\frac{c_\gamma A}{N^2} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} \frac{\left(F_t^a\left(\tfrac{x}{N},\tfrac{y}{N}\right)\right)^2}{|\tfrac{x}{N}-\tfrac{y}{N}|^{1+\gamma}} + \frac{2N^{\gamma-1}}{A}D_N^0(\sqrt{f},\nu_h^N).
\end{aligned}$$

Since h is Lipschitz we have that $\sup_{\eta\in\Omega_N}|\theta^{x,y}(\eta)-1| = O\left(\frac{|x-y|}{N}\right)$. By Young's inequality and the fact that f is a density, for any $A^{'}>0$, the last term in (3.3.17) is bounded from above by

$$
\begin{aligned}
&\frac{N^{\gamma-1}}{A^{'}}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\left(F_t^a\left(\tfrac{x}{N},\tfrac{y}{N}\right)\right)^2 p(y-x)\;+\;A^{'}N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}p(y-x)\left(\tfrac{|x-y|}{N}\right)^2\\
&=\frac{c_\gamma}{A^{'}N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{\left(F_t^a\left(\frac{x}{N},\frac{y}{N}\right)\right)^2}{|\frac{x}{N}-\frac{y}{N}|^{1+\gamma}}+\frac{A^{'}c_\gamma}{N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{1}{|\frac{x}{N}-\frac{y}{N}|^{\gamma-1}}.
\end{aligned}
$$

Recall (3.3.14), so that by choosing $A=8$ and $B=1$ and using the two results above we have just proved that (3.3.16) is bounded from above by C_0 plus

$$
\frac{c_\gamma(8+\frac{1}{A^{'}})}{N^2}\sum_{x\neq y\in\Lambda_N}\frac{\left[F_t^a(\frac{x}{N},\frac{y}{N})\right]^2}{|\frac{x}{N}-\frac{y}{N}|^{1+\gamma}}+C(\kappa+1)+c_\gamma A^{'}A^{''},
$$

where

$$
A^{''}:=\sup_{\varepsilon>0}\sup_{N\geq1}\frac{1}{N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{1}{|\frac{x}{N}-\frac{y}{N}|^{\gamma-1}}<\infty
$$

since $\gamma<2$. Therefore, we have proved that there exist constants $A^{'''}$ and $B^{'}$ (independent of $\varepsilon>0$, $N\geq1$, and $F\in C_c^\infty(I\times[0,1]^2)$) such that

$$
\begin{aligned}
&\mathbb{E}_{\mu_N}\left[\int_I N^{\gamma-1}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}F_t(\tfrac{x}{N},\tfrac{y}{N})p(y-x)(\eta_y^N(t)-\eta_x^N(t))\,dt\right]\\
&=\mathbb{E}_{\mu_N}\left[\int_I -2c_\gamma\langle\pi_t^N,g_t^N\rangle\,dt\right]\\
&\leq\int_I\frac{A^{'''}}{N^2}\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}}\frac{c_\gamma\left(F_t^a(\frac{x}{N},\frac{y}{N})\right)^2}{|\frac{x}{N}-\frac{y}{N}|^{1+\gamma}}dt+B^{'}|I|(\kappa+1).
\end{aligned}
\tag{3.3.18}
$$

Above the function g^N is defined on $I\times[0,1]$ by

$$
g_t^N(u)=\frac{1}{N}\sum_{y\in\Lambda_N}\mathbf{1}_{\left|\frac{y}{N}-u\right|\geq\varepsilon}\frac{F_t^a\left(u,\frac{y}{N}\right)}{|u-\frac{y}{N}|^{1+\gamma}}
$$

and it is a discretization of the smooth function g defined on $(t,u)\in I\times[0,1]$ by

$$
g_t(u)=\int_0^1\mathbf{1}_{\{|v-u|\geq\varepsilon\}}\frac{F_t^a(u,v)}{|u-v|^{1+\gamma}}dv.
$$

Let $Q_\varepsilon = \{(u,v) \in [0,1]^2\ ;\ |u-v| \geq \varepsilon\}$. Observe first that for symmetry reasons we have that for any integrable function π,

$$\int_0^1 \pi(u) g_t(u) du = \iint_{Q_\varepsilon} \frac{(\pi(v)-\pi(u)) F_t^a(u,v)}{|u-v|^{1+\gamma}} du dv.$$

By taking the limit as $N \to \infty$ in (3.3.18), we conclude that there exist constants $C > 0$ independent of $F \in C_c^{0,\infty}(I \times [0,1]^2)$ and $\varepsilon > 0$ such that

$$\mathbb{E}_{\mathbb{Q}} \left[\int_I \iint_{Q_\varepsilon} \frac{(\pi_t^\kappa(v) - \pi_t^\kappa(u)) F_t^a(u,v)}{|u-v|^{1+\gamma}} - C \frac{\left(F_t^a(u,v)\right)^2}{|u-v|^{1+\gamma}} du dv dt \right] \lesssim |I|(\kappa+1).$$

From Lemma 7.5 in [41] we can insert the supremum over F inside the expectation above, so that

$$\mathbb{E}_{\mathbb{Q}} \left[\sup_F \left\{ \int_I \iint_{Q_\varepsilon} \frac{(\pi_t^\kappa(v) - \pi_t^\kappa(u)) F_t^a(u,v)}{|u-v|^{1+\gamma}} - C \frac{\left(F_t^a(u,v)\right)^2}{|u-v|^{1+\gamma}} du dv dt \right\} \right] \lesssim |I|(\kappa+1).$$

Since the function $(u,v) \in [0,1]^2 \to \pi(v) - \pi(u)$ is antisymmetric we may replace F^a by F in the previous variational formula, i.e.

$$\mathbb{E}_{\mathbb{Q}} \left[\sup_F \left\{ \int_I \iint_{Q_\varepsilon} \frac{(\pi_t^\kappa(v) - \pi_t^\kappa(u)) F_t(u,v)}{|u-v|^{1+\gamma}} - C \frac{\left(F_t(u,v)\right)^2}{|u-v|^{1+\gamma}} du dv dt \right\} \right] \lesssim |I|(\kappa+1). \quad (3.3.19)$$

Consider the Hilbert space $\mathbb{L}^2([0,1]^2, d\mu_\varepsilon)$ where μ_ε is the measure whose density with respect to Lebesgue measure is given by $(u,v) \in [0,1]^2 \to \mathbf{1}_{|u-v|\geq\varepsilon} |u-v|^{-(1+\gamma)}$. By taking

$$\Pi^\kappa : (t; u, v) \in I \times [0,1]^2 \to \pi_t^\kappa(v) - \pi_t^\kappa(u),$$

the previous formula implies that

$$\mathbb{E}_{\mathbb{Q}} \left[\int_I \iint_{[0,1]^2} \left(\Pi_t^\kappa(u,v)\right)^2 d\mu_\varepsilon(u,v) dt \right] \lesssim |I|(\kappa+1). \quad (3.3.20)$$

Letting $\varepsilon \to 0$, by the monotone convergence theorem, we conclude that

$$\int_I \iint_{[0,1]^2} \frac{(\pi_t^\kappa(v) - \pi_t^\kappa(u))^2}{|u-v|^{1+\gamma}} du dv dt < \infty$$

$\mathbb{Q}$ almost surely.

Second step: $\int_I \int_0^1 \left\{ \frac{(\alpha - \pi_t^\kappa(u))^2}{u^\gamma} + \frac{(\beta - \pi_t^\kappa(u))^2}{(1-u)^\gamma} \right\} du\ dt < \infty$ $\mathbb{Q}$ almost surely. Now we have to prove that the function $(t,u) \to \pi_t^\kappa(u) - \alpha$ is in the space $L^2(I \times (0,1), dt \otimes d\mu)$, where μ is the measure whose density with respect to the Lebesgue measure is given by

$$u \in (0,1) \to \frac{1}{u^\gamma}.$$

A similar argument shows that the function $(t,u) \to \pi_t^\kappa(u) - \beta$ belongs to $L^2([0,T] \times (0,1), dt \otimes d\mu')$, where μ' is the measure whose density with respect to the Lebesgue measure is given by

$$u \in [0,1] \to \frac{1}{(1-u)^\gamma}.$$

Let ν_h^N be the Bernoulli product measure corresponding to a profile h which is Lipschitz such that $h(0) = \alpha \leq h(u) \leq \beta = h(1)$ for all $u \in [0,1]$. Let $G \in C_c^\infty(I \times [0,1])$. Using the entropy and Jensen's inequalities and the Feynman-Kac's formula we get that

$$\begin{aligned} &\mathbb{E}_{\mu_N}\left[\int_I N^{\gamma-1} \sum_{x \in \Lambda_N} G_t r_N^-\left(\tfrac{x}{N}\right)(\eta_x^N(t) - \alpha) \right] dt \\ &\leq C_0 + \int_I \sup_f \left\{ N^{\gamma-1} \sum_{x\in\Lambda_N} (G_t r_N^-)\left(\tfrac{x}{N}\right) \langle \eta_x - \alpha, f \rangle_{\nu_h^N} + \Theta(N) N^{-1} \left\langle L_N \sqrt{f}, \sqrt{f} \right\rangle_{\nu_h^N} \right\} dt, \end{aligned} \tag{3.3.21}$$

where the supremun is taken over all the densities f on Ω_N with respect to ν_h^N. Using (3.3.14) with $B = 1$ we can bound from above the second term on the right hand side of (3.3.21) by

$$-\frac{\Theta(N)}{4N} D_N(\sqrt{f}, \nu_h^N) + C\Theta(N) N^{-\gamma}(\kappa N^{-\theta} + 1),$$

and from 3.3.15 with $A_x = \frac{G_t\left(\frac{x}{N}\right)}{\kappa}$ the term on the right side of (3.3.21) is bounded from above by

$$\frac{CN^{\gamma-1}}{\kappa} \sum_{x\in\Lambda_N} r_N^-\left(\tfrac{x}{N}\right)\left(G_t\left(\tfrac{x}{N}\right)\right)^2 + C(\kappa+1).$$

Taking $N \to \infty$ we can conclude that there exists a constant $C' > 0$ independent of G and of t such that

$$\mathbb{E}_{\mathbb{Q}}\left[\int_I \int_0^1 \left(\frac{(\pi_t^\kappa(u) - \alpha) G_t(u)}{|u|^\gamma} - \frac{C'}{\kappa} \frac{G_t^2(u)}{|u|^\gamma} \right) du dt \right] \lesssim |I|(\kappa+1).$$

From Lemma 7.5 in [41] we can insert the supremum over G inside the expectation above, and we get

$$\mathbb{E}_{\mathbb{Q}}\left[\sup_G \left\{ \int_I \int_0^1 \left(\frac{(\pi_t^\kappa(u) - \alpha) G_t(u)}{|u|^\gamma} - \frac{C'}{\kappa} \frac{G_t^2(u)}{|u|^\gamma} \right) du dt \right\} \right] \lesssim |I|(\kappa+1). \tag{3.3.22}$$

The previous formula implies that

$$\int_I \int_0^1 \frac{(\pi_t^\kappa(u)-\alpha)^2}{|u|^\gamma}\, du dt < \infty$$

$\mathbb{Q}$ almost surely. Similarly, we get

$$\int_I \int_0^1 \frac{(\pi_t^\kappa(u)-\beta)^2}{|u|^\gamma}\, du dt < \infty$$

$\mathbb{Q}$ almost surely.

Final step. By Definition 3.2.3, the two steps above allow us to show that $\mathbb{Q}$ is concentrated on trajectories of measures whose density is a weak solution of the corresponding hydrodynamic equation (see Proposition 3.3.6). By uniqueness of the weak solution (see Lemma 3.2.8) we get that $\mathbb{Q}$ is unique. Indeed, we have that $\mathbb{Q} = \delta_{\{\rho_t^\kappa(u)du\}}$ (Dirac mass). Then, by using the latter, we compute the expectation in (3.3.20) and (3.3.22) and we are done.

□

3.3.4 Characterization of limit points

In the present subsection we characterize all limit points $\mathbb{Q}$ of the sequence $\{\mathbb{Q}_N\}_{N\geq 1}$, which we know that exist from the results of Subsection 3.3.2. Let us assume without lost of generality, that $\{\mathbb{Q}_N\}_{N\geq 1}$ converges to $\mathbb{Q}$. Since there is at most one particle per site, it is easy to show that $\mathbb{Q}$ is concentrated on trajectories of measures absolutely continuous with respect to the Lebesgue measure, i.e. $\pi_t^\kappa(du) = \rho_t^\kappa(u)du$ (for details see [40]). In Proposition 3.3.6 below we prove, for each range of θ, that $\mathbb{Q}$ is concentrated on trajectories of measures whose density satisfies a weak form of the corresponding hydrodynamic equation. Moreover, we have seen in Theorem 3.3.3 that $\mathbb{Q}$ is concentrated on trajectories of measures whose density satisfies the energy estimate, i.e. $\rho^\kappa \in L^2(0,T;\mathscr{H}^{\gamma/2})$ and

$$\int_0^T \int_0^1 \left\{ \frac{(\alpha-\rho_t^\kappa(u))^2}{u^\gamma} + \frac{(\beta-\rho_t^\kappa(u))^2}{(1-u)^\gamma} \right\} du dt < \infty.$$

Since a weak solution of the hydrodynamic equation (3.2.4) is unique we have that $\mathbb{Q}$ is unique and takes the form of a Dirac mass.

Proposition 3.3.6. *If $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\geq 1}$ then*

1. *if $\theta < 0$:*

$$\mathbb{Q}\Big(\pi_\cdot \in \mathscr{D}^T_{\mathscr{M}^+} : F_{Reac}(t,\rho^\kappa,G,g) = 0, \forall t \in [0,T],\ \forall G \in C_c^{1,2}([0,T]\times[0,1])\Big) = 1.$$

2. *if $\theta = 0$:*

$$\mathbb{Q}\Big(\pi_\cdot \in \mathscr{D}^T_{\mathscr{M}^+} : F_{Dir}(t,\rho^\kappa,G,g) = 0, \forall t \in [0,T],\ \forall G \in C_c^{1,2}([0,T]\times[0,1])\Big) = 1.$$

Proof. Note that in order to prove the proposition, it is enough to verify, for $\delta > 0$ and G in the corresponding space of test functions, that

$$\mathbb{Q}\left(\pi_\cdot \in \mathcal{D}^T_{\mathcal{M}^+} : \sup_{0\le t\le T} |F_\theta(t,\rho^\kappa,G,g)| > \delta\right) = 0,$$

for each θ, where F_θ stands for F_{Reac} if $\theta < 0$ and F_{Dir} if $\theta = 0$. Indeed, we have that

$$\begin{aligned} F_\theta(t,\rho^\kappa,G,g) = &\left\langle \rho^\kappa_t, G_t\right\rangle - \langle g, G_0\rangle - \int_0^t \left\langle \rho^\kappa_s, \left(\partial_s + \mathbf{1}_{\{\theta=0\}}\mathbb{L}\right) G_s\right\rangle ds \\ &+ \mathbf{1}_{\{\theta\le 0\}}\kappa \int_0^t \left\langle \rho^\kappa_s, G_s\right\rangle_{V_1} ds - \mathbf{1}_{\{\theta\le 0\}}\kappa \int_0^t \langle G_s, V_0\rangle\, ds = 0. \end{aligned} \tag{3.3.23}$$

From here on, in order to simplify notation, we will erase $\pi_\cdot$ from the sets that we have to look at.

By definition of F_θ above we can bound from above the previous probability by the sum of

$$\mathbb{Q}\left(\sup_{0\le t\le T} |F_\theta(t,\rho^\kappa,G,\rho_0)| > \frac{\delta}{2}\right) \tag{3.3.24}$$

and

$$\mathbb{Q}\left(|\langle \rho_0 - g, G_0\rangle| > \frac{\delta}{2}\right).$$

We note that last probability is equal to zero since $\mathbb{Q}$ is a limit point of $\{\mathbb{Q}_N\}_{N\ge 1}$ and $\mathbb{Q}_N$ is induced by μ_N which is associated to g. Now we deal with (3.3.24). Since for $\theta \le 0$ the function G_s has compact support included in $(0,1)$ the singularities of V_0 and V_1 are not present, thus from Proposition A.3 of [33], the set inside the probability in (3.3.24) is an open set in the Skorohod topology. Therefore, from Portmanteau's Theorem we bound (3.3.24) from above by

$$\liminf_{N\to\infty} \mathbb{Q}_N\left(\sup_{0\le t\le T} |F_\theta(t,\rho^\kappa,G,\rho_0)| > \frac{\delta}{2}\right).$$

Summing and subtracting $\int_0^t \Theta(N) L_N \langle \pi^N_s, G_s\rangle ds$ to the term inside the previous absolute value, recalling (3.3.1) and the definition of $\mathbb{Q}_N$, we can bound the previous probability from above by the sum of the next two terms

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T} \left|M^N_t(G)\right| > \frac{\delta}{4}\right)$$

and

$$\begin{aligned} \mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Bigg| &\int_0^t \Theta(N) L_N \langle \pi^N_s, G_s\rangle ds - \int_0^t \left\langle \pi^N_s, \mathbf{1}_{\{\theta=0\}}\mathbb{L} G_s\right\rangle ds \\ &+ \mathbf{1}_{\{\theta\le 0\}}\kappa \int_0^t \langle \rho_s, G_s\rangle_{V_1} ds - \mathbf{1}_{\{\theta\le 0\}}\kappa \int_0^t \langle G_s, V_0\rangle\, ds\Bigg| > \frac{\delta}{4}\Bigg). \end{aligned} \tag{3.3.25}$$

By Doob's inequality we have that

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\left|M_t^N(G)\right|>\frac{\delta}{4}\right)\lesssim\frac{1}{\delta^2}\mathbb{E}_{\mu_N}\left[\int_0^T\Theta(N)\left[L_N\langle\pi_s^N,G\rangle^2-2\langle\pi_s^N,G\rangle L_N\langle\pi_s^N,G\rangle\right]ds\right].$$

In the proof of Proposition 3.3.2 we have proved that the term inside the time integral in the previous expression is $O(N^{\gamma-2})$. Then, using the fact that $\gamma<2$ we have that last probability vanishes as $N\to\infty$. It remains to prove that (3.3.25) vanishes as $N\to\infty$. For that purpose, we recall (3.3.2) and we bound (3.3.25) from above by the sum of the following terms

$$\mathbb{P}_{\mu_N}\left(\sup_{0\le t\le T}\left|\int_0^t\frac{\Theta(N)}{N-1}\sum_{x\in\Lambda_N}\mathscr{L}_N G_s(\tfrac{x}{N})\eta_x^N(s)ds-\int_0^t\left\langle\pi_s^N,\mathbf{1}_{\{\theta=0\}}\mathbb{L}G_s\right\rangle ds\right|>\frac{\delta}{2^4}\right),\tag{3.3.26}$$

$$\begin{aligned}\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t\Bigg\{&\frac{\kappa\Theta(N)}{N^\theta(N-1)}\sum_{x\in\Lambda_N}(G_s r_N^-)(\tfrac{x}{N})(\alpha-\eta_x^N(s))\\&-\mathbf{1}_{\{\theta\le0\}}\kappa\int_0^1(G_s r^-)(u)(\alpha-\rho_s^\kappa(u))du\Bigg\}ds\Bigg|>\frac{\delta}{2^4}\Bigg)\end{aligned}\tag{3.3.27}$$

and

$$\begin{aligned}\mathbb{P}_{\mu_N}\Bigg(\sup_{0\le t\le T}\Bigg|\int_0^t\Bigg\{&\frac{\kappa\Theta(N)}{N^\theta(N-1)}\sum_{x\in\Lambda_N}(G_s r_N^+)(\tfrac{x}{N})(\beta-\eta_x^N(s))\\&-\mathbf{1}_{\{\theta\le0\}}\kappa\int_0^1(G_s r^+)(u)(\beta-\rho_s^\kappa(u))du\Bigg\}ds\Bigg|>\frac{\delta}{2^4}\Bigg).\end{aligned}\tag{3.3.28}$$

For $\theta=0$ from (3.3.4) we have that (3.3.26) goes to 0 as $N\to\infty$. For $\theta\le0$ we have that from Lemma 3.3.1 the boundary terms (3.3.27) and (3.3.28) go to 0 as $N\to\infty$. This finishes the proof Proposition 3.3.6. □

3.4 Proof of Theorem 3.2.10

For an easy understanding of the proof of items i) and ii) of Theorem 3.2.10, we first introduce some notation and prove some lemmata.

Recall the function $\bar\rho^\infty$ introduced in Remark 3.2.7 which can be rewritten as

$$\bar\rho^\infty(u)=\frac{\beta u^\gamma+\alpha(1-u)^\gamma}{u^\gamma+(1-u)^\gamma}.$$

We have that $\bar\rho^\infty(0)=\alpha$ and $\bar\rho^\infty(1)=\beta$. Moreover, it is not difficult to see that $\bar\rho^\infty\in C^1([0,1])$ and that

$$\lim_{u\to0}(\bar\rho^\infty(u))'u^{2-\gamma}=\lim_{u\to1}(\bar\rho^\infty(u))'(1-u)^{2-\gamma}=0,$$

and from Lemma 7.2 of [37] we conclude that

$$\|\bar{\rho}^{\infty}\|_{\gamma/2} < \infty. \tag{3.4.1}$$

Hereinafter, we simplify the notation of $\rho^{\infty,1}$ (see Remark 3.2.7) by denoting it by ρ^{∞}.

By the fractional Hardy's inequality (see e.g. [28]) and the fact that $V_1(\frac{1}{2}) \leq V_1(u)$ for all $u \in (0,1)$ we know that

$$\|g\| \lesssim \|g\|_{V_1} \lesssim \|g\|_{\gamma/2} \tag{3.4.2}$$

for any $g \in \mathscr{H}_0^{\gamma/2}$.

In order to prove items i) and ii) of Theorem 3.2.10 we first guarantee the existence of weak solutions of equation (3.2.4) with $\hat{\kappa} = 0$ and (3.2.6), (see Lemmas 3.4.1 and 3.4.3 below), then we establish the convergence in $L^2(0,T;L^2)$ (see Lemmas 3.4.2 and 3.4.4 below) which will allow us to conclude.

Lemma 3.4.1. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. Then, there exists a weak solution of (3.2.4) with $\hat{\kappa} = 0$ and initial condition ρ_0.*

Proof. The strategy of the proof is to construct the solution as the limit of ρ^{κ}, as $\kappa \to 0$, where ρ^{κ} is the weak solution of (3.2.4) with initial condition ρ_0 and $\hat{\kappa} = \kappa$.

By item i) in Theorem 3.3.3 and since $\kappa > 0$ we know that

$$\int_I \|\rho_t^{\kappa}\|_{\gamma/2}^2 dt \lesssim |I|(\kappa + 1) \tag{3.4.3}$$

for any interval $I \subset [0,T]$. We define

$$\forall t \in [0,T], \quad \forall u \in [0,1], \quad \varphi_t^{\kappa}(u) := \rho_t^{\kappa}(u) - \bar{\rho}^{\infty}(u). \tag{3.4.4}$$

Since we are interested in small values of κ, say $\kappa \leq 1$, from (3.4.3), (3.4.1) and the fact $(a+b)^2 \leq 2a^2 + 2b^2$, it is not difficult to see that

$$\int_I \|\varphi_t^{\kappa}\|_{\gamma/2}^2 dt \lesssim |I|, \tag{3.4.5}$$

thus we have that $\varphi^{\kappa} \in L^2(0,T;\mathscr{H}_0^{\gamma/2})$. It is also easy to see that φ^{κ} satisfies

$$\langle \varphi_t^{\kappa}, G_t \rangle - \langle \varphi_0, G_0 \rangle - \int_0^t \left\langle \varphi_s^{\kappa}, (\mathbb{L} + \partial_s) G_s \right\rangle ds + \kappa \int_0^t \langle \varphi_s^{\kappa}, G_s \rangle_{V_1} ds - \int_0^t \langle \bar{\rho}^{\infty}, \mathbb{L} G_s \rangle ds = 0 \tag{3.4.6}$$

for all $t \in [0,T]$, for any function $G \in C_c^{1,\infty}([0,T] \times (0,1))$ and where $\varphi_0(u) = \rho_0(u) - \bar{\rho}^{\infty}(u)$. From (3.4.5) we conclude that there exists a subsequence of $\{\varphi^{\kappa}\}_{\kappa \in (0,1)}$ converging weakly to some element $\varphi^0 \in L^2(0,T;\mathscr{H}_0^{\gamma/2})$ as $\kappa \to 0$. We claim that $\rho^0 := \bar{\rho}^{\infty} + \varphi^0$ is the desired solution. Indeed, first note that since the norm $\|\cdot\|_{\gamma/2}$ is weakly lower-semicontinuous we have that

$$\int_I \|\varphi_t^0\|_{\gamma/2}^2 dt \lesssim |I|. \tag{3.4.7}$$

By using $(a+b)^2 \le 2a^2 + 2b^2$ we have that

$$\int_I \|\rho_t^0\|_{\gamma/2}^2 dt \le 2\int_I \|\bar{\rho}^\infty\|_{\gamma/2}^2 dt + 2\int_I \|\varphi_t^0\|_{\gamma/2}^2 dt \lesssim |I|.$$

Taking $I=[0,T]$, we have that ρ^0 satisfies item i) of Definition 3.2.3. Since $\varphi^0 \in L^2(0,T;\mathscr{H}_0^{\gamma/2})$, it is easy to see that $\rho_t^0(0) = \bar{\rho}^\infty(0) = \alpha$ and $\rho_t^0(1) = \bar{\rho}^\infty(1) = \beta$ for almost every $t \in [0,T]$. Then, item ii) for $\hat{\kappa}=0$ in Definition 3.2.3 is satisfied. In order to verify that ρ^0 satisfies item iii) in Definition 3.2.3 we first integrate (3.4.6) over $[0,t]$. Thus we have that

$$\begin{aligned}\int_0^t \langle \varphi_s^\kappa, G_s\rangle ds - t\langle \varphi_0, G_0\rangle - \int_0^t \int_0^s \left\langle \varphi_r^\kappa, (\mathbb{L}+\partial_r)\, G_r \right\rangle drds \\ +\kappa \int_0^t \int_0^s \langle \varphi_r^\kappa, G_r\rangle_{V_1} drds - \int_0^t \int_0^s \langle \bar{\rho}^\infty, \mathbb{L}G_r\rangle drds = 0\end{aligned}$$

for any function $G \in C_c^{1,\infty}([0,T]\times(0,1))$. Taking $\kappa \to 0$, by weak convergence and Lebesgue's dominated convergence theorem we get from the previous equality that

$$\int_0^t \langle \varphi_s^0, G_s\rangle ds - t\langle \varphi_0, G_0\rangle - \int_0^t \int_0^s \left\{ \left\langle \varphi_r^0, (\mathbb{L}+\partial_r)\, G_r \right\rangle - \langle \bar{\rho}^\infty, \mathbb{L}G_r\rangle \right\} drds = 0.$$

Now, taking the derivative with respect to t in the previous equality we get that φ^0 satisfies

$$\langle \varphi_t^0, G_t\rangle - \langle \varphi_0, G_0\rangle - \int_0^t \langle \varphi_s^0, \left(\mathbb{L}+\partial_s\right) G_s\rangle\, ds - \int_0^t \langle \bar{\rho}^\infty, \mathbb{L}G_s\rangle ds = 0, \tag{3.4.8}$$

for all $t \in [0,T]$. Then, item iii) with $\kappa = 0$ in Definition 3.2.3 follows from (3.4.8), the definition of ρ^0 and $\bar{\rho}^\infty$. □

Lemma 3.4.2. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. Let ρ^κ be the weak solution of (3.2.4) with initial condition ρ_0 and $\hat{\kappa} = \kappa$. Then, ρ^κ converges strongly to ρ^0 in $L^2(0,T;L^2)$ as κ goes to 0, where ρ^0 is the weak solution of (3.2.4) with $\hat{\kappa}=0$ and initial condition ρ_0.*

Proof. Note that it is enough to show that

$$\int_0^t \|\rho_s^\kappa - \rho_s^0\|^2\, ds \lesssim t^2\kappa,$$

for all $t \in [0,T]$. By Lemma 3.4.1 we know that $\rho^0 = \bar{\rho}^\infty + \varphi^0$. Then, last inequality is equivalent to

$$\int_0^t \|\varphi_s^\kappa - \varphi_s^0\|^2\, ds \lesssim t^2\kappa. \tag{3.4.9}$$

By subtracting (3.4.8) from (3.4.6) and calling $\delta_t^k := \varphi_t^\kappa - \varphi_t^0$ we obtain that

$$\langle \delta_t^\kappa, G_t\rangle - \int_0^t \left\langle \delta_s^\kappa, (\mathbb{L}+\partial_s)\, G_s \right\rangle\, ds = -\kappa \int_0^t \langle \varphi_s^\kappa, G_s\rangle_{V_1} ds \tag{3.4.10}$$

for any function $G \in C_c^{1,\infty}([0,T]\times(0,1))$. Let $\{H_n^\kappa\}_{n\geq 1}$ be a sequence of functions in the space $C_c^{1,\infty}([0,T]\times(0,1))$ converging to δ^κ as $n\to\infty$ with respect to the norm of $L^2(0,T;\mathscr{H}_0^{\gamma/2})$ and for $n\geq 1$, let $G_n^\kappa(s,u)=\int_s^t H_n^\kappa(r,u)dr$. We claim that by plugging G_n into (3.4.10) and taking $n\to\infty$ we get that

$$\int_0^t \|\delta_s^\kappa\|^2\,ds+\frac{1}{2}\left\|\int_0^t \delta_s^\kappa ds\right\|_{\gamma/2}^2=-\kappa\int_0^t\left\langle \varphi_s^\kappa,\int_s^t \delta_r^\kappa dr\right\rangle_{V_1} ds. \tag{3.4.11}$$

We postpone the justification of the equality above to the end of the proof. Now, by using successively the Cauchy-Schwarz's inequality we have that

$$\begin{aligned}\int_0^t \|\delta_s^\kappa\|^2\,ds+\frac{1}{2}\left\|\int_0^t \delta_s^\kappa ds\right\|_{\gamma/2}^2 &\leq \kappa\int_0^t \|\varphi_s^\kappa\|_{V_1}\left\|\int_s^t \delta_r^\kappa dr\right\|_{V_1} ds\\ &\lesssim \kappa\sqrt{\int_0^t \|\varphi_s^\kappa\|_{\gamma/2}^2 ds}\sqrt{\int_0^t\left\|\int_s^t \delta_r^\kappa dr\right\|_{\gamma/2}^2 ds.}\end{aligned} \tag{3.4.12}$$

In the last inequality of the previous expression we used (3.4.2). By the triangular inequality we have that $\sqrt{\int_0^t\left\|\int_s^t \delta_r^\kappa dr\right\|_{\gamma/2}^2 ds}$ is bounded from above by

$$\sqrt{\int_0^t\left(\int_s^t \|\delta_r^\kappa\|_{\gamma/2}dr\right)^2 ds}\leq\sqrt{t\int_0^t\int_0^t \|\delta_r^\kappa\|_{\gamma/2}^2 dr ds}\lesssim\sqrt{t^2\int_0^t\left(\|\varphi_r^\kappa\|_{\gamma/2}^2+\|\varphi_r^0\|_{\gamma/2}^2\right)dr.} \tag{3.4.13}$$

In the first inequality in the previous display we used the Cauchy-Schwarz's inequality and in the second inequality we used the Minkowski's inequality and the inequality $(a+b)^2\leq 2(a^2+b^2)$. Using (3.4.5) and (3.4.7), we get from (3.4.12) and (3.4.13) the result.

We conclude this proof justifying (3.4.11). Note that it is enough to show that

i) $\displaystyle\lim_{n\to\infty}\int_0^t \langle\delta_s^\kappa,(\partial_s G_n^\kappa)(s,\cdot)\rangle ds=-\int_0^t \|\delta_s^\kappa\|^2 ds.$

ii) $\displaystyle\lim_{n\to\infty}\int_0^t \langle\delta_s^\kappa,\mathbb{L}G_n^\kappa(s,\cdot)\rangle ds=-\frac{1}{2}\left\|\int_0^t \delta_s^\kappa ds\right\|_{\gamma/2}^2.$

iii) $\displaystyle\lim_{n\to\infty}\int_0^t \left\langle\varphi_s^\kappa,G_n^\kappa(s,\cdot)\right\rangle_{V_1} ds=\int_0^t\left\langle\varphi_s^\kappa,\int_s^t \delta_r^\kappa dr\right\rangle_{V_1} ds.$

For i) we rewrite $\int_0^t\langle\delta_s^\kappa,(\partial_s G_n^\kappa)(s,\cdot)\rangle ds$ as

$$-\int_0^t\langle\delta_s^\kappa\,,\,H_n^\kappa(s,\cdot)\rangle\,ds=-\int_0^t\left\langle\delta_s^\kappa\,,\,H_n^\kappa(s,\cdot)-\delta_s^\kappa\right\rangle ds-\int_0^t \|\delta_s^\kappa\|^2\,ds.$$

Observe then that by the Cauchy-Schwarz's inequality we have

$$\begin{aligned}
&\left|\int_0^T \left\langle \delta_s^\kappa , H_n^\kappa(s,\cdot)-\delta_s^\kappa \right\rangle ds\right| \leq \int_0^T \|\delta_s^\kappa\| \, \|H_n^\kappa(s,\cdot)-\delta_s^\kappa\| \, ds \\
&\leq \sqrt{\int_0^T \|\delta_s^\kappa\|^2 \, ds} \, \sqrt{\int_0^T \|H_n^\kappa(s,\cdot)-\delta_s^\kappa\|^2 \, ds}
\end{aligned}$$

which goes to 0 as $n\to\infty$ since $H_n^\kappa \to \delta_s^\kappa$ in $L^2(0,T;\mathcal{H}_0^{\gamma/2})$. For ii), since G_n has compact support included in $(0,1)$, we can use the integration by parts formula for the regional fractional Laplacian (see Theorem 3.3 in [37]) which permits to write

$$\int_0^t \langle \delta_s^\kappa, \mathbb{L} G_n^\kappa(s,\cdot)\rangle ds = -\int_0^t \left\langle \delta_s^\kappa, G_n^\kappa(s,\cdot) \right\rangle_{\gamma/2} ds.$$

Then we have $\int_0^t \left\langle \delta_s^\kappa , G_n^\kappa(s,\cdot) \right\rangle_{\gamma/2} ds$ is equal to

$$\begin{aligned}
&\int_0^t \left\langle \delta_s^\kappa , \int_s^t \delta_r^\kappa dr \right\rangle_{\gamma/2} ds + \int_0^t \left\langle \delta_s^\kappa , G_n^\kappa(s,\cdot) - \int_s^t \delta_r^\kappa dr \right\rangle_{\gamma/2} ds \\
=&\iint_{0\leq s<r\leq t} \langle \delta_s^\kappa , \delta_r^\kappa \rangle_{\gamma/2} \, ds dr \; + \; \int_0^t \left\langle \delta_s^\kappa , \int_s^t \left(H_n^\kappa(r,\cdot)-\delta_r^\kappa\right) dr \right\rangle_{\gamma/2} ds \\
=&\frac{1}{2}\iint_{[0,t]^2} \langle \delta_s^\kappa , \delta_r^\kappa \rangle_{\gamma/2} \, ds dr \; + \; \int_0^t \left\langle \delta_s^\kappa , \int_s^t \left(H_n^\kappa(r,\cdot)-\delta_r^\kappa\right) dr \right\rangle_{\gamma/2} ds \\
=&\frac{1}{2}\left\| \int_0^t \delta_s^\kappa ds \right\|_{\gamma/2}^2 + \int_0^t \left\langle \delta_s^\kappa , \int_s^t \left(H_n^\kappa(r,\cdot)-\delta_r^\kappa\right) dr \right\rangle_{\gamma/2} ds.
\end{aligned}$$

To conclude the proof of ii) it is sufficient to show that the term at the right hand side of last expression vanishes as n goes to ∞. Indeed, such a term is bounded from above by

$$\begin{aligned}
&\int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} \left\| \int_s^t \left(H_n^\kappa(r,\cdot)-\delta_r^\kappa\right) dr \right\|_{\gamma/2} ds \leq \int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} \int_s^t \left\|H_n^\kappa(r,\cdot)-\delta_r^\kappa\right\|_{\gamma/2} dr\, ds \\
&\leq \int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} \int_0^t \left\|H_n^\kappa(r,\cdot)-\delta_r^\kappa\right\|_{\gamma/2} dr\, ds = \left(\int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2} ds\right)\left(\int_0^t \left\|H_n^\kappa(r,\cdot)-\delta_r^\kappa\right\|_{\gamma/2} dr\right) \\
&\leq t\sqrt{\int_0^t \left\|\delta_s^\kappa\right\|_{\gamma/2}^2 ds}\,\sqrt{\int_0^t \left\|H_n^\kappa(r,\cdot)-\delta_r^\kappa\right\|_{\gamma/2}^2 dr} \xrightarrow[n\to\infty]{} 0.
\end{aligned} \tag{3.4.14}$$

Note that we obtained the inequalities above as consequence of a successive use of Cauchy-Schwarz's inequalities.

To prove iii) we rewrite $\int_0^t \langle \varphi_s^\kappa, G_n^\kappa(s,\cdot)\rangle_{V_1} ds$ as

$$\int_0^t \left\langle \varphi_s^\kappa, \int_s^t \left(H_n^\kappa(r,\cdot) - \delta_r^\kappa\right) dr \right\rangle_{V_1} ds + \int_0^t \left\langle \varphi_s^\kappa, \int_s^t \delta_r^\kappa dr \right\rangle_{V_1} ds$$

and, to conclude the proof it is sufficient to show that the term at the left hand side of last expression vanishes as $n \to \infty$. Indeed, as consequence of a successive use of the Cauchy-Schwarz's inequality such a term is bounded from above by

$$\begin{aligned}\int_0^t \left\|\varphi_s^\kappa\right\|_{V_1} \left\| \int_s^t \left(H_n^\kappa(r\cdot) - \delta_r^\kappa\right) dr \right\|_{V_1} ds &\leq t \sqrt{\int_0^t \left\|\varphi_s^\kappa\right\|_{V_1}^2 ds} \sqrt{\int_0^t \left\|H_n^\kappa(r,\cdot) - \delta_r^\kappa\right\|_{V_1}^2 dr} \\ &\leq C t \sqrt{\int_0^t \left\|\varphi_s^\kappa\right\|_{\gamma/2}^2 ds} \sqrt{\int_0^t \left\|H_n^\kappa(r,\cdot) - \delta_r^\kappa\right\|_{\gamma/2}^2 dr} \xrightarrow[n\to\infty]{} 0\end{aligned}$$

where in the last inequality we used the fractional Hardy's inequality (see (3.4.2)).

□

Lemma 3.4.3. *Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. Consider the function $\rho_t^\infty = \bar{\rho}^\infty + (\rho_0 - \bar{\rho}^\infty)e^{-tV_1}$. If $g^\infty := \rho_0 - \bar{\rho}^\infty \in \mathcal{H}^{\gamma/2}$, then*

i) $\rho^\infty \in L^2(0,T;\mathcal{H}^{\gamma/2})$.

ii) ρ^∞ is a weak solution of (3.2.6) with initial condition ρ_0.

Proof. For i) note that by using the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ we get that

$$\int_0^T \|\rho_t^\infty\|_{\gamma/2}^2 dt \leq 2T\|\bar{\rho}^\infty\|_{\gamma/2}^2 + 2\int_0^T \left\|g^\infty e^{-tV_1}\right\|_{\gamma/2}^2 dt.$$

Since $\|\bar{\rho}^\infty\|_{\gamma/2} < \infty$ (see (3.4.1)) it is enough to prove that the term on the right hand side of last expression is finite. Note that $\left\|g^\infty e^{-tV_1}\right\|_{\gamma/2}^2$ is equal to

$$\begin{aligned}&\frac{c_\gamma}{2}\iint_{[0,1]^2} \frac{\left(g^\infty(u)e^{-tV_1(u)} - g^\infty(v)e^{-tV_1(v)}\right)^2}{|u-v|^{\gamma+1}} dudv \\ =&\frac{c_\gamma}{2}\iint_{[0,1]^2} \frac{\left(g^\infty(u)\left(e^{-tV_1(u)} - e^{-tV_1(v)}\right) + \left(g^\infty(u) - g^\infty(v)\right)e^{-tV_1(v)}\right)^2}{|u-v|^{\gamma+1}} dudv.\end{aligned}$$

Using the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ and that $|g^\infty(u)| \leq 2$ for any $u \in [0,1]$ we get that last expression is less than $8\|e^{-tV_1}\|_{\gamma/2}^2 + 2\|g^\infty\|_{\gamma/2}^2$. Note that the term $8\|e^{-tV_1}\|_{\gamma/2}^2$ can be written as

$$\begin{aligned}&4c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u -tV_1'(w)e^{-tV_1(w)}dw\right)^2}{|u-v|^{\gamma+1}} dudv \\ =&4c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u t\left(\frac{\gamma}{w}r^-(w) - \frac{\gamma}{1-w}r^+(w)\right)e^{-tV_1(w)}dw\right)^2}{|u-v|^{\gamma+1}} dudv.\end{aligned}$$

Using again $(a+b)^2 \leq 2a^2+2b^2$ and the fact that $e^{-tV_1(w)} \leq e^{-tr^{\pm}(w)}$ for any $w \in [0,1]$, we get that last expression is bounded from above by

$$\begin{aligned}
&8c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u \frac{\gamma}{w} tr^-(w) e^{-tr^-(w)} dw\right)^2}{|u-v|^{\gamma+1}} + \frac{\left(\int_v^u \frac{\gamma}{1-w} tr^+(w) e^{-tr^+(w)} dw\right)^2}{|u-v|^{\gamma+1}} dudv \\
=&16c_\gamma \iint_{[0,1]^2} \frac{\left(\int_v^u \frac{\gamma}{w} tr^-(w) e^{-tr^-(w)} dw\right)^2}{|u-v|^{\gamma+1}} dudv.
\end{aligned}$$

In the last equality we used a symmetry argument. We can write last expression as

$$C_\gamma t^{\frac{2-2\gamma}{\gamma}} \iint_{[0,1]^2} \frac{\left(\int_v^u w^{\gamma-2} (tr^-(w))^{\frac{2\gamma-1}{\gamma}} e^{-tr^-(w)} dw\right)^2}{|u-v|^{\gamma+1}} dudv,$$

where $C_\gamma = 16c_\gamma^{\frac{2-\gamma}{\gamma}} \gamma^{\frac{4\gamma-2}{\gamma}}$. Since the function $E_\gamma : [0,\infty) \to [0,\infty)$ defined as $E_\gamma(z) = z^{\frac{2\gamma-1}{\gamma}} e^{-z}$ is bounded from above by $E_\gamma\left(\frac{2\gamma-1}{\gamma}\right)$, we can bound last expression from above by

$$\begin{aligned}
&C_\gamma t^{\frac{2-2\gamma}{\gamma}} E_\gamma^2(\tfrac{2\gamma-1}{\gamma}) \iint_{[0,1]^2} \frac{\left(\int_v^u w^{\gamma-2} dw\right)^2}{|u-v|^{\gamma+1}} dudv \\
&= C_\gamma t^{\frac{2-2\gamma}{\gamma}} E_\gamma^2(\tfrac{2\gamma-1}{\gamma})(\gamma-2)^{-2} \iint_{[0,1]^2} \frac{\left(u^{\gamma-1} - v^{\gamma-1}\right)^2}{|u-v|^{\gamma+1}} dudv,
\end{aligned}$$

which is finite from (7.2) in the proof of Lemma 7.2 of [37]. Thus, we have that

$$8\|e^{-tV_1}\|_{\gamma/2}^2 \lesssim t^{\frac{2-2\gamma}{\gamma}}. \tag{3.4.15}$$

Therefore, if $g^\infty \in \mathscr{H}^{\gamma/2}$ then we conclude that

$$\int_0^T \|\rho_t^\infty\|_{\gamma/2}^2 dt \lesssim T\|\bar{\rho}^\infty\|_{\gamma/2}^2 + T\,\|g^\infty\|_{\gamma/2}^2 + \int_0^T t^{\frac{2-2\gamma}{\gamma}} dt \lesssim T\|\bar{\rho}^\infty\|_{\gamma/2}^2 + T\,\|g^\infty\|_{\gamma/2}^2 + T^{\frac{2-\gamma}{\gamma}},$$

which is finite since $\gamma < 2$.

For ii), since ρ^∞ is the solution of (3.2.6) then it satisfies item ii) of Definition 3.2.6. In order to see that ρ^∞ satisfies item i) of Definition 3.2.6, note that using $(a+b)^2 \leq 2a^2+2b^2$

we have that

$$
\begin{aligned}
&\int_0^T\int_0^1\left(\frac{\big(\alpha-\rho_t^{\infty}(u)\big)^2}{u^\gamma}+\frac{\big(\beta-\rho_t^{\infty}(u)\big)^2}{(1-u)^\gamma}\right)du\,dt\\
&\leq 2T\int_0^1\left(\frac{(\alpha-\bar\rho^{\infty}(u))^2}{u^\gamma}+\frac{(\beta-\bar\rho^{\infty}(u))^2}{(1-u)^\gamma}\right)du+\frac{8\gamma}{c_\gamma}\int_0^T\|e^{-tV_1}\|_{V_1}^2\,dt\\
&=2T(\beta-\alpha)^2\int_0^1(u^\gamma+(1-u)^\gamma)\,du+\frac{8\gamma}{c_\gamma}\int_0^T\|e^{-tV_1}\|_{V_1}^2\,dt\\
&\leq 2^\gamma(\beta-\alpha)^2T+\frac{8\gamma}{c_\gamma}\int_0^T\|e^{-tV_1}\|_{V_1}^2\,dt.
\end{aligned}
$$

For the term on the right hand side of last expression we first see that we can extend continuously the function e^{-tV_1} in such a way that it vanishes at 0 and at 1. There exists a constant C_2 (see 3.4.2) such that the previous expression is bounded from above by

$$2^\gamma(\beta-\alpha)^2T+\frac{8\gamma C_2^2}{c_\gamma}\int_0^T\|e^{-tV_1}\|_{\gamma/2}^2\,dt.$$

Thus, we obtain the desired result by using (3.4.15). □

Lemma 3.4.4. *Let $\rho_0:[0,1]\to[0,1]$ be a measurable function, such that $\rho_0-\bar\rho^{\infty}\in\mathscr{H}^{\gamma/2}$. Furthermore, let ρ^κ and ρ^∞ be the weak solutions of (3.2.4) with $\hat\kappa=\kappa$ and (3.2.6), respectively, and with the same initial condition ρ_0. Let $\hat\rho_t^\kappa:=\rho_{t/\kappa}^\kappa$, for all $t\in[0,T]$. Then $\hat\rho^\kappa$ converges strongly to ρ^∞ in $L^2(0,T;L^2)$, as κ goes to ∞.*

Proof. It is enough to show that

$$\int_0^t\|\hat\rho_s^\kappa-\rho_s^\infty\|^2\,ds=\int_0^t\|\hat\varphi_s^\kappa-\varphi_s^\infty\|^2\,ds\lesssim\frac{1}{\sqrt{\kappa}},\tag{3.4.16}$$

for all $t\in[0,T]$ where $\hat\varphi_t^\kappa=\hat\rho_t^\kappa-\bar\rho^\infty$ and $\varphi_t^\infty=(\rho_0-\bar\rho^\infty)e^{-tV_1}$. It is not difficult to see that $\hat\varphi_t^\kappa$ satisfies

$$\langle\hat\varphi_t^\kappa,G_t\rangle-\langle\varphi_0,G_0\rangle-\int_0^t\langle\hat\varphi_s^\kappa,\partial_sG_s\rangle\,ds+\int_0^t\langle\hat\varphi_s^\kappa,G_s\rangle_{V_1}ds-\frac{1}{\kappa}\int_0^t\langle\hat\rho_s^\kappa,\mathbb{L}G_s\rangle ds=0\tag{3.4.17}$$

for all functions $G\in C_c^{1,\infty}([0,T]\times(0,1))$. Then, calling $\hat\delta^k:=\hat\varphi^\kappa-\varphi^\infty$ we have that

$$\langle\hat\delta_t^\kappa,G_t\rangle-\int_0^t\left\langle\hat\delta_s^\kappa,\left(\tfrac{1}{\kappa}\mathbb{L}+\partial_s\right)G_s\right\rangle ds+\int_0^t\left\langle\hat\delta_s^\kappa,G_s\right\rangle_{V_1}=\frac{1}{\kappa}\int_0^t\langle\rho_s^\infty,G_s\rangle_{\gamma/2}ds\tag{3.4.18}$$

for any function $G\in C_c^{1,\infty}([0,T]\times(0,1))$. Let $\{\hat H_n^\kappa\}_{n\geq1}$, be a sequence of functions in the space $C_c^{1,\infty}([0,T],(0,1))$ converging to $\hat\delta^\kappa$ with respect to the norm of $L^2(0,T;\mathscr{H}_0^{\gamma/2})$. Now, for $n \geq 1$ we define the test function $\hat{G}_n^\kappa(s,u) = \int_s^t \hat{H}_n^\kappa(r,u)dr$. Plugging $\hat{G}_n^\kappa$ into (3.4.18) and using a similar argument as in proof of Lemma 3.4.2 we get that

$$\int_0^t \|\hat{\delta}_s^\kappa\|^2\,ds + \frac{1}{2\kappa}\left\|\int_0^t \hat{\delta}_s^\kappa ds\right\|_{\gamma/2}^2 + \frac{1}{2}\left\|\int_0^t \hat{\delta}_s^\kappa ds\right\|_{V_1}^2 = \frac{1}{\kappa}\int_0^t \left\langle \rho_s^\infty, \int_s^t \hat{\delta}_r^\kappa dr\right\rangle_{\gamma/2} ds.$$

By neglecting terms we get that

$$\int_0^t \|\hat{\rho}_s^\kappa - \rho_s^\infty\|^2\,ds = \int_0^t \|\hat{\delta}_s^\kappa\|^2\,ds \leq \frac{1}{\kappa}\int_0^t \left\langle \rho_s^\infty, \int_s^t \hat{\delta}_r^\kappa dr\right\rangle_{\gamma/2} ds.$$

Then it suffices to show that

$$\frac{1}{\kappa}\int_0^t \left\langle \rho_s^\infty, \int_s^t \hat{\delta}_r^\kappa dr\right\rangle_{\gamma/2} ds \lesssim \frac{1}{\sqrt{\kappa}}.$$

Indeed, by using twice the Cauchy-Schwarz's inequality we have that the term at the left hand side of the previous expression is bounded from above by

$$\frac{1}{\kappa}\int_0^t \|\rho_s^\infty\|_{\gamma/2}\left\|\int_s^t \hat{\delta}_r^\kappa dr\right\|_{\gamma/2} ds \leq \frac{1}{\kappa}\sqrt{\int_0^t \|\rho_s^\infty\|_{\gamma/2}^2 ds}\sqrt{\int_0^t \left\|\int_s^t \hat{\delta}_r^\kappa dr\right\|_{\gamma/2}^2 ds}.$$

Since by hypothesis $\rho_0 - \bar{\rho}^\infty \in \mathcal{H}^{\gamma/2}$ we know from item $i)$ of Lemma 3.4.3 that $\rho^\infty \in L^2(0,T;\mathcal{H}^{\gamma/2})$. Thus, from the latter and by the triangular inequality, the right hand side in the previous expression can be bounded from above by a constant times

$$\frac{1}{\kappa}\sqrt{\int_0^t \left(\int_s^t \|\hat{\delta}_r^\kappa\|_{\gamma/2} dr\right)^2 ds} \lesssim \frac{1}{\kappa}\sqrt{t\left(\int_0^t \|\hat{\delta}_r^\kappa\|_{\gamma/2} dr\right)^2}.$$

By using again the Cauchy-Schwarz's inequality, the term on the right hand side in the last expression is bounded from above by

$$\frac{1}{\kappa}\sqrt{t^2\int_0^t \|\hat{\delta}_r^\kappa\|_{\gamma/2}^2 dr} = \frac{1}{\kappa}\sqrt{t^2\int_0^t \|\hat{\rho}_r^\kappa - \rho_r^\infty\|_{\gamma/2}^2 dr} \lesssim \frac{1}{\kappa}\sqrt{2t^2\int_0^t \|\hat{\rho}_r^\kappa\|_{\gamma/2}^2 + \|\rho_r^\infty\|_{\gamma/2}^2 dr}.$$

In the last inequality we used the Minkowski's inequality and the fact that $(a+b)^2 \leq 2a^2+2b^2$. Now, since $\int_0^t \|\hat{\rho}_r^\kappa\|_{\gamma/2}^2 dr \lesssim \kappa$ (this is due to item i) of Theorem 3.3.3 and a change of variables) and $\rho^\infty \in L^2(0,T;\mathcal{H}^{\gamma/2})$ we can see that

$$\frac{1}{\kappa}\sqrt{2t^2\int_0^t \|\hat{\rho}_r^\kappa\|_{\gamma/2}^2 + \|\rho_r^\infty\|_{\gamma/2}^2 dr} \lesssim \frac{1}{\kappa}\sqrt{\kappa+1} \lesssim \frac{1}{\sqrt{\kappa}},$$

and we are done. □

3.4.1 Proof of item i) of Theorem 3.2.10.

Recall φ^κ_t defined in (3.4.4). Note that it is enough to show (3.4.9) with $\|\cdot\|$ replaced with $\|\cdot\|_{\gamma/2}$. From (3.4.10) we obtain, for $\varepsilon > 0$, that

$$\langle \delta^\kappa_{t+\varepsilon}, G_{t+\varepsilon}\rangle - \langle \delta^\kappa_t, G_t\rangle - \int_t^{t+\varepsilon} \langle \delta^\kappa_s, (\mathbb{L}+\partial_s)G_s\rangle\, ds = -\kappa \int_t^{t+\varepsilon} \langle \varphi^\kappa_s, G_s\rangle_{V_1} ds \tag{3.4.19}$$

for any function $G \in C_c^{1,\infty}([0,T]\times[0,1])$. Let $\{H^\kappa_n\}_{n\geq 1}$ be a sequence of functions in the space $C_c^{1,\infty}([0,T],(0,1))$ converging to δ^κ with respect to the norm of $L^2(0,T;\mathscr{H}_0^{\gamma/2})$ as $n \to \infty$. Now, for $n \geq 1$, we define the test function $G^\kappa_n(t,u) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} H^\kappa_n(r,u)dr$. Plugging G^κ_n into last equality and taking $n \to \infty$, a similar argument to the one of the proof of Lemma 3.4.2 allows to get

$$\frac{1}{\varepsilon}\left\langle \delta^\kappa_{t+\varepsilon} - \delta^\kappa_t, \int_t^{t+\varepsilon} \delta^\kappa_r dr \right\rangle + \varepsilon \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\|^2_{\gamma/2} = \kappa \int_t^{t+\varepsilon} \left\langle \varphi^\kappa_s, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\rangle_{V_1} ds.$$

Integrating last equality over $[0,\tilde{t}]$ we get:

$$\begin{aligned}\varepsilon \int_0^{\tilde t} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\|^2_{\gamma/2} dt =& \kappa \int_0^{\tilde t}\int_t^{t+\varepsilon} \left\langle \varphi^\kappa_s, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\rangle_{V_1} ds\, dt \\ & - \frac{1}{\varepsilon}\int_0^{\tilde t} \left\langle \delta^\kappa_{t+\varepsilon} - \delta^\kappa_t, \int_t^{t+\varepsilon} \delta^\kappa_r dr \right\rangle dt.\end{aligned} \tag{3.4.20}$$

Now we use the Cauchy-Schwarz's inequality, Hardy's inequality (see (3.4.2)) and (3.4.5) to get that

$$\begin{aligned}&\kappa \int_0^{\tilde t}\int_t^{t+\varepsilon} \left\langle \varphi^\kappa_s, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\rangle_{V_1} ds\, dt \lesssim \kappa \int_0^{\tilde t}\int_t^{t+\varepsilon} \|\varphi^\kappa_s\|_{\gamma/2} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\|_{\gamma/2} ds\, dt \\ &\lesssim \kappa \sqrt{\int_0^{\tilde t}\int_t^{t+\varepsilon} \|\varphi^\kappa_s\|^2_{\gamma/2} ds\, dt}\sqrt{\int_0^{\tilde t}\int_t^{t+\varepsilon} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\|^2_{\gamma/2} ds\, dt} \\ &\lesssim \kappa\varepsilon\sqrt{\tilde t}\sqrt{\int_0^{\tilde t} \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \delta^\kappa_r dr \right\|^2_{\gamma/2} dt}.\end{aligned} \tag{3.4.21}$$

Let us estimate the second term on the right hand side (3.4.20). First note that by changing variables we have that

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_0^{\tilde t}\left\langle \delta^\kappa_{t+\varepsilon}-\delta^\kappa_t,\int_t^{t+\varepsilon}\delta^\kappa_r dr\right\rangle dt\\
&=\frac{1}{\varepsilon}\int_0^{\tilde t}\int_t^{t+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dr dt-\frac{1}{\varepsilon}\int_0^{\tilde t}\int_t^{t+\varepsilon}\langle\delta^\kappa_{t+\varepsilon},\delta^\kappa_r\rangle dr dt \qquad (3.4.22)\\
&=\frac{1}{\varepsilon}\int_0^{\tilde t}\int_r^{r+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt dr-\frac{1}{\varepsilon}\int_\varepsilon^{\tilde t+\varepsilon}\int_{t-\varepsilon}^{t}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dr dt.
\end{aligned}$$

The first term first term at the right hand side of the last equality can be split as

$$\frac{1}{\varepsilon}\left(\int_0^{\varepsilon}\int_r^{\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt dr+\int_0^{\varepsilon}\int_\varepsilon^{r+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt dr+\int_\varepsilon^{\tilde t}\int_r^{r+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt dr\right).$$

By Fubini's theorem, we have that the second term at the right hand side of (3.4.22) is equal to

$$\frac{1}{\varepsilon}\left(\int_0^{\varepsilon}\int_\varepsilon^{r+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt dr+\int_\varepsilon^{\tilde t}\int_r^{r+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt dr+\int_{\tilde t}^{\tilde t+\varepsilon}\int_r^{\tilde t+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt dr\right).$$

Therefore we can write the second term on the right hand side of (3.4.20) as

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_{\tilde t}^{\tilde t+\varepsilon}\int_r^{\tilde t+\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt\,dr+\frac{1}{\varepsilon}\int_0^{\varepsilon}\int_r^{\varepsilon}\langle\delta^\kappa_t,\delta^\kappa_r\rangle dt\,dr\\
&\le\frac{1}{\varepsilon}\int_{\tilde t}^{\tilde t+\varepsilon}\int_{\tilde t}^{\tilde t+\varepsilon}\|\delta^\kappa_t\|\|\delta^\kappa_r\|dt\,dr+\frac{1}{\varepsilon}\int_0^{\varepsilon}\int_0^{\varepsilon}\|\delta^\kappa_t\|\|\delta^\kappa_r\|dt\,dr\\
&=\frac{1}{\varepsilon}\left(\int_{\tilde t}^{\tilde t+\varepsilon}\|\delta^\kappa_t\|\,dt\right)^2+\frac{1}{\varepsilon}\left(\int_0^{\varepsilon}\|\delta^\kappa_t\|\,dt\right)^2 \qquad (3.4.23)\\
&\le\int_{\tilde t}^{\tilde t+\varepsilon}\|\delta^\kappa_t\|^2 dt+\int_0^{\varepsilon}\|\delta^\kappa_t\|^2 dt,
\end{aligned}$$

where in the inequalities above we used the Cauchy-Schwarz's inequality. Then, using (3.4.21) and (3.4.23) in (3.4.20) we obtain that

$$\int_0^{\tilde t}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\delta^\kappa_r dr\right\|^2_{\gamma/2}dt\lesssim\kappa\sqrt{\tilde t}\sqrt{\int_0^{\tilde t}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\delta^\kappa_r dr\right\|^2_{\gamma/2}dt}+\frac{1}{\varepsilon}\int_{\tilde t}^{\tilde t+\varepsilon}\|\delta^\kappa_t\|^2 dt+\frac{1}{\varepsilon}\int_0^{\varepsilon}\|\delta^\kappa_t\|^2 dt.$$

Taking $\varepsilon\to 0$, using Lebesgue's differentiation Theorem (see Theorem 1.35 in [53]) and the fact that $\delta^\kappa_0=0$ (since the initial condition for ρ^κ and ρ^0 is the same) we get that

$$\int_0^{\tilde t}\|\delta^\kappa_t\|^2_{\gamma/2}dt\lesssim\kappa\sqrt{\tilde t}\sqrt{\int_0^{\tilde t}\|\delta^\kappa_t\|^2_{\gamma/2}dt}+\|\delta^\kappa_{\tilde t}\|^2,$$

for all $\tilde{t} \in [0,T]$. Integrating last inequality over $[0,T]$ and using the Cauchy-Schwarz's inequality and (3.4.9) we conclude that

$$\int_0^T \int_0^{\tilde{t}} \|\delta_t^\kappa\|_{\gamma/2}^2 dt d\tilde{t} \lesssim \kappa T \sqrt{\int_0^T \int_0^{\tilde{t}} \|\delta_t^\kappa\|_{\gamma/2}^2 dt d\tilde{t}} + \kappa T^2, \tag{3.4.24}$$

where in the last inequality we have used (3.4.9). Then, by a simple computation we have that

$$\int_0^T \int_0^{\tilde{t}} \|\delta_t^\kappa\|_{\gamma/2}^2 dt d\tilde{t} \lesssim \kappa T^2. \tag{3.4.25}$$

By Fubini's Theorem, we get that

$$\int_0^T \int_0^{\tilde{t}} \|\delta_t^\kappa\|_{\gamma/2}^2 \, dt d\tilde{t} = \int_0^T (T-t)\|\delta_t^\kappa\|_{\gamma/2}^2 \, dt \geq \frac{T}{2} \int_0^{T/2} \|\delta_t^\kappa\|_{\gamma/2}^2 \, dt. \tag{3.4.26}$$

The result now follows from (3.4.25) and (3.4.26).

□

3.4.2 Proof of item ii) of Theorem 3.2.10

Recall $\hat{\varphi}_t^\kappa$ and φ_t^∞ defined in Lemma 3.4.4. It is enough to show (3.4.16) with $\|\cdot\|$ replaced with $\|\cdot\|_{V_1}$:

$$\int_0^T \|\hat{\varphi}_t^\kappa - \varphi_t^\infty\|_{V_1}^2 \, dt \lesssim \frac{1}{\sqrt{\kappa}}. \tag{3.4.27}$$

From (3.4.18), we obtain, for $\varepsilon > 0$, that

$$\langle \hat{\delta}_{t+\varepsilon}^\kappa, G_{t+\varepsilon}\rangle - \langle \hat{\delta}_t^\kappa, G_t\rangle - \int_t^{t+\varepsilon} \langle \hat{\delta}_s^\kappa, \Big(\frac{1}{\kappa}\mathbb{L} + \partial_s\Big) G_s\rangle \, ds + \int_t^{t+\varepsilon} \langle \hat{\delta}_s^\kappa, G_s\rangle_{V_1} \, ds = \frac{1}{\kappa} \int_t^{t+\varepsilon} \langle \rho_s^\infty, G_s\rangle_{\gamma/2} ds \tag{3.4.28}$$

for any function $G \in C_c^{1,\infty}([0,T]\times[0,1])$. Let $\{\hat{H}_n^\kappa\}_{n\geq 1}$ be a sequence of functions in the space $C_c^{1,\infty}([0,T],(0,1))$ converging to $\hat{\delta}^\kappa$ with respect to the norm of $L^2(0,T;\mathcal{H}_0^{\gamma/2})$ as $n \to \infty$. Now, for $n \geq 1$ we define the test functions $\hat{G}_n^\kappa(u) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{H}_n^\kappa(r,u) dr$. Plugging $\hat{G}_n^\kappa$ into (3.4.28) and taking $n \to \infty$, a similar argument to the one of the proof of Lemma 3.4.2 allows to get

$$\frac{1}{\varepsilon}\left\langle \hat{\delta}_{t+\varepsilon}^\kappa - \hat{\delta}_t^\kappa, \int_t^{t+\varepsilon} \hat{\delta}_r^\kappa dr \right\rangle + \frac{\varepsilon}{\kappa}\left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa dr \right\|_{\gamma/2}^2 + \varepsilon \left\| \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa dr \right\|_{V_1}^2 = \frac{1}{\kappa}\int_t^{t+\varepsilon} \left\langle \rho_s^\infty, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \hat{\delta}_r^\kappa dr \right\rangle_{\gamma/2} ds. \tag{3.4.29}$$

By neglecting the term $\frac{\varepsilon}{\kappa}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\|_{\gamma/2}^2$ in (3.4.29) and then integrating that equality over $[0,\tilde{t}]$ we get that

$$\varepsilon\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\|_{V_1}^2 dt \le \frac{1}{\kappa}\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\left\langle \rho_s^\infty, \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\rangle_{\gamma/2} ds\, dt$$
$$-\frac{1}{\varepsilon}\int_0^{\tilde{t}}\left\langle \hat{\delta}_{t+\varepsilon}^\kappa-\hat{\delta}_t^\kappa, \int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\rangle dt. \quad (3.4.30)$$

Now we use twice the Cauchy-Schwarz's inequality in order to get that the first term on the right hand side in the previous expression is bounded from above by

$$\begin{aligned}
&\frac{1}{\kappa}\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\|\rho_s^\infty\|_{\gamma/2}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\|_{\gamma/2} ds\, dt\\
\le&\frac{1}{\kappa}\sqrt{\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\|\rho_s^\infty\|_{\gamma/2}^2 ds dt}\sqrt{\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\|_{\gamma/2}^2 ds\, dt} \qquad (3.4.31)\\
\le&\frac{\sqrt{\varepsilon}}{\kappa}\sqrt{\int_0^{\tilde{t}}\int_t^{t+\varepsilon}\|\rho_s^\infty\|_{\gamma/2}^2 ds dt}\sqrt{\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\|_{\gamma/2}^2 dt}.
\end{aligned}$$

By a similar argument as the one in the proof of item i) of Theorem 3.2.10 we have that the second term on the right hand side in (3.4.30) is bounded from above by

$$\frac{1}{\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\hat{\delta}_t^\kappa\|^2 dt+\frac{1}{\varepsilon}\int_0^{\varepsilon}\|\hat{\delta}_t^\kappa\|^2 dt. \quad (3.4.32)$$

Therefore, by using (3.4.31) and (3.4.32) in (3.4.30) we get that

$$\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\|_{V_1}^2 dt \le\frac{1}{\kappa}\sqrt{\int_0^{\tilde{t}}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\|\rho_s^\infty\|_{\gamma/2}^2 ds dt}\sqrt{\int_0^{\tilde{t}}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\hat{\delta}_r^\kappa dr\right\|_{\gamma/2}^2 dt}$$
$$+\frac{1}{\varepsilon}\int_{\tilde{t}}^{\tilde{t}+\varepsilon}\|\hat{\delta}_t^\kappa\|^2 dt+\frac{1}{\varepsilon}\int_0^{\varepsilon}\|\hat{\delta}_t^\kappa\|^2 dt.$$

Taking $\varepsilon\to 0$, using Lebesgue's differentiation Theorem (see Theorem 1.35 in [53]) and the fact that $\hat{\delta}_0^\kappa=0$ we get that

$$\int_0^{\tilde{t}}\|\hat{\delta}_t^\kappa\|_{V_1}^2 dt\le\frac{1}{\kappa}\sqrt{\int_0^{\tilde{t}}\|\rho_t^\infty\|_{\gamma/2}^2 dt}\sqrt{\int_0^{\tilde{t}}\|\hat{\delta}_t^\kappa\|_{\gamma/2}^2 dt+\|\hat{\delta}_{\tilde{t}}^\kappa\|^2},$$

for all $\tilde{t} \in [0,T]$. Integrating the previous expression over $[0,T]$ and using the Cauchy-Schwarz's inequality we get that

$$\begin{aligned}\int_0^T\int_0^{\tilde{t}} \|\hat{\delta}_t^\kappa\|_{V_1}^2 dt d\tilde{t} \leq & \frac{1}{\kappa}\sqrt{\int_0^T\int_0^{\tilde{t}} \|\rho_t^\infty\|_{\gamma/2}^2 dt d\tilde{t}}\sqrt{\int_0^T\int_0^{\tilde{t}} \|\hat{\delta}_t^\kappa\|_{\gamma/2}^2 dt d\tilde{t}} + \int_0^T \|\hat{\delta}_{\tilde{t}}^\kappa\|^2 d\tilde{t} \\ \lesssim & \frac{1}{\kappa}\sqrt{\int_0^T\int_0^T \|\hat{\delta}_t^\kappa\|_{\gamma/2}^2 dt d\tilde{t}} + \frac{1}{\sqrt{\kappa}}, \\ \lesssim & \frac{1}{\kappa}\sqrt{2T\int_0^T \|\hat{\rho}_t^\kappa\|_{\gamma/2}^2 + \|\rho_t^\infty\|_{\gamma/2}^2 dt} + \frac{1}{\sqrt{\kappa}}, \\ \lesssim & \frac{1}{\kappa}\sqrt{(\kappa+2)} + \frac{1}{\sqrt{\kappa}}.\end{aligned} \tag{3.4.33}$$

In the second inequality above we used the fact that $\rho^\infty \in L^2(0,T;\mathcal{H}^{\gamma/2})$ (see item i) of Lemma 3.4.3) and (3.4.27), while in the third inequality of we used Minkoski's inequality and the fact that $(a+b)^2 \leq 2a^2 + 2b^2$. And finally, the last inequality of (3.4.33) is true since $\rho^\infty \in L^2(0,T;\mathcal{H}^{\gamma/2})$ and by item i) of Theorem 3.3.3. Then, by a simple computation we have that

$$\int_0^T\int_0^{\tilde{t}} \|\hat{\delta}_t^\kappa\|_{V_1}^2 dt d\tilde{t} \lesssim \frac{1}{\sqrt{\kappa}}. \tag{3.4.34}$$

By Fubini's Theorem, we have that

$$\int_0^T\int_0^{\tilde{t}} \|\hat{\delta}_t^\kappa\|_{V_1}^2 \, dt d\tilde{t} = \int_0^T (T-t)\|\hat{\delta}_t^\kappa\|_{V_1}^2 \, dt \geq \frac{T}{2}\int_0^{T/2} \|\hat{\delta}_t^\kappa\|_{V_1}^2 \, dt. \tag{3.4.35}$$

The result now follows from (3.4.34) and (3.4.35).

□

3.5 Proof of Theorem 3.2.17

In this section we prove items i) and ii) of Theorem 3.2.17. We are interested in analyzing the convergence of the stationary solution $\bar{\rho}^\kappa$ as $\kappa \to 0$ and $\kappa \to \infty$. From Definition 3.2.12, for $\kappa \geq 0$, and for $\bar{\varphi}^\kappa = \bar{\rho}^\kappa - \bar{\rho}^\infty$ we have that $\bar{\varphi}^\kappa \in \mathcal{H}_0^{\gamma/2}$ and

$$\langle \bar{\varphi}^\kappa, -\mathbb{L}G\rangle + \kappa\langle \bar{\varphi}^\kappa, G\rangle_{V_1} = I_{\bar{\rho}^\infty}(G), \tag{3.5.1}$$

for any test function G of compact support included in $(0,1)$. Above $I_{\bar{\rho}^\infty} : \mathcal{H}_0^{\gamma/2} \to \mathbb{R}$ is a linear form defined by $I_{\bar{\rho}^\infty}(G) = \langle \bar{\rho}^\infty, \mathbb{L}G\rangle$. Moreover, this linear form is continuous. Indeed, using integration by parts given in Proposition 3.3 in [37] we have that

$$|I_{\bar\rho^\infty}(G)| = \left|\int_0^1 \bar\rho^\infty(u)\mathbb{L}G(u)du\right| = \frac{c_\gamma}{2}\left|\iint_{[0,1]^2} \frac{(\bar\rho^\infty(u)-\bar\rho^\infty(v))(G(u)-G(v))}{|u-v|^{\gamma+1}}dvdu\right|$$
$$\leq \|\bar\rho^\infty\|_{\gamma/2}\|G\|_{\gamma/2} < \infty. \tag{3.5.2}$$

Above we used the Cauchy-Schwarz's inequality and the fact that $\|\bar\rho^\infty\|_{\gamma/2}$ is finite (see ((3.4.1))). Therefore, $|I_{\rho^\infty}(G)| \lesssim \|G\|_{\mathscr{H}_0^{\gamma/2}}$.

Then it is enough to analyze the behavior of $\bar\varphi^\kappa$. We claim that we can take $G = \bar\varphi^\kappa$ in (3.5.1). The justification is postponed to the end of the proof. Whence, from (3.5.2) we have that

$$\|\bar\varphi^\kappa\|^2_{\gamma/2} + \kappa\|\bar\varphi^\kappa\|^2_{V_1} = I_{\bar\rho^\infty}(\bar\varphi^\kappa) \lesssim \|\bar\varphi^\kappa\|_{\gamma/2}, \tag{3.5.3}$$

from where we conclude that $\|\bar\varphi^\kappa\|_{\gamma/2} < \infty$. Plugging this back into (3.5.3) we get that

$$\|\bar\varphi^\kappa\|_{V_1} \lesssim \frac{1}{\sqrt{\kappa}}. \tag{3.5.4}$$

Now, note that $\bar\varphi^0 \in \mathscr{H}_0^{\gamma/2}$ satisfies $\langle\bar\varphi^0, -\mathbb{L}G\rangle = I_{\bar\rho^\infty}(G)$, for any function $G \in C_c^\infty((0,1))$. Then $\bar\varphi^\kappa - \bar\varphi^0$ satisfies

$$\langle\bar\varphi^\kappa - \bar\varphi^0, -\mathbb{L}G\rangle + \kappa\langle\bar\varphi^\kappa, G\rangle_{V_1} = 0,$$

for any function $G \in C_c^\infty((0,1))$. We claim that we can take $G = \bar\varphi^\kappa - \bar\varphi^0$ in the previous equality. The proof is analogous to the one done at the end of this section. Thus, we get that

$$\|\bar\varphi^\kappa - \bar\varphi^0\|^2_{\gamma/2} = k\langle\bar\varphi^\kappa, \bar\varphi^0 - \bar\varphi^\kappa\rangle_{V_1} \leq \kappa\|\bar\varphi^\kappa\|_{V_1}\|\bar\varphi^\kappa - \bar\varphi^0\|_{V_1}.$$

From (3.5.4) and fractional Hardy's inequality given in (3.4.2) we have that

$$\|\bar\varphi^\kappa - \bar\varphi^0\|^2_{\gamma/2} \lesssim \sqrt{\kappa}\|\bar\varphi^\kappa - \bar\varphi^0\|_{V_1} \lesssim \sqrt{\kappa}\|\bar\varphi^\kappa - \bar\varphi^0\|_{\gamma/2},$$

from where we conclude that $\|\bar\varphi^\kappa - \bar\varphi^0\|_{\gamma/2} \lesssim \sqrt{\kappa}$. Then $\bar\varphi^\kappa$ converges to $\bar\varphi^0$, as $k \to 0$ in the $\|\cdot\|_{\gamma/2}$ norm. So far we proved item i).

Remark 3.5.1. *From fractional Hardy's inequality (see 3.4.2) the convergence is also true in $L^2_{V_1}$ and since*

$$\|\bar\varphi^\kappa - \bar\varphi^0\|_{V_1} \geq V_1(\tfrac{1}{2})\|\bar\varphi^\kappa - \bar\varphi^0\|$$

we conclude that the convergence also holds in L^2.

For item ii), by (3.5.4) we get that $\|\bar\varphi^\kappa\|_{V_1} \to 0$ and so $\|\bar\varphi^\kappa\| \to 0$ as $k \to \infty$.

We conclude this proof by showing that we can take $G = \bar\varphi^\kappa$ in (3.5.1). Indeed, since $C_c^\infty((0,1))$ is dense in $\mathscr{H}_0^{\gamma/2}$, there exists a sequence $\{\bar H_n^\kappa\}_{n\geq 1}$ in $C_c^\infty((0,1))$ converging to $\bar\varphi^\kappa$, i.e, $\|\bar H_n^\kappa - \bar\varphi^\kappa\|_{\gamma/2} \to 0$ as $n \to \infty$. Observe that as a result of the latter and (3.4.2) we also have $\|\bar H_n^\kappa - \bar\varphi^\kappa\|_{V_1} \to 0$ as $n \to \infty$. Using the Cauchy-Schwarz's inequality we have that

$$
\begin{aligned}
\langle \bar\varphi^\kappa, \bar H_n^\kappa - \bar\varphi^\kappa\rangle_{\gamma/2} &\le \|\bar\varphi^\kappa\|_{\gamma/2}\|\bar H_n^\kappa - \bar\varphi^\kappa\|_{\gamma/2},\\
\langle \bar\varphi^\kappa, \bar H_n^\kappa - \bar\varphi^\kappa\rangle_{V_1} &\le \|\bar\varphi^\kappa\|_{V_1}\|\bar H_n^\kappa - \bar\varphi^\kappa\|_{V_1},\\
I_{\bar\rho^\infty}(\bar H_n^\kappa - \bar\varphi^\kappa) &\le \|\bar\rho^\infty\|_{\gamma/2}\|\bar H_n^\kappa - \bar\varphi^\kappa\|_{\gamma/2},
\end{aligned}
$$

all going to 0 as $n \to \infty$. Thus, we can rewrite (3.5.1) as

$$
\langle \bar\varphi^\kappa, -\mathbb{L}\bar\varphi^\kappa\rangle + \langle \bar\varphi^\kappa, -\mathbb{L}(\bar H_n^\kappa - \bar\varphi^\kappa)\rangle + \kappa(\langle \bar\varphi^\kappa, \bar\varphi^\kappa\rangle_{V_1} + \langle \bar\varphi^\kappa, \bar H_n^\kappa - \bar\varphi^\kappa\rangle_{V_1}) = I_{\bar\rho^\infty}(\bar\varphi^\kappa) + I_{\bar\rho^\infty}(\bar H_n^\kappa - \bar\varphi^\kappa).
$$

Now it is enough to take $n \to \infty$.

□

3.6 Proofs of the Hydrostatic limit and Fractional Fick's law

Here we write the proof for the case $\theta = 0$ and $\kappa = 1$, but it can be extended for $\kappa > 0$. The first step in the proof consists to obtain a sharp upper bound on the average current in the non-equilibrium stationary state (see Lemma 3.6.1). This bound will be used to derive an estimate of the entropy production (Lemma 3.6.2) which is the key estimate to obtain by a coarse graining argument and entropy bounds, that the empirical density at each extremity of Λ_N is given by α and β (Corollary 3.6.4). To identify the form of the stationary profile in the bulk, we use a method introduced in [42] for boundary driven diffusive systems (Lemma 3.6.6). Fractional Fick's law is then derived.

3.6.1 Entropy production bounds

Recall the definition of the current W_x (see (2.2.10)) introduced in Chapter 2.

Lemma 3.6.1. *We have that $\langle W_1\rangle_N = O(N^{1-\gamma})$, for any $N \ge 2$.*

Proof. By stationarity we have that for any $x \in \Lambda_N$, $\langle W_1\rangle_N = \langle W_x\rangle_N$. It follows that

$$
\begin{aligned}
\langle W_1\rangle_N = \frac{1}{N-1}\sum_{x=1}^{N-1}\langle W_x\rangle_N = \frac{1}{N-1}&\sum_{y<z} p(z-y)[\langle \eta_y\rangle_N - \langle \eta_z\rangle_N]\theta(y,z)\\
&+ (\beta-\alpha)\sum_{\substack{y\le 0\\ z\ge N}} p(z-y)
\end{aligned}
$$

where $\theta(y,z) = \text{Card}\{x \in \Lambda_N \ ; \ y+1 \le x \le z\}$. Considering the different positions of y,z in Λ_N, we get

$$\begin{aligned}
\langle W_1 \rangle_N &= \frac{1}{N-1} \sum_{z=1}^{N-1} z[\alpha - \langle \eta_z \rangle_N] \sum_{y \le 0} p(z-y) \\
&+ \frac{1}{N-1} \sum_{y=1}^{N-1} (N-1-y)[\langle \eta_y \rangle_N - \beta] \sum_{z \ge N} p(z-y) \\
&+ \frac{1}{N-1} \sum_{\substack{y<z \\ z,y \in \Lambda_N}} p(z-y)(z-y)[\langle \eta_y \rangle_N - \langle \eta_z \rangle_N] \\
&= (I) + (II) + (III).
\end{aligned} \tag{3.6.1}$$

We have that

$$|(I)| \le \frac{2}{N-1} \sum_{z=1}^{N-1} z \sum_{y \ge z} p(y) = O(N^{1-\gamma})$$

since $\sum_{y \ge z} p(y) = O(z^{-\gamma})$ as $z \to \infty$. A similar upper bound is valid for (II). For the last term we observe that

$$(III) = -\frac{1}{N-1} \sum_{y=1}^{N-2} \sum_{k=1}^{N-1-y} kp(k)[\langle \eta_{y+k} \rangle_N - \langle \eta_y \rangle_N].$$

Now, using Fubini's Theorem we get

$$(III) = -\frac{1}{N-1} \sum_{k=1}^{N-2} kp(k) \sum_{y=1}^{N-1-k} [\langle \eta_{y+k} \rangle_N - \langle \eta_y \rangle_N].$$

Recall (2.4.12). It follows that

$$|(III)| = \frac{1}{N-1} \sum_{k=1}^{N-2} kp(k) \sum_{y=1}^{k} |\langle \eta_{N-y} \rangle_N - \langle \eta_y \rangle_N| \le \frac{2}{N-1} \sum_{k=1}^{N-2} k^2 p(k) = O(N^{1-\gamma}).$$

□

A simple consequence of this lemma is the following bound on the Dirichlet form with respect to the stationary state. Recall from Section 3.2 that for any $\rho \in (0,1)$ the density of $\bar{\mu}_N$ with respect to ν_ρ is denoted by $f_{N,\rho}$.

Lemma 3.6.2. *Let $\rho \in (0,1)$. There exists a constant $C := C(\rho,\alpha,\beta) > 0$ such that for any $N \geq 2$*

$$\sum_{x,y\in\Lambda_N} p(y-x)\Big\langle\Big[\sqrt{f_{N,\rho}(\sigma^{x,y}\eta)}-\sqrt{f_{N,\rho}(\eta)}\Big]^2\Big\rangle_\rho \leq \frac{C}{N^{\gamma-1}},$$

$$\sum_{x\in\Lambda_N}\sum_{y\leq 0} p(y-x)\Big\langle\Big[\sqrt{f_{N,\alpha}(\sigma^{x}\eta)}-\sqrt{f_{N,\alpha}(\eta)}\Big]^2\Big\rangle_\alpha \leq \frac{C}{N^{\gamma-1}},$$

$$\sum_{x\in\Lambda_N}\sum_{y\geq N} p(y-x)\Big\langle\Big[\sqrt{f_{N,\beta}(\sigma^{x}\eta)}-\sqrt{f_{N,\beta}(\eta)}\Big]^2\Big\rangle_\beta \leq \frac{C}{N^{\gamma-1}}.$$

Proof. To simplify the notation we denote $f_{N,\rho}$ by f_N. By definition of stationary state we have:

$$\begin{aligned}0 &= \langle f_N L_N \log f_N\rangle_\rho\\ &= \langle f_N L_N^0 \log f_N\rangle_\rho + \langle f_N L_N^r \log f_N\rangle_\rho + \langle f_N L_N^\ell \log f_N\rangle_\rho.\end{aligned} \tag{3.6.2}$$

We first obtain an upper bound for the second and the third term on the right hand side of the previous equality. For any $R > 0$, the second term is equal to

$$\begin{aligned}&\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\langle f_N(\eta)\eta_x(1-\beta)[\log f_N(\sigma^x\eta)-\log f_N(\eta)]\rangle_\rho\\
&+\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\langle f_N(\eta)(1-\eta_x)\beta[\log f_N(\sigma^x\eta)-\log f_N(\eta)]\rangle_\rho\\
&=\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\Big\langle f_N(\eta)\eta_x(1-\beta)\Big[\log\frac{Rf_N(\sigma^x\eta)}{f_N(\eta)}\Big]\Big\rangle_\rho\\
&+\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\Big\langle f_N(\eta)(1-\eta_x)\beta\Big[\log\frac{f_N(\sigma^x\eta)}{Rf_N(\eta)}\Big]\Big\rangle_\rho\\
&-\log R\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\langle f_N(\eta)(\eta_x(1-\beta)-(1-\eta_x)\beta)\rangle_\rho.\end{aligned} \tag{3.6.3}$$

Now by the change of variable $w=\sigma^x\eta$ we have that (3.6.3) is equal to

$$\begin{aligned}&-\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\Big\langle f_N(\sigma^x w)(1-w_x)(1-\beta)\Big[\log\frac{f_N(\sigma^x w)}{Rf_N(w)}\Big]\Big(\frac{\rho}{1-\rho}\Big)\Big\rangle_\rho\\
&+\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\Big\langle f_N(\eta)(1-\eta_x)\beta\Big[\log\frac{f_N(\sigma^x\eta)}{Rf_N(\eta)}\Big]\Big\rangle_\rho\\
&-\log R\sum_{\substack{x\in\Lambda_N\\ y\geq N}} p(x-y)\langle f_N(\eta)(\eta_x(1-\beta)-(1-\eta_x)\beta)\rangle_\rho.\end{aligned}$$

Now, choosing $R = \dfrac{\beta}{1-\beta}\dfrac{1-\rho}{\rho}$ and using $(x-y)\log(\frac{y}{x}) < 0$, we have that the last expression is equal to

$$
\begin{aligned}
& \frac{\beta}{R} \sum_{\substack{x \in \Lambda_N \\ y \geq N}} p(x-y) \Big\langle (1-w_x)(R f_N(w) - f_N(\sigma^x w)) \Big[\log \frac{f_N(\sigma^x w)}{R f_N(w)} \Big] \Big\rangle_\rho \\
& - \log R \sum_{\substack{x \in \Lambda_N \\ y \geq N}} p(x-y) \langle f_N(\eta)(\eta_x(1-\beta) - (1-\eta_x)\beta) \rangle_\rho \\
\leq \; & -\log\left(\frac{\beta}{1-\beta} \frac{1-\rho}{\rho} \right) \sum_{\substack{x \in \Lambda_N \\ y \geq N}} p(x-y) \langle f_N(\eta)(\eta_x - \beta) \rangle_\rho \,.
\end{aligned}
$$

We proved therefore that

$$
\langle f_N L_N^r \log f_N \rangle_\rho \leq -\log\left(\frac{\beta}{1-\beta} \frac{1-\rho}{\rho} \right) \langle W_N \rangle_N \,.
$$

Similar computations give that

$$
\langle f_N L_N^\ell \log f_N \rangle_\rho \leq -\log\left(\frac{1-\alpha}{\alpha} \frac{\rho}{1-\rho} \right) \langle W_1 \rangle_N \,.
$$

By Lemma 3.6.1, we get that there exists a constant $C' > 0$ such that

$$
\langle f_N L_N^r \log f_N \rangle_\rho \leq C' N^{1-\gamma}, \quad \langle f_N L_N^\ell \log f_N \rangle_\rho \leq C' N^{1-\gamma}.
$$

Therefore, by (3.6.2), we have that $-\langle f_N L_N^0 \log f_N \rangle_\rho \leq C N^{1-\gamma}$. Now, using the simple inequality $a(\log b - \log a) \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a})$, we obtain that $-\langle \sqrt{f_N} L_N^0 \sqrt{f_N} \rangle_\rho \leq C N^{1-\gamma}$. This gives the first inequality in Lemma 3.6.2 since the left hand side of the previous inequality is equal to the left hand side of the first inequality of Lemma 3.6.2 because L_N^0 is reversible with respect to ν_ρ for any ρ. Choosing now $\rho = \alpha$, and using again the simple inequality $a(\log b - \log a) \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a})$, we have that

$$
-\langle \sqrt{f_{N,\alpha}}\, L_N^\ell \sqrt{f_{N,\alpha}} \rangle_\alpha \leq C' N^{1-\gamma}.
$$

Since L_N^ℓ is reversible with respect to ν_α we have that

$$
\begin{aligned}
& -\langle \sqrt{f_{N,\alpha}}\, L_N^\ell \sqrt{f_{N,\alpha}} \rangle_\alpha \\
& = \frac{1}{2} \sum_{x \in \Lambda_N} \sum_{y \leq 0} p(y-x) \Big\langle [\eta_x(1-\alpha) + (1-\eta_x)\alpha] \Big[\sqrt{f_{N,\alpha}(\sigma^x \eta)} - \sqrt{f_{N,\alpha}(\eta)} \Big]^2 \Big\rangle_\alpha .
\end{aligned}
$$

Since $\alpha \wedge 1-\alpha \leq \eta_x(1-\alpha) + (1-\eta_x)\alpha$, the term above is bigger or equal to a constant times the left hand side of the second inequality of Lemma 3.6.2. The third inequality of Lemma 3.6.2 is obtained similarly by choosing $\rho = \beta$. □

3.6.2 Proof of Theorem 3.2.15

Let $\mathcal{M}_2^+$, be the space of positive measures on $[0,1]^2$ with total mass bounded by 1 equipped with the weak topology. For any $\eta \in \Omega^N$ the empirical measure $\hat{\pi}^N(\eta) \in \mathcal{M}_2^+$ is defined by

$$\hat{\pi}^N(\eta) = \frac{1}{(N-1)^2} \sum_{x,y=1}^{N-1} \eta_x \eta_y \delta_{(x/N,y/N)}$$

where $\delta_{(u,v)}$ is the Dirac mass on $(u,v) \in [0,1]^2$. Recall $\mathcal{M}^+$ introduced in Subsection 2.2.3. Let $\mathbb{P}^N$ be the law on $\mathcal{M}^+ \times \mathcal{M}_2^+$ induced by $(\pi^N, \hat{\pi}^N) : \Omega^N \to \mathcal{M}^+ \times \mathcal{M}_2^+$ when Ω^N is equipped with the non-equilibrium stationary state $\bar{\mu}_N$. To simplify notations, we denote $\hat{\pi}^N(\eta)$ by $\hat{\pi}^N$ and the action of $\pi \in \mathcal{M}_2^+$ on a continuous function $G : [0,1]^2 \to \mathbb{R}$ by $\langle \pi, G \rangle = \int_{[0,1]^2} G(u)\pi(du)$.

The sequence $\{\mathbb{P}^N\}_{N\geq 2}$ is tight on $\mathcal{M}^+ \times \mathcal{M}_2^+$. This is obvious since it is a family of probabilities over the compact set $\mathcal{M}^+ \times \mathcal{M}_2^+$. Our goal is to prove that every limit point $\mathbb{P}^*$ of this sequence is concentrated on the set of measures $(\pi, \hat{\pi})$ of $\mathcal{M}^+ \times \mathcal{M}_2^+$ such that π (resp. $\hat{\pi}$) is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ (resp. $[0,1]^2$) and with a density $\bar{\rho}^1(u)$ (resp. $\bar{\rho}^1(u)\bar{\rho}^1(v)$) where $\bar{\rho}^1$ is a weak solution of (3.2.7).

Lemma 3.6.3. *Let $\mathbb{P}^*$ be a limit point of the sequence $\{\mathbb{P}^N\}_N$. Then $\mathbb{P}^*$ is concentrated on measures $(\pi, \hat{\pi})$ such that π (resp. $\hat{\pi}$) is absolutely continuous with respect to Lebesgue measure on $[0,1]$ (resp. $[0,1]^2$). The density ρ of π is a continuous function on $[0,1]$ and the density of $\hat{\pi}$ is equal to $\rho \otimes \rho : (u,v) \in [0,1]^2 \to \rho(u)\rho(v)$.*

Proof. See Appendix 4.8. □

With some abuse of notation we denote by $\{\mathbb{P}^N\}_N$ a fixed subsequence converging to a limit point $\mathbb{P}^*$. A generic element of $\mathcal{M}^+ \times \mathcal{M}_2^+$ is denoted by $(\pi, \hat{\pi})$ with the convention that π and $\hat{\pi} = \pi \otimes \pi$ denotes the probability measure as well as its density with respect to the Lebesgue measure.

Proposition 3.6.4. *We have that $\mathbb{P}^*$ almost surely $\pi(0) = \alpha$ and $\pi(1) = \beta$.*

Proof. For small $\varepsilon > 0$ and small $\lambda \in \mathbb{R}$, let B be the box $B := \{[N\varepsilon], \dots, N-1\}$ in Λ_N and let u be the function defined by

$$u = e^{\lambda \sum_{x\in B} \eta_x}.$$

We recall that the action of the generator L_N^ℓ on a function $f : \Omega_N \to \mathbb{R}$ can be rewritten as

$$(L_N^\ell f)(\eta) = \sum_{z\in\Lambda_N} r_N^-\left(\tfrac{z}{N}\right)\left[\eta_z(1-\alpha) + (1-\eta_z)\alpha\right]\left[f(\sigma^z\eta) - f(\eta)\right]$$

where $r_N^-\left(\frac{z}{N}\right) = \sum_{y\geq z} p(y)$. An elementary computation shows that

$$\begin{aligned} -\frac{L_N^\ell u}{u} = &\left[(e^\lambda - 1) - 2(1-\alpha)(\cosh\lambda - 1)\right] \sum_{z\in B} r_N^-\left(\tfrac{z}{N}\right)(\eta_z - \alpha) \\ &- 2\alpha(1-\alpha)(\cosh\lambda - 1)\sum_{z\in B} r_N^-\left(\tfrac{z}{N}\right). \end{aligned} \tag{3.6.4}$$

Multiplying (3.6.4) by $f_{N,\alpha}$, integrating with respect to ν_α and using the variational formula of the Dirichlet form (see Theorem A.10.2 in [40]) we deduce that

$$
\begin{aligned}
&\Big[(e^{\lambda}-1)-2(1-\alpha)(\cosh\lambda-1)\Big]\sum_{z\in B} r_N^-\big(\tfrac{z}{N}\big)(\langle \eta_z\rangle_N-\alpha)\\
&\le \sum_{z\in\Lambda_N} r_N^-\big(\tfrac{z}{N}\big)\Big\langle\Big[\sqrt{f_{N,\alpha}(\sigma^z\eta)}-\sqrt{f_{N,\alpha}(\eta)}\Big]^2\Big\rangle_\alpha\\
&+2\alpha(1-\alpha)(\cosh\lambda-1)\sum_{z\in B} r_N^-\big(\tfrac{z}{N}\big)\\
&\le CN^{1-\gamma}+2\alpha(1-\alpha)(\cosh\lambda-1)\sum_{z\in B} r_N^-\big(\tfrac{z}{N}\big)
\end{aligned}
$$

where the last inequality is a consequence of Lemma 3.6.2. Observe that for $\lambda\to 0$, the term $(e^{\lambda}-1)-2(1-\alpha)(\cosh\lambda-1)$ is equivalent to λ and has therefore the sign of λ for sufficiently small λ. The term $\cosh\lambda-1$ is of order λ^2. Assume first that $\lambda>0$ is small. Then there exists a constant $C>0$ independent of λ,ε and N such that

$$
\mu_N\left(\langle \pi^N-\alpha\,,\,N^\gamma\,\mathbf{1}_{[\varepsilon,1]}\big(\tfrac{z}{N}\big)\,r_N^-\big(\tfrac{z}{N}\big)\rangle\right)=N^{\gamma-1}\sum_{z\in B} r_N^-\big(\tfrac{z}{N}\big)(\langle\eta_z\rangle_N-\alpha)\le\frac{C}{\lambda}+C\lambda N^{-1}\sum_{z\in B}N^\gamma r_N^-\big(\tfrac{z}{N}\big).
$$

By Lemma 3.3.1 we have that

$$
N^{-1}\sum_{z\in B}N^\gamma r_N^-\big(\tfrac{z}{N}\big)\lesssim\int_\varepsilon^1 u^{-\gamma}du=O(\varepsilon^{1-\gamma}).
$$

Therefore we conclude that

$$
\limsup_{\varepsilon\to 0}\varepsilon^{\gamma-1}\limsup_{N\to\infty}\mu_N\left(\langle\pi^N-\alpha\,,\,\mathbf{1}_{[\varepsilon,1]}\big(\tfrac{z}{N}\big)N^\gamma r_N^-\big(\tfrac{z}{N}\big)\rangle\right)\le 0.
$$

Similarly, by considering small $\lambda<0$, we deduce that

$$
\liminf_{\varepsilon\to 0}\varepsilon^{\gamma-1}\liminf_{N\to\infty}\mu_N\left(\langle\pi^N-\alpha\,,\,\mathbf{1}_{[\varepsilon,1]}\big(\tfrac{z}{N}\big)N^\gamma r_N^{-1}\big(\tfrac{z}{N}\big)\rangle\right)\ge 0.
$$

By using Lemma 3.3.1 we deduce that $\mathbb{P}^*$ a.s. we have

$$
\lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1\frac{\pi(u)-\alpha}{u^\gamma}du=0.
$$

But since by Lemma 3.6.3 π is a continuous function on $[0,1]$, if $\pi(0)\neq\alpha$, we have that

$$
\lim_{\varepsilon\to 0}\varepsilon^{\gamma-1}\int_\varepsilon^1\frac{\pi(u)-\alpha}{u^\gamma}du=\frac{\pi(0)-\alpha}{\gamma-1}\neq 0
$$

and we get a contradiction. We deduce thus that $\pi(0)=\alpha$. Similarly $\pi(1)=\beta$. $\square$

Remark 3.6.5. *The usual proof of this proposition for driven diffusive systems is quite different and based on the so-called two-blocks estimate ([30], [41]). It turns out that in the context of exclusion process with long jumps in contact with stochastic reservoirs this approach does not work since the control of the entropy production is not sufficient to cancel the heavy tails of p, even by using the arguments of [38].*

Lemma 3.6.6. *Let $\bar{\rho}^1$ be the unique weak solution of (3.2.7). For any F, G in $C_c^{\infty}([0,1])$ we have*

$$\int_{[0,1]^2} \left[G(u)((-\Delta)^{\gamma/2}F)(v) + F(v)((-\Delta)^{\gamma/2}G)(u)\right] I(u,v)du dv = 0$$

where $I(u,v) = \mathbb{E}^*\left[(\pi(u)-\bar{\rho}^1(u))(\pi(v)-\bar{\rho}^1(v))\right]$.

Proof. Recall (3.3.3). We have that

$$\begin{aligned} L_N(\langle \pi^N, F\rangle) &= \frac{1}{N-1}\sum_{x\in\Lambda_N}\sum_{y\in\mathbb{Z}} F(\tfrac{x}{N})p(y-x)(\eta_y-\eta_x) \\ &= \langle \pi^N, \mathscr{K}_N F\rangle + \frac{\alpha}{N-1}\sum_{x\in\Lambda_N}(Fr_N^-)(\tfrac{x}{N}) + \frac{\beta}{N-1}\sum_{x\in\Lambda_N}(Fr_N^+)(\tfrac{x}{N}). \end{aligned} \tag{3.6.5}$$

We then multiply (3.6.5) by N^γ and take the expectation with respect to $\bar{\mu}_N$ on both sides, the left hand side being then equal to 0 by stationarity. By using Lemma 3.3.1 and weak convergence we conclude that

$$\mathbb{E}^*\left[\int_0^1 \left\{\mathbb{L}F - r^-F - r^+F\right\}(u)\,\pi(u)du\right] + \int_0^1 \left\{\alpha r^-F + \beta r^+F\right\}(u)\,du = 0.$$

We compute now $L_N(\langle \hat{\pi}^N, J\rangle)$ where $J:[0,1]^2\to\mathbb{R}$ is a smooth test function with compact support strictly included in $[0,1]^2$ and which is identically equal to 0 on the diagonal. Consider a small $\delta>0$ and take a smooth even function $H_\delta:\mathbb{R}\to[0,1]$ which is equal to 0 on $[-\delta,\delta]$ and equal to 1 outside of $[-2\delta,2\delta]$. Let then $J_\delta(u,v) = F(u)G(v)H_\delta(v-u)$, $(u,v)\in[0,1]^2$.

Recall (2.4.4). By using Lemma 4.1.1 we get that

$$
\begin{aligned}
L_N(\langle \hat{\pi}^N, J_\delta\rangle) = & \frac{1}{N-1}\sum_{x\in\Lambda_N}\eta_x F(\tfrac{x}{N})\langle \pi^N, \mathcal{K}_N G_{\delta,x/N}\rangle \\
& + \frac{1}{N-1}\sum_{x\in\Lambda_N}\eta_x G(\tfrac{x}{N})\langle \pi^N, \mathcal{K}_N F_{\delta,x/N}\rangle \\
& + \frac{\alpha}{N-1}\sum_{x\in\Lambda_N}\eta_x G(\tfrac{x}{N})\left\{\frac{1}{N-1}\sum_{y\in\Lambda_N}F_{\delta,x/N}(\tfrac{y}{N})r_N^-(\tfrac{y}{N})\right\} \\
& + \frac{\alpha}{N-1}\sum_{x\in\Lambda_N}\eta_x F(\tfrac{x}{N})\left\{\frac{1}{N-1}\sum_{y\in\Lambda_N}G_{\delta,x/N}(\tfrac{y}{N})r_N^-(\tfrac{y}{N})\right\} \\
& + \frac{\beta}{N-1}\sum_{x\in\Lambda_N}\eta_x G(\tfrac{x}{N})\left\{\frac{1}{N-1}\sum_{y\in\Lambda_N}F_{\delta,x/N}(\tfrac{y}{N})r_N^+(\tfrac{y}{N})\right\} \\
& + \frac{\beta}{N-1}\sum_{x\in\Lambda_N}\eta_x F(\tfrac{x}{N})\left\{\frac{1}{N-1}\sum_{y\in\Lambda_N}G_{\delta,x/N}(\tfrac{y}{N})r_N^+(\tfrac{y}{N})\right\} \\
& - \frac{1}{(N-1)^2}\sum_{x,y\in\Lambda_N}p(y-x)(\eta_y-\eta_x)^2 J_\delta(\tfrac{x}{N}\tfrac{y}{N}).
\end{aligned}
\tag{3.6.6}
$$

Since $J_\delta(u,v)$ is equal to 0 for $|u-v|\le\delta$, we have that

$$
N^\gamma \bar{\mu}_N\left(\frac{-1}{(N-1)^2}\sum_{x,y\in\Lambda_N}p(y-x)(\eta_y-\eta_x)^2 J_\delta(\tfrac{x}{N}\tfrac{y}{N})\right) = O(N^{-1}).
$$

We multiply (3.6.6) by N^γ and take the expectation with respect to $\bar{\mu}_N$ on both sides, the left hand side being then equal to 0 by stationarity. By using Lemma 3.3.1 and weak convergence we conclude that

$$
\begin{aligned}
& -\mathbb{E}^*\left[\int_{[0,1]^2}\left\{G(u)((-\Delta)^{\gamma/2}F_{\delta,u})(v) + F(v)((-\Delta)^{\gamma/2}G_{\delta,v})(u)\right\}\pi(u)\pi(v)du dv\right] \\
& + \mathbb{E}^*\left[\int_{[0,1]^2}\left\{G(u)\,\alpha r^-(v)F_{\delta,u}(v) + G(u)\,\beta r^+(v)F_{\delta,u}(v)\right\}\pi(u)du dv\right] \\
& + \mathbb{E}^*\left[\int_{[0,1]^2}\left\{F(u)\,\alpha r^-(v)G_{\delta,u}(v) + F(u)\,\beta r^+(v)G_{\delta,u}(v)\right\}\pi(u)du dv\right] = 0.
\end{aligned}
$$

We can take the limit $\delta\to 0$ and since H_δ converges to the function identically equal to 1, we get

$$-\mathbb{E}^*\left[\int_{[0,1]^2}\left\{G(u)((-\Delta)^{\gamma/2}F)(v)+F(v)((-\Delta)^{\gamma/2}G)(u)\right\}\pi(u)\pi(v)dudv\right]$$
$$+\mathbb{E}^*\left[\int_{[0,1]^2}\left\{G(u)\,\alpha r^-(v)F(v)+G(u)\,\beta r^+(v)F(v)\right\}\pi(u)dudv\right]$$
$$+\mathbb{E}^*\left[\int_{[0,1]^2}\left\{F(u)\,\alpha r^-(v)G(v)+F(u)\,\beta r^+(v)G(v)\right\}\pi(u)dudv\right]=0.$$

We have also proved that for any smooth compactly supported function H

$$-\mathbb{E}^*\left[\int_0^1((-\Delta)^{\gamma/2}H)(u)\pi(u)du\right]+\int_0^1\left\{\alpha r^-H+\beta r^+H\right\}(u)\,du=0.$$

Let $\bar{\rho}$ be the unique weak solution of (3.2.7). Then we have

$$-\int_0^1((-\Delta)^{\gamma/2}H)(u)\bar{\rho}(u)du+\int_0^1\left\{\alpha r^-H+\beta r^+H\right\}(u)\,du=0.$$

It follows that

$$\int_{[0,1]^2}\left[G(u)((-\Delta)^{\gamma/2}F)(v)+F(v)((-\Delta)^{\gamma/2}G)(u)\right]I(u,v)dudv=0.$$

□

Since $\mathbb{P}^*$ almost surely $\pi(0)=\bar{\rho}^1(0)=\alpha$ and $\pi(1)=\bar{\rho}^1(1)=\beta$ and that $\pi,\bar{\rho}^1$ are continuous functions, by extending then to $\mathbb{R}$ by $\pi(u)=\bar{\rho}^1(u)=\alpha$ if $u\le 0$ and $\pi(u)=\bar{\rho}^1(u)=\beta$ if $u\ge 1$, we get that for any F,G in $C_c^\infty([0,1]^2)$,

$$\int_{\mathbb{R}^2}\left[G(u)((-\Delta)^{\gamma/2}F)(v)+F(v)((-\Delta)^{\gamma/2}G)(u)\right]I(u,v)dudv=0.$$

By using Theorem 3.12 in [9] we deduce that I is a.s. constant with respect to the Lebesgue measure on $[0,1]^2$. Since by Proposition 3.6.4, we have $I(0,0)=I(1,1)=0$, we deduce that I is identically equal to 0. Thus $\mathbb{P}^*$ almost surely $\pi=\bar{\rho}^1$. Thus, we have proved the following proposition.

Proposition 3.6.7. *The sequence $\{\mathbb{P}^N\}_N$ converges in law to the delta measure concentrated on*

$$(\bar{\rho}^1(u)du,\bar{\rho}^1(u)\bar{\rho}^1(v)dudv)$$

where $\bar{\rho}^1$ is the unique weak solution of (3.2.7).

Theorem 3.2.15 is a trivial consequence of this proposition.

3.6.3 Proof of Fick's law (Theorem 3.2.16)

Let us define for $z \in \Lambda_N$

$$\tilde{r}_N^-\big(\tfrac{z}{N}\big) = \sum_{y \geq z} y p(y), \quad \tilde{r}_N^+\big(\tfrac{z}{N}\big) = -\sum_{y \leq z-N} y p(y)$$

which are, up to a multiplicative constant, defined as $r_N^{\pm}$ with γ replaced by $\gamma - 1 \in (0,1)$. Recalling (3.6.1) we see that

$$N^{\gamma-1}\langle W_1\rangle_N = \mu_N\left(\langle \pi^N, \varphi_N\rangle\right) + N^{\gamma-1}\theta_N$$

where $\varphi_N : (0,1) \to \mathbb{R}$ defined by

$$\begin{aligned}
\varphi_N(\tfrac{z}{N}) &= -N^\gamma \sum_{y \leq 0} \tfrac{z}{N}\, p(z-y) + N^\gamma \sum_{y \geq N} \big[1 - \tfrac{1}{N} - \tfrac{z}{N}\big] p(y-z) \\
&\quad + N^\gamma \sum_{\substack{y>z\\ y\in\Lambda_N}} p(y-z)\big(\tfrac{y-z}{N}\big) - N^\gamma \sum_{\substack{y<z\\ y\in\Lambda_N}} p(y-z)\big(\tfrac{z-y}{N}\big) \\
&= -\tfrac{z}{N} N^\gamma r_N^-\big(\tfrac{z}{N}\big) + \big(1 - \tfrac{1}{N} - \tfrac{z}{N}\big) N^\gamma r_N^+\big(\tfrac{z}{N}\big) + N^\gamma \sum_{y\in\Lambda_N} p(y-z)\big(\tfrac{y-z}{N}\big) \\
&= -\tfrac{z}{N} N^\gamma r_N^-\big(\tfrac{z}{N}\big) + \big(1 - \tfrac{1}{N} - \tfrac{z}{N}\big) N^\gamma r_N^+\big(\tfrac{z}{N}\big) + N^{\gamma-1}\tilde{r}_N^-\big(\tfrac{z}{N}\big) - N^{\gamma-1}\tilde{r}_N^+\big(\tfrac{z}{N}\big)
\end{aligned}$$

is a discrete approximation of the function $\varphi : (0,1) \to \mathbb{R}$ given by

$$\varphi(u) = \frac{c_\gamma}{\gamma(1-\gamma)}\{(1-u)^{1-\gamma} - u^{1-\gamma}\}$$

and

$$\theta_N = \frac{\alpha}{N-1}\sum_{z=1}^{N-1}\sum_{y\leq 0} z\, p(z-y) - \frac{\beta}{N-1}\sum_{y=1}^{N-1}\sum_{z \geq N}(N-1-y)\, p(z-y).$$

It is easy to compute the limit of $N^{\gamma-1}\theta_N$ by writing it as a Riemann sum:

$$\begin{aligned}
\lim_{N\to\infty} N^{\gamma-1}\theta_N &= \alpha c_\gamma \lim_{N\to\infty} \frac{N}{N-1}\frac{1}{N^2}\sum_{z=1}^{N-1}\sum_{y\leq 0} \frac{\frac{z}{N}}{\left|\frac{z}{N} - \frac{y}{N}\right|^{1+\gamma}} \\
&\quad - \beta c_\gamma \lim_{N\to\infty} \frac{N}{N-1}\frac{1}{N^2}\sum_{y=1}^{N-1}\sum_{z\geq N} \frac{\left(1 - \frac{1}{N} - \frac{y}{N}\right)}{\left|\frac{z}{N} - \frac{y}{N}\right|^{1+\gamma}} \\
&= \alpha c_\gamma \int_0^1 \left(\int_{-\infty}^0 \tfrac{dy}{|z-y|^{1+\gamma}}\right) z\, dz - \beta c_\gamma \int_0^1 \left(\int_1^{+\infty} \tfrac{dz}{|z-y|^{1+\gamma}}\right)(1-y)\, dy \\
&= \frac{c_\gamma(\alpha-\beta)}{\gamma(2-\gamma)}.
\end{aligned}$$

Let us now compute the limit of $\mu_N\Big(\langle \pi^N, \varphi_N\rangle\Big) = \frac{1}{N-1}\sum_{z=1}^{N-1}\varphi_N(\frac{z}{N})\langle\eta_z\rangle_N$. Observe that the function φ is singular at $u=0$ and $u=1$ but it is integrable on $[0,1]$. Lemma 3.3.1 implies that $\lim_{N\to\infty}|\varphi_N([Nu]/N)-\varphi(u)|=0$ uniformly in $u\in[a,1-a]$,for any $a\in(0,1)$. Therefore we fix some small $a\in(0,1)$ and we split the sum in three sums, one over $z<aN$, one over $aN\le z\le(1-a)N$ and the last one over $z>(1-a)N$. By using the estimate (4.3.1) for r_N^- and similar ones for $r_N^+, \tilde{r}_N^\pm$ it is easy to get that

$$\Big|\varphi_N\big(\tfrac{z}{N}\big)\Big| \lesssim \Big[\big(\tfrac{z}{N}\big)^{1-\gamma} + \big(1-\tfrac{z}{N}\big)^{1-\gamma}\Big]$$

so that (use $\langle\eta_z\rangle_N\le 1$)

$$\Bigg|\frac{1}{N-1}\sum_{\substack{z<aN\\ z>(1-a)N}}\varphi_N(\tfrac{z}{N})\langle\eta_z\rangle_N\Bigg| \lesssim [a^{2-\gamma}+(1-a)^{2-\gamma}].$$

By using the uniform convergence of φ_N to φ over $[a,1-a]$, as $N\to\infty$, we get that

$$\lim_{N\to\infty}\frac{1}{N-1}\sum_{aN\le z\le(1-a)N}\varphi_N(\tfrac{z}{N})\langle\eta_z\rangle_N = \int_a^{1-a}\varphi(u)\bar{\rho}^1(u)du.$$

Thus sending first $N\to\infty$ and then $a\to 0$ we conclude that

$$\lim_{N\to\infty}\mu_N\big(\langle\pi^N,\varphi_N\rangle\big) = \int_0^1\varphi(u)\bar{\rho}^1(u)du.$$

Then Theorem 3.2.16 follows by simple integral computations and using the fact that $\bar{\rho}^1$ is the stationary solution of the fractional diffusion equation with Dirichlet boundary conditions.

Chapter 4

Appendix

Contents

4.1 Computations involving the generator

Lemma 4.1.1. *For any $x \neq y \in \Lambda_N$, we have*

i) $L_N^0(\eta_x\eta_y) = \eta_x L_N^0\eta_y + \eta_y L_N^0\eta_x - p(y-x)(\eta_y-\eta_x)^2$,

ii) $L_N^r(\eta_x\eta_y) = \eta_x L_N^r\eta_y + \eta_y L_N^r\eta_x$,

iii) $L_N^\ell(\eta_x\eta_y) = \eta_x L_N^\ell\eta_y + \eta_y L_N^\ell\eta_x$.

Proof. For i) we have, by definition of L_N^0, that

$$
\begin{aligned}
L_N^0(\eta_x\eta_y) =& \frac{1}{2}\sum_{\bar{x},\bar{y}\in\Lambda_N} p(\bar{y}-\bar{x})\big[(\sigma^{\bar{x},\bar{y}}\eta)_x(\sigma^{\bar{x},\bar{y}}\eta)_y - \eta_x\eta_y\big]\\
=& \frac{1}{2}\sum_{\bar{x},\bar{y}\in\Lambda_N} p(\bar{y}-\bar{x})\big[((\sigma^{\bar{x},\bar{y}}\eta)_x\eta_y - \eta_x\eta_y) + ((\sigma^{\bar{x},\bar{y}}\eta)_y\eta_x - \eta_x\eta_y) +\\
&+ (\sigma^{\bar{x},\bar{y}}\eta)_x(\sigma^{\bar{x},\bar{y}}\eta)_y - (\sigma^{\bar{x},\bar{y}}\eta)_x\eta_y - (\sigma^{\bar{x},\bar{y}}\eta)_y\eta_x + \eta_x\eta_y\big]\\
=& \eta_x L_N^0\eta_y + \eta_y L_N^0\eta_x + \frac{1}{2}\sum_{\bar{x},\bar{y}\in\Lambda_N} p(\bar{y}-\bar{x})\big[(\sigma^{\bar{x},\bar{y}}\eta)_x - \eta_x\big]\big[(\sigma^{\bar{x},\bar{y}}\eta)_y - \eta_y\big]\\
=& \eta_x L_N^0\eta_y + \eta_y L_N^0\eta_x - p(y-x)(\eta_y-\eta_x)^2.
\end{aligned}
$$

In order to prove ii), note that $\big[(\sigma^{\bar{x}}\eta)_x - \eta_x\big]\big[(\sigma^{\bar{x}}\eta)_y - \eta_y\big]$ is equal to zero, for all $\bar{x}\in\mathbb{Z}$. Thus, by definition of L_N^r, we have that

$$
\begin{aligned}
L_N^r(\eta_x\eta_y) =& \sum_{\bar{x}\in\Lambda_N,\bar{y}\geq N} p(\bar{y}-\bar{x})[\eta_{\bar{x}}(1-\beta)+(1-\eta_{\bar{x}})\beta]\big[(\sigma^{\bar{x}}\eta)_x(\sigma^{\bar{x}}\eta)_y - \eta_x\eta_y\big]\\
=& \eta_x L_N^r\eta_y + \eta_y L_N^r\eta_x +\\
& \sum_{\bar{x}\in\Lambda_N,\bar{y}\geq N} p(\bar{y}-\bar{x})[\eta_{\bar{x}}(1-\beta)+(1-\eta_{\bar{x}})\beta]\big[(\sigma^{\bar{x}}\eta)_x - \eta_x\big]\big[(\sigma^{\bar{x}}\eta)_y - \eta_y\big]\\
=& \eta_x L_N^r\eta_y + \eta_y L_N^r\eta_x.
\end{aligned}
$$

The proof of the third expression is analogous. □

4.2 Proof of Lemma 2.3.1

Let $\varepsilon>0$ be fixed. We have that $N^2(K_NG)(\frac{x}{N})$ is equal to

$$
N^2\sum_{|y|\geq\varepsilon N}(G(\tfrac{x+y}{N}) - G(\tfrac{x}{N}))p(y) + N^2\sum_{|y|<\varepsilon N}(G(\tfrac{x+y}{N}) - G(\tfrac{x}{N}))p(y). \tag{4.2.1}
$$

The first term at the left hand side in (4.2.1) goes to zero with N, since we have that

$$
\left|N^2\sum_{|y|\geq\varepsilon N}(G(\tfrac{x+y}{N}) - G(\tfrac{x}{N}))p(y)\right| \lesssim \frac{\|G\|_\infty N^2}{(\varepsilon N)^\gamma}.
$$

On the term at the right hand side of (4.2.1) we perform a Taylor expansion of G and we have that

$$
N^2\sum_{|y|<\varepsilon N}[G(\tfrac{x+y}{N}) - G(\tfrac{x}{N})]p(y) = N^2\sum_{|y|<\varepsilon N}\left[G'(\tfrac{x}{N})\tfrac{y}{N} + \frac{1}{2}G''(\tfrac{x}{N})(\tfrac{y}{N})^2\right]p(y),
$$

plus lower-order terms (with respect to N). Now, we use the fact that p is symmetric to see that $\sum_{|y|<\varepsilon N} yp(y)=0$. Since p has finite second moment, $\sum_{|y|<\varepsilon N} y^2p(y)\to\sigma^2$ as N goes to ∞, so that the proof ends.

4.3 Proof of Lemma 2.3.2

Let us prove the first item, the second one being similar. It is sufficient to prove it for u in the form z/N, $z \geq aN$. We have, by performing an integration by parts, that

$$
\begin{aligned}
N^{\gamma} r_N^{-}(\tfrac{z}{N}) - r^{-}(\tfrac{z}{N}) &= N^{\gamma} \sum_{y \geq z} p(y) - c_{\gamma} \int_{z/N}^{\infty} u^{-\gamma-1} du = c_{\gamma} \sum_{y \geq z} \left[\tfrac{1}{N} (\tfrac{y}{N})^{-\gamma-1} - \int_{y/N}^{(y+1)/N} u^{-\gamma-1}\, du \right] \\
&= c_{\gamma} \sum_{y \geq z} \int_{y/N}^{(y+1)/N} \left[(\tfrac{y}{N})^{-\gamma-1} - u^{-\gamma-1} \right] du = c_{\gamma} \sum_{y \geq z} \int_{y/N}^{(y+1)/N} \frac{d}{du} \big[u - (\tfrac{y+1}{N}) \big] \left[(\tfrac{y}{N})^{-\gamma-1} - u^{-\gamma-1} \right] du \\
&= -(\gamma+1) c_{\gamma} \sum_{y \geq z} \int_{y/N}^{(y+1)/N} u^{-(\gamma+2)} \big(u - \tfrac{y+1}{N} \big)\, du.
\end{aligned}
$$

Therefore we have that

$$
\left| N^{\gamma} r_N^{-}(\tfrac{z}{N}) - r^{-}(\tfrac{z}{N}) \right| \leq c_{\gamma} N^{-1} (\tfrac{z}{N})^{-\gamma-1} \tag{4.3.1}
$$

which is of order $O(N^{-1})$ since $z/N \geq a$.

4.4 Proof of Lemma 3.3.1

Note that it is sufficient to prove it for $u = \frac{x}{N}$. By using the symmetry of p we can rewrite

$$
(\mathscr{K}_N G)(\tfrac{x}{N}) = \frac{1}{2} \sum_{z \in \mathbb{Z}} p(z) \big[G(\tfrac{x+z}{N}) + G(\tfrac{x-z}{N}) - 2G(\tfrac{x}{N}) \big].
$$

We split the sum over $z \in \mathbb{Z}$ into a sum over $z \geq 1$ and over $z \leq -1$ (recall that $p(0) = 0$) and we treat separately the convergence of these two sums. Since the study is the same we consider only the sum over $z \geq 1$. Then, by a discrete integration by parts, we have

$$
N^{\gamma} \sum_{z \geq 1} p(z) \big[G(\tfrac{x+z}{N}) + G(\tfrac{x-z}{N}) - 2G(\tfrac{x}{N}) \big] = \sum_{z=2}^{\infty} N^{\gamma} r_N^{-}(\tfrac{z}{N}) \Big\{ \theta_{\frac{x}{N}}(\tfrac{z}{N}) - \theta_{\frac{x}{N}}(\tfrac{z-1}{N}) \Big\} + N^{\gamma} r_N^{-}(\tfrac{1}{N})\, \theta_{\frac{x}{N}}(\tfrac{1}{N})
$$

where

$$
\theta_v(u) = G(u+v) + G(v-u) - 2G(v).
$$

By a second order Taylor expansion of G, which is uniform over x since G has compact support, we see that since $\gamma < 2$,

$$
\lim_{N \to \infty} N^{\gamma} r_N^{-}(\tfrac{1}{N})\, \theta_{\frac{x}{N}}(\tfrac{1}{N}) = 0
$$

uniformly over x. Our aim is now to replace in the remaining sum the term $N^{\gamma} r_N^{-}(\frac{z}{N})$ by $r^{-}(\frac{z}{N})$. Recall that we have seen in Appendix 4.3 that for any $a \in (0,1)$ there exists a constant $C_a > 0$ such that

$$
|N^{\gamma} r_N^{-}(\tfrac{z}{N}) - r^{-}(\tfrac{z}{N})| \leq C_a N^{-1}.
$$

We rewrite the sum

$$\sum_{z=2}^{\infty}\left\{N^{\gamma}r_N^-(\tfrac{z}{N})-r^-(\tfrac{z}{N})\right\}\left\{\theta_{\frac{x}{N}}(\tfrac{z}{N})-\theta_{\frac{x}{N}}(\tfrac{z-1}{N})\right\}$$

as the sum over $2\le z\le aN$ and the sum over $z>aN$. In fact the sum over $z>aN$ is equal to the sum over $3N>z>aN$ since for $z\ge 3N$, $\theta_{\frac{x}{N}}(\frac{z}{N})-\theta_{\frac{x}{N}}(\frac{z-1}{N})=0$. Moreover, we have that $|\theta_{\frac{x}{N}}(\frac{z}{N})-\theta_{\frac{x}{N}}(\frac{z-1}{N})|=O(N^{-1})$ uniformly in x and z. The sum over $3N>z>aN$ is thus bounded from above by C'_a/N for some positive constant C'_a (going to ∞ as a goes to 0). Since $\theta_v(u)\le Cu^2$ for some positive constant uniformly in v, by using the estimate (4.3.1) obtained in the proof of the first item, we have also that

$$\left|\sum_{z=2}^{[aN]}\left\{N^{\gamma}r_N^-(\tfrac{z}{N})-r^-(\tfrac{z}{N})\right\}\left\{\theta_{\frac{x}{N}}(\tfrac{z}{N})-\theta_{\frac{x}{N}}(\tfrac{z-1}{N})\right\}\right|\le C'\sum_{z=2}^{[aN]}(\tfrac{z}{N})^2N^{-1}(\tfrac{z}{N})^{-\gamma-1}\le C''a^{2-\gamma}$$

for constants C',C'' which do not depend on a and x. In conclusion, the replacement of the term $N^{\gamma}r_N^-(\frac{z}{N})$ by $r^-(\frac{z}{N})$ costs $C''a^{2-\gamma}+C'_a/N$. Therefore, by sending $N\to\infty$ and then $a\to 0$, we are reduced to estimate

$$\sum_{z=2}^{\infty}r^-(\tfrac{z}{N})\left\{\theta_{\frac{x}{N}}(\tfrac{z}{N})-\theta_{\frac{x}{N}}(\tfrac{z-1}{N})\right\}=\frac{1}{N}\sum_{z=2}^{\infty}r^-(\tfrac{z}{N})\theta'_{\frac{x}{N}}(\tfrac{z}{N})\;+\;\varepsilon_N(x).$$

By a second Taylor expansion, and using that $\gamma<2$, it is easy to see that

$$\lim_{N\to\infty}\sup_{x\in\Lambda_N}|\varepsilon_N(x)|=0.$$

To conclude we observe that there exists $C>0$ such that

$$|r^-(u)\theta'_v(u)-r^-(u')\theta'_v(u')|\le C|u-u'|(u\wedge u')^{-\gamma},$$

uniformly in v. This is because $\theta'_v(0)=0$. It follows that

$$\left|\frac{1}{N}\sum_{z=2}^{\infty}r^-(\tfrac{z}{N})\theta'_{\frac{x}{N}}(\tfrac{z}{N})-\int_{2/N}^{\infty}r^-(u)\theta'_{\frac{x}{N}}(u)du\right|=\left|\sum_{z=2}^{\infty}\int_{\frac{z}{N}}^{\frac{z+1}{N}}(r^-(\tfrac{z}{N})\theta'_{\frac{x}{N}}(\tfrac{z}{N})-r^-(u)\theta'_{\frac{x}{N}}(u))du\right|$$
$$\lesssim N^{\gamma-2}\sum_{z=2}^{\infty}z^{-\gamma}$$

where the last term goes to 0, as N goes to ∞.

4.5 Proof of Lemma 3.2.14

Recall (3.5.1). As we will see below, by Lax-Milgram's Theorem (see [11]), there exists a unique function $\bar{\varphi}^{\hat{\kappa}}\in\mathscr{H}_0^{\gamma/2}$ which is solution of (3.5.1). Then, it is not difficult to see that $\bar{\rho}^{\hat{\kappa}}:=$

$\bar{\varphi}^{\hat{\kappa}} + \bar{\rho}^{\infty}$ is the desired weak solution of (3.2.7). For that purpose, let $a^{\hat{\kappa}} : \mathcal{H}_0^{\gamma/2} \times \mathcal{H}_0^{\gamma/2} \to \mathbb{R}$ be the bilinear form defined, for $G, F \in \mathcal{H}_0^{\gamma/2}$, as

$$a^{\hat{\kappa}}(F,G) = \langle F, G \rangle_{\gamma/2} + \hat{\kappa} \langle F, G \rangle_{V_1}. \tag{4.5.1}$$

From Lax-Milgram Theorem, in order to conclude the existence and uniqueness it is enough to prove that $a^{\hat{\kappa}}$ is coercive and continuous. For $\hat{\kappa} > 0$, we can easily see that

$$a^{\hat{\kappa}}(G,G) \geq \min\{1, \hat{\kappa} V_1(\tfrac{1}{2})\} \left(\|G\|^2_{\gamma/2} + \|G\|^2 \right) = \min\{1, \hat{\kappa} V_1(\tfrac{1}{2})\} \|G\|^2_{\mathcal{H}_0^{\gamma/2}}.$$

For $\hat{\kappa} = 0$, since on $\mathcal{H}_0^{\gamma}$ the norms $\|\cdot\|_{\gamma/2}$ and $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ are equivalent we have that

$$a^0(G,G) = \|G\|^2_{\gamma/2} \gtrsim \|G\|^2_{\mathcal{H}_0^{\gamma/2}}.$$

Therefore $a^{\hat{\kappa}}$ is coercive for $\hat{\kappa} \geq 0$. Moreover, by using the Cauchy-Schwarz inequality we obtain that

$$|a^{\hat{\kappa}}(F,G)| \leq \|F\|_{\gamma/2} \|G\|_{\gamma/2} + \hat{\kappa}(\|F\|_{V_1} \|G\|_{V_1}).$$

From the fractional Hardy's inequality (see (3.4.2)) we have that

$$|a^{\hat{\kappa}}(F,G)| \lesssim (\hat{\kappa} + 1)(\|F\|_{\gamma/2} \|G\|_{\gamma/2})$$

and since on $\mathcal{H}_0^{\gamma/2}$ the norms $\|\cdot\|_{\gamma/2}$ and $\|\cdot\|_{\mathcal{H}^{\gamma/2}}$ are equivalent, we conclude that the bilinear form $a^{\hat{\kappa}}$ is continuous for $\hat{\kappa} \geq 0$. This end the proof.

4.6 Uniqueness of weak solutions

The uniqueness of the weak solutions of the partial differential equations given in Chapter 2 and 3 is fundamental for the proof of the hydrodynamic limit.

4.6.1 Diffusive case

The uniqueness of weak solutions of (2.2.1) is standard if $\hat{\kappa} = 0$. Since we were not able to find in the literature a proof in the case $\hat{\kappa} > 0$ we give a complete proof below. The proof of uniqueness of weak solutions of (2.2.2) can be found in, for example, [1].

Now we prove the uniqueness of weak solutions of (2.2.1). We assume that $\hat{\sigma} > 0$ and $\hat{\kappa} > 0$ first and then we consider the case $\hat{\sigma} = 0$ and $\hat{\kappa} > 0$.

Let ρ^1 and ρ^2 be two weak solutions of (2.2.1) with the same initial condition and let us denote $\tilde{\rho} = \rho^1 - \rho^2$. By assumption we have that

$$\tilde{\rho} \in L^2\Big(0, T; \mathcal{H}^1([0,1])\Big) \cap L^2\Big(0, T; L^2_{V_1}([0,1])\Big),$$

recall that $V_1(u) = u^{-\gamma} + (1-u)^{-\gamma}$ and $\langle \cdot, \cdot \rangle_{V_1}$ (resp. $\|\cdot\|_{V_1}$) is the scalar product (resp. the norm) corresponding to the Hilbert space $L^2_{V_1}([0,1])$.

For almost every $t \in [0,T]$, we identify $\tilde{\rho}_t$ with its continuous representation in $[0,1]$. Therefore, from Remark 2.2.5, we have that $\tilde{\rho}_t(0) = \tilde{\rho}_t(1) = 0$ for all $t \in [0,T]$. Since $\mathscr{H}_0^1([0,1])$ is equal to the set of functions in $\mathscr{H}^1([0,1])$ vanishing at 0 and 1 we have that for a.e. time $t \in [0,T]$, $\tilde{\rho}_t \in \mathscr{H}_0^1([0,1])$ and in fact $\tilde{\rho} \in L^2\Big(0,T;\mathscr{H}_0^1([0,1])\Big)$. From ii) in Definition 2.2.3, for any $t \in [0,T]$ and any $G \in C_c^{1,2}([0,T]\times[0,1])$ we have

$$\int_0^1 \tilde{\rho}_t(u)G_t(u)\,du - \int_0^t\int_0^1 \tilde{\rho}_s(u)\Big(\partial_s + \tfrac{\hat{\sigma}^2}{2}\Delta\Big)G_s(u)\,du\,ds + \hat{\kappa}\int_0^t\int_0^1 V_1(u)G_s(u)\tilde{\rho}_s(u)\,du\,ds = 0. \tag{4.6.1}$$

We know that $C_c^{1,\infty}([0,T]\times(0,1))$ is dense in $L^2\Big(0,T;\mathscr{H}_0^1([0,1])\Big)\cap L^2\Big(0,T;L^2_{V1}([0,1])\Big)$. Therefore, let $\{H_n\}_{n\geq 0}$ be a sequence of functions in $C_c^{1,\infty}([0,T]\times(0,1))$ converging to $\tilde{\rho}$ with respect to the norms of $L^2\Big(0,T;\mathscr{H}_0^1([0,1])\Big)$ and $L^2\Big(0,T;L^2_{V1}([0,1])\Big)$. We define G_n in $C_c^{1,\infty}([0,T]\times[0,1])$ by $\forall t \in [0,T]$, $\forall u \in [0,1]$,

$$G_n(t,u) = \int_t^T H_n(s,u)\,ds. \tag{4.6.2}$$

Plugging G_n into (4.6.1) and letting $n\to\infty$ we conclude, by Lemma 4.6.1 below, that

$$\int_0^T\int_0^1 \tilde{\rho}_s^2(u)\,du\,ds + \tfrac{\hat{\sigma}^2}{4}\Big\|\int_0^T \tilde{\rho}_s ds\Big\|_1^2 + \tfrac{\hat{\kappa}}{2}\Big\|\int_0^T \tilde{\rho}_s ds\Big\|_{V_1}^2 = 0.$$

It follows that for almost every time $s \in [0,T]$ the continuous function $\tilde{\rho}_s$ is equal to 0 and we conclude the uniqueness of weak solution to (2.2.1) in the case $\hat{\sigma} > 0$.

Lemma 4.6.1. *Let $\{G_n\}_{n\geq 0}$ be defined as in (4.6.2). We have*

i) $\lim_{n\to\infty}\int_0^T\int_0^1 \tilde{\rho}_s(u)\,(\partial_s G_n)(s,u)\,du ds = -\int_0^T\int_0^1 \tilde{\rho}_s^2(u)\,du ds.$

ii) $\lim_{n\to\infty}\int_0^T\int_0^1 \tilde{\rho}_s(u)\Delta G_n(s,u)\,du ds = -\tfrac{1}{2}\Big\|\int_0^T \tilde{\rho}_s ds\Big\|_1^2.$

iii) $\lim_{n\to\infty}\int_0^T\int_0^1 V_1(u)G_n(s,u)\tilde{\rho}_s(u)\,du\,ds = \tfrac{1}{2}\Big\|\int_0^T \tilde{\rho}_s ds\Big\|_{V_1}^2 < \infty.$

Proof. For i) we write

$$\begin{aligned} -\int_0^T\int_0^1 \tilde{\rho}_s(u)\,(\partial_s G_n)(s,u)\,du ds &= \int_0^T\int_0^1 \tilde{\rho}_s(u) H_n(s,u)\,du ds \\ = \int_0^T \langle \tilde{\rho}_s\,,\, H_n(s,\cdot)\rangle\,ds &= \int_0^T \Big\langle \tilde{\rho}_s\,,\, H_n(s,\cdot) - \tilde{\rho}_s\Big\rangle\,ds \;+\; \int_0^T \|\tilde{\rho}_s\|^2\,ds. \end{aligned}$$

Observe then that by the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\left|\int_0^T \left\langle \tilde{\rho}_s\,,\, H_n(s,\cdot)-\tilde{\rho}_s\right\rangle ds\right| \le \int_0^T \|\tilde{\rho}_s\|\,\|H_n(s,\cdot)-\tilde{\rho}_s\|\,ds \\
&\le \sqrt{\int_0^T \|\tilde{\rho}_s\|^2\,ds}\,\sqrt{\int_0^T \|H_n(s,\cdot)-\tilde{\rho}_s\|^2\,ds}
\end{aligned}
\tag{4.6.3}
$$

which goes to 0 as $n \to \infty$. Above we have used the fact that $\{H_n\}_{n\ge 0}$ converges to $\tilde{\rho}$ as $N \to \infty$ with respect to the norm of $L^2(0,T;\mathscr{H}_0^1([0,1]))$.

For ii) we first use the integration by parts formula for $\mathscr{H}_1([0,1])$ functions which permits to write

$$
\begin{aligned}
&\int_0^T\int_0^1 \tilde{\rho}_s(u)\,\Delta G_n(s,u)\,duds = -\int_0^T \Big\langle \tilde{\rho}_s\,,G_n(s,\cdot)\Big\rangle_1 ds \\
&= \int_0^T \Big\langle \tilde{\rho}_s\,,\int_s^T \tilde{\rho}_r dr\Big\rangle_1 ds + \int_0^T \Big\langle \tilde{\rho}_s\,,G_n(s,\cdot)-\int_s^T \tilde{\rho}_r dr\Big\rangle_1 ds \\
&= \iint_{0\le s<r\le T} \langle \tilde{\rho}_s\,,\,\tilde{\rho}_r\rangle_1\,dr\,ds \;+\; \int_0^T \Big\langle \tilde{\rho}_s\,,\int_s^T \{H_n(r,\cdot)-\tilde{\rho}_r\}dr\Big\rangle_1 ds \\
&= \frac{1}{2}\iint_{[0,T]^2} \langle \tilde{\rho}_s\,,\,\tilde{\rho}_r\rangle_1\,drds \;+\; \int_0^T \Big\langle \tilde{\rho}_s\,,\int_s^T \{H_n(r,\cdot)-\tilde{\rho}_r\}dr\Big\rangle_1 ds \\
&= \frac{1}{2}\left\|\int_0^T \tilde{\rho}_s ds\right\|_1^2 + \int_0^T \Big\langle \tilde{\rho}_s\,,\int_s^T \{H_n(r,\cdot)-\tilde{\rho}_r\}dr\Big\rangle_1 ds.
\end{aligned}
$$

To conclude the proof of ii) it is sufficient to prove that the last term in the previous expression vanish as $n \to \infty$. Indeed, the absolute value of such term is bounded from above by

$$
\begin{aligned}
\int_0^T \left\|\tilde{\rho}_s\right\|_1 \left\|\int_s^T \{H_n(r,\cdot)-\tilde{\rho}_r\}dr\right\|_1 ds &\le \int_0^T \left\|\tilde{\rho}_s\right\|_1 \int_s^T \left\|H_n(r,\cdot)-\tilde{\rho}_r\right\|_1 dr\,ds \\
&\le \left(\int_0^T \left\|\tilde{\rho}_s\right\|_1 ds\right)\left(\int_0^T \left\|H_n(r,\cdot)-\tilde{\rho}_r\right\|_1 dr\right) \\
&\le T\sqrt{\int_0^T \left\|\tilde{\rho}_s\right\|_1^2 ds}\,\sqrt{\int_0^T \left\|H_n(r,\cdot)-\tilde{\rho}_r\right\|_1^2 dr} \xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

Above we have used the Cauchy-Schwarz inequality and the fact that $\{H_n\}_{n\ge 0}$ converges to $\tilde{\rho}$ as $N \to \infty$ with respect to the norm of $L^2(0,T;\mathscr{H}_0^1([0,1]))$.

The proof of iii) is similar. We have that $\int_0^T \left\langle \tilde{\rho}_s\,, G_n(s,\cdot) \right\rangle_{V_1} ds$ is equal to

$$
\begin{aligned}
&\int_0^T \left\langle \tilde{\rho}_s\,, \int_s^T \tilde{\rho}_r dr \right\rangle_{V_1} ds + \int_0^T \left\langle \tilde{\rho}_s\,, G_n(s,\cdot) - \int_s^T \tilde{\rho}_r dr \right\rangle_{V_1} ds \\
=& \iint_{0\leq s<r\leq T} \langle \tilde{\rho}_s\,, \tilde{\rho}_r \rangle_{V_1} dr\, ds \;+\; \int_0^T \left\langle \tilde{\rho}_s\,, \int_s^T \{H_n(r,\cdot) - \tilde{\rho}_r\} dr \right\rangle_{V_1} ds \\
=& \frac{1}{2} \iint_{[0,T]^2} \langle \tilde{\rho}_s\,, \tilde{\rho}_r \rangle_{V_1} dr\, ds \;+\; \int_0^T \left\langle \tilde{\rho}_s\,, \int_s^T \{H_n(r,\cdot) - \tilde{\rho}_r\} dr \right\rangle_{V_1} ds \\
=& \frac{1}{2} \left\| \int_0^T \tilde{\rho}_s ds \right\|_{V_1}^2 + \int_0^T \left\langle \tilde{\rho}_s\,, \int_s^T \{H_n(r,\cdot) - \tilde{\rho}_r\} dr \right\rangle_{V_1} ds.
\end{aligned}
$$

To conclude the proof of iii) it is sufficient to show that

$$
\lim_{n\to\infty} \int_0^T \left\langle \tilde{\rho}_s\,, \int_s^T \{H_n(r,\cdot) - \tilde{\rho}_r\} dr \right\rangle_{V_1} ds = 0.
$$

This is a consequence of the Cauchy-Schwarz inequality:

$$
\begin{aligned}
\left| \int_0^T \left\langle \tilde{\rho}_s\,, \int_s^T \{H_n(r,\cdot) - \tilde{\rho}_r\} dr \right\rangle_{V_1} ds \right| &\leq \int_0^T \left\| \tilde{\rho}_s \right\|_{V_1} \left\| \int_s^T \{H_n(r,\cdot) - \tilde{\rho}_r\} dr \right\|_{V_1} ds \\
&\leq \int_0^T \left\| \tilde{\rho}_s \right\|_{V_1} \int_s^T \left\| H_n(r,\cdot) - \tilde{\rho}_r \right\|_{V_1} dr\, ds \\
&\leq \left(\int_0^T \left\| \tilde{\rho}_s \right\|_{V_1} ds \right) \left(\int_0^T \left\| H_n(r,\cdot) - \tilde{\rho}_r \right\|_{V_1} dr \right) \\
&\leq T \sqrt{\int_0^T \left\| \tilde{\rho}_s \right\|_{V_1}^2 ds} \sqrt{\int_0^T \left\| H_n(r,\cdot) - \tilde{\rho}_r \right\|_{V_1}^2 dr} \xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

□

Note that when $\hat{\sigma} > 0$ and $\hat{\kappa} = 0$ the proof above also shows uniqueness of the weak solution of the heat equation with Dirichlet boundary conditions.

Now we look at the case $\hat{\sigma} = 0$. In this case we do not have any regularity assumption on $\tilde{\rho}$. However, it can be proved that

$$
\int_0^T \int_0^1 \tilde{\rho}_s^2(u)\, du ds + \tfrac{\hat{\kappa}}{2} \left\| \int_0^T \tilde{\rho}_s ds \right\|_{V_1}^2 = 0
$$

holds by showing only the first and third item of the previous lemma. This requires only the density of $C_c^{1,\infty}([0,T]\times(0,1))$ in $L^2\Big(0,T; L^2_{V_1}([0,1])\Big)$. We also note that in the proof of item i) in Lemma 4.6.1, in order to conclude the convergence in (4.6.3), before applying the Cauchy-Schwarz inequality, we multiply and divide the integrand function by V_1 and since V_1^{-1} is bounded we get that $\|\tilde{\rho}_s V_1^{-1}\|^2 < \infty$ and the result follows.

4.6.2 Super-diffusive case: Proof of Lemma 3.2.8

We only focus in the proof of the uniqueness of weak solutions of (3.2.4) for $\hat{\kappa} = \kappa$. In the end of the section we comment the other cases.

Let $\rho^{\kappa,1}$ and $\rho^{\kappa,2}$ two weak solutions of (3.2.4) with the same initial condition and let us denote $\tilde{\rho}^{\kappa} = \rho^{\kappa,1} - \rho^{\kappa,2}$. For almost every $t \in [0, T]$, we identify $\tilde{\rho}^{\kappa}_t$ with its continuous representation on $[0,1]$. Therefore, by Remark 3.2.4 we have $\tilde{\rho}^{\kappa}_t(0) = \tilde{\rho}^{\kappa}_t(1) = 0$. Since $\mathscr{H}_0^{\gamma/2}$ is equal to the set of functions in $\mathscr{H}^{\gamma/2}$ vanishing at 0 and 1 we have that $\tilde{\rho}^{\kappa}_t \in \mathscr{H}_0^{\gamma/2}$ for a.e. time $t \in [0, T]$ and, in fact, $\tilde{\rho}^{\kappa} \in L^2(0, T; \mathscr{H}_0^{\gamma/2})$. Moreover, for any $t \in [0, T]$ and all functions $G \in C_c^{1,\infty}([0, T] \times (0,1))$ we have

$$\langle \tilde{\rho}^{\kappa}_t, G_t \rangle - \int_0^t \left\langle \tilde{\rho}^{\kappa}_s, \left(\partial_s + \mathbb{L}\right) G_s \right\rangle ds + \kappa \int_0^t \left\langle \tilde{\rho}^{\kappa}_s, G_s \right\rangle_{V_1} ds = 0. \tag{4.6.4}$$

Note that $C_c^{1,\infty}([0, T] \times (0,1))$ is dense in $L^2(0, T; \mathscr{H}_0^{\gamma/2})$. Let $\{H_n^{\kappa}\}_{n \geq 1}$ be a sequence of functions in $C_c^{1,\infty}([0, T] \times (0,1))$ converging to $\tilde{\rho}^{\kappa}$ with respect to the norm of $L^2(0, T; \mathscr{H}_0^{1/2})$ as $n \to \infty$. For $n \geq 1$, we define the test functions $\forall t \in [0, T]$, $\forall u \in [0,1]$, $G_n^{\kappa}(t, u) = \int_t^T H_n^{\kappa}(s, u)\, ds$. Plugging G_n^{κ} into (4.6.4) and letting $n \to \infty$ we conclude by Lemma 4.6.2 below that

$$\int_0^T \|\tilde{\rho}^{\kappa}_s\|^2 ds + \frac{1}{2} \left\| \int_0^T \tilde{\rho}^{\kappa}_s ds \right\|^2_{\gamma/2} + \frac{\kappa}{2} \left\| \int_0^T \tilde{\rho}^{\kappa}_s ds \right\|^2_{V_1} = 0.$$

Recall that $\langle \cdot, \cdot \rangle_{V_1}$ (resp. $\| \cdot \|_{V_1}$) is the scalar product (resp. the norm) corresponding to the Hilbert space $L^2_{V_1}$.

Then, it follows that for almost every time $s \in [0, T]$ the continuous function $\tilde{\rho}^{\kappa}_s$ is equal to 0 and we conclude the uniqueness of the weak solutions to (3.2.4).

Lemma 4.6.2. *Let $\{G_n^{\kappa}\}_{n \geq 1}$ be defined as above. We have*

i) $$\lim_{n \to \infty} \int_0^T \left\langle \tilde{\rho}^{\kappa}_s, (\partial_s G_n^{\kappa})(s, \cdot) \right\rangle ds = -\int_0^T \|\tilde{\rho}^{\kappa}_s\|^2 ds.$$

ii) $$\lim_{n \to \infty} \int_0^T \left\langle \tilde{\rho}^{\kappa}_s, \mathbb{L} G_n^{\kappa}(s, \cdot) \right\rangle ds = -\frac{1}{2} \left\| \int_0^T \tilde{\rho}^{\kappa}_s ds \right\|^2_{\gamma/2}.$$

iii) $$\lim_{n \to \infty} \int_0^T \left\langle \tilde{\rho}^{\kappa}_s, G_n^{\kappa}(s, \cdot) \right\rangle ds = \frac{1}{2} \left\| \int_0^T \tilde{\rho}^{\kappa}_s ds \right\|^2_{V_1} < \infty.$$

Proof. For i) we write

$$-\int_0^T \left\langle \tilde{\rho}^{\kappa}_s, (\partial_s G_n^{\kappa})(s, \cdot) \right\rangle ds = \int_0^T \langle \tilde{\rho}^{\kappa}_s, H_n^{\kappa}(s, \cdot) \rangle ds = \int_0^T \left\langle \tilde{\rho}^{\kappa}_s, H_n^{\kappa}(s, \cdot) - \tilde{\rho}^{\kappa}_s \right\rangle ds + \int_0^T \|\tilde{\rho}^{\kappa}_s\|^2 ds.$$

Observe then that by the Cauchy-Schwarz inequality we have that

$$\begin{aligned}\left|\int_0^T \left\langle \tilde\rho_s^\kappa, H_n^\kappa(s,\cdot)-\tilde\rho_s^\kappa\right\rangle ds\right| &\le \int_0^T \|\tilde\rho_s^\kappa\|\,\|H_n^\kappa(s,\cdot)-\tilde\rho_s^\kappa\|\,ds\\ &\le \sqrt{\int_0^T \|\tilde\rho_s^\kappa\|^2\,ds}\sqrt{\int_0^T \|H_n^\kappa(s,\cdot)-\tilde\rho_s^\kappa\|^2\,ds}\end{aligned}$$

which goes to 0 as $n\to\infty$.

For ii) we first use the integration by parts formula for the regional fractional Laplacian (see Theorem 3.3 in [37]) which permits to write

$$\int_0^T \left\langle \tilde\rho_s^\kappa, \mathbb{L}G_n^\kappa(s,\cdot)\right\rangle ds = -\int_0^T \left\langle \tilde\rho_s^\kappa, G_n^\kappa(s,\cdot)\right\rangle_{\gamma/2} ds.$$

Then we have that $\int_0^T \left\langle \tilde\rho_s^\kappa, G_n^\kappa(s,\cdot)\right\rangle_{\gamma/2} ds$ is equal to

$$\begin{aligned}&\int_0^T \left\langle \tilde\rho_s^\kappa, \int_s^T \tilde\rho_t^\kappa dt\right\rangle_{\gamma/2} ds + \int_0^T \left\langle \tilde\rho_s^\kappa, G_n^\kappa(s,\cdot)-\int_s^T \tilde\rho_t^\kappa dt\right\rangle_{\gamma/2} ds\\ &= \iint_{0\le s<t\le T} \langle \tilde\rho_s^\kappa, \tilde\rho_t^\kappa\rangle_{\gamma/2}\, ds dt \;+\; \int_0^T \left\langle \tilde\rho_s^\kappa, \int_s^T \left(H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right) dt\right\rangle_{\gamma/2} ds\\ &= \frac{1}{2}\iint_{[0,T]^2} \langle \tilde\rho_s^\kappa, \tilde\rho_t^\kappa\rangle_{\gamma/2}\, ds dt \;+\; \int_0^T \left\langle \tilde\rho_s^\kappa, \int_s^T \left(H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right) dt\right\rangle_{\gamma/2} ds\\ &= \frac{1}{2}\left\|\int_0^T \tilde\rho_s^\kappa ds\right\|_{\gamma/2}^2 + \int_0^T \left\langle \tilde\rho_s^\kappa, \int_s^T \left(H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right) dt\right\rangle_{\gamma/2} ds.\end{aligned}$$

To conclude the proof of ii) it is sufficient to show that the last term in the previous expression vanish as $n\to\infty$. This is a consequence of a successive use of Cauchy-Schwarz inequalities. Indeed, the last term in the previous expression is bounded from above by

$$\begin{aligned}&\int_0^T \left\|\tilde\rho_s^\kappa\right\|_{\gamma/2} \left\|\int_s^T \left(H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right) dt\right\|_{\gamma/2} ds \le \int_0^T \left\|\tilde\rho_s^\kappa\right\|_{\gamma/2} \int_s^T \left\|H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right\|_{\gamma/2} dt\, ds\\ &\le \int_0^T \left\|\tilde\rho_s^\kappa\right\|_{\gamma/2} \int_0^T \left\|H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right\|_{\gamma/2} dt\, ds = \left(\int_0^T \left\|\tilde\rho_s^\kappa\right\|_{\gamma/2} ds\right)\left(\int_0^T \left\|H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right\|_{\gamma/2} dt\right)\\ &\le T\sqrt{\int_0^T \left\|\tilde\rho_s^\kappa\right\|_{\gamma/2}^2 ds}\sqrt{\int_0^T \left\|H_n^\kappa(t,\cdot)-\tilde\rho_t^\kappa\right\|_{\gamma/2}^2 dt} \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

The proof of iii) is similar to the proof of ii) by using the fractional Hardy's inequality (see (3.4.2)) and since $C_c^\infty((0,1))$ is dense in $H_0^{\gamma/2}$ we have that any $g\in H_0^{\gamma/2}$ is also in the space

$L^2_{V_1}$ and that (3.4.2) remains valid for g. In particular, we have that the right hand side of iii) is finite.

We have that $\int_0^T \left\langle \tilde{\rho}^\kappa_s, G^\kappa_n(s,\cdot) \right\rangle_{V_1} ds$ is equal to

$$\begin{aligned}
&\int_0^T \left\langle \tilde{\rho}^\kappa_s, \int_s^T \tilde{\rho}^\kappa_t dt \right\rangle_{V_1} ds + \int_0^T \left\langle \tilde{\rho}^\kappa_s, G^\kappa_n(s,\cdot) - \int_s^T \tilde{\rho}^\kappa_t dt \right\rangle_{V_1} ds \\
&= \iint_{0\le s<t\le T} \langle \tilde{\rho}^\kappa_s, \tilde{\rho}^\kappa_t \rangle_{V_1} ds dt \; + \int_0^T \left\langle \tilde{\rho}^\kappa_s, \int_s^T \left(H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right) dt \right\rangle_{V_1} ds \\
&= \frac{1}{2} \iint_{[0,T]^2} \langle \tilde{\rho}^\kappa_s, \tilde{\rho}^\kappa_t \rangle_{V_1} ds dt \; + \int_0^T \left\langle \tilde{\rho}^\kappa_s, \int_s^T \left(H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right) dt \right\rangle_{V_1} ds \\
&= \frac{1}{2} \left\| \int_0^T \tilde{\rho}^\kappa_s ds \right\|^2_{V_1} + \int_0^T \left\langle \tilde{\rho}^\kappa_s, \int_s^T \left(H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right) dt \right\rangle_{V_1} ds.
\end{aligned}$$

As a consequence of a successive use of the Cauchy-Schwarz inequalities and Hardy's inequality we have that the term at the right hand side in the previous expression is bounded from above by

$$\begin{aligned}
&\int_0^T \left\| \tilde{\rho}^\kappa_s \right\|_{V_1} \left\| \int_s^T \left(H^\kappa_n(t\cdot) - \tilde{\rho}^\kappa_t \right) dt \right\|_{V_1} ds \le \int_0^T \left\| \tilde{\rho}^\kappa_s \right\|_{V_1} \int_s^T \left\| H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right\|_{V_1} dt\, ds \\
&\le \int_0^T \left\| \tilde{\rho}^\kappa_s \right\|_{V_1} \int_0^T \left\| H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right\|_{V_1} dt\, ds = \left(\int_0^T \left\| \tilde{\rho}^\kappa_s \right\|_{V_1} ds \right) \left(\int_0^T \left\| H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right\|_{V_1} dt \right) \\
&\le T \sqrt{\int_0^T \left\| \tilde{\rho}^\kappa_s \right\|^2_{V_1} ds} \sqrt{\int_0^T \left\| H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right\|^2_{V_1} dt} \\
&\le C T \sqrt{\int_0^T \left\| \tilde{\rho}^\kappa_s \right\|^2_{\gamma/2} ds} \sqrt{\int_0^T \left\| H^\kappa_n(t,\cdot) - \tilde{\rho}^\kappa_t \right\|^2_{\gamma/2} dt} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

□

The proof of the uniqueness of the weak solutions of (3.2.4) for $\kappa = 0$ is analogous, the difference is that only the first two items in Lemma 4.6.2 above are required. The uniqueness of the weak solutions of (3.2.6) is analogous as well, in this case only items i) and iii) in Lemma 4.6.2 above are required.

4.7 Energy estimates for Lemma 2.4.1

In this section we prove that the density π belongs to $\mathcal{H}^1([0,1])$ and satisfies $\int_0^1 \left\{ \frac{(\alpha-\pi(u))^2}{u^\gamma} + \frac{(\beta-\pi(u))^2}{(1-u)^\gamma} \right\} du < \infty$.

In order to prove that $\pi \in \mathcal{H}^1([0,1])$, we adapt the proof of the Proposition A.1.1 in [42]. Let $G \in C_c^\infty([0,1])$ and denote by $\{\eta^N(t)\}_{t\ge0}$ the boundary driven symmetric long-range exclusion with generator $N^2 L_N$. By stationarity of $\bar{\mu}_N$ and the entropy inequality (2.3.19) we have

$$\bar{\mu}_N\left(c_\gamma\sum_{x=1}^{N-2}G(\tfrac{x}{N})(\eta_x-\eta_{x+1})\right)=\mathbb{E}_{\bar{\mu}_N}\left(\int_0^1\left\{\sum_{x=1}^{N-2}c_\gamma G(\tfrac{x}{N})(\eta_x^N(t)-\eta_{x+1}^N(t))\right\}dt\right)$$
$$\leq C_0+\frac{1}{N}\log\left\{\mathbb{E}_{\nu_h^N}\left(e^{N\int_0^1\left\{c_\gamma\sum_{x=1}^{N-2}G(\frac{x}{N}))(\eta_x^N(t)-\eta_{x+1}^N(t))\right\}dt}\right)\right\}$$

where C_0 is a bound on the relative entropy of $\bar{\mu}_N$ with respect to ν_h^N. By Feynman-Kac's formula the last expression is bounded by from above by

$$C_0+\sup_f\left\{c_\gamma\sum_{x=1}^{N-2}G(\tfrac{x}{N})\int_{\Omega_N}(\eta_x-\eta_{x+1})f(\eta)d\,\nu_h^N(\eta)+N\left\langle L_N\sqrt{f},\sqrt{f}\right\rangle_{\nu_h^N}\right\}$$
$$\leq C_0+\sup_f\left\{c_\gamma\sum_{x=1}^{N-2}G(\tfrac{x}{N})\int_{\Omega_N}(\eta_x-\eta_{x+1})f(\eta)d\,\nu_h^N(\eta)-\frac{N}{4}D_N^0(\sqrt{f},\nu_h^N)+C\right\}$$

where the supremum is taken over all densities f on Ω_N with respect to ν_h^N. In the last inequality we used (2.3.22). Note that $\sum_{x=1}^{N-2}G(\frac{x}{N})p(1)\int_{\Omega_N}(\eta_x-\eta_{x+1})f(\eta)d\,\nu_h^N(\eta)$ is equal to

$$\frac{c_\gamma}{2}\sum_{x=1}^{N-2}G(\tfrac{x}{N})\int(\eta_x-\eta_{x+1})(f(\eta)-f(\sigma^{x,x+1}\eta))d\,\nu_h^N(\eta)$$
$$+\frac{c_\gamma}{2}\sum_{x=1}^{N-2}G(\tfrac{x}{N})\int(\eta_x-\eta_{x+1})f(\sigma^{x,x+1}\eta)(1-\theta^{x,x+1}(\eta))d\,\nu_h^N(\eta)$$
$$\leq\frac{C}{N}\sum_{x=1}^{N-2}(G(\tfrac{x}{N}))^2+\frac{N}{4}D_N^0(\sqrt{f},\nu_h^N)+C.$$

In the last inequality we used the facts that for $a,b\geq 0$, $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$, Young's inequality and (2.3.25). Then we have that

$$\bar{\mu}_N\left(\frac{c_\gamma}{N}\sum_{x\in\Lambda_N}\nabla_N G(\tfrac{x-1}{N})\eta_x\right)\leq\frac{C}{N}\sum_{x=1}^{N-2}(G(\tfrac{x}{N}))^2+C,$$

where $\nabla_N G(\frac{x-1}{N})=N(G(\frac{x}{N})-G(\frac{x-1}{N}))$. Taking the limit $N\to\infty$, we conclude that there exist constants $C>0$ independent of $G\in C_c^\infty([0,1])$ such that

$$\mathbb{E}^*\left[\int_0^1 c_\gamma G'(u)\,\pi(u)\,du-C\|G\|^2\right]\leq C.$$

It is easy to see that the supremum over G can be inserted in the expectation (see Lemma 7.5 in [41]) so that

$$\mathbb{E}^*\left[\sup_G\left\{\int_0^1 c_\gamma G'(u)\,\pi(u)\,du-C\|G\|^2\right\}\right]<\infty.$$

Then, we get $\pi \in \mathscr{H}^1([0,1])$.

Now, in order to prove that $\int_0^1 \left\{\frac{(\alpha-\pi(u))^2}{u^\gamma} + \frac{(\beta-\pi(u))^2}{(1-u)^\gamma}\right\} du < \infty$ we note that it is enough to prove that the function $u \to \pi(u) - \alpha$ belongs to $L^2_{r^-}([0,1])$ and the function $u \to \pi(u) - \beta$ belongs to $L^2_{r^+}([0,1])$.

By stationarity of $\bar{\mu}_N$, entropy inequality and the Feynman-Kac's formula, we have that

$$\begin{aligned}
&\mathbb{E}_{\mu_N}\left(\int_0^1 dt\, N^{\gamma-1} \sum_{x\in\Lambda_N} G(\tfrac{x}{N}) r_N^-\Big(\tfrac{x}{N}\Big)(\eta_x^N(t)-\alpha)\right) \\
&\leq C_0 + \sup_f \left\{ N^{\gamma-1} \sum_{x\in\Lambda_N} G(\tfrac{x}{N}) r_N^-\Big(\tfrac{x}{N}\Big)\langle t_x^\alpha, f\rangle_{\nu_h^N} + N\left\langle L_N\sqrt{f}, \sqrt{f}\right\rangle_{\nu_h^N}\right\}
\end{aligned} \tag{4.7.1}$$

where $\langle t_x^\alpha, f\rangle_{\nu_h^N} = \int_{\Omega_N} (\eta_x - \alpha) f(\eta)\, d\nu_h^N$ and the supremum is taken over all the densities f on Ω_N with respect to ν_h^N. Since the profile h is Hölder of parameter $\gamma/2$ at the boundaries and Lipschitz inside, and from (2.3.22) the term at the right hand side of last expression is bounded from above by

$$-\frac{N}{4} D_N(\sqrt{f}, \nu_h^N) + C.$$

By using Lemma 2.3.10 with $A_x = (4\kappa)^{-1} G(\frac{x}{N})$ it is easy to show that the last expression in (4.7.1) is bounded from above by

$$CN^{\gamma-1} \sum_{x\in\Lambda_N} r_N^-\Big(\tfrac{x}{N}\Big)(G(\tfrac{x}{N}))^2 + C.$$

We take the limit $N \to \infty$ and we conclude that there exists a constant $C > 0$ independent of G such that

$$\mathbb{E}^*\left[\int_0^1 (\pi(u)-\alpha) G(u) r^-(u)\, du \; - \; C\int_0^1 (G(u))^2 r^-(u)\, du\right] \lesssim 1.$$

By using a similar method as in the proof of the previous subsection we see that the supremum over G can be inserted in the expectation so that

$$\mathbb{E}^*\left[\sup_G \left\{\int_0^1 (\pi(u)-\alpha) G(u) r^-(u)\, du \; - \; C\int_0^1 (G(u))^2 r^-(u)\, du\right\}\right] \lesssim 1.$$

The previous formula implies that $\mathbb{E}^*\left[\int_0^1 (\pi(u)-\alpha)^2 r^-(u)\, du\right] \lesssim 1$. Similarly we prove that the function $u \to \pi(u) - \beta$ belongs to $L^2_{r^+}([0,1])$.

4.8 Proof of Lemma 3.6.3

The fact that $\mathbb{P}^*$ is concentrated on absolutely continuous measures is obvious since for any continuous function $G : [0,1] \to \mathbb{R}$ we have

$$|\langle \pi^N, G\rangle| \leq \frac{1}{(N-1)} \sum_{x=1}^{N-1} |G(\tfrac{x}{N})|$$

and similarly for $\hat{\pi}^N$. Since for any continuous function G, the functional $\pi \in \mathscr{M}_d^+ \to \langle \pi, G\rangle$ is continuous, by weak convergence, we have that $\mathbb{P}^*$ is concentrated on measures $(\pi, \hat{\pi})$ such that for any continuous function $G : [0,1] \to \mathbb{R}$, $\hat{G} : [0,1]^2 \to \mathbb{R}$

$$|\langle \pi, G\rangle| \leq \int_{[0,1]} |G(u)|du, \quad |\langle \hat{\pi}, \hat{G}\rangle| \leq \int_{[0,1]^2} |\hat{G}(u,v)|dudv$$

which implies that such a π and $\hat{\pi}$ are absolutely continuous with respect to the Lebesgue measure. The densities are denoted by π and $\hat{\pi}$. Since $\hat{\pi}^N$ is a product measure whose marginals are given by π^N, by weak convergence, we have that $\hat{\pi}(u,v) = \pi(u)\pi(v)$ for any $(u,v) \in [0,1]^2$.

To prove that π is continuous we adapt the proof of [42] Proposition A.1.1. Recall ν_h^N defined in (2.3.18), for $h : [0,1] \to [0,1]$ a smooth function such that $\alpha \leq h(u) \leq \beta$, for all $u \in [0,1]$, and $h(0) = \alpha$ and $h(1) = \beta$.

Let $\varepsilon > 0$ be a small real number. Let $F \in C_c^\infty([0,1]^2)$ be a smooth test function and denote by $\{\eta(t)\}_{t\geq 0}$ the boundary driven symmetric long-range exclusion process with generator $N^\gamma L_N$. By stationarity of $\bar{\mu}_N$ and the entropy inequality we have

$$\begin{aligned}
&\bar{\mu}_N\Bigg(N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F(\tfrac{x}{N}, \tfrac{y}{N})p(y-x)(\eta_y - \eta_x)\Bigg)\\
&= \mathbb{E}_{\mu_N}\Bigg(\int_0^1 dt\, N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F(\tfrac{x}{N}, \tfrac{y}{N})p(y-x)(\eta_y^N(t) - \eta_x^N(t))\Bigg)\\
&\leq C_0 + \frac{1}{N}\log\left\{\mathbb{E}_{\nu_h^N}\left(e^{\left[N^\gamma \int_0^1 dt\, \sum F(\frac{x}{N}, \frac{y}{N})p(y-x)(\eta_y^N(t) - \eta_x^N(t))\right]}\right)\right\}
\end{aligned}$$

where the sum is over the same domain as before and C_0 is a constant resulting from the bound of the relative entropy of μ_N with respect to ν_h^N.

By Feynman-Kac's formula the last expression is bounded by $\dfrac{\lambda_N}{N} + C_0$ where the eigenvalue λ_N is given by the variational formula

$$\lambda_N = \sup_f\left\{N^\gamma \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F(\tfrac{x}{N}, \tfrac{y}{N})p(y-x)\langle(\eta_y - \eta_x)f(\eta)\rangle_{\nu_h^N} + N^\gamma\left\langle L_N\sqrt{f}, \sqrt{f}\right\rangle_{\nu_h^N}\right\} \tag{4.8.1}$$

and the supremum is taken over all the densities f on Ω_N with respect to ν_h^N. Let F^a be the antisymmetric (resp. symmetric) part of F, i.e. $\forall (u,v) \in [0,1]^2$,

$$F^a(u,v) = \frac{1}{2}\Big[F(u,v) - F(v,u)\Big], \quad F^s(u,v) = \frac{1}{2}\Big[F(u,v) + F(v,u)\Big].$$

Observe that $F^a(u,u) = 0$ and that $F = F^a + F^s$. We can rewrite

$$\sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F(\tfrac{x}{N}, \tfrac{y}{N})p(y-x)\langle(\eta_y - \eta_x)f(\eta)\rangle_{\nu_h^N} = \sum_{\substack{x,y\in\Lambda_N\\|x-y|\geq\varepsilon N}} F^a(\tfrac{x}{N}, \tfrac{y}{N})p(y-x)\langle(\eta_y - \eta_x)f(\eta)\rangle_{\nu_h^N} \tag{4.8.2}$$

as

$$
\begin{aligned}
& \sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} F^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\langle \eta_y\,(f(\eta)-f(\sigma^{x,y}\eta))\rangle_{\nu_h^N}\\
+ & \sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} F^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\langle \eta_y f(\sigma^{x,y}\eta)(1-\theta^{xy}(\eta))\rangle_{\nu_h^N}\\
= & \sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} F^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\langle \eta_y\,(f(\eta)-f(\sigma^{x,y}\eta))\rangle_{\nu_h^N}\\
+ & \sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} F^a(\tfrac{x}{N},\tfrac{y}{N})p(y-x)\langle \eta_x f(\eta)(\theta^{xy}(\eta)-1)\rangle_{\nu_h^N}\\
= & \ (I)+(II)
\end{aligned}
$$

where $\theta^{xy}(\eta)=\frac{d\nu_h^N(\sigma^{x,y}\eta)}{d\nu_h^N(\eta)}$. By Cauchy-Schwarz inequality, the fact that f is a density and $|\eta_y|\leq 1$, we have that (I) is bounded above by

$$
\sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} \left|F^a(\tfrac{x}{N},\tfrac{y}{N})\right| p(y-x)\sqrt{\left\langle [\sqrt{f(\sigma^{x,y}\eta)}-\sqrt{f(\eta)}]^2\right\rangle_{\nu_h^N}}.
$$

Since ρ is Lipshitz we have that $\sup_{\eta\in\Omega_N}|\theta^{xy}(\eta)-1| = O\Big(\frac{|x-y|}{N}\Big)$. Therefore, by using the elementary inequality $|ab|\leq \frac{a^2}{2C}+\frac{Cb^2}{2}$, and the fact that f is a density, we have that (II) is bounded above by a constant (independent of N,ε,F) times

$$
\begin{aligned}
&\sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} p(y-x)\Big[F^a\Big(\tfrac{x}{N},\tfrac{y}{N}\Big)\Big]^2 + \sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} p(y-x)\Big(\tfrac{|x-y|}{N}\Big)^2\\
&= c_\gamma N^{1-\gamma}\Bigg\{\frac{1}{N^2}\sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} \frac{\Big[F^a\Big(\frac{x}{N},\frac{y}{N}\Big)\Big]^2}{|\frac{x}{N}-\frac{y}{N}|^{1+\gamma}} + \frac{1}{N^2}\sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} |\tfrac{x}{N}-\tfrac{y}{N}|^{1-\gamma}\Bigg\}.
\end{aligned}
$$

Observe that

$$
\sup_{\varepsilon>0}\sup_{N\geq 1}\frac{1}{N^2}\sum_{\substack{x,y\in\Lambda_N\\ |x-y|\geq \varepsilon N}} |\tfrac{x}{N}-\tfrac{y}{N}|^{1-\gamma}<\infty
$$

since $1-\gamma>-1$.

By using (4.8.1), (3.3.14), Cauchy-Schwarz inequality and the previous upper bound for (4.8.2) it follows that there exist constants C', C'', C''', K (independent of $\varepsilon>0$, $N\geq 1$ and

$F \in C_c^\infty([0,1]^2)$) such that

$$\frac{\lambda_N}{N} \le N^{\gamma-1} \sup_f \Bigg[\sum_{\substack{x,y\in\Lambda_N\\|x-y|\ge\varepsilon N}} p(y-x)\Bigg(\big|F^a(\tfrac{x}{N},\tfrac{y}{N})\big| \sqrt{\Big\langle [\sqrt{f(\sigma^{x,y}\eta)} - \sqrt{f(\eta)}]^2 \Big\rangle_{\nu_h^N}}$$

$$-C' \Big\langle [\sqrt{f(\sigma^{x,y}\eta)} - \sqrt{f(\eta)}]^2 \Big\rangle_{\nu_h^N} \Bigg) \Bigg] + \frac{C''}{N^2} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\ge\varepsilon N}} \frac{\Big[F^a\Big(\tfrac{x}{N},\tfrac{y}{N}\Big)\Big]^2}{|\tfrac{x}{N}-\tfrac{y}{N}|^{1+\gamma}} + K$$

$$\le C''' \frac{1}{N^2} \sum_{x\neq y\in\Lambda_N} \frac{c_\gamma}{|\tfrac{x}{N}-\tfrac{y}{N}|^{1+\gamma}} \big[F^a(\tfrac{x}{N},\tfrac{y}{N})\big]^2 + K.$$

We have proved that

$$\bar\mu^N\Bigg(N^{\gamma-1} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\ge\varepsilon N}} F(\tfrac{x}{N},\tfrac{y}{N}) p(y-x)(\eta_y-\eta_x)\Bigg) = -2c_\gamma\, \bar\mu^N\left(\langle \pi^N, g_N\rangle\right)$$

$$\lesssim \frac{1}{N^2} \sum_{\substack{x,y\in\Lambda_N\\|x-y|\ge\varepsilon N}} \frac{c_\gamma}{|\tfrac{x}{N}-\tfrac{y}{N}|^{1+\gamma}} \big[F^a(\tfrac{x}{N},\tfrac{y}{N})\big]^2 + 1.$$

Above g_N is the function defined by

$$\forall u\in[0,1], \quad g_N(u) = \frac{1}{N} \sum_{\substack{y\in\Lambda_N\\ \left|\frac{y}{N}-u\right|\ge\varepsilon}} \frac{F^a\big(u,\tfrac{y}{N}\big)}{|u-\tfrac{y}{N}|^{1+\gamma}}$$

and it is a discretization of the smooth function g defined by

$$\forall u\in[0,1], \quad g(u) = \int_{\substack{y\in[0,1],\\|y-u|\ge\varepsilon}} \frac{F^a(u,y)}{|y-u|^{1+\gamma}}\,dy.$$

Let $Q_\varepsilon = \{(u,v)\in[0,1]^2\ ;\ |u-v|\ge\varepsilon\}$.Observe first that for symmetry reasons we have that for any integrable function π,

$$\int_0^1 \pi(u)g(u)du = \frac{1}{2}\iint_{Q_\varepsilon} \frac{(\pi(v)-\pi(u))F^a(u,v)}{|u-v|^{1+\gamma}}\,dudv.$$

We take the limit $N\to\infty$ and we conclude that there exists a constant $C>0$ independent of $F\in C_c^\infty([0,1]^2)$ and $\varepsilon>0$ such that

$$\mathbb{E}^*\left[\iint_{Q_\varepsilon} \frac{(\pi(v)-\pi(u))F^a(u,v)}{|u-v|^{1+\gamma}}\,dudv \;-\; C\iint_{Q_\varepsilon} \frac{\big[F^a(u,v)\big]^2}{|u-v|^{1+\gamma}}\,dudv \right] \lesssim 1.$$

It is easy to see that the supremum over F can be inserted in the expectation (see Lemma 7.5 in [41]) so that

$$\mathbb{E}^*\left[\sup_F\left\{\iint_{Q_\varepsilon}\frac{(\pi(v)-\pi(u))F^a(u,v)}{|u-v|^{1+\gamma}}\,dudv\;-\;C\iint_{Q_\varepsilon}\frac{\left[F^a(u,v)\right]^2}{|u-v|^{1+\gamma}}\,dudv\right\}\right]\lesssim 1.$$

By writing $F = F^a + F^s$, and observing that the function $(u,v)\in[0,1]^2\to\pi(v)-\pi(u)$ is antisymmetric, we have that

$$\iint_{Q_\varepsilon}\frac{(\pi(v)-\pi(u))F^a(u,v)}{|u-v|^{1+\gamma}}\,dudv = \iint_{Q_\varepsilon}\frac{(\pi(v)-\pi(u))F(u,v)}{|u-v|^{1+\gamma}}\,dudv.$$

Moreover, by using the definition of F^a and using the inequality $(\frac{a+b}{2})^2\le\frac{a^2+b^2}{2}$, it is easy to see that

$$\iint_{Q_\varepsilon}\frac{\left[F^a(u,v)\right]^2}{|u-v|^{1+\gamma}}\,dudv \le \iint_{Q_\varepsilon}\frac{\left[F(u,v)\right]^2}{|u-v|^{1+\gamma}}\,dudv.$$

It follows that

$$\mathbb{E}^*\left[\sup_F\left\{\iint_{Q_\varepsilon}\frac{(\pi(v)-\pi(u))F(u,v)}{|u-v|^{1+\gamma}}\,dudv\;-\;C\iint_{Q_\varepsilon}\frac{\left[F(u,v)\right]^2}{|u-v|^{1+\gamma}}\,dudv\right\}\right]\lesssim 1.$$

Consider the Hilbert space $\mathbb{L}^2([0,1]^2,d\mu_\varepsilon)$ where μ_ε is the measure whose density with respect to Lebesgue measure is $(u,v)\in[0,1]^2\to\mathbf{1}_{|u-v|\ge\varepsilon}\,|u-v|^{-(1+\gamma)}$. By letting $\Pi:(u,v)\in[0,1]^2\to\pi(v)-\pi(u)$ the previous formula implies that

$$\mathbb{E}^*\left[\iint_{[0,1]^2}(\Pi(u,v))^2\,d\mu_\varepsilon(u,v)\right]\lesssim 1.$$

Letting $\varepsilon\to 0$, by the monotone convergence theorem, we conclude that

$$\iint_{[0,1]^2}\frac{(\pi(v)-\pi(u))^2}{|u-v|^{1+\gamma}}\,dudv$$

is finite $\mathbb{P}^*$ a.s.. It follows from Theorem 8.2 of [23] that $\mathbb{P}^*$ almost surely π is $\frac{\gamma-1}{2}$-Hölder. This concludes the proof of Lemma 3.6.3.

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