Observation and control of nonlinear time-delay systems

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Observation et Commande des Systèmes Non-linéaires à Retard

THÈSE

présentée et soutenue publiquement le 07 Nov 2013

pour l'obtention du

Doctorat de l'Université de Lorraine
(Spécialité automatique)

par

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Acknowledgments

First and foremost, I would like to thank my supervisors Prof. Mohamed BOUTAYEB, University de Lorraine for giving me the opportunity to join his research group and for his guidance and encouragement for Dr. Ali ZEMOUCHE, Maître de conférence, University of Lorraine for being a source of inspiration and for his constant support scientifically and personally.

Additionally, I would like to thank all the members of my Ph.D. evaluation committee: Prof. Sophie TARBOURIECH, director of research CNRS, LAAS-CNRS, Toulouse, Prof. Dr. Wim MICHIELS, K.U. Leuven for having accepted to report my thesis and for their thorough review. It is also a pleasure to thank Prof. Yacine CHITOUR, Université de Paris XIII for his participation in the evaluation of the quality of my work.

I wish to express my gratitude to the staff of CRAN-Longwy: Michel Zasadzinski, Latifa-Boutat-Baddas, Harouna Souley Ali, Hugues Rafaralahy, Cédric Delattre, Mohamed Darouach, Joëlle Pinelli, Marouane Alma, Nathalie Clément and Allison Bordier. I also like to thank all the PHD students whom I have encountered during the last four years: Mohamed Benallouch, Bertrand Grandvallet, Arnaud Koehl, Mohamed Zerrougui, Houda Meddeb, Asma Barbata, Assem Thabet, Adrien Drouot, Mohamed Ghattassi, Nan Gao, Hao Nguyen Dang, Florian Seve, Yassine Boukal and Nejla El-Gharbi.

Finally, I want to seize the opportunity to express my deep gratitude to my family, for their love, encouragement and support.

To my family...

Contents

Notation and acronyms

Sets and Norms

$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of positive real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$
$\mathbb{R}^n$	Set of n-dimensional real vectors
$\mathbb{R}^{n\times m}$	Set of n$\times$m-dimensional real matrices
$\mathbb{Z}_+$	Set of positive integers
$\mathbb{S}^n$	Set of symmetric matrices in $\mathbb{R}^{n\times n}$, i.e., $\mathbb{S}^n = \{X \in \mathbb{R}^{n\times n} \mid X = X^T\}$
$L_2[0,+\infty)$	Set of square integrable functions on $[0,+\infty)$
$l_2[0,+\infty)$	Set of square summable functions on $[0,+\infty)$
$\mathcal{C}_{n,\tau}$	or $\mathcal{C}([-\tau,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$ with the topology of uniform convergence
$\|.\|$	The Euclidean vector norm
$\|\phi\|_c$	Stands for the norm of a function $\phi \in \mathcal{C}_{n,\tau}$, i.e., $\|\phi\|_c = \sup\limits_{-\tau\leq t\leq 0} \|\phi(t)\|$
$\mathcal{C}^v_{n,\tau}$	Set defined by : $\{\phi \in \mathcal{C}_{n,\tau} : \|\phi\|_c < v\}$ where v is a positive real number

Matrices and Operators

$A > 0$ $(A \geq 0)$	Real symmetric (semi)positive-definite matrix A
$A < 0$ $(A \leq 0)$	Real symmetric (semi)negative-definite matrix A
I_n	Identity matrix of dimension $n \times n$
A^{-1}	Inverse of matrix $A \in \mathbb{R}^{n\times n}$, $\det A \neq 0$
A^T	Transpose of matrix A
$A^\perp$	Any matrix such that $A^\perp A = 0$ and $A^\perp A^{\perp^T} > 0$
I_r	Identity matrix of dimension r.

$(\star)$	Block induced by symmetry
$\det A$	Determinant of matrix $A \in \mathbb{R}^{n \times n}$
$rankA$	Rank of matrix $A \in \mathbb{R}^{n \times m}$
$\lambda(A)$	Set of eigenvalues of matrix $A \in \mathbb{R}^{n \times n}$
$\|A\|$	Induced Euclidian norm of matrix $A \in \mathbb{R}^{n \times n}$
$\mathrm{Co}(x, y)$	Convex hull of the set x, y, $\mathrm{Co}(x, y) = \{\lambda\, x + (1 - \lambda)\, y, 0 \leq \lambda \leq 1\}$

Acronyms

ARE	Algebraic Riccati Equations
BMI	Bilinear Matrix Inequality
BRL	Bounded Real Lemma
DCP	Delay Central-Point
DDE	Delay Differential Equations
DLF	Discretized Lyapunov Functional
$DMVT$	Differential Mean Value Theorem
FDE	Functional Differential Equations
FSA	Finite Spectrum Assignment
FWM	Free Weighting Matrix
LKF	Lyapunov Krasovskii Functional
LMI	Linear Matrix Inequality
LPV	Linear Parameter Varying
LTI	Linear Time-Invariant
$NFDE$	Neutral FDE
ODE	Ordinary Differential Equations
PAM	Piecewise Analysis Method
PDE	Partial Differential Equations
$RFDE$	Retarded FDE
TDS	Time Delay System

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Introduction

In systems theory, controlling a process usually requires good informations on this process. The variables that are not directly measurable do not generally cover all the variables likely to describe the behavior of the process (states). This led to the problem of reconstructing the informations (the role of an observer or a state estimator) that are not directly measured by available sensors. In the last decades, a large part of the research activity in the field of automatic control focused on the problem of observing the state of nonlinear dynamic systems. This is motivated by the fact that state estimation is not only important for control design but also for the purpose of diagnosis and/or monitoring industrial systems.

On the other hand, assuming that all systems depend only on their present states and that anterior states has no influence on the behaviour of the systems is erroneous. Such type of systems are called time-delay systems and they have attracted for decades a great deal of interest. The delay is a natural component of different processes in biology, chemistry and communications to mention a few. The effect of the delay varies depending on the process, in some systems, the presence of the delay might have negative impact on the stability. Whereas, in other cases, the delay is introduced intentionally to enhance performances.

In this thesis, we concentrate on the estimation and control of systems with delays. Different methods are proposed to ensure the stability of the observer/controller for different classes of delay systems (linear, nonlinear, singular) with different types of delay (constant, time varying, unknown). The Lyapunov-Krasovskii approach is the main approach used for the stability analysis.

1 Objectives of the thesis

The main objectives of this dissertation:

- Developing new methods for the synthesis of observers and observer-based controllers for the class of nonlinear time-delay systems, including Lipschitz systems. The objective is to establish sufficient conditions for the design of less restrictive terms of feasibility compared with existing results in the literature.

- Proposing new observer structures based on those recently developed in the literature. The aim is to extend the applicability of the obtained methods to larger classes of dynamical systems with non differentiable nonlinearities.

- Searching for new Lyapunov functions that provide less restrictive synthesis conditions: generalization of the standard quadratic Lyapunov-Krasovskii functional to taking into account the disturbances.

2 Outline of the thesis

This thesis consists of five principal chapters and two appendices. In what follows, more details about the content of each part:

Chapter 1: devoted to present some useful preliminaries and to remind of some essential definitions needed to better understand the manuscript. In addition, it contains a description background to time-delay systems as well as a brief overview of some existing methods.

Chapter 2: presents a synthesis of an observer for nonlinear time-delay systems. The main idea relies on transforming the non-convex problem into a convex LPV one by exploiting the Lipschitz property. The proposed method exploits all the properties of the nonlinearities of the system, without approximating them by their norms, and thus offers less restrictive conditions. The delay-independent and delay-dependent cases are discussed respectively. A comparison is made between the proposed LPV-based methods and those based on the classical Lipschitz property. In particular, a method where the nonlinearity is rewritten in a more detailed form depending only on the states that actually appear in the nonlinear part. Finally by mean of some examples, we show the superiority of our method over the latter. Indeed, using the classical Lipschitz property leads to restrictive conditions that does not tolerate large Lipschitz constants nor large delay bounds in the case of delay-dependent conditions.

Chapter 3: a controller is proposed based on the observer discussed in Chapter 2. Contrarily to previous chapter, we distinguish two sources of non-convexity to be solved. The first is caused by the nonlinearity which is treated by reformulating the Lipschitz property such that to get an LPV system. The other is related to the controller which we propose to treat using some mathematical artifacts. In order to compare our results to existing ones, we chose one of the classical methods in the literature [Lie04]. In fact, in spite of the simplicity of the solution, the chosen approach imposes a condition in terms of a Linear Matrix Equality (LME) in order to solve the non-convexity caused by the controller. Obviously, this equality constraint is considered conservative when compared to our LMI. In addition, an extension to discrete time-delay systems is also given for the delay-dependent case. For sake of comparison, another method is chosen, involving the separation between the observer and the controller design problems. The idea consist in decomposing the original non-convex problem into two separate convex issues by introducing some free scalar variables to deduce at the end three LMIs conditions, that had to be hold simultaneously in order to ensure the stability, which is more computationally complex when compared to our approach with one LMI condition.

Chapter 4: two approaches to design observers for singular delay systems with disturbances are presented and compared in this chapter. The main difficulty lies in the presence of the derivative of the disturbances when developing the dynamics of the estimation error. The first proposed solution is a $\mathcal{H}_\infty$ criterion associated with a special Lyapunov-Krasovskii functional depending on the disturbances. The results are developed in a unified form for both continuous and discrete versions of the system. The second involves the use of a less conventional approach by using a $\mathcal{W}^{1,2}$ criterion based on Sobolev norms, which can be considered as an alternative solution to the $\mathcal{H}_\infty$ method when the derivative of the disturbance is difficult or impossible to avoid (for instance, imposed pseudo measurements).

Chapter 5: a controller is presented based on the observer of Chapter 4. The singularity of the system adds another type of difficulty to our problem, a difficulty that can not be treated with methods designed for regular systems. Inspired by some existing works, involving the use of some free matrices to make the singular system more exploitable, a sufficient condition in terms of LMI is presented. The used Lyapunov-Krasovskii functional is dependant on the disturbance vector as described in Chapter 4 but slightly modified in order to be applied on singular systems. The nonlinear part is treated using the Differential Mean Value theorem (DMVT) which leads to less restrictive results due to the reformulation of the nonlinearity in a more detailed form that take into consideration only the states on which the nonlinearity depends.

Chapter 6: contains some results on observer-based control design for unknown time-delay systems. Since the observer requires the knowledge of the delay which in this case is unknown, we introduce an estimation over the interval of definition. The delay interval is divided into r subintervals and the estimate of the delay is calculated on each segment as the mean value of the subinterval bounds. The stability of the estimated error is guaranteed by the use of Free Weighting Matrix (FWM) method. This approach introduces some free matrices in order to formulate a sufficient condition in terms of LMI.

Appendix A: presents a few useful lemmas and reminds of some theories and concepts used in the manuscript.

Appendix B: lists the articles based on the developed results in this thesis.

Introduction et résumé détaillé de la thèse

Au cours des dernières décennies, une grande partie des activités de recherche dans le domaine de l'automatique s'est focalisée sur le problème de l'observation de l'état des systèmes dynamiques non linéaires. Ceci est motivé par le fait que l'estimation d'état est une étape importante voire essentielle pour la conception des lois de commande et également pour faire du diagnostic et/ou de la surveillance des systèmes industriels.
Contrairement aux systèmes ordinaires dont l'évolution est déterminée à partir de la valeur de l'état à l'instant présent, l'évolution des systèmes à retard dépend des valeurs passées de l'état. Dans ce cas, il est nécessaire de mémoriser une partie de l'historique du système pour connaître son évolution. Ce type de systèmes a attiré un grand intérêt depuis des décennies et devient de plus en plus un sujet de recherche en constante évolution.
Le retard est un composant naturel qui apparaît dans de nombreux procédés de différents domaines tels que la biologie, la chimie et la communication. L'effet du retard varie en fonction du procédé. Dans certains systèmes, la présence du retard pourrait avoir un impact négatif sur la stabilité, alors que, dans d'autres cas, le retard est introduit intentionnellement pour améliorer la performance ou pour rendre le comportement du système complexe, comme c'est le cas dans des systèmes de communication chaotique. En effet, le retard introduit d'une façon convenable dans un système peut créer un comportement chaotique qui est très utile pour le cryptage/décryptage chaotique. Cette notion est appelée "la chaotification".
Dans cette thèse, nous nous intéressons à l'estimation et la commande des systèmes à retard. Différentes méthodes ont été proposées pour assurer la stabilité de l'observateur/contrôleur pour différentes classes de systèmes à retard (linéaires, non linéaires, singuliers) avec différents types de retards (constants, variables dans le temps, inconnus). L'approche de Lyapunov-Krasovskii est l'approche principale utilisée pour l'analyse de la stabilité.

3 Les objectifs de la thèse

Les principaux objectifs de cette thèse sont :

- Développer de nouvelles méthodes de synthèse d'observateurs et de contrôleurs basés sur un observateur pour des classes de systèmes non-linéaires à retard, notamment les non-linéarités de type Lipschitz. L'objectif consiste à établir des conditions de synthèse moins contraignantes par rapport à des résultats existants dans la littérature.

- Proposer de nouvelles structures d'observateurs en se basant sur celles développées récemment dans la littérature. Le but est d'étendre l'applicabilité des méthodes que nous avons obtenues à des classes plus larges de systèmes dynamiques, à savoir les systèmes avec des non-linéarités non différentiables.

- Rechercher de nouvelles fonctions de Lyapunov qui offrent des conditions de synthèse moins conservatives : généralisation de la forme quadratique de la fonctionnelle de Lyapunov-Krasovskii en tenant compte des perturbations.

4 La classe des systèmes considérés

En général, il existe trois façons de représenter les systèmes à retard [KNG99], [Nic01a] :

- *Comme des équations différentielles sur des espaces linéaires abstraits de dimension infinie (systèmes de dimension infinie)* : dans cette approche, le système à retard est considéré comme une partie d'une classe plus large de systèmes, à savoir "la classe des systèmes linéaires de dimension infinie" décrite par des équations différentielles abstraites. Cependant, cette approche nécessite une généralisation de certaines propriétés telles que les concepts de contrôlabilité, de stabilisabilité, d'observabilité et de détectabilité [KR99].

- *Comme des équations différentielles sur des espaces fonctionnels (FDE)* : dans ce cas, les systèmes à retard peuvent être considérés comme des évolutions dans un espace de dimension finie ou dans un espace fonctionnel. La première utilise la finitude de l'espace vectoriel pour analyser le comportement du système alors que la deuxième reflète le caractère de dimension infinie du système [HL93]. Bien que la manipulation des problèmes de dimension infinie en utilisant des outils de dimension finie a ses avantages, les résultats obtenus sont conservatifs. Dans cette thèse, nous considérons les systèmes non linéaires à retard décrits par des systèmes d'équations différentielles ordinaires non linéaires sous la forme :

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t) + f(x(t), x_t(\theta)), & t \geq t_0, \\ x_{t_0}(\theta) = \phi(\theta), & \forall \theta \in [-d, 0]. \end{cases} \tag{1}$$

où $x(t) \in \mathbb{R}^n$ et x_t désigne l'opérateur de translation agissant sur la trajectoire : $x_t(\theta) = x(t+\theta)\ \forall \theta \in [-d, 0]$, $\phi(t)$ est la condition initiale sur $[-d, 0]$. La fonction f est continue et Lipschitzienne par rapport à ses arguments, i.e.,

$$\|f(x, x_d) - f(y, y_d)\| \leq \gamma_f \left\| \begin{matrix} x - y \\ x_d - y_d \end{matrix} \right\|.$$

.

- *Comme des équations différentielles sur des anneaux d'opérateurs* : Les FDEs associées sont exprimées comme des équations différentielles vectorielles définies sur des anneaux d'opérateurs. Cette méthode a été appliquée avec succès pour résoudre certains problèmes de contrôle tels que le découplage et le rejet de perturbations [Sen01].

Chaque méthode a ses avantages et ses inconvénients en fonction du problème à traiter.

Une autre classe de systèmes qui ne sera pas abordée dans cette thèse, mais qui est largement étudiée dans la littérature, est la classe des systèmes décrits par des équations différentielles fonctionnelles de type neutre. Dans ce cas, l'évolution du système dépend à la fois des valeurs passées de son état ainsi que de ses variations, impliquant une complexité mathématique accrue. Ce type peut être décrit par les équations suivantes :

$$\begin{cases} \dot{x}(t) = f(x_t(\theta), \dot{x}_t(\theta)), \quad t \geq t_0, \\ x_{t_0}(\theta) = \phi(\theta), \quad \forall \theta \in [-d, 0]. \end{cases} \tag{2}$$

Cette catégorie de systèmes à retard est largement utilisée pour décrire les phénomènes de propagation sans perte [NRG03].
Un bref examen de certaines propriétés de base des systèmes à retard est donné dans les sections suivantes.

5 Catégories du retard

La façon de traiter les systèmes à retard diffère selon le type de retard. Ainsi, il est important de présenter les différentes catégories ou types du retard qui peuvent être rencontrés dans la littérature :

- **Retard discret ou ponctuel** :
  des conditions différentes pour la stabilité des systèmes linéaires à retard discret ont été développées [KNR99], [HLT11]. De toute évidence, les systèmes à retard unique ne sont pas toujours suffisants pour décrire des systèmes réels et la représentation du système peut impliquer de nombreux retard. Ci-après, nous présentons une forme de systèmes à retard discret qui est largement utilisée :

  $$\dot{x} = \sum_{i=0}^{n} A_i x(t - d_i).$$

  Cela a conduit à distinguer deux types de retard discret ou ponctuel : retard proportionnel et retard incommensurable.

  - Retard proportionnel : $d_i \in \mathbb{R}, i \in \mathbb{N}$ sont dits proportionnels si d_i/d_j est rationnel, ce qui correspond à la recherche d'un délai minimal d tels que $d_i = id$, alors le système devient, avec une réorganisation appropriée des indices, comme suit :

    $$\dot{x} = \sum_{i=0}^{n} A_i x(t - id).$$

    Cette classe de systèmes est considérée comme très conservative, mais le problème de la stabilité de cette classe peut être traité de façon similaire au cas du retard simple puisque l'équation caractéristique associée au système aura les mêmes propriétés algébriques [Nic01a]. Pour ce type de retard, une condition de stabilisabilité indépendante de la taille du retard a été présentée dans [Kam82].

  - Retard incommensurable : les retards $d_i, i \in \mathbb{N}$ sont des paramètres libres. Pour ce type de retard, un observateur utilisant un changement de coordonnées de sorte que tous les termes liés au retard, dans les nouvelles coordonnées, dans la description du système soient associés à la sortie seulement a été conçu dans [HZP02]. Le problème de la stabilisation d'une classe de systèmes à retard différentiel de type neutre avec plusieurs retards fixes, incommensurables, en utilisant des compensateurs causaux a été abordé [EK84].

- **Retard variant dans le temps** :

  - Retard borné : $0 < d_1 \leq d(t) \leq d_2$. Ce type de retard, souvent traité dans la littérature, implique des critères de stabilité indépendants du retard [XCP04]. Différent class de systèmes ont été traité dans la littérature, à savoir les systèmes singuliers [CZZ11] et les systèmes non linéaires [Bou07]. Ce type de retard est également très exploité pour la conception de la lois de commande pour des systèmes avec des retards incertains sur l'entrée [BEBC99]. Le cas des systèmes présentant des retards avec des bornes supérieures inconnues a également été étudié [SFRS07],
  - Retard à dérivée bornée : $\dot{d}(t) \leq \mu < 1$. Signifie que la fonction $g(t) = t - d(t)$ est monotone. Généralement, ce type de retard est associé au précédent pour obtenir des critères de stabilité dépendants du retard. Par exemple, dans le cas des systèmes linéaires incertains [SPP99], [PT09] et des systèmes incertains de type neutre [Lie07].
  - Retards variables arbitraires : le retard $d(t)$ et sa dérivé $\dot{d}$ ne sont pas limités.

- **Retard distribué** $\int\limits_{s-\tau}^{t} x(s)\mathrm{d}s$: ce type de retard a été traité dans la littérature, pour la conception d'observateurs pour des systèmes non linéaires [GP05]; pour démontrer la stabilité des systèmes linéaires de type neutre [Han03]; la stabilisation robuste des systèmes neutres incertains [CZ07] et la commande robuste [ZF02], [FT09]. D'autre part, l'une des méthodes pour traiter les termes intégraux de la fonctionnelle de Lyapunov-Krasovskii est d'utiliser certaines transformations (comme nous le verrons plus tard) qui modifient le système d'origine avec un retard discret en un nouveau système avec un retard distribué.

- **Retard dépendant de l'état** : le retard est présenté comme une fonction de l'état du système [Mur01], [BHJ$^+$10].

La classification précédente du retard n'est pas suffisante dans le sens ou, plus d'informations sur le retard sont nécessaire afin d'élaborer des critères de stabilité non contraignants, ceci a conduit à la définition de nouvelles catégories en fonction de la dérivée du retard comme suit :

- **Retard variant lentement dans le temps** : $d(t)$ est une fonction dérivable presque partout, satisfaisant $\dot{d}(t) \leq \mu < 1$.

- **Retard variant modérément dans le temps** : $d(t)$ est une fonction dérivable presque partout, satisfaisant $\dot{d}(t) \leq \mu$ avec $\mu \geq 1$.

- **Retard variant rapidement dans le temps** : $d(t)$ est une fonction mesurable (par exemple, continue par morceaux) sans aucune contrainte sur sa dérivée.

Un phénomène important lié aux systèmes à retard est appelé "quenching" qui se produit lorsque la stabilité (resp. instabilité) d'un système, à retard constant dans un certain intervalle, est perdue quand le retard est supposé variable dans le temps à l'intérieur du même intervalle et vice-versa. Ce problème a été mentionné dans [Lou99]. Évidemment, ce problème nous empêche d'appliquer les résultats obtenus dans le cas des retards fixes pour le cas des retards variables dans le temps. Toutefois, des efforts ont été faits pour prendre ce phénomène en considération [PPN07].

6 Sur la stabilité des systèmes à retard

Le retard peut limiter et réduire les performances des systèmes commandés. Parfois, le retard induit à l'instabilité du système. Par conséquence, la stabilité des systèmes à retard a été largement discutée dans de nombreuses monographies [GKC03], [WHS10], [Nic01a], [MN08].
Considérons l'équation différentielle fonctionnelle (RFDE) suivantes :

$$\begin{cases} x(t) = f(t, x_t), & t \geq t_0, \\ x_{t_0}(\theta) = \phi(\theta), & \forall \theta \in [-d, 0], \end{cases} \tag{3}$$

où $x_t(.)$, pour $t \geq t_0$, désigne la restriction de $x(.)$ sur l'intervalle $[t-d, t]$ converti en $[-d, 0]$, i.e.,

$$x_t(\theta) = x(t+\theta), \quad \forall \theta \in [-d, 0].$$

On suppose que $\phi \in \mathcal{C}^v_{n,d}$ et l'application $f(t, \phi) \ : \ \mathbb{R}^+ \times \mathcal{C}^v_{n,d} \mapsto \mathbb{R}^n$ est continue et Lipschitzienne en ϕ et que $f(t, 0) = 0$. On désigne par $x(t_0, \phi)$ la solution de l'équation différentielle fonctionnelle (3) avec la condition initiale $(t_0, \phi) \in \mathbb{R}^+ \times \mathcal{C}^v_{n,d}$.

6.1 Concept de stabilité et définitions

Cette sous-section présente quelques informations sur les systèmes à retard, en particulier, des concepts fondamentaux et différents types de stabilité.
Soit $\bar{x}(t)$ une solution de la RFDE (3). La stabilité de la solution concerne le comportement du système lorsque la trajectoire $x(t)$ du système s'écarte de $\bar{x}(t)$. Sans perte de généralité, nous supposons que la RFDE (3) admet la solution $x(t) = 0$, qui sera dénommée la solution triviale. Si la stabilité d'une solution non triviale, $\bar{x}(t)$, doit être étudiée, on peut utiliser la transformation de la variable $z(t) = x(t) - \bar{x}(t)$ pour produire le nouveau système

$$\dot{z}(t) = f(t, z_t + \bar{x}_t) - f(t, \bar{x}_t), \tag{4}$$

qui a la solution triviale $z(t) = 0$. Pour la fonction $\phi \in \mathcal{C}([a, b], \mathbb{R}^n)$, on définit la norme continue $\|.\|_c$ par

$$\|\phi\|_c = \sup_{a \leq \theta \leq b} \|\phi(\theta)\|$$

Dans cette définition, la norme $\|.\|$ représente la norme-2 $\|.\|_2$.
Il existe plusieurs type de définitions de la stabilité de la solution triviale des systèmes à retard.

Définitions [HL93] :

- **Stabilité** : si, $\forall t_0 \in \mathbb{R}$ et $\epsilon > 0, \exists \delta = \delta(t_0, \epsilon) > 0$ tel que

  $$\|x_{t_0}\|_c < \delta \Rightarrow \|x(t)\| < \epsilon, \;\; t \geq t_0,$$

  alors la solution triviale de (3) est stable.

- **Stabilité asymptotique** : si la solution triviale de (3) est stable, et si, $\forall t_0 \in \mathbb{R}, \exists \delta_a = \delta_a(t_0) > 0$ tel que

  $$\|x_{t_0}\|_c < \delta_a \Rightarrow \lim_{t \to \infty} x(t) = 0,$$

  alors la solution triviale de (3) est asymptotiquement stable.

- **Stabilité uniforme** : si la solution triviale de (3) est stable et si $\delta(t_0, \epsilon)$ peut être choisi indépendamment de t_0, alors la solution triviale de (3) est uniformément stable.

- **Stabilité asymptotique uniforme** : si la solution triviale de (3) est uniformément stable et si $\exists \delta_a > 0$ tel que, $\forall \eta > 0$, $\exists T = T(\delta_a, \eta)$ tel que

  $$\|x_{t_0}\|_c < \delta_a \Rightarrow \|x(t)\| < \eta, \;\; \forall t \geq t_0 + T \text{and } t_0 \in \mathbb{R},$$

  alors la solution triviale de (3) est uniformément asymptotiquement stable.

- **Stabilité asymptotique globale (uniforme)** : si la solution triviale de (3) est (uniformément) asymptotiquement stable est si δ_a peut être un nombre fini arbitrairement grand alors la solution triviale de (3) est globalement (uniformément) asymptotiquement stable.

- **Stabilité exponentielle globale** : s'il existe des constantes $\alpha > 0$ et $\beta > 0$ telles que $\|x(t)\| \leq \beta \sup_{-h \leq \theta \leq 0} \|x(\theta)\| \, e^{-\alpha t}$, alors la solution triviale (3) est globalement exponentiellement stable et α est appelé le taux de convergence exponentielle.

En outre, si le système est linéaire, les propriétés de « stabilité asymptotique uniforme », « stabilité asymptotique », « stabilité exponentielle » sont équivalentes.
Les méthodes principales pour examiner la stabilité peuvent être classées en deux types : approches fréquentielles et temporelles. Les méthodes dans le domaine fréquentiel déterminent la stabilité d'un système à partir de la distribution des racines de l'équation caractéristique [Nic01a] ou à partir des solutions d'une équation fonctionnelle matricielle complexe de Lyapunov [BCLZ82]. Elles ne conviennent que pour les systèmes à retard constant. Dans le domaine temporel, les approches les plus courantes pour étudier la stabilité des systèmes à retard sont liées à la fonctionnelle de Lyapunov-Krasovskii [Bli01] et à la fonction de Razumikhin [HL93]. Jusqu'aux années 1990, les critères de stabilité obtenus par ces deux approches étaient généralement sous la forme de conditions d'existence à cause de la difficulté de construire des fonctionnelles de Lyapunov-Krasovskii et des fonctions de Lyapunov. Depuis lors, en raison de l'utilisation des équations de Riccati [HL99], des inégalités matricielles linéaires (LMI) et des boîtes à outils Matlab [BGFB94], des solutions générales ont été développées et des résultats significatifs ont été établis (voir [Ric03], [Zho06] et les références qui s'y trouvent). Parmi ces conditions suffisantes, deux catégories ont reçu beaucoup d'attention. La première est indépendante de la taille du retard. La deuxième utilise des informations sur la taille du retard, et les conditions obtenues sont donc dépendantes du retard.

6.2 Méthodes fréquentielles

Dans cette thèse, nous nous concentrons sur les méthodes dans le domaine temporel pour l'analyse de stabilité. Pourtant, il nous semble important de souligner quelques-unes des méthodes dans le domaine fréquentiel développées dans la littérature.
Certains critères sont une généralisation directe de la méthode de Hurwitz pour des systèmes à retard en vérifiant si les racines d'une équation caractéristique du système sont dans le demi-plan gauche. Citons les trois critères connus ci-dessous :

- Critère de Pontryagin : ce critère a étendu les méthodes utilisées pour prouver le critère de Routh-Hurwitz pour que les zéros d'un polynôme soient dans le demi-plan gauche [HL93].
- Critère de Chebotarev : ce critère est une généralisation du critère de Routh-Hurwitz aux quasi-polynômes en cas de retards proportionnels. Son inconvénient est de calculer un grand nombre de déterminants.
- Critère de Yesupovisch-Svirskii : ce critère est une version simplifiée du critère de Pontryagin, mais reste fortement dépendante de la géométrie et de l'application du principe de l'argument. De plus, il est généralement appliqué pour les systèmes à retard unique ponctuel.

D'autres critères dépendent de la méthode de lieu de racines. il est basé sur la détermination des valeurs des paramètres pour lesquels l'équation caractéristique a des racines sur l'axe imaginaire, tels que : la méthode de décomposition-$\mathcal{D}$ [Nei49], la méthode de décomposition-τ [Hsu70] et la méthode de principe de l'argument [KN86].

6.3 Théorème de stabilité de Lyapunov-Krasovskii

La méthode de Lyapunov a été utilisée efficacement dans l'analyse de la stabilité des systèmes sans retard. Ainsi, il est naturel d'essayer cette méthode sur des systèmes à retard. Évidemment, cela nécessite quelques adaptations sur la fonction de Lyapunov. En présence du retard, la fonction de Lyapunov $V(t, x_t)$ dépend de x_t (la valeur de la variable d'état dans l'intervalle $[t-h, t]$) et devient donc fonctionnelle. Cette fonctionnelle est appelée «fonctionnelle de Lyapunov-Krasovskii».

Soit $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ une fonction continue, et $x(t_0, \phi)$ est une solution de (3) à l'instant t avec la condition initiale $x_{t0} = \phi$. Alors, la dérivée supérieure à droite au sens de Dini est définie comme suit:

$$\dot{V}(t, \phi) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t + d, x_{t+d}(t_0, \phi) - V(t, \phi)].$$

Théorème 0.6.1. *(Théorème de Lyapunov-Krasovskii) [Nic01a] : Supposons que la fonction* $f : \mathbb{R} \times \mathcal{C}_{n,d} \mapsto \mathbb{R}^n$ *soit bornée et supposons que les fonctions* $u(s), v(s)$ *et* $w(s)$ *sont continues, non négatives et non décroissantes avec* $u(s), v(s) > 0$ *pour* $s \neq 0$ *et* $u(0) = v(0) = 0$*. S'il existe une fonction continue* $V : \mathbb{R} \times \mathcal{C}_{n,d} \mapsto \mathbb{R}$ *telle que*

(i) $u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|_c)$,

(ii) $\dot{V}(t, \phi) \leq -w(\|\phi(0)\|)$,

alors la solution (triviale) $x = 0$ *de l'équation* (3) *est uniformément stable.*
Si $u(s) \to \infty$ *pour* $s \to \infty$ *alors la solution est uniformément bornée .*
If $w(s) > 0$ *pour* $s > 0$*, alors la solution* $x = 0$ *est uniformément asymptotiquement stable.*

6.4 Théorème de stabilité de Razumikhin

Le théorème de Lyapunov-Krasovskii nécessite la manipulation des fonctionnelles, ce qui le rend difficile à appliquer. Cela a conduit à l'utilisation d'une approche alternative impliquant des fonctions au lieu des fonctionnelles. Cette approche est appelée l'approche par fonctions de Razumikhin. Elle est considérée comme l'outil d'analyse classique d'interprétation dans l'espace de dimension finie. L'idée principale derrière ce théorème est l'utilisation d'une fonction de Lyapunov, $V(x)$, dont la dérivée n'est pas négative pour toutes les trajectoires, mais seulement pour les trajectoires de l'état qui s'éloignent du point d'équilibre. La définition précise est donnée dans le théorème suivant.

Théorème 0.6.2. ***(Théorème de Razumikhin)*** *[Nic01a] : Supposons que la fonction $f : \mathbb{R} \times \mathcal{C}_{n,d} \mapsto \mathbb{R}^n$ soit bornée et que $u, v, w : \mathbb{R}^+ \mapsto \mathbb{R}^+$ sont des fonctions continues, non décroissantes telles que $u(s), v(s), w(s) > 0$ pour $s \neq 0$ et $u(0) = v(0) = 0$. Supposons qu'il existe une fonction continue $V : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ telle que*

$$u(\|x\|) \leq V(t,x) \leq v(\|x\|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n. \tag{5}$$

Alors, les assertions suivantes sont vérifiées :

a) $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$ si $V(t+\theta, x(t+\theta)) < p(V(t,x(t))), \quad \forall \theta \in [-d,0]$ alors la solution triviale de (3) est uniformément stable.

b) S'il existe une fonction continue non décroissante $p : \mathbb{R}^+ \mapsto \mathbb{R}^+, p(s) > s$, telle que $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$ si $V(t+\theta, x(t+\theta)) < p(V(t,x(t))), \quad \forall \theta \in [-d,0]$, alors la solution triviale de (3) est uniformément asymptotiquement stable.

Si $u(s) \to \infty$ quand $s \to \infty$ alors la solution triviale est globalement asymptotiquement stable.

Auparavant, les conditions de stabilité des systèmes à retard variant dans le temps ont été obtenues uniquement par des fonctions de Lyapunov-Razumikhin. Néanmoins, l'approche Razumikhin mène à des conditions plus conservatives et elle est inapplicable dans le cas de la commande $\mathcal{H}_\infty$ [XL94], [LM07].
Il existe deux types de stabilité asymptotique des systèmes de la forme (3), en fonction de l'information sur la taille du retard : indépendante du retard et dépendante du retard. Ces deux types seront traités plus en détail dans les paragraphes suivants.

6.5 Analyse indépendant du retard

Les conditions indépendantes du retard ne contiennent aucune information sur le retard. Ainsi, elles sont trop conservatives, en particulier lorsque le retard est faible. La fonctionnelle de Lyapunov-Krasovskii candidate est généralement choisie comme suit :

$$V_1(x_t) = x^T(t)\mathcal{P}x(t) + \int_{t-d}^{t} x^T(s)\mathcal{Q}x(s)\mathrm{d}s. \tag{6}$$

où $\mathcal{P} > 0$ et $\mathcal{Q} > 0$ sont deux matrices à déterminer et appelées matrices de Lyapunov. De nombreux auteurs ont utilisé cette méthode pour prouver la stabilité et commander différents types de systèmes à retard : des systèmes linéaires avec retard fixe [LD02], [Bli01], des systèmes avec des retards proportionnels [Kam82], des systèmes discrets incertains à grande échelle [LH97], des systèmes neutres incertains avec des retards variants dans le temps [Lie07], des systèmes à commutation [KCL08] et des systèmes non linéaires commandés en réseaux [MHB06].

6.6 Analyse dépendant du retard

Le conservatisme des conditions indépendantes du retard a produit une autre classe importante de conditions de stabilité, à savoir, les conditions dépendantes du retard. Car cela nécessite des informations sur la taille du retard. Dans ce cas, le système (3) est supposé être stable lorsque $d = 0$, i.e., $A + A_d$ est Hurwitz. En outre, le système (3) s'avère stable pour tout $d \in [0,\ \bar{d}]$, où $\bar{d}$ est la valeur maximale du retard. Pour étudier la stabilité dépendante du retard, un terme quadratique à intégrale double est ajouté à la fonctionnelle de Lyapunov-Krasovskii (6) :

$$V(x_t) = V_1(x_t) + V_2(x_t), \tag{7}$$

avec

$$V_2(x_t) = \int_{-d}^{0} \int_{t+\theta}^{t} x^T(s) Z x(s) ds d\theta. \tag{8}$$

La dérivé de $V_2(x_t)$ est

$$\dot{V}_2(x_t) = d x^T(t) Z x(t) - \int_{t-d}^{t} x^T(s) Z x(s) \mathrm{d}s. \tag{9}$$

Comme nous l'avons remarqué, le terme à intégrale double, dans la fonctionnelle de Lyapunov-Krasovskii (8), entraîne des termes quadratiques à intégrales apparaissant dans la dérivée de cette fonction comme on peut le voir dans (9). L'un des principaux défis à relever lors de l'étude des problèmes de stabilité par l'approche dépendante du retard est de savoir comment faire face à ce terme intégral pour obtenir des résultats moins restrictifs. De nombreuses méthodes ont été mises au point pour résoudre ce problème : l'utilisation de la fonctionnelle de Lyapunov-Krasovskii discrétisée, l'utilisation des transformations de modèles fixes et les transformations de modèles paramétrés [Nic99] et enfin les méthodes utilisant des matrices de pondérations libres (FWM).

7 Synthèse d'observateurs et contrôleur des systèmes à retard

Au cours des dernières décennies, le problème de la synthèse des observateurs a suscité l'intérêt de beaucoup de chercheurs et a fait l'objet d'un grand nombre de travaux depuis l'article primitif de Luenberger [Lue71]. L'observateur permet d'estimer la partie non mesurée de l'état à partir d'un modèle du système dynamique et des mesures d'autres grandeurs, cette estimation est fondamentale pour la commande ou pour le diagnostic. En effet, l'état du système n'est pas toujours complètement accessible et ceci est dû essentiellement à deux raisons. D'une part, en raison des contraintes technologiques, on ne dispose pas toujours de capteurs pour mesurer certaines grandeurs physiques. D'autre part, pour des contraintes économiques, on cherche à minimiser le coût en s'affranchissant de certains capteurs. La synthèse des observateurs dépend essentiellement de la classe de systèmes considérée (systèmes à retard, systèmes bilinéaires, systèmes singuliers, etc). La synthèse d'observateurs devient plus délicate lors de la prise en compte des retards. La prise en compte de ces retards est essentielle pour une bonne description du fonctionnement du système. Leur présence peut rendre délicate l'estimation de l'état, puisqu'ils conduisent à des dimensions infinie dans les équations caractéristiques du système.
Plusieurs travaux ont été réalisés concernant la synthèse d'observateur et contrôleur en présence des retards par example la decomposition spectrale [HL93], [Sal82]; la representation matricielle fractionnelle [EK82], l'approche du changement de coordonnées [HZP02], la méthode LMI [FSD00], la technique de transformation réduite [PF89], l'approche de factorisation

[YZ96], et l'approche polynomiale [Sen97]. Un autre type de méthodes utilisées pour stabiliser les systèmes à retard est les méthodes basées sur la prédiction qui transforment le problème dans un système sans retard comme prédicteur de Smith [Smi59]; Finite Spectrum Assignment (FSA) [MO79], [Zho06] et adaptative Posicast [NA03].

8 Structure du mémoire

Cette thèse se compose de cinq chapitres principaux et de deux annexes. Dans ce qui suit, un récapitulatif du contenu de chaque partie est présenté :

Chapitre 1 : ce chapitre est consacré à la présentation de quelques préliminaires utiles ainsi qu'au rappels de quelques définitions essentielles et nécessaires pour mieux comprendre le manuscrit. En outre, il contient un état de l'art sur les systèmes à retard ainsi qu'un bref aperçu de quelques méthodes existantes.

Chapitre 2 : ce chapitre présente une synthèse d'observateurs pour les systèmes non linéaires à retard. L'idée principale repose sur la transformation de la non linéarité dans la dynamique de l'erreur d'observation en un système LPV en exploitant la propriété de Lipschitz. La méthode proposée exploite toutes les propriétés de la non-linéarité du système, sans l'approximer par ses normes, et offre ainsi des conditions moins conservatives. Deux cas ont été étudiés (indépendant et dépendant de la taille du retard). Une comparaison entre la méthode proposée et la méthode basée sur les propriétés de Lipschitz classiques également abordées dans ce chapitre a été effectuée. En effet, dans cette dernière, avec l'aide de quelques matrices libres, la non-linéarité est réécrite sous une forme plus détaillée en fonction des états qui apparaissent réellement dans la partie non-linéaire. Ces matrices jouent un rôle important sur la faisabilité des conditions de synthèse. Néanmoins, les résultats montrent la supériorité de notre méthode par rapport à celle-ci. En effet, l'utilisation de la propriété de Lipschitz classique conduit à des conditions restrictives qui ne tolèrent pas les constantes de Lipschitz importantes ni des bornes importantes du retard dans le cas de la synthèse dépendant du retard.

Chapitre 3 : cette partie est consacrée à la synthèse de lois de commande basées sur l'utilisation de l'observateur construit dans le chapitre 2. Une nouvelle méthode de synthèse de lois de commande basée sur un observateur est proposée. Une relaxation importante des conditions de synthèse LMI classique a été obtenue grace à une nouvelle utilisation de l'inégalité de Young de façon judicieuse. Pour comparer notre méthode aux résultats existants, nous avons choisi l'une des méthodes classiques de la littérature, à savoir l'approche imposant une contrainte d'égalité matricielle linéaire (LME) afin de résoudre la non-convexité causée par le contrôleur. De toute évidence, cette condition d'égalité est jugée contraignante par comparaison à notre LMI. Une extension aux systèmes temps-discrets à retard a été donnée pour le cas dépendant du retard. Une autre méthode de la littérature est choisie, impliquant la séparation du problème de la conception de l'observateur de celui de la conception du contrôleur. L'idée consiste à décomposer le problème non convexe original en deux problèmes convexes dépendantes en introduisant des variables scalaires libres. Ceci mène à trois conditions LMI dépendantes, qui doivent être valides, simultanément, afin d'assurer la stabilité. En revanche, cette technique complexifie les calculs comparé à notre approche avec une seule condition LMI. Notre méthode demeure supérieure et fournie des conditions de synthèse moins contraignantes.

Chapitre 4 : deux approches de conception d'observateur pour les systèmes singuliers à retard avec perturbations ont été présentées et comparées. La difficulté principale réside dans la présence de la dérivée des perturbations lors de l'élaboration de la dynamique de l'erreur d'estimation. La première solution proposée utilise un critère $\mathcal{H}_\infty$ associé à une fonctionnelle de Lyapunov-Krasovskii particulière dépendante des perturbations. Les résultats ont été développés pour les deux versions temps-continu et temps-discret du système. La seconde solution implique l'utilisation d'une approche moins conventionnelle en utilisant un critère $\mathcal{W}^{1,2}$ basé sur les normes de Sobolev, qui peut être considérée comme une solution alternative à la méthode $\mathcal{H}_\infty$ lorsque la dérivée de la perturbation est difficile, voire impossible à éviter (par exemple, des pseudo mesures imposées). Des comparaisons numériques entre les deux méthodes ont été données.

Chapitre 5 : un contrôleur $\mathcal{H}_\infty$ basé sur l'observateur du chapitre 4 est présenté. La singularité du système ajoute un autre type de difficulté à notre problème, une difficulté qui ne peux pas être traitée avec des méthodes conçues pour les systèmes réguliers. En se basant sur certains ouvrages existants, impliquant l'utilisation de certaines matrices libres pour rendre le système singulier plus exploitable, une condition suffisante sous forme LMI est présentée. La fonctionnelle de Lyapunov-Krasovskii et dépendante du vecteur de perturbation comme décrite dans le chapitre 4 et légèrement modifiée pour être appliquée aux systèmes singuliers. La partie non linéaire est traitée en utilisant le théorème des accroissements finis (DMVT) qui conduit à des résultats moins restrictifs en raison de la reformulation de la non-linéarité sous une forme plus détaillée qui prend en considération uniquement les états intervenant dans la non-linéarité.

Chapitre 6 : contient des résultats concernant la conception d'un contrôleur basé sur un observateur pour les systèmes à retard inconnu. En raison de la difficulté du problème, nous avons travaillé sur une classe de systèmes non linéaires où la non linéarité ne dépende pas de l'état retardé. Puisque l'observateur nécessite la connaissance du retard qui dans ce cas est inconnu, on remplace sa valeur par son estimation sur un intervalle de définition. L'intervalle du retard est divisé en r sous-intervalles et l'estimation du retard est calculée sur chaque segment comme la valeur moyenne. La stabilité de l'erreur d'estimation est garantie par l'utilisation de la méthode des matrices de pondération libres (FWM). Cette approche présente certaines matrices libres afin de formuler une condition suffisante sous forme LMI.

Annexe A : présente quelques lemmes et rappels mathématiques utiles pour le développement des résultats établis dans ce manuscrit.

Annexe B : présente une liste complète des publications issues de la thèse.

CHAPTER

1

State of the art

Contents

The purpose of this chapter is to recall some basic and necessary definitions and concepts related to time-delay systems to help understanding this document, and partly a state of the art on time-delay systems, observability and controllability as well.

1.1 Introduction

When describing the evolution of a system, it is sometimes insufficient to assume that the system depends only on its present states and ignore its dependency of past states. Such type of systems are called time-delay systems. Time-delay systems has for decades been an active area of scientific research in mathematics, biology, ecology, economics, and in engineering, under such terms as hereditary systems, systems with aftereffect, or systems with time-lag, and more generally as a subclass of functional differential equations and infinite dimensional systems. The delay, whether it is in the state and/or input, affects the stability of the systems. Its presence is considered a problem of recurring interest since it may have contradictory effects on the system. On the one hand, it can degrade the system performance and induce complex behavior e.g., instability, oscillations, bad performances. For example, small delays are capable of destabilizing some systems [Hal97]. Furthermore, if the delayed state is a nonlinear function, this may induce a chaotic behavior. On the other hand, delay can enhance the stability, i.e., large delays may stabilize some system [BM75], [Mac86]. In other cases, a delayed output is used to stabilize chaotic systems [KNG99].

1.2 Practical examples

Before discussing time-delay systems in more details, let us start by giving some examples to give the reader a glimpse on how widely time-delays may occur in practice.

1.2.1 Example 1

Regenerative chatter in metal cutting [SN97]: Regenerative effect has been widely studied in the literature because it is one of the most important causes of instability in the cutting process. Figure 1.1 shows the metal cutting process in a typical machine tool. Where a workpiece rotates with constant angular velocity ω and the cutting tool translates along the axis of the workpiece with constant linear velocity $\frac{\omega f}{2\pi}$, with f is the feed rate in length per revolution corresponding to the normal thickness of the chip removed. Under regenerative cutting, a displacement y of the tool can result in a vibration of the tool relative to the workpiece, and the surface of the workpiece becomes wavy. After a round of the workpiece (or tool), due to the non-uniformity of the surface, a variation of the chip thickness is generated. As a result, the cutting force depends on the actual and delayed values of the relative displacement of the tool and the workpiece, where the length of the delay d is equal to the time period of the workpiece (or tool). A model often used in studying such a process is shown in Figure 1.1, where the surface generated by the previous pass becomes the upper surface of the chip on the subsequent pass. Notice that the delay is the result of an intrinsic property of the system. This time-delay system can be described by the equation

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = F_t(f + y(t) - y(t-\tau)) \tag{1.1}$$

where m, c, and k represent the inertia, damping and stiffness characteristics of the machine tool respectively, the delay time $\tau = \frac{\omega f}{2\pi}$ corresponds to the time for the workpiece to complete

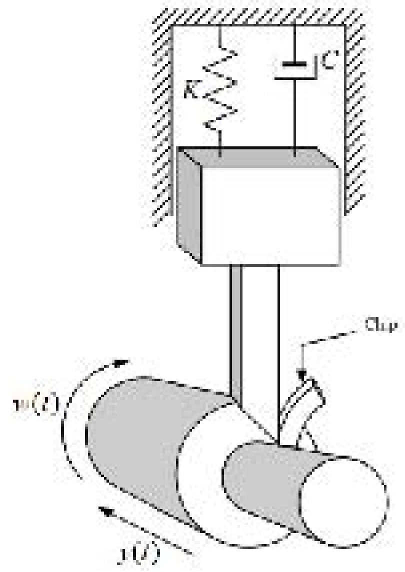

Figure 1.1: Regenerative chatter

one revolution, and $F_t(.)$ is the thrust force depending on the instantaneous chip thickness $f + y(t) - y(t - \tau)$. It is often sufficient to consider $F_t(.)$ to be linear, and techniques for linear time-delay systems are often used. Notice that the delay occurs because of an intrinsic property of the system.

1.2.2 Example 2

Delayed Resonator [GKC03]: This example is retrieved by modifying the classical vibration absorber Figure 1.2, which is a device mounted in structures (frequently used in power transmission, automobiles, and buildings) to reduce the amplitude of mechanical vibrations. Its application can enhance comfort and performance. In addition, it can reduce outright structural failure. The vibration absorber system consists of a mass m_a, a spring with constant c_a, and a damper with damping factor k_a attached to a main structure consisting of an object with m, k, and c, subject to a harmonic excitation $f(t)$.
However, the classical resonators are sensitive to the excitation frequency leading to a mediocre performance, to enhance the performance additional force proportional to the delayed displacement of M is introduced as shown in Figure 1.3. Consequently, the entire system can be described by the the following equations:

$$\begin{aligned} m_a\ddot{x}_a(t) + c_a(\dot{x}_a(t) - \dot{x}(t)) + k_a(x_a(t) - x(t)) - gx(t-\tau) = 0 \\ M\ddot{x}(t) + (c + c_a)\dot{x}(t) - c_a\dot{x}_a(t) + (k + k_a)x(t) - k_a x_a(t) + gx(t-\tau) = f(t) \end{aligned} \tag{1.2}$$

This system is an example of applications where the delay is intentionally introduced to enhance the system performance.

1.3 The considered class of systems

In general, based on the differential state interpretation, there are mainly three ways to represent delay systems [KNG99], [Nic01a]:

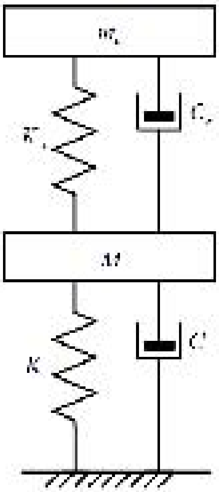

Figure 1.2: Classical vibration absorber

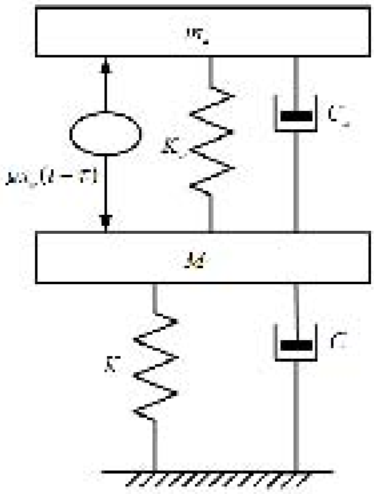

Figure 1.3: Delayed resonator

1. As differential equations on abstract spaces of infinite-dimension (infinite-dimensional systems): In this approach, the delay system class is considered to be a part of a larger class of systems: the infinite-dimensional system class, described by abstract differential equations. However, this approach requires a generalization of some finite-dimensional properties to infinite-dimensional case such as the concepts of controllability, stabilizability, observability, detectability [KR99], [CP78].

2. As differential equations on functional spaces (FDE): time-delay system can be considered as evolutions in a finite-dimensional space or in a function space. The former uses the finiteness of the vector space to analyze the system's behaviour while the latter reflects the infinite-dimensional character of the system [HL93]. Although the method of treating infinite-dimensional problems using finite dimensional tools has its advantage, the obtained results are conservative.

   This representation is widely used for time-delay systems and we can distinguish three different types:

   - System with pointwise or discrete delay: The delay may act on the state x, input u or/and output y:

     $$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t) + B_d u(t-d_u) + f(x(t), x_t(\theta)), \quad t \geq t_0 \\ y(t) = Cx(t) + C_d x(t-d_y) \\ x_{t_0}(\theta) = \phi(\theta), \quad \forall \theta \in [-d, 0] \end{cases}$$

     where $x(t) \in \mathbb{R}^n$, and x_t denotes the translation operator acting on the trajectory: $x_t(\theta) = x(t+\theta)\ \forall \theta \in [-d, 0]$, u is the input vector, y is the output; d, d_h, d_y are respectively the state delay, the input delay and the measurement delay; $\phi(t)$ is the initial condition on $[-d, 0]$; A, A_d, B, B_d C, and C_d are matrices of appropriate dimensions.

     In this dissertation, we are interested in nonlinear time-delay systems described by functional differential equations of retarded type (RFDE) with delays acting on the state only:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d) + Bu(t) + f(x(t), x_t(\theta)), & t \geq t_0, \\ x_{t_0}(\theta) = \phi(\theta), & \forall \theta \in [-d, 0]. \end{cases} \tag{1.3}$$

In addition, in the following chapters, the function f is presumed to be continuous and also Lipschitzian with respect to its arguments, i.e.,

$$\|f(x, x_d) - f(y, y_d)\| \leq \gamma_f \left\| \begin{matrix} x - y \\ x_d - y_d \end{matrix} \right\|.$$

- Distributed delay systems: this class contains a particular form of the delay which acts in a distributed manner over a whole interval, e.g., consider the following model:

$$\begin{cases} \dot{x}(t) = Ax(t) + \int_0^{\tau} A_d(\theta) x(t+\theta) d\theta + Bu(t) + \int_0^{d} B_d(\theta) u(t+\theta) d\theta, \\ x_{t_0}(\theta) = \phi(\theta), \quad \forall \theta \in [-d, 0]. \end{cases}$$

  Remark 1.3.1. *It is always possible, by means of a model transformation (as we will see later), to transform a system with pointwise (or discrete) delays into a system with distributed delays, however the reverse transformation (distributed to pointwise) is not always possible.*

- Another class of systems that will not be addressed in this thesis, but is widely studied in the literature is neutral functional differential equations in which the function f includes, in addition to the previous properties, information on the derivative of x_t, which implies an increased mathematical complexity. This type can be described as follows:

$$\begin{cases} \dot{x}(t) = f(x_t(\theta), \dot{x}_t(\theta)), & t \geq t_0, \\ x_{t_0}(\theta) = \phi(\theta), & \forall \theta \in [-d, 0]. \end{cases} \tag{1.4}$$

  This category of time-delay systems is largely used for describing lossless propagation phenomena [NRG03].

3. As differential equations over rings of operators: The associated FDE is expressed as a vector differential equation defined over a ring of operators. Although this method was successfully applied to solve some interesting control issues such as decoupling, disturbance rejection [Sen01] less contributions were devoted to observation problems. Nevertheless, some important results have been provided for observer design issue [LO81], [PC86].

   Let us consider a continuous-time system with delays:

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x(t - id), \tag{1.5}$$

   with $x(t) \in \mathbb{R}^n$ is the state vector, matrices A_i are real matrices of appropriate dimensions, $d \in \mathbb{R}^+$ is the delay and N represents the maximal state delay. The same time-delay system can be also represented by the following linear differential equation with coefficients over a module:

$$\dot{x}(t) = A(\nabla) x(t), \tag{1.6}$$

   where ∇ is the delay operator defined for any continuous function $f(t)$ by

$$\nabla f(t) = f(t-d),$$

and A is a polynomial matrix in ∇ given by $\sum_{i=0}^{N} \nabla^i A_i$.

Each method has its own advantages and disadvantages depending on the problem to be treated. A brief review of some basic properties of time-delay systems is given in the following sections.

1.4 Time categories

The way to treat time-delay systems differs according to the type of delay. So, it is important to present the different categories or types of delays that can be encountered in the literature:

- **Discrete or pointwise**:
  Obviously, systems with single delay are not always sufficient to describe real systems and systems representation might involve many delays. Hereafter, we present a form of a system with discrete delays that has been largely used:

  $$\dot{x} = \sum_{i=0}^{n} A_i x(t - d_i).$$

  Different conditions for stability and stabilizability for systems with pointwise delays have been developed over the years such as [KNR99], [DPR99], [MAN05], [HLT11]. Based on the relation between the delays, we can distinguish two types of discrete or pointwise delays: commensurate delays and incommensurate delays.

  - Commensurate: $d_i \in \mathbb{R}, i \in \mathbb{N}$ are commensurate if d_i/d_j is rational, which corresponds to finding a minimal delay d such that $d_i = id$, then the system becomes, with an appropriate reordering of the indices, as follows:

    $$\dot{x} = \sum_{i=0}^{n} A_i x(t - id)$$

    This class is considered very conservative. The stability problem for this class can be treated similarly to the single delay case, since the characteristic equation associated to the system will have the same algebraic properties [Nic01a]. Different results related to this type of delay have been presented. For example, The stability and the ill-posedness with respect to small perturbations in the delay parameters of systems were discussed in [Dat98], [Lou95]. For different stability criteria in the frequency domain, one can refer to [GKC03], [KL95], [BCLZ82], [Kam82]. A method to compute a delay interval such that the system under consideration is stable for all delays in that computed interval was presented in [CGN12].
  - Incommensurate: the delays $d_i, i \in \mathbb{N}$ are free parameters. Different ideas have been proposed to deal with this kind of delay, e.g., a change in coordinates such that all time-delay terms, in the new coordinates, in the system description are associated with the output only [HZP02]. A causal dynamic compensators to solve the problem of stabilizing a class of neutral delay-differential systems with several fixed, non commensurate point delays has been discussed in [EK84]. The descriptor model transformation condition was exploited to deal with systems having two delays in [FS02b], and later an ameliorated condition which accepts delays with lower bounds different than zero was presented in [HWS06].

- **Time-varying delay**: which can belong to different categories:
  - Bounded delay: $0 < d_1 \leq d(t) \leq d_2$. This type of delay is frequently treated in the literature and it implies delay-independent criteria. For example, one can refer to [XCP04], [KB93] for a stability analysis; [SFRS07] for the case of unknown upper bound; [CZZ11] for singular systems; [Bou07] for nonlinear systems; [CS06] for uncertain neutral differential systems; and [BEBC99] for a controller design for systems with uncertainty in the input delay.
  - Derivative bounded delays: $\dot{d}(t) \leq \mu < 1$. This condition means that $f(t) = t - d(t)$ is monotonic. Usually, this type is associated with the previous one to get delay-dependent criterion. For instance, for the case of uncertain linear systems [SPP99], [PT09], [CSL98]; for uncertain neutral systems [Lie07].
  - Arbitrary varying delays: in this case, the delay d and its derivative $\dot{d}$ are not bounded.

- **Distributed delay** $\int\limits_{s-\tau}^{t} x(s)\mathrm{d}s$: this type of delays have been treated in many papers e.g., observer design for nonlinear systems [GP05], stability of linear neutral systems [Han03], robust stabilization for uncertain neutral systems [CZ07] and robust control [ZF02], [FT09]. One of the methods to treat the integral terms in the Lyapunov-Krasovskii functional is to use some transformations (as we will discuss later) which change the original system with a discrete delay into a new system with a distributed delay.

- **State-dependent delay**: the delay is presented as a function of the state of the system [Mur01], [BHJ+10].

The previous classification was not enough, and more informations on the delay were needed to develop less conservative criteria. Which motivated defining new categories depending on the delay derivative as follows:

- Slowly varying delay: $d(t)$ is a differentiable almost everywhere function, satisfying $\dot{d}(t) \leq \mu < 1$.

- Moderately varying delay: $d(t)$ is a differentiable almost everywhere function, satisfying $\dot{d}(t) \leq \mu$ with $\mu \geq 1$.

- Fast varying delay: $d(t)$ is a measurable (e.g., piecewise-continuous) function without any constraints on the delay derivative.

An important phenomenon related to time-delay systems is called "quenching" which occurs when the stability (resp. instability) of a system with constant delays within a certain interval is lost when that delay is assumed to be time-varying inside the same interval and vice-versa. This problem was first mentioned in [Lou99]. Obviously, this problem prevent us from applying the obtained results in the case of fixed delays to the case of time-varying delays. However, some efforts were made to take this phenomenon into consideration in [PPN07].

1.5 Stability-related topics of TDS

Since time-delays can limit and degrade the achievable performance of controlled systems, and even induce instability. Stability of time-delay systems has been extensively discussed in many monographs [GKC03], [WHS10], [Nic01a], [MN08].

Consider the general functional differential equation of retarded type (RFDE):

$$\begin{cases} x(t) = f(t, x_t), & t \geq t_0 \\ x_{t_0}(\theta) = \phi(\theta), & \forall\theta \in [-d, 0] \end{cases} \tag{1.7}$$

where $x_t(.)$, for a given $t \geq t_0$, denotes the restriction of $x(.)$ to the interval $[t-d, t]$ translated to $[-d, 0]$, i.e.,

$$x_t(\theta) = x(t+\theta), \quad \forall\theta \in [-d, 0]$$

It is assumed that $\phi \in \mathcal{C}^v_{n,d}$ $(\mathcal{C}([-d, 0], \mathbb{R}^n))$ (refer to the glossary for the definition) and the map $f(t, \phi) : \mathbb{R}^+ \times \mathcal{C}^v_{n,d} \mapsto \mathbb{R}^n$ is continuous and Lipschitz in ϕ and $f(t, 0) = 0$. Let us denote by $x(t_0, \phi)$ the solution of the functional differential equation (1.7) with the initial condition $(t_0, \phi) \in \mathbb{R}^+ \times \mathcal{C}^v_{n,d}$.

1.5.1 Concept of stability

This subsection presents some basic information on time-delay systems, specifically, fundamental concepts, descriptions, and types of stability.
Let $\bar{x}(t)$ be a solution of the RFDE (1.7). The stability of the solution concerns the system's behavior when the system trajectory $x(t)$ deviates from $\bar{x}(t)$. Without loss of generality, we assume that RFDE (1.7) admits the solution $x(t) = 0$, which will be referred to as the trivial solution. If the stability of a nontrivial solution, $\bar{x}(t)$, needs to be studied, then we can use the variable transformation $z(t) = x(t) - \bar{x}(t)$ to produce the new system

$$\dot{z}(t) = f(t, z_t + \bar{x}_t) - f(t, \bar{x}_t) \tag{1.8}$$

which has the trivial solution $z(t) = 0$. For the function $\phi \in \mathcal{C}([a, b], \mathbb{R}^n)$, define the continuous norm $\|.\|_c$ to be

$$\|\phi\|_c = \sup_{a \leq \theta \leq b} \|\phi(\theta)\|$$

In this definition, the vector norm $\|.\|$ represents the 2-norm $\|.\|_2$.
There are various types of stability for the trivial solution of time-delay systems. In what follows, the definition of some stability types:

Definition 1.5.1. *[HL93]*

- ***Stability***: *if* $\forall t_0 \in \mathbb{R}$ *and* $\epsilon > 0, \exists\delta = \delta(t_0, \epsilon) > 0$ *such that*

  $$\|x_{t_0}\|_c < \delta \Rightarrow \|x(t)\| < \epsilon, \quad t \geq t_0,$$

  then the trivial solution of (1.7) *is stable.*

- ***Asymptotic stability***: *if the trivial solution of* (1.7) *is stable, and if,* $\forall t_0 \in \mathbb{R}, \exists\delta_a = \delta_a(t_0) > 0$ *such that*

  $$\|x_{t_0}\|_c < \delta_a \Rightarrow \lim_{t\to\infty} x(t) = 0,$$

  then the trivial solution of (1.7) *is asymptotically stable.*

- ***Uniform stability***: *if the trivial solution of* (1.7) *is stable and if* $\delta(t_0, \epsilon)$ *can be chosen independently of* t_0*, then the trivial solution of* (1.7) *is uniformly stable.*

- ***Uniform asymptotic stability***: *if the trivial solution of* (1.7) *is uniformly stable and if* $\exists \delta_a > 0$ *such that,* $\forall \eta > 0$, $\exists T = T(\delta_a, \eta)$ *such that*

$$\|x_{t_0}\|_c < \delta_a \Rightarrow \|x(t)\| < \eta, \ \ \forall t \geq t_0 + Tand \ t_0 \in \mathbb{R},$$

then the trivial solution of (1.7) *is uniformly asymptotically stable.*

- ***Global (uniform) asymptotic stability***: *if the trivial solution of* (1.7) *is (uniformly) asymptotically stable and if* δ_a *can be an arbitrarily large, finite number, then the trivial solution of* (1.7) *is globally (uniformly) asymptotically stable.*

- ***Global exponential stability***: *if there exist constants* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\|x(t)\| \leq \beta \sup_{-h \leq \theta \leq 0} \|x(\theta)\| e^{-\alpha t}$$

then the trivial solution of (1.7) *is globally exponentially stable and* α *is called the exponential convergence rate.*

Furthermore, if the system is linear and time-invariant, the 'uniform asymptotic stability' property is equivalent to the 'asymptotic stability' or to the 'exponential stability' property.
The main methods of examining the stability can be classified into two types: frequency domain and time-domain methods. Frequency-domain methods determine the stability of a system from the distribution of the roots of its characteristic equation [Nic01a] or from the solutions of a complex Lyapunov matrix function equation [BCLZ82]. They are suitable only for systems with constant delays. In the time-domain, the most common approaches to study the stability of time-delay systems are the Lyapunov-Krasovskii functional [Bli01] and Razumikhin function methods [HL93]. Until the 1990s, the stability criteria obtained using these two approaches were generally in the form of existence conditions because of the difficulty of constructing Lyapunov-Krasovskii functionals and Lyapunov functions. Since then, due to the use of Riccati equations [HL99], linear matrix inequalities (LMIs) [BGFB94], and Matlab toolboxes, general solutions were derived and significant results have continued to appear one after another (see [Ric03] and references therein). Among them, two classes of sufficient conditions have received a great deal of attention. One class is independent of the length of the delay, and its members are called delay-independent conditions. The other class makes use of the information on the length of the delay, and its members are called delay-dependent conditions.

1.5.2 Frequency-domain methods

In this dissertation, we focus on time-domain methods for the stability analysis. Yet, we find it important to outline some of the frequency-domain methods developed in the literature.
Some criteria generalized the Hurwitz method to delay systems by verifying if the roots of a characteristic equation of the system are in the left half-plane, such as:

- Pontryagin criterion: extended the methods used in proving the Routh-Hurwitz criterion for the zeros of a polynomial to be in the left half-plane [HL93].

- Chebotarev criterion: generalization of the Routh-Hurwitz criterion to the quasipolynomials in commensurate delays case. It has a drawback of computing a large number of determinants.

- Yesupovisch-Svirskii criterion: which is a simpler version of the Pontryagin criterion but still relies heavily on the geometry and the application of the argument principle. It was usually applied for single (pointwise) delay systems.

Other criteria depend on the Root locus method to determine the values of the parameters for which the characteristic equation has roots on the imaginary axis such as:

- $\mathcal{D}$-decomposition method: this method consists in decomposing the parameter space into several regions, each region is bounded by a hypersurface which has the property that at least one root of the characteristic equation lies on the imaginary axis [Nei49].

- τ-decomposition method: this method involves the use of a ratio of two given polynomials D_0. The behaviour of the contour D_0 is then analyzed with respect to the unit circle in the complex plane. It is worth mentioning that this method is applied only for delay systems with a single delay [Hsu70].

- Argument principle methods: since the number of unstable roots of the associated characteristic equations is finite, criteria based on this method, such as Nyquist or Michailov-Leonhard criteria, can be extended to time-delay systems [KN86].

1.5.3 Lyapunov-Krasovskii stability theorem

Lyapunov method has been used effectively in stability analysis of delay-free systems. So, it is natural to try this method on time-delay systems. Clearly, this requires a few adaptations on the Lyapunov function since the state space of Delay Differential Equations (DDEs) is infinite dimensional. In time-delay case, the Lyapunov function $V(t, x_t)$ depends on x_t (the value of state variable in the interval $[t-d, t]$) and thus becomes functional. This type of functional is called a Lyapunov-Krasovskii functional. In this section, we give sufficient conditions for the stability of the solution $x = 0$ of equation (1.7) which generalize the second method of Lyapunov for Ordinary Differential Equations (ODEs).
Before proceeding, let $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ be continuous, and $x(t_0, \phi)$ be a solution of (1.7) at time t with the initial condition $x_{t0} = \phi$. Then, we may calculate the derivative of $V(t, x_t)$ with respect to t as:

$$\dot{V}(t, \phi) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+d, x_{t+d}(t_0, \phi) - V(t, \phi)]$$

The function $\dot{V}(t, \phi)$ is the upper right-hand Dini's derivate of $V(t, \phi)$ along the solution of equation (1.7).

Theorem 1.5.2. ***(Lyapunov-Krasovskii stability theorem)*** *[Nic01a]: Suppose that the function $f : \mathbb{R} \times \mathcal{C}_{n,d} \mapsto \mathbb{R}^n$ takes bounded sets of $\mathcal{C}_{n,d}$ in bounded sets of $\mathbb{R}^n$ and suppose that $u(s), v(s)$ and $w(s)$ are continuous, non-negative and nondecreasing functions with $u(s), v(s) > 0$ for $s \neq 0$ and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R} \times \mathcal{C}_{n,d} \mapsto \mathbb{R}$ such that:*

(i) $u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|_c)$

(ii) $\dot{V}(t, \phi) \leq -w(\|\phi(0)\|)$

then the (trivial) solution $x = 0$ of the equation (1.7) is uniformly stable.
If $u(s) \to \infty$ as $s \to \infty$ the solution is uniformly bounded.
If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

1.5.4 Razumikhin stability theorem

The Lyapunov-Krasovskii theorem requires the manipulation of functionals, which makes it difficult to apply. This motivated the use of an alternative approach that involves only functions instead of functionals. This approach is called the Razumikhin theorem and is considered as the classical analysis tool in the finite-dimensional space interpretation. The key idea behind the Razumikhin theorem is the use of a Lyapunov function, $V(x)$, whose derivative is not negative for all trajectories, but only for special solutions of the system. The precise statement is given in the next theorem.

Theorem 1.5.3. ***(Razumikhin theorem)*** *[Nic01a]: Suppose that the function* $f : \mathbb{R} \times \mathcal{C}_{n,d} \mapsto \mathbb{R}^n$ *takes bounded sets of* $\mathcal{C}_{n,d}$ *in bounded sets of* $\mathbb{R}^n$ *and suppose that* $u, v, w : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *are continuous, nondecreasing functions such that* $u(s), v(s), w(s) > 0$ *for* $s \neq 0$ *and* $u(0) = v(0) = 0$*. Assume that there exists a continuous function* $V : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ *such that:*

$$u(\|x\|) \leq V(t,x) \leq v(\|x\|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n \tag{1.9}$$

The following statements hold:

a) $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$ *if* $V(t+\theta, x(t+\theta)) < p(V(t,x(t))), \quad \forall \theta \in [-d, 0]$ *then the trivial solution of* (1.7) *is uniformly stable.*

b) If there exists a continuous nondecreasing function $p : \mathbb{R}^+ \mapsto \mathbb{R}^+, p(s) > s$*, such that* $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$ *if* $V(t+\theta, x(t+\theta)) < p(V(t,x(t))), \quad \forall \theta \in [-d, 0]$ *then the trivial solution of* (1.7) *is uniformly asymptotically stable.*

If $u(s) \to \infty$ *as* $s \to \infty$ *then the trivial solution is globally asymptotically stable.*

Previously, the stability conditions for time-varying delay systems were derived only via Lyapunov-Razumikhin functions. Nevertheless, the Razumikhin approach leads to more conservative conditions and it can not be applied to the case of $\mathcal{H}_\infty$ control [XL94], [LM07].
There are two different kinds of asymptotic stability for systems of the form (1.7), depending on the information on the delay size: delay-independent and delay-dependent. These two types will be discussed in more details in the following subsections.

1.5.5 Delay-independent analysis

Delay-independent conditions contain no information on the delay. Thus, they are overly conservative, especially when the delay is small. The Lyapunov-Krasovskii functional candidate is generally chosen to be:

$$V_1(x_t) = x^T(t)\mathcal{P}x(t) + \int_{t-d}^{t} x^T(s)\mathcal{Q}x(s)\mathrm{d}s \tag{1.10}$$

where $\mathcal{P} > 0$ and $\mathcal{Q} > 0$ are to be determined. Many authors have used this approach to prove stability and control of different types of delay systems such as linear systems with a fixed time-delay [LD02], [Bli01]; with commensurate time-delays [Kam82]; uncertain discrete large-scale systems [LH97]; uncertain neutral systems with time-varying delays [Lie07]; switching systems [KCL08]; nonlinear networked control systems [MHB06].

1.5.6 Delay-dependent analysis

The conservatism of the delay-independent conditions has produced another important class of stability conditions, namely, delay-dependent conditions. This type contains information on the length of a delay.
In this case, the system (1.7) is assumed to be stable when $d = 0$, i.e., $A + A_d$ is Hurwitz. In addition, the system (1.7) is proved to be stable for all $d \in [0,\ \bar{d}]$, where $\bar{d}$ is the upper bound on the delay. To study the delay-dependent stability, a quadratic double-integral term is added to the Lyapunov-Krasovskii functional (1.11). Such construction was proposed by Krasovskii in the 1950s and was used, or variations of it, to establish delay-dependent conditions [NRG03], [CW04], [LTP04]:

$$V(x_t) = V_1(x_t) + V_2(x_t) \tag{1.11}$$

with

$$V_2(x_t) = \int_{-d}^{0}\int_{t+\theta}^{t} x^T(s)Zx(s)\mathrm{d}s\mathrm{d}\theta \tag{1.12}$$

The derivative of $V_2(x_t)$ is

$$\dot{V}_2(x_t) = dx^T(t)Zx(t) - \int_{t-d}^{t} x^T(s)Zx(s)\mathrm{d}s \tag{1.13}$$

Besides the previous candidate, more complicated functionals have been proposed, involving the various terms:

$$\begin{aligned}
V_3(x_t) &= x^T(t)\int_{-d}^{0} Sx(t+\theta)\mathrm{d}\theta \\
V_4(x_t) &= \dot{x}^T(t)\int_{-d}^{0} Ux(t+\theta)\mathrm{d}\theta \\
V_5(x_t) &= \int_{-d}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\
V_6(x_t) &= d\int_{-d}^{0}\int_{t+\theta}^{t} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} T_{11} & T_{12} \\ (\star) & T_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ \dot{x}(s) \end{bmatrix} x(s)\mathrm{d}s\mathrm{d}\theta
\end{aligned}$$

V_3, V_4 and V_5 were used to establish delay-dependent stability for neutral systems in [WHS04] and V_5 also appeared in a control stabilization problem in [PCE$^+$05], [Lie07], and in a delay-dependent stability analysis based on fractioning the delay in [GP06]. V_6 is used in [PT09] to present a delay-dependent Bounded Real Lemma (BRL) and in [SKM10] to treat a robust stability analysis.
Important topics in control theory are delay-dependent problems, for instance, stability analysis [Nic99], [PT08], [AG07], [BHJ$^+$10]; robust control, stabilisation, [CZ07], [MPKL01]; $\mathcal{H}_\infty$ control [PT09], [FS03], [LMKP04], [Fan09a], [JG11]; guaranteed-cost control [Lie07]; saturation input control [TPGQ02], [FD09]. The main criterion for judging the conservatism of the existing delay-dependent conditions is the maximum delay possible. As we notice, the double-integral term, in the Lyapunov-Krasovskii functional (1.12), results in quadratic integral terms appearing in the derivative of that functional (1.13).
One of the main challenges when studying delay-dependent problems is how to deal with this integral term to get less restrictive results. So far, many methods have been devised to solve this

problem: the discretized Lyapunov-Krasovskii functional method, fixed model transformations, and parameterized model transformations [Nic99], Free Weighting Matrix methods (FWM). In what follows, we will give a brief idea about each method, its advantages and drawbacks.

1.5.6.1 Discretized Lyapunov functional

The Discretized Lyapunov Functional (DLF) method proposed by Gu [Gu06] is mainly used to study the stability of linear systems and neutral systems with a constant delay. By using this method, the obtained maximum allowable delay that guarantees the stability of the system is very close to the actual value, and the results can be written in the form of LMIs. Although this method has not been widely used because it is computationally expensive, it has been applied to robust stability analysis of linear retarded and neutral type systems in [GN01], [HYG04], non-linear neutral systems in [Han08b], and was also generalized by using a more general Lyapunov functional to the case where the delay may be time-varying in [GN00], [HG01]. Another type of LKFs is the discretized Lyapunov functional proposed by [Fri06] which combines between the aforementioned discretization method of Gu and the descriptor model transformation by Fridman which will be discussed later. Before presenting the DLF, let us first present the complete quadratic Lyapunov-Krasovskii functional needed to get a necessary and sufficient stability condition for linear systems.

$$\begin{aligned} V(\phi) = {} & \phi^T(0)P\phi(0) + 2\phi^T(0)\int_{-d}^{0} Q(\xi)\phi(\xi)\mathrm{d}\xi + \int_{-d}^{0}\int_{-d}^{0} \phi^T(\xi)R(\xi,\eta)\phi(\xi)\mathrm{d}\eta\mathrm{d}\xi \\ & + \int_{-d}^{0} \phi^T(\xi)S(\xi)\phi(\xi)\mathrm{d}\xi \end{aligned} \tag{1.14}$$

where $P = P^T$ and $R(\xi,\eta) = R^T(\xi,\eta)$, $S(\xi) = S^T(\xi)$ for all $-d \leq \xi \leq 0$ and $-d \leq \eta \leq 0$. The basic idea of this method is to divide the domain of definition of matrix functions Q, R and S into smaller regions, and choose these matrix functions to be continuous piecewise linear, in other words, divide the delay interval $[-d, 0]$ into N segments, thus reducing the choice of the Lyapunov-Krasovskii functional V into choosing a finite number of parameters [GKC03]. So, the DLF can be written of the form:

$$\begin{aligned} V(\phi) = {} & \phi^T(0)P\phi(0) + 2\phi^T(0)\sum_{p=1}^{N}\int_{0}^{1} Q^{(p)}(\alpha)\phi^{(p)}(\alpha)h\mathrm{d}\alpha \\ & + h^2\sum_{p=1}^{N}\sum_{q=1}^{N}\int_{0}^{1}\mathrm{d}\alpha\int_{0}^{1} \phi^{(p)T}(\alpha)R^{(pq)}(\alpha,\beta)\phi^{(q)}(\beta)\mathrm{d}\beta \\ & + \sum_{p=1}^{N}\int_{0}^{1} \phi^{(p)T}(\alpha)S^{(q)}(\alpha)\phi^{(q)}(\alpha)h\mathrm{d}\alpha \end{aligned} \tag{1.15}$$

where $\phi^{(p)} = \phi(\theta_p + \alpha h)$ The segments can be chosen to form a uniform grid size [GN00] or non-uniform grid size [Gu99] which is more appropriate to treat multiple delays. Consequently, the extent of conservatism depends on the grid size.

1.5.6.2 Model transformations

The primary way of dealing with the quadratic integral term on the right-hand side of equation (1.13) is by using one of the three fixed model transformation [KR99] to bring the integral terms into the system equation so as to produce cross terms and quadratic integral terms in the derivative of the Lyapunov-Krasovskii functional. Then, the bounding of the cross terms eliminates the quadratic integral terms.

- **Model transformation I**:

$$\dot{x}(t) = (A + A_d)x(t) - A_d \int_{t-d}^{t} [Ax(s) + A_d x(s-d)]\mathrm{d}s \tag{1.16}$$

- **Model transformation II**:

$$\frac{d}{dt}\Big[x(t) + A_d \int_{t-d}^{t} x(s)\mathrm{d}s\Big] = (A + A_d)x(t) \tag{1.17}$$

  Model transformations I and II introduce additional dynamics into the transformed system, the transformed system is not equivalent to the original one. Thus, these transformations were soon replaced by others.

- **Model transformation III**: this model was presented in [PK99]. In this case, the transformed system is equivalent to the original one

$$\dot{x}(t) = (A + A_d)x(t) - A_d \int_{t-d}^{t} \dot{x}(s)\mathrm{d}s \tag{1.18}$$

- **Model transformation IV**: or the descriptor model transformation [Fri01], which has attracted a great deal of attention in subsequent years.

$$\begin{cases} \dot{x}(t) = y(t) \\ y(t) = (A + A_d)x(t) - A_d \int\limits_{t-d}^{t} y(s)\mathrm{d}s \end{cases} \tag{1.19}$$

All these model transformations produce cross terms in the derivative of the Lyapunov-Krasovskii functional [FS02b], [FS02a], [LMKP04], [Nic01b]. However, the bounding of those cross terms is usually the major source of conservatism and some new analysis methods to reduce the conservatism have focused on this point. Such as matrix inequality methods which depend on some novel bounds for the inner product of two vectors mainly Park's inequality [PK99] (an extension of the basic inequality $-2a^T b \leq a^T Ra + b^T R^{-1} b$) and on Moon et al.'s inequality [MPKL01] which has greater generality (see Appendix A for definitions). These two inequalities played an important role in deriving a series of delay dependent stability criteria and many efforts have been concentrated on constructing less conservative Lyapunov Krasovskii functional. The first attempts to study systems with bounded delay supposed that the delay has a zero lower bound and a bounded delay derivative. Some of the drawbacks of these methods are the fact that the delay derivative upper bound has to be less than one (slowly varying delay), and that the analysis includes ignoring some terms because they basically use the Newton-Leibnitz formula to replace delay terms in the derivative of the Lyapunov-Krasovskii functional; but not all the delay terms are necessarily replaced which gave partial information on the relationship among

delay-related terms. To solve this problems, some authors treated these main difficulties either by manipulating the integral terms differently using Jensen inequality [CZ07], or by exploiting the delay upper-bounded state [PK07], or by trying to describe the integral terms in a way to bring out new LMI conditions that are less conservative.
Later on, more delay dependent methods were developed to cover the cases of fast varying delay [Sha08], moderately varying delay [SF07], and for interval time-varying delay where the lower bound is not restricted to zero (non-small delay or interval-delay systems) [PT08], [ZY10]. Another important method for dealing with time-varying delay is called piecewise analysis method (PAM) where the variation interval of the time-delay is firstly divided into several subintervals [YTZ09]. By checking the variation of the Lyapunov functional in every subinterval, some new delay-dependent stability criteria are derived. It is also worth mentioning, the methodologies based on Finsler lemma [SFS04], [GP06], [AG07] which were successfully applied on time-varying delays. So many LMI conditions were proposed in the literature using different techniques, which brings up for discussion the issue of the differences and the resemblance features between these methods. [XL07] tried to tackle this problem by comparing between seven stability criteria and proved them to be mathematically equivalent and even showed that, based on the least number of unknown variables to be determined in the LMI, the work presented in [XL05] is the most computationally efficient.

1.5.6.3 Parameterized model transformations

In this type of methods, a new matrix C is introduced and the delay term is divided into two parts: a delay-independent one and one to which a fixed model transformation is applied. Consequently, transforming the system into

$$\dot{x}(t) = Ax(t) + (A_d - C)x(t-d) + Cx(t-d) \tag{1.20}$$

where C is a matrix parameter to be determined. Combining the parameterized model transformation with a fixed model transformation (e.g., Model transformation I), the transformed system becomes a discrete plus a distributed delay system of the form:

$$\dot{x}(t) = (A+C)x(t) + (A_d - C)x(t-d) - C\int_{-d}^{0}[Ax(t+s) + A_d x(t+s-d)]\mathrm{d}s \tag{1.21}$$

Certainly, the limitations of the fixed model transformation remain a source of conservatism. The parameterized model transformation can be realized in many different ways as summarized in [Nic01a], however, all these methods involves the injection of a free matrix C:

$$\dot{x}(t) = Ax(t) + (I - C)A_d x(t-d) + CA_d x(t-d) \tag{1.22}$$
$$\dot{x}(t) = Ax(t) - Cx(t-d) + (A_d + C)x(t-d) \tag{1.23}$$
$$\dot{x}(t) = Ax(t) + M_d(N_d + C)x(t-d) - M_d Cx(t-d) \tag{1.24}$$

with $A_d = M_d N_d, M_d \in \mathbb{R}^{n\times n_d}$ and $N_d \in \mathbb{R}^{n_d \times n}$ is a full rank matrix. In [Han03], an effective approach to matrix decomposition was presented, based on the descriptor model transformation and the decomposition technique of discrete-delay term matrix. Yet, three undetermined matrices have to be equal, which leads to unavoidable conservatism.

1.5.6.4 Free weighting matrix methods

In a time-delay system with a bounded delay $d < h$ and a bounded delay derivative, instead of using fixed weighting matrices to express the relationships among the terms of the Newton-Leibnitz formula (refer to Appendix A for definition), [WHS10] suggested introducing free variables into the Lyapunov-Krasovskii functional. From the Newton-Leibnitz formula, the following equation is true for any matrices N_1 and N_2 with appropriate dimensions:

$$2\left[x^T(t)N_1 + x^T(t-h)N_2\right]\left[x(t) - \int\limits_{t-h}^{t} \dot{x}(s)\mathrm{d}s - x(t-h)\right] = 0 \tag{1.25}$$

The left side of this equation is added to the derivative of the Lyapunov-Krasovskii functional. Another inequality which plays an important role in this method is the following:

$$h\begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}^T X \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix} - \int\limits_{t-d(t)}^{t} \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}^T X \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix} \geq 0 \tag{1.26}$$

Choosing the following Lyapunov-Krasovskii functional candidate

$$V(x_t) = x^T(t)Px(t) + \int\limits_{t-d(t)}^{t} x^T(s)Qx(s)\mathrm{d}s + \int\limits_{-h}^{0}\int\limits_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta, \tag{1.27}$$

then adding the equations (1.25), (1.26) to the derivative of the Lyapunov Functional, one can get the following LMIs

$$\begin{bmatrix} PA + A^TP + N_1 + N_1^T + Q + hX_{11} & PA_d + N_1 + N_2^T + hX_{12} & hA^TZ \\ * & N_2 + N_2^T + (1-\eta)Q + hX_{22} & hA_d^TZ \\ * & * & -hZ \end{bmatrix} < 0 \tag{1.28}$$

$$\begin{bmatrix} X_{11} & X_{12} & N_1 \\ * & X_{22} & N_2 \\ * & * & Z \end{bmatrix} \geq 0 \tag{1.29}$$

The optimal values of the free matrices N_1, N_2 and $X = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$ can be obtained by solving the two aforementioned LMIs and their presence overcomes the conservatism arising from the use of fixed weighting matrices. Generally speaking, there are two forms for free weighting matrices: the one with a null summing term added to the Lyapunov functional derivative [HWSL04], [WHS04], [XL05] and the one with free matrices item added to the Lyapunov functional combined with the descriptor model transformation [LMKP04]. This method shows that the descriptor model transformation of Fridman is a special case of the FWM approach. Furthermore, this treatment in combination with a parameter-dependent Lyapunov-Krasovskii functional is easily extended to deal with the delay-dependent stability of systems with polytopic-type uncertainties [WH04], [HWS05]. The free weighting matrices method has been shown to be less conservative than previous methods, due to the avoidance of using any bounding technique. Yet, many researchers criticized the fact that this method introduces many free variables which complicates the system synthesis and consequently lead to a significant increase in the computational burden [PT08], [PG05]. On the other hand, when applying this method to design delay-dependent $\mathcal{H}_\infty$ controllers with memoryless state feedback, the resulting conditions cannot be expressed strictly in terms of LMIs. This NLMI problem is usually solved either by an iterative algorithm devised by [MPKL01], or by a parameter-tuning method often used by Fridman [FS02b], [FS03], which transforms the NLMI into an LMI.

1.6 Observers and controllers design for TDS

The problem of observer design was studied recurrently ever since the original work of Luenberger [Lue71] first appeared. The observation problem is to construct internal information of the system depending on external measurements. The need for designing an observer can be motivated by various purposes: modeling, fault detection and control. This makes the problem of observer design the core of a general control problem. Obviously, due to the diversity of the signals in a system, one cannot use as many sensors as signals characterizing the behavior of the system. Roughly speaking, those signals can include:

- Time-varying signals characterizing the system or what we call state variables.
- constant signals representing the system parameters.
- unmeasured external signals representing the disturbances.

In short, An observer depends on a model in order to reconstruct the internal state, based on available measurements.
Obviously, in the case of time-delay systems, it is difficult to analyze the system and synthesise an observer and/or controller using classical methods since delay terms lead to infinite dimensionality in the characteristic equations. Thus, many methods have been developed over the last decades for time-delay systems. In the next subsection, we present some definitions related to the observability of TDS.

1.6.1 Observability and controllability

The observability and controllability are two fundamental attributes of a dynamical system. Such properties of TDS have been studied extensively (see e.g., [Ric03], [YUN10] and the references therein). Unlike systems of ODEs, the problem of observability/controllability of delay systems depends mainly on the nature of the considered system, which was the reason for developing numerous definitions (e.g., approximate, spectral, weak, strong, point-wise and absolute). According to [Che84], the observability/controllability for linear ODEs systems, is equivalent to the arbitrary assignability of the eigenvalues of the observer/controller. However, the generalization of this property does not necessary hold for Delay Differential Equations (DDEs). The relationship between eigenvalue assignment and observabilty/controllabilty is still an open issue in spite of the extensive research activities regarding this problem.

First, let us start by introducing some important definitions and notation.
Consider the space $\mathcal{L}_2([-d,0];\mathbb{R}^n)$ of all maps $I(-d,0) \to \mathbb{R}^n$ which are square integrable in $I(-d,0)$ endowed with the seminorm:

$$\|y\|_{M^2} = \left[|y(0)|^2_{\mathbb{R}^n} + \int_{-d}^{0} |y(\theta)|^2_{\mathbb{R}^n} d\theta \right]^{\frac{1}{2}}$$

The quotient space of $L^2([-d,0];\mathbb{R}^n)$ by the linear subspace of all y such that $\|y\|_{M^2} = 0$ is a Hilbert space which is isometrically isomorphic to the product space $\mathbb{R}^n \times \mathcal{L}_2([-d,0];\mathbb{R}^n)$. It will be denoted by $\mathcal{M}_2([-d,0];\mathbb{R}^n)$ and its norm by $\|.\|_{M^2}$.

$$\kappa : \mathcal{M}_2([-d,0];\mathbb{R}^n) \to \mathbb{R}^n \times \mathcal{L}_2([-d,0];\mathbb{R}^n) : \kappa(h) = \big(h^0 = x(0), h^1 = x(\theta)\big)$$

Let Y be a Hilbert space which might be thought of as the observation space. We can observe the map $x(.,h,u)$ with an observer $Z \in L_\infty([0,T];\mathcal{L}(\mathbb{R}^n,Y))$; the observation at time t is defined by

$$z(t;h,u) = Z(t)x(t;h,u)$$

We can also observe the map $\tilde{x}(t;h,u) = x(t+\theta,h,u)$ with $\mathcal{M}_2$-observer $\tilde{Z} \in L_\infty([0,T];\mathcal{L}(\mathbb{M}_2,Y))$

$$\tilde{z}(t;h,u) = \tilde{Z}(t)\tilde{x}(t;h,u)$$

Since $\mathcal{M}_2$ is isomorphic to $\mathbb{R}^n \times \mathcal{L}_2(-d,0;\mathbb{R}^n)$, there exist $\tilde{Z}^0(t) \in \mathcal{L}(\mathbb{R}^n,Y)$ and $\tilde{Z}^1(t) \in \mathcal{L}(\mathcal{L}_2([-d,0];\mathbb{R}^n),Y)$ such that:

$$\tilde{Z}(t)(\kappa^{-1}(h^0,0)) = \tilde{Z}^0(t)h^0$$

and

$$\tilde{Z}(t)(\kappa^{-1}(0,h^1)) = \tilde{Z}^1(t)h^1$$

Let us now present some of the most important nontrivial definitions of observability. Evidently, each definition leads to different characteristics of state observation:

Definition 1.6.1. ***(Observability)*** *[DM72]*
System (1.7) *is observable in* $[0,T]$ *if for all* $h \in \mathcal{M}_2([-d,0];\mathbb{R}^n)$ *and* $u \in \mathcal{L}_2([0,T];\mathbb{R}^m)$*, the point* $h^0 \in \mathbb{R}^n$ *can be uniquely determined from a knowledge of* u*,* h^1 *and the observation map* $z(.;h,u)$*.*

Definition 1.6.2. ***(Strong observability)*** *[DM72]*
System (1.7) *is strongly observable in* $[0,T]$ *if for all* $h \in \mathcal{M}_2([-d,0];\mathbb{R}^n)$ *and* $u \in \mathcal{L}_2([0,t];\mathbb{R}^m)$*, the state* h *can be uniquely determined from a knowledge of* u *and the observation map* $z(.;h,u)$*.*

Definition 1.6.3. ***($\mathcal{M}_2$-observability)*** *[DM72]*
System (1.7) *is* $\mathcal{M}_2$*-observable in* $[0,T]$ *if for all* $h \in \mathcal{M}_2([-d,0];\mathbb{R}^n)$ *and* $u \in \mathcal{L}_2([0,t];\mathbb{R}^m)$*, the state* h *can be uniquely determined from a knowledge of* u *and the observation map* $\tilde{z}(.;h,u)$*.*

Remark 1.6.1. *Strong observability* $\Longrightarrow$ $\mathcal{M}_2$*-observability and observability*

Definition 1.6.4. ***(F-observability)*** *[Olb81]*
Let F denote a class of initial functions for system (1.7)*. System* (1.7) *is F-observable on* $[0,T]$ *(respectively. F-observable) iff* $x_0(.) \in F$ *and the output* $y(t) = 0$ *for all* $t \in [0,T]$ *(respectively, for all* $t \geq 0$*) implies that* $x(t) = 0$ *for all* $t \geq 0$*.*

Definition 1.6.5. ***(finally F-observability)*** *[Olb81]*
System (1.7) *is said to be finally F-observable iff for any continuous initial function* $x_0(.) \in F$ *and* $y(t) = 0$ *for all* $t \in [0,T]$ *implies* $x(T) = 0$*.*

Notice that F-observability on $[0,T]$ implies final F-observability on $[0,T]$.

Definition 1.6.6. ***(Infinite-time-observability)*** *[Olb81]*
System (1.7) *is said to be infinite-time observable iff for any continuous initial function* $x_0(.)$ *the condition* $x(t) = 0$ *for all* $t \geq 0$ *implies that for some* $t_1 \geq 0$ *the identity* $x(t) = 0$ *for all* $t \geq t_1$ *holds.*

Definition 1.6.7. ***(Point wise observability)*** *[DM72]*
The system (1.7) *is point-wise observable, (or equivalently, observable in* $[0, T]$ *if the initial point* x_0 *can be uniquely determined from the knowledge of the input* $u(t)$*,* $\phi(t)$*, and the output* $y(t)$*.*

These notions can be transposed to the controllability as mentioned in [Olb73].
Compared to ODEs, the observability and controllability of delay systems present three main differences [Ric03], [DM72]:

- First, in the case of ODEs, the controllability describes the ability to move the internal state of a system from any initial state to any other final state at a time t_1. Whereas, for any functional model, the actual notion of controllability means to derive the system to reach a function (which here means to assign the state vector $x(t)$ from time t_1 to time $t_1 + h$.

- Second, delays introduce the notion of a minimum reaching time. Thus, one must define how many units of delays the system needs in order to reach the target therewith the usual controllability indices that correspond to reachable spaces.

- Lastly, the nature of the control law to be implemented: choosing a state-feedback law of the form $u(t) = g(x_t)$, implies that the controller has infinite dimension. Different types of possible controllers for DDEs are available; "memoryless" controls $u(t) = g(x(t))$ or point-wise delayed controls $u(t) = g(x(t), x(t - h_i))$.

1.6.2 A brief overview of some existing results

Observer and controller design theory for time-delay systems has been most widely considered in the last decade. Various methods have been used, for example, spectral decomposition [HL93], [Sal82]; matrix fractional representation [EK82], coordinate change approach [HZP02], LMI method [FSD00], reducing transformation technique [PF89], factorization approach [YZ96], and polynomial approach [Sen97]. Another type of methods used to stabilize time-delay systems is prediction-based methods which transform the problem into a non-delay system such as Smith predictor [Smi59]; Finite Spectrum Assignment (FSA) [MO79], [Zho06]; and adaptive Posicast [NA03]. However, this methods require model-based calculations which may cause unexpected errors when applied to a real system. Some of these methods will be discussed in more details in the following subsections.

1.6.2.1 Finite dimension approximations

Earlier time-delay problems in engineering systems are often solved indirectly by using approximation and/or prediction methods [Ric03] in order to treat an infinite-dimensional system as a finite-dimensional one. One of the widely used approximation method is the Padé approximation, which is a rational approximation involving the truncation of some infinite series and results in a shortened fraction for the approximation of the delay term. However, such approaches have many drawbacks e.g., their high order which can complicate the control problem. Even in the case of constant time-delay which is relatively simple, this approximation constitute a limitation inaccuracy, that can lead to unstable behaviors of the true system and induce non-minimum phase and, thus, high-gain problems. In addition, the problem of choosing the order of the approximation a priori can be difficult.

1.6.2.2 Finite spectrum assignment

Motivated by the fact that delay systems have infinite eigenvalues which makes it practically impossible to control their location in feedback stabilising problems. Spectrum assignment method has been used in order to design a closed-loop delay system with finite spectrum [MM03]. The key is to construct a linear feedback law such that the corresponding closed-loop system has a finite number (equal to the dimension of the differential equation describing the system) of eigenvalues located at an arbitrarily preassigned set of points in the complex plane while the others are automatically eliminated. Compared to the method of spectral decomposition which consist in decomposing the observer equation into a finite- and an infinite-dimensional component [Sal82] or shifting an arbitrary but finite number of eigenvalues [Pan75], the advantage of the spectrum assignment method is that it does not need the calculation of the open-loop system spectrum, which significantly simplifies the design procedure. However, one of the disadvantages stems from the fact that it is a prediction-based method, thereby it is restricted to problems with input/output delays that can be countered by a prediction. Besides, it is considerably complicated in the case of MIMO systems.

1.6.2.3 Coordinate change approach

The main idea of this method is to find a generalized coordinate change such that in the new coordinates, all the time-delay terms in the system description are associated with the output only. The condition for such coordinate change to exist is guaranteed by a rank condition on the observability matrix [HZP02].

1.6.2.4 Lyapunov framework

Observers and controllers have also been designed using the Lyapunov framework involving Algebraic Riccati Equations (AREs) [RA06], Bounded Real Lemmas (BRLs) [PT09], [XLZ06] and/or Linear Matrix Inequalities (LMIs). Over the last decades, the LMI formulation has been widely used to express sufficient conditions for the stability and stabilization of time-delay systems. Combined with other methods, different results were given e.g., [ZB07] for a class of Lipschitz nonlinear discrete-time systems independent of delay; [PCE+05], [SPP99] for uncertain linear systems; [HBC98], [ES03] for sliding-mode design; static output feedback control [CT99]; and stability analysis of time-delay systems [YW12].

1.6.2.5 factorization approach

This method has been used to solve different control problems in the finite-dimension framework. Whereas, for infinite-dimension systems, the problem becomes more complicated, because a proper stable Bezout factorisation does not, in general, exist for this class of system [KS82]. Later, it was shown that the existence of the proper stable Bezout factorisations is equivalent to the spectral controllability (or spectral observability) of the cocanonical (or canonical) realisation of a transfer function matrix and an explicit procedure for computing such a factorization was giving in [KKT86].
In the literature, one can find many methods discussing observer design problem for time-delay systems, where the observer parametrisation is achieved using the factorisation approach [YZ96], [FSD00].

1.6.2.6 Reducing transformation

Employs a 'reducing transformation' on the state of the differential delay equation (1.7) such that the transformed state satisfies an ordinary differential equation whose spectrum contains the unstable poles of the delay system. Thus, a prior condition is that the set of poles or eigenvalues for (1.7) be computed relative to any specified right-half plane. This is admittedly an onerous task, but the fact that (1.7) is known to possess at most a finite number of poles in any right-half plane, notwithstanding the countably infinite spectrum in general, hints at the possibility of using finite-dimensional state variable techniques once the unstable pole set has been delineated [PF89].

1.6.2.7 Polynomial approach

This method was used for strongly observable system to design observers by assigning the coefficient of the characteristic polynomial. The observer is constructed by solving some Bezout equations [EK82].

1.7 Conclusion

In this chapter, some important definitions and principles concerning time-delay systems in general were presented and some existing methods concerning the stability were summarized. In addition, a few reminders and definitions of observability and controllability in the case of time-delay systems were presented.

CHAPTER

2

LPV formulation to design state observers for Lipschitz systems

Contents

2.1 Introduction

Observer design for time-delay systems had received a lot of attention in the last decades. Different methods for different types of systems have been proposed in the literature. For example, we can mention, studying the eigenvalues and the eigenvectors of the linear stability matrix [Raj98]; parameterising all observers by means of the factorisation approach [YZ96]; finding a coordinate change such that in the new coordinates all the time-delay terms in the system description are associated with the output only [HZP02]; and finally, using the Lyapunov-Krasovskii approach to deduce conditions expressed in terms of Riccati equations [FSD98], [ABD99]; or in terms of LMIs [Ibr07].
On the other hand, different attempts with different time-delay types have been made. We start by constant delays [LD02], [TAN04], [ZB07]; delay varying in a range [Sha08]; delays acting on the output [GMP02]; and unknown time-varying delays [SFRS07], [GLB12].
We can distinguish two types of observers when dealing with time-delay systems depending on whether the additional information on the delay is considered or not: observers with memory [BK76] and memoryless observers [HL99], [Wu09]. At first sight, it seems that observers involving the delay value in their realizations are more important. Nevertheless, memoryless observers do not require additional memory space nor exact knowledge of the delay, thereby, they are preferred in some applications such as embedded systems.
Observers for special classes of nonlinear systems, namely Lipschitz systems, has been an ongoing subject of interest for decades. Different methods have been developed to treat this type of systems starting by systems verifying the global Lipschitz property leading to exclude common nonlinearities such as e^x or x^3. Later, this limitation was replaced by the local property provided that the operating range of the state is bounded [Raj98]. An alternative method, presented in [AK01], removes this restriction by exploiting the nonlinearity and injecting a matrix representing only the state components on which depends the nonlinearity. However, it is only applicable on systems with monotonic nonlinearities. Generalizations of this technique were proposed in [Ibr07], [ZB09a]. A different and significant idea is the output injection approach, based on canceling the nonlinearity by means of an output injection term [KI83], [KR85], [HP98], [HK96] in this case only nonlinearities depending on the measured output are considered. Another important technique is the high-gain methodology which is characterized by dominating the state-dependent nonlinearities by high-gain linear terms [GHO92], [BH91]. All of the aforementioned approaches were soon extended for time-delay systems [GMP02], [GP05], [GW03], [GLB12], [KB09], [XLZY04], [RPN04], [MMM04], [AAAHLL11], [CMM13]. In this chapter, we are interested in those involving the use of the Lipschitz property. Various conditions have been provided in terms of LMI [AM07] or algebraic Riccati equations [PR10b]. Nevertheless, the use of the Lipschitz property was proven to be restrictive and different approaches has been proposed aiming to relax the existing results. From the relatively recent results, we can mention, using the differential mean value theorem to transform the nonlinear system into a Linear Parameter Varying system [Ibr09], [ZBB08], [PR10a]. The main objective of all these results is reducing the high gain and accept classes with larger Lipschitz constants.
In this chapter, we investigate the observer design problem and we propose a less conservative observer synthesis method for nonlinear time-delay systems that exploits all the properties of the nonlinearities of the system. The idea consists in transforming the problem of estimating the state of a nonlinear system to the stability problem of a Linear Parameter Varying (LPV) system. This transformation is based on a reformulation of the Lipschitz property leading to change the dynamics of the estimation error into an LPV system. The stability analysis is fulfilled using the convexity principle and the Lyapunov stability theory. The particularity of the studied system is

that the nonlinearity is non differentiable as opposed to existing results.
In order to compare our results to the existing ones, we chose to develop an LMI condition based on the use of the classical Lipschitz property largely used in the literature to study the stability of nonlinear observers. The comparison is made for two cases, delay-independent and delay-dependent respectively. Two examples were given in which we show the superiority of our methods in terms of the tolerance to larger Lipschitz constants and/or larger delays upper bounds.

2.2 System presentation and preliminaries

In this chapter, the considered class of time-delay systems is described by the following equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x\left(t - d(t)\right) + Bf\left(x(t), x\left(t - d(t)\right)\right), \\ y(t) &= Cx(t), \\ x(t) &= x_0(t), \quad -\bar{d} \leq t \leq 0. \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ represent the state and the output vectors respectively. A, A_d, B, and C are constant matrices of appropriate dimensions. The nonlinear function $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^q$ is assumed to be γ_f-Lipschitz, i.e.,:

$$\|f(x, x_d) - f(y, y_d)\| \leq \gamma_f \left\| \begin{array}{c} x - y \\ x_d - y_d \end{array} \right\| \quad \forall\, x, y \in \mathbb{R}^n, \tag{2.2}$$

and $d(t)$ is a bounded time-varying delay verifying:

$$\begin{aligned} 0 < d(t) &\leq \bar{d}, \\ \dot{d}(t) &\leq \mu < 1. \end{aligned} \tag{2.3}$$

Let us start by introducing some important definitions and preliminaries needed for the development of the sequel sections.

Definition 2.2.1. *Consider two vectors:*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \ \ \textit{and} \ \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n.$$

For all $i = 1, ..., n$, we define an auxiliary vector $x^{y_i} \in \mathbb{R}^n$ corresponding to x and y as follows:

$$x^{y_i} = \begin{pmatrix} y_1(t) \\ \vdots \\ y_i(t) \\ x_{i+1}(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \textit{for} \quad i = 1, ..., n, \quad \textit{with} \quad x^{y_0} = x. \tag{2.4}$$

Definition 2.2.2. *Consider a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q$. Then, for all vectors $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$ and $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$, we define two functions ψ_j and ψ_j^d as follows:*

$$\psi_j\left(x^{y_{j-1}}, x^{y_j}\right) = \begin{cases} 0 & \text{if} \quad x_j = y_j \\ \frac{f\left(x^{y_{j-1}}, z\right) - f\left(x^{y_j}, z\right)}{x_j - y_j} & \text{if} \quad x_j \neq y_j \end{cases}, \tag{2.5}$$

$$\psi_j^d\left(z^{y_{j-1}}, z^{y_j}\right) = \begin{cases} 0 & \text{if} \quad z_j = y_j \\ \frac{f\left(x, z^{y_{j-1}}\right) - f\left(x, z^{y_j}\right)}{z_j - y_j} & \text{if} \quad z_j \neq y_j \end{cases}. \tag{2.6}$$

It worth noting that $\psi_j\left(x^{y_{j-1}}, x^{y_j}\right)$ and $\psi_j^d\left(z^{y_{j-1}}, z^{y_j}\right)$ depend on z and x respectively, but they were dropped out intentionally from the notation for simplicity since their absence will not effect the development of the sequel results.

Notice that the Definition 2.2.1 allows to write the nonlinear function f in the system (2.1) under the following form which is needed subsequently in the rest of this chapter:

$$\begin{aligned} f(x, x_d) - f(y, y_d) &= f(x, x_d) - f(x^{y_1}, x_d) + \ldots + f(x^{y_{j-1}}, x_d) - f(x^{y_j}, x_d) + \ldots \\ &\quad + f(x^{y_{n-1}}, x_d) - f(y, x_d) + f(y, x_d) - f(y, x_d^{y_1}) + \ldots \\ &\quad + f(y, x_d^{y_{j-1}}) - f(y, x_d^{y_j}) + \ldots + f(y, x_d^{y_{n-1}}) - f(y, y_d), \\ &= \sum_{j=1}^{j=n} \left(f\left(x^{y_{j-1}}, x_d\right) - f\left(x^{y_j}, x_d\right)\right) + \sum_{j=1}^{j=n} \left(f\left(y, x_d^{y_{j-1}}\right) - f\left(y, x_d^{y_j}\right)\right). \end{aligned}$$

Consequently, from Definition 2.2.2 we can write:

$$\begin{aligned} f(x, x_d) - f(y, y_d) &= \sum_{j=1}^{j=n} \psi_j\left(x^{y_{j-1}}, x^{y_j}\right)(x_j - y_j) + \sum_{j=1}^{j=n} \psi_j^d\left(x_d^{y_{j-1}}, x_d^{y_j}\right)(x_{dj} - y_{dj}), \\ &= \left(\sum_{j=1}^{j=n} \psi_j\left(x^{y_{j-1}}, x^{y_j}\right) e_n^T(j)\right)(x - y) + \left(\sum_{j=1}^{j=n} \psi_j^d\left(x_d^{y_{j-1}}, x_d^{y_j}\right) e_n^T(j)\right)(x_d - y_d). \end{aligned} \tag{2.7}$$

In the following, we introduce a new property, necessary to develop the rest of the chapter. The idea consists in reformulating the Lipschitz condition (2.2) to get an appropriate form for the application of our LPV-based approach.

Lemma 2.2.3. *Consider a function $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^q$. Then, the following statements are equivalent:*

- *The function f is γ_f-Lipschitz as defined in* (2.2).
- *For all $i = 1, \ldots, q$, $j = 1, \ldots, n$, there exist functions*

$$\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},$$

$$\psi_{ij}^d \ : \ \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \ \mathbb{R},$$

and constants a_{ij}, b_{ij}, a_{ij}^d, and b_{ij}^d, such that:

$$f(x, x_d) - f(y, y_d) = \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij} H_{ij} \right) (x - y) + \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij}^d H_{ij} \right) (x_d - y_d), \tag{2.8}$$

where the functions ψ_{ij} and ψ_{ij}^d are bounded as follows:

$$\begin{aligned} a_{ij} \leq \psi_{ij} \leq b_{ij}, \\ a_{ij}^d \leq \psi_{ij}^d \leq b_{ij}^d, \end{aligned} \tag{2.9}$$

and

$$\psi_{ij} \triangleq \psi_{ij}\left(x^{y_{j-1}}, x^{y_j}\right), \ \psi_{ij}^d \triangleq \psi_{ij}\left(x_d^{y_{j-1}}, x_d^{y_j}\right) \ \textit{ and } \ H_{ij} = e_q(i) e_n^T(j).$$

Proof. The proof of this lemma is inspired from [ZB13].

1) ***Sufficiency***: First, let us prove the sufficiency. Assume that for all $i = 1, \ldots, q$, $j = 1, \ldots, n$ there exist functions $\psi_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ $\psi_{ij}^d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and constants a_{ij}, a_{ij}^d, b_{ij} and b_{ij}^d so that equation (2.8) with conditions (2.9) hold for all $x, y \in \mathbb{R}^n$.

   Notice from (2.8) that:

$$f(x, x_d) - f(y, y_d) = \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij} H_{ij} \right) (x - y) + \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij}^d H_{ij} \right) (x_d - y_d).$$

   Thus

$$\begin{aligned} \|f(x, x_d) - f(y, y_d)\|^2 &= \left\| \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij} H_{ij} \right) (x - y) + \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij}^d H_{ij} \right) (x_d - y_d) \right\|^2, \\ &\leq \left[\left\| \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij} H_{ij} \right) (x - y) \right\| + \left\| \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij}^d H_{ij} \right) (x_d - y_d) \right\| \right]^2, \\ &\leq \left[\left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} |\psi_{ij}|^2 \right)^{\frac{1}{2}} \|x - y\| + \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} |\psi_{ij}^d|^2 \right)^{\frac{1}{2}} \|x_d - y_d\| \right]^2, \\ &\leq \left[2 \sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \max\left(|\gamma_{ij}|^2, |\gamma_{ij}^d|^2 \right) \right] \left\| \begin{matrix} x - y \\ x_d - y_d \end{matrix} \right\|^2, \end{aligned}$$

   where $\gamma_{ij} = \max(|a_{ij}|, |b_{ij}|)$ and $\gamma_{ij}^d = \max(|a_{ij}^d|, |b_{ij}^d|)$. Hence, the nonlinear function f is γ_f-Lipschitz with γ_f verifying:

$$\gamma_f \leq \sqrt{2 \sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \max\left(|\gamma_{ij}|^2, |\gamma_{ij}^d|^2 \right)}. \tag{2.10}$$

2) ***Necessity***: Any function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be rewritten in the form:

$$f(x, x_d) = \begin{bmatrix} f_1(x, x_d) \\ \vdots \\ f_q(x, x_d) \end{bmatrix} = \sum_{i=1}^{i=q} e_q(i) f_i(x, x_d), \quad \forall x \in \mathbb{R}^n.$$

Then, since f is γ_f-Lipschitz, we have

$$\begin{aligned} \|f(x, x_d) - f(y, y_d)\|^2 &= \sum_{i=1}^{i=q} |f_i(x, x_d) - f_i(y, y_d)|^2, \\ &\leq \gamma_f \left\| \begin{matrix} x - y \\ x_d - y_d \end{matrix} \right\|^2. \end{aligned}$$

The last inequality leads to having:

$$\|f_i(x, x_d) - f_i(y, y_d)\| \leq \gamma_f \left\| \begin{matrix} x - y \\ x_d - y_d \end{matrix} \right\|. \tag{2.11}$$

which means that each component f_i is γ_{f_i}-Lipschitz with $\gamma_{f_i} \leq \gamma_f$. In this case, functions f_i can be written in terms of two functions $\psi_{ij}, \psi_{ij}^d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as given in Definition 2.2.2 (similarly to (2.7)).

$$\begin{aligned} f_i(x, x_d) - f_i(y, y_d) = &\left(\sum_{j=1}^{j=n} \psi_{ij}\left(x^{y_{j-1}}, x^{y_j}\right) e_n^T(j) \right)(x - y) \\ &+ \left(\sum_{j=1}^{j=n} \psi_{ij}^d\left(x_d^{y_{j-1}}, x_d^{y_j}\right) e_n^T(j) \right)(x_d - y_d). \end{aligned} \tag{2.12}$$

Since f_i is γ_{f_i}-Lipschitz. Then from (2.11):

$$|\psi_{ij}\left(x^{y_{j-1}}, x^{y_j}\right)| = \frac{|f_i\left(x^{y_{j-1}}, x^{y_j}\right)|}{|x_j - y_j|} \leq \gamma_{f_i} \frac{|x_j - y_j|}{|x_j - y_j|} = \gamma_{f_i}.$$

Hence, we have the existence of (2.9) with $a_{ij} = -\gamma$ and $b_{ij} = \gamma$. In the same manner, we get $-\gamma_{f_i} \leq \psi_{ij}^d \leq \gamma_{f_i}$ and $a_{ij}^d = -\gamma$ and $b_{ij}^d = \gamma$.

This ends the proof.

□

The proposed lemma provides a detailed version of the Lipschitz property. This detailed version allows to use the LPV approach, and will lead to less conservative observer synthesis conditions. Indeed, the new notation (2.8) and (2.9) enable us to exploit all the properties of the nonlinearity and to take into consideration the bounds, a_{ij} and b_{ij}, of each component of the investigated nonlinearity, hence providing a better precision when compared to the classical Lipschitz form (2.2). Due to this manipulation of the nonlinearity, the resulting condition becomes able of distinguishing between different nonlinearities having the same Lipschitz constant with different bounds, e.g., $\sin(x)$ and $\arctan(x)$.

2.3 Observer design

In this section, we consider a Luenberger observer of the form:

$$\dot{\hat{x}} = A\hat{x}(t) + A_d\hat{x}\left(t - d(t)\right) + Bf(\hat{x}, \hat{x}_d) + K\Big(y - C\hat{x}\Big) + K_d\Big(y_d - C\hat{x}_d\Big). \tag{2.13}$$

where $\hat{x}$ is the estimate of the x. Our objective consists in finding the gain matrices (decision variables) K and K_d so that the estimation error $e = x - \hat{x}$ converges asymptotically towards zero.
The dynamics of the estimation error is then given by:

$$\dot{e} = (A - KC)e(t) + (A_d - K_dC)e\left(t - d(t)\right) + B\underbrace{\left[f(x, x_d) - f(\hat{x}, \hat{x}_d)\right]}_{\delta f}. \tag{2.14}$$

Since $f(.)$ is γ_f-Lipschitz, then following Lemma 2.2.3, there exist functions $\psi_{ij},\ \psi_{ij}^d$ where

$$\psi_{ij} \triangleq \psi_{ij}\left(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\right), \quad \psi_{ij}^d \triangleq \psi_{ij}\left(x_d^{\hat{x}_{d_{j-1}}}, x_d^{\hat{x}_{d_j}}\right),$$

and constants a_{ij}, b_{ij}, a_{ij}^d, and b_{ij}^d such that

$$f(x, x_d) - f(\hat{x}, \hat{x}_d) = \left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=n}\psi_{ij}H_{ij}\right)(x - \hat{x}) + \left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=n}\psi_{ij}^dH_{ij}\right)(x_d - \hat{x}_d),$$

and notation (2.9) is verified.
Now, define the affine matrix functions:

$$\mathcal{A}\left(\rho\right) = A + B\sum_{i=1}^{i=q}\sum_{j=1}^{j=n}\psi_{ij}H_{ij}, \quad \mathcal{A}(\rho_d) = A_d + B\sum_{i=1}^{i=q}\sum_{j=1}^{j=n}\psi_{ij}^dH_{ij}, \tag{2.15}$$

with the matrices

$$\rho = \left(\psi_{ij}\right)_{ij}, \quad \rho_d = \left(\psi_{ij}^d\right)_{ij}. \tag{2.16}$$

Consequently, the dynamics (2.14) can be rewritten as:

$$\dot{e} = \left(\mathcal{A}(\rho) - KC\right)e + \left(\mathcal{A}(\rho_d) - K_dC\right)e_d. \tag{2.17}$$

Notice that according to (2.9), the matrix parameters ρ and ρ_d belongs to bounded convex sets $\mathcal{H}_n$ and $\mathcal{H}_n^d$ respectively for which the sets of vertices are defined by:

$$\mathcal{V}_{\mathcal{H}_n} = \left\{\Phi \in \mathbb{R}^{q\times n}\ :\ \Phi_{ij} \in \{a_{ij}, b_{ij}\}\right\}, \tag{2.18}$$

$$\mathcal{V}_{\mathcal{H}_n^d} = \left\{\Phi_d \in \mathbb{R}^{q\times n}\ :\ \Phi_{ij}^d \in \left\{a_{ij}^d, b_{ij}^d\right\}\right\}. \tag{2.19}$$

In the literature, the first attempts treating nonlinear systems used the Lipschitz property to provide solutions to the stability problem. Here, we present three different methods, the first treats the nonlinearity using the Lipschitz property directly as in [Raj98] whereas the second uses a reformulation of the latter that will help us exploiting the nonlinear part more efficiently [ZB09b]. Finally, these two methods will be compared with our LPV-based approach. In the next sections, we will develop these different methods for delay-independent and delay-dependent analysis cases.

2.3.1 Delay-independent synthesis

In the delay-independent analysis, the size of the delay is not taken into consideration, and the stability is verified for all delay values. For this section, we chose to perform an analysis independent of the upper bound of the delay $\bar{d}$ but dependent on the rate μ.

2.3.1.1 Standard LMI approach

Before presenting our LPV-based design methodology, we start by this subsection where we give a generalization of the standard LMI design technique to time-delay systems. We provide a sufficient LMI condition to ensure the asymptotic convergence of the estimation error towards zero.

Theorem 2.3.1. *The system* (2.13) *is an asymptotic observer for system* (2.1) *if there exist symmetric and positive definite matrices $\mathcal{P}$ and $\mathcal{Q}$, and matrices R, R_d of appropriate dimensions and a positive scalar α so that the following LMI condition holds:*

$$\begin{bmatrix} A^T\mathcal{P} + \mathcal{P}A - C^TR^T - RC + \mathcal{Q} + \alpha\gamma_f^2 I & \mathcal{P}A - R_dC & \mathcal{P}B \\ (\star) & -(1-\mu)\mathcal{Q} + \alpha\gamma_f^2 I & 0 \\ (\star) & (\star) & -\alpha I \end{bmatrix} < 0. \tag{2.20}$$

Hence, the observer gains are given by:

$$K = \mathcal{P}^{-1}R, \quad K_d = \mathcal{P}^{-1}R_d. \tag{2.21}$$

Proof. Consider the following delay-independent and rate-dependent Lyapunov-Krasovskii functional:

$$V(e(t)) = e^T(t)\mathcal{P}e(t) + \int_{t-d(t)}^{t} e^T(s)\mathcal{Q}e(s)\mathrm{d}s. \tag{2.22}$$

The derivative of V along the trajectories of system (2.14) is of the form:

$$\begin{aligned} \dot{V} = e^T &\Big((A-KC)^T\mathcal{P} + \mathcal{P}(A-KC) + \mathcal{Q}\Big)e + 2e^T\mathcal{P}(A_d - K_dC)e_d \\ &+ 2e^T\mathcal{P}B\delta f - \Big(1-\dot{d}\Big)e_d^T\mathcal{Q}e_d. \end{aligned} \tag{2.23}$$

Since the delay verifies (2.3), we can write

$$\begin{aligned} \dot{V} \leq e^T &\Big((A-KC)^T\mathcal{P} + \mathcal{P}(A-KC) + \mathcal{Q}\Big)e + 2e^T\mathcal{P}(A_d - K_dC)e_d \\ &+ 2e^T\mathcal{P}B\delta f - (1-\mu)e_d^T\mathcal{Q}e_d. \end{aligned} \tag{2.24}$$

In addition, since the function f is Lipschitz, then for any $\alpha > 0$, we can write:

$$\alpha\gamma_f^2 \begin{bmatrix} e(t) \\ e(t-d) \end{bmatrix}^T \begin{bmatrix} e(t) \\ e(t-d) \end{bmatrix} - \alpha\delta f^T\delta f \geq 0. \tag{2.25}$$

Adding (2.25) to (2.24), we get:

$$\dot{V} \leq \begin{bmatrix} e \\ e_d \\ \delta f \end{bmatrix}^T \begin{bmatrix} (A-KC)^T\mathcal{P} + \mathcal{P}(A-KC) + \mathcal{Q} + \alpha\gamma_f^2 I & \mathcal{P}(A_d - K_dC) & \mathcal{P}B \\ (\star) & -(1-\mu)\mathcal{Q} + \alpha\gamma_f^2 I & 0 \\ (\star) & (\star) & -\alpha I \end{bmatrix} \begin{bmatrix} e \\ e_d \\ \delta f \end{bmatrix}. \tag{2.26}$$

Using the change of variables $R = \mathcal{P}K$, $R_d = \mathcal{P}K_d$, we deduce that in order to get $\dot{V} < 0$, LMI (2.20) should be verified. □

Remark 2.3.1. *The main disadvantage of the previous LMI condition lies in its limitation to nonlinearities with Lipschitz constants lesser than one. Thereby, this approach fails to provide a solution in certain applications characterized by large Lipschitz constants such as chaos synchronization in digital and analog communication systems (Chua's circuit) or in vehicle systems, in the case of aerodynamic drag force.*

It was proven that writing the nonlinearity f in a more detailed form to bring out only the states on which the nonlinearity really depends, instead of considering that the nonlinearity depends on all the states and that the nonlinearity lies in all the components of the system [ZB09a]. In this case, we can introduce two matrices $\mathcal{H}$ and $\mathcal{H}_d$ such that:

$$f(x, x_d) = f\left(\begin{bmatrix}\mathcal{H} & \mathcal{H}_d\end{bmatrix}\begin{bmatrix}x \\ x_d\end{bmatrix}\right). \tag{2.27}$$

To show how to choose these matrices, take for example the function $f(x, x_d) = (\sin(x_1)x_2 - x_{d2})$. This function can be written in the form (2.27) with $\mathcal{H} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$ and $\mathcal{H}_d = \begin{bmatrix}0 & -1\end{bmatrix}$. Notice that the choice of $\mathcal{H}$ and $\mathcal{H}_d$ is not unique but it plays an important role on the feasibility of the resulting sufficient LMI conditions (this will be discussed in more details in chapters 4 and 5).

Then we can write the Lipschitz property as:

$$\alpha\gamma_f^2 \begin{bmatrix}e(t) \\ e(t-d)\end{bmatrix}^T \begin{bmatrix}\mathcal{H}^T \\ \mathcal{H}_d^T\end{bmatrix} \begin{bmatrix}\mathcal{H} & \mathcal{H}_d\end{bmatrix} \begin{bmatrix}e(t) \\ e(t-d)\end{bmatrix} - \alpha \delta f^T \delta f \geq 0. \tag{2.28}$$

So, by adding the previous inequality to (2.24) instead of the inequality (2.25), we can get a less conservative and more detailed LMI condition as follows:

Theorem 2.3.2. *The system* (2.13) *is an asymptotic observer for system* (2.1) *if there exist symmetric and positive definite matrices $\mathcal{P}$ and $\mathcal{Q}$, and matrices R, R_d of appropriate dimensions and a positive scalar α so that the following LMI condition holds:*

$$\begin{bmatrix} A^T\mathcal{P} + \mathcal{P}A - C^TR^T - RC + \mathcal{Q} + \alpha\gamma_f^2\mathcal{H}^T\mathcal{H} & \mathcal{P}A - R_dC + \alpha\gamma_f^2\mathcal{H}^T\mathcal{H}_d & \mathcal{P}B \\ (\star) & -(1-\mu)\mathcal{Q} + \alpha\gamma_f^2\mathcal{H}_d^T\mathcal{H}_d & 0 \\ (\star) & (\star) & -\alpha I \end{bmatrix} < 0. \tag{2.29}$$

Hence, the observer gains are given by:

$$K = \mathcal{P}^{-1}R, \quad K_d = \mathcal{P}^{-1}R_d. \tag{2.30}$$

These previous theorems will be compared against a method based on reformulating the nonlinear system in the form of LPV one as we will see in the next section.

2.3.1.2 LPV-based approach

At this stage, we can state the following theorem, which provides an LMI condition for an observer design of Lipschitz systems. This condition is obtained by rewriting the nonlinearities in a more exploitable form as demonstrated earlier in Lemma 2.2.3.

Theorem 2.3.3. *The system* (2.17) *is asymptotically stable if there exist symmetric and positive definite matrices $\mathcal{P}$ and $\mathcal{Q}$, and matrices R, R_d of appropriate dimensions so that the following LMI condition holds:*

$$\begin{bmatrix} \mathcal{A}(\Phi)^T \mathcal{P} + \mathcal{P}\mathcal{A}(\Phi) - C^T R^T - RC + \mathcal{Q} & \mathcal{P}\mathcal{A}(\Phi_d) - R_d C \\ (\star) & -(1-\mu)\mathcal{Q} \end{bmatrix} < 0, \qquad \forall\, \Phi \in \mathcal{V}_{\mathcal{H}_n},\ \Phi_d \in \mathcal{V}_{\mathcal{H}_n^d}. \tag{2.31}$$

Hence, the observer gains are given by:

$$K = \mathcal{P}^{-1}R, \quad K_d = \mathcal{P}^{-1}R_d. \tag{2.32}$$

Proof. Consider the same delay-independent Lyapunov-Krasovskii functional. By calculating the derivative of V along the trajectories of (2.17), we obtain

$$\dot{V} = e^T \left((\mathcal{A}(\rho) - KC)^T \mathcal{P} + \mathcal{P}(\mathcal{A}(\rho) - KC) + \mathcal{Q} \right) e - \left(1 - \dot{d}\right) e_d^T \mathcal{Q} e_d + 2e^T \mathcal{P}(\mathcal{A}(\rho_d) - K_d C) e_d. \tag{2.33}$$

Since the delay satisfies (2.3), we can write

$$\dot{V} \le \begin{bmatrix} e \\ e_d \end{bmatrix}^T \underbrace{\begin{bmatrix} \mathcal{A}^T(\rho)\mathcal{P} - C^T K^T \mathcal{P} + \mathcal{P}\mathcal{A}(\rho) - \mathcal{P}KC + \mathcal{Q} & \mathcal{P}\mathcal{A}(\rho_d) - \mathcal{P}K_d C \\ (\star) & -(1-\mu)\mathcal{Q} \end{bmatrix}}_{\Omega(\rho,\rho_d)} \begin{bmatrix} e \\ e_d \end{bmatrix}. \tag{2.34}$$

Consequently, $\dot{V} < 0$ for all $e \neq 0$ if

$$\Omega(\rho, \rho_d) < 0, \ \ \forall\, \rho \in \mathcal{H}_n, \ \ \forall\, \rho_d \in \mathcal{H}_n^d. \tag{2.35}$$

Therefore, from the convexity principle [BGFB94] and the notation (2.32), we deduce that $\dot{V} < 0$ for all $e \neq 0$ holds if the LMI (2.31) is verified for all $\Phi \in \mathcal{V}_{\mathcal{H}_n}$ and $\Phi_d \in \mathcal{V}_{\mathcal{H}_n^d}$. □

In the last theorem, we presented an LMI condition ensuring the asymptotic stability of the estimation error around zero. The developed approach enables the design of observers for a larger class of nonlinear systems with a larger Lipschitz constant. A comparison will be provided later by means of an example.

2.3.2 Delay-dependent synthesis

Similarly to the previous subsection 2.3.1, this part presents two delay-dependant methods based on a generalization of the classical Lipschitz property and the LPV transformation proposed in Lemma 2.2.3 respectively.

2.3.2.1 Standard LMI approach

In this subsection, we use the modified Lipschitz property as we did in the time-independent case (inequality (2.28)). However, a straightforward application of the modified property is not enough in the delay-dependant case due to the presence of additional nonlinear terms in the Lyapunov-Krasovskii functional. Those terms lead to having a condition in the form of nonlinear matrix inequalities. The main contribution of this subsection is to combine the later method with the use of the Young's inequality to linearize those unwanted nonlinear terms and to get more relaxed conditions in terms of LMI ensuring the stability of the estimation error. First, let us start by presenting the main result in the following theorem:

Theorem 2.3.4. *The system* (2.13) *is an asymptotic observer for system* (2.1) *if for a predefined scalars* $\alpha > 0$ *and* $\epsilon > 0$ *there exist symmetric and positive definite matrices* $\mathcal{P}$, $\mathcal{Q}$ *and* Z, *and matrices* R, R_d *of appropriate dimensions so that the following LMI condition holds:*

$$\begin{bmatrix} A^T\mathcal{P}+\mathcal{P}A-C^TR^T-RC+\mathcal{Q}-Z+\alpha\gamma_f^2\mathcal{H}^T\mathcal{H} & \mathcal{P}A_d-R_dC+Z+\alpha\gamma_f^2\mathcal{H}^T\mathcal{H}_d & \mathcal{P}B & 0 & A^T\mathcal{P}-C^TR^T & 0 \\ (\star) & -(1-\mu)\mathcal{Q}-Z+\alpha\gamma_f^2\mathcal{H}_d^T\mathcal{H}_d & 0 & 0 & A_d^T\mathcal{P}-C^TR_d^T & 0 \\ (\star) & (\star) & -\alpha I & B^TZ & 0 & 0 \\ (\star) & (\star) & (\star) & -\frac{1}{\bar{d}^2}Z & 0 & Z \\ (\star) & (\star) & (\star) & (\star) & -\epsilon\mathcal{P} & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\frac{1}{\epsilon}\mathcal{P} \end{bmatrix} < 0. \quad (2.36)$$

Hence, the observer gains are given by:

$$K = \mathcal{P}^{-1}R, \quad K_d = \mathcal{P}^{-1}R_d. \quad (2.37)$$

Proof. Consider the delay-dependent Lyapunov-Krasovskii functional:

$$V(e(t)) = e^T(t)\mathcal{P}e(t) + \int_{t-d(t)}^{t} e^T(s)\mathcal{Q}e(s)\mathrm{d}s + \bar{d}\int_{-\bar{d}}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s\mathrm{d}\theta. \quad (2.38)$$

Calculating the derivative of V along the trajectories of (2.14), we obtain

$$\begin{aligned} \dot{V} = e^T\left((A-KC)^T\mathcal{P}+\mathcal{P}(A-KC)+\mathcal{Q}\right)e + 2e^T\mathcal{P}(A_d-K_dC)e_d + 2e^T\mathcal{P}B\delta f \\ -\left(1-\dot{d}\right)e_d^T\mathcal{Q}e_d + \bar{d}^{\,2}\dot{e}^T(t)Z\dot{e}(t) - \bar{d}\int_{t-\bar{d}}^{t}\dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s. \end{aligned} \quad (2.39)$$

The integral term in the derivative can be treated using Jensen's inequality (refer to Appendix A). The Lyapunov-Krasovskii functional candidate V is frequently used in the literature, when combined with Jensen's inequality it does not require any model transformation nor bounding techniques of cross terms which are known to induce conservatism. Such a solution was considered, for studying the stability, in different papers [GN00], [Han08a], [GP06].
Since the delay verify (2.3), we can write:

$$\begin{aligned} -\bar{d}\int_{t-\bar{d}}^{t}\dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s &\le -d(t)\int_{t-d(t)}^{t}\dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s \\ &\le \begin{bmatrix} e \\ e_d \end{bmatrix}^T \begin{bmatrix} -Z & Z \\ (\star) & -Z \end{bmatrix} \begin{bmatrix} e \\ e_d \end{bmatrix}. \end{aligned} \quad (2.40)$$

To make full use of the nonlinearity, we add the inequality (2.28) to $\dot{V}$.

In addition, by using the inequality (2.40), we get:

$$\dot{V} \leq \begin{bmatrix} e \\ e_d \\ \delta f \end{bmatrix}^T \Gamma \begin{bmatrix} e \\ e_d \\ \delta f \end{bmatrix}, \tag{2.41}$$

where

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \mathcal{P}A_d - \mathcal{P}K_dC + Z + +\alpha\gamma_f^2\mathcal{H}^T\mathcal{H}_d & \mathcal{P}B \\ (\star) & -(1-\mu)\mathcal{Q} - Z + \alpha\gamma_f^2\mathcal{H}_d^T\mathcal{H}_d & 0 \\ (\star) & (\star) & -\alpha I \end{bmatrix}$$
$$+ \bar{d}^{\,2} \begin{bmatrix} (A-KC)^T \\ (A_d - K_dC)^T \\ B^T \end{bmatrix} Z \begin{bmatrix} A - KC & A_d - K_dC & B \end{bmatrix}, \tag{2.42}$$

where $\Gamma_{11} = A^T\mathcal{P} - C^TK^T\mathcal{P} + \mathcal{P}A - \mathcal{P}KC + \mathcal{Q} - Z + \alpha\gamma_f^2\mathcal{H}^T\mathcal{H}$.
Using Schur lemma, one can see that $\Gamma < 0$ is equivalent to having:

$$\Omega = \begin{bmatrix} A^T\mathcal{P} - C^TK^T\mathcal{P} + \mathcal{P}A - \mathcal{P}KC + \mathcal{Q} - Z + \alpha\gamma_f^2\mathcal{H}^T\mathcal{H} & \mathcal{P}A_d - \mathcal{P}K_dC + Z + \alpha\gamma_f^2\mathcal{H}^T\mathcal{H}_d & \mathcal{P}B & (A-KC)^T Z \\ (\star) & -(1-\mu)\mathcal{Q} - Z + \alpha\gamma_f^2\mathcal{H}_d^T\mathcal{H}_d & 0 & (A_d - K_dC)^T Z \\ (\star) & (\star) & -\alpha I & B^T Z \\ (\star) & (\star) & (\star) & -\frac{1}{\bar{d}^2} Z \end{bmatrix} < 0. \tag{2.43}$$

Notice, the matrix Ω contains some bilinearities K^TZ and K_d^TZ. To linearize these terms, we propose to use Young's inequality (Appendix A) to separate the gains K and K_d from the matrix Z and replace the latter by $\mathcal{P}$:

$$\Omega = \Omega_1 + \underbrace{\begin{bmatrix} (A-KC)^T \\ (A_d - K_dC)^T \\ 0 \\ 0 \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 0 & 0 & 0 & Z \end{bmatrix}}_{Y} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ Z \end{bmatrix}}_{Y^T} \underbrace{\begin{bmatrix} A - KC & A - K_dC & 0 & 0 \end{bmatrix}}_{X}, \tag{2.44}$$

where

$$\Omega_1 = \begin{bmatrix} A^T\mathcal{P} - C^TK^T\mathcal{P} + \mathcal{P}A - \mathcal{P}KC + \mathcal{Q} - Z + \alpha\gamma_f^2\mathcal{H}^T\mathcal{H} & \mathcal{P}A_d - \mathcal{P}K_dC + Z + \alpha\gamma_f^2\mathcal{H}^T\mathcal{H}_d & \mathcal{P}B & 0 \\ (\star) & -(1-\mu)\mathcal{Q} - Z + \alpha\gamma_f^2\mathcal{H}_d^T\mathcal{H}_d & 0 & 0 \\ (\star) & (\star) & -\alpha I & B^T Z \\ (\star) & (\star) & (\star) & -\frac{1}{\bar{d}^2} Z \end{bmatrix}.$$

So, by using Young's Inequality, we can write:

$$\Omega \leq \Omega_1 - \begin{bmatrix} (A-KC)^T \Pi & 0 \\ (A_d - K_dC)^T \Pi & 0 \\ 0 & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} -\epsilon\Pi & 0 \\ 0 & -\frac{1}{\epsilon}\Pi \end{bmatrix}^{-1} \begin{bmatrix} \Pi(A-KC) & \Pi(A-K_dC) & 0 & 0 \\ 0 & 0 & 0 & Z \end{bmatrix}. \tag{2.45}$$

where Π is a symmetric positive definite matrix.
Using Schur lemma and choosing $\Pi = \mathcal{P}$, we can see that the condition $\Omega < 0$ is satisfied if the LMI (2.36) is fulfilled. □

In the next section, a new LMI condition based on the LPV formulation is proposed. The nonlinearity is treated differently due to the use of the reformulated Lipschitz property proposed in Lemma 2.2.3.

2.3.2.2 LPV-based approach

In the following theorem, the stability of the estimation error is guaranteed if a set of LMI conditions are feasible. The main idea of this approach is to use the reformulation of the nonlinearity f into the LPV form as demonstrated in Lemma 2.2.3 and combine it with the Young's inequality to relax the remaining bilinearities. Those bilinearities stem from different sources, in our case, the choice of the Lyapunov-Krasovskii functional and the presence of the observer.

Theorem 2.3.5. *The system* (2.17) *is asymptotically stable if for a predefined scalar* $\epsilon > 0$ *there exist symmetric and positive definite matrices* $\mathcal{P}$*,* $\mathcal{Q}$ *and* Z*, and matrices* R*,* R_d *of appropriate dimensions such that the following LMI conditions hold:*

$$\begin{bmatrix} (1,1) & \mathcal{P}\mathcal{A}(\rho_d) - R_dC + Z & 0 & \mathcal{A}^T(\rho)\mathcal{P} - C^TR^T & 0 \\ (\star) & -(1-\mu)\mathcal{Q} - Z & 0 & \mathcal{A}^T(\rho_d)\mathcal{P} - C^TR_d^T & 0 \\ (\star) & (\star) & -\frac{1}{\bar{d}^2}Z & 0 & Z \\ (\star) & (\star) & (\star) & -\epsilon\mathcal{P} & 0 \\ (\star) & (\star) & (\star) & 0 & -\frac{1}{\epsilon}\mathcal{P} \end{bmatrix} < 0,$$

$$\forall\ \rho, \varrho \in \mathcal{V}_{\mathcal{H}_n},\ \rho_d,\ \varrho \in \mathcal{V}_{\mathcal{H}_n^d}. \tag{2.46}$$

where

$$(1,1) = \mathcal{A}^T(\rho)\mathcal{P} + \mathcal{P}\mathcal{A}(\rho) - C^TR^T - RC + \mathcal{Q} - Z.$$

Hence, the observer gains are given by:

$$K = \mathcal{P}^{-1}R, \quad K_d = \mathcal{P}^{-1}R_d. \tag{2.47}$$

Proof. Consider the same delay-dependent Lyapunov-Krasovskii functional (2.38). Then, calculating the derivative of V along the trajectories of (2.17), we obtain:

$$\dot{V} = e^T\Big((\mathcal{A}(\rho) - KC)^T\mathcal{P} + \mathcal{P}(\mathcal{A}(\rho) - KC) + \mathcal{Q}\Big)e - \big(1-\dot{d}\big)e_d^T\mathcal{Q}e_d \tag{2.48}$$

$$+ 2e^T\mathcal{P}(\mathcal{A}(\rho_d) - K_dC)e_d + \bar{d}^{\,2}\dot{e}^T(t)Z\dot{e}(t) - \bar{d}\int_{t-\bar{d}}^{t}\dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s. \tag{2.49}$$

Using the inequality (2.40), $\dot{V}$ can be rewritten in the form:

$$\dot{V} \leq \begin{bmatrix} e \\ e_d \end{bmatrix}^T \Gamma(\rho, \rho_d) \begin{bmatrix} e \\ e_d \end{bmatrix}, \tag{2.50}$$

where

$$\begin{aligned}\Gamma(\rho, \rho_d) = &\begin{bmatrix} \mathcal{A}^T(\rho)\mathcal{P} - C^T K^T \mathcal{P} + \mathcal{P}\mathcal{A}(\rho) - \mathcal{P}KC + \mathcal{Q} - Z & \mathcal{P}\mathcal{A}(\rho_d) - \mathcal{P}K_d C + Z \\ (\star) & -(1-\mu)\mathcal{Q} - Z \end{bmatrix} \\ &+ \bar{d}^{\,2} \begin{bmatrix} (\mathcal{A}(\rho) - KC)^T \\ (\mathcal{A}(\rho_d) - K_d C)^T \end{bmatrix} Z \begin{bmatrix} (\mathcal{A}(\rho) - KC) & (\mathcal{A}(\rho_d) - K_d C) \end{bmatrix}. \end{aligned} \tag{2.51}$$

Using Schur lemma, we can see that $\Gamma(\rho, \rho_d) < 0$ is equivalent to having:

$$\Omega(\rho, \rho_d) < 0, \tag{2.52}$$

with $\Omega(\rho, \rho_d)$ in the form:

$$\Omega(\rho, \rho_d) = \begin{bmatrix} \mathcal{A}^T(\rho)\mathcal{P} - C^T K^T \mathcal{P} + \mathcal{P}\mathcal{A}(\rho) - \mathcal{P}KC + \mathcal{Q} - Z & \mathcal{P}\mathcal{A}(\rho_d) - \mathcal{P}K_d C + Z & (\mathcal{A}(\rho) - KC)^T Z \\ (\star) & -(1-\mu)\mathcal{Q} - Z & (\mathcal{A}(\rho_d) - K_d C)^T Z \\ (\star) & (\star) & -\frac{1}{\bar{d}^2} Z \end{bmatrix}. \tag{2.53}$$

The matrix $\Omega(\rho, \rho_d)$ contains some bilinear matrix terms that can be linearized using Young's inequality (Appendix A) as will be seen in the following notation:

$$\Omega(\rho, \rho_d) = \Omega_1(\rho, \rho_d) + \underbrace{\begin{bmatrix} -(KC)^T \\ -(K_d C)^T \\ 0 \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 0 & 0 & Z \end{bmatrix}}_{Y} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix}}_{Y^T} \underbrace{\begin{bmatrix} -KC & -K_d C & 0 \end{bmatrix}}_{X}, \tag{2.54}$$

where

$$\Omega_1 = \begin{bmatrix} \mathcal{A}^T(\rho)\mathcal{P} - C^T K^T \mathcal{P} + \mathcal{P}\mathcal{A}(\rho) - \mathcal{P}KC + \mathcal{Q} - Z & \mathcal{P}\mathcal{A}(\rho_d) - \mathcal{P}K_d C + Z & \mathcal{A}^T(\rho) Z \\ (\star) & -(1-\mu)\mathcal{Q} - Z & \mathcal{A}^T(\rho_d) Z \\ (\star) & (\star) & -\frac{1}{\bar{d}^2} Z \end{bmatrix}.$$

In this case, and for any symmetric and positive definite matrix Π, we have:

$$\Omega(\rho, \rho_d) \leq \Omega_1(\rho, \rho_d) + \frac{1}{\epsilon} \begin{bmatrix} (KC)^T \\ (K_d C)^T \\ 0 \end{bmatrix} \Pi \begin{bmatrix} KC & K_d C & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix} \Pi^{-1} \begin{bmatrix} 0 & 0 & Z \end{bmatrix}, \tag{2.55}$$

$$= \underbrace{\Omega_1(\rho, \rho_d) - \begin{bmatrix} (KC)^T \Pi & 0 \\ (K_d C)^T \Pi & 0 \\ 0 & Z \end{bmatrix} \left(\begin{bmatrix} -\epsilon \Pi & 0 \\ (\star) & -\frac{1}{\epsilon}\Pi \end{bmatrix} \right)^{-1} \begin{bmatrix} \Pi KC & \Pi K_d C & 0 \\ 0 & 0 & Z \end{bmatrix}}_{\Lambda_1(\rho, \rho_d)}. \tag{2.56}$$

By using Schur lemma again, the condition $\Lambda_1(\rho, \rho_d) < 0$ is equivalent to having:

$$\begin{bmatrix} \Omega_1(\rho, \rho_d) & \begin{bmatrix} (KC)^T\Pi & 0 \\ (K_dC)^T\Pi & 0 \\ 0 & Z \end{bmatrix} \\ (\star) & \begin{bmatrix} -\epsilon\Pi & 0 \\ (\star) & -\frac{1}{\epsilon}\Pi \end{bmatrix} \end{bmatrix} < 0. \tag{2.57}$$

Now, by choosing $\Pi = \mathcal{P}$, one can easily get LMI (2.46). Thus, in order to have $\dot{V} < 0$ for all $e \neq 0$ the LMI (2.46) should be verified for all $\Phi \in \mathcal{V}_{\mathcal{H}_n}$ and $\Phi_d \in \mathcal{V}_{\mathcal{H}_n^d}$. □

2.3.3 Comments on the LPV-based approach

In the following, we discuss in more details the proposed LPV-based approach in order to highlight the main advantages and clarify some related points.

Numerical aspects

The inequalities (2.36) and (2.46) are not considered as LMIs unless the positive scalar variable ϵ is fixed a priori. The choice of such a variable can be arbitrary, but an adequate manner to chose this scalar variable is the use of the gridding method [LF97, Remark 5]. The method consists in scaling ϵ by defining $\nu = \frac{\epsilon}{1+\epsilon}$. In order to have $\epsilon > 0$, we need to define ν in the interval $]0, 1[$. Then, we assign a uniform grid on that interval and for each grid point we look for a solution.
On the other hand, the proposed LPV-based method is more complex computation-wise, since we have to solve 2^{n^2} LMIs for an n dimensional nonlinear vector. Nevertheless, the results obtained by this method are less conservative. In addition, from the feasibility point of view, the computation of the gains of the observer is done off-line which make this method suitable for real-time applications despite the computational complexity.

Details on the LPV reformulation

We should emphasize that this chapter does not treat LPV systems. The objective is to develop an observer for nonlinear systems using a reformulation of the Lipschitz property in order to transform the problem of nonlinear observer design to the stability of quasi LPV systems.
The proposed LPV-based approaches can be looked at as an extension of the bounded jacobean based approach which is only valid for systems with differentiable nonlinearities or a generalization of the recent work of [ZB13] for time-delay systems. Nevertheless, we thought that it is important to be presented in a detailed manner including comparison with classical observer design methods based on LMI approaches. Furthermore, the presented results are essential to the next chapter, in which an observer-based controller is proposed.
Moreover, it is possible to generalize our approach by proposing a more general observer that groups LPV systems with nonlinear systems. The structure of this general observer is the following:

$$\dot{\hat{x}} = A\hat{x} + A_d\hat{x}_d + Bf(\hat{x}, \hat{x}_d) + K(y, \rho(t), t)\Big(y - C\hat{x}\Big) + K_d(y, \varrho(t), t)\Big(y_d - C\hat{x}_d\Big), \tag{2.58}$$

where ρ is a time-varying parameter and:

$$K(y, \rho(t), t) = K_0 + \sum_{(i,j)\in S} K_{ij} \frac{\delta f_i}{\delta x_j}(z),$$

$$K_d(y, \varrho(t), t) = K_{0d} + \sum_{(i,j)\in S_d} K_{dij} \frac{\delta f_i}{\delta x_{dj}}(z).$$

where

$$S = \{(i,j) : \frac{\delta f_i}{\delta x_j}(z) = g_{ij}(y, \rho(t), t),\ g_{ij} \not\equiv 0,\ \forall z \in \mathbb{R}^n\},$$

$$S_d = \{(i,j) : \frac{\delta f_i}{\delta x_{dj}}(z_d) = g^d_{ij}(y, \varrho(t), t),\ g^d_{ij} \not\equiv 0,\ \forall z_d \in \mathbb{R}^n\}.$$

In this case, a parameter-dependent Lyapunov matrix can also be used probably under some additional assumptions on the bounds of the derivatives of the parameters.
To resume, it is up to the user to choose (depending on the studied system) the adequate observer structure in order to obtain the desired performances. In addition, any method in the LPV bibliography can be applied on this general observer in order to reduce the conservatism and/or ameliorate the performance.

Details on the Lipschitz reformulation

The Lipschitz property and consequently the continuity of the nonlinearity is a fundamental assumption. Indeed, the proposed method is not applicable on discontinuous nonlinearities. However, nonlinearities with large Lipschitz constants are not considered discontinuous.
Compared to high gain approaches, both methods admit large Lipschitz constants. However, high gain observers are very sensitive to noisy output measurements and less robust than our proposed type of observers.
The proposed reformulation of Lipschitz property (Lemma 2.2.3) is able of distinguishing between functions having the same Lipschitz constant but with different bounds.

2.4 Numerical examples and comparisons

This section is devoted to show the superiority of our proposed design methodology. We consider two examples for the delay-independent and delay-dependent cases respectively. The goal is to show that the LPV-based method tolerates nonlinearities with larger values of Lipschitz constant. In the following examples, there are no optimization criteria, only the feasibility of the LMI conditions is tested.

2.4.1 Example 1: Delay-independent case

Consider the system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x\left(t - d(t)\right) + Bf(x(t), x\left(t - d(t)\right)), \\ y(t) = Cx(t). \end{cases} \tag{2.59}$$

with

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

$$f(x, x_d) = \rho\Big(|x_{d2} - x_1 - 1| - |x_{d2} - x_1 + 1|\Big)$$

Following Lemma 2.2.3, there exist functions $\psi_{ij},\ \psi_{ij}^d$ such that the nonlinearity satisfies the reformulated Lipschitz condition with

$$-2\rho \leq \psi_{11} \leq 2\rho, \quad -2\rho \leq \psi_{12}^d \leq 2\rho$$

Notice that $\gamma_f = 2\rho$ which coincides with (2.10).
The delay is time-varying in the form:

$$d(t) = \frac{\bar{d}}{2}\sin(2t) + \frac{\bar{d}}{2}.$$

To compare between the different approaches presented in the previous section, we used Matlab toolbox to solve the LMI conditions and increase the value of the Lipschitz constant until the LMIs can not return solutions. The results of the comparisons are summarized in Table 2.1. We notice that the LPV-based methodology does tolerate larger Lipschitz constants.

Method	LMI (2.20)	LMI (2.29)	LMI (2.31)
γ_f	< 1	$\approx 10^9$	$\approx 10^{12}$

Table 2.1: Comparison between the proposed delay-independent methods

2.4.2 Example 2: Delay-dependent case

In this example, we study a nonlinear system in the form (2.1), described by the following parameters:

$$A = \begin{bmatrix} 0 & 0 \\ -2 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 0.4 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

The nonlinear part is the same of Example 1. In the following table, we compare between the maximum Lipschitz constant γ_f allowable for different delay bounds $\bar{d}$. The computation of γ_f is done using two conditions, LMI (2.36) and LMI (2.46), with $\epsilon = 200$. We notice that the admissible Lipschitz constant decreases for larger values of $\bar{d}$. However, the LPV-based method tolerates larger values of γ_f as presented in Table 2.2.

Method	$\bar{d} = 0.1$	$\bar{d} = 0.5$	$\bar{d} = 0.7$
LMI (2.36)	0.98	0.94	0.88
LMI (2.46)	1.98	1.76	1.52

Table 2.2: Comparison between the proposed delay-dependent methods

2.5 Conclusion

In order to improve existing results based on classical Lipschitz property, we proposed two observer design methods, delay-independent and delay-dependent respectively. Our methods provide a relaxation to the existing results by reformulating the nonlinear system into an LPV one. This method is compared to more classical ones where the Lipschitz property was used directly to get LMI conditions ensuring the asymptotic stability of the error dynamics. We conclude from this chapter that the new reformulation of the Lipschitz property provides less restrictive LMI synthesis conditions. Indeed, from the feasibility point of view, the LPV-based method provides solutions for larger Lipschitz constants. The superiority of the LPV design method was shown through two numerical examples.

CHAPTER

3

Observer-based controller design via LPV approach

Contents

3.1 Introduction

When using the observer in closed-loop configurations, the problem of stability analysis becomes more complicated. The most common approaches for stability analysis of time-delay systems are the Lyapunov-Krasovskii functional and Razumikhin function methods. Initially, Lyapunov-Razumikhin functions were the only tool to study the stability for time-varying delay systems. However, the Razumikhin approach leads to more conservative conditions and it is inapplicable to the case of $\mathcal{H}_\infty$ control [XL94], [LM07].
There exist different sources of conservatism stemming from the choice of: Lyapunov functional, model transformations and bounding techniques, etc. In the control framework (especially the observer-based control), additional sources appear due to the presence of cross terms and multiple products. The available solutions to this problem generally involve iterative linear matrix inequality conditions [CC96], [WHS10]; constrained convex optimization conditions that involve some equality constraints to be satisfied conjointly with an LMI condition [Lie04], [SB06]; the feasibility of different dependent LMI conditions simultaneously [ID08].
In this chapter, we consider the same system presented in Chapter 2. Depending on the previous results, we aim to design a controller based on the same proposed observer. Using the Lyapunov-Krasovskii approach, delay-independent and delay-dependent stability conditions are formulated in terms of LMI. The proposed method is based on the new representation of the Lipschitz property, as presented in the previous chapter, combined with linearizing technique based on Young's inequality.
In order to show the effectiveness of our results, we compare them with other existing methods in the literature. For the continuous time case, we chose the method of [Lie04] which provides a solution to the non-convexity problem of observer-based controllers on the expense of imposing an additional condition in the form of equality. This equality condition might be considered restrictive especially under the presence of significant uncertainties. A comparison between our proposed design methodology and other existing results is then provided and validated by means of numerical examples.
Different solutions proposing purely LMI conditions were also developed. In the case of discrete-time systems, we choose to compare our results with the discrete version of the result of [Lie04] and with the work of [ID08] as well who introduced an interesting approach, of which the main idea relies on decoupling the observer design problem from the controller design one by introducing some free scalar variables that link the two separate problems. As a result, the original non-convex problem is decomposed into two separate convex issues and the stability is ensured by solving three LMI conditions simultaneously as opposed to one LMI in our case.

3.2 System presentation

In this chapter, we will study continuous nonlinear systems with delayed state, on the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x\left(t - d(t)\right) + Bf\left(x(t), x\left(t-d(t)\right)\right) + B_u u(t), \\ y(t) &= Cx(t), \\ x(t) &= x_0(t), \quad -\bar{d} \le t \le 0. \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^p$ is the output. A, A_d, B, B_u and C are constant matrices of adequate dimensions.

The delay is assumed to be time-varying and satisfying both of the following conditions:

$$\begin{aligned} 0 \le d(t) \le \bar{d}, \\ \dot{d}(t) \le \mu. \end{aligned} \tag{3.2}$$

The function $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^q$ is assumed to be γ_f-Lipschitz. Without loss of generality we assume that $f(0,0) = 0$.
Following Definition 2.2.2 there exist functions

$$\bar{\psi}_{ij} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},$$

$$\bar{\psi}_{ij}^d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},$$

and constants a_{ij}, b_{ij}, a_{ij}^d, and b_{ij}^d, such that we can always rewrite the function f as follows:

$$f(x, x_d) - f(0,0) = f(x, x_d) = \left(\sum_{j=1}^{j=n} \bar{\psi}_j e_n^T(j) \right) x + \left(\sum_{j=1}^{j=n} \bar{\psi}_j^d e_n^T(j) \right) x_d, \tag{3.3}$$

with

$$\begin{aligned} a_{ij} \le \bar{\psi}_{ij} \le b_{ij}, \\ a_{ij}^d \le \bar{\psi}_{ij}^d \le b_{ij}^d. \end{aligned} \tag{3.4}$$

and

$$\bar{\psi}_{ij} \triangleq \bar{\psi}_{ij} \left(x^{0_{j-1}}, x^{0_j} \right), \quad \psi_{ij}^d \triangleq \bar{\psi}_{ij} \left(x_d^{0_{j-1}}, x_d^{0_j} \right) \text{ and } H_{ij} = e_q(i) e_n^T(j).$$

Define

$$\mathcal{A}(\varrho) = A + B \sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \bar{\psi}_{ij} H_{ij}, \text{ with } \quad \varrho = \left(\bar{\psi}_{ij} \right)_{ij}, \tag{3.5}$$

$$\mathcal{A}(\varrho_d) = A_d + B \sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \bar{\psi}_{ij}^d H_{ij}, \text{ with } \quad \varrho_d = \left(\bar{\psi}_{ij}^d \right)_{ij}, \tag{3.6}$$

where ϱ and ϱ_d are parameters belonging to bounded convex sets respectively $\mathcal{H}_n$ and $\mathcal{H}_n^d$ for which the sets of vertices are defined by:

$$\mathcal{V}_{\mathcal{H}_n} = \left\{ \Phi \in \mathbb{R}^{q \times n} : \Phi_{ij} \in \{a_{ij}, b_{ij}\} \right\}, \tag{3.7}$$

$$\mathcal{V}_{\mathcal{H}_n^d} = \left\{ \Phi_d \in \mathbb{R}^{q \times n} : \Phi_{ij}^d \in \left\{ a_{ij}^d, b_{ij}^d \right\} \right\}. \tag{3.8}$$

3.3 Observer-based controller design

In this section, we propose to design an observer-based controller with memory for system (3.1) on the form:

$$u(t) = -L\hat{x}(t) - L_d \hat{x}(t-d), \tag{3.9}$$

with the observer dynamics given by:

$$\dot{\hat{x}} = A\hat{x}(t) + A_d \hat{x}(t - d(t)) + Bf(\hat{x}, \hat{x}_d) + K\Big(y - C\hat{x}\Big) + K_d\Big(y_d - C\hat{x}_d\Big) + B_u u(t), \tag{3.10}$$

where the gain matrices K, K_d, L and L_d are to be determined.
As we saw earlier in Chapter 2, the error dynamics ($e = x - \hat{x}$) is expressed as follows:

$$\dot{e} = \left(\mathcal{A}(\rho) - KC\right) e + \left(\mathcal{A}(\rho_d) - K_d C\right) e_d, \tag{3.11}$$

with

$$\mathcal{A}\left(\rho\right) = A + B \sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij} H_{ij}, \quad \mathcal{A}(\rho_d) = A_d + B \sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \psi_{ij}^{d} H_{ij}, \tag{3.12}$$

where ρ and ρ_d (not to be confounded with ϱ and ϱ_d) defined as:

$$\rho = \Big(\psi_{ij}\left(x^{y_{j-1}}, x^{y_j}\right)\Big)_{ij}, \quad \rho_d = \Big(\psi_{ij}^{d}\left(x_d^{y_{j-1}}, x_d^{y_j}\right)\Big)_{ij}, \tag{3.13}$$

and ρ and ρ_d belong to the bounded convex sets $\mathcal{H}_n$ and $\mathcal{H}_n^d$ respectively.

At first, we consider an augmented form containing both the dynamical model of the system and the error, in order to guarantee the convergence of the observation error and the stability of the closed-loop system simultaneously as will be seen undermentioned:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\varrho) - B_u L & B_u L \\ 0 & \mathcal{A}(\rho) - KC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} \mathcal{A}(\varrho_d) - B_u L_d & B_u L_d \\ 0 & \mathcal{A}(\rho_d) - K_d C \end{bmatrix} \begin{bmatrix} x_d \\ e_d \end{bmatrix}, \tag{3.14}$$

By defining

$$\xi = \begin{bmatrix} x \\ e \end{bmatrix}, \tag{3.15}$$

$$\mathbb{A} = \begin{bmatrix} \mathcal{A}(\varrho) & 0 \\ 0 & \mathcal{A}(\rho) \end{bmatrix}, \tag{3.16}$$

$$\mathbb{A}_d = \begin{bmatrix} \mathcal{A}(\varrho_d) & 0 \\ 0 & \mathcal{A}(\rho_d) \end{bmatrix}, \tag{3.17}$$

$$\mathbb{K} = \begin{bmatrix} B_u L & -B_u L \\ 0 & KC \end{bmatrix}, \tag{3.18}$$

$$\mathbb{K}_d = \begin{bmatrix} B_u L_d & -B_u L_d \\ 0 & K_d C \end{bmatrix}, \tag{3.19}$$

the system can be rewritten in the form:

$$\dot{\xi}(t) = \left(\mathbb{A} - \mathbb{K}\right) \xi(t) + \left(\mathbb{A}_d - \mathbb{K}_d\right) \xi\left(t - d(t)\right). \tag{3.20}$$

We aim to determine the matrices K, K_d, L and L_d simultaneously such that the new defined vector ξ converges asymptotically to zero.

As we notice, equation (3.20) has the same form of (2.17). Consequently, we can use the same methodologies of subsections (2.3.1 and 2.3.2) to retrieve delay-independent and delay-dependent stabilization criteria, respectively. Nevertheless, a straightforward application of the aforementioned methods does not directly lead to tractable LMIs. In fact, the presence of multiple product terms is one of the main challenges in the observer-based control design. In the literature, so many methods were developed to treat this kind of problems [CYLH08], [Lie04], [GLCS03], [ID08], [Ibr11]. In the next sections, we will discuss some of these solutions. At first, we will present an approach involving an equality constraint [Lie04] in addition to our own method. Furthermore, we will extend the results for the discrete-time case and present a method based on the decoupling approach developed in [ID08]. At the end, we will provide a comparison between all these solutions.

3.3.1 Delay-independent synthesis

We start by the delay-independent case. For this purpose, we consider the following Lyapunov-Krasovskii candidate:

$$V(\xi(t)) = \xi^T(t)\mathcal{P}\xi(t) + \int\limits_{t-d(t)}^{t} \xi^T(s)\mathcal{Q}\xi(s)\mathrm{d}s. \tag{3.21}$$

One of the common methodologies is to assume that the Lyapunov matrix $\mathcal{P}$ is a block-diagonal and each block corresponds to a specific part of the augmented system that is the system state and the estimation error. In this case, we can define the matrix $\mathcal{P}$ on the form:

$$\mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}. \tag{3.22}$$

The derivative of V along the trajectories of system (3.20) is:

$$\dot{V} = \xi^T \left((\mathbb{A} - \mathbb{K})^T \mathcal{P} + \mathcal{P} (\mathbb{A} - \mathbb{K}) + \mathcal{Q} \right) \xi + 2\xi^T \mathcal{P} (\mathbb{A}_d - \mathbb{K}_d) \xi_d - \left(1 - \dot{d}\right) \xi_d^T \mathcal{Q} \xi_d. \tag{3.23}$$

Thus, $\dot{V} < 0$ is verified if the following inequality condition is satisfied:

$$\begin{bmatrix} (\mathbb{A} - \mathbb{K})^T \mathcal{P} + \mathcal{P} (\mathbb{A} - \mathbb{K}) + \mathcal{Q} & \mathcal{P} (\mathbb{A}_d - \mathbb{K}_d) \\ (\star) & -(1-\mu)\mathcal{Q} \end{bmatrix} < 0 \quad \forall\, \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}. \tag{3.24}$$

By developing (3.24) we get

$$\begin{bmatrix} \phi_{11} + \phi_{11}^T + \mathcal{Q} & \phi_{12} \\ (\star) & -(1-\mu)\mathcal{Q} \end{bmatrix} < 0 \quad \forall\, \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}, \tag{3.25}$$

with

$$\phi_{11} = \begin{bmatrix} P_1\mathcal{A}(\varrho) - P_1 B_u L & P_1 B_u L \\ 0 & P_2\mathcal{A}(\rho) - P_2 K C \end{bmatrix}, \tag{3.26}$$

$$\phi_{12} = \begin{bmatrix} P_1\mathcal{A}(\varrho_d) - P_1 B_u L_d & P_1 B_u L_d \\ 0 & P_2\mathcal{A}(\rho_d) - P_2 K_d C \end{bmatrix}. \tag{3.27}$$

From this point on, we can apply different methods to solve the stabilization problem. The main difficulty is that the condition (3.25) is non-convex due to the multiple product terms P_2K, P_2K_d, P_1B_uL and $P_1B_uL_d$. The first two terms can be treated by simple change of variables $R = P_2K$, $R_d = P_2K_d$. For the last two terms, different methods were suggested in the literature. Out of these methods, we chose the one presented in [Lie04], which proposes an LMI condition subject to an equality constraint to solve the aforementioned bilinearities. Whereas, we propose a solution to avoid the equality constraint using the inequality of Young in a different way, as it will be shown later.

3.3.1.1 First method: LMI conditions under equality constraint

In this subsection, the provided results are a straightforward generalization of [Lie04] to time-delay systems. This method proposes a simple solution that involves an equality constraint besides the sufficient condition in form of LMI, from which comes the restrictiveness of such a method.

Theorem 3.3.1. *System* (3.1) *is asymptotically stable under the action of the observer-based controller* (3.9) *if there exist symmetric and positive definite matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$ and $\mathcal{Q} \in \mathbb{R}^{2n\times 2n}$, and matrices $\hat{P} \in \mathbb{R}^{m\times m}$, S, S_d, R and R_d of appropriate dimensions so that the following conditions hold:*

$$P_1 B_u = B_u \hat{P}, \tag{3.28}$$

$$\begin{bmatrix} \tilde{\Phi}_{11} + \tilde{\Phi}_{11}^T + \mathcal{Q} & \tilde{\Phi}_{12} \\ (\star) & -(1-\mu)\mathcal{Q} \end{bmatrix} < 0, \forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}, \tag{3.29}$$

with

$$\tilde{\Phi}_{11} = \begin{bmatrix} P_1\mathcal{A}(\varrho) - B_u S & B_u S \\ 0 & P_2\mathcal{A}(\rho) - RC \end{bmatrix}, \tag{3.30}$$

$$\tilde{\Phi}_{12} = \begin{bmatrix} P_1\mathcal{A}_d(\varrho) - B_u S_d & B_u S_d \\ 0 & P_2\mathcal{A}_d(\rho) - R_d C \end{bmatrix}. \tag{3.31}$$

Then, the gain matrices of the observer-based controller are given by:

$$K = P_2^{-1}R, \quad K_d = P_2^{-1}R_d, \tag{3.32}$$

$$L = \hat{P}^{-1}S, \quad L_d = \hat{P}^{-1}S_d. \tag{3.33}$$

Proof. The proof is straightforward. Remember that the inequality (3.25) implies the condition $\dot{V} < 0$. In addition, if (3.28) is verified then the cross multiple terms containing P_1 in (3.25) are replaced by new ones involving the matrix $\hat{P}$. In this case, the matrices ϕ_{11} and ϕ_{12} defined in (3.26) and (3.27) become:

$$\phi_{11} = \begin{bmatrix} P_1\mathcal{A}(\varrho) - B_u\hat{P}L & B_u\hat{P}L \\ 0 & P_2\mathcal{A}(\rho) - P_2KC \end{bmatrix},$$

$$\phi_{12} = \begin{bmatrix} P_1\mathcal{A}_d(\varrho) - B_u\hat{P}L_d & B_u\hat{P}L_d \\ 0 & P_2\mathcal{A}_d(\rho) - P_2K_dC \end{bmatrix}.$$

The new terms can be treated using the change of variables $S = \hat{P}L$ and $S_d = \hat{P}L_d$. Similarly, using $R = P_2K$ and $R_d = P_2K_d$, the cross terms containing P_2 are also linearized. Consequently, inequality (3.25) is equivalent to LMI condition (3.29), which ends the proof. □

Remark 3.3.1. *In the Theorem 3.3.1, we suppose, without loss of generality, that the matrix $B_u \in \mathbb{R}^{n\times m}$ is full column rank, this implies that the columns of matrices B_u and P_1B_u are all linear independent with $P_1 > 0$. Hence, if the* (3.28) *is satisfied for some $P_1 > 0$, the matrix $\hat{P}$ must be nonsingular.*

In the next section, we will propose an LPV-based method which involves LMI conditions only.

3.3.1.2 Second method: Judicious use of Young's relation

This subsection is devoted to present a different method based on the use of the Young's inequality in order to relax the equality condition mentioned in the first method. First, let us state the following theorem, in which only strict LMI conditions ensuring the stability of the system (3.1) via the observer-based controller (3.9) are provided.

Theorem 3.3.2. *System* (3.1) *is asymptotically stable under the action of the observer-based controller* (3.9) *if for a predefined scalar $\epsilon > 0$, there exist symmetric and positive definite matrices W_1, P_2 and $\bar{\mathcal{Q}}$, and matrices S, S_d, R and R_d of appropriate dimensions so that the following LMI conditions hold:*

$$\begin{bmatrix} \bar{\Gamma}_{11} + \bar{\Gamma}_{11}^T + \bar{\mathcal{Q}} & \bar{\Gamma}_{12} & \begin{bmatrix} B_u S & B_u S_d \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \\ (\star) & -(1-\mu)\bar{\mathcal{Q}} & 0 & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\ (\star) & (\star) & \begin{bmatrix} -\epsilon W_1 & 0 \\ 0 & -\epsilon W_1 \end{bmatrix} & 0 \\ (\star) & (\star) & (\star) & \begin{bmatrix} -\frac{1}{\epsilon} W_1 & 0 \\ 0 & -\frac{1}{\epsilon} W_1 \end{bmatrix} \end{bmatrix} < 0,$$

$$\forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d} \quad (3.34)$$

with

$$\bar{\Gamma}_{11} = \begin{bmatrix} \mathcal{A}(\varrho) W_1 - B_u S & 0 \\ 0 & P_2 \mathcal{A}(\rho) - RC \end{bmatrix}, \quad (3.35)$$

$$\bar{\Gamma}_{12} = \begin{bmatrix} \mathcal{A}(\varrho_d) W_1 - B_u S_d & 0 \\ 0 & P_2 \mathcal{A}(\rho_d) - R_d C \end{bmatrix}. \quad (3.36)$$

Then, the gain matrices of the observer-based controller are given by:

$$K = P_2^{-1} R, \quad K_d = P_2^{-1} R_d, \quad (3.37)$$

$$L = S W_1^{-1}, \quad L_d = S_d W_1^{-1}. \quad (3.38)$$

Proof. Using the Lyapunov-Krasovskii functional (3.21), notice that in condition (3.25), the gain matrices of the observer K and K_d are not multiplied in the same manner by the system matrices as the gains of the controller L and L_d, i.e., K and K_d are multiplied from the left by P_2 whereas L and L_d are multiplied by $P_1 B_u$ which leads to unavoidable bilinearities that can not be solved by change of variables without severe conservatism. Hence, in order to overcome this problem and commute P_1 and B_u, we perform a congruence transformation, on the controller part only, by pre and post-multiplying (3.25) by:

$$\begin{bmatrix} \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 \\ (\star) & \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}. \quad (3.39)$$

As a result we get:

$$\Gamma = \begin{bmatrix} \Gamma_{11} + \Gamma_{11}^T + \bar{\mathcal{Q}} & \Gamma_{12} \\ (\star) & -(1-\mu)\bar{\mathcal{Q}} \end{bmatrix} < 0, \tag{3.40}$$

with

$$\Gamma_{11} = \begin{bmatrix} \mathcal{A}(\varrho)P_1^{-1} - B_u L P_1^{-1} & B_u L \\ 0 & P_2\mathcal{A}(\rho) - RC \end{bmatrix}, \tag{3.41}$$

$$\Gamma_{12} = \begin{bmatrix} \mathcal{A}(\varrho_d)P_1^{-1} - B_u L_d P_1^{-1} & B_u L_d \\ 0 & P_2\mathcal{A}(\rho_d) - R_d C \end{bmatrix}, \tag{3.42}$$

$$\bar{\mathcal{Q}} = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} \mathcal{Q} \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix}. \tag{3.43}$$

In this case, the condition $\dot{V} < 0$ is verified if $\Gamma < 0$. But the last step and the change of variables $R = P_2 K$, $R_d = P_2 K_d$, $S = LP_1^{-1}$ and $S_d = L_d P_1^{-1}$ are not enough to get rid of all the nonlinearities in Γ. As a solution, we propose to separate the remaining multiple product terms and replace the products $B_u L$ and $B_u L_d$ by LP_1^{-1} and $L_d P_1^{-1}$ through judicious use of the Young's inequality.
Indeed, to apply the Young's relation with an appropriate manner, we write Γ as follows:

$$\Gamma = \underbrace{\begin{bmatrix} \bar{\Gamma}_{11} + \bar{\Gamma}_{11}^T + \bar{\mathcal{Q}} & \bar{\Gamma}_{12} \\ (\star) & -(1-\mu)\bar{\mathcal{Q}} \end{bmatrix}}_{\Upsilon} + \underbrace{\begin{bmatrix} B_u L & B_u L_d \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}^T}_{Y} + \underbrace{\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix}}_{Y^T} \underbrace{\begin{bmatrix} B_u L & B_u L_d \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T}_{X}. \tag{3.44}$$

Then, using the Young's inequality, we can write:

$$\Gamma \le \Upsilon + \frac{1}{\epsilon} \begin{bmatrix} B_u L & B_u L_d \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Pi \begin{bmatrix} L^T B_u^T & 0 & 0 & 0 \\ L_d^T B_u^T & 0 & 0 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \Pi^{-1} \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{3.45}$$

where Π is a symmetric positive definite matrix.
In addition, using Schur lemma, the right-hand term of the last inequality is negative (thus $\Gamma < 0$) if the following equivalent term is negative:

$$\begin{bmatrix} \Upsilon & \begin{bmatrix} B_u L \Pi_1 & B_u L_d \Pi_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \\ (\star) & \begin{bmatrix} -\epsilon\Pi & 0 \\ 0 & -\frac{1}{\epsilon}\Pi \end{bmatrix} \end{bmatrix} < 0. \tag{3.46}$$

Choosing $\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_1^{-1} \end{bmatrix}$ and defining $W_1 = P_1^{-1}$ lead to having the LMI conditions (3.34). $\square$

In this section, we presented two methods to ensure the stability of the system in the closed loop independently of the delay. In the next section, the size of the delay will be taken into consideration in the stability analysis.

3.3.2 Delay-dependent synthesis

Similarly to the previous chapter, we present two methods for the delay-dependent case. For that reason, we use the following Lyapunov-Krasovskii functional:

$$V(\xi(t)) = \xi^T(t)\mathcal{P}\xi(t) + \int\limits_{t-d(t)}^{t} \xi^T(s)\mathcal{Q}\xi(s)\mathrm{d}s + \bar{d}\int\limits_{-\bar{d}}^{0}\int\limits_{t+\theta}^{t} \dot{\xi}^T(s)Z\dot{\xi}(s)\mathrm{d}\theta. \tag{3.47}$$

The derivative of V along the trajectories of system (3.20) is:

$$\begin{aligned}\dot{V} =&\xi^T\Big((\mathbb{A}(\rho)-\mathbb{K})^T\mathcal{P} + \mathcal{P}(\mathbb{A}(\rho)-\mathbb{K}) + \mathcal{Q}\Big)\xi - (1-\dot{d})\xi_d^T\mathcal{Q}\xi_d \\ &+ 2\xi^T\mathcal{P}(\mathbb{A}(\rho_d)-\mathbb{K}_d)\xi_d + \bar{d}^{\,2}\dot{\xi}^T(t)Z\dot{\xi}(t) - \bar{d}\int\limits_{t-\bar{d}}^{t}\dot{\xi}^T(s)Z\dot{\xi}(s)\mathrm{d}s.\end{aligned} \tag{3.48}$$

Consequently, using Jensen's inequality, the integral term is bounded as follows:

$$\begin{aligned}-\bar{d}\int\limits_{t-\bar{d}}^{t}\dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s &\leq -d(t)\int\limits_{t-d(t)}^{t}\dot{e}^T(s)Z\dot{e}(s)\mathrm{d}s, \\ &\leq \begin{bmatrix} e \\ e_d\end{bmatrix}^T \begin{bmatrix} -Z & Z \\ (\star) & -Z\end{bmatrix}\begin{bmatrix} e \\ e_d\end{bmatrix}.\end{aligned} \tag{3.49}$$

From (3.49), we have:

$$\dot{V} \leq \begin{bmatrix}\xi \\ \xi_d\end{bmatrix}^T \begin{bmatrix}(\mathbb{A}(\rho)-\mathbb{K})^T\mathcal{P} + \mathcal{P}(\mathbb{A}(\rho)-\mathbb{K}) + \mathcal{Q} - Z & (\star) \\ (\mathbb{A}(\rho_d)-\mathbb{K}_d)^T\mathcal{P} + Z & -(1-\mu)\mathcal{Q} - Z\end{bmatrix}\begin{bmatrix}\xi \\ \xi_d\end{bmatrix} + \bar{d}^{\,2}\dot{\xi}^T(t)Z\dot{\xi}(t),$$

$$\forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}. \tag{3.50}$$

Now by applying Schur lemma to separate the quadratic term $\bar{d}^{\,2}\dot{\xi}^T(t)Z\dot{\xi}(t)$, we conclude that in order to get $\dot{V} < 0$, the following sufficient conditions should be verified:

$$\Omega(\rho,\rho_d) = \begin{bmatrix}(\mathbb{A}(\rho)-\mathbb{K})^T\mathcal{P} + \mathcal{P}(\mathbb{A}(\rho)-\mathbb{K}) + \mathcal{Q} - Z & (\star) & (\star) \\ (\mathbb{A}(\rho_d)-\mathbb{K}_d)^T\mathcal{P} + Z & -(1-\mu)\mathcal{Q} - Z & (\star) \\ Z(\mathbb{A}(\rho)-\mathbb{K}) & Z(\mathbb{A}(\rho_d)-\mathbb{K}_d) & -\frac{1}{\bar{d}^2}Z\end{bmatrix} < 0,$$

$$\forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d} \tag{3.51}$$

Theses conditions are not suited for stabilization purposes due to the multiple bilinear terms. In the next sections, the same solutions presented in the delay-independent case, will be applied here to get pure LMIs.

3.3.2.1 First method: LMI conditions under equality constraint

The method of [Lie04] will be implemented in the delay-dependent case. We start by introducing a theorem that ensures the stability of the system (3.1) in the closed loop.

Theorem 3.3.3. *System* (3.1) *is asymptotically stable under the action of the observer-based controller* (3.9) *if for a predefined scalar $\epsilon > 0$ there exist symmetric and positive definite matrices $\mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, $\mathcal{Q}$ and Z, and a nonsingular matrix $\hat{P}$ and matrices R, R_d, S and S_d of appropriate dimensions so that the following conditions hold:*

$$P_1 B_u = B_u \hat{P}, \tag{3.52}$$

$$\begin{bmatrix} \tilde{\Phi}_{11} + \tilde{\Phi}_{11}^T + \mathcal{Q} - Z & \tilde{\Phi}_{12} + Z & 0 & \tilde{\Phi}_{11}^T & 0 \\ (\star) & -(1-\mu)\mathcal{Q} - Z & 0 & \tilde{\Phi}_{12} & 0 \\ (\star) & (\star) & -\frac{1}{d^2} Z & 0 & Z \\ (\star) & (\star) & (\star) & -\epsilon \mathcal{P} & 0 \\ (\star) & (\star) & (\star) & 0 & -\frac{1}{\epsilon}\mathcal{P} \end{bmatrix} < 0,$$
$$\forall\, \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}, \tag{3.53}$$

with $\tilde{\Phi}_{11}$ and $\tilde{\Phi}_{12}$ are defined in (3.30) *and* (3.31) *respectively.*
Then, the gain matrices of the observer-based controller are given by:

$$K = P_2^{-1} R, \quad K_d = P_2^{-1} R_d, \tag{3.54}$$
$$L = \hat{P}^{-1} S, \quad L_d = \hat{P}^{-1} S_d. \tag{3.55}$$

Proof. Using the Lyapunov-Krasovskii candidate (3.47), the condition $\dot{V} < 0$ is verified if Ω defined in (3.51) is definite negative. We Notice that Ω can be separated into the following matrices:

$$\Omega(\rho, \rho_d) = \underbrace{\begin{bmatrix} (\mathbb{A}(\rho) - \mathbb{K})^T \mathcal{P} + \mathcal{P}(\mathbb{A}(\rho) - \mathbb{K}) + \mathcal{Q} - Z & (\star) & (\star) \\ (\mathbb{A}(\rho_d) - \mathbb{K}_d)^T \mathcal{P} + Z & -(1-\mu)\mathcal{Q} - Z & (\star) \\ 0 & 0 & -\frac{1}{d^2} Z \end{bmatrix}}_{\Omega_1}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix} \begin{bmatrix} (\mathbb{A}(\rho) - \mathbb{K}) & (\mathbb{A}(\rho_d) - \mathbb{K}_d) & 0 \end{bmatrix} + \begin{bmatrix} (\mathbb{A}(\rho) - \mathbb{K})^T \\ (\mathbb{A}(\rho_d) - \mathbb{K}_d)^T \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & Z \end{bmatrix}. \tag{3.56}$$

By Young's inequality with a symmetric matrix $\tilde{\Pi} > 0$, we have:

$$\Omega \le \Omega_1 + \frac{1}{\epsilon} \begin{bmatrix} (\mathbb{A}(\rho) - \mathbb{K})^T \\ (\mathbb{A}(\rho_d) - \mathbb{K}_d)^T \\ 0 \end{bmatrix} \tilde{\Pi} \begin{bmatrix} (\mathbb{A}(\rho) - \mathbb{K}) & (\mathbb{A}(\rho_d) - \mathbb{K}_d) & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ 0 \\ Z \end{bmatrix} \tilde{\Pi}^{-1} \begin{bmatrix} 0 & 0 & Z \end{bmatrix}. \tag{3.57}$$

Hence, $\Omega < 0$ if the right term of (3.57) is definite negative, which is equivalent, by Schur lemma, to the following inequality:

$$\begin{bmatrix} \Omega_1 & \begin{bmatrix} (\mathbb{A}(\rho) - \mathbb{K})^T \tilde{\Pi} & 0 \\ (\mathbb{A}(\rho_d) - \mathbb{K}_d)^T \tilde{\Pi} & 0 \\ 0 & Z \end{bmatrix} \\ (\star) & \begin{bmatrix} -\epsilon \tilde{\Pi} & 0 \\ 0 & -\frac{1}{\epsilon}\tilde{\Pi} \end{bmatrix} \end{bmatrix} < 0. \tag{3.58}$$

Choosing $\tilde{\Pi} = \mathcal{P}$ and using the change of variables $R = P_2 K$, $R_d = P_2 K_d$, $S = \hat{P} L$ and $S_d = \hat{P} L_d$, the inequality (3.58) becomes identical to the LMI (3.53). This means that $\dot{V} < 0$ if the LMI (3.53) is fulfilled. This ends the proof of the theorem. □

3.3.2.2 Second method: Judicious use of Young's relation

This subsection is devoted to one of the main contribution of this chapter, namely the introduction of our new delay-dependent observer-based controller design method. Indeed, as in the delay-independent case, this method is based on a judicious use of the Young's relation to linearize a classical BMI problem. Our methodology leads to less restrictive delay-dependent synthesis conditions expressed in terms of LMIs. This statement is illustrated through a numerical example given in section 3.6. Our methodology is stated in the following theorem, which provides sufficient LMI conditions ensuring the asymptotic stability of the system (3.1) in closed loop.

Theorem 3.3.4. *System* (3.1) *is asymptotically stable under the action of the observer-based controller* (3.9) *if for predefined scalars* $\epsilon, \bar{\epsilon} > 0$ *there exist symmetric and positive definite matrices* W_1, P_2, $\bar{\mathcal{Q}}$ *and* $\bar{Z} = \begin{bmatrix} \bar{Z}_1 & \bar{Z}_2 \\ (\star) & \bar{Z}_3 \end{bmatrix}$, *and matrices* R, R_d, S *and* S_d *of appropriate dimensions so that the following LMI conditions hold:*

$$
\left[\begin{array}{c:c:c:c:c}
 & \begin{bmatrix} B_uS & B_uS_d \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} & \bar{\Gamma}_{11}^T & 0 \\
\bar{\Upsilon} & 0 & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} & \bar{\Gamma}_{12}^T & 0 \\
 & 0 & 0 & 0 & \bar{Z} \\
\hdashline
(\star) & \begin{bmatrix} -\epsilon W_1 & 0 \\ 0 & -\epsilon W_1 \end{bmatrix} & 0 & \begin{bmatrix} B_uS & B_uS_d \\ 0 & 0 \end{bmatrix} & 0 \\
(\star) & (\star) & \begin{bmatrix} -\frac{1}{\epsilon}W_1 & 0 \\ 0 & -\frac{1}{\epsilon}W_1 \end{bmatrix} & 0 & 0 \\
\hdashline
(\star) & (\star) & (\star) & \begin{bmatrix} -\bar{\epsilon}W_1 & 0 \\ 0 & -\bar{\epsilon}P_2 \end{bmatrix} & 0 \\
(\star) & (\star) & (\star) & (\star) & \begin{bmatrix} -\frac{1}{\bar{\epsilon}}W_1 & 0 \\ 0 & -\frac{1}{\bar{\epsilon}}P_2 \end{bmatrix}
\end{array}\right] < 0,
$$

$$\forall\, \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}, \quad (3.59)$$

where

$$\bar{\Upsilon} = \begin{bmatrix} \bar{\Gamma}_{11} + \bar{\Gamma}_{11}^T + \bar{\mathcal{Q}} - \bar{Z} & \bar{\Gamma}_{12} + \bar{Z} & 0 \\ (\star) & -(1-\mu)\bar{\mathcal{Q}} - \bar{Z} & 0 \\ (\star) & (\star) & -\frac{1}{d^2}\bar{Z} \end{bmatrix}, \tag{3.60}$$

with $\bar{\Gamma}_{11}$, $\bar{\Gamma}_{12}$, $\bar{\mathcal{Q}}$ *already defined in Theorem 3.3.2 (equations* (3.35), (3.36) *and* (3.43) *respectively).*
Then, the gain matrices of the observer-based controller are given by:

$$K = P_2^{-1}R, \quad K_d = P_2^{-1}R_d, \tag{3.61}$$

$$L = SW_1^{-1}, \quad L_d = S_dW_1^{-1}. \tag{3.62}$$

Proof. Using the Lyapunov-Krasovskii candidate defined in (3.47), we find that the condition $\dot{V} < 0$ is verified if Ω defined in (3.51) is definite negative. To treat a part of the bilinearities, we perform a congruence transformation to commute the diagonal terms containing the control matrix B_u and the decision matrix P_1, i.e. pre and post-multiplying the right term of (3.51) by:

$$\begin{bmatrix} \underbrace{\begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix}}_{\bar{P}} & 0 & 0 \\ (\star) & \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 \\ (\star) & (\star) & \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}.$$

As a result, we obtain the equivalence between (3.51) and the following inequality:

$$\Gamma_1 = \begin{bmatrix} \Gamma_{11} + \Gamma_{11}^T + \bar{Q} - \bar{Z} & \Gamma_{12} + \bar{Z} & \Gamma_{11}^T Z \bar{P} \\ (\star) & -(1-\mu)\bar{Q} - \bar{Z} & \Gamma_{12}^T Z \bar{P} \\ (\star) & (\star) & -\frac{1}{d^2}\bar{Z} \end{bmatrix} < 0. \tag{3.63}$$

with $\bar{Z} = \bar{P} Z \bar{P}$ and Γ_{11}, Γ_{12}, $\bar{Q}$ already defined in Theorem 3.3.2 (equations (3.41), (3.42) and (3.43) respectively). Define $W_1 = P_1^{-1}$ and use the change of variables $R = P_2 K$, $R_d = P_2 K_d$, $S = L W_1$ and $S_d = L_d W_1$. Then all the diagonal terms in Γ_{11} and Γ_{12} become linear. Notice that the nonlinear terms stemming from the presence of the controller, the off-diagonal entries in Γ_{11} and Γ_{12} to be more specific, do not contain the matrix W_1 like the entries on the diagonal. The presence of W_1 is necessary to perform a change of variables and linearize those terms. In order to introduce this matrix, we will use the Young's inequality. So let us rewrite Γ_1 as follows:

$$\Gamma_1 = \bar{\Upsilon} + \bar{\Upsilon}_1 + \underbrace{\begin{bmatrix} B_u L & B_u L_d \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \bar{P} Z \begin{bmatrix} B_u L & B_u L_d \\ 0 & 0 \end{bmatrix} \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}}_{Y}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{Y^T} \underbrace{\begin{bmatrix} L^T B_u^T & 0 & 0 & 0 & \begin{bmatrix} L^T B_u^T & 0 \\ L_d^T B_u^T & 0 \end{bmatrix} Z \bar{P} \\ L_d^T B_u^T & 0 & 0 & 0 & \end{bmatrix}}_{X} < 0, \tag{3.64}$$

where $\bar{\Upsilon}$ is defined in (3.60) and

$$\bar{\Upsilon}_1 = \begin{bmatrix} 0 & 0 & \Gamma_{11}^T Z \bar{P} \\ (\star) & 0 & \Gamma_{12}^T Z \bar{P} \\ (\star) & (\star) & 0 \end{bmatrix}. \tag{3.65}$$

Using the Young's inequality with a symmetric matrix $\Pi > 0$, we can write

$$\Gamma_1 \leq \bar{\Upsilon} + \bar{\Upsilon}_1 + \frac{1}{\epsilon} \begin{bmatrix} B_u L & B_u L_d \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \bar{P} Z \begin{bmatrix} B_u L \\ 0 \end{bmatrix} & \begin{bmatrix} B_u L_d \\ 0 \end{bmatrix} \end{bmatrix} \Pi \begin{bmatrix} L^T B_u^T & 0 & 0 & 0 & \begin{bmatrix} L^T B_u^T & 0 \\ L_d^T B_u^T & 0 \end{bmatrix} Z \bar{P} \\ L_d^T B_u^T & 0 & 0 & 0 & \end{bmatrix}$$

$$+ \epsilon \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Pi^{-1} \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}. \quad (3.66)$$

Using Schur lemma, the last condition can be written as:

$$\Gamma_1 \leq \underbrace{\begin{bmatrix} \bar{\Upsilon} + \bar{\Upsilon}_1 & \begin{bmatrix} B_u L \Pi_1 & B_u L_d \Pi_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ \bar{P} Z \begin{bmatrix} B_u L \Pi_1 \\ 0 \end{bmatrix} & \begin{bmatrix} B_u L_d \Pi_2 \\ 0 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} \\ (\star) & \begin{bmatrix} -\epsilon \Pi & 0 \\ (\star) & -\frac{1}{\epsilon} \Pi \end{bmatrix} \end{bmatrix}}_{\bar{\Gamma}_1} < 0, \quad (3.67)$$

Choosing $\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & W_1 \end{bmatrix}$, help us getting rid of a part of the nonlinearities in the inequality (3.67), the one related to the controller. The other part, which is due to the presence of the matrix Z in the double integral term of V, is linearized in the same manner by rewriting $\bar{\Gamma}_1$ as follows::

$$\bar{\Gamma}_1 = \underbrace{\begin{bmatrix} \bar{\Upsilon} & \begin{bmatrix} B_u S W_1 & B_u S_d & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ (\star) & \begin{bmatrix} -\epsilon W_1 & 0 & 0 & 0 \\ (\star) & -\epsilon W_1 & 0 & 0 \\ (\star) & (\star) & -\frac{1}{\epsilon} W_1 & 0 \\ (\star) & (\star) & (\star) & -\frac{1}{\epsilon} W_1 \end{bmatrix} \end{bmatrix}}_{\bar{\Gamma}_2} + X_1^T Y_1 + Y_1^T X_1, \quad (3.68)$$

with

$$X_1^T = \begin{bmatrix} 0 \\ 0 \\ \bar{P} Z \begin{bmatrix} I & 0 \\ 0 & P_2^{-1} \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}, \quad Y_1^T = \begin{bmatrix} \bar{\Gamma}_{11}^T \\ \bar{\Gamma}_{12}^T \\ 0 \\ \begin{bmatrix} B_u S & B_u S_d \\ 0 & 0 \end{bmatrix} \\ 0 \end{bmatrix}. \quad (3.69)$$

Applying Young's inequality on (3.68), we can replace the term $\begin{bmatrix} I & 0 \\ 0 & P_2^{-1} \end{bmatrix}$ by $\bar{P} = \begin{bmatrix} W_1 & 0 \\ 0 & I \end{bmatrix}$ such that to obtain $\bar{P} Z \bar{P} = \bar{Z}$.

$$\bar{\Gamma}_1 \leq \bar{\Gamma}_2 + \begin{bmatrix} Y_1^T & X_1^T \bar{\Pi} \end{bmatrix} \begin{bmatrix} -\bar{\epsilon}\bar{\Pi} & 0 \\ (\star) & -\frac{1}{\bar{\epsilon}}\bar{\Pi} \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ \bar{\Pi} X_1 \end{bmatrix}, \tag{3.70}$$

where $\bar{\Pi}$ is a symmetric positive definite matrix. Letting $\bar{\Pi} = \begin{bmatrix} W_1 & 0 \\ (\star) & P_2 \end{bmatrix}$, our nonlinear problem can be now transformed into a linear one. Using the Schur lemma, we deduce from (3.70) that $\bar{\Gamma}_1 < 0$ if the following inequality holds:

$$\begin{bmatrix} \bar{\Gamma}_2 & \begin{bmatrix} Y_1^T & X_1^T \bar{\Pi} \end{bmatrix} \\ (\star) & \begin{bmatrix} -\bar{\epsilon}\bar{\Pi} & 0 \\ (\star) & -\frac{1}{\bar{\epsilon}}\bar{\Pi} \end{bmatrix} \end{bmatrix} < 0. \tag{3.71}$$

Consequently, the condition $\bar{\Gamma}_1 < 0$ is verified if the LMI (3.59) holds. This ends the proof. □

3.4 Extension to discrete time systems

This section is devoted to extend the previous results on observer-based controller to discrete-time case. Let us start by presenting the discrete-time version of the augmented system since all prior steps are the same.

$$\begin{aligned} \xi(t+1) &= (\mathbb{A} - \mathbb{K})\, \xi(t) + (\mathbb{A}_d - \mathbb{K}_d)\, \xi\left(t - d(t)\right), \\ \xi(t) &= \xi_0(t), \quad t \in [-\bar{d}, 0]. \end{aligned} \tag{3.72}$$

The delay $d(t)$ is a nonnegative integer and time-varying that satisfies:

$$0 \leq d(t) \leq \bar{d}, \quad \bar{d} \in \mathbb{N}^\star,$$

with the parameters defined in section 3.3 (equations (3.15)-(3.19)).
In this section, we aim to investigate the stabilization of the system (3.72) by the same type of controller $u(t)$ defined in (3.9).
We will present the delay-dependent stabilizability only. The delay-independent case can be extended easily from the continuous conditions previously presented.

3.4.1 Delay-dependent synthesis

The Lyapunov-Krasovskii functional candidate is chosen as follows:

$$V(\xi(t)) = V_1(t) + V_2(t) + V_3(t),$$

where

$$V_1(t) = \xi^T(t) \mathcal{P} \xi(t), \tag{3.73}$$

$$V_2(t) = \sum_{i=1}^{i=d(t)} \xi^T(t-i) \mathcal{Q} \xi(t-i), \tag{3.74}$$

$$V_3(t) = \sum_{j=0}^{j=\bar{d}-1} \sum_{i=1}^{i=j} \xi^T(t-i)\mathcal{Q}\xi(t-i). \tag{3.75}$$

Studying the $\Delta V = V(\xi(t+1)) - V(\xi(t))$ we get:

$$\begin{aligned}
\Delta V_1(t) &= \xi^T(t+1)\mathcal{P}\xi(t+1) - \xi^T(t)\mathcal{P}\xi(t),\\
\Delta V_2(t) &= \sum_{i=1}^{i=d(t+1)} \xi^T(t+1-i)\mathcal{Q}\xi(t+1-i) - \sum_{i=1}^{i=d(t)} \xi^T(t-i)\mathcal{Q}\xi(t-i),\\
&= \xi^T(t)\mathcal{Q}\xi(t) - \xi^T(t-d(t))\mathcal{Q}\xi(t-d(t)) + \sum_{i=1}^{i=d(t+1)-1} \xi^T(t-i)\mathcal{Q}\xi(t-i)\\
&\quad - \sum_{i=1}^{i=d(t)-1} \xi^T(t-i)\mathcal{Q}\xi(t-i),\\
&\leq \xi^T(t)\mathcal{Q}\xi(t) - \xi^T(t-d(t))\mathcal{Q}\xi(t-d(t)) + \sum_{i=1}^{i=\bar{d}-1} \xi^T(t-i)\mathcal{Q}\xi(t-i),\\
\Delta V_3(t) &= \sum_{j=0}^{j=\bar{d}-1} \sum_{i=1}^{i=j} \xi^T(t+1-i)\mathcal{Q}\xi(t+1-i) - \sum_{j=0}^{j=\bar{d}-1} \sum_{i=1}^{i=j} \xi^T(t-i)\mathcal{Q}\xi(t-i),\\
&= \bar{d}\xi^T(t)\mathcal{Q}\xi(t) - \sum_{i=1}^{i=\bar{d}-1} \xi^T(t-i)\mathcal{Q}\xi(t-i).
\end{aligned}$$

Hence, in order to get $\Delta V < 0$, the following condition should be verified:

$$\begin{bmatrix} (\mathbb{A}-\mathbb{K})^T P (\mathbb{A}-\mathbb{K}) - \mathcal{P} + \left(\bar{d}+1\right)\mathcal{Q} & (\mathbb{A}-\mathbb{K})^T P (\mathbb{A}_d - \mathbb{K}_d) \\ (\star) & -\mathcal{Q} + (\mathbb{A}_d - \mathbb{K}_d)^T P (\mathbb{A}_d - \mathbb{K}_d) \end{bmatrix} < 0,$$
$$\forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}. \tag{3.76}$$

By applying Schur lemma, (3.76) is equivalent to:

$$\begin{bmatrix} -\mathcal{P} + \left(\bar{d}+1\right)\mathcal{Q} & 0 & \begin{bmatrix} \mathcal{A}^T(\varrho)P_1 - L^T B_u^T P_1 & 0 \\ L^T B_u^T & \mathcal{A}^T(\rho)P_2 - C^T R^T \end{bmatrix} \\ (\star) & -\mathcal{Q} & \begin{bmatrix} \mathcal{A}_d^T(\varrho)P_1 - L_d^T B_u^T P_1 & 0 \\ L_d^T B_u^T & \mathcal{A}_d^T(\rho)P_2 - C^T R_d^T \end{bmatrix} \\ (\star) & (\star) & -\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{bmatrix} < 0,$$
$$\forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}. \tag{3.77}$$

Similarly to the continuous case, we will apply different methods to linearize the nonlinearities in (3.77) and solve the stabilizability problem as we will see in the next subsections. The chosen approaches are: firstly, the method of [Lie04] with the imposed equality constraint. Secondly, a method developed by [ID08] which decomposes the untractable condition (3.76) into three pure LMIs by separating the observer design issue from the controller one. Finally, our method based on Young's inequality as a relaxation technique.

3.4.1.1 First method: LMI conditions under equality constraint

In this subsection, we extend the method of [Lie04] to time-delay systems in discrete-time. Let us start by presenting sufficient conditions including an equality constraint to ensure the stability of the system (3.72).

Theorem 3.4.1. *System* (3.72) *is asymptotically stable under the action of the observer-based controller* (3.9) *if there exist symmetric and positive definite matrices* $\mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ *and* $\mathcal{Q}$ *and a nonsingular matrix* $\hat{P}$ *and matrices* S*,* S_d*,* R*, and* R_d *of appropriate dimensions so that the following conditions hold:*

$$P_1 B_u = B_u \hat{P}, \tag{3.78}$$

$$\begin{bmatrix} -\mathcal{P} + \left(\bar{d}+1\right)\mathcal{Q} & 0 & \Theta_{11}^T \\ (\star) & -\mathcal{Q} & \Theta_{12}^T \\ (\star) & (\star) & -\mathcal{P} \end{bmatrix} < 0, \forall\, \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}, \tag{3.79}$$

with

$$\Theta_{11} = \begin{bmatrix} P_1\mathcal{A}(\varrho) - B_u S & B_u S \\ 0 & P_2\mathcal{A}(\rho) - RC \end{bmatrix}, \tag{3.80}$$

$$\Theta_{12} = \begin{bmatrix} P_1\mathcal{A}_d(\varrho) - B_u S_d & B_u S_d \\ 0 & P_2\mathcal{A}_d(\rho) - R_d C \end{bmatrix}. \tag{3.81}$$

Then, the gain matrices of the observer-based controller are given by:

$$K = P_2^{-1} R, \quad K_d = P_2^{-1} R_d, \tag{3.82}$$

$$L = \hat{P}^{-1} S, \quad L_d = \hat{P}^{-1} S_d. \tag{3.83}$$

Proof. Starting by the condition (3.77) which implies $\Delta V < 0$, we find that if equation (3.78) is verified, then by simple change of variables $R = P_2 K$, $R_d = P_2 K_d$, $S = \hat{P} L$ and $S_d = \hat{P} L_d$, the inequality (3.77) become equal to the LMI (3.79), That ends the proof. □

3.4.1.2 Second method: decoupling approach

In this subsection, we extend the method developed in [ID08] to time-delay system. This method consists in introducing some free parameters in order to decompose the non-convex problem into two convex ones linked by a third LMI. In other words, the observer design problem is separated from the controller's. Before proceeding, we present a lemma that plays an important role in the analysis.

Lemma 3.4.2. *Let* P *be a symmetric and positive definite matrix, and let* α *and* β *be two positive real scalars. Then* $P > \frac{\alpha}{\beta^2} I$ *holds, if the following linear matrix inequality holds:*

$$\begin{bmatrix} P & I \\ I & (2\beta - \alpha) I \end{bmatrix} > 0. \tag{3.84}$$

Now, we will present the main result retrieved by this approach.

Theorem 3.4.3. *System (3.72) is asymptotically stable under the action of the observer-based controller (3.9) if there exist symmetric and positive definite matrices W_1, P_2 and $\bar{\mathcal{Q}}$, and matrices S, R, R_d of appropriate dimensions and α and β two positive real scalars, so that the following conditions hold:*

$$\begin{bmatrix} W_1 & I \\ I & (2\beta-\alpha)I \end{bmatrix} > 0, \tag{3.85}$$

$$\begin{bmatrix} -W_1 + \left(\bar{d}+1\right)\bar{Q}_1 & W_1\mathcal{A}^T(\varrho) - S^T B_u^T & 0 & 0 \\ (\star) & -W_1 & \mathcal{A}_d(\varrho)W_1 - B_u S_d & 0 \\ (\star) & (\star) & -\bar{Q}_1 & 0 \\ (\star) & (\star) & (\star) & -\alpha I \end{bmatrix} < 0, \tag{3.86}$$

$$\begin{bmatrix} -P_2 + \left(\bar{d}+1\right)\bar{Q}_2 & \mathcal{A}^T(\rho)P_2 - C^T R^T & 0 & \beta I \\ (\star) & -P_2 & P_2\mathcal{A}_d(\rho) - R_d C & 0 \\ (\star) & (\star) & -\bar{Q}_2 & 0 \\ (\star) & (\star) & (\star) & -W_1 \end{bmatrix} < 0, \tag{3.87}$$

$$\forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}.$$

Then, the gain matrices of the observer-based controller are given by:

$$K = P_2^{-1}R, \quad K_d = P_2^{-1}R_d, \tag{3.88}$$

$$L = SW_1^{-1}, \quad L_d = S_d W_1^{-1}. \tag{3.89}$$

Proof. We found that $\Delta V < 0$ if condition (3.77) is fulfilled. But, this condition contains different bilinear terms. Thus, to treat the terms related to the controller and commute P_1 and B_u, pre and post-multiply (3.77) by:

$$\begin{bmatrix} \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 & 0 \\ (\star) & \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 \\ (\star) & (\star) & \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}.$$

By using the change of variables $R = P_2K$, $R_d = P_2K_d$, $S = LP^{-1}$ and $S_d = L_dP^{-1}$, we get the equivalence between (3.77) and the following inequality:

$$\Lambda = \begin{bmatrix} -\bar{\mathcal{P}} + \left(\bar{d}+1\right)\bar{\mathcal{Q}} & 0 & \begin{bmatrix} P_1^{-1}\mathcal{A}^T(\varrho) - S^T B_u^T & 0 \\ L^T B_u^T & \mathcal{A}^T(\rho)P_2 - C^T R^T \end{bmatrix} \\ (\star) & -\bar{\mathcal{Q}} & \begin{bmatrix} P_1^{-1}\mathcal{A}_d^T(\varrho) - S_d^T B_u^T & 0 \\ L_d^T B_u^T & \mathcal{A}_d^T(\rho)P_2 - C^T R_d^T \end{bmatrix} \\ (\star) & (\star) & -\bar{\mathcal{P}} \end{bmatrix} < 0, \tag{3.90}$$

with

$$\bar{\mathcal{P}} = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{bmatrix},$$

$$\bar{\mathcal{Q}} = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix} \mathcal{Q} \begin{bmatrix} P_1^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

The matrix (3.90) is not totally linear yet. But before proceeding, we presume some additional conditions so that this method become applicable on our system in closed loop:

- The matrix $L_d = 0$.
- The matrix $\bar{Q}$ is diagonal and equal to $\begin{bmatrix}\bar{Q}_1 & 0\\ 0 & \bar{Q}_2\end{bmatrix}$.

Implementing those conditions in matrix (3.90), we can rearrange the elements of the matrix and rewrite it in the form:

$$\begin{bmatrix} -P_1^{-1} + \left(\bar{d}+1\right)\bar{Q}_1 & P_1^{-1}\mathcal{A}^T(\varrho) - S^T B_u^T & 0 & 0 & 0 & 0 \\ (\star) & -P_1^{-1} & \mathcal{A}_d(\varrho)P_1^{-1} - B_u S_d & B_u L & 0 & 0 \\ (\star) & (\star) & -\bar{Q}_1 & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & -P_2 + \left(\bar{d}+1\right)\bar{Q}_2 & \mathcal{A}^T(\rho)P_2 - C^T R^T & 0 \\ (\star) & (\star) & (\star) & (\star) & -P_2 & P_2\mathcal{A}_d(\rho) - R_d C \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\bar{Q}_2 \end{bmatrix} < 0. \quad (3.91)$$

According to [ID08], for any $\alpha > 0$, the left term of inequality (3.91) can be rewritten as follows:

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_1 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \Gamma \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_1 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}^T, \quad (3.92)$$

with

$$\Gamma = \begin{bmatrix} -P_1^{-1} + \left(\bar{d}+1\right)\bar{Q}_1 & P_1^{-1}\mathcal{A}^T(\varrho) - S^T B_u^T & 0 & 0 & 0 & 0 & 0 \\ (\star) & -P_1^{-1} & \mathcal{A}_d(\varrho)P_1^{-1} - B_u S_d & 0 & 0 & 0 & 0 \\ (\star) & (\star) & -\bar{Q}_1 & 0 & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & -\alpha I & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & -P_2 + \left(\bar{d}+1\right)\bar{Q}_2 + \alpha P_1 P_1 & \mathcal{A}^T(\rho)P_2 - C^T R^T & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & -P_2 & P_2\mathcal{A}_d(\rho) - R_d C \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\bar{Q}_2 \end{bmatrix}. \quad (3.93)$$

Consequently, the sufficient conditions to fulfill inequality (3.91) is $\Gamma < 0$, which holds if the following conditions are satisfied:

$$\begin{bmatrix} -P_1^{-1} + \left(\bar{d}+1\right)\bar{Q}_1 & P_1^{-1}\mathcal{A}^T(\varrho) - S^T B_u^T & 0 & 0 \\ (\star) & -P_1^{-1} & \mathcal{A}_d(\varrho)P_1^{-1} - B_u S_d & 0 \\ (\star) & (\star) & -\bar{Q}_1 & 0 \\ (\star) & (\star) & (\star) & -\alpha I \end{bmatrix} < 0, \tag{3.94}$$

$$\begin{bmatrix} -P_2 + \left(\bar{d}+1\right)\bar{Q}_2 + \alpha P_1 P_1 & \mathcal{A}^T(\rho)P_2 - C^T R^T & 0 \\ (\star) & -P_2 & P_2\mathcal{A}_d(\rho) - R_d C \\ (\star) & (\star) & -\bar{Q}_2 \end{bmatrix} < 0. \tag{3.95}$$

By defining $W_1 = P_1^{-1}$, inequality (3.94) is equivalent to the LMI (3.86). In order to linearize the inequality (3.95) with respect to its variables, let $\beta > 0$ be some independent constant such that:

$$W_1 = P_1^{-1} > \frac{\alpha}{\beta^2} I. \tag{3.96}$$

Then, by the use the result of Lemma 3.4.2, we can deduce that (3.96) holds if the following LMI holds:

$$\begin{bmatrix} P_1^{-1} & I \\ I & (2\beta - \alpha)I \end{bmatrix} > 0. \tag{3.97}$$

From conditions (3.96) and (3.95), we derive a new sufficient condition to fulfill (3.95), that is:

$$\begin{bmatrix} -P_2 + \left(\bar{d}+1\right)\bar{Q}_2 + \beta^2 P_1 & \mathcal{A}^T(\rho)P_2 - C^T R^T & 0 \\ (\star) & -P_2 & P_2\mathcal{A}_d(\rho) - R_d C \\ (\star) & (\star) & -\bar{Q}_2 \end{bmatrix} < 0. \tag{3.98}$$

By Schur lemma, the last matrix inequality is equivalent to (3.87) which ends the proof. □

Remark 3.4.1. *In this method, the decomposition approach is similar to that of Ibrir in [ID08] but the nonlinearity is treated differently for sake of comparison. Indeed, the transformation into an LPV system using Definition 2.2.2 is applied instead of the differential mean value theorem adopted by Ibrir to treat the nonlinearities as uncertainties affecting the linear dynamics. In this case, the LPV approach gives less conservative conditions. In addition, The LMI formulation using Young's inequality reduces the conservatism even more as we will see later in the example.*

Now, we proceed to present our own method to obtain an LMI condition for the observer-based controller design problem.

3.4.1.3 Third method: Judicious use of Young's relation

This subsection is devoted to extend our method to discrete-time case. The method exploits the Young's inequality to provide only one set of strict LMI conditions as opposed to the first two methods. First, we will state the following theorem summarizing the main results of the method.

Theorem 3.4.4. *System* (3.72) *is asymptotically stable under the action of the observer-based controller* (3.9) *if for a predefined scalar $\epsilon > 0$ there exist symmetric and positive definite matrices W_1, P_2 and $\bar{\mathcal{Q}}$, and matrices S, S_d, R, R_d of appropriate dimensions so that the following LMI condition holds:*

$$\begin{bmatrix} -\begin{bmatrix} W_1 & 0 \\ 0 & P_2 \end{bmatrix} + \left(\bar{d}+1\right)\bar{\mathcal{Q}} & 0 & \bar{\Theta}_{11}^T & 0 & \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ (\star) & -\bar{\mathcal{Q}} & \bar{\Theta}_{12}^T & 0 & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\ (\star) & (\star) & -\bar{\mathcal{P}} & \begin{bmatrix} B_uS & B_uS_d \\ 0 & 0 \end{bmatrix} & 0 \\ (\star) & (\star) & (\star) & \begin{bmatrix} -\epsilon W_1 & 0 \\ 0 & -\epsilon W_1 \end{bmatrix} & 0 \\ (\star) & (\star) & (\star) & (\star) & \begin{bmatrix} -\frac{1}{\epsilon}W_1 & 0 \\ 0 & -\frac{1}{\epsilon}W_1 \end{bmatrix} \end{bmatrix} < 0,$$

$$\forall\ \rho,\ \varrho \in \mathcal{V}_{\mathcal{H}_n};\ \rho_d,\ \varrho_d \in \mathcal{V}_{\mathcal{H}_n^d}, \tag{3.99}$$

with

$$\bar{\Theta}_{11} = \begin{bmatrix} \mathcal{A}(\varrho)W_1 - B_uS & 0 \\ 0 & P_2\mathcal{A}(\rho) - RC \end{bmatrix}, \tag{3.100}$$

$$\bar{\Theta}_{12} = \begin{bmatrix} \mathcal{A}(\varrho_d)W_1 - B_uS_d & 0 \\ 0 & P_2\mathcal{A}(\rho_d) - R_dC \end{bmatrix}. \tag{3.101}$$

Then, the gain matrices of the observer-based controller are given by:

$$K = P_2^{-1}R, \quad K_d = P_2^{-1}R_d, \tag{3.102}$$

$$L = SW_1^{-1}, \quad L_d = S_dW_1^{-1}. \tag{3.103}$$

Proof. Similarly to the proof of the Theorem 3.4.4, we start by the matrix (3.77). Then we perform a congruence transformation and use the change of variables $R = P_2K$, $R_d = P_2K_d$, $S = LP^{-1}$ and $S_d = L_dP^{-1}$, to obtain the matrix (3.90). The transformation acted only on diagonal terms. So, we still need to treat the off-diagonal terms B_uL and B_uL_d. For that reason, we rewrite the matrix (3.90) in such a way to bring out those terms and then apply the Young's inequality to introduce the matrix P^{-1}:

$$\Lambda = \underbrace{\begin{bmatrix} -\bar{\mathcal{P}} + \left(\bar{d}+1\right)\bar{\mathcal{Q}} & 0 & \bar{\Theta}_{11}^T \\ (\star) & -\bar{\mathcal{Q}} & \bar{\Theta}_{12}^T \\ (\star) & (\star) & -\bar{\mathcal{P}} \end{bmatrix}}_{\Upsilon} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ B_uL & B_uL_d \\ 0 & 0 \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}}_{Y}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{Y^T} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & L^TB_u^T & 0 \\ 0 & 0 & 0 & 0 & L_d^TB_u^T & 0 \end{bmatrix}}_{X}. \tag{3.104}$$

Using Young's inequality we can write

$$\Lambda \leq \Upsilon + \frac{1}{\epsilon}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ B_uL & B_uL_d \\ 0 & 0 \end{bmatrix} \Pi \begin{bmatrix} 0 & 0 & 0 & 0 & L^TB_u^T & 0 \\ 0 & 0 & 0 & 0 & L_d^TB_u^T & 0 \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Pi^{-1} \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}. \tag{3.105}$$

By using Schur lemma, the condition $\Lambda < 0$ is satisfied if the following inequality holds:

$$\begin{bmatrix} \Upsilon & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ B_uL\Pi_1 & B_uL_d\Pi_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ (\star) & \begin{bmatrix} -\epsilon\Pi & 0 \\ 0 & -\frac{1}{\epsilon}\Pi \end{bmatrix} \end{bmatrix} < 0. \tag{3.106}$$

To transform the last condition into an LMI, choose $\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix} = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & P_1^{-1} \end{bmatrix}$. Then by defining $W_1 = P_1^{-1}$ and using the change of variables $R = P_2K$, $R_d = P_2K_d$, $S = LW_1$ and $S_d = L_dW_1$, one can get the LMI (3.99). This means that the LMI (3.99) ensures the asymptotic stabilization of the system (3.72) under the action of the observer-based controller (3.9). □

3.5 Discussions: Comments and comparisons

This section provides a few comments on the used methodologies. In addition, it discusses the main contribution of this chapter and compare it with some existing results in the literature.

3.5.1 On the use of the Young's relation

It is worth noticing that by using the congruence transformation (3.39), we do not need to extract all the components containing the terms B_uL and B_uL_d, only off-diagonal entries in Γ_{11} and Γ_{12} pose problems of nonlinearity. Whereas diagonal terms $\mathcal{A}(\varrho)P_1^{-1} - B_uLP_1^{-1}$ and $\mathcal{A}(\varrho_d)P_1^{-1} - B_uL_dP_1^{-1}$ were treated easily by change of variables $S = LP_1^{-1}$ and $S_d = L_d^{-1}P_1$ respectively, without affecting the stabilizability nor the detectability of the system contrarily to some existing works in the literature [Bri08]. In fact, the authors did not use any congruence transformation, so they were obliged to face similar terms to the ones in (3.26), where the cross terms are on and off the diagonal simultaneously. So, in order to get rid of this kind of nonlinearity and treat the terms P_1B_uL and $P_1B_uL_d$, they proposed to use Young's inequality to bound all controller-related terms. This unsuitable usage of the inequality lead to separating the diagonal terms $P_1\mathcal{A}(\varrho) - P_1B_uL$ and $P_1\mathcal{A}(\varrho_d) - P_1B_uL_d$. Thus, influencing the stabilizability of the system. In other words, the system should be stable in order to find a solution to their LMI condition. In order to demonstrate this remark, and for sake of simplicity, we consider an LTI system with no delays nor perturbations. Therefore, according to the technique developed in [Bri08, Theorem 6.2.2], their proposed LMI condition is feasible if the following block-diagonal matrix is negative-definite:

$$\begin{bmatrix} -(X+X^T) & P+\begin{bmatrix} X_o^T(A-KC) & 0 \\ 0 & X_c^T A \end{bmatrix} \\ (\star) & -P \end{bmatrix}.$$

Hence, it is obvious that the matrix A need to be stable in order to find a solution to the LMI. In addition, the proposed technique helps relaxing the results of the existing LMI approaches such as using additional constraints in form of equalities and choosing the matrix of Lyapunov a priori [Lie04], [Che07] or solving several dependent LMIs simultaneously [ID08].
On the other hand, Young's inequality has been frequently used to separate multiple product terms. There are different sources for such coupling terms, i.e., the presence of uncertainties [SPP99], [CZ07]; observer-based control design methods; complicated Lyapunov-Krasovskii functionals. However, this bounding technique has not been exploited in an appropriate manner which lead to restrictive results. This technique consists in bounding nonlinear terms as follows: $\forall a, b \in \mathbb{R}^n$ and a scalar $\epsilon > 0,\ \forall R$ invertible matrix

$$2a^T b \leq \epsilon a^T R a + \frac{1}{\epsilon} b^T R^{-1} b. \tag{3.107}$$

To our knowledge, most of the papers using this inequality presume $R = I$. This choice provides a good solution in the case of norm-bounded uncertainties; in the observer design framework [AM07] or for observer-based output feedback controllers [JLM09] but it is not suitable for the bilinearities in the observer-based controller case [PSBP13]. A Different value of the matrix R was used in [CSL98] to deal with a memoryless state feedback controller but it involves a model transformation which introduce additional dynamics into the transformed system. Another interesting result involves $R = P^2$ was presented in [BBZ12] in the case of observers for one sided Lipschitz systems, providing additional condition on the Lyapunov matrix.

3.5.2 On the LMI conditions under equality constraint

In this part, we will provide some remarks on the feasibility of the LMI condition (3.29) subject to the equality constraint (3.28). For sake of simplicity, let us consider a linear time-delay system in the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x\left(t - d(t)\right) + B_u u(t), \\ y(t) &= Cx(t), \\ x(t) &= x_0(t), \quad -\bar{d} \leq t \leq 0, \end{aligned} \tag{3.108}$$

where $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ and $B_u = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$. For a matrix $P_1 = \begin{bmatrix} p_1 & p_2 \\ (\star) & p_3 \end{bmatrix}$ ((3.22)), the equality constraint proposed by [Lie04] ($P_1 B_u = B_u \hat{P}$) lead to having $p_2 = 0$. Thus, the matrix P_1 is diagonal.
If the LMI (3.29) is verified, we deduce that $\tilde{\Phi}_{11} + \tilde{\Phi}_{11}^T + \mathcal{Q} < 0$. Since $\mathcal{Q} > 0$, the feasibility of the LMI (3.29) implies $\tilde{\Phi}_{11} + \tilde{\Phi}_{11}^T < 0$.
Let $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$. By computing the last condition, we get:

$$\tilde{\Phi}_{11} + \tilde{\Phi}_{11}^T = \begin{bmatrix} (1.1) & (1.2) \\ (\star) & (2.2) \end{bmatrix} < 0, \tag{3.109}$$

where

$$(1.1) = p_1(A_1 - B_1L_1) + (A_1 - B_1L_1)^T p_1, \tag{3.110}$$
$$(1.2) = p_2(A_2 - B_1L_2) + A_3^T p_3, \tag{3.111}$$
$$(2.2) = p_3A_4 + A_4^T p_3. \tag{3.112}$$

Thus, A_4 should be Hurwitz diagonally stable in order to verify $p_3A_4 + A_4^T p_3 < 0$. As a result, with the solution of [Lie04], we find that in addition to the stabilizability and the detectability of the studied system, the matrix A_4 is Hurwitz diagonally stable is also a necessary condition for the feasibility of the LMI (3.29).

3.6 Numerical Examples

This section is devoted to provide a comparaison between the diffrent proposed methods. We will concentrate on the maximum Lipschitz constant allowed by each method. The value of γ_f will be incresed until the LMIs give no solutions.

3.6.1 Example 1: Delay-independent case

In this part, we present a nonlinear time-delay system and we will show how our designed controller is able to stabilize the system:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 0.4 & 0 \end{bmatrix} \tag{3.113}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The delay is time-varying $d(t) = \frac{\bar{d}}{2}\sin(2t) + \frac{\bar{d}}{2}$. The nonlinearity is on the form:

$$f(x, x_d) = \rho\left(|x_{d1} - x_1 - 1| - |x_{d1} - x_1 + 1|\right). \tag{3.114}$$

The nonlinearity f has the bounds:

$$|\psi_{11}| \leq 2\rho, \quad |\psi_{11}^d| \leq 2\rho.$$

Thus $\gamma_f = 2\rho$. The LMI condition (3.29) is infeasible for the proposed system. This can be proven in light of the subsection 3.5.2. The matrix $B_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the same form as discussed in 3.5.2. Then, the condition $p_3A_4 + A_4^T p_3 < 0$ should be verified in order to find a solution to the LMI (3.29). Since we have $A_4 = 0.1$, then p_3 should be definite negative which contradicts the definition of the matrix $P_1 > 0$.

On the other hand, the LMI (3.34) is solvable in Matlab. In the following table, we present the maximum Lipschitz constant γ_f allowable for different delay rates μ with a fixed $\epsilon = 200$.

Method	$\mu = 0.1$	$\mu = 0.5$	$\mu = 0.7$
LMI (3.34)	47	39	32

Table 3.1: Comparison of Lipschitz constant γ_f of the proposed LPV-based method

3.6.2 Example 2: Delay-dependent case

This example is inspired by [MPKL01], which considers a liquid monopropellant rocket motor with a pressure feedback system with the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.115}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Similarly to Example 1, we use the same delay d and nonlinearity f functions. By solving the two LMI conditions (3.53) and (3.59), for different values of $\bar{d}$ and $\epsilon = \bar{\epsilon} = 100$, we can compute the maximum allowable Lipschitz constant γ_f for each method as can be seen in Table 3.2.

Method	$\bar{d} = 0.1$	$\bar{d} = 0.5$	$\bar{d} = 0.7$
LMI (3.53)	0.98	0.94	0.88
LMI (3.59)	71	59	49

Table 3.2: Comparison between the proposed delay-dependent methods

The results presented in the table show the superiority of the new method over the ones retrieved by means of the equality constraint.

3.6.3 Example 3: Discrete-time case

In this example, we consider a discrete nonlinear time delay system of the form:

$$A = \begin{bmatrix} 0.9 & 0.5 \\ 0.8 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.3 & 0 \\ 0.8 & 0.5 \end{bmatrix} \tag{3.116}$$

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_u = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly to previous examples, we use the same delay d but the nonlinearity f of the form $\rho \sin(x_1 + x_{1d})$. In this case, the nonlinearity f has the bounds:

$$|\psi_{11}| \leq \rho, \quad |\psi_{11}^d| \leq \rho, \quad \gamma_f = \rho.$$

A simple comparison between the proposed methods in sections 3.4.1.2 and 3.4.1.3, for $\epsilon = 300$, provides us with the following results, which show the superiority of the LPV-based method.

Method	$\bar{d} = 0.3$	$\bar{d} = 0.5$	$\bar{d} = 0.7$
Theorem 3.4.3	0.21	0.19	0.17
Theorem 3.4.4	0.25	0.23	0.22

Table 3.3: Comparison between the proposed delay-dependent methods

It is worth reminding that in Theorem 3.4.3, for sake of comparison, the nonlinearity was treated in the same manner used in our approach, using the LPV formulation (Remark 3.4.1), to which due the close results of both methods.

3.7 Conclusion

In this chapter, new observer-based controller methods for nonlinear time-delay systems have been presented. In this method, the system with nonlinearity is transformed into an LPV system and the non-convex problem was treated using Young's inequality in a different way than that proposed in the literature. By proposing an appropriate Lyapunov-Krasovskii functional, delay-independent and delay-dependent conditions were given respectively in the form of LMIs. In addition, the conditions were extended to treat discrete time systems. Furthermore, in order to to show the effectiveness of our work, we proposed a comparison with two different methods. The first involves LMI conditions subject to an equality constraint. The second, introduces some free parameters to separate the observer design issue from the controller design issue leading to three dependent LMI conditions needed to be valid simultaneously. Finally, the results of the two proposed conditions were illustrated through some examples.

CHAPTER

4

$\mathcal{W}^{1,2}$ Observer design for singular time-delay systems

Contents

4.1 Introduction

Great attention has been paid toward singular systems with and without time-delay. In general, several design procedures have been proposed to study the problem of stability, stabilization and robustness [XL06], [DZH96], [DBB08], [JSC06], [XL04], [WM94]. One of the related difficulties lies in the fact that the standard Lyapunov stability theory cannot be applied directly to singular systems. The reason is that the quadratic Lyapunov function or functional for singular systems may not be positive definite. In addition, while studying the stability problem for singular systems, one should investigate the regularity and absence of impulses (for continuous systems) and causality (for discrete systems) simultaneously. Nevertheless, many results developed for regular systems have been extended to singular systems. Some of the available results made use of the LMI formulation by incorporating either non-strict LMIs [LH05] or strict LMIs [Pea07], [UI99].

In this chapter, we address the problem of designing an observer for a class of nonlinear time-delay systems in descriptor form in the presence of disturbances in both the dynamics of the system and the output vector. Both discrete-time and continuous-time cases are investigated. In addition, two different approaches to deal with the state estimation are proposed. Our interest in the descriptor form is due to the related difficulty when dealing with such a representation and the desire of providing a different solution from existing results. In addition, the sensitivity of descriptor systems to slight input changes, and the bad effect of the presence of disturbances or unknown inputs on the design of observers have motivated us to propose two approaches in order to work-around the presence of the disturbances in the term of the estimation error. Like most of the preceding works on this subject, the nonlinear function should verify the Lipschitz property. Furthermore, the construction of the LMI is based on a recent methodology for Lipschitz systems, firstly proposed by [AK01] and later generalized by [ZB09a]. This modified condition results in new LMI conditions capable of treating functions with larger Lipschitz constants. Indeed, it is worth mentioning that large Lipschitz constants represent the main limitation of the existing results. The proposed observer design method leads to solve an LMI condition ensuring the robust convergence of the estimation error to zero. The first proposed method presents a special Lyapunov-Krasovskii functional depending on the disturbances to avoid the presence of the disturbance's derivatives, which ensures, when a certain LMI is satisfied, the $\mathcal{H}_\infty$ convergence of the estimation error. The second method proposes a new criterion of robustness based on Sobolev norm, inspired by [BC95], in which the authors used the Sobolev space in place of the Lebesgue space $\mathcal{L}_2$ for obtaining local input-output stability results. Moreover, they defined "local $\mathcal{W}$-stability" and showed that all nice properties of $\mathcal{L}_2$ are still satisfied by Sobolev space $\mathcal{W}^{1,2}$. In addition, they studied the relationship between what they called $\mathcal{W}$-stability and asymptotic stability. Resorting to Sobolev spaces has already been reported in the literature, for example, in [Ale07] an optimal estimation problem and the relationship between the internal stability and the input-output stability of the Sobolev type is studied.

4.2 Problem formulation

Many popular physical process in different fields are described in descriptor form. In this chapter, we are proposing an observer design method for singular time-delay systems. So, the systems under consideration have the following form:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-d) + Bf\left(x(t), x(t-d)\right) + E_\omega \omega(t), \\ y(t) &= Cx(t) + D_\omega \omega(t), \\ x(t) &= x_0(t), \quad -d \leq t \leq 0. \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^{n_x}$, $\omega \in L_2^r$, and $y \in \mathbb{R}^p$, are the state, the disturbances and the output vectors respectively. $E \in \mathbb{R}^{n \times n_x}$, $A, A_d \in \mathbb{R}^{n \times n_x}$, $B \in \mathbb{R}^{n \times q}$, $E_\omega \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{p \times n_x}$, and $D_\omega \in \mathbb{R}^{p \times r}$ are constant matrices of appropriate dimensions. d is a known positive delay, $x_0(t)$ is the initial condition. The nonlinear function $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \longrightarrow \mathbb{R}^q$ is assumed, without loss of generality, to be differentiable with respect to its arguments. Indeed, if the f is not differentiable, but only Lipschitz, we can use the reformulated Lipschitz property introduced in Chapter 2. Without loss of generality, we can always write f under the following detailed form:

$$\begin{aligned} f\left(x, x(t-d)\right) &= \begin{pmatrix} f_1\left(\mathcal{H}_1 x(t), \mathcal{H}_1^d x(t-d)\right) \\ \vdots \\ f_q\left(\mathcal{H}_q x(t), \mathcal{H}_q^d x(t-d)\right) \end{pmatrix}, \\ &= \sum_{i=1}^{i=q} e_q(i) f_i \left(\mathcal{H}_i x(t), \mathcal{H}_i^d x(t-d)\right). \end{aligned} \tag{4.2}$$

Remark 4.2.1. *The matrices $\mathcal{H}_i$ and $\mathcal{H}_i^d$ have been injected into the system to deal with the more general case where the considered nonlinearity does not depend on all the states and/or does not lies in all the components of the system [ZB09a]. Indeed, they were introduced to select only the state components on which depends the nonlinear function f_i. Obviously, if each component of the system has different nonlinear components and each nonlinear component depends on every state variable then the matrices $\mathcal{H}_i$ and $\mathcal{H}_i^d$ are equal to the identity matrix of size n_x.*

Remark 4.2.2. *There is no condition required on how to choose the matrices $\mathcal{H}_i$ and $\mathcal{H}_i^d$. They represent selection matrices, therefore, and without loss of generality, we choose them to be full row rank.*
For example, let us consider f of the form:

$$\begin{aligned} f\left(x, x(t-d)\right) &= \begin{pmatrix} \sin(x_1(t)x_2(t-d)) \\ \sin(x_2(t)) \end{pmatrix}, \\ &= \sum_{i=1}^{i=2} e_2(i) f_i \left(\mathcal{H}_i x(t), \mathcal{H}_i^d x(t-d)\right). \end{aligned}$$

with

$$\mathcal{H}_1 = [1 \quad 0],\ \mathcal{H}_2 = [0 \quad 1],$$

$$\mathcal{H}_1^d = [0 \quad 1],\ \mathcal{H}_2^d = [0 \quad 0].$$

$$f_1\left(\mathcal{H}_1 x(t), \mathcal{H}_1^d x(t-d)\right) = \sin(x_1(t)x_2(t-d)),$$

$$f_2\left(\mathcal{H}_2 x(t), \mathcal{H}_2^d x(t-d)\right) = \sin(x_2(t)).$$

Before proceeding to the design of the observer, we state two necessary assumptions the studied system should verify:

Assumption 4.2.1.

- *The following rank condition is assumed to be fulfilled:*

$$\text{rank}\begin{pmatrix} E \\ C \end{pmatrix} = n_x \text{ with } n_x \leq n + p. \tag{4.3}$$

- *The nonlinear function f is assumed to be Lipschitz, i.e.,*

$$\|f(x(t), x(t-d)) - f(\hat{x}(t), \hat{x}(t-d))\| \leq \gamma_f \left\| \begin{pmatrix} x(t) - \hat{x}(t) \\ x(t-d) - \hat{x}(t-d) \end{pmatrix} \right\|. \tag{4.4}$$

Since f is differentiable with respect to its arguments, then we can reformulate the condition (4.4) *as follows:*

$$a_{ij} \leq \frac{\partial f_i}{\partial \zeta_j(t)}(\zeta(t), w(t)) \leq b_{ij}, \ \ \forall\, \zeta(t) \in \mathbb{R}^{s_i}, \ \ \forall\, w(t) \in \mathbb{R}^{r_i}, \tag{4.5}$$

$$a_{ij}^d \leq \frac{\partial f_i}{\partial \zeta_j(t)}(v(t), \zeta(t)) \leq b_{ij}^d, \ \ \forall\, \zeta(t) \in \mathbb{R}^{r_i}, \ \ \forall\, v(t) \in \mathbb{R}^{s_i}. \tag{4.6}$$

Those conditions coincides with what has been presented in the Lemma 2.2.3 of Chapter 2, on the reformulation of the Lipschitz property. Here, the function f is differential, so ψ_{ij} and ψ_{ij}^d defined in (2.5)-(2.6) *become the partial derivative of f with respect to the first and the second arguments respectively.*

As proven in the Lemma 2.2.3, conditions (4.5)-(4.6) *imply that the differentiable function f is γ_f-Lipschitz with*

$$\gamma_f \leq \sqrt{2 \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \max\left(\max\left(|a_{ij}|^2, |b_{ij}|^2\right), \max\left(|a_{ij}^d|^2, |b_{ij}^d|^2\right)\right)}. \tag{4.7}$$

Remark 4.2.3. *Descriptor form* (4.1) *can be found, for instance, in the case of systems with unknown inputs where the estimation of the unknown inputs and the states is done simultaneously. This can be employed in the domain of fault diagnosis (fault detection, fault isolation) [CS03], [GH06]. Indeed, let us consider the following Lipschitz nonlinear time-delay system with unknown inputs and disturbances:*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + B_u u(t) + B_u^d u(t-d) \\ &\quad + Bf\left(x(t), u(t), x(t-d), u(t-d)\right) + E_\omega \omega(t), \\ y(t) &= Cx(t) + Du(t) + D_\omega \omega(t), \\ x(t) &= x_0(t), \quad -d \leq t \leq 0. \end{aligned} \tag{4.8}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the unknown input, and $\omega(t) \in L_2^r$ is the vector of bounded disturbances. $A, A_d \in \mathbb{R}^{n\times n}; B_u, B_u^d \in \mathbb{R}^{n\times m}; B \in \mathbb{R}^{n\times q}, E_\omega \in \mathbb{R}^{n\times r}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$ and $D_\omega \in \mathbb{R}^{p\times r}$ are constant matrices of adequate dimensions. d is a positive delay, $x_0(t)$ is the initial condition. In the above system, the unknown input u can refer to a sensor or actuator fault to be detected.

Let us define the following notation:

$$E = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & D \end{bmatrix}, \tag{4.9a}$$

$$\Xi_1 = \begin{bmatrix} A & B_u \end{bmatrix}, \quad \Xi_1^d = \begin{bmatrix} A_d & B_u^d \end{bmatrix}, \tag{4.9b}$$

$$\mathcal{H}_i = \begin{bmatrix} H_i & 0 \\ 0 & F_i \end{bmatrix}, \quad \mathcal{H}_i^d = \begin{bmatrix} H_i^d & 0 \\ 0 & F_i^d \end{bmatrix}, \tag{4.9c}$$

$$\xi(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{4.9d}$$

In this case, in order to the condition rank $\begin{pmatrix} E \\ \mathcal{C} \end{pmatrix} = n + m$ *be verified, the matrix* D *must be of full column rank, i.e. rank*$(D) = m$.
We can rewrite the dynamics (4.8) *under the aforementioned descriptor form* (4.1) *and the nonlinear function* f *in the form* (4.2)*:*

$$\begin{aligned}
&E\dot{\xi}(t) = \Xi_1\xi(t) + \Xi_1^d\xi(t-d) + B\sum_{i=1}^{i=q} e_q(i) f_i\left(\mathcal{H}_i\xi(t), \mathcal{H}_i^d\xi(t-d)\right) + E_\omega\omega(t),\\
&y(t) = \mathcal{C}\xi(t) + D_\omega\omega(t),\\
&\xi(t) = \xi_0(t), \quad -d \le t \le 0.
\end{aligned}$$

Remark 4.2.4. *Notice that* s_i *and* r_i *represent the number of rows of the matrices* $\mathcal{H}_i$ *and* $\mathcal{H}_i^d$*, respectively. Hence, each function* f_i *depends on* $s_i + r_i$ *variables* $\zeta_j(t)$ *and* $w_j(t)$*, for* $i = 1, \ldots, q$ *and* $j = 1, \ldots, s_i$*. Indeed, we have* $f_i(\mathcal{H}_i x(t), \mathcal{H}_i^d x(t-d))$.

Remark 4.2.5. *We assume, without loss of generality, that* f *satisfies* (4.5) *and* (4.6) *with* $a_{ij} = 0$, $\forall i = 1, ..., q$, $\forall j = 1, ..., s$ *where* $s = \max_{1\le i\le q}(s_i)$ *and* $a_{kl}^d = 0$, $\forall k = 1, ..., q$, $\forall l = 1, ..., r$*, where* $r = \max_{1\le k\le q}(r_k)$*. Indeed, if there exist subsets* $S_1, S_1^d \subset \{1, ..., q\}$, $S_2 \subset \{1, ..., s\}$ *and* $S_2^d \subset \{1, ..., r\}$ *such that* $a_{ij} \neq 0$, $\forall (i,j) \in S_1 \times S_2$ *and* $a_{kl}^d \neq 0$, $\forall (k,l) \in S_1^d \times S_2^d$*. Then, we can rewrite the system* (4.1) *on the form:*

$$\begin{aligned}
E\dot{x}(t) &= \underbrace{\left(A + B\sum_{(i,j)\in S_1\times S_2} a_{ij}H_{ij}\mathcal{H}_i\right)}_{\tilde{A}} x(t)\\
&+ \underbrace{\left(A_d + B\sum_{(k,l)\in S_1^d\times S_2^d} a_{kl}^d H_{kl}^d\mathcal{H}_k^d\right)}_{\tilde{A}_d} x(t-d) + E_\omega\omega(t)\\
&+ \underbrace{B\left(f(x, x_d) - \left(\sum_{(i,j)\in S_1\times S_2} a_{ij}H_{ij}\mathcal{H}_i\right)x(t) - \left(\sum_{(k,l)\in S_1^d\times S_2^d} a_{kl}^d H_{kl}^d\mathcal{H}_k^d\right)x(t-d)\right)}_{\tilde{f}(x(t),x(t-d))},\\
&= \tilde{A}x(t) + \tilde{A}_d x(t-d) + E_\omega\omega(t) + B\tilde{f}(x(t), x(t-d)).
\end{aligned} \tag{4.10}$$

where

$$H_{ij} = e_q(i)e_{s_i}^T(j) \ \textit{ and } \ H_{kl}^d = e_q(k)e_{r_k}^T(l).$$

In this case, the nonlinearity $\tilde{f}$ satisfies the inequalities (4.5) *and* (4.6) *with:*

$$\tilde{a}_{ij} = 0,\ \tilde{a}_{ij}^d = 0, \tilde{b}_{ij} \ = \ b_{ij} - a_{ij} \textit{ and } \tilde{b}_{ij}^d \ = \ b_{ij}^d - a_{ij}^d.$$

since we have:

$$\begin{aligned}
\frac{\partial \tilde{f}_i}{\partial \zeta_j(t)}(\zeta(t), w(t)) &= \frac{\partial f_i}{\partial \zeta_j(t)}(\zeta(t), w(t)) - \frac{\partial}{\partial \zeta_j(t)}\left(\sum_{(i,j)\in S_1\times S_2} a_{ij}H_{ij}\mathcal{H}_i\right),\\
&\geq \frac{\partial f_i}{\partial \zeta_j(t)}(\zeta(t), w(t)) - a_{ij} \geq a_{ij} - a_{ij} = 0.
\end{aligned}$$

We get the same result for $\frac{\partial f_i}{\partial \zeta_j(t)}(v(t), \zeta(t))$.
This remark will be of use, subsequently, when we prove the existence of the LMI condition.

As mentioned above, our aim is to design a state observer in order to estimate robustly asymptotically the state $x(t)$ despite the presence of the disturbances $\omega(t)$. Thus, taking into consideration the descriptor form of the system, we propose an observer with the following structure:

$$\begin{cases}
\dot{\upsilon}(t) = & \Pi_1\upsilon(t) + \Pi_1^d\upsilon(t-d) + \Pi_2 y(t) + \Pi_2^d y(t-d)\\
& +\digamma \sum\limits_{i=1}^{i=q} Be_q(i)f_i\left(\mathcal{H}_i\hat{x}(t), \mathcal{H}_i^d\hat{x}(t-d)\right),\\
\hat{x}(t) = & \upsilon(t) + \daleth y(t).
\end{cases} \tag{4.11}$$

where $\hat{x}$ denotes the estimate of x. The gain matrices Π_1, Π_1^d, Π_2 and Π_2^d are to be determined. A necessary condition for the existence of the matrices $\digamma$ and $\daleth$ (and consequently Π_1, Π_1^d, Π_2 and Π_2^d as we will see later in Theorem 4.3.1) is that the matrix $\begin{bmatrix} E \\ C \end{bmatrix}$ is of full column rank (4.3). The objective is to determine the matrices Π_1, Π_1^d, Π_2 and Π_2^d, so that the estimation error $e(t) = \hat{x}(t) - x(t)$ converges robustly asymptotically towards zero, i.e.,

$$\lim_{t\to\infty} e(t) = 0 \qquad \text{for } \ \omega(.) = 0 \tag{4.12}$$

$$\|e\|_{L_2^{n_x}} \leq \lambda \|\omega\|_{L_2^r} \qquad \forall\, \omega(t) \neq 0,\ e(0) = 0 \tag{4.13}$$

where $\lambda > 0$ is the disturbance attenuation level to be minimized. Now, let us compute $\dot{e}$:

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = \dot{\upsilon}(t) + \daleth\dot{y}(t) - \dot{x}(t). \tag{4.14}$$

If the matrix $\begin{bmatrix} E \\ C \end{bmatrix}$ is of full column rank then there exist matrices $\digamma \in \mathbb{R}^{n_x\times n}, \daleth \in \mathbb{R}^{n_x\times p}$ such that:

$$\digamma E + \daleth C = I_{n_x}.$$

Those matrices are not unique and can be computed using the pseudo-inverse of $\begin{bmatrix} E \\ C \end{bmatrix}$:

$$\begin{bmatrix} \digamma & \daleth \end{bmatrix} = \left(\begin{bmatrix} E \\ C \end{bmatrix}^T \begin{bmatrix} E \\ C \end{bmatrix}\right)^{-1} \begin{bmatrix} E \\ C \end{bmatrix}^T.$$

Hence, since $\dot{y}(t) = C\dot{x}(t) + D_\omega \dot{\omega}(t)$, we can write

$$\begin{aligned} \dot{e}(t) &= \dot{v}(t) + \daleth \dot{y}(t) - (\digamma E + \daleth C)\,\dot{x}(t), \\ &= \dot{v}(t) - \digamma E\dot{x}(t) + \daleth D_\omega \dot{\omega}(t). \end{aligned} \tag{4.15}$$

By exploiting (4.1) and (4.11), we obtain

$$\begin{aligned} \dot{e}(t) = {} & \Pi_1 e(t) + \Pi_1^d e(t-d) + \sum_{i=1}^{i=q} \digamma B e_q(i) \delta f_i + \daleth D_\omega \dot{\omega}(t) \\ & + (\Pi_1 + (\Pi_2 - \Pi_1 \daleth)\, C - \digamma A)\, x(t) + \left(\Pi_1^d + \left(\Pi_2^d - \Pi_1^d \daleth\right) C - \digamma A_d\right) x(t-d) \\ & + ((\Pi_2 - \Pi_1 \daleth)\, D_\omega - \digamma E_\omega)\, \omega(t) + \left(\left(\Pi_2^d - \Pi_1^d \daleth\right) D_\omega\right) \omega(t-d). \end{aligned} \tag{4.16}$$

where

$$\delta f_i = f_i\left(\hat{v}(t), \hat{w}(t)\right) - f_i\left(v(t), w(t)\right).$$

Using the DMVT (Differential Mean Value theorem), we deduce that there exist $z_i \in Co(v, \hat{v})$, $z_i^d \in Co(w, \hat{w})$ so that:

$$\delta f_i = \sum_{j=1}^{j=s_i} h_{ij}(t) e_{s_i}^T(j) \mathcal{H}_i e(t) + \sum_{j=1}^{j=r_i} h_{ij}^d(t) e_{r_i}^T(j) \mathcal{H}_i^d e(t-d). \tag{4.17}$$

where

$$h_{ij}(t) = \frac{\partial f_i}{\partial v_j}\left(z_i(t), w(t)\right), \tag{4.18}$$

$$h_{ij}^d(t) = \frac{\partial f_i}{\partial w_j}\left(v(t), z_i^d(t)\right). \tag{4.19}$$

This form matches the one we obtained in Chapter 2 (Lemma 2.2.3). According to (4.17), and using the following notation:

$$\mathcal{G} = \begin{bmatrix} \daleth D_\omega & 0_{(n_x)\times r} \end{bmatrix}, \tag{4.20}$$

$$\bar{\omega} = \begin{bmatrix} \omega(t) \\ \omega(t-d) \end{bmatrix}, \tag{4.21}$$

and

$$\daleth_\omega = \begin{bmatrix} (\Pi_2 - \Pi_1 \daleth) D_\omega - \digamma E_\omega & (\Pi_2^d - \Pi_1^d \daleth) D_\omega \end{bmatrix}, \tag{4.22}$$

we can rewrite (4.16) as follows:

$$\begin{aligned} \dot{e}(t) = {} & \Pi_1 e(t) + \Pi_1^d e(t-d) + \daleth_\omega \bar{\omega}(t) + \mathcal{G}\dot{\bar{\omega}}(t) \\ & + (\Pi_1 + (\Pi_2 - \Pi_1 \daleth)\, C - \digamma A)\, x(t) + \left(\Pi_1^d + \left(\Pi_2^d - \Pi_1^d \daleth\right) C - \digamma A_d\right) x(t-d) \\ & + \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} h_{ij}(t) \digamma B H_{ij} \mathcal{H}_i e(t) + \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} h_{ij}^d(t) \digamma B H_{ij}^d \mathcal{H}_i^d e(t-d). \end{aligned} \tag{4.23}$$

Since (4.23) depends on $\bar{\omega}$, then we need to rewrite the condition (4.13) with respect to $\bar{\omega}$. In fact, it is easy to show that:

$$\|\bar{\omega}\|_{L_2^{2r}}^2 = \|\omega\|_{L_2^r}^2 + \|\omega_d\|_{L_2^r}^2 = 2\,\|\omega\|_{L_2^r}^2 + \int_{-d}^{0} \omega^T(t)\omega(t)\mathrm{d}t.$$

Hence, the inequality (4.13) becomes:

$$\|e\|_{L_2^{nx}} \leq \frac{\lambda}{\sqrt{2}} \left(\|\bar{\omega}\|^2_{L_2^{2r}} - \int_{-d}^{0} \omega^T(t)\omega(t)\mathrm{d}t \right)^{\frac{1}{2}}, \quad \forall\, \omega(t) \neq 0,\; e(0) = 0. \tag{4.24}$$

To simplify, assume that $\omega(t) = 0, \forall t \in [-d, 0]$. Thus: $\int_{-d}^{0} \omega^T(t)\omega(t)\mathrm{d}t = 0$.

If this condition is not verified, inequality (4.13) should be replaced by the following to get rid of the integral part.

$$\|e\|_{L_2^{nx}} \leq \lambda \sqrt{\|\omega\|^2_{L_2^r} + \frac{1}{2}\int_{-d}^{0} \omega^T(t)\omega(t)\mathrm{d}t}, \quad \forall\, \omega(t) \neq 0,\; e(0) = 0.$$

Consequently, given the system (4.8) and the observer (4.11), the $\mathcal{H}_\infty$ filtering design is to determine the matrices Π_1, Π_1^d, Π_2 and Π_2^d so that:

$$\lim_{t\to\infty} e(t) = 0, \text{for } \bar{\omega}(t) = 0 \tag{4.25}$$

$$\|e\|_{L_2^{nx}} \leq \frac{\lambda}{\sqrt{2}} \|\bar{\omega}\|_{L_2^{2r}}, \forall\, \bar{\omega}(t) \neq 0,\; e(0) = 0. \tag{4.26}$$

The problem of $\mathcal{H}_\infty$ filtering design can be reduced to finding a Lyapunov function $V(t)$ such that

$$W(t) = \dot{V}(t) + e^T(t)e(t) - \frac{\lambda^2}{2}\bar{\omega}^T(t)\bar{\omega}(t) < 0. \tag{4.27}$$

It is easy to show that (4.27) implies (4.25) and (4.26). Notice that for $\bar{\omega}(t) = 0$, if (4.27) is verified then $\dot{V} < 0$ and consequently the estimation error converges asymptotically to zero based on the Lyapunov theory and thus we get (4.25).

Whereas, if $\bar{\omega}(t) \neq 0$ and $e(0) = 0$, then from (4.27) we get:

$$V(t) + \int_0^t e^T(s)e(s)\mathrm{d}s - \frac{\lambda^2}{2}\int_0^t \bar{\omega}^T(s)\bar{\omega}(s)\mathrm{d}s \; < 0. \tag{4.28}$$

Since $V(e(t)) \geq 0$ for all $t \geq 0$, then for $t \to +\infty$, we obtain:

$$\int_0^\infty e^T(s)e(s)\mathrm{d}s \leq \frac{\lambda^2}{2}\int_0^\infty \bar{\omega}^T(s)\bar{\omega}(s)\mathrm{d}s. \tag{4.29}$$

which is equivalent to (4.26).

4.3 $\mathcal{H}_\infty$ performance analysis

The content of this section consists of proposing a new observer synthesis method for a class of nonlinear time-delay system in continuous case. The following theorem summarizes the main result.

Theorem 4.3.1. *For a prescribed $\lambda > 0$, the $\mathcal{H}_\infty$ filtering design problem corresponding to the system (4.8) and the observer (4.11) is solvable, with the $\mathcal{H}_\infty$ performance level less than λ, if there exist matrices $\mathcal{P} = \mathcal{P}^T > 0$, $\mathcal{Q} = \mathcal{Q}^T > 0$, R, and R_d of adequate dimensions so that the following conditions are feasible.*

1. *The observer gain matrices Π_1, Π_1^d, Π_2 and Π_2^d are given by:*

$$\Pi_1 = \digamma A - \mathcal{P}^{-1} R^T C, \tag{4.30}$$

$$\Pi_1^d = \digamma A_d - \mathcal{P}^{-1} R_d^T C, \tag{4.31}$$

$$\Pi_2 = \mathcal{P}^{-1} R^T + \Pi_1 \daleth, \tag{4.32}$$

$$\Pi_2^d = \mathcal{P}^{-1} R_d^T + \Pi_1^d \daleth, \tag{4.33}$$

2. *The following LMI condition is feasible:*

$$\begin{bmatrix} \mathcal{A} + \mathcal{Q} + I_{n_x} & \mathcal{A}_d & \mathbb{N}_{13} & \mathcal{M} + \mathcal{P}\digamma\Sigma & \mathcal{P}\digamma\Sigma_d \\ (\star) & -\mathcal{Q} & -\mathcal{A}_d^T \mathcal{G} & 0 & \mathcal{N} \\ (\star) & (\star) & \Theta - \frac{\mu}{2} I_{2r} & -\mathcal{G}^T \mathcal{P}\digamma\Sigma & -\mathcal{G}^T \mathcal{P}\digamma\Sigma_d \\ (\star) & (\star) & (\star) & -\Upsilon & 0 \\ (\star) & (\star) & (\star) & (\star) & -\Upsilon_d \end{bmatrix} < 0, \tag{4.34}$$

where

$$\begin{aligned} \lambda &= \sqrt{\mu}, \\ \mathcal{A} &= \mathcal{P}\digamma A - R^T C + \left(\mathcal{P}\digamma A - R^T C\right)^T, \\ \mathcal{A}_d &= \mathcal{P}\digamma A_d - R_d^T C, \\ \mathbb{N}_{13} &= \left[R^T D_\omega - \mathcal{P}\digamma E_\omega - \left(\mathcal{P}\digamma A - R^T C\right)^T \daleth D_\omega \quad R_d^T D_\omega \right], \\ \Theta &= -\begin{bmatrix} \Theta_{1,1}(\mathcal{P}, R) & D_\omega^T \daleth^T R_d^T D_\omega \\ D_\omega^T R_d \daleth D_\omega & 0_{r\times r} \end{bmatrix}, \\ \Theta_{1,1}(\mathcal{P}, R) &= D_\omega^T \daleth^T \left(R^T D_\omega - \mathcal{P}\digamma, E_\omega\right) + \left(R^T D_\omega - \mathcal{P}\digamma E_\omega\right)^T \daleth D_\omega. \end{aligned}$$

The constant matrices are defined as follows:

$$\mathcal{M} = [\mathcal{M}_1 \cdots \mathcal{M}_q], \ \textit{where} \ \mathcal{M}_i = \left[\underbrace{\mathcal{H}_i^T ... \mathcal{H}_i^T}_{s_i \text{ times}} \right], \tag{4.35}$$

$$\mathcal{N} = [\mathcal{N}_1 \cdots \mathcal{N}_q], \ \textit{where} \ \mathcal{N}_i = \left[\underbrace{(\mathcal{H}_i^d)^T ... (\mathcal{H}_i^d)^T}_{r_i \text{ times}} \right], \tag{4.36}$$

$$\Sigma = B \left[H_{11} \cdots H_{1s_1} \ H_{21} \cdots H_{qs_q} \right], \tag{4.37}$$

$$\Sigma_d = B \left[H_{11}^d \cdots H_{1r_1}^d \ H_{21}^d \cdots H_{qr_q}^d \right] \tag{4.38}$$

$$\Upsilon = \textit{diag} \left(\beta_{11} I_{s_1}, ..., \beta_{1s_1} I_{s_1}, \beta_{21} I_{s_2}, ..., \beta_{qs_q} I_{s_q} \right), \tag{4.39}$$

$$\Upsilon_d = \textit{diag} \left(\beta_{11}^d I_{r_1}, ..., \beta_{1r_1}^d I_{r_1}, \beta_{21}^d I_{r_2}, ..., \beta_{qr_q}^d I_{r_q} \right), \tag{4.40}$$

$$\beta_{ij} = \frac{2}{b_{ij}}, \ \ \beta_{ij}^d = \frac{2}{b_{ij}^d}. \tag{4.41}$$

Proof. First, notice that for any $\mathcal{P}$, R and R_d, conditions (4.30)-(4.33) lead to reduce (4.23) to the following:

$$\begin{aligned}\dot{e}(t) = & \left(\digamma A - \mathcal{P}^{-1}R^TC\right)e(t) + \left(\digamma A_d - \mathcal{P}^{-1}R_d^TC\right)e(t-d) \\ & + \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} h_{ij}(t)\digamma BH_{ij}\mathcal{H}_i e(t) + \sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} h_{ij}^d(t)\digamma BH_{ij}^d\mathcal{H}_i^d e(t-d) \\ & + \daleth_\omega\bar{\omega}(t) + \mathcal{G}\dot{\bar{\omega}}(t)\end{aligned} \tag{4.42}$$

where $\daleth_\omega$ becomes:

$$\daleth_\omega = \left[\mathcal{P}^{-1}R^TD_\omega - \digamma E_\omega \quad \mathcal{P}^{-1}R_d^TD_\omega\right] \tag{4.43}$$

Now, it suffices to show that the matrices $\mathcal{P}$, R and R_d, provided by the LMI (4.34), ensure the robustness of the proposed observer.
At first, let us use the following Lyapunov-Krasovskii functional:

$$V(e(t)) = (e(t) - \mathcal{G}\bar{\omega}(t))^T\mathcal{P}(e(t) - \mathcal{G}\bar{\omega}(t)) + \int_{t-d}^{t} e^T(s)\mathcal{Q}e(s)\mathrm{d}s \tag{4.44}$$

The fact to introduce the term $e(t) - \mathcal{G}\bar{\omega}(t)$ into the Lyapunov-Krasovskii-like function allows avoiding the presence of the quadratic term $\dot{\bar{\omega}}^T\mathcal{G}^T\mathcal{P}\mathcal{G}\dot{\bar{\omega}}$ in $\dot{V}$. In other words, this allows to avoid deriving the disturbances, i.e., the presence of $\dot{\bar{\omega}}$ into $\dot{V}$.
Considering $\dot{V}$ along the system (4.42), we have:

$$\begin{aligned}\dot{V} = & e^T(t)\left[\left(\digamma A - \mathcal{P}^{-1}R^TC\right)^T\mathcal{P} + \mathcal{P}\left(\digamma A - \mathcal{P}^{-1}R^TC\right) + \mathcal{Q}\right]e(t) - e_d^T\mathcal{Q}e_d \\ & + 2e^T(t)\mathcal{P}\left(\digamma A_d - \mathcal{P}^{-1}R_d^TC\right)e_d + 2e^T(t)\left(\mathcal{P}\daleth_\omega - \left(\digamma A - \mathcal{P}^{-1}R^TC\right)^T\mathcal{P}\mathcal{G}\right)\bar{\omega}(t) \\ & + 2e^T(t)\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\digamma BH_{ij}\zeta_{ij}\right) + 2e^T(t)\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}\digamma BH_{ij}^d\zeta_{ij}^d\right) \\ & - 2\bar{\omega}^T(t)\mathcal{G}^T\mathcal{P}\left(\digamma A_d - \mathcal{P}^{-1}R_d^TC\right)e_d - \bar{\omega}^T(t)\left(\mathcal{G}^T\mathcal{P}\daleth_\omega + \daleth_\omega^T\mathcal{P}\mathcal{G}\right)\bar{\omega}(t) \\ & - 2\bar{\omega}^T(t)\mathcal{G}^T\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\digamma BH_{ij}\zeta_{ij}\right) - 2\bar{\omega}^T(t)\mathcal{G}^T\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}\digamma BH_{ij}^d\zeta_{ij}^d\right)\end{aligned} \tag{4.45}$$

where

$$\zeta_{ij} = h_{ij}(t)\mathcal{H}_ie(t), \;\; \zeta_{ij}^d = h_{ij}^d(t)\mathcal{H}_i^de(t-d)$$

From (4.5), (4.6), and the Remark 4.2.5 the following inequalities always hold:

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\zeta_{ij}^T\left(\frac{1}{h_{ij}} - \frac{1}{b_{ij}}\right)\zeta_{ij} \geq 0 \tag{4.46}$$

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}(\zeta_{ij}^d)^T\left(\frac{1}{h_{ij}^d} - \frac{1}{b_{ij}^d}\right)\zeta_{ij}^d \geq 0 \tag{4.47}$$

The inequalities (4.46) and (4.47) become respectively:

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} e^T(t)\mathcal{H}_i^T\zeta_{ij} - \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\frac{1}{b_{ij}}\zeta_{ij}^T\zeta_{ij} \geq 0 \tag{4.48}$$

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} e^T(t-d)(\mathcal{H}_i^d)^T\zeta_{ij}^d - \sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}\frac{1}{b_{ij}^d}(\zeta_{ij}^d)^T\zeta_{ij}^d \geq 0 \tag{4.49}$$

Consequently, by adding the aforementioned inequalities to (4.45), $W(t)$ defined in (4.27) becomes:

$$W(t) \leq \begin{bmatrix} e(t) \\ e(t-d) \\ \bar{\omega}(t) \\ \zeta(t) \\ \zeta^d(t) \end{bmatrix}^T \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} \\ (\star) & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} \\ (\star) & (\star) & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} \\ (\star) & (\star) & (\star) & \Gamma_{44} & \Gamma_{45} \\ (\star) & (\star) & (\star) & (\star) & \Gamma_{55} \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-d) \\ \bar{\omega}(t) \\ \zeta(t) \\ \zeta^d(t) \end{bmatrix} \tag{4.50}$$

where

$$\Gamma_{11} = \left(\digamma A - \mathcal{P}^{-1}R^TC\right)^T\mathcal{P} + \mathcal{P}\left(\digamma A - \mathcal{P}^{-1}R^TC\right) - \mathcal{P} + \mathcal{Q} + I_{n_x}, \tag{4.51}$$
$$\Gamma_{12} = \mathcal{P}\left(\digamma A_d - \mathcal{P}^{-1}R^T\lceil C\right), \tag{4.52}$$
$$\Gamma_{13} = \mathcal{P}\daleth_\omega - \left(\digamma A - \mathcal{P}^{-1}R^TC\right)^T\mathcal{P}\mathcal{G}, \tag{4.53}$$
$$\Gamma_{14} = \mathcal{M} + \mathcal{P}\digamma\Sigma, \tag{4.54}$$
$$\Gamma_{15} = \mathcal{P}\digamma\Sigma_d, \tag{4.55}$$
$$\Gamma_{22} = -\mathcal{Q}, \tag{4.56}$$
$$\Gamma_{23} = -\left(\digamma A_d - \mathcal{P}^{-1}R_d^TC\right)^T\mathcal{P}\mathcal{G}, \tag{4.57}$$
$$\Gamma_{24} = 0, \tag{4.58}$$
$$\Gamma_{25} = \mathcal{N}, \tag{4.59}$$
$$\Gamma_{33} = \mathcal{G}^T\mathcal{P}\daleth_\omega + \daleth_\omega^T\mathcal{P}\mathcal{G} - \frac{\lambda^2}{2}I_{2r}, \tag{4.60}$$
$$\Gamma_{34} = -\mathcal{G}^T\mathcal{P}\digamma\Sigma, \tag{4.61}$$
$$\Gamma_{35} = -\mathcal{G}^T\mathcal{P}\digamma\Sigma_d, \tag{4.62}$$
$$\Gamma_{44} = -\Upsilon, \tag{4.63}$$
$$\Gamma_{45} = 0, \tag{4.64}$$
$$\Gamma_{55} = -\Upsilon_d, \tag{4.65}$$
$$\zeta(t) = [\zeta_{11}^T, ..., \zeta_{1s_1}^T, \zeta_{21}^T, ..., \zeta_{qs_q}^T]^T, \tag{4.66}$$
$$\zeta^d(t) = [(\zeta_{11}^d)^T, ..., (\zeta_{1r_1}^d)^T, (\zeta_{21}^d)^T, ..., (\zeta_{qr_q}^d)^T]^T. \tag{4.67}$$

which is identical to (4.34). Consequently, we deduce that under the condition (4.34), the estimation error converges robustly asymptotically towards zero. This ends the proof of Theorem 4.3.1. $\square$

Remark 4.3.1. *If* $\begin{bmatrix} E \\ C \end{bmatrix}$ *is of full column rank, for any square matrix* A *with the same dimension as* E *then* (E, A, C) *is impulse observable.*

Remark 4.3.2. *If* (E, A, C) *is impulse observable then* $(\digamma A, C)$ *is observable with* $\digamma$ *verifying* $\digamma E + \daleth C = I_{n_x}$.

Remark 4.3.3. *The assumption of observability of the system* (4.1) *is implicitly hidden in the sufficient LMI condition* (4.34)*. In fact, the necessary condition for the feasibility of the LMI* (4.34) *is the* *observability of the pair* $(\mathcal{F}A, C)$*, i.e.,* $\Gamma_{11} < 0$*. Indeed, if the pair* $(\mathcal{F}A, C)$ *is not observable, then the LMI* (4.34) *cannot be solved. It should be noticed that in all LMI approaches, the observability condition is hidden in the sufficient synthesis condition.*

Remark 4.3.4. *The detailed form of* f_i *(see* (4.2)*) and the insertion of* $\mathcal{H}_i$ *and* $\mathcal{H}_i^d$ *is not without interest, on the contrary, it plays an important role on the feasibility of the synthesis condition. Indeed, these matrices appear in the LMIs* (4.34) *in the terms* $\mathcal{M}$ *and* $\mathcal{N}$*.*

Remark 4.3.5. *Conditions* (4.30)-(4.33) *imply the unbiasedness of the observer, i.e., the estimation error does not depend explicitly on system states and thus the estimation error is decoupled from the measured input.*

Remark 4.3.6. *This method can be extended to systems with known time-varying delay and to systems with delayed output.*

Remark 4.3.7. *The* $\mathcal{H}_\infty$ *method can be applied if the assumption on the rank of matrix* D *is verified. But if* D *is not of full column rank, pseudo measurements, namely, the derivatives of the measurements [ZB09a], can be used. In fact, in this case, the presence of the derivative of the disturbances in the dynamics of the error is unavoidable, which brings out the need for an alternative method.*

In this section, we proposed a Lyapunov functional dependent on the disturbances in order to get rid of the derivative $\dot{\omega}$ in the dynamics of the error. In the next section, we propose an alternative method to the latter, the $\mathcal{W}^{1,2}$ approach. Depending on the nature of the studied space, the use of this criterion requires differentiability of the disturbances. By means of this type of approaches the stability of the estimation error can be ensured and the presence of the derivatives of the disturbances becomes no longer an obstacle.

4.4 $\mathcal{W}^{1,2}$ performance analysis

In this section we introduce a different criterion based on Sobolev norm [BC95], [Ale07]. In this approach the signals and their derivatives are taken into account. Although this solution was connivently chosen for a purely technical reason to get rid of the the derivatives of the disturbances, nevertheless, we will show through an example that this method provides better results than the one proposed in the previous section. Let us start by defining Sobolev spaces and Sobolev norms.

4.4.1 Sobolev space and Sobolev norm

- **Sobolev space**: is a vector space of functions equipped with a norm that is a combination of L_p norms of the function itself as will as its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space. This space is defined as follows:

$$\mathcal{W}_r^{k,p}([0,+\infty]) = \{z : [0,+\infty] \to \mathbb{R}^r \text{ such that } \frac{\partial^i z}{\partial t^i} \in L_p^r([0,+\infty]), i = 0,\ldots,k\} \quad (4.68)$$

- **Sobolev norm**: the aforementioned Sobolev space $\mathcal{W}_r^{k,p}([0,+\infty])$ admits a natural norm defined as follows:

$$\begin{aligned}\|z\|_{k,p}^{r} &= \left[\sum_{i=0}^{i=k}\left(\left\|\frac{\partial^i z}{\partial t^i}\right\|_{L_p^r}\right)^p\right]^{\frac{1}{p}} \\ &= \left(\sum_{i=0}^{i=k}\int_0^{+\infty}\left\|\frac{\partial^i z}{\partial t^i}\right\|^p dt\right)^{\frac{1}{p}}\end{aligned} \tag{4.69}$$

The space $\mathcal{W}_r^{k,p}([0,+\infty])$ equipped with the norm $\|.\|_{k,p}^r$ is a Banach space.

Remark 4.4.1. *To avoid complication, the same notations used in the previous sections will be used here.*

4.4.2 $\mathcal{W}^{1,2}$ performance criterion

We assume that $\omega \in \mathcal{W}_r^{1,2}$. Then, following the same developments as in the previous section, the criterion can be stated as follows:
Given the system (4.1), the observer (4.11) and the disturbances $\bar{\omega}$, then the $\mathcal{W}^{1,2}$ filtering design is to determine the matrices Π_1, Π_1^d, Π_2 and Π_2^d so that

$$\lim_{t\to\infty} e(t) = 0 \qquad for\ \ \bar{\omega}(.) = 0 \tag{4.70}$$

$$\|e\|_{1,2}^r \leq \frac{\lambda_{1,2}}{\sqrt{2}}\|\bar{\omega}\|_{1,2}^r \qquad \forall\, \bar{\omega}(t) \neq 0,\ \ t \geq 0;\ e(0) = 0 \tag{4.71}$$

In fact, we have

$$\left(\|\bar{\omega}\|_{1,2}^r\right)^2 = 2\left(\|\omega\|_{1,2}^r\right)^2 + \int_{-d}^{0}\omega^T(s)\omega(s)\mathrm{d}s + \int_{-\mathrm{d}}^{0}\dot{\omega}^{\mathrm{T}}(\mathrm{s})\dot{\omega}(\mathrm{s})\mathrm{ds}$$

Then it suffices to suppose, to simplify the notation, that $\omega(t) = 0,\ \forall\ t \in [-d, 0]$ to get the criterion (4.71). Indeed, in this case, we have

$$\int_{-d}^{0}\omega^T(s)\omega(s)\mathrm{d}s = \int_{-\mathrm{d}}^{0}\dot{\omega}^{\mathrm{T}}(\mathrm{s})\dot{\omega}(\mathrm{s})\mathrm{ds} = 0.$$

Now that the criterion is well defined, then following the Lyapunov theory, the problem of $\mathcal{W}^{1,2}$ filtering design can be reduced to finding a Lyapunov function $V(t)$ such that:

$$\vartheta(t) = \dot{V}(t) + e^T(t)e(t) + \dot{e}^T(t)\dot{e}(t) - \frac{\lambda_{1,2}^2}{2}\bar{\omega}^T(t)\bar{\omega}(t) - \frac{\lambda_{1,2}^2}{2}\dot{\bar{\omega}}^T(t)\dot{\bar{\omega}}(t) < 0 \tag{4.72}$$

Notice that condition (4.72) implies (4.70) and (4.71). In fact, if $\bar{\omega} = 0$ ($\dot{\bar{\omega}} = 0$) then (4.72) implies $\dot{V} < 0$ and then the estimation error converges asymptotically to zero which leads to (4.70). Now, if $\bar{\omega} \neq 0$ and $e(0) = 0$, then from (4.72) we get:

$$V(t) + \int_0^t e^T(s)e(s)\mathrm{d}s + \int_0^t \dot{e}^T(s)\dot{e}(s)\mathrm{d}s - \frac{\lambda^2}{2}\int_0^t \bar{\omega}^T(s)\bar{\omega}(s)\mathrm{d}s - \frac{\lambda_{1,2}^2}{2}\int_0^t \dot{\bar{\omega}}^T(s)\dot{\bar{\omega}}(s)\mathrm{d}s < 0. \tag{4.73}$$

Since $V(e(t)) \geq 0$ for all $t \geq 0$, then for $t \to +\infty$, we obtain:

$$\int_0^\infty e^T(s)e(s)\mathrm{d}s + \int_0^\infty \dot{e}^T(s)\dot{e}(s)\mathrm{d}s \leq \frac{\lambda_{1,2}^2}{2}\int_0^\infty \bar{\omega}^T(s)\bar{\omega}(s)\mathrm{d}s + \frac{\lambda_{1,2}^2}{2}\int_0^\infty \dot{\bar{\omega}}^T(s)\dot{\bar{\omega}}(s)\mathrm{d}s, \tag{4.74}$$

which is equivalent to (4.71).
The aforementioned condition (4.72), needed to ensure the $\mathcal{W}^{1,2}$ observation design problem, contains two additional terms when compared to (4.27), the condition for the $\mathcal{H}_\infty$ approach. The first is the quadratic term of $\dot{\bar{\omega}}$ which add no particular difficulty to our problem, and the second is the term $\dot{e}^T(t)\dot{e}(t)$ which leads to an unavoidable Bilinear Matrix Inequality (BMI). So, in order to counter this problem, we propose a solution based on relaxing the previous inequality by replacing the term $\dot{e}^T(t)\dot{e}(t)$ by $\epsilon\dot{e}^T(t)\mathcal{P}\dot{e}(t)$. In this case, the modified criterion is as follows:

$$\vartheta_{\mathcal{P}}(t) = \dot{V} + e^T(t)e(t) + \epsilon\dot{e}^T(t)\mathcal{P}\dot{e}(t) - \frac{\lambda_{\mathcal{P}}^2}{2}\bar{\omega}^T\bar{\omega} - \frac{\lambda_{\mathcal{P}}^2}{2}\dot{\bar{\omega}}^T\dot{\bar{\omega}} < 0. \tag{4.75}$$

Now, in the following theorem, we state the conditions ensuring the $\mathcal{W}^{1,2}$-stability of the estimation error.

Theorem 4.4.1. *For prescribed scalars $\lambda_{1,2} > 0$ and $\epsilon > 0$, the $\mathcal{W}^{1,2}$ filtering design problem corresponding to the system* (4.1) *and the observer* (4.11) *is solvable, with the $\mathcal{W}^{1,2}$ performance level less than $\lambda_{1,2}$, if there exist matrices $\mathcal{P} = \mathcal{P}^T > 0$, $\mathcal{Q} = \mathcal{Q}^T > 0$, R, and R_d of adequate dimensions so that the conditions below are fulfilled*

1. *The observer gain matrices Π_1, Π_1^d, Π_2 and Π_2^d are given by:*

$$\begin{aligned} \Pi_1 &= \digamma A - \mathcal{P}^{-1}R^TC, & (4.76)\\ \Pi_1^d &= \digamma A_d - \mathcal{P}^{-1}R_d^TC, & (4.77)\\ \Pi_2 &= \mathcal{P}^{-1}R^T + \Pi_1\daleth, & (4.78)\\ \Pi_2^d &= \mathcal{P}^{-1}R_d^T + \Pi_1^d\daleth. & (4.79)\end{aligned}$$

2. *The following LMI condition is feasible:*

$$\begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} & \mathbb{M}_{13} & \mathcal{P}\mathcal{G} & \mathcal{M}+\mathcal{P}\digamma\Sigma & \mathcal{P}\digamma\Sigma_d & \left(\mathcal{P}\digamma A - R^TC\right)^T \\ (\star) & -\mathcal{Q} & 0 & 0 & 0 & \mathcal{N} & \left(\mathcal{P}\digamma A_d - R_d^TC\right)^T \\ (\star) & (\star) & -\frac{\lambda_{\mathcal{P}}^2}{2}I_{2r} & 0 & 0 & 0 & \begin{bmatrix} D_\omega^TR - E_\omega^T\digamma^T\mathcal{P} \\ D_\omega^TR_d \end{bmatrix} \\ (\star) & (\star) & (\star) & -\frac{\lambda_{\mathcal{P}}^2}{2}I_{2r} & 0 & 0 & \mathcal{G}^T\mathcal{P} \\ (\star) & (\star) & (\star) & (\star) & -\Upsilon & 0 & (\digamma\Sigma)^T\mathcal{P} \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\Upsilon_d & (\digamma\Sigma_d)^T\mathcal{P} \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\frac{1}{\epsilon}\mathcal{P} \end{bmatrix} < 0, \tag{4.80}$$

$$\begin{aligned} \lambda_{1,2} &= \frac{\lambda_{\mathcal{P}}}{\sqrt{\min\left(1, \epsilon\lambda_{\min}(\mathcal{P})\right)}}, \\ \mathbb{M}_{11} &= \mathcal{P}\digamma A - R^TC + \left(\mathcal{P}\digamma A - R^TC\right)^T + \mathcal{Q} + I_{n_x}, \\ \mathbb{M}_{12} &= \mathcal{P}\digamma A_d - R_d^TC, \\ \mathbb{M}_{13} &= \begin{bmatrix} R^TD_\omega - \mathcal{P}\digamma E_\omega & R_d^TD_\omega \end{bmatrix}. \end{aligned}$$

Proof. Let us consider the following Lyapunov-Krasovskii functional:

$$V(e(t)) = e^T(t)\mathcal{P}e(t) + \int_{t-d}^{t} e^T(\theta)\mathcal{Q}e(\theta)\mathrm{d}\theta. \tag{4.81}$$

Considering $\dot{V}$ along the system (4.42), we have:

$$\begin{aligned}
\dot{V} &= \dot{e}^T(t)\mathcal{P}e(t) + e^T(t)\mathcal{Q}e(t) - e^T(t-d)\mathcal{Q}e(t-d) \\
&= e^T(t)\left[\left(\digamma A - \mathcal{P}^{-1}R^TC\right)^T\mathcal{P} + \mathcal{P}\left(\digamma A - \mathcal{P}^{-1}R^TC\right) + \mathcal{Q}\right]e(t) \\
&\quad + 2e^T(t)\mathcal{P}\left(\digamma A_d - \mathcal{P}^{-1}R_d^TC\right)e(t-d) + 2e^T(t)\mathcal{P}\daleth_\omega + 2e^T(t)\mathcal{P}\mathcal{G}\dot{\bar{\omega}}(t) \\
&\quad + 2e^T(t)\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\digamma BH_{ij}\zeta_{ij}\right) + 2e^T(t)\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}\digamma BH_{ij}^d\zeta_{ij}^d\right) \\
&\quad - e^T(t-d)\mathcal{Q}e(t-d).
\end{aligned} \tag{4.82}$$

By taking into consideration the additive terms of (4.75), we deduce that:

$$\vartheta_{\mathcal{P}}(t) \leq \begin{bmatrix} e(t) \\ e_d(t) \\ \bar{\omega}(t) \\ \dot{\bar{\omega}}(t) \\ \zeta(t) \\ \zeta^d(t) \end{bmatrix}^T \left(\mathcal{Q}_1 - \mathcal{Q}_2\mathcal{Q}_3^{-1}\mathcal{Q}_2^T\right) \begin{bmatrix} e(t) \\ e_d(t) \\ \bar{\omega}(t) \\ \dot{\bar{\omega}}(t) \\ \zeta(t) \\ \zeta^d(t) \end{bmatrix}, \tag{4.83}$$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$ are matrices of appropriate dimensions such that

$$\mathcal{Q}_1 = \begin{bmatrix}
\mathcal{A} + \mathcal{Q} + I_{n_x} & \mathcal{A}_d & \mathcal{P}\daleth_\omega & \mathcal{P}\mathcal{G} & \mathcal{M} + \mathcal{P}\digamma\Sigma & \mathcal{P}\digamma\Sigma_d \\
(\star) & -\mathcal{Q} & 0 & 0 & 0 & \mathcal{N} \\
(\star) & (\star) & -\frac{\lambda_{\mathcal{P}}^2}{2}I_{2r} & 0 & 0 & 0 \\
(\star) & (\star) & (\star) & -\frac{\lambda_{\mathcal{P}}^2}{2}I_{2r} & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & -\Upsilon & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & -\Upsilon_d
\end{bmatrix}. \tag{4.84}$$

with

$$\mathcal{A} = \mathcal{P}\digamma A - R^TC + \left(\mathcal{P}\digamma A - R^TC\right)^T, \quad \mathcal{A}_d = \mathcal{P}\digamma A_d - R_d^TC,$$

$$\mathcal{Q}_2 = \begin{bmatrix} \left(\mathcal{P}\digamma A - R^TC\right)^T \\ \left(\mathcal{P}\digamma A_d - R_d^TC\right)^T \\ \daleth_\omega^T\mathcal{P} \\ \mathcal{G}^T\mathcal{P} \\ (\digamma\Sigma)^T\mathcal{P} \\ (\digamma\Sigma_d)^T\mathcal{P} \end{bmatrix}, \tag{4.85}$$

$$\mathcal{Q}_3 = -\frac{1}{\epsilon}\mathcal{P}. \tag{4.86}$$

Using Schur lemma we deduce that the inequality $\mathcal{Q}_1 - \mathcal{Q}_2\mathcal{Q}_3^{-1}\mathcal{Q}_2^T < 0$ is equivalent to the LMI (4.80). Hence, the feasibility of (4.80) leads to (4.75).

Notice that

$$\vartheta_{\mathcal{P}}(t) > \dot{V} + \min\left(1, \epsilon\lambda_{\min}(\mathcal{P})\right)\left[e^T(t)e(t) + \dot{e}^T(t)\dot{e}(t)\right] - \frac{\lambda_{\mathcal{P}}^2}{2}\left\|\bar{\omega}\right\|_{1,2}. \tag{4.87}$$

In this case, the criterion (4.72) is also fulfilled with a performance level

$$\lambda_{1,2} = \frac{\lambda_{\mathcal{P}}}{\sqrt{\min\left(1, \epsilon\lambda_{\min}(\mathcal{P})\right)}}. \tag{4.88}$$

It is clear that if $\epsilon\lambda_{\min}(\mathcal{P}) \geq 1$, then $\lambda_{1,2} = \lambda_{\mathcal{P}}$. This ends the proof of Theorem 4.4.1. □

Remark 4.4.2. *If the disturbances are differentiable, both $\mathcal{H}_\infty$ and $\mathcal{W}^{1,2}$ methods can be applied if the rank condition on matrix D is satisfied. If the disturbances are differentiable and the rank condition is not verified, the $\mathcal{W}^{1,2}$ approach may still be used if the rank condition is satisfied by using pseudo measurements. On the other hand, in some cases even if the rank condition holds, the $\mathcal{W}^{1,2}$ approach may be more robust than the $\mathcal{H}_\infty$ method as we will see in the undermentioned examples.*

Remark 4.4.3. *Sobolev space preserves all the nice properties of L_2, and $\mathcal{W}$-stability is well suited to formulate a local version of the small gain theorem and of the passivity theorem [BC95].*

4.5 Some extensions

This section is divided into two parts. In the first part, we present the extension of the developed results to time-varying delay. The second will be devoted to the discrete-time case of the $\mathcal{H}_\infty$ approach.

4.5.1 Systems with time-varying delays

The results of this chapter can be extended to systems with time-varying delay. In this case, a few modifications are expected. First, the delay is assumed to be bounded with a bounded derivative $\dot{d}$, with $\dot{d} \leq \mu < 1$. Second, the integral term in the Lyapunov-Krasovskii function become

$$V_2(t) = \int_{t-d(t)}^{t} e^T(\theta)\mathcal{Q}e(\theta)\mathrm{d}\theta. \tag{4.89}$$

Hence, the derivative become dependent on the rate $\dot{d}$:

$$\dot{V}_2 = e^T\mathcal{Q}e - \left(1 - \dot{d}\right)e_d^T\mathcal{Q}e_d \leq e^T\mathcal{Q}e - (1-\mu)e_d^T\mathcal{Q}e_d. \tag{4.90}$$

Following the same steps of sections 4.3 and 4.4, we get the following delay-independent and rate-dependent conditions in terms of LMIs:

- **In the $\mathcal{H}_\infty$ context:**

  Theorem 4.5.1. *For a prescribed $\lambda > 0$, the $\mathcal{H}_\infty$ filtering design problem corresponding to the system* (4.8) *and the observer* (4.11) *is solvable, with the $\mathcal{H}_\infty$ performance level less than λ, if there exist matrices $\mathcal{P} = \mathcal{P}^T > 0$, $\mathcal{Q} = \mathcal{Q}^T > 0$, R, and R_d of adequate dimensions so that the following conditions are feasible.*

1. *The observer gain matrices Π_1, Π_1^d, Π_2 and Π_2^d are given by:*

$$\Pi_1 = \digamma A - \mathcal{P}^{-1} R^T C, \tag{4.91}$$
$$\Pi_1^d = \digamma A_d - \mathcal{P}^{-1} R_d^T C, \tag{4.92}$$
$$\Pi_2 = \mathcal{P}^{-1} R^T + \Pi_1 \daleth, \tag{4.93}$$
$$\Pi_2^d = \mathcal{P}^{-1} R_d^T + \Pi_1^d \daleth. \tag{4.94}$$

2. *The following LMI condition is feasible:*

$$\begin{bmatrix} \mathcal{A} + \mathcal{Q} + I_{n_x} & \mathcal{A}_d & \mathbb{N}_{13} & \mathcal{M} + \mathcal{P}\digamma\Sigma & \mathcal{P}\digamma\Sigma_d \\ (\star) & -(1-\mu)\mathcal{Q} & \mathbb{N}_{23} & 0 & \mathcal{N} \\ (\star) & (\star) & \Theta - \frac{\lambda^2}{2} I_{2r} & -\mathcal{G}^T\mathcal{P}\digamma\Sigma & -\mathcal{G}^T\mathcal{P}\digamma\Sigma_d \\ (\star) & (\star) & (\star) & -\Upsilon & 0 \\ (\star) & (\star) & (\star) & (\star) & -\Upsilon_d \end{bmatrix} < 0, \tag{4.95}$$

$$\mathbb{N}_{13} = \left[R^T D_\omega - \mathcal{P}\digamma E_\omega - \left(\mathcal{P}\digamma A - R^T C\right)^T \daleth D_\omega \quad R_d^T D_\omega \right],$$
$$\mathbb{N}_{23} = -\left(\mathcal{P}\digamma A_d - R_d^T C\right)^T \mathcal{G}.$$

- **In the $\mathcal{W}^{1,2}$ context:**

Theorem 4.5.2. *For prescribed scalars $\lambda_{1,2} > 0$ and $\epsilon > 0$, the $\mathcal{W}^{1,2}$ filtering design problem corresponding to the system* (4.1) *and the observer* (4.11) *is solvable, with the $\mathcal{W}^{1,2}$ performance level less than $\lambda_{1,2}$, if there exist matrices $\mathcal{P} = \mathcal{P}^T > 0$, $\mathcal{Q} = \mathcal{Q}^T > 0$, R, and R_d of adequate dimensions so that the conditions below are fulfilled*

1. *The observer gain matrices Π_1, Π_1^d, Π_2 and Π_2^d are given by:*

$$\Pi_1 = \digamma A - \mathcal{P}^{-1} R^T C, \tag{4.96}$$
$$\Pi_1^d = \digamma A_d - \mathcal{P}^{-1} R_d^T C, \tag{4.97}$$
$$\Pi_2 = \mathcal{P}^{-1} R^T + \Pi_1 \daleth, \tag{4.98}$$
$$\Pi_2^d = \mathcal{P}^{-1} R_d^T + \Pi_1^d \daleth. \tag{4.99}$$

2. *The following LMI condition is feasible:*

$$\begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} & \mathbb{M}_{13} & \mathcal{P}\mathcal{G} & \mathcal{M} + \mathcal{P}\digamma\Sigma & \mathcal{P}\digamma\Sigma_d & \left(\mathcal{P}\digamma A - R^T C\right)^T \\ (\star) & -(1-\mu)\mathcal{Q} & 0 & 0 & 0 & \mathcal{N} & \left(\mathcal{P}\digamma A_d - R_d^T C\right)^T \\ (\star) & (\star) & -\frac{\lambda_{\mathcal{P}}^2}{2} I_{2r} & 0 & 0 & 0 & \begin{bmatrix} D_\omega^T R - E_\omega^T \digamma^T \mathcal{P} \\ D_\omega^T R_d \end{bmatrix} \\ (\star) & (\star) & (\star) & -\frac{\lambda_{\mathcal{P}}^2}{2} I_{2r} & 0 & 0 & \mathcal{G}^T\mathcal{P} \\ (\star) & (\star) & (\star) & (\star) & -\Upsilon & 0 & (\digamma\Sigma)^T\mathcal{P} \\ (\star) & (\star) & (\star) & (\star) & (\star) & -\Upsilon_d & (\digamma\Sigma_d)^T\mathcal{P} \\ (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & -\frac{1}{\epsilon}\mathcal{P} \end{bmatrix} < 0, \tag{4.100}$$

$$\lambda_{1,2} = \frac{\lambda_{\mathcal{P}}}{\sqrt{\min\left(1, \epsilon\lambda_{\min}(\mathcal{P})\right)}},$$

$$\begin{aligned}
\mathbb{M}_{11} &= \mathcal{P}F A - R^T C + \left(\mathcal{P}F A - R^T C\right)^T + \mathcal{Q} + I_{n_x},\\
\mathbb{M}_{12} &= \mathcal{P}F A_d - R_d^T C,\\
\mathbb{M}_{13} &= \begin{bmatrix} R^T D_\omega - \mathcal{P}F E_\omega & R_d^T D_\omega \end{bmatrix}.
\end{aligned}$$

The proofs of these theorems are omitted, because the extension is straightforward and same steps of Theorem 4.3.1 and Theorem 4.4.1 are followed.
All the previous results were developed in continuous time due to the interest of the subject. Nevertheless, in the next section, we will discuss the $\mathcal{H}_\infty$ method in the discrete-time case and show the advantage of such a solution.

4.5.2 The discrete-time case

In this section, we propose an extension of the introduced $\mathcal{H}_\infty$ approach with constant delay to discrete-time case.

The $\mathcal{W}^{1,2}$ criterion has no sense in the discrete case. If we define the vector $\bar{\omega} = \begin{bmatrix} \omega(t+1) \\ \omega(t) \\ \omega(t-d) \end{bmatrix}$, then the $\mathcal{W}^{1,2}$ filtering problem defined in (4.70)-(4.71) is reduced to an $\mathcal{H}_\infty$ filtering one. This is because

$$\left(\|\bar{\omega}\|_{1,2}^r\right)^2 = \left(\|\omega_{t+1}\|_{1,2}^r\right)^2 + \left(\|\omega_t\|_{1,2}^r\right)^2 + \left(\|\omega_{t-d}\|_{1,2}^r\right)^2.$$

For simplicity, let us presume that $\omega(t) = 0$, $\forall t \in [-d, 0]$. Thus, we get

$$\left(\|\bar{\omega}\|_{1,2}^r\right)^2 = 3\left(\|\omega_t\|_{1,2}^r\right)^2.$$

One of the used methods when dealing with the studied class of systems in discrete-time, is to augment the state to obtain a new system without delay but this is a quite different philosophy of what we have proposed in this chapter. In fact, we have chosen to use the non augmented state for different reasons:

- For large values of delays, we risk of having systems of large dimensions which leads to problems in case of real-time control.
- For time-varying delays, this type of solutions is unsuitable, whereas our method can be easily extended to this type of delay.
- This method implies re-estimating the state at every iteration which adds a computation burden to this technique.
- Augmenting the state of the system to avoid the delays will not be of crucial help if the observer is combined with memoryless state feedback [IXCY06].

In addition, In certain applications such as fault diagnosis, in particular $\mathcal{H}_-/\mathcal{H}_\infty$ performance problem, the presence of $\omega(t+1)$ presents a real obstacle in the analysis and defining a new vector $\begin{bmatrix} \omega(t+1) \\ \omega(t) \end{bmatrix}$ does not lead to a feasible LMI. For these reasons, we believe that the $\mathcal{H}_\infty$ method proposed in 4.3 provides an interesting solution in the discrete case.

The studied system is on the form:

$$
\begin{aligned}
Ex(t+1) &= Ax(t) + A_d x(t-d) + Bf\left(x(t), x(t-d)\right) + E_\omega \omega(t) \\
y(t) &= Cx(t) + D_\omega \omega(t) \\
x(t) &= x_0(t), \quad t \in [-\bar{d}, 0]
\end{aligned}
\tag{4.101}
$$

The delay d is a nonnegative integer. For simplicity, we will use the same notation, same hypothesis on the system matrices and the nonlinearity (see Assumption 4.2.1) and the observer has the form:

$$
\left\{
\begin{aligned}
v(t+1) = \;& \Pi_1 v(t) + \Pi_1^d v(t-d) + \Pi_2 y(t) + \Pi_2^d y(t-d) \\
& + \digamma \sum_{i=1}^{i=q} Be_q(i) f_i\left(\mathcal{H}_i \hat{x}(t), \mathcal{H}_i^d \hat{x}(t-d)\right), \\
\hat{x}(t) = \;& v(t) + \daleth y(t),
\end{aligned}
\right.
\tag{4.102}
$$

where Π_1, Π_1^d, Π_2 and Π_2^d are gain matrices to be determined.
The next theorem proposes a new condition ensuring the robust asymptotic stability of the estimation error in discrete case.

Theorem 4.5.3. *For a prescribed $\lambda > 0$, the $\mathcal{H}_\infty$ filtering design problem corresponding to the system* (4.101) *and the observer* (4.102) *is solvable, with the $\mathcal{H}_\infty$ performance level less than λ, if there exist matrices $\mathcal{P} = \mathcal{P}^T > 0$, $\mathcal{Q} = \mathcal{Q}^T > 0$, and matrices R, and R_d of adequate dimensions so that:*

1. *The observer gain matrices Π_1, Π_1^d, Π_2 and Π_2^d are given by:*

$$\Pi_1 = \digamma A - \mathcal{P}^{-1} R^T C, \tag{4.103}$$

$$\Pi_1^d = \digamma A_d - \mathcal{P}^{-1} R_d^T C, \tag{4.104}$$

$$\Pi_2 = \mathcal{P}^{-1} R^T + \Pi_1 \daleth, \tag{4.105}$$

$$\Pi_2^d = \mathcal{P}^{-1} R_d^T + \Pi_1^d \daleth. \tag{4.106}$$

2. *The following LMI condition is feasible:*

$$
\begin{bmatrix}
-\mathcal{P} + \mathcal{Q} + I_{n_\xi} & 0 & \mathcal{P}\mathcal{G} & \mathcal{M} & 0 & (\digamma \Xi_1)^T \mathcal{P} - \mathcal{C}^T R \\
(\star) & -\mathcal{Q} & 0 & 0 & \mathcal{N} & (\digamma \Xi_1^d)^T \mathcal{P} - \mathcal{C}^T R_d \\
(\star) & (\star) & -\mathcal{G}^T \mathcal{P} \mathcal{G} - \frac{\mu}{2} I_{2r} & 0 & 0 & \begin{bmatrix} D_\omega^T R - E_\omega^T \digamma^T \mathcal{P} \\ D_\omega^T R_d \end{bmatrix} \\
(\star) & (\star) & (\star) & -\Upsilon & 0 & (\digamma \Sigma)^T \mathcal{P} \\
(\star) & (\star) & (\star) & (\star) & -\Upsilon^d & (\digamma \Sigma^d)^T \mathcal{P} \\
(\star) & (\star) & (\star) & (\star) & (\star) & -\mathcal{P}
\end{bmatrix} < 0,
\tag{4.107}
$$

where the matrices are already defined in Theorem 4.34 (equations (4.35)-(4.41)*).*

Proof. First, notice that for any $\mathcal{P}$, R and R_d, conditions (4.103)-(4.106) lead to reduce the error dynamics to the following:

$$
\begin{aligned}
e(t+1) = \;& \left(\digamma \Xi_1 - \mathcal{P}^{-1} R^T \mathcal{C}\right) e(t) + \left(\digamma \Xi_1^d - \mathcal{P}^{-1} R_d^T \mathcal{C}\right) e(t-d) \\
& + \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} h_{ij}(t) \digamma B H_{ij} \mathcal{H}_i e(t) + \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} h_{ij}^d(t) \digamma B H_{ij}^d \mathcal{H}_i^d e(t-d) \\
& + \daleth_\omega \bar{\omega}(t) + \mathcal{G} \bar{\omega}(t+1).
\end{aligned}
\tag{4.108}
$$

where $\daleth_\omega$ becomes:

$$\daleth_\omega = \begin{bmatrix} \mathcal{P}^{-1} R^T D_\omega - \digamma E_\omega & \mathcal{P}^{-1} R_d^T D_\omega \end{bmatrix}. \tag{4.109}$$

Now, it suffices to show that the matrices $\mathcal{P}$, R and R_d provided by the LMI (4.107), ensure the $\mathcal{H}_\infty$ robustness of the proposed observer. At first, let us use the following Lyapunov-Krasovskii functional candidate:

$$V(e(t)) = (e(t) - \mathcal{G}\bar{\omega}(t))^T \mathcal{P}(e(t) - \mathcal{G}\bar{\omega}(t)) + \sum_{i=1}^{i=d} e^T(t-i)\mathcal{Q}e(t-i). \tag{4.110}$$

The fact to introduce the term $e(t) - \mathcal{G}\bar{\omega}(t)$ into the Lyapunov-Krasovskii-like function allows avoiding the presence of the quadratic term $\bar{\omega}^T(t+1)\mathcal{G}^T\mathcal{P}\mathcal{G}\bar{\omega}(t+1)$ into $\Delta V = V(t+1) - V(t)$. After calculation of the difference ΔV, we have:

$$\begin{aligned}
\Delta V = {} & e^T(t)\left[\left(\digamma \Xi 1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T \mathcal{P}\left(\digamma \Xi_1 - \mathcal{P}^{-1}R^T\mathcal{C}\right) - \mathcal{P} + \mathcal{Q}\right] e(t) \\
& + e^T(t-d)\left[\left(\digamma \Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T \mathcal{P}\left(\digamma \Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right) - \mathcal{Q}\right] e(t-d) \\
& + \bar{\omega}^T(t)\left[\daleth_\omega^T \mathcal{P} \daleth_\omega - \mathcal{G}^T\mathcal{P}\mathcal{G}\right]\bar{\omega}(t) \\
& + 2e^T(t)\left(\digamma \Xi 1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T \mathcal{P}\left(\digamma \Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right) e(t-d) \\
& + 2e^T(t)\left(\digamma \Xi 1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T \mathcal{P}\daleth_\omega\bar{\omega}(t) \\
& + 2e^T(t)\left(\digamma \Xi 1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T \mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \digamma B H_{ij}\zeta_{ij}\right) \\
& + 2e^T(t)\left(\digamma \Xi 1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T \mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} \digamma B H_{ij}^d\zeta_{ij}^d\right) \\
& + 2e^T(t-d)\left(\digamma \Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T \mathcal{P}\daleth_\omega\bar{\omega}(t) \\
& + 2e^T(t-d)\left(\digamma \Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T \mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \digamma B H_{ij}\zeta_{ij}\right) \\
& + 2e^T(t-d)\left(\digamma \Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T \mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} \digamma B H_{ij}^d\zeta_{ij}^d\right) \\
& + 2\bar{\omega}^T(t)\daleth_\omega^T\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \digamma B H_{ij}\zeta_{ij}\right) + 2\bar{\omega}^T(t)\daleth_\omega^T\mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} \digamma B H_{ij}^d\zeta_{ij}^d\right) \\
& + \left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \digamma B H_{ij}\zeta_{ij}\right)^T \mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \digamma B H_{ij}\zeta_{ij}\right) \\
& + \left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} \digamma B H_{ij}^d\zeta_{ij}^d\right)^T \mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} \digamma B H_{ij}^d\zeta_{ij}^d\right) \\
& + 2\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \digamma B H_{ij}\zeta_{ij}\right)^T \mathcal{P}\left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} \digamma B H_{ij}^d\zeta_{ij}^d\right).
\end{aligned} \tag{4.111}$$

where

$$\zeta_{ij} = h_{ij}(t)\mathcal{H}_i e(t), \;\; \zeta_{ij}^d = h_{ij}^d(t)\mathcal{H}_i^d e(t-d).$$

From (4.5), (4.6), and the Remark 4.2.5 the following inequalities always hold:

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\zeta_{ij}^T\left(\frac{1}{h_{ij}}-\frac{1}{b_{ij}}\right)\zeta_{ij} \geq 0 \tag{4.112}$$

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}(\zeta_{ij}^d)^T\left(\frac{1}{h_{ij}^d}-\frac{1}{b_{ij}^d}\right)\zeta_{ij}^d \geq 0 \tag{4.113}$$

The inequalities (4.112) and (4.113) become respectively:

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} e^T(t)\mathcal{H}_i^T\zeta_{ij} - \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i}\frac{1}{b_{ij}}\zeta_{ij}^T\zeta_{ij} \geq 0, \tag{4.114}$$

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} e^T(t-d)(\mathcal{H}_i^d)^T\zeta_{ij}^d - \sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}\frac{1}{b_{ij}^d}(\zeta_{ij}^d)^T\zeta_{ij}^d \geq 0. \tag{4.115}$$

Consequently, adding the aforementioned inequalities to (4.111), we get:

$$W(t) \leq \begin{bmatrix} e(t) \\ e(t-d) \\ \bar{\omega}(t) \\ \zeta(t) \\ \zeta^d(t) \end{bmatrix}^T \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} \\ (\star) & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} \\ (\star) & (\star) & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} \\ (\star) & (\star) & (\star) & \Gamma_{44} & \Gamma_{45} \\ (\star) & (\star) & (\star) & (\star) & \Gamma_{55} \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-d) \\ \bar{\omega}(t) \\ \zeta(t) \\ \zeta^d(t) \end{bmatrix}, \tag{4.116}$$

where

$$\Gamma_{11} = \left(\digamma\Xi_1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T\mathcal{P}\left(\digamma\Xi_1 - \mathcal{P}^{-1}R^T\mathcal{C}\right) - \mathcal{P} + \mathcal{Q} + I_{n_\xi}, \tag{4.117}$$

$$\Gamma_{12} = \left(\digamma\Xi_1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T\mathcal{P}\left(\digamma\Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right), \tag{4.118}$$

$$\Gamma_{13} = \left(\digamma\Xi_1 - \mathcal{P}^{-1}R^T\mathcal{C}\right)^T\mathcal{P}\daleth_w, \tag{4.119}$$

$$\Gamma_{14} = \mathcal{M} + \left(\digamma\Xi_1 - \mathcal{P}^{-1}(R)^T\mathcal{C}\right)^T\mathcal{P}\digamma\Sigma, \tag{4.120}$$

$$\Gamma_{15} = \left(\digamma\Xi_1 - \mathcal{P}^{-1}(R)^T\mathcal{C}\right)^T\mathcal{P}\digamma\Sigma^d, \tag{4.121}$$

$$\Gamma_{22} = \left(\digamma\Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T\mathcal{P}\left(\digamma\Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right) - \mathcal{Q}, \tag{4.122}$$

$$\Gamma_{23} = \left(\digamma\Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T\mathcal{P}\daleth_\omega, \tag{4.123}$$

$$\Gamma_{24} = \left(\digamma\Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T\mathcal{P}\digamma\Sigma, \tag{4.124}$$

$$\Gamma_{25} = \mathcal{N} + \left(\digamma\Xi_1^d - \mathcal{P}^{-1}R_d^T\mathcal{C}\right)^T\mathcal{P}\digamma\Sigma^d, \tag{4.125}$$

$$\Gamma_{33} = \daleth_\omega^T\mathcal{P}\daleth_\omega - \mathcal{G}^T\mathcal{P}\mathcal{G} - \frac{\lambda^2}{2}I_{2r}, \tag{4.126}$$

$$\Gamma_{34} = \daleth_\omega^T\mathcal{P}\digamma\Sigma, \tag{4.127}$$

$$\Gamma_{35} = \daleth_\omega^T\mathcal{P}\digamma\Sigma^d, \tag{4.128}$$

$$\Gamma_{44} = (\digamma\Sigma)^T\mathcal{P}\digamma\Sigma - \Upsilon, \tag{4.129}$$

$$\Gamma_{45} = (\digamma\Sigma)^T\mathcal{P}\digamma\Sigma^d, \tag{4.130}$$

$$\Gamma_{55} = (\digamma\Sigma^d)^T\mathcal{P}\digamma\Sigma^d - \Upsilon^d, \tag{4.131}$$

$$\zeta(t) = [\zeta_{11}^T, ..., \zeta_{1s_1}^T, \zeta_{21}^T, ..., \zeta_{qs_q}^T]^T, \tag{4.132}$$

$$\zeta^d(t) = [(\zeta_{11}^d)^T, ..., (\zeta_{1r_1}^d)^T, (\zeta_{21}^d)^T, ..., (\zeta_{qr_q}^d)^T]^T. \tag{4.133}$$

We notice that the following matric can be rewritten in the form:

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \Gamma_{15} \\ (\star) & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} \\ (\star) & (\star) & \Gamma_{33} & \Gamma_{34} & \Gamma_{35} \\ (\star) & (\star) & (\star) & \Gamma_{44} & \Gamma_{45} \\ (\star) & (\star) & (\star) & (\star) & \Gamma_{55} \end{bmatrix} = \mathcal{Q}_1 - \mathcal{Q}_2\mathcal{Q}_3^{-1}\mathcal{Q}_2^T, \tag{4.134}$$

where $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$ are matrices of appropriate dimensions such that:

$$\mathcal{Q}_1 = \begin{bmatrix} -\mathcal{P} + \mathcal{Q} + I_{n_\xi} & 0 & \mathcal{P}\mathcal{G} & \mathcal{M} & 0 \\ (\star) & -\mathcal{Q} & 0 & 0 & \mathcal{N} \\ (\star) & (\star) & -\mathcal{G}^T\mathcal{P}\mathcal{G} - \frac{\lambda^2}{2}I_{2r} & 0 & 0 \\ (\star) & (\star) & (\star) & -\Upsilon & 0 \\ (\star) & (\star) & (\star) & (\star) & -\Upsilon^d \end{bmatrix}, \tag{4.135}$$

$$\mathcal{Q}_2 = \begin{bmatrix} (\digamma\Xi_1)^T\mathcal{P} - \mathcal{C}^T R \\ (\digamma\Xi_1^d)^T\mathcal{P} - \mathcal{C}^T R_d \\ \daleth_\omega^T\mathcal{P} \\ (\digamma\Sigma)^T\mathcal{P} \\ (\digamma\Sigma^d)^T\mathcal{P} \end{bmatrix}, \tag{4.136}$$

$$\mathcal{Q}_3 = -\mathcal{P} < 0. \tag{4.137}$$

The condition $W(t) < 0$ is fulfilled if the matrix defined in (4.134) is negative definite which is equivalent, using Schur lemma, to the following:

$$\begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_2 \\ \mathcal{Q}_2^T & \mathcal{Q}_3 \end{bmatrix} < 0, \tag{4.138}$$

which is identical to (4.107). Consequently, under the condition (4.107), the estimation error converges robustly asymptotically towards zero. This ends the proof of Theorem 4.5.3. □

4.6 Numerical examples and comparisons

In this section, we present some numerical examples to illustrate the performance of the proposed methods. The first two examples concern the $\mathcal{H}_\infty$ method stated in sections 4.5.2 and 4.3 in discrete-time and continuous-time cases respectively. The other two examples compare between the $\mathcal{H}_\infty$ and $\mathcal{W}^{1,2}$ (section 4.4) methods.

4.6.1 Example 1

In this example we consider the system:

$$Ex(t+1) = Ax(t) + A_d x_d(t) + B\tanh(0.1x(t-d)),$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.9 & 1 \\ 0 & 0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad d = 2,$$

$$\mathcal{H}_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \mathcal{H}_1^d = \begin{bmatrix} 0.1 & 0 \end{bmatrix},$$

$$E_\omega = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D_\omega = 1.$$

Moreover, the disturbance ω is a Gaussian distributed random signal with mean zero and standard deviation $\sigma = 0.5$, which is added on two finite intervals of time $I = \{20, \ldots, 50\} + \{100, \ldots, 200\}$. In other words, the disturbance is $\chi_k \omega_k$, where χ_k is defined by:

$$\chi_k = \begin{cases} 1 & \text{if } k \in I, \\ 0 & \text{otherwise.} \end{cases}$$

We chose to put the disturbances on the aforementioned form in order to show simultaneously the robustness and the asymptotic convergence to zero of the proposed observer, respectively with and without disturbances. The bounds of the partial derivatives of f are:

$$a_{11} = 0, \; a_{11}^d = 0, \; b_{11} = 1, \; b_{11}^d = 1.$$

According to Remark 4.2.5 our system does fulfill the required condition.
By taking the initial conditions $x_0 = [1, 2]$, $\hat{x}_0 = [-5, -1]$, and using Matlab tools to solve the LMI (4.34), we obtain the following solution:

$$\mathcal{P} = 10^4 \begin{bmatrix} 8.60 & 8.60 \\ 8.60 & 8.60 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 8.26 & 1.16 \\ 1.16 & 0.164 \end{bmatrix},$$

$$\Pi_1 = \begin{bmatrix} -0.84 & -0.74 \\ 0.68 & 0.58 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\Pi_1^d = 10^{-3} \begin{bmatrix} 0.81 & 0.81 \\ -0.81 & -0.81 \end{bmatrix}, \quad \Pi_2^d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

and the optimal value of the disturbance attenuation level is $\lambda = 1.5258$.
The simulation results depicted in Figure 4.1 and Figure 4.2 show the good estimation of all state components.

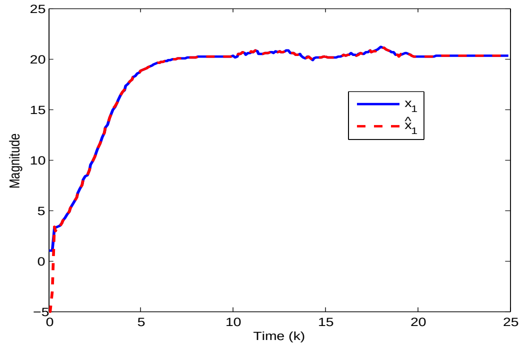

Figure 4.1: x_1 and $\hat{x}_1$ with respect to t

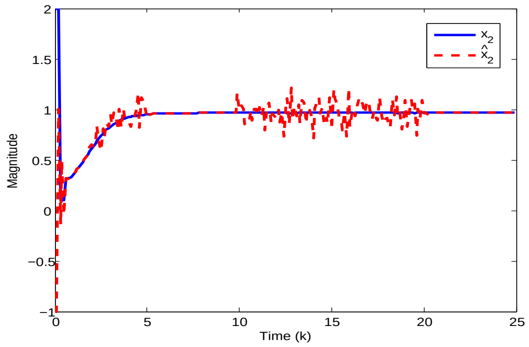

Figure 4.2: x_2 and $\hat{x}_2$ with respect to t

4.6.2 Example 2

The considered system is chaotic (Ikeda-like system [SD09]) with unknown input u that can be used in the field of synchronization and input recovery. We show that the proposed observer is able of estimating the state and the unknown input as well.
The nonlinearity is on the form:

$$f(x(t), u(t), x(t-d), u(t-d)) = \tanh(0.1x(t-d)) \text{ with } d = 2$$

and the system is described as follows:

$$\dot{x}(t) = -10x(t) + u(t) + 10\tanh(0.1x(t-d))$$

The unknown input u is a sinusoidal signal $u(t) = \sin(0.2t)$. The previous equations can be transformed into the form (4.1) with:

$$E = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -10 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$B = 10, \quad C = 1, \quad D = 1$$
$$\mathcal{H}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{H}_1^d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$E_\omega = 0.5, \quad D_\omega = 0.$$

The disturbance ω is defined as shown in the previous example but on $I = [50 \quad 100] \cup [200 \quad 400]$. Solving the LMI condition (4.34), we obtain the following solutions:

$$\mathcal{P} = 10^8 \begin{bmatrix} 2.3521 & 2.3521 \\ 2.3521 & 2.3521 \end{bmatrix}, \quad \mathcal{Q} = 10^8 \begin{bmatrix} 2.7803 & 2.7803 \\ 2.7803 & 2.7803 \end{bmatrix},$$

$$\Pi_1 = \begin{bmatrix} -5.7379 & 5.2621 \\ 4.1867 & -6.8133 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$\Pi_1^d = \begin{bmatrix} -0.0255 & -0.0255 \\ 0.0258 & 0.0258 \end{bmatrix}, \quad \Pi_2^d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

and the optimal value of the disturbance attenuation level is $\lambda = 0.10006$.
The simulation results are shown in Figure 4.3 and Figure 4.4 for the initial conditions $x_0 = [1, 0]$, $\hat{x}_0 = [-2, -1]$.

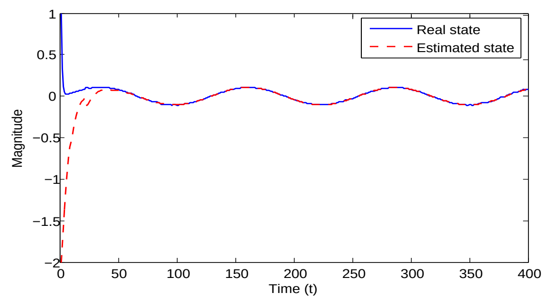

Figure 4.3: x and $\hat{x}$ with respect to t

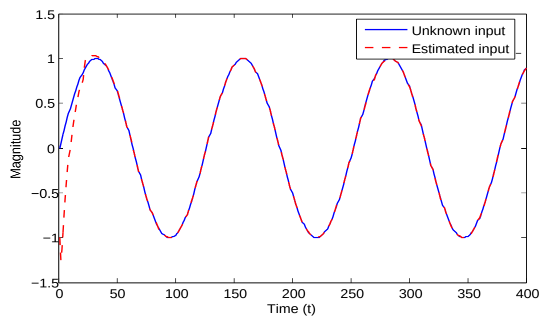

Figure 4.4: $u(t)$ and its estimate with respect to t

It is clear from the simulation that the estimates of both the input and system state settle quickly to the actual responses of these signals.

4.6.3 Example 3

In this example, we compare between $\mathcal{H}_\infty$ and $\mathcal{W}^{1,2}$ methods. The considered system is on the form:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\dot{x}(t)=\begin{bmatrix}0&3&1\\0&0&1\\0&0&-1\end{bmatrix}x(t)+\begin{bmatrix}1&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\frac{1}{3}&0\\0&\frac{1}{6}\\1&0\end{bmatrix}\begin{bmatrix}\tanh(0.5x_1(t))\\\tanh(0.7x_2(t-d))\end{bmatrix}+E_\omega\omega(t)$$
$$y(t)=\begin{bmatrix}2&0&0.5\end{bmatrix}x(t)+D_\omega\omega(t),\quad d=1 \tag{4.139}$$

The bounds of the partial derivatives of f are:

$$b_{11}=0.5,\ b_{22}^d=0.7$$

and $a_{ij}=0,\ a_{ij}^d=0,\ \ b_{ij}=0,\ b_{ij}^d=0$ otherwise.

$$\mathcal{H}_1=\begin{bmatrix}0.5&0&0\end{bmatrix},\ =\mathcal{H}_2^d=\begin{bmatrix}0&0.7&0\end{bmatrix},\ \mathcal{H}_1^d=\mathcal{H}_2=0$$

$$E_\omega=\begin{bmatrix}0.5\\0.5\\0.5\end{bmatrix},\quad D_\omega=0.1$$

The proposed methods are applied on the studied system and the following solutions are founded:

- $\mathcal{H}_\infty$ **Method:** solving the LMI condition (4.34), we get the following matrices.

$$\mathcal{P}=\begin{bmatrix}429.26\times10^6&30.68&107.32\times10^6\\30.68&10.86&8.36\\107.32\times10^6&8.36&268.29\times10^6\end{bmatrix},\quad \mathcal{Q}=\begin{bmatrix}74.27&24.72&11.59\\24.72&9.95&3.87\\11.59&3.87&1.80\end{bmatrix},$$

$$\Pi_1=\begin{bmatrix}9.02&3.00&3.25\\-3.20&0&0.19\\-40.64&-12.00&-14.16\end{bmatrix},\quad \Pi_2=\begin{bmatrix}2\\2\\-8\end{bmatrix},$$

$$\Pi_1^d=\begin{bmatrix}5.10&1.00&1.02\\0.03&0&0.01\\-20.43&-4.00&-4.10\end{bmatrix},\quad \Pi_2=0_{3\times1}.$$

and the optimal value of the disturbance attenuation level is $\lambda\ =\ 0.7286$.

- $\mathcal{W}^{1,2}$ **Method:** Solving the LMI (4.80), we find the following solution.

$$\mathcal{P}=\begin{bmatrix}114.23&-19.019&18.498\\-19.019&14.045&-3.7907\\18.498&-3.7907&3.0972\\0.32\times10^6&4.55&0.08\times10^6\end{bmatrix},\quad \mathcal{Q}=\begin{bmatrix}74.27&24.72&11.59\\24.72&9.95&3.87\\11.59&3.87&1.80\end{bmatrix},$$

$$\Pi_1=\begin{bmatrix}17.47&3.0&5.36\\-24.02&0&-5.00\\-149.6&-12.0&-41.39\end{bmatrix},\quad \Pi_2=\begin{bmatrix}2\\2\\-8\end{bmatrix},$$

$$\Pi_1^d = 10^{-2}\begin{bmatrix} 2.67 & 1.0 & 0.41 \\ 0.0047 & 0 & 0.0012 \\ -10.68 & -4.00 & -1.67 \end{bmatrix}, \quad \Pi_2 = 0_{3\times 1}.$$

and for $\epsilon = 0.05$ the optimal value of the disturbance attenuation level is $\lambda_{1,2} = 14.4306$.

The simulation results are shown in Figure 4.5-Figure 4.7 where the initial conditions are: $x_0 = [0.4, 0.6, 0.1]$ and $\hat{x}_0 = [0.7, 0.1, 1]$, and $\omega = 0.2\sin(0.2t)$ is defined on the interval $I = \begin{bmatrix}2 & 5\end{bmatrix} \cup \begin{bmatrix}10 & 20\end{bmatrix}$. We notice that the observer estimates successfully all the states of the system.

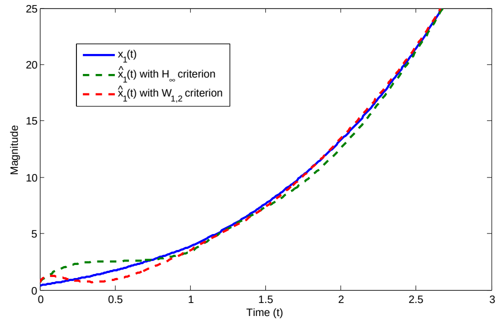

Figure 4.5: x_1 and $\hat{x}_1$ with respect to t

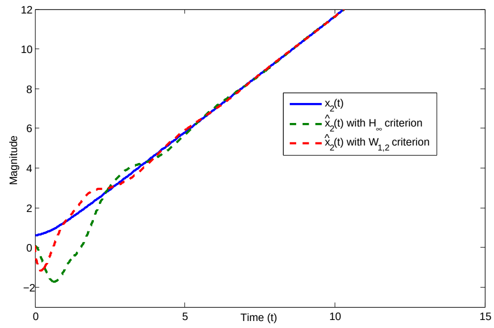

Figure 4.6: x_2 and $\hat{x}_2$ with respect to t

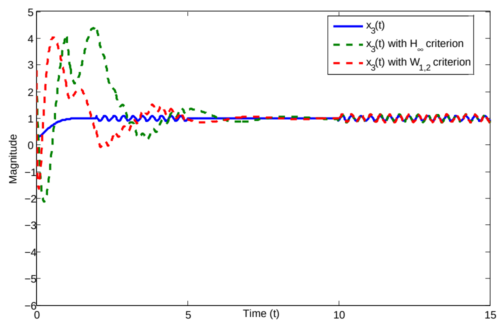

Figure 4.7: x_3 and $\hat{x}_3$ with respect to t

4.6.4 Example 4

In this example, we also compare between the $\mathcal{H}_\infty$ and $\mathcal{W}^{1,2}$ methods but here we consider a nonlinear time-delay system with unknown input u.

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}-10 & 0\\ 0 & -10\end{bmatrix}x(t) + \begin{bmatrix}1\\1\end{bmatrix}u(t) + \begin{bmatrix}0.2 & -0.01\\ -0.5 & 0.45\end{bmatrix}\begin{bmatrix}\tanh(x_1(t))\\ \tanh(x_2(t))\end{bmatrix}\\ &\quad + \begin{bmatrix}-1.5 & -0.1\\ -0.2 & -4\end{bmatrix}\begin{bmatrix}\tanh(x_1(t-d))\\ \tanh(x_2(t-d))\end{bmatrix} + E_\omega\omega(t),\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}x(t) + u(t) + D_\omega\omega(t), \quad d = 2.\end{aligned} \tag{4.140}$$

The unknown input u is a sinusoidal signal of the form $u(t) = \sin(0.5t)$.
According to Remark 4.2.3, the previous equations can be transformed into the form (4.1) with:

$$E = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \quad A = \begin{bmatrix}-10 & 0 & 1\\ 0 & -10 & 1\end{bmatrix}, \quad A_d = 0_{2\times3},$$

$$\mathcal{H}_1 = \mathcal{H}_1^d = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad \mathcal{H}_2 = \mathcal{H}_2^d = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$C = \begin{bmatrix}1 & 0 & 1\end{bmatrix}, \quad E_\omega = \begin{bmatrix}1\\1\end{bmatrix}, \quad D_\omega = 0.1.$$

Moreover, the disturbance ω is a sinusoidal signal $\omega(t) = 0.2\sin(0.2t)$, which is added on two finite intervals of time $I = \begin{bmatrix}2 & 5\end{bmatrix} \cup \begin{bmatrix}10 & 20\end{bmatrix}$.
The bounds of the partial derivatives of f are:

$$a_{ij} = 0,\ a_{ij}^d = 0,\ b_{ij} = 1,\ b_{ij}^d = 1,\ \ i,j = 1,2$$

According to Remark 4.2.5 our system does fulfill the required condition.
By taking the initial conditions $x_0 = [0.4, 0.5, 0]$, $\hat{x}_0 = [0.7, 0.1, 1]$, we obtain the following solutions:

- $\mathcal{H}_\infty$ **Method:** the LMI condition (4.34) is solved and we obtained the following matrices:

$$\mathcal{P} = \begin{bmatrix}7.46\times10^6 & 0.15 & 7.46\times10^6\\ 0.15 & 0.39 & 0.21\\ 7.46\times10^6 & 0.21 & 7.46\times10^6\end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix}26.57 & 2.72 & 15.45\\ 2.72 & 2.45 & 3.02\\ 15.45 & 3.02 & 10.97\end{bmatrix}$$

$$\Pi_1 = \begin{bmatrix}4.55 & 015.55 & \\ -6.17 & -10 & -5.17\\ -5.05 & 0 & -16.05\end{bmatrix}, \quad \Pi_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix},$$

$$\Pi_1^d = \begin{bmatrix}-0.799 & 0 & -0.799\\ -4.634 & 0 & -4.634\\ 0.799 & 0 & 0.799\end{bmatrix}, \quad \Pi_2 = 0_{3\times1}.$$

and the optimal value of the disturbance attenuation level is $\lambda = 0.3537$.

- $\mathcal{W}^{1,2}$ **Method:** solving the LMI condition (4.80), the following values are founded:

$$\mathcal{P} = \begin{bmatrix} 12.21 & 0.08 & 6.21 \\ 0.08 & 0.47 & 0.25 \\ 6.21 & 0.25 & 3.63 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 73.81 & 2.74 & 39.40 \\ 2.74 & 2.54 & 3.30 \\ 39.40 & 3.30 & 23.62 \end{bmatrix},$$

$$\Pi_1 = \begin{bmatrix} -13.09 & 0 & -2.09 \\ -3.11 & -10 & -2.11 \\ 4.58 & 0 & -6.42 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

$$\Pi_1^d = \begin{bmatrix} -0.85 & 0 & -0.85 \\ -5.09 & 0 & -5.09 \\ 0.85 & 0 & 0.85 \end{bmatrix}, \quad \Pi_2^d = 0_{3\times 1}.$$

and for $\epsilon = 0.1$ the optimal value of the disturbance attenuation level is $\lambda_{1,2} = 3.2044$.

The simulation results are shown in Figure 4.8 and Figure 4.9.

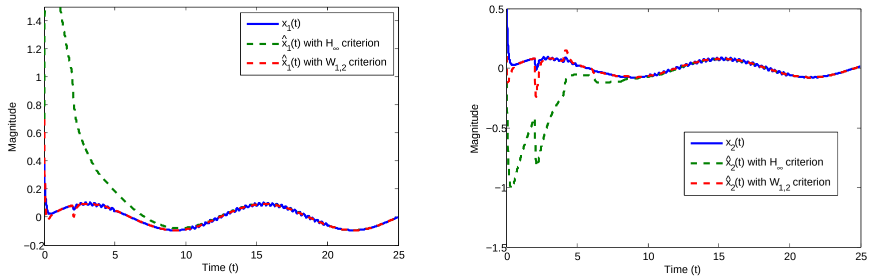

Figure 4.8: x and its estimate

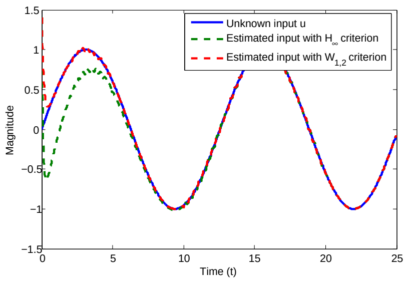

Figure 4.9: u and its estimate

Remark 4.6.1. *By analyzing the results of both methods, we notice that the $\mathcal{W}^{1,2}$ is better than $\mathcal{H}_\infty$. Clearly, involving conditions on the derivatives of the disturbances leads to better performance. However, it is worth mentioning that the disturbance attenuation levels λ and $\lambda_{1,2}$ do not have the same sense.*

4.7 Conclusion

In this chapter, we presented a new observer design method for a class of singular nonlinear time-delay systems. This method can be applied on nonlinear systems with unknown inputs, which can be confronted when dealing with failure detection and fault diagnosis problems. The nonlinearity of the considered system is assumed to be Lipschitz with respect to its arguments. By use of the DMVT and a particular Lyapunov-Krasovskii functional, the robustness of the proposed observer was treated using two approaches: $\mathcal{H}_\infty$ and $\mathcal{W}^{1,2}$ criteria. The resulting conditions were expressed in terms of LMI. In the last section, four numerical examples demonstrating the effectiveness of the established design method were provided, and comparing between the $\mathcal{H}_\infty$ and $\mathcal{W}^{1,2}$ methods.

CHAPTER

5

$\mathcal{H}_\infty$ observer-based controller for singular time-delay systems

Contents

5.1 Introduction

The study of control and regulation of singular systems have received considerable attention in the past decades and interesting results were developed, for example, the problem of $\mathcal{H}_2$ control was addressed in [ILU00]; $\mathcal{H}_\infty$ control method was presented in [WYC06] based on Youla parameterization. Although stabilisation and control of descriptor time-delay systems have been treated in the literature abundantly [WH03], [WZXL08], [Wan12], [KLP02], [Fan09b], [CZZ11], [HL99], observer-based controllers were rarely treated as opposed to state feedback [Bou07] or output feedback [MKO97]. On the other hand, singular systems in discrete time are more difficult to treat especially in the case of designing observer-based controllers due to the presence of multiple product terms and the fact that the matrix appearing in the study of singular systems is indefinite which render the generalization of some state space methods to descriptor systems a very difficult task.
Over the last decades, different techniques were adapted for singular systems in order to ameliorate the existing results. One of the interesting results is the introduction of some slack variables to induce a relaxation of resulting conditions, leading to increase the degree of freedom of the design variables, hence deriving less restrictive conditions [OBG99], [PABB00], [CD12]. This method consists in eliminating the decoupling between the Lyapunov matrix and replace it by a coupling between the slack variable and the unknown synthesis matrices. Another important

contribution is the result of [ZXS08], which lead to replace the semi definite matrix inequality condition known to distinguish singular systems [XY00], [LJDY08] by a strict tractable LMI condition ensuring the admissibility of the system.
In this chapter, we consider the same class of nonlinear system presented in Chapter 4. we choose to study the system in discrete-time because of its difficulty and the rarety of the related results when compared to the continuous-time. Based on the same observer model presented in the previous chapter, we aim to design a controller. The proposed method, make use of some existing results related to singular systems in order to develop a new strict LMI condition ensuring the stability of the system in closed loop. The idea consists in combining the slack variables technique proposed by [OBG99] with a perturbation dependant Lyapunov functional to solve the non-convex problem.

5.2 Problem statement

Let us start by reminding the reader of the considered system and the designed observer. The system is on the form:

$$\begin{cases} Ex_{k+1} &= Ax_k + A_d x_{k-d} + B_u u_k + B \sum\limits_{i=1}^{i=q} e_q(i) f_i \left(\mathcal{H}_i x_k, \mathcal{H}_i^d x_{k-d}\right) + E_\omega \omega_k, \\ y_k &= Cx_k + D_\omega \omega_k, \\ x(k) &= \phi(k), \quad k \in [-d, 0]. \end{cases} \tag{5.1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the input, $y_k \in \mathbb{R}^p$ is the output, and $\omega_k \in \mathbb{R}^r$ is the disturbance vector. The matrices $E, A, A_d \in \mathbb{R}^{n \times n_x}$; $B_u \in \mathbb{R}^{n \times n_u}, B \in \mathbb{R}^{n \times q}, C \in \mathbb{R}^{p \times n_x}$; $E_\omega \in \mathbb{R}^{n \times r}, D_\omega \in \mathbb{R}^{p \times r}$ are constant. $d > 0$ is a known delay. In order to design an observer, we assumed that the following conditions are verified:

Assumption 5.2.1.

- $rank\left(\begin{bmatrix} E \\ C \end{bmatrix}\right) = n_x$, *with* $n_x \leq n + p$.
- *The nonlinear function f is assumed to have uniformly bounded partial derivatives.*

$$a_{ij} \leq \frac{\partial f_i}{\partial \zeta_k^j}(\zeta_k, w_k) \leq b_{ij}, \quad \forall\, \zeta_k \in \mathbb{R}^{s_i}, \quad \forall\, w_k \in \mathbb{R}^{r_i}, \tag{5.2}$$

$$a_{ij}^d \leq \frac{\partial f_i}{\partial \zeta_k^j}(v_k, \zeta_k) \leq b_{ij}^d, \quad \forall\, \zeta_k \in \mathbb{R}^{r_i}, \quad \forall\, v_k \in \mathbb{R}^{s_i}. \tag{5.3}$$

On the other hand, the designed observer-based controller is given by

$$\upsilon_{k+1} = \Pi_1 \upsilon_k + \Pi_1^d \upsilon_{k-d} + \Pi_2 y_k + \Pi_2^d y_{k-d} + B_u u_k + \digamma B \sum_{i=1}^{i=q} e_q(i) f_i \left(\mathcal{H}_i \hat{x}_k, \mathcal{H}_i^d \hat{x}_{k-d}\right), \tag{5.4}$$

$$\hat{x}_k = \upsilon_k + \daleth y_k, \tag{5.5}$$

$$u_k = K\hat{x}_k + K_d \hat{x}_{k-d}. \tag{5.6}$$

where $\hat{x}$ is the estimate of x. The matrices Π_1, Π_1^d, Π_2, Π_2^d, K and K_d are gain matrices to be determined. The matrices $\digamma$ and $\daleth$ are constant matrices calculated from the pseudo-inverse of $\begin{bmatrix} E \\ C \end{bmatrix}$:

$$[\digamma \quad \daleth] = \left(\begin{bmatrix} E \\ C \end{bmatrix}^T \begin{bmatrix} E \\ C \end{bmatrix}\right)^{-1} \begin{bmatrix} E \\ C \end{bmatrix}^T.$$

Using the DMVT, we deduce that there exist $z_i \in Co(x_k, \hat{x}_k)$, $z_i^d \in Co(x_{k-d}, \hat{x}_{k-d})$ such that:

$$f_i(\mathcal{H}_i x_k, \mathcal{H}_i^d x_{k-d}) - f_i(\mathcal{H}_i \hat{x}_k, \mathcal{H}_i^d \hat{x}_{k-d}) = \sum_{j=1}^{j=s_i} h_{ij} e_{s_i}^T(j) \mathcal{H}_i \left[x_k - \hat{x}_k\right] + \sum_{j=1}^{j=r_i} h_{ij}^d e_{r_i}^T(j) \mathcal{H}_i^d \left[x_{k-d} - \hat{x}_{k-d}\right] \tag{5.7}$$

where

$$h_{ij} = \frac{\partial f_i}{\partial v_j}\left(z_i, w\right), \quad h_{ij}^d = \frac{\partial f_i}{\partial w_j}\left(v, z_i^d\right). \tag{5.8}$$

and the estimation error can be written as:

$$\begin{aligned} e_{k+1} &= \Pi_1 e_k + \Pi_1^d e_{k-d} + \daleth_\omega \bar{\omega}_k + \mathcal{G}\bar{\omega}_{k+1} \\ &\quad + \left(\Pi_1 + \left(\Pi_2 - \Pi_1 \daleth\right) C - \digamma A\right) x_k + \left(\Pi_1^d + \left(\Pi_2^d - \Pi_1^d \daleth\right) C - \digamma A_d\right) x_{k-d} \\ &\quad + \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} h_{ij}(t) \digamma B H_{ij} \mathcal{H}_i e_k + \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} h_{ij}^d(t) \digamma B H_{ij}^d \mathcal{H}_i^d e_{k-d}. \end{aligned} \tag{5.9}$$

where

$$\mathcal{G} = \begin{bmatrix} \daleth D_\omega & 0_{(n_x)\times r} \end{bmatrix}, \tag{5.10}$$

$$\bar{\omega}_k = \begin{bmatrix} \omega_k \\ \omega_{k-d} \end{bmatrix}, \tag{5.11}$$

$$\daleth_\omega = \begin{bmatrix} (\Pi_2 - \Pi_1 \daleth) D_\omega - \digamma E_\omega & (\Pi_2^d - \Pi_1^d \daleth) D_\omega . \end{bmatrix} \tag{5.12}$$

Without loss of generality, we assume that $f(0,0) = 0$. Thus, similarly to (5.7), there exist $\bar{z}_i \in Co(0, x_k)$, $\bar{z}_i^d \in Co(0, x_{k-d})$ such that the function f can be rewritten on the form:

$$f_i(\mathcal{H}_i x_k, \mathcal{H}_i^d x_{k-d}) = \sum_{j=1}^{j=s_i} g_{ij} e_{s_i}^T(j) \mathcal{H}_i x_k + \sum_{j=1}^{j=r_i} g_{ij}^d e_{r_i}^T(j) \mathcal{H}_i^d x_{k-d}, \tag{5.13}$$

where

$$g_{ij} = \frac{\partial f_i}{\partial v_j}\left(\bar{z}_i, w\right), \quad g_{ij}^d = \frac{\partial f_i}{\partial w_j}\left(v, \bar{z}_i^d\right). \tag{5.14}$$

At first, we use an interesting technique proposed by [CDD08] in which the system state is augmented with the control. Then we add the estimation error to the augmented vector, in order to guarantee the convergence of the observation error and the stability of the closed-loop system simultaneously. Thus, we get the following:

$$
\begin{aligned}
\underbrace{\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_{k+1} \\ e_{k+1} \end{bmatrix}}_{\xi(k+1)} &= \begin{bmatrix} A & B_u & 0 \\ K & -I_{n_u} & K \\ \Pi_1 + (\Pi_2 - \Pi_1 \daleth)\, C - \digamma A & 0 & \Pi_1 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ e_k \end{bmatrix} \\
&+ \begin{bmatrix} A_d & 0 & 0 \\ K_d & 0 & K_d \\ \Pi_1^d + \left(\Pi_2^d - \Pi_1^d \daleth\right) C - \digamma A_d & 0 & \Pi_1^d \end{bmatrix} \begin{bmatrix} x_{k-d} \\ u_{k-d} \\ e_{k-d} \end{bmatrix} \\
&+ \begin{bmatrix} E_\omega & 0 \\ 0 & 0 \\ (\Pi_2 - \Pi_1 \daleth) D_\omega - \digamma E_\omega & (\Pi_2^d - \Pi_1^d \daleth) D_\omega \end{bmatrix} \begin{bmatrix} \omega_k \\ \omega_{k-d} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \daleth D_\omega & 0 \end{bmatrix} \begin{bmatrix} \omega_{k+1} \\ \omega_{k+1-d} \end{bmatrix} \\
&+ \begin{bmatrix} \sum\limits_{i=1}^{i=q} \sum\limits_{j=1}^{j=s_i} g_{ij} B H_{ij} \mathcal{H}_i x_k \\ 0 \\ \sum\limits_{i=1}^{i=q} \sum\limits_{j=1}^{j=s_i} h_{ij} \digamma B H_{ij} \mathcal{H}_i e_k \end{bmatrix} + \begin{bmatrix} \sum\limits_{i=1}^{i=q} \sum\limits_{j=1}^{j=r_i} g_{ij}^d B H_{ij}^d \mathcal{H}_i^d x_{k-d} \\ 0 \\ \sum\limits_{i=1}^{i=q} \sum\limits_{j=1}^{j=r_i} h_{ij}^d \digamma B H_{ij}^d \mathcal{H}_i^d e_{k-d} \end{bmatrix}.
\end{aligned}
\tag{5.15}
$$

Remark 5.2.1. *Augmenting the state with the controller does not affect the regularity nor the stability of the original descriptor system [CD12, Lemma 9]. Furthermore, This augmentation does not change the finite modes nor impulsive modes of the original system.*

By using this model, the usual multiple product terms containing the gain of the controller and the decision matrix (PB_uK) are avoided. Thus, there is no need for a congruence transformation. So the problem to be addressed in this chapter is to determine the gains of the observer Π_1, Π_2, Π_1^d and Π_2^d along with the controller gains K and K_d such that the designed observer-based controller $u_k = K\hat{x}_k + K_d\hat{x}_{k-d}$ is robust and verify:

- The closed loop system is asymptotically stable.
- For the zero state response $\phi(k) = 0$, $k \in [-d, 0]$ and for any arbitrary $\omega(k) \in l_2[0, +\infty)$, the following condition holds:

$$
\sum_{k=0}^{k=\infty} \left(\xi^T(k)\xi(k) - \lambda^2 \omega^T(k)\omega(k)\right) < 0 \tag{5.16}
$$

5.3 Observer-based controller design

In the this section, we provide sufficient condition in the form of LMI ensuring the stability of the system (5.1) in the closed loop and guaranteeing a minimal attenuation level in the $\mathcal{H}_\infty$ sense defined in (5.16). First, let us start by summarizing the main result in the following theorem.

Theorem 5.3.1. *System* (5.1) *is asymptotically stable under the action of the observer-based controller* (5.6) *with $\mathcal{H}_\infty$ performance λ, if there exist symmetric and positive definite matrices S_2*

and $\mathcal{Q}$, $S_1 = \begin{bmatrix} S_{11} & S_{12} \\ (\star) & S_{22} \end{bmatrix}$, and matrices $\mathcal{F}$, $\mathcal{F}_d$, $\mathcal{F}_\omega$, M and M_d, $L = L^T$, R and R_d of adequate dimensions so that the following conditions hold.

1. *The LMI condition* (5.25) *in the next page is feasible.*

2. *The following conditions are fulfilled:*

$$\begin{aligned}
\Pi_1 &= \digamma A - S_2^{-1} R^T C, \\
\Pi_1^d &= \digamma A_d - S_2^{-1} R_d^T C, \\
\Pi_2 &= S_2^{-1} R^T + \Pi_1 \daleth, \\
\Pi_2^d &= S_2^{-1} R_d^T + \Pi_1^d \daleth.
\end{aligned} \tag{5.17}$$

and the gains of the controller are given by:

$$K = S_2^{-1} \bar{K}, \quad K_d = S_2^{-1} \bar{K}_d. \tag{5.18}$$

where the variables are defined as follows:

$$\bar{\mathcal{M}} = \begin{bmatrix} \bar{\mathcal{M}}_1 & \cdots & \bar{\mathcal{M}}_q \end{bmatrix}, \quad \textit{where} \quad \bar{\mathcal{M}}_i = \begin{bmatrix} \underbrace{\begin{bmatrix} \mathcal{H}_i^T & 0 \\ 0 & 0 \\ 0 & \mathcal{H}_i^T \end{bmatrix} \cdots \begin{bmatrix} \mathcal{H}_i^T & 0 \\ 0 & 0 \\ 0 & \mathcal{H}_i^T \end{bmatrix}}_{s_i \text{ times}} \end{bmatrix}, \tag{5.19}$$

$$\bar{\mathcal{N}} = \begin{bmatrix} \bar{\mathcal{N}}_1 & \cdots & \bar{\mathcal{N}}_q \end{bmatrix}, \quad \textit{where} \quad \bar{\mathcal{N}}_i = \begin{bmatrix} \underbrace{\begin{bmatrix} (\mathcal{H}_i^d)^T & 0 \\ 0 & 0 \\ 0 & (\mathcal{H}_i^d)^T \end{bmatrix} \cdots \begin{bmatrix} (\mathcal{H}_i^d)^T & 0 \\ 0 & 0 \\ 0 & (\mathcal{H}_i^d)^T \end{bmatrix}}_{r_i \text{ times}} \end{bmatrix}, \tag{5.20}$$

$$\begin{aligned}
\breve{\Sigma} &= \begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & \digamma B \end{bmatrix} \left[\begin{bmatrix} H_{11} & 0 \\ 0 & H_{11} \end{bmatrix} \cdots \begin{bmatrix} H_{1s_1} & 0 \\ 0 & H_{1s_1} \end{bmatrix}, \ldots, \begin{bmatrix} H_{q1} & 0 \\ 0 & H_{q1} \end{bmatrix} \cdots \begin{bmatrix} H_{qs_q} & 0 \\ 0 & H_{qs_q} \end{bmatrix} \right], \\
&\triangleq \begin{bmatrix} \Sigma_1 \\ 0 \\ \Sigma_2 \end{bmatrix}.
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
\breve{\Sigma}_d &= \begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & \digamma B \end{bmatrix} \left[\begin{bmatrix} H_{11}^d & 0 \\ 0 & H_{11}^d \end{bmatrix} \cdots \begin{bmatrix} H_{1s_1}^d & 0 \\ 0 & H_{1s_1}^d \end{bmatrix}, \ldots, \begin{bmatrix} H_{q1}^d & 0 \\ 0 & H_{q1}^d \end{bmatrix} \cdots \begin{bmatrix} H_{qs_q}^d & 0 \\ 0 & H_{qs_q}^d \end{bmatrix} \right], \\
&\triangleq \begin{bmatrix} \Sigma_{d1} \\ 0 \\ \Sigma_{d2} \end{bmatrix}.
\end{aligned} \tag{5.22}$$

$$\bar{\Upsilon} = \mathit{diag}\left(\begin{bmatrix} \beta_{11} I_{s_1} & 0 \\ 0 & \beta_{11} I_{s_1} \end{bmatrix} \cdots \begin{bmatrix} \beta_{1s_1} I_{s_1} & 0 \\ 0 & \beta_{1s_1} I_{s_1} \end{bmatrix}, \ldots, \begin{bmatrix} \beta_{q1} I_{s_q} & 0 \\ 0 & \beta_{q1} I_{s_q} \end{bmatrix} \cdots \begin{bmatrix} \beta_{qs_q} I_{s_q} & 0 \\ 0 & \beta_{qs_q} I_{s_q} \end{bmatrix} \right), \tag{5.23}$$

$$\bar{\Upsilon}_d = \mathit{diag}\left(\begin{bmatrix} \beta_{11}^d I_{r_1} & 0 \\ 0 & \beta_{11}^d I_{r_1} \end{bmatrix} \cdots \begin{bmatrix} \beta_{1r_1}^d I_{r_1} & 0 \\ 0 & \beta_{1r_1}^d I_{r_1} \end{bmatrix}, \ldots, \begin{bmatrix} \beta_{q1}^d I_{r_q} & 0 \\ 0 & \beta_{q1}^d I_{r_q} \end{bmatrix} \cdots \begin{bmatrix} \beta_{qr_q}^d I_{r_q} & 0 \\ 0 & \beta_{qr_q}^d I_{r_q} \end{bmatrix} \right), \tag{5.24}$$

$$\Phi = \begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \begin{bmatrix} 0 \\ 0 \\ \left(S_2 F A - R^T C\right)^T \end{bmatrix} & \mathcal{F} - \bar{\mathcal{A}}^T G^T & \begin{bmatrix} 0 & I_{n\times n_u} M & \bar{K}^T & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & \bar{K}^T & 0 \end{bmatrix} \\
(\star) & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \begin{bmatrix} 0 \\ 0 \\ \left(S_2 F A_d - R_d^T C\right)^T \end{bmatrix} & \mathcal{F}_d - \bar{\mathcal{A}}_d^T G^T & \begin{bmatrix} I_{n\times n_u} M_d & 0 & 0 & \bar{K}_d^T \\ M_d & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{K}_d^T \end{bmatrix} \\
(\star) & (\star) & \Phi_{33} & \Phi_{34} & \Phi_{35} & \begin{bmatrix} \left(R^T D_\omega - S_2 F E_\omega\right)^T \\ \left(R_d^T D_\omega\right)^T \end{bmatrix} & \mathcal{F}_\omega - \bar{\Xi}_\omega^T G^T & \begin{bmatrix} F_{\omega 12} & F_{\omega 12} & 0 & 0 \\ F_{\omega 14} & F_{\omega 14} & 0 & 0 \end{bmatrix} \\
(\star) & (\star) & (\star) & \Phi_{44} & \Phi_{45} & \begin{bmatrix} 0 \\ 0 \\ \Sigma_2 S_2 \end{bmatrix} & 0 & \begin{bmatrix} \Sigma_1^T (S_{12} - L_{12}) & \Sigma_1^T (S_{12} - L_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
(\star) & (\star) & (\star) & (\star) & \Phi_{55} & \begin{bmatrix} 0 \\ 0 \\ \Sigma_{d2} S_2 \end{bmatrix} & 0 & \begin{bmatrix} \Sigma_{d1}^T (S_{12} - L_{12}) & \Sigma_{d1}^T (S_{12} - L_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
(\star) & (\star) & (\star) & (\star) & (\star) & -S_2 & 0 & 0 \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & \Phi_{77} & \begin{bmatrix} G_{12} & G_{12} & 0 & 0 \\ G_{14} & G_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
(\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & \begin{bmatrix} -\epsilon_1 M & 0 & 0 & 0 \\ 0 & -\epsilon_2 M_d & 0 & 0 \\ 0 & 0 & -\frac{1}{\epsilon_1} M & 0 \\ 0 & 0 & 0 & -\frac{1}{\epsilon_2} M_d \end{bmatrix}
\end{bmatrix} < 0 \quad (5.25)$$

$$\Phi_{11} = -\bar{E}^T S \bar{E} + \mathcal{Q} + I_{2n} + \underbrace{\begin{bmatrix} F_{11}A + I_{n\times n_u}\bar{K} & F_{11}B_u - I_{n\times n_u}M & I_{n\times n_u}\bar{K} \\ F_{13}A + \bar{K} & F_{13}B_u - M & \bar{K} \\ 0 & 0 & 0 \end{bmatrix}}_{Z_1} + Z_1^T, \tag{5.26}$$

$$\Phi_{12} = \begin{bmatrix} F_{11}A_d & 0 & 0 \\ F_{13}A_d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} F_{d11}A & F_{d11}B_u - I_{n\times n_u}M_d & 0 \\ F_{d13}A & F_{d13}B_u - M_d & 0 \\ 0 & 0 & 0 \end{bmatrix}^T, \tag{5.27}$$

$$\Phi_{13} = \bar{E}^T S \Theta + \begin{bmatrix} F_{11}E_\omega & 0 \\ F_{13}E_\omega & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F_{\omega 11}A & F_{\omega 11}B_u - F_{\omega 12} & 0 \\ F_{\omega 13}A & F_{\omega 13}B_u - F_{\omega 14} & 0 \end{bmatrix}^T, \tag{5.28}$$

$$\Phi_{14} = \bar{\mathcal{M}} + \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \begin{bmatrix} A^T(S_{11} - L_{11})\Sigma_1 \\ \left(B_u^T S_{11} - S_{21}\right)\Sigma_1 \\ 0 \end{bmatrix}, \tag{5.29}$$

$$\Phi_{15} = \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \begin{bmatrix} A^T(S_{11} - L_{11})\Sigma_{d1} \\ \left(B_u^T S_{11} - S_{21}\right)\Sigma_{d1} \\ 0 \end{bmatrix}, \tag{5.30}$$

$$\Phi_{22} = -Q + \underbrace{\begin{bmatrix} F_{d11}A_d + I_{n\times n_u}\bar{K}_d & 0 & I_{n\times n_u}\bar{K}_d \\ F_{d13}A_d + \bar{K}_d & 0 & \bar{K}_d \\ 0 & 0 & 0 \end{bmatrix}}_{Z_2} + Z_2^T, \tag{5.31}$$

$$\Phi_{23} = \begin{bmatrix} F_{d11}E_\omega & 0 \\ F_{d13}E_\omega & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F_{\omega 11}A_d & F_{\omega 11}B_u - F_{\omega 12} & 0 \\ F_{\omega 13}A_d & F_{\omega 13}B_u - F_{\omega 14} & 0 \end{bmatrix}^T, \tag{5.32}$$

$$\Phi_{24} = \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \begin{bmatrix} A_d^T(S_{11} - L_{11})\Sigma_1 \\ 0 \\ 0 \end{bmatrix}, \tag{5.33}$$

$$\Phi_{25} = \bar{\mathcal{N}} + \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \begin{bmatrix} A_d^T(S_{11} - L_{11})\Sigma_{d1} \\ 0 \\ 0 \end{bmatrix}, \tag{5.34}$$

$$\Phi_{33} = \begin{bmatrix} F_{\omega 12}E_\omega & 0 \\ F_{\omega 14}E_\omega & 0 \end{bmatrix} - \Theta^T\left(S - E^{\perp T}LE^{\perp}\right)\Theta - \mu I_{2r}, \tag{5.35}$$

$$\Phi_{34} = \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \begin{bmatrix} E_\omega^T S_{11}\Sigma_1 \\ 0 \end{bmatrix} - \bar{\Xi}_\omega^T E^{\perp T}LE^{\perp}\bar{\Sigma}, \tag{5.36}$$

$$\Phi_{35} = \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \begin{bmatrix} E_\omega^T S_{11}\Sigma_{d1} \\ 0 \end{bmatrix} - \bar{\Xi}_\omega^T E^{\perp T}LE^{\perp}\bar{\Sigma}_d, \tag{5.37}$$

$$\Phi_{44} = \bar{\Sigma}^T\left(S - E^{\perp T}LE^{\perp}\right)\bar{\Sigma} - \bar{\Upsilon}, \tag{5.38}$$

$$\Phi_{45} = \bar{\Sigma}^T\left(S - E^{\perp T}LE^{\perp}\right)\bar{\Sigma}_d, \tag{5.39}$$

$$\Phi_{55} = \bar{\Sigma}_d^T\left(S - E^{\perp T}LE^{\perp}\right)\bar{\Sigma}_d - \bar{\Upsilon}_d, \tag{5.40}$$

$$\Phi_{77} = S - E^{\perp T}LE^{\perp} - G - G^T, \tag{5.41}$$

$$\beta_{ij} = \frac{2}{b_{ij}}, \quad \beta_{ij}^d = \frac{2}{b_{ij}^d}, \tag{5.42}$$

Proof. Taking into considerations the gains of the observer (5.17), we can simplify the augmented system as following:

$$\begin{aligned}
\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} x_{k+1} \\ u_{k+1} \\ e_{k+1} \end{bmatrix} &= \begin{bmatrix} A & B_u & 0 \\ K & -I_{n_u} & K \\ 0 & 0 & \digamma A - P_2^{-1} R^T C \end{bmatrix} \begin{bmatrix} x_k \\ u_k \\ e_k \end{bmatrix} \\
&+ \begin{bmatrix} A_d & 0 & 0 \\ K_d & 0 & K_d \\ 0 & 0 & \digamma A_d - P_2^{-1} R_d^T C \end{bmatrix} \begin{bmatrix} x_{k-d} \\ u_{k-d} \\ e_{k-d} \end{bmatrix} \\
&+ \begin{bmatrix} E_\omega & 0 \\ 0 & 0 \\ P_2^{-1} R^T D_\omega - \digamma E_\omega & P_2^{-1} R_d^T D_\omega \end{bmatrix} \begin{bmatrix} \omega_k \\ \omega_{k-d} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \neg D_\omega & 0 \end{bmatrix} \begin{bmatrix} \omega_{k+1} \\ \omega_{k+1-d} \end{bmatrix} \\
&+ \begin{bmatrix} \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} g_{ij} B H_{ij} \mathcal{H}_i x_k \\ 0 \\ \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} h_{ij} \digamma B H_{ij} \mathcal{H}_i e_k \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} g_{ij}^d B H_{ij}^d \mathcal{H}_i^d x_{k-d} \\ 0 \\ \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} h_{ij}^d \digamma B H_{ij}^d \mathcal{H}_i^d e_{k-d} \end{bmatrix}.
\end{aligned} \tag{5.43}$$

or more easily

$$\begin{aligned}
\bar{E}\xi_{k+1} = \mathcal{A}\xi_k + \mathcal{A}_d \xi_{k-d} + \Xi_\omega \bar{\omega}_k + \Theta \bar{\omega}_{k+1} + \sum_{i=1}^{i=q} \sum_{j=1}^{j=s_i} \begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & \digamma B \end{bmatrix} \begin{bmatrix} H_{ij} & 0 \\ 0 & H_{ij} \end{bmatrix} \bar{\zeta}_{ij} \\
+ \sum_{i=1}^{i=q} \sum_{j=1}^{j=r_i} \begin{bmatrix} B & 0 \\ 0 & 0 \\ 0 & \digamma B \end{bmatrix} \begin{bmatrix} H_{ij}^d & 0 \\ 0 & H_{ij}^d \end{bmatrix} \bar{\zeta}_{ij}^d,
\end{aligned} \tag{5.44}$$

where

$$\xi_k = \begin{bmatrix} x_k \\ u_k \\ e_k \end{bmatrix}, \tag{5.45}$$

$$\bar{\omega}_k = \begin{bmatrix} \omega_k \\ \omega_{k-d} \end{bmatrix}, \tag{5.46}$$

$$\mathcal{A} = \begin{bmatrix} A & B_u & 0 \\ K & -I_{n_u} & K \\ 0 & 0 & \digamma A - P_2^{-1} R^T C \end{bmatrix}, \tag{5.47}$$

$$\mathcal{A}_d = \begin{bmatrix} A_d & 0 & 0 \\ K_d & 0 & K_d \\ 0 & 0 & \digamma A_d - P_2^{-1} R_d^T C \end{bmatrix}, \tag{5.48}$$

$$
\Xi_\omega = \begin{bmatrix} E_\omega & 0 \\ 0 & 0 \\ P_2^{-1}R^TD_\omega - \varGamma E_\omega & P_2^{-1}R_d^TD_\omega \end{bmatrix}, \tag{5.49}
$$

$$
\Theta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \daleth D_\omega & 0 \end{bmatrix}, \tag{5.50}
$$

$$
\bar{\zeta}_{ij} = \begin{bmatrix} g_{ij}I_{s_i} & 0 \\ 0 & h_{ij}I_{s_i} \end{bmatrix} \begin{bmatrix} \mathcal{H}_i & 0 \\ 0 & \mathcal{H}_i \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix}, \tag{5.51}
$$

$$
\bar{\zeta}_{ij}^d = \begin{bmatrix} g_{ij}^dI_{r_i} & 0 \\ 0 & h_{ij}^dI_{r_i} \end{bmatrix} \begin{bmatrix} \mathcal{H}_i^d & 0 \\ 0 & \mathcal{H}_i^d \end{bmatrix} \begin{bmatrix} x_{k-d} \\ e_{k-d} \end{bmatrix}. \tag{5.52}
$$

According to the results of Chapter 4, the presence of the disturbances and consequently their derivatives can be treated using a Lyapunov-Krasovskii functional depending on the disturbance vector. In this chapter, we will use the same technique but slightly modified to be more suitable for the considered problem. In fact, the system in the augmented form (5.44) is singular unlike the case of Chapter 4, where the dynamics of the estimation error is regular. Indeed, the matrix $\bar{E}$ is now introduced as follows:

$$
V_k = (\bar{E}\xi_k - \Theta\bar{\omega}_k)^T\mathcal{P}(\bar{E}\xi_k - \Theta\bar{\omega}_k) + \sum_{i=1}^{i=d} \xi_{k-i}^T \mathcal{Q}\xi_{k-i}. \tag{5.53}
$$

with

$$
\mathcal{P} = \begin{bmatrix} \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ (\star) & P_{22} \end{bmatrix}}_{P_1} & 0 \\ 0 & P_2 \end{bmatrix}.
$$

By calculating $\Delta V = V_{k+1} - V_k$, we get

$$
\begin{aligned}
\Delta V = {} & \xi_k^T \left(\mathcal{A}^T\mathcal{P}\mathcal{A} - \bar{E}^T\mathcal{P}\bar{E} + \mathcal{Q} - I_{2n}\right)\xi_k + 2\xi_k^T\mathcal{A}^T\mathcal{P}\mathcal{A}_d\xi_{k-d} \\
& + 2\xi_k^T\left(\mathcal{A}^T\mathcal{P}\Xi_\omega + \bar{E}^T\mathcal{P}\Theta\right)\bar{\omega}_k + 2\xi_k^T\mathcal{A}^T\mathcal{P}\breve{\Sigma}\bar{\zeta}_k + 2\xi_k^T\mathcal{A}^T\mathcal{P}\breve{\Sigma}^d\bar{\zeta}_k^d \\
& + \xi_{k-d}^T\left(\mathcal{A}_d^T\mathcal{P}\mathcal{A}_d - \mathcal{Q}\right)\xi_{k-d} + 2\xi_{k-d}^T\mathcal{A}_d^T\mathcal{P}\Xi_\omega\bar{\omega}_k \\
& + 2\xi_{k-d}^T\mathcal{A}_d^T\mathcal{P}\breve{\Sigma}\bar{\zeta}_k + 2\xi_{k-d}^T\mathcal{A}_d^T\mathcal{P}\breve{\Sigma}^d\bar{\zeta}_k^d \\
& + \bar{\omega}_k^T\left(\Xi_\omega^T\mathcal{P}\Xi_\omega - \Theta^T\mathcal{P}\Theta\right)\bar{\omega}_k + 2\bar{\omega}_k^T\Xi_\omega^T\mathcal{P}\breve{\Sigma}\bar{\zeta}_k + 2\bar{\omega}_k^T\Xi_\omega^T\mathcal{P}\breve{\Sigma}^d\bar{\zeta}_k^d \\
& + \bar{\zeta}_k^T\breve{\Sigma}^T\mathcal{P}\breve{\Sigma}\bar{\zeta}_k\bar{\zeta}_k^d + \bar{\zeta}_k^T\breve{\Sigma}^T\mathcal{P}\breve{\Sigma}^d\bar{\zeta}_k^d + (\bar{\zeta}_k^d)^T(\breve{\Sigma}^d)^T\mathcal{P}\breve{\Sigma}^d.
\end{aligned} \tag{5.54}
$$

Notice that

$$
\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \left(\bar{\zeta}_{ij}\right)^T \begin{bmatrix} (\frac{1}{g_{ij}} - \frac{1}{b_{ij}})I_{s_i} & 0 \\ 0 & (\frac{1}{h_{ij}} - \frac{1}{b_{ij}})I_{s_i} \end{bmatrix} \bar{\zeta}_{ij} \geq 0, \tag{5.55}
$$

$$
\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i} \left(\bar{\zeta}_{ij}^d\right)^T \begin{bmatrix} (\frac{1}{g_{ij}^d} - \frac{1}{b_{ij}^d})I_{s_i} & 0 \\ 0 & (\frac{1}{h_{ij}^d} - \frac{1}{b_{ij}^d})I_{s_i} \end{bmatrix} \bar{\zeta}_{ij}^d \geq 0. \tag{5.56}
$$

The inequalities (5.55) and (5.56) become respectively:

$$
\sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \xi_k^T \begin{bmatrix} \mathcal{H}_i^T & 0 \\ 0 & 0 \\ 0 & \mathcal{H}_i^T \end{bmatrix} \bar{\zeta}_{ij} - \sum_{i=1}^{i=q}\sum_{j=1}^{j=s_i} \left(\bar{\zeta}_{ij}\right)^T \begin{bmatrix} \frac{1}{b_{ij}}I_{s_i} & 0 \\ 0 & \frac{1}{b_{ij}}I_{s_i} \end{bmatrix} \bar{\zeta}_{ij} \geq 0, \tag{5.57}
$$

$$\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}\xi_{k-d}^T\begin{bmatrix}(\mathcal{H}_i^d)^T & 0\\ 0 & 0\\ 0 & (\mathcal{H}_i^d)^T\end{bmatrix}\bar{\zeta}_{ij}^d-\sum_{i=1}^{i=q}\sum_{j=1}^{j=r_i}\left(\bar{\zeta}_{ij}^d\right)^T\begin{bmatrix}\frac{1}{b_{ij}^d}I_{s_i} & 0\\ 0 & \frac{1}{b_{ij}^d}I_{s_i}\end{bmatrix}\bar{\zeta}_{ij}^d\geq 0. \tag{5.58}$$

To ensure a robust behavior of both the observer and the controller, we show that we need to calculate

$$W_k=\Delta V+\xi_k{}^T\xi_k-\frac{\lambda^2}{2}\bar{\omega}_k^T\bar{\omega}_k<0. \tag{5.59}$$

By adding the left-hand side terms of the inequalities (5.57) and (5.58) to ΔV, we obtain

$$W_k\leq\begin{bmatrix}x_k\\u_k\\e_k\\\omega_k\\\bar{\zeta}_k\\\bar{\zeta}_k^d\end{bmatrix}^T\Omega\begin{bmatrix}x_k\\u_k\\e_k\\\omega_k\\\bar{\zeta}_k\\\bar{\zeta}_k^d\end{bmatrix}<0, \tag{5.60}$$

where Ω is on the form:

$$\Omega=\left[\begin{matrix}\mathcal{A}^T\mathcal{P}\mathcal{A}-\bar{E}^T\mathcal{P}\bar{E}+\mathcal{Q}+I_{2n} & \mathcal{A}^T\mathcal{P}\mathcal{A}_d\\ (\star) & \mathcal{A}_d^T\mathcal{P}\mathcal{A}_d-\mathcal{Q}\\ (\star) & (\star)\\ (\star) & (\star)\\ (\star) & (\star)\end{matrix}\right.$$

$$\left.\begin{matrix}\mathcal{A}^T\mathcal{P}\Xi_\omega+\bar{E}^T\mathcal{P}\Theta & \mathcal{A}^T\mathcal{P}\breve{\Sigma}+\bar{\mathcal{M}} & \mathcal{A}^T\mathcal{P}\breve{\Sigma}_d\\ \mathcal{A}_d^T\mathcal{P}\Xi_\omega & \mathcal{A}_d^T\mathcal{P}\breve{\Sigma} & \mathcal{A}_d^T\mathcal{P}\breve{\Sigma}_d+\bar{\mathcal{N}}\\ \Xi_\omega^T\mathcal{P}\Xi_\omega-\Theta^T\mathcal{P}\Theta-\mu I_{2r} & \Xi_\omega^T\mathcal{P}\breve{\Sigma} & \Xi_\omega^T\mathcal{P}\breve{\Sigma}_d\\ (\star) & \breve{\Sigma}^T\mathcal{P}\breve{\Sigma}-\bar{\Upsilon} & \breve{\Sigma}\mathcal{P}\breve{\Sigma}_d\\ (\star) & (\star) & \breve{\Sigma}_d^T\mathcal{P}\breve{\Sigma}_d-\bar{\Upsilon}_d\end{matrix}\right]. \tag{5.61}$$

From (5.60), we conclude that $\Omega<0$. The last matrix (5.61) is not linear and because of the singularity of the system $E^TP_1E\geq 0$, the analysis results developed for state-space systems, exploiting the symmetry and the positive-definiteness of the matrix $\mathcal{P}$ in the Lyapunov functional, cannot be applied on our descriptor system. In other words, if we try to use Schur lemma to decouple the cross terms, we end up with a non-strict linear inequality (zeros on the diagonal). For that reason, we adopt a different approach. The developed method consist of the following steps:

- Decoupling the cross terms related to the observer, i.e., the terms containing P_2 with R or R_d.
- Introducing a free matrix G in order to relax the resulting condition and replace the existing multiple product terms containing the matrix $\mathcal{P}$ by others easier to manipulate.
- Using the inequality of Young to treat the non-convex condition and transform it into a tractable LMI.

We start by applying Schur lemma on the observer part only. Thus, we can treat the bilinearity in that part separately. As a result, the cross terms stemmed from the presence of the observer are decoupled and the condition $\Omega < 0$ can be replaced by an equivalent inequality (5.62) as follows

$$\Omega_1 = \begin{bmatrix} \bar{\mathcal{A}}^T\mathcal{P}\bar{\mathcal{A}} - \bar{E}^T\mathcal{P}\bar{E} + \mathcal{Q} + I_{2n} & \bar{\mathcal{A}}^T\mathcal{P}\bar{\mathcal{A}}_d & \bar{\mathcal{A}}^T\mathcal{P}\bar{\Xi}_\omega + \bar{E}^T\mathcal{P}\Theta & \bar{\mathcal{A}}^T\mathcal{P}\bar{\Sigma} + \bar{\mathcal{M}} & \mathcal{A}^T\mathcal{P}\bar{\Sigma}_d & \begin{bmatrix} 0 \\ 0 \\ \left(P_2\mathcal{F}A - R^TC\right)^T \end{bmatrix} \\ (\star) & \bar{\mathcal{A}}_d^T\mathcal{P}\bar{\mathcal{A}}_d - \mathcal{Q} & \mathcal{A}_d^T\mathcal{P}\bar{\Xi}_\omega & \bar{\mathcal{A}}_d^T\mathcal{P}\bar{\Sigma} & \bar{\mathcal{A}}_d^T\mathcal{P}\bar{\Sigma}_d + \bar{\mathcal{N}} & \begin{bmatrix} 0 \\ 0 \\ \left(P_2\mathcal{F}A_d - R_d^TC\right)^T \end{bmatrix} \\ (\star) & (\star) & \bar{\Xi}_\omega^T\mathcal{P}\bar{\Xi}_\omega - \Theta^T\mathcal{P}\Theta - \mu I_{2r} & \bar{\Xi}_\omega^T\mathcal{P}\bar{\Sigma} & \bar{\Xi}_\omega^T\mathcal{P}\bar{\Sigma}_d & \begin{bmatrix} \left(R^TD_\omega - P_2\mathcal{F}E_\omega\right)^T \\ \left(R_d^TD_\omega\right)^T \end{bmatrix} \\ (\star) & (\star) & (\star) & \bar{\Sigma}^T\mathcal{P}\bar{\Sigma} - \bar{\Upsilon} & \bar{\Sigma}\mathcal{P}\bar{\Sigma}_d & \begin{bmatrix} 0 \\ 0 \\ \Sigma_2P_2 \end{bmatrix} \\ (\star) & (\star) & (\star) & (\star) & \bar{\Sigma}_d^T\mathcal{P}\bar{\Sigma}_d - \bar{\Upsilon}_d & \begin{bmatrix} 0 \\ 0 \\ \Sigma_{d2}P_2 \end{bmatrix} \\ (\star) & (\star) & (\star) & (\star) & (\star) & -P_2 \end{bmatrix} < 0, \quad (5.62)$$

with

$$\bar{\mathcal{A}} = \begin{bmatrix} A & B_u & 0 \\ K & -I_{n_u} & K \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{A}}_d = \begin{bmatrix} A_d & 0 & 0 \\ K_d & 0 & K_d \\ 0 & 0 & 0 \end{bmatrix}, \quad (5.63)$$

$$\bar{\Xi}_\omega = \begin{bmatrix} E_\omega & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{\Sigma} = \begin{bmatrix} \Sigma_1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\Sigma}_d = \begin{bmatrix} \Sigma_{d1} \\ 0 \\ 0 \end{bmatrix}. \quad (5.64)$$

According to the pioneering work of [OBG99] and some attempts in the literature [LH05], [CD12], it is possible when treating singular systems to use a relaxation technique based on injecting some free matrices. In this section, we will show how to generalize this method so it can be applied to singular time-delay systems with nonlinearities.
In order to do so, notice that $\Omega_1 < 0$ implies $\dot{V} < 0$. In addition, we can always find a matrix G such that $\mathcal{P} - G - G^T < 0$. Thus, from (5.62) we can write

$$\Omega_2 = \begin{bmatrix} \Omega_1 & 0 \\ 0 & \mathcal{P} - G - G^T \end{bmatrix} < 0. \quad (5.65)$$

From the structure of $\mathcal{P}$ and to reduce the number of cross terms, the matrix G can be chosen on the form:

$$G = \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{13} & G_{14} & 0 \\ 0 & 0 & G_4 \end{bmatrix}. \quad (5.66)$$

Define $\Lambda = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & -\bar{\mathcal{A}}^T \\ (\star) & I & 0 & 0 & 0 & 0 & -\bar{\mathcal{A}}_d^T \\ (\star) & (\star) & I & 0 & 0 & 0 & -\bar{\Xi}_\omega^T \\ (\star) & (\star) & (\star) & I & 0 & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & I & 0 & 0 \\ (\star) & (\star) & (\star) & (\star) & (\star) & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$.

Pre-multiplying (5.65) by Λ and post-multiplying by Λ^T and then by making these change of variables:

$$\mathcal{F} = \bar{\mathcal{A}}^T\mathcal{P} - \bar{\mathcal{A}}^T G, \tag{5.67}$$

$$\mathcal{F}_d = \bar{\mathcal{A}}_d^T\mathcal{P} - \bar{\mathcal{A}}_d^T G, \tag{5.68}$$

$$\mathcal{F}_\omega = \bar{\Xi}_\omega^T\mathcal{P} - \bar{\Xi}_\omega^T G, \tag{5.69}$$

we can get the inequality (5.70), in which the matrix $\mathcal{P}$ is not involved in any product with the matrices $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}_d^T$ and $\bar{\Xi}_\omega$.

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \mathcal{F}\bar{\mathcal{A}}_d + \bar{\mathcal{A}}^T\mathcal{F}_d^T & \mathcal{F}\Xi_\omega + \bar{\mathcal{A}}^T\mathcal{F}_\omega^T + \bar{E}^T S\Theta & \bar{\mathcal{A}}^T\mathcal{P}\bar{\Sigma} + \bar{\mathcal{M}} & \bar{\mathcal{A}}^T\mathcal{P}\bar{\Sigma}_d & \begin{bmatrix} 0 \\ 0 \\ \left(P_2 F A - R^T C\right)^T \end{bmatrix} & \mathcal{F} - \bar{\mathcal{A}}^T G^T \\ (\star) & -\mathcal{Q} + \mathcal{F}_d\bar{\mathcal{A}}_d + \bar{\mathcal{A}}_d^T\mathcal{F}_d^T & \mathcal{F}_d\bar{\Xi}_\omega + \bar{\mathcal{A}}_d^T\mathcal{F}_\omega^T & \bar{\mathcal{A}}_d^T\mathcal{P}\bar{\Sigma} & \bar{\mathcal{A}}_d^T\mathcal{P}\bar{\Sigma}_d + \bar{\mathcal{N}} & \begin{bmatrix} 0 \\ 0 \\ \left(P_2 F A_d - R_d^T C\right)^T \end{bmatrix} & \mathcal{F}_d - \bar{\mathcal{A}}_d^T G^T \\ (\star) & (\star) & \mathcal{F}_\omega\bar{\Xi}_\omega^T + \bar{\Xi}_\omega\mathcal{F}_\omega^T - \Theta^T\mathcal{P}\Theta - \mu I_{2r} & \bar{\Xi}_\omega^T\mathcal{P}\bar{\Sigma} & \bar{\Xi}_\omega^T\mathcal{P}\bar{\Sigma}_d & \begin{bmatrix} \left(R^T D_\omega - P_2 F E_\omega\right)^T \\ \left(R_d^T D_\omega\right)^T \end{bmatrix} & \mathcal{F}_\omega - \bar{\Xi}_\omega^T G^T \\ (\star) & (\star) & (\star) & \bar{\Sigma}^T\mathcal{P}\bar{\Sigma} - \bar{\Upsilon} & \bar{\Sigma}\mathcal{P}\bar{\Sigma}_d & \begin{bmatrix} 0 \\ 0 \\ \Sigma_2 P_2 \end{bmatrix} & 0 \\ (\star) & (\star) & (\star) & (\star) & \bar{\Sigma}_d^T\mathcal{P}\bar{\Sigma}_d - \bar{\Upsilon}_d & \begin{bmatrix} 0 \\ 0 \\ \Sigma_{d2} P_2 \end{bmatrix} & 0 \\ & & & (\star) & (\star) & -P_2 & 0 \\ & & & (\star) & (\star) & (\star) & \mathcal{P} - G - G^T \end{bmatrix} < 0.$$

$$\Gamma_{11} = -\bar{E}^T\mathcal{P}\bar{E} + \mathcal{F}\bar{\mathcal{A}} + \bar{\mathcal{A}}^T\mathcal{F}^T + \mathcal{Q} + I_{2n} \tag{5.70}$$

This technique relaxes the stability condition due to the presence of the extra degree of freedom provided by the introduction of the matrix G. Following [ZXS08], one can use a very interesting result ever since used to get rid of the restrictive equality constraint that for years characterize singular systems. The method consists in finding the matrices $S > 0$ and $L = L^T$ such that:

$$\mathcal{P} = S - \bar{E}^{\perp^T} L \bar{E}^{\perp}, \tag{5.71}$$

where $E^{\perp}$ is any matrix verifying $E^{\perp}E = 0$ and $E^{\perp}E^{\perp^T} > 0$.

In this case, the notation $\bar{E}^T \mathcal{P} \bar{E} \geq 0$ leads to a new condition in terms of a strict inequality $\bar{E}^T S \bar{E} > 0$ with S on the form:

$$S = \begin{bmatrix} \underbrace{\begin{bmatrix} S_{11} & S_{12} \\ (\star) & S_{22} \end{bmatrix}}_{S_1} & 0 \\ 0 & S_2 \end{bmatrix}.$$

Although the introduction of G has countered the problem of singularity, it added different multiple product terms that need to be decoupled. Due to the number of variables, the problem is difficult to treat. For that reason, we choose the matrices $\mathcal{F}$, $\mathcal{F}_d$ and $\mathcal{F}_\omega$ as follows:

$$\mathcal{F} = \begin{bmatrix} F_{11} & I_{n\times n_u}M & 0 \\ F_{13} & M & 0 \\ 0 & 0 & F_4 \end{bmatrix}, \tag{5.72}$$

$$\mathcal{F}_d = \begin{bmatrix} F_{d11} & I_{n\times n_u}M_d & 0 \\ F_{d13} & M_d & 0 \\ 0 & 0 & F_{d4} \end{bmatrix}, \tag{5.73}$$

$$\mathcal{F}_\omega = \begin{bmatrix} F_{\omega 11} & F_{\omega 12} & F_{\omega 21} \\ F_{\omega 13} & F_{\omega 14} & F_{\omega 22} \end{bmatrix}. \tag{5.74}$$

where $M,\ M_d \in \mathbb{R}^{n_u}$ are free positive matrices.
Notice that

$$\Gamma = \Psi + X^T Y + Y^T X, \tag{5.75}$$

where the components of Ψ are given by:

$$\Psi_{ij} = \Phi_{ij} \quad i, j \in [1, 5], \tag{5.76}$$

and Φ_{ij} are defined in equations (5.26)-(5.41).

$$X = \begin{bmatrix} 0 & I_{n\times n_u}M \\ 0 & M \\ 0 & 0 \\ I_{n\times n_u}M_d & 0 \\ M_d & 0 \\ 0 & 0 \\ F_{\omega 12} & F_{\omega 12} \\ F_{\omega 14} & F_{\omega 14} \\ G_{12} & G_{12} \\ G_{14} & G_{14} \\ 0 & 0 \\ \Sigma_1^T(S_{12} - L_{12}) & \Sigma_1^T(S_{12} - L_{12}) \\ \Sigma_{d1}^T(S_{12} - L_{12}) & \Sigma_{d1}^T(S_{12} - L_{12}) \end{bmatrix}, \quad Y = \begin{bmatrix} K^T & 0 \\ 0 & 0 \\ K^T & 0 \\ 0 & K_d^T \\ 0 & 0 \\ 0 & K_d^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.77}$$

Applying Young's inequality with symmetric and positive definite matrix Π, leads to

$$\Gamma \leq \Psi + \frac{1}{\epsilon} X^T \Pi^{-1} X + \epsilon Y^T \Pi Y,$$

which can be written of the form:

$$\Gamma \leq \Psi + \begin{bmatrix} X^T & Y^T\Pi \end{bmatrix} \begin{bmatrix} \frac{1}{\epsilon}\Pi^{-1} & 0 \\ 0 & \epsilon\Pi^{-1} \end{bmatrix} \begin{bmatrix} X \\ \Pi Y \end{bmatrix}.$$

Thus, according to Schur lemma, the condition $\Gamma < 0$ is verified if the following condition is fulfilled.

$$\begin{bmatrix} \Psi & \begin{bmatrix} X^T \\ Y^T\Pi \end{bmatrix} \\ (\star) & \begin{bmatrix} \epsilon\Pi & 0 \\ 0 & \frac{1}{\epsilon}\Pi \end{bmatrix}^{-1} \end{bmatrix} < 0. \tag{5.78}$$

By choosing $\Pi = \begin{bmatrix} M & 0 \\ 0 & M_d \end{bmatrix}$, and defining $K = M^{-1}\bar{K}$ and $K_d = M_d^{-1}\bar{K}_d$, we can easily get (5.25). □

5.4 Numerical example

In this section, we present a numerical example to show the performances of the proposed controller. We will consider a simple example of a discrete system of the form (5.1), where:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1.6 & 2 \\ 1 & 2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.24 & 0.32 \\ 0.16 & -0.27 \end{bmatrix}, \quad B_u = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.15 \end{bmatrix}, \quad E_\omega = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}, \quad D_\omega = 0.1.$$

and $f(x_k, x_{k-d}) = \begin{bmatrix} \sin(x_1(k)) \\ \sin(0.5x_2(k-d)) \end{bmatrix}$ with a delay $d = 5$.

The nonlinear function can be rewritten in the form $f(x_k, x_{k-d}) = \sum_{i=1}^{i=2} e_q(i) f_i\Big(\mathcal{H}_i x_k, \mathcal{H}_i^d x_{k-d}\Big)$, with

$$\mathcal{H}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathcal{H}_2^d = \begin{bmatrix} 0 & 0.5 \end{bmatrix}.$$

The disturbance ω is a Gaussian distributed random signal with mean zero and standard deviation $\sigma = 0.1$, which we will be added on two intervals of time only, as demonstrated later in the simulation, in order to show simultaneously the robustness and the asymptotic stabilisation of the system in closed loop, respectively with and without disturbances.
The bounds of the partial derivatives of f are:

$$a_{11} = -1, \; a_{22}^d = -0.5, \; b_{11} = 1, \; b_{22}^d = 0.5.$$

According to the Remark 4.2.5 we need to solve the LMI (5.25) with

$$\tilde{b}_{11} = b_{11} - a_{11} = 2, \; \tilde{b}_{22}^d = b_{22}^d - a_{22}^d = 1,$$

and

$$\tilde{A} = \begin{bmatrix} 1.5 & 2 \\ 1 & 2 \end{bmatrix}, \quad \tilde{A}_d = \begin{bmatrix} 0.24 & 0.32 \\ 0.16 & -0.26 \end{bmatrix}.$$

Hence, we obtain the following solutions:

$$L = 10^3 \begin{bmatrix} 1.5305 & 0.0044 \\ 0.0057 & 4.6440 \end{bmatrix}, \quad S_{12} = \begin{bmatrix} -12.9946 \\ 1.9365 \end{bmatrix},$$

$$S_{11} = 10^3 \begin{bmatrix} 0.0055 & -0.0010 \\ -0.0010 & 1.2872 \end{bmatrix}, \quad S_{22} = 1314.7,$$

$$S_2 = \begin{bmatrix} 897.6945 & 902.3377 \\ 902.3377 & 924.6203 \end{bmatrix},$$

$$G = 10^2 \begin{bmatrix} 0.990 & -0.409 & 0.001 & 0 & 0 \\ -0.409 & 0.059 & -0.058 & 0 & 0 \\ 0.050 & 0.107 & 0.006 & 0 & 0 \\ 0 & 0 & 0 & 10.924 & 4.512 \\ 0 & 0 & 0 & 4.512 & 11.058 \end{bmatrix}.$$

The observer-based controller gain matrices are given as follows:

$$\Pi_1 = \begin{bmatrix} -1.2763 & -0.7763 \\ 1.2788 & 0.7788 \end{bmatrix}, \quad \Pi_1^d = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} -0.0018 & 0.0782 \\ 0.0023 & -0.0777 \end{bmatrix}, \quad \Pi_2^d = \begin{bmatrix} 0.32 \\ -0.32 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.9605 & -1.1661 \end{bmatrix}, \quad K_d = \begin{bmatrix} -0.2762 & -0.3566 \end{bmatrix}.$$

and the optimal value of the disturbance attenuation level is $\lambda = 1.41$.
The simulation results represent the behaviour of the system subject to two types of control:

1. The control law: $u_k = 0.1 \sin(Tk)$, $T = 1ms$, (Figure 5.1-Figure 5.2).

2. The proposed controller $u_k = K\hat{x}(k) + K_d\hat{x}_d$ (Figure 5.3-Figure 5.4).

In both cases, we notice a good estimation of the state, and the robustness of the system to the presence of disturbances.

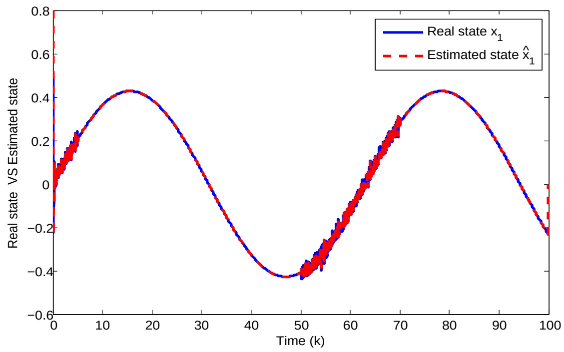

Figure 5.1: x_1 and its estimate

5.5 Conclusion

We presented in this chapter a new observer-based controller design method for a class of non-linear time-delay singular systems with disturbances. The nonlinearity of the considered system is assumed to be Lipschitz with respect to its arguments. A new sufficient LMI condition was

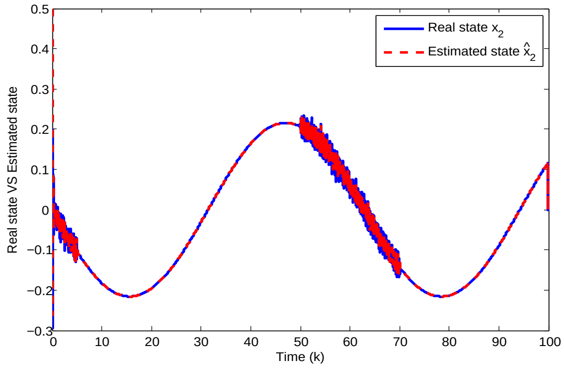

Figure 5.2: x_2 and its estimate

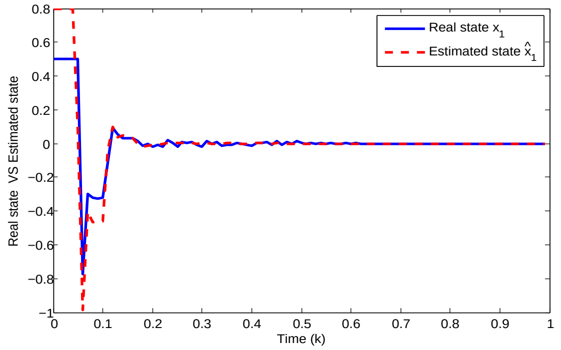

Figure 5.3: x_1 and its estimate

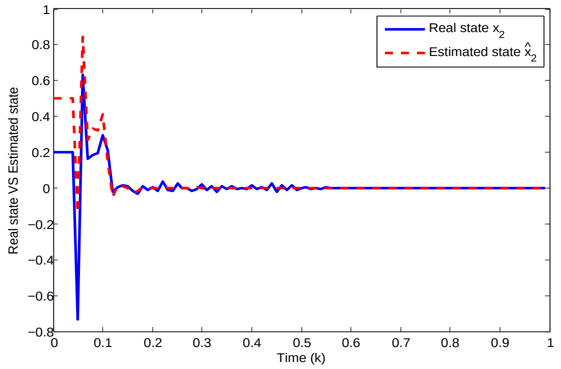

Figure 5.4: x_2 and its estimate

proposed to ensure the $\mathcal{H}_\infty$ robustness of the proposed observer-based controller despite the presence of disturbances. Different techniques were used to treat this class of systems. First, the approach of [OBG99] was used to transform the system in closed loop into a more exploitable form by means of injecting some free matrices. Second, the method of [ZXS08] was performed to get rid of the restrictive equality constraint which characterizes singular systems. Finally, the DMVT method was applied to linearize the nonlinear Lipschitz part. The main contribution of this chapter lies in using a particular Lyapunov-Krasovskii functional disturbance-dependent to get rid of the derivatives of the disturbance and using the Young's inequality to transform the non-convex problem into an LMI stability condition.

CHAPTER

6

Observer-based controller for unknown time-delay systems

Contents

6.1 Introduction

When dealing with time-delay systems, the choice of the Lyapunov Krasovskii functional presents one of the main difficulties. Considering a complete quadratic Lyapunov-Krasovskii functional, where the matrices are continuous functions, is not practical. Thus, some researchers choose what is known as the Discretized Lyapunov Functional (DLF) method [GKC03], where the matrices chosen to be piecewise linear functions. Discretization techniques were used in the literature accompanied with Jensen's inequality or Finsler's lemma to get LMI conditions. Inspired by the DLF method and based on the Lyapunov functional method, some stability criteria were derived by checking the variation of the Lyapunov functional in the whole interval of the delay. To study the delay-dependent stability and stabilization for systems with interval time-varying delay, the so called Delay Central-Point DCP method is employed. This method consists in dividing the delay interval into two subintervals [Yue04] and the variation of the Lyapunov functional is checked in both subintervals. The main advantage of these methods is that more information on the variation interval of the delay is employed and the delay central-point DCP state $x(t-d_0)$, (where $d_0 = \frac{d_M - d_m}{2}$ and d_M, d_m are the maximum and minimum delay respectively) is introduced. As an extension of the DCP method, the variation interval of the delay is divided into $l > 2$ parts with equal length in [YTZ09]. This method has been used to investigate the $\mathcal{H}_\infty$ control and filtering for networked control systems and stabilization with unknown input delay [Yue04]. In addition, it was exploited to design observers with unknown delay, the estimate of the delay is chosen to be the average of the delay [SFRS07] which is equivalent to the DCP method with two subintervals.

The study of linear systems with known delays has received considerable attention in the last decades [GKC03], [ZB07], [VA98], [Kha99] and [IXCY06], [PT08], [LdS97] for known time-varying delay in particular. The abundance of the existing results leads sometimes to the resemblance or even the equivalence between the proposed LMI conditions [XL07], [GP06]. The case of unknown delays has received less attention, due to the difficulty of the problem. However, Some attempts using sliding mode observer and adaptive observers were suggested to treat this problem [SOS00], [SFRS09].

On the other hand, Free Weighting Matrix (FWM) method was proposed to reduce conservatism and introduce delay-derivative-dependent stability conditions [WHS10], [HWLW07]. This method uses some free matrices to express the relationships among the terms of the Newton-Leibnitz formula. The results obtained by this technique generalizes some of the existing work such as [FS02b] combined with Park's inequality (presented in Appendix A).

When using the observer in closed-loop configurations, the problem of stability analysis becomes more complicated. The available solutions to this problem generally involve iterative linear matrix inequality conditions as in the case of FWM method [WHS10] or constrained convex optimization conditions that involve some equality constraints [Lie04].

In this chapter, we extend the DCP method and combine it with the FWM method to design an observer-based controller for unknown time-delay systems. In our case, we make use of the DCP method in the design of the observer itself. We divide the delay interval into different subintervals. The estimate of the delay over each segment will be defined as the mean value of that segment. In addition, motivated by the FWM method and using an appropriate Lyapunov-Krasovskii functional, we obtain a delay-dependent stability condition. This condition will be written in the form of LMI as opposed to some existing result in the literature. It is worth mentioning that the method is developed for time-delay systems where the nonlinearity depends only on the present state.

6.2 Problem formulation

In this section, we aim to design an observer-based controller to estimate the state x and stabilize the system in the closed loop without knowing the value of the delay at each instant.

6.2.1 System presentation

First, let us consider the following class of continuous nonlinear systems with delayed state:

$$\begin{aligned} \dot{x}(t) =& Ax(t) + A_d x\left(t - d(t)\right) + B_u u(t) + Bf(x(t)), \\ y(t) =& Cx(t). \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^p$ is the output. A, A_d, B_u, B, and C are constant matrices of adequate dimensions.
The nonlinear function $f \ : \ \mathbb{R}^n \longrightarrow \ \mathbb{R}^q$ is assumed to be γ_f-Lipschitz, i.e.,

$$\|f(x) - f(y)\| \leq \gamma_f \|x - y\| \quad \forall\, x, y \in \mathbb{R}^n. \tag{6.2}$$

Without loss of generality we assume that $f(0) = 0$.
The delay is assumed to be unknown time-varying and satisfies both of the following conditions:

$$\begin{aligned} 0 \leq d(t) \leq \bar{d}, \\ \dot{d}(t) \leq \mu. \end{aligned} \tag{6.3}$$

Following Chapter 2-Lemma 2.2.3, the Lipschitz property is equivalent to the existence of functions

$$\bar{\psi}_{ij} \ : \ \mathbb{R}^n \longrightarrow \ \mathbb{R}$$

and constants a_{ij} and b_{ij} such that we can rewrite the function f as follows:

$$f(x) - f(0) = f(x) = \left(\sum_{i=1}^{i=q} \sum_{j=1}^{j=n} \bar{\psi}_{ij} H_{ij}(j) \right) x. \tag{6.4}$$

with $a_{ij} \leq \bar{\psi}_{ij} \leq b_{ij}$, $\bar{\psi}_{ij} \triangleq \bar{\psi}_{ij}\left(x^{0_{j-1}}, x^{0_j}\right)$ and $H_{ij} = e_q(i) e_n^T(j)$.

6.2.2 Observer-based controller design

In this subsection, we address the problem of observer-based control design for continuous time-delay systems with unknown delay (6.1). The method consists in dividing the maximum delay interval $[0, \bar{d}]$ into r divisions $[d_{i-1},\ d_i]$ with $i = 1, \ldots, r$, not necessarily of equal length, then considering the mean value of each subinterval as the estimate of the delay. The delay estimate is chosen to be the arithmetic mean over each subinterval $\hat{d}_i = \frac{d_{i-1}+d_i}{2}$. In addition, we define the constants α_i which verify $\sum\limits_{i=1}^{i=r} \alpha_i = 1$, and $\alpha_i \ \geq \ 0$.
A delay-dependent stability criteria is given first in the form of BMI then using a method based on Young's inequality, the condition is transformed into LMI.
We propose to use a memoryless controller depending on the mean value of each interval instead on the exact value of the delay:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + A_d \sum_{i=1}^{i=r} \alpha_i \hat{x}\left(t - \hat{d}_i\right) + K\left(y(t) - \hat{y}(t)\right) + B_u u(t) + Bf(\hat{x}(t)),$$

$$+\sum_{i=1}^{i=r}\alpha_i K_{di}\left(y\left(t-\hat{d}_i\right)-\hat{y}\left(t-\hat{d}_i\right)\right), \tag{6.5}$$

$$u(t) = -L\hat{x}(t), \tag{6.6}$$

Where K, K_{di} $(i = 1, \ldots, r)$ and L are gain matrices to be determined.
By calculating the error $e(t) = x(t) - \hat{x}(t)$, we find:

$$\dot{e}(t) = (A - KC)e(t) + \sum_{i=1}^{i=r}\alpha_i\left(A_d - K_{di}C\right)e\left(t-\hat{d}_i\right) + B(f(x(t)) - f(\hat{x}(t)))$$
$$+ A_d\sum_{i=1}^{i=r}\alpha_i\left(x\left(t-d(t)\right) - x\left(t-\hat{d}_i\right)\right). \tag{6.7}$$

The nonlinear function can be written, as seen in Chapter 2-Lemma 2.2.3, in the form:

$$f(x) - f(\hat{x}) = \left(\sum_{i=1}^{i=q}\sum_{j=1}^{j=n}\psi_{ij}H_{ij}\right)(x-\hat{x}), \tag{6.8}$$

where the function ψ_{ij} is bounded as follows: $a_{ij} \leq \psi_{ij} \leq b_{ij}$ and $\psi_{ij} \triangleq \psi_{ij}\left(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\right)$.
Define the parameter matrices ϱ and ρ as:

$$\varrho = \left(\bar{\psi}_{ij}\left(x^{0_{j-1}}, x^{0_j}\right)\right)_{ij}, \quad \rho = \left(\psi_{ij}\left(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\right)\right)_{ij}.$$

Then, ϱ and ρ belong to a bounded convex set $\mathcal{H}_n$ for which the sets of vertices are defined by:

$$\mathcal{V}_{\mathcal{H}_n} = \left\{\Phi \in \mathbb{R}^{q\times n} \ : \ \Phi_{ij} \in \{a_{ij}, b_{ij}\}\right\}. \tag{6.9}$$

Consequently, the dynamics of the system and the error can be rewritten as follows:

$$\dot{x}(t) = \mathcal{A}(\varrho)x(t) + A_d x\left(t-d(t)\right) + B_u u(t), \tag{6.10}$$

$$\dot{e}(t) = (\mathcal{A}(\rho) - KC)e(t) + \sum_{i=1}^{i=r}\alpha_i\left(A_d - K_{di}C\right)e\left(t-\hat{d}_i\right) + A_d\sum_{i=1}^{i=r}\alpha_i\left(x\left(t-d(t)\right) - x\left(t-\hat{d}_i\right)\right), \tag{6.11}$$

with

$$\mathcal{A}(\varrho) = A + B\sum_{i=1}^{i=q}\sum_{j=1}^{j=n}\bar{\psi}_{ij}H_{ij}, \quad \mathcal{A}(\rho) = A + B\sum_{i=1}^{i=q}\sum_{j=1}^{j=n}\psi_{ij}H_{ij}. \tag{6.12}$$

The systems in the closed loop can be described as follows:

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}\mathcal{A}(\varrho) - B_u L & B_u L\\ 0 & \mathcal{A}(\rho) - KC\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} + \begin{bmatrix}A_d & 0\\ A_d & 0\end{bmatrix}\begin{bmatrix}x_{d(t)}\\ e_{d(t)}\end{bmatrix} + \begin{bmatrix}0\\ A_{d2} - A_{d1} - K_d C\end{bmatrix}\begin{bmatrix}x_{\hat{d}_1}\\ e_{\hat{d}_1}\\ \vdots\\ x_{\hat{d}_r}\\ e_{\hat{d}_r}\end{bmatrix}, \tag{6.13}$$

with

$$A_{d1} = \left[\begin{bmatrix}\alpha_1 A_d & 0\end{bmatrix} \quad \dots \quad \begin{bmatrix}\alpha_r A_d & 0\end{bmatrix}\right], \tag{6.14}$$
$$A_{d2} = \left[\begin{bmatrix}0 & \alpha_1 A_d\end{bmatrix} \quad \dots \quad \begin{bmatrix}0 & \alpha_r A_d\end{bmatrix}\right], \tag{6.15}$$
$$K_d = \left[\begin{bmatrix}0 & \alpha_1 K_{d1}\end{bmatrix} \quad \dots \quad \begin{bmatrix}0 & \alpha_r K_{dr}\end{bmatrix}\right]. \tag{6.16}$$

By defining

$$\xi = \begin{bmatrix} x \\ e \end{bmatrix}, \tag{6.17}$$
$$\xi\left(t-\hat{d}\right) = \begin{bmatrix} x(t-\hat{d}_1) \\ e(t-\hat{d}_1) \\ \vdots \\ x(t-\hat{d}_r) \\ e(t-\hat{d}_r) \end{bmatrix}, \tag{6.18}$$
$$\mathbb{A} = \begin{bmatrix} \mathcal{A}(\varrho) & 0 \\ 0 & \mathcal{A}(\rho) \end{bmatrix}, \tag{6.19}$$
$$\mathbb{A}_d = \begin{bmatrix} A_d & 0 \\ A_d & 0 \end{bmatrix}, \tag{6.20}$$
$$\mathbb{A}_{\hat{d}} = \begin{bmatrix} 0 \\ A_{d2} - A_{d1} \end{bmatrix}, \tag{6.21}$$
$$\mathbb{K} = \begin{bmatrix} B_u L & -B_u L \\ 0 & KC \end{bmatrix}, \tag{6.22}$$
$$\mathbb{K}_d = \begin{bmatrix} 0 \\ K_d C \end{bmatrix}. \tag{6.23}$$

the system can be rewritten as follows:

$$\dot{\xi}(t) = (\mathbb{A} - \mathbb{K})\,\xi(t) + \mathbb{A}_d \xi\left(t - d(t)\right) + \left(\mathbb{A}_{\hat{d}} - \mathbb{K}_d\right)\xi\left(t-\hat{d}\right). \tag{6.24}$$

The objective is to find the matrices K, K_d and L such that the system in the closed loop is asymptotically stable.

6.3 Main results: Observer-based controller synthesis

In this section, we shall study and derive sufficient conditions ensuring the asymptotic stability of the system under the action of an observer-based feedback. The results are summarized in the following theorem.

Theorem 6.3.1. *System* (6.1) *is asymptotically stable under the action of the observer-based controller* (6.6)*, if there exist:*

- *two predefined positive scalars: ϵ and $\bar{\epsilon}$;*
- *symmetric and positive definite matrices: $\bar{P}$, $\bar{T}_{12}$, W_{11}, $\bar{Z}_1$, and $\bar{Z}_{2l}$ $(l = 1, \dots, r)$;*

- *symmetric positive semi-definite matrices:* Q_1, Q_2, $\bar{X} = \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} & \bar{X}_{13} \\ (\star) & \bar{X}_{22} & \bar{X}_{23} \\ (\star) & (\star) & \bar{X}_{33} \end{bmatrix}$ *and* $\bar{Y}_l = \begin{bmatrix} \bar{Y}_{l11} & \bar{Y}_{l12} & \bar{Y}_{l13} \\ (\star) & \bar{Y}_{l22} & \bar{Y}_{l23} \\ (\star) & (\star) & \bar{Y}_{l33} \end{bmatrix}$ $(l = 1, \ldots, r)$*;*

- *any appropriately dimensioned matrices* R, R_d, S, $\bar{M}_i$, $\bar{N}_i$ $(i = 1, \ldots, 3)$, $\bar{T}_i = \begin{bmatrix} \bar{T}_{i1} & 0 \\ 0 & T_{i2} \end{bmatrix}$ $(i = 2, 3)$*, and* $\bar{T}_4 = \begin{bmatrix} \bar{T}_{41} & T_{42} \end{bmatrix}$ *with* $\bar{T}_{41} = \begin{bmatrix} \bar{T}_{41}^1 \\ 0 \\ \vdots \\ \bar{T}_{41}^r \\ 0 \end{bmatrix}$, $T_{42} = \begin{bmatrix} 0 \\ T_{42}^1 \\ \vdots \\ 0 \\ T_{42}^r \end{bmatrix}$

so that the following LMI conditions hold:

$$\left[\begin{array}{ccccccccc}
 & W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T & 0 & 0 & 0 & 0 & 0 & B_u S & 0 \\
 & 0 & \mathcal{A}(\rho)^T T_{12} - C^T R^T & 0 & 0 & I_n & 0 & 0 & T_{12} \\
 & W_{11} A_d^T & 0 & \bar{T}_{21} & 0 & 0 & W_{11} A_d^T & 0 & 0 \\
\bar{\Gamma} & 0 & 0 & 0 & T_{22} & 0 & 0 & 0 & 0 \\
 & 0 & 0 & \bar{T}_{31} & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & T_{32} & 0 & 0 & 0 & 0 \\
 & 0 & 0 & \bar{T}_{41} & 0 & 0 & -W_{11} A_{d1}^T & 0 & 0 \\
 & 0 & A_{d2}^T T_{12} - C^T R_d^T & 0 & T_{42} & 0 & 0 & 0 & 0 \\
 & -\epsilon W_{11} & 0 & 0 & 0 & 0 & 0 & B_u S & 0 \\
(\star) & & -\epsilon T_{12}^T & 0 & 0 & 0 & 0 & 0 & T_{12} \\
 & & & -\frac{1}{\epsilon} W_{11} & 0 & 0 & 0 & 0 & 0 \\
 & & & & -\frac{1}{\epsilon} T_{12}^T & 0 & 0 & 0 & 0 \\
 & & & & & -\bar{\epsilon} W_{11} & 0 & 0 & 0 \\
 & & (\star) & & & & -\bar{\epsilon} I_n & 0 & 0 \\
 & & & & & & & -\frac{1}{\bar{\epsilon}} W_{11} & 0 \\
 & & & & & & (\star) & & -\frac{1}{\bar{\epsilon}} I_n
\end{array}\right] < 0,$$

$$\forall\ \varrho,\ \rho \in \mathcal{V}_{\mathcal{H}_n}, \quad (6.25)$$

$$\bar{\Psi} = \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} & \bar{X}_{13} & \bar{M}_1 \\ (\star) & \bar{X}_{22} & \bar{X}_{23} & \bar{M}_2 \\ (\star) & (\star) & \bar{X}_{33} & \bar{M}_3 \\ (\star) & (\star) & (\star) & \bar{Z}_1 \end{bmatrix} \geq 0, \quad (6.26)$$

$$\bar{\Phi}_l = \begin{bmatrix} \bar{Y}_{l11} & \bar{Y}_{l12} & \bar{Y}_{l13} & \bar{N}_{1l} \\ (\star) & \bar{Y}_{l22} & \bar{Y}_{l23} & \bar{N}_{2l} \\ (\star) & (\star) & \bar{Y}_{l33} & \bar{N}_{3l} \\ (\star) & (\star) & (\star) & \bar{Z}_{2l} \end{bmatrix} \geq 0 \quad l = 1, \ldots, r. \quad (6.27)$$

Then, the observer-based controller gains are given by:

$$K = T_{12}^{-1} R, \quad K_d = T_{12}^{-1} R_d, \quad (6.28)$$

$$L = SW_{11}^{-1}. \tag{6.29}$$

where

$$\begin{aligned}
\bar{\Gamma}_{11} &= \bar{Q}_1 + \bar{Q}_2\bar{I} + \bar{M}_1 + \bar{M}_1^T + \bar{N}_1\bar{I} + \bar{I}^T\bar{N}_1^t + \bar{d}\bar{X}_{11} + \bar{Y}_{11}\bar{I}, \\
&\quad + \begin{bmatrix} -T_{11}^{-1}\mathcal{A}(\varrho)^T + S^TB_u^T - \mathcal{A}(\varrho)T_{11}^{-1} + B_uS & 0 \\ 0 & -\mathcal{A}(\rho)^TT_{12}^T + C^TR^T - T_{12}\mathcal{A}(\rho) + RC \end{bmatrix}, \\
\bar{\Gamma}_{12} &= \bar{M}_2^T - \bar{M}_1 + \bar{d}\bar{X}_{12} - \begin{bmatrix} A_dT_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\
\bar{\Gamma}_{13} &= \bar{P} + \bar{M}_3^T + \bar{I}^T\bar{N}_3^t + T_1 + \bar{d}\bar{X}_{13} + \bar{Y}_{13}\bar{I}, \\
\bar{\Gamma}_{14} &= -\bar{N}_1 + \bar{N}_2^T + \bar{Y}_{12} - \begin{bmatrix} 0 \\ T_{12}A_{d2} - R_dC \end{bmatrix}, \\
\bar{\Gamma}_{22} &= -(1-\mu)\bar{Q}_1 - \bar{M}_2 - \bar{M}_2^T + \bar{d}\bar{X}_{22}, \\
\bar{\Gamma}_{23} &= -\bar{M}_3^T + \bar{T}_2 + \bar{d}\bar{X}_{23}, \\
\bar{\Gamma}_{33} &= \bar{d}\bar{Z}_1 + \bar{Z}_2\bar{I} + \bar{T}_3 + \bar{T}_3^T + \bar{d}\bar{X}_{33} + \bar{Y}_{33}\bar{I}, \\
\bar{\Gamma}_{34} &= -\bar{N}_3 + \bar{Y}_{32} + \bar{T}_4^T, \\
\bar{\Gamma}_{44} &= -\tilde{Q}_2 - \tilde{N}_2 - \tilde{N}_2^T + \tilde{Y}_2, \\
\bar{\Gamma}_{ij} &= 0 \quad \textit{Otherwise}.
\end{aligned} \tag{6.30}$$

with

$$\bar{I} = \begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix}, \tag{6.31}$$

$$\bar{Y}_{ij} = \begin{bmatrix} \alpha_1\hat{d}_1\bar{Y}_{1ij} & \dots & \alpha_r\hat{d}_r\bar{Y}_{rij} \end{bmatrix}, \tag{6.32}$$

$$\tilde{Y}_2 = \begin{pmatrix} \alpha_1\hat{d}_1\bar{Y}_{122} & \dots & 0 \\ \vdots & \alpha_i\hat{d}_i\bar{Y}_{i22} & \vdots \\ 0 & \dots & \alpha_r\hat{d}_r\bar{Y}_{r22} \end{pmatrix}, \tag{6.33}$$

$$\bar{N}_i = \begin{bmatrix} \alpha_1\bar{N}_{i1} & \dots & \alpha_r\bar{N}_{ir} \end{bmatrix} \quad i = 1, \dots, 3, \tag{6.34}$$

$$\bar{N}_i^t = \begin{bmatrix} \alpha_1\bar{N}_{i1}^T & \dots & \alpha_r\bar{N}_{ir}^T \end{bmatrix} \quad i = 1, \dots, 3, \tag{6.35}$$

$$\tilde{N}_2 = \begin{pmatrix} \alpha_1\bar{N}_{21} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_i\bar{N}_{2i} & 0 \\ 0 & \dots & 0 & \alpha_r\bar{N}_{2r} \end{pmatrix}, \tag{6.36}$$

$$\bar{Q}_2 = \begin{bmatrix} \alpha_1\bar{Q}_{21} & \dots & \alpha_r\bar{Q}_{2r} \end{bmatrix}, \tag{6.37}$$

$$\tilde{Q}_2 = \begin{pmatrix} \alpha_1\bar{Q}_{21} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_i\bar{Q}_{2i} & 0 \\ 0 & \dots & 0 & \alpha_r\bar{Q}_{2r} \end{pmatrix}, \tag{6.38}$$

$$\bar{Z}_2 = \begin{bmatrix} \alpha_1\hat{d}_1\bar{Z}_{21} & \dots & \alpha_r\hat{d}_r\bar{Z}_{2r} \end{bmatrix}. \tag{6.39}$$

Proof. In order to simplify the proof, we chose to divide it into two main parts. The first contains the Lyapunov stability which lead to a BMI condition ensuring the stability of the studied system in closed loop. The second part focuses on linearizing the BMI to reach a tractable LMI.

1. *Sufficient condition in the form of BMI:*
At first, let us use the following delay-dependent Lyapunov-Krasovskii functional:

$$V(\xi(t)) = V_1(t) + V_2(t) + \sum_{i=1}^{i=r} \alpha_i V_{3,i}(t) + V_4(t) + \sum_{i=1}^{i=r} \alpha_i V_{5,i}(t). \tag{6.40}$$

where

$$\begin{aligned} V_1(t) =& \xi^T(t)P\xi(t), \\ V_2(t) =& \int_{t-d(t)}^{t} \xi^T(s)Q_1\xi(s)\mathrm{d}s, \\ V_{3,i}(t) =& \int_{t-\hat{d}_i}^{t} \xi^T(s)Q_{2i}\xi(s)\mathrm{d}s, \\ V_4(t) =& \int_{-d(t)}^{0}\int_{t+\theta}^{t} \dot{\xi}^T(s)Z_1\dot{\xi}(s)dsd\theta, \\ V_{5,i}(t) =& \int_{-\hat{d}_i}^{0}\int_{t+\theta}^{t} \dot{\xi}^T(s)Z_{2i}\dot{\xi}(s)dsd\theta, \\ & \sum_{i=1}^{i=r} \alpha_i = 1. \end{aligned} \tag{6.41}$$

Consequently

$$\begin{aligned} \dot{V}_1 &= 2\xi^T(t)P\dot{\xi}(t), \\ \dot{V}_2 &= \xi^T(t)Q_1\xi(t) - (1-\dot{d}(t))\xi^T(t-d(t))Q_1\xi(t-d(t)), \\ \dot{V}_{3,i} &= \xi^T(t)Q_{2i}\xi(t) - \xi^T\left(t-\hat{d}_i\right)Q_{2i}\xi\left(t-\hat{d}_i\right), \\ \dot{V}_4 &= d(t)\dot{\xi}^T(t)Z_1\dot{\xi}(t) - (1-\dot{d}(t))\int_{t-d(t)}^{t} \dot{\xi}^T(s)Z_1\dot{\xi}(s), \\ \dot{V}_{5,i} &= \hat{d}_i\dot{\xi}^T(t)Z_{2i}\dot{\xi}(t) - \int_{t-\hat{d}_i}^{t} \dot{\xi}^T(s)Z_{2i}\dot{\xi}(s). \end{aligned} \tag{6.42}$$

From notation (6.3), we get

$$\begin{aligned} \dot{V} \leq& \xi^T(t)\left(Q_1 + Q_2\bar{I}\right)\xi(t) + 2\xi^T(t)P\dot{\xi}(t) - (1-\mu)\,\xi^T\left(t-d(t)\right)Q_1\xi\left(t-d(t)\right) \\ &- \xi^T\left(t-\hat{d}\right)\bar{Q}_2\xi\left(t-\hat{d}\right) + \dot{\xi}^T(t)\left(\bar{d}Z_1 + Z_2\bar{I}\right)\dot{\xi}(t) \\ &- (1-\mu)\int_{t-d(t)}^{t} \dot{\xi}^T(s)Z_1\dot{\xi}(s) - \sum_{i=1}^{i=r}\alpha_i\int_{t-\hat{d}_i}^{t} \dot{\xi}^T(s)Z_{2i}\dot{\xi}(s). \end{aligned} \tag{6.43}$$

From the dynamics of the system in closed loop (6.24), we can write:

$$2\left[\xi^T(t)T_1 + \xi^T\left(t-d(t)\right)T_2 + \dot{\xi}^T(t)T_3 + e^T\left(t-\hat{d}\right)T_4\right] \times \left[\dot{\xi}(t) - (\mathbb{A}-\mathbb{K})\,\xi(t)\right.$$

$$- \mathbb{A}_d \xi\left(t - d(t)\right) - \left(\mathbb{A}_{\hat{d}} - \mathbb{K}_d\right) \xi\left(t - \hat{d}\right)\Big]. \quad (6.44)$$

with T_i $(i = 1, \ldots, 4)$ are matrices of adequate dimensions.
In addition, using the Leibnitz-Newton formula, we can write

$$\begin{aligned} \xi\left(t - d(t)\right) &= \xi(t) - \int_{t-d(t)}^{t} \dot{\xi}(s)\mathrm{d}s, \\ \xi\left(t - \hat{d}_i\right) &= \xi(t) - \int_{t-\hat{d}_i}^{t} \dot{\xi}(s)\mathrm{d}s \quad i = 1, \ldots, r. \end{aligned} \quad (6.45)$$

Rather than using the Newton-Leibnitz formula to directly replace the delay term, we use the Free Weighting Matrices (FWM) method discussed in [WHS10] to take into account the relationships among the terms of the Newton-Leibnitz formula in order to derive a delay-dependent stability criteria for system (6.24).
Hence, for any appropriately dimensioned matrices M_1, M_2 and M_3, we have

$$2\left[\xi^T(t)M_1 + \xi^T\left(t - d(t)\right)M_2 + \dot{\xi}^T(t)M_3\right] \times \left[\xi(t) - \int_{t-d(t)}^{t} \dot{\xi}(s)\mathrm{d}s - \xi\left(t - d(t)\right)\right] = 0. \quad (6.46)$$

Also, for any N_{1l}, N_{2l} and N_{3l} $(l = 1, \ldots, r)$, we have:

$$2\alpha_l\left[\xi^T(t)N_{1l} + \xi^T\left(t - \hat{d}_l\right)N_{2l} + \dot{\xi}^T(t)N_{3l}\right] \times \left[\xi(t) - \int_{t-\hat{d}_l}^{t} \dot{\xi}(s)\mathrm{d}s - \xi\left(t - \hat{d}_l\right)\right] = 0. \quad (6.47)$$

On the other hand, for any matrices $X \geq 0$, $Y_i \geq 0$ $(i = 1, \ldots, 3)$, the following holds:

$$\begin{aligned} &\bar{d}\xi_1^T(t)X\xi_1(t) - \int_{t-d(t)}^{t} \xi_1^T(t)X\xi_1(t)\mathrm{d}s \geq 0, \\ &\hat{d}_i\xi_2^T(t)Y_i\xi_2(t) - \int_{t-\hat{d}_i}^{t} \xi_2^T(t)Y_i\xi_2(t)\mathrm{d}s \geq 0 \quad i = 1, \ldots, 3. \end{aligned} \quad (6.48)$$

where $\xi_1 = [\ \xi^T(t) \quad \xi^T\left(t - d(t)\right) \quad \dot{\xi}^T(t)\ ]^T$ and $\xi_2 = [\ \xi^T(t) \quad \xi^T\left(t - \hat{d}_i\right) \quad \dot{\xi}^T(t)\ ]^T$.
Finally, by adding equations (6.44) and (6.46)-(6.48) to (6.43), we conclude that

$$\dot{V}(t) \leq \zeta^T(t)\Omega\zeta(t) - \int_{t-d(t)}^{t} {\zeta_1}^T(t,s)\Psi\zeta_1(t,s)\mathrm{d}s - \sum_{i=1}^{i=r}\alpha_i \int_{t-\hat{d}_i}^{t} {\zeta_2}^T(t,s)\Phi_i\zeta_2(t,s)\mathrm{d}s, \quad (6.49)$$

where

$$\zeta_1(t,s) = \left[\xi_1^T(t) \quad \dot{\xi}^T(s)\right]^T, \quad (6.50)$$

$$\zeta_2(t,s) = \left[\xi_2^T(t) \quad \dot{\xi}^T(s)\right]^T, \quad (6.51)$$

$$\zeta(t) = \left[\xi(t) \quad \xi\left(t - d(t)\right) \quad \dot{\xi}(t) \quad \xi\left(t - \hat{d}\right)\right]^T, \quad (6.52)$$

and the matrices:

$$\Psi = \begin{bmatrix} X_{11} & X_{12} & X_{13} & M_1 \\ (\star) & X_{22} & X_{23} & M_2 \\ (\star) & (\star) & X_{33} & M_3 \\ (\star) & (\star) & (\star) & Z_1 \end{bmatrix}, \quad (6.53)$$

$$\Phi_l = \begin{bmatrix} Y_{l11} & Y_{l12} & Y_{l13} & N_{1l} \\ (\star) & Y_{l22} & Y_{l23} & N_{2l} \\ (\star) & (\star) & Y_{l33} & N_{3l} \\ (\star) & (\star) & (\star) & Z_{2l} \end{bmatrix} \quad l = 1, \ldots, r, \tag{6.54}$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ (\star) & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ (\star) & (\star) & \Omega_{33} & \Omega_{34} \\ (\star) & (\star) & (\star) & \Omega_{44} \end{bmatrix}, \tag{6.55}$$

where

$$\begin{aligned} \Omega_{11} =& Q_1 + Q_2\bar{I} + M_1 + M_1^T + N_1\bar{I} + \bar{I}^T N_1^t - T_1(\mathbb{A} - \mathbb{K}) - (\mathbb{A} - \mathbb{K})^T T_1 \\ &+ \bar{d}X_{11} + Y_{11}\bar{I}, \end{aligned} \tag{6.56}$$

$$\Omega_{12} = M_2^T - M_1 - T_1\mathbb{A}_d - (\mathbb{A} - \mathbb{K})^T T_2^T + \bar{d}X_{12}, \tag{6.57}$$

$$\Omega_{13} = P + M_3^T + \bar{I}^T N_3^t + T_1 - (\mathbb{A} - \mathbb{K})^T T_3^T + \bar{d}X_{13} + Y_{13}\bar{I}, \tag{6.58}$$

$$\Omega_{14} = -N_1 + N_2^T - T_1\left(\mathbb{A}_{\hat{d}} - \mathbb{K}_d\right) - (\mathbb{A} - \mathbb{K})^T T_4^T + Y_{12}, \tag{6.59}$$

$$\Omega_{22} = -(1-\mu)Q_1 - M_2 - M_2^T + \bar{d}X_{22} - T_2\mathbb{A}_d - \mathbb{A}_d^T T_2^T, \tag{6.60}$$

$$\Omega_{23} = -M_3^T - \mathbb{A}_d^T T_3^T + T_2 + \bar{d}X_{23}, \tag{6.61}$$

$$\Omega_{24} = -\mathbb{A}_d^T T_4^T - T_2\left(\mathbb{A}_{\hat{d}} - \mathbb{K}_d\right), \tag{6.62}$$

$$\Omega_{33} = \bar{d}Z_1 + Z_2\bar{I} + T_3 + T_3^T + \bar{d}X_{33} + Y_{33}\bar{I}, \tag{6.63}$$

$$\Omega_{34} = -N_3 + Y_{32} + T_4^T - T_3\left(\mathbb{A}_{\hat{d}} - \mathbb{K}_d\right), \tag{6.64}$$

$$\Omega_{44} = -\breve{Q}_2 - \breve{N}_2 - \breve{N}_2^T + \breve{Y}_2 - T_4\left(\mathbb{A}_{\hat{d}} - \mathbb{K}_d\right) - \left(\mathbb{A}_{\hat{d}} - \mathbb{K}_d\right)^T T_4^T, \tag{6.65}$$

with

$$Y_{ij} = \begin{bmatrix} \alpha_1\hat{d}_1 Y_{1ij} & \ldots & \alpha_r\hat{d}_r Y_{rij} \end{bmatrix}, \tag{6.66}$$

$$\tilde{Y}_2 = \begin{pmatrix} \alpha_1\hat{d}_1 Y_{122} & \ldots & 0 \\ \vdots & \alpha_i\hat{d}_i Y_{i22} & \vdots \\ 0 & \ldots & \alpha_r\hat{d}_r Y_{r22} \end{pmatrix}, \tag{6.67}$$

$$N_i = \begin{bmatrix} \alpha_1 N_{i1} & \ldots & \alpha_r N_{ir} \end{bmatrix} \quad i = 1, \ldots, 3, \tag{6.68}$$

$$N_i^t = \begin{bmatrix} \alpha_1 N_{i1}^T & \ldots & \alpha_r N_{ir}^T \end{bmatrix} \quad i = 1, \ldots, 3, \tag{6.69}$$

$$\breve{N}_2 = \begin{pmatrix} \alpha_1 N_{21} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_i N_{2i} & 0 \\ 0 & \ldots & 0 & \alpha_r N_{2r} \end{pmatrix}, \tag{6.70}$$

$$Q_2 = \begin{bmatrix} \alpha_1 Q_{21} & \ldots & \alpha_r Q_{2r} \end{bmatrix}, \tag{6.71}$$

$$\breve{Q}_2 = \begin{pmatrix} \alpha_1 Q_{21} & 0 & \ldots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \alpha_i Q_{2i} & 0 \\ 0 & \ldots & 0 & \alpha_r Q_{2r} \end{pmatrix}, \tag{6.72}$$

$$Z_2 = \begin{bmatrix} \alpha_1\hat{d}_1 Z_{21} & \ldots & \alpha_r\hat{d}_r Z_{2r} \end{bmatrix}. \tag{6.73}$$

If we can prove that $\Omega < 0$, $\Psi \geq 0$ and $\Phi_i \geq 0$ $(l = 1, \ldots, r)$ then $\dot{V}(\xi) < -\varsigma \|\xi\|^2$ holds for any sufficiently small $\varsigma > 0$, which ensures the asymptotic stability of system (6.1) in closed loop.

We notice that the inequality $\Omega < 0$ is a BMI due to the presence of the terms T_iK, T_iK_d and T_iB_uL for $i = 1, \ldots, 4$ which makes it difficult to determine the gains of the observer and the controller. To solve this problem and to linearize the previous condition, we have two possibilities: the first is to fix the matrices T_1, T_2, T_3, T_4 and as a result loose the degree of freedom presented by these matrices. Besides, this method is not systematic which is a major inconvenient. The second and the better technique consist in separating T_1, T_2, T_3, T_4 from K, K_d and L using a congruence transformation and Young's inequality. This method will be discussed in more details in the next part.

2. *Linearization of the BMI*:
In this part we will transform the BMI (6.55) into an LMI by using some matrices manipulation and Young's inequality (Appendix A). We start first by commuting T_{11} and the control matrix B_u in the term T_1B_uL. For this purpose, pre- and post multiply Ω by:

$$\begin{bmatrix} \begin{bmatrix} T_{11}^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 & 0 & 0 \\ (\star) & \begin{bmatrix} T_{11}^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 & 0 \\ (\star) & (\star) & \begin{bmatrix} T_{11}^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 \\ (\star) & (\star) & (\star) & \underbrace{\begin{bmatrix} \begin{bmatrix} T_{11}^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 & 0 \\ (\star) & \begin{bmatrix} T_{11}^{-1} & 0 \\ 0 & I \end{bmatrix} & 0 \\ (\star) & (\star) & \begin{bmatrix} T_{11}^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}}_{\bar{T}} \end{bmatrix}. \tag{6.74}$$

Notice that by defining

$$W_{11} = T_{11}^{-1}, \tag{6.75}$$

$$T_{12}K = R, \tag{6.76}$$

$$T_{12}K_d = R_d, \tag{6.77}$$

$$W_{11}L = S, \tag{6.78}$$

$$\begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} T_i \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \bar{T}_i, \quad i = 2, 3 \tag{6.79}$$

$$\bar{T}T_4\bar{T} = \bar{T}_4 \tag{6.80}$$

$$\begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} Q_{ij} \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \bar{Q}_{ij} \tag{6.81}$$

$$\begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} Z_{ij} \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \bar{Z}_{ij} \tag{6.82}$$

$$\begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} M_i \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \bar{M}_i \tag{6.83}$$

$$\begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} N_{ij} \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \bar{N}_{ij} \tag{6.84}$$

$$\begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} X_{ij} \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \bar{X}_{ij} \tag{6.85}$$

$$\begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} Y_{lij} \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \bar{Y}_{lij}, \quad i,j = 1,\dots,3, \quad l = 1,\dots,r, \tag{6.86}$$

and rewriting the resulting matrix, we get the following equation:

$$\bar{\Omega} = \bar{\Gamma} - \Pi - \Pi^T \tag{6.87}$$

where $\bar{\Gamma}$ is defined in (6.30) and the matrix Π is given as follows:

$$\Pi = \begin{bmatrix}
0 & 0 & (W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T)T_{21}^T W_{11} & 0 & (W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T)T_{31}^T W_{11} & 0 & (W_{11}\mathcal{A}(\varrho) - S^T B_u^T)T_{41}^T W_{11} \\
T_{11} S^T B_u^T & 0 & L^T B_u^T T_{21}^T W_{11} & (\mathcal{A}(\rho) - KC)^T T_{22}^T & L^T B_u^T T_{31}^T W_{11} & (\mathcal{A}(\rho) - KC)^T T_{32}^T & L^T B_u^T T_{41}^T W_{11} + (\mathcal{A}(\rho) - KC)^T T_{42}^T \\
0 & W_{11} A_d^T T_{12}^T & W_{11} A_d^T T_{21}^T W_{11} & W_{11} A_d^T T_{22}^T & W_{11} A_d^T T_{31}^T W_{11} & W_{11} A_d^T T_{32}^T & W_{11} A_d^T T_{41}^T W_{11} + W_{11} A_d^T T_{42}^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -W_{11} A_{d1}^T T_{12}^T & 0 & -W_{11} A_{d1}^T T_{22}^T & 0 & -W_{11} A_{d1}^T T_{32}^T & -W_{11} A_{d1}^T T_{42}^T \\
0 & 0 & 0 & (A_{d2} - K_d C)^T T_{22}^T & 0 & (A_{d2} - K_d C)^T T_{32}^T & (A_{d2} - K_d C)^T T_{42}^T
\end{bmatrix} \tag{6.88}$$

The matrix Π contains all the multiple product terms which prevent us from providing a solution to $\bar{\Omega} < 0$. To linearize this matrix, the cross terms will be separated and treated using Young's inequality. The problem is that one bounding is not enough to decouple all the elements, so we will have to perform this technique twice. We start first by treating the matrices T_i $(i = 2, \dots, 4)$ and we leave T_1 to the next step:

$$\bar{\Omega} = \tilde{\Gamma} - \underbrace{\begin{bmatrix} (W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T)T_{11} & 0 \\ L^T B_u^T T_{11} & (\mathcal{A}(\rho) - KC)^T \\ W_{11}A_d^T T_{11} & W_{11}A_d^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -W_{11}A_{d1}^T \\ 0 & (A_{d2} - K_d C)^T \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} 0 & 0 & \bar{T}_{21}^T & 0 & \bar{T}_{31}^T & 0 & \bar{T}_{41}^T & 0 \\ 0 & 0 & 0 & T_{22}^T & 0 & T_{32}^T & 0 & T_{42}^T \end{bmatrix}}_{Y}$$

$$- \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \bar{T}_{21} & 0 \\ 0 & T_{22} \\ \bar{T}_{31} & 0 \\ 0 & T_{32} \\ \bar{T}_{41} & 0 \\ 0 & T_{42} \end{bmatrix}}_{Y^T} \underbrace{\begin{bmatrix} T_{11}(\mathcal{A}(\varrho)W_{11} - B_u S) & T_{11}B_u L & T_{11}A_d W_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{A}(\rho) - KC & A_d W_{11} & 0 & 0 & -A_{d1}W_{11} & A_{d2} - K_d C \end{bmatrix}}_{X} \tag{6.89}$$

with $\tilde{\Gamma}$ contains the rest of the cross terms that can not be decoupled directly from the first time.

$$\tilde{\Gamma} = \bar{\Gamma} + \begin{bmatrix} 0 & 0 \\ T_{11} & 0 \\ 0 & -W_{11}A_d^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & W_{11}A_{d1}^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S^T B_u^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T_{12}^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} B_u S & 0 \\ 0 & T_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & T_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_d W_{11} & 0 & 0 & 0 & A_{d1}W_{11} & 0 \end{bmatrix} \tag{6.90}$$

Applying Young's inequality on (6.89) with a symmetric positive definite matrix Θ, we get:

$$\bar{\Omega} \leq \tilde{\Gamma} + \frac{1}{\epsilon} \begin{bmatrix} (W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T)T_{11} & 0 \\ L^T B_u^T T_{11} & (\mathcal{A}(\rho) - KC)^T \\ W_{11} A_d^T T_{11} & W_{11} A_d^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -W_{11} A_{d1}^T \\ 0 & (A_{d2} - K_d C)^T \end{bmatrix} \Theta$$

$$\begin{bmatrix} T_{11}(\mathcal{A}(\varrho)W_{11} - B_u S) & T_{11} B_u L & T_{11} A_d W_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{A}(\rho) - KC & A_d W_{11} & 0 & 0 & -A_{d1} W_{11} & A_{d2} - K_d C \end{bmatrix}$$

$$+ \epsilon \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \bar{T}_{21} & 0 \\ 0 & T_{22} \\ \bar{T}_{31} & 0 \\ 0 & T_{32} \\ \bar{T}_{41} & 0 \\ 0 & T_{42} \end{bmatrix} \Theta^{-1} \begin{bmatrix} 0 & 0 & \bar{T}_{21}^T & 0 & \bar{T}_{31}^T & 0 & \bar{T}_{41}^T & 0 \\ 0 & 0 & 0 & T_{22}^T & 0 & T_{32}^T & 0 & T_{42}^T \end{bmatrix} \quad (6.91)$$

Choosing:

$$\Theta = \begin{bmatrix} W_{11} & 0 \\ 0 & T_{12} \end{bmatrix}, \quad (6.92)$$

then the last inequality can be rewritten as follows:

$$\bar{\Omega} \leq \tilde{\Gamma} - \underbrace{\begin{bmatrix} W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T & 0 & 0 & 0 \\ L^T B_u^T & \mathcal{A}(\rho)^T T_{12} - C^T R^T & 0 & 0 \\ W_{11} A_d^T & W_{11} A_d^T T_{12} & \bar{T}_{21} & 0 \\ 0 & 0 & 0 & T_{22} \\ 0 & 0 & \bar{T}_{31} & 0 \\ 0 & 0 & 0 & T_{32} \\ 0 & -W_{11} A_{d1}^T T_{12} & \bar{T}_{41} & 0 \\ 0 & A_{d2}^T T_{12} - C^T R_d^T & 0 & T_{42} \end{bmatrix}}_{Q_2^T} \Bigg(\underbrace{\begin{bmatrix} -\epsilon\Theta & 0 \\ 0 & -\frac{1}{\epsilon}\Theta \end{bmatrix}}_{Q_3} \Bigg)^{-1}$$

$$\underbrace{\begin{bmatrix} \mathcal{A}(\varrho)W_{11} - B_u S & B_u L & A_d W_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & T_{12}\mathcal{A}(\rho) - RC & T_{12} A_d W_{11} & 0 & 0 & 0 & -T_{12} A_{d1} W_{11} & T_{12} A_{d2} - R_d C \\ 0 & 0 & \bar{T}_{21}^T & 0 & \bar{T}_{31}^T & 0 & \bar{T}_{41}^T & 0 \\ 0 & 0 & 0 & T_{22}^T & 0 & T_{32}^T & 0 & T_{42}^T \end{bmatrix}}_{Q_2}$$

(6.93)

then, using Schur lemma, we conclude that $\bar{\Omega} < 0$ is verified if $\Upsilon = \begin{bmatrix} \tilde{\Gamma} & Q_2^T \\ Q_2 & Q_3 \end{bmatrix} < 0$.

Hence

$$\Upsilon = \begin{bmatrix} \begin{matrix} & W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T & 0 & 0 & 0 \\ & L^T B_u^T & \mathcal{A}(\rho)^T T_{12} - C^T R^T & 0 & 0 \\ & W_{11} A_d^T & W_{11} A_d^T T_{12} & \bar{T}_{21} & 0 \\ \tilde{\Gamma} & 0 & 0 & 0 & T_{22} \\ & 0 & 0 & \bar{T}_{31} & 0 \\ & 0 & 0 & 0 & T_{32} \\ & 0 & -W_{11} A_{d1}^T T_{12} & \bar{T}_{41} & 0 \\ & 0 & A_{d2}^T T_{12} - C^T R_d^T & 0 & T_{42} \\ \hdashline & -\epsilon W_{11} & 0 & 0 & 0 \\ (\star) & 0 & -\epsilon T_{12} & 0 & 0 \\ & 0 & 0 & -\frac{1}{\epsilon} W_{11} & 0 \\ & 0 & 0 & 0 & -\frac{1}{\epsilon} T_{12} \end{matrix} \end{bmatrix} \tag{6.94}$$

From (6.90), we notice that Υ can be also decomposed as follows:

$$\Upsilon = \underbrace{\begin{bmatrix} \begin{matrix} & W_{11}\mathcal{A}(\varrho)^T - S^T B_u^T & 0 & 0 & 0 \\ & 0 & \mathcal{A}(\rho)^T T_{12} - C^T R^T & 0 & 0 \\ & W_{11} A_d^T & 0 & \bar{T}_{21} & 0 \\ \bar{\Gamma} & 0 & 0 & 0 & T_{22} \\ & 0 & 0 & \bar{T}_{31} & 0 \\ & 0 & 0 & 0 & T_{32} \\ & 0 & 0 & \bar{T}_{41} & 0 \\ & 0 & A_{d2}^T T_{12} - C^T R_d^T & 0 & T_{42} \\ \hdashline & -\epsilon W_{11} & 0 & 0 & 0 \\ (\star) & 0 & -\epsilon T_{12} & 0 & 0 \\ & 0 & 0 & -\frac{1}{\epsilon} W_{11} & 0 \\ & 0 & 0 & 0 & -\frac{1}{\epsilon} T_{12} \end{matrix} \end{bmatrix}}_{\bar{\Upsilon}} + X_1^T Y_1 + Y_1^T X_1 \tag{6.95}$$

with

$$X_1 = \begin{bmatrix} 0 & W_{11}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_d W_{11} & 0 & 0 & 0 & -A_{d1} W_{11} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.96}$$

$$Y_1 = \begin{bmatrix} S^T B_u^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S^T B_u^T & 0 & 0 & 0 \\ 0 & T_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{12} & 0 & 0 \end{bmatrix} \tag{6.97}$$

Then, by Young's inequality with a symmetric and positive definite matrix $\bar{\Theta}$, it is easy to show that: $\Upsilon \leq \bar{\Upsilon} + \frac{1}{\bar{\epsilon}} X_1^T \bar{\Theta} X_1 + \bar{\epsilon} Y_1^T \bar{\Theta}^{-1} Y_1$

Choosing $\bar{\Theta} = \begin{bmatrix} W_{11} & 0 \\ 0 & I_n \end{bmatrix}$, we can write:

$$\Upsilon \leq \bar{\Upsilon} - \begin{bmatrix} 0 & 0 & B_u S & 0 \\ I_n & 0 & 0 & T_{12} \\ 0 & W_{11}A_d^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -W_{11}A_{d1}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B_u S & 0 \\ 0 & 0 & 0 & T_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\bar{\epsilon}\bar{\Theta} & 0 \\ 0 & -\frac{1}{\bar{\epsilon}}\bar{\Theta} \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_d W_{11} & 0 & 0 & 0 & -A_{d1}W_{11} & 0 & 0 & 0 & 0 & 0 \\ S^T B_u^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S^T B_u^T & 0 & 0 & 0 \\ 0 & T_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{12} & 0 & 0 \end{bmatrix} \quad (6.98)$$

Then by applying Schur lemma on (6.98), we conclude that $\Upsilon < 0$ is verified if the inequality (6.25) is fulfilled. That completes the proof.

□

6.3.1 Comments on the approach

In this section, we provide some comments on the method and the choice of the nonlinearity.

The FWM method

As seen in the previous sections, the FWM method provides a solution for our controller problem by introducing some free matrices instead of eliminating an integral term in the derivative of the Lyapunov-Krasovskii functional. Nevertheless, computing all these decision variables is considered a source of conservatism but when dealing with time-delay system, there is always a trade-off between the complexity and precision.

The nonlinearity

The main idea in the design of the observer is that it does not require the exact value of the delay but its estimates over different intervals. For that reason, nonlinearities dependent on the delayed state was not used so that to avoid the presence of the delayed state in the dynamics of the observer.
Although the proposed approach imposes a specific form of the nonlinearity, yet it proposes a new sufficient condition in terms of LMI based on the existence of some free variables contrary to the existing methods which involve nonlinear conditions.

6.4 Conclusion

We presented in this chapter a new observer-based feedback for nonlinear unknown time-delay systems. In this method, the maximum delay interval is divided into r subintervals; then the mean value over each subinterval is assumed to be an estimate of the delay. The proposed observer depends on those estimates instead of the exact value of the delay. By using the FWM method with an appropriate Lyapunov-Krasovskii functional a delay-dependent stability of the error was studied and a sufficient condition in the form of LMI was provided.

Conclusion and Perspectives

In this thesis we have tackled the problem of observation and control for time delay systems. We tried to treat different types of systems (linear, nonlinear, singular) with different types of delays (constant, time varying, unknown). Different approaches were proposed to design state observers/observer-based controllers for nonlinear systems that improve the existing results in the literature. These methods reduce the conservatism of some of the techniques mentioned in Chapter 1, in the sense that the synthesis conditions are less restrictive and applicable to a larger class of nonlinear systems.

Different methods were presented depending on the studied class of delay systems. Chapter 2 was devoted to the design of observers for nonlinear systems with varying delay. The method consists in transforming the non-convex problem into a problem of stability of an LPV system, by reformulating the Lipschitz property. The Lyapunov-Krasovskii approach was used to deduce sufficient conditions ensuring the stability of the considered systems for the delay-independent and delay-dependent cases, respectively. The superiority of our methods were proved by comparing them with other classical methods already existing in the literature involving the use of the Lipschitz property directly in the dynamics of the error. Indeed, we demonstrated that with the reformulation of the Lipschitz property the resulting stability condition can tolerate nonlinearities with larger constants and delays with larger upper bounds.

Chapter 3 concentrated on designing a controller based on the observer already developed in Chapter 4 using the LPV approach. The non-convexity issue related to the presence of the controller was treated using some mathematical tools leading to less conservative conditions than the existing results. A comparison with two different methods were giving for continuous and discrete time respectively. The first provides an easy solution on the expense of imposing additional condition in form of equality. The second, consists in separating between the observer and the controller issues resulting in producing three LMI conditions needed to ensure the stability of the closed loop system.

Chapter 4 treated the problem of designing observers for singular nonlinear systems with constant delays and subject to disturbances. The main difficulty is to get rid of the derivatives of the disturbances in the error dynamics. The first suggested method involved the use of the classical $\mathcal{H}_\infty$ criterion associated with a Lyapunov-Krasovskii functional dependant of disturbances. Another less conventional approach based on a $\mathcal{W}^{1,2}$ criterion based on Sobolev norms, which can be looked at as an alternative solution to the $\mathcal{H}_\infty$ method when the derivative of the disturbance is difficult or impossible to avoid. These two methods were extended to time-varying delays. In addition, the $\mathcal{H}_\infty$ approach was also discussed in the discrete-time case.

Chapter 5 considered the problem of designing observer-based controllers for the aforementioned class of systems. The proposed method inspired by some works in the literature for linear singular systems but was adapted to take into consideration Lipschitz nonlinearities. A new Lyapunov-Krasovskii functional dependent on disturbances was used to retrieve a sufficient condition in terms of LMI.

Another important contribution were developed in Chapter 6, concerning the problem of observer-based controller design for unknown time-delay systems. The key idea is to replace the delay by its estimate. For that reason, the delay interval is divided into r subintervals and the estimate of the delay is considered as the mean on each segment. The stability of the estimated error is guaranteed by the use of Free Weighting Matrix (FWM) method which consists in injecting some free matrices in order to formulate a sufficient condition in terms of LMI.

Perspectives

As a future work on the presented results in this thesis, we can mention a few points that can be developed and improved:

- Considering different classes of systems such as neutral systems and/or systems with distributed delays [Han03], hybrid systems with delays [XWY03], fuzzy systems [CH06].
- All of the results were developed for a delay with zero lower bound $0 = \underline{d} \leq d \leq \bar{d}$. One of the possible perspectives is to study systems with delays in a range with $0 \leq \underline{d}$, namely delay-range analysis. This type of delay is already tackled in the literature, for instance, the work of [HG01], [HWLW07], [Sha08].
- Developing different controllers: dynamic output feedback control laws or sliding mode controllers.
- Trying to obtain more efficient and less restrictive conditions by using different Lyapunov Krasovskii functionals, changing the used bounding techniques, minimising the number of decision variables, and exploiting some existing techniques such as Finsler lemma [GP06].
- Applying our techniques on real applications and different physical systems [CL07], for example, consensus of multi-agent systems under communication delay [SDJ08].

APPENDIX

A

Fundamental elements

Contents

A.1 Schur Complement

Lemma A.1.1. *(Schur Complement) [BGFB94]*
Given the matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times m}$ and the matrix bloc $M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$, then the following representations are equivalents:

1. *M is negative-definite.*
2. *$R < 0$ and $Q - SR^{-1}S^T < 0$.*
3. *$Q < 0$ and $R - S^TQ^{-1}S < 0$.*

Lemma A.1.2. *(Schur complement for the non-strict inequalities) [BGFB94]*
Given the matrices $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{n\times m}$ and the matrix bloc $M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$, then the following representations are equivalents:

1. *M is negative semi-definite.*
2. *$R \leq 0$, $Q - SR^{-1}S^T \leq 0$ and $S(I - RR^{-1}) = 0$.*
3. *$Q \leq 0$, $R - S^TQ^{-1}S \leq 0$ and $(I - Q^{-1}Q)S = 0$.*

A.2 Useful inequalities

The following inequalities play an important role in the stability problem of time-delay systems.

Young's inequality: $\forall a, b \in \mathbb{R}^n$ and $\epsilon > 0,\ \forall R > 0$

$$2a^Tb \leq \epsilon a^TRa + \frac{1}{\epsilon}b^TR^{-1}b. \tag{A.1}$$

Park's inequality [PK99]: $\forall a, b \in \mathbb{R}^n$, $\forall R > 0$ and $\forall M \in \mathbb{R}^{n\times n}$

$$2a^Tb \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} R & RM \\ (\star) & (M^TR + I)R^{-1}(RM + I) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A.2}$$

Moons et al.'s inequality [MPKL01]: $\forall a \in \mathbb{R}^n$, $\forall b \in \mathbb{R}^m$, $\forall N \in \mathbb{R}^{n\times m}$, and for $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times m}$, and $Z \in \mathbb{R}^{m\times m}$, if $\begin{bmatrix} X & Y \\ (\star) & Z \end{bmatrix} \geq 0$, then

$$2a^TNb \leq \begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} X & Y - N \\ (\star) & Z \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A.3}$$

Jensen's inequality [GKC03]: For any constant matrix $M \in \mathbb{R}^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^m$ such that the integrations concerned are well defined, we have

$$\gamma \int_0^\gamma \omega^T(\beta)M\omega(\beta)\mathrm{d}\beta \geq \left(\int_0^\gamma \omega(\beta)\mathrm{d}\beta\right)^T M \left(\int_0^\gamma \omega(\beta)\mathrm{d}\beta\right) \tag{A.4}$$

A.3 Some elements on the convexity

In this section, we present a few definitions and properties on the convex sets, convex functions and the principle of convexity.

Definition A.3.1. *(convex set)*
A set E is said convex if:

$$\lambda x_1 + (1-\lambda)x_2 \in E \tag{A.5}$$

for all $x_1, x_2 \in E$ and for all $0 \leq \lambda \leq 1$.

Geometrically, this means that every segment between any two points belonging to a convex set is included in this set.

Definition A.3.2. *(convex function)*
A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said convex if:

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) \tag{A.6}$$

for all $x_1, x_2 \in \mathbb{R}^n$ and for all $0 \leq \lambda \leq 1$.

The function φ is strictly convex if and only if:

$$\varphi(\lambda x_1 + (1-\lambda)x_2) < \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2) \tag{A.7}$$

for all $x_1 \neq x_2$ and for all $0 < \lambda < 1$.

A.4 Differential Mean Value theorem

In this section, the differential mean value theorem (DMVT) approach for vector functions, which allows to write the dynamics of the estimation error as an LPV system, is presented.
Let

$$M_s = \{e_s(i) \mid e_s(i) = (0, ..., 0, 1, 0, ...0)^T, i = 1, ..., s\} \tag{A.8}$$

be the canonical basis of the vectorial space $\mathbb{R}^s$ for all $s \geq 1$. Let $f : \mathbb{R}^n \to \mathbb{R}^q$ be a vector function. Then we have $f(x) = [f_1(x), ..., f_q(x)]^T$ where f_i: $\mathbb{R}^n \to \mathbb{R}$ is the i^{th} component of f. The vectorial space $\mathbb{R}^q$ is generated by the canonical basis M_q, so we can write:

$$f(x) = \sum_{i=1}^{q} e_q(i) f_i(x) \tag{A.9}$$

Now we give the following theorem concerning DMVT for vector functions.

Theorem A.4.1. *Let f: $\mathbb{R}^n \to \mathbb{R}^q$ and $a, b \in \mathbb{R}^n$. We assume that f is differentiable on Co(a, b). Then, there exist constant vectors $c_1, ..., c_q \in$ Co$(a, b), c_i \neq a, c_i \neq b$ for $i = 1, ..., q$ such that:*

$$f(a) - f(b) = \sum_{i,j=1}^{q,n} e_q(i) e_{n(j)}^T \frac{\partial f_i}{\partial x_j}(c_i)(a-b) \tag{A.10}$$

Appendix B

Publication list

Peer-reviewed journal papers

- **Hassan, L.** and Zemouche, A. and Boutayeb, M. "$\mathcal{H}_\infty$ design for a class of nonlinear time-delay systems in descriptor form", *International Journal of Control*, 84:10, 1653-1663, 2011.
- **Hassan, L.** and Zemouche, A. and Boutayeb, M. "Robust unknown input observers for nonlinear time-delay systems", *SIAM: Journal on Control and Optimization* , 51:4, 2735-2752, 2013.
- **Hassan, L.** and Zemouche, A. and Boutayeb, M. "Robust observer and observer-based controller for time-delay singular systems", *Asian Journal of Control*, In press, 2013.
- **Hassan, L.** and Zemouche, A. and Boutayeb, M. "LMI Synthesis Method to Design Observer-based Controller for Nonlinear Time-Varying Delay Systems", *Systems & control letters*, In review, 2013.

International Conferences

- **Hassan, L.** and Zemouche, A. and Boutayeb, M. "$\mathcal{H}_\infty$ Unknown Input Observers Design for a Class of Nonlinear Time-Delay Systems", *18th IFAC World Congress*, Milan, Italy, 2011.

Nationales Conferences

- **Hassan, L.** and Zemouche, A. and Boutayeb, M. "Synthèse d'observateurs $\mathcal{H}_\infty$ pour une classe de systèmes non linéaires à retard à entrées inconnues", *Les Journées Doctorales MACS*, Marseille, France, 2011.

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Résumé

L'objectif de cette thèse est de développer des méthodes de synthèses d'observateurs et des contrôleurs basés sur un observateur pour les systèmes à retard. Différentes classes de systèmes ont été traitées avec différents types de retard. Trois méthodes ont été développées. La première méthode traite des systèmes non linéaires avec des non-linéarités lipschitziennes et consiste à transformer le système d'origine à un système LPV grâce à une reformulation de la propriété classique de Lipschitz. Cette technique est formulée pour les cas continu et discret, respectivement. Nous avons démontré, à travers des exemples numériques, que cette technique offre des conditions de synthèse moins restrictives par rapport aux résultats existants dans la littérature. La seconde méthode est développée pour une classe de systèmes singuliers avec des perturbations. La principale difficulté résidait dans la présence des dérivées des perturbations qui entravent l'analyse de la stabilité et pour laquelle deux approches ont été proposées: une approche $\mathcal{H}_\infty$ en utilisant une fonctionnelle de Lyapunov-Krasovskii spéciale dépendante des perturbations et une approche basée sur l'utilisation d'un critère de performance $\mathcal{W}^{1,2}$. La dernière méthode est basée sur l'utilisation des matrices de pondération libres pour résoudre le problème de contrôle des systèmes non-linéaires à retards inconnus. La solution proposée fournit une condition de synthèse LMI garantissant la stabilisation du système en boucle fermée malgré la présence du retard inconnu, au lieu d'une inégalité matricielle linéaire itérative ILMI trouvée habituellement dans la littérature.

Mots-clés: Systèmes à retard, observateurs non linéaires, contrôleurs basés sur un observateur, stabilité de Lyapunov-Krasovskii, stabilité de Lyapunov-Razumikhin, systèmes singuliers, théorème des accroissements finis (Differential Mean Value Theorem, pour le sigle anglais), systèmes LPV, LMIs.

Abstract

The objective of this dissertation is to develop observers and observer-based controllers synthesis methods for time-delay systems. Different classes of systems were treated with different types of delay. Three different methods were developed. The first one treats nonlinear systems with Lipschitz nonlinearities and consists in transforming the original system into an LPV system based on a reformulation of the classical Lipschitz property. This technique was formulated for continuous and discrete cases respectively and it was proven to provide less restrictive synthesis conditions when compared to the existing results in the literature. The second method deals with singular systems with disturbances. The main difficulty lay in the presence of the derivatives of the disturbances which hinder the stability analysis and for which two approaches are proposed: a $\mathcal{H}_\infty$ criterion combined with a special Lyapunov-Krasovskii functional depending on disturbances and a $\mathcal{W}^{1,2}$ criterion based on the use of Sobolev norms. The last method is based on the Free Weighting Matrices technique to solve the observation and control problems of a class of nonlinear systems with unknown delays. The proposed solution provides a sufficient LMI synthesis condition ensuring the asymptotic stabilization of the closed loop system, instead of the iterative LMI condition usually found in the literature.

Keywords: Time-delay systems, nonlinear observers, observer-based controllers, Lyapunov-Krasovskii stability, Lyapunov-Razumikhin stability, singular systems, Differential Mean Value Theorem (DMVT), LPV systems, LMIs.